Stability of Some Functional Equations using Fixed Point Approach

A Thesis

submitted in partial fulfillment of the requirements for the award of the degree of

Doctor of Philosophy

in

Mathematics

by

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under the supervision of

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THAPAR INSTITUTE
OF ENGINEERING & TECHNOLOGY
(Deemed to be University)

DEDICATED

TO

THE ALMIGHTY

AND

MY MOTHER

Certificate

I hereby certify that the work, which is being presented in the thesis, entitled **Stability of Some Functional Equations using Fixed Point Approach**, in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** and submitted to the institution is an authentic record of my own work carried out during the period **February 2019** to **August 29, 2023** under the supervision of **Sumit Chandok**, Associate Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

The matter presented in this thesis has not been submitted elsewhere for the award of any other degree or diploma from any institution.

Date: 27/02/2023

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It is certified that the above statement made by the candidate is correct to the best of our knowledge.

Date: 27/02/2023

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Acknowledgements

First and foremost, I earnestly like to express my deep sense of gratitude and indebtedness to my respected teacher and Guide, Prof. Sumit Chandok. I would like to express my heartfelt gratitude to him for being a great advisor with brilliant ideas and never-failing support. Working with my supervisor, Prof. Sumit Chandok, has been the experience of a lifetime. I joined my Ph.D. without having much idea of what research means, and all the credit for my understanding of how to pursue a research problem goes to him. To express their gratitude is no less difficult than the problems dealt with in this dissertation. I would like to thank Prof. Chandok from the bottom of my heart for teaching me the depth of subjects like analysis.

I am thankful to Prof. M.K. Sharma, Head, School of Mathematics, and Prof. Satish Kumar Sharma, Former Head, School of Mathematics for providing me with the best facilities for research work. I thank Dr. Jatinderdeep Kaur, Dr. Meenu Rani, and Dr. Amit Munjal as members of my Ph.D. doctoral committee for their support and encouragement. I am thankful to the Ph.D. coordinators, all respected faculty members, and the staff of School of Mathematics for their kind cooperation. I am thankful to the Director, Registrar, and Dean (Research and Sponsored Project) of Thapar Institute of Engineering and Technology, Patiala for providing the necessary facilities and financial support through Teaching Associateship (TA ship) to carry out the present research work.

I owe my deep gratitude to Prof. P. Pradhan, Prof. D.C. Gupta, and specially Lecturer N. D. Ahamad for the way they molded my life.

I will not be able to thank Mrs. Kalavati Devi and Mr. Satya Narayan Verma in Patiala enough for their love and care, especially the good food and care.

I thank Mr. Amit Pandey, Dr. Bhuvaneswar, Dr. Vijil, and Mr. Kamal Kant for helping me. They have always been available for discussion and offered all assistance required. I am grateful to him for believing in me and giving me enough confidence to pursue the research problems that I have worked on. Not only Mathematics, but they have also taught me a lot about having the right attitude in life for which I can never thank them enough.

I would also like to thank my childhood friends Mr. Bhanu Tyagi and Mr. Abhishek Tyagi for their care and support.

Now comes the turn of friends (who may read this and think would be offended if thanked.). It is difficult to imagine my life at TIET without friends like Mr. Mohit, Mr. Watanjeet Singh, Mr. Anshul Rana, Ms. Shivani, Ms. Shagun and Mr. Kallal Pal. These creatures have left no stone unturned to bug me throughout my Ph.D., and I see no hopes of any relief in the future (which I wish should never happen). Apart from this, these are the friends whom I have always relied on since I came to Patiala.

I am much indebted to my dear friend and colleague, Ms. Meenu, whose dedication and support have helped me a lot, especially during my hard time. I owe her for unselfishly helping me throughout my Ph.D. journey. Your support and encouragement were worth more than I can express on a piece of paper.

Finally, my family has been my biggest support all my life. My Mother (Mrs. Kamlesh Sharma), Elder Sister (Mrs. Anuradha Sharma), Brother in Law (Mr. Parbhat Sharma) & elder brother (Mr. Manoj Sharma) who have been there for me in my good and bad times. My friend Mr. Amit Panday, Ms. Meenu, Dr. Bhuvneshwar, and Dr. Vijil Kumar support me unconditionally. My mother and Sister for their love, care, and support. Last but not least, my Father for being my strength.

I am thankful to My Father (Late Mr. Sunil Kumar Sharma), Lord Shiva, and Maa Naina Devi (Kuldevi) for giving me the strength to complete my work.

Ravinder Kumar Sharma

Abstract

The thesis has been split into seven chapters, the first of which includes an introduction to the subject matter and a review of the literature, followed by a summary of the thesis's contents. In the second chapter, we obtain a few sufficient conditions for the existence of fixed point in the framework of $\mathcal{F}$-metric space, orthogonal $\mathcal{F}$-metric space, orthogonal metric space, and complete quasi-2-normed space. In the third chapter, we investigate the Hyers Ulam stability of fixed point and Cauchy functional equations in the context of $\mathcal{F}$-metric space. We study properties, equivalence results, and Ulam-type stability for different forms of quadratic functional equations in the fourth chapter. In the fifth chapter, we study the stability of a quartic functional equation in non-Archmedean β-normed space and complete (β, p)-normed space. We study the hyperstability of a general linear functional equation in a complete quasi-2-normed space in the sixth chapter. In the last chapter, we study the stability of integral equations in the setting $\mathcal{F}$-metric space and provide a solution for a Caputo-type nonlinear fractional integro-differential equation in the framework of orthogonal metric space.

Keywords: Stability, Fixed Point Methods, Functional Equations, Quasi-Normed Space, (β, p)***-Normed Space, Non-Archimedean*** β***-Normed Space, Quasi-2-Normed Space.***

List of Publications

1. **R. K. Sharma** and S. Chandok "Multivalued problems, orthogonal mappings, and fractional integro-differential equation" *Journal of Mathematics*, vol. 2020, pp. 1–8, Article ID 6615478, 2020 **(SCIE, IF 1.4)**.

2. **R. K. Sharma** and S. Chandok "Existence, stability, and well-posedness of fixed point problem with application to integral equation," *Plotehnica of Bucharest Scientific Bulletin-Series A- Applied Mathematics and Physics*, vol. 83, no. 2, pp. 59–68, 2021 **(SCIE, IF 1.160)**.

3. **R. K. Sharma** and S. Chandok: "The generalized hyperstability of general linear equation in quasi-2-Banach space", *Acta Mathematica Scientia*, vol. 42 (4), pp. 1357–1372, 2022 **(SCIE, IF 1.0)**.

4. **R. K. Sharma** and S. Chandok "Quartic functional equation: Ulam-type stability in (β, p)-Banach space and non-Archimedean β-normed space," *Journal of Mathematics*, vol. 2022, pp. 1–12, Article ID 9908530, 2022 **(SCIE, IF 1.4)**.

5. **R. K. Sharma** and S. Chandok "Well-posedness and Ulam's stability of functional equation in $\mathcal{F}$-metric space with an application," *Filomat*, vol. 36, no. 16, pp. 5573–5589. 2022. **(SCIE, IF 0.8)**.

6. **R. K. Sharma** and S. Chandok "Ulam stability of a quadratic-type functional equation in 3 variables in the quasi-Banach spaces," *Functiones et Approximatio*, pp. 1-19, (2022). DOI: 10.7169/facm/1934. **(ESCI, IF 0.5)**.

7. **R. K. Sharma** and S. Chandok "Approximation of Drygas functional equation in quasi-Banach space," *Journal of Applied Mathematics and Informatics*, vol. 41, no. 3, pp. 469 - 485, 2023. https://doi.org/10.14317/jami.2023.469 **(ESCI, IF 0.3)**.

8. **R. K. Sharma** and S. Chandok "Equivalence for various forms of quadratic functional equations and their generalized Hyers–Ulam–Rassias stability," *International Journal of Mathematics and Mathematical Sciences*, vol. 2023, pp. 1–15, Article ID 1721273, 2023. https://doi.org/10.1155/2023/1721273 **(ESCI, IF 1.2)**.

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Chapter 1

Introduction

1.1 Foundation of Functional Equations

The analysis of functional equations is most important in a modern branch of mathematics. It offers a powerful technique for working with critical concepts and relationships in analysis and algebra. It also aided in the development of powerful tools in a variety of allied disciplines. Various mathematicians, including Euler, Poisson, Cauchy, Abel, Darboux, and Hilbert, studied functional equations in various forms. Even, odd, and periodic functions are the most basic types of functional equations (see [1, 2, 119]).

A Hungarian mathematician Aczél (see [1, 2]) defined functional equation as follows:
A Functional equation is an equation

$$A_1 = A_2$$

between two terms A_1 and A_2, which contain k independent variable's $\varpi_1, \varpi_2, \cdots, \varpi_k$ and $n \geq 1$ unknown functions $F_1, F_2, \cdots, F_n$ of $j_1, j_2, \cdots, j_n$ variables respectively, as well as a finite number of known functions.

An equation of the form $g(\varpi, \vartheta, \gamma, \cdots) = 0$, where the function g contains a finite number of independent variables, unknown functions, and known functions is called a functional equation. That is to say, a functional equation is any equation that specifies a function implicitly. Functional equations include differential, integral, difference, and iterative equations.

Oresme (see [36]) defined linear functions in an indirect manner. Although Oresme's definition of linearity can be acknowledged as an early version of a functional equation, it does not demonstrate the functional equation's theory's starting point.

Although D'Alembert was the first researcher to consider functional equations, the topic of functional equations can be traced back to Cauchy's work (see [119]). The functional equation associated with Cauchy is

$$g(\varpi + \vartheta) = g(\varpi) + g(\vartheta). \tag{1.1.1}$$

For a category of continuous real-valued functions, Cauchy solved (1.1.1). Concept of additive functional equations is an extremely useful tool in the development of natural and social sciences. It is well understood that an additive function g has linearity (see [107]) if g fulfills any of the following conditions:

(i) g is monotonic or bounded on an interval of positive length;

(ii) g is measurable.

(iii) g is integrable;

(iv) g is continuous at a point;

In 1906, Jensen (see [107], [119]) give the following functional equation

$$g\left(\frac{\varpi + \vartheta}{2}\right) = \frac{g(\varpi) + g(\vartheta)}{2}. \tag{1.1.2}$$

It is believed that Jensen's functional equation is the average of (1.1.1).

General linear functional equation (see [66]) has the form

$$g(a\varpi + b\vartheta + c) = Ag(\varpi) + Bg(\vartheta) + C, \quad abAB \neq 0, \tag{1.1.3}$$

where $g : \mathbb{R}^N \to \mathbb{R}$ is unknown function and $c \in \mathbb{R}^N$, $b, a, C, B, A, \in \mathbb{R}$. The solution of (1.1.3) has been discussed in Kuczma [66].
Note that (1.1.1) is a specific case of (1.1.3) for $c = 0 = C, b = a = 1 = B = A$. Moreover, (1.1.3) also includes (1.1.2) for $c = 0 = C, B = A = \dfrac{1}{2} = b = a$.

A quadratic equation is a characteristic equation of a second-order linear differential equation. Quadratic equations are used when gravity is present, such as the path of a ball or the shape of cables in a suspension bridge. Quadratic functional equation (see [3]) is the well known functional equation of the following form

$$g(\varpi + \vartheta) + g(\varpi - \vartheta) = 2g(\varpi) + 2g(\vartheta). \tag{1.1.4}$$

(1.1.4) is solved by the function $g(\varpi) = C\varpi^2$, where C is a constant. (1.1.4) is also said to be Euler-Lagrange functional equation introduced by Rassias [94].
There are different forms of quadratic functional equations in which some forms of

quadratic functional equations are given below:
In 1995, Kannappan [61] solved the following functional equation

$$g(\varpi+\vartheta+\gamma)+g(\varpi)+g(\vartheta)+g(\gamma)=g(\varpi+\vartheta)+g(\vartheta+\gamma)+g(\gamma+\varpi). \tag{1.1.5}$$

In 2000, Bae [11] showed that

$$g(\varpi+\vartheta+\gamma)+g(\varpi-\vartheta)+g(\varpi-\gamma)+g(\vartheta-\gamma)=3[g(\varpi)+g(\vartheta)+g(\gamma)] \tag{1.1.6}$$

is equivalent to (1.1.4).
In 1987, Drygas [43] considered the following functional equation

$$g(\varpi)+g(\vartheta)=g(\varpi-\vartheta)+2\left\{g\left(\frac{\varpi+\vartheta}{2}\right)-g\left(\frac{\varpi-\vartheta}{2}\right)\right\}, \tag{1.1.7}$$

which is closely related to a quadratic functional equation. It can be morphed into the following equation (see Remark 9.1, pp 131, [107])

$$g(\varpi+\vartheta)+g(\varpi-\vartheta)=2g(\varpi)+g(\vartheta)+g(-\vartheta), \tag{1.1.8}$$

for all $\vartheta, \varpi \in \mathbb{R}$. (1.1.8) is a generalization of (1.1.4).
Ebanks et al. [45] provided a solution of (1.1.8) and has the following form

$$g(\varpi)=l(\varpi)+m(\vartheta) \tag{1.1.9}$$

where $m:\mathbb{R}\to\mathbb{R}$ and $l:\mathbb{R}\to\mathbb{R}$ are quadratic and additive functions respectively.
A quartic equation is a fourth-order linear differential or difference equation's characteristic equation. Quartic equations are frequently encountered in computational geometry and allied fields such as optics, computer-aided manufacturing, computer-aided design, and computer graphics. There are many different types of quartic equations, and some forms of quartic functional equations are given below:
In 1999, Rassias [96] investigated some properties of the following quartic functional equation:

$$g(\varpi-2\vartheta)+g(\varpi+2\vartheta)+6g(\varpi)=4\left(g(\varpi-\vartheta)+g(\varpi+\vartheta)\right)+24g(\vartheta). \tag{1.1.10}$$

Chung and Sahoo [35] showed that $g(\varpi)=l(\varpi,\varpi,\varpi,\varpi)$ if and only if $g:\mathbb{R}\to\mathbb{R}$ is the solution of (1.1.10), where $l:\mathbb{R}^4\to\mathbb{R}$ is additive and symmetric. The premise that every solution of (1.1.10) is even and can be expressed as follows:

$$g(2\varpi-\vartheta)+g(2\varpi+\vartheta)=4\left(g(\varpi-\vartheta)+g(\varpi+\vartheta)\right)+24g(\varpi)-6g(\vartheta). \tag{1.1.11}$$

The function $g(\varpi)=C\varpi^4$ is easily recognized as a solution of (1.1.11), where C is a constant. In 2014, Bodaghi [18] solved the following quartic equation

$$\begin{aligned}g(\varpi+m\vartheta)+g(\varpi-m\vartheta)=2(m-1)(7m-9)g(\varpi)+2(m^2-1)m^2g(\vartheta)-(m-1)^2g(2\varpi)\\+m^2\{g(\varpi-\vartheta)+g(\varpi+\vartheta)\}.\end{aligned} \tag{1.1.12}$$

where m is a constant integer.

1.2 Stability of Functional Equations

Stability problem of functional equations arose in 1940 as a result of a question Ulam raised in his talk at the University of Wisconsin [123, 124].

"Is it correct to say that a function that approximates a functional equation F must be near to an exact solution of F? "

Hyers [53] provided a completely positive response to the question of Ulam in complete normed space in 1941, and this form of the problem is referred as HU stability problem.

Theorem 1.2.1. *[53] Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be complete normed spaces, $g:\mathscr{E}_1\to\mathscr{E}_2$ is a map such that*

$$\| g(\vartheta+\varpi)-g(\vartheta)-g(\varpi)\| \le \delta, \forall\ \varpi,\vartheta\in\mathscr{E}_1,\ \textit{for some } \delta>0. \tag{1.2.1}$$

Then the limit

$$H(\varpi)=\lim_{n\to\infty}2^{-n}f(2^n\varpi) \tag{1.2.2}$$

exists, and $H:\mathscr{E}_1\to\mathscr{E}_2$ is a unique additive function such that

$$\| g(\varpi)-H(\varpi)\|\le\delta,$$

for every $\varpi\in\mathscr{E}_1$.

Besides that, it is possible that there is no such alternative, that is, all solutions of stability inequality are exact solutions of functional equation. In this situation, there is a need for hyperstability. In 1949, Bourgin [21] gave the first hyperstability result concerning ring homomorphism. The term hyperstability was most likely first used in [71]. Aoki [8] and Rassias [99] generalize Hyers Theorem 1.2.1 for linear and additive maps in 1950 and 1978, respectively. Later, Găvurta [50] generalized the corresponding results of Aoki [8] and Rassias [99] for additive Cauchy linear functional equation.

HU stability concept is very beneficial in various branches of Mathematics, viz., optimization, differential equations, numerical analysis, and other allied areas. Recently, many researchers (see, [9,19,55,69,75,108]) have investigated the HU stability of various forms of differential equations and their solutions.

A most essential branch of nonlinear analysis is fixed point theory. A well-known mathematician Stefan Banach [16] stated and proved an important result known as Banach contraction principle in 1922. Since the introduction of this theory, many generalizations have been made in various frameworks. In 1968, Margolis and Diaz [42] generalized Banach contraction principle. Thereafter many researchers generalized contraction principle and obtained a few interesting outcomes (see [27,31,49,51,80,101,102,104,125–127]. Over the last two decades, fixed point methods have emerged as an effective and necessary way of investigating the HU stability of functional equations.

Theorem 1.2.2. *[42] Let $(\mathscr{E}_1, \mathfrak{D})$ be a CGMS and $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ is a contraction map fulfilling:*
(C1) $\forall\, \vartheta, \varpi \in \mathscr{E}_1$, there exists a constant $M (0 < M < 1)$, such that as often as $\mathfrak{D}(\varpi, \vartheta) < \infty$ one has

$$\mathfrak{D}(\mathscr{T}\varpi, \mathscr{T}\vartheta) \leq M\mathfrak{D}(\varpi, \vartheta).$$

Then for each $\varpi \in \mathscr{E}_1$, we have either of the following conditions:

(A) either $\forall\, n \in \mathbb{N}$, $\mathfrak{D}(\mathscr{T}^n\varpi, \mathscr{T}^{n+1}\varpi) = \infty$,

(B) or

(i) $\lim_{n\to\infty} \mathscr{T}^n\varpi = \varpi^*$, where ϖ^* is a fixed point of $\mathscr{T}$; ‘

(ii) $\mathfrak{D}(\varpi, \varpi^*) \leq \left(\frac{1}{1-M}\right)\mathfrak{D}(\varpi, \mathscr{T}\varpi)$.

Radu [92] presented a new approach in 2003 for determining the existence of extract solutions and error estimations using Margolis and Diaz fixed point approach and he used this method to investigate Jensen's functional equation's stability. Thereafter, many authors investigated Ulam-type stability using this method (see, [31, 32, 41, 74]).

In 2011, Brzdęk et al. [22] proposed a fixed point theorem for some operators and obtained numerous general outcomes on the stability of functional equations's class. Thereafter, various researchers investigated Ulam-type stability using the fixed point approach provided in [22] (see, [25, 46, 89, 90]).

In 2018, Dung and Hang [44] extended fixed point method of [22] to the setting of a complete QNS. Later, using Dung and Hang fixed point approach, Kim [65] studied the Fréchet functional equation's stability.

In 2018, Brzdęk and Ciepliński [27] extended the approach of [22] in complete 2NS. Almahalebi and Chahbi [6] extended the approach of [22] to the setting of a complete 2NS, in a similar way. Thereafter, several authors studied Ulam-type stability using fixed point methods provided in [6] and [27] (see, [5, 29, 81]).

Using fixed point methods and direct methods, various mathematicians have studied the stability of functional equations (like Cauchy, quartic, general linear, quadratic functional equations) in different spaces (see [4, 5, 11–14, 20, 22–25, 28, 29, 33, 34, 37, 39, 40, 44, 46, 47, 54, 57–60, 67, 68, 70, 73, 77–79, 83, 85, 89, 91, 93, 95, 96, 100, 105, 106, 117, 118, 120, 121] and references cited therein)

So motivated by these works, utilizing direct methods and fixed point, we obtain some stability results of some functional equations in various spaces.

1.3 Preliminaries

Here, we present the notions, definitions, results, and properties that will be beneficial throughout the thesis.

For the topology of *quasi-normed* space (QNS) and a sequence to be convergent, Cauchy, complete in QNS, p-norm (see [7, 17, 30, 72, 76] and references cited therein).

Based on the Aoki-Rolewicz theorem (see [72, 76]), each QN is equivalent to some p-norm (see also [17]).

Every NS is a type of QNS (for modulus of concavity $K = 1$).

Remark 1.3.1. *The sequence space* $\mathscr{E}_1 = \ell^p$, $0 < p < 1$, *with the function*

$$\|\varpi\| = \left(\sum_{i=1}^{\infty} \mid \varpi_i \mid^p \right)^{\frac{1}{p}}$$

is a QNS with $K = 2^{(1/p)-1}$ *(see [7, 30]), but if we take* $p = \frac{1}{2}$ *and* $\vartheta = \{\vartheta_i\} =$

$\{0,0,2,0,0,...\} \in \mathscr{E}_1,\ \varpi = \{\varpi_i\} = \{0,1,0,0,0,...\} \in \mathscr{E}_1$.
Then we have

$$\|\varpi + \vartheta\| = \left(\sum_{i=1}^{\infty} \mid \varpi_i + \vartheta_i \mid^{1/2}\right)^2 = (1+\sqrt{2})^2,$$

and

$$\|\varpi\| + \|\vartheta\| = \left(\sum_{i=1}^{\infty} \mid \varpi_i \mid^{1/2}\right)^2 + \left(\sum_{i=1}^{\infty} \mid \vartheta_i \mid^{1/2}\right)^2 = 3.$$

It's obvious that

$$\|\varpi\| + \|\vartheta\| < \|\varpi + \vartheta\|.$$

Thus, $\mathscr{E}_1$ is not a NS.

Theorem 1.3.2. *(see [72]) Let $(\mathscr{E}_1, \|.\|, K \geq 1)$ be a QNS, $p = \log_{2K} 2$, and*

$$|||\varpi||| = \inf\{\left(\sum_{j=1}^{n} \|\varpi_j\|^p\right)^{\frac{1}{p}} | \varpi = \sum_{j=1}^{n} \varpi_j, \varpi_j \in \mathscr{E}_1, n \geq 1\}.$$

Then $|||.|||$ is a p-norm on $\mathscr{E}_1$, that is,

$$|||\vartheta + \varpi|||^p \leq |||\vartheta|||^p + |||\varpi|||^p,\ \forall\ \vartheta, \varpi \in \mathscr{E}_1. \tag{1.3.1}$$

Moreover,$\forall\ \varpi \in \mathscr{E}_1$,

$$\frac{1}{2K}\|\varpi\| \leq |||\varpi||| \leq \|\varpi\|. \tag{1.3.2}$$

If $\|.\|$ is a norm, then $p = 1$ and $|||.||| = \|.\|$.

In 2009, Rassias and Kim [97] introduced QβN and QβNS. For the topology of quasi-β-normed space, (β, p)-normed space ((β, p)-NS) and a sequence to be Cauchy and convergent in QβNS (see [97, 103] and references cited therein).

For the topology of b-metric space (BMS) (see [30, 38] and references cited therein).

For the topology of generalized b-metric space (GBMS) (see [10] and references cited therein).

The following proposition of Paluszyński and Stempak [84] will be used in the sequel.

Proposition 1.3.3. *Suppose that* $(\mathscr{E}_1, \delta, K \geq 1)$ *is a b-metric space,* $p = \log_{2K} 2$*, and*

$$\mathfrak{D}(\varpi, \vartheta) = \inf\{\sum_{i=1}^{n} \delta^p(\varpi_i, \varpi_{i+1}) : \varpi_1 = \varpi, \varpi_2, \cdots, \varpi_n, \varpi_{n+1} = \vartheta \in \mathscr{E}_1, \ n \geq 1\} \quad (1.3.3)$$

Then $\mathfrak{D}$ *is a metric on* $\mathscr{E}_1$ *fulfilling*

$$\frac{1}{4}\delta^p(\varpi, \vartheta) \leq \mathfrak{D}(\varpi, \vartheta) \leq \delta^p(\varpi, \vartheta), \ \forall \ \varpi, \vartheta \in \mathscr{E}_1. \quad (1.3.4)$$

In especially, if δ *is a metric then* $p = 1$ *and* $\mathfrak{D} = \delta$.

Hensel [52] proposed a NS without the Archimedean property in 1897. Non-Archimedean spaces were later discovered to have a wide variety of useful applications (see [62,63,82]). For the toplogy of non-Archimedean β-normed space (NAβ-NS) (see [98,113]).

A sequence $\{\varpi_{n+1} - \varpi_n\}$ converge to zero iff sequence $\{\varpi_n\}$ is a Cauchy sequence in a NAβ-NS $\mathscr{E}_1$.

For the topology of quasi-2-normed space (Q2NS), quasi-p-norm, 2-normed space (2NS), Quasi-2-norm (Q2N), and a sequence to be quasi-2-convergent, quasi-2-Cauchy sequence, and quasi-2-bounded sequence in Q2NS, see [48,86,112].

Here, we present the following results as a lemma (details in [87]), which will be used in the sequel.

Lemma 1.3.4. *Assume that* $\mathscr{E}_1$ *is a 2NS. Then,*

(1) $|\|\vartheta, \gamma\| - \|\varpi, \gamma\|| \leq \|\vartheta - \varpi, \gamma\| \ \forall \ \varpi, \gamma, \vartheta \in \mathscr{E}_1$,

(2) $\|\varpi, \gamma\| = 0 = \|\gamma, w\|$, *then* $\varpi = 0$, *where* $\varpi, w, \gamma \in \mathscr{E}_1$ *and* γ, w *are linearly independent (LI),*

(3) for a convergent sequence $\varpi_n \in \mathscr{E}_1$,

$$\lim_{n\to\infty} ||\varpi_n, \gamma|| = || \lim_{n\to\infty} \varpi_n, \gamma||, \quad \forall \ \gamma \in \mathscr{E}_1.$$

Now, we show that every 2NS is a special situation of Q2NS (for modulus of concavity $K = 1$).

Motivated by the examples given in [64], we give the following two examples.

Example 1.3.1. *Assume that* $\mathscr{E}_1 = \mathbb{R}^3$ *and* $\varpi = \varpi_1 i + \varpi_2 j + \varpi_3 k, \ \vartheta = \vartheta_1 i + \vartheta_2 j + \vartheta_3 k \in$

$\mathbb{R}^3$. *Define*

$$\|\varpi,\vartheta\| = a|\varpi_{i_0}\vartheta_{i_0+1} - \varpi_{i_0+1}\vartheta_{i_0}| + \sum_{i\neq i_0}^{3} |\varpi_i\vartheta_{i+1} - \varpi_{i+1}\vartheta_i|,$$

where $|\varpi_{i_0}\vartheta_{i_0+1} - \varpi_{i_0+1}\vartheta_{i_0}| = \min\{|\varpi_i\vartheta_{i+1} - \varpi_{i+1}\vartheta_i| : 1 \leq i \leq 3\}, \varpi_4 = \varpi_1,\ \vartheta_4 = \vartheta_1$ *and* $a > 1$. *Then* $(\mathbb{R}^3, \|\varpi,\vartheta\|)$ *is a Q2NS.*

Now, we'll show that $(\mathbb{R}^3, \|\varpi,\vartheta\|)$ *is not a 2NS.*

For $\vartheta = (1,0,0), \gamma = (0,1,2), \varpi = (0,-1,1),$ *we have*

$$\|\varpi,\vartheta\| = 2,\ \|\varpi,\gamma\| = 3,\ \textit{and}\ \|\varpi,\vartheta+\gamma\| = a+5,$$

and $\|\varpi,\vartheta+\gamma\| = a+5 > \|\varpi,\vartheta\| + \|\varpi,\gamma\| = 2+3 = 5.$
Therefore for every $a > 1$ *condition* (N_4) *is not satisfy. Hence for every* $a > 1$, $(\mathbb{R}^3, \|\varpi,\vartheta\|)$ *is not a 2NS.*

Example 1.3.2. *On interval* $I = [0,1]$, *assume that* $\mathbb{P}_2$ *is the set of real polynomials of degree* ≤ 2. $\mathbb{P}_2$ *is linear vector space when scalar multiplication and addition are used. In [0, 1], suppose* $\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$ *are different fixed points. Consider a 2NS on* $\mathbb{P}_2$ *as*

$$\|f,g\| = \frac{22}{9}a|f(\varpi_{i_0})g'(\varpi_{i_0}) - f'(\varpi_{i_0})g(\varpi_{i_0})| + \sum_{i\neq i_0}^{4} |f(\varpi_i)g'(\varpi_i) - f'(\varpi_i)g(\varpi_i)|,$$

where $|f(\varpi_{i_0})g'(\varpi_{i_0}) - f'(\varpi_{i_0})g(\varpi_{i_0})| = \min\{|f(\varpi_i)g'(\varpi_i) - f'(\varpi_i)g(\varpi_i)| : 1 \leq i \leq 4\}$ *and* $a > 1$. *Then* $(\mathbb{P}_2, \|f,g\|)$ *is a Q2NS.*
We now demonstrate that $(\mathbb{P}_2, \|f,g\|)$ *is not a 2NS.*
Let us take $\varpi_1 = 1, \varpi_2 = \frac{1}{2}, \varpi_3 = \frac{1}{3}$ *and* $\varpi_4 = 0$. *For* $f = \varpi, h = (\varpi-1)^2,$ *and* $g = \varpi^2$, *we get*

$$\begin{aligned}
|\ f(\varpi_1)g'(\varpi_1) - f'(\varpi_1)g(\varpi_1)\ | &= 1\\
|\ f(\varpi_2)g'(\varpi_2) - f'(\varpi_2)g(\varpi_2)\ | &= \frac{1}{4}\\
|\ f(\varpi_3)g'(\varpi_3) - f'(\varpi_3)g(\varpi_3)\ | &= \frac{1}{9}\\
|\ f(\varpi_4)g'(\varpi_4) - f'(\varpi_4)g(\varpi_4)\ | &= 0,
\end{aligned}$$

and $\|f,g\| = \dfrac{49}{36}$.

In, a similar way, we get

$$
\begin{aligned}
| f(\varpi_1)h'(\varpi_1) - f'(\varpi_1)h(\varpi_1) | &= 0 \\
| f(\varpi_2)h'(\varpi_2) - f'(\varpi_2)h(\varpi_2) | &= \frac{3}{4} \\
| f(\varpi_3)h'(\varpi_3) - f'(\varpi_3)h(\varpi_3) | &= \frac{8}{9} \\
| f(\varpi_4)h'(\varpi_4) - f'(\varpi_4)h(\varpi_4) | &= 1,
\end{aligned}
$$

and $\|f, h\| = \dfrac{95}{36}$.
Also, we get

$$
\begin{aligned}
| f(\varpi_1)(g+h)'(\varpi_1) - f'(\varpi_1)(g+h)(\varpi_1) | &= 1 \\
| f(\varpi_2)(g+h)'(\varpi_2) - f'(\varpi_2)(g+h)(\varpi_2) | &= \frac{1}{2} \\
| f(\varpi_3)(g+h)'(\varpi_3) - f'(\varpi_3)(g+h)(\varpi_3) | &= \frac{7}{9} \\
| f(\varpi_4)(g+h)'(\varpi_4) - f'(\varpi_4)(g+h)(\varpi_4) | &= 1,
\end{aligned}
$$

and $\|f, (g+h)\| = \dfrac{11}{9}a + \dfrac{25}{9}$.
We can infer from the above results that

$$\|f, g+h\| = \frac{11}{9}a + \frac{25}{9} > \|f, g\| + \|f, h\| = \frac{49}{36} + \frac{95}{36} = \frac{36}{9}.$$

Hence, $(\mathbb{P}_2, \|f, g\|)$ *is not a 2NS, for every* $a > 1$.

Definition 1.3.5. *Suppose that* $\mathscr{E}_1$ *is a real linear space of dimension greater than 1. Q2N's* $|||\cdot,\cdot|||$ *and* $||\cdot,\cdot||$ *define on* $\mathscr{E}_1$ *are equivalent if there exist* $m, M > 0$ *such that*

$$m||\vartheta, \varpi|| \leq |||\vartheta, \varpi||| \leq M||\vartheta, \varpi||, \ \forall \ \varpi, \vartheta \in \mathscr{E}_1.$$

Let $\gamma \in \mathscr{E}_1$ *be fixed and define a equivalent Q2N by*

$$|||\varpi, \gamma||| = \inf\{\Big(\sum \|\varpi_j, \gamma\|^r\Big)^{\frac{1}{r}} \mid \varpi = \sum \varpi_j\}. \tag{1.3.5}$$

It is obvious that $|||\alpha\varpi, \gamma||| =| \alpha | \, |||\varpi, \gamma|||$, *that is* $|||\varpi, \gamma||| \leq \|\varpi, \gamma\|$, *and* $|||\cdot,\cdot|||$ *satisfied* *the inequality,*

$$\|\sum_{j=1}^{n}\varpi_j,\gamma\| \leq 4^{\frac{1}{r}}\left(\sum_{j=1}^{n}\|\varpi_j,\gamma\|^r\right)^{\frac{1}{r}}$$

for all $\{\varpi_j\}_{j=1}^{n}$. *It means that* $\|\varpi,\gamma\| \leq 4^{\frac{1}{r}}|||\varpi,\gamma|||$, *that is,*

$$\frac{1}{4}\parallel \varpi,\gamma \parallel^r \leq |||\varpi,\gamma|||^r \leq \parallel \varpi,\gamma \parallel^r, \tag{1.3.6}$$

where $2^{\frac{1}{r}} = 2K$ *(or* $r = \log_{2K} 2$*) for* $K > 1$.

The following result of Park [86] will be used in the sequel.

Lemma 1.3.6. *Assume that* $(\mathscr{E}_1, \|\cdot,\cdot\|)$ *is a Q2NS. There is a* $p(0 < p \leq 1)$ *and equivalent Q2N* $|||\cdot,\cdot|||$ *on* $\mathscr{E}_1$ *fulfilling*

$$|||\vartheta+\varpi,\gamma|||^p \leq |||\vartheta,\gamma|||^p + |||\varpi,\gamma|||^p,\ \forall\ \varpi,\gamma,\vartheta \in \mathscr{E}_1.$$

Here it is to note that Q2N is not continuous, but its equivalent norm defined by (1.3.5) is continuous.

Lemma 1.3.7. *[112] If* $\mathscr{E}_1$ *is a Q2NS, then the equivalent Q2N defined as*

$$|||\varpi,\gamma||| = \inf\{\left(\sum\|\varpi_j,\gamma\|^r\right)^{\frac{1}{r}} \mid \varpi = \sum\varpi_j\},$$

$\forall\ \varpi,\gamma \in \mathscr{E}_1$ *and* $r = \log_{2K} 2$, *is continuous on* $\mathscr{E}_1$.

Assume that $\mathfrak{F}$ is the set of function $f : (0,+\infty) \to \mathbb{R}$ fulfilling the following assumptions:

(Θ_1) f is non-decreasing, that is $0 < \lambda < \mu$ implies $f(\lambda) \leq f(\mu)$,

(Θ_2) for every sequence $\{\mu_n\} \subset (0,+\infty)$, we get

$$\lim_{n\to\infty}\mu_n = 0 \text{ iff } \lim_{n\to\infty} f(\mu_n) = -\infty.$$

For the toplogy of $\mathfrak{F}$-metric space ($\mathfrak{F}$MS) and a sequence to be $\mathfrak{F}$-convergent, $\mathfrak{F}$-Cauchy, and $\mathfrak{F}$-complete in $\mathfrak{F}$MS (see [56]).

Definition 1.3.8. *[125, 127] Let* $F : (0,+\infty) \to \mathbb{R}$ *be a map fulfilling:*

(F1) $\forall\ \varpi, \vartheta > 0$, $\varpi > \varkappa$ *implies* $F(\varpi) > F(\vartheta)$;

(F2) for every sequence $\{\varpi_n\}$ *in* $\mathbb{R}^+$ *we get* $\lim_{n\to+\infty} \varpi_n = 0$ *iff* $\lim_{n\to+\infty} F(\varpi_n) = -\infty$,

(F3) there exists a number $k \in (0,1)$ *such that* $\lim_{a\to 0^+} \varpi^k F(\varpi) = 0$.

If $\lim_{t\to 0^+} F(t) = -\infty$, then using (F1), we get $F(t_n) \to -\infty \Rightarrow t_n \to 0$ (see [122,127]).

Following Wardowski's work [125–127], we designate $\overline{\mathcal{F}}$ as the family of all the functions $F : (0,+\infty) \to \mathbb{R}$ fulfilling (F3) and (F1).

We designate $\mathcal{F}'$ as the family of all the functions $F : (0,+\infty) \to \mathbb{R}$ fulfilling (F3), (F1) and

(F4) $F(\inf A) = \inf F(A)\ \forall\ A \subset (0,\infty)$ with $\inf A > 0$.

Here, $\lim_{c\to d^-} F(c) = F(d-0) = \lim_{\varepsilon\to 0^+} F(d-\varepsilon)$ (left limit at d) and $\lim_{c\to d^+} F(c) = F(d+0) = \lim_{\varepsilon\to 0^+} F(d+\varepsilon)$ (right limit at d) $\forall\ d \in (0,+\infty)$. According to mathematical analysis, $\forall$ $d \in (0,+\infty)$, following is true

$$F(d-0) \leq F(d) \leq F(d+0). \tag{1.3.7}$$

Example 1.3.3. *Let functions* $F_1, F_2, F_3 : (0,+\infty) \to \mathbb{R}$ *defined by:*

(1) $F_1(\varpi) = \frac{-1}{\sqrt{\varpi}}, \forall\ \varpi > 0$;

(2) $F_2(\varpi) = \ln \varpi,\ \forall\ \varpi > 0$;

(3) $F_3(\varpi) = \varpi + \ln \varpi,\ \forall\ \varpi > 0$.

Then $F_1, F_2, F_3 \in \overline{\mathcal{F}}$

Let $(\mathscr{E}_1, d)$ be a metric space. Assume that H is a Hausdorff-Pompeiu metric generated by metric d on a set $\mathscr{E}_1$, $\mathcal{P}(\mathscr{E}_1)$ represent the family of all nonempty subset of $\mathscr{E}_1$, $\mathcal{CB}(\mathscr{E}_1)$ represent the family of all nonempty, bounded and closed subsets of $\mathscr{E}_1$ and $\mathcal{K}(\mathscr{E}_1)$ denotes the family of all nonempty compact subsets of $\mathscr{E}_1$. $H : \mathcal{CB}(\mathscr{E}_1) \times \mathcal{CB}(\mathscr{E}_1) \to \mathbb{R}$ defined by, for every $B, A \in \mathcal{CB}(\mathscr{E}_1)$,

$$H(A,B) = \max\left\{\sup_{\varpi\in A} d(\varpi, B), \sup_{\vartheta\in B} d(\vartheta, A)\right\},$$

where $d(\varpi, A) = \inf\{d(\varpi,\vartheta) : \vartheta \in A\}$.

Suppose that Φ represent the family of functions $\varphi : [0,\infty) \to [0,\infty)$ fulfilling the following assumptions:

(Wi) φ is non-decreasing, that is $0 < \lambda < \mu$ implies $\varphi(\lambda) \leq \varphi(\mu)$.

(Wii) $\varphi^r(t) \to 0$ as $r \to \infty$, for $t \in [0, \infty)$.

1.4 Objectives of thesis

The main objectives of this thesis are

- To prove some results on the existence of fixed points.
- To prove some results on the stability or hyperstability of some functional equations using the fixed point approach.

1.5 Structure of the thesis

The thesis has been split into seven chapters, the first of which includes an introduction to the subject matter and a review of literature, followed by a brief summary of thesis's contents. In the second chapter, we obtain a few sufficient conditions for the existence of fixed point in the framework of $\mathfrak{F}$MS, O$\mathfrak{F}$MS, OMS, and complete Q2NS. In the third chapter, we study HU stability of Cauchy and fixed point functional equations in the context of $\mathfrak{F}$MS. We study properties, equivalence results, and Ulam-type stability for different forms of quadratic functional equations in the fourth chapter. In the fifth chapter, we discuss stability of a quartic functional equation in NAβ-NS and complete (β, p)-NS. We discuss hyperstability of a general linear functional equation in complete Q2NS in the sixth chapter. We discuss the stability of integral equations in the setting $\mathfrak{F}$MS and provide a solution for a Caputo-type nonlinear fractional integro-differential equation in the framework of OMS, in the last chapter.

The thesis's contents are as follows:

Chapter 1: Introduction

This chapter includes an introduction to the subject matter and a review of literature, as well as preliminary, thesis objectives, and a brief summary of contents of the various chapters of thesis.

Chapter 2: Existence of Fixed Point

In this chapter, we discuss some sufficient conditions for the existence of fixed point in the framework of $\mathfrak{F}$MS, O$\mathfrak{F}$MS, OMS, and complete Q2NS. Results of this chapter are published in Sharma and Chandok [109–112].

Chapter 3: Stability of Cauchy and fixed point functional equations

We study HU stability of Cauchy and fixed point functional equations in this chapter. Also, we present well-posedness of a fixed point functional equation. Results of this chapter are published in Sharma and Chandok [111].

Chapter 4: Generalized HUR stability for different forms of quadratic functional equations

This chapter deals with equivalence and generalized HUR stability of quadratic functional equations. It has been split into four sections. We investigate the equivalence for different forms of quadratic functional equations in the first section. We study the generalized HUR stability of a 3-variables quadratic functional equation in the setting of complete 2NS and complete QNS, respectively, in the second section and the third section. The results of the first section and the second section are proved in Sharma and Chandok [116], and results of the third section are published in Sharma and Chandok [114]. In the last section, we study the generalized HUR stability of a Drygas functional equation in the framework complete QNS. Results of this chapter are accepted in Sharma and Chandok [115].

Chapter 5: Ulam-type stability of a quartic functional equation

This Chapter has been split into three sections. In the first two sections, we discuss generalized HU stability of a quartic functional equation utilizing direct and fixed point methods in the setting of (β, p)-NS. In the last section, we use a direct method to investigate the generalized HU stability of a quartic functional equation in the framework of NAβ-NS. Results of this chapter are published in Sharma and Chandok [113].

Chapter 6: Generalized hyperstability of a general linear equation in a complete quasi-2-normed space

We study the generalized hyperstability of a general linear functional equation in a complete Q2NS using a fixed point approach in this chapter. Results of this chapter are published in Sharma and Chandok [112].

Chapter 7: Application to Hyers-Ulam Stability

This chapter has been split into two sections. We study the stability of integral equation in $\mathcal{F}$MS, in the first section. The findings of this section are proved in Sharma and Chandok [111]. In the last section, we provide a solution for a Caputo-type nonlinear fractional integro-differential equation in OMS. The findings of this section are published in Sharma and Chandok [109].

Chapter 2

Existence of fixed point

Introduction

In this chapter, we get a few sufficient conditions for the existence of fixed point in the framework of $\mathfrak{F}$MS, O$\mathfrak{F}$MS, OMS, complete Q2NS. Outcomes of this chapter are published in Sharma and Chandok [109–112].

2.1 Existence of fixed point in $\mathfrak{F}$-metric space

Here, we study the existence of a fixed point in the framework of $\mathfrak{F}$MS. Results of this section are accepted in Sharma and Chandok [111].

Theorem 2.1.1. *Let $(\mathscr{E}_1, \mathscr{D})$ be a $\mathfrak{F}$-complete $\mathfrak{F}$MS and $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ be a self map fulfilling*

$$\mathscr{D}(\mathscr{T}\varpi, \mathscr{T}\vartheta) \leq \varphi(\mathscr{D}(\varpi, \vartheta)),\ \forall\ \varpi, \vartheta \in \mathscr{E}_1,\ and\ \varphi \in \Phi. \tag{2.1.1}$$

Then $\mathscr{T}$ has a unique fixed point.

Proof. Choose ϖ_0 is an arbitrary element. Define $\{\varpi_r\} \subset \mathscr{E}_1$ by $\varpi_{r+1} = \mathscr{T}\varpi_r = \mathscr{T}^r\varpi_0$ for all $r \in \mathbb{N} \cup \{0\}$. We may assume that $\mathscr{D}(\varpi_0, \varpi_1) > 0$. From (2.1.1), we obtain

$$\mathscr{D}(\varpi_r, \varpi_{r+1}) = \mathscr{D}(\mathscr{T}\varpi_{r-1}, \mathscr{T}\varpi_r) \leq \varphi(\mathscr{D}(\varpi_{r-1}, \varpi_r)) \leq \ldots \leq \varphi^r(\mathscr{D}(\varpi_0, \varpi_1)).$$

Taking limit $r \to \infty$ and using the condition (Wii), we have

$$\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi_{r+1}) \leq \lim_{r\to\infty} \varphi^r(D(\varpi_0, \varpi_1)) \to 0. \tag{2.1.2}$$

Suppose that $\varepsilon > 0$, and $(f, \alpha) \in \mathfrak{F} \times [0, \infty)$ such that ([56], D_3, pp. 3) is fulfilled. By (Θ_1), there exists a $\eta > 0$ such that for $0 < t < \eta$, we get

$$f(t) < f(\varepsilon) - \alpha. \tag{2.1.3}$$

Using (2.1.2), we have $\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi_{r+1}) = 0$. Further,

$$\sum_{i=r}^{s-1} \mathscr{D}(\varpi_i, \varpi_{i+1}) = \mathscr{D}(\varpi_r, \varpi_{r+1}) + \mathscr{D}(\varpi_{r+1}, \varpi_{r+2}) + \ldots + \mathscr{D}(\varpi_{s-1}, \varpi_s). \tag{2.1.4}$$

It implies that

$$\sum_{i=r}^{s-1} \mathscr{D}(\varpi_i, \varpi_{i+1}) \leq \varphi^r(\mathscr{D}(\varpi_0, \varpi_1)) + \varphi^{r+1}(\mathscr{D}(\varpi_0, \varpi_1)) + \ldots + \varphi^{s-1}(\mathscr{D}(\varpi_0, \varpi_1)). \tag{2.1.5}$$

Hence, we have

$$\sum_{i=r}^{s-1} \mathscr{D}(\varpi_i, \varpi_{i+1}) \leq \frac{\varphi^r(\mathscr{D}(\varpi_0, \varpi_1))}{1 - \varphi(\mathscr{D}(\varpi_0, \varpi_1))}.$$

Since $\lim_{r\to\infty} \frac{\varphi^r(\mathscr{D}(\varpi_0, \varpi_1))}{1 - \varphi(\mathscr{D}(\varpi_0, \varpi_1))} = 0$, for a given $\eta > 0$ there exists $N \in \mathbb{N}$ such that $0 < \frac{\varphi^r(\mathscr{D}(\varpi_0, \varpi_1))}{1 - \varphi(\mathscr{D}(\varpi_0, \varpi_1))} < \eta$, for $r \geq N$. Hence by (2.1.3) and (Θ_1), we obtain

$$f\left(\sum_{i=n}^{m-1} \mathscr{D}(\varpi_i, \varpi_{i+1})\right) \leq f\left(\frac{\varphi^r(\mathscr{D}(\varpi_0, \varpi_1))}{1 - \varphi(\mathscr{D}(\varpi_0, \varpi_1))}\right) < f(\varepsilon) - \alpha, \quad s > r \geq N. \tag{2.1.6}$$

From ([56], D_3, pp. 3) and (2.1.6), we get

$$f(\mathscr{D}(\varpi_r, \varpi_s)) \leq f\left(\sum_{i=r}^{s-1} \mathscr{D}(\varpi_i, \varpi_{i+1})\right) + \alpha < f(\varepsilon).$$

Using Θ_1, we have $\mathscr{D}(\varpi_r, \varpi_s) < \varepsilon$, for $s, r \geq N$. Hence $\{\varpi_r\}$ is $\mathfrak{F}$- Cauchy. Due to $\mathfrak{F}$-completeness, there exists $\varpi^* \in \mathscr{E}_1$ such that $\{\varpi_r\}$ is $\mathfrak{F}$- convergent to ϖ^*, that is,

$$\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi^*) = 0. \tag{2.1.7}$$

Now, we have to demonstrate that ϖ^* is a fixed point of $\mathscr{T}$. We argue by paradox, let $\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*) > 0$. From ([56], D_3, pp. 3), we obtained

$$\begin{aligned} f(\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*)) &\leq f\left[\mathscr{D}(\mathscr{T}\varpi^*, \mathscr{T}\varpi_r) + \mathscr{D}(\mathscr{T}\varpi_r, \varpi^*)\right] + \alpha \\ &\leq f\left[\varphi(\mathscr{D}(\varpi^*, \varpi_r)) + \mathscr{D}(\varpi_{r+1}, \varpi^*)\right] + \alpha. \end{aligned}$$

Taking $r \to \infty$, using (Θ_2) and (2.1.7), we have

$$\lim_{r\to\infty} f\left[\varphi(\mathscr{D}(\varpi^*, \varpi_r)) + \mathscr{D}(\varpi_{r+1}, \varpi^*)\right] + \alpha = -\infty.$$

Therefore, $f(\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*)) \leq -\infty$ or $\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*) \leq -\infty$, which is a contradiction. So, we get $\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*) = 0$, that is $T\varpi^* = \varpi^*$.
Now to demonstrate uniqueness of fixed point. Suppose that $\mathscr{T}$ has two distinct fixed points say $\vartheta_1, \vartheta_2 \in \mathscr{E}_1$, such that $\mathscr{D}(\vartheta_1, \vartheta_2) > 0$. utilising (2.1.1) we get

$$\begin{aligned}\mathscr{D}(\vartheta_1, \vartheta_2) = \mathscr{D}(\mathscr{T}\vartheta_1, \mathscr{T}\vartheta_2) \leq \varphi(\mathscr{D}(\vartheta_1, \vartheta_2)) = \varphi(\mathscr{D}(\mathscr{T}\vartheta_1, \mathscr{T}\vartheta_2)) \leq \varphi^2(\mathscr{D}(\vartheta_1, \vartheta_2)) \leq ...\\ \leq \varphi^r(\mathscr{D}(\vartheta_1, \vartheta_2)).\end{aligned}$$

Taking $r \to \infty$, and using (Wii), we have, $\mathscr{D}(\vartheta_1, \vartheta_2) \leq 0$. Hence $\mathscr{D}(\vartheta_1, \vartheta_2) = 0$, that is, $\vartheta_1 = \vartheta_2$. ■

Example 2.1.1. *Suppose that $\mathscr{E}_1 = [0, 3]$, $\mathscr{D} : \mathscr{E}_1 \times \mathscr{E}_1 \to \mathscr{E}_1$ is a mapping defined by*

$$\mathscr{D}(\varpi, \vartheta) = (\varpi - \vartheta)^2, \ \forall\ (\varpi, \vartheta) \in \mathscr{E}_1 \times \mathscr{E}_1$$

is $\mathfrak{F}$-complete $\mathfrak{F}MS$ with $\alpha = ln(3)$ and $f(t) = ln(t)$.
Define $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ as

$$\mathscr{T}\varpi = \frac{\varpi}{2} + 1, \ \forall\ \varpi \in \mathscr{E}_1.$$

Define $\varphi : [0, 3] \to [0, 3]$ as $\varphi(t) = \frac{t}{4} + a$, where $a \leq \frac{1}{5}$, so it fulfills $\varphi \in \Phi$. Then $\mathscr{T}$ has a unique fixed point.

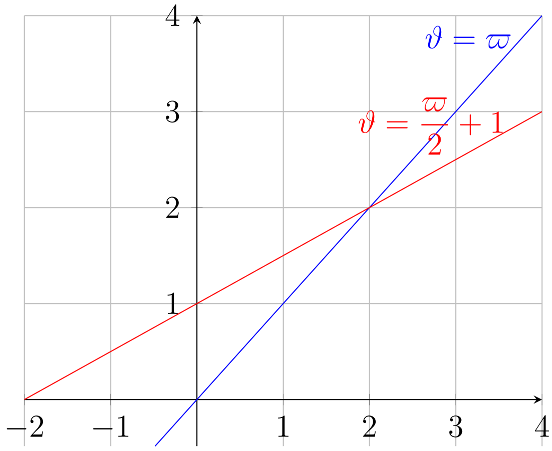

Figure 2.1: Graph of $\varpi = \frac{\varpi}{2} + 1$.

Proof. Choose $\mathscr{T}\varpi = \frac{\varpi}{2} + 1$, for each $\varpi \in \mathscr{E}_1$. Consider

$$\begin{aligned}\mathscr{D}(\mathscr{T}\varpi, \mathscr{T}\vartheta) &= \left(\frac{\varpi}{2} + 1 - \frac{\vartheta}{2} - 1\right)^2 \\ &= \left(\frac{\varpi}{2} - \frac{\vartheta}{2}\right)^2 \\ &= \frac{(\varpi - \vartheta)^2}{4} \\ &= \frac{\mathscr{D}(\varpi, \vartheta)}{4} \\ &\leq \frac{\mathscr{D}(\varpi, \vartheta)}{4} + a.\end{aligned}$$

Therefore, $\mathscr{D}(\mathscr{T}\varpi, \mathscr{T}\vartheta) \leq \varphi(\mathscr{D}(\varpi, \vartheta))$, $\varphi(t) = \frac{t}{4} + a$ is non decreasing function. Hence $\mathscr{T}$ has a unique fixed point (see Fig. 2.1). ∎

2.2 Existence of fixed point in orthogonal $\mathfrak{F}$-metric space

Gordji et al. [51] defined an O-set and showed a few fixed point results in OMS. So motivated by the work of Gordji et al. [51], we extend our results from $\mathfrak{F}$MS to O$\mathfrak{F}$MS and study the existence of fixed point in the framework of O$\mathfrak{F}$MS. Results of this section are published in Sharma and Chandok [110].

Definition 2.2.1. *Suppose that $(\mathscr{E}_1, \perp, D)$ is an O$\mathfrak{F}$MS ($(\mathscr{E}_1, \perp)$ is an O-set and $(\mathscr{E}_1, D)$ is a $\mathfrak{F}$MS).*

Example 2.2.1. *Suppose that $(\mathscr{E}_1 = [0,1], D)$ is an $\mathfrak{F}$MS with $\mathfrak{F}$-metric defined as*

$$D(\varpi, \vartheta) = \begin{cases} e^{|\varpi - \vartheta|}, \vartheta \neq \varpi \\ 0, \vartheta = \varpi, \ \forall\ \vartheta, \varpi \in \mathscr{E}_1, \end{cases}$$

$f(t) = \frac{-1}{t}, t > 0$ and $a = 1$. Define $\varpi \perp \vartheta$ as $\varpi\vartheta \leq \varpi$ or $\varpi\vartheta \leq \vartheta$. Then $\forall\ \vartheta \in \mathscr{E}_1$, $0 \perp \varpi$, so $(\mathscr{E}_1, \perp)$ is an O-set and $(\mathscr{E}_1, \perp, D)$ is an O$\mathfrak{F}$MS.

For a sequence to be Orthogonal-sequence (briefly, O-sequence) orthogonally $\mathfrak{F}$-complete (briefly, O-$\mathfrak{F}$-complete) and a self map to be $\perp$-preserving and $\perp$-continuous in O$\mathfrak{F}$MS, see [110].

Theorem 2.2.2. *Let $(\mathscr{E}_1, \perp, D)$ be an O-complete O$\mathfrak{F}$MS and $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ be a self map* *fulfilling*

$$D(\mathscr{T}\varpi, \mathscr{T}\vartheta) \leq \varphi(D(\varpi, \vartheta)), \forall\ \varpi, \vartheta \in \mathscr{E}_1, \tag{2.2.1}$$

with $\varpi \perp \vartheta$, $D(\mathscr{T}\varpi, \mathscr{T}\vartheta) > 0$ *and* $\varphi \in \Phi$. *Suppose that* $\mathscr{T}$ *is* $\perp$*-preserving and* $\perp$*-continuous. If there exists* $\varpi_0 \in \mathscr{E}_1$ *such that* $\varpi_0 \perp \mathscr{T}\varpi_0$ *or* $\mathscr{T}\varpi_0 \perp \varpi_0$*, then* $\mathscr{T}$ *has a unique fixed point in* $\mathscr{E}_1$.

Proof. Choose $\varpi_0 \in \mathscr{E}_1$ such that $\varpi_0 \perp \mathscr{T}(\varpi_0)$ or $\mathscr{T}\varpi_0 \perp \varpi_0$. Take $\varpi_1 := \mathscr{T}\varpi_0$, $\varpi_2 := \mathscr{T}\varpi_1 = \mathscr{T}^2\varpi_0$. We define sequence $\{\varpi_n\} \in \mathscr{E}_1$ by $\varpi_{n+1} = \mathscr{T}\varpi_n = \mathscr{T}^{n+1}\varpi_0\ \forall\ n \in \mathbb{N} \cup \{0\}$. Due to the $\perp$-preserving of $\mathscr{T}$, we get $\varpi_{n+1} \perp \varpi_n$ or $\varpi_n \perp \varpi_{n+1}\ \forall\ n \in \mathbb{N} \cup \{0\}$. It means that $\{\varpi_n\}$ is an orthogonal sequence.
If there exists $n_0 \in \mathbb{N}$ such that $\varpi_{n_0+1} = \varpi_{n_0}$, then ϖ_{n_0} is a fixed point of $\mathscr{T}$. Thus, we may suppose that $D(\varpi_n, \varpi_{n+1}) > 0$. On the similar lines of Theorem 2.1.1, we get $\forall\ n \in \mathbb{N}$, $D(\varpi_n, \varpi_{n+1}) \leq \varphi^n(D(\varpi_0, \varpi_1))$. Letting limit $n \to \infty$ and from (Wii) we have

$$\lim_{n\to\infty} D(\varpi_n, \varpi_{n+1}) \leq \lim_{n\to\infty} \varphi^n(D(\varpi_0, \varpi_1) \to 0. \tag{2.2.2}$$

Suppose that $\varepsilon > 0$, and $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that (D_3, pp. 3, [56]) is fulfilled.

By (Θ_1), there exists a $\delta > 0$ such that for $\delta > t > 0$, we have

$$f(t) < f(\varepsilon) - \alpha. \tag{2.2.3}$$

Suppose that $D(\varpi_0, \varpi_1) > 0$. From (2.2.2), we get $\lim\limits_{n\to\infty} D(\varpi_n, \varpi_{n+1}) = 0$. Further, we have

$$\sum_{i=n}^{m-1} D(\varpi_i, \varpi_{i+1}) = D(\varpi_n, \varpi_{n+1}) + D(\varpi_{n+1}, \varpi_{n+2}) + \ldots + D(\varpi_{m-1}, \varpi_m). \tag{2.2.4}$$

It implies that

$$\sum_{i=n}^{m-1} D(\varpi_i, \varpi_{i+1}) \leq \varphi^n(D(\varpi_0, \varpi_1)) + \varphi^{n+1}(D(\varpi_0, \varpi_1)) + \ldots + \varphi^{m-1}(D(\varpi_0, \varpi_1)). \tag{2.2.5}$$

Hence, we have

$$\sum_{i=n}^{m-1} D(\varpi_i, \varpi_{i+1}) \leq \frac{\varphi^n(D(\varpi_0, \varpi_1))}{1 - \varphi(D(\varpi_0, \varpi_1))}.$$

Since $\lim_{n\to\infty} \dfrac{\varphi^n(D(\varpi_0,\varpi_1))}{1-\varphi(D(\varpi_0,\varpi_1))} = 0$, for a given $\delta > 0$ there exists $N \in \mathbb{N}$ such that $0 < \dfrac{\varphi^n(D(\varpi_0,\varpi_1))}{1-\varphi(D(\varpi_0,\varpi_1))} < \delta$, for $n \geq N$. Hence by (2.2.3) and (Θ_1), we obtain

$$f\left(\sum_{i=n}^{m-1} D(\varpi_i,\varpi_{i+1})\right) \leq f\left(\frac{\varphi^n(D(\varpi_0,\varpi_1))}{1-\varphi(D(\varpi_0,\varpi_1))}\right) < f(\varepsilon)-\alpha, N \leq n < m. \tag{2.2.6}$$

From (D_3, pp. 3 [56]) and (2.2.6), we obtain

$$f(D(\varpi_n,\varpi_m)) \leq f\left(\sum_{i=n}^{m-1} D(\varpi_i,\varpi_{i+1})\right) + \alpha < f(\varepsilon).$$

Using Θ_1, for $n,m \geq N$, we have

$$D(\varpi_n,\varpi_m) < \varepsilon.$$

Hence $\{\varpi_n\}$ is an orthogonal $\mathfrak{F}$- Cauchy.
Here $(\mathscr{E}_1, D)$ is an orthogonal $\mathfrak{F}$-complete, there exists $\varpi^* \in \mathscr{E}_1$ such that $\{\varpi_n\}$ is orthogonal $\mathfrak{F}$- convergent to ϖ^*, that is

$$\lim_{n\to\infty} D(\varpi_n,\varpi^*) = 0. \tag{2.2.7}$$

Utilizing $\perp$-continuity of $\mathscr{T}$, we get $\mathscr{T}\varpi^* = \lim_{n\to+\infty} \mathscr{T}\varpi_n = \lim_{n\to\infty} \varpi_{n+1} = \varpi^*$. Thus ϖ^* is a fixed point of $\mathscr{T}$.

To demonstrate the fixed point's uniqueness, suppose κ^* be another fixed point of $\mathscr{T}$. Then $\forall\ n \in \mathbb{N}$, we get $\mathscr{T}^n\kappa^* = \kappa^*$. By our choice of ϖ_0, we get $\varpi_0 \perp \kappa^*$ or $\kappa^* \perp \varpi_0$. Since $\mathscr{T}$ is $\perp$-preserving, $\forall\ n \in \mathbb{N}$, we get $\mathscr{T}^n x_0 \perp \mathscr{T}^n\kappa^*$ or $\mathscr{T}^n\kappa^* \perp \mathscr{T}^n\varpi_0$.
Suppose that $D(\mathscr{T}^n\varpi_0,\kappa^*) > 0$. From ($D_3$, pp. 3, [56]), we get

$$D(\mathscr{T}^n\varpi_0,\kappa^*) \leq \varphi(D(\mathscr{T}^{n-1}\varpi_0,\kappa^*)) \leq \cdots \leq \varphi^n(D(\varpi_0,\kappa^*)).$$

Taking $n \to \infty$, we have $\lim_{n\to\infty} \varpi_n = \kappa^*$. Further, uniqueness of limit implies that $\varpi^* = \kappa^*$. Hence the result. ■

Corollary 2.2.3. *Assume that $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ is a self map on an O$\mathfrak{F}$MS $(\mathscr{E}_1,\perp,D)$ and fulfilled the following assumptions:*

(i) $\mathscr{E}_1$ is an orthogonal $\mathfrak{F}-$complete;

(ii) $\mathscr{T}$ is $\perp$-preserving and $\perp$-continuous;

(iii) there exists $k(0 < k < 1)$ *such that* $\forall\ (\varpi, \vartheta) \in \mathscr{E}_1 \times \mathscr{E}_1$ *with* $\varpi \perp \vartheta$, $D(\mathscr{T}\varpi, \mathscr{T}\vartheta) > 0$, *we get*

$$D(\mathscr{T}\varpi, \mathscr{T}\vartheta) \leq kD(\varpi, \vartheta).$$

If there exists $\varpi_0 \in \mathscr{E}_1$ *such that* $\varpi_0 \perp \mathscr{T}\varpi_0$ *or* $\mathscr{T}\varpi_0 \perp \varpi_0$*, then* $\mathscr{T}$ *has a unique fixed point.*

Example 2.2.2. *Assume that* $\mathscr{E}_1 = \{\varpi_n = \ln\left(\frac{n(n+1)}{2}\right) : n \in \mathbb{N}\}$ *fitted with* $\mathfrak{F}$*-metric provided by*

$$D(\varpi, \vartheta) = \begin{cases} e^{|\varpi - \vartheta|}, \varpi \neq \vartheta \\ 0, \varpi = \vartheta, \end{cases}$$

with $f(\mu) = \frac{-1}{\mu}$ *and* $\alpha = 1$*. For all* $\varpi_n, \varpi_w \in \mathscr{E}_1$*, define* $\varpi_n \perp \varpi_w$ *if and only if* $(w \geq 2 \wedge n = 1)$*. Then* $(\mathscr{E}_1, \perp, D)$ *is an O*$\mathfrak{F}$*MS.*

Define $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ *as*

$$\mathscr{T}(\varpi_n) = \begin{cases} \varpi_1, n = 1 \\ \varpi_{n-1}, n > 1. \end{cases}$$

Take $\phi(t) = t, t \geq 0$*. Here* $D(\mathscr{T}\varpi_n, \mathscr{T}\varpi_w) > 0$*, so for every* $w \geq 2$*, we get*

$$\begin{aligned} \frac{D(\mathscr{T}\varpi_1, \mathscr{T}\varpi_w)}{D(\varpi_1, \varpi_w)} e^{D(\mathscr{T}\varpi_1, \mathscr{T}\varpi_w) - D(\varpi_1, \varpi_w)} &= \frac{e^{\varpi_{w-1} - \varpi_1}}{e^{\varpi_w - \varpi_1}} e^{\varpi_{w-1} - \varpi_w} \\ &= \frac{w-1}{w+1} e^{-w} < e^{-1}. \end{aligned}$$

Hence, all of Theorem 2.2.2's hypotheses are fulfilled, and $\mathscr{T}$ *has a unique fixed point.*

2.3 Existence of fixed point in orthogonal metric space

We discuss existence of fixed point in OMS in this section. Results of this section are published in Sharma and Chandok [109]. Throughout this section, $(\mathscr{E}_1, \perp, d)$ is an OMS.

Definition 2.3.1. *Let* B *and* A *be two nonempty subsets of an O-set* $(\mathscr{E}_1, \perp)$*. The set* B *orthogonal to set* A *is defined as follows:*
$B \perp_1 A$*, if for every* $b \in B$ *and* $a \in A$*,* $b \perp a$*.*

The following outcomes can be easily seen.

Lemma 2.3.2. *Assume that* $(\mathscr{E}_1, \perp, d)$ *is an OMS and* $a \in A$*,* $B, A \in \mathcal{K}(\mathscr{E}_1)$*. Then*

- *there exists* $c \in A$ *such that* $d(\varpi, A) = d(\varpi, c)$*.*

- *there exists $b \in B$ such that $d(a,b) \leq H(A,B)$.*

Theorem 2.3.3. *Assume that $\mathscr{T} : \mathscr{E}_1 \to \mathcal{K}(\mathscr{E}_1)$ is a multivalued map on $\mathscr{E}_1$ and following assumptions are fulfilled:*

(i) there exists $\varpi_0 \in \mathscr{E}_1$ such that $\{\varpi_0\} \perp_1 \mathscr{T}\varpi_0$ or $\mathscr{T}\varpi_0 \perp_1 \{\varpi_0\}$,

(ii) $\forall\ \vartheta, \varpi \in \mathscr{E}_1$, $\varpi \perp \vartheta$ implies $\mathscr{T}\varpi \perp_1 \mathscr{T}\vartheta$,

(iii) $\forall\ n \in \mathbb{N}$, if $\{\varpi_n\}$ is an orthogonal sequence in $\mathscr{E}_1$ such that $\varpi_n \to \varpi^* \in \mathscr{E}_1$, then $\varpi_n \perp \varpi^*$ or $\varpi^* \perp \varpi_n$.

(iv) if $F \in \overline{\mathfrak{F}}$, there exists $\tau > 0$ such that $\forall\ \varpi, \vartheta \in \mathscr{E}_1$ with $\varpi \perp \vartheta$ fulfilling the following:

$H(\mathscr{T}\varpi, \mathscr{T}\vartheta) > 0$, $\tau + F(H(\mathscr{T}\varpi, \mathscr{T}\vartheta)) \leq F(d(\varpi, \vartheta))$.

Then $\mathscr{T}$ has at least a fixed point.

Proof. From (i), there exists $\varpi_1 \in \mathscr{T}\varpi_0$ such that $\varpi_0 \perp \varpi_1$ or $\varpi_1 \perp \varpi_0$. By (ii), we have $\mathscr{T}\varpi_0 \perp_1 \mathscr{T}\varpi_1$, that is there exists $\varpi_2 \in \mathscr{T}\varpi_1$ such that $\varpi_1 \perp \varpi_2$ or $\varpi_2 \perp \varpi_1$. If $\varpi_1 \in \mathscr{T}\varpi_1$, then ϖ_1 is a fixed point of $\mathscr{T}$. Let $\varpi_1 \notin \mathscr{T}\varpi_1$. Since $\mathscr{T}\varpi_1$ is compact, $d(\varpi_1, \mathscr{T}\varpi_1) > 0$. As $d(\varpi_1, \mathscr{T}\varpi_1) \leq H(\mathscr{T}\varpi_0, \mathscr{T}\varpi_1)$, using (F1), we have $F(d(\varpi_1, \mathscr{T}\varpi_1)) \leq F(H(\mathscr{T}\varpi_0, \mathscr{T}\varpi_1))$. Therefore, using (iv), we get

$$F(d(\varpi_1, \mathscr{T}\varpi_1)) \leq F(H(\mathscr{T}\varpi_0, \mathscr{T}\varpi_1)) \leq F(d(\varpi_0, \varpi_1)) - \tau. \tag{2.3.1}$$

Continuing this process inductively, we construct an orthogonal sequence $\{\varpi_n\}$ in $\mathscr{E}_1$ such that $\varpi_{n+1} \in \mathscr{T}\varpi_n$, for all $n \in \mathbb{N} \cup \{0\}$. Thus we have $\varpi_{n+1} \perp \varpi_n$ or $\varpi_n \perp \varpi_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

For some $k \in \mathbb{N} \cup \{0\}$, if $\varpi_k \in \mathscr{T}\varpi_k$ then ϖ_k is a fixed point of $\mathscr{T}$.
Thus, $\forall\ n \in \mathbb{N} \cup \{0\}$, we assume that $\varpi_n \notin \mathscr{T}\varpi_n$. Since $\mathscr{T}\varpi_n$ is closed, $\forall\ n \in \mathbb{N} \cup \{0\}$, we get $d(\varpi_n, \mathscr{T}\varpi_n) > 0$. Also $d(\varpi_n, \mathscr{T}\varpi_n) \leq H(\mathscr{T}\varpi_{n-1}, \mathscr{T}\varpi_n)$. So using (F1), we get $F(d(\varpi_n, \mathscr{T}\varpi_n)) \leq F(H(\mathscr{T}\varpi_{n-1}, \mathscr{T}\varpi_n))$. Further from (iv) and for every $n \geq 1$, we get

$$F(d(\varpi_n, \mathscr{T}\varpi_n)) \leq F(H(\mathscr{T}\varpi_{n-1}, \mathscr{T}\varpi_n)) \leq F(d(\varpi_{n-1}, \varpi_n)) - \tau. \tag{2.3.2}$$

Thus from the strictly increasing property of F, we get $H(\mathscr{T}\varpi_n, \mathscr{T}\varpi_{n-1}) < d(\varpi_n, \varpi_{n-1})$. We know that $\varpi_{n+1} \in \mathscr{T}\varpi_n$, $d(\varpi_n, \varpi_{n+1}) = d(\varpi_n, \mathscr{T}\varpi_n) \leq H(\mathscr{T}\varpi_{n-1}, \mathscr{T}\varpi_n) < d(\varpi_{n-1}, \varpi_n)$. Therefore, the sequence $\{d(\varpi_{n+1}, \varpi_n)\}$ is strictly decreasing sequence. Suppose that $t_n = d(\varpi_{n+1}, \varpi_n) \to t$, for some $t \geq 0$.

Further, $\forall\ n_0 \leq n$, we get

$$\tau + F(d(\varpi_{n+1}, \varpi_n)) \leq \tau + F(H(\mathscr{T}\varpi_n, \mathscr{T}\varpi_{n-1})) \leq F(d(\varpi_n, \varpi_{n-1})). \tag{2.3.3}$$

Taking $n \to +\infty$ in (2.3.3), we have $F(t+0) + \tau \leq F(t+0)$, which is a paradox and hence $t_n = d(\varpi_{n+1}, \varpi_n) \to 0$. By (F3), there exists $k(0 < K < 1)$ such that

$$\lim_{n \to +\infty} t_n^k F(t_n) = 0. \tag{2.3.4}$$

Using (2.3.2), we get

$$F(t_n) \leq F(t_{n-1}) - \tau \leq F(t_{n-2}) - 2\tau \leq \ldots \leq F(t_0) - n\tau. \tag{2.3.5}$$

From (2.3.5), following holds

$$t_n^k F(t_n) - t_n^k F(t_0) \leq -t_n^k\ n\tau \leq 0, \forall\ n \in \mathbb{N}. \tag{2.3.6}$$

Choosing $n \to \infty$ in (2.3.6), we have $\lim\limits_{n \to +\infty} n t_n^k = 0$. Therefore there exists $n_1 \in \mathbb{N}$ such that $n t_n^k \leq 1$, $\forall\ n_1 \leq n$. Thus, we get

$$t_n \leq \frac{1}{n^{\frac{1}{k}}}. \tag{2.3.7}$$

We now need to demonstrate that $\{\varpi_n\}$ is a Cauchy orthogonal sequence. Consider $m, n \in \mathbb{N}$ such that $n_1 \leq n < m$. From (2.3.7) and triangle inequality, we have

$$\begin{aligned} d(\varpi_n, \varpi_m) &\leq d(\varpi_n, \varpi_{n+1}) + d(\varpi_{n+1}, \varpi_{n+2}) + \ldots + d(\varpi_{m-1}, \varpi_m) \\ &= t_n + t_{n+1} + \ldots + t_{m-1} \\ &= \sum_{i=n}^{m-1} t_i \\ &\leq \sum_{i=n}^{+\infty} t_i \\ &\leq \sum_{i=n}^{+\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

By the convergence of series, $\sum_{i=n}^{+\infty} \frac{1}{i^{\frac{1}{k}}}$, passing to limit $n \to +\infty$, we have $d(\varpi_n, \varpi_m) \to 0$. This proves that $\{\varpi_n\}$ is a Cauchy orthogonal sequence. Due to the orthogonal completeness of $\mathscr{E}_1$, there exists $\varpi^* \in \mathscr{E}_1$ such that $\lim\limits_{n \to +\infty} \varpi_n = \varpi^*$.

We assert that $\varpi^* \in \mathscr{T}\varpi^*$. Suppose the contrary that $\varpi^* \notin \mathscr{T}\varpi^*$. So there exists $n_1 \in \mathbb{N}$

such that $\varpi^* \notin \{\varpi_n\}_{n\geq n_1}$, $H(\mathscr{T}\varpi_n, \mathscr{T}\varpi^*) > 0$. Therefore, further by our assumption, $\varpi_n \perp \varpi^*$ or $\varpi^* \perp \varpi_n$ and using (iv), we have

$$\begin{aligned} F(d(\varpi_{n+1}, \mathscr{T}\varpi^*)) &\leq \tau + F(H(\mathscr{T}\varpi_n, \mathscr{T}\varpi^*)) \\ &\leq F(d(\varpi_n, \varpi^*)). \end{aligned}$$

Using the strict increasing property of F and $\tau > 0$, we have $d(\varpi_{n+1}, \mathscr{T}\varpi^*) < d(\varpi_n, \varpi^*)$. Taking $n \to +\infty$, we get $\varpi^* \in \overline{\mathscr{T}\varpi^*} = \mathscr{T}\varpi^*$. Hence the result. ■

It should be noted that in Theorem 2.3.3, $\mathscr{T}\varpi$ is compact $\forall\ \varpi \in \mathscr{E}_1$. In 1974, Reich [101,102] asked whether a nonempty compact set can be replaced by a nonempty bounded and closed set. We provide a partial answer to Reich's problem below.

Theorem 2.3.4. *Suppose that $\mathscr{T} : \mathscr{E}_1 \to \mathcal{CB}(\mathscr{E}_1)$ is a multivalued map on $\mathscr{E}_1$ and following assumptions are fulfilled:*

(i) there exists $\varpi_0 \in \mathscr{E}_1$ such that $\{\varpi_0\} \perp_1 \mathscr{T}\varpi_0$ or $\mathscr{T}\varpi_0 \perp_1 \{\varpi_0\}$,

(ii) $\forall\ \vartheta, \varpi \in \mathscr{E}_1$, $\varpi \perp \vartheta$ implies $\mathscr{T}\varpi \perp_1 \mathscr{T}\vartheta$,

(iii) $\forall\ n \in \mathbb{N}$, if $\{\varpi_n\}$ is an orthogonal sequence in $\mathscr{E}_1$ such that $\varpi_n \to \varpi^* \in \mathscr{E}_1$, then $\varpi_n \perp \varpi^*$ or $\varpi^* \perp \varpi_n$.

(iv) if $F \in \mathfrak{F}'$, there exists $\tau > 0$ such that $\forall\ \varpi, \vartheta \in \mathscr{E}_1$ with $\varpi \perp \vartheta$ fulfilling the following:

$H(\mathscr{T}\varpi, \mathscr{T}\vartheta) > 0$, $\tau + F(H(\mathscr{T}\varpi, \mathscr{T}\vartheta)) \leq F(d(\varpi, \vartheta))$.

Then $\mathscr{T}$ has at least a fixed point.

Proof. Choose $\varpi_0 \in \mathscr{E}_1$. Since $\mathscr{T}\varpi$ is nonempty $\forall\ \varpi \in \mathscr{E}_1$, by (i), we can take $\varpi_1 \in \mathscr{T}\varpi_0$ such that $\varpi_0 \perp \varpi_1$ or $\varpi_1 \perp \varpi_0$. If $\varpi_1 \in \mathscr{T}\varpi_1$, then ϖ_1 is a fixed point of $\mathscr{T}$. Suppose that $\varpi_1 \notin \mathscr{T}\varpi_1$. Then $d(\varpi_1, \mathscr{T}\varpi_1) > 0$ since $\mathscr{T}\varpi_1$ is closed. Since $d(\varpi_1, \mathscr{T}\varpi_1) \leq H(\mathscr{T}\varpi_0, \mathscr{T}\varpi_1)$, then from (F1), we get

$$F(d(\varpi_1, \mathscr{T}\varpi_1)) \leq F(H(\mathscr{T}\varpi_0, \mathscr{T}\varpi_1)).$$

Using (iv), we get

$$F(d(\varpi_1, \mathscr{T}\varpi_1)) \leq F(H(\mathscr{T}\varpi_0, \mathscr{T}x_1)) \leq F(d(\varpi_0, \varpi_1)) - \tau. \tag{2.3.8}$$

From (F4), we gave $F(d(\varpi_1, \mathscr{T}\varpi_1)) = \inf_{\vartheta \in \mathscr{T}\varpi_1} F(d(\varpi_1, \vartheta))$. So from (2.3.8), we get

$$\begin{aligned} F(d(\varpi_1, \mathscr{T}\varpi_1)) = \inf_{\vartheta \in \mathscr{T}\varpi_1} F(d(\varpi_1, \vartheta)) &\leq F(H(\mathscr{T}\varpi_0, \mathscr{T}\varpi_1)) \\ &\leq F(d(\varpi_0, \varpi_1)) - \tau \\ &< F(d(\varpi_0, \varpi_1)) - \frac{\tau}{2}. \end{aligned} \tag{2.3.9}$$

By (ii), we get $\mathscr{T}\varpi_0 \perp_1 \mathscr{T}\varpi_1$. Proceeding this process, $\forall\ n \in \mathbb{N} \cup \{0\}$, we make an orthogonal sequence $\{\varpi_n\}$ in $\mathscr{E}_1$ such that $\varpi_{n+1} \in \mathscr{T}\varpi_n$. So, $\forall\ n \in \mathbb{N} \cup \{0\}$, we get $\varpi_n \perp \varpi_{n+1}$ or $\varpi_{n+1} \perp \varpi_n$.

If $\varpi_k \in \mathscr{T}\varpi_k$, then ϖ_k is a fixed point of $\mathscr{T}$ for some $k \in \mathbb{N} \cup \{0\}$. Thus the proof is completed.

Therefore, we may suppose that $\varpi_n \notin \mathscr{T}\varpi_n\ \forall\ n \in \mathbb{N} \cup \{0\}$. Since $\mathscr{T}\varpi_n$ is closed, we get $d(\varpi_n, \mathscr{T}\varpi_n) > 0$, $\forall\ n \in \mathbb{N} \cup \{0\}$. Also $d(\varpi_n, \mathscr{T}\varpi_n) \leq H(\mathscr{T}\varpi_{n-1}, \mathscr{T}\varpi_n)$ and from (F1), we get $F(d(\varpi_n, \mathscr{T}\varpi_n)) \leq F(H(\mathscr{T}\varpi_{n-1}, \mathscr{T}\varpi_n))$.

Further, using (iv), we get

$$F(d(\varpi_n, \mathscr{T}\varpi_n)) \leq F(H(\mathscr{T}\varpi_n, \mathscr{T}\varpi_{n+1})) \leq F(d(\varpi_n, \varpi_{n+1})) - \tau < F(d(\varpi_n, \varpi_{n+1})) - \frac{\tau}{2}. \tag{2.3.10}$$

Since $F(d(\varpi_n, \mathscr{T}\varpi_n)) = \inf_{\vartheta \in \mathscr{T}\varpi_n} F(d(\varpi_n, \vartheta))$. Therefore, using (2.3.10), we get

$$\begin{aligned} F(d(\varpi_n, \mathscr{T}\varpi_n)) = \inf_{\vartheta \in \mathscr{T}\varpi_n} F(d(\varpi_n, \vartheta)) &\leq F(H(\mathscr{T}\varpi_{n-1}, \mathscr{T}\varpi_n)) \\ &< F(d(\varpi_{n-1}, \varpi_n)) - \frac{\tau}{2}. \end{aligned} \tag{2.3.11}$$

So from (2.3.11), we can get a sequence $\{\varpi_n\}$ in $\mathscr{E}_1$ such that there exists $\varpi_{n+1} \in \mathscr{T}\varpi_n$ and $F(d(\varpi_n, \varpi_{n+1})) < F(d(\varpi_{n-1}, \varpi_n))$ for all $n \in \mathbb{N}$. Now, continuing along the lines of Theorem 2.3.3, we have the result. ■

Corollary 2.3.5. *Suppose that a map $\mathscr{T} : \mathscr{E}_1 \to \mathcal{K}(\mathscr{E}_1)$ fulfilling the following assumptions:*

(i) there exists $\varpi_0 \in \mathscr{E}_1$ such that $\{\varpi_0\} \perp_1 \mathscr{T}\varpi_0$ or $\mathscr{T}\varpi_0 \perp_1 \{\varpi_0\}$,

(ii) $\forall\ \varpi, \vartheta, \in \mathscr{E}_1$, $\varpi \perp \vartheta$ implies $\mathscr{T}\varpi \perp_1 \mathscr{T}\vartheta$,

(iii) if $\{\varpi_n\}$ is an orthogonal sequence in $\mathscr{E}_1$ such that $\varpi_n \to \varpi^* \in \mathscr{E}_1$, then $\varpi_n \perp \varpi^*$ or $\varpi^* \perp \varpi_n$, $\forall\ n \in \mathbb{N}$.

(iv) *there exists some* $\tau_i > 0$, $i = 1, 2, 3$ *such that* $\forall\, \varpi, \vartheta \in \mathscr{E}_1$ *with* $\varpi \perp \vartheta$, $H(\mathscr{T}\varpi, \mathscr{T}\vartheta) > 0$, *either of the following contractive assumptions hold*

$$\begin{aligned}
&\tau_1 + H(\mathscr{T}\varpi, \mathscr{T}\vartheta) \leq d(\varpi, \vartheta);\\
&\tau_2 - \frac{1}{H(\mathscr{T}\varpi, \mathscr{T}\vartheta)} \leq -\frac{1}{d(\varpi, \vartheta)};\\
&\tau_3 + \frac{1}{1 - e^{H(\mathscr{T}\varpi, \mathscr{T}\vartheta)}} \leq \frac{1}{1 - e^{d(\varpi, \vartheta)}}.
\end{aligned}$$

Then $\mathscr{T}$ *has at least a fixed point in each of these cases.*

Proof. As each functions $F_1(r) = r$, $F_2(r) = \frac{-1}{r}$, $F_3(r) = \frac{1}{1-e^r}$ and , where $r = d(\varpi, \varkappa) > 0$ is strictly increasing on $(0, +\infty)$, hence the proof instantly follows from Theorem 2.3.3. ■

By replacing (iii) with $\mathscr{T}$ is $\perp$-continuous, we have the following outcome for single valued map as a result of Theorem 2.3.3.

Corollary 2.3.6. *Suppose that* $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ *fulfilling the following assumptions:*

(i) *there exists some* $\tau > 0$, *such that* $\forall\, \varpi, \vartheta \in \mathscr{E}_1$ *with* $\varpi \perp \vartheta$, $d(\mathscr{T}\varpi, \mathscr{T}\vartheta) > 0$,

$$\tau + F(d(\mathscr{T}\varpi, \mathscr{T}\vartheta)) \leq F(d(\varpi, \vartheta)), \tag{2.3.12}$$

where $F \in \overline{\mathfrak{F}}$,

(ii) *there exists* $\varpi_0 \in \mathscr{E}_1$ *such that* $\varpi_0 \perp \mathscr{T}\varpi_0$ *or* $\mathscr{T}\varpi_0 \perp \varpi_0$,

(iii) $\mathscr{T}$ *is* $\perp$*-continuous,*

(iv) $\forall\, \vartheta, \varpi \in \mathscr{E}_1$, $\varpi \perp \vartheta$ *implies* $\mathscr{T}\varpi \perp \mathscr{T}\vartheta$.

Then $\mathscr{T}$ *has a fixed point.*

Proof. By considering $\mathscr{T}\varpi$ is a singleton set for every $\varpi \in \mathscr{E}_1$, we can use $\mathscr{T}$ as a multivalued map. Asserting along the same lines as Theorem 2.3.3, we consider $\{\varpi_n\}$ is a Cauchy orthogonal sequence and $\lim_{n\to\infty} \varpi_n = \varpi^*$. As $\mathscr{T}$ is $\perp$-continuous, we get

$$d(\varpi^*, \mathscr{T}\varpi^*) = \lim_{n\to+\infty} d(\mathscr{T}\varpi_n, \mathscr{T}\varpi^*) = 0,$$

that is, ϖ^* is a fixed point of $\mathscr{T}$. ■

Taking $F(r) = \ln r$, $r > 0$, we have following outcome as a consequence of Corollary 2.3.6.

Corollary 2.3.7. *Suppose that $\mathscr{T}:\mathscr{E}_1\to\mathscr{E}_1$ fulfilling the following assumptions:*

(i) there exists some $\tau>0$, such that $\forall\ \varpi,\vartheta\in\mathscr{E}_1$ with $\varpi\perp\vartheta$, $d(\mathscr{T}\varpi,\mathscr{T}\vartheta)>0$,

$$d(\mathscr{T}\varpi,\mathscr{T}\vartheta)\le e^{-\tau}d(\varpi,\vartheta), \tag{2.3.13}$$

where $F\in\overline{\mathfrak{F}}$,

(ii) there exists $\varpi_0\in\mathscr{E}_1$ such that $\varpi_0\perp\mathscr{T}\varpi_0$ or $\mathscr{T}\varpi_0\perp\varpi_0$,

(iii) $\mathscr{T}$ is $\perp$-continuous,

(iv) $\forall\ \vartheta,\varpi\in\mathscr{E}_1$, $\varpi\perp\vartheta$ implies $\mathscr{T}\varpi\perp\mathscr{T}\vartheta$.

Then $\mathscr{T}$ has a fixed point.

Here, we provide an example, which proves that $\mathscr{T}$ is a multivalued orthogonal map and fulfills (iv) of Theorem 2.3.3, but it is not multivalued orthogonal contraction ($H(\mathscr{T}\varpi,\mathscr{T}\vartheta)\le kd(\varpi,\vartheta)$, for $k\in[0,1)$ with $\varpi\perp\vartheta$).

Example 2.3.1. *Assume that $\mathscr{E}_1=\left\{S_n=\frac{n(n+1)}{2}:n\in\mathbb{N}\right\}$ and $d:\mathscr{E}_1\times\mathscr{E}_1\to[0,\infty)$ is a map defined by $d(\varpi,\vartheta)=|\varpi-\vartheta|\ \forall\ \vartheta,\varpi\in\mathscr{E}_1$.*

Define a relation $\perp$ on $\mathscr{E}_1$ by $\varpi\perp\vartheta$ if and only if $\varpi\vartheta\in\{\varpi,\vartheta\}\subseteq\mathscr{E}_1=\{S_n\}$.

Thus $(\mathscr{E}_1,\perp,d)$ is an O-complete OMS.
We define a map $\mathscr{T}:\mathscr{E}_1\to\mathcal{K}(\mathscr{E}_1)$ by

$$\mathscr{T}\varpi=\begin{cases}\{\varpi_1\}, & \varpi=\varpi_1\\ \{\varpi_1,\ldots,\varpi_{n-1}\}, & \varpi=\varpi_n, n\ge 1.\end{cases}$$

We assert that $\mathscr{T}$ is a multivalued orthogonal map fulfilling (iv) of Theorem 2.3.3 with respect to $F(\alpha)=\ln(\alpha)+\alpha,\alpha>0$ and $\tau=1$. To prove this, we have the following situations.

We notice that $\forall\ n,m\in\mathbb{N}$, [$H(\mathscr{T}\varpi,\mathscr{T}\vartheta)>0$ iff (($n=1$ and $m>2$) or $(1<n<m)$)].

Case 1. For $n = 1$ and $m > 2$, we get

$$\begin{aligned}
&\frac{H(\mathscr{T}\varpi_m, \mathscr{T}\varpi_1)}{d(\varpi_m, \varpi_1)} e^{H(\mathscr{T}\varpi_m, \mathscr{T}\varpi_1) - d(\varpi_m, \varpi_1)} \\
&= \frac{\varpi_{m-1} - \varpi_1}{\varpi_m - \varpi_1} e^{\varpi_{m-1} - \varpi_1} \\
&= \frac{m^2 - m - 2}{m^2 + m - 2} e^{-m} < e^{-m} < e^{-1}.
\end{aligned}$$

Case 2. For $1 < n < m$, we have

$$\begin{aligned}
&\frac{H(\mathscr{T}\varpi_m, \mathscr{T}\varpi_n)}{d(\varpi_m, \varpi_n)} e^{H(\mathscr{T}\varpi_m, \mathscr{T}\varpi_n) - d(\varpi_m, \varpi_n)} \\
&= \frac{\varpi_{m-1} - \varpi_{n-1}}{\varpi_m - \varpi_n} e^{\varpi_{m-1} - \varpi_{n-1} - \varpi_m + \varpi_n} \\
&= \frac{m + n - 1}{m + n + 1} e^{n-m} < e^{n-m} \leq e^{-1}.
\end{aligned}$$

This proves that $\mathscr{T}$ fulfills (iv) of Theorem 2.3.3. Hence $\mathscr{T}$ has a fixed point.

In contrast, $\mathscr{T}$ is not multivalued orthogonal contraction ($H(\mathscr{T}\varpi, \mathscr{T}\vartheta) \leq kd(\varpi, \vartheta)$, $k \in [0, 1)$), as

$$\lim_{n \to +\infty} \frac{H(\mathscr{T}\varpi_n, \mathscr{T}\varpi_1)}{d(\varpi_n, \varpi_1)} = \lim_{n \to +\infty} \frac{\varpi_{n-1} - 1}{\varpi_n - 1} = 1.$$

2.4 Existence of fixed point in complete quasi-2-normed space

In this section, we extend Brzdęk and Ciepliński's fixed point result [27] to Q2NS. Results of this section are published in Sharma and Chandok [112].

Theorem 2.4.1. *(1) Let $\mathscr{E}_2$ be a complete Q2NS, $\mathscr{E}_1$ be a nonempty set, and Z_0 be a subset of $\mathscr{E}_2$ containing two LI vectors, $f_i : \mathscr{E}_1 \to \mathscr{E}_1$, $g_i : Z_0 \to Z_0$ and $l_i : \mathscr{E}_1 \times Z_0 \to \mathbb{R}_+$ for $i = 1, \cdots, k, k \in \mathbb{N}$.*

(2) Assume that $\mathscr{T} : \mathscr{E}_2^{\mathscr{E}_1} \to \mathscr{E}_2^{\mathscr{E}_1}$ is an operator satisfies

$$\|\mathscr{T}\xi(\varpi) - \mathscr{T}\mu(\varpi), z\| \leq \sum_{i=1}^{k} l_i(\varpi, z)\|(\xi - \mu)f_i(\varpi), g_i(z)\|, \forall\ \mu, \xi \in \mathscr{E}_2^{\mathscr{E}_1}, z \in Z_0, \varpi \in \mathscr{E}_1. \tag{2.4.1}$$

(3) Further suppose that $\varepsilon : \mathscr{E}_1 \times Z_0 \to \mathbb{R}_+$ and $\varphi : \mathscr{E}_1 \to \mathscr{E}_2$ satisfying the following

assumptions

$$\|\mathscr{T}\varphi(\varpi)-\varphi(\varpi),z\|\leq\varepsilon(\varpi,z), \tag{2.4.2}$$

for every $\varpi\in\mathscr{E}_1$ *and* $r=\log_{2K}2$,

$$\varepsilon^*(\varpi,z)=\sum_{n=0}^{\infty}\left(\Lambda^n\varepsilon\right)^r(\varpi,z)<\infty, \tag{2.4.3}$$

where $\Lambda:\mathbb{R}_+^{\mathscr{E}_1\times Z_0}\to\mathbb{R}_+^{\mathscr{E}_1\times Z_0}$ *is a linear operator defined by*

$$\Lambda\delta(\varpi,z):=\sum_{i=1}^{k}l_i(\varpi,z)\delta(f_i(\varpi),g_i(z)), \tag{2.4.4}$$

for $\delta\in\mathbb{R}_+^{\mathscr{E}_1\times Z_0}$, $z\in Z_0$ *and* $\varpi\in\mathscr{E}_1$.
Then
(1) there exists a fixed point ψ *of* Υ *with*

$$\|\varphi(\varpi)-\psi(\varpi),z\|^r\leq 4\varepsilon^*(\varpi,z),\ \forall\ \varpi\in\mathscr{E}_1,z\in Z_0. \tag{2.4.5}$$

Additionally, for every $\varpi\in\mathscr{E}_1$,

$$\lim_{n\to\infty}\Upsilon^n\varphi(\varpi)=\psi(\varpi). \tag{2.4.6}$$

(2) For every $\varpi\in\mathscr{E}_1$, $z\in Z_0$ *if*

$$\varepsilon^*(\varpi,z)\leq\left(M\sum_{n=0}^{\infty}\left(\Lambda^n\varepsilon\right)(\varpi,z)\right)^r<\infty, \tag{2.4.7}$$

then fixed point of $\mathscr{T}$ *is unique for a few positive real number* M.

Proof. We demonstrate by using induction that, for every $n\in\mathbb{N}_0$, $\varpi\in\mathscr{E}_1$ and $z\in Z_0$

$$\|(\mathscr{T}^n\varphi)(\varpi)-(\mathscr{T}^{n+1}\varphi)(\varpi),z\|\leq\left(\Lambda^n\varepsilon\right)(\varpi,z). \tag{2.4.8}$$

In fact, by (2.4.2), we have

$$\|\varphi(\varpi)-(\mathscr{T}\varphi)(\varpi),z\|\leq\left(\Lambda^0\varepsilon\right)(\varpi,z)=\varepsilon(\varpi,z).$$

Then the condition $n=0$ is correct. Assume that, for some $n\in\mathbb{N}$, (2.4.8) is correct.

Then from (2.4.1) and using induction, we have

$$\begin{aligned}\|(\mathscr{T}^{n+1}\varphi)(\varpi)-(\mathscr{T}^{n+2}\varphi)(\varpi),z\| &\le \sum_{i=1}^{k} l_i(\varpi,z)\|(\mathscr{T}^{n}\varphi)(f_i(\varpi))-(\mathscr{T}^{n+1}\varphi)(f_i(\varpi)),(g_i(z))\| \\ &\le \sum_{i=1}^{k} l_i(\varpi,z)\left(\Lambda^n\varepsilon\right)\left(f_i(\varpi),g_i(z)\right)\le\left(\Lambda^{n+1}\varepsilon\right)(\varpi,z).\end{aligned}$$

Therefore (2.4.8) holds $\forall\ n\in\mathbb{N}_0$.

For $l\ge 1$ and $n\in\mathbb{N}$, by (2.4.8) and (1.3.6) and using Lemma 1.3.6, we have

$$\begin{aligned}|||(\mathscr{T}^{n}\varphi)(\varpi)-(\mathscr{T}^{n+l}\varphi)(\varpi),z|||^r &\le \sum_{i=0}^{l-1}|||(\mathscr{T}^{n+i}\varphi(\varpi))-(\mathscr{T}^{n+i+1}\varphi(\varpi)),g_i(z)|||^r \\ &\le \sum_{i=0}^{l-1}\|\ (\mathscr{T}^{n+i}\varphi(\varpi))-(\mathscr{T}^{n+i+1}\varphi(\varpi)),g_i(z)\ \|^r \\ &= \sum_{i=0}^{l-1}\left(\Lambda^{n+i}\varepsilon\right)^r(\varpi,z) \\ &\le \sum_{i=n}^{n+l-1}\left(\Lambda^{i}\varepsilon\right)^r(\varpi,z)\le\sum_{i=n}^{\infty}\left(\Lambda^{i}\varepsilon\right)^r(\varpi,z).\end{aligned}\tag{2.4.9}$$

By combining (2.4.9) and (2.4.3), the sequence $\{(\mathscr{T}^n\phi)(x)\}$ is a quasi-2-Cauchy sequence in the equivalent Q2NS $(\mathscr{E}_2,|||.,.|||)$. Using (1.3.6), $\{(\mathscr{T}^n\phi)(\varpi)\}$ is also quasi-2-Cauchy in $(\mathscr{E}_2,\|.,.\|,K)$. By the completeness of $\mathscr{E}_2$, there exists the following limit in $(\mathscr{E}_2,\|.,.\|,K)$

$$\lim_{n\to\infty}(\mathfrak{T}^n\phi)(\varpi)=\psi(\varpi).\tag{2.4.10}$$

Thus (2.4.6) holds. From (2.4.9), we have $|||(\mathscr{T}^{n}\varphi)(\varpi)-(\mathscr{T}^{n+l}\varphi)(\varpi),z|||^r\le\varepsilon^*(\varpi,z)$. Therefore for all $l\ge 1$ and $n\in\mathbb{N}$, we get

$$|||\varphi(\varpi)-(\mathscr{T}^{l}\varphi)(\varpi),z|||^r\le\varepsilon^*(\varpi,z).\tag{2.4.11}$$

Letting $l\to\infty$ in (2.4.11) and using (2.4.10), and continuity of the equivalent Q2N $|||.,.|||$ by Lemma 1.3.7, we have

$$|||\varphi(\varpi)-\psi(\varpi),z|||^r\le\varepsilon^*(\varpi,z).\tag{2.4.12}$$

According to (1.3.6), $\|\ \varphi(\varpi)-\psi(\varpi),z\ \|^r\le 4\varepsilon^*(\varpi,z)$. So (2.4.5) valid. From (2.4.1) and

(1.3.6) we get $\forall\, n \in \mathbb{N}$, that

$$\begin{aligned} &\| (\mathscr{T}^{n+1}\varphi)(\varpi) - (\mathscr{T}\psi)(\varpi), z \| \\ =&\| (\mathscr{T}\mathscr{T}^{n}\varphi)(\varpi) - (\mathscr{T}\psi)(\varpi), z \| \\ \le& \sum_{i=1}^{k} l_i(\varpi, z) \| (\mathscr{T}^{n}\varphi)(f_i(\varpi)) - (\mathscr{T}\psi)(f_i(\varpi)), g_i(z) \| \\ \le& 4^{\frac{1}{r}} \sum_{i=1}^{k} l_i(\varpi, z) |||(\mathscr{T}^{n}\varphi)(f_i(\varpi)) - (\mathscr{T}\psi)(f_i(\varpi)), g_i(z)|||. \end{aligned} \tag{2.4.13}$$

Taking $n \to \infty$ in (2.4.13) and from (2.4.10), we get

$$\lim_{n\to\infty} \| (\mathscr{T}^{n+1}\varphi)(\varpi) - (\mathscr{T}\psi)(\varpi), z \| = 0,$$

that is $(\mathscr{T}^{n+1}\varphi)(\varpi) = (\mathscr{T}\psi)(\varpi)$. Using this with (2.4.10), we have $(\mathscr{T}\psi)(\varpi) = \psi(\varpi)$, $\forall\, \varpi \in \mathscr{E}_1$. This demonstrates that $\mathscr{T}\psi = \psi$. Thus ψ is a fixed point of $\mathscr{T}$ that fulfills (2.4.5).

Furthermore, assume that ϕ is also a fixed point of $\mathscr{T}$ fulfilling (2.4.5). Now, we have to demonstrate that for every $m \in \mathbb{N}$ and $\varpi \in \mathscr{E}_1, z \in Z_0$,

$$\| \psi(\varpi) - \phi(\varpi), z \| \equiv \| (\mathscr{T}^{m}\psi(\varpi) - (\mathscr{T}^{m}\phi)(\varpi), z \| \le 8^{\frac{1}{r}} M \sum_{i=m}^{\infty} \left(\Lambda^i \varepsilon\right)(\varpi, z). \tag{2.4.14}$$

Indeed, for $m = 0$, and using (2.4.12), we have

$$|||\psi(\varpi) - \phi(\varpi), z|||^r \le |||\psi(\varpi) - \varphi(\varpi), z|||^r + |||\varphi(\varpi) - \phi(\varpi), z|||^r \le 2\varepsilon^*(\varpi, z). \tag{2.4.15}$$

From (1.3.6) and (2.4.7), we get

$$\| \psi(\varpi) - \phi(\varpi), z \|^r \le 8\varepsilon^*(\varpi, z) \le 8 \left(M \sum_{i=0}^{\infty} (\Lambda^i \varepsilon)(\varpi, z) \right)^r.$$

Thus (2.4.14) valid for $m = 0$. Suppose that (2.4.14) valid for some $m \in \mathbb{N}$. From (2.4.1) and (1.3.6), we get

$$\begin{aligned}
&|||(\mathscr{T}^{m+1}\psi)(\varpi) - (\mathscr{T}^{m+1}\phi)(\varpi), z|||\\
&\leq\| (\mathscr{T}^{m+1}\psi)(\varpi) - (\mathscr{T}^{m+1}\phi)(\varpi), z \|\\
&=\| (\mathscr{T}\mathscr{T}^{m}\psi)(\varpi) - (\mathscr{T}\mathscr{T}^{m}\phi)(\varpi), z \|\\
&\leq \sum_{i=1}^{k} l_i(\varpi, z) \| (\mathfrak{T}^{m}\psi)(f_i(\varpi)) - (\mathfrak{T}^{m}\phi)(f_i(\varpi)), g_i(z) \|\\
&\leq \sum_{i=1}^{k} l_i(\varpi, z) \left(8^{\frac{1}{r}} M \sum_{j=m}^{\infty} (\Lambda^j \varepsilon)(f_j(\varpi)), g_j(z) \right)\\
&= 8^{\frac{1}{r}} M \sum_{j=m}^{\infty} \sum_{i=1}^{k} l_i(\varpi, z)(\Lambda^j\varepsilon)\left((f_j(\varpi)), g_j(z)\right)\\
&= 8^{\frac{1}{r}} M \sum_{j=m}^{\infty} \Lambda \left(\Lambda^j \varepsilon\right)(\varpi, z)\\
&= 8^{\frac{1}{r}} M \sum_{j=m+1}^{\infty} \left(\Lambda^j \varepsilon\right)(\varpi, z).
\end{aligned}$$

Therefore, (2.4.14) valid $\forall\ m \in \mathbb{N}$. Taking $m \to \infty$ and from (2.4.7), we have $|||\psi(\varpi) - \phi(\varpi), z|||^{\frac{1}{r}} = 0$, that is, $\psi = \phi$. Thus fixed point fulfilling (2.4.5) of $\mathscr{T}$ is unique. ■

Remark 2.4.2. *If $\mathscr{E}_2$ is complete 2NS in Theorem 2.4.1, then $K = 1$, $r = 1$ and we get [Theorem 1, p. 380, [27]]*

Chapter 3

Stability of fixed point and Cauchy functional equations

Introduction

We study HU stability of fixed point and Cauchy functional equations in this chapter. Also, we present the well-posedness of a fixed point functional equation. Results of this chapter are published in Sharma and Chandok [111].

3.1 Stability of fixed point functional equation

We study well-posedness and HU stability of fixed point functional equation in this section. Results of this section are published in Sharma and Chandok [111].

Definition 3.1.1. *Assume that $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ is an operator on an $\mathfrak{F}$MS $(\mathscr{E}_1, \mathscr{D})$. The fixed point equation*

$$\varpi = \mathscr{T}(\varpi), \quad \varpi \in \mathscr{E}_1 \tag{3.1.1}$$

is HU stable if there exists a strictly increasing and surjective map $\beta : [0, \infty) \to [0, \infty)$ with $\beta(t) = t - \varphi(t)$, $t \in [0, \infty)$, where $\varphi \in \Phi$ and such that for each $\varepsilon > 0$ and each solution ϑ^* of $\mathscr{D}(\vartheta, \mathscr{T}(\vartheta)) < \varepsilon$, for each $\vartheta \in \mathscr{E}_1$, there exists a solution ϖ^* of (3.1.1) *such that*

$$\mathscr{D}(\vartheta^*, \varpi^*) < \beta^{-1}(\varepsilon).$$

Definition 3.1.2. *If a fixed point problem (3.1.1) for $\mathscr{T}$ meets the following conditions, then it is well-posed*

(p1) $\mathscr{T}$ has a unique fixed point $\varpi^* \in \mathscr{E}_1$,

(p2) if for any sequence $\{\varpi_r\}$ *in* $\mathscr{E}_1$ *such that*

$$\lim_{r\to\infty} \mathscr{D}(\mathscr{T}\varpi_r, \varpi_r) = 0,$$

then

$$\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi^*) = 0.$$

Theorem 3.1.3. *Assume that all of Theorem 2.1.1's hypotheses are true. Then the following conditions are valid:*

$(A1)$ *The fixed point equation (3.1.1) is HU stable, that is, if for each* $\varepsilon > 0$ *and each solution* ϑ^* *of* $\mathscr{D}(\vartheta, \mathscr{T}(\vartheta)) < \varepsilon$*, for each* $\vartheta \in \mathscr{E}_1$*, there exists a solution* ϖ^* *of (3.1.1) such that*

$$\mathscr{D}(\vartheta^*, \varpi^*) < \beta^{-1}(\varepsilon).$$

$(A2)$ *If* $\{\varpi_r\}$ *is a sequence in* $\mathscr{E}_1$ *such that* $\lim_{r\to\infty} \mathscr{D}(\mathscr{T}\varpi_r, \varpi_r) = 0$ *and* ϖ^* *is a fixed point of* $\mathscr{T}$*, then (3.1.1) is well-posed.*

Proof. $(A1)$ Using Theorem 2.1.1, there is a unique $\varpi^* \in \mathscr{E}_1$ such that $\varpi^* = \mathscr{T}\varpi^*$ that is $\varpi^* \in \mathscr{E}_1$ is solution of fixed point equation ($\varpi = \mathscr{T}\varpi$). Assume that $\varepsilon > 0$ and $\vartheta^* \in \mathscr{X}$. Using ([56], D_3, pp. 3), we get

$$\begin{aligned} f(\mathscr{D}(\vartheta^*, \varpi^*)) &\leq f[\mathscr{D}(\vartheta^*, \mathscr{T}\vartheta^*) + \mathscr{D}(\mathscr{T}\vartheta^*, \varpi^*)] + \alpha \\ &\leq f[\varepsilon + \mathscr{D}(\mathscr{T}\vartheta^*, \mathscr{T}\varpi^*)] + \alpha \\ &\leq f[\varepsilon + \varphi(\mathscr{D}(\vartheta^*, \varpi^*))] + \alpha. \end{aligned}$$

Hence using property of (Θ_1), we have $\mathscr{D}(\vartheta^*, \varpi^*) \leq \varepsilon + \varphi(\mathscr{D}(\vartheta^*, \varpi^*))$, or $\mathscr{D}(\vartheta^*, \varpi^*) - \varphi\mathscr{D}(\vartheta^*, \varpi^*) \leq \varepsilon$. Further, we have $\beta(\mathscr{D}(\vartheta^*, \varpi^*)) \leq \varepsilon$. Hence

$$\mathscr{D}(\vartheta^*, \varpi^*) \leq \beta^{-1}(\varepsilon),$$

which completes the proof.
$(A2)$ If $\{\xi_r\}$ is a sequence in $\mathscr{E}_1$ such that $\lim_{r\to\infty} \mathscr{D}(\mathscr{T}\xi_r, \xi_r) = 0$ and ϖ^* is a unique fixed point of $\mathscr{T}$ (using Theorem 2.1.1). From the contractive condition and triangle inequality,

we have

$$\begin{aligned} f(\mathscr{D}(\xi_r, \varpi^*)) &\le f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \mathscr{D}(\mathscr{T}\xi_r, \varpi^*)] + \alpha \\ &\le f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \mathscr{D}(\mathscr{T}\xi_r, \mathscr{T}\varpi^*)] + \alpha \\ &\le f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \varphi(\mathscr{D}(\xi_r, \varpi^*))] + \alpha. \end{aligned}$$

On the same lines of above cases, we have $\beta(\mathscr{D}(\xi_r, \varpi^*)) \le \mathscr{D}(\xi_r, \mathscr{T}\xi_r)$. Taking limit $r \to \infty$, we get

$$\lim_{r\to\infty} \beta(\mathscr{D}(\xi_r, \varpi^*)) \le \lim_{r\to\infty} \mathscr{D}(\xi_r, \mathscr{T}\xi_r).$$

Therefore, $\lim_{r\to\infty} \beta(\mathscr{D}(\xi_r, \varpi^*)) = 0$. Hence $\mathscr{D}(\xi_r, \varpi^*) = 0$. This shows that fixed point functional equation (3.1.1) is well-posed. ■

Theorem 3.1.4. *Assume that all of Theorem 2.1.1's hypotheses are true. If* $\mathscr{R} : \mathscr{E}_1 \to \mathscr{E}_1$ *is a map such that there exists* $\Lambda > 0$ *with*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{R}\xi) < \Lambda, \forall\ \xi \in \mathscr{E}_1,$$

then for any fixed point ϑ^* *of* $\mathscr{R}$*, we get*

$$\mathscr{D}(\varpi^*, \vartheta^*) \le \beta^{-1}(\Lambda).$$

Proof. Choose ϑ^* be the fixed point of $\mathscr{R}$ then by the property of ([56], D_3, pp. 3), we get

$$\begin{aligned} f(\mathscr{D}(\varpi^*, \vartheta^*)) &\le f(\mathscr{D}(\varpi^*, \vartheta^*)) + \alpha \\ &\le f(\mathscr{D}(\mathscr{T}\varpi^*, \mathscr{R}\vartheta^*)) + \alpha \\ &\le f[\mathscr{D}(\mathscr{T}\varpi^*, \mathscr{T}\vartheta^*) + \mathscr{D}(\mathscr{T}\vartheta^*, \mathscr{R}\vartheta^*)] + \alpha \\ &\le f[\varphi(\mathscr{D}(\varpi^*, \vartheta^*)) + \mathscr{D}(\mathscr{T}\vartheta^*, \mathscr{R}\vartheta^*)] + \alpha \\ &\le f[\varphi(\mathscr{D}(\varpi^*, \vartheta^*)) + \Lambda] + \alpha. \end{aligned}$$

Using the property of Θ_1, we have

$$\mathscr{D}(\varpi^*, \vartheta^*) \le \varphi(\mathscr{D}(\varpi^*, \vartheta^*)) + \Lambda.$$

It implies $\mathscr{D}(\varpi^*, \vartheta^*) - \varphi(\mathscr{D}(\varpi^*, \vartheta^*)) \le \Lambda$. Therefore, we get $\beta(\mathscr{D}(\varpi^*, \vartheta^*)) \le \Lambda$ or $\mathscr{D}(\varpi^*, \vartheta^*) \le \beta^{-1}(\Lambda)$. Hence the result. ■

3.2 Stability of Cauchy functional equation

We study stability of Cauchy functional equation in $\mathfrak{F}$-linear metric space ($\mathfrak{F}$LMS) in this section. Results of this section are published in Sharma and Chandok [111].

Definition 3.2.1. *Let $(\mathscr{E}_1, \mathscr{D})$ is an $\mathfrak{F}$MS. If $\mathscr{D}$ satisfying the following condition:*

$(\mathscr{D}_4)$ $\quad \mathscr{D}(\lambda\vartheta, \lambda\varpi) \leq |\lambda| \mathscr{D}(\vartheta, \varpi), \quad \lambda \in \mathbb{R}.$

Then $\mathscr{D}$ is an $\mathfrak{F}$-linear metric on $\mathscr{E}_1$, and pair $(\mathscr{E}_1, \mathscr{D})$ is said to be an $\mathfrak{F}$LMS.

Theorem 3.2.2. *Assume that $(\mathscr{E}_1, \mathscr{D})$ is an $\mathfrak{F}$-complete $\mathfrak{F}$LMS and $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ is a self map satisfying 2.1.1. Assume that $R : \mathscr{E}_1 \to \mathscr{E}_1$ be a map such that for each $\varepsilon_1 > 0$*

$$\mathscr{D}(R(\vartheta + \varpi), R(\vartheta) + R(\varpi)) < \varepsilon_1, \ \forall \ \varpi, \vartheta \in \mathscr{E}_1. \tag{3.2.1}$$

Then there exists a unique map $q : \mathscr{E}_1 \to \mathscr{E}_1$ fulfills

$$\mathscr{D}(R(\vartheta), q(\vartheta)) < \beta^{-1}(\varepsilon), \tag{3.2.2}$$

where $\varepsilon = \dfrac{\varepsilon_1}{2}$.

Proof. Substitute $\vartheta = \varpi$ in (3.2.1) and using Definition 3.2.1, we have

$$\mathscr{D}\left(R(2\vartheta), 2R(\vartheta)\right) \leq |2| \mathscr{D}\left(\frac{1}{2}R(2\vartheta), R(\vartheta)\right) < \varepsilon_1.$$

So, we get

$$\mathscr{D}\left(\frac{1}{2}R(2\vartheta), R(\vartheta)\right) < \frac{\varepsilon_1}{2}. \tag{3.2.3}$$

We define an operator $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ by

$$\mathscr{T}R(\vartheta) = \frac{1}{2}R(\vartheta). \tag{3.2.4}$$

Then (3.2.3) becomes

$$\mathscr{D}\left(\mathscr{T}R(\vartheta), R(\vartheta)\right) < \varepsilon, \tag{3.2.5}$$

where $\dfrac{\varepsilon_1}{2} = \varepsilon$. Now we have to demonstrate that there exists a unique map $q : \mathscr{E}_1 \to \mathscr{E}_1$

fulfills

$$\mathscr{D}(R(\vartheta), q(\vartheta)) < \beta^{-1}(\varepsilon).$$

To prove this, using Theorem 2.1.1, there is a unique $q(\vartheta)$ such that $q(\vartheta) = \mathscr{T}q(\vartheta)$. Assume that $\varepsilon > 0$ and $\vartheta \in \mathscr{E}_1$. Using ([56], D_3, pp. 3), we have

$$\begin{aligned} f(\mathscr{D}(q(\vartheta), R(\vartheta))) &\leq f[\mathscr{D}(q(\vartheta), \mathscr{T}R(\vartheta)) + \mathscr{D}(\mathscr{T}R(\vartheta), R(\vartheta))] + \alpha \\ &\leq f[\mathscr{D}(\mathscr{T}q(\vartheta), \mathscr{T}R(\vartheta)) + \varepsilon] + \alpha \\ &\leq f[\varphi(\mathscr{D}(q(\vartheta), R(\vartheta))) + \varepsilon] + \alpha. \end{aligned}$$

Hence using property of (Θ_1), we have $\mathscr{D}(q(\vartheta), R(\vartheta)) \leq \varepsilon + \varphi(\mathscr{D}(q(\vartheta), R(\vartheta)))$, or $\mathscr{D}(q(\vartheta), R(\vartheta)) - \varphi(\mathscr{D}(q(\vartheta), R(\vartheta))) \leq \varepsilon$. Further, we have $\beta(\mathscr{D}(q(\vartheta), R(\vartheta))) \leq \varepsilon$. Hence

$$\mathscr{D}(q(\vartheta), R(\vartheta)) \leq \beta^{-1}(\varepsilon).$$

Finally, we have to demonstrate the uniqueness part. To prove this, assume that there exists a map $q_1 : \mathscr{E}_1 \to \mathscr{E}_1$ $(q \neq q_1)$ such that

$$\mathscr{D}(R(\vartheta)), q_1(\vartheta)) < \beta^{-1}(\varepsilon).$$

Now, Using ([56], D_3, pp. 3), we have

$$\begin{aligned} f(\mathscr{D}(q(\vartheta), q_1(\vartheta))) &\leq f[\mathscr{D}(q(\vartheta), R(\vartheta)) + \mathscr{D}(R(\vartheta), q_1(\vartheta))] \\ &\leq f[\beta^{-1}(\varepsilon) + \beta^{-1}(\varepsilon)]. \end{aligned}$$

Using (Θ_1), we have

$$\mathscr{D}(q(\vartheta), q_1(\vartheta)) \leq \beta^{-1}(\varepsilon) + \beta^{-1}(\varepsilon). \tag{3.2.6}$$

Since $\beta : [0, \infty) \to [0, \infty)$ is a strictly increasing and surjective function, so for any given $\varepsilon \geq 0$ there exists a $\delta > 0$ such that $\beta(\frac{\delta}{2}) = \varepsilon$ or $\beta^{-1}(\varepsilon) = \frac{\delta}{2}$. Therefore equation (3.2.6) implies that

$$\mathscr{D}(q(\vartheta), q_1(\vartheta)) \leq \delta,$$

which completes the proof. ■

Chapter 4

Generalized HUR stability for different forms of quadratic functional equations

Introduction

We study equivalence and generalized HUR stability of quadratic functional equations in this chapter. It has been split into four sections. We investigate the equivalence for different forms of quadratic functional equations in the first section. In the second and third sections, we discuss generalized HUR stability of a 3-variables quadratic functional equation in the setting of complete 2NS and complete QNS, respectively. The results of the first section and the second section are proved in Sharma and Chandok [116], and results of the third section are published in Sharma and Chandok [114]. We study the generalized HUR stability of a Drygas functional equation in the framework of complete QNS in the last section. Outcomes of this section are proved in Sharma and Chandok [115].
Throughout this chapter $\mathscr{E}_1$ is a nonempty set, we write $\mathscr{E}_1\backslash\{0\} := \mathscr{E}_0$. $Aut(\mathscr{E}_1)$ stands for the family of all automorphisms of $\mathscr{E}_1$. $Id_{\mathscr{E}_1}$ will be used to represent the identity function on $\mathscr{E}_1$.

4.1 Equivalence of quadratic functional equations

We study some properties and equivalence of functional equations (1.1.4), (1.1.5), and (1.1.6) in this section. Throughout this section, $\mathscr{E}_2$ and $\mathscr{E}_1$ are vector spaces. Results of this section are proved in Sharma and Chandok [116].

Lemma 4.1.1. *Suppose that a function $g : \mathscr{E}_1 \to \mathscr{E}_2$ fulfills (1.1.4) $\forall\ \varpi, \vartheta \in \mathscr{E}_1$. Then g satisfies*

$$g(r\varpi) = r^2 g(\varpi), \tag{4.1.1}$$

for every $\varpi \in \mathscr{E}_1$ and r is a scalar.

Proof. Letting $\varpi = \vartheta = 0$ in (1.1.4), we have $g(0) = 0$. Thus we can say that equation g satisfies (4.1.1) for $r = 0$ when $\varpi = \vartheta = 0$.

By taking $\vartheta = 0$ in (1.1.4) and using $g(0) = 0$, we have $g(\varpi) = g(\varpi)$. Thus we can say that equation g satisfies (4.1.1) for $r = 1$ when $\vartheta = 0$.

Letting $\varpi = 0$ in (1.1.4) and using $g(0) = 0$, we have $g(-\vartheta) = g(\vartheta)$. Thus we can say that equation g satisfies (4.1.1) for $r = -1$ when $\varpi = 0$.

Letting $\varpi = \vartheta$ in (1.1.4) and using $g(0) = 0$, we have $g(2\vartheta) = 4g(\vartheta)$. Thus we can say that equation g satisfies (4.1.1) for $r = 2$ when $\varpi = \vartheta$.

Suppose that

$$g(k\varpi) = k^2 g(\varpi) \tag{4.1.2}$$

holds for $k = 1, 2, 3, \cdots, r-1, r$, where r is a positive integer.

If we take $k = r + 1$, we get

$$\begin{aligned} g((r+1)\varpi) = g(r\varpi + \varpi) &= 2g(r\varpi) + 2g(\varpi) - g(r\varpi - \varpi) \\ &= 2r^2 g(\varpi) + 2g(\varpi) - g((r-1)\varpi) \\ &= 2r^2 g(\varpi) + 2g(\varpi) - (r-1)^2 g(\varpi) \\ &= [2r^2 + 2 - (r^2 + 1 - 2r)]g(\varpi) \\ &= [r^2 + 1 + 2r]g(\varpi). \end{aligned}$$

So we have $g((r+1)\varpi) = (r+1)^2 g(\varpi)$. Now, assume that (4.1.2) is correct for $k \le r - l$, where l is a positive integer. Further take $k = r + l$,

$$\begin{aligned} g((r+l)\varpi) = g(r\varpi + l\varpi) &= 2g(r\varpi) + 2g(l\varpi) - g(rx - lx) \\ &= 2r^2 g(\varpi) + 2l^2 g(\varpi) - (r-l)^2 g(\varpi) \\ &= [2r^2 + 2l^2 - (r^2 + l^2 - 2rl)]g(\varpi) \\ &= [r^2 + l^2 + 2rl]g(\varpi). \end{aligned}$$

So we have $g((r+l)\varpi) = (r+l)^2 g(\varpi)$. Therefore, $g : \mathscr{E}_1 \to \mathscr{E}_2$ is quadratic then $g(r\varpi) = r^2 g(\varpi)$, for a positive integer r.

Since g is a even function $(g(-\varpi) = (-1)^2 g(\varpi))$, so $g(r\varpi) = r^2 g(\varpi)$ is true for a integer r. ■

Remark 4.1.2. *Assume that a function $g : \mathscr{E}_1 \to \mathscr{E}_2$ satisfies (1.1.5), $\forall\ \varpi, \vartheta, \gamma \in \mathscr{E}_1$.*

Case A1. *If* $\varpi = \vartheta = \gamma = 0$ *in (1.1.5), we get* $g(0) = 0$.

Case A2. *If* $\vartheta = \gamma = 0$ *in (1.1.5) and using case A1, we have*

$$g(\varpi) = g(\varpi).$$

So we have $g(1\varpi) = 1^2 g(\varpi)$.

Case A3. *If* $\vartheta + \gamma = -\varpi$ *in (1.1.5) and using case A1, we see*

$$g(\varpi) + g(\vartheta) + g(\gamma) = g(-\gamma) + g(-\varpi) + g(-\vartheta).$$

Take $\gamma = \varpi = 0$, *we have* $g(-1\vartheta) = (-1)^2 g(\vartheta)$.

Case A4. *If* $\varpi + \vartheta = 0$, *in (1.1.5), we get*

$$g(\gamma) + g(\varpi) + g(-\varpi) + g(\gamma) = g(0) + g(-\varpi + \gamma) + g(\gamma + \varpi).$$

Take $\varpi = \gamma$ *and using Cases A1 and A3, we get*

$$g(\varpi) + g(\varpi) + g(-\varpi) + g(\varpi) = g(0) + g(\varpi + \varpi).$$

So we have $g(2\varpi) = 2^2 g(\varpi)$.

Proposition 4.1.3. *A function* $g : \mathscr{E}_1 \to \mathscr{E}_2$ *fulfills (1.1.5) iff* $g : \mathscr{E}_1 \to \mathscr{E}_2$ *is a quadratic mapping, that is,*

$$g(\varpi - \vartheta) + g(\vartheta + \varpi) = 2g(\vartheta) + 2g(\varpi), \ \forall \ \varpi, \vartheta, \gamma \in \mathscr{E}_1.$$

Proof. Take $\gamma = -\vartheta$ in (1.1.5), we get

$$g(\varpi) + g(\varpi) + g(\vartheta) + g(-\vartheta) = g(\varpi + \vartheta) + g(\vartheta - \vartheta) + g(-\vartheta + \varpi).$$

Using Remark 4.1.2, we have

$$2g(\vartheta) + 2g(\varpi) = g(\varpi - \vartheta) + g(\varpi + \vartheta), \text{ for all } \varpi, \vartheta \in \mathscr{E}_1.$$

Conversely, suppose a mapping $g : \mathscr{E}_1 \to \mathscr{E}_2$ is quadratic, then

$$g(\varpi + \vartheta + \gamma) + g(\varpi) + g(\vartheta) + g(\gamma) = g(\varpi + \vartheta) + g(\vartheta + \gamma) + g(\gamma + \varpi).$$

Take $-\vartheta = \gamma$, we get

$$g(\varpi) + g(\varpi) + g(\vartheta) + g(-\vartheta) = g(\varpi + \vartheta) + g(0) + g(-\vartheta + \varpi).$$

Since g is quadratic and using Lemma 4.1.1, we get

$$2g(\varpi) + 2g(\vartheta) = 2g(\varpi) + 2g(\vartheta), \forall \varpi, \vartheta \in \mathscr{E}_1.$$

■

Remark 4.1.4. *Let a function $g : \mathscr{E}_1 \to \mathscr{E}_2$ fulfills (1.1.6), $\forall\ \varpi, \vartheta, \gamma \in \mathscr{E}_1$.*

Case B1. *If $\gamma = \vartheta = \varpi = 0$ in (1.1.6), we have $g(0) = 0$.*

Case B2. *If $\vartheta = \gamma = 0$ in (1.1.6) and using case B1, we get*

$$g(\varpi) + g(\varpi) + g(\varpi) - 3g(\varpi) = 0.$$

Thus we have $g(1\varpi) = 1^2 g(\varpi)$.

Case B3. *If $\varpi = \gamma = 0$ in (1.1.6) and using case B1 we have*

$$g(\vartheta) + g(-\vartheta) + g(0) + g(\vartheta) - 3[g(0) + g(\vartheta) + g(0)] = 0.$$

So we have $g(-1\vartheta) = (-1)^2 g(\vartheta)$.

Case B4. *If $\gamma = 0$, $\vartheta = \varpi$, in (1.1.6) and using case B1, we have*

$$g(2\varpi) + g(0) + g(\varpi) + g(\varpi) - 3[g(\varpi) + g(\varpi) + g(0)] = 0.$$

So we have $g(2\varpi) = 2^2 g(\varpi)$.

Case B5. *If $\varpi = \vartheta = \gamma$ in (1.1.6) and using case B1, we have*

$$g(3\varpi) + g(0) + g(0) + g(0) - 3[g(\varpi) + g(\varpi) + g(\varpi)] = 0.$$

So we have $g(3\varpi) = 3^2 g(\varpi)$.

Case B6. *If $\gamma = \varpi, \vartheta = 2\varpi$ in (1.1.6), we have*

$$g(4\varpi) + g(-\varpi) + g(0) + g(\varpi) - 3[g(\varpi) + g(2\varpi) + g(\varpi)] = 0.$$

Using Cases B1, B3, and B4, we have

$$g(4\varpi) + g(\varpi) + g(\varpi) - 3[g(\vartheta) + 4g(\varpi) + g(\varpi) = 0.$$

So $g(4\varpi) = 4^2 g(\varpi)$.

Along similar lines, we can prove $g(5\varpi) = 5^2 g(\varpi),\ g(6\varpi) = 6^2 g(\varpi)$ *and so on.*

Proposition 4.1.5. *A function* $g : \mathscr{E}_1 \to \mathscr{E}_2$ *fulfills (1.1.6) iff* $g : \mathscr{E}_1 \to \mathscr{E}_2$ *is quadratic, that is,*

$$g(\varpi - \vartheta) + g(\varpi + \vartheta) = 2g(\vartheta) + 2g(\varpi),\ \forall\ \varpi, \vartheta, \gamma \in \mathscr{E}_1.$$

Proof. Take $\gamma = 0$ in (1.1.6), we have

$$g(\varpi + \vartheta) + g(\varpi - \vartheta) + g(\varpi) + g(\vartheta) - 3[g(\varpi) + g(\vartheta) + g(0)] = 0.$$

Using Remark 4.1.4, we have

$$g(\varpi - \vartheta) + g(\varpi + \vartheta) = 2g(\vartheta) + 2g(\varpi),\ \forall\ \varpi, \vartheta \in \mathscr{E}_1.$$

Conversely, let a mapping $g : \mathscr{E}_1 \to \mathscr{E}_2$ is quadratic, then

$$g(\varpi + \vartheta + \gamma) + g(\varpi - \vartheta) + g(\varpi - \gamma) + g(\vartheta - \gamma) - 3[g(\varpi) + g(\vartheta) + g(\gamma)] = 0,$$

$\forall\ \gamma, \vartheta, \varpi \in \mathscr{E}_1$. Take $\vartheta = -\gamma$, we get

$$g(\varpi) + g(\varpi - \vartheta) + g(\varpi + \vartheta) + g(2\vartheta) - 3[g(\varpi) + g(\vartheta) + g(-\vartheta)] = 0,$$

or

$$g(\varpi - \vartheta) + g(\varpi + \vartheta) + g(2\vartheta) - 2g(\varpi) - 3g(\vartheta) - 3g(-\vartheta) = 0.$$

Using Lemma 4.1.1, we get

$$g(\varpi - \vartheta) + g(\varpi + \vartheta) + 4g(\vartheta) - 2g(\varpi) - 3g(\vartheta) - 3g(\vartheta) = 0,$$

or

$$g(\varpi - \vartheta) + g(\varpi + \vartheta) - 2g(\vartheta) - 2g(\vartheta) = 0.$$

As g is quadratic, we have

$$2g(\varpi) + 2g(\vartheta) - 2g(\vartheta) - 2g(\varpi) = 0,\ \forall\ \vartheta, \varpi \in \mathscr{E}_1.$$

■

Using the Propositions 4.1.3 and 4.1.5, we get the following outcome:

Proposition 4.1.6. *A function $g : \mathscr{E}_1 \to \mathscr{E}_2$ fulfills (1.1.5) iff it satisfies (1.1.6),* $\forall$ $\varpi, \vartheta, \gamma \in \mathscr{E}_1$.

Proof. Suppose $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a map satisfying (1.1.5). Then using Proposition 4.1.3, we have

$$\begin{aligned}&g(\varpi+\vartheta+\gamma)+g(\varpi)+g(\vartheta)+g(\gamma)\\&=2g(\varpi)+2g(\vartheta)-g(\varpi-\vartheta)+2g(\vartheta)+2g(\gamma)-g(\vartheta-\gamma)+2g(\gamma)+2g(\varpi)-g(\gamma-\varpi).\end{aligned}$$

Further, after rewrite, we get

$$g(\varpi+\vartheta+\gamma)+g(\varpi-\vartheta)+g(\vartheta-\gamma)+g(\gamma-\varpi)=3[g(\varpi)+g(\vartheta)+g(\gamma)].$$

Conversely, Suppose $g : \mathscr{E}_1 \to \mathscr{E}_2$ satisfying (1.1.6). Using Proposition 4.1.5, we have

$$\begin{aligned}&g(\varpi+\vartheta+\gamma)+2g(\varpi)+2g(\vartheta)-g(\varpi+\vartheta)+2g(\vartheta)+2g(\gamma)-g(\vartheta+\gamma)\\&+2g(\gamma)+2g(\varpi)-g(\gamma+\varpi)=3g(\varpi)+3g(\vartheta)+3g(\gamma).\end{aligned}$$

Further, after rewrite, we get

$$g(\varpi+\vartheta+\gamma)+g(\varpi)+g(\vartheta)+g(\vartheta)=g(\varpi+\vartheta)+g(\vartheta+\gamma)+g(\gamma+\varpi),\ \forall\ \gamma,\vartheta,\varpi\in\mathscr{E}_1$$

■

4.2 Generalized HUR stability of 3-variables quadratic functional equation in complete 2-normed space

Here, we study generalized HUR stability of (1.1.6). Also, we obtain some hyperstability results for this equation. This section's results extend various previously known outcomes in the framework of complete 2NS. Throughout this section, $\mathscr{E}_2$ is a complete 2NS, $(\mathscr{E}_1, +)$ is an abelian group, and Z_0 is a subset of $\mathscr{E}_2$ containing two LI vectors. Results of this section are proved in Sharma and Chandok [116].

Theorem 4.2.1. *Let $g : \mathscr{E}_1 \to \mathscr{E}_2$ be a map such that*

$$||g(\varpi + \vartheta + \gamma) + g(\varpi - \vartheta) + g(\varpi - \gamma) + g(\vartheta - \gamma) - 3[g(\varpi) + g(\vartheta) + g(\gamma)], z_1|| \\ \leq \varepsilon(\varpi, \vartheta, \gamma, z_1), \tag{4.2.1}$$

where $\varepsilon : \mathscr{E}_0^3 \times Z_0 \to [0, \infty]$, $\varpi, \vartheta, \gamma \in \mathscr{E}_0$, $z_1 \in Z_0$, and $\varpi - \gamma, \varpi - \vartheta, \vartheta - \gamma, \varpi + \vartheta + \gamma \neq 0$. Assume that

$$l(\mathscr{E}_1) := \{\eta \in Aut(\mathscr{E}_1) : 2\eta, 2\eta - Id_{\mathscr{E}_1}, -\eta, \eta, \eta - Id_{\mathscr{E}_1}, \in Aut(\mathscr{E}_1), \alpha_\eta < 1\} \tag{4.2.2}$$

is a nonempty set, where

$$\alpha_\eta := 2\lambda(\eta - Id_{\mathscr{E}_1}) + 3\lambda(\eta) + 3\lambda(-\eta) + \lambda(2\eta) + \lambda(2\eta - Id_{\mathscr{E}_1}), \tag{4.2.3}$$

$$\lambda(\eta) := \inf\Big\{ \mathfrak{t} \in \mathbb{R}_+ : \varepsilon(\eta\varpi, \eta\vartheta, \eta\gamma, z_1) \leq \mathfrak{t}\varepsilon(\varpi, \vartheta, \gamma, z_1), \forall \varpi, \vartheta, \gamma \in \mathscr{E}_0, z_1 \in Z_0 \Big\}, \tag{4.2.4}$$

for $\eta \in Aut(\mathscr{E}_1)$.
Then, there exists a unique $\mathscr{G} : \mathscr{E}_1 \to \mathscr{E}_2$ fulfilling (1.1.6) and

$$\|\mathscr{G}(\varpi) - g(\varpi), z_1\| \leq \varepsilon^*(\varpi, z_1), \ \forall \ \varpi \in \mathscr{E}_0, z_1 \in Z_0, \tag{4.2.5}$$

where

$$\varepsilon^*(\varpi, z_1) := \inf\left\{ \frac{\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1)}{1 - \alpha_\eta} : \eta \in l(\mathscr{E}_1) \right\}.$$

Proof. Fix $\eta \in l(\mathscr{E}_1)$. Replacing $(\varpi, \vartheta, \gamma)$ by $(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi)$ in (4.2.1), we have

$$\|2g((\eta - Id_{\mathscr{E}_1})\varpi) + 3g(\eta\varpi) + 3g(-\eta\varpi) - g(2\eta\varpi) - g((2\eta - Id_{\mathscr{E}_1})\varpi) - g(\varpi), z_1\| \\ \leq \varepsilon(\eta\varpi, (\eta - 1)\varpi, -\eta\varpi, z_1) := \varepsilon_\eta(\varpi, z_1), \text{ for all } \varpi \in \mathscr{E}_0. \tag{4.2.6}$$

We define the operators $\mathscr{T}_\eta : \mathscr{E}_2^{\mathscr{E}_0} \to \mathscr{E}_2^{\mathscr{E}_0}$ and $\Lambda_\eta : \mathbb{R}_+^{\mathscr{E}_0 \times Z_0} \to \mathbb{R}_+^{\mathscr{E}_0 \times Z_0}$ by

$$\begin{aligned} &\mathscr{T}_\eta\xi(\varpi) := 2\xi((\eta - Id_{\mathscr{E}_1})\varpi) + 3\xi(\eta\varpi) + 3\xi(-\eta\varpi) - \xi(2\eta\varpi) - \xi((2\eta - Id_{\mathscr{E}_1})\varpi), \\ &\text{and} \\ &\Lambda_\eta\delta(\varpi, z_1) := 2\delta((\eta - Id_{\mathscr{E}_1})\varpi, z_1) + 3\delta(\eta\varpi, z_1) + 3\delta(-\eta\varpi, z_1) + \delta(2\eta x, z_1) \\ &\qquad\qquad + \delta((2\eta - Id_{\mathscr{E}_1})\varpi, z_1), \ \forall \ \varpi \in \mathscr{E}_0, \ \xi \in \mathscr{E}_2^{\mathscr{E}_0}, \ \delta \in \mathbb{R}_+^{\mathscr{E}_0}. \end{aligned} \tag{4.2.7}$$

Then (4.2.6) becomes

$$\|g(\varpi) - \mathscr{T}_\eta g(\varpi), z_1\| \leq \varepsilon_\eta(\varpi, z_1), \forall\ \varpi \in \mathscr{E}_0, z_1 \in Z_0.$$

The operator Λ_η has the form given by [[27], (3.1), pp. 380] with $j = 5$ and $g_1(\varpi) = (\eta - Id_{\mathscr{E}_1})\varpi$, $g_2(\varpi) = \eta\varpi$, $g_3(\varpi) = -\eta\varpi$, $g_4(\varpi) = 2\eta\varpi$, $g_5(\varpi) = (2\eta - Id_{\mathscr{E}_1})\varpi$, $l_1(\varpi, z_1) = 2$, $l_2(\varpi, z_1) = 3$, $l_3(\varpi, z_1) = 3$, $l_4(\varpi, z_1) = 1$, and $l_5(\varpi, z_1) = 1\ \forall\ \varpi \in \mathscr{E}_0$. Further, we have

$$\begin{aligned}
&\|\mathscr{T}_\eta\xi(\varpi) - \mathscr{T}_\eta\mu(\varpi), z_1\| \\
&= \|2(\xi-\mu)g_1(\varpi) + 3(\xi-\mu)g_2(\varpi) + 3(\xi-\mu)g_3(\varpi) - (\xi-\mu)g_4(\varpi) - (\xi-\mu)g_5(\varpi), z_1\| \\
&\leq 2\|(\xi-\mu)g_1(\varpi), z_1\| + 3\|(\xi-\mu)g_2(\varpi), z_1\| + 3\|(\xi-\mu)g_3(\varpi), z_1\| + \|(\xi-\mu)g_4(\varpi), z_1\| \\
&\quad + \|(\xi-\mu)g_5(\varpi), z_1\| \\
&= \sum_{i=1}^{5} l_i(\varpi, z)\|(\xi-\mu)g_i(\varpi), z_1\|, \forall\ \varpi \in \mathscr{E}_0, z_1 \in Z_0 \text{ and } \xi, \mu \in \mathscr{E}_2^{\mathscr{E}_0}, \eta \in l(\mathscr{E}_1).
\end{aligned}$$

Using the definition of $\lambda(\eta)$, $\varepsilon(\eta\varpi, \eta\vartheta, \eta\gamma, z_1) \leq \lambda(\eta)\varepsilon(\varpi, \vartheta, \gamma, z_1)\ \forall\ \vartheta, \gamma, \varpi \in \mathscr{E}_0, z_1 \in Z_0, \eta \in l(\mathscr{E}_1)$. Using induction on r, we have $\Lambda_\eta^r\varepsilon_\eta(\varpi, z_1) \leq \alpha_\eta^r\varepsilon(\eta\varpi, (\eta-1)\varpi, -\eta\varpi, z_1)\ \forall\ \varpi \in \mathscr{E}_0$, where $\alpha_\eta = 2\lambda(\eta - Id_{\mathscr{E}_1}) + 3\lambda(\eta) + 3\lambda(-\eta) + \lambda(2\eta) + \lambda(2\eta - Id_{\mathscr{E}_1})$. For $r = 1$, we have

$$\begin{aligned}
&\Lambda_\eta\varepsilon_\eta(\varpi, z_1) \\
&= 2\varepsilon_\eta((\eta - Id_{\mathscr{E}_1})\varpi, z_1) + 3\varepsilon_\eta(\eta\varpi, z_1) + 3\varepsilon_\eta(-\eta\varpi, z) + \varepsilon_\eta(2\eta\varpi, z_1) + \varepsilon_\eta((2\eta - Id_{\mathscr{E}_1})\varpi, z_1) \\
&\leq 2\lambda((\eta - Id_{\mathscr{E}_1}))\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z) + 3\lambda(\eta)\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1) + \\
&\ 3\lambda(-\eta)\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1) + \lambda(2\eta)\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1) + \\
&\ \lambda(2\eta - Id_{\mathscr{E}_1})\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1) \\
&= \left(2\lambda(\eta - Id_{\mathscr{E}_1}) + 3\lambda(\eta) + 3\lambda(-\eta) + \lambda(2\eta) + \lambda(2\eta - Id_{\mathscr{E}_1})\right)\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1) \\
&= \alpha_\eta\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1).
\end{aligned} \tag{4.2.8}$$

As the Λ operator is linear, we obtain

$$\begin{aligned}
\varepsilon^*(\varpi, z_1) &= \sum_{r=0}^{\infty} \left(\Lambda_\eta^r\varepsilon_\eta\right)(\varpi, z_1) \\
&\leq \varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1)\sum_{r=0}^{\infty}\alpha_\eta^r \\
&= \frac{\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1)}{1 - \alpha_\eta} < \infty,
\end{aligned}$$

$\forall\ \varpi \in \mathscr{E}_0, z_1 \in Z_0, \eta \in l(\mathscr{E}_1)$.
Therefore by [[27], Theorem 1, pp. 380], there exists a unique solution $\mathscr{G}_\eta : \mathscr{E}_1 \to \mathscr{E}_2$ of

$$\mathscr{G}_\eta(\varpi) = 2\mathscr{G}_\eta((\eta - Id_{\mathscr{E}_1})\varpi) + 3\mathscr{G}_\eta(\eta\varpi) + 3\mathscr{G}_\eta(-\eta\varpi) - \mathscr{G}_\eta(2\eta\varpi) - \mathscr{G}_\eta((2\eta - Id_{\mathscr{E}_1})\varpi), \tag{4.2.9}$$

$\forall\ \varpi \in \mathscr{E}_0$, which is a fixed point of $\mathscr{T}_\eta$ such that

$$\|\mathscr{G}_\eta(\varpi) - g(\varpi), z_1\| \le \varepsilon^*(\varpi, z_1), \forall\ \varpi \in \mathscr{E}_0, z_1 \in Z_0, \eta \in l(\mathscr{E}_1). \tag{4.2.10}$$

Moreover,

$$\mathscr{G}_\eta(\varpi) = \lim_{r\to\infty} \mathscr{T}_\eta^r g(\varpi),\ \forall\ \varpi \in \mathscr{E}_0, \eta \in l(\mathscr{E}_1). \tag{4.2.11}$$

Now, to show that $\mathscr{G}_\eta$ fulfills (1.1.6) on $\mathscr{E}_1$, we have to show the following inequality

$$\begin{aligned} &||\mathscr{T}_\eta^r g(\varpi + \vartheta + \gamma) + \mathscr{T}_\eta^r g(\varpi - \vartheta) + \mathscr{T}_\eta^r g(\varpi - \gamma) + \mathscr{T}_\eta^r g(\vartheta - \gamma) \\ &\quad - 3[\mathscr{T}_\eta^r g(\varpi) + \mathscr{T}_\eta^r g(\vartheta) + \mathscr{T}_\eta^r g(\gamma)], z_1|| \le \quad \alpha_\eta^r \varepsilon(\varpi, \vartheta, \gamma, z_1), \end{aligned} \tag{4.2.12}$$

$\forall\ \vartheta, \gamma, \varpi \in \mathscr{E}_0, z_1 \in Z_0, \eta \in l(\mathscr{E}_1)$.
Indeed, if $r = 0$ then (4.2.12) is simply (4.2.1). So we suppose that (4.2.12) valid for $r \in \mathbb{N}$, $\eta \in l(\mathscr{E}_1)$ and $\varpi, \vartheta, \gamma \in \mathscr{E}_0, z_1 \in Z_0$. Then from (4.2.7) and the triangle inequality, we have

$$\begin{aligned} &||\mathscr{T}_\eta^{r+1} g(\varpi + \vartheta + \gamma) + \mathscr{T}_\eta^{r+1} g(\varpi - \vartheta) + \mathscr{T}_\eta^{r+1} g(\varpi - \gamma) + \mathscr{T}_\eta^{r+1} g(\vartheta - \gamma) \\ &- 3[\mathscr{T}_\eta^{r+1} g(\varpi) + \mathscr{T}_\eta^{r+1} g(\vartheta) + \mathscr{T}_\eta^{r+1} g(\gamma)], z_1|| \\ &= ||2\mathscr{T}_\eta^r g((\eta - Id_{\mathscr{E}_1})(\varpi + \vartheta + \gamma)) + 3\mathscr{T}_\eta^r g(\eta(\varpi + \vartheta + \gamma)) + 3\mathscr{T}_\eta^r g(-\eta(\varpi + \vartheta + \gamma)) \\ &- \mathscr{T}_\eta^r g(2\eta(\varpi + \vartheta + \gamma)) - \mathscr{T}_\eta^r g((2\eta - Id_{\mathscr{E}_1})(\varpi + \vartheta + \gamma)) + 2\mathscr{T}_\eta^r g((\eta - Id_{\mathscr{E}_1})(\varpi - \vartheta)) \\ &+ 3\mathscr{T}_\eta^r g(\eta(\varpi - \vartheta)) + 3\mathscr{T}_\eta^r g(-\eta(\varpi - \vartheta)) - \mathscr{T}_\eta^r g(2\eta(\varpi - \vartheta)) - \mathscr{T}_\eta^r g((2\eta - Id_{\mathscr{E}_1})(\varpi - \vartheta)) \\ &+ 2\mathscr{T}_\eta^r g((\eta - Id_{\mathscr{E}_1})(\varpi - \gamma)) + 3\mathscr{T}_\eta^r g(\eta(\varpi - \gamma)) + 3\mathscr{T}_\eta^r g(-\eta(\varpi - \gamma)) - \mathscr{T}_\eta^r g(2\eta(\varpi - \gamma)) \\ &- \mathscr{T}_\eta^r g((2\eta - Id_{\mathscr{E}_1})(\varpi - \gamma)) + 2\mathfrak{T}_\eta^r g((\eta - Id_{\mathscr{E}_1})(\vartheta - \gamma)) + 3\mathscr{T}_\eta^r g(\eta(\vartheta - \gamma)) \\ &+ 3\mathscr{T}_\eta^r g(-\eta(\vartheta - \gamma)) - \mathscr{T}_\eta^r g(2\eta(\vartheta - \gamma)) - \mathscr{T}_\eta^r g((2\eta - Id_{\mathscr{E}_1})(\vartheta - \gamma)) \\ &- 3[2\mathscr{T}_\eta^r g((\eta - Id_{\mathscr{E}_1})(\varpi)) + 3\mathfrak{T}_\eta^r g(\eta(\varpi)) + 3\mathscr{T}_\eta^r g(-\eta(\varpi)) - \mathscr{T}_\eta^r g(2\eta(\varpi)) \\ &- \mathscr{T}_\eta^r g((2\eta - Id_{\mathscr{E}_1})(\varpi)) + 2\mathscr{T}_\eta^r g((\eta - Id_{\mathscr{E}_1})(\vartheta)) + 3\mathscr{T}_\eta^r g(\eta(\vartheta)) \\ &+ 3\mathscr{T}_\eta^r g(-\eta(\vartheta)) - \mathscr{T}_\eta^r g(2\eta(\vartheta)) - \mathscr{T}_\eta^r g((2\eta - Id_{\mathscr{E}_1})(\vartheta)) + 2\mathscr{T}_\eta^r g((\eta - Id_{\mathscr{E}_1})(\gamma)) \\ &+ 3\mathscr{T}_\eta^r g(\eta(\gamma)) + 3\mathscr{T}_\eta^r g(-\eta(\gamma)) - \mathscr{T}_\eta^r g(2\eta(\gamma)) - \mathscr{T}_\eta^r g((2\eta - Id_{\mathscr{E}_1})(\gamma))], z_1|| \end{aligned}$$

$$
\begin{aligned}
&\le 2\alpha_\eta^r \varepsilon((\eta - Id_{\mathscr{E}_1})\varpi, (\eta - Id_{\mathscr{E}_1})\vartheta, (\eta - Id_{\mathscr{E}_1})\gamma, z_1) + 3\alpha_\eta^r \varepsilon(\eta\varpi, \eta\vartheta, \eta\gamma, z_1) \\
&\quad + 3\alpha_\eta^r \varepsilon(-\eta\varpi, -\eta\vartheta, -\eta\gamma, z_1) + \alpha_\eta^r \varepsilon(2\eta\varpi, 2\eta\vartheta, 2\eta\gamma, z_1) \\
&\quad + \alpha_\eta^r \varepsilon((2\eta - Id_{\mathscr{E}_1})\varpi, (2\eta - Id_{\mathscr{E}_1})\vartheta, (2\eta - Id_{\mathscr{E}_1})\gamma, z_1) \\
&\le 2\alpha_\eta^r \lambda(\eta - Id_{\mathscr{E}_1})\varepsilon(\varpi, \vartheta, \gamma, z_1) + 3\alpha_\eta^r \lambda(\eta)\varepsilon(\varpi, \vartheta, \gamma, z_1) + 3\alpha_\eta^r \lambda(-\eta)\varepsilon(\varpi, \vartheta, \gamma, z_1) \\
&\quad + \alpha_\eta^r \lambda(2\eta)\varepsilon(\varpi, \vartheta, \gamma, z_1) + \alpha_\eta^r \lambda(2\eta - Id_{\mathscr{E}_1})\varepsilon(\varpi, \vartheta, \gamma, z_1) \\
&= \alpha_\eta^r [2\lambda(\eta - Id_{\mathscr{E}_1}) + 3\ \lambda(\eta) + 3\lambda(-\eta) + \lambda(2\eta) + \lambda(2\eta - Id_{\mathscr{E}_1})]\varepsilon(\varpi, \vartheta, \gamma, z_1) \\
&= \alpha_\eta^{r+1} \varepsilon(\varpi, \vartheta, \gamma, z_1).
\end{aligned}
$$

So we have

$$
\begin{aligned}
&||\mathscr{T}_\eta^{r+1} g(\varpi + \vartheta + \gamma) + \mathscr{T}_\eta^{r+1} g(\varpi - \vartheta) + \mathfrak{T}_\eta^{r+1} g(\varpi - \gamma) + \mathscr{T}_\eta^{r+1} g(\vartheta - \gamma) \\
&- 3[\mathscr{T}_\eta^{r+1} g(\varpi) + \mathscr{T}_\eta^{r+1} g(\vartheta) + \mathscr{T}_\eta^{r+1} g(\gamma)], z_1|| \\
&\le \alpha_\eta^{(r+1)} \varepsilon(\varpi, \vartheta, \gamma, z_1). \qquad (4.2.13)
\end{aligned}
$$

Letting $r \to \infty$ in (4.2.13), using (4.2.11) and definition of $l(\mathscr{E}_1)$, we get

$$
\mathscr{G}_\eta(\varpi + \vartheta + \gamma) + \mathscr{G}_\eta(\varpi - \vartheta) + \mathscr{G}_\eta(\varpi - \gamma) + \mathscr{G}_\eta(\vartheta - \gamma) = 3[\mathscr{G}_\eta(\varpi) + \mathscr{G}_\eta(\vartheta) + \mathscr{G}_\eta(\gamma)], \quad (4.2.14)
$$

$\forall\ \varpi, \vartheta, \gamma \in \mathscr{E}_0, \eta \in l(\mathscr{E}_1)$. Therefore we have proven that for every for $\eta \in l(\mathscr{E}_1)$ there exists a $\mathscr{G}_\eta : \mathscr{E}_1 \to \mathscr{E}_2$, that is the solution of (1.1.6) on $\mathscr{E}_0$ and fulfills $\|g(\varpi) - \mathscr{G}_\eta(\varpi), z_1\| \le \frac{\varepsilon_\eta(\varpi, z_1)}{1 - \alpha_\eta} = \varepsilon^*(\varpi, z_1)$, $\forall\ \varpi \in \mathscr{E}_0$.
Now, we prove that $\mathscr{G}_\eta = \mathscr{G}_q$ for all $\eta, q \in l(\mathscr{E}_1)$. Fix η, q and note that $\mathscr{G}_q$ satisfies (4.2.10) with η replaced by q. Hence by replacing $(\varpi, \vartheta, \gamma)$ with $(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi)$ in (4.2.14), we get $\mathscr{T}\mathscr{G}_j = \mathscr{G}_j$, for $j = \eta, q$ and

$$
\begin{aligned}
||\mathscr{G}_\eta(\varpi) - \mathscr{G}_q(\varpi), z_1|| &\le ||\mathscr{G}_\eta(\varpi) - g(\varpi), z_1|| + ||\mathscr{G}_q(\varpi) - g(\varpi), z_1|| \\
&\le \left(\frac{\varepsilon_\eta(\varpi, z_1)}{1 - \alpha_\eta} \right) + \left(\frac{\varepsilon_q(\varpi, z_1)}{1 - \alpha_q} \right),
\end{aligned}
$$

for every $\varpi \in \mathscr{E}_0$. Using the linearity of Λ and (4.2.8), we get

$$
\begin{aligned}
||\mathscr{G}_\eta(\varpi) - \mathscr{G}_q(\varpi), z_1|| &= ||\mathscr{T}^r \mathscr{G}_\eta(\varpi) - \mathscr{T}^r \mathscr{G}_q(\varpi), v|| \\
&\le \left(\frac{\Lambda^r \varepsilon_\eta(\varpi, z_1)}{1 - \alpha_\eta} \right) + \left(\frac{\Lambda^r \varepsilon_q(\varpi, z_1)}{1 - \alpha_q} \right) \\
&\le \alpha_\eta^r U_\eta(\varpi, z_1) + \alpha_q^r U_q(\varpi, v),
\end{aligned}
$$

where $U_k(\varpi, z_1) = \frac{\varepsilon_k(\varpi, z_1)}{1 - \alpha_k}$ for every $\varpi \in \mathscr{E}_0$, $k = \eta, q$ and $r \in \mathbb{N}$. Letting $r \to \infty$, we get

$\mathscr{G}_\eta = \mathscr{G}_q = \mathscr{G}$. Thus, we have

$$\|g(\varpi) - \mathscr{G}(\varpi), z_1\| \leq U_\eta(\varpi, z_1),$$

for all $\varpi \in \mathscr{E}_0, \eta \in l(\mathscr{E}_1)$. Therefore, we get (4.2.5). Using (4.2.14), it is obvious that $\mathscr{G}$ is a solution of (1.1.6). Now to demonstrate the uniqueness of map $\mathscr{G}$, suppose that there exists a map $\mathscr{G}' : \mathscr{E}_1 \to \mathscr{E}_2$ satisfying (1.1.6) and inequality

$$\|g(\varpi) - \mathscr{G}'(\varpi), z_1\| \leq \varepsilon^*(\varpi, z_1).$$

Then

$$||\mathscr{G}(\varpi) - \mathscr{G}'(\varpi), z_1|| \leq 2\varepsilon^*(\varpi, z_1).$$

Further $\mathscr{T}\mathscr{G}'(\varpi) = \mathscr{G}'(\varpi)$ for every $\varpi \in \mathscr{E}_0$. As a result, with a fixed $\eta \in l(\mathscr{E}_1)$, we get

$$\begin{aligned} ||\mathscr{G}(\varpi) - \mathscr{G}'(\varpi), z_1|| = ||\mathscr{T}^r\mathscr{G}(\varpi) - \mathscr{T}^r\mathscr{G}'(\varpi), z_1|| &\leq 2\Lambda^r\varepsilon^*(\varpi, z_1) \\ &= \frac{2\Lambda^r\varepsilon_\eta(\varpi, z_1)}{1-\alpha_\eta} \\ &\leq \frac{2\alpha_\eta^r\varepsilon_\eta(\varpi, z_1)}{1-\alpha_\eta}, \end{aligned}$$

$\forall\ \varpi \in \mathscr{E}_0$ and $r \in \mathbb{N}$. Taking $r \to \infty$, we get $\mathscr{G} = \mathscr{G}'$. ∎

Theorem 4.2.2. *Let $g : \mathscr{E}_1 \to \mathscr{E}_2$ and $\varepsilon : \mathscr{E}_0^3 \times Z_0 \to [0, \infty)$ be functions, and the conditions (4.2.2), (4.2.3) and (4.2.4) hold. Assume that*

$$\inf\{\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1) : \eta \in l(\mathscr{E}_1)\} = 0, \tag{4.2.15}$$

$\forall\ \varpi \in \mathscr{E}_0, z_1 \in Z_0$. Then g fulfills (1.1.6) on $\mathscr{E}_0$.

Proof. Suppose that

$$\inf\{\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1) : \eta \in l(\mathscr{E}_1)\} = 0, \text{ for all } \varpi \in \mathscr{E}_0, z_1 \in Z_0. \tag{4.2.16}$$

Therefore, from Theorem 4.2.1, we obtain $\varepsilon^*(\varpi, z_1) = 0$. Then g satisfies (1.1.6) on $\mathscr{E}_1$.

∎

Remark 4.2.3. *In Theorem 4.2.1, if*

$$\inf\{2\lambda(\eta - Id_{\mathscr{E}_1}) + 3\lambda(\eta) + 3\lambda(-\eta) + \lambda(2\eta) + \lambda(2\eta - Id_{\mathscr{E}_1}) : \eta \in l(\mathscr{E}_1)\} = 0,$$

(this is the case when, that is, $\lim_{|\eta|\to\infty} \lambda(\eta) = 0$*), then (4.2.2) holds and*

$$\varepsilon^*(\varpi, z_1) = \inf_{\eta\in l(\mathscr{E}_1)} \varepsilon(\eta\varpi, (\eta - 1)\varpi, -\eta\varpi, z_1), \ \forall \ \varpi \in \mathscr{E}_0, z_1 \in Z_0.$$

Now, we show that a function satisfying (1.1.6) on $\mathscr{E}_0$ fulfills it on the whole space $\mathscr{E}_1$.

Proposition 4.2.4. *If* $g : \mathscr{E}_1 \to \mathscr{E}_2$ *satisfies (1.1.6)* $\forall\ \gamma, \vartheta, \varpi \in \mathscr{E}_0$*, then* $g(0) = 0$*.*

Proof. It is sufficient to demonstrate that $g(0) = 0$.
Letting $\gamma = \vartheta = \varpi$ in (1.1.6), we get

$$g(3\varpi) + 3g(0) = 9g(\varpi), \quad \varpi \in \mathscr{E}_0. \tag{4.2.17}$$

Taking $\vartheta = -\varpi, \gamma = -\varpi$ in (1.1.6), we have

$$-5g(-\varpi) + 2g(2\varpi) + g(0) - 3g(\varpi) = 0, \quad \varpi \in \mathscr{E}_0. \tag{4.2.18}$$

Taking $\varpi = \vartheta, \gamma = -\varpi$ in (1.1.6), we have

$$-5g(\varpi) + 2g(2\varpi) + g(0) - 3g(-\varpi) = 0, \quad \varpi \in \mathscr{E}_0. \tag{4.2.19}$$

From (4.2.18) and (4.2.19)

$$g(\varpi) = g(-\vartheta), \quad \varpi \in \mathscr{E}_0. \tag{4.2.20}$$

Taking $\gamma = -\varpi, \vartheta = 2\varpi$ in (1.1.6), we have

$$2g(2\varpi) + g(-\varpi) + g(3\vartheta) = 3g(\varpi) + 3g(2\varpi) + 3g(-\varpi).$$

Using (4.2.20), we get

$$g(2\varpi) + 5g(\varpi) = g(3\varpi).$$

Using (4.2.17), we get

$$g(2\varpi) = 4g(\varpi) - 3g(0). \tag{4.2.21}$$

Taking $\gamma = \varpi$ and $\vartheta = -\varpi$ in (1.1.6), we have

$$-5g(\varpi) + g(2\varpi) + g(-2\varpi) + g(0) - 3g(-\varpi) = 0.$$

Using (4.2.20), we get

$$-5g(\varpi) + g(2\varpi) + g(2\varpi) + g(0) - 3g(\varpi) = 0.$$

Using (4.2.21), we have

$$-5g(\varpi) + 8g(\varpi) - 6g(0) + g(0) - 3g(\varpi) = 0.$$

Thus, we get $g(0) = 0$. ■

It is obvious that if $g(0) = 0$, then g satisfies (1.1.6) on the whole $\mathscr{E}_1$. So we get the following outcome.

Theorem 4.2.5. *Let $g : \mathscr{E}_1 \to \mathscr{E}_2$ be a map such that*

$$\begin{aligned} &||g(\varpi + \vartheta + \gamma) + g(\varpi - \vartheta) + g(\varpi - \gamma) + g(\vartheta - \gamma) - 3[g(\varpi) + g(\vartheta) + g(\gamma)], z_1|| \\ &\quad \leq \varepsilon(\varpi, \vartheta, \gamma, z_1), \end{aligned} \tag{4.2.22}$$

where $\varepsilon : \mathscr{E}_0^3 \times Z_0 \to [0, \infty]$, $\varpi, \vartheta, \gamma \in \mathscr{E}_0, z_1 \in Z_0$.
Assume that

$$l(\mathscr{E}_1) := \{\eta \in Aut(\mathscr{E}_1) : 2\eta, 2\eta - Id_{\mathscr{E}_1}, -\eta, \eta, \eta - Id_{\mathscr{E}_1} \in Aut(\mathscr{E}_1), \alpha_\eta < 1\} \tag{4.2.23}$$

is a nonempty set, where

$$\alpha_\eta := 2\lambda(\eta - Id_{\mathscr{E}_1}) + 3\lambda(\eta) + 3\lambda(-\eta) + \lambda(2\eta) + \lambda(2\eta - Id_{\mathscr{E}_1}), \tag{4.2.24}$$

$$\lambda(\eta) := \inf\Big\{\mathrm{t} \in \mathbb{R}_+ : \varepsilon(\eta\varpi, \eta\vartheta, \eta\gamma, z_1) \leq \mathrm{t}\varepsilon(\varpi, \vartheta, \gamma, z_1),\ \forall \gamma, \vartheta, \varpi \in \mathscr{E}_0, z_1 \in Z_0\Big\}, \tag{4.2.25}$$

for $\eta \in Aut(\mathscr{E}_1)$.

Then, there exists a unique $\mathscr{G} : \mathscr{E}_1 \to \mathscr{E}_2$ fulfilling (1.1.6) and

$$\|\mathscr{G}(\varpi) - g(\varpi), z_1\| \leq \varepsilon^*(\varpi, z_1),\ \forall\ \varpi \in \mathscr{E}_1,\ z_1 \in Z_0, \tag{4.2.26}$$

where

$$\varepsilon^*(\varpi, z_1) := \inf\left\{\frac{\varepsilon(\eta\varpi, (\eta - Id_{\mathscr{E}_1})\varpi, -\eta\varpi, z_1)}{1-\alpha_\eta} : \eta \in l(\mathscr{E}_1)\right\}.$$

Now, we provide a few results of Theorem 4.2.1 for the following two cases:

$$\varepsilon_1(\varpi, \vartheta, \gamma, z_1) = \phi\left(\|\varpi\|^{\mathfrak{p}_1}\|\vartheta\|^{\mathfrak{p}_2}\|\gamma\|^{\mathfrak{p}_3}\right)\|z_1\|,$$

$\mathfrak{p}_1 + \mathfrak{p}_3 < 0$ and $\mathfrak{p}_2 < 0$,

$$\varepsilon_2(\varpi, \vartheta, \gamma, z_1) = \phi\left(\|\varpi\|^{\mathfrak{p}_1} + \|\vartheta\|^{\mathfrak{p}_1} + \|\gamma\|^{\mathfrak{p}_1}\right)\|z_1\|, \quad \mathfrak{p}_1 < 0,$$

where $\varepsilon(\varpi, \vartheta, \gamma, z_1) = \varepsilon_j(\varpi, \vartheta, \gamma, z_1)$ for $j \in \{1, 2\}, \phi \in \mathbb{R}_+, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \in \mathbb{R}$ and $\varpi, \vartheta, \gamma \neq 0$.

Corollary 4.2.6. *Assume that* $\mathcal{S} := (\mathcal{S}, +)$ *is a nonempty subset of* $(\mathscr{E}_1, +)$*. Suppose that* $g : \mathcal{S} \to \mathscr{E}_2$*, fulfills*

$$\begin{aligned}&||g(\varpi+\vartheta+\gamma) + g(\varpi-\vartheta) + g(\varpi-\gamma) + g(\vartheta-\gamma) - 3[g(\varpi) + g(\vartheta) + g(\gamma)], z_1||\\&\leq \phi\left(\|\varpi\|^{\mathfrak{p}_1}\|\vartheta\|^{\mathfrak{p}_2}\|\gamma\|^{\mathfrak{p}_3}\right)\|z_1\|,\ \forall\ \varpi, \vartheta, \gamma \in \mathcal{S}_0,\end{aligned}$$

with $\varpi + \vartheta + \gamma, \varpi - \varkappa, \varpi - \gamma, \vartheta - \gamma \neq 0,$ $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \in \mathbb{R},$ $\mathfrak{p}_1 + \mathfrak{p}_3 < 0, \mathfrak{p}_2 < 0,$ *and* $\phi > 0, z_1 \in Z_0$*. Then* g *is a solution of (1.1.6) on* $\mathcal{S}_0$*.*

Proof. Proof comes from Theorem 4.2.1 by choosing

$$\varepsilon_1(\varpi, \vartheta, \gamma, z_1) = \phi\left(\|\varpi\|^{\mathfrak{p}_1}\|\vartheta\|^{\mathfrak{p}_2}\|\gamma\|^{\mathfrak{p}_3}\right)\|z_1\|, \forall\ \varpi, \vartheta, \gamma \in \mathcal{S}_0, z_1 \in Z_0$$

with some real number $\phi > 0, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \in \mathbb{R},\ \mathfrak{p}_1 + \mathfrak{p}_3 < 0, \mathfrak{p}_2 < 0$. For each $j \in \mathbb{N}$, define $\eta_j : \mathcal{S}_0 \to \mathcal{S}_0$ by $\eta_j a = ja$. For all $\varpi, \vartheta, \gamma \in \mathcal{S}_0, z_1 \in Z_0$ and $j \in \mathbb{N}$, we have

$$\begin{aligned}\varepsilon_1(\eta_j\varpi, \eta_j\vartheta, \eta_j\gamma, z_1) &= \varepsilon_1(j\varpi, j\vartheta, j\gamma, z_1)\\&= \phi\left(\|j\varpi\|^{\mathfrak{p}_1}\|j\vartheta\|^{\mathfrak{p}_2}\|j\gamma\|^{\mathfrak{p}_3}\right)\|z_1\|\\&= |j|^{\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3}\phi\left(\|\varpi\|^{\mathfrak{p}_1}\|\vartheta\|^{\mathfrak{p}_2}\|\gamma\|^{\mathfrak{p}_3}\right)\|z_1\|\\&= |j|^{\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3}\varepsilon_1(\varpi, \vartheta, \gamma, z_1).\end{aligned}$$

Taking limit $j \to \infty$, we have $\lim_{j\to\infty} \varepsilon_1(\eta_j\varpi, \eta_j\vartheta, \eta_j\gamma, z_1) = \lim_{j\to\infty} |j|^{\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3}\varepsilon_1(\varpi, \vartheta, \gamma, z_1) = 0$. So (4.2.15) is valid with $\lambda(\eta_j) = |j|^{\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3}$ for $j \in \mathbb{N}$, and there exists $\eta_0 \in \mathbb{N}$ with

$\eta_0 > 1$ such that $j \geq \eta_0$ and

$$\alpha_{\eta_j} = 2 \mid \eta - 1 \mid^{\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3} + (6 + 2^{\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3}) \mid \eta \mid^{\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3} + \mid 2\eta - 1 \mid^{\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{p}_3} < 1.$$

Therefore we can say that (4.2.2) is satisfied with $l(\mathscr{E}_1) := \{\eta \in Aut(\mathscr{E}_1) : j \in \mathbb{N}_{\eta_0}\}$. Hence by the Theorem 4.2.2, every function $g : \mathcal{S} \to \mathscr{E}_2$, satisfies (1.1.6) on $\mathcal{S}_0$. ■

Corollary 4.2.7. *Assume that* $\mathcal{S} := (\mathcal{S}, +)$ *is a nonempty subset of* $(\mathscr{E}_1, +)$. *Suppose that* $g : \mathcal{S} \to \mathscr{E}_2$ *fulfills*

$$\begin{aligned} &||g(\varpi + \vartheta + \gamma) + g(\varpi - \vartheta) + g(\varpi - \gamma) + g(\vartheta - \gamma) - 3[g(\varpi) + g(\vartheta) + g(\gamma)], z_1|| \\ &\quad \leq \phi \left(\|\varpi\|^{\mathfrak{p}_1} + \|\vartheta\|^{\mathfrak{p}_1} + \|\gamma\|^{\mathfrak{p}_1}\right) \|z_1\|, \end{aligned}$$

$\forall\, \vartheta, \varpi, \gamma \in \mathcal{S}_0$, *with* $\varpi+\vartheta+\gamma, \varpi-\vartheta, \varpi-\gamma, \vartheta-\gamma \neq 0$, $\mathfrak{p}_1 \in \mathbb{R}$, $\mathfrak{p}_1 < 0$, *and* $\phi > 0, z_1 \in Z_0$. *Then* g *is a solution of (1.1.6) on* $\mathcal{S}\backslash\{0\}$, *such that* $\varpi + \vartheta + \gamma, \varpi - \vartheta, \varpi - \gamma, \vartheta - \gamma \neq 0$.

Proof. The proof comes from Theorem 4.2.1 by choosing

$$\varepsilon_1(\varpi, \vartheta, \gamma, z_1) = \phi \left(\|\varpi\|^{\mathfrak{p}_1}\|\vartheta\|^{\mathfrak{p}_1}\|\gamma\|^{\mathfrak{p}_1}\right) \|z_1\|,$$

$\forall\, \varpi, \vartheta, \gamma \in \mathcal{S}_0, z_1 \in Z_0$ with some real number $\phi > 0, \mathfrak{p}_1 \in \mathbb{R}$, $\mathfrak{p}_1 < 0$. For each $j \in \mathbb{N}$, define $\eta_j : \mathcal{S}_0 \to \mathcal{S}_0$ by $\eta_j a = ja$. For all $\varpi, \vartheta, \gamma \in \mathcal{S}_0, z_1 \in Z_0$ and $j \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon_1(\eta_j\varpi, \eta_j\vartheta, \eta_j\gamma, z_1) &= \varepsilon_1(j\varpi, j\vartheta, j\gamma, z_1) \\ &= \phi \left(\|j\varpi\|^{\mathfrak{p}_1}\|j\vartheta\|^{\mathfrak{p}_1}\|j\gamma\|^{\mathfrak{p}_1}\right) \|z_1\| \\ &= |j|^{\mathfrak{p}_1}\phi \left(\|\varpi\|^{\mathfrak{p}_1}\|\vartheta\|^{\mathfrak{p}_1}\|\gamma\|^{\mathfrak{p}_1}\right) \|z_1\| \\ &= |j|^{\mathfrak{p}_1}\varepsilon_1(\varpi, \vartheta, \gamma, z_1). \end{aligned}$$

Taking limit $j \to \infty$, we get $\lim\limits_{j\to\infty} \varepsilon_1(\eta_j\varpi, \eta_j\vartheta, \eta_j\gamma, z_1) = \lim\limits_{j\to\infty} |j|^{\mathfrak{p}_1}\varepsilon_1(\varpi, \vartheta, \gamma, z_1) = 0$.
So (4.2.15) is correct with $\lambda(\eta_j) = |j|^{\mathfrak{p}_1}$ for $j \in \mathbb{N}$, and there exists $\eta_0 \in \mathbb{N}$ with $\eta_0 > 1$ such that $j \geq \eta_0$ and

$$\alpha_{\eta_j} = 2 \mid \eta - 1 \mid^{\mathfrak{p}_1} + (6 + 2^{\mathfrak{p}_1}) \mid \eta \mid^{\mathfrak{p}_1} + \mid 2\eta - 1 \mid^{\mathfrak{p}_1} < 1.$$

Therefore we can say that (4.2.2) is satisfied with $l(\mathscr{E}_1) := \{\eta \in Aut(\mathscr{E}_1) : j \in \mathbb{N}_{\eta_0}\}$. Hence by the Theorem 4.2.2, every function $g : \mathcal{S} \to \mathscr{E}_2$, fulfills (1.1.6) on $\mathcal{S}_0$. ■

Corollary 4.2.8. *Let* $g : \mathscr{E}_1 \to \mathscr{E}_2, \varepsilon : \mathscr{E}_1^3 \times Z_0 \to [0, \infty)$ *and the conditions (4.2.2), (4.2.4) and (4.2.15) hold. Further, if* $F : \mathscr{E}_1^3 \to \mathscr{E}_2$ *is a map such that* $F(\varpi_0, \vartheta_0, \gamma_0) \neq 0$

for some $\varpi_0, \vartheta_0, \gamma_0 \in \mathscr{E}_1$ *and*

$$||F(\varpi, \vartheta, \gamma), z_1|| \leq \varepsilon^*(\varpi, \vartheta, \gamma, z_1),$$

$\forall \ \varpi, \vartheta, \gamma \in \mathscr{E}_1, z_1 \in Z_0$*, then the following function equation for all* $\vartheta, \varpi, \gamma \in \mathscr{E}_1$,

$$g(\varpi + \vartheta + \gamma) + g(\varpi - \vartheta) + g(\varpi - \gamma) + g(\vartheta - \gamma) = F(\varpi, \vartheta, \gamma) + 3[g(\varpi) + g(\vartheta) + g(\gamma)], \tag{4.2.27}$$

has no solution in the class of function $g : \mathscr{E}_1 \to \mathscr{E}_2$.

Proof. Assume that $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a solution of (4.2.27). So (4.2.1) valid, and as a result, as per Theorem 4.2.2, g fulfills (1.1.6) on $\mathscr{E}_1$, that is, $F(\varpi_0, \vartheta_0, \gamma_0) = 0$, which is a contradiction. ■

4.3 Generalized HUR stability of a quadratic type functional equations in complete quasi-normed space

Here, we study the generalized HUR stability of (1.1.6) in complete QNS. Also, we obtain some hyperstability results for this equation. This section's results extend a number of previously established results to the setting of complete QNS. Throughout this section, $\mathscr{E}_2$ is a complete QNS, $(\mathscr{E}_1, +)$ is an abelian group. Results of this section are published in Sharma and Chandok [114].

Theorem 4.3.1. *Let* $g : \mathscr{E}_1 \to \mathscr{E}_2$ *be a map such that*

$$||g(\varpi + \vartheta + \gamma) + g(\varpi - \vartheta) + g(\varpi - \gamma) + g(\vartheta - \gamma) - 3[g(\varpi) + g(\vartheta) + g(\gamma)]|| \leq \varepsilon(\varpi, \vartheta, \gamma), \tag{4.3.1}$$

where $\varepsilon : \mathscr{E}_1 \times \mathscr{E}_1 \times \mathscr{E}_1 \to [0, \infty]$, $\varpi, \vartheta, \gamma \in \mathscr{E}_1$.
Assume that

$$l(\mathscr{X}) := \{m \in Aut(\mathscr{E}_1) : 2m, 2m - Id_{\mathscr{E}_1}, m, -m, m - Id_{\mathscr{E}_1} \in Aut(\mathscr{E}_1), \alpha_m < 1\} \tag{4.3.2}$$

is a nonempty set, where

$$\alpha_m := 2K\lambda(m - Id_{\mathscr{E}_1}) + 3K^2\lambda(m) + 3K^3\lambda(-m) + K^4\lambda(2m) + K^4\lambda(2m - Id_{\mathscr{E}_1}), \tag{4.3.3}$$

$$\lambda(m) := \inf\{t \in \mathbb{R}_+ : \varepsilon(m\varpi, m\vartheta, m\gamma) \le t\varepsilon(\varpi, \vartheta, \gamma)\ \forall \varpi, \vartheta, \gamma \in \mathscr{E}_1\}, \textit{for } m \in Aut(\mathscr{E}_1). \tag{4.3.4}$$

Then, there exists a unique $\mathscr{G} : \mathscr{E}_1 \to \mathscr{E}_2$ *fulfilling (1.1.6) and*

$$\|\mathscr{G}(\varpi) - g(\varpi)\|^\theta \le 4\varepsilon^*(\varpi), \tag{4.3.5}$$

$\forall\ \varpi \in \mathscr{E}_1$*, where* $\theta = \log_{2K} 2$ *with* $K \ge 1$, $\varepsilon^*(\varpi) := \inf\left\{\frac{\varepsilon^\theta(m\varpi,(m-Id_{\mathscr{E}_1})\varpi,-m\varpi)}{1-\alpha_m^\theta} : m \in l(\mathscr{E}_1)\right\}$.

Proof. Fix $m \in l(\mathscr{E}_1)$. Replacing $(\varpi, \vartheta, \gamma)$ by $(m\varpi, (m - Id_{\mathscr{E}1})\varpi, -m\varpi)$ in (4.3.1), we have

$$\begin{aligned}&\|2g((m - Id_{\mathscr{E}1})\varpi) + 3g(m\varpi) + 3g(-m\varpi) - g(2m\varpi) - g((2m - Id_{\mathscr{E}1})\varpi) - g(\varpi)\|\\ &\le \varepsilon(m\varpi, (m - Id_{\mathscr{E}1})\varpi, -m\varpi) := \varepsilon_m(\varpi),\end{aligned} \tag{4.3.6}$$

$\forall\ \varpi \in \mathscr{E}_1$. We define the operators $\mathscr{T}_m : \mathscr{E}_2^{\mathscr{E}_1} \to \mathscr{E}_2^{\mathscr{E}_1}$, and $\Lambda_m : \mathbb{R}_+^{\mathscr{E}_1} \to \mathbb{R}_+^{\mathscr{E}_1}$ by

$$\begin{aligned}&\mathcal{T}_m\xi(\varpi) := 2\xi((m - Id_{\mathscr{E}1})\varpi) + 3\xi(m\varpi) + 3\xi(-m\varpi) - \xi(2m\varpi) - \xi((2m - Id_{\mathscr{E}1})\varpi)\\ &\text{and}\\ &\Lambda_m\delta(\varpi) := 2K\delta((m - Id_{\mathscr{E}_1})\varpi) + 3K^2\delta(m\varpi) + 3K^3\delta(-m\varpi) + K^4\delta(2m\varpi)\\ &\qquad\qquad + K^4\delta((2m - Id_{\mathscr{E}_1})\varpi)],\end{aligned} \tag{4.3.7}$$

$\forall\ \varpi \in \mathscr{E}_1$, $\xi \in \mathscr{E}_2^{\mathscr{E}_1}$, and $\delta \in \mathbb{R}_+^{\mathscr{E}_1}$. Then (4.3.6) becomes

$$\|g(\varpi) - \mathscr{T}_m g(\varpi)\| \le \varepsilon_m(\varpi),$$

$\forall\ \varpi \in \mathscr{E}_1$. The operator Λ_m has the form given by [[44], (2.17), pp. 138] and $\mathscr{T}$ has the form given by [[44], (2.15), pp. 138] with $k = 5$ and $g_1(\varpi) = (m - Id_{\mathscr{E}_1})\varpi,\ g_2(\varpi) = m\varpi,\ g_3(\varpi) = -m\varpi,\ g_4(\varpi) = 2m\varpi,\ g_5(\varpi) = (2m - Id_{\mathscr{E}_1})\varpi,\ L_1(\varpi) = 2K,\ L_2(\varpi) = 3K^2,\ L_3(\varpi) = 3K^3,\ L_4(\varpi) = K^4,\ L_5(\varpi) = K^4$, for all $\varpi \in \mathscr{E}_1$. Further, we have

$$\begin{aligned}&\|\mathscr{T}_m\xi(\varpi) - \mathscr{T}_m\mu(\varpi)\|\\ &= \|2(\xi - \mu)g_1(\varpi) + 3(\xi - \mu)g_2(\varpi) + 3(\xi - \mu)g_3(\varpi) - (\xi - \mu)g_4(\varpi) - (\xi - \mu)g_5(\varpi)\|\\ &\le 2K\|(\xi - \mu)g_1(\varpi)\| + 3K^2\|(\xi - \mu)g_2(\varpi)\| + 3K^3\|(\xi - \mu)g_3(\varpi)\| + K^4\|(\xi - \mu)g_4(\varpi)\|\\ &\quad + K^4\|(\xi - \mu)g_5(\varpi)\|\\ &= \sum_{i=1}^{5} L_i(x)\|(\xi - \mu)g_i(\varpi)\|,\end{aligned}$$

$\forall\ \varpi \in \mathscr{E}_1$ and $\xi, \mu \in \mathscr{E}_2^{\mathscr{E}_1}, m \in l(\mathscr{E}_1)$. Using the definition of $\lambda(m)$, $\varepsilon(m\varpi, m\vartheta, m\gamma) \leq \lambda(m)\varepsilon(\varpi, \vartheta, \gamma)\ \forall\ \ \varpi, \vartheta, \gamma \in \mathscr{E}_1, m \in l(\mathscr{E}_1)$. Using induction on n, we have $\Lambda_m^n \varepsilon_m(\varpi) \leq \alpha_m^n \varepsilon(m\varpi, (m - Id_{\mathscr{E}_1})\varpi, -m\varpi)\ \forall\ \varpi \in \mathscr{E}_1$, where $\alpha_m = 2K\lambda(m - Id_{\mathscr{E}_1}) + 3K^2\lambda(m) + 3K^3\lambda(-m) + K^4\lambda(2m) + K^4\lambda(2m - Id_{\mathscr{E}_1})$. For $n = 1$, we have

$$\begin{aligned}
&\Lambda_m \varepsilon(\varpi) \\
&= 2K\varepsilon((m - Id_{\mathscr{E}1})\varpi) + 3K^2\varepsilon(m\varpi) + 3K^3\varepsilon(-m\varpi) + K^4\varepsilon(2\varpi) + K^4\varepsilon((2m - Id_{\mathscr{E}_1})\varpi) \\
&\leq 2K\lambda((m - Id_{\mathscr{E}1}))\varepsilon(m\varpi, (m - Id_{\mathscr{E}1})\varpi, -m\varpi) + 3K^2\lambda(m)\varepsilon(m\varpi, (m - Id_{\mathscr{E}1})\varpi, -m\varpi) \\
&\quad + 3K^3\lambda(-m)\varepsilon(m\varpi, (m - Id_{\mathscr{E}_1})\varpi, -m\varpi) + K^4\lambda(2m)\varepsilon(m\varpi, (m - Id_{\mathscr{E}1})\varpi, -m\varpi) \\
&\quad + K^4\lambda(2m - Id_{\mathscr{E}_1})\varepsilon(m\varpi, (m - Id_{\mathscr{E}_1})\varpi, -m\varpi) \\
&= \big(2K\lambda(m - Id_{\mathscr{E}_1}) + 3K^2\lambda(m) + 3K^3\lambda(-m) + K^4\lambda(2m) + K^4\lambda(2m - Id_{\mathscr{E}_1})\big) \\
&\quad \varepsilon(m\varpi, (m - Id_{\mathscr{E}_1})\varpi, -m\varpi) \\
&= \alpha_m \varepsilon(m\varpi, (m - Id_{\mathscr{E}_1})\varpi, -m\varpi).
\end{aligned} \tag{4.3.8}$$

Due to the operator Λ being linear, we get

$$\begin{aligned}
\varepsilon^*(\varpi) &= \sum_{n=0}^{\infty} \left(\Lambda_m^n \varepsilon_m\right)^\theta (\varpi) \\
&\leq \varepsilon^\theta(m\varpi, (m - Id_{\mathscr{E}_1})\varpi, -m\varpi) \sum_{n=0}^{\infty} \alpha_m^{n\theta} \\
&= \frac{\varepsilon^\theta(m\varpi, (m - Id_{\mathscr{E}_1})\varpi, -m\varpi)}{1 - \alpha_m^\theta} \\
&< \infty,
\end{aligned}$$

$\forall\ \varpi \in \mathscr{E}_1, m \in l(\mathscr{E}_2)$. Therefore by [[44], Corollary 2.2, p. 138], there exists a unique solution $\mathscr{G}_m : \mathscr{E}_1 \to \mathscr{E}_2$ of

$$\mathscr{G}_m(\varpi) = 2\mathscr{G}_m((m - Id_{\mathscr{E}_1})\varpi) + 3\mathscr{G}_m(m\varpi) + 3\mathscr{G}_m(-m\varpi) - \mathscr{G}_m(2m\varpi) - \mathscr{G}_m((2m - Id_{\mathscr{E}_1})\varpi), \tag{4.3.9}$$

$\forall\ \varpi \in \mathscr{E}_1$, which is a fixed point of $\mathscr{T}_m$ such that

$$\|\mathscr{G}_m(\varpi) - g(\varpi)\|^\theta \leq 4\varepsilon^*(\varpi),\ \forall\ \varpi \in \mathscr{E}_1, m \in l(\mathscr{E}_1). \tag{4.3.10}$$

Moreover,

$$\mathscr{G}_m(\varpi) = \lim_{r \to \infty} \mathscr{T}_m^r g(\varpi), \ \ \forall\ \varpi \in \mathscr{E}_1, m \in l(\mathscr{E}_1). \tag{4.3.11}$$

Now, to show that $\mathscr{G}_m$ fulfills the functional equation (1.1.6) on $\mathscr{E}_1$, we have to show the following inequality

$$\begin{aligned}&\|\mathscr{T}_m^r g(\varpi+\vartheta+\gamma)+\mathscr{T}_m^r g(\varpi-\vartheta)+\mathscr{T}_m^r g(\varpi-\gamma)+\mathscr{T}_u^r g(\vartheta-\gamma)-3[\mathscr{T}_m^r g(\varpi)\\&+\mathscr{T}_m^r g(\vartheta)+\mathscr{T}_m^r g(\gamma)]\| \leq \quad \alpha_m^r \varepsilon(\varpi,\vartheta,\gamma), \forall\, \varpi,\vartheta,\gamma\in\mathscr{E}_1,\ m\in l(\mathscr{E}_1).\end{aligned} \tag{4.3.12}$$

Indeed if $r=0$ then (4.3.12) is simply (4.3.1). So we assume that (4.3.12) holds for $r\in\mathbb{N}$, $m\in l(\mathscr{E}_1)$ and $\varpi,\vartheta,\gamma\in\mathscr{E}_1$. Then from (4.3.7) and the triangle inequality, we have

$$\begin{aligned}&||\mathscr{T}_m^{r+1}g(\varpi+\vartheta+\gamma)+\mathscr{T}_m^{r+1}g(\varpi-\vartheta)+\mathscr{T}_m^{r+1}g(\varpi-\gamma)+\mathscr{T}_m^{r+1}g(\vartheta-\gamma)\\
&-3[\mathscr{T}_m^{r+1}g(\varpi)+\mathscr{T}_m^{r+1}g(\vartheta)+\mathscr{T}_m^{r+1}g(\gamma)]||\\
&=||2\mathscr{T}_m^r g((m-Id_{\mathscr{E}1})(\varpi+\vartheta+\gamma))+3\mathscr{T}_m^r g(m(\varpi+\vartheta+\gamma))+3\mathscr{T}_m^r g(-m(\varpi+\vartheta+\gamma))-\\
&\mathscr{T}_m^r g(2m(\varpi+\varkappa+\gamma))-\mathscr{T}_m^r g((2m-Id_{\mathscr{E}1})(\varpi+\vartheta+\gamma))+2\mathscr{T}_m^r g((m-Id_{\mathscr{E}1})(\varpi-\vartheta))+\\
&3\mathscr{T}_m^r g(m(\varpi-\vartheta))+3\mathscr{T}_m^r g(-m(\varpi-\vartheta))-\mathscr{T}_m^r g(2m(\varpi-\vartheta))-\mathscr{T}_m^r g((2m-Id_{\mathscr{E}1})(\varpi-\vartheta))+\\
&2\mathscr{T}_m^r g((m-Id_{\mathscr{E}1})(\varpi-\gamma))+3\mathscr{T}_m^r g(m(\varpi-\gamma))+3\mathscr{T}_m^r g(-m(\varpi-\gamma))-\mathscr{T}_m^r g(2m(\varpi-\gamma))-\\
&\mathscr{T}_m^r g((2m-Id_{\mathscr{E}1})(\varpi-\gamma))+2\mathscr{T}_m^r g((m-Id_{\mathscr{E}1})(\vartheta-\gamma))+3\mathscr{T}_m^r g(m(\vartheta-\gamma))\\
&\quad+3\mathscr{T}_m^r g(-m(\vartheta-\gamma))-\mathscr{T}_m^r g(2m(\vartheta-\gamma))-\mathscr{T}_m^r g((2m-Id_{\mathscr{E}1})(\vartheta-\gamma))\\
&\quad-3[2\mathscr{T}_m^r g((m-Id_{\mathscr{E}1})(\varpi))+3\mathscr{T}_m^r g(m(\varpi))+3\mathscr{T}_m^r g(-m(\varpi))-\mathscr{T}_m^r g(2m(\varpi))\\
&\quad-\mathscr{T}_m^r g((2m-Id_{\mathscr{E}1})(\varpi))+2\mathscr{T}_m^r g((m-Id_{\mathscr{E}1})(\vartheta))+3\mathscr{T}_m^r g(m(\vartheta))+3\mathscr{T}_m^r g(-m(\vartheta))\\
&\quad-\mathscr{T}_m^r g(2m(\vartheta))-\mathscr{T}_m^r g((2m-Id_{\mathscr{E}1})(\vartheta))+2\mathscr{T}_m^r g((m-Id_{\mathscr{E}1})(\gamma))+3\mathscr{T}_m^r g(m(\gamma))\\
&\quad+3\mathscr{T}_m^r g(-m(\gamma))-\mathscr{T}_m^r g(2m(\gamma))-\mathscr{T}_m^r g((2m-Id_{\mathscr{E}1})(\gamma))]||\\
&\leq 2K\alpha_m^r\varepsilon((m-Id_{\mathscr{E}1})\varpi,(m-Id_{\mathscr{E}1})\vartheta,(m-Id_{\mathscr{E}1})\gamma)\\
&\quad+3K^2\alpha_m^r\varepsilon(m\varpi,m\vartheta,m\gamma)+3K^3\alpha_m^r\varepsilon(-m\varpi,-m\vartheta,-m\gamma)\\
&\quad+K^4\alpha_m^r\varepsilon(2m\varpi,2m\vartheta,2m\varpi)+K^4\alpha_m^r\varepsilon((2m-Id_{\mathscr{E}_1})\varpi,(2m-Id_{\mathscr{E}_1})\vartheta,(2m-Id_{\mathscr{E}_1})\gamma)\\
&\leq 2K\alpha_m^r\lambda(m-Id_{\mathscr{E}_1})\varepsilon(\varpi,\vartheta,\gamma)+3K^2\alpha_m^r\lambda(m)\varepsilon(\varpi,\vartheta,\gamma)+3K^3\alpha_m^r\lambda(-m)\varepsilon(\varpi,\vartheta,\gamma)\\
&\quad+K^4\alpha_m^r\lambda(2m)\varepsilon(\varpi,\vartheta,\gamma)+K^4\alpha_m^r\lambda(2m-Id_{\mathscr{E}_1})\varepsilon(\varpi,\vartheta,\gamma)\\
&=\alpha_m^r[2K\lambda(m-Id_{\mathscr{E}_1})+3K^2\ \lambda(m)+3K^3\lambda(-m)+K^4\lambda(2m)+K^4\lambda(2m-Id_{\mathscr{E}1})]\varepsilon(\varpi,\vartheta,\gamma)\\
&\leq \alpha_m^{r+1}\varepsilon(\varpi,\vartheta,\gamma).\end{aligned}$$

So we get

$$\begin{aligned}&||\mathscr{T}_m^{r+1}g(\varpi+\vartheta+\gamma)+\mathscr{T}_m^{r+1}g(\varpi-\vartheta)+\mathscr{T}_m^{r+1}g(\varpi-\gamma)+\mathscr{T}_m^{r+1}g(\vartheta-\gamma)\\&-3[\mathscr{T}_m^{r+1}g(\varpi)+\mathscr{T}_m^{r+1}g(\vartheta)+\mathscr{T}_m^{r+1}g(\gamma)]||^\theta\leq \alpha_m^{(r+1)\theta}\varepsilon^\theta(\varpi,\vartheta,\gamma).\end{aligned}$$

By induction, we have proven that (4.3.12) valid $\forall\, \varpi,\vartheta,\gamma\in\mathscr{E}_1$.

From (4.3.12) and (1.3.2), we get

$$\begin{aligned}&|||\mathscr{T}_m^{r+1}g(\varpi+\vartheta+\gamma)+\mathscr{T}_m^{r+1}g(\varpi-\vartheta)+\mathscr{T}_m^{r+1}g(\varpi-\gamma)+\mathscr{T}_m^{r+1}g(\vartheta-\gamma)\\&-3[\mathscr{T}_m^{r+1}g(\varpi)+\mathscr{T}_m^{r+1}g(\vartheta)+\mathscr{T}_m^{r+1}g(\gamma)]\ |||^\theta\leq\alpha_m^{(r+1)\theta}\varepsilon^\theta(\varpi,\vartheta,\gamma).\end{aligned}\tag{4.3.13}$$

Letting $r\to\infty$ in (4.3.13), using (4.3.11) and definition of $l(\mathscr{E}_1)$, we get

$$\mathscr{G}_\eta(\varpi+\vartheta+\gamma)+\mathscr{G}_\eta(\varpi-\vartheta)+\mathscr{G}_\eta(\varpi-\gamma)+\mathscr{G}_\eta(\vartheta-\gamma)=3[\mathscr{G}_\eta(\varpi)+\mathscr{G}_\eta(\vartheta)+\mathscr{G}_\eta(\gamma)]\tag{4.3.14}$$

$\forall\ \varpi,\vartheta,\gamma\in\mathscr{E}_1,m\in l(\mathscr{E}_1)$. Therefore we have proven that for every for $m\in l(\mathscr{E}_1)$ there exists a $\mathscr{G}_m:\mathscr{E}_1\to\mathscr{E}_2$ which is the solution of (1.1.6) on $\mathscr{E}_1$ and fulfills

$$\|g(\varpi)-\mathscr{G}_m(\varpi)\|^\theta\leq 4\left(\frac{\varepsilon_m^\theta(\varpi)}{1-\alpha_m^\theta}\right)=4\varepsilon^*(\varpi),\forall\ \varpi\in\mathscr{E}_1.$$

Now, we prove that $\mathscr{G}_m=\mathscr{G}_q\ \forall\ m,q\in l(\mathscr{E}_1)$. Fix m,q and note that $\mathscr{G}_q$ satisfies (4.3.10) with m replaced by q. Therefore by replacing $(\varpi,\vartheta,\gamma)$ with $(m\varpi,(m-Id_{\mathscr{E}_1})\varpi,-m\varpi)$ in (4.3.14), and using (1.3.1) and (1.3.2), we get $\mathscr{T}D_j=D_j$, for $j=m,q$ and

$$\begin{aligned}|||\mathscr{G}_m(\varpi)-\mathscr{G}_q(\varpi)|||^\theta&\leq|||\mathscr{G}_m(\varpi)-g(\varpi)|||^\theta+|||\mathscr{G}_q(\varpi)-g(\varpi)|||^\theta\\&\leq\left(\frac{4\varepsilon_m^\theta(\varpi)}{1-\alpha_m^\theta}\right)+\left(\frac{4\varepsilon_q^\theta(x)}{1-\alpha_q^\theta}\right),\ \forall\ \varpi\in\mathscr{E}_1.\end{aligned}$$

It follows from (4.3.8) and linearity of Λ that

$$\begin{aligned}|||\mathscr{G}_m(\varpi)-\mathscr{G}_q(\varpi)|||^\theta&=|||\mathscr{T}^n\mathscr{G}_m(\varpi)-\mathscr{T}^n\mathscr{G}_q(\varpi)|||^\theta\\&\leq 4\left(\frac{\Lambda^n\varepsilon_m^\theta(\varpi)}{1-\alpha_m^\theta}\right)+4\left(\frac{\Lambda^n\varepsilon_q^\theta(\varpi)}{1-\alpha_q^\theta}\right)\\&\leq(\alpha_m)^nU_m(\varpi)+(\alpha_q)^nU_q(\varpi),\end{aligned}$$

where $U_m(\varpi)=4\frac{\varepsilon_m^\theta(\varpi)}{1-\alpha_m^\theta}\ \forall\ \varpi\in\mathscr{E}_1$, and $n\in\mathbb{N}$. Taking $n\to\infty$, we have $\mathscr{G}_m=\mathscr{G}_q=\mathscr{G}$. Thus, we get

$$\|g(\varpi)-\mathscr{G}(\varpi)\|^\theta\leq U_m(\varpi),\ \forall\ \varpi\in\mathscr{E}_1,m\in l(\mathscr{E}_1).$$

Thus, we derive (4.3.5). Due to (4.3.14), it is obvious that $\mathscr{G}$ is a solution of (1.1.6). Now to demonstrate the uniqueness of map $\mathscr{G}$, assume that there exists a map $\mathscr{G}':\mathscr{E}_1\to\mathscr{E}_2$

which fulfills (1.1.6) and

$$\|g(\varpi) - \mathscr{G}'(\varpi)\|^{\theta} \leq 4\varepsilon^{*}(\varpi).$$

Using (1.3.2), we get

$$|||g(\varpi) - \mathscr{G}'(\varpi)|||^{\theta} \leq \|g(\varpi) - \mathscr{G}'(\varpi)\|^{\theta} \leq 4\varepsilon^{*}(\varpi), \ \forall \ \varpi \in \mathscr{E}_1.$$

Then

$$|||\mathscr{G}(\varpi) - \mathscr{G}'(\varpi)|||^{\theta} \leq 8\varepsilon^{*}(\varpi).$$

Further $\mathscr{T}\mathscr{G}'(\varpi) = \mathscr{G}'(\varpi) \ \forall \ \varpi \in \mathscr{E}_1$. Consequently, with a fixed $m \in l(\mathscr{E}_1)$

$$\begin{aligned} |||\mathscr{G}(\varpi) - \mathscr{G}'(\varpi)|||^{\theta} &= |||\mathscr{T}^{n}\mathscr{G}(\varpi) - \mathscr{T}^{n}\mathscr{G}'(\varpi)|||^{\theta} \\ &\leq 8\Lambda^{n}\varepsilon^{*}(\varpi) \\ &\leq \frac{8\Lambda^{n}\varepsilon_m^{\theta}(\varpi)}{1-\alpha_m^{\theta}} \\ &\leq \frac{8\alpha_m^{n}\varepsilon_m^{\theta}(\varpi)}{1-\alpha_m^{\theta}}, \end{aligned}$$

$\forall \varpi \in \mathscr{E}_1, m \in l(\mathscr{E}_1)$, and $n \in \mathbb{N}$. Taking $n \to \infty$, we have $\mathscr{G} = \mathscr{G}'$. ■

Theorem 4.3.2. *Let $g : \mathscr{E}_1 \to \mathscr{E}_2$, and $\varepsilon : \mathscr{E}_1 \times \mathscr{E}_1 \times \mathscr{E}_1 \to [0, \infty)$ be functions, and the conditions (4.3.2), (4.3.3) and (4.3.4) be valid. Suppose that*

$$\inf\{\varepsilon^{\theta}(m\varpi, (m - Id_{\mathscr{E}_1})\varpi, -m\varpi) : m \in l(\mathscr{E}_1)\} = 0, \ \forall \ \varpi \in \mathscr{E}_1. \tag{4.3.15}$$

Then g fulfills (1.1.6) on $\mathscr{E}_1$.

Proof. Assume that

$$\inf\{\varepsilon^{\theta}(m\varpi, (m - Id_{\mathscr{E}1})\varpi, -m\varpi) : m \in l(\mathscr{E}_1)\} = 0, \forall \ \varpi \in \mathscr{E}_1. \tag{4.3.16}$$

Hence, from Theorem 4.3.1, we have $\varepsilon(\varpi) = 0$ for all $\varpi \in \mathscr{E}_1$. Then g fulfills (1.1.6) on $\mathscr{E}_1$. ■

Remark 4.3.3. *In Theorem 4.3.1, if*

$$\inf\{2K\lambda(m - Id_{\mathscr{E}_1}) + 3K^2\lambda(m) + 3K^3\lambda(-m) + K^4\lambda(2m) + K^4\lambda(2m - Id_{\mathscr{E}1}) : m \in l(\mathscr{E}_1)\} = 0$$

(this is the case when $\lim_{|m|\to\infty} \lambda(m) = 0$), then (4.3.2) holds,

$$\varepsilon^*(\varpi) = \inf_{m\in l(\mathscr{E}_1)} \varepsilon^\theta(m\varpi, (m - Id_{\mathscr{E}1})\varpi, -m\varpi), \forall\ \varpi \in \mathscr{E}_1.$$

Remark 4.3.4. *As the main result of Brzdęk [Theorem 7, p.5, [30]] is a partial generalization of main result of Dung and Hang [Corollary 2.2, p. 138, [44]]. So using the result of Brzdęk [30], we can partially generalized our Theorem 4.3.1, equation (4.3.5) can be*

$$\|\mathscr{G}(\varpi) - g(\varpi)\|^\theta \leq 4\varepsilon^*(\varpi),$$

$\forall\ \varpi \in \mathscr{E}_0$, where $\theta = \log_{2K} 2$ with $K \geq 1$, $\varepsilon^*(\varpi) := \sum_{r=0}^{\infty} \left(\Lambda^r_{\theta m}\varepsilon^\theta_m\right)^\theta(\varpi)$ (because $\Lambda = \Lambda_\theta$ and $\varepsilon = \varepsilon^\theta$).

Similarly, we can show the following result on $\mathscr{E}_0\backslash\{0\} = \mathscr{E}_0$.

Theorem 4.3.5. *Let $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a map such that*

$$||g(\varpi + \vartheta + \gamma) + g(\varpi - \vartheta) + g(\varpi - \gamma) + g(\vartheta - \gamma) - 3[g(\varpi) + g(\vartheta) + g(\gamma)]|| \leq \varepsilon(\varpi, \vartheta, \gamma), \tag{4.3.17}$$

where $\varepsilon : \mathscr{E}_0 \times \mathscr{E}_0 \times \mathscr{E}_0 \to [0, \infty]$, $\gamma, \vartheta, \varpi \in \mathscr{E}_0$, and $\varpi - \gamma, \varpi + \vartheta + \gamma, \varpi - \vartheta, \vartheta - \gamma \neq 0$.

Assume that

$$l(\mathscr{X}) := \{m \in Aut(\mathscr{E}_1) : m, -m, m - Id_{\mathscr{E}1}, 2m, 2m - Id_{\mathscr{E}1} \in Aut(\mathscr{E}_1), \alpha_m < 1\} \tag{4.3.18}$$

is a nonempty set, where

$$\alpha_m := 2K\lambda(m - Id_{\mathscr{E}1}) + 3K^2\lambda(m) + 3K^3\lambda(-m) + K^4\lambda(2m) + K^4\lambda(2m - Id_{\mathscr{E}1}), \tag{4.3.19}$$

$$\lambda(m) := \inf\{t \in \mathbb{R}_+ : \varepsilon(m\varpi, m\vartheta, m\gamma) \leq t\varepsilon(\varpi, \vartheta, \gamma)\ \forall\varpi, \vartheta, \gamma \in \mathscr{E}_0\}, \tag{4.3.20}$$

for $m \in Aut(\mathscr{E}_1)$.

Then, there exists a unique $\mathscr{G} : \mathscr{E}_0 \to \mathscr{E}_2$ fulfilling (1.1.6) on $\mathscr{E}_0$ and

$$\|\mathscr{G}(\varpi) - g(\varpi)\|^\theta \leq 4\varepsilon^*(\varpi), \tag{4.3.21}$$

$\forall\ \varpi \in \mathscr{E}_0$, where $\theta = \log_{2K} 2$ with $K \geq 1$, $\varepsilon^*(\varpi) := \inf\left\{\frac{\varepsilon^\theta(m\varpi,(m-Id_{\mathscr{E}_1})\varpi,-m\varpi)}{1-\alpha^\theta_m} : m \in l(\mathscr{E}_1)\right\}$.

Now, we provide a few consequences of Theorem 4.3.5 for the following two cases:

$$\varepsilon_1(\varpi,\vartheta,\gamma) = \left(\phi\|\varpi\|^p\|\vartheta\|^q\|\gamma\|^r\right)^\theta,$$

$p+r<0$ and $q<0$,

$$\varepsilon_2(\varpi,\vartheta,\gamma) = \left(\phi(\|\varpi\|^p+\|\vartheta\|^p+\|\gamma\|^p)\right)^\theta, \quad p<0,$$

where $\varepsilon(\varpi,\vartheta,\gamma)=\varepsilon_n(\varpi,\vartheta,\gamma)$ for $n\in\{1,2\}$, $\phi>0$, $r,q,p\in\mathbb{R}$, $\varpi,\vartheta\neq 0$, and $\theta=\log_{2K}2$ with $K\geq 1$.

g is a solution of (1.1.6) on $S\backslash\{0\}=S_0$ if (1.1.6) holds for every $\varpi,\vartheta,\gamma\in S_0$ with $\varpi-\gamma,\varpi+\vartheta+\gamma,\varpi-\vartheta,\vartheta-\gamma\neq 0$. So we have the following results.

Corollary 4.3.6. *Assume that $S:=(S,+)$ is a nonempty subgroup of $(\mathscr{E}_1,+)$. Let $g:S\to\mathscr{E}_2$ satisfy*

$$\begin{aligned}&||g(\varpi+\vartheta+\gamma)+g(\varpi-\vartheta)+g(\varpi-\gamma)+g(\vartheta-\gamma)-3[g(\varpi)+g(\vartheta)+g(\gamma)]||^\theta\\&\leq\left(\phi\|\varpi\|^p\|\vartheta\|^q\|\gamma\|^r\right)^\theta,\end{aligned}$$

$\forall\varpi,\vartheta,\gamma\in S_0$, with $\varpi-\gamma,\varpi+\vartheta+\gamma,\varpi-\vartheta,\vartheta-\gamma\neq 0$, $r,q,p\in\mathbb{R}$, $q<0, p+r<0$, $\phi>0$ and $\theta=\log_{2K}2$. Then g is a solution of (1.1.6) on S_0.

Proof. We apply Theorem 4.3.5 with $\varepsilon_1(\varpi,\vartheta,\gamma)=\left(\phi\|\varpi\|^p\|\vartheta\|^q\|\gamma\|^r\right)^\theta$, $\forall\ \varpi,\vartheta,\gamma\in S_0$ with few real numbers $\phi>0$, $r,q,p\in\mathbb{R},q<0,p+r<0$, $\theta=\log_{2K}2$. For each $j\in\mathbb{N}$, define $m_j:S_0\to S_0$ by $m_j\varpi=j\varpi$. Then

$$\begin{aligned}\varepsilon_1(m_j\varpi,m_j\vartheta,m_j\gamma)&=\varepsilon_1(j\varpi,j\vartheta,j\gamma)\\&=\left(\phi\|j\varpi\|^p\|j\vartheta\|^q\|j\gamma\|^r\right)^\theta\\&=|j|^{(p+q+r)\theta}\left(\phi\|\varpi\|^p\|\vartheta\|^q\|\gamma\|^r\right)^\theta\\&\leq|j|^{p+q+r}\varepsilon_1(\varpi,\vartheta,\gamma),\end{aligned}$$

$\forall\ \varpi,\vartheta,\gamma\in S_0$ and $j\in\mathbb{N}$. Hence

$$\lim_{j\to\infty}\varepsilon_1(m_j\varpi,m_j\vartheta,m_j\gamma)\leq\lim_{j\to\infty}|j|^{p+q+r}\varepsilon_1(\varpi,\vartheta,\gamma)=0,\forall\ \varpi,\vartheta,\gamma\in S_0 \text{ and } j\in\mathbb{N}.$$

Then (4.3.15) is correct with $\lambda(m_j)=|j|^{p+q+r}$ for $j\in\mathbb{N}$, there exists $m_0\in\mathbb{N}$ with $m_0>1$ such that $j\geq m_0$ and

$$\alpha_{m_j}=2\mid j-1\mid^{p+q+r}+(6+2^{p+q+r})\mid j\mid^{p+q+r}+\mid 2j-1\mid^{p+q+r}<1.$$

Therefore we can say that (4.3.18) is satisfied with $l(\mathscr{E}_1) := \{m_j \in Aut(\mathscr{E}_1) : j \in \mathbb{N}_{m_0}\}$. Hence by the Theorem 4.3.2, every function $g : S \to \mathscr{E}_2$ satisfies (1.1.6) on S_0. ■

Corollary 4.3.7. *Assume that $S := (S, +)$ be a nonempty subgroup of $(\mathscr{E}_1, +)$. Let $g : S \to \mathscr{E}_2$ satisfy*

$$\begin{aligned}&||g(\varpi + \vartheta + \gamma) + g(\varpi - \vartheta) + g(\varpi - \gamma) + g(\vartheta - \gamma) - 3[g(\varpi) + g(\vartheta) + g(\gamma)]||^\theta \\ &\leq (\phi(\|\varpi\|^p + \|\vartheta\|^p + \|\gamma\|^p))^\theta, \ \forall \ \varpi, \vartheta, \gamma \in S_0,\end{aligned}$$

with $\varpi - \gamma, \vartheta + \varpi + \gamma, \varpi - \vartheta, \vartheta - \gamma \neq 0$, $\phi > 0$, $p < 0$, and $\theta = \log_{2K} 2$. Then g is a solution of (1.1.6) on S_0.

Proof. We apply Theorem 4.3.5 with $\varepsilon_2(\varpi, \vartheta, \gamma) = (\phi\|\varpi\|^p + \|\vartheta\|^p + \|\gamma\|^p)^\theta$, $\forall \ \varpi, \vartheta, \gamma \in S_0$ with a few real numbers $\phi > 0$, $p < 0$, $\theta = \log_{2K} 2$. For each $j \in \mathbb{N}$, define $m_j : S_0 \to S_0$ by $m_j\varpi = j\varpi$. Then

$$\begin{aligned}\varepsilon_2(m_j\varpi, m_j\vartheta, m_j\gamma) &= \varepsilon_2(j\varpi, j\vartheta, j\gamma) \\ &= (\phi\|j\varpi\|^p + \|j\vartheta\|^p + \|j\gamma\|^p)^\theta \\ &= |j|^{(p)\theta} (\phi\|\varpi\|^p + \|\vartheta\|^p + \|\gamma\|^p)^\theta \\ &\leq |j|^p \varepsilon_2(\varpi, \vartheta, \gamma), \ \forall \ \varpi, \vartheta, \gamma \in S_0, j \in \mathbb{N}.\end{aligned}$$

Hence

$$\lim_{j\to\infty} \varepsilon_2(m_j\varpi, m_j\vartheta, m_j\gamma) \leq \lim_{j\to\infty} |j|^p \varepsilon_2(\varpi, \vartheta, \gamma) = 0, \ \forall \ \varpi, \vartheta, \gamma \in S_0, j \in \mathbb{N}.$$

Then (4.3.15) is true with $\lambda(m_j) = |j|^p$, there exists $m_0 \in \mathbb{N}$ with $m_0 > 1$ such that $j \geq m_0$ and

$$\alpha_{m_j} = 2 \mid j - 1 \mid^p + (6 + 2^p) \mid j \mid^p + \mid 2j - 1 \mid^p < 1.$$

Therefore we can say that (4.3.18) is satisfied with $l(\mathscr{E}_1) := \{m_j \in Aut(\mathscr{E}_1) : j \in \mathbb{N}_{m_0}\}$. Hence by the Theorem 4.3.2, every function $g : S \to \mathscr{E}_2$ satisfies (1.1.6) on S_0. ■

Now, we prove that Corollaries 4.3.6 and 4.3.7 yield a characterization of IPS.

Corollary 4.3.8. *Suppose that $(\mathscr{E}_1, K, \|.\|)$ is a QNS, $\mathscr{E}_0 = \mathscr{E}_1 \backslash \{0\}$, and write*

$$\Delta(\varpi, \vartheta, \gamma) = \big| ||\varpi + \vartheta + \gamma||^2 + ||\varpi - \vartheta||^2 + ||\varpi - \gamma||^2 + ||\vartheta - \gamma||^2 - 3[||\varpi||^2 + ||\vartheta||^2 + ||\gamma||^2] \big|,$$

$\forall \ \varpi, \vartheta, \gamma \in \mathscr{E}_1$. *let either of the following two conditions remain* *(i)* $\sup_{\varpi,\vartheta,\gamma\in\mathscr{E}_0} \dfrac{\Delta(\varpi,\vartheta,\gamma)}{\varepsilon_1(\varpi,\vartheta,\gamma)} < \infty;$

(ii) $\sup_{\varpi,\vartheta,\gamma\in\mathscr{E}_0} \dfrac{\Delta(\varpi,\vartheta,\gamma)}{\varepsilon_2(\varpi,\vartheta,\gamma)} < \infty;$

Then $\mathscr{E}_1$ is an IPS.

Proof. Write $g(\varpi) = ||\varpi||^2$. Then from Corollaries 4.3.6 and 4.3.7, we simply find that g is a solution of (1.1.6). This means that $\Delta(\varpi,\vartheta) = 0$. So, QN $||.||$ on $\mathscr{E}_1$ fulfills the parallelogram law:

$$||\varpi+\vartheta||^2 + ||\varpi-\vartheta||^2 = 2||\varpi||^2 + 2||\vartheta||^2, \ \forall\ \varpi,\vartheta \in \mathscr{E}_1.$$

Therefore, $\mathscr{E}_1$ is an IPS. ■

The following example proves that the assumption in the above corollaries is essential.

Example 4.3.1. *Let $\mathscr{E}_1 = \mathscr{E}_2 = L^{\frac{1}{2}}[0,1]$, and $\| \varpi \|_{\mathscr{E}_1} = \| \varpi \|_{\mathscr{E}_2} = \left(\int_0^1 \mid \varpi(t) \mid^{\frac{1}{2}} dt\right)^2$, $\forall$ $\varpi \in \mathscr{E}_1$, where $L^{\frac{1}{2}}[0,1] = \{g : [0,1] \to \mathbb{R} : \mid g \mid^{1/2}$ is Lebesgue integrable$\}$. Let $g(\varpi) = \varpi$, for all $\varpi \in \mathscr{E}_1$ and $\varepsilon(\varpi,\vartheta,\gamma) = \left(\int_0^1 (2 \mid \vartheta(t) \mid - 4 \mid \gamma(t) \mid)^{\frac{1}{2}}\right)^2$, $\forall\ \varpi,\vartheta,\gamma \in \mathscr{E}_1$ such that $\varpi-\gamma, \varpi+\vartheta+\gamma, \varpi-\vartheta, \vartheta-\gamma \neq 0$. Then*

$$||g(\varpi+\vartheta+\gamma) + g(\varpi-\vartheta) + g(\varpi-\gamma) + g(\vartheta-\gamma) - 3[g(\varpi)+g(\vartheta)+g(\gamma)]|| \le \varepsilon(\varpi,\vartheta,\gamma), \tag{4.3.22}$$

holds $\forall\ \varpi,\vartheta,\gamma \in \mathscr{E}_1$ but g does not satisfy (1.1.6).

Proof. Note that $\mathscr{E}_1$ and $\mathscr{E}_2$ are quasi-Banach spaces.

$$\begin{aligned}
&||g(\varpi+\vartheta+\gamma) + g(\varpi-\vartheta) + g(\varpi-\gamma) + g(\vartheta-\gamma) - 3[g(\varpi)+g(\vartheta)+g(\gamma)]||\\
&= \Bigg(\int_0^1 ((\varpi(t)+\vartheta(t)+\gamma(t)) + (\varpi(t)-\vartheta(t)) + (\varpi(t)-\gamma(t)) + (\vartheta(t)-\gamma(t))\\
&\quad - 3\varpi(t - 3\vartheta(t) - 3\gamma(t))^{\frac{1}{2}}\Bigg)^2\\
&= \left(\int_0^1 (2 \mid \vartheta(t) \mid - 4 \mid \gamma(t) \mid)^{\frac{1}{2}}\right)^2\\
&= \varepsilon(\varpi,\vartheta,\gamma).
\end{aligned}$$

This proves that (4.3.22) holds, but g does not satisfy the (1.1.6). ■

4.4 Approximation of Drygas functional equation in complete quasi-normed space

Here, We study the generalized HUR stability problem for (1.1.8) in the setting of complete QNS. For this equation, We also obtain some hyperstability results. This section's results extend various previously known results in the framework of complete QNS. Results of this section are proved in Sharma and Chandok [115].

Throughout this section, $\mathscr{E}_2$ is a complete QNS, $\mathscr{E}_1$ is a QNS and for each $m \in \mathscr{E}_1^{\mathscr{E}_1}$ we write $m\varpi = m(\varpi)$ for $\varpi \in \mathscr{E}_1$ and we defined $-m$ by $-m\varpi := -m(\varpi), 2m\varpi = m\varpi + m\varpi$ and $m' = m'\varpi := (Id_{\mathscr{E}_1} - m)\varpi = \varpi - m\varpi$ for $\varpi \in \mathscr{E}_1$.

Theorem 4.4.1. *Suppose that $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a map such that*

$$||g(\varpi + \vartheta) + g(\varpi - \vartheta) - 2\ g(\varpi) - \ g(\vartheta) - g(-\vartheta)|| \leq \varepsilon(\varpi, \vartheta), \tag{4.4.1}$$

where $\varepsilon : \mathscr{E}_0 \times \mathscr{E}_0 \to [0, \infty]$, $\vartheta, \varpi \in \mathscr{E}_0$ such that $\varpi - \vartheta \neq 0$ and $\varpi + \vartheta \neq 0$.

Assume that

$$l(\mathscr{E}_1) := \{m \in Aut(\mathscr{E}_1) : (Id_{\mathscr{E}_1} - 2m), m', -m, m, \in Aut(\mathscr{E}_1), \alpha_m < 1\} \tag{4.4.2}$$

is an nonempty set, where

$$\alpha_m := 2K\lambda(m') + K^2\lambda(m) + K^3\lambda(-m) + K^3\lambda(Id_{\mathscr{E}_1} - 2m),$$

$$\lambda(m) := \inf\{t \in \mathbb{R}_+ : \varepsilon(m\varpi, m\vartheta) \leq t\varepsilon(\varpi, \vartheta)\ \forall \varpi, \vartheta \in \mathscr{E}_0\},$$

for $m \in Aut(\mathscr{E}_1), K \geq 1$.

Then, for each nonempty subset $\mathcal{A} \subset l(\mathscr{E}_1)$ such that

$$b \circ a = a \circ b =, (b, a \in \mathcal{A}), \tag{4.4.3}$$

there exists a unique $\mathscr{G} : \mathscr{E}_1 \to \mathscr{E}_2$ fulfilling (1.1.8) and

$$\| \mathscr{G}(\varpi) - g(\varpi) \|^{\theta} \leq 4\varepsilon^*(\varpi),\ \forall\ \varpi \in \mathscr{E}_0, \tag{4.4.4}$$

where $\theta = \log_{2K} 2$ and $\varepsilon^*(\varpi) := \inf\left\{\frac{\varepsilon^{\theta}(m'\varpi, m\varpi)}{1-\alpha_m^{\theta}} : m \in \mathcal{A}\right\}$.

Proof. Fix $m \in \mathcal{A}$. Replacing ϑ by $m\varpi$ and ϖ by $m'\varpi$ in (4.4.1), we have

$$\|g(\varpi) + g((Id_{\mathscr{E}_1} - 2m)\varpi) - 2g(m'\varpi) - g(m\varpi) - g(-m\varpi)\| \leq \varepsilon(m'\varpi, m\varpi) := \varepsilon_m(\varpi), \tag{4.4.5}$$

$\forall\ \varpi \in \mathscr{E}_0$. We define the operators $\mathscr{T}_m : \mathscr{E}_2^{\mathscr{E}_0} \to \mathscr{E}_2^{\mathscr{E}_0}$ and $\Lambda_m : \mathbb{R}_+^{\mathscr{E}_0} \to \mathbb{R}_+^{\mathscr{E}_0}$ by

$$\mathscr{T}_m\xi(\varpi) := 2\xi(m'\varpi) + \xi(m\varpi) + \xi(-m\varpi) - \xi((Id_{\mathscr{E}_1} - 2m)\varpi) \tag{4.4.6}$$

and

$$\Lambda_m\delta(\varpi) := 2K\delta(m'\varpi) + K^2\delta(m\varpi) + K^3\delta(-m\varpi) + K^3\delta((Id_{\mathscr{E}_1} - 2m)\varpi), \tag{4.4.7}$$

$\forall\ \varpi \in \mathscr{E}_0$, $\xi \in \mathscr{E}_2^{\mathscr{E}_0}$ and $\delta \in \mathbb{R}_+^{\mathscr{E}_0}$.
Then (4.4.5) becomes $\|g(\varpi) - \mathscr{T}_m g(\varpi)\| \leq \varepsilon_m(\varpi)$, $\forall\ \varpi \in \mathscr{E}_0$. The operator Λ_m has the form given by [[44], (2.17), pp. 138] with $k = 4$ and $g_1(\varpi) = m'\varpi, g_2(\varpi) = m\varpi, g_3(\varpi) = -m\varpi, g_4(\varpi) = (Id_{\mathscr{E}_1} - 2m)\varpi$, $L_1(\varpi) = 2K$, $L_2(x) = K^2$ and $L_3(\varpi) = L_4(\varpi) = K^3$ $\forall$ $\varpi \in \mathscr{E}_0$. Further, we have

$$\begin{aligned}
&||\mathfrak{T}_m\xi(\varpi) - \mathfrak{T}_m\mu(\varpi)|| \\
&= ||2\xi(m'\varpi) + \xi(m\varpi) + \xi(-m\varpi) - \xi((Id_{\mathscr{E}_1} - 2m)\varpi) - 2\mu(m'\varpi) - \mu(m\varpi) - \mu(-m\varpi) + \\
&\quad \mu((Id_{\mathscr{E}_1} - 2m)\varpi)|| \\
&\leq 2K||\xi(m'\varpi) - \mu(m'\varpi)|| + K^2||\xi(m\varpi) - \mu(m\varpi)|| + K^3||\xi(-m\varpi) - \mu(-m\varpi)|| \\
&\quad + K^3||\xi((Id_{\mathscr{E}_1} - 2m)\varpi) - \mu((Id_{\mathscr{E}_1} - 2m)\varpi)||, \\
&= \sum_{i=0}^{4} L_i(u) \parallel \xi(g_i(\varpi)) - \mu(g_i(\varpi)) \parallel, \forall\ \varpi \in \mathscr{E}_0, \xi, \mu \in \mathscr{E}_2^{\mathscr{E}_0}.
\end{aligned}$$

Using the definition of $\lambda(m)$, $\varepsilon(m\varpi, m\vartheta) \leq \lambda(m)\varepsilon(m'\varpi, m\varpi)$ $\forall\ \varpi, \vartheta \in \mathscr{E}_0$ we have to show that $\Lambda_m^n\varepsilon_m(\varpi) \leq \alpha_m^n\varepsilon(m'\varpi, m\varpi)$ $\forall\ \varpi \in \mathscr{E}_0$, where $\alpha_m = 2K\lambda(m') + K^2\lambda(m) + K^3\lambda(-m) + K^3\lambda(Id_{\mathscr{E}_1} - 2m)$.

If $n=0$, then $\varepsilon_m(\varpi)=\varepsilon(m'\varpi,m\varpi)$. If $n=1$, we have

$$\begin{aligned}
&\Lambda_m\varepsilon(\varpi)\\
&=2K\varepsilon_m(m'\varpi)+K^2\varepsilon_m(m\varpi)+K^3\varepsilon_m(-m\varpi)+K^3\varepsilon_m((Id_{\mathscr{E}_1}-2m)\varpi)\\
&=2K\varepsilon\left(m'(m'\varpi),m(m'\varpi)\right)+K^2\varepsilon\left(m'(m\varpi),m(m\varpi)\right)+K^3\varepsilon\left(m'(-m\varpi,m(-m\varpi)\right)\\
&\quad+K^3\varepsilon\left(m'((Id_{\mathscr{E}_1}-2m)\varpi),m((Id_{\mathscr{E}_1}-2m)\varpi)\right)\\
&=2K\varepsilon\left(m'(m'\varpi),m'(m\varpi)\right)+K^2\varepsilon\left(m(m'\varpi),m(m\varpi)\right)+K^3\varepsilon\left(-m(m'\varpi),-m(m\varpi)\right)\\
&\quad+K^3\varepsilon\left((Id_{\mathscr{E}_1}-2m)(m'(\varpi),(Id_{\mathscr{E}_1}-2m)(m\varpi)\right)\\
&\le 2K\lambda(m')\varepsilon(m'\varpi,m\varpi)+K^2\lambda(m)\varepsilon(m'\varpi,m\varpi)+K^3\lambda(-m)\varepsilon(m'\varpi,m\varpi)\\
&\quad+K^3\lambda(Id_{\mathscr{E}_1}-2m)\varepsilon(m'\varpi,m\varpi)\\
&=2K\lambda(m')+K^2\lambda(m)+K^3\lambda(-m)+K^3\lambda(Id_{\mathscr{E}_1}-2m)\varepsilon(m'\varpi,m\varpi)=\alpha_m\varepsilon(m'\varpi,m\varpi).
\end{aligned}$$

Now, further, if $r=2$, we have

$$\begin{aligned}
&\Lambda^2\varepsilon_m(\varpi)\\
&=\Lambda\left[\Lambda\varepsilon_m(\varpi)\right]\\
&=2K\Lambda_m\varepsilon_m(m'\varpi)+K^2\Lambda_m\varepsilon_m(m\varpi)+K^3\Lambda_m\varepsilon_m(-m\varpi)+K^3\Lambda_m\varepsilon_m((Id_{\mathscr{E}_1}-2m)\varpi)\\
&=2K\alpha_m\varepsilon\left(m'(m'\varpi),m(m'\varpi)\right)+K^2\alpha_m\varepsilon\left(m'(m\varpi),m(m\varpi)\right)\\
&\quad+K^3\alpha_m\varepsilon\left(m'(-m\varpi,m(-m\varpi)\right)+K^3\varepsilon\left(m'((Id_{\mathscr{E}_1}-2m)\varpi),m((Id_{\mathscr{E}_1}-2m)\varpi)\right)\\
&=2K\alpha_m\varepsilon\left(m'(m'\varpi),m'(m\varpi)\right)+K^2\alpha_m\varepsilon\left(m(m'\varpi),m(m\varpi)\right)\\
&\quad+K^3\alpha_m\varepsilon\left(-m(m'\varpi),-m(m\varpi)\right)+K^3\alpha_m\varepsilon\left((Id_{\mathscr{E}_1}-2m)(m'x),(Id_{\mathscr{E}_1}-2m)(m\varpi)\right)\\
&\le 2K\alpha_m\lambda(m')\varepsilon(m'\varpi,m\varpi)+K^2\alpha_m\lambda(m)\varepsilon(m'\varpi,m\varpi)+K^3\alpha_m\lambda(-m)\varepsilon(m'\varpi,m\varpi)\\
&\quad+K^3\lambda(Id_{\mathscr{E}_1}-2m)\varepsilon(m'\varpi,m\varpi)\\
&=\alpha_m\left(2K\lambda(m')+K^2\lambda(m)+K^3\lambda(-m)+K^3\lambda(Id_{\mathscr{E}_1}-2m)\right)\varepsilon(m'\varpi,m\varpi)\\
&=\alpha_m^2\varepsilon(m'\varpi,m\varpi).
\end{aligned}$$

Proceeding on similar lines, we get

$$\Lambda_m^n\varepsilon_m(\varpi)\le\alpha_m^n\varepsilon(m'\varpi,m\varpi),\forall\ \varpi\in\mathscr{E}_0,n\in\mathbb{N}_0. \tag{4.4.8}$$

Hence

$$\varepsilon^*(\varpi)=\sum_{n=0}^{\infty}\left(\Lambda_m^n\varepsilon_m\right)^\theta(x)\le\varepsilon^\theta(m'\varpi,m\varpi)\sum_{r=0}^{\infty}\alpha_m^{n\theta}=\frac{\varepsilon^\theta(m'\varpi,m\varpi)}{1-\alpha_m^\theta}<\infty,\ \forall\ \varpi\in\mathscr{E}_0.$$

Therefore by [[44], Corollary 2.2, pp. 138], there exists a solution $\mathscr{G}_m : \mathscr{E}_1 \to \mathscr{E}_2$ of

$$\mathscr{G}_m(\varpi) = 2\mathscr{G}_m(m'x) + \mathscr{G}_m(m\varpi) + \mathscr{G}_m(-m\varpi) - \mathscr{G}_m((Id_{\mathscr{E}_1} - 2m)\varpi), \ \forall \ \varpi \in \mathscr{E}_0, \quad (4.4.9)$$

which is a fixed point of $\mathfrak{T}_m$ such that

$$\|\mathscr{G}_m(\varpi) - g(\varpi)\|^\theta \leq 4\varepsilon^*(\varpi), \quad (4.4.10)$$

$\forall \ \varpi \in \mathscr{E}_0$. Moreover, $\mathscr{G}_m(\varpi) = \lim_{r\to\infty} \mathfrak{T}_m^r g(\varpi)$ for all $\varpi \in \mathscr{E}_0$.
Now, to show that $\mathscr{G}_m$ fulfills the functional equation (1.1.8) on $\mathscr{E}_0$, we have to show the following inequality

$$\| \ \mathscr{T}_m^r g(\varpi + \vartheta) + \mathscr{T}_m^r g(\varpi - \vartheta) - 2\mathscr{T}_m^r g(\varpi) - \mathscr{T}_m^r g(\vartheta) - \mathscr{T}_m^r g(-\vartheta) \ \| \leq \ \alpha_m^r \varepsilon(\varpi, \vartheta), \quad (4.4.11)$$

$\forall \ \vartheta, \varpi \in \mathscr{E}_0$ such that $\varpi - \vartheta \neq 0$, $\varpi + \vartheta \neq 0$, and $r \in \mathbb{N}_0$. In fact, if $r = 0$, then (4.4.11) is simply (4.4.1). Thus we suppose that (4.4.11) holds for $\varpi, \vartheta \in \mathscr{E}_0$ and $r \in \mathbb{N}$ such that $\varpi - \vartheta \neq 0$, $\varpi + \vartheta \neq 0$. Then from (4.4.6) and the triangle inequality, we have

$$\begin{aligned}
&\| \ \mathscr{T}_m^{r+1} g(\varpi + \vartheta) + \mathscr{T}_m^{r+1} g(\varpi - \vartheta) - 2\mathscr{T}_m^{r+1} g(\varpi) - \mathscr{T}_m^{r+1} g(\vartheta) - \mathscr{T}_m^{r+1} g(-\vartheta) \ \| \\
&= \| \ 2\mathscr{T}_m^r g(m'(\varpi + \vartheta)) + \mathscr{T}_m^r g(m(\varpi + \vartheta)) + \mathscr{T}_m^r g(-m(\varpi + \vartheta)) \\
&\quad - \mathscr{T}_m^r g((Id_{\mathscr{E}_1} - 2m)(\varpi + \vartheta)) + 2\mathscr{T}_m^r g(m'(\varpi - \vartheta)) + \mathscr{T}_m^r g(m(\varpi - \vartheta)) \\
&\quad + \mathscr{T}_m^r g(-m(\varpi - \vartheta)) - \mathscr{T}_m^r g((Id_{\mathscr{E}_1} - 2m)(\varpi - \vartheta)) - 4\mathscr{T}_m^r g(m'(\varpi)) - 2\mathscr{T}_m^r g(m(\varpi)) \\
&\quad - 2\mathscr{T}_m^r g(-m(\varpi)) + 2\mathscr{T}_m^r g((Id_{\mathscr{E}_1} - 2m)(\varpi)) - 2\mathscr{T}_m^r g(m'(\vartheta)) - \mathscr{T}_m^r g(m(\vartheta)) \\
&\quad - \mathscr{T}_m^r g(-m(\vartheta)) + \mathscr{T}_m^r g((Id_{\mathscr{E}_1} - 2m)(\varkappa)) - 2\mathscr{T}_m^r g(m'(-\vartheta)) - \mathscr{T}_m^r g(m(-\vartheta)) \\
&\quad - \mathscr{T}_m^r g(-m(-\vartheta)) + \mathscr{T}_m^r g((Id_{\mathscr{E}_1} - 2m)(-\vartheta)) \ \| \\
&\leq 2K \ \| \ \mathscr{T}_m^r g(m'(\varpi + \vartheta)) + \mathscr{T}_m^r g(m'(\varpi - \vartheta)) - 2\mathscr{T}_m^r g(m'(\varpi)) \\
&\quad - \mathscr{T}_m^r g(m'(\vartheta)) - \mathscr{T}_m^r g(m'(-\vartheta)) \ \| \\
&\quad + K^2 \ \| \ \mathscr{T}_m^r g(m(\varpi + \vartheta)) + \mathscr{T}_m^r g(m(\varpi - \varkappa)) - 2\mathscr{T}_m^r g(m(\varpi)) \\
&\quad - \mathscr{T}_m^r g(m(\vartheta)) - \mathscr{T}_m^r g(m(-\vartheta) \ \| \\
&\quad + K^3 \ \| \ \mathscr{T}_m^r g(-m(\varpi + \vartheta)) + \mathscr{T}_m^r g(-m(\varpi - \vartheta)) - 2\mathscr{T}_m^r g(-m(\varpi)) \\
&\quad - \mathscr{T}_m^r g(-m(\vartheta)) - \mathscr{T}_m^r g(-m(-\vartheta) \ \| \\
&\quad + K^3 \ \| \ \mathscr{T}_m^r g((Id_{\mathscr{E}_1} - 2m)(\varpi + \vartheta)) + \mathscr{T}_m^r g((Id_{\mathscr{E}_1} - 2m)(\varpi - \vartheta)) \\
&\quad - 2\mathscr{T}_m^r g((Id_{\mathscr{E}_1} - 2m)(\varpi)) - \mathscr{T}_m^r g((Id_{\mathscr{E}_1} - 2m)(\vartheta)) - \mathscr{T}_m^r g((Id_{\mathscr{E}_1} - 2m)(-\vartheta) \ \|
\end{aligned}$$

$$\leq \alpha_m^r[2K\varepsilon(m'\varpi, m'\vartheta) + K^2\varepsilon(m\varpi, m\vartheta) + K^3\varepsilon(-m\varpi, -m\vartheta)$$
$$+ K^3\varepsilon((Id_{\mathscr{E}_1} - 2m)\varpi, (Id_{\mathscr{E}_1} - 2m)\vartheta)]$$
$$\leq \alpha_m^r[2K\lambda(m') + K^2\lambda(m) + K^3\lambda(-m) + K^3\lambda(Id_{\mathscr{E}_1} - 2m)]\varepsilon(\varpi, \vartheta)$$
$$= \alpha_m^{r+1}\varepsilon(\varpi, \vartheta).$$

We have demonstrated by induction that (4.4.11) is correct $\forall\ r \in \mathbb{N}$. Therefore from (1.3.2) and (4.4.11), we have

$$|||\mathscr{T}_m^r g(\varpi + \vartheta) + \mathscr{T}_m^r g(\varpi - \vartheta) - 2\mathscr{T}_m^r g(\varpi) - \mathscr{T}_m^r g(\vartheta) - \mathscr{T}_m^r g(-\vartheta)|||^\theta \ \leq\ K^{r\theta}\alpha_m^{r\theta}\varepsilon^\theta(\varpi, \vartheta). \tag{4.4.12}$$

Letting $r \to \infty$ in (4.4.12) and using the definition of $l(\mathscr{X})$, we have

$$\mathscr{G}_m(\varpi + \vartheta) + \mathscr{G}_m(\varpi - \vartheta) = 2\mathscr{G}_m(\varpi) + \mathscr{G}_m(\vartheta) + \mathscr{G}_m(-\vartheta), \forall\ \varpi, \vartheta \in \mathscr{E}_0. \tag{4.4.13}$$

Therefore we have prove that for every for $m \in \mathcal{A}$ there exists a $\mathscr{G}_m : \mathscr{E}_0 \to \mathscr{E}_2$ which is the solution of (1.1.8) on $\mathscr{E}_0$ and satisfies

$$||g(\varpi) - \mathscr{G}_m(\varpi)||^\theta \leq 4\left(\frac{\varepsilon^\theta(m'\varpi, m\varpi)}{1 - \alpha_m^\theta}\right) = 4\varepsilon^*(\varpi), \forall\ \varpi \in \mathscr{E}_0.$$

Now, we show that $\mathscr{G}_m = \mathscr{G}_q\ \forall\ m, q \in \mathcal{A}$. Fix m, q and note that $\mathscr{G}_q$ satisfies (4.4.10) with m replaced by q. Hence by replacing (ϖ, ϑ) with $(m'\varpi, m\varpi)$ in (4.4.13) and using (1.3.1) and (1.3.2), we get $T\mathscr{G}_j = \mathscr{G}_j$, for $j = m, q$ and

$$\begin{aligned}|||\mathscr{G}_m(\varpi) - \mathscr{G}_q(\varpi)|||^\theta &\leq |||\mathscr{G}_m(\varpi) - g(\varpi)|||^\theta + |||\mathscr{G}_q(\varpi) - g(\varpi)|||^\theta \\ &\leq \left(\frac{4\varepsilon_m^\theta(\varpi)}{1 - \alpha_m^\theta}\right) + \left(\frac{4\varepsilon_q^\theta(\varpi)}{1 - \alpha_q^\theta}\right), \forall\ \varpi \in \mathscr{E}_0.\end{aligned}$$

It follows from (4.4.8) and linearity of Λthat

$$\begin{aligned}|||\mathscr{G}_m(\varpi) - \mathscr{G}_q(\varpi)|||^\theta &= |||\mathscr{T}^n\mathscr{G}_m(\varpi) - \mathscr{T}^n\mathscr{G}_q(\varpi)|||^\theta \\ &\leq 4\left(\frac{\Lambda^n\varepsilon_m^\theta(\varpi)}{1 - \alpha_m^\theta}\right) + 4\left(\frac{\Lambda^n\varepsilon_q^\theta(\varpi)}{1 - \alpha_q^\theta}\right) \\ &\leq (\alpha_m)^n U_m(\varpi) + (\alpha_q)^n U_q(\varpi),\end{aligned}$$

where $U_m(\varpi) = 4\frac{\varepsilon_m^\theta(\varpi)}{1-\alpha_m^\theta}\ \forall\ \varpi \in \mathscr{E}_0$ and $n \in \mathbb{N}$. Letting $n \to \infty$, we have $\mathscr{G}_m = \mathscr{G}_q = \mathscr{G}$.

Thus, we get

$$\|g(\varpi) - \mathscr{G}(\varpi)\|^\theta \leq U_m(\varpi), \ \forall \ \varpi \in \mathscr{E}_0, m \in \mathcal{A}.$$

So, we derive (4.4.4). Due to (4.4.13), it is obvious that $\mathscr{G}$ is a solution of (1.1.8). Now to show uniqueness of map $\mathscr{G}$, let's suppose that there exists a map $\mathscr{G}' : \mathscr{E}_1 \to \mathscr{E}_2$ which fulfills (1.1.8) and

$$\|g(\varpi) - \mathscr{G}'(\varpi)\|^\theta \leq 4\varepsilon^*(\varpi).$$

Using(1.3.2), we get

$$|||\mathscr{G}(\varpi) - \mathscr{G}'(\varpi)|||^\theta \leq 8\varepsilon^*(\varpi).$$

Further $\mathscr{T}\mathscr{G}'(\varpi) = \mathscr{G}'(\varpi) \ \forall \ \varpi \in \mathscr{E}_0$. Consequently, with a fixed $m \in \mathcal{A}$

$$\begin{aligned} |||\mathscr{G}(\varpi) - \mathscr{G}'(\varpi)|||^\theta = |||\mathscr{T}^n\mathscr{G}(\varpi) - \mathscr{T}^n\mathscr{G}'(\varpi)|||^\theta &\leq 8\Lambda^n\varepsilon^*(\varpi) \\ &\leq \frac{8\Lambda^n\varepsilon_m^\theta(\varpi)}{1-\alpha_m^\theta} \\ &\leq \frac{8\alpha_m^n\varepsilon_m^\theta(\varpi)}{1-\alpha_m^\theta}, \end{aligned}$$

$\forall \ \varpi \in \mathscr{E}_0, n \in \mathbb{N}, m \in \mathcal{A}$. Taking $n \to \infty$, we get $\mathscr{G} = \mathscr{G}'$. ■

The hyperstability of (1.1.8) in the quasi-Banach space is demonstrated in the following outcome.

Theorem 4.4.2. *Let ε as in the above Theorem 4.4.1, there exists a nonempty set $\mathcal{A} \in l(\mathscr{E}_1)$ such that $b \ o \ a = a \ o \ b \ \forall \ b, a \in \mathcal{A}$ and*

$$\begin{cases} \inf_{m\in\mathcal{A}} \varepsilon^\theta(m'\varpi, m\varpi) & = 0 \\ \sup_{m\in\mathcal{A}} \alpha_m < 1. \end{cases} \tag{4.4.14}$$

$\varpi \in \mathscr{E}_0$, then every $g : \mathscr{E}_1 \to \mathscr{E}_2$ satisfying (4.4.1) is a solution of (1.1.8) on $\mathscr{E}_0$.

Proof. Assume that $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a map which is fulfills (4.4.1). Then, by the Theorem 4.4.1, there exists a map $\mathscr{G} : \mathscr{E}_1 \to \mathscr{E}_2$, which fulfills (1.1.8) and $||g(\varpi) - \mathscr{G}(\varpi)||^\theta \leq \varepsilon^*(\varpi)$ $\forall \ \varpi \in \mathscr{E}_0$. Since, from (4.4.14), $\varepsilon^*(\varpi) = 0$ for all $\varpi \in \mathscr{E}_0$. This means that $g(\varpi) = \mathscr{G}(\varpi) \forall$ $\varpi \in \mathscr{E}_0$, where

$$g(\varpi + \vartheta) + g(\varpi - \vartheta) = 2g(\varpi) + g(\vartheta) + g(-\vartheta), \ \forall \ \varpi, \vartheta \in \mathscr{E}_0.$$

It implies that g fulfills (1.1.8) on $\mathscr{E}_0$. ■

As natural consequences of Theorems 4.4.1 and 4.4.2, we can derive the following corollaries.

Corollary 4.4.3. *Suppose that $q < 0,\ p < 0$ and φ is a positive number. If $g : \mathscr{E}_1 \to \mathscr{E}_2$ satisfies*

$$\| g(\varpi + \vartheta) + g(\varpi - \vartheta) - 2g(\varpi) - g(\vartheta) - g(-\vartheta) \|^\theta \le \varphi^\theta \left(\| \varpi \|^p + \| \vartheta \|^q\right)^\theta, \quad (4.4.15)$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_0$, then g is a solution of (1.1.8) on $\mathscr{E}_0$.

Proof. We apply Theorem 4.4.2 with $\varepsilon^\theta(\varpi, \vartheta) = \varphi^\theta \left(\| \varpi \|^p + \| \vartheta \|^q\right)^\theta\ \forall\ \varpi, \vartheta \in \mathscr{E}_0$ with some real numbers $\varphi \geqslant 0$, $q < 0$ and $p < 0$. Define m_j: $\mathscr{E}_0 \to \mathscr{E}_0$ by $m_j\varpi := m_j(\varpi) = -j\varpi$ and m'_j:$\mathscr{E}_0 \to \mathscr{E}_0$ by $m'_j\varpi := m'_j(\varpi) = (1 + j)\varpi$, for each $j \in \mathbb{N}$. Then

$$\begin{aligned}
\varepsilon^\theta(m_j\varpi, m_k\vartheta) &= \varepsilon^\theta(-j\varpi, -k\vartheta)\\
&= [\varphi\,(\| -j\varpi \|^p + \| -k\vartheta \|^q)]^\theta\\
&= [\varphi j^p \| \varpi \|^p + \varphi k^q \| \vartheta \|^q]^\theta\\
&\le [(j^p + k^q)\varphi(\| \varpi \|^p + \| \vartheta \|^q)]^\theta\\
&= (j^p + k^q)^\theta \varepsilon^\theta(\varpi, \vartheta),\ \forall\ \varpi, \vartheta \in \mathscr{E}_0, k, j \in \mathbb{N}.
\end{aligned}$$

Hence

$$\lim_{j\to\infty} \varepsilon^\theta(m'_j\varpi, m_j\vartheta) \le \lim_{j\to\infty} \left((1 + j)^p + j^q)\right)^\theta \varepsilon^\theta(\varpi, \vartheta) = 0,\ \forall\ \varpi, \vartheta \in \mathscr{E}_0, k, j \in \mathbb{N}.$$

Then (4.4.14) is holds with $\lambda(m_j) = j^p + j^q$ for $j \in \mathbb{N}$, and there exists $n_0 \in \mathbb{N}$ such that $j \ge n_0$ and

$$\alpha_{m_j} = 2\left((1 + j)^p + (1 + j)^q\right) + 2(j^p + j^q) + (1 + 2j)^p + (1 + 2j)^q < 1.$$

Therefore, (4.4.2) is satisfied with $\mathcal{A} := \{m_j \in Aut(\mathscr{E}_1) : j \in \mathbb{N}_{n0}\}$ and by Theorem 4.4.2, every $g : \mathscr{E}_1 \to \mathscr{E}_2$ satisfying (4.4.15) is a solution of (1.1.8) on $\mathscr{E}_0$. ■

We expand on the findings of Piszczek et al. [88] in the setting of quasi-Banach space.

Corollary 4.4.4. *If $g : \mathscr{E}_1 \to \mathscr{E}_2$ satisfies*

$$\| g(\varpi + \vartheta) + g(\varpi - \vartheta) - 2g(\varpi) - g(\vartheta) - g(-\vartheta) \|^\theta \le \varphi^\theta \left(\| \varpi \|^p + \| \vartheta \|^p\right)^\theta, \quad (4.4.16)$$

for $p < 0$ and $\varphi > 0$ and $\forall\ \varpi, \vartheta \in \mathscr{E}_0$, then g is a solution of (1.1.8) on $\mathscr{E}_0$.

Proof. It is obvious that the function ε given by

$$\varepsilon^{\theta}(\varpi, \vartheta) = \left[\varphi\left(\| \varpi \|^{p} + \| \vartheta \|^{p}\right)\right]^{\theta},$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_0$ satisfies (4.4.14), since

$$\varepsilon^{\theta}(j\varpi, k\vartheta) = \left[\varphi \| j\varpi \|^{p} + \varphi \| k\vartheta \|^{p}\right]^{\theta} \leq \left[\varphi(j^{p} + k^{p})\left(\| \varpi \|^{p} + \| \vartheta \|^{p}\right)\right]^{\theta} = (j^{p} + k^{p})^{\theta}\varepsilon^{\theta}(\varpi, \vartheta),$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_0, k, j \in \mathbb{N}$ and $kj \neq 0$. Remaining part of the proof is the same as the Corollary 4.4.3. ■

Remark 4.4.5. *Piszczek et al. [88] obtained Corollary 4.4.4 in the framework of a complete normed space.*

If $g : \mathscr{E}_1 \to \mathscr{E}_2$ satisfies (4.4.16) for $\varpi, \vartheta \in \mathscr{E}_0$, with $p < 0$, then , according to Theorem 4.4.2, g fulfills (1.1.8) on $\mathscr{E}_0$. It is obvious that if $g(0) = 0$, then g satisfies (1.1.8) on $\mathscr{E}_1$.

Corollary 4.4.6. *Let φ be a positive number and $p + q < 0$. If $g : \mathscr{E}_1 \to \mathscr{E}_2$ satisfies*

$$\| g(\varpi + \vartheta) + g(\varpi - \vartheta) - 2g(\varpi) - g(\vartheta) - g(-\vartheta) \|^{\theta} \leq \varphi^{\theta}\left(\| \varpi \|^{p}\| \vartheta \|^{q}\right)^{\theta}, \quad (4.4.17)$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_0$, *then g is a solution of the (1.1.8) on $\mathscr{E}_0$.*

Proof. It is obvious that the function ε given by

$$\varepsilon^{\theta}(\varpi, \vartheta) = \left(\varphi\left(\| \varpi \|^{p}\| \vartheta \|^{q}\right)\right)^{\theta},$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_0$ satisfies (4.4.14), since

$$\varepsilon^{\theta}(j\varpi, k\vartheta) = \varphi^{\theta}\left(\| j\varpi \|^{p}\| k\vartheta \|^{q}\right)^{\theta} \leq \varphi^{\theta}(j^{p}k^{q})^{\theta}\left(\| \varpi \|^{p}\| \vartheta \|^{q}\right)^{\theta} = (j^{p}k^{q})^{\theta}\varepsilon^{\theta}(\varpi, \vartheta),$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_0$, $k, j \in \mathbb{N}$ and $kj \neq 0$. On the similar lines of the Corollary 4.4.3, we get the results. ■

By a similar conclusion, the function ε given by

$$\varepsilon^{\theta}(\varpi, \vartheta) = \varphi^{\theta}\left(\| \varpi \|^{p} + \| \vartheta \|^{q} + \| \varpi \|^{p}\| \vartheta \|^{q}\right)^{\theta},$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_0$ satisfies (4.4.14), since

$$\begin{aligned}\varepsilon^\theta(j\varpi, k\vartheta &= \varphi^\theta \left(\| j\varpi \|^p + \| k\vartheta \|^q + \| j\varpi \|^p \| k\vartheta \|^q\right)^\theta \\ &= \varphi^\theta \left(j^p \| \varpi \|^p + k^q \| \vartheta \|^q + j^p k^q \| \varpi \|^p \| \vartheta \|^q\right)^\theta \\ &\leq \left(j^p + k^q + j^p k^q\right)^\theta \varepsilon^\theta(\varpi, \vartheta),\end{aligned}$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_0, k, j \in \mathbb{N}$ and $kj \neq 0$. Thus we get the following result.

Corollary 4.4.7. *Suppose that $q < 0,\ p < 0,\ p + q < 0$ and φ is a positive number. If $g : \mathscr{E}_1 \to \mathscr{E}_2$ satisfies*

$$\| g(\varpi + \vartheta) + g(\varpi - \vartheta) - 2g(\varpi) - g(\vartheta) - g(-\vartheta) \|^\theta \leq \varphi^\theta \left(\| \varpi \|^p + \| \vartheta \|^q + \| \varpi \|^p \| \vartheta \|^q\right)^\theta, \tag{4.4.18}$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_0$, then g is a solution of (1.1.8) on $\mathscr{E}_0$.

The following outcome match result on the non-homogeneous Drygas functional equation (4.4.19).

Corollary 4.4.8. *Assume that ε as in Theorem 4.4.1 and $H : \mathscr{E}_1^2 \to \mathscr{E}_2$. Assume that $\| H(\varpi, \vartheta) \|^\theta \leq \varepsilon^\theta(\varpi, \vartheta)\ \forall\ \varpi, \vartheta \in \mathscr{E}_0$, where $H(\varpi_0, \vartheta_0) \neq 0$ for some $\varpi_0, \vartheta_0 \in \mathscr{E}_0$, there exists a nonempty $\mathcal{A} \in l(\mathscr{E}_1)$ such that (4.4.3) and (4.4.14) satisfies. Then the non-homogeneous equation*

$$g(\varpi + \vartheta) + g(\varpi - \vartheta) - 2g(\varpi) - g(\vartheta) - g(-\vartheta) + H(\varpi, \vartheta), \tag{4.4.19}$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_0$, has no solution in the class of functions $g : \mathscr{E}_1 \to \mathscr{E}_2$.

Proof. Assume that $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a solution to (4.4.19). Then

$$\begin{aligned}&\| g(\varpi + \vartheta) + g(\varpi - \vartheta) - 2g(\varpi) - g(\vartheta) - g(-\vartheta) \|^\theta \\ &= \| 2g(\varpi) + g(\vartheta) + g(-\vartheta) + H(\varpi, \vartheta) - 2g(\varpi) - g(\vartheta) - g(-\vartheta) \|^\theta \\ &= \| H(\varpi, \vartheta) \|^\theta \\ &\leq \varepsilon^\theta(\varpi, \vartheta), \quad \forall\ \varpi, \vartheta \in \mathscr{E}_0.\end{aligned}$$

Consequently, by Theorem 4.4.2, g is a solution of (1.1.8). Therefore, we have

$$H(\varpi_0, \vartheta_0) = g(\varpi_0 + \vartheta_0) + g(\varpi_0 - \vartheta_0) - 2g(\varpi_0) + g(\vartheta_0) - g(-\vartheta_0) = 0,$$

which is contradiction. Hence non-homogeneous equation (4.4.19) has no solution in the class of functions $g : \mathscr{E}_1 \to \mathscr{E}_2$. ∎

The following example demonstrates the importance of the assumption in the preceding Corollary.

Example 4.4.1. *Let* $\mathscr{E}_1 = \mathscr{E}_2 = L^{\frac{1}{2}}[0,1]$ *and* $\parallel \varpi \parallel_{\mathscr{E}_1} = \parallel \varpi \parallel_{\mathscr{E}_2} = \left(\int_0^1 \mid \varpi(t) \mid^{\frac{1}{2}} dt\right)^2$ *for all* $\varpi \in \mathscr{E}_1$, *where* $L^{\frac{1}{2}}[0,1] = \{g : [0,1] \to \mathbb{R} : \mid g \mid^{\frac{1}{2}}$ *is Lebesgue integrable*$\}$ *and* $g(\varpi) = \varpi^4 + \varpi^2\ \forall\ \varpi \in \mathscr{E}_2$, $\varepsilon(\varpi, \vartheta) = \left(\int_0^1 \left(2\sqrt{3} \mid \varpi(t)\vartheta(t) \mid\right) dt\right)^2\ \forall\ \varpi, \vartheta \in \mathscr{E}_1$ *such that* $\varpi - \vartheta, \varpi + \vartheta \neq 0$.

$$||g(\varpi+\vartheta) + g(\varpi-\vartheta) - 2g(\varpi) - g(\vartheta) - g(-\vartheta)|| \leq \varepsilon(\varpi, \vartheta) \tag{4.4.20}$$

but g *does not satisfy (1.1.8).*

Proof. Note that $\mathscr{E}_1$ and $\mathscr{E}_2$ be complete QNSs

$$\begin{aligned}
&||g(\varpi+\vartheta) + g(\varpi-\vartheta) - 2g(\varpi) - g(\vartheta) - g(-\vartheta)|| \\
&= \left(\int_0^1 \left((\varpi(t)+\vartheta(t))^4 + (\varpi(t)+\vartheta(t))^2 + (\varpi(t)-\vartheta(t))^4 + (\varpi(t)-\vartheta(t))^2\right.\right. \\
&\quad \left.\left. - 2\varpi^4(t) - 2\varpi^2(t) - \vartheta^4(t) - \vartheta^2(t) - \vartheta^4(t) - \vartheta^2(t)\right)^{\frac{1}{2}} dt\right)^2 \\
&= \left(\int_0^1 \left(\vartheta^4(t) + 4\vartheta^3(t)\varpi(t) + 6\varpi^2(t)\vartheta^2(t) + 4\vartheta^3(t)\varpi(t) + \vartheta^4(t) + \varpi^2(t) + \vartheta^2(t)\right.\right. \\
&\quad + 2\varpi(t)\vartheta(t) + \varpi^4(t) - 4\varpi^3(t)\vartheta(t) + 6\varpi^2(t)\vartheta^2(t) - 4\varpi(t)\vartheta^3(t) + \vartheta^4(t) + \varpi^2(t) \\
&\quad \left.\left. + \vartheta^2(t) - 2\varpi(t)\vartheta(t) - 2\varpi^4(t) - 2\varpi^2(t) - \vartheta^4(t) - \vartheta^2(t) - \vartheta^4(t) - \vartheta^2(t)\right)^{\frac{1}{2}} dt\right)^2 \\
&= \left(\int_0^1 \left(12\varpi^2(t)\vartheta^2(t)\right)^{\frac{1}{2}} dt\right)^2 \\
&= \left(\int_0^1 \left(2\sqrt{3} \mid \varpi(t)\vartheta(t) \mid\right) dt\right)^2 \\
&= \varepsilon(\varpi, \vartheta).
\end{aligned}$$

This proves that (4.4.20) holds but g does not fulfill (1.1.8). ■

Remark 4.4.9. *If* $\mathscr{E}_2$ *is a complete NS,* $\mathscr{E}_1$ *is a NS and* $K = 1$ *in Theorem 4.4.1, we obtain the corresponding results of Sirouni et al. [117].*

Chapter 5

Ulam-type stability of a quartic functional equation

Introduction

This chapter has been divided into three sections. In the first two sections, we use direct and fixed point methods to discuss the generalized HU stability of a quartic functional equation in the framework of (β, p)-NS. In the last section, we use a direct method to study the generalized HU stability of a quartic functional equation in the framework of NAβ-NS. Results of this chapter are published in Sharma and Chandok [113].

5.1 Ulam-type stability in (β, p)-normed space by fixed point approach

Here, we study the generalized HU stability of (1.1.12) for a fixed integer m in the framework of a complete (β, p)-NS by utilising Theorem 1.2.2 using the following Remark 5.1.1.

Remark 5.1.1. *Let $\mathscr{E}_2$ and $\mathscr{E}_1$ be two vector spaces over the same field, $g : \mathscr{E}_1 \to \mathscr{E}_2$ satisfies functional equation (1.1.12). Then for all $\varpi, \vartheta \in \mathscr{E}_1$ and $l, k \geq 0$, we get*
Case 1. *if $\varpi = \vartheta = 0$ in (1.1.12), then $g(0) = 0$;*
Case 2. *if $\vartheta = 0$ in (1.1.12), then $g(2\varpi) = 2^4 g(\varpi)$;*
Case 3. *if we replace ϖ with $k\varpi$ and ϑ with $l\varpi$ in (1.1.12), then $g(k\varpi + l\varpi) = (k+l)^4 g(\varpi)$. If $k + l = r$, so we have $g(r\varpi) = r^4 g(\varpi)$.*
Case 4. *if $\varpi = 0$ in (1.1.12), then $g(-1\varpi) = (-1)^4 g(\varpi)$.*

Assume that $m(m \geq 2)$ is an integer, we use the abbreviation for map $g : \mathscr{E}_1 \to \mathscr{E}_2$ as follows:

$$B_m g(\varpi, \vartheta) := g(\varpi + m\vartheta) + g(\varpi - m\vartheta) - 2(m-1)(7m-9)g(\varpi) - 2(m^2-1)m^2 g(\vartheta) \\ + (m-1)^2 g(2\varpi) - m^2\{g(\varpi + \vartheta) + g(\varpi - \vartheta)\}.$$

From here, throughout this section, $(\mathscr{E}_1, \|\cdot\|_{\mathscr{E}_1}, K_{\mathscr{E}_1} \geq 1)$ is a (β, p)-NS, $(\mathscr{E}_2, \|\cdot\|_{\mathscr{E}_2}, K_{\mathscr{E}_2} \geq 1)$ is a complete (β, p)-NS over the same field with $\mathscr{E}_1$,

Theorem 5.1.2. *Assume that $\Theta : \mathscr{E}_1^2 \to [0.\infty)$ is a map and $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a map with $g(0) = 0$. Assume that following assumptions are fulfill, $\forall\ \varpi, \vartheta \in \mathscr{E}_1$*

$$(a) \qquad \Theta(m\varpi, m\vartheta) \leq Mm^{4\beta}\Theta(\varpi, \vartheta), \text{ where, } 0 \leq M < 1, 0 < \beta \leq 1, \tag{5.1.1}$$

$$(b) \qquad \|B_m g(\varpi, \vartheta)\|_{\mathscr{E}_2} \leq \Theta(\varpi, \vartheta). \tag{5.1.2}$$

Then there exists a unique map $G : \mathscr{E}_1 \to \mathscr{E}_2$ fulfilling (1.1.12) and

$$\|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \left(\frac{4}{1-M}\right)^{\frac{1}{p}} \frac{1}{2^{4\beta}|m-1|^{2\beta}}\Theta(\varpi, 0), \forall\ \varpi \in \mathscr{E}_1,\ p = \log_{2K_{\mathscr{E}_2}} 2. \tag{5.1.3}$$

Proof. Choose $\sigma = \{f : \mathscr{E}_1 \to \mathscr{E}_2\}$. Define a function $\delta : \sigma \times \sigma \to [0, \infty)$, as

$$\delta(f, J) = \inf\{C \geq 0 \mid \|f(\varpi) - J(\varpi)\|_{\mathscr{E}_2} \leq C\Theta(\varpi, 0), \forall \varpi \in \mathscr{E}_1\}$$

$\forall\ f, J \in \sigma$, where $\inf \emptyset = \infty$. Firstly, we'll prove that (σ, δ) is a GBM. It's easy to see $\delta(f, J) = \delta(J, f)$. If $f = J$, then $\delta(f, J) = 0$. Note that $\|f(\varpi) - J(\varpi)\|_{\mathscr{E}_2} \leq \delta(f, J)\Theta(\varpi, 0)$ $\forall\ \varpi \in \mathscr{E}_1$. If $\delta(f, J) = 0$, then $\|f(\varpi) - J(\varpi)\|_{\mathscr{E}_2} = 0$, that is, $f(\varpi) = J(\varpi)$ $\forall \varpi \in \mathscr{E}_1$. Then $f = J$. For each $f, J, a \in \sigma$ and $\varpi \in \mathscr{E}_1$, we have
$\|f(\varpi) - a(\varpi)\|_{\mathscr{E}_2} \leq \delta(f, a)\Theta(\varpi, 0)$, $\|a(\varpi) - J(\varpi)\|_{\mathscr{E}_2} \leq \delta(a, J)\Theta(\varpi, 0)$, and $\|f(\varpi) - J(\varpi)\|_{\mathscr{E}_2} \leq \delta(f, J)\Theta(\varpi, 0)$.
Now, from [[113], ($\delta 3$), pp. 3], it follows that for all $\varpi \in \mathscr{E}_1$,

$$\begin{aligned}\|f(\varpi) - J(\varpi)\|_{\mathscr{E}_2} &\leq K_{\mathscr{E}_2}\left(||f(\varpi) - a(\varpi)||_{\mathscr{E}_2} + ||a(\varpi) - J(\varpi)||_{\mathscr{E}_2}\right) \\ &\leq \left(K_{\mathscr{E}_2}\delta(f, a) + K_{\mathscr{E}_2}\delta(a, J)\right)\Theta(\varpi, 0),\end{aligned}$$

then we can write

$$\|f(\varpi) - J(\varpi)\| \leq \left(K_{\mathscr{E}_2}\delta(f, a) + K_{\mathscr{E}_2}\delta(a, J)\right)\Theta(\varpi, 0). \tag{5.1.4}$$

So we have

$$\delta(f, J) \leq K_{\mathscr{E}_2}\delta(f, a) + K_{\mathscr{E}_2}\delta(a, J).$$

Therefore, $(\sigma, \delta, K_{\mathscr{E}_2})$ is a GBM with a coefficient $K_{\mathscr{E}_2}$ on σ.

Now, we will prove that GBM $(\sigma, \delta, K_{\mathscr{E}_2})$ is complete. Assume that $\{g_n\}$ is a Cauchy sequence in $(\sigma, \delta, K_{\mathscr{E}_2})$. Then we get $\lim_{n,m\to\infty} \delta(g_n, g_m) = 0$. Note that $\forall\ \varpi \in \mathscr{E}_1$, we get

$$\|g_n(\varpi) - g_m(\varpi)\|_{\mathscr{E}_2} \leq \delta(g_n, g_m)\Theta(x, 0). \tag{5.1.5}$$

Then $\lim_{n,m\to\infty} \|g_n(\varpi) - g_m(\varpi)\|_{\mathscr{E}_2} = 0$. It means that $\{g_n(\varpi)\}$ is a Cauchy sequence in $(\mathscr{E}_2, \|\cdot\|_{\mathscr{E}_2}, K_{\mathscr{E}_2})$. Since $(\mathscr{E}_2, \|\cdot\|_{\mathscr{E}_2}, K_{\mathscr{E}_2})$ is a complete (β, p)-NS, there exists $\lim_{n\to\infty} g_n(\varpi) = \vartheta$ in $(\mathscr{E}_2, \|\cdot\|_{\mathscr{E}_2}, K_{\mathscr{E}_2})$. Put $f(\varpi) = \vartheta$, where $f : \mathscr{E}_1 \to \mathscr{E}_2$ is a mapping. We'll show that $\lim_{n\to\infty} g_n = f$ in $(\sigma, \delta, K_{\mathscr{E}_2})$. Indeed, for each $\epsilon > 0$, there exists n_0 such that $\delta(g_n, g_m) < \epsilon$ $\forall\ m, n \geq n_0$. So from (5.1.5), $\forall\ \varpi \in \mathscr{E}_1$ and $n \geq n_0$,

$$\|g_n(\varpi) - g_m(\varpi)\|_{\mathscr{E}_2} \leq \epsilon\Theta(\varpi, 0). \tag{5.1.6}$$

Taking $m \to \infty$ in (5.1.6), we get $\forall\ \varpi \in \mathscr{E}_1$ and $n \geq n_0$,

$$\|g_n(\varpi) - f(\varpi)\|_{\mathscr{E}_2} \leq \epsilon\Theta(\varpi, 0).$$

This implies that $\delta(g_n, f) \leq \epsilon\ \forall\ n \geq n_0$. Thus $\lim_{n\to\infty} g_n = f$ in $(\sigma, \delta, K_{\mathscr{E}_2})$. Then $(\sigma, \delta, K_{\mathscr{E}_2})$ is complete.

Next, letting $\varpi = 0$ in (5.1.2) and using $g(0) = 0$, we have

$$\|(m-1)^2 g(2\varpi) - (m-1)^2 16 g(\varpi)\|_{\mathscr{E}_2} \leq \Theta(\varpi, 0).$$

Further, after rewrite, we have

$$\left\|\frac{g(2\varpi)}{16} - g(\varpi)\right\|_{\mathscr{E}_2} \leq \frac{\Theta(\varpi, 0)}{2^{4\beta}|m-1|^{2\beta}},\ \forall\ \varpi \in \mathscr{E}_1.$$

It yields that

$$\left\|\frac{g(2\varpi)}{2^4} - g(\varpi)\right\|_{\mathscr{E}_2} \leq \frac{\Theta(\varpi, 0)}{2^{4\beta}|m-1|^{2\beta}},\ \forall\ \varpi \in \mathscr{E}_1. \tag{5.1.7}$$

Define a map $\mathscr{T} : \sigma \to \sigma$ by $(\mathscr{T}f)(\varpi) = \dfrac{f(2\varpi)}{2^4}$ for all $f \in \sigma$ and $\forall\ \varpi \in \mathscr{E}_1$.

Now, we have to show that $\delta(\mathscr{T}f, \mathscr{T}J) \leq M\delta(f, J)$. By (5.1.1), and $f, J \in \sigma$, we have

$$\begin{aligned}
\|(\mathscr{T}f)(\varpi)-(\mathscr{T}J)(\varpi)\|_{\mathscr{E}_2} &= \Big\|\frac{f(2\varpi)}{2^4}-\frac{J(2\varpi)}{2^4}\Big\|_{\mathscr{E}_2}\\
&= \frac{1}{2^{4\beta}}\|f(2\varpi)-J(2\varpi)\|_{\mathscr{E}_2}\\
&\le \frac{\delta(f,J)}{2^{4\beta}}\Theta(2\varpi,0)\\
&\le \frac{\delta(f,J)}{2^{4\beta}}M2^{4\beta}\Theta(\varpi,0)\\
&= M\delta(f,J)\Theta(\varpi,0).
\end{aligned}$$

So, we get

$$\delta(\mathscr{T}f,\mathscr{T}J)\le M\delta(f,J),\forall\ f,J\in\sigma,\ 0\le M<1.$$

Using Theorem 1.2.2, for the mapping $\mathscr{T}$ on the CGMS $(\mathscr{E}_1,\mathfrak{D})$, we get
(A) either $\mathfrak{D}(\mathscr{T}^n f,\mathscr{T}^{n+1}f)=\infty\ \forall\ n\in\mathbb{N}$,
(B) or

(1) $\lim_{n\to\infty}\mathscr{T}^n f=G$, where G is a fixed point of $\mathscr{T}$;

(2) $\mathfrak{D}(f,G)\le\left(\frac{1}{1-M}\right)\mathfrak{D}(f,\mathscr{T}f)$.

By (1.3.4) we get $\forall\ n\in\mathbb{N}$,

$$\delta(f,G)\le 4^{\frac{1}{p}}\mathfrak{D}^{\frac{1}{p}}(f,G)\le\left(\frac{4}{1-M}\right)^{\frac{1}{p}}\mathfrak{D}^{\frac{1}{p}}(f,\mathscr{T}f)\le\left(\frac{4}{1-M}\right)^{\frac{1}{p}}\delta(f,\mathscr{T}f).$$

From (5.1.7), we have $\forall\ \varpi\in\mathscr{E}_1$, $\|\mathscr{T}g(\varpi)-g(\varpi)\|_{\mathscr{E}_2}\le\frac{\Theta(\varpi,0)}{2^{4\beta}|m-1|^{2\beta}}$. So $\delta(\mathscr{T}g,g)\le\frac{1}{2^{4\beta}|m-1|^{2\beta}}<\infty$. This proves that if we take $f=g$, then

(1) $\lim_{n\to\infty}\mathscr{T}^n g=G$.

(2) $\delta(g,G)\le\left(\frac{4}{1-M}\right)^{\frac{1}{p}}\delta(g,Tg)$.

So, we find that

$$\delta(g,G)\le\left(\frac{4}{1-M}\right)^{\frac{1}{p}}\delta(g,\mathscr{T}g)\le\left(\frac{4}{1-M}\right)^{\frac{1}{p}}\frac{1}{2^{4\beta}|m-1|^{2\beta}}.$$

Then $\forall\ \varpi \in \mathscr{E}_1,\ \|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \left(\frac{4}{1-M}\right)^{\frac{1}{p}} \frac{1}{2^{4\beta}|m-1|^{2\beta}}\Theta(\varpi, 0)$. That is (5.1.3) holds.

Next, we have to show that G is quartic by using continuity of $\|\cdot\|_{\mathscr{E}_2}$. Note that $(\mathscr{T} g)(\varpi) = \frac{g(2\varpi)}{2^4}$, for each $\varpi \in \mathscr{E}_1$. So,

$$(\mathscr{T}^2 g)(\varpi) = \frac{\mathscr{T} g(2\varpi)}{2^4} = \frac{g(2^2\varpi)}{2^{4.2}}, \cdots, (\mathscr{T}^n g)(\varpi) = \frac{g(2^n\varpi)}{2^{4n}}.$$

So (5.1.1) and (5.1.2), we have $\forall\ \varpi, \vartheta \in \mathscr{E}_1$,

$$\begin{aligned}
&||G(\varpi + m\vartheta) + G(\varpi - m\vartheta) - 2(m-1)(7m-9)G(\varpi) - 2(m^2-1)m^2G(\vartheta) \\
&\quad + (m-1)^2G(2\varpi) - m^2\{G(\varpi+\vartheta) + G(\varpi-\vartheta)\}||^p_{\mathscr{E}_2} \\
&= ||\lim_{n\to\infty}(\mathscr{T}^n g)(\varpi + m\vartheta) + \lim_{n\to\infty}(\mathscr{T}^n g)(\varpi - m\vartheta) \\
&\quad - 2(m-1)(7m-9)\lim_{n\to\infty}(\mathscr{T}^n g)(\varpi) - 2(m^2-1)m^2\lim_{n\to\infty}(\mathscr{T}^n g)(\vartheta) \\
&\quad + (m-1)^2\lim_{n\to\infty}(\mathscr{T}^n g)(2\varpi) - m^2\lim_{n\to\infty}(\mathscr{T}^n g)(\varpi+\vartheta) - m^2\lim_{n\to\infty}(\mathscr{T}^n g)(\varpi-\vartheta)||^p_{\mathscr{Y}} \\
&= \lim_{n\to\infty}||(\mathscr{T}^n g)(\varpi + m\vartheta) + (\mathscr{T}^n g)(\varpi - m\vartheta) - 2(m-1)(7m-9)(\mathscr{T}^n g)(\varpi) \\
&\quad - 2(m^2-1)m^2(\mathscr{T}^n g)(\vartheta) + (m-1)^2(\mathscr{T}^n g)(2\varpi) - m^2(\mathscr{T}^n g)(\varpi+\vartheta) \\
&\quad - m^2(\mathscr{T}^n g)(\varpi-\vartheta)||^p_{\mathscr{E}_2} \\
&= \lim_{n\to\infty}\left|\left|\frac{g(2^n(\varpi + m\vartheta))}{2^{4n}} + \frac{g(2^n(\varpi - m\vartheta))}{2^{4n}} - 2(m-1)(7m-9)\frac{g(2^n(\varpi))}{2^{4n}}\right.\right. \\
&\quad \left.\left. - 2(m^2-1)m^2\frac{g(2^n(\vartheta))}{2^{4n}} + (m-1)^2\frac{g(2^n(2\varpi))}{2^{4n}} - m^2\frac{g(2^n(\varpi+\vartheta)}{2^{4n}} - m^2\frac{g(2^n(\varpi-\vartheta)}{2^{4n}}\right|\right|^p_{\mathscr{E}_2} \\
&= \lim_{n\to\infty}\frac{1}{2^{4np\beta}}||g(2^n\varpi + 2^n m\vartheta) + g(2^n\varpi - 2^n m\vartheta) - 2(m-1)(7m-9)g(2^n\varpi) \\
&\quad - 2(m^2-1)m^2g(2^n\vartheta) + (m-1)^2g(2^n(2\varpi)) - m^2g(2^n\varpi + 2^n\vartheta) - m^2g(2^n\varpi - 2^n\vartheta)||^p_{\mathscr{E}_2} \\
&\leq \lim_{n\to\infty}\frac{1}{2^{4np\beta}}\Theta^p(2^n\varpi, 2^n\vartheta) \\
&\leq \lim_{n\to\infty}\frac{1}{2^{4np\beta}}M^n 2^{4np\beta}\Theta^p(\varpi, \vartheta) \\
&= \lim_{n\to\infty}M^n\Theta^p(\varpi, \vartheta) = 0.
\end{aligned}$$

It implies that $\forall\ \varpi, \vartheta \in \mathscr{E}_1$,

$$\begin{aligned}
&G(\varpi + m\vartheta) + G(\varpi - m\vartheta) - 2(m-1)(7m-9)G(\varpi) \\
&\quad - 2(m^2-1)m^2G(\vartheta) + (m-1)^2G(2\varpi) - m^2\{G(\varpi+\vartheta) + G(\varpi-\vartheta)\} = 0.
\end{aligned}$$

So G satisfies functional equation (1.1.12). By Remark 5.1.1, G is a quartic map.

Lastly, we demonstrate the uniqueness of G. Assume that $J : \mathscr{E}_1 \to \mathscr{E}_2$ is also a quartic map fulfilling (5.1.3). We must demonstrate that $J = G$. It follows from Remark 5.1.1, $G(2\varpi) = 2^4 G(\varpi)$ and $J(2\varpi) = 2^4 J(\varpi)$. By using (5.1.1) and (5.1.3), for each $n \in \mathbb{N}$, we get

$$\begin{aligned}
||G(\varpi) - J(\varpi)||^p_{\mathscr{E}_2} &= \left|\left|\frac{G(2^n\varpi)}{2^{4n}} - \frac{J(2^n\varpi)}{2^{4n}}\right|\right|^p_{\mathscr{E}_2} \\
&= \frac{1}{2^{4np\beta}}||G(2^n\varpi) - J(2^n\varpi)||^p_{\mathscr{E}_2} \\
&= \frac{1}{2^{4np\beta}}||G(2^n\varpi) - g(2^n\varpi) + g(2^n\varpi) - J(2^n\varpi)||^p_{\mathscr{E}_2} \\
&\leq \frac{1}{2^{4np\beta}}\left(||G(2^n\varpi) - g(2^n\varpi)||^p_{\mathscr{E}_2} + ||g(2^n\varpi) - J(2^n\varpi)||^p_{\mathscr{E}_2}\right) \\
&\leq \frac{1}{2^{4np\beta}}\left(||G(2^n\varpi) - g(2^n\varpi)||^p_{\mathscr{E}_2} + ||g(2^n\varpi) - J(2^n\varpi)||^p_{\mathscr{E}_2}\right) \\
&\leq \frac{1}{2^{4np\beta}}\Big(\frac{4}{1-M}\frac{1}{2^{4p\beta}|m-1|^{2p\beta}}\Theta^p(2^n\varpi, 0) \\
&\quad + \frac{4}{1-M}\frac{1}{2^{4p\beta}|m-1|^{2p\beta}}\Theta^p(2^n\varpi, 0)\Big) \\
&\leq \frac{1}{2^{4np\beta+4p\beta}|m-1|^{2p\beta}}\frac{8}{1-M}\left(\Theta^p(2^n\varpi, 0)\right) \\
&\leq \frac{1}{2^{4np\beta+4p\beta}|m-1|^{2p\beta}}\frac{8}{1-M}M^{np}2^{4np\beta}\Theta^p(\varpi, 0) \\
&= \frac{1}{2^{4p\beta}|m-1|^{2p\beta}}\frac{8M^{np}}{1-M}\Theta^p(\varpi, 0).
\end{aligned}$$

Since $0 \leq M < 1$ and $p = \log_{2K_{\mathscr{E}_2}} 2$, taking $n \to \infty$, we have $||G(\varpi) - J(\varpi)||^p_{\mathscr{E}_2} = 0 \ \forall \ \varpi \in \mathscr{E}_1$. This proves $J = G$. ■

Now, we give some consequences of the above result in complete (β, p)-NS.

Corollary 5.1.3. *Let $g : \mathscr{E}_1 \to \mathscr{E}_2$ be a map with $g(0) = 0$. Assume that there are real numbers $\lambda > 0$ and $r_1 < 4$ such that*

$$\|B_m g(\varpi, \vartheta)\|_{\mathscr{E}_2} \leq \lambda\left(\|\varpi\|^{r_1} + \|\vartheta\|^{r_1}\right), \forall \ \varpi, \vartheta \in \mathscr{E}_1\backslash\{0\}. \tag{5.1.8}$$

Then there exists a unique map $G : \mathscr{E}_1 \to \mathscr{E}_2$ fulfilling (1.1.12) and

$$\|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \left(\frac{1}{1-M}\right)^{\frac{1}{p}}\frac{1}{2^{4\beta}|m-1|^{2\beta}}\|\varpi\|^{r_1}, \tag{5.1.9}$$

$\forall \ \varpi \in \mathscr{E}_1, \ p = \log_{2K_{\mathscr{E}_2}} 2$ and $0 \leq M < 1$.

Proof. Define a map $\Theta : \mathscr{E}_1^2 \to [0, \infty)$ by

$$\Theta(\varpi, \vartheta) := \begin{cases} 0, & \text{if } \varpi = 0 \text{ or } \vartheta = 0; \\ \lambda\left(\|\varpi\|^{r_1} + \|\vartheta\|^{r_1}\right), & \text{otherwise.} \end{cases}$$

Next, we will show that

$$\Theta(m\varpi, m\vartheta) \leq Mm^{4\beta}\Theta(\varpi, \vartheta), \forall\ \varpi, \vartheta \in \mathscr{E}_1, \text{ where } M := m^{r_1\beta - 4\beta} \in [0, 1).$$

Let $\varpi, \vartheta \in \mathscr{E}_1$. If $\vartheta = 0$ or $\varpi = 0$ then

$$\Theta(m\varpi, m\vartheta) = 0 \leq Mm^{4\beta}\Theta(\varpi, \vartheta).$$

If $\vartheta \neq 0$ and $\varpi \neq 0$, then we have

$$\begin{aligned} \Theta(m\varpi, m\vartheta) &= \lambda\left(\|m\varpi\|^{r_1} + \|m\vartheta\|^{r_1}\right) \\ &= m^{r_1\beta}\lambda\left(\|\varpi\|^{r_1} + \|\vartheta\|^{r_1}\right) \\ &= Mm^{4\beta}\Theta(\varpi, \vartheta). \end{aligned}$$

So, all the assumptions of Theorem 5.1.2 are correct. So we obtain the result. ∎

Corollary 5.1.4. *Assume that $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a map with $g(0) = 0$, and there are real numbers $\lambda > 0$ and $r_1 < 4$ such that*

$$\|B_m g(\varpi, \vartheta)\|_{\mathscr{E}_2} \leq \lambda\left(\|\varpi\|^{r_1}\|\vartheta\|^{r_1}\right), \forall\ \varpi, \vartheta \in \mathscr{E}_1 \backslash \{0\}. \tag{5.1.10}$$

Then there exists a unique map $G : \mathscr{E}_1 \to \mathscr{E}_2$ fulfilling (1.1.12) and

$$\|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \left(\frac{1}{1 - M}\right)^{\frac{1}{p}} \frac{1}{2^{4\beta}|m - 1|^{2\beta}}\|\varpi\|^{r_1}, \tag{5.1.11}$$

$\forall\ \varpi \in \mathscr{E}_1,\ p = \log_{2K_{\mathscr{E}_2}} 2$ *and* $0 \leq M < 1$.

Proof. Define a map $\Theta : \mathscr{E}_1^2 \to [0, \infty)$ by

$$\Theta(\varpi, \vartheta) := \begin{cases} 0, & \text{if } \varpi = 0 \text{ or } \vartheta = 0; \\ \lambda\left(\|\varpi\|^{r_1}\|\vartheta\|^{r_1}\right), & \text{otherwise.} \end{cases}$$

Next, we will show that

$$\Theta(m\varpi, m\vartheta) \leq M m^{4\beta}\Theta(\varpi, \vartheta), \forall\ \varpi, \vartheta \in \mathscr{E}_1, \text{ where } M := m^{2r_1\beta-4\beta} \in [0,1).$$

Let $\varpi, \vartheta \in \mathscr{E}_1$. If $\vartheta = 0$ or $\varpi = 0$ then

$$\Theta(m\varpi, m\vartheta) = 0 \leq M m^{4\beta}\Theta(\varpi, \vartheta).$$

If $\vartheta \neq 0$ and $\varpi \neq 0$, then we have

$$\begin{aligned}\Theta(m\varpi, m\vartheta) &= \lambda\left(\|mx\|^{r_1}\|m\vartheta\|^{r_1}\right)\\ &= m^{2r_1\beta}\lambda\left(\|\varpi\|^{r_1}\|\vartheta\|^{r_1}\right)\\ &= M m^{4\beta}\Theta(\varpi, \vartheta).\end{aligned}$$

All of the assumptions in Theorem 5.1.2 are now fulfilled. Hence, we obtain the result.

■

Corollary 5.1.5. *Let $g : \mathscr{E}_1 \to \mathscr{E}_2$ be a map with $g(0) = 0$. Assume that there are real numbers $\lambda > 0$ and $r_1 < 0$ with $m^{r_1\beta} < m^{\beta}$ such that*

$$\|B_m g(\varpi, \vartheta)\|_{\mathscr{E}_2} \leq \lambda\left(\|\varpi\|^{r_1}\|\vartheta\|^{r_1} + \|\varpi\|^{r_1} + \|\vartheta\|^{r_1}\right), \forall\ \varpi, \vartheta \in \mathscr{E}_1\backslash\{0\}. \tag{5.1.12}$$

Then there exists a unique map $G : \mathscr{E}_1 \to \mathscr{E}_2$ fulfilling (1.1.12) and

$$\|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \left(\frac{1}{1-M}\right)^{\frac{1}{p}} \frac{1}{2^{4\beta}|m-1|^{2\beta}}\|\varpi\|^{r_1},\ \forall\ \varpi \in \mathscr{E}_1,\ p = \log_{2K_{\mathscr{E}_2}} 2. \tag{5.1.13}$$

Proof. Define a map $\Theta : \mathscr{E}_1^2 \to [0, \infty)$ by

$$\Theta(\varpi, \vartheta) := \begin{cases} 0, & \text{if } \varpi = 0 \text{ or } \vartheta = 0;\\ \lambda\left(\|\varpi\|^{r_1}\|\vartheta\|^{r_1} + \|\varpi\|^{r_1} + \|\vartheta\|^{r_1}\right), & \text{otherwise.}\end{cases}$$

Next, we will show that

$$\Theta(m\varpi, m\vartheta) \leq M m^{4\beta}\Theta(\varpi, \vartheta), \forall\ \varpi, \vartheta \in \mathscr{E}_1, \text{ where } M := m^{r_1\beta-4\beta} \in [0,1).$$

Let $\varpi, \vartheta \in \mathscr{E}_1$. If $\vartheta = 0$ or $\varpi = 0$ then

$$\Theta(m\varpi, m\vartheta) = 0 \leq Mm^{4\beta}\Theta(\varpi, \vartheta).$$

If $\vartheta \neq 0$ and $\varpi \neq 0$, then we have

$$\begin{aligned}
\Theta(m\varpi, m\vartheta) &= \lambda\left(\|m\varpi\|^{r_1}\|m\vartheta\|^{r_1} + \|m\varpi\|^{r_1} + \|m\vartheta\|^{r_1}\right) \\
&= m^{r_1\beta}\lambda\left(m^{r_1\beta}(\|\varpi\|^{r_1}\|\vartheta\|^{r_1}) + \|\varpi\|^{r_1} + \|\vartheta\|^{r_1}\right) \\
&\leq m^{r_1\beta}\lambda\left(\|\varpi\|^{r_1}\|\vartheta\|^{r_1} + \|\varpi\|^{r_1} + \|\vartheta\|^{r_1}\right) \\
&= Lm^{4\beta}\Theta(\varpi, \vartheta).
\end{aligned}$$

All the assumptions of Theorem 5.1.2 are valid. Therefore, we obtain the result. ■

5.2 Ulam Type stability in (β, p)-normed space by direct method

Utilizing the direct method, we discuss the stability of (1.1.12). Throughout this section, $(\mathscr{E}_1, \|\cdot\|_{\mathscr{E}_1}, K_{\mathscr{E}_1})$ is a (β, p)-NS, $(\mathscr{E}_2, \|\cdot\|_{\mathscr{E}_2}, K_{\mathscr{E}_2})$ is a complete (β, p)-NS over the same field with $\mathscr{E}_1$,

Theorem 5.2.1. *Assume that $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a map for which there exists a $\Theta : \mathscr{E}_1 \times \mathscr{E}_1 \to [0, \infty)$ such that*

$$\Theta'(\varpi, \vartheta) := \sum_{k=0}^{\infty} \frac{1}{2^{4kp\beta}} \Theta^p(2^k\varpi, 2^k\vartheta) < \infty, \tag{5.2.1}$$

$$\|B_m g(\varpi, \vartheta)\|_{\mathscr{E}_2} \leq \Theta(\varpi, \vartheta), \forall\ \varpi, \vartheta \in \mathscr{E}_2. \tag{5.2.2}$$

Then there exists a unique quartic map $G : \mathscr{E}_1 \to \mathscr{E}_2$ such that

$$\|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \frac{1}{2^{4\beta}} \frac{(\Theta'(\varpi, 0))^{\frac{1}{p}}}{(m-1)^{2\beta}}, \forall\ \varpi \in \mathscr{E}_1. \tag{5.2.3}$$

Proof. Taking $\vartheta = 0$ in (5.2.2), we have

$$\|(m-1)^2 g(2\varpi) - (m-1)^2 16 g(\varpi)\|_{\mathscr{E}_2} \leq \Theta(\varpi, 0), \tag{5.2.4}$$

$\forall\ \varpi \in \mathscr{E}_1$. Thus

$$\Big\|\frac{g(2\varpi)}{16} - g(\varpi)\Big\|_{\mathscr{E}_2}^p \le \frac{\Theta^p(\varpi,0)}{2^{4p\beta}|m-1|^{2p\beta}}, \tag{5.2.5}$$

$\forall\ \varpi \in \mathscr{E}_1$. Replacing ϖ by 2ϖ in (5.2.5) and continuing this method, we get

$$\Big\|\frac{g(2^n\varpi)}{2^{4n}} - g(\varpi)\Big\|_{\mathscr{E}_2}^p \le \frac{1}{2^{4p\beta}|m-1|^{2p\beta}} \sum_{k=0}^{n-1} \frac{\Theta^p(2^k\varpi,0)}{2^{4kp\beta}}. \tag{5.2.6}$$

On the other hand, we can apply induction to find

$$\Big\|\frac{g(2^n\varpi)}{2^{4n}} - \frac{g(2^l\varpi)}{2^{4l}}\Big\|_{\mathscr{E}_2}^p \le \frac{1}{2^{4p\beta}|m-1|^{2p\beta}} \sum_{k=0}^{n-1} \frac{\Theta^p(2^k\varpi,0)}{2^{4kp\beta}}, \tag{5.2.7}$$

$\forall\ \varpi \in \mathscr{E}_1$, and $0 \le l < n$. So sequence $\left\{\dfrac{g(2^n\varpi)}{2^{4n}}\right\}$ is a Cauchy by (5.2.1) and (5.2.7). Since $\mathscr{E}_2$ is complete, there exists a mapping G so that

$$\lim_{n\to\infty} \frac{g(2^n\varpi)}{2^{4n}} = G(\varpi). \tag{5.2.8}$$

We can see that inequality (5.2.3) holds if we take the limit as $n \to \infty$ in (5.2.6) and use (5.2.8). Now, we replace ϖ, ϑ by $2^n\varpi, 2^n\vartheta$, respectively, in (5.2.2), then

$$\frac{1}{2^{4np\beta}}\|B_m g(2^n\varpi, 2^n\vartheta)\|_{\mathscr{E}_2}^p \le \frac{\Theta^p(2^n\varpi, 2^n\vartheta)}{2^{4np\beta}}. \tag{5.2.9}$$

Letting the limit as $n \to \infty$, we have $B_m G(x,y) = 0$, all $\varpi, \vartheta \in \mathscr{E}_1$. Therefore, by [[18], Theorem 3, pp-2], $G : \mathscr{E}_1 \to \mathscr{E}_2$ is a quartic map. Now, suppose that $G' : \mathscr{E}_1 \to \mathscr{E}_2$ is another quartic map fulfilling (5.2.3). Then we get

$$\begin{aligned}
\|G(\varpi) - G'(\varpi)\|_{\mathscr{E}_2}^p &= \frac{1}{2^{4np\beta}}\|G(2^n\varpi) - G'(2^n\varpi)\|_{\mathscr{E}_2}^p \\
&\le \frac{1}{2^{4np\beta}}\left(\|G(2^n\varpi) - g(2^n\varpi)\|_{\mathscr{E}_2}^p + \|g(2^n\varpi) - G'(2^n\varpi)\|_{\mathscr{E}_2}^p\right) \\
&\le \frac{1}{2^{4np\beta}}\left[\frac{\Theta'(2^n\varpi,0)}{(2^4|m-1|^2)^{p\beta}}\right] \\
&= \frac{1}{(2^4|m-1|^2)^{p\beta}} \sum_{k=0}^{\infty} \frac{1}{2^{4(n+k)p\beta}}\Theta^p(2^{n+k}\varpi,0) \\
&= \frac{1}{(2^4|m-1|^2)^{p\beta}} \sum_{n=0}^{\infty} \frac{1}{2^{4np\beta}}\Theta^p(2^n\varpi,0),
\end{aligned} \tag{5.2.10}$$

$\forall\ \varpi \in \mathscr{E}_1$. Taking $n \to \infty$ in the preceding inequality, it is obvious that G is unique. ■

Now we have a result similar to Theorem 5.2.1 for (1.1.12).

Theorem 5.2.2. *Assume that* $g : \mathscr{E}_1 \to \mathscr{E}_2$ *is a map for which there exists a* $\Theta : \mathscr{E}_1 \times \mathscr{E}_1 \to [0, \infty)$ *such that*

$$\Theta'(\varpi, \vartheta) := \sum_{k=0}^{\infty} 2^{4kp\beta} \Theta^p \left(\frac{\varpi}{2^k}, \frac{\vartheta}{2^k} \right) < \infty, \tag{5.2.11}$$

$$\|B_m(g(\varpi, \vartheta)\|_{\mathscr{E}_2} \leq \Theta(\varpi, \vartheta), \tag{5.2.12}$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_1$.
Then there exists a unique quartic map $G : \mathscr{E}_1 \to \mathscr{E}_2$ *such that*

$$\|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \left(\frac{\Theta'(\varpi, 0)}{(m-1)^{2\beta}} \right)^{\frac{1}{p}}, \tag{5.2.13}$$

$\forall\ \varpi \in \mathscr{E}_1$.

Proof. It follows from (5.2.11) that $\Theta(0, 0) = 0$. So from (5.2.12) we get $g(0) = 0$. Put $\vartheta = 0$ in (5.2.12), we have

$$\|(m-1)^2 g(2\varpi) - (m-1)^2 16 g(\varpi)\|_{\mathscr{E}_2} \leq \Theta(\varpi, 0), \tag{5.2.14}$$

$\forall\ \varpi \in \mathscr{E}_1$. In the previous inequality, if we replace ϖ by $\frac{\varpi}{2}$ and divide both sides by $(m-1)^{2\beta}$, we get

$$\left\| g(\varpi) - 16 g\left(\frac{\varpi}{2}\right) \right\|_{\mathscr{E}_2}^p \leq \frac{\Theta^p(\varpi/2, 0)}{(m-1)^{2p\beta}}, \tag{5.2.15}$$

$\forall\ \varpi \in \mathscr{E}_1$. Using triangular inequality and moving forward in this manner

$$\left\| g(\varpi) - 2^{4n} g\left(\frac{\varpi}{2^n}\right) \right\|_{\mathscr{E}_2}^p \leq \frac{1}{(m-1)^{2p\beta}} \sum_{k=1}^{n} 2^{4kp\beta} \Theta^p \left(\frac{\varpi}{2^k}, 0 \right), \tag{5.2.16}$$

$\forall\ \varpi \in \mathscr{E}_1$. If we show that the sequence $\{2^{4n} g(\varpi/2^n)\}$ is Cauchy, then the completeness of $\mathscr{E}_2$ will imply that it is convergent. For this, if we substitute ϖ in (5.2.16) with $\varpi/2^l$

and then multiply both sides by 2^{4l}, then we have

$$\left\|2^{4(l+n)}g\left(\frac{\varpi}{2^n}\right)-2^{4l}g\left(\frac{\varpi}{2^l}\right)\right\|_{\mathscr{E}_2}^p \leq \frac{1}{(m-1)^{2p\beta}}\sum_{k=1}^{n}2^{4p(k+l)\beta}\Theta^p\left(\frac{\varpi}{2^{k+l}},0\right)$$
$$=\frac{1}{(m-1)^{2p\beta}}\sum_{k=l+1}^{l+n}2^{4pk\beta}\Theta^p\left(\frac{\varpi}{2^k},0\right), \tag{5.2.17}$$

$\forall\ \varpi \in \mathscr{E}_1$, and $0 < l < n$.
Therefore sequence $\{2^{4n}g(\varpi/2^n)\}$ is convergent to the map G, that is,

$$G(\varpi) = \lim_{n\to\infty} 2^{4n}g\left(\frac{\varpi}{2^n}\right). \tag{5.2.18}$$

Then as in the view of Theorem 5.2.1, it is to see that G is a unique quartic mapping. ■

Now, we give some consequences of the above result in complete (β, p)- normed space.

Corollary 5.2.3. *Let λ, γ, r_1 and s_1 be a nonnegative real numbers such that $s_1 > 0$ and $r_1, s_1 < 4$. Let $g : \mathscr{E}_1 \to \mathscr{E}_2$ be a map fulfilling*

$$\|B_m g(\varpi,\vartheta)\|_{\mathscr{E}_2} \leq \lambda\|\varpi\|^{r_1} + \gamma\|\vartheta\|^{s_1}, \tag{5.2.19}$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_1$.
Then there exists a unique quartic map $G : \mathscr{E}_1 \to \mathscr{E}_2$ such that

$$\|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \frac{1}{(m-1)^{2\beta}}\frac{\lambda\|\varpi\|^{r_1}}{(2^{4p\beta}-2^{r_1p\beta})^{\frac{1}{p}}} \tag{5.2.20}$$

$\forall\ \varpi \in \mathscr{E}_1$ *and all* $\varpi \in \mathscr{E}_1\backslash\{0\}$ *if* $r_1 < 0$.

Proof. If we put $\varpi = \vartheta = 0$ in (5.2.19), we have $g(0) = 0$. Using $\Theta(\varpi,\vartheta) = \lambda\|\varpi\|^{r_1} + \gamma\|\vartheta\|^{s_1}$ in Theorem 5.2.1, we obtain the result. ■

Corollary 5.2.4. *Assume that λ, γ, r_1 and s_1 are nonnegative real numbers such that $r_1, s_1 > 4$. Let $g : \mathscr{E}_1 \to \mathscr{E}_2$ be a map fulfilling*

$$\|B_m g(\varpi,\vartheta)\|_{\mathscr{E}_2} \leq \lambda\|\varpi\|^{r_1} + \gamma\|\vartheta\|^{s_1}, \tag{5.2.21}$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_1$.
Then there exists a unique quartic map $G : \mathscr{E}_1 \to \mathscr{E}_2$ such that

$$\|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \frac{2^{\beta r_1}\lambda\|\varpi\|^{r_1}}{(m-1)^{2\beta}(2^{4p\beta}-2^{r_1p\beta})^{\frac{1}{p}}}, \tag{5.2.22}$$

$\forall\ \varpi \in \mathscr{E}_1$.

Proof. If we put $\vartheta = \varpi = 0$ in (5.2.21), we have $g(0) = 0$. Using $\Theta(\varpi, \vartheta) = \lambda\|\varpi\|^{r_1} + \gamma\|\vartheta\|^{s_1}$ in Theorem 5.2.2, we get the wanted result. ■

Remark 5.2.5. *If $\beta = 1 = p$ in Theorems 5.2.1, 5.2.2 and Corollaries 5.2.3, 5.2.4, we get the corresponding results of [18] (Theorems 4, 6 and Corollaries 5, 7).*

5.3 Ulam-type stability in non-Archimedean β-normed space by direct method

In 1897, Hensel [52] proposed a NS without the Archimedean property. Non-Archimedean spaces were later found to have a broad range of useful applications (see [62, 63, 82]). Here, we demonstrate the stability of (1.1.12) in the context of NAβ-NS. Throughout this section, $\mathscr{E}_2$ is a complete NAβ-NS, $\mathscr{E}_1$ is a NAβ-NS.

Theorem 5.3.1. *Let $\Theta : \mathscr{E}_1 \times \mathscr{E}_1 \to [0, \infty)$ such that*

$$\lim_{k\to\infty} \frac{1}{2^{4k\beta}} \Theta\left(2^k\varpi, 2^k\vartheta\right) = 0. \tag{5.3.1}$$

Suppose that $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a map satisfying equality

$$\|B_m g(\varpi, \vartheta)\|_{\mathscr{E}_2} \leq \Theta(\varpi, \vartheta), \forall\ \varpi, \vartheta \in \mathscr{E}_1. \tag{5.3.2}$$

Then there exists a unique quartic map $G : \mathscr{E}_1 \to \mathscr{E}_2$ such that

$$\|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \frac{\Theta'(\varpi, 0)}{|2^4(m-1)^2|^\beta}, \forall\ \varpi \in \mathscr{E}_1, \tag{5.3.3}$$

where $\Theta'(\varpi, 0) = \sup\{\Theta(2^j\varpi, 0)/|2^4|^{j\beta} : j \in \mathbb{N} \cup \{0\}\}$.

Proof. Putting $\vartheta = 0$ in (5.3.2), we have

$$\|(m-1)^2 g(2\varpi) - (m-1)^2 16 g(\varpi)\|_{\mathscr{E}_2} \leq \Theta(\varpi, 0), \forall\ \varpi \in \mathscr{E}_1. \tag{5.3.4}$$

Thus we get

$$\|g(2\varpi) - 16g(\varpi)\|_{\mathscr{E}_2} \leq \frac{1}{|m-1|^{2\beta}} \Theta(\varpi, 0), \forall\ \varpi \in \mathscr{E}_1. \tag{5.3.5}$$

Replacing ϖ by $2^r\varpi$ in (5.3.5) and then dividing both sides by $|2^4|^{(r+1)\beta}$, we have

$$\Big\|\frac{1}{2^{4(r+1)}}g(2^{r+1}\varpi)-\frac{1}{2^{4(r)}}g(2^r\varpi)\Big\|_{\mathscr{E}_2}\le\frac{1}{|m-1|^{2\beta}|2^4|^{(r+1)\beta}}\Theta(2^r\varpi,0),\forall\ \varpi\in\mathscr{E}_1 \quad (5.3.6)$$

and all nonnegative integers r. Thus sequence $\left\{\frac{g(2^r\varpi)}{2^{4r}}\right\}$ is Cauchy by (5.3.1) and (5.3.6). Because of the completeness of $\mathscr{E}_2$ as a NAβ-NS, there exists a map G so that

$$\lim_{r\to\infty}\frac{g(2^r\varpi)}{2^{4r}}=G(\varpi). \quad (5.3.7)$$

For each $\varpi\in\mathscr{E}_1$ and nonnegative integers r, we have

$$\begin{aligned}\Big\|g(\varpi)-\frac{g(2^r\varpi)}{2^{4r}}\Big\|_{\mathscr{E}_2}&=\Big\|\sum_{j=0}^{r-1}\left(\frac{g(2^j\varpi)}{2^{4j}}-\frac{g(2^{j+1}\varpi)}{2^{4(j+1)}}\right)\Big\|_{\mathscr{E}_2}\\&\le\max\left\{\Big\|\sum_{j=0}^{r-1}\left(\frac{g(2^j\varpi)}{2^{4j}}-\frac{g(2^{j+1}\varpi)}{2^{4(j+1)}}\right)\Big\|_{\mathscr{E}_2}:0\le j<r\right\}\\&\le\frac{1}{|2^4(m-1)^2|^\beta}\max\left\{\frac{\Theta(2^j\varpi,0)}{|2^4|^{j\beta}}:0\le j<r\right\}. \quad (5.3.8)\end{aligned}$$

Taking $r\to\infty$ in (5.3.8) and using (5.3.7), we can see that the inequality (5.3.3) valid when $m\ge 2$. It follows from (5.3.1), (5.3.2), and (5.3.7) that

$$\|B_mG(\varpi,\vartheta)\|_{\mathscr{E}_2}=\lim_{r\to\infty}\frac{1}{|2|^{4r\beta}}\|B_mg(2^r\varpi,2^r\vartheta)\|_{\mathscr{E}_2}\le\lim_{r\to\infty}\frac{1}{|2|^{4r\beta}}\Theta(2^r\varpi,2^r\vartheta)=0. \quad (5.3.9)$$

Hence, the mapping G satisfies (1.1.12). Now, let $G':\mathscr{E}_1\to\mathscr{E}_2$ be another quartic map satisfying (5.3.3). Then we get

$$\begin{aligned}\|G(\varpi)-G'(\varpi)\|_{\mathscr{E}_2}&=\lim_{k\to\infty}\frac{1}{2^{4k}}\|G(2^k\varpi)-G'(2^k\varpi)\|_{\mathscr{E}_2}\\&\le\lim_{k\to\infty}\frac{1}{2^{4k}}\max\{\|G(2^k\varpi)-g(2^k\vartheta)\|_{\mathscr{E}_2},\|g(2^k\vartheta)-G'(2^k\varpi)\|_{\mathscr{E}_2}\}\\&\le\frac{1}{|2^4(m-1)^2|^\beta}\lim_{k\to\infty}\lim_{r\to\infty}\max\left\{\frac{\Theta(2^j\varpi,0)}{|2^4|^{j\beta}}:k\le j<r+k\right\}\\&=\frac{1}{|2^4(m-1)^2|^\beta}\lim_{k\to\infty}\sup\left\{\frac{\Theta(2^j\varpi,0)}{|2^4|^{j\beta}}:k\le j<\infty\right\}=0,\ \forall\ \varpi,\vartheta\in\mathscr{E}_1.\end{aligned}$$

■

Following outcome, which is similar to Theorem 5.3.1 for (1.1.12).

Theorem 5.3.2. *Let* $\Theta : \mathscr{E}_1 \times \mathscr{E}_1 \to [0, \infty)$ *such that*

$$\lim_{k\to\infty} |2|^{4k\beta}\Theta\left(\frac{\varpi}{2^k}, \frac{\vartheta}{2^k}\right) = 0. \tag{5.3.10}$$

Suppose that $g : \mathscr{E}_1 \to \mathscr{E}_2$ *is a map satisfying equality*

$$\|B_m(g(\varpi, \vartheta)\|_{\mathscr{E}_2} \leq \Theta(\varpi, \vartheta), \ \forall\ \varpi, \vartheta \in \mathscr{E}_1. \tag{5.3.11}$$

Then there exists a unique quartic map $R : \mathscr{E}_1 \to \mathscr{E}_2$ *such that*

$$\|g(\varpi) - R(\varpi)\|_{\mathscr{E}_2} \leq \frac{\Theta'(\varpi, 0)}{|2^4(m-1)^2|^{\beta}}, \ \forall\ \varpi \in \mathscr{E}_1, \tag{5.3.12}$$

where $\Theta'(\varpi, 0) = \sup\{|2^4|^{j\beta}\Theta(\varpi/2^j, 0) : j \in \mathbb{N} \cup \{0\}\}$.

Proof. In the same way as Theorem 5.3.1, we have

$$\Big|\Big|g(2\varpi) - 16g(\varpi)\Big|\Big|_{\mathscr{E}_2} \leq \frac{1}{|m-1|^{2\beta}}\Theta(\varpi, 0), \ \forall\ \varpi \in \mathscr{E}_1. \tag{5.3.13}$$

Replacing ϖ by $\varpi/2^{r+1}$ in (5.3.13) and then multiply both sides by $|2|^{4r\beta}$, we get

$$\Big|\Big|2^{4r}g\left(\frac{\varpi}{2^n}\right) - 2^{4(r+1)}g\left(\frac{\varpi}{2^{r+1}}\right)\Big|\Big|_{\mathscr{E}_2} \leq \frac{2^{4r\beta}}{|m-1|^{2\beta}}\Theta\left(\frac{\varpi}{2^{r+1}}, 0\right), \ \forall\ \varpi \in \mathscr{E}_1 \tag{5.3.14}$$

and all nonnegative integers r. Thus the sequence $\left\{2^r g\left(\frac{\varpi}{2^r}\right)\right\}$ is Cauchy by (5.3.10) and (5.3.14). Because of the completeness of $\mathscr{E}_2$ as a NAβ-NS, there exists a map R so that

$$\lim_{r\to\infty} 2^{4r}g\left(\frac{\varpi}{2^r}\right) = R(\varpi). \tag{5.3.15}$$

For each $\varpi \in \mathscr{E}_1$ and nonnegative integers r, we have

$$\Big|\Big|g(\varpi) - 2^{4r}g\left(\frac{\varpi}{2^r}\right)\Big|\Big|_{\mathscr{E}_2} \leq \frac{1}{|(m-1)|^{2\beta}} \max\left\{|2|^{4j\beta}\Theta\left(\frac{\varpi}{2^{j+1}}, 0\right) : 0 \leq j < r\right\}, \forall\ \varpi \in \mathscr{E}_1 \tag{5.3.16}$$

and nonnegative integers r. Since right hand side of (5.3.16) tends to 0 as $r \to \infty$, using (5.3.15), we deduce inequality (5.3.12). We can now finish the proof in the same way that we did with Theorem 5.3.1. ■

Now, we give some consequences of the results in NAβ-NS.

Corollary 5.3.3. *Let $\lambda > 0$, and $\omega : [0,\infty) \to [0,\infty)$ is a function satisfying $\omega(|e|z) \leq \omega(|e|)\omega(z)\ \forall\ z,e \in [0,\infty)$ for which $\omega(|2|) \leq |16|$. Assume that $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a map fulfilling the inequality*

$$\|B_m g(\varpi,\vartheta)\|_{\mathscr{E}_2} \leq \lambda(\omega(\|\varpi\|) + \omega(\|\vartheta\|)),\ \forall\ \varpi,\vartheta \in \mathscr{E}_1. \tag{5.3.17}$$

Then there exists a unique quartic mappings $G : \mathscr{E}_1 \to \mathscr{E}_2$ such that

$$\|g(\varpi) - G(\varpi)\|_{\mathscr{E}_2} \leq \frac{\lambda\omega(\|\varpi\|)}{|2^4(m-1)^2|^\beta},\ \forall\ \varpi \in \mathscr{E}_1. \tag{5.3.18}$$

Proof. Define $\varphi : \mathscr{E}_1 \times \mathscr{E}_1 \to [0,\infty)$ by $\varphi(\varpi,\vartheta) = C(\omega(\|\varpi\|) + \omega(\|\vartheta\|))$, we have

$$\lim_{n\to\infty} \frac{1}{|2|^{4n\beta}}\varphi(2^n\varpi, 2^n\vartheta) \leq \lim_{n\to\infty} \left(\frac{\omega(|2|)}{|2^4|}\right)^{n\beta} \varphi(\varpi,\vartheta) = 0,\ \forall\ \varpi,\vartheta \in \mathscr{E}_1. \tag{5.3.19}$$

We also get

$$\varphi'(\varpi,0) = \sup\left\{\frac{\varphi(2^j\varpi,0)}{|2|^{4\beta}} : 0 \leq j < \infty\right\} = \varphi(\varpi,0) = C(\omega(\|\varpi\|)), \forall\ \varpi \in \mathscr{E}_1. \tag{5.3.20}$$

Now, Theorem 5.3.1 implies the wanted result. ∎

Following consequence is a direct consequence of Theorem 5.3.2, and its proof is similar to that of Corollary 5.3.3.

Corollary 5.3.4. *Suppose that $\lambda > 0$, and $\omega : [0,\infty) \to [0,\infty)$ is a function satisfying $\omega(|e|z) \leq \omega(|e|)\omega(z) \forall\ e,z \in [0,\infty)$ for which $\omega(|2|^{-1}) \leq |16|^{-1}$. Assume that $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a map fulfilling the inequality*

$$\|B_m g(\varpi,\vartheta)\|_{\mathscr{E}_2} \leq \lambda(\omega(\|\varpi\|) + \omega(\|\vartheta\|)), \tag{5.3.21}$$

$\forall\ \varpi,\vartheta \in \mathscr{E}_1$. *Then there exists a unique quartic map $R : \mathscr{E}_1 \to \mathscr{E}_2$ such that*

$$\|g(\varpi) - R(\varpi)\|_{\mathscr{E}_2} \leq \frac{\lambda\omega(\|\varpi/2\|)}{|(m-1)|^{2\beta}}, \tag{5.3.22}$$

$\forall\ \varpi \in \mathscr{E}_1$.

Remark 5.3.5. *If $\beta = 1$ in Theorems 5.3.1, 5.3.2 and Corollaries 5.3.3, 5.3.4 we get the corresponding results of Bodagi [18] (Theorems 12, 14 and Corollaries 13, 15).*

Chapter 6

Generalized hyperstability of a general linear functional equation in complete quasi-2-normed space

Introduction

In this chapter, we study the generalized hyperstability of a general linear functional equation in complete Q2NS using a fixed point approach. The results of this chapter are published in Sharma and Chandok [112].

6.1 Hyperstability of functional equation

Here, we study the hyperstability of (1.1.3) in the setting of complete Q2NS using Theorem 2.4.1. Throughout this section, $(\mathscr{E}_2, \|\cdot,\cdot\|, K \geq 1)$ is a complete Q2NS over the field $\mathbb{K}$, $(\mathscr{E}_1, \|\cdot,\cdot\|, K \geq 1)$ is a Q2NS over the field $\mathbb{F}$, and Z_0 is a subset of $\mathscr{E}_2$ containing two LI vectors.

Theorem 6.1.1. *(1)If $g : \mathscr{E}_1 \to \mathscr{E}_2,\ f : Z_0 \to Z_0$ are given function and there exist $B, A \in \mathbb{K}\backslash\{0\}$, $b, a \in \mathbb{F}\backslash\{0\}$, and $v, u : \mathscr{E}_1 \times Z_0 \to \mathbb{R}_+$ such that $l(\mathscr{E}_1) = \{n \in \mathbb{N} : \alpha_u < 1\}$ is an infinite set, where*

$$\alpha_u = K\left(\left|\frac{1}{A}\right|\lambda_1(a+bn)\lambda_2(a+bn) + \left|\frac{B}{A}\right|\lambda_1(n)\lambda_2(n)\right),$$

$r = \log_{2K} 2$,

$$\lambda_1(n) = \inf\{t \in \mathbb{R}_+ : u(n\varpi) \leq tu(\varpi)\ \forall\ \varpi \in \mathscr{E}_1\}$$
$$\lambda_2(n) = \inf\{t \in \mathbb{R}_+ : v(n\varpi) \leq tv(\varpi)\ \forall\ \varpi \in \mathscr{E}_1\}$$

for $n \in \mathbb{F}\backslash\{0\}$, and λ_1, λ_2 fulfill the following two assumptions, (where $n \to \infty$ in $\mathbb{F}$ if and only if $\mid n \mid \to \infty$),

(a) $\lim_{n\to\infty} \lambda_1(\pm n)\lambda_2(\pm n) = 0$

(b) $\lim_{n\to\infty} \lambda_1(n) = 0$ *or* $\lim_{n\to\infty} \lambda_2(n) = 0$.

(2) Assume that $g : \mathscr{E}_1 \to \mathscr{E}_2$ *fulfills the inequality*

$$\|g(a\varpi + b\vartheta) - Ag(\varpi) - Bg(\vartheta), z\| \leq u(\varpi, z)v(\vartheta, z), \tag{6.1.1}$$

for $\varpi, \vartheta \in \mathscr{E}_1\backslash\{0\}, z \in Z_0$ *with* $a\varpi + b\vartheta \neq 0$.
Then g *fulfills*

$$g(a\varpi + b\vartheta) = Ag(\varpi) + Bg(\vartheta), \tag{6.1.2}$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_1\backslash\{0\}$ *with* $a\varpi + b\vartheta \neq 0$.

Proof. Note that for $n \in \mathbb{F}\backslash\{0\}$, and $\varpi \in \mathscr{E}_1\backslash\{0\}, z \in Z_0$, we have

$$u(n\varpi, z) \leq \lambda_1(n)u(\varpi, z) \text{ and } v(n\varpi, z) \leq \lambda_2(n)v(\varpi, z). \tag{6.1.3}$$

Case 1. $\lim_{n\to\infty} \lambda_2(n) = 0$.

For $m \in l(\mathscr{E}_1)$, replacing ϑ by $m\varpi$ in (6.1.1), we have

$$\|g(a\varpi + bm\varpi) - Ag(\varpi) - Bg(m\varpi), z\| \leq u(\varpi, z)v(m\varpi, z),$$

for $\varpi \in \mathscr{E}_1\backslash\{0\}$ and $z \in Z_0$.

$$\left\|\frac{1}{A}g(a\varpi + bm\varpi) - g(\varpi) - \frac{B}{A}g(m\varpi), z\right\| \leq \left|\frac{1}{A}\right|u(\varpi, z)v(m\varpi, z), \tag{6.1.4}$$

for $\varpi \in \mathscr{E}_1\backslash\{0\}$ and $z \in Z_0$. Define a function $\mathscr{T}_m : \mathscr{E}_2^{\mathscr{E}_1\backslash\{0\}} \to \mathscr{E}_2^{\mathscr{E}_1\backslash\{0\}}$ by

$$(\mathscr{T}_m\xi)(\varpi) = \frac{1}{A}\xi\left((a + bm)\varpi\right) - \frac{B}{A}\xi(mx), \tag{6.1.5}$$

$\forall\ \varpi \in \mathscr{E}_1\backslash\{0\}$ and $\xi \in \mathscr{E}_2^{\mathscr{E}_1\backslash\{0\}}$.
First, we have to show that $\mathscr{T}_m$ fulfills all the conditions of Theorem 2.4.1, where the function $\mathscr{T}_m$ and the set $\mathscr{E}_1\backslash\{0\}$ play the roles of $\mathscr{T}$ and $\mathscr{E}_1$ respectively.
Take $\varepsilon_m(\varpi, z) = \left|\frac{1}{A}\right|u(\varpi, z)v(m\varpi, z)$ for all $\varpi \in \mathscr{E}_1\backslash\{0\}, z \in Z_0$. Using (6.1.3), we have

$$\varepsilon_m(\varpi, z) = \left|\frac{1}{A}\right|\lambda_2(m)u(\varpi, z)v(\varpi, z). \tag{6.1.6}$$

Using (6.1.4) and (6.1.5), we get

$$\|(\mathscr{T}_m g)(\varpi) - g(\varpi), z\| \leq \varepsilon_m(\varpi, z),$$

$\forall\ \varpi \in \mathscr{E}_1\backslash\{0\}$ and $z \in Z_0$. This prove that (2.4.2) is fulfilled with $\varphi = g$ and $\varepsilon = \varepsilon_m$. Now, we define a function $\Lambda_m : \mathbb{R}_+^{\mathscr{E}_1\backslash\{0\}\times Z_0} \to \mathbb{R}_+^{\mathscr{E}_1\backslash\{0\}\times Z_0}$ by

$$\Lambda_m\delta(\varpi, z) = K\Big|\frac{1}{A}\Big|\delta\left((a+bm)\varpi, z\right) + K\Big|\frac{B}{A}\Big|\delta(m\varpi, z),$$

$\forall\ \varpi \in \mathscr{E}_1\backslash\{0\}$ has the shape given in (2.4.4) with $k = 2,\ g_1(\varpi) = (a+bm)\varpi,\ g_2(\varpi) = m\varpi,\ l_1(\varpi, z) = K\Big|\frac{1}{A}\Big|,\ l_2(\varpi, z) = K\Big|\frac{B}{A}\Big|\ \forall\ \varpi \in \mathscr{E}_1$. Furthermore, for each $\xi, \mu \in \mathscr{E}_2^{\mathscr{E}_1\backslash\{0\}}$ and $\varpi \in \mathscr{E}_1\backslash\{0\}, z \in Z_0$, we have

$$\begin{aligned}
&\|(\mathscr{T}_m\xi)(\varpi) - (\mathscr{T}_m\mu)(\varpi), z\| \\
&= \Big\|\frac{1}{A}\xi\left((a+bm)\varpi\right) - \frac{B}{A}\xi(m\varpi) - \frac{1}{A}\mu\left((a+bm)\varpi\right) + \frac{B}{A}\mu(m\varpi), z\Big\| \\
&\leq K\Big|\frac{1}{A}\Big|\|\xi\left((a+bm)\varpi\right) - \mu\left((a+bm)\varpi\right), z\| + K\Big|\frac{B}{A}\Big|\|\xi(m\varpi) - \mu(m\varpi), z\| \\
&= \sum_{i=1}^{2} l_i(\varpi, z)\|\xi(g_i(\varpi)) - \mu(g_i(\varpi)), f_i(z)\|.
\end{aligned}$$

This shows that (2.4.1) is fulfilled. Next, we have to prove that for each $n \in \mathbb{N}$ and $\varpi \in \mathscr{E}_1\backslash\{0\}$,

$$\begin{aligned}
&\Lambda_m^n\varepsilon_m(\varpi, z) \\
&\leq \Big|\frac{1}{A}\Big|K^n\left(\Big|\frac{1}{A}\Big|\lambda_1(a+bm)\lambda_2(a+bm) + \Big|\frac{B}{A}\Big|\lambda_1(n)\lambda_2(n)\right)^n\lambda_2(m)u(\varpi, z)v(\varpi, z).
\end{aligned} \tag{6.1.7}$$

If $n = 0$ then (6.1.7) holds by (6.1.6). Also by (6.1.6) and the definition of Λ and λ_1, λ_2 we have

$$\begin{aligned}
&\Lambda_m\varepsilon_m(\varpi, z) \\
&= K\Big|\frac{1}{A}\Big|\varepsilon_m((a+bm)\varpi, z) + K\Big|\frac{B}{A}\Big|\varepsilon_m(m\varpi, z) \\
&\leq K\Big|\frac{1}{A}\Big|\Big|\frac{1}{A}\Big|\lambda_2(m)u((a+mb)\varpi, z)v((a+mb)\varpi, z) + K\Big|\frac{1}{A}\Big|\Big|\frac{B}{A}\Big|\lambda_2(m)u(m\varpi, z)v(m\varpi, z) \\
&\leq K\Big|\frac{1}{A}\Big|\Big|\frac{1}{A}\Big|\lambda_2(m)\lambda_1(a+mb)u(\varpi, z)\lambda_2(a+mb)v(\varpi, z) \\
&\ + K\Big|\frac{1}{A}\Big|\Big|\frac{B}{A}\Big|\lambda_2(m)\lambda_1(m)u(\varpi, z)\lambda_2(m)v(\varpi, z) \\
&= \Big|\frac{1}{A}\Big|K\left(\Big|\frac{1}{A}\Big|\lambda_1(a+mb)\lambda_2(a+mb) + \Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\right)\lambda_2(m)u(\varpi, z)v(\varpi, z).
\end{aligned} \tag{6.1.8}$$

Using (6.1.8) and iterating argument, we get

$$\begin{aligned}\Lambda_m^2\varepsilon_m(\varpi,z) &= K\Big|\frac{1}{A}\Big|(\Lambda_m\varepsilon_m)((a+bm)\varpi,z)+K\Big|\frac{B}{A}\Big|(\Lambda_m\varepsilon_m)(m\varpi,z)\\ &\leq K\Big|\frac{1}{A}\Big|\Big[\ \Big|\frac{1}{A}\Big|K\left(\Big|\frac{1}{A}\Big|\lambda_1(a+mb)\lambda_2(a+mb)+\Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\right)\\ &\quad\lambda_2(m)u((a+bm)\varpi,z)v((a+bm)\varpi,z)\ \Big]\\ &\quad+K\Big|\frac{B}{A}\Big|\Big[\ \Big|\frac{1}{A}\Big|K\left(\Big|\frac{1}{A}\Big|\lambda_1(a+mb)\lambda_2(a+mb)+\Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\right)\\ &\quad\lambda_2(m)u(m\varpi,z)v(m\varpi,z)\ \Big].\end{aligned}$$

By using (6.1.3), we get

$$\begin{aligned}&\Lambda_m^2\varepsilon_m(\varpi,z)\\ &\leq\Big|\frac{1}{A}\Big|\Big[\ \Big|\frac{1}{A}\Big|K^2\left(\Big|\frac{1}{A}\Big|\lambda_1(a+mb)\lambda_2(a+mb)+\Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\right)\\ &\quad\lambda_2(m)\lambda_1(a+bm)u(\varpi,z)\lambda_2(a+bm)v(\varpi,z)\ \Big]\\ &\quad+\Big|\frac{B}{A}\Big|\Big[\ \Big|\frac{1}{A}\Big|K^2\left(\Big|\frac{1}{A}\Big|\lambda_1(a+mb)\lambda_2(a+mb)+\Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\right)\\ &\quad\lambda_2(m)\lambda_1(m)u(\varpi,z)\lambda_2(m)v(\varpi,z)\ \Big]\\ &=\Big|\frac{1}{A}\Big|K^2\Bigg(\ \Big|\frac{1}{A}\Big|^2\lambda_1^2(a+mb)\lambda_2^2(a+mb)+2\Big|\frac{1}{A}\Big|\Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\lambda_1(a+bm)\lambda_2(a+bm)\\ &\quad\Big|\frac{B}{A}\Big|^2\lambda_1(m)^2\lambda_2(m)^2\ \Bigg)\lambda_2(m)u(\varpi,z)v(\varpi,z)\\ &=\Big|\frac{1}{A}\Big|K^2\left(\Big|\frac{1}{A}\Big|\lambda_1(a+mb)\lambda_2(a+mb)+\Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\right)^2\lambda_2(m)u(\varpi,z)v(\varpi,z).\end{aligned}$$

In the same way, using induction we see that (6.1.7) holds $\forall\ n\in\mathbb{N},\varpi\in\mathscr{E}_1\backslash\{0\}$ and $z\in Z_0$. By definition $l(\mathscr{E}_1)$ and adding the geometric progression, we obtain that for each $\varpi\in\mathscr{E}_1\backslash\{0\},z\in Z_0$ and $m\in l(\mathscr{E}_1)$,

$$\begin{aligned}&\sum_{n=0}^{\infty}(\Lambda_m^n\varepsilon_m)^r(\varpi)\\ &\leq\sum_{n=0}^{\infty}\Big|\frac{1}{A}\Big|^rK^{nr}\left(\Big|\frac{1}{A}\Big|\lambda_1(a+mb)\lambda_2(a+mb)+\Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\right)^{nr}\lambda_2^r(m)u^r(\varpi,z)v^r(\varpi,z)\\ &=\Big|\frac{1}{A}\Big|^r\frac{\lambda_2^r(m)u^r(\varpi,z)v^r(\varpi,z)}{1-K^r\left(\Big|\frac{1}{A}\Big|\lambda_1(a+mb)\lambda_2(a+mb)+\Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\right)^r}.\end{aligned}$$

This proves that (2.4.3) is satisfied, where

$$\varepsilon^*(\varpi, z) = \Big|\frac{1}{A}\Big|^r \frac{\lambda_2^r(m)u^r(\varpi, z)v^r(\varpi, z)}{1 - K^r\left(\Big|\frac{1}{A}\Big|\lambda_1(a+mb)\lambda_2(a+mb) + \Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\right)^r}.$$

By the above, all assumptions of Theorem 2.4.1 are fulfilled. Therefore, there exists a fixed point $Q_m : \mathscr{E}_1\backslash\{0\} \to \mathscr{E}_2$ of the function $\mathcal{T}_m$ fulfilling

$$\|g(\varpi) - Q_m(\varpi), z\|^r \leq 4\Big|\frac{1}{A}\Big|^r \frac{\lambda_2^r(m)u^r(\varpi, z)v^r(\varpi, z)}{1 - K^r\left(\Big|\frac{1}{A}\Big|\lambda_1(a+mb)\lambda_2(a+mb) + \Big|\frac{B}{A}\Big|\lambda_1(m)\lambda_2(m)\right)^r}, \tag{6.1.9}$$

$\forall\ \varpi \in \mathscr{E}_1\backslash\{0\}, z \in Z_0$. That is

$$Q_m(\varpi) = \frac{1}{A}Q_m\left((a+bm)\varpi\right) + \frac{B}{A}Q_m(m\varpi)$$

and (6.1.9) holds $\forall\ \varpi \in \mathscr{E}_1\backslash\{0\}, z \in Z_0$. Moreover

$$\lim_{n\to\infty} \mathcal{T}_m^n g(\varpi) = Q_m(\varpi). \tag{6.1.10}$$

We also get that $\forall\ \varpi, \vartheta \in \mathscr{E}_1\backslash\{0\}, z \in Z_0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\|A\mathscr{T}_m^n g(\varpi) + B\mathscr{T}_m^n g(\vartheta) - AQ_m(\varpi) - BQ_m(\vartheta), z\| \\
&\leq K\left(\|A\mathscr{T}_m^n g(\varpi) - AQ_m(\varpi), z\| + \|B\mathscr{T}_m^n g(\vartheta) - BQ_m(\vartheta), z\|\right) \\
&= K\left(|\ A\ |\ \|\mathscr{T}_m^n g(\varpi) - Q_m(\varpi), z\| + |\ B\ |\ \|\mathscr{T}_m^n g(\vartheta) - Q_m(\vartheta), z\|\right).
\end{aligned} \tag{6.1.11}$$

Taking $n \to \infty$ in (6.1.11) and using (6.1.10), we get

$$\lim_{n\to\infty} \|A\mathscr{T}_m^n g(\varpi) + B\mathscr{T}_m^n g(\vartheta) - AQ_m(\varpi) - BQ_m(\vartheta), z\| = 0;$$

that is, $\forall\ \varpi \in \mathscr{E}_1\backslash\{0\}, z \in Z_0$,

$$\lim_{n\to\infty}\left(A\mathscr{T}_m^n g(\varpi) + B\mathscr{T}_m^n g(\vartheta)\right) = AQ_m(\varpi) + BQ_m(\vartheta). \tag{6.1.12}$$

Next, we have to show that for each $n \in \mathbb{N}$ and all $\varpi, \vartheta \in \mathscr{E}_1\backslash\{0\}, z \in Z_0$ with $a\varpi + b\vartheta \neq$

0,

$$\|\mathscr{T}_m^n g(a\varpi + b\vartheta) - A\mathscr{T}_m^n g(\varpi) - B\mathscr{T}_m^n g(\vartheta), z\|$$
$$\leq K^n \left(\left|\frac{1}{A}\right|\lambda_1(a+mb)\lambda_2(a+mb) + \left|\frac{B}{A}\right|\lambda_1(m)\lambda_2(m)\right)^n u(\varpi, z)v(\vartheta, z). \tag{6.1.13}$$

Indeed, if $n = 0$ then (6.1.13) holds by (6.1.1). Suppose that (6.1.13) holds for some n and all $\varpi, \vartheta \in \mathscr{E}_1 \backslash \{0\}, z \in Z_0$ with $a\varpi + b\vartheta \neq 0$.
By the induction assumption, definition of $\mathscr{T}_m$ and (6.1.3), we obtain that

$$\|\mathscr{T}_m^{n+1} g(a\varpi + b\vartheta) - A\mathscr{T}_m^{n+1} g(\varpi) - B\mathscr{T}_m^{n+1} g(\vartheta), z\|$$
$$= \Bigg\| \left|\frac{1}{A}\right| \mathscr{T}_m^n g((a+bm)(a\varpi + b\vartheta)) + \left|\frac{B}{A}\right| \mathscr{T}_m^n g((m)(a\varpi + b\vartheta))$$
$$- A\left(\left|\frac{1}{A}\right| \mathscr{T}_m^n g((a+bm)\varpi)+)) + \left|\frac{B}{A}\right| \mathscr{T}_m^n g((m\varpi)\right)$$
$$- B\left(\left|\frac{1}{A}\right| \mathscr{T}_m^n g((a+bm)\vartheta)+)) + \left|\frac{B}{A}\right| \mathscr{T}_m^n g((m\varpi)\right), z\Bigg\|$$
$$\leq K\left|\frac{1}{A}\right| ||\mathscr{T}_m^n g((a+bm)(a\varpi + b\vartheta)) - A\mathscr{T}_m^n g((a+bm)\varpi) - B\mathscr{T}_m^n g((a+bm)\vartheta), z||$$
$$+ K\left|\frac{B}{A}\right| ||\mathscr{T}_m^n g(m(a\varpi + b\vartheta)) - A\mathfrak{T}_m^n g(m\varpi) - B\mathscr{T}_m^n g(m\vartheta), z||$$
$$\leq K\left|\frac{1}{A}\right| K^n \left(\left|\frac{1}{A}\right|\lambda_1(a+mb)\lambda_2(a+mb) + \left|\frac{B}{A}\right|\lambda_1(m)\lambda_2(m)\right)^n$$
$$u((a+mb)\varpi, z)v((a+mb)\vartheta, z)$$
$$+ K\left|\frac{B}{A}\right| K^n \left(\left|\frac{1}{A}\right|\lambda_1(a+mb)\lambda_2(a+mb) + \left|\frac{B}{A}\right|\lambda_1(m)\lambda_2(m)\right)^n u(m\varpi, z)v(m\vartheta, z)$$
$$\leq K\left|\frac{1}{A}\right| K^n \left(\left|\frac{1}{A}\right|\lambda_1(a+mb)\lambda_2(a+mb) + \left|\frac{B}{A}\right|\lambda_1(m)\lambda_2(m)\right)^n$$
$$\lambda_1(a+mb)u(\varpi, z)\lambda_2(a+mb)v(\vartheta, z)$$
$$+ K\left|\frac{B}{A}\right| K^n \left(\left|\frac{1}{A}\right|\lambda_1(a+mb)\lambda_2(a+mb) + \left|\frac{B}{A}\right|\lambda_1(m)\lambda_2(m)\right)^n$$
$$\lambda_1(m)u(\varpi, z)\lambda_2(m)v(\vartheta, z)$$
$$= K^n \left(\left|\frac{1}{A}\right|\lambda_1(a+mb)\lambda_2(a+mb) + \left|\frac{B}{A}\right|\lambda_1(m)\lambda_2(m)\right)^n u(\varpi, z)v(\vartheta, z)$$
$$\times K\left(\left|\frac{1}{A}\right|\lambda_1(a+mb)\lambda_2(a+mb) + \left|\frac{B}{A}\right|\lambda_1(m)\lambda_2(m)\right)$$
$$= K^{n+1} \left(\left|\frac{1}{A}\right|\lambda_1(a+mb)\lambda_2(a+mb) + \left|\frac{B}{A}\right|\lambda_1(m)\lambda_2(m)\right)^{n+1} u(\varpi, z)v(\vartheta, z).$$

Hence by induction, (6.1.13) holds $\forall\ n \in \mathbb{N}$.
Now, from (6.1.13) and (1.3.6) we find that

$$\begin{aligned} & |||\mathscr{T}_m^n g(a\varpi + b\vartheta) - A\mathscr{T}_m^n g(\varpi) - B\mathscr{T}_m^n g(\vartheta), z|||^r \\ & \leq \|\mathscr{T}_m^n g(a\varpi + b\vartheta) - A\mathscr{T}_m^n g(\varpi) - B\mathscr{T}_m^n g(\vartheta), z\|^r \\ & \leq K^{rn} \left(\left|\frac{1}{A}\right| \lambda_1(a+mb)\lambda_2(a+mb) + \left|\frac{B}{A}\right| \lambda_1(m)\lambda_2(m) \right)^{rn} u^r(\varpi, z) v^r(\varpi, z). \end{aligned} \tag{6.1.14}$$

Since equivalent Q2NS is continuous, by letting $n \to \infty$ in (6.1.14), using (6.1.10), (6.1.12) and definition of $l(\mathscr{E}_1)$ we have for $\varpi, \vartheta \in \mathscr{E}_1 \backslash \{0\}, z \in Z_0$ with $a\varpi + b\vartheta \neq 0$,

$$\begin{aligned} & |||Q_m(a\varpi + b\vartheta) - AQ_m(\varpi) - BQ_m(\vartheta), z||| \\ & = \lim_{n\to\infty} |||\mathscr{T}_m^n g(a\varpi + b\vartheta) - A\mathscr{T}_m^n g(\varpi) - B\mathscr{T}_m^n g(\vartheta), z||| = 0. \end{aligned} \tag{6.1.15}$$

Note that $\lim_{n\to\infty} \lambda_1(\pm n)\lambda_2(\pm n) = 0$, $\lim_{n\to\infty} \lambda_2(n) = 0$ and the set $l(\mathscr{E}_1)$ is nonempty set. Then we can take $m \to \infty$ in (6.1.9) to obtain $\lim_{m\to\infty} \|g(\varpi) - Q_m(\varpi), z\| = 0$; that is,

$$\lim_{m\to\infty} Q_m(\varpi) = g(\varpi). \tag{6.1.16}$$

Using (6.1.16), fashion similar to proof of (6.1.12) we also have

$$\lim_{m\to\infty} (AQ_m(\varpi) + BQ_m(\vartheta)) = Ag(\varpi) + Bg(\vartheta). \tag{6.1.17}$$

Letting $m \to \infty$ in (6.1.15), using (6.1.16) and (6.1.17) and the continuity of equivalent Q2N, we have

$$|||g(a\varpi + b\vartheta) - Ag(\varpi) - Bg(\vartheta), z||| = \lim_{m\to\infty} |||Q_m(a\varpi + b\vartheta) - AQ_m(\varpi) - BQ_m(\vartheta), z||| = 0$$

that is

$$g(a\varpi + b\vartheta) = Ag(\varpi) + Bg(\vartheta),$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_1 \backslash \{0\}, z \in Z_0$ with $a\varpi + b\vartheta \neq 0$. Hence, g is a solution of (6.1.2).
Case 2. $\lim_{m\to\infty} \lambda_1(n) = 0$.
For $m \in l(\mathscr{E}_1)$, replacing ϖ by $m\vartheta$ in (6.1.2), we perform similar calculations to obtain the result. ■

By Brzdęk [[26], Lemma 3.1], if $g : \mathscr{E}_1 \to \mathscr{E}_2$ fulfills (6.1.2) on $\mathscr{E}_1 \backslash \{0\}$, then it fulfills (6.1.2) on $\mathscr{E}_1$. Based on this argument and Theorem 6.1.1, we obtain the outcome on $\mathscr{E}_1$, which is an extension of the corresponding outcome of Aiemsomboon et al. [[4], Theorem

2.3] in the setting of complete Q2NS.

By using Theorem 6.1.1, we have an extension of Piszczek [[90], Theorem 2.1] to the framework of complete Q2NS as follows:

Corollary 6.1.2. *If there exist* $B, A \in \mathbb{K}\backslash\{0\}$, $b, a \in \mathbb{F}\backslash\{0\}$, *and* $q, p \in \mathbb{R}$ *with* $q + p < 0, c \geq 0$ *and* $g : \mathscr{E}_1 \to \mathscr{E}_2$ *is a given function such that*

$$\|g(a\varpi + b\vartheta) - Ag(\varpi) - Bg(\vartheta), z\| \leq c\|\varpi, z\|^p\|\vartheta, z\|^q, \tag{6.1.18}$$

$\forall\ \varpi, \vartheta \in \mathscr{X}\backslash\{0\}, z \in Z_0$ *and* $a\varpi + b\vartheta \neq 0$ *with* $\|\varpi, z\|\|\vartheta, z\| \neq 0$. *Then* g *fulfills*

$$g(a\varpi + b\vartheta) = Ag(\varpi) + Bg(\vartheta),$$

$\forall\ \varpi, \vartheta \in \mathscr{E}_1\backslash\{0\}$ *with* $a\varpi + b\vartheta \neq 0$.

Proof. Define $v, u : \mathscr{E}_1 \to \mathbb{R}_+$ by

$$v(\varpi, z) = t\|\varpi, z\|^q, u(\varpi, z) = s\|\varpi, z\|^p,$$

where $s, t \in \mathbb{R}_+$ and $st = c$.
If $c > 0$ then, $s, t > 0$. Thus we have

$$\lambda_1(n) = \inf\{t \in \mathbb{R}_+ : u(n\varpi) \leq tu(\varpi)\ \forall\ \varpi \in \mathscr{E}_1\} =\mid n \mid^p$$
$$\lambda_2(n) = \inf\{t \in \mathbb{R}_+ : v(n\varpi) \leq tv(\varpi)\ \forall\ \varpi \in \mathscr{E}_1\} =\mid n \mid^q .$$

It means that

$$\lim_{n\to\infty} \lambda_1(\pm n)\lambda_2(\pm n) = \lim_{n\to\infty} \mid n \mid^{p+q} = 0.$$

Since $q, p \in \mathbb{R}$ with $q + p < 0$, we obtain that $q < 0$ or $p < 0$. Then $\lim_{n\to\infty} \lambda_1(n) = 0$ or $\lim_{n\to\infty} \lambda_2(n) = 0$.
Hence, we get that

$$K\left(\left|\frac{1}{A}\right|\lambda_1(a + bn)\lambda_2(a + bn) + \left|\frac{B}{A}\right|\lambda_1(n)\lambda_2(n)\right) = K\left(\left|\frac{1}{A}\right| \mid a + bn \mid^{p+q} + \left|\frac{B}{A}\right| \mid n \mid^{p+q}\right).$$

Since $p + q < 0$, we have

$$\lim_{n\to\infty} \mid a + bn \mid^{p+q} = \lim_{n\to\infty} \mid n \mid^{p+q} = 0.$$

Thus the set $l(\mathscr{E}_1)$ is infinite. Hence all the conditions of the Theorem 2.4.1 are fulfilled.

Therefore g fulfills $g(a\varpi + b\vartheta) = Ag(\varpi) + Bg(\vartheta), \forall\ \varpi, \vartheta \in \mathscr{E}_1\backslash\{0\}$ with $a\varpi + b\vartheta \neq 0$.
■

Corollary 6.1.3. *If $g : \mathscr{E}_1 \to \mathscr{E}_2$ is a given function and $q, p \in \mathbb{R}$ with $q + p < 0, c \geq 0$ such that*

$$\|g(\varpi + \vartheta) - g(\varpi) - g(\vartheta), z\| \leq c\|\varpi, z\|^p \|\vartheta, z\|^q, \tag{6.1.19}$$

$\forall\ \vartheta, \varpi \in \mathscr{E}_1\backslash\{0\}, z \in Z_0$ and $\varpi + \vartheta \neq 0$ with $\|\varpi, z\|\|\vartheta, z\| \neq 0$. Then g fulfills $g(\varpi + \vartheta) = g(\varpi) + g(\vartheta)$, $\forall\ \vartheta, \varpi \in \mathscr{E}_1\backslash\{0\}$ with $\varpi + \vartheta \neq 0$.

Remark 6.1.4. *Corollary 6.1.3 has been proved in n-Banach space by Brzdęk et al. [[28], Theorem 4]*

Chapter 7

Application to Hyers-Ulam stability

Introduction

This chapter has been split into two sections. We study the stability of integral equation in $\mathcal{F}$MS, in the first section. The findings of this section are proved in Sharma and Chandok [111]. In the last section, we provide a solution for a Caputo-type nonlinear fractional integro-differential equation in OMS. The findings of this section are published in Sharma and Chandok [109].

7.1 Stability of integral equation in $\mathcal{F}$-metric space

Here, using the results proved in Chapter 2 and 3 we investigte the stability of integral equation in $\mathcal{F}$-metric space. Results of this section are published in Sharma and Chandok [111].

Let $\mathscr{E}_1 = C[\eta_1, \eta_2]$ be the set of all real continuous functions on $[\eta_1, \eta_2]$ fitted with $\mathcal{F}$-metric $\mathscr{D}(\varpi, \vartheta) = \|\varpi - \vartheta\|_\infty$. Then $(\mathscr{E}_1, \mathscr{D})$ is an $\mathcal{F}$-complete $\mathcal{F}$-metric space with $f(t) = \ln(t)$ and $\alpha = 0$. We consider an integral equation

$$\varpi(q) = \int_{\eta_1}^{\eta_2} K(q, p, \varpi(p))d(p), \tag{7.1.1}$$

where $K : [\eta_1, \eta_2] \times [\eta_1, \eta_2] \times \mathbb{R} \to \mathbb{R}$. Let $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ be a map defined by

$$\mathscr{T}\varpi(q) = \int_{s}^{t} K(q, p, \varpi(p))d(p), \tag{7.1.2}$$

$\forall\ \varpi \in \mathscr{E}_1, q, p \in [\eta_1, \eta_2]$.

Theorem 7.1.1. *Suppose that $\mathscr{E}_1 = C[\eta_1, \eta_2]$ equipped with $\mathcal{F}$-metric $\mathscr{D}(\varpi, \vartheta) = \|\varpi - \vartheta\|_\infty$ and $K : [\eta_1, \eta_2] \times [\eta_1, \eta_2] \times \mathbb{R} \to \mathbb{R}$ is a continuous map fulfilling the following axiom*

$$\mid K(q, p, \varpi(p)) - K(q, p, \vartheta(p)) \mid^2 \leq F(q, p) \ln\left(\frac{\mid \varpi(p) - \vartheta(p) \mid^2}{4} + 1\right), \tag{7.1.3}$$

where $F : [\eta_1, \eta_2] \times [\eta_1, \eta_2] \to \mathbb{R}$ is a continuous map such that

$$\int_{\eta_1}^{\eta_2} F(q,p)dp \leq \frac{1}{\eta_1 - \eta_2}, \quad \forall\ \varpi \in \mathscr{E}_1, p, q \in [\eta_1, \eta_2].$$

Then, (7.1.1) has a solution in $\mathscr{E}_1$.

Proof. Suppose that $\varpi, \vartheta \in \mathscr{E}_1$. Using Cauchy Schwarz inequality and equation (7.1.3), we have

$$\begin{aligned}
\mid \mathscr{T}\varpi(q) - \mathscr{T}\vartheta(q) \mid^2 &= \left(\int_{\eta_1}^{\eta_2} \mid K(q,p,\varpi(p)) - K(q,p,\vartheta(p)) \mid dp \right)^2 \\
&\leq \int_{\eta_1}^{\eta_2} 1^2 da \int_{\eta_1}^{\eta_2} \mid K(q,p,\varpi(p)) - K(q,p,\vartheta(p)) \mid^2 dp \\
&\leq (\eta_1 - \eta_2) \int_{\eta_1}^{\eta_2} F(q,p) \ln \left(\frac{\mid \varpi(p) - \vartheta(a) \mid^2}{4} + 1 \right) dp \\
&= (\eta_1 - \eta_2) \int_{\eta_1}^{\eta_2} F(q,p) \ln \left(\frac{\mathscr{D}(\varpi,\vartheta)^2}{4} + 1 \right) dp \\
&= (\eta_1 - \eta_2) \ln \left(\frac{\mathscr{D}(\varpi,\vartheta)^2}{4} + 1 \right) \int_{\eta_1}^{\eta_2} F(q,p) dp \\
&\leq \ln \left(\frac{\mathscr{D}(\varpi,\vartheta)^2}{4} + 1 \right) \\
&\leq \frac{\mathscr{D}(\varpi,\vartheta)^2}{4}.
\end{aligned}$$

So, we get

$$\mathscr{D}(\mathscr{T}\varpi, \mathscr{T}\vartheta) \leq \frac{\mathscr{D}(\varpi,\vartheta)}{2}.$$

Further, after rewrite, we have $\varphi(\mathscr{D}(\mathscr{T}\varpi, \mathscr{T}\vartheta)) \leq k\varphi(\mathscr{D}(\varpi,\vartheta))$, where $\varphi(t) = t$ is non-decreasing function and $k = \frac{1}{2}$. Therefore by Theorem 2.1.1 $\mathscr{T}$ has a fixed point. ■

Theorem 7.1.2. *Assume that all of Theorem 7.1.1's hypotheses are true. Then the following conditions hold:*
(A1) (7.1.1) is HU stable, that is, if for each $\varepsilon > 0$ and each solution ϑ^* of the inequality $\mathscr{D}(\vartheta, \mathscr{T}(\vartheta)) < \frac{\varepsilon}{2}$, for each $\vartheta \in \mathscr{E}_1$, there exists a solution ϖ^* of (7.1.1) such that

$$\mathscr{D}(\vartheta^*, \varpi^*) < \varepsilon.$$

(A2) If $\{\varpi_n\}$ is a sequence in $\mathscr{E}_1$ such that $\lim_{n\to\infty} \mathscr{D}(\mathscr{T}\varpi_n, \varpi_n) = 0$ and ϖ^* is a unique fixed point of $\mathscr{T}$ then (7.1.1) is a well-posed.

Proof. (A1) Using Theorem 7.1.1, there is a unique $\varpi^* \in \mathscr{E}_1$ such that $\varpi^* = \mathscr{T}\varpi^*$ that is $\varpi^* \in \mathscr{E}_1$ is solution of the integral equation ($\varpi = \mathscr{T}\varpi$). Assume that $\varepsilon > 0$ and $\vartheta^* \in \mathscr{E}_1$. Using ([56], D_3, pp. 3), we get

$$\begin{aligned} f(\mathscr{D}(\vartheta^*, \varpi^*)) &\leq f[\mathscr{D}(\vartheta^*, \mathscr{T}\vartheta^*) + \mathscr{D}(\mathscr{T}\vartheta^*, \varpi^*)] + \alpha \\ &\leq f\left[\frac{\varepsilon}{2} + \mathscr{D}(\mathscr{T}\vartheta^*, \mathscr{T}\varpi^*)\right] + \alpha. \end{aligned}$$

Hence using property of (Θ_1), we have

$$\begin{aligned} \mathscr{D}(\vartheta^*, \varpi^*) &\leq \frac{\varepsilon}{2} + \mathscr{D}(\mathscr{T}\vartheta^*, \mathscr{T}\varpi^*), \\ &\leq \frac{\varepsilon}{2} + \frac{\mathscr{D}(\vartheta^*, \varpi^*)}{2}. \end{aligned}$$

So we have $\mathscr{D}(\vartheta^*, \varpi^*) \leq \varepsilon$, which completes the proof.

(A2) If $\{\xi_r\}$ is a sequence in $\mathscr{E}_1$ such that $\lim_{r\to\infty} \mathscr{D}(\mathscr{T}\xi_r, \xi_r) = 0$ and ϖ^* is a unique fixed point of $\mathscr{T}$ (using Theorem 7.1.1). From the contractive condition and property ([56], D_3, pp. 3), we get

$$\begin{aligned} f(\mathscr{D}(\xi_r, \varpi^*)) &\leq f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \mathscr{D}(\mathscr{T}\xi_r, \varpi^*)] + \alpha \\ &\leq f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \mathscr{D}(\mathscr{T}\xi_r, \mathscr{T}\varpi^*)] + \alpha. \end{aligned}$$

Hence using property of (Θ_1), we have $\frac{1}{2}(\mathscr{D}(\xi_r, \varpi^*)) \leq \mathscr{D}(\xi_r, \mathscr{T}\xi_r)$. Taking limit $r \to \infty$, we get

$$\lim_{r\to\infty} \frac{1}{2}(\mathscr{D}(\xi_r, \varpi^*)) \leq \lim_{r\to\infty} \mathscr{D}(\xi_r, \mathscr{T}\xi_r).$$

Therefore, $\lim_{r\to\infty} \frac{1}{2}(\mathscr{D}(\xi_r, \varpi^*)) = 0$. Hence $\mathscr{D}(\xi_r, \varpi^*) = 0$. This shows that integral equation (7.1.1) is well-posed. ■

7.2 Application to nonlinear fractional integro-differential equation

Caputo derivative of a continuous map $g : [0, \infty) \to \mathbb{R}$, (order $\delta > 0$) is provided by

$${}^{C}D^{\delta} g(\eta_2) := \frac{1}{\Gamma(n-\delta)} \int_0^{\eta_2} \frac{g^{(n)}(\eta_1)\, d\eta_1}{(\eta_2 - \eta_1)^{\delta-n+1}}, \ n = [\delta] + 1, n - 1 \leq \delta < n,$$

where $[\delta]$ denotes the integer part of the positive real number δ and Γ denotes the gamma function.
Using the results proved in Chapter 3, we investigate the nonlinear fractional integro-differential equation of Caputo type. The findings of this section have been published in Sharma and Chandok [109].

$$\begin{cases} {}^{C}D^{\delta}\varpi(\eta_2) = \mathscr{G}(\eta_2, \varpi(\eta_2)), \eta_2 \in I = [0,1], 1 < \delta \leq 2 \\ \varpi(0) = 0, \varpi(1) = \int_0^{\theta} \varpi(\eta_1)\, d\eta_1, \end{cases} \tag{7.2.1}$$

where $\varpi \in (C[0,1],\mathbb{R})$, $\theta \in (0,1)$ and $\mathscr{G} : I \times \mathbb{R} \to \mathbb{R}$ is a continuous map (for more information, see [15]).
We consider $\mathscr{E}_1 = \{\varpi : \varpi \in (C[0,1],\mathbb{R})\}$ with supremum norm $||\varpi|| = \sup_{t\in[0,1]} |\varpi(t)|$. So $(\mathscr{E}_1, ||.||)$ is a Banach space.
$\mathscr{E}_1 := C([0,1],\mathbb{R})$ fitted with metric $d : \mathscr{E}_1 \times \mathscr{E}_1 \to [0,\infty)$ defined as $d(\varpi,\vartheta) = ||\varpi - \vartheta|| = \sup_{\eta_2\in[0,1]} |\varpi(\eta_2) - \vartheta(\eta_2)|$ and define an orthogonal relation $\varpi \perp \vartheta$ if and only if $\varpi\vartheta \geq 0$, $\forall\ \varpi, \vartheta \in \mathscr{E}_1$. Then $(\mathscr{E}_1, \perp, d)$ is an OMS.
It is clear that a solution of (7.2.1) is a fixed point of

$$\begin{aligned} &\mathscr{T}\varpi(\eta_2) \\ &= \frac{1}{\Gamma(\delta)} \int_0^{\eta_2} (\eta_2 - \eta_1)^{\delta-1} \mathscr{G}(\eta_1, \varpi(\eta_1))\, d\eta_1 - \frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)} \int_0^1 (1-\eta_1)^{\delta-1} \mathscr{G}(\eta_1, \varpi(\eta_1))\, d\eta_1 \\ &\quad + \frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)} \int_0^{\theta} \left(\int_0^{\eta_1} (\eta_1 - m)^{\delta-1} \mathscr{G}(s, \varpi(m))\, dm \right) ds. \end{aligned} \tag{7.2.2}$$

Theorem 7.2.1. *Suppose that $\mathscr{G} : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and fulfilling*

$$|\mathscr{G}(\eta_1, \varpi(\eta_1)) - \mathscr{G}(\eta_1, \vartheta(\eta_1))| \leq \frac{\Gamma(\delta+1)}{5} e^{-\tau}\ |\varpi(\eta_1) - \vartheta(\eta_1)|,$$

for each $\eta_1 \in [0,1]$, $\forall\ \varpi, \vartheta \in C([0,1],\mathbb{R})$ and for some $\tau > 0$. Then (7.2.1) with given boundary conditions has a solution.

Proof. $\mathscr{E}_1 := C([0,1],\mathbb{R})$ fitted with metric $d : \mathscr{E}_1 \times \mathscr{E}_1 \to [0,\infty)$ defined as $d(\varpi,\vartheta) = \sup_{t\in[0,1]} |\varpi(t) - \vartheta(t)|$, $\forall\ \varpi, \vartheta \in \mathscr{E}_1$. Define an orthogonal relation $\varpi \perp \vartheta$ if and only if $\varpi\vartheta \geq 0$, $\forall\ \varpi, \vartheta \in \mathscr{E}_1$. Then $(\mathscr{E}_1, \perp, d)$ is an OMS. Define $\mathscr{T} : \mathscr{E}_1 \to \mathscr{E}_1$ as in (7.2.2). Thus $\mathscr{T}$ is $\perp$-continuous. First, we demonstrate that $\mathscr{T}$ is $\perp$-preserving, assume that

$\varpi(\eta_2) \perp \vartheta(\eta_2)\ \forall\ \eta_2 \in [0,1]$. Now, from (7.2.2), we find

$$\begin{aligned}
&\mathscr{T}\varpi(\eta_2)\\
=&\frac{1}{\Gamma(\delta)}\int_0^{\eta_2}(\eta_2-\eta_1)^{\delta-1}\mathscr{G}(s,\varpi(\eta_1))\,d\eta_1-\frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)}\int_0^1(1-\eta_1)^{\delta-1}\mathscr{G}(s,\varpi(\eta_1))\,ds\\
&+\frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^{\eta_1}(\eta_1-m)^{\delta-1}\mathscr{G}(\eta_1,\varpi(m))\,dm\right)\,d\eta_1>0,
\end{aligned}$$

that means $\mathscr{T}\varpi \perp \mathscr{T}\vartheta$.

Now, for $F(r)=\ln r, r>0$, we have to prove that $\mathscr{T}$ fulfills (i) of Corollary 2.3.6. For all $\eta_2 \in [0,1]$, $\varpi(\eta_2) \perp \vartheta(\eta_2)$, we have

$$\begin{aligned}
&\left|\mathscr{T}\varpi(\eta_2)-\mathscr{T}\vartheta(\eta_2)\right|\\
&=\Bigg|\frac{1}{\Gamma(\delta)}\int_0^{\eta_2}(\eta_2-\eta_1)^{\delta-1}\mathscr{G}(\eta_1,\varpi(\eta_1))\,d\eta_1-\frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)}\int_0^1(1-\eta_1)^{\delta-1}\mathscr{G}(\eta_1,\varpi(\eta_1))\,d\eta_1\\
&+\frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^{\eta_1}(\eta_1-m)^{\delta-1}\mathscr{G}(\eta_1,\varpi(m))\,dm\right)\,d\eta_1\\
&-(\frac{1}{\Gamma(\delta)}\int_0^{\eta_2}(\eta_2-\eta_1)^{\delta-1}\mathscr{G}(\eta_1,\vartheta(\eta_1))\,d\eta_1-\frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)}\int_0^1(1-\eta_1)^{\delta-1}\mathscr{G}(\eta_1,\vartheta(\eta_1))\,d\eta_1\\
&+\frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^{\eta_1}(\eta_1-m)^{\delta-1}\mathscr{G}(\eta_1,\vartheta(m))\,dm\right)\,d\eta_1)\Bigg|\\
&\leq\frac{1}{\Gamma(\delta)}\int_0^{\eta_2}(\eta_2-\eta_1)^{\delta-1}|\mathscr{G}(\eta_1,\varpi(\eta_1))-\mathscr{G}(\eta_1,\vartheta(\eta_1))|\,d\eta_1\\
&-\frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)}\int_0^1(1-\eta_1)^{\delta-1}|\mathscr{G}(\eta_1,\varpi(\eta_1))-\mathscr{G}(\eta_1,\vartheta(\eta_1))|\,ds\\
&+\frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^{\eta_1}(\eta_1-m)^{\delta-1}|\mathscr{G}(\eta_1,\varpi(m))-\mathscr{G}(\eta_1,\vartheta(m))|\,dm\right)\,d\eta_1\\
&\leq\frac{1}{\Gamma(\delta)}\int_0^{\eta_2}(\eta_2-\eta_1)^{\delta-1}\left[\frac{\Gamma(\delta+1)}{5}e^{-\tau}\sup_{\eta_1\in[0,1]}|\varpi(\eta_1)-\vartheta(\eta_1)|\right]\,d\eta_1\\
&-\frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)}\int_0^1(1-\eta_1)^{\delta-1}\left[\frac{\Gamma(\delta+1)}{5}e^{-\tau}\sup_{s\in[0,1]}|\varpi(\eta_1)-\vartheta(\eta_1)|\right]\,d\eta_1\\
&+\frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^{\eta_1}(\eta_1-m)^{\delta-1}\left[\frac{\Gamma(\delta+1)}{5}e^{-\tau}\sup_{s\in[0,1]}|\varpi(\eta_1)-\vartheta(\eta_1)|\right]\,dm\right)\,d\eta_1
\end{aligned}$$

$$\leq \left[\frac{\Gamma(\delta+1)}{5} e^{-\tau} \sup_{\eta_1 \in [0,1]} |\varpi(\eta_1) - \vartheta(\eta_1)|\right] \times \sup_{\eta_2 \in [0,1]} (\frac{1}{\Gamma(\delta)} \int_0^{\eta_2} (\eta_2 - \eta_1)^{\delta-1} \, d\eta_1$$

$$- \frac{2\eta_2}{(2-\theta^2)\Gamma(\delta)} \int_0^1 (1-\eta_2)^{\delta-1} \, d\eta_2 + \frac{2\eta_1}{(2-\theta^2)\Gamma(\delta)} \int_0^{\theta} \left(\int_0^{\eta_1} (\eta_1 - m)^{\delta-1} dm \right) d\eta_2)$$

$$\leq e^{-\tau} \sup_{\eta_1 \in [0,1]} |\varpi(\eta_1) - \vartheta(\eta_1)|$$

$$= e^{-\tau} d(\varpi, \vartheta),$$

$\forall\, \varpi, \vartheta \in \mathscr{E}_1$. Hence, (i) of Corollary 2.3.6 holds. Accordingly, all assumptions of Corollary 2.3.6 are fulfilled, and $\mathscr{T}$ has a fixed point. It demonstrates that (7.2.1) possesses a solution. ■

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