BIVARIATE EXTENSION OF DURRMEYER OPERATORS BY D. D. STANCU

A Thesis

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Shivani Rani

Roll No. 301703029

Under the guidance of

Dr. Meenu Rani

THAPAR INSTITUTE
OF ENGINEERING & TECHNOLOGY
(Deemed to be University)

School of Mathematics

Thapar Institute of Engineering and Technology

Patiala- 174004(Punjab)

INDIA

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**Bivariate extension of Durrmeyer operators by D. D. Stancu**" in partial fulfillment of the requirement for the award of degree of Master of Science, School of Mathematics (SOM), Thapar Institute of Engineering and Technology, Patiala is an authentic record of my own carried out under the supervision of Dr. Meenu Rani, Assistant Professor, SoM, Thapar Institute of Engineering and Technology, Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Shivani Rani
(SHIVANI RANI)
Reg. No. 301703029

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

Date: July 31, 2019

(**Dr. Meenu Rani**)
Supervisor

Abstract

In this report, we review some basic definitions related to approximation theory. Then, we study some approximation properties and rate of convergence for certain bivariate linear positive operators.
In chapter 1, we recall some linear positive operators in two variables i.e Bernstein polynomials, Schurer-Stancu operators, Baskakov Kantorovich operators, and properties of the Baskakov-Kantorovich operators, Generalized Baskakov-Kantorovich operators, Durrmeyer operators etc. We review the main results for these operators.

In chapter 2, we investigated the Bivariate extension of Durrmeyer operators by D.D. Stancu. We obtain auxiliary results for these operators. Then, we study the rate of convergence in terms of second order modulus of continuity, basic convergence theorem and asymptotic formula for these operators.

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Contents

Chapter 1

Introduction

1.0.1 Bernstein polynomial in two variables

The Bernstein polynomials for functions of two variables of class $C^{(k)}$ is defined in [1]. Let $\phi(x, y)$ be a continuous function in a closed region $R : 0 \leq x \leq 1, 0 \leq y \leq 1$. The Bernstein polynomials $B_{m,n}(x, y)$ associated with the function $\phi(x, y)$ is given by:

$$B_{m,n}(x, y) = \sum_{p=0}^{n} \sum_{q=0}^{m} \phi\left(\frac{p}{n}, \frac{q}{m}\right) \lambda_{n,p}(x) \lambda_{m,q}(y).$$

For fixed $m \in \mathbb{N}$, let $C^m(R)$ be the space of all continuous function f and having the partial derivatives $\frac{\partial^k}{\partial x^s \partial y^{k-s}} \in C(R), s = 1, 2, ...k; k = 1, 2, ...m.$
Taking into account that

$$\sum_{p=0}^{n} \lambda_{n,p}(x) = 1.$$

and

$$\sum_{p=0}^{n} (nx - p)^2 \lambda_{n,p}(x) = nx(1 - x),$$

if we define for $K \geq 0, i = 0, 1, 2, ...K,$

$$A_{p,q}^{i,k-i} = \sum_{\alpha=0}^{i}\sum_{\beta=0}^{k-i}(-1)^{\alpha+\beta}.$$

Lemma 1.0.1 *If $f(x,y)$ is bounded in the square R, then at every point of continuous (x,y) of f*

$$\lim_{n_1,n_2\to\infty} B_{n_1,n_2}^{f}(x,y) = f(x,y),$$

the result holding uniformly in x and y if $f(x,y)$ is continuous in R.

Lemma 1.0.2 *If all the partial derivatives of $f(x,y)$ of order $\leq p$ exist and are continuous in R, then*

$$\frac{\partial^p}{\partial x^q \partial y^{p-q}} B_{n_1,n_2}^{f}(x,y) \to \frac{\partial^p}{\partial x^q \partial y^{p-q}} f(x,y),$$

uniformly in R as n_1, n_2 approach infinity.

In 1989, Martinez [3] studied some approximation properties of two dimensional Bernstein polynomials. The convergence of the polynomials with integral coefficients $B_{m,n}^{(i,k-i),e} f$ to $f_{i,k-i}^{(k)}$ both in the uniform and L_p norm. Here the superscript "e" denotes a polynomial with integral coefficients sense and $\| \, . \, \|$ denotes the $L_p[R]$ norm $(1 \leq p \leq \infty)$.

Continuity of two variable function

A function $f(x,y)$ is continuous at (x_0,y_0) for every $\epsilon > 0 \;\exists\; \delta > 0$ such that

$$|f(x,y) - f(x_0,y_0)| < \epsilon \quad \forall \; |x - x_0| < \delta \;\text{ and }\; |y - y_0| < \delta.$$

A function $f(x,y)$ is said to be continuous at (x_0,y_0) if the simultaneous limits exist and is equal to it's functional value $f(x_0,y_0)$ at (x_0,y_0).

1.0.2 Convergence:-

Pointwise convergence in two variables

A sequence of functions $\{f_n(x,y)\}$ is said to convergence pointwise to $f(x,y)$ in $I = [a,b]\times[a,b]$ if for each $\epsilon > 0$, and each $(x,y) \in I$ exists a natural number N (depending on ϵ and (x,y) such that $|f_n(x,y) - f(x,y)| < \epsilon \;\forall\; n \geqslant N$.

Uniform convergence in two variables

A sequence of function $f_n(x,y)$ is said to converge uniformly in $[a,b]\times[a,b]$ to a function $f(x,y)$ if for each $\epsilon>0$ $\exists$ a natural number N (independent of (x,y) but dependent of ϵ) such that:-

$$|f_n(x,y)-f(x,y)|<\epsilon \quad \forall n\geqslant \mathbb{N} \quad \forall (x,y)\in I.$$

Modulus of Continuity

The modulus of continuity of a function $f(x,y)$ is defined by

$$\omega(\delta_1,\delta_2)=\sup|f(x_2,y_2)-f(x_1,y_1)|,$$

where $\delta_1\geq 0,\delta_2\geq 0$ are real numbers, whereas (x_1,y_1) and (x_2,y_2) are points of $\triangle : x>0, y>0, y-x>0$ in manner that:$|x_2-x_1|\leq\delta_1, |y_2-y_1|\leq\delta_2$.

Alternately, the complete modulus of continuity of f which we denoted by $\omega(f;\delta)$, is defined as

$$\omega(f;\delta)=\sup_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\leq\delta}|f(x_1,y_1)-f(x_2,y_2)|.$$

Taking into account that on $\triangle$ we have

$$\lambda_m(x,y)\geq 0, \sum_{i=0}^{m}\sum_{j=0}^{m-i}\lambda_m^{i,j}(x,y)=1,$$

$$|f(x_2,y_2)-f(x_1,y_1)|\leq\omega(|x_2-x_1|,|y_2-y_1|)\leq\omega(\delta_1,\delta_2)$$

and the inequality

$$\omega(\lambda_1\delta_1,\lambda_2\delta_2)\leq(\lambda_1+\lambda_2+1)\omega(\delta_1,\delta_2), \lambda_1\geq 0, \lambda_2\geq 0,$$

Partial moduli of continuity

The partial moduli of continuity of f is represented by:

$$\omega^{(1)}(f;\delta) = \omega(f;\delta,0) = sup_y sup_{|x_1-x_2|\leq\delta}|f(x_1,y) - f(x_2,y)|$$

$$\omega^{(2)}(f;\delta) = \omega(f;\delta,0) = sup_x sup_{|y_1-y_2|\leq\delta}|f(x,y_1) - f(x,y_2)|$$

and $\omega(f;\delta,\epsilon)$ is the complete modulus of continuity of f.

Complete modulus of continuity

The complete modulus of continuity $\omega(f;\delta,\epsilon)$ of the function f is connected with its partial moduli of continuity $\omega(f;\delta,0)$ and $\omega(f;0,\epsilon)$ by the inequalities

$$\omega(f;\delta,\epsilon) \leq \omega(f;\delta,0) + \omega(f;0,\epsilon) \leq 2\omega(f;\delta,\epsilon).$$

Pop [2] obtained the rate of convergence in terms of modulus of continuity.

Schurer-Stancu Operators

Barbosu [4] invented the Schurer-Stancu bivariate operators. The Bernstein Schurer-Stancu operators $S_{m,p}^{\alpha,\beta} : C[0,1+p] \to C[0,1]$, represented any $m \in \mathbb{N}$ and any $f \in C[0,1+p]$ by

$$S_{m,p}^{\alpha,\beta}(f;x) = \sum_{k=0}^{m+p} \lambda_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right),$$

where $\lambda_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}$.

For $\alpha = \beta = 0$, these operators are to similar to Schurer operators and for $p = 0$, the operators similar to Stancu operators.

The bivariate Schurer-Stancu type operators $S_{m,n,p,q}^{\alpha_1,\beta_1,\alpha_2,\beta_2} : C([0,1+p]\times[0,1+q]) \to C([0,1]\times[0,1])$, where $p, q \geq 0$ are given integers and $0 \leq \alpha_1 \leq \beta_1, 0 \leq \alpha_2 \leq \beta_2$ are real parameters, let

$$S_{m,p}^{\alpha_1,\beta_1} : C([0,1+p]) \to C([0,1]),$$

$$S_{n,p}^{\alpha_2,\beta_2} : C([0,1+q]) \to C([0,1])$$

The parametric extension of Schurer-Stancu operator is defined by

$$x^{S_{m,p}^{\alpha_1,\beta_1}} y^{S_{n,q}^{\alpha_2,\beta_2}} : C([0,1+p] \times [0,1+q])$$

$$(x^{S_{m,p}^{(\alpha_1,\beta_1)}}f)(x,y) : \sum_{k=0}^{m+p} \lambda_{m,k}(x) f((k+\alpha_1)/(m+\beta_1), y),$$

$$(y^{S_{n,q}^{(\alpha_2,\beta_2)}}f)(x,y) : \sum_{j=0}^{n+q} \lambda_{n,j}(y) f(x, (j+\alpha_2)/(n+\beta_2)).$$

Lemma 1.0.3 *The parametric extension* $(x^{S_{m,p}^{(\alpha_1,\beta_1)}}f)(x,y) : \sum_{k=0}^{m+p} \lambda_{m,k}(x) f((k+\alpha_1)/(m+\beta_1), y)$, $(y^{S_{n,q}^{(\alpha_2,\beta_2)}}f)(x,y) : \sum_{j=0}^{n+q} \lambda_{n,j}(y) f(x, (j+\alpha_2)/(n+\beta_2))$*, are linear positive operators.*

Lemma 1.0.4 *The parametric extensions* $x^{S_{m,p}^{\alpha_1,\beta_1}} y^{S_{n,q}^{\alpha_2,\beta_2}}$ *of Schurer-Stancu operator commute on* $C([0,1+p]\times[0,1+q])$*, their product of Schurer-stancu type operators is*

$$S_{m,n,p,q}^{\alpha_1,\beta_1,\alpha_2,\beta_2} : C([0,1+p]\times[0,1+q]) \to C([0,1]\times[0,1])$$

defined for any $f \varepsilon C([0,1+p]\times[0,1+q])$ *and any* $m,n \varepsilon N$ *by*

$$(S_{m,n,p,q}^{\alpha_1,\beta_1,\alpha_2,\beta_2} f)(x,y) = \sum_{k=0}^{m+p}\sum_{j=0}^{n+q} \lambda_{m,k}(x)\lambda_{n,j}(y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{n+\beta_2}\right). \tag{1.0.1}$$

Lemma 1.0.5 *The bivariate Schurer-Stancu operator (1.0.1) is linear and positive.*

Baskakov Operators

In Gurdek et al. [5] the Baskakov operators, as positive linear operators on the unbounded interval $[0,+\infty)$, are defined by

$$K_n(f;x) = \sum_{k=0}^{\infty} f\binom{k}{n} q_{n,k}(x), x \in [0,\infty),$$

where

$$q_{n,k}(x) = \frac{(n+k-1)!}{(n-1)!k!}\frac{x^k}{(1+x)^{n+k}}.$$

For the $K_n(f;x)$, there have been plentiful researches on approximation and preservation.

Properties of the Baskakov Operators

(i). (Monotonicity) If $f(x)$ is increasing, then for each $n \geq 1$, $K_n f(x)$ is increasing.

(ii). (Convexity) If $f(x)$ is convex, then for each $n \geq 1$, $K_n f(x)$ is convex.
(iii). (Monotonic approximation) If $f(x)$ is convex, then for any $x \in [0, \infty)$ and $n \in \mathbb{N}$, $K_n(f; x) \geq K_{n+1}(f; x)$.

Kantorovich Operators

for $f \in L_1[0, 1]$ (the class of a Lebesgue integrable function on $[0, 1]$, Kantorovich introduced the operators

$$K_n^*(f; x) = (n+1) \sum_{k=0}^{n} \lambda_{n,k}(x) \int_0^1 \chi_{n,k}(t) f(t) dt,$$

where

$$\lambda_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, x \in [0, 1],$$

is the Bernstein basis function and $\chi_{n,k}(t)$ is the characteristic function of the interval $\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$.

Generalized Baskakov Operators In 1998, Mihesan [7] introduced the generalised Baskakov operators $B_{n,a}^*$ defined as

$$B_{n,a}^*(f; x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) f\left(\frac{k}{k+1}\right),$$

where

$$W_{n,k}^a(x) = e^{\frac{-ax}{1+x}} \frac{P_K(n, a)}{k!} \frac{x^k}{(1+x)^{n+k}}, P_k(n, a) = \sum_{i=0}^{k} \binom{k}{i} (n)_i a^{k-i},$$

and $(n)_0 = 1, (n)_i = n(n+1).......(n+i-1)$, for $i \geq 1$.

Baskakov-Durrmeyer Operators

Erencin [6] defined the Durrrmeyer type modification of these operators as

$$L_n^a(f; x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) \frac{1}{B(K+1, n)} \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} f(t) dt, x \geq 0,$$

and discussed some approximation properties. The author studied the simultaneously approximation and approximation of function having derivatives of bounded variation by these operators.

A new kind of Durrmeyer type modification of the general Baskakov operators having weights of Szasz basis function and studied some approximation properties of these operators.

Modified Bernstein-Kantorovich operators for functions of two variables

for $f \in C(I)$, with $I = [0,1] \times [0,1]$, the classical Bernstein polynomials are represented as follows:

$$B_{n_1,n_2}(f;x,y) = \sum_{k_1=0}^{n_1} \lambda_{n_1,k_1}(x) \sum_{k_2=0}^{n_2} \lambda_{n_2,k_2}(y) f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right), (x,y)\epsilon I,$$

Also for $f : [0,1] \times [0,1] \to \mathbb{R}$ an integrable function, the classical Bernstein-Kantorovich operators are represented by

$$K_{n_1,n_2}(f;x,y) = \sum_{k_1=0}^{n_1} \lambda_{n_1,k_1}(x) \sum_{k_2=0}^{n_2} \lambda_{n_2,k_2}(y) \int_{\frac{k_1}{(n_1+1)}}^{\frac{(k_1+1)}{(n_1+1)}} \int_{\frac{k_2}{(n_2+1)}}^{\frac{(k_2+1)}{(n_2+1)}} f(t,s)dt\, ds, (x,y) \in I, n \in \mathbb{N}.$$

Stancu [8] first introduced new linear positive operators in two and several dimensional variables.

Many papers were published on approximation by modified Szasz-Mirakyan and Baskakov operators for functions of one or two variables cf. [9] which deal with convergence, degree of approximation and Voronovskaja type theorems as well as convergence of partial derivatives of these operators. Wafi and Khatoon [10] defined the generalized Baskakov operators for functions of two variables in polynomial and exponential weighted spaces and discussed the rate of convergence and direct results. Later, in [11], the convergence of first order derivatives of these operators and a Voronovskaja type theorem were studied. Very recently, Agrawal et al. [12] constructed a bivariate extension of Bernstein-Schurer-Kantorovich operators and discussed the rate of convergence and the asymptotic approximation.

Chapter 2

Bivariate extension of Durrmeyer operators by D.D. Stancu

The Bernstein polynomials B_n are given by:-

$$(B_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\binom{\nu}{n},$$

where $p_{n,\nu}(x)$ denote the Bernstein basis polynomials which are given as:-

$$p_{n,\nu}(x) = \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}.$$

If a function f is bounded on $[0,1]$ then $(B_n f)(x) \to f(x)$ pointwise and if $f \epsilon C[0,1]$, then $(B_n f)(x) \to f(x)$ uniformly in $f \epsilon C[0,1]$.

In 1984, D. D. Stancu gave the following Bernstein-type operators by using the parameters $r, s \epsilon N_0$:-

$$(S_{n,r,s} f)(x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) f\left(\frac{\mu+\nu r}{n}\right). \tag{2.0.1}$$

If $r = s = 0$, in equation (2.0.1) reduces to Bernstein polynomials B_n then, $S_{n,0,0} = B_n$. Now, the Durrmeyer variant of the operators (2.0.1) are defined as:

$$(G_{n,r,s} f)(x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{\nu=0}^{s} p_{s,\nu}(x)(n+1) \int_0^1 p_{n,\mu+\nu r}(t) f(t)dt,$$

$$(G_{n,r,s}f)(x) = (n+1)\sum_{\mu=0}^{n-sr} Q_{n-sr,\mu,s,\nu}(x)\int_0^1 p_{n,\mu+\nu r}(t)f(t)dt.$$

where $Q_{n-sr,\mu,s,\nu}(x) = p_{n-sr,\mu}(x)\sum_{\nu=0}^{s} p_{s,\nu}(x)$,
Now, we introduce the bivariate modification of Durrmeyer operators:-

$$\begin{aligned}(G^{s_1,s_2}_{n_1,n_2,r_1,r_2}f)(x,y) &= (n_1+1)(n_2+1)\sum_{\mu_1=0}^{n_1-s_1r_1}\sum_{\mu_2=0}^{n_2-s_2r_2} Q^{s_1,s_2,\mu_1,\mu_2}_{n_1-s_1r_1,n_2-s_2r_2}(x,y) \\ &\quad\times\int_0^1\int_0^1 P_{n_1,n_2,\mu_1+\nu_1r_1,\mu_2+\nu_2r_2}(u,v)dudv \qquad (2.0.2)\end{aligned}$$

where:-

$$Q^{s_1,s_2,\mu_1,\mu_2}_{n_1-s_1r_1,n_2-s_2r_2}(x,y) = p_{n_1-s_1r_1,\mu_1}(x)\sum_{\nu_1=0}^{s_1} p_{s_1,\nu_1}(x)p_{n_2-s_2r_2,\mu_2}(y)\sum_{\nu_2=0}^{s_2} p_{s_2,\nu_2}(y).$$

2.1 Auxiliary Results

Lemma 2.1.1 *Let $e_l(l = 0,1,2.....)$denote the monomials given by $e_l(x) = x^l$ and for $x\epsilon\mathbb{R}$, we have*

(i) $(G^{s_1,s_2}_{n_1,n_2,r_1,r_2}e_{00})(1;x,y) = 1,$

(ii) $(G^{s_1,s_2}_{n_1,n_2,r_1,r_2}e_{10})(u;x,y) = \frac{1}{n_1+2}[x(n_1-s_1\nu_1)+xr_1+1];$

(iii) $(G^{s_1,s_2}_{n_1,n_2,r_1,r_2}e_{01})(v;x,y) = \frac{1}{n_2+2}[y(n_2-s_2\nu_2)+yr_2+1];$

(iv) $(G^{s_1,s_2}_{n_1,n_2,r_1,r_2}e_{20})(u^2;x,y) = \frac{1}{(n_1+3)(n_1+2)}[x^2(n_1-s_1\nu_1)(n_1-s_1\nu_1-1)+x(n_1-s_1\nu_1)+$ $2xr_1+4x(n_1-s_1\nu_1)+x^2(s_1)(s_1-1)r_1^2+3xr_1+2];$

(v) $(G^{s_1,s_2}_{n_1,n_2,r_1,r_2}e_{02})(v^2;x,y) = \frac{1}{(n_2+3)(n_2+2)}[y^2(n_2-s_2\nu_2)(n_2-s_2\nu_2-1)+y(n_2-s_2\nu_2)+$ $2yr_2+4y(n_2-s_2\nu_2)+y^2(s_2)(s_2-1)r_2^2+3yr_2+2].$

Proof (i) Let $f(u,v) = 1$ in equation (2.0.2), then we have

$G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(u;x,y)=$

$$(n_1+1)(n_2+1)\sum_{\mu_1=0}^{n_1-s_1r_1}\sum_{\mu_2=0}^{n_2-s_2r_2} Q^{s_1,s_2,\mu_1,\mu_2}_{n_1-s_1r_1,n_2-s_2r_2}(x,y)\int_0^1 P_{n_1,\mu_1+\nu_1r_1,}1.udu\int_0^1 P_{n_2,\mu_2+\gamma_2r_2}(v)dv=1$$

Proof (ii) Let $f(u,v)=u$ in equation (2.0.2), then we have
$G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(u;x,y)$

$$=(n_1+1)(n_2+1)\sum_{\mu_1=0}^{n_1-s_1r_1}\sum_{\mu_2=0}^{n_2-s_2r_2} Q^{s_1,s_2,\mu_1,\mu_2}_{n_1-s_1r_1,n_2-s_2r_2}(x,y)\int_0^1 P_{n_1,\mu_1+\nu_1r_1,}(u)udu\int_0^1 P_{n_2,\mu_2+\nu_2r_2}(v)dv \tag{2.1.1}$$

Now,

$$\int_0^1 P_{n_1,\mu_1+\nu_1r_1}(u)udu=\int_0^1\binom{n_1}{\mu_1+\nu_1r_1}u^{\mu_1+\nu_1r_1+1}(1-u)^{n_1-(\mu_1+\nu_1r_1)}du$$

$$\begin{aligned}
&= \binom{n_1}{\mu_1+\nu_1r_1}B(\mu_1+\nu_1r_1+2,n_1-\mu_1-\nu_1r_1+1)\\
&= \frac{n_1!(\mu_1+\nu_1r_1+1)!(n_1-\mu_1-\nu_1r_1)!}{(\mu_1+\nu_1r_1)!(n_1-\mu_1-\nu_1r_1)!(\mu_1+\nu_1r_1+2+n_1-\mu_1-\nu_1r_1)!}\\
&= \frac{n_1!(\mu_1+\nu_1r_1+1)!}{(\mu_1+\nu_1r_1)!(n_1+2)!}=\frac{n_1!(\mu_1+\nu_1r_1+1)(\mu_1+\nu_1r_1)!}{(\mu_1+\nu_1r_1)!(n_1+2)!}\\
&= \frac{n_1!(\mu_1+\nu_1r_1+1)}{(n_1+2)!}=\frac{(\mu_1+\nu_1r_1+1)}{(n_1+2)(n_1+1)}.
\end{aligned}$$

Putting the above values in the eqn (2.1.1)

$$G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(u;x,y) = (n_1+1)(n_2+1)\sum_{\mu_1=0}^{n_1-s_1\nu_1}\sum_{\mu_2=0}^{n_2-s_2\nu_2} Q^{s_1,s_2,\mu_1,\mu_2}_{n_1-s_1\nu_1,n_2-s_2\nu_2}(x,y)\frac{(\mu_1+\nu_1r_1+1)}{(n_1+2)(n_1+1)}.$$

$$=\frac{(n_2+1)}{(n_1+2)}\sum_{\mu_1=0}^{n_1-s_1\nu_1} Q^{s_1,\mu_1}_{n_1-s_1\nu_1}(x)(\mu_1+\nu_1r_1+1)\sum_{\mu_2=0}^{n_2-s_2\nu_2} Q^{s_2,\mu_2}_{n_2-s_2r_2}(y)$$

where

$$\sum_{\mu_2=0}^{n_2-s_2\nu_2} Q_{n_2-s_2r_2}^{s_2,\mu_2}(y) = 1$$

$$\begin{aligned}
G_{n_1,n_2,r_1,r_2}^{s_1,s_2}(u;x,y) &= \frac{1}{(n_1+2)} \sum_{\mu_1=0}^{n_1-s_1\nu_1} Q_{n_1-s_1\nu_1}^{s_1,\mu_1}(x)(\mu_1+\nu_1 r_1+1) \\
&= \frac{1}{(n_1+2)}\Bigg(\sum_{\mu_1=0}^{n_1-s_1\nu_1} P_{n_1-s_1\nu_1,\mu_1}(x) \sum_{\nu_1=0}^{s_1} P_{s_1,\nu_1}(x)(\mu) \\
&\quad + P_{n_1-s_1\nu_1,\mu_1}(x) \sum_{\nu_1=0}^{s_1} P_{s_1,\nu_1}(x)(\nu_1 r_1) + P_{n_1-s_1\nu_1,\mu_1}(x) \sum_{\nu_1=0}^{s_1} P_{s_1,\nu_1}(x)(1)\Bigg) \qquad (2.1.2) \\
&= \frac{1}{(n_1+2)}(S_1+S_2+S_3),
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \frac{1}{(n_1+2)} \sum_{\mu_1=0}^{n_1-s_1\nu_1} P_{n_1-s_1\nu_1,\mu_1}(x) \sum_{\nu_1=0}^{s_1} P_{s_1,\nu_1}(x)(\mu_1) \\
&= \frac{1}{(n_1+2)} \sum_{\mu_1=1}^{n_1-s_1\nu_1} \binom{n_1-s_1\nu_1}{\mu_1} x^{\mu_1}(1-x)^{n_1-s_1\nu_1-\mu_1} \sum_{\nu_1=0}^{s} \binom{s}{\nu_1} x^{\nu}(1-x)^{\nu_1-s}\mu_1 \\
&= \frac{1}{(n_1+2)} \sum_{\mu_1=1}^{n_1-s_1\nu_1} \frac{(n_1-s_1\nu_1)!x^{\mu_1}(1-x)^{n_1-s_1\nu_1-\mu_1}\mu_1}{(\mu_1)!(n_1-s_1\nu_1-\mu_1)!} \\
&= \frac{1}{(n_1+2)} \sum_{\mu_1=0}^{n_1-s_1\nu_1-1} \frac{(n_1-s_1\nu_1)!x^{\mu_1+1}(1-x)^{n_1-s_1\nu_1-\mu_1-1}}{(\mu_1)!(n_1-s_1\nu_1-\mu_1-1)!} \\
&= x\frac{1}{(n_1+2)} \sum_{\mu_1=0}^{n_1-s_1\nu_1-1} (n_1-s_1\nu_1) \frac{(n_1-s_1\nu_1-1)!x^{\mu_1}(1-x)^{n_1-s_1\nu_1-1-\mu_1}}{(\mu_1)!(n_1-s_1\nu_1-1-\mu_1)!} \\
&= x\frac{1}{(n_1+2)}(n_1-s_1\nu_1) \sum_{\mu_1=0}^{n_1-s_1\nu_1-1} \binom{n_1-s_1\nu_1-1}{\mu_1} x^{\mu_1}(1-x)^{n_1-s_1\nu_1-1-\mu_1}
\end{aligned}$$

where:-

$$\sum_{\mu_1=0}^{n_1-s_1\nu_1-1} \binom{n_1-s_1\nu_1-1}{\mu_1} x^{\mu_1}(1-x)^{n_1-s_1\nu_1-1-\mu_1} = 1$$

$$S_1 = x\frac{(n_1 - s_1\nu_1)}{(n_1+2)},$$

and

$$\begin{aligned}
S_2 &= \frac{1}{(n_1+2)}\sum_{\mu_1=0}^{n_1-s_1\nu_1} P_{n_1-s_1\nu_1,\mu_1}(x)\sum_{\nu=1}^{s_1} P_{s_1,\nu_1}(x).\nu_1 r_1\\
&= \frac{1}{(n_1+2)}\sum_{\mu_1=0}^{n_1-s_1\nu_1}\binom{n_1-s_1\nu_1}{\mu_1}x^{\mu_1}(1-x)^{n_1-s_1\nu_1-\mu_1}\sum_{\nu_1=0}^{s_1}\binom{s_1}{\nu_1}x^{\nu_1}(1-x)^{\nu_1-s_1}\nu_1 r_1\\
&= \frac{1}{(n_1+2)}\sum_{\mu_1=0}^{n_1-s_1\nu_1}\frac{(n_1-s_1\nu_1)!x^{\mu_1}(1-x)^{n_1-s_1\nu_1-\mu_1}}{(\mu_1)!(n_1-s_1\nu_1-\mu_1)!}\sum_{\nu=1}^{s_1}\binom{s_1}{\nu_1}x^{\nu_1}(1-x)^{\nu_1-s_1}(\nu_1 r_1)\\
&= \frac{1}{(n_1+2)}\sum_{\mu_1=0}^{n_1-s_1\nu_1}\frac{(n_1-s_1\nu_1)!x^{\mu_1}(1-x)^{n_1-s_1\nu_1-\mu_1}}{(\mu_1)!(n_1-s_1\nu_1-\mu_1)!}\sum_{\nu_1=0}^{s_1-1}\frac{(s_1-1)!}{\nu_1!(s_1-\nu_1-1)!}x^{\nu_1+1}(1-x)^{\nu_1+1-s_1}(r_1)\\
&= xr_1\sum_{\nu_1=0}^{s_1-1}\binom{s_1-1}{\nu_1}x^{\nu_1}(1-x)^{\nu_1+1-s_1}
\end{aligned}$$

where:-

$$\sum_{\nu_1=0}^{s_1-1}\binom{s_1-1}{\nu_1}x^{\nu_1}(1-x)^{\nu_1+1-s_1} = 1$$

$$S_2 = xr_1,$$

$$S_3 = P_{n_1-s_1\nu_1,\mu_1}(x)\sum_{\nu_1=0}^{s_1} P_{s_1,\nu_1}(x)(1) = 1.$$

Putting the above values S_1, S_2, S_3 in the eqn (2.1.2) we get:-

$$G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(u;x,y) = \frac{1}{(n_1+2)}[x(n_1-s_1\nu_1)+xr_1+1].$$

Proof (iii) Similarly for $f(u,v) = v$, we have $G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(v;x,y)$.

$$= (n_1+1)(n_2+1)\sum_{\mu_1=0}^{n_1-s_1r_1}\sum_{\mu_2=0}^{n_2-s_2r_2} Q^{s_1,s_2,\mu_1,\mu_2}_{n_1-s_1r_1,n_2-s_2r_2}(x,y)\int_0^1 P_{n_1,\mu_1+\nu_1r_1,}(u)du\int_0^1 P_{n_2,\mu_2+\nu_2r_2}(v)vdv$$

$$G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(v;x,y) = \frac{1}{(n_2+2)}[y(n_2 - s_2\nu_2) + yr_2 + 1].$$

proof (iv) Let $f(u,v) = u^2$ in equation (2.0.2), then we have
$G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(u^2;x,y)$

$$= (n_1+1)(n_2+1)\sum_{\mu_1=0}^{n_1-s_1r_1}\sum_{\mu_2=0}^{n_2-s_2r_2} Q^{s_1,s_2,\mu_1,\mu_2}_{n_1-s_1r_1,n_2-s_2r_2}(x,y)\int_0^1 P_{n_1,\mu_1+\nu_1r_1,}(u)u^2du\int_0^1 P_{n_2,\mu_2+\nu_2r_2}(v)dv \tag{2.1.3}$$

Now,

$$\int_0^1 P_{n_1,\mu_1+\nu_1r_1,}(u)u^2du = \int_0^1 \binom{n_1}{\mu_1+\nu_1r_1} u^{\mu_1+\nu_1r_1+2+1-1}(1-u)^{n_1-(\mu_1+\nu_1r_1)}du$$

$$= \binom{n_1}{\mu_1+\nu_1r_1} B(\mu_1+\nu_1r_1+3, n_1-\mu_1-\nu_1r_1+1)$$

$$= \frac{n_1!(\mu_1+\nu_1r_1+2)!(n_1-\mu_1-\nu_1r_1)!}{(\mu_1+\nu_1r_1)!(n_1-\mu_1-\nu_1r_1)!(\mu_1+\nu_1r_1+3+n_1-\mu_1-\nu_1r_1+1-1)!}$$

$$= \frac{n_1!(\mu_1+\nu_1r_1+2)!}{(\mu_1+\nu_1r_1)!(n_1+3)!} = \frac{n_1!(\mu_1+\nu_1r_1+2)(\mu_1+\nu_1r_1+1)(\mu_1+\nu_1r_1)!}{(\mu_1+\nu_1r_1)!(n_1+3)(n_1+2)(n_1+1)(n)!}$$

$$= \frac{(\mu_1+\nu_1r_1+2)(\mu_1+\nu_1r_1+1)}{(n_1+3)(n_1+2)(n_1+1)}.$$

Putting above the values in the eqn (2.1.3)
$G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(u^2;x,y)$

$$= (n_1+1)(n_2+1)\sum_{\mu_1=0}^{n_1-s_1r_1}\sum_{\mu_2=0}^{n_2-s_2r_2} Q^{s_1,s_2,\mu_1,\mu_2}_{n_1-s_1r_1,n_2-s_2r_2}(x,y)\frac{(\mu_1+\nu_1r_1+2)(\mu_1+\nu_1r_1+1)}{(n_1+3)(n_1+2)(n_1+1)}$$

$$= \frac{(n_2+1)}{(n_1+2)}\sum_{\mu_1=0}^{n_1-s_1\nu_1} Q^{s_1,\mu_1}_{n_1-s_1\nu_1}(x)(\mu_1+\nu_1r_1+2)(\mu_1+\nu_1r_1+1)\sum_{\mu_2=0}^{n_2-s_2\nu_2} Q^{s_2,\mu_2}_{n_2-s_2r_2}(y)$$

where

$$\sum_{\mu_2=0}^{n_2-s_2\nu_2} Q_{n_2-s_2r_2}^{s_2,\mu_2}(y) = 1$$

$$\begin{aligned}
G_{n_1,n_2,r_1,r_2}^{s_1,s_2}(u^2;x,y) &= \frac{(n_2+1)}{(n_1+3)(n_1+2)} \sum_{\mu_1=0}^{n_1-s_1\nu_1} Q_{n_1-s_1\nu_1}^{s_1,\mu_1}(x)(\mu_1+\nu_1r_1+2)(\mu_1+\nu_1r_1+1) \\
&= \frac{(n_2+1)}{(n_1+3)(n_1+2)}\Bigg(\sum_{\mu_1=0}^{n_1-s_1\nu_1} P_{n_1-s_1\nu_1,\mu_1}(x)\sum_{\nu_1=0}^{s_1} P_{s_1,\nu_1}(x)(\mu_1^2) \\
&\quad + \sum_{\mu_1=0}^{n_1-s_1\nu_1} P_{n_1-s_1\nu_1,\mu_1}(x)\sum_{\nu_1=0}^{s_1} P_{s_1,\nu_1}(x).\mu_1(2\nu_1r_1+4) \\
&\quad + \sum_{\mu_1=0}^{n_1-s_1\nu_1} P_{n_1-s_1\nu_1,\mu_1}(x)\sum_{\nu_1=0}^{s_1} P_{s_1,\nu_1}(x)(\nu_1r_1+1)(\nu_1r_1+2)\Bigg) \qquad (2.1.4) \\
&= \frac{(n_2+1)}{(n_1+3)(n_1+2)}(S_1+S_2+S_3),
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \sum_{\mu_1=0}^{n_1-s_1\nu_1} P_{n_1-s_1\nu_1,\mu_1}(x)\sum_{\nu_1=0}^{s_1} P_{s_1,\nu_1}(x)\mu_1^2 \\
&= \frac{(n_2+1)}{(n_1+3)(n_1+2)} \sum_{\mu_1=1}^{n_1-s_1\nu_1} \binom{n_1-s_1\nu_1}{\mu_1} x^{\mu_1}(1-x)^{n_1-s_1\nu_1-\mu_1}.(\mu_1(\mu_1-1)+\mu_1) \\
&= \frac{(n_2+1)}{(n_1+3)(n_1+2)} \sum_{\mu_1=2}^{n_1-s_1\nu_1} \frac{(n_1-s_1\nu_1)!x^{\mu_1}(1-x)^{n_1-s_1\nu_1-\mu_1}.(\mu_1(\mu_1-1)+\mu_1)}{\mu_1(\mu_1-1)(\mu_2-2)!(n_1-s_1\nu_1-\mu_1)!} \\
&= \frac{(n_2+1)}{(n_1+3)(n_1+2)} \sum_{\mu_1=0}^{n_1-s_1\nu_1-2} \frac{(n_1-s_1\nu_1)!x^{\mu_1+2}(1-x)^{n_1-s_1\nu_1-\mu_1-2}}{\mu_1!(n_1-s_1\nu_1-\mu_1-2)!} \\
&= \frac{(n_2+1)}{(n_1+3)(n_1+2)} \sum_{\mu_1=0}^{n_1-s_1\nu_1-2} \frac{(n_1-s_1\nu_1)(n_1-s_1\nu_1-1)(n_1-s_1\nu_1-2)!x^{\mu_1+2}(1-x)^{n_1-s_1\nu_1-\mu_1-2}}{\mu_1!(n_1-s_1\nu_1-\mu_1-2)!} \\
&= x^2\frac{(n_2+1)}{(n_1+3)(n_1+2)} \sum_{\mu_1=0}^{n_1-s_1\nu_1-2} (n_1-s_1\nu_1)(n_1-s_1\nu_1-1)\frac{(n_1-s_1\nu_1-2)!x^{\mu_1}(1-x)^{n_1-s_1\nu_1-2-\mu_1}}{\mu_1!(n_1-s_1\nu_1-\mu_1-2)!} \\
&= x^2\frac{(n_1-s_1\nu_1)(n_1-s_1\nu_1-1)}{(n_1+3)(n_1+2)} \sum_{\mu_1=0}^{n_1-s_1\nu_1-2} \binom{n_1-s_1\nu_1-2}{\mu_1} x^{\mu_1}(1-x)^{n_1-s_1\nu_1-2-\mu_1}
\end{aligned}$$

where

$$\sum_{\mu_1=0}^{n_1-s_1\nu_1-2} \binom{n_1-s_1\nu_1-2}{\mu_1} x^{\mu_1}(1-x)^{n_1-s_1\nu_1-2-\mu_1} = 1$$

$$S_1 = x^2 \frac{(n_1-s_1\nu_1)(n_1-s_1\nu_1-1)}{(n_1+3)(n_1+2)}$$

and

$$\begin{aligned} S_2 &= \frac{(n_2+1)}{(n_1+3)(n_1+2)} \sum_{\mu_1=0}^{n_1-s_1\nu_1} P_{n_1-s_1\nu_1,\mu_1}(x) \sum_{n_1=0}^{s_1,\mu_1} P_{s_1,\nu_1}(x).\mu_1(2\nu_1 r_1+4) \\ &= \frac{1}{(n_1+3)(n_1+2)}(x(n_1-s_1\nu_1)+2xr_1) + \frac{4x(n_1-s_1\nu_1)}{(n_1+3)(n_1+2)} \end{aligned}$$

and

$$\begin{aligned} S_3 &= \sum_{\mu_1=0}^{n_1-s_1\nu_1} P_{n_1-s_1\nu_1,\mu_1}(x) \sum_{\nu_1=0}^{s_1} P_{s_1,\nu_1}(x)(\nu_1 r_1+1)(\nu_1 r_1+2) \\ &= \frac{1}{(n_1+3)(n_1+2)}(x^2(s_1)(s_1-1)r_1^2+3xr_1+2). \end{aligned}$$

Putting the above values S_1, S_2, S_3 in the eqn (2.1.4) we get:-

$$\begin{aligned} G_{n_1,n_2,r_1,r_2}^{s_1,s_2}(u^2;x,y) &= \frac{1}{(n_1+3)(n_1+2)}\Big((x^2(n_1-s_1\nu_1)(n_1-s_1\nu_1-1)+x(n_1-s_1\nu_1) \\ &\quad +2xr_1+4x(n_1-s_1\nu_1)+x^2(s_1)(s_1-1)r_1^2+3xr_1+2)\Big). \end{aligned}$$

Proof of (v) Similarly for $f(u,v)=v^2$, we have
$G_{n_1,n_2,r_1,r_2}^{s_1,s_2}(v^2;x,y)$

$$= (n_1+1)(n_2+1)\sum_{\mu_1=0}^{n_1-s_1r_1}\sum_{\mu_2=0}^{n_2-s_2r_2} Q_{n_1-s_1r_1,n_2-s_2r_2}^{s_1,s_2,\mu_1,\mu_2}(x,y)\int_0^1 P_{n_1,\mu_1+\nu_1r_1}(u)du\int_0^1 P_{n_2,\mu_2+\nu_2r_2}(v)\,v^2dv$$

$$= \frac{1}{(n_2+3)(n_2+2)}\Big((y^2(n_2-s_2\nu_2)(n_2-s_2\nu_2-1)+y(n_2-s_2\nu_2)$$

$$+2yr_1+4y(n_2-s_2\nu_2)+y^2(s_2)(s_2-1)r_2^2+3yr_2+2)\Big).$$

Lemma 2.1.2 *For $n_1, n_2 \epsilon \mathbb{N}$,we have*

(i) $G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(u-x;x,y) = \frac{1}{n_1+2}[x(n_1-s_1\nu_1)+xr_1+1)-x]$;

(ii) $G^{s_1,s_2}_{n_1,n_2,r_1,r_2}((u-x)^2;x,y) = \frac{1}{(n_1+3)(n_1+2)}[x^2(n_1-s_1\nu_1)(n_1-s_1\nu_1-1)+x(n_1-s_1\nu_1)+2xr_1+4x(n_1-s_1\nu_1)+x^2(s_1)(s_1-1)r_1^2+3xr_1+2)+x^2-\frac{2x}{n_1+2}(x(n_1-s_1\nu_1)+xr_1+1)-x]$.

(iii) $G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(v-y;x,y) = \frac{1}{n_2+2}[y(n_2-s_2\nu_2)+yr_2+1)-y]$;

(iv) $G^{s_1,s_2}_{n_1,n_2,r_1,r_2}((v-y)^2;x,y) = \frac{1}{(n_2+3)(n_2+2)}[y^2(n_2-s_2\nu_2)(n_2-s_2\nu_2-1)+y(n_2-s_2\nu_2)+2yr_2+4y(n_2-s_2\nu_2)+y^2(s_2)(s_2-1)r_2^2+3yr_2+2)+y^2-\frac{2y}{n_2+2}(y(n_2-s_2\nu_2)+yr_2+1)-y]$.

2.1.1 Convergence of $G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f(x,y);x,y)$

Theorem 2.1.1 *For the fumction $f \in C_B(I)$ the following inequality:-*
$|G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f;x,y)|$

$$\begin{aligned}
\leq\ & 4K(f;M^{s_1,s_2}_{n_1,n_2,r_1,r_2}(x,y)) \\
& +\omega(f;\sqrt{\frac{1}{n_1+2}(-x+x(n_1-s_1\nu_1)+xr_1+1)^2+\frac{1}{n_2+2}(-y+y(n_2-s_2\nu_2+yr_2+1))^2} \\
\leq\ & (\omega_2(f;\sqrt{M^{s_1,s_2}_{n_1,n_2,r_1,r_2}(x,y))}+min\{1,M^{s_1,s_2}_{n_1,n_2,r_1,r_2}(x,y)\}\|f\|C^2_B(I)) \\
& +\omega(f;\sqrt{\frac{1}{n_1+2}(-x+x(n_1-s_1\nu_1)+xr_1+1)^2+\frac{1}{n_2+2}(-y+y(n_2-s_2\nu_2+yr_2+1))^2}
\end{aligned}$$

holds. The constant $M \geq 0$ is independent of f and $M^{s_1,s_2}_{n_1,n_2,r_1,r_2}(x,y))$.
where:-

$$M^{s_1,s_2}_{n_1,n_2,r_1,r_2}(x,y)) = \frac{(\delta^{s_1}_{n_1,r_1}(x))^2}{n_1+2}+\frac{(\delta^{s_2}_{n_2,r_2}(y))^2}{n_2+2}.$$

Proof 2.1.1 *We define the auxiliary operator as follows:-*

$$\bar{G}^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f;x,y) = G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f;x,y) - f\Bigg(\frac{1}{n_1+2}[-x + x(n_1 - s_1\nu_1) + xr_1 + 1],$$

$$\frac{1}{n_2+2}[-y + y(n_2 - s_2\nu_2) + xr_2 + 1]\Bigg) + f(x,y).$$

Then, from Lemma (2.1.2),

$$\bar{G}^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f;x,y)((u-x);x,y) = 0.$$

and

$$\bar{G}^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f;x,y)((v-y);x,y) = 0.$$

Let $g \in C^2_B(I)$ and $(u,v) \in (I)$, using Taylor's theorem, we have

$$g(u,v) - g(x,y) = \frac{\partial g(x,y)}{\partial x}(u-x) + \int_x^u (u-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha + \frac{\partial g(x,y)}{\partial y}(v-y) + \int_y^v (v-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta.$$

Operating $\bar{G}^{s_1,s_2}_{n_1,n_2,r_1,r_2}(.;x,y)$ on both sides:-

$\bar{G}^{s_1,s_2}_{n_1,n_2,r_1,r_2}(g;x,y) - g(x,y)$

$$\begin{aligned}
&= \bar{G}^{s_1,s_2}_{n_1,n_2,r_1,r_2}\left(\int_x^u (u-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha;x,y\right) + \bar{G}^{s_1,s_2}_{n_1,n_2,r_1,r_2}\left(\int_y^v (v-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta;x,y\right) \\
&= G^{s_1,s_2}_{n_1,n_2,r_1,r_2}\left(\int_x^u (u-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha;x,y\right) \\
&\quad -\int_x^{\frac{1}{n_1+2}(x(n_1-s_1\nu_1+xr_1+1)} \left(\frac{1}{n_1+2}(x(n_1-s_1\nu_1)+xr_1+1)-\alpha\right)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha \\
&\quad +G^{s_1,s_2}_{n_1,n_2,r_1,r_2}\left(\int_y^v (v-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta;x,y\right) \\
&\quad -\int_y^{n_2+1(y(n_2-s_2\nu_2)+yr_1+1-1)} \left(\frac{1}{n_2+1}(y(n_2-s_2\nu_2)+yr_2+1-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta\right).
\end{aligned}$$

Hence

$$
\begin{aligned}
&|G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(g;x,y)-g(x,y)|\\
&\leq G^{s_1,s_2}_{n_1,n_2,r_1,r_2}\left(\int_x^u |(u-\alpha)\left|\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}\right| d\alpha;x,y\right)\\
&+\int_x^{\frac{1}{n_1+2}(x(n_1-s_1\nu_1)+xr_1+1}\left|\frac{1}{n_1+2}(x(n_1-s_1\nu_1)+xr_1+1)-\alpha\right|\left|\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}\right| d\alpha\\
&+G^{s_1,s_2}_{n_1,n_2,r_1,r_2}\left(\int_y^v |v-\beta|\left|\frac{\partial^2 g(x,\beta)}{\partial\beta^2}\right| d\beta;x,y\right)\\
&+\int_y^{\frac{1}{n_2+2}(x(n_2-s_2\nu_2)+yr_2+1}\left|\frac{1}{n_2+2}(x(n_2-s_2\nu_2+yr_2+1)\right|\left|\frac{\partial^2 g(x,\beta)}{\partial\beta^2}\right| d\beta\\
&\leq \left\{G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(u-x)^2;x,y)+\left(\frac{1}{n_1+2}(x(n_1-s_1\nu_1)+xr_1+1)-x\right)^2\right\}\|g\|C_B^2(I)\\
&+\left\{G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(v-y)^2;x,y)+\left(\frac{1}{n_2+2}(y(n_2-s_2\nu_2)+yr_2+1-y\right)^2\right\}|g\|C_B^2(I)\\
&\leq \left\{\frac{1}{n_1+2}\delta^{s_1}_{n_1,r_1}(x))^2+\left(\frac{1}{n_1+2}(x(n_1-s_1\nu_1)+xr_1+1)\right)^2\right.\\
&\left.+\frac{1}{n_2+2}\delta^{s_2}_{n_2,r_2}(y))^2+\left(\frac{1}{n_2+2}(y(n_2-s_2\nu_2)+yr_2+1)\right)^2\right\}\|g\|C_B^2(I).
\end{aligned}
$$

Thus, we get

$$
|G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(g;x,y)-g(x,y)|\leq\left\{\frac{2}{n_1+2}\delta^{s_1}_{n_1,r_1}(x))^2+\frac{2}{n_2+2}\delta^{s_2}_{n_2,r_2}(y))^2\right\}\|g|C_B^2(I).
$$

Also,

$$
\begin{aligned}
|G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f;x,y)| &\leq |G^a_{n_1,n_2}(f;x,y)|+|f(\frac{1}{n_1+2}(x(n_1-s_1\nu_1)+xr_1+1),\\
&\quad \frac{1}{n_2+2}(y(n_2-s_2\nu_2)+yr_2+1)|+|f(x,y)|.\\
&\leq 3\|f\|C_B(I)
\end{aligned}
$$

$|G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f;x,y)-f(x,y)|$

$$\begin{aligned}
&\leq |G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f-g;x,y)|+|G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(g;x,y)-g(x,y)|\\
&\quad +|g(x,y)-f(x,y)|\\
&\quad +\left|f\left(\frac{1}{n_1+2}(x(n_1-s_1\nu_1)+xr_1+1),\frac{1}{n_2+2}(y(n_2-s_2\nu_2)+yr_2+1)\right)-f(x,y)\right|\\
&\leq 3\|f-g|C^2_B(I)+|f-g|C^2_B(I)+|G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(g;x,y)-g(x,y)|\\
&\quad +\left|f\left(\frac{1}{n_1+2}(x(n_1-s_1\nu_1)+xr_1+1),\frac{1}{n_2+2}(y(n_2-s_2\nu_2)+yr_2+1)\right)-f(x,y)\right|\\
&\leq (4\|f-g\|C^2_B(I)+2M^{s_1,s_2}_{n_1,n_2,r_1,r_2}(x,y)||g||C^2_B(I))\\
&+\ \omega\left(f,\sqrt{\frac{1}{n_1+2}(-x+x(n_1-s_1\nu_1)+xr_1+1)^2+\frac{1}{n_2+2}(-y+y(n_2-s_2\nu_2)+yr_2+1)^2}\right).
\end{aligned}$$

Taking infimum on right hand side

$|G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f;x,y)|$

$$\begin{aligned}
&\leq 4K(f;M^{s_1,s_2}_{n_1,n_2,r_1,r_2}(x,y))\\
&\quad +\omega(f;\sqrt{\frac{1}{n_1+2}(-x+x(n_1-s_1\nu_1)+xr_1+1)^2+\frac{1}{n_2+2}(-y+y(n_2-s_2\nu_2+yr_2+1))^2}.
\end{aligned}$$

Thus, the proof is completed.

Lemma 2.1.3 *For each*$(x,y)\epsilon I$;

1. $G^{s_1,s_2}_{n_1,n_2,r_1,r_2}((u-x)^4;x)=O\left(\frac{1}{n^2}\right)$;

2. $G^{s_1,s_2}_{n_1,n_2,r_1,r_2}((v-y)^4;y)=O\left(\frac{1}{n^2}\right)$;

3. $\lim_{n\to\infty} nG^{s_1,s_2}_{n_1,n_2,r_1,r_2}(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y)=0.$

Theorem 2.1.2 *Let* $f\in C_B(I)$, *Then for every* $(x,y)\in I$,

$$\lim_{n_1,n_2\to\infty} G^{s_1,s_2}_{n_1,n_2,r_1,r_2}(f;x,y)=f(x,y).$$

Proof 2.1.2 *From lemma (2.1), and by using Krovkin theorem, we get the desired result.*

2.1.2 Voronovskaja type theorem

Theorem 2.1.3 *Let* $f \in C_B^2(I)$. *Then for every* $(x,y) \in I$

$$\begin{aligned}\lim_{n\to\infty} n(G_{n,r_1,r_2}^{s_1,s_2}(f;x,y) - f(x,y)) &= f_x(x,y)(xr_1+1) + f_y(x,y)(yr_2+1)\\ &+ \frac{x}{2}(1+x)f_{xx}(x,y) + \frac{y}{2}(1+y)f_{yy}(x,y).\end{aligned}$$

Proof 2.1.3 *Let* $(x,y) \in I$ *be fixed. By Taylor formula, we have*

$$\begin{aligned}f(u,v) &= f(x,y) + f_x(x,y)(u-x) + f_y(x,y)(v-y) + \frac{1}{2}f_{xx}(x,y)(u-x)^2\\ &+ 2f_{x,y}(x,y)(u-x)(v-y) + f_{yy}(x,y)(v-y)^2 + \psi(u,v,x,y)\sqrt{(u-x)^4+(v-y)^4}.\end{aligned}$$

where $\psi(..;x,y) = \psi(.,.) \in C_B^2(I)$ *and* $\psi(x,y) = 0$. *Thus, we get*

$$\begin{aligned}G_{n,r_1,r_2}^{s_1,s_2}(f(u,v);x,y) &= f(x,y) + f_x(x,y)G_{n,r_1,r_2}^{s_1,s_2}(u-x;x) + f_y(x,y)G_{n,r_1,r_2}^{s_1,s_2}(v-y;y)\\ &+\frac{1}{2}f_{xx}(x,y)G_{n,r_1,r_2}^{s_1,s_2}(u-x)^2\\ &+2f_{x,y}(x,y)G_{n,r_1,r_2}^{s_1,s_2}(u-x;x)G_{n,r_1,r_2}^{s_1,s_2}(v-y;y)\\ &+f_{yy}(x,y)G_{n,r_1,r_2}^{s_1,s_2}((v-y)^2;y)\\ &\times G_{n,r_1,r_2}^{s_1,s_2}(\psi(u,v,x,y)\sqrt{(u-x)^4+(v-y)^4};x,y).\end{aligned}$$

By using (2.1.3), *the lemma*

$$\begin{aligned}lim_{n\to\infty} n(G_{n,r_1,r_2}^{s_1,s_2}(f;x,y) - f(x,y)) &= f_x(x,y)(xr_1+1) + f_y(x,y)(yr_2+1)\\ &+\frac{1}{2}(x(1+x)f_{xx}(x,y) + y(1+y)f_{yy}(x,y))\\ &+\lim_{n\to\infty} nG_{n,r_1,r_2}^{s_1,s_2}(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y).\end{aligned}$$

Applying the holder inequality
$|G_{n,r_1,r_2}^{s_1,s_2}(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y|$

$$\begin{aligned}&\leq [G_{n,r_1,r_2}^{s_1,s_2}(\psi^2(u,v);x,y)]^{\frac{1}{2}}[G_{n,r_1,r_2}^{s_1,s_2}((u-x)^4+(v-y)^4;x,y)]^{\frac{1}{2}}\\ &\leq [G_{n,r_1,r_2}^{s_1,s_2}(\psi^2(u,v);x,y)]^{\frac{1}{2}}[G_{n,r_1,r_2}^{s_1,s_2}((u-x)^4;x) + G_{n,r_1,r_2}^{s_1,s_2}((v-y)^4;y)]^{\frac{1}{2}}.\end{aligned}$$

By theorem (2.1.2)

$$\lim_{n\to\infty} G_{n,r_1,r_2}^{s_1,s_2}(\psi^2(u,v);x,y) = \psi^2(x,y) = 0.$$

By combining (2.1.1) *and* (2.1.2)*, we obtained the desired result.*

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