INTEGRABILITY AND L^1-CONVERGENCE OF COSINE TRIGONOMETRIC SERIES WITH SPECIAL COEFFICIENTS

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CERTIFICATE

Certified that the dissertation entitled, **"Integrability and L^1-Convergence of Cosine trigonometric Series with special coefficients"**, which is being submitted by **Kiran Bala** (Roll No. 301103007), in the fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of **Dr. Jatinderdeep Kaur** during the period from January 2013 to July 2013.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University/Institute for the award of any degree.

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ABSTRACT

The present dissertation entitled, **"Integrability and L^1-Convergence of Cosine trigonometric Series with special coefficients",** contains a brief account of investigations carried out by me on L^1–convergence of trigonometric cosine series under the supervision of **Dr. Jatinderdeep Kaur, Assistant Professor**, School of Mathematics and Computer Applications, Thapar University, Patiala.

The work presented in this dissertation has been divided into three chapters. The first chapter is introductory. In this chapter, apart from setting up the notations and terminology to be used in sequel, we have presented some well-known results interrelated to our results along with a brief plan of our results presented in the subsequent chapters.

The purpose of chapter II is to study the integrability and L^1-convergence of cosine trigonometric series under the Class ***S*** of Sidon and Telyakovskii.

In chapter III, I have studied the results concerning the L^1–convergence of modified cosine sums introduced by C.S. Rees and C.V. Stanojevic by using Cesaro means of integral and non-integral orders and obtained L^1–convergence Of Fourier cosine series as corollary.

Towards the end, references of various publications cited in the present dissertation have been reported.

Contents

CHAPTER I

INTRODUCTION

1.1 The present dissertation comprises certain results studied by the author "**Integrability and L^1-Convergence of Cosine Trigonometric Series with Special Coefficients**". It is well known that if a trigonometric series converges in L^1–metric to a function $f \in L^1(T)$, then it is the Fourier series of the function *f*. Riesz {[27], Vol. II, Ch VIII article 22} gave a counter example to show that converse of above result does not hold good in L^1-metric. This motivated various authors to study the L^1-convergence of the trigonometric series.

Integrability and L^1-convergence of trigonometric series have been studied by number of authors. The work on this topic was initiated in 1913 by W.H. Young [40] and A.N. Kolmogorov [1] in 1923 by taking the classes of convex sequences $\left(\Delta^2 a_n > 0\right)$ and quasi–convex sequences $\left(\sum_{n=1}^{\infty} n\left|\Delta^2 a_n\right| < \infty\right)$ respectively.

In 1973, S.A. Telijakovskii [35] yet studied another class S which was introduced by Sidon [37] in 1939 for L^1-convergence of trigonometric series. The results obtained by these authors were further generalized and extended by G.H. Hardy and J.E. Littlewood [14], T. Kano [39], J.W. Garrett and C.V. Stanojevic ([17], [18], [19]), B. Ram ([4], [6]), N. Singh and K.M. Sharma ([29], [30], [31]), R. Bojanic and C.V. Stanojevic [32], C.P. Chen [9], R. Bala and B. Ram [33], F. Moricz [13], S.S. Bhatia and B. Ram [38], Z. Tomovski ([41], [42], [43], [44]), N. Hooda and B. Ram [26], K. Kaur, S.S. Bhatia and B. Ram ([21], [22]), J. Kaur and S.S. Bhatia ([15], [16]) and others by considering various generalizations of classes of sequences mentioned above for one-dimensional trigonometric cosine and sine series.

During investigation, I found that many authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that these sums converge in L^1- metric to the sum of trigonometric series whereas the classical series itself may not. In this concern, various authors like C.S. Rees and C.V. Stanojevic [11], C.P. Chen [10], S. Kumari and B. Ram [36], B. Ram and S. Kumari [7], N. Hooda and B. Ram [26], K. Kaur, S.S. Bhatia and B. Ram [22], J. Kaur and S.S. Bhatia ([15], [16]) have introduced various new modified trigonometric cosine and sine sums and have studied their L^1-convergence under different classes of coefficient sequences.

To provide sufficient background for later chapters, a summary of basic definitions and notations, known results and a brief chapter wise resume of the results contained in the dissertation has been given in this introductory chapter. However, some of the definitions will be repeated occasionally in chapters for the sake of convenience.

1.2 DEFINITIONS AND NOTATIONS

Let $\{a_n\}$ be a sequence. Then we write

$$\Delta a_n = a_n - a_{n+1}$$

$$\Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}$$

Abel's transformation ([27], Vol. I, p.1). If $a_0, a_1, \ldots, a_n, v_1, v_2, \ldots, v_n \ldots$ are any real numbers and assume that

$$V_n = v_0 + v_1 + \ldots\ldots + v_n,$$

Then for any values of *m* and *n*, we find that

$$\sum_{k=m}^{n} a_k v_k = \sum_{k=m}^{n-1} \Delta a_k V_k + a_n V_n - a_m V_{m-1}$$

is called Abel's transformation, where $\Delta a_k = a_k - a_{k+1}$.

If $m = 0$ and $V_{-1} = 0$, then

$$\sum_{k=0}^{n} a_k v_k = \sum_{k=0}^{n-1} \Delta a_k V_k + a_n V_n.$$

Trigonometric Series: A series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called one dimensional trigonometric series, where $a_0, a_1, a_2, \ldots, b_1, b_2, b_3, \ldots,$ are the coefficients (may be real or complex).

Fourier Series: A Fourier series may be defined as an expansion of a periodic and intergrable function $f(x)$ over interval $(-\pi, \pi)$ in series of sines and cosines as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad \text{for all } n = 1,2,3,\ldots$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad \text{for all } n = 1,2,3,\ldots.$$

Convex sequence. A sequence $\{a_n\}$ is said to be convex if $\Delta^2 a_n \geq 0$.

Quasi-Convex sequence ([27], Vol. II, p.202). A sequence $\{a_n\}$ is said to be quasi-convex if $\sum_{n=1}^{\infty} n\left|\Delta^2 a_n\right| < \infty$.

Semi-Convex sequence [39]. A null sequence $\{a_n\}$ is said to be semi–convex if

$$\sum_{n=1}^{\infty} n\left|\Delta^2 a_{n-1} + \Delta^2 a_n\right| < \infty \ , \qquad (a_0 = 0).$$

Class of Bounded Variation [27]. A sequence $\{a_n\}$ belongs to the class of Bounded Variation (BV) if $\sum_{n=1}^{\infty} \left|\Delta\, a_n\right| < \infty$.

Sequence of bounded variation of order $m \geq 1$ [20]. A null sequence $\{a_n\}$ is said to be of bounded variation $(BV)^m$ of order $m \geq 1$, if $\sum_{n=1}^{\infty} \left|\Delta^{m} a_n\right| < \infty$,

where

$$\Delta^m a_n = \Delta\left(\Delta^{m-1} a_n\right) = \Delta^{m-1} a_n - \Delta^{m-1} a_{n+1} \ .$$

Clearly for $m = 1$, the class $(BV)^m$ reduces to the class BV.

Also $\qquad (BV)^m \subset (BV)^{m+1}; \qquad$ for $m = 1,2,3,...$

but the converse is not true.

<u>*Example:*</u> Let $\Delta a_n = \frac{1}{n}$ and $\Delta^2 a_n = \frac{1}{n(n+1)}$

$$\sum_{n=1}^{\infty}\left|\Delta^2 a_n\right| = \sum_{n=1}^{\infty}\left|\frac{1}{n(n+1)}\right| < \infty \quad \text{(by p-series test)}$$

However, the series $\sum_{n=1}^{\infty}\left|\Delta a_n\right| = \sum_{n=1}^{\infty}\left|\frac{1}{n}\right|$ is not convergent.

Therefore, the converse of $(BV)^m \subset (BV)^{m+1}$ is not true.

Class S of coefficient sequence ([35], [37]). A null sequence $\{a_n\}$ belongs to class S, if there exists a sequence $\{A_n\}$ such that

(i) $A_n \downarrow 0 \quad as \quad n \to \infty,$

(ii) $\sum_{n=0}^{\infty} A_n < \infty$,

(iii) $|\Delta a_n| \le A_n$, for all n.

The class S is usually called as Sidon-Teljakovskii class. Since, firstly, Teljakovskii expressed Sidon's conditions [37] in a succinct equivalent form and secondly, he showed that the class S is also a class of L^1-convergence.

Obviously, $S \subset BV$. Further letting $A_n = \sum_{k=\mathrm{n}}^{\infty} |\Delta^2 a_k|$; we observe that every quasi-convex null sequence satisfies the class S.

In 1978, Singh and Sharma [29] introduced new class S' in the following way:

Class S' [29]. A sequence $\{a_n\}$ of numbers is said to belongs to class S', if $a_n \to 0 \quad as \quad n \to \infty$ and there exists a sequence $\{A_n\}$ such that $\{A_n\}$ is quasi-monotone, $\sum_{n=0}^{\infty} A_n < \infty$ and $|\Delta a_n| \le A_n$, for all n.

However, In 2000, Leindler [23] proved that the class S and class S' are equivalent.

In 1980, Garrett, Rees and Stanojevic [20] gave an equivalent definition of class S in the following way:

Class S^2 of coefficient sequence [20]. A null sequence $\{a_n\}$ belongs to the class S^2 if there exists a null sequence $\{A_n\}$ of non-negative numbers such that $\sum_{\mathrm{n}=1}^{\infty} n|\Delta\, A_n| < \infty$ and $|\Delta a_n| \le A_n$ for all n.

Class C **[18].** A null sequence $\{a_n\}$ belongs to the class C if for every $\in > 0$, there exists a $\delta(\in) > 0$, independent of *n* and such that for all $n \geq 0$,

$$\int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \in .$$

Class R **[39].** A null sequence $\{a_n\}$ belongs to the class R, if $\sum_{n=1}^{\infty} n^2 \left| \Delta^2 \left(\frac{a_n}{n} \right) \right| < \infty$.

O-o Relation. Let u_n and v_n be sequence of real numbers. Then u_n be of order v_n

i.e, $u_n = o(v_n)$ *if* $\lim_{n \to \infty} \frac{u_n}{v_n} = 0$ and if $\frac{u_n}{v_n}$ is bounded, then $u_n = O(v_n)$.

Dirichlet kernel ([27], Vol. I, p.85). Let

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + \dots\dots\dots + \cos nx$$

Moreover,

$$2\sin\frac{x}{2} D_n(x) = \sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + 2\sin\frac{x}{2}\cos 2x + \dots\dots\dots + 2\sin\frac{x}{2}\cos nx$$

$$= \sin\left(n + \frac{1}{2}\right)x$$

Hence

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

This expression is known as Dirichlet's kernel. Moreover,

$$\widetilde{D}_n(x) = \sin x + \sin 2x + \dots\dots\dots\dots + \sin nx$$

$$= \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

is called the kernel conjugate to Dirichlet kernel.

If $x \neq 0 (\bmod 2\pi)$, then

$$\left|D_n(x)\right| \leq \frac{\pi}{2x} \;, \qquad \text{for} \quad 0 < |x| \leq \pi$$

and

$$\left|\widetilde{D}_n(x)\right| \leq \frac{\pi}{x} \;, \qquad \text{for} \quad 0 < |x| \leq \pi \,.$$

Fejer kernel ([3], [27]). The Fejer kernel $K_n(x)$ is defined as

$$K_n(x) \;=\; \frac{1}{n+1}\sum_{j=0}^{n} D_j(x)$$

$$= \frac{1}{n+1}\sum_{j=0}^{n} \frac{\sin\left(j+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

Using $|D_n(x)| \leq n+1$, it follows that $|K_n(x)| \leq n+1$.

It has the properties:

(i) $K_n(x) \geq 0 \quad \mathit{for}\ 0 < x < \pi,$

(ii) $\frac{1}{\pi}\int_{-\pi}^{\pi} |K_n(x)|\,dx = 1.$

The Conjugate Fejer kernel is defined as

$$\widetilde{K}_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} \widetilde{D}_j(x)$$

We have

$$\widetilde{K}_n(x) > 0 \quad \text{for} \quad 0 < x < \pi, \quad n = 1,2,3,\ldots$$

And

$$\left|\widetilde{K}_n(x)\right| < \frac{1}{2}n.$$

Cesaro Means. Let a sequence $S_0, S_1, S_2, \ldots,$ we define for every $\alpha = 0,1,2,\ldots$, the sequence $S_0^\alpha, S_1^\alpha, S_2^\alpha, \ldots,$ by the condition:

$$S_n^0 = S_n$$

$$S_n^\alpha = S_0^{\alpha-1} + S_1^{\alpha-1} + S_2^{\alpha-1} + \ldots + S_n^{\alpha-1} \qquad (\alpha = 1,2,3,\ldots,\ n = 0,1,2,\ldots).$$

Similarly, for $\alpha = 0,1,2,\ldots$, we define the sequence of numbers $A_0^\alpha, A_1^\alpha, A_2^\alpha, \ldots,$ by the conditions

$$A_n^0 = 1,$$

$$A_n^\alpha = A_0^{\alpha-1} + A_1^{\alpha-1} + A_2^{\alpha-1} + \ldots + A_n^{\alpha-1} \qquad (\alpha = 1,2,3,\ldots,\ n = 0,1,2,\ldots).$$

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series. For any real number α the Cesaro sums of order α of $\sum_{n=0}^{\infty} a_n$ are defined by

$$S_n^\alpha(a_p) = S_n^\alpha = \sum_{p=0}^{n} A_{n-p}^\alpha a_p = \sum_{p=0}^{n} A_{n-p}^{\alpha-1} S_p$$

where $S_n = S_n^0 = a_0 + a_1 + \ldots + a_n$ and A_p^α denotes the binomial coefficients.

And $S_0, S_1, S_2, \ldots,$ are defined by

$$\sum_{p=0}^{\infty} S_p^\alpha x^p = (1-x)^{-\alpha} \sum_{p=0}^{\infty} S_p x^p$$

Also

$$A_n^\alpha = \sum_{p=0}^{\infty} A_p^{\alpha-1}, \qquad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$$

$$A_n^\alpha = \binom{n+\alpha}{\Gamma(\alpha+1)} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)} \qquad (\alpha \neq -1,-2,-3,\ldots).$$

The Cesaro means T_n^α of order α of the series $\sum_{n=0}^{\infty} a_n$ be defined as

$$T_n^\alpha = S_n^\alpha / A_n^\alpha .$$

1.3 The following results about the behavior of cosine and sine series read as:

Theorem 1.1 ([1], [27], [40]). *If $\{a_n\}$ is a quasi-convex null sequence, then*

(1.3.1) $$f(x) = \sum_{n=1}^{\infty} a_n \cos nx \in L^1[0,\pi]$$

Theorem 1.2 ([27], [34]). *If $\{a_n\}$ is a quasi-convex null sequence, then*

(1.3.2) $$\sum_{n=1}^{\infty} a_n \sin nx$$

is a Fourier series if and only if $\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$.

Kano [39] generalized Theorem 1.1 and Theorem 1.2 in 1968 as follows:

Theorem 1.3 [39]. *If $\{a_n\}$ is a null sequence and $\sum_{n=1}^{\infty} n^2 \left| \Delta^2 \left(\frac{a_n}{n} \right) \right| < \infty$, then (1.3.1) and (1.3.2) are the Fourier series, or equivalently they represent integrable functions.*

Concerning the integrability of trigonometric series belonging to the class S ([35], [37] (introduced already in article 1.2)), Teljakovaskii [35] established the following theorem:

Theorem 1.4. *Let the cosine series* $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ *belongs to the class S ([35], [37]). Then this is a Fourier series and the following relation holds:*

$$\int_0^{\pi}\left|\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx\right|dx \le C\sum_{n=1}^{\infty}A_n\ ,$$

where C is an absolute constant.

In 1979, Ram [5] showed that the condition S is sufficient for the integrability of Rees-Stanojevic [11] sums

(1.3.3) $$g_n(x)=\frac{1}{2}\sum_{k=0}^{n}\Delta a_k+\sum_{k=1}^{n}\left(\sum_{j=k}^{n}\Delta a_j\right)\cos kx$$

and proved the following theorems:

Theorem 1.5 [6]. *Let the sequence $\{a_n\}$ satisfy the condition S. Then $g(x)=\lim_{n\to\infty}g_n(x)$ exists for $x\in(0,\pi]$ and $\int_0^{\pi}|g(x)|dx \le C\sum_{n=0}^{\infty}A_n$, where C is absolute constant.*

Theorem 1.6 [6]. *Let $\{a_n\}$ be a sequence satisfying the condition S. Then*

$$\frac{1}{x}\sum_{k=1}^{\infty}\Delta a_k\sin\left(k+\frac{1}{2}\right)x=\frac{h(x)}{x} \text{ converges for } x\in(0,\pi] \text{ and } \frac{h(x)}{x}\in L^1[0,\pi].$$

The above theorems were further generalized by Ram [6], under a weaker condition where the monotonicity of the sequence in the definition of the class S is replaced by quasi-monotonicity. But in 2000, Leinder [23] proved that the class S and class S' are equivalent.

Consider the cosine series

(1.3.4) $$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

Let the partial sums of (1.3.4) is denoted by $S_n(x)$ and $f(x)=\lim_{n\to\infty}S_n(x)$. Denote the class of sequence of Fourier coefficients $\{a_n\}$ by **F**. There are subclasses of **F** for which $a_n \log n = o(1), \quad n \to \infty$ is a necessary and sufficient condition for $\|S_n - f\| = o(1), n \to \infty$.

A subclass **G** of **F** is called a class of L^1-convergence if $\|S_n - f\| = o(1), n \to \infty$ if and only if $a_n \log n = o(1), \quad n \to \infty$.

Concerning the L^1-convergence of the cosine series, we have the following classical result of Kolmogorov [1].

Theorem 1.7 [1]. *If $a_n \downarrow 0$ and $\{a_n\}$ is convex or quasi-convex, then for the convergence of the series (1.3.4) in the metric space L^1, it is necessary and sufficient that $a_n \log n = o(1), \quad n \to \infty$.*

The case, in which the sequence $\{a_n\}$ is convex, of Theorem 1.7 was established by Young [40].

Generalizing the above classical result, Teljakovskii [35] proved the following result:

Theorem 1.8. *If the coefficient sequence $\{a_n\}$ of the cosine series (1.3.4) belongs to the class S, Then a necessary and sufficient condition for L^1-convergence of (1.3.4) is $a_n \log n = o(1), \quad n \to \infty$.*

Rees and Stanojevic [11] introduced modified cosine sums (1.3.3) and obtained an analogue of Theorem 1.7 for these sums. These modified cosine sums approximate their limits better than the classical cosine series as they converge in L^1-metric to the sum of the cosine series whereas the classical cosine series itself may not. They proved the following result:

Theorem 1.9. *Let f be the sum of the cosine series (1.3.4). Then $g_n(x)$ converges to f in L^1- metric if and only if $\{a_n\}$ belongs to the class C .*

Further, In 1977, Ram [4] proved the following result on L^1-convergence of Rees-Stanojevic sums (1.3.3).

Theorem 1.10. If *(1.3.4) belongs to class S. Then* $\|f-g_n\|_{L^1}=o(1),\ n\to\infty$.

In 1978, Singh and Sharma [29] proved the theorem by replacing the monotonicity of sequence $\{A_n\}$ in the definition of class S by quasi-monotonocity of $\{A_n\}$. Their result reads as:

Theorem 1.11. *Let $a_n \in S'$, then $f_n(x)$ converges to $f(x)$ in L^1-metric.*

Further, Ram and Kumari [7] introduced new modified cosine and sine sums as

$$(1.3.5)\qquad f_n(x) = \frac{a_0}{2}+\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\cos kx$$

and

$$(1.3.6)\qquad g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\sin kx$$

and studied their L^1-convergence under the condition that the cosine series and sine series belongs to the classes R and S. They also deduced the results about L^1-convergence of cosine and sine series. Their results state as below:

Theorem 1.12. Let $\{a_n\}$ belongs to the class S. If $\lim_{n\to\infty}|a_{n+1}|\log n=0,$ then

$$\|f-f_n\|_{L^1}=o(1),\quad n\to\infty .$$

Theorem 1.13. *Let $\{a_n\}$ belongs to the class **R**. If $t_n(x)$ represents $f_n(x)$ and $g_n(x)$ then*

$$\|f-t_n\|_{L^1}=o(1),\quad n\to\infty .$$

Later, N. Hooda and B. Ram [26] have proved the following theorem:

Theorem 1.14. *Let the sequence* $\{a_n\}$ *belongs to class* S' *. Then*

$$\|f - g_n\|_{L^1} = o(1), \quad n \to \infty .$$

Here, we discuss brief summary of chapter II and chapter III.

The aim of chapter II, is to study the integrability and L^1-convergence of cosine series of under class S of L^1-convergence.

In chapter III, we have studied the L^1-convergence of modified cosine sums with generalized quasi-convex coefficients by using Cesaro means of integral and non-integral order and obtained the necessary and sufficient conditions for the L^1-convergence of cosine trigonometric series.

CHAPTER II

INTEGRABILITY AND L^1–CONVERGENCE OF TRIGONOMETRIC COSINE SERIES

2.1 Introduction

The problem of L^1–convergence, via Fourier coefficients, consists of finding the properties of Fourier coefficients such that the cosine series

(2.1.1) $$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$$

is a Fourier series of some $f \in L^1(0,\pi)$ and $\|S_n - f\| = o(1),\ n \to \infty$ if and only if $a_n \log n = o(1),\ n \to \infty$. Here, $S_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n} a_k \cos kx$, denotes the n-th partial sum of the series (2.1.1) and $\|.\|$ is the L^1 – norm.

Integrability and L^1 – convergence of trigonometric series have been studied by number of authors. The work on this topic was initiated in 1913 by W.H. Young [40] and A.N. Kolmogorov [1] in 1923 by taking the classes of convex sequences and quasi–convex sequences defined as follows:

Definition: *A sequence $\{a_n\}$ is said to be convex if $\Delta^2 a_n \geq 0$.*

Definition: *The sequence $\{a_n\}$ of real numbers is said to be quasi-convex if* $\sum_{\mathrm{n}=1}^{\infty}(\mathrm{n}+1)\left|\Delta^2 a_n\right| < \infty$, *where* $\Delta^2 a_n = \Delta(\Delta a_n) = \Delta a_n - \Delta a_{n+1} = a_n - 2a_{n+1} + a_{n+2}$.

A classical result concerning the integrability and L^1 – convergence of a series (2.1.1) is the following well-known theorem of Kolmogorov [12].

Theorem 2.1 [1]. *If $\{a_n\}$ is a quasi-convex null-sequence, then the series (2.1.1) is the Fourier series of some $f \in L^1(0,\pi)$ and $\|S_n - f\| = o(1), \quad n \to \infty$ if and only if $a_n \log n = o(1), \quad n \to \infty$ holds.*

In 1973, S.A. Telyakovskii [35] studied another class S which was introduced by Sidon [37] in 1939 for L^1-convergence of trigonometric series, which is defined as:

Definition: *A null-sequence $\{a_n\}$ belongs to the class S if there exists a monotonically decreasing null sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta a_n| \leq A_n$ for all n.*

Telyakovskii [35] proved the following result concerning the L^1-convergence of Fourier series.

Theorem 2.2 [35]. *Let $\{a_n\} \in S$. Then the series (2.1.1) is the Fourier series of some $f \in L^1(0,\pi)$ and $\|S_n - f\| = o(1), n \to \infty$ if and only if $a_n \log n = o(1), n \to \infty$ holds.*

Further, in 1933, C.N. Moore [8] generalized quasi-convexity of null-sequences $\{a_n\}$ in the following way:

Definition: *A null sequence $\{a_n\}$ belongs to class of generalized quasi-convexity if*

(2.1.2) $\sum_{n=1}^{\infty} n^k \left|\Delta^{k+1} a_n\right| < \infty$, *for some $k > 0$ holds.*

<u>*Remark:*</u> for $k = 1$; this class reduces to class of quasi-convexity.

The aim of this chapter is to study the Integrability and L^1-convergence of cosine trigonometric series under the coefficient of generalized quasi-convexity.

2.2 Lemmas: The following lemmas will be required in the proof of main theorem:

Lemma 2.2.1 [24]. If $\{a_n\}$ is a null-sequence satisfying the condition (2.1.2), then

(2.2.1) $\sum_{n=1}^{\infty} n^r \left|\Delta^{r+1} a_n\right| < \infty$, for $0 \le r < k$, holds.

Lemma 2.2.2. If $\{a_n\}$ is a null sequence, then the series

(2.2.2) $$\Delta^k a_n = \sum_{m=0}^{\infty} \binom{m+\alpha}{m} a_{n+m} \qquad (n = 0,1,2,\ldots \text{ and for } k \ge 0)$$

where

$$\binom{m+\alpha}{m} = \frac{(1+\alpha)\ldots(m+\alpha)}{m!}$$

is convergent and $\lim_{n\to\infty} \Delta^k a_n = 0$ holds.

Moreover, for $\alpha > -1$, we have

$$\binom{\alpha+n}{\alpha} = \frac{(\alpha+1)(\alpha+2)\ldots.(\alpha+n)}{n!} > 0,$$

and for $0 < \alpha \le 1$,

$$\binom{\alpha+n}{\alpha} = \frac{n^{\alpha}}{\Gamma(\alpha+1)} + O(1),$$

and for any real no. α *and* $n \in \mathrm{N}$,

$$\sum_{i=0}^{n} \binom{i+\alpha}{\alpha} = \binom{\alpha+n+1}{n}.$$

2.3 Main Result

The main result of this chapter reads as:

Theorem 2.3.1 [31]. *Let k be a real number such that $k > 0$. If*

(2.3.1) $$\lim_{n \to \infty} a_n = 0,$$

and

(2.3.2) $$\sum_{n=1}^{\infty} n^k \left|\Delta^{k+1} a_n\right| < \infty ,$$

holds, then the series (2.1.1) is the Fourier series of some function $f \in L^1(0,\pi)$ and $\|S_n - f\| = o(1),\ n \to \infty$ if and only if $a_n \log n = o(1),\ n \to \infty$.

Proof: The proof of main result is based on Theorem 2.2. We need to show that the conditions (2.3.1) and (2.3.2) of Theorem 2.3.1 implies the condition *S*. Firstly, we suppose that for some k, $0 < k \le 1$, the series in (2.3.2) converges.

For this, we consider, $0 < k \le 1$; and we construct the sequence

$$A_n = \sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} \left|\Delta^{k+1} a_i\right|$$

Then, by using the properties for binomial coefficients and lemma (2.2.2), we have

$$\sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} \left|\Delta^{k+1} a_i\right|$$

$$= \sum_{i=0}^{\infty} \left|\Delta^{k+1} a_i\right| \sum_{n=0}^{i} \binom{i-n+k-1}{i-n}$$

$$= \sum_{i=0}^{\infty} \left|\Delta^{k+1} a_i\right| \sum_{n=0}^{i} \binom{n+k-1}{n}$$

$$= \sum_{i=0}^{\infty} \binom{i+k}{k} \left|\Delta^{k+1} a_i\right|$$

$$= \frac{1}{\Gamma(k+1)} \sum_{i=0}^{\infty} i^k \left|\Delta^{k+1} a_i\right| + O\left(\sum_{i=0}^{\infty} \left|\Delta^{k+1} a_i\right|\right)$$

Again, by using lemma (2.2.1) we get

$$\left(\sum_{i=0}^{\infty} \left|\Delta^{k+1} a_i\right|\right) = \left|\Delta^{k+1} a_0\right| + \left(\sum_{i=1}^{\infty} \left|\Delta^{k+1} a_i\right|\right)$$

$$\leq \sum_{m=0}^{\infty} \binom{m-k-2}{m} a_m + \sum_{i=1}^{\infty} i^k \left|\Delta^{k+1} a_i\right| < \infty .$$

Therefore, $\sum_{n=0}^{\infty} A_n < \infty$ and $A_n \downarrow 0$.

Also, we note that

$$\Delta a_n = \sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} \Delta^{k+1} a_i ,$$

and

$$\left|\Delta a_n\right| \leq \sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} \left|\Delta^{k+1} a_i\right| = A_n , \quad \text{for all } n.$$

Further, thus all the conditions of class S holds.

for $k > 1$, by using (2.2.1), put $r = 1 < k$, we get

$$\sum_{n=1}^{\infty} n\left|\Delta^2 a_n\right| < \infty \,, \quad \text{i.e.} \quad \{a_n\} \in S \,.$$

Thus, the conditions of Theorem 2.2 holds.

Therefore, the series (2.1.1) is the Fourier series and $\|S_n - f\| = o(1), \ n \to \infty \ \Leftrightarrow$ $a_n \log n = o(1), \ n \to \infty$.

CHAPTER III

L^1–CONVERGENCE OF MODIFIED COSINE SUMS WITH GENERALIZED QUASICONVEX COEFFICIENTS

3.1 Introduction

Let

(3.1.1) $$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$$

be the cosine series and $S_n(x)$ be the nth partial sums of the series (3.1.1).

Let $$g(x) = \lim_{n\to\infty} S_n(x).$$

In 1973, C.S. Rees and C.V.Stanojevic [11] introduced modified cosine sums as:

(3.1.2) $$g_n(x) = \frac{1}{2}\sum_{i=0}^{n}\Delta a_i+\sum_{i=1}^{n}\left(\sum_{j=i}^{n}\Delta a_j\right)\cos ix$$

and studied the necessary and sufficient conditions for integrability of trigonometric cosine series.

Further in 1976, J.W. Garrett and C.V. Stanojevic [19] proved the necessary and sufficient conditions for L^1 –convergence of trigonometric cosine series as:

Theorem A. *Let $\{a_n\}$ be a null-sequence of bounded variation. Then the sequence of modified cosine sums*

$$g_n(t) = S_n(t)-a_{n+1}D_n(t),$$

where $D_n(x)$ *is the Dirichlet kernel, converges in* $L(0,\pi)-$ *norm to* g, *the pointwise sum of the cosine series, if and only if for every* $\in > 0$, *there exists* $\partial(\in) > 0$, *independent of n, such that*

(3.1.3) $$\int_0^{\partial}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(t)\right|dt < \in, \text{ for every } n.$$

In 1923, A.N. Kolmogorov [1] have proved the following result concering the L^1-convergence of trigonometric cosine series:

Theorem B. *Let* $\{a_n\}$ *be a null sequence of bounded variation satisfying the condition (3.1.3). Then the cosine series is the Fourier series of its sum g and* $\|S_n(x)-g\| = o(1)$, $n \to \infty$ *is equivalent to* $a_n \log n = o(1), \quad n \to \infty$.

we note that Theorem A is the corollary to Theorem B. Thus, Theorem A contains special case of classical results.

In 1923, A.N. Kolmogorov introduced quasi-convex sequences and proved well-known result.

Definition: *The sequence* $\{a_n\}$ *of real numbers is said to be quasi-convex if*

$$\sum_{\mathrm{n}=1}^{\infty}(\mathrm{n}+1)\left|\Delta^2 a_n\right| < \infty, \text{ where } \Delta^2 a_n = \Delta(\Delta a_n) = \Delta a_n - \Delta a_{n+1} = a_n - 2a_{n+1} + a_{n+2}.$$

Theorem C. *If* $\{a_n\}$ *is a quasi-convex null sequence, then for the convergence of the series (3.1.1) in the metric space* L^1 *it is necessary and sufficient that* $\lim\limits_{n\to\infty} a_n \log n = 0$.

In 1933, C. N. Moore [8] generalized quasiconvexity of null sequences in the following manner:

(3.1.4) $$\sum_{n=1}^{\infty} n^k\left|\Delta^{k+1}a_n\right| < \infty, \qquad \text{for } k > 0,$$

where the order of differences is fractional.

L.S. Bosanquet [24] showed that if $\{a_n\}$ is a null sequence satisfying the condition (3.1.4), then

(3.1.5) $$\sum_{n=1}^{\infty} n^r \left|\Delta^{r+1} a_n\right| < \infty \text{ for } 0 \le r < k.$$

For $r = 0$; (3.1.5) implies bounded variation.

The aim of this chapter is to study the L^1-convergence of modified cosine sums with generalized quasi-convex coefficients. Before proving the main result, we first define some notations and formulae.

3.2 Notations and Formulae:

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series. For any real number k the Cesaro sums of order k of $\sum_{n=0}^{\infty} a_n$ are defined by

(3.2.1) $$S_n^k(a_p) = S_n^k = \sum_{p=0}^{n} A_{n-p}^k a_p = \sum_{p=0}^{n} A_{n-p}^{k-1} S_p$$

where $S_n = S_n^0 = a_0 + a_1 + \ldots + a_n$ and A_p^k denotes the binomial coefficients.

The Cesaro means T_n^k of order k of the series $\sum_{n=0}^{\infty} a_n$ be defined as

(3.2.2) $$T_n^k = S_n^k / A_n^k .$$

The following formulae will be required in the proof of main result:

(3.2.3) $$S_n^k\left(S_p^r\right) = S_n^{k+r+1} ,$$

(3.2.4) $$S_n^{k+1} - S_{n-1}^{k+1} = S_n^k , \qquad \sum_{p=0}^{n} A_{n-p}^{\alpha} A_p^{\beta} = \sum_{p=0}^{n} A_n^{\alpha+\beta+1} .$$

For any positive integer k, the differences of order k of the sequence $\{a_n\}$ are defined by the equations

$$\Delta^1 a_n = a_n - a_{n+1} , \qquad \Delta^k a_n = \Delta^1\left(\Delta^{k-1} a_n\right), \quad n = 0,1,2,\ldots .$$

For these differences, note that

(3.2.5) $$\Delta^k a_n = \sum_{m=0}^{k} A_m^{-k-1} a_{n+m} = \sum_{m=0}^{\infty} A_m^{-k-1} a_{n+m} ,$$

Since $A_m^{-k-1} = 0 \;\; for \;\; m \geq k+1$.

If the series (3.2.5) are convergent for some k which is not a positive integer, we define the differences as

(3.2.6) $$\Delta^k a_n = \sum_{m=0}^{\infty} A_m^{-k-1} a_{n+m} \qquad (n = 0,1,2,......).$$

The broken differences $\Delta_n^k a_p$ are defined by

(3.2.7) $$\Delta_n^k a_p = \sum_{m=0}^{n-p} A_m^{-k-1} a_{p+m}.$$

By means of the broken differences we can apply the Abel formula of order k,

(3.2.8) $$\sum_{p=0}^{n} a_p b_p = \sum_{p=0}^{n} S_p^{k-1}(a_p) \Delta_n^k b_p .$$

If k is positive integer, we get

(3.2.9) $$\sum_{p=0}^{n} a_p b_p = \sum_{p=0}^{n-k} S_p^{k-1}(a_p) \Delta^k b_p + \sum_{p=n-k+1}^{n} S_p^{k-1}(a_p) \Delta_n^k b_p .$$

3.3 Lemmas

The following lemmas will be required in the proof of main results:

Lemma 3.3.1 [24]. If $\alpha \geq 0$, $p \geq 0$,

(i) $\varepsilon_n = o(n^{-p})$,

(ii) $\sum_{n=0}^{\infty} A_n^{\alpha+p} \left|\Delta^{\alpha+1} \varepsilon_n\right| < \infty$, then

(iii) $\sum_{n=0}^{\infty} A_n^{\lambda+p} \left|\Delta^{\lambda+1} \varepsilon_n\right| < \infty, \quad for \quad -1 \leq \lambda \leq \alpha$ and

(iv) $A_n^{\lambda+p} \Delta^{\lambda} \varepsilon_n$ is bounded variation for $0 \leq \lambda \leq \alpha$ and tends to zero as $n \to \infty$.

Lemma 3.3.2 [2]. Let $r \geq 0$ be a real number and let S denote the integral part and δ the fractional part of $r (0 \leq \delta < 1)$. Then, if the sequence $\{\varepsilon_n\}$ satisfies the conditions

(i) $\varepsilon_n = o(1)$, and

(ii) $\sum_{n=1}^{\infty} n^r \left|\Delta^{r+1} \varepsilon_n\right| < \infty$, We have

(iii) $\Delta^q \varepsilon_n = o\left(n^{-q}\right), \quad for \quad q = 1,2,......,S,$

(iv) $\Delta^{S+1}\varepsilon_n = \sum_{m=0}^{\infty} A_m^{\delta-1}\Delta^{r+1}\varepsilon_{n+m}$.

Lemma 3.3.3 [25]. If $0 \le \delta \le 1$ *and* $0 \le m < n$, then

$$\left|\sum_{i=0}^{m} A_{n-i}^{\delta-1} S_i\right| \le \max_{0\le p\le m} \left|S_p^b\right|.$$

Lemma 3.3.4 [8]. If the set of constants $a_0, a_1, a_2, ..., a_n, ...$ satisfy the conditions 3.1.4, then the series (3.1.1) will converge in the interval $0 < x \le \pi$ and will represent there an L–integrable function whose Fourier cosine development is given by (3.1.1). Further,

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=0}^{\infty} S_n^k(x)\Delta^{k+1}a_n ,$$

where $S_n^k(x)$ denotes the Cesaro sum of order k of the series $\frac{1}{2} + \cos x + \cos 2x + \ldots + \cos nx + \ldots$

Lemma 3.3.5 [3]. Let $S_n(x)$ and $T_n^k(x)$ be the nth partial sum and Cesaro mean of order $k > 0$, respectively, of the series $\frac{1}{2} + \cos x + \cos 2x + \ldots + \cos nx + \ldots$. Then

(i) $\int_0^{\pi} \left|S_n(x)\right| dx \approx \log n$,

(ii) $\int_0^{\pi} \left|T_n^k(x)\right| dx$ remains bounded for all *n*.

3.4 Main Results:

Theorem 3.4.1. *Let* $k > 0$ *be a real number. If*

(i) $\lim_{n\to\infty} a_n = 0,$

(ii) $\sum_{n=1}^{\infty} n^k \left|\Delta^{k+1}a_n\right| < \infty,$

then $g_n(x)$ *converges to* $g(x)$ *in the* L^1-*metric.*

If we take $k=1$, then this theorem reduces to the theorem of Garrett and Stanojevic [19].

Theorem 3.4.2. *Let $\{a_n\}$ be a null sequence satisfying the condition (3.1.4). Then it satisfying the condition (3.1.3).*

In other words, the Moore's class M is a subclass of the class C. This theorem supplies an interesting and nontrivial example for Theorem B.

Theorem 3.4.3. *If $\{a_n\}$ is a null quasiconvex sequence, then $g_n(x)$ converges to $g(x)$ in the L^1–norm.*

As a consequence of Theorem 3.4.2 and Theorem B we obtain a generalization of Theorem C in the form of the following theorem:

Theorem 3.4.4. *Let k be a real number such that $k>0$. If*

(i) $\lim_{n\to\infty} a_n = 0,$

(ii) $\sum_{n=1}^{\infty} n^k \left|\Delta^{k+1} a_n\right| < \infty,$

then for the convergence of the series (3.1.1) in the metric space L^1 it is necessary and sufficient that $\lim_{n\to\infty} a_n \log n = 0$.

Proof of Theorem 3.4.1.

Consider

$$g_n(x) = \frac{1}{2}\sum_{i=0}^{n} \Delta a_i + \sum_{i=1}^{n}\left(\sum_{j=i}^{n} \Delta a_j\right)\cos ix$$

By using partial summation formula, we have

$$= \sum_{i=0}^{n} \Delta a_i S_i(x).$$

We prove the theorem in two parts depending on the value of k (i.e, Integral and Non-Integral).

Part 1. Let k be integral. Applying Abel's transformation of order k to $g_n(x)$, we have

(3.4.1)
$$g_n(x) = \sum_{i=0}^{n-k} S_i^k(x)\Delta^{k+1}a_i + S_{n-k+1}^k(x)\Delta^k a_{n-k+1}$$
$$+ S_{n-k+2}^{k-1}(x)\Delta^{k-1}a_{n-k+2} + \ldots + S_{n-1}^2(x)\Delta^2 a_{n-1} + S_n^1(x)\Delta a_n.$$

Now by lemma (3.3.4),

(3.4.2) $$g(x) = \sum_{i=0}^{n} S_i^k(x)\Delta^{k+1}a_i,$$

So, by (3.4.1) and (3.4.2),

$$\int_0^\pi |g(x) - g_n(x)|dx$$

$$= \int_0^\pi \left| \sum_{i=n-k+1}^{\infty} S_i^k(x)\Delta^{k+1}a_i - S_{n-k+1}^k(x)\Delta^k a_{n-k+1} - S_{n-k+2}^{k-1}(x)\Delta^{k-1}a_{n-k+2} - \ldots - S_n^1(x)\Delta a_n \right| dx$$

$$= \int_0^\pi \left| \sum_{i=n-k+1}^{\infty} S_i^k(x)\Delta^{k+1}a_i - \sum_{i=1}^{k} S_{n-i+1}^i(x)\Delta^i a_{n-i+1} \right| dx$$

$$\le \int_0^\pi \left| \sum_{i=n-k+1}^{\infty} S_i^k(x)\Delta^{k+1}a_i \right| dx + \int_0^\pi \left| \sum_{i=1}^{k} S_{n-i+1}^i(x)\Delta^i a_{n-i+1} \right| dx$$

$$\le \sum_{i=n-k+1}^{\infty} \int_0^\pi |S_i^k(x)||\Delta^{k+1}a_i| dx + \sum_{i=1}^{k} \int_0^\pi |S_{n-i+1}^i(x)||\Delta^i a_{n-i+1}| dx$$

Using (3.2.2) , we have

$$= \sum_{i=n-k+1}^{\infty} A_i^k |\Delta^{k+1}a_i| \int_0^\pi |T_i^k(x)|\, dx + \sum_{i=1}^{k} A_{n-i+1}^i |\Delta^i a_{n-i+1}| \int_0^\pi |T_{n-i+1}^i(x)|\, dx,$$

$$\le C \sum_{i=n-k+1}^{\infty} A_i^k |\Delta^{k+1}a_i| + C\sum_{i=1}^{k} A_{n-i+1}^i |\Delta^i a_{n-i+1}|,$$ (C is an absolute constant);

by lamma (3.3.1) and lemma (3.3.5) and the hypothesis of the theorem,

$$\int_0^\pi |g(x) - g_n(x)|dx = o(1) + o(1) = o(1), \text{ as } n \to \infty$$

Thus we have

(3.4.3) $$\lim_{n\to\infty}\int_0^{\pi}|g(x)-g_n(x)|\,dx = 0.$$

<u>Part 2</u>. Let k be non-integral. Let $k = r+\delta$, r is the integral part of k, and δ is its fractional part, 0 < $0<\delta<1$.

Case (i). Let $r=0$. Applying Abel's transformation of order $-\delta$ and using (3.2.8), we have

$$\sum_{i=0}^{n} S_i^{\delta}(x)\Delta^{\delta+1}a_i = \sum_{i=0}^{n} S_i(x)\sum_{m=0}^{n-i} A_m^{\delta-1}\,\Delta^{\delta+1}a_{i+m}$$

Again by the result (iv) of Lemma (3.3.2), this formula can be transformed into

$$\sum_{i=0}^{n} S_i^{\delta}(x)\Delta^{\delta+1}a_i = \sum_{i=0}^{n} S_i(x)\Delta a_i - R_n(x),$$

where

$$R_n(x) = \sum_{i=0}^{n} S_i(x)\left(A_{n-i+1}^{\delta-1}\Delta^{\delta+1}a_{n+1} + A_{n-i+2}^{\delta-1}\Delta^{\delta+1}a_{n+2} + \ldots\right).$$

This implies that

$$\sum_{i=0}^{n} S_i(x)\Delta a_i = \sum_{i=0}^{n} S_i^{\delta}(x)\Delta^{\delta+1}a_i + R_n(x),$$

And consequently,

$$g_n(x) = \sum_{i=0}^{n} S_i^{\delta}(x)\Delta^{\delta+1}a_i + R_n(x),$$

When $r=0$, it is clear that $k=\delta$ and

$$g(x) = \sum_{i=0}^{n} S_i^{\delta}(x)\Delta^{\delta+1}a_i\,.$$

Therefore,

$$\int_0^{\pi}|g(x)-g_n(x)|\,dx = \int_0^{\pi}\left|\sum_{i=n+1}^{\infty} S_i^{\delta}(x)\Delta^{\delta+1}a_i - R_n(x)\right|dx$$

$$\leq \sum_{i=n+1}^{\infty}\left|\Delta^{\delta+1}a_i\right|\int_0^{\pi}\left|S_i^{\delta}(x)\right|dx + \int_0^{\pi}|R_n(x)|\,dx$$

$$= C\sum_{i=n+1}^{\infty} A_i^{\delta}\left|\Delta^{\delta+1}a_i\right|\int_0^{\pi}\left|T_i^{\delta}(x)\right|dx+\int_0^{\pi}\left|R_n(x)\right|dx$$

$$\leq C\sum_{i=n+1}^{\infty} A_i^{\delta}\left|\Delta^{\delta+1}a_i\right|+\int_0^{\pi}\left|R_n(x)\right|dx$$

using lemma (3.3.1) and lemma (3.3.5),

(3.4.4) $$\int_0^{\pi}\left|g(x)-g_n(x)\right|dx = o(1)+\int_0^{\pi}\left|R_n(x)\right|dx$$

now we estimate

$$\int_0^{\pi}\left|R_n(x)\right|dx = \int_0^{\pi}\left|\left(\sum_{i=0}^{n}S_i(x)A_{n-i+1}^{\delta-1}\right)\Delta^{\delta+1}a_{n+1}+\left(\sum_{i=0}^{n}S_i(x)A_{n-i+2}^{\delta-1}\right)\Delta^{\delta+1}a_{n+2}+\ldots\right|dx$$

$$\leq \int_0^{\pi}\left|\Delta^{\delta+1}a_{n+1}\right|\left|\sum_{i=0}^{n}S_i(x)A_{n-i+1}^{\delta-1}\right|dx+\int_0^{\pi}\left|\Delta^{\delta+1}a_{n+2}\right|\left|\sum_{i=0}^{n}S_i(x)A_{n-i+2}^{\delta-1}\right|dx+\ldots$$

$$\leq \int_0^{\pi}\left|\Delta^{\delta+1}a_{n+1}\right|\max_{0\leq p\leq n+1}\left|S_p^{\delta}(x)\right|dx+\int_0^{\pi}\left|\Delta^{\delta+1}a_{n+2}\right|\max_{0\leq p\leq n+2}\left|S_p^{\delta}(x)\right|dx+\ldots,$$

by lemma (3.3.3),

$$=\left|\Delta^{\delta+1}a_{n+1}\right|A_{n+1}^{\delta}\int_0^{\pi}\max_{0\leq p\leq n+1}\left|T_p^{\delta}(x)\right|dx+\left|\Delta^{\delta+1}a_{n+2}\right|A_{n+2}^{\delta}\int_0^{\pi}\max_{0\leq p\leq n+2}\left|T_p^{\delta}(x)\right|dx+\ldots$$

$$\leq CA_{n+1}^{\delta}\left|\Delta^{\delta+1}a_{n+1}\right|+CA_{n+2}^{\delta}\left|\Delta^{\delta+1}a_{n+2}\right|+\ldots$$

$$= o(1)+o(1)+\ldots$$

$$= o(1),$$ by lemmas (3.3.1) and (3.3.5).

Thus, using (3.4.4), we have

(3.4.5) $$\int_0^{\pi}\left|g(x)-g_n(x)\right|dx = o(1).$$

Case (ii). Let $r\geq 1$. Applying Abel's transformation of order r,

$$g_n(x)=\sum_{i=0}^{n}\Delta a_i S_i(x)$$

(3.4.6) $$=\sum_{i=0}^{n-r}S_i^{r}(x)\Delta^{r+1}a_i+\sum_{i=0}^{r}S_{n-i+1}^{i}(x)\Delta^{i}a_{n-i+1}$$

Again using an Abel transformation of order $-\delta$, we obtain

$$\sum_{i=0}^{n} S_i^k(x)\Delta^{r+1}a_i = \sum_{i=0}^{n} S_i^r(x)\sum_{p=0}^{n-i} A_p^{\delta-1}\Delta^{k+1}a_{i+p}.$$

By using the result (iv) of Lemma (3.3.2), this formula can be transformed into

$$\sum_{i=0}^{n} S_i^k(x)\Delta^{r+1}a_i = \sum_{i=0}^{n} S_i^r(x)\Delta^{r+1}a_i - R_n(x), \tag{3.4.7}$$

where

$$R_n(x) = \sum_{i=0}^{n} S_i^r(x)\left(A_{n-i+1}^{\delta-1}\,\Delta^{k+1}a_{n+1} + A_{n-i+2}^{\delta-1}\,\Delta^{k+1}a_{n+2} + \ldots\right)$$

$$= \left(\sum_{i=0}^{n} S_i^r(x)A_{n-i+1}^{\delta-1}\right)\Delta^{k+1}a_{n+1} + \left(\sum_{i=0}^{n} S_i^r(x)A_{n-i+2}^{\delta-1}\right)\Delta^{k+1}a_{n+2} + \ldots$$

Replacing n by $n-r$ in (3.4.7), we have

$$\sum_{i=0}^{n-r} S_i^k(x)\Delta^{k+1}a_i = \sum_{i=0}^{n-r} S_i^r(x)\Delta^{r+1}a_i - R_{n-r}(x). \tag{3.4.8}$$

Now by (3.4.6) and (3.4.8),

$$g_n(x) = \sum_{i=0}^{n-r} S_i^k(x)\Delta^{k+1}a_i + R_{n-r}(x) + \sum_{i=0}^{r} S_{n-i+1}^i(x)\Delta^i a_{n-i+1}. \tag{3.4.9}$$

Therefore, by lemma (3.3.4),

$$\int_0^\pi |g(x)-g_n(x)|\,dx = \int_0^\pi \left| \sum_{i=n-r+1}^{\infty} S_i^k(x)\Delta^{k+1}a_i - R_{n-r}(x) - \sum_{i=1}^{r} S_{n-i+1}^i(x)\Delta^i a_{n-i+1}\right| dx$$

$$\leq \sum_{i=n-r+1}^{\infty}\left|\Delta^{k+1}a_i\right|\int_0^\pi\left|S_i^k(x)\right|dx + \int_0^\pi |R_{n-r}(x)|\,dx + \sum_{i=1}^{r}\left|\Delta^i a_{n-i+1}\right|\int_0^\pi\left|S_{n-i+1}^i(x)\right|dx$$

$$= \sum_{i=n-r+1}^{\infty} A_i^k\left|\Delta^{k+1}a_i\right|\int_0^\pi\left|T_i^k(x)\right|dx + \int_0^\pi |R_{n-r}(x)|\,dx + \sum_{i=1}^{r} A_{n+1-i}^i\left|\Delta^i a_{n-i+1}\right|\int_0^\pi\left|T_{n-i+1}^i(x)\right|dx$$

Using lemma (3.3.5),

$$\leq C\sum_{i=n-r+1}^{\infty} A_i^k\left|\Delta^{k+1}a_i\right| + \int_0^\pi |R_{n-r}(x)|\,dx + C\sum_{i=1}^{r} A_{n+1-i}^i\left|\Delta^i a_{n-i+1}\right|$$

$$= o(1) + o(1) + \int_0^\pi |R_{n-r}(x)|\,dx$$

By the hypothesis of the theorem and Lemma (3.3.1),

$$\int_0^\pi |g(x) - g_n(x)|\,dx = o(1) + \int_0^\pi |R_{n-r}(x)|\,dx, \tag{3.4.10}$$

Now we estimate

$$\int_0^\pi |R_{n-r}(x)|\,dx = \int_0^\pi \left| \left(\sum_{i=0}^{n-r} S_i^r(x) A_{n-i+1-r}^{\delta-1} \right) \Delta^{k+1} a_{n-r+1} + \left(\sum_{i=0}^{n-r} S_i^r(x) A_{n-i-r+2}^{\delta-1} \right) \Delta^{k+1} a_{n-r+2} + \ldots \right| dx$$

$$\leq \sum_{i=0}^{n-r} A_{n-i+1-r}^{\delta-1} A_i^r \int_0^\pi |T_i^r(x)|\,dx \left|\Delta^{k+1} a_{n-r+1}\right| + \sum_{i=0}^{n-r} A_{n-i-r+2}^{\delta-1} A_i^r \int_0^\pi |T_i^r(x)|\,dx \left|\Delta^{k+1} a_{n-r+2}\right| + \ldots$$

$$\leq C \sum_{i=0}^{n-r} A_{n-i+1-r}^{\delta-1} A_i^r \left|\Delta^{k+1} a_{n-r+1}\right| + C \sum_{i=0}^{n-r} A_{n-i-r+2}^{\delta-1} A_i^r \left|\Delta^{k+1} a_{n-r+2}\right| + \ldots,$$

Again, by lemma (3.3.5),

$$\leq C \sum_{i=0}^{n-r+1} A_{n-i+1-r}^{\delta-1} A_i^r \left|\Delta^{k+1} a_{n-r+1}\right| + C \sum_{i=0}^{n-r+2} A_{n-i-r+2}^{\delta-1} A_i^r \left|\Delta^{k+1} a_{n-r+2}\right| + \ldots$$

$$= C A_{n+1-r}^{r+\delta} \left|\Delta^{k+1} a_{n-r+1}\right| + C A_{n-r+2}^{r+\delta} \left|\Delta^{k+1} a_{n-r+2}\right| + \ldots, \qquad \text{by (3.2.4)},$$

$$= C A_{n+1-r}^{k} \left|\Delta^{k+1} a_{n-r+1}\right| + C A_{n-r+2}^{k} \left|\Delta^{k+1} a_{n-r+2}\right| + \ldots$$

$$= o(1) + o(1) + \ldots$$

$$= o(1), \quad \text{by the hypothesis of the Theorem 3.4.1.}$$

Hence

$$\int_0^\pi |R_{n-r}(x)|\,dx = o(1) \quad as \quad n \to \infty.$$

Using (3.4.10), this implies that

$$\int_0^\pi |g(x) - g_n(x)|\,dx = o(1), \; n \to \infty. \tag{3.4.11}$$

Thus, by (3.4.5) and (3.4.11), for k is non-integral

(3.4.12) $$\int_0^{\pi} |g(x) - g_n(x)|\, dx = o(1), \; n \to \infty..$$

Hence, in view of (3.4.3) and (3.4.12), for any $k > 0$, we have

$$\lim_{n\to\infty} \int_0^{\pi} |g(x) - g_n(x)|\, dx = 0.$$

This implies that $g_n \to g$ in the L-metric.

This proves Theorem 3.4.1.

Proof of Theorem 3.4.2.

As if $\{a_n\}$ is a null sequence satisfying the condition (3.1.4), then $\{a_n\}$ is of bounded variation. This implies the conditions of Theorem A and Theorem 3.4.1 are satisfied. Therefore, the conclusion of Theorem 3.4.2 holds.

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