KING - TYPE MODIFICATION OF SOME POSITIVE LINEAR OPERATORS

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Certificate

This is to certify that the work which is being presented in this thesis entitled " **King-type modification of some positive linear operators** " in partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics and Computing to the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala is a record of my own work studied under the supervision of Dr. Meenu Rani. The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other University/Institute.

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Abstract

In this report, we review some definitions related to approximation theory, which are important for convergence point of view. Then, we study some properties and convergence behaviour for certain linear positive operators.

In chapter 1, we recall some linear positive operators i.e Bernstein polynomials, Szász-Mirakjan operators, Baskakov operators, Phillips operators, genuine Durrmeyer-type operators etc. We review the main results concerning certain King-type modifications of well known sequences of these operators.

In chapter 2, we introduce the King type modification of generalized Lupas operators. We obtain auxiliary results for these operators. Then, we study asymptotic formula and error estimation for these operators in terms of second order modulus of continuity.

In chapter 3, we show the convergence of King type modification of Lupas operators and comparison with generalised Lupas operators through Graphics in Matlab. Finally, we observe that our modified operators is better than existing Lupas operators.

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Table of Contents

Chapter 1

Introduction

1.1 General Introduction

Approximation theory is an important area of research in last few years. It deals with Weierstrass Approximation theorem, Korovkin theorem and linear positive operators. The convergence of the operators is given by Korovkin theorem. For this, the moments of these operators play a very important role. There are many approximating operators like Bernstein polynomials, Szász - Mirakjan operators, Kantorovich operators, Phillips operators which preserves the constant and linear functions i.e.$L_m(e_j, y) = e_j(y)$ where $e_j(y) = y^j (j = 0, 1)$. King [1] modify the Bernstein polynomials $P_m(f, y)$ so that it preserve two test functions e_0 and e_2.

1.1.1 Weierstrass Approximation Theorem

The Weierstrass Approximation theorem was given by Weierstrass in 1885. This theorem states that every continous function defined on a closed interim $[c, d]$ can be approximated by a sequence of polynomials with real coefficients.
In other words, suppose that $f(y)$ is continous function defined on interval $[c, d]$ so that for $\in > 0$, there exist a polynomial $p(y)$ on interval $[c, d]$ such that
$|f(y) - p(y)| < \in, \forall y \in [c, d]$.

Many mathematicians proved this theorem. Some of them are : Runge (1885), Lerch (1892 and 1903), Bernstein (1912) and Picard(1891).
Now, we recall some basic definitions i.e first modulus of continuity, second modulus of continuity related to approximation theory:-

Definition 1.1.1. First Modulus of Continuity
If g is continuous on a bounded and closed interval $I = [c, d]$, then for each $\delta > 0$, the first modulus of continuity $w(g, \delta)$ is defined as :

$$w(g, \delta) = \sup_{x,y \in I, |x-y|<\delta} |g(x) - g(y)|. \tag{1.1.1}$$

Properties : 1) For every natural number n, then we have

$$w(g, n\delta) \leq nw(g, \delta).$$

2) For every positive and real value of λ, then we have

$$w(g, \lambda\delta) \leq (\lambda + 1)w(g, \delta).$$

3) Assuming that $0 < \delta_1 < \delta_2$ and $w(g, \delta)$ is an increasing function of δ, we have

$$w(g, \delta_1) \leq w(g, \delta_2).$$

4) Let g is uniformly continous in (c, d), then the neccessary condition is:

$$\lim_{\delta \to 0} w(g, \delta) = 0.$$

Definition 1.1.2. Second Modulus of Continuity
The second modulus of continuity is defined as :

$$w^2(g, \delta) = \sup_{y,u,v \geq 0} \sup_{|u-v| \leq \delta} |g(y + 2u) - 2g(y + u + v) + g(y + 2v)|, \tag{1.1.2}$$

for $\delta \geq 0$,

Definition 1.1.3. K **- functional**

K - functional is defined as:

$$K_2(g, \delta) \;=\; \inf_{f \in W^2} \{||g - f|| + \delta ||f''||\} \tag{1.1.3}$$

where $W^2 = f \in C_B[0, \infty) : f'' \in C_B[0, \infty)$, $\delta > 0$.
where $C_B[0, \infty)$ is the space of all uniformly and bounded continous function on $[0, \infty)$. There exist a constant $A > 0$ so that

$$K_2(g, \delta) \;\leq\; Aw_2(g, \sqrt{\delta})$$

where second order modulus of smoothness of g is given by

$$w^2(g, \sqrt{\delta}) = \sup_{0<|h|\leq\sqrt{\delta}} \sup_{y \in [0,\infty)} |g(y + 2h) - 2g(y + h) + g(y)|.$$

1.1.2 Linear Positive Operator

Suppose non empty set be X and H be the linear space of all the real functions. Then $V : H \to H$ is called linear positive operator if for $\alpha, \beta \in \mathbb{R}$ and $f, g \in H$;

1. $V(\alpha f + \beta g) = \alpha V(f) + \beta V(g)$
2. $V(f) \geq 0$ for all non negative $f \in H$.

The convergence of the sequences of positive linear operators was proposed by H. Bohman [2], P.P. Korovkin [10] and T. Popoviciu [9].

1.1.3 Korovkin Theorem

If $(T_m)_{m\geq 1}$ is a sequence of positive linear operators from $C[c,d]$ into $C[c,d]$ such that for each $g \in (e_0, e_1, e_2)$, where $e_j(t) = t^j, j = 0, 1, 2, 3$, we have

$$\lim_{m\to\infty} T_m(g) = g$$

uniformly in the $[c,d]$, then for each $f \in [c,d]$, condition hold

$$\lim_{m\to\infty} T_m(f) = f$$

uniformly in the $[c,d]$.

1.2 Linear positive operators

We discuss some operators like Bernstein operators, Bernstein - Durrrmeyer operators, Szász - Mirakjan - Beta operator, Integrated Szász - Mirakjan - Beta operators, Phillips operators.

1.2.1 Bernstein operators

Bernstein polynomials are defined for $f \in B[0,1]$:

$$P_m(f,y) = \sum_{k=0}^{m} v_{m,k}(y) f\left(\frac{k}{m}\right), 0 \leq y \leq 1 \tag{1.2.1}$$

where $v_{m,k}(y) = \binom{m}{k} y^k (1-y)^{m-k}$ and $B[0,1]$ is the space of all the bounded functions in $[0,1]$.

Properties:- (i) If $f(y) \geq g(y), y \in [0,1]$ then $P_m(f,y) \geq P_m(g,y)$ i.e. P_m is a monotone operators.

(ii)For degree n, Bernstein polynomials are non negative.

(iii) The Bernstein operator preserve linear functions.

(iv) $\sum_{k=0}^{m} v_{m.k}(y) = 1.$

(v) Only the first and last Bernstein polynomial are non zero at the interval endpoints 0 and 1.

(vi) $v_{m,k}(1-y) = v_{m,m-k}(y).$

Moments:

1. $P_m(1, y) = 1,$

2. $P_m(t, y) = y,$

3. $P_m(t^2, y) = y^2 + \frac{y(1-y)}{m}.$

Hence, $P_m(e_j(t), y)$ converges uniformly to $e_j(y)$ in $[0, 1]$,where $e_j(t) = t^j, j = 0, 1, 2.$ As, Bernstein polynomials are positive - linear operators, so all conditions of Korovkin theorem are satisfied. Hence, $P_m(f, y)$ converges uniformly to the continous function $f(y)$ in the $[0, 1]$.

After Bernstein, many mathematicans have introduced many sequence of the positive linear operators and examined their properties. Now, we list some expert mathematicians like Durrmeyer [3], Phillips [4], Szász - Mirakjan [7] who involved in these activities and made so many efforts to improve the approximation of these operators.

1.2.2 Bernstein - Durrrmeyer operators

Gupta and Duman [3] altered the Bernstein - Durrmeyer operators as follows :

$$K_m(g; y) = \sum_{k=0}^{m} v_{m,k}(y) \int_0^1 d_{m,k}(t)g(t)dt \tag{1.2.2}$$

for $\Phi_m(y) = (1-y)^m$ where $y \in [0,1], m \in \mathbb{N}$,

$$v_{m,k}(y) = \binom{m}{k} y^k (1-y)^{m-k},$$
$$d_{m,k}(t) = (-1)^{k+1} \frac{y^k}{k!} \Phi_m^{(k+1)}(t).$$

We use property e_j as $e_j(y) = y^j, j = 0,1,2$ and its moment function $\Phi_y = t - y$. For $y \in [0,1]$ and $m \in \mathbb{N}$, the moments of Bernstein-Durrmeyer operators are:

1. $K_m(e_0, y) = 1,$
2. $K_m(e_1, y) = \dfrac{my+1}{n+1},$
3. $K_m(e_2, y) = \dfrac{m^2y^2 - m(y-4) + 2}{(m+1)(m+2)}.$

For $y \in [0,1]$ and $m \in \mathbb{N}$, the central moments are :

4. $K_m(\Phi_y, y) = \dfrac{1-y}{m+1},$
5. $K_m(\Phi_y^2, y) = \dfrac{-2y^2(m-1) + 2y(m-2)y + 2}{(m+1)(m+2)}.$

1.2.3 Bernstein - Kantorovich operators

Bernstein - Kantorovich operator is defined as :

$$D_m(g, y) = (m+1) \sum_{k=0}^{m} v_{m,k}(y) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} g(t)dt, m \in \mathbb{N} \tag{1.2.3}$$

where $g \in C[0,1]$,

$$v_{m,k}(y) = \binom{m}{k} y^k (1-y)^{m-k},$$

and $C[0,1]$ is space of continous functions on $[0,1]$.
The moments of Bernstein-Kantorovich operators for $y \in [0,1]$, $m \in \mathbb{N}$ are given by:

1. $D_m(e_0, y) = 1,$
2. $D_m(e_1, y) = \dfrac{1}{2(m+1)} + \dfrac{m}{(m+1)}y,$
3. $D_m(e_2, y) = \dfrac{1}{3(m+1)^2} + \dfrac{2m}{(m+1)^2}y + \dfrac{m(m-1)}{(m+1)^2}y^2.$

The central moments of Bernstein-Kantorovich operator for $y \in [0,1]$, $m \in \mathbb{N}$ and $\Phi_y(t) = t - y$ are given by :

1. $D_m(\Phi_y, y) = \dfrac{1}{2(m+1)} - \dfrac{y}{(m+1)},$
2. $D_m(\Phi_y^2, y) = \dfrac{1}{3(m+1)^2} - \dfrac{(1-m)}{(m+1)^2}y + \dfrac{(1-m)}{(m+1)^2}y^2.$

1.2.4 Szász - Mirakjan - Beta operators

Gupta and Noor [7] introduced the sequence of the mixed form of summation - integral operators, we known as Szász - Mirakjan - Beta operators which is defined as :

$$S_m(g, y) = e^{-my} \sum_{k=1}^{\infty} \frac{(my)^k}{k! B(m+1, k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{m+k+1}} g(t) dt + e^{-my} g(0), \quad (1.2.4)$$

where $g \in C[0, \infty)$ so that $|g(t)| \leq M(1+t)^{\gamma}$ for any $M > 0, \gamma > 0$.
Moments of Szász Mirakjan operators are :

1. $S_m(e_0, y) = 1,$
2. $S_m(e_1, y) = y,$
3. $S_m(e_2, y) = \dfrac{my^2 + 2y}{m-1}.$

1.2.5 Integrated Szász - Mirakjan - Beta operators

Integrated Szász - Mirakjan operators were introduced by Dubey and Jam in [8] with constant $d > 0$. They modify operators for constant $d > 0$ as :

$$L_{n,d}(g, y) = \sum_{v=0}^{\infty} t_{n,v}(y) \int_0^{\infty} d_{n,v,d}(t)g(t)dt, y \in [0, \infty) \tag{1.2.5}$$

where

$$t_{n,v}(y) = e^{-ny}\frac{(ny)^v}{v!}, d_{n,v,d}(t) = d\frac{\Gamma(\frac{n}{d} + v + 1)}{\Gamma(v + 1)\Gamma(\frac{n}{d})}\frac{(dt)^v}{(1 + dt)^{(\frac{n}{d}+v+1)}}.$$

These operators reduce to the Szász -Beta operators when $d = 1$. It preserve only constant function.
Moments of Integrated Szász - Mirakjan - Beta operators are :

1. $L_{n,d}(e_0, y) = 1$,
2. $L_{n,d}(e_1, y) = \dfrac{(1 + ny)}{n - d}$,
3. $L_{n,d}(e_2, y) = \dfrac{ny^2 + 4ny + 2}{(n - d)(n - 2d)}$.

1.2.6 Phillips operators

Gupta [4] modified the Phillips operators :

$$U_m(g, y) = m\sum_{v=1}^{\infty} k_{m,v}(y) \int_0^{\infty} k_{m,v-1}(t)g(t)dt + e^{-my}g(0), y \in [0, \infty) \tag{1.2.6}$$

where

$$k_{m,v}(y) = e^{-my}\frac{(my)^v}{v!}.$$

Moments of Phillips operator are :

1. $U_m(e_0, y) = 1$,
2. $U_m(e_1, y) = y$,
3. $U_m(e_2, y) = \dfrac{my^2 + 2y}{m}$.

1.3 Modification of operators

King [1] modified the classical Bernstein operators to preserve constant and quadratic functions. These types of operators are called as King type operators. In Particular, we recall the main results concerning certain King- type modifications of some linear positive operators like the Bernstein operators, Bernstein - Durrmeyer operators, Szász- Mirakjan - Beta operators, Phillips operators. We review the better properties by modifying sequences of linear positive operators. King [1] modify the Bernstein polynomials $P_m(f, y)$ to reproduce two test functions e_0 and e_2. Other authors modify different operators to reproduce the test functions and give better approximation.

1.3.1 Modified Bernstein polyomial

King [1] considerded $r_m(y)$ as :

$$r_m(y) = -\frac{1}{2(m-2)} + \sqrt{\left(\frac{m}{m-1}\right) y^2 + \frac{1}{4(m-1)^2}}, m = 2, 3......$$

The modified Bernstein polynomial :

$$P_m^*(f; y) = \sum_{k=0}^{m} \binom{m}{k} (r_m(y))^k (1 - r_m(y))^{m-k} f\left(\frac{k}{m}\right), \tag{1.3.1}$$

for $0 \leq r_m(y) \leq 1, m = 1, 2, ..., 0 \leq y \leq 1$.

Moments of modified Bernstein polynomial are :

1. $P_m^*(e_0, y) = 1$,
2. $P_m^*(e_1, y) = r_m(y)$,
3. $P_m^*(e_2, y) = y_2$.

The error estimation of $P_m^*(f; y)$ to $f(y)$ is better than the Bernstein polynomial to $f(y)$ when $0 \leq y \leq \dfrac{1}{3}$.

1.3.2 Modified Bernstein kantorovich operators

Let $r_m(y)$ defined on $[0, 1]$ is a sequence of real - valued continous functions with $0 \leq r_m(y) \leq 1$. Now, if y is replaced by $r_m(y)$ defined as :

$$r_m(y) = \frac{2(m+1)y - 1}{2m}, m \in \mathbb{N}, 1/4 \leq y \leq 3/4, \tag{1.3.2}$$

$$\tag{1.3.3}$$

then, the modified Bernstein - Kantorovich operators as :

$$D_m^*(g, y) = (m+1)\sum_{k=0}^{m}\binom{m}{k}\left(\frac{2(m+1)y-1}{2m}\right)^k\left[\frac{(2m+1)-2(m+1)y}{2m}\right]^{m-k}\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} g(t)dt \tag{1.3.4}$$

$$\begin{aligned} D_m^*(g, y) &= (m+1)\sum_{k=0}^{m}\binom{m}{k} v_{m,k}(r_m(y)) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} g(t)dt \\ &= (m+1)\sum_{k=0}^{m}\binom{m}{k} d_{m,k}(y) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} g(t)dt \end{aligned} \tag{1.3.5}$$

For convenient, $v_{m,k}(r_m(y))$ is denoted by $d_{m,k}(y)$.

For $y \in [1/4, 3/4]$, $m \in \mathbb{N}$, then the moments of modified Bernstein - Kantorovich operators are :

1. $D^*_m(e_0, y) = 1$,

2. $D^*_m(e_1, y) = y$,

3. $D^*_m(e_2, y) = \dfrac{12(m-1)(m+1)^2y^2 + 12(m+1)^2y - (5m+3)}{12m(m+1)^2}$.

For $y \in [1/4, 3/4]$, $\Phi_y(t) = t - y$, we have central moments of modified Bernstein - Kantorovich operators :

1. $D^*_m(\Phi_y; y) = 0$,

2. $D^*_m(\Phi_y^2; y) = \dfrac{y(1-y)}{m} - \dfrac{(5m+3)}{12m(m+1)^2}$.

Remark 1. For all $\delta > 0$ and each $x, y \in [0, 1]$, we can write

$$|g(x) - g(y)| \leq w(g, \delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{1.3.6}$$

Theorem 1.3.1. *If $g \in L[0, 1], y \in [1/4, 3/4]$ and $m \in \mathbb{N}$. Then*

$$|D^*_m(g; y) - g(y)| \leq 2w(g; \delta_y), \tag{1.3.7}$$

where

$$\delta_y = \sqrt{\frac{y(1-y)}{m} - \frac{(5m+3)}{12m(m+1)^2}}$$

Remark 2. For every $g \in L[0, 1]$ and $y \in [1/4, 3/4], m \in \mathbb{N}$, the operator D_m satisfy

$$|D_m(g; y) - g(y)| \leq 2w(g, v_y) \tag{1.3.8}$$

where

$$v_y = \sqrt{\frac{1}{3(m+1)^2} - \frac{(1-m)}{(m+1)^2}y + \frac{(1-m)}{(m+1)^2}y^2}$$

The error approximation obtained in 1.3.7 is better than 1.3.8 for $y \in [1/4, 3/4], m \in \mathbb{N}$.
To prove that $\delta_y \leq v_y$. Now

$$\delta_y \leq v_y \Leftrightarrow \frac{y(1-y)}{m} - \frac{(5m+3)}{12m(m+1)^2} \leq \frac{1}{3(m+1)^2} - \frac{(1-m)}{(m+1)^2}y + \frac{(1-m)}{(m+1)^2}y^2$$
$$\Leftrightarrow -(36m+12)y^2 + (36m+12)y - (9m+3) \leq 0$$
$$\Leftrightarrow (3m+1)[-12y^2 + 12y - 3] \leq 0 \Leftrightarrow 12y^2 - 12y + 3 \geq 0 \Leftrightarrow (2y-1)^2 \geq 0.$$

Thus, it is true for $y \in [1/4, 3/4]$.

1.3.3 Modified Phillips operator

Gupta [4] modified Phillips operators in order to preserve the two test functions e_0 and e_2.
Let $r_m(y)$ defined on the interval $[0, \infty)$ is sequence of real -valued continous functions for $0 \leq r_m(y) < \infty$.
Now y is replaced by $r_m(y)$, which is :

$$r_m(y) = \frac{-1 + \sqrt{1 + m^2y^2}}{m}$$

and the modified Phillips operator is:

$$U_m^*(g, r_m(y)) = me^{-mr_m(y)} \sum_{v=1}^{\infty} \frac{(mr_m(y))^v}{v!} \int_0^{\infty} k_{m,v-1}(t)g(t)dt + e^{-mr_m(y)}g(0). \tag{1.3.9}$$

Moments of Phillips operator for $y \in [0, \infty)$ are :

1. $U_m^*(e_0, y) = 1$,
2. $U_m^*(e_1, y) = \dfrac{-1 + \sqrt{1 + m^2y^2}}{m}$,
3. $U_m^*(e_2, y) = y^2$.

Theorem 1.3.2. *[4] For all $g \in C_B[0,\infty)$ and $y \geq 0$, we have modified Phillips operator:*

$$|U_m^*(g,y) - g(y)| \leq 2w(g, \delta_{m,y}), \tag{1.3.10}$$

where

$$\delta_{m,y} = \sqrt{2y(y - r_m(y))}. \tag{1.3.11}$$

and $w(g,\delta)$ is the modulus of continuity which is defined as :

$$w(g,\delta) = \sup_{|t-y|\leq\delta} |g(t) - g(y)|$$

where $y, t \in [0,\infty)$.

Remark 3. For all $g \in C_B[0.\infty)$ and $y \geq 0$, we have Phillips operator $U_m(g,y)$

$$|U_m(g,y) - g(y)| \leq 2w(g, \beta_{m,y}) \tag{1.3.12}$$

where

$$\beta_{m,y} = \sqrt{\frac{2y}{m}}.$$

Now by theorem 1.3.10 and remark 1.3.12, we noticed $2y(y - r_m(y)) \leq \dfrac{2y}{m}$, for $y \geq 0$. We proved $\delta_{m,y} \leq \beta_{m,y}$. Hence, we claim that the modified Phillips operators 1.3.9 give better approximation than the original Phillips operators in 1.2.6.

1.3.4 Modified Szász - Mirakjan - Beta operators

Duman et al. [5] examined that modified operator give better approximation on $[0,2]$. Let y defined on the interval $[0,\infty)$ be sequence of real -valued continous functions for $0 \leq y \leq \infty$. Now, y is replaced by $r_m(y)$ as :

$$r_m(y) = \frac{1}{m}(-1 + \sqrt{1 + m(m-1)y^2}), y \geq 0, m \in \mathbb{N}$$

Then the modified operators becomes :

$$S_m^*(g, r_m(y)) = e^{-mr_m(y)} \sum_{k=1}^{\infty} \frac{(mr_m(y))^k}{k!B(n+1,k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} g(t)dt + e^{-mr_m(y)} g(0). \tag{1.3.13}$$

The modified form preserve the two test function e_0 and e_2.

Theorem 1.3.3. *[5] For each $g \in C_B[0.\infty), y \geq 0$ and $m > 1$, we have modified Szász - Mirakjan - beta operators:*

$$|S_m^*(g, y) - g(y)| \leq 2w(g, \delta_{m,y}) \tag{1.3.14}$$

where $\delta_{m,y} = \sqrt{2y(y - r_m(y)}$ and $w(g, \delta)$ is the modulus of continuity defined as :

$$w(g, \delta) = \sup_{|t-y| \leq \delta; y,t \in [0,\infty)} |g(t) - g(y)|$$

Remark: For all $y \geq 0, m > 1$, and $g \in C_B[0, \infty)$, we have Szász- Mirakjan -Beta operators $U_m(g, y)$:

$$|S_m(g, y) - g(y)| \leq 2w(g, \beta_{m,y}) \tag{1.3.15}$$

where

$$\beta_{m,y} = \sqrt{\frac{y(2+y)}{m-1}}.$$

The error estimation in 1.3.13 is better than 1.2.4 provided $g \in C_B[0,\infty)$ and $y \in [0,2]$ according to the Duman et al [5]. We have $y^2 \leq 1$ for $0 \leq y \leq 2$.
As $(m-1/2)^2 - m(m-1) = 1/4$, we have :
$y^2[(m-1/2^2)] - m(m-1)] \leq 1$
Hence,

$$-\frac{1}{m} + \frac{1}{m}\sqrt{1+m(m-1)y^2} \geq -\frac{1}{m} + \left(\frac{2m-1}{2m}\right)y.$$

Using this inequality, it become:

$$y - r_m(y) \leq \frac{2+y}{2m}$$

or we say that

$$2y(y - r_m(y)) \leq \frac{y(2+y)}{m} \leq \frac{y(2+y)}{m-1}.$$

This shows that $\delta_{m,y} \leq \beta_{m,y}$ for $m > 1$ and $y \in [0,2]$.

1.3.5 Modified Integrated Szász - Mirakjan - Beta operators

Let $r_n^*(y)$ defined on $[0,\infty)$ is a sequence of real - continous functions. As $n > d$, we have:

$$r_n^*(y) = \frac{(n-d)y - 1}{n}.$$

Gupta and Deo replaced y by $r_n^*(y) \in [0.\infty)$ in $L_{n,d}(g,y)$1.2.5 for $n \in \mathbb{N}$. Then the modified operators become:

$$L_{n,d}^*(g,y) = \sum_{v=0}^{\infty} t_{n,v}(r_n^*(y)) \int_0^{\infty} d_{n,v,d}(t) g(t) dt, \tag{1.3.16}$$

where

$$t_{n,v}(r_n^*(y)) = e^{-nr_n^*(y)} \frac{(nr_n^*(y))^v}{v!}$$

for $y \in [0, \infty), n \in \mathbb{N}$.

$L_{n,d}^*(g, y)$ preserve the constant and linear functions.

Theorem 1.3.4. *[6] For each $g \in C[0, \infty)$, $n \in \mathbb{N}$, the operator $L_{n,d}^*$ is :*

$$|L_{n,d}^*(g; y) - g(y)| \leq 2w(g, \delta_y) \tag{1.3.17}$$

where

$$\delta_y = \sqrt{\frac{d(n-d)y^2 + 2(n-d)y - 1}{(n-d)(n-2d)}}$$

Remark : For each $g \in C[0, \infty)$ and $n \in \mathbb{N}$, we have the operator $L_{n,d}$ given by 1.2.5,

$$|L_{n,d}(g; y) - g(y)| \leq 2w(g, d_y) \tag{1.3.18}$$

where

$$d_y = \sqrt{\frac{(4dy + 2d^2y^2 + 2) + ny(dy + 2)}{(n-d)(n-2d)}}$$

In order to get better approximation, we need to show that $\delta_y \leq d_y$. One can attain

$$\begin{aligned} \delta_y \leq d_y \Leftrightarrow \frac{d(n-d)y^2 + 2(n-d)y - 1}{(n-d)(n-2d)} \\ \Leftrightarrow 3d^2y^2 + 6dy + 3 \geq 0 \\ \Leftrightarrow (dy + 1)^2 \geq 0, \end{aligned}$$

that holds. This shows that $\delta_y \leq d_y$.

Chapter 2

King modification of generalized Lupas operator

2.1 Lupas Operator

Lupas - type operators are the modifications of Bernstein polynomials to infinite intervals. Integral modifications of many operators was introduced in last years. For $g : [0, \infty) \to \mathbb{R}$, Lupas [12] proposed the linear positive operators :

$$L_m(g, y) = (1-u)^{my} \sum_{k=0}^{\infty} \frac{(my)_k}{k!} g\left(\frac{k}{m}\right) u^k, |u| < 1, m \in \mathbb{N}, y \in [0, \infty), \quad (2.1.1)$$

with the help of this identity $\frac{1}{(1-u)^{my}} = \sum_{k=0}^{\infty} \frac{(my)_k}{k!} u^k$ with the factorial $(my)_j = (my)(my+1)(my+2)\cdots(my+j-1)$. Afterwards, Agratini [11] introduced and modify Lupas operators which is given below :

$$I_m^a(g, y) = \sum_{k=0}^{\infty} S_{m,k}(y, a) g\left(\frac{k}{m}\right), a \geq 0, y \geq 0, \quad (2.1.2)$$

where

$$S_{m,k}(y, a) = \frac{e^{-ay}}{2^{my}} \frac{D_{k,m}^a(y) y^k}{k!}$$

such that

$$\sum_{k=0}^{\infty} S_{m,k}(y,a) = 1,$$

and

$$D^a_{m,k}(y) = \sum_{j=0}^{k} \binom{k}{j} \frac{a^{k-j}(my)_j}{(2y)^j}.$$

2.1.1 Auxiliary results

Lemma 2.1.1. *For $y \in [0,\infty)$ and $a \geq 0$, then :*

(i) $\sum_{k=0}^{\infty} \frac{D^a_{k,m}(y)y^k}{k!} = 2^{my}e^{ay}$,

(ii) $\sum_{k=0}^{\infty} \frac{D^a_{k+1,m}(y)y^{k+1}}{k!} = 2^{my}e^{ay}(a+m)y$,

(iii) $\sum_{k=0}^{\infty} \frac{D^a_{k+2,m}(y)y^{k+2}}{k!} = (a^2y^2 + 2ay^2m + my(my+1))2^{my}e^{ay}$.

Lemma 2.1.2. *For operators $I^a_m(s^p,x), p = 0,1,2,...,$ Moments are :*

(i) $I^a_m(1,y) = 1$,

(ii) $I^a_m(s,y) = y + \frac{ay}{m}$,

(iii) $I^a_m(s^2,y) = y^2 + \frac{a^2y^2}{m^2} + \frac{ay}{m^2} + \frac{2y}{m} + \frac{2ay^2}{m}$.

Lemma 2.1.3. *For the operators I^a_m, Central Moments are :*

(i) $I^a_m((s-y),y) = \frac{ay}{m}$,

(ii) $I^a_m((s-y)^2,y) = \frac{ay}{m^2} + \frac{a^2y^2}{m^2} + \frac{2y}{m} + \frac{ay^2}{m}$.

2.2 King Modification of generalized Lupas operators

Now, we apply King modification to generalization of Lupas operator,we obtain new results after modification. Thus, we have:

$$\begin{aligned} r(y) &= y + \frac{ay}{m} \\ mr(y) &= ym + ay \\ mr(y) &= y(m+a) \\ y &= \frac{mr(y)}{m+a}. \end{aligned}$$

Now, we replace y by $\dfrac{mr(y)}{m+a}$ in the definition of $I_m^a(g,y)$.

$$I_m^a(g,y) = \sum_{k=0}^{\infty} S_{m,k}(y,a) g\left(\frac{k}{m}\right), a \geq 0, y \geq 0,$$

The modified operators $T_m^a(g,y)$ become:

$$T_m^a(g,y) = I_m^a\left(g, \frac{mr(y)}{m+a}\right) = \sum_{k=0}^{\infty} S_{m,k}\left(\frac{mr(y)}{m+a}, a\right) g\left(\frac{k}{m}\right), r(y) \geq 0, \quad (2.2.1)$$

where

$$W_{m,k}(y,a) = S_{m,k}\left(\frac{mr(y)}{m+a}, a\right) = \frac{e^{\frac{-amr(y)}{m+a}}\left[D_{k,m}^a\left(\frac{mr(y)}{m+a}\right)\right]\left(\frac{mr(y)}{m+a}\right)}{2^{\left(\frac{m^2 r(y)}{m+a}\right)} k!}$$

such that

$$\sum_{k=0}^{\infty} W_{m,k}(r(y),a) = 1$$

and

$$H_{m,k}^{a}(y)=D_{m,k}^{a}\left(\frac{mr(y)}{m+a}\right)=\sum_{j=0}^{k}\binom{k}{j}\frac{a^{k-j}\left(\frac{m^2r(y)}{m+a}\right)_j}{\left(\frac{2mr(y)}{m+a}\right)^j}. \tag{2.2.2}$$

2.2.1 Auxilliary results

Lemma 2.2.1. *For $y\in[0,\infty)$ and $a\geq 0$, then :*

(i) $$\sum_{k=0}^{\infty}H_{m,k}^{a}(y)\frac{y^k}{k!}=\sum_{j=0}^{k}\binom{k}{j}\frac{a^{j}\left(\frac{m^2r(y)}{m+a}\right)_j}{\left(\frac{2mr(y)}{m+a}\right)^j}\frac{(mr(y))^k}{k!(m+a)^k}=e^{\left(\frac{mar(y)}{m+a}\right)}{}_2\left(\frac{m^2r(y)}{m+a}\right),$$

(ii) $$\sum_{k=0}^{\infty}H_{m,k+1}^{a}(y)\frac{y^{k+1}}{k!}=\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{a^{k+1-j}\left(\frac{m^2r(y)}{m+a}\right)_j}{\left(\frac{2mr(y)}{m+a}\right)^j}\frac{(mr(y))^{k+2}}{k!(m+a)^{k+2}}\frac{(mr(y))^{k+1}}{k!(m+a)^{k+1}}$$
$$=e^{\left(\frac{mar(y)}{m+a}\right)}{}_2\left(\frac{m^2r(y)}{m+a}\right)(a+m)\frac{mr(y)}{m+a},$$

(iii) $$\sum_{k=0}^{\infty}H_{m,k+2}^{a}(y)\frac{y^{k+2}}{k!}=\sum_{j=0}^{k+2}\binom{k+2}{j}\frac{a^{k+2-j}\left(\frac{m^2r(y)}{m+a}\right)_j}{\left(\frac{2mr(y)}{m+a}\right)^j}$$
$$=e^{\left(\frac{mar(y)}{m+a}\right)}{}_2\left(\frac{m^2r(y)}{m+a}\right)\left(\frac{a^2(mr(y))^2}{(m+a)^2}+\frac{2am^2r(y)^2}{(m+a)^2}+\frac{m^2r(y)}{m+a}\left(\frac{m^2r(y)}{m+a}+1\right)\right).$$

Proof. (i) For $a \geq 0$ and $y \in [0,\infty)$

$$\sum_{k=0}^{\infty}\frac{H^a_{m,k}(y)}{k!}y^k = \sum_{k=0}^{\infty}\frac{D^a_{m,k}\left(\frac{mr(y)}{m+a}\right)}{k!}\frac{(mr(y))^k}{(m+a)^k}$$

$$= \sum_{k=0}^{\infty}\frac{(mr(y))^k}{(m+a)^k k!}\sum_{j=0}^{k}\binom{k}{j}\frac{a^{k-j}\left(\frac{m^2r(y)}{m+a}\right)_j}{\left(\frac{2mr(y)}{m+a}\right)^j}$$

$$= e^{a\left(\frac{mr(y)}{m+a}\right)} + \frac{m}{2}\left(\frac{mr(y)}{m+a}\right)e^{a\left(\frac{mr(y)}{m+a}\right)} + e^{a\left(\frac{mr(y)}{m+a}\right)}\left(\frac{mr(y)}{m+a}\right)\frac{\left(\frac{mr(y)}{m+a}+1\right)}{2!2^2}$$

$$+\cdots+e^{a\left(\frac{mr(y)}{m+a}\right)}\frac{\left(\frac{m^2r(y)}{m+a}\right)_k}{2^k}$$

$$= e^{a\left(\frac{mr(y)}{m+a}\right)}\left(1+m\left(\frac{mr(y)}{m+a}\right)+m\left(\frac{mr(y)}{m+a}\right)\frac{\left(\frac{mr(y)}{y+a}+1\right)}{2!2^2}+\cdots\right).$$

$$\sum_{k=0}^{\infty}\frac{H^a_{m,k}(y)}{k!}y^k = e^{a\left(\frac{mr(y)}{m+a}\right)}2^{m\left(\frac{mr(y)}{m+a}\right)}.$$

Proof (ii) For $a \geq 0$ and $y \in [0,\infty)$

$$\sum_{k=0}^{\infty}\frac{H^a_{m,k+1}(y)}{k!}y^{k+1} = \sum_{k=0}^{\infty}\frac{D^a_{m,k+1}\left(\frac{mr(y)}{m+a}\right)}{k!}\frac{(mr(y))^{k+1}}{(m+a)^{k+1}}$$

$$= \sum_{k=0}^{\infty}\frac{(mr(y))^{k+1}}{(m+a)^{k+1}k!}\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{a^{k+1-j}\left(\frac{m^2r(y)}{m+a}\right)_j}{\left(\frac{2mr(y)}{m+a}\right)^j}$$

$$= \frac{mar(y)}{m+a}\left(2^{\frac{m^2r(y)}{m+a}} e^{\frac{mar(y)}{m+a}}\right) + \sum_{k=0}^{\infty}\frac{(mr(y))^{k+1}}{k!(m+a)^{k+1}}$$

$$\times \sum_{j=0}^{k}\binom{k}{j}\frac{a^{k-j}\frac{m^2r(y)}{m+a}\left(\frac{m^2r(y)}{m+a}+1\right)_j}{\frac{(2mr(y))^{j+1}}{(m+a)^{j+1}}}$$

$$= \frac{mar(y)}{n+a}\left(2^{\frac{m^2r(y)}{m+a}} e^{\frac{mar(y)}{m+a}}\right) + \frac{m^2r(y)}{2(m+a)}2^{\left(\frac{m^2r(y)}{m+a}+1\right)} e^{\left(\frac{mar(y)}{m+a}\right)}$$

$$\sum_{k=0}^{\infty}\frac{H^a_{m,k+1}(y)}{k!}y^{k+1} = 2^{\left(\frac{m^2r(y)}{m+a}\right)} e^{a\left(\frac{mr(y)}{m+a}\right)} mr(y).$$

In a similar way, we prove (iii).

□

Lemma 2.2.2. *For the operators $T^a_m(s^p;y), p = 0,1,2,....,$ Moments are:*

(i) $T^a_m(1;y) = I^a_m\left(1;\frac{mr(y)}{n+a}\right) = 1,$

(ii) $T^a_m(s,y) = I^a_m\left(s;\frac{mr(y)}{m+a}\right) = r(y),$

(iii) $T^a_m(s^2;y) = I^a_m\left(s^2;\frac{mr(y)}{m+a}\right) = \frac{a^2r(y)^2}{(m+a)^2} + \frac{2amr(y)^2}{(m+a)^2} + \frac{m^2r(y)^2}{(m+a)^2} + \frac{r(y)}{m+a} + \frac{r(y)}{m}.$

Proof. (i)

$$
\begin{aligned}
T_m^a(1,y) = I_m^a\left(1 : \frac{mr(y)}{m+a}\right) &= \sum_{k=0}^{\infty} e^{\left(\frac{-amr(y)}{m+a}\right)} 2^{\left(\frac{-m^2r(y)}{m+a}\right)} \frac{D_{k,m}^a\left(\frac{mr(y)}{m+a}\right)}{k!} g(1) \frac{(mr(y))^k}{(m+a)^k} \\
&= \sum_{k=0}^{\infty} e^{\left(\frac{-amr(y)}{m+a}\right)} 2^{\left(\frac{-m^2r(y)}{m+a}\right)} \frac{D_{k,m}^a\left(\frac{mr(y)}{m+a}\right)}{k!} \frac{(mr(y))^k}{(m+a)^k} \\
&= e^{\left(\frac{-amr(y)}{m+a}\right)} 2^{\left(\frac{-m^2r(y)}{m+a}\right)} \frac{D_{k,m}^a\left(\frac{mr(y)}{m+a}\right)}{k!} \frac{(mr(y))^k}{(m+a)^k} \\
&= e^{\left(\frac{-amr(y)}{m+a}\right)} 2^{\left(\frac{-m^2r(y)}{m+a}\right)} \left(e^{\frac{amr(y)}{m+a}} 2^{\frac{m^2r(y)}{m+a}}\right) = 1.
\end{aligned}
$$

Proof of (ii) $T_m^a(s,y) = I_m^a\left(s; \frac{mr(y)}{m+a}\right)$

$$
\begin{aligned}
&= e^{\left(\frac{-amr(y)}{m+a}\right)} 2^{\left(\frac{-m^2r(y)}{m+a}\right)} \sum_{k=0}^{\infty} \frac{D_{k,m}^a\left(\frac{mr(y)}{m+a}\right)}{k!} \frac{k}{m} \frac{(mr(y))^k}{(m+a)^k} \\
&= e^{\left(\frac{-amr(y)}{m+a}\right)} 2^{\left(\frac{-m^2r(y)}{m+a}\right)} \sum_{k=1}^{\infty} \frac{D_{k,m}^a\left(\frac{mr(y)}{m+a}\right)}{(k-1)!} \frac{1}{m} \frac{(mr(y))^k}{(m+a)^k} \\
&= e^{\left(\frac{-amr(y)}{m+a}\right)} 2^{\left(\frac{-m^2r(y)}{m+a}\right)} \sum_{k=0}^{\infty} \frac{D_{k+1,m}^a\left(\frac{mr(y)}{m+a}\right)}{k!} \frac{1}{m} \frac{(mr(y))^{k+1}}{(m+a)^{k+1}} \\
&= e^{\left(\frac{-amr(y)}{m+a}\right)} 2^{\left(\frac{-m^2r(y)}{m+a}\right)} e^{\left(\frac{amr(y)}{m+a}\right)} 2^{\left(\frac{m^2r(y)}{m+a}\right)} \frac{1}{m}(m+a)\left(\frac{mr(y)}{m+a}\right)
\end{aligned}
$$

$$T_m^a(s,y) = r(y).$$

Proof (iii)

$$
\begin{aligned}
T_m^a(s^2,y) = I_m^a\left(s^2; \frac{mr(y)}{m+a}\right) \quad &= \sum_{k=0}^{\infty} e^{\left(\frac{-amr(y)}{m+a}\right)} {}_2\left(\frac{-m^2r(y)}{m+a}\right) \frac{D_{k,m}^a\left(\frac{mr(y)}{m+a}\right)}{k!} \frac{k^2}{m^2} \frac{(mr(y))^k}{(m+a)^k} \\
&+ \quad e^{\left(\frac{-amr(y)}{m+a}\right)} {}_2\left(\frac{-m^2r(y)}{m+a}\right) \sum_{k=1}^{\infty} \frac{D_{k,m}^a\left(\frac{mr(y)}{m+a}\right)}{k(k-1)!} \frac{k}{m^2} \frac{(mr(y))^k}{(m+a)^k} \\
&= e^{\left(\frac{-amr(y)}{m+a}\right)} {}_2\left(\frac{-m^2r(y)}{m+a}\right) \sum_{k=2}^{\infty} \frac{D_{k,m}^a\left(\frac{mr(y)}{m+a}\right)}{(k-2)!m^2} \frac{(mr(y))^k}{(m+a)^k} \\
&+ e^{\left(\frac{-amr(y)}{m+a}\right)} {}_2\left(\frac{-m^2r(y)}{m+a}\right) \sum_{k=1}^{\infty} \frac{D_{k,m}^a\left(\frac{mr(y)}{m+a}\right)}{(k-1)!m^2} \frac{(mr(y))^k}{(m+a)^k} \\
&= e^{\left(\frac{-amr(y)}{m+a}\right)} {}_2\left(\frac{-m^2r(y)}{m+a}\right) \sum_{k=0}^{\infty} \frac{D_{k+2,m}^a\left(\frac{mr(y)}{m+a}\right)}{k!} \frac{(mr(y))^{k+2}}{m^2(m+a)^{k+2}} \\
&+ \frac{1}{m^2}(m+a)\left(\frac{mr(y)}{m+a}\right) \\
&= \frac{1}{m^2}\left(\frac{a^2m^2r(y)^2}{(m+a)^2} + \frac{2am(m^2r(y)^2)}{(m+a)^2} + \frac{m^2(m^2r(y)^2)}{(m+a)^2} + \frac{m^2r(y)}{m+a}\right) \\
&+ \frac{r(y)}{m} \\
T_m^a(s^2,y) \quad &= \frac{a^2r(y)^2}{(m+a)^2} + \frac{2amr(y)^2}{(m+a)^2} + \frac{m^2r(y)^2}{(m+a)^2} + \frac{r(y)}{m+a} + \frac{r(y)}{m}.
\end{aligned}
$$

□

Lemma 2.2.3. *Central moments for T_m^a are :*

(i) $T_m^a((s-r(y)); r(y)) = 0,$

(ii) $T_m^a((s-r(y))^2; r(y)) = \frac{a^2r(y)^2}{(m+a)^2} + \frac{2amr(y)^2}{(m+a)^2} + \frac{m^2r(y)^2}{(m+a)^2} + \frac{r(y)}{m+a} + \frac{r(y)}{m} - r(y)^2.$

Proof. (i) Apply $T_m(g;y)$ on above equation

$$\begin{aligned} T_m^a((s-r(y));r(y)) &= T_m^a(s;r(y)) - T_m^a(r(y);r(y)) \\ &= T_m^a(s;r(y)) - r(y)T_m^a(1;r(y)) \\ &= T_m^a(s;r(y)) - r(y)(1) = 0. \end{aligned}$$

Proof (ii) Apply $T_m(g;y)$ on above equation

$$\begin{aligned} T_m^a((s-r(y))^2;r(y)) &= T_m^a(s^2+r(y)^2-2sr(y);r(y)) \\ &= T_m^a(s^2;r(y)) + T_m^a(r(y)^2;r(y)) - 2T_m^a(sr(y);r(y)) \\ &= T_m^a(s^2;r(y)) + r(y)^2T_m^a(1,r(y)) - 2r(y)(s;r(y)) \\ &= \frac{a^2r(y)^2}{(m+a)^2} + \frac{2amr(y)^2}{(m+a)^2} + \frac{m^2r(y)^2}{(m+a)^2} + \frac{r(y)}{m+a} + \frac{r(y)}{m} + r(y)^2(1) - 2r(y)^2 \\ &= \frac{a^2r(y)^2}{(m+a)^2} + \frac{2amr(y)^2}{(m+a)^2} + \frac{m^2r(y)^2}{(m+a)^2} + \frac{r(y)}{m+a} + \frac{r(y)}{m} - r(y)^2. \end{aligned}$$

□

2.2.2 Main results

Theorem 2.2.4. *Suppose $g \in C_B[0,\infty)$ and $y \in I_n$ .Then, there exists a positive constant A such as*

$$|(T_ng)(y) - g(y)| \leq Aw^2\left(g, \sqrt{\delta_n^a(y)}\right)$$

where

$$\delta_m^a(y) = \frac{a^2r(y)^2}{(m+a)^2} + \frac{2amr(y)^2}{(m+a)^2} + \frac{m^2r(y)^2}{(m+a)^2} + \frac{r(y)}{m+a} + \frac{r(y)}{m} - r(y)^2.$$

Proof. Let $f \in W^2, y \in I_n$ and $s \in [0,\infty)$. Using Taylor's theorem, we get :

$$f(s) = f(y) + (s-y)f'(y) + \int_y^s (s-v)f''(v)dv$$

Apply T_m on this equation

$$\begin{aligned}(T_m f)(y) - f(y) &= \left(T_m \int_y^s (s-v) f''(v)dv\right)(y)| \left(\int_y^s (s-v) f''(v)dv\right)| \leq (s-v)^2 ||f''|| \\ |(T_m f)(y) - f(y)| &\leq (T_m (s-y)^2 (y))||f''|| = \delta_m^a(y)||f''||\end{aligned}$$

Since $|(T_m g)(y)| \leq ||g||$, then we have

$$\begin{aligned}|(T_m f)(y) - f(y)| &\leq |(T_m)(y) - (T_m f)(y) - g(y) + f(y) + (T_m f)(y) - f(y)| \\ &\leq |T_m(g-f)(y)| + |(g-f)(x)| + |(T_m f)(x) - f(x)| \\ &\leq ||g-f|| + ||g-f|| + \delta_m^a(y)||f''|| \\ &\leq 2||g-f|| + \delta_m^a(y)||f''||.\end{aligned}$$

Now, take infimum over all $f \in W^2$ and by definition of K - functional :

$$K_2(g,\delta) = \inf_{f\in W^2}\{||g-f|| + \delta||f''||\} \tag{2.2.3}$$

where $W^2 = f \in C_B[0,\infty) : f'' \in C_B[0,\infty),\ \delta > 0$.

There exist a constant $A > 0$ so that

$$K_2(g,\delta) \leq Aw_2(g,\sqrt{\delta})$$

and $\delta = \delta_m^a(y)$

$$|T_m g(y) - g(y)| \leq Aw_2(g,\sqrt{\delta_m^a(y)})$$

where

$$\delta_m^a(y) = \frac{a^2 r(y)^2}{(m+a)^2} + \frac{2amr(y)^2}{(m+a)^2} + \frac{m^2 r(y)^2}{(m+a)^2} + \frac{r(y)}{m+a} + \frac{r(y)}{m} - r(y)^2.$$

Hence proved. □

Theorem 2.2.5. *Suppose $g \in C_\gamma[0,\infty)$. Let $g''(x)$ exists at $y \in I_n$, then prove that*

$$\lim_{m\to\infty}(m)\left[(\tilde{L}_{m,k,1}^{\alpha,\beta} g)(y) - g(y)\right] = \frac{1}{2} y^2 g''(y)$$

Proof. Using the Taylor's formula, we get :

$$g(s) = g(y) + (s-y)^2 g'(y) + \frac{1}{2} g''(y)(s-y)^2 + p(s,y)(s-y)^2,$$

where $p(s,y)$ is the Peano form of the remainder.
and

$$\lim_{s \to y} p(s,y) = 0.$$

Apply $(\tilde{L}^{\alpha,\beta}_{m,k,1} g)(y)$ to both sides, we have :

$$\begin{aligned}
m\left[(\tilde{L}^{\alpha,\beta}_{m,k,1} g)(y) - g(y)\right] &= mg'(y)(\tilde{L}^{\alpha,\beta}_{m,k,1}(s-y))(y) + \frac{1}{2}(n) g''(y)(\tilde{L}^{\alpha,\beta}_{m,k,1}(s-y)^2)(y) \\
&\quad + m(\tilde{L}^{\alpha,\beta}_{m,k,1}(s-y)^2 p(s,y))(y). \\
m(\tilde{L}^{\alpha,\beta}_{m,k,1}(s-y)^2))(y) &= \left(\frac{(m-k+2)m - (m-k+1)(m-1)}{(m-k+1)(m-1)}\right) y^2 \\
&\quad + \frac{2\alpha}{m+\beta}\left(\frac{(m-k+1)(m-1) - (m-k+2)(m)}{(m-k+1)(m-1)}\right) y \\
&\quad + \frac{\alpha^2}{(m+\beta)^2}\left(\frac{(m-k+2)(m) - (m-k+1)(m-1)}{(m-k+1)(m-1)}\right) \\
&= \frac{m(2m-k+1)}{(m-k+1)(m-1)}\left(y^2 - \frac{2\alpha}{m+\beta} y + \frac{\alpha^2}{(m+\beta)^2}\right) \\
\lim_{m \to \infty} m(\tilde{L}^{\alpha,\beta}_{m,k,1}(s-y)^2))(y) &= y^2
\end{aligned}$$

We know,

$$\lim_{m \to \infty} m(\tilde{L}^{\alpha,\beta}_{m,k,1}(s-y))(y) = 0$$

Now, we need to prove that:

$$m(\tilde{L}^{\alpha,\beta}_{m,k,1}(s-y)^2 p(s,y))(y) \to 0,$$

when $m \to \infty$. With the help of Cauchy - Schwarz inequality,

$$(\tilde{L}_{m,k,1}^{\alpha,\beta})(s-y)^2 p(s,y)) \quad \leq \quad \sqrt{(\tilde{L}_{m,k,1}^{\alpha,\beta} p^2(s,y))(y)}\sqrt{(\tilde{L}_{m,k,1}^{\alpha,\beta}(s-y)^4)(y)}$$

we examined that $p^2(y,y) = 0$, $p^2(y,y) \in C_\gamma[0.\infty)$.We have:

$$\lim_{m\to\infty} (\tilde{L}_{m,k,1}^{\alpha,\beta} p^2(s,y))(y) \quad = \quad p^2(y,y) = 0$$

In fact,

$$(\tilde{L}_{m,k,1}^{\alpha,\beta}(s-y)^4)(y) \quad = \quad O\left(\frac{1}{m^2}\right).$$

From the above equations,

$$\lim_{m\to\infty} m(L_{m,k,1}^{\alpha,\beta}(s-y)^2 p(s,y))(y) = 0$$

we have:

$$\begin{aligned}
m\left[(\tilde{L}_{m,k,1}^{\alpha,\beta} g)(y) - g(y)\right] &= mg'(y)(\tilde{L}_{m,k,1}^{\alpha,\beta}(s-y))(y) + \frac{1}{2}mg''(y)(\tilde{L}_{m,k,1}^{\alpha,\beta}(s-y)^2)(y) \\
&\quad + m\tilde{L}_{m,k,1}^{\alpha,\beta}(s-y)^2 p(s,y))(y) \\
m\left[(\tilde{L}_{m,k,1}^{\alpha,\beta} g)(y) - g(y)\right] &= 0 + \frac{1}{2}(m)g''(y)y^2 + 0 \\
\left[(\tilde{L}_{m,k,1}^{\alpha,\beta} g)(y) - g(y)\right] &= \frac{1}{2}y^2 g''(y)
\end{aligned}$$

Hence, the proof is completed. □

Hence, our modified operators give better approximation than generalised Lupas operators.

Chapter 3

Convergence results

3.1 Introduction

We study asymptotic formula and error estimation for these operators in terms of second modulus of continuity.

Theorem 3.1.1. *For operator* T_m*, provided* $y \in [0,\infty), g \in C[0,\infty), m \in \mathbb{N}$ *and* $r(y) = y$*, we get*

$$|(T_m g)(y) - g(y)| \leq A w_2(g, \sqrt{\delta_m^a(y)})$$

where

$$\delta_m^a(y) = \frac{a^2y^2}{(m+a)^2} + \frac{2amy^2}{(m+a)^2} + \frac{m^2y^2}{(m+a)^2} + \frac{y}{m+a} + \frac{y}{m} - y^2$$

Proof. For $g \in C[0,\infty), y \in [0,\infty)$, we get

$$|I_m^a(g,y) - g(y)| \leq 5w(g, \sqrt{U_m^a(y)}) + \frac{13}{2} w_2(g, \sqrt{U_m^a(y)})$$

where

$$U_m^a(y) = I_m^a((s-y)^2; y)$$

where

$$I_n^a((s-y)^2;y) \quad = \quad \frac{a^2y^2}{m^2}+\frac{ay}{m^2}+\frac{2y}{m}+\frac{ay^2}{m}$$

After applying King modification we get the result,

$$|(T_m g)(x)-g(x)| \leq Aw_2(g,\sqrt{\delta_m^a(y)})$$

where

$$\delta_m^a(y) \quad = \quad \frac{a^2y^2}{(m+a)^2}+\frac{2amy^2}{(m+a)^2}+\frac{m^2y^2}{(m+a)^2}+\frac{y}{m+a}+\frac{y}{m}-y^2$$

For better approximation,we need to prove

$$\delta_m^a(y) \leq U_m^a(y)$$

$$\frac{a^2y^2}{(m+a)^2}+\frac{2amy^2}{(m+a)^2}+\frac{m^2y^2}{(m+a)^2}+\frac{y}{m+a}+\frac{y}{m}-y^2 \leq \frac{a^2y^2}{m^2}+\frac{ay}{m^2}+\frac{2y}{m}+\frac{ay^2}{m}$$

$$\frac{a^2y^2}{(m+a)^2}+\frac{2amy^2}{(m+a)^2}-\frac{a^2y^2}{(m+a)^2}-\frac{2amy^2}{(m+a)^2}+\frac{y}{m+a}+\frac{y}{m} \leq \frac{a^2y^2}{m^2}+\frac{ay}{m^2}+\frac{2y}{m}+\frac{ay^2}{m}$$

$$\frac{y}{m+a}+\frac{y}{m} \leq \frac{a^2y^2}{m^2}+\frac{ay}{m^2}+\frac{2y}{m}+\frac{ay^2}{m}$$

$$\frac{y}{m+a}+\frac{y}{m} \leq y^2\left(\frac{a^2}{m^2}+\frac{a}{m}\right)+\frac{y}{m}\left(2+\frac{a}{m}\right)$$

$$\frac{y}{m+a}+\frac{y}{m} \leq \left(\frac{a}{m}+2\right)\left(\frac{ay^2}{n}+\frac{y}{n}\right)$$

After solving,we get,

$$ay \geq \frac{m-m-a}{m+a} \Rightarrow y \geq \frac{-1}{m+a} \Rightarrow \left(y+\frac{1}{m+a}\right) \geq 0.$$

Hence , we show $\delta_m^a(y) \leq U_m^a(y)$. We proved that our King type modification of generalized operators gives better approximation than the generalized Lupas operators. Now we show the convergence of our operators and comparison of the operators with generalized Lupas operators. □

3.1.1 Convergence and comparison

With the help of Matlab, we try to show the better convergence for the generalized Lupas operators. Now, we show the convergence and comparison of these operators which give better result than original operators.

Example 1. The convergence of the King type modification of the Lupas operators $T_m^a(g;y) = \frac{a^2y^2}{(m+a)^2} + \frac{2amy^2}{(m+a)^2} + \frac{m^2y^2}{(m+a)^2} + \frac{y}{m+a} + \frac{y}{m} - y^2$ (red) to $g(y) = y^2 + 3y - 2$(blue) is shown in figure 1,2,3.We observed that the function g converges to the King type modification of the Lupas operators when n changes from 50 to 200.

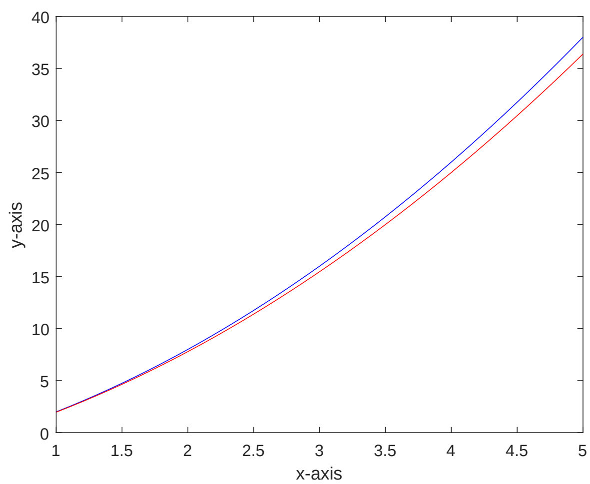

Figure 1

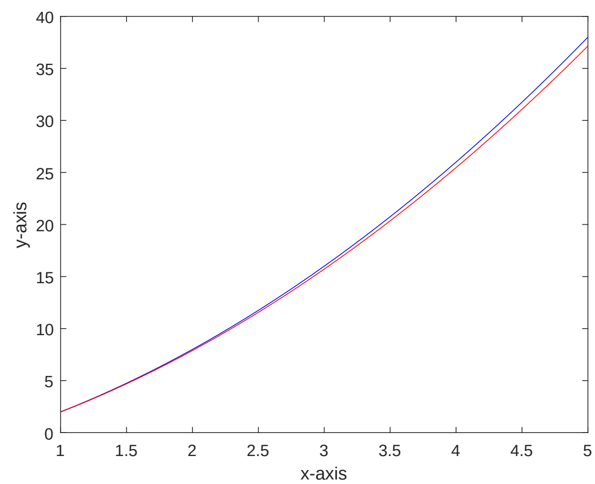

Figure 2

Figure 1 shows the convergence of $T_{50}^2(g;y)$(red) to $g(y)$ (blue)

Figure 2 shows the convergence of $T_{100}^2(g;y)$(red) to $g(y)$ (blue).

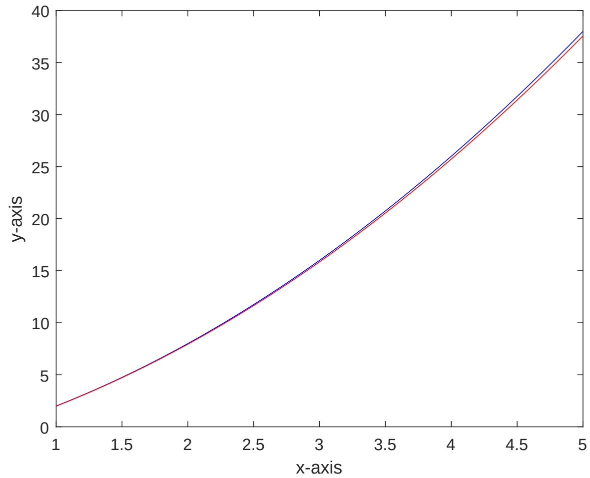

Figure 3

The convergence of $T^2_{200}(g;y)$(red) to $g(y)$ (blue).

Example 2. The comparison of the generalised Lupas operators $I^a_m(g;y)$ (green)and King type modification of the Lupas operators$T^a_m(g;y)$ (red) to $g(y) = y^2 + 3y - 2$(blue) is shown in figure 4,5,6.We observed that the error estimation of g by the King type modification of the Lupas operators is smaller than the generalised Lupas operator when n changes from 50 to 200.

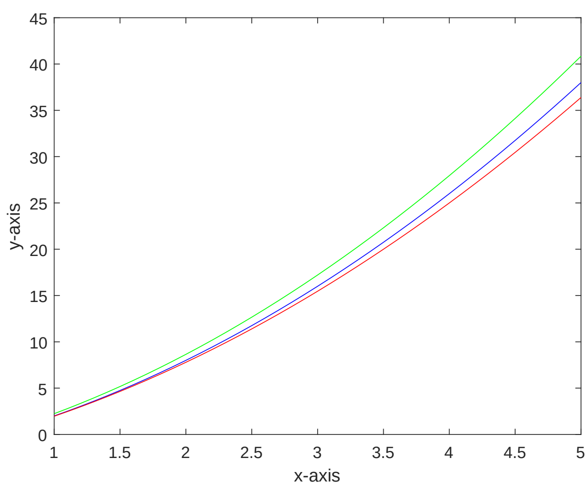

Figure 4

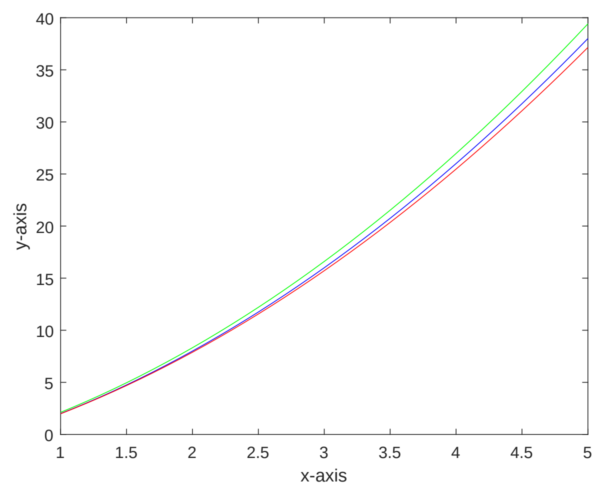

Figure 5

Figure 4 shows the comparison of $I^2_{50}(g;y)$ (green) and $T^2_{50}(g;y)$(red) to $g(y)$ (blue).
Figure 5 shows the comparison of $I^2_{100}(g;y)$ (green) and $T^2_{100}(g;y)$(red) to $g(y)$ (blue).

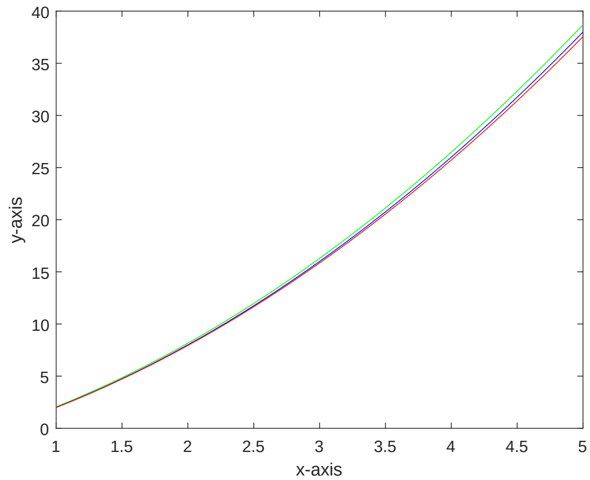

Figure 6

The comparison of $I^2_{200}(g;y)$ (green) and $T^2_{200}(g;y)$(red) to $g(y)$ (blue).

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