DUALITY FOR MINIMAX FRACTIONAL PROGRAMMING PROBLEMS

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Certificate

I hereby declare that the work which is being presented here in the dissertation entitled **"DUALITY FOR MINIMAX FRACTIONAL PROGRAMMING PROBLEMS"** in partial fulfillment of the requirement for the award of degree of **Master of Science in Mathematics and Computing** submitted in School of Mathematics, Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of **Dr. Vikas Sharma**, Assistant Professor, SOM and **Dr. Navdeep Kailey**, Lecturer, SOM and refer other researcher's work which are duly listed in the reference section.

The matter presented in this thesis has not been submitted to any other University/Institute for the award of my degree.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

The work being presented in the present thesis is devoted to the study of duality results for minimax fractional programming problems under exponential (p, r)-invexity assumptions.

The chapterwise summary of the thesis is as follows:

Chapter 1 is introductory in nature. This chapter includes minimax fractional programming problem, definitions, notations that are used throughout the work and detailed review of minimax fractional programming problems and summary of the thesis has also been discussed.

In Chapter 2, we have studied a minimax fractional programming problem involving exponential (p, r)-invex functions [16] and weak, strong and strict converse duality theorems are established under exponential $(p, r)-$invex assumptions.

In Chapter 3, we have reviewed a mixed-type dual problem with exponential (p, r)-invexity considered by Lai and Ho [12] and proved the duality theorems related to the primal problem and the mixed-type dual problem.

In Chapter 4, motivated by Lai and Ho [10], we have established nonparametric necessary and sufficient optimality conditions for a minimax fractional programming problem with B-(p, r)-invexity. These optimality conditions are deduced to two parameter-free type dual models: Mond-Weir type dual and Wolfe type dual problems. On these duality types, we have established the duality theorems under exponential B-(p, r) invexities including weak duality, strong duality and strict converse duality theorems.

Contents

Chapter 1

Introduction

Optimization problems occur frequently in day to day life in relation to taking the best possible decision from the given set of possible alternatives. The decision making situation could be something like buying a certain product from market, deciding on means of transport to reach a particular destination etc. In such situations a fundamental question arises in one's mind is "What decision should I take now?". Interestingly, finding an answer to such situations is equivalent of solving an optimization problem. Mathematically an optimization problem can be constructed by specifying a constraint set S, which consist of available decision x and an objective function $f(x)$ that maps each $x \in S$ into a scalar and represents a measure of understandability of choosing a decision x. Hence, the general form of an optimization problem is defined as:

$$\begin{aligned} &\min\ f(x) \\ &\text{subject to } x \in S, \end{aligned}$$

where $f : R^m \to R$ and S is a nonempty subset of R^m. The aim is to find an $x^* \in S$ such that $f(x^*) \leq f(x)$ for all $x \in S$. If either the objective function or at least one of the constraints of an optimization problem are nonlinear then, the problem is called nonlinear programming problem. A class of nonlinear programming problem in which the objective function is a ratio of two numerical functions is called a fractional programming problem. The main reason for interest in fractional programming stems from the fact that programming models could better fit the real problems if we consider the maximization of return on investment, maximization of return/risk, minimization of cost/time etc. An important class of fractional programming is the class of minimax fractional programming problem. Mathematically a minimax fractional programming problem is formulated as:

$$\textbf{(P)} \qquad \min_{x\in X} \max_{y\in Y} \frac{f(x,y)}{g(x,y)}$$
$$\text{subject to} \quad X = \{x \in R^n \mid h(x) \in -R^p_+\}$$
$$\text{and } Y \text{ is a compact subset of } R^m,$$

where $f(\cdot,\cdot)$ and $g(\cdot,\cdot) : R^n \times R^m \to R$ are continuous functions. These problems have arisen in multiobjective programming, game theory, economics and minimum risk problem. Minimax programming problems are in general difficult to study, one possible reason is that this problem falls in the category of nonsmooth optimization. In the last few years, researchers have studied minimax fractional programming under convexity, generalized convexity[1, 11], invexity, generalized invexity[14, 18] etc.

Another important concept related to minimax fractional programming is duality which means that a problem may be viewed from either of two perspectives, the primal problem or the dual problem. Since the dual of dual is primal, the solution of a programming problem can be obtained by solving either the primal or the dual. Duality theory has been well-developed with extensive applications in economics, control theory, scientific computation etc. Recently, many authors have established the necessary optimality conditions and proved various duality results for minimax fractional programming problem under different assumptions. These optimality conditions were employed to search an optimal solution for minimax programming problem.

The present chapter is divided into three sections. The first section gives important preliminaries. The second section contains a review of various developments in minimax fractional programming which are relevant to the thesis and the last one presents a summary of the thesis.

1.1 Preliminaries

Notations

We will use the following notations throughout the thesis. We denote by R^n the n-dimensional

Euclidean space, $R^1 = R$ the set of all real numbers, $R^p_+ = \{x \in R^p : x_j \geq 0, j = 1, 2, \ldots, p\}$ the non-negative orthant of R^p.

A small Latin letter with an integer subscript will denote a component of a vector. For example, if $x \in R^3$, then x_3 and x_i denote respectively the third and ith components of x. For any vector v in R^n, the generalized directional derivative of f at x in the direction $v \in R^n$ in Clarke's sense is defined by

$$f^o(x; v) = \lim \sup_{y \to x \lambda \to 0^+} \frac{f(y + \lambda v) - f(y)}{\lambda}$$

The generalized subdifferential of f at $x \in S$ is defined by the set

$$\partial^c f(x) = \{\xi \in R^n : f^o(x; v) \geq \langle \xi, v \rangle \quad \text{for all } v \in R^n\}$$

where $\langle \xi, v \rangle$ stands for the dual pair. The vector $\nabla f(\bar{x})$ denotes the gradient of a scalar differentiable function $f : R^n \to R$ at $\bar{x}$, and is defined as

$$\nabla f(\bar{x}) = \left[\frac{\partial}{\partial x_1} f(\bar{x}), \frac{\partial}{\partial x_2} f(\bar{x}), \ldots, \frac{\partial}{\partial x_n} f(\bar{x}) \right]^T.$$

Generalized convex functions

In this section, some basic definitions are discussed. Let $X \subset R^n$ be a convex set. Then, the function $\psi : X \to R$ is said to be

(i) **Convex** if for all $x, x^* \in X$

$$\psi(\lambda x + (1 - \lambda)x^*) \leq \lambda \psi(x) + (1 - \lambda)\psi(x^*) \text{ for all } 0 \leq \lambda \leq 1.$$

The function ψ is said to be strictly convex if the above condition holds as strict inequality for $x \neq x^*$, $0 < \lambda < 1$.

(ii) **Quasiconvex** if for all $x, x^* \in X$

$$\psi(\lambda x + (1 - \lambda)x^*) \leq \max\{\psi(x), \psi(x^*)\} \text{ for all } 0 \leq \lambda \leq 1.$$

The function ψ is said to be strictly quasiconvex if the above condition holds as strict inequality with $f(x) \neq f(x^*)$ and for all $0 < \lambda < 1$.

Every convex function is quasiconvex function but converse is not true eg. $f(x) = x^3$ is quasiconvex but not convex.

(iii) **Pseudoconvex** if ψ is differentiable at x^* and for all $x, x^* \in X$

$$\nabla\psi(x^*)^T(x - x^*) \geq 0 \Rightarrow \psi(x) \geq \psi(x^*),$$

or equivalently, if

$$\psi(x) < \psi(x^*) \Rightarrow \nabla\psi(x^*)^T(x - x^*) < 0.$$

Every pseudoconvex function is quasiconvex function but converse is not true eg. $f(x) = x^3$ is quasiconvex but not pseudoconvex. Also every convex function is pseudoconvex function under differentiability but converse is not true eg. $f(x) = x+x^3$ is pseudoconvex but not convex.

(iv) **Invex** if there exists a function $\eta : X \times X \to R^n$ such that for all $x, x^* \in X$

$$\psi(x) - \psi(x^*) \geq \eta(x, x^*)^T\nabla\psi(x^*).$$

(v) **(p,r)-invex (strictly)** if for all $x \in X$, the following inequalities hold:

$\frac{1}{r}e^{r\psi(x)} \geq \frac{1}{r}e^{r\psi(x^*)}\left[1 + \frac{r}{p}\left\langle \nabla\psi(x^*), (e^{p\eta(x,x^*)} - \mathbf{1})\right\rangle\right] (> \text{ if } x \neq x^*)$ for $p \neq 0,\ r \neq 0$,

$e^{r\psi(x)} - e^{r\psi(x^*)} \geq re^{r\psi(x^*)}\big\langle \nabla\psi(x^*), \eta(x, x^*)\big\rangle \quad (> \text{ if } x \neq x^*)$ for $p = 0,\ r \neq 0$,

$\psi(x) - \psi(x^*) \geq \frac{1}{p}\big\langle \nabla\psi(x^*), (e^{p\eta(x,x^*)} - \mathbf{1})\big\rangle \quad (> \text{ if } x \neq x^*)$ for $p \neq 0,\ r = 0$,

$\psi(x) - \psi(x^*) \geq \big\langle \nabla\psi(x^*), \eta(x, x^*)\big\rangle \quad (> \text{ if } x \neq x^*)$ for $p = 0,\ r = 0$,

(vi) **Locally Lipschitz** if there exists a positive real constant K and a neighbourhood L of $x \in X$ such that

$$|\psi(y) - \psi(z)| \leq K\|y - z\| \quad \text{for all} \quad z, y \in L,$$

where $\|.\|$ is an arbitrary norm in R^n.

1.2 Review of the Related Work

In this thesis, we study a minimax fractional programming problem and then prove weak, strong and strict converse duality theorems for the problem and its dual problem under exponential (p, r)-invexity assumptions.

Fractional programming problem

Amongst various important applications, one important application of nonlinear programming is to maximize or minimize the ratio of two functions. The mathematical programming problem in which the objective function is a ratio of two numerical functions is called a fractional programming problem. We can find applications of fractional programming problem in the articles given by Schaible [22], Schaible and Ibarki [23] and Stancu-Minasian [25]. An important class of nonlinear programming problems, namely, fractional programming problems, described as follows

(FP)
$$\max \frac{f(x)}{g(x)}$$
subject to
$$x \in S = \{x \in R^n : \; h_i(x) \leq 0 \; (i = 1, \ldots, m)\}.$$

Here S is a non empty compact convex set and $f, g : R^n \to R$ are continuous functions on S with $g(x) > 0$ for all $x \in S$.

If all the functions $f, \; g, \; h_i \; (i = 1, \ldots, m)$, involved in problem (FP) are linear functions then the fractional programming problem is called the linear fractional programming problem, otherwise it is called nonlinear fractional programming problem.

The (FP) problem can be solved by using the following auxiliary problem with parameter $q \in R$

(NFP)
$$\text{Max } f(x) - qg(x)$$
subject to $x \in S$.

By denoting
$$F(q) = \text{Max}\{f(x) - qg(x) : \; x \in S\}, q \in R,$$
we can find the optimal solution of (FP) because we know that $x^* \in S$ is an optimal solution of (FP) if and only if x^* is an optimal solution of $F(q^*)$ with optimal objective value $F(q^*) = 0$, where, $q^* = \dfrac{f(x^*)}{g(x^*)}$. If we observe that f is concave, g is convex and $q \geq 0$, then $f - qg$ is a concave function, whereas, $\dfrac{f}{g}$ is not a concave function. This makes (NFP) easier to solve than (FP).

Duality for a class of minimax programming problem has been a subject of immense interest for researchers in recent past few years. Duality in fractional programming is an important class of duality theory, and several contributions have been made in past [1, 3, 4] for its development. Recently, Liang et al. [17] obtained some duality results for a nonlinear fractional programming problem by defining a new class of generalized convexity, called (F, α, ρ, d)-convexity. Bector and Chandra [5] studied the duality for the following fractional programming problem

Primal Problem

$$\text{Minimize } \frac{f(x)}{h(x)}$$
$$\text{subject to } x \in X,\ g(x) \leq 0.$$

Dual Problem

$$\text{Maximize } \frac{f(x) + \mu^T g(y)}{h(y)}$$
$$\text{subject to } \nabla\left(\frac{f(x) + \mu^T g(y)}{h(y)}\right) = 0,$$
$$y \in X, \mu \geq 0.$$

where X is an open subset of R^n and $f,\ h : X \to R,\ g : X \to R^m$, and for all $x,\ h(x) > 0$ and $f(x) \geq 0$ (if h is nonlinear).

Mond [21] considered the following pair of nondifferentiable fractional programming problems:

Primal Problem

$$\text{Maximize} \frac{f(x) - (x^T Bx)^{1/2}}{g(x) + (x^T Dx)^{1/2}}$$
$$\text{subject to } h(x) \leq 0.$$

Dual Problem

$$\text{Minimize } G(u, y, v, w, p) = p$$
$$\text{subject to } \nabla y^T h(u) + p\nabla g(u) + Bu + pDw = \nabla f(u),$$
$$- f(u) + u^T Bv + pg(u) + pu^T Dw + y^T h(u) \geq 0,$$
$$y, p \geq 0,$$
$$v^T Bv \leq 1,\ w^T Dw \leq 1.$$

and further necessary and sufficient conditions for optimality as well as appropriate duality theorems are also established.

Minimax fractional programming

Fractional programming problem is an important class of nonlinear programming problem.

The necessary and sufficient conditions for generalized minimax programming were first developed by Schmitendorf [24]. Tanimoto [26] applied these optimality conditions to define a dual problem and derived duality theorems. Crouzeix et al. [7] have given a variety of applications of generalized fractional programming and have shown that the minimax fractional programming can be solved by solving a minimax parametric program.

Lai et al. [9] established necessary and sufficient optimality conditions for non-differentiable minimax fractional problem with generalized convexity and applied these optimality conditions to construct a parametric dual model and also discussed duality theorems. Ho and Lai [16] introduced exponential (p, r)-invexity for Lipschitz function as well as the definition of differentiable (p, r)-invex function given by Antczak [2]. For the case of convex differentiable minimax fractional programming, Yadav and Mukherjee [27] formulated two dual models and derived duality theorems. Chandra and Kumar [6] pointed out certain omissions and inconsistencies in the dual formulation of Yadav and Mukherjee; they constructed two modified dual problems for fractional minimax programming problem and proved duality results. Liu and Wu [18, 19] derived the sufficient optimality conditions and duality theorems for the minimax fractional programming in the framework of invexity and (F, α, ρ, d)-convex functions.

1.3 Summary of the thesis

The aim of the present thesis is to study the duality for minimax fractional programming problem using exponential (p, r)-invex assumptions. We discuss a new concept of invexity for a locally Lipschitz function named as exponential (p, r)-invexity. In this thesis, we consider the following nonsmooth minimax fractional programming problem

$$\textbf{(P)} \qquad \min_{x\in X} \max_{y\in Y} \frac{f(x,y)}{g(x,y)}$$

$$\text{subject to} \quad X = \{x \in R^n \mid h(x) \in -R^p_+\}$$

$$\text{and } Y \text{ is a compact subset of } R^m,$$

where $f(\cdot,\cdot)$ and $g(\cdot,\cdot) : R^n \times R^m \to R$ are continuous functions and for each $y \in Y$, $f(\cdot, y), g(\cdot, y)$ and $h : R^n \to R^p$ are locally Lipschitz functions. Without loss of generality, we may assume that $g(x, y) > 0$ and $f(x, y)$ is nonnegative for all $(x, y) \in X \times Y$. The problem (P) is equivalent to a nonfractional parametric programming problem (P_λ):

$$(P_\lambda) \qquad v(\lambda) = \min_{x\in X} \sup_{y\in Y} (f(x,y) - \lambda g(x,y)) \ (\leq 0)$$

If $\lambda = \lambda^*$ is an optimal value of (P), then (P_λ) is a minimization problem with objective function:

$$F_\lambda(x) = \sup_{y\in Y} \big[f(x,y) - \lambda g(x,y)\big].$$

with the defined set for $(s,t,y) \in N \times R^s_+ \times R^{ms}$,

$K_\lambda(x) = \Big\{(s,t,y) \in N \times R^s_+ \times R^{ms} \Big|\ t = (t_1,\dots,t_s) \in R^s_+,\ \sum_{i=1}^{s} t_i = 1,\ y = (y_1,y_2,\dots,y_s),\ y_i \in Y_\lambda(x),\ i = 1,2,\dots,s\Big\}.$

In this thesis, we discuss the following necessary and sufficient conditions for nonsmooth minimax fractional programming problem (P) established by Lai [16]:

Theorem 1.1 (Parametric Necessary Optimality Conditions). Let x^* be a (P)-optimal solution and the constraint qualification hold at x^*. Then there exist $(s^*,t^*,y^*) \in K_{\lambda^*}(x^*)$, $\lambda^* \in R_+$, and a p-vector Lagrange multiplier $\mu^* \in R^p_+$ such that

$$0 \in \sum_{i=1}^{s^*} t^*_i\{\partial^c f(x^*,y^*_i) + \lambda^*\partial^c(-g(x^*,y^*_i))\} + \big\langle \mu^*, \partial^c h(x^*)\big\rangle_p,$$

$$f(x^*,y^*_i) - \lambda^* g(x^*,y^*_i) = 0, \quad i = 1,2,\dots,s^*$$

$$\mu^*_j h_j(x^*) = 0, \quad j = 1,2,\dots,p,$$

$$\mu^* \in R^p_+,\ t^*_i \geq 0,\ \sum_{i=1}^{s^*} t^*_i = 1,\ y^*_i \in Y(x^*), \quad i = 1,2,\dots,s^*,$$

Theorem 1.2 (Sufficient Optimality Conditions). Let x^* be a feasible solution of (P) satisfying the necessary conditions. Suppose that $A(\cdot) = \sum_{i=1}^{s^*} t^*_i\{f(\cdot,y^*_i) - \lambda^* g(\cdot,y^*_i)\}$ and $B(\cdot) = \langle\mu^*, h(\cdot)\rangle_p$ are exponential (p,r)-invex at $x^* \in X$ w.r.t the same function η. Then x^* is an optimal solution of (P).

In Chapter 2, the results obtained in [16] has been discussed. Lai and Ho [16] considered the following parametric type dual:

Parametric Type Duality Model

(D) $$\max_{u} \sup_{(s,t,y,\lambda,\mu)\in K(u)} \lambda$$

subject to

$$0 \in \sum_{i=1}^{s} t_i\big(\partial^c f(u, y_i) + \lambda \partial^c(-g(u, y_i))\big) + \sum_{j=1}^{p} \mu_j \partial^c h_j(u),$$
$$\sum_{i=1}^{s} t_i\big(f(u, y_i) - \lambda g(u, y_i)\big) = 0,$$
$$\sum_{j \in J} \mu_j h_j(u) = 0, \qquad j = 1, 2, \ldots, p,$$

and proved that the dual problem has the same optimal value as the primal problem under some reasonable conditions.

In Chapter 3, we review the results obtained in the paper [12]. Lai and Ho [12] focused on a nonsmooth minimax fractional programming problem involving exponential (p, r)-invexity and constructed a mixed-type dual problem:

Mixed-Type Duality Model

(MD) $$\max_{u} \max_{(\mu,s,t,y)\in K_1(u)} \left(\frac{\sum\limits_{i=1}^{s} t_i f(u, y_i) + \big\langle\big\langle \mu, h(u) \big\rangle\big\rangle_{J_0}}{\sum\limits_{i=1}^{s} t_i g(u, y_i)} \equiv \tilde{F}(u; \mu, s, t, y) \right)$$

subject to

$$0 \in \sum_{i=1}^{s} t_i g(u, y_i) \left[\sum_{i=1}^{s} t_i \partial^c f(u, y_i) + \big\langle\big\langle \mu, \partial^c h(u) \big\rangle\big\rangle_{J_0} \right]$$
$$+ \left[\sum_{i=1}^{s} t_i f(u, y_i) + \big\langle\big\langle \mu, h(u) \big\rangle\big\rangle_{J_0} \right] \sum_{i=1}^{s} t_i \partial^c(-g(u, y_i)),$$
$$+ \sum_{i=1}^{s} t_i g(u, y_i) \sum_{\alpha=1}^{k} \big\langle\big\langle \mu, \partial^c h(u) \big\rangle\big\rangle_{J_\alpha}$$
$$\sum_{j \in J_\alpha} \mu_j h_j(u) = 0, \qquad \alpha = 1, 2, \ldots, k,$$
$$\mu \in R^p_+, \; t \in I, \; y_i \in Y(u), \quad i = 1, 2, \ldots, s,$$

Section 3.4 contains weak and strong duality results under exponential (p, r)-invexity.

In Chapter 4, we consider a minimax fractional programming problem with exponential B-(p, r) invexity. We establish a nonparametric necessary and sufficient optimality conditions. The nonparametric necessary and sufficient optimality conditions deduce to two parameter-free type dual models: Mond-Weir type dual and Wolfe type dual problems.

Mond-Weir Type Duality Model

(MWD) $$\max_{u} \max_{(\mu,s,t,y)\in K_1(u)} \frac{f(u, y)}{g(u, y)}$$

subject to

$$0 \in \sum_{i=1}^{s} t_i \{ g(u, y_i) \partial^c f(u, y_i) + f(u, y_i) \partial^c (-g(u, y_i)) \} + \langle \mu, \partial^c h(u) \rangle_p,$$

$$\sum_{j \in J} \mu_j h_j(u) = 0, \qquad j = 1, 2, \ldots, p,$$

$$\mu \in R^p_+, \ t \in I, \ y_i \in Y(u), \quad i = 1, 2, \ldots, s,$$

and

Wolfe Type Duality Model

(WD)
$$\max_{u \in X} \max_{(\mu, s, t, y) \in K_2(u)} \frac{\sum_{i=1}^{s} t_i f(u, y_i) + \langle \mu, h(u) \rangle_p}{\sum_{i=1}^{s} t_i g(u, y_i)}$$

subject to

$$0 \in \sum_{i=1}^{s} t_i g(u, y_i) \left[\sum_{i=1}^{s} t_i \partial^c f(u, y_i) + \langle \mu, \partial^c h(u) \rangle_p \right] + \left[\sum_{i=1}^{s} t_i f(u, y_i) + \langle \mu, h(u) \rangle_p \right] \sum_{i=1}^{s} t_i \partial^c (-g(u, y_i)),$$

$$\mu \in R^p_+, \ t \in I, \ y_i \in Y(u), \quad i = 1, 2, \ldots, s,$$

With these duality models, Lai and Ho [10] established the duality theorems under exponential (p, r) invexities including weak duality, strong duality and strict converse duality. In this dissertation, we establish the duality theorems under B-(p, r) invexities in order to relate the primal and dual problems.

Chapter 2

Optimality and Duality for Nonsmooth Minimax Fractional Programming

2.1 Introduction

The importance of minimax models and methods is well-known in a great variety of optimal decision making situations. Recently, optimality conditions and various duality results have been obtained for minimax fractional programming problems involving the optimization of several ratios in the objective function. The necessary and sufficient conditions for generalized minimax programming were first presented by Schmitendorf [24]. Many authors proved duality theorems involving in convexity, generalized convexity (cf. [18], [6], [11]) invexity for mathematical programming problem.

In this chapter, we study a nonsmooth minimax fractional programming problem under the assumptions of exponential (p, r)-invexity. Section 2.2 contains notations and preliminaries. In section 2.3, we consider a parametric type dual model and prove appropriate duality relations using the notion of Exp. (p, r)-invexity assumptions.

2.2 Notations and preliminaries

Definition 2.2.1 [16] Let $p,\ r$ be arbitrary real numbers. A differentiable function $f : S \subset R^n \to R$ is said to be exponential (p, r)-invex (strictly) at $u \in S$ if there exists a function $\eta : S \times S \to R^n$ with property $\eta(x, u) = 0$ only if $u = x$ in S such that for each $x \in S$, the following inequalities holds for $\xi \in \partial^c f(u)$:

$$\begin{aligned}
&\frac{1}{r}e^{rf(x)} \geq \frac{1}{r}e^{rf(u)}\left[1+\frac{r}{p}\left\langle \xi, (e^{p\eta(x,u)}-\mathbf{1})\right\rangle\right] \quad (> \text{ if } x \neq u) \quad \text{for} \quad p \neq 0,\ r \neq 0,\\
&e^{rf(x)} - e^{rf(u)} \geq re^{rf(u)}\big\langle \xi, \eta(x,u)\big\rangle \qquad (> \text{ if } x \neq u) \quad \text{for} \quad p = 0,\ r \neq 0,\\
&f(x) - f(u) \geq \frac{1}{p}\big\langle \xi, (e^{p\eta(x,u)}-\mathbf{1})\big\rangle \quad (> \text{ if } x \neq u) \quad \text{for} \quad p \neq 0,\ r = 0,\\
&f(x) - f(u) \geq \big\langle \xi, \eta(x,u)\big\rangle \qquad (> \text{ if } x \neq u) \quad \text{for} \quad p = 0,\ r = 0,
\end{aligned}$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in R^n$, $(e^{p\eta(x,u)} - \mathbf{1})$ stands for the n-vector $(e^{p\eta_1(x,u)} - 1,\ e^{p\eta_2(x,u)} - 1, \ldots,\ e^{p\eta_n(x,u)} - 1)$, and $\langle \cdot, \cdot \rangle$ stands for the inner product in R^n throughout the chapter.
In this chapter, we consider the following minimax fractional programming problem:

$$\textbf{(P)} \qquad \min_{x\in X}\ \max_{y\in Y} \frac{f(x,y)}{g(x,y)}$$
$$\text{subject to} \quad X = \{x \in R^n \mid h(x) \in -R^p_+\}$$
$$\text{and } Y \text{ is a compact subset of } R^m,$$

where $f(\cdot, \cdot)$ and $g(\cdot, \cdot) : R^n \times R^m \to R$ are continuous functions and for each $y \in Y$, $f(\cdot, y), g(\cdot, y)$ and $h : R^n \to R^p$ are locally Lipschitz functions. Without loss of generality, we may assume that $g(x, y) > 0$ and $f(x, y)$ is nonnegative for all $(x, y) \in X \times Y$. Then for $x \in X$, we denote

$$J(x) = \{j \in J \mid h_j(x) = 0\} \text{ where } J = \{1, 2, \ldots, p\}, \text{ and}$$
$$Y(x) = \left\{ y \in Y \,\middle|\, \frac{f(x,y)}{g(x,y)} = \sup_{z\in Y} \frac{f(x,z)}{g(x,z)} \right\}.$$

By the compactness of Y, the continuous function has finite points attended to its maximum points say s. Thus we can set
$K(x) = \big\{(s, t, y) \in N \times R^s_+ \times R^{ms} \big| t = (t_1, \ldots, t_s) \in R^s_+,\ \sum_{i=1}^{s} t_i = 1,\ y = (y_1, y_2, \ldots, y_s),$
$y_i \in Y(x),\ i = 1, 2, \ldots, s\big\}.$

2.3 Parametric type duality model

Denote the set:

$K(u) = \big\{(s,t,y,\lambda,\mu)\big|\ t=(t_1,\dots,t_s)\in R^s_+,\ \sum_{i=1}^{s} t_i = 1,\ y=(y_1,y_2,\dots,y_s),\ y_i\in Y_\lambda(u),\ i = 1,2,\dots,s,\ \lambda\in R_+,\ \mu\in R^p_+\big\}$,

in which μ is p-vector multiplier for the constraint vector function $h(u)\le 0$ and λ is the objective parameter of the dual problem which depends on the feasible variable u of problem (P). We will use the parameter λ as the objective of a parametric dual problem (D) [10] with respect to the primal problem (P) which we constitute a parametric dual problem as maximization problem as following:

(D) $\quad \max_{u} \sup_{(s,t,y,\lambda,\mu)\in K(u)} \lambda$

subject to

$$0 \in \sum_{i=1}^{s} t_i\big(\partial^c f(u,y_i) + \lambda\partial^c(-g(u,y_i))\big) + \sum_{j=1}^{p} \mu_j \partial^c h_j(u), \tag{2.1}$$

$$\sum_{i=1}^{s} t_i\big(f(u,y_i) - \lambda g(u,y_i)\big) = 0, \tag{2.2}$$

$$\sum_{j\in J} \mu_j h_j(u) = 0, \qquad j = 1,2,\dots,p, \tag{2.3}$$

In order to show that the problem (D) is a dual problem w.r.t. the primal problem (P), we denote L by the set of all feasible solutions of (D). Moreover, denote by elements satisfying the necessary optimality conditions of (P) which are defined by projective-like from L into the feasible solutions of (P):

$$L_1 = \{u\in X\subset R^n |\ (u;\mu,s,t,y,\lambda)\in L\}.$$

Now we prove the weak duality theorem related to (P) and (D).

Theorem 2.3.1 (Weak Duality). Let x and $(u;\mu,s,t,y,\lambda)$ be (P)-feasible and (D)-feasible, respectively. Assume that $D(\cdot) = \sum_{i=1}^{s} t_i\{f(\cdot,y_i) - \lambda g(\cdot,y_i)\}$ and $E(\cdot) = \langle \mu, h(\cdot)\rangle_p$ are exponential (p,r)-invex at $u\in L_1$ w.r.t the same function η.

Then

$$\sup_{y\in Y} \frac{f(x,y)}{g(x,y)} \ge \lambda.$$

Proof. Suppose on the contrary, that

$$\sup_{y\in Y} \frac{f(x,y)}{g(x,y)} < \lambda.$$

Then for any $y \in Y$,

$$\frac{f(x,y)}{g(x,y)} < \lambda.$$

This is equivalent to

$$f(x,y) - \lambda g(x,y) < 0 \text{ for all } y \in Y.$$

Multiplying the above expression respectively by $t_i \geq 0, i = 1, 2, \dots, s$ and then summing up, it would yield

$$D(x) = \sum_{i=1}^{s} t_i(f(x,y_i) - \lambda g(x,y_i) < 0 = \sum_{i=1}^{s} t_i(f(u,y_i) - \lambda g(u,y_i)) = D(u).$$

That is

$$\frac{1}{r}e^{rD(x)} < \frac{1}{r}e^{rD(u)}. \tag{2.4}$$

By the expression (2.1), there exists $\xi_i \in \partial^c f(u,y_i), \zeta_i \in \partial^c(-g(u,y_i))$ for all $i = 1, 2, \dots, s$, and $\rho_j \in \partial^c h_j(x)$ for all $j \in J$, such that

$$\langle a \rangle \equiv \sum_{i=1}^{s} t_i\{\xi_i + \lambda\zeta_i\} + \langle \mu, \rho \rangle_p = 0,$$

This implies that the inner product

$$\tfrac{1}{p}\Big\langle \langle a \rangle, (e^{p\eta(x,u)} - \mathbf{1}) \Big\rangle = 0. \tag{2.5}$$

Since D is Exp (p,r)-invex function w.r.t. η at u, we have

$$\frac{1}{r}e^{rD(x)} - \frac{1}{r}e^{rD(u)} \geq \tfrac{1}{p}e^{rD(u)}\Big\langle \sum_{i=1}^{s} t_i(\xi_i + \lambda\zeta_i), (e^{p\eta(x,u)} - \mathbf{1}) \Big\rangle. \tag{2.6}$$

Taking the above inequality together with (2.4), we have

$$\tfrac{1}{p}\Big\langle \sum_{i=1}^{s} t_i\{\xi_i + \lambda\zeta_i\}, (e^{p\eta(x,u)} - \mathbf{1}) \Big\rangle < 0. \tag{2.7}$$

From (2.5) and (2.7), we get

$$\tfrac{1}{p}\Big\langle \sum_{j=1}^{p} \mu_j\rho_j, (e^{p\eta(x,u)} - \mathbf{1}) \Big\rangle > 0. \tag{2.8}$$

By the exponential (p,r)-invexity of $E(.)$ at u w.r.t. the same function η and the inequality (2.8), we get

$$E(x) > E(u). \tag{2.9}$$

On the other hand, from $h_j(x) \leq 0$, $j \in J$ and (2.3), we obtain

$$E(x) = \langle \mu, h(x) \rangle_p \leq 0 = \langle \mu, h(u) \rangle_p = E(u),$$

Which contradicts (2.9). Hence

$$\sup_{y\in Y}\frac{f(x,y)}{g(x,y)}\geq\lambda.$$

Theorem 2.3.2 (Strong Duality). Let x^* be the optimal solution of problem (P) and the constraint qualification be satisfied at x^*. Then there exists $(s^*,t^*,y^*,\lambda^*,\mu^*)\in K(x^*)$, such that $(x^*;\mu^*,s^*,t^*,y^*,\lambda^*)$ is a feasible point for (D). If the hypotheses of Theorem 2.3.1 are fulfilled, then $(x^*;\mu^*,s^*,t^*,y^*,\lambda^*)$ is an efficient solution to problem (D), and the two problems (P) and (D) have the same optimal values.

Proof. By assumption, x^* is an optimal point of (P) and the constraint qualification holds at x^*. Then by necessary optimality conditions of (P), there exist $\lambda^*\in R_+$, $(s^*,t^*,y^*)\in K_{\lambda^*}(x^*)$, and a p-vector Lagrange multiplier $\mu^*\in R^p_+$ such that

$$0\in\sum_{i=1}^{s^*}t_i^*\{\partial^c f(x^*,y_i^*)+\lambda^*\partial^c(-g(x^*,y_i^*))\}+\left\langle\mu^*,\partial^c h(x^*)\right\rangle_p,$$

$$f(x^*,y_i^*)-\lambda^*g(x^*,y_i^*)=0,\quad i=1,2,\ldots,s^*$$

$$\mu_j^*h_j(x^*)=0,\quad j=1,2,\ldots,p,$$

$$\mu^*\in R^p_+,\ t_i^*\geq 0,\ \sum_{i=1}^{s^*}t_i^*=1,\ y_i^*\in Y(x^*),\quad i=1,2,\ldots,s^*,$$

Thus, there exists $(s^*,t^*,y^*,\lambda^*,\mu^*)\in K(x^*)$, such that $(x^*;\mu^*,s^*,t^*,y^*,\lambda^*)$ is a feasible point for (D). From $\lambda^*=\dfrac{f(x^*,y_i^*)}{g(x^*,y_i^*)}$ and Theorem 2.3.1, we have

$$\min(P)=\frac{f(x^*,y_i^*)}{g(x^*,y_i^*)}=\sup_{y\in Y}\frac{f(x^*,y)}{g(x^*,y)}\geq\lambda^*=\frac{f(x^*,y_i^*)}{g(x^*,y_i^*)}.$$

This shows that min(P)=max(D). The proof is complete.

Theorem 2.3.3 (Strict Converse Duality). Let x^* and $(u^*;\mu^*,s^*,t^*,y^*,\lambda^*)$ be the efficient solutions of problem (P) and (D), respectively, and the constraint qualification be satisfied at x^*. Assume that $A(\cdot)=\sum_{i=1}^{s^*}t_i^*\{f(\cdot,y_i^*)-\lambda^*g(\cdot,y_i^*)\}$ is strictly exponential (p,r)-invex and $B(\cdot)=\langle\mu^*,h(\cdot)\rangle_p$ is exponential (p,r)-invex at $u^*\in L_1$ w.r.t η for all optimal vectors x^* for (P), and $(u^*;\mu^*,s^*,t^*,y^*,\lambda^*)$ for (D), respectively. Then $x^*=u^*$, and the optimal values of problems (P) and (D) are equal.

Proof. Suppose on the contrary that $x^*\neq u^*$. Then by (2.1), there exists $\xi_i\in\partial^c f(u^*,y_i^*),\zeta_i\in\partial^c(-g(u^*,y_i^*))$ for all $i=1,2,\ldots,s^*$, and $\rho_j\in\partial^c h_j(u^*)$ for all $j\in J$, such that

$$\langle b\rangle\equiv\sum_{i=1}^{s^*}t_i^*\{\xi_i+\lambda^*\zeta_i\}+\langle\mu^*,\rho\rangle_p=0,$$

that is, $\langle b\rangle$ is a zero vector, where $\langle\mu^*,\rho\rangle_p=\sum_{j=1}^{p}\mu_j^*\rho_j$ and $\rho=(\rho_1,\rho_2,\ldots,\rho_p)$.

This implies that

$$\tfrac{1}{p}\Big\langle \langle b\rangle, (e^{p\eta(x^*,u^*)}-\mathbf{1})\Big\rangle = 0. \tag{2.10}$$

Since x^* and $(u^*;\mu^*,s^*,t^*,y^*,\lambda^*)$ are optimal solutions of (P) and (D), respectively, we know

$$B(x^*) = \sum_{j=1}^{p}\mu_j^* h_j(x^*) \leq 0$$

and

$$B(u^*) = \sum_{j=1}^{p}\mu_j^* h_j(u^*) = 0.$$

This implies that

$$B(x^*) \leq B(u^*).$$

By the exponential function, we get

$$\frac{1}{r}e^{rB(x^*)} - \frac{1}{r}e^{rB(u^*)} \leq 0 \text{ for any } r \neq 0. \tag{2.11}$$

Since $B(.)$ is exponential $(p,r)-$invex w.r.t. η at u^*,we get

$$\frac{1}{r}e^{rB(x^*)} - \frac{1}{r}e^{rB(u^*)} \geq \tfrac{1}{p}e^{rB(u^*)}\Big\langle \sum_{j=1}^{p}\mu_j^*\rho_j, (e^{p\eta(x^*,u^*)}-\mathbf{1})\Big\rangle. \tag{2.12}$$

From above inequality and (2.11), we have

$$\tfrac{1}{p}\Big\langle \sum_{j=1}^{p}\mu_j^*\rho_j, (e^{p\eta(x^*,u^*)}-\mathbf{1})\Big\rangle \leq 0. \tag{2.13}$$

Also from (2.10) and (2.13), we have

$$\tfrac{1}{p}\Big\langle \sum_{i=1}^{s^*} t_i^*\{\xi_i + \lambda^*\zeta_i\}, (e^{p\eta(x^*,u^*)}-\mathbf{1})\Big\rangle \geq 0. \tag{2.14}$$

Moreover, from the exponential type $(p,r)-$invexity with respect to the same function η at u^* of $A(.)$ and the inequality (2.14), we get

$$\frac{1}{r}e^{rA(x^*)} - \frac{1}{r}e^{rA(u^*)} > 0 \text{ for any } r \neq 0.$$

This implies

$$A(x^*) > A(u^*). \tag{2.15}$$

By Theorem 2.3.2, we see that

$$\sup_{y\in Y}\frac{f(x^*,y)}{g(x^*,y)} = \lambda^*.$$

and so

$$\frac{f(x^*,y)}{g(x^*,y)} \leq \lambda^* \text{ for all } y \in Y.$$

This implies that

$$f(x^*,y) - \lambda^* g(x^*,y) \leq 0 \text{ for all } y \in Y.$$

Multiplying the above expression respectively by $t_i^* \geq 0, i = 1, 2, \ldots, s^*$ and then summing up, it would yield

$$\begin{aligned} A(x^*) &= \sum_{i=1}^{s^*} t_i^*(f(x^*, y_i^*) - \lambda^* g(x^*, y_i^*)) \\ &\leq 0 = \sum_{i=1}^{s^*} t_i^*(f(u^*, y_i^*) - \lambda^* g(u^*, y_i^*)) = A(u^*). \end{aligned}$$

This contradicts the inequality (2.15). Hence the proof is complete.

Chapter 3

Mixed-Type Duality for Nonsmooth Minimax Fractional Programming

3.1 Introduction

Minimax mathematical programming and deriving the duality theorems for them have been of much interest in the recent past. The mathematical programming problem in which the objective function is a ratio of two numerical functions is called a fractional programming problem. It is used in business and economic situations, mainly in the situations of deficit of financial resources. Several researchers have made the efforts to study this minimax fractional programming problem (cf. [1], [5], [8], [10], [13]-[15], [20]).

In this chapter, we consider a nonsmooth minimax fractional programming problem involving exponential (p, r)-invexity. We introduce a new class of Lipschitz functions, namely exponential (p, r)-invex Lipschitz functions. This chapter is divided into three sections. Section 3.2 contains notations and preliminaries. In section 3.3, we consider a nonparametric mixed type dual model and prove the duality results .

3.2 Notations and preliminaries

Definition 3.2.1 Let $p,\ r$ be arbitrary real numbers. A differentiable function $f : S \subset R^n \to R$ is said to be exponential (p, r)-invex (strictly) at $u \in S$ if there exists a function

$\eta : S \times S \to R^n$ with property $\eta(x,u) = 0$ only if $u = x$ in S such that for each $x \in S$, the following inequality holds for $\xi \in \partial^c f(u)$:

$$\begin{aligned}
&\frac{1}{r}e^{rf(x)} \geq \frac{1}{r}e^{rf(u)}\left[1 + \frac{r}{p}\left\langle \xi, (e^{p\eta(x,u)} - \mathbf{1})\right\rangle\right] \quad (> \text{ if } x \neq u) \quad \text{for} \quad p \neq 0,\ r \neq 0,\\
&e^{rf(x)} - e^{rf(u)} \geq re^{rf(u)}\big\langle \xi, \eta(x,u)\big\rangle \qquad (> \text{ if } x \neq u) \quad \text{for} \quad p = 0,\ r \neq 0,\\
&f(x) - f(u) \geq \frac{1}{p}\big\langle \xi, (e^{p\eta(x,u)} - \mathbf{1})\big\rangle \quad (> \text{ if } x \neq u) \quad \text{for} \quad p \neq 0,\ r = 0,\\
&f(x) - f(u) \geq \big\langle \xi, \eta(x,u)\big\rangle \qquad (> \text{ if } x \neq u) \quad \text{for} \quad p = 0,\ r = 0,
\end{aligned}$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in R^n$, $(e^{p\eta(x,u)} - \mathbf{1})$ stands for the n-vector $(e^{p\eta_1(x,u)} - 1,\ e^{p\eta_2(x,u)} - 1, \ldots,\ e^{p\eta_n(x,u)} - 1)$, and $\langle \cdot, \cdot \rangle$ stands for the inner product in R^n throughout the chapter.

If h is a function from R^n to R^p, then $h(x)$ is a p-vector where $h = (h_1, \ldots, h_p)$. R^n denotes the n-dimensional Euclidean space, $R^p_+ = \{x \in R^p : x_j \geq 0,\ j = 1, 2, \ldots, p\}$ the non-negative orthant of R^p and R^+ the set of nonnegative real numbers.

In this chapter, we consider the following minimax fractional programming problem:

$$\begin{aligned}
\textbf{(P)} \qquad & \min_{x \in X}\ \max_{y \in Y} \frac{f(x,y)}{g(x,y)}\\
& \text{subject to} \quad X = \{x \in R^n \mid h(x) \in -R^p_+\}\\
& \text{and } Y \text{ is a compact subset of } R^m,
\end{aligned}$$

where $f(\cdot,\cdot)$ and $g(\cdot,\cdot) : R^n \times R^m \to R$ are continuous functions and for each $y \in Y$, $f(\cdot, y), g(\cdot, y)$ and $h : R^n \to R^p$ are locally Lipschitz functions. Without loss of generality, we may assume that $g(x,y) > 0$ and $f(x,y)$ is nonnegative for all $(x,y) \in X \times Y$. Then for $x \in X$, we denote

$$\begin{aligned}
&J(x) = \{j \in J \mid h_j(x) = 0\} \text{ where } J = \{1, 2, \ldots, p\}, \text{ and}\\
&Y(x) = \left\{ y \in Y \,\middle|\, \frac{f(x,y)}{g(x,y)} = \sup_{z \in Y} \frac{f(x,z)}{g(x,z)} \right\}.
\end{aligned}$$

By the compactness of Y, the continuous function has finite points attended to its maximum points say s. Thus we can set

$K(x) = \Big\{(s,t,y) \in N \times R^s_+ \times R^{ms} \Big| t = (t_1, \ldots, t_s) \in R^s_+,\ \sum_{i=1}^{s} t_i = 1,\ y = (y_1, y_2, \ldots, y_s),$
$y_i \in Y(x),\ i = 1, 2, \ldots, s\Big\}.$

3.3 Optimality conditions

The necessary and sufficient optimality conditions for problem (P) using exponential (p, r)-invexity are as follows:

Theorem 3.3.1 [10](Parameter-Free Necessary Optimality Conditions). Let x^* be a (P)-optimal solution and the constraint qualification hold at x^*. Then there exist $(s^*, t^*, y^*) \in K(x^*)$ and a p-vector Lagrange multiplier $\mu^* \in R^p_+$ such that

$$0 \in \sum_{i=1}^{s^*} t_i^* g(x^*, y_i^*) \left[\sum_{i=1}^{s^*} t_i^* \partial^o f(x^*, y_i^*) + \left\langle \mu^*, \partial^o h(x^*) \right\rangle_p \right] + \left[\sum_{i=1}^{s^*} t_i^* f(x^*, y_i^*) + \left\langle \mu^*, h(x^*) \right\rangle_p \right] \sum_{i=1}^{s^*} t_i^* \partial^o(-g(x^*, y_i^*)), \tag{3.1}$$

$$\mu_j^* h_j(x^*) = 0, \quad j = 1, 2, \ldots, p, \tag{3.2}$$

$$\mu^* \in R^p_+, \ t^* \in I, \ y_i^* \in Y(x^*), \quad i = 1, 2, \ldots, s^*, \tag{3.3}$$

where $I = \{t^* \in R^{s^*} : t^* = (t_1^*, t_2^*, \ldots, t_{s^*}^*), \ t_i^* \geq 0, \ i = 1, 2, \ldots, s^* \text{ with } \sum_{i=1}^{s^*} t_i^* = 1\}$ and $\left\langle \mu^*, \partial^o h(x^*) \right\rangle_p \equiv \sum_{j=1}^{p} \mu_j^* \partial^o h_j(x^*)$.

Theorem 3.3.2 [10](Sufficient Optimality Conditions). Let x^* be a feasible solution of (P) and the necessary conditions (3.1) $-$ (3.3) hold. Denote a function $C : X \to R$ by

$$C(\cdot) = \sum_{i=1}^{s} t_i g(x^*, y_i) \left[\sum_{i=1}^{s} t_i f(\cdot, y_i) + \left\langle \mu, h(\cdot) \right\rangle_p \right] + \left[\sum_{i=1}^{s} t_i f(x^*, y_i) + \left\langle \mu, h(x^*) \right\rangle_p \right] \sum_{i=1}^{s} t_i(-g(\cdot, y_i))$$

with $C(x^*) = 0$. Assume that $C(.)$ is Exp $(p, r)-$invex w.r.t. η at x^* in X. Then x^* is an optimal solution of (P).

Remark 3.3.1 The objective function of problem (P) satisfies

$$F(x) = \sup_{y \in Y} \frac{f(x, y)}{g(x, y)} = \sup_{t \in I} \frac{\sum_{i=1}^{s} t_i f(x, y_i)}{\sum_{i=1}^{s} t_i g(x, y_i)} \text{ for } y_i \in Y(x).$$

3.4 Mixed type duality model

A mixed-type dual model can be considered as an incomplete Lagrangian duality form and, employing Remark 3.3.1, we can perform a mixed-type dual form w.r.t. the primal fractional programming problem (P) as follows. First, we partition the index set $J = \{1, 2, \ldots, p\}$ of the constraint function $h = (h_1, \ldots, h_p) : R^n \to R^p$ of (P) to be $J_0, J_1, \ldots, J_k$ $(k < p)$ with $\bigcup_{\alpha=0}^{k} J_\alpha = J$ and $J_\alpha \bigcap J_\beta = \phi$ if $\alpha \neq \beta$. Then the mixed-type problem is performed as the following form:

$$\textbf{(MD)} \qquad \max_{u} \max_{(\mu,s,t,y)\in K_1(u)} \left(\frac{\sum_{i=1}^{s} t_i f(u, y_i) + \langle\langle \mu, h(u)\rangle\rangle_{J_0}}{\sum_{i=1}^{s} t_i g(u, y_i)} \equiv \tilde{F}(u; \mu, s, t, y) \right)$$

subject to

$$\begin{aligned} 0 \in & \sum_{i=1}^{s} t_i g(u, y_i) \left[\sum_{i=1}^{s} t_i \partial^c f(u, y_i) + \langle\langle \mu, \partial^c h(u)\rangle\rangle_{J_0} \right] \\ & + \left[\sum_{i=1}^{s} t_i f(u, y_i) + \langle\langle \mu, h(u)\rangle\rangle_{J_0} \right] \sum_{i=1}^{s} t_i \partial^c(-g(u, y_i)), \\ & + \sum_{i=1}^{s} t_i g(u, y_i) \sum_{\alpha=1}^{k} \langle\langle \mu, \partial^c h(u)\rangle\rangle_{J_\alpha}, \end{aligned} \tag{3.4}$$

$$\sum_{j\in J_\alpha} \mu_j h_j(u) = 0, \qquad \alpha = 1, 2, \ldots, k, \tag{3.5}$$

$$\mu \in R^p_+,\ t \in I,\ y_i \in Y(u), \quad i = 1, 2, \ldots, s, \tag{3.6}$$

Here

(i) $\langle\langle \mu, h(u)\rangle\rangle_{J_0} \equiv \sum_{j\in J_0} \mu_j h_j(u)$ and $\langle\langle \mu, \partial^c h(u)\rangle\rangle_{J_\alpha} \equiv \sum_{j\in J_\alpha} \mu_j \partial^c h_j(u)$ for $\alpha = 0, 1, 2, \ldots, k$.

(ii) For $u \in X$, the (P)-feasible solution, we denote $K_1(u) = \big\{(\mu, s, t, y) \in R^p_+ \times N \times R^s_+ \times R^{ms} \big|\ t = (t_1, \ldots, t_s) \in R^s_+,\ \sum_{i=1}^{s} t_i = 1,\ y = (y_1, y_2, \ldots, y_s),\ y_i \in Y(u),\ i = 1, 2, \ldots, s\big\}$,

where elements in $K_1(u)$ satisfy the expressions (3.1) and (3.2) in the necessary conditions of (P) in Theorem 3.3.1.

Without loss of generality, we assume that

$$\sum_{i=1}^{s} t_i f(u, y_i) + \langle\langle \mu, h(u)\rangle\rangle_{J_0} \geq 0 \text{ and } \sum_{i=1}^{s} t_i g(u, y_i) > 0.$$

In order to show problem (MD) [12] is surely a dual problem w.r.t. the primal problem (P), we denote L as the constraint set of (MD). Moreover, we denote by elements satisfying the necessary optimality conditions of (P), which is defined to be projective-like by the feasible solutions of problem (P) to be

$$L_1 = \big\{u \in X \subseteq R^n \big|\ (u : \mu, s, t, y) \in L\big\}.$$

First, we show the weak duality theorem related to problems (P) and (MD) as following:

Theorem 3.4.1 (Weak Duality). Let x and $(u;\mu,s,t,y)$ be (P)-feasible and (MD)-feasible, respectively. Denote a function $D: X \to R$ by

$$D(\cdot) = \sum_{i=1}^{s} t_i g(u,y_i)\left[\sum_{i=1}^{s} t_i f(\cdot,y_i) + \langle\langle \mu, h(\cdot)\rangle\rangle_{J_0}\right] - \left[\sum_{i=1}^{s} t_i f(u,y_i) + \langle\langle \mu, h(u)\rangle\rangle_{J_0}\right]\sum_{i=1}^{s} t_i(g(\cdot,y_i))$$

with $D(u)=0$. Assume that $D(\cdot)$ and $\langle\langle \mu, h(\cdot)\rangle\rangle_{J_\alpha}$ for $\alpha = 1,2,\ldots,k$ are exponential (p,r)-invex at $u \in L_1$ w.r.t the same function η.

Then $F(x) \geq \tilde{F}(u;\mu,s,t,y)$.

Proof. Suppose that $F(x) \geq \tilde{F}(u;\mu,s,t,y)$ was not true. Then there would be a feasible solution, $x \in X$ such that

$$F(x) < \tilde{F}(u;\mu,s,t,y) \tag{3.7}$$

for any $(u;\mu,s,t,y) \in L$.

Let $i = 1,2,\ldots,s$, we have

$$\tilde{F}(u;\mu,s,t,y) = \frac{\sum_{i=1}^{s} t_i f(u,y_i) + \langle\langle \mu, h(u)\rangle\rangle_{J_0}}{\sum_{i=1}^{s} t_i g(u,y_i)} \text{ for all } y_i \in Y(u), \tag{3.8}$$

and

$$F(x) \geq \frac{f(x,y)}{g(x,y)} \text{ for all } y \in Y. \tag{3.9}$$

By expressions (3.7), (3.8) and (3.9), it yields

$$f(x,y)\sum_{i=1}^{s} t_i g(u,y_i) - g(x,y)\left[\sum_{i=1}^{s} t_i f(u,y_i) + \langle\langle \mu, h(u)\rangle\rangle_{J_0}\right] < 0$$

for all $i = 1,2,\ldots,s$, and $y \in Y$.

Multiplying the above expression respectively, by $t_i,\ i = 1,2,\ldots,s$ and adding them, it yields

$$\sum_{i=1}^{s} t_i f(x,y_i)\sum_{i=1}^{s} t_i g(u,y_i) - \sum_{i=1}^{s} t_i g(x,y_i)\left[\sum_{i=1}^{s} t_i f(u,y_i) + \langle\langle \mu, h(u)\rangle\rangle_{J_0}\right] < 0,$$

that is,

$$\begin{aligned} &\sum_{i=1}^{s} t_i g(u,y_i)\left[\sum_{i=1}^{s} t_i f(x,y_i) + \langle\langle \mu, h(x)\rangle\rangle_{J_0}\right] \\ &\quad - \sum_{i=1}^{s} t_i g(x,y_i)\left[\sum_{i=1}^{s} t_i f(u,y_i) + \langle\langle \mu, h(u)\rangle\rangle_{J_0}\right] \\ &\qquad = D(x) < \sum_{i=1}^{s} t_i g(u,y_i)\langle\langle \mu, h(x)\rangle\rangle_{J_0}. \end{aligned} \tag{3.10}$$

Since the relations (3.6), $g(u, y_i) > 0, i = 1, 2, ..., s$, and $h_j(x) \leq 0$, we get

$$\sum_{i=1}^{s} t_i g(u, y_i) \big\langle \langle \mu, h(x) \rangle \big\rangle_{J_0} \leq 0.$$

Therefore, from (3.10), we obtain

$$D(x) < 0 = D(u). \tag{3.11}$$

Let x and $(u; \mu, s, t, y)$ be (P)-feasible and (MD)-feasible, respectively. According to the expression (3.4), there exist $\xi_i \in \partial^c f(u, y_i)$, $\zeta_i \in \partial^c(-g)(u, y_i)$, $i = 1, 2, ..., s$, and $\rho_j \in \partial^c h_j(u)$, $j \in J$ such that the vector

$$\langle a \rangle \equiv \sum_{i=1}^{s} t_i g(u, y_i) \left[\big\langle t, \xi \big\rangle_s + \big\langle \langle \mu, \rho \rangle \big\rangle_{J_0} \right] + \left[\sum_{i=1}^{s} t_i f(u, y_i) + \big\langle \langle \mu, h(u) \rangle \big\rangle_{J_0} \right] \big\langle t, \zeta \big\rangle_s$$
$$+ \sum_{i=1}^{s} t_i g(u, y_i) \sum_{\alpha=1}^{k} \big\langle \langle \mu, \rho \rangle \big\rangle_{J_\alpha} = 0,$$

that is, the vector $\langle a \rangle$ is a zero vector, where

$$\big\langle t, \xi \big\rangle_s \equiv \sum_{i=1}^{s} t_i \xi_i, \quad \big\langle t, \zeta \big\rangle_s \equiv \sum_{i=1}^{s} t_i \zeta_i, \quad \big\langle \langle \mu, \rho \rangle \big\rangle_{J_\alpha} \equiv \sum_{j=1}^{p} \mu_j \rho_j,$$

for $\alpha = 1, 2, \ldots, k$, $\xi = (\xi_1, \ldots, \xi_s)$, $\zeta = (\zeta_1, \ldots, \zeta_s)$, and $\rho = (\rho_1, \ldots, \rho_p)$.

This implies that

$$\frac{1}{p} \Big\langle < a >, (e^{p\eta(x,u)} - \mathbf{1}) \Big\rangle = 0. \tag{3.12}$$

since $x \in X$ and equality (3.5), we have

$$\big\langle \langle \mu, h(x) \rangle \big\rangle_{J_\alpha} \leq 0 = \big\langle \langle \mu, h(u) \rangle \big\rangle_{J_\alpha} \text{ for } \alpha = 1, 2, \ldots, k.$$

Because of the fundamental property of exponential functions, we reduce

$$\frac{1}{r} e^{r \langle \langle \mu, h(x) \rangle \rangle_{J_\alpha}} - \frac{1}{r} e^{r \langle \langle \mu, h(u) \rangle \rangle_{J_\alpha}} \leq 0 \text{ for } \alpha = 1, 2, \ldots, k. \tag{3.13}$$

Using the Exp. (p, r)-invexity of $\big\langle \langle \mu, h(x) \rangle \big\rangle_{J_\alpha}$, $\alpha = 1, 2, \ldots, k$ at u in L_1 w.r.t. the same η, we see that

$$\frac{1}{r} e^{r \langle \langle \mu, h(x) \rangle \rangle_{J_\alpha}} - \frac{1}{r} e^{r \langle \langle \mu, h(u) \rangle \rangle_{J_\alpha}} \geq \frac{1}{p} e^{r \langle \langle \mu, h(u) \rangle \rangle_{J_\alpha}} \big\langle \big\langle \langle \mu, \rho \rangle \big\rangle_{J_\alpha}, (e^{p\eta(x,u)} - \mathbf{1}) \big\rangle$$

for $\alpha = 1, 2, \ldots, k$.

The inequalities together with inequality (3.13), yield

$$0 \geq \frac{1}{p} \big\langle \big\langle \langle \mu, \rho \rangle \big\rangle_{J_\alpha} (e^{p\eta(x,u)} - \mathbf{1}) \big\rangle \text{ for } \alpha = 1, 2, \ldots, k.$$

Multiplying $\sum_{i=1}^{s} t_i g(u, y_i)$, the above α-inequalities, respectively, and adding them, we have

$$0 \geq \frac{1}{p} \left\langle \sum_{i=1}^{s} t_i g(u, y_i) \sum_{\alpha=1}^{k} \big\langle \langle \mu, \rho \rangle \big\rangle_{J_\alpha}, (e^{p\eta(x,u)} - \mathbf{1}) \right\rangle. \tag{3.14}$$

From equality (3.12) and (3.14), we know

$$\frac{1}{p}\Big\langle < \hat{a} >, (e^{p\eta(x,u)} - \mathbf{1})\Big\rangle \geq 0. \tag{3.15}$$

where $< \hat{a} >$ denotes the expression as follows

$$\langle \hat{a} \rangle \equiv \sum_{i=1}^{s} t_i g(u, y_i)\Big[\big\langle t, \xi\big\rangle_s + \big\langle\big\langle \mu, \rho\big\rangle\big\rangle_{J_0}\Big] + \Big[\sum_{i=1}^{s} t_i f(u, y_i) + \big\langle\big\langle \mu, h(u)\big\rangle\big\rangle_{J_0}\Big]\big\langle t, \zeta\big\rangle_s.$$

If D is Exp. (p, r)-invex w.r.t. the same η at u in L_1, we get

$$\frac{1}{r}e^{rD(x)} - \frac{1}{r}e^{rD(u)} \geq \frac{1}{p}e^{rD(u)}\Big\langle < \hat{d} >, (e^{p\eta(x,u)} - \mathbf{1})\Big\rangle.$$

According to the above relation and (3.15), we have

$$\frac{1}{r}e^{rD(x)} \geq \frac{1}{r}e^{rD(u)}.$$

By the exponential function, we obtain

$$D(x) \geq D(u) = 0 \text{ for } r \neq 0;$$

this contradicts inequality (3.11). Hence, the proof is complete.

Theorem 3.4.2 (Strong Duality). Let x^* be an optimal solution of (P) satisfying the constraint qualification. Then there exists $(\mu^*, s^*, t^*, y^*) \in K_1(x^*)$, such that $(x^*; \mu^*, s^*, t^*, y^*)$ is a feasible solution for (MD). In addition, if the hypothesis of Theorem 3.4.1 holds for all (MD)-feasible points $(x; \mu, s, t, y)$, then $(x^*; \mu^*, s^*, t^*, y^*)$ is an optimal solution of (MD) and the two problems (P) and (MD) have the same optimal values.

Proof. By Theorem 3.3.1, there exists $(s^*, t^*, y^*) \in K(x^*)$, and a p-vector Lagrange multiplier $\mu^* \in R_+^p$ such that

$$\begin{aligned}0 \in \sum_{i=1}^{s^*} t_i^* g(x^*, y_i^*)\Big[\sum_{i=1}^{s^*} t_i^* \partial^o f(x^*, y_i^*) + \big\langle \mu^*, \partial^o h(x^*)\big\rangle_p\Big] \\ + \Big[\sum_{i=1}^{s^*} t_i^* f(x^*, y_i^*) + \big\langle \mu^*, h(x^*)\big\rangle_p\Big]\sum_{i=1}^{s^*} t_i^* \partial^o(-g(x^*, y_i^*)),\end{aligned}$$

and

$$\mu_j^* h_j(x^*) = 0, \quad j = 1, 2, \ldots, p,$$

where

$$\mu^* \in R_+^p,\ t^* \in I,\ y_i^* \in Y(x^*), \qquad i = 1, 2, \ldots, s^*,$$

Then there exists $(\mu^*, s^*, t^*, y^*) \in K_1(x^*)$ such that $(x^*; \mu^*, s^*, t^*, y^*)$ turns to a feasible solution of (MD). Actually, $(x^*; \mu^*, s^*, t^*, y^*)$ is also an efficient solution of (MD).

If $(x^*; \mu^*, s^*, t^*, y^*)$ were not an efficient solution of (MD), then there must be some feasible solution $(x;\mu,s,t,y)$ of (MD), such that

$$\frac{\sum_{i=1}^{s^*} t_i^* f(x^*,y_i^*) + \left\langle \left\langle \mu^*, h(x^*) \right\rangle \right\rangle_{J_0}}{\sum_{i=1}^{s^*} t_i^* g(x^*,y_i^*)} < \frac{\sum_{i=1}^{s} t_i f(x,y_i) + \left\langle \left\langle \mu, h(x) \right\rangle \right\rangle_{J_0}}{\sum_{i=1}^{s} t_i g(x,y_i)}.$$

It follows from the above inequality and equality (3.2) that results in a contradiction

$$F(x^*) < \tilde{F}(x;\mu,s,t,y).$$

Hence, $(x^*;\mu^*,s^*,t^*,y^*)$ is an efficient solution of (MD).

Now, we prove the strict converse duality theorem as follows.

Theorem 3.4.3 (Strict Converse Duality). Let x^* and $(u^*;\mu^*,s^*,t^*,y^*)$ be optimal solutions of (P) and (MD), respectively, and the constraint qualification be satisfied at x^*. Denote a function $E : X \to R$ by

$$\begin{aligned} E(\cdot) = {} & \sum_{i=1}^{s^*} t_i g(u^*,y_i^*) \left[\sum_{i=1}^{s^*} t_i^* f(\cdot,y_i^*) + \left\langle \left\langle \mu^*, h(\cdot) \right\rangle \right\rangle_{J_0} \right] \\ & - \left[\sum_{i=1}^{s^*} t_i^* f(u^*,y_i^*) + \left\langle \left\langle \mu^*, h(u^*) \right\rangle \right\rangle_{J_0} \right] \sum_{i=1}^{s^*} t_i^* (g(\cdot,y_i^*)) \end{aligned}$$

If any one of the following conditions holds:

(i) $E(\cdot)$ is a strictly Exp. (p,r)-invex and $\left\langle \left\langle \mu^*, h(\cdot) \right\rangle \right\rangle_{J_\alpha}$ for $\alpha = 1,2,\ldots,k$ are Exp. (p,r)-invex w.r.t. the same η at u^* in L_1.

(ii) $E(\cdot)$ is an Exp. (p,r)-invex and at least one of $\left\langle \left\langle \mu^*, h(\cdot) \right\rangle \right\rangle_{J_\alpha}$ for $\alpha = 1,2,\ldots,k$ are strictly Exp. (p,r)-invex w.r.t. the same η at u^* in L_1.

Then $x^* = u^*$; that is, u^* solves the problem (P) and

$$\sup_{y \in Y} \frac{f(u^*,y)}{g(u^*,y)} = \frac{\sum_{i=1}^{s^*} t_i^* f(u^*,y_i^*) + \left\langle \left\langle \mu^*, h(u^*) \right\rangle \right\rangle_{J_0}}{\sum_{i=1}^{s^*} t_i^* g(u^*,y_i^*)}.$$

Proof. Since x^* and $(u^*;\mu^*,s^*,t^*,y^*)$ are optimal solutions of (P) and (MD), respectively, we obtain

$$\left\langle \left\langle \mu^*, h(x^*) \right\rangle \right\rangle_{J_\alpha} \leq 0 \text{ for } \alpha = 1,2,\ldots,k,$$

and

$$\left\langle \left\langle \mu^*, h(u^*) \right\rangle \right\rangle_{J_\alpha} = 0 \text{ for } \alpha = 1,2,\ldots,k.$$

This implies that

$$\big\langle\big\langle\mu^*, h(x^*)\big\rangle\big\rangle_{J_\alpha} \leq 0 = \big\langle\big\langle\mu^*, h(u^*)\big\rangle\big\rangle_{J_\alpha} \text{ for } \alpha = 1, 2, \ldots, k.$$

This inequality together with the property of exponential functions, it yields

$$\frac{1}{r} e^{r\big\langle\big\langle\mu^*, h(x^*)\big\rangle\big\rangle_{J_\alpha}} - \frac{1}{r} e^{r\big\langle\big\langle\mu^*, h(u^*)\big\rangle\big\rangle_{J_\alpha}} \leq 0 \text{ for } \alpha = 1, 2, \ldots, k. \tag{3.16}$$

According to the expression (3.4), there exist $\xi_i \in \partial^c f(u^*, y_i^*)$, $\zeta_i \in \partial^c(-g)(u^*, y_i^*)$, $i = 1, 2, ..., s^*$, and $\rho_j \in \partial^c h_j(u^*)$, $j \in J$ such that the vector

$$\begin{aligned} \langle b\rangle \equiv & \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) \Big[\big\langle t^*, \xi\big\rangle_{s^*} + \big\langle\big\langle\mu^*, \rho\big\rangle\big\rangle_{J_0}\Big] + \Big[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \big\langle\big\langle\mu^*, h(u^*)\big\rangle\big\rangle_{J_0}\Big]\big\langle t^*, \zeta\big\rangle_{s^*} \\ & + \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) \sum_{\alpha=1}^{k} \big\langle\big\langle\mu^*, \rho\big\rangle\big\rangle_{J_\alpha} = 0, \end{aligned} \tag{3.17}$$

that is, the vector $\langle b\rangle$ is a zero vector, where

$$\big\langle t^*, \xi\big\rangle_{s^*} \equiv \sum_{i=1}^{s^*} t_i^* \xi_i, \quad \big\langle t^*, \zeta\big\rangle_{s^*} \equiv \sum_{i=1}^{s^*} t_i^* \zeta_i, \quad \big\langle\big\langle\mu^*, \rho\big\rangle\big\rangle_{J_\alpha} \equiv \sum_{j=1}^{p} \mu_j^* \rho_j,$$

for $\alpha = 1, 2, \ldots, k$, $\xi = (\xi_1, \ldots, \xi_{s^*})$, $\zeta = (\zeta_1, \ldots, \zeta_{s^*})$, and $\rho = (\rho_1, \ldots, \rho_p)$.

This implies that

$$\frac{1}{p}\Big\langle < b >, (e^{p\eta(x^*, u^*)} - \mathbf{1})\Big\rangle = 0. \tag{3.18}$$

If hypothesis (i) holds, $\big\langle\big\langle\mu^*, h(.)\big\rangle\big\rangle_{J_\alpha}$ for $\alpha = 1, 2, \ldots, k$ are Exp. (p, r)-invex w.r.t. the same function η at $u^* \in L_1$, we obtain

$$\frac{1}{r} e^{r\big\langle\big\langle\mu^*, h(x^*)\big\rangle\big\rangle_{J_\alpha}} - \frac{1}{r} e^{r\big\langle\big\langle\mu^*, h(u^*)\big\rangle\big\rangle_{J_\alpha}} \geq \frac{1}{p} e^{r\big\langle\big\langle\mu^*, h(u^*)\big\rangle\big\rangle_{J_\alpha}} \Big\langle\big\langle\big\langle\mu^*, \rho\big\rangle\big\rangle_{J_\alpha}, (e^{p\eta(x^*, u^*)} - \mathbf{1})\Big\rangle$$

for $\alpha = 1, 2, \ldots, k$.

Taking the above inequality together with inequality (3.16), yield

$$0 \geq \frac{1}{p}\Big\langle\big\langle\big\langle\mu^*, \rho\big\rangle\big\rangle_{J_\alpha} (e^{p\eta(x^*, u^*)} - \mathbf{1})\Big\rangle \leq 0 \text{ for } \alpha = 1, 2, \ldots, k.$$

Multiplying $\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)$, the above α-inequalities, respectively, and adding them, we have

$$0 \geq \frac{1}{p}\Big\langle \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) \sum_{\alpha=1}^{k} \big\langle\big\langle\mu, \rho\big\rangle\big\rangle_{J_\alpha}, (e^{p\eta(x^*, u^*)} - \mathbf{1})\Big\rangle. \tag{3.19}$$

We want to prove $x^* = u^*$.

Suppose $x^* \neq u^*$. This will results in a contradiction.

Actually, it follows from the relations (3.18) and (3.19), we know

$$\frac{1}{p}\Big\langle < \hat{b} >, (e^{p\eta(x^*, u^*)} - \mathbf{1})\Big\rangle \geq 0. \tag{3.20}$$

where $< \hat{b} >$ denotes the expression as follows

$$\langle \hat{b}\rangle \equiv \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) \Big[\big\langle t^*, \xi\big\rangle_{s^*} + \big\langle\big\langle\mu^*, \rho\big\rangle\big\rangle_{J_0}\Big] + \Big[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \big\langle\big\langle\mu^*, h(u^*)\big\rangle\big\rangle_{J_0}\Big]\big\langle t^*, \zeta\big\rangle_{s^*}.$$

If E is strictly Exp. (p, r)-invex w.r.t. the same η at u^* in L_1, we get

$$\frac{1}{r}e^{rE(x^*)} - \frac{1}{r}e^{rE(u^*)} > \frac{1}{p}e^{rE(u^*)}\Big\langle < \hat{b} >, (e^{p\eta(x^*,u^*)} - \mathbf{1})\Big\rangle.$$

According to the above relation and (3.20), we have

$$\frac{1}{r}e^{rE(x^*)} > \frac{1}{r}e^{rE(u^*)}.$$

By the exponential function, we obtain

$$E(x^*) > E(u^*) = 0 \text{ for } r \neq 0; \tag{3.21}$$

It follows from Theorem 3.4.2, we see that

$$\sup_{y\in Y}\frac{f(x^*, y)}{g(x^*, y)} = \frac{\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \Big\langle\Big\langle \mu^*, h(u^*)\Big\rangle\Big\rangle_{J_0}}{\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)}.$$

and so

$$\frac{f(x^*, y)}{g(x^*, y)} \leq \frac{\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \Big\langle\Big\langle \mu^*, h(u^*)\Big\rangle\Big\rangle_{J_0}}{\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)} \text{ for all } y \in Y.$$

This implies that

$$f(x^*, y)\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) - g(x^*, y)\left[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \Big\langle\Big\langle \mu^*, h(u^*)\Big\rangle\Big\rangle_{J_0}\right] \leq 0$$

for all $i = 1, 2, \ldots, s^*$, and $y \in Y$.

By $t^* \in I$ and $y_i^* \in Y(u^*), i = 1, 2, \ldots, s^*$, we obtain

$$\sum_{i=1}^{s^*} t_i^* f(x^*, y_i^*)\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) - \sum_{i=1}^{s^*} t_i^* g(x^*, y_i^*)\left[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \Big\langle\Big\langle \mu^*, h(u^*)\Big\rangle\Big\rangle_{J_0}\right] \leq 0. \tag{3.22}$$

that is,

$$\begin{aligned}
&\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)\left[\sum_{i=1}^{s^*} t_i^* f(x^*, y_i^*) + \Big\langle\Big\langle \mu^*, h(x^*)\Big\rangle\Big\rangle_{J_0}\right] \\
&\quad - \sum_{i=1}^{s^*} t_i^* g(x^*, y_i^*)\left[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \Big\langle\Big\langle \mu^*, h(u^*)\Big\rangle\Big\rangle_{J_0}\right] \\
&\qquad = E(x^*) \leq \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)\Big\langle\Big\langle \mu^*, h(x^*)\Big\rangle\Big\rangle_{J_0}.
\end{aligned} \tag{3.23}$$

Since the relations (3.6), $g(u^*, y_i^*) > 0, i = 1, 2, \ldots, s^*$, and $h_j(x^*) \leq 0$, we get

$$\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)\Big\langle\Big\langle \mu^*, h(x^*)\Big\rangle\Big\rangle_{J_0} \leq 0.$$

Therefore, from (3.23), we obtain

$$E(x^*) \leq 0 = E(u^*). \tag{3.24}$$

Consequently, expression (3.24) contradicts (3.21). Hence, u^* is an optimal solution to (P) and $E(x^*) = E(u^*)$ deduces $x^* = u^*$. Thus,

$$\max_{y\in Y} \frac{f(u^*,y)}{g(u^*,y)} = \frac{\sum_{i=1}^{s^*} t_i^* f(u^*,y_i^*) + \big\langle\big\langle \mu^*, h(u^*)\big\rangle\big\rangle_{J_0}}{\sum_{i=1}^{s^*} t_i^* g(u^*,y_i^*)}.$$

If (ii) holds, it can be shown by the same as in the case (i).

Chapter 4

Mond Weir and Wolfe Type Duality for Nonsmooth Minimax Fractional Programming

4.1 Introduction

Fractional programming can be used in various fields of study. It can be used in engineering and economics to minimize a ratio of functions between a given period of time and a utilized resource in order to measure the efficiency or productivity of a system. The necessary and sufficient conditions for generalized minimax programming were first developed by Schmitendorf [24]. These conditions were employed by Yadav and Mukherjee [27] to construct two dual problems and derived duality theorems for fractional minimax programming problem.

In this chapter, we study a minimax fractional programming problem with exponential B-(p, r)-invexity. Section 4.2 contains notations and preliminaries. Section 4.3 contains optimality conditions. In section 4.4, we consider a Mond Weir type dual and prove appropriate duality relations using the notion of exp. B-(p, r)-invexity assumptions. In section 4.5, we consider a Wolfe type dual and prove duality results.

4.2 Notations and preliminaries

Definition 4.2.1 The differentiable function $f : S \subset R^n \to R$ is said to be (strictly) exponential B-(p, r)-invex with respect to η and b at $u \in S$ if there exists a function $\eta : S \times S \to R^n$ and a function $b : S \times S \to R_+$, such that, for all $x \in S$, the following inequalities hold:

$$\frac{1}{r} b(x,u)(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{p}\big\langle \xi, (e^{p\eta(x,u)} - \mathbf{1})\big\rangle \ (> \text{ if } x \neq u) \ \text{ for } \ p \neq 0, \ r \neq 0,$$
$$\frac{1}{r} b(x,u)(e^{r(f(x)-f(u))} - 1) \geq \big\langle \xi, \eta(x,u)\big\rangle \ (> \text{ if } x \neq u) \ \text{ for } \ p = 0, \ r \neq 0,$$
$$b(x,u)(f(x) - f(u)) \geq \frac{1}{p}\big\langle \xi, (e^{p\eta(x,u)} - \mathbf{1})\big\rangle \ (> \text{ if } x \neq u) \ \text{ for } \ p \neq 0, \ r = 0,$$
$$b(x,u)(f(x) - f(u)) \geq \big\langle \xi, \eta(x,u)\big\rangle \ (> \text{ if } x \neq u) \ \text{ for } \ p = 0, \ r = 0.$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in R^n$, $(e^{p\eta(x,u)} - \mathbf{1})$ stands for the n-vector $(e^{p\eta_1(x,u)} - 1, \ e^{p\eta_2(x,u)} - 1, \ldots, \ e^{p\eta_n(x,u)} - 1)$, and $\big\langle \cdot, \cdot \big\rangle$ stands for the inner product in R^n throughout the chapter. In this chapter, we consider the following minimax fractional programming problem:

$$\textbf{(P)} \qquad \min_{x \in X} \ \max_{y \in Y} \frac{f(x,y)}{g(x,y)}$$
$$\text{subject to} \quad X = \{x \in R^n \mid h(x) \in -R^p_+\}$$
$$\text{and } Y \text{ is a compact subset of } R^m,$$

where $f(\cdot, \cdot)$ and $g(\cdot, \cdot) : R^n \times R^m \to R$ are continuous functions and for each $y \in Y$, $f(\cdot, y), g(\cdot, y)$ and $h : R^n \to R^p$ are locally Lipschitz functions. Without loss of generality, we may assume that $g(x, y) > 0$ and $f(x, y)$ is nonnegative for all $(x, y) \in X \times Y$. Then for $x \in X$, we denote

$$J(x) = \{j \in J \mid h_j(x) = 0\} \text{ where } J = \{1, 2, \ldots, p\}, \text{ and}$$
$$Y(x) = \left\{ y \in Y \middle| \frac{f(x,y)}{g(x,y)} = \sup_{z \in Y} \frac{f(x,z)}{g(x,z)} \right\}$$

By the compactness of Y, the continuous function has finite points attended to its maximum points say s. Thus we can set
$K(x) = \Big\{(s, t, y) \in N \times R^s_+ \times R^{ms} \Big| t = (t_1, \ldots, t_s) \in R^s_+, \ \sum_{i=1}^{s} t_i = 1, \ y = (y_1, y_2, \ldots, y_s),$
$y_i \in Y(x), \ i = 1, 2, \ldots, s\Big\}.$

4.3 Optimality conditions

We derive the necessary and sufficient optimality conditions for problem (P) using exponential B-(p,r)-invexity.

Theorem 4.3.1 (Necessary Optimality Conditions). Let x^* be a (P)-optimal solution and the constraint qualification hold at x^*. Then there exist $(s^*, t^*, y^*) \in K(x^*)$, and a p-vector Lagrange multiplier $\mu^* \in R^p_+$ such that

$$0 \in \sum_{i=1}^{s^*} t_i^* \{g(x^*, y_i^*)\partial^c f(x^*, y_i^*) + f(x^*, y_i^*)\partial^c(-g(x^*, y_i^*))\} + \left\langle \mu^*, \partial^c h(x^*)\right\rangle_p, \quad (4.1)$$

$$\mu_j^* h_j(x^*) = 0, \quad j = 1, 2, \ldots, p, \quad (4.2)$$

$$\mu^* \in R^p_+,\ t^* \in I,\ y_i^* \in Y(x^*), \quad i = 1, 2, \ldots, s^*, \quad (4.3)$$

where $I = \{t^* \in R^{s^*} : t^* = (t_1^*, t_2^*, \ldots, t_{s^*}^*),\ t_i^* \geq 0,\ i = 1, 2, \ldots, s^* \text{ with } \sum_{i=1}^{s^*} t_i^* = 1\}$ and $\left\langle \mu^*, \partial^o h(x^*)\right\rangle_p \equiv \sum_{j=1}^{p} \mu_j^* \partial^o h_j(x^*)$.

Actually the expression (4.1) is the same as

$$0 \in \sum_{i=1}^{s^*} t_i^* g(x^*, y_i^*)\left[\sum_{i=1}^{s^*} t_i^* \partial^c f(x^*, y_i^*) + \left\langle \mu^*, \partial^c h(x^*)\right\rangle_p\right] + \left[\sum_{i=1}^{s^*} t_i^* f(x^*, y_i^*) + \left\langle \mu^*, h(x^*)\right\rangle_p\right] \sum_{i=1}^{s^*} t_i^* \partial^c(-g(x^*, y_i^*)), \quad (4.4)$$

The sufficient optimality conditions follow from the converse of the necessary optimality conditions with extra assumptions. Now, we employ the necessary optimality conditions and Exp (p,r)-invexity to establish sufficient optimality conditions.

Theorem 4.3.2 (Sufficient Optimality Conditions). Let x^* be a feasible solution of (P) satisfying the necessary conditions (4.1) – (4.3). Let a function $A : X \to R$ be

$$A(\cdot) = \sum_{i=1}^{s} t_i \{g(x^*, y_i) f(\cdot, y_i) - f(x^*, y_i) g(\cdot, y_i)\}$$

and $B : X \to R$ be

$$B(\cdot) = \langle \mu, h(\cdot)\rangle_p$$

with $A(x^*) = 0$ and $B(x^*) = 0$. Assume that $A(\cdot)$ and $B(\cdot)$ are exponential B-(p,r) invex at $x^* \in X$ w.r.t the same function η and b satisfying $b(x, x^*) > 0$ for all $x \in X$. Then x^* is an optimal solution of (P).

Proof. Suppose that x^* is not an optimal solution of (P). Then there exists a (P)-feasible solution x, such that

$$\frac{f(x^*,y_i)}{g(x^*,y_i)} > \max_{y\in Y}\frac{f(x,y)}{g(x,y)} \qquad \text{for all } i=1,2,\ldots,s.$$

This implies that

$$f(x,y)g(x^*,y_i) - g(x,y)f(x^*,y_i) < 0 \text{ for all } y\in Y. \tag{4.5}$$

By relations (4.3) and (4.5), we have

$$\sum_{i=1}^{s} t_i(f(x,y_i)g(x^*,y_i) - g(x,y_i)f(x^*,y_i)) < 0$$

$$= \sum_{i=1}^{s} t_i(f(x^*,y_i)g(x^*,y_i) - g(x^*,y_i)f(x^*,y_i)) \tag{4.6}$$

On the other hand, from $h_j(x) \le 0$,$j\in J$ and $\mu \in R^p_+$, we obtain

$$B(x) = \langle \mu, h(x)\rangle_p \le 0 = \langle \mu, h(x^*)\rangle_p = B(x^*), \tag{4.7}$$

This implies that

$$\tfrac{1}{r}b(x,x^*)(e^{r(B(x)-B(x^*))}-1) \le 0. \tag{4.8}$$

By the relation (4.1), there exists $\xi_i \in \partial^c f(x^*,y_i), \zeta_i \in \partial^c(-g(x^*,y_i))$ for all $i=1,2,\ldots,s$, and $\rho_j \in \partial^c h_j(x^*)$ for all $j\in J$, such that

$$\langle a\rangle \equiv \sum_{i=1}^{s} t_i\{g(x^*,y_i)\xi_i + f(x^*,y_i)\zeta_i\} + \langle \mu,\rho\rangle_p = 0, \tag{4.9}$$

that is, $\langle a\rangle$ is a zero vector, where $\langle \mu,\rho\rangle_p = \sum_{j=1}^{p}\mu_j\rho_j$ and $\rho = (\rho_1,\rho_2,...,\rho_p)$.
It follows the inner product

$$\tfrac{1}{p}\Big\langle \langle a\rangle, (e^{p\eta(x,x^*)} - \mathbf{1})\Big\rangle = 0. \tag{4.10}$$

Since B is Exp B-(p,r) invex function w.r.t. η and b at x^* in X, we have
$\tfrac{1}{r}b(x,x^*)(e^{r(B(x)-B(x^*))}-1) \ge \tfrac{1}{p}\big\langle \sum_{j=1}^{p}\mu_j\rho_j, (e^{p\eta(x,x^*)}-\mathbf{1})\big\rangle.$
This inequality together with equality (4.8) gives

$$0 \ge \tfrac{1}{p}\big\langle \sum_{j=1}^{p}\mu_j\rho_j, (e^{p\eta(x,x^*)}-\mathbf{1})\big\rangle. \tag{4.11}$$

From relations (4.10) and (4.11), we have

$$0 \le \tfrac{1}{p}\big\langle \sum_{i=1}^{s} t_i\{g(x^*,y_i)\xi_i + f(x^*,y_i)\zeta_i\}, (e^{p\eta(x,x^*)}-\mathbf{1})\big\rangle. \tag{4.12}$$

Moreover, since $A(.)$ is an exponential $B-(p,r)-$invex at x^* w.r.t the same function η, it reduces

$$\tfrac{1}{r}b(x,x^*)(e^{r(A(x)-A(x^*))}-1) \ge \tfrac{1}{p}\big\langle \sum_{i=1}^{s} t_i\{g(x^*,y_i)\xi_i + f(x^*,y_i)\zeta_i\}, (e^{p\eta(x,x^*)}-\mathbf{1})\big\rangle$$

According to inequality (4.12) and the above inequality, we obtain

$A(x) \geq A(x^*)$,

which contradicts (4.6). Hence x^* is an optimal solution to (P).

Theorem 4.3.3 (sufficient Optimality Conditions). If x^* is a feasible solution of (P) and the necessary conditions (4.2) – (4.4) hold. Denote a function $C : X \to R$ by

$$C(\cdot) = \sum_{i=1}^{s} t_i g(x^*, y_i)\left[\sum_{i=1}^{s} t_i f(\cdot, y_i) + \langle \mu, h(\cdot)\rangle_p\right] + \left[\sum_{i=1}^{s} t_i f(x^*, y_i) + \langle \mu, h(x^*)\rangle_p\right]\sum_{i=1}^{s} t_i(-g(\cdot, y_i))$$

with $C(x^*) = 0$. Assume that $C(\cdot)$ is an Exp B-$(p,r)-$invex w.r.t. η and b satisfying $b(x, x^*) > 0$ for all $x, x^* \in X$. Then x^* is an optimal solution of (P).

Proof. It follows along the lines of the proof of Thoerem 4.3.2.

4.4 Mond-Weir type duality model

The Mond-Weir type duality contains no constraint of problem (P) in the objective fractional functional of (MWD), as the following form

(MWD) $$\max_{u} \max_{(\mu,s,t,y)\in K_1(u)} \frac{f(u,y)}{g(u,y)}$$

subject to

$$0 \in \sum_{i=1}^{s} t_i\{g(u, y_i)\partial^c f(u, y_i) + f(u, y_i)\partial^c(-g(u, y_i))\} + \langle \mu, \partial^c h(u)\rangle_p, \tag{4.13}$$

$$\sum_{j\in J} \mu_j h_j(u) = 0, \qquad j = 1, 2, \ldots, p, \tag{4.14}$$

$$\mu \in R^p_+,\ t \in I,\ y_i \in Y(u), \quad i = 1, 2, \ldots, s, \tag{4.15}$$

where $u \in X$ is the (P)-feasible solution and the $K_1(u)$ stand in (MWD) is represented by the set

$K_1(u) = \{(\mu, s, t, y) \in R^p_+ \times N \times R^s_+ \times R^{ms} \big| t \in I,\ y = (y_1, y_2, \ldots, y_s),\ y_i \in Y(u),\ i = 1, 2, \ldots, s\}$,

which means that elements in $K_1(u)$ satisfy the expression (4.1) and (4.2) in the necessary optimality conditions of (P) in theorem 4.3.1.

In order to show that (MWD) is the dual problem w.r.t. the primal problem (P), we denote L_1 the constraint set of (MWD). Moreover, denote by the elements satisfying the necessary optimality conditions of (P) which is defined by a projective-like set as the feasible solutions of problem (P) to be

$$L_2 = \{u \in X \subseteq R^n \big| (u; \mu, s, t, y) \in L_1\}.$$

Now, we show the weak duality theorem related to problems (P) and (MWD), as following.

Theorem 4.4.1 (Weak Duality). Let x and $(u; \mu, s, t, y)$ be (P)-feasible and (MWD)-feasible, respectively. Assume that

$D(\cdot) = \sum_{i=1}^{s} t_i\{g(u, y_i)f(\cdot, y_i) - f(u, y_i)g(\cdot, y_i)\}$ and $E(\cdot) = \langle \mu, h(\cdot)\rangle_p$ are exponential B-(p, r) invex at $u \in L_2$ w.r.t the same function η and b satisfying $b(x, u) > 0$.

Then

$$\max_{y\in Y} \frac{f(x, y)}{g(x, y)} \geq \frac{f(u, y)}{g(u, y)}.$$

Proof. Suppose on the contrary, that

$$\max_{y\in Y} \frac{f(x, y)}{g(x, y)} < \frac{f(u, y)}{g(u, y)}.$$

Then there is a feasible solution $x \in X$ such that

$$\max_{y\in Y} \frac{f(x, y)}{g(x, y)} < \frac{f(u, y)}{g(u, y)} \text{ for any } (u; \mu, s, t, y) \in L_1.$$

Thus the inequality would reduce to

$$\frac{f(x, y)}{g(x, y)} < \frac{f(u, y)}{g(u, y)} \text{ for all } y \in Y.$$

This is equivalent to

$$f(x, y)g(u, y) - f(u, y)g(x, y) < 0 \text{ for all } y \in Y.$$

Multiplying the above expression respectively by $t_i \geq 0, i = 1, 2, \ldots, s$ and then summing up, it would yield

$$\begin{aligned} D(x) &= \sum_{i=1}^{s} t_i(f(x, y_i)g(u, y_i) - g(x, y_i)f(u, y_i)) < 0 \\ &= \sum_{i=1}^{s} t_i(f(u, y_i)g(u, y_i) - g(u, y_i)f(u, y_i)) = D(u). \end{aligned} \tag{4.16}$$

By the expression (4.13), there exists $\xi_i \in \partial^c f(u, y_i), \zeta_i \in \partial^c(-g(u, y_i))$ for all $i = 1, 2, \ldots, s$, and $\rho_j \in \partial^c h_j(x)$ for all $j \in J$, such that

$$\langle a\rangle \equiv \sum_{i=1}^{s} t_i\{g(x, y_i)\xi_i + f(x, y_i)\zeta_i\} + \langle \mu, \rho\rangle_p = 0,$$

that is, $\langle a\rangle$ is a zero vector, where $\langle \mu, \rho\rangle_p = \sum_{j=1}^{p} \mu_j\rho_j$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_p)$.

This implies that the inner product

$$\tfrac{1}{p}\Big\langle \langle a\rangle, (e^{p\eta(x,u)} - \mathbf{1})\Big\rangle = 0. \tag{4.17}$$

Since D is Exp B-(p,r)-invex function w.r.t. η and b at u, we have

$$\frac{1}{r}b(x,u)(e^{r(D(x)-D(u))}-1) \geq \frac{1}{p}\left\langle \sum_{i=1}^{s} t_i\{g(x,y_i)\xi_i + f(x,y_i)\zeta_i\}, (e^{p\eta(x,u)} - \mathbf{1})\right\rangle. \tag{4.18}$$

Taking the above inequality together with (4.16), we have

$$\frac{1}{p}\left\langle \sum_{i=1}^{s} t_i\{g(x,y_i)\xi_i + f(x,y_i)\zeta_i\}, (e^{p\eta(x,u)} - \mathbf{1})\right\rangle < 0. \tag{4.19}$$

From (4.17) and (4.19), we get

$$\frac{1}{p}\left\langle \sum_{j=1}^{p} \mu_j\rho_j, (e^{p\eta(x,u)} - \mathbf{1})\right\rangle > 0. \tag{4.20}$$

By the exponential B-(p,r)-invexity of $E(.)$ at u w.r.t. the same function η and b and the inequality (4.20), we get

$$E(x) > E(u). \tag{4.21}$$

On the other hand, from $h_j(x) \leq 0$, $j \in J$ and (4.15) we obtain

$$E(x) = \langle \mu, h(x)\rangle_p \leq 0 = \langle \mu, h(u)\rangle_p = E(u),$$

Which contradicts (4.21). Hence

$$\max_{y\in Y} \frac{f(x,y)}{g(x,y)} \geq \frac{f(u,y)}{g(u,y)}.$$

Theorem 4.4.2 (Strong Duality). Let x^* be the efficient solution of problem (P) satisfying the constraint qualification at x^*. Then there exists $(\mu^*, s^*, t^*, y^*) \in K_1(x^*)$, such that $(x^*; \mu^*, s^*, t^*, y^*)$ is a feasible point for (MWD). If the hypotheses of Theorem 4.4.1 are fulfilled, then $(x^*; \mu^*, s^*, t^*, y^*)$ is an efficient solution to problem (MWD), and the two problems (P) and (MWD) have the same optimal values.

Proof. By assumption, x^* is an efficient point of (P) and the constraint qualification holds at x^*. Then by conditions (4.1) $-$ (4.3), we conclude that $(x^*; \mu^*, s^*, t^*, y^*)$ is a feasible for (MWD). Since

$$\frac{f(x^*, y_i^*)}{g(x^*, y_i^*)} = \max_{y\in Y} \frac{f(x^*, y)}{g(x^*, y)}$$

then, by using the weak duality theorem (Theorem 4.4.1), we conclude that $(x^*; \mu^*, s^*, t^*, y^*)$ is an efficient solution for (MWD). Consequently, the two problems (P) and (MWD) have the same optimal values.

Theorem 4.4.3 (Strict Converse Duality). Let x^* and $(u^*; \mu^*, s^*, t^*, y^*)$ be the efficient solutions of problem (P) and (MWD), respectively, and the constraint qualification be satisfied at x^*. Assume that $A(\cdot) = \sum_{i=1}^{s^*} t_i^*\{g(u^*, y_i^*)f(\cdot, y_i^*) - f(u^*, y_i^*)g(\cdot, y_i^*)\}$ and $B(\cdot) = \langle \mu^*, h(\cdot)\rangle_p$

are exponential B-(p,r) invex at $u^* \in L_2$ w.r.t η and b satisfying $b(x^*,u^*) > 0$ for all optimal vectors x^* for (P), and $(u^*;\mu^*,s^*,t^*,y^*)$ for (MWD), respectively. Then $x^* = u^*$ and the optimal values of problems (P) and (MWD) are equal.

Proof. Suppose on the contrary that $x^* \neq u^*$. Then by (4.13), there exists $\xi_i \in \partial^c f(u^*,y_i^*), \zeta_i \in \partial^c(-g(u^*,y_i^*))$ for all $i = 1,2,\ldots,s^*$, and $\rho_i \in \partial^c h_j(u^*)$ for all $j \in J$, such that

$$\langle b\rangle \equiv \sum_{i=1}^{s^*} t_i^*\{g(u^*,y_i^*)\xi_i + f(u^*,y_i^*)\zeta_i\} + \langle \mu^*,\rho\rangle_p = 0,$$

that is, $\langle b\rangle$ is a zero vector, where $\langle \mu^*,\rho\rangle_p = \sum_{j=1}^{p} \mu_j^*\rho_j$ and $\rho = (\rho_1,\rho_2,\ldots,\rho_p)$.
This implies that

$$\frac{1}{p}\left\langle \langle b\rangle, (e^{p\eta(x^*,u^*)} - \mathbf{1})\right\rangle = 0. \tag{4.22}$$

Since x^* and $(u^*;\mu^*,s^*,t^*,y^*)$ are optimal solutions of (P) and (MWD), respectively, we know

$$B(x^*) = \sum_{j=1}^{p} \mu_j^* h_j(x^*) \leq 0.$$

and

$$B(u^*) = \sum_{j=1}^{p} \mu_j^* h_j(u^*) = 0.$$

This implies that

$$B(x^*) \leq B(u^*).$$

By the exponential function, we get

$$\frac{1}{r}b(x^*,u^*)(e^{r(B(x^*)-B(u^*))} - 1) \leq 0 \text{ for any } r \neq 0. \tag{4.23}$$

Since $B(.)$ is exponential $B-(p,r)-$invex w.r.t. η and b at u^*,we get

$$\frac{1}{r}b(x^*,u^*)(e^{r(B(x^*)-B(u^*))} - 1) \geq \frac{1}{p}\left\langle \sum_{j=1}^{p} \mu_j^*\rho_j, (e^{p\eta(x^*,u^*)} - \mathbf{1})\right\rangle. \tag{4.24}$$

From above inequality and (4.23), we have

$$\frac{1}{p}\left\langle \sum_{j=1}^{p} \mu_j^*\rho_j, (e^{p\eta(x^*,u^*)} - \mathbf{1})\right\rangle \leq 0. \tag{4.25}$$

Also from (4.22) and (4.25), we have

$$\frac{1}{p}\left\langle \sum_{i=1}^{s^*} t_i^*\{g(u^*,y_i^*)\xi_i + f(u^*,y_i^*)\zeta_i\}, (e^{p\eta(x^*,u^*)} - \mathbf{1})\right\rangle \geq 0. \tag{4.26}$$

Moreover, from the exponential type B-$(p,r)-$invexity with respect to the same function η and b at u^* of $A(.)$ and the inequality (4.26), we get

$$\frac{1}{r}b(x^*,u^*)(e^{r(A(x^*)-A(u^*))} - 1) \geq 0 \text{ for any } r \neq 0.$$

This implies

$$A(x^*) \geq A(u^*). \tag{4.27}$$

By Theorem (4.4.2), we see that

$$\max_{y \in Y} \frac{f(x^*, y)}{g(x^*, y)} = \frac{f(u^*, y_i^*)}{g(u^*, y_i^*)}.$$

and so

$$\frac{f(x^*, y)}{g(x^*, y)} \leq \frac{f(u^*, y_i^*)}{g(u^*, y_i^*)} \text{ for all } y \in Y.$$

This implies that

$$f(x^*, y)g(u^*, y_i^*) - f(u^*, y_i^*)g(x^*, y) \leq 0 \text{ for all } y \in Y.$$

Multiplying the above expression respectively by $t_i^* \geq 0$ and $y_i^* \in Y(u^*),\ i = 1, 2, \ldots, s^*$ and then summing up, it would yield

$$\begin{aligned} A(x^*) &= \sum_{i=1}^{s^*} t_i^*(f(x^*, y_i^*)g(x^*, y_i^*) - g(x^*, y_i^*)f(x^*, y_i^*)) \\ &< 0 = \sum_{i=1}^{s^*} t_i^*(f(u^*, y_i^*)g(u^*, y_i^*) - g(u^*, y_i^*)f(u^*, y_i^*)) = A(u^*). \end{aligned}$$

This contradicts the inequality (4.27). Hence $x^* = u^*$ and the optimal values of problems (P) and (MWD) are equal.

4.5 Wolfe type duality model

The Wolfe type duality in fractional programming problem can be considered by the objective of fractional functional of (P) by adding the constraint scalarization with a multiplier μ into the numerator of the fractional functional in (P), precisely the Wolfe type dual is stated as follows

(WD) $$\max_{u \in X} \max_{(\mu, s, t, y) \in K_2(u)} \frac{\sum_{i=1}^{s} t_i f(u, y_i) + \langle \mu, h(u) \rangle_p}{\sum_{i=1}^{s} t_i g(u, y_i)}$$

subject to

$$\begin{aligned} 0 \in \sum_{i=1}^{s} t_i g(u, y_i) &\left[\sum_{i=1}^{s} t_i \partial^c f(u, y_i) + \langle \mu, \partial^c h(u) \rangle_p \right] \\ &+ \left[\sum_{i=1}^{s} t_i f(u, y_i) + \langle \mu, h(u) \rangle_p \right] \sum_{i=1}^{s} t_i \partial^c(-g(u, y_i)), \end{aligned} \tag{4.28}$$

$$\mu \in R_+^p,\ t \in I,\ y_i \in Y(u), \quad i = 1, 2, \ldots, s, \tag{4.29}$$

Here $u \in X$ is a (P)-feasible solution, and denote the set

$K_2(u) = \Big\{(\mu, s, t, y) \in R^p_+ \times N \times R^s_+ \times R^{ms} \Big| \ t \in I, \ y = (y_1, y_2, \ldots, y_s), \ y_i \in Y(u), \ i = 1, 2, ..., s\Big\}$,

as elements in $K_2(u)$ satisfying the expressions (4.2) and (4.4) which is equivalent to $K_1(u)$ for the necessary optimality conditions of (P) in Theorem 4.3.1 taken as the constraint given in the problem (MWD).

In order to show the problem (WD) being surely a dual problem w.r.t. the primal problem (P), we denote L_3 as the constraint set of (WD). Actually L_3 is also defined by projective like as the feasible solutions of problem (P).

$$L_4 = \Big\{u \in X \subseteq R^n \Big| \ (u; \mu, s, t, y) \in L_4\Big\}.$$

Now we show the duality theorems related to (P) and (WD):

Theorem 4.5.1 (Weak Duality). Let x and $(u; \mu, s, t, y)$ be (P)-feasible and (WD)-feasible, respectively. Denote a function $D : X \to R$ by

$$D(\cdot) = \sum_{i=1}^{s} t_i g(u, y_i)\Big[f(\cdot, y_i) + \big\langle \mu, h(\cdot)\big\rangle_p\Big] - \Big[f(u, y_i) + \big\langle \mu, h(u)\big\rangle_p\Big] \sum_{i=1}^{s} t_i(g(\cdot, y_i))$$

with $D(u) = 0$. Assume that $D(\cdot)$ is exponential B-(p, r) invex at $u \in L_4$ w.r.t the same function η. Then

$$\max_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq \frac{\sum_{i=1}^{s} t_i f(u, y_i) + \big\langle \mu, h(u)\big\rangle_p}{\sum_{i=1}^{s} t_i g(u, y_i)}.$$

Proof. Suppose that

$$\max_{y \in Y} \frac{f(x, y)}{g(x, y)} \geq \frac{\sum_{i=1}^{s} t_i f(u, y_i) + \big\langle \mu, h(u)\big\rangle_p}{\sum_{i=1}^{s} t_i g(u, y_i)}$$

were not true.

Then there is a feasible solution $x \in X$ such that

$$\max_{y \in Y} \frac{f(x, y)}{g(x, y)} < \frac{\sum_{i=1}^{s} t_i f(u, y_i) + \big\langle \mu, h(u)\big\rangle_p}{\sum_{i=1}^{s} t_i g(u, y_i)}.$$

for any $(u; \mu, s, t, y) \in L_3$.

This implies that

$$\frac{f(x,y)}{g(x,y)} < \frac{\sum_{i=1}^{s} t_i f(u,y_i) + \big\langle \mu, h(u)\big\rangle_p}{\sum_{i=1}^{s} t_i g(u,y_i)} \text{ for all } y \in Y,$$

Or equivalently,

$$f(x,y)\sum_{i=1}^{s} t_i g(u,y_i) - g(x,y)\left[\sum_{i=1}^{s} t_i f(u,y_i) + \big\langle \mu, h(u)\big\rangle_p\right] < 0 \text{ for all } y \in Y.$$

Multiplying the above respectively by $t_i \geq 0, i = 1,2,\ldots,s$ and summing up, it yields

$$\sum_{i=1}^{s} t_i f(x,y)\sum_{i=1}^{s} t_i g(u,y_i) - \sum_{i=1}^{s} t_i g(x,y)\left[\sum_{i=1}^{s} t_i f(u,y_i) + \big\langle \mu, h(u)\big\rangle_p\right] < 0,$$

it reduces to

$$\begin{aligned}\sum_{i=1}^{s} t_i g(u,y_i)&\left[\sum_{i=1}^{s} t_i f(x,y_i) + \big\langle \mu, h(x)\big\rangle_p\right]\\ &- \sum_{i=1}^{s} t_i g(x,y_i)\left[\sum_{i=1}^{s} t_i f(u,y_i) + \big\langle \mu, h(u)\big\rangle_p\right]\\ &\qquad = D(x) < \sum_{i=1}^{s} t_i g(u,y_i)\big\langle \mu, h(x)\big\rangle_p. \end{aligned} \tag{4.30}$$

Since the relations (4.29), $g(u,y_i) > 0, i = 1,2,\ldots,s$, and $h_j(x) \leq 0$, we get

$$\sum_{i=1}^{s} t_i g(u,y_i)\big\langle \mu, h(x)\big\rangle_p \leq 0.$$

Therefore, from (4.30), we obtain

$$D(x) < 0 = D(u). \tag{4.31}$$

Let x and $(u;\mu,s,t,y)$ be (P)-feasible and (WD)-feasible, respectively. According to the expression (4.28), there exist $\xi_i \in \partial^c f(u,y_i), \zeta_i \in \partial^c(-g)(u,y_i), i = 1,2,\ldots,s$, and $\rho_j \in \partial^c h_j(u), j \in J$ such that the vector

$$\langle a\rangle \equiv \sum_{i=1}^{s} t_i g(u,y_i)\left[\big\langle t,\xi\big\rangle_s + \big\langle \mu,\rho\big\rangle_p\right] + \left[\sum_{i=1}^{s} t_i f(u,y_i) + \big\langle \mu, h(u)\big\rangle_p\right]\big\langle t,\zeta\big\rangle_s = 0,$$

that is, the vector $\langle a\rangle$ is a zero vector, where

$$\big\langle t,\xi\big\rangle_s \equiv \sum_{i=1}^{s} t_i\xi_i,\ \ \big\langle t,\zeta\big\rangle_s \equiv \sum_{i=1}^{s} t_i\zeta_i,\ \ \big\langle \mu,\rho\big\rangle_p \equiv \sum_{j=1}^{p} \mu_j\rho_j,$$

$$\xi = (\xi_1,\ldots,\xi_s),\ \zeta = (\zeta_1,\ldots,\zeta_s), \text{ and } \rho = (\rho_1,\ldots,\rho_p).$$

This implies that

$$\frac{1}{p}\Big\langle < a >, (e^{p\eta(x,u)} - \mathbf{1})\Big\rangle = 0. \tag{4.32}$$

If D is an Exp $B-(p,r)-$invexity w.r.t η at u in L_4, we get

$$\tfrac{1}{r}b(x,u)(e^{r(D(x)-D(u))} - 1) \geq \tfrac{1}{p}\Big\langle < a >, (e^{p\eta(x,u)} - \mathbf{1})\Big\rangle.$$

According to the above relation and (4.32), we have

$\frac{1}{r}b(x,u)(e^{r(D(x)-D(u))}-1)\geq 0.$

By the exponential function, we obtain

$$D(x)\geq D(u)=0 \text{ for } r\neq 0,$$

which contradicts the inequality (4.31). Hence the inequality

$$\max_{y\in Y}\frac{f(x,y)}{g(x,y)}\geq \frac{\sum\limits_{i=1}^{s}t_i f(u,y_i)+\langle \mu, h(u)\rangle_p}{\sum\limits_{i=1}^{s}t_i g(u,y_i)}$$

is true, and the theorem is proved.

Theorem 4.5.2 (Strong Duality). Let x^* be the efficient solution of problem (P) satisfying the constraint qualification at x^*. Then there exists $(\mu^*, s^*, t^*, y^*)\in K_2(x^*)$, such that $(x^*;\mu^*, s^*, t^*, y^*)$ is a feasible point for (WD). If the hypotheses of Theorem 4.5.1 are fulfilled, then $(x^*;\mu^*, s^*, t^*, y^*)$ is an efficient solution to problem (WD), and the two problems (P) and (WD) have the same optimal values.

Proof. By assumption, x^* is an efficient point of (P) and the constraint qualification holds at x^*. Then by conditions (4.2) – (4.4), we conclude that $(x^*;\mu^*, s^*, t^*, y^*)$ is a feasible for (WD). Since

$$\max_{y\in Y}\frac{f(x^*,y)}{g(x^*,y)}=\frac{\sum\limits_{i=1}^{s^*}t_i^* f(x^*,y_i^*)+\langle \mu^*, h(x^*)\rangle_p}{\sum\limits_{i=1}^{s^*}t_i^* g(x^*,y_i^*)},$$

then, by using the weak duality theorem (Theorem 4.5.1), we conclude that $(x^*;\mu^*, s^*, t^*, y^*)$ is an efficient solution for (WD). Consequently, the two problems (P) and (WD) have the same optimal values.

Theorem 4.5.3 (Strict Converse Duality). Let x^* and $(x^*;\mu^*, s^*, t^*, y^*)$ be the efficient solutions of problem (P) and (WD), respectively, and the constraint qualification be satisfied at x^*. Denote a function $E : X\to R$ by

$$E(\cdot)=\sum_{i=1}^{s^*}t_i^* g(u^*,y_i^*)\big[f(\cdot,y_i^*)+\langle \mu^*, h(\cdot)\rangle_p\big]$$
$$-\big[f(u^*,y_i^*)+\langle \mu^*, h(u^*)\rangle_p\big]\sum_{i=1}^{s^*}t_i^*(g(\cdot,y_i^*))$$

with $E(u^*)=0$. Assume that $E(\cdot)$ is strictly exponential B-(p,r) invex at $u^*\in L_4$ w.r.t the same function η for all optimal vectors x^* for (P) and $(u^*; s^*, t^*, y^*)$ for (WD), respectively. Then $x^*=u^*$, and the efficient values of (P) and (WD) are equal.

Proof. We want to prove $x^* = u^*$.

Suppose on contrary that $x^* \neq u^*$. It will deduce to a contradiction. According to relation (5.1), there exist $\xi_i \in \partial^c f(u^*, y_i^*), \zeta_i \in \partial^c(-g)(u^*, y_i^*), i = 1, 2, ..., s^*$, and $\rho_j \in \partial^c h_j(u^*), j \in J$ such that the vector

$$\langle a \rangle \equiv \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) \big[\langle t^*, \xi \rangle_{s^*} + \langle \mu^*, \rho \rangle_p \big] + \big[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^*, h(u^*) \rangle_p \big] \langle t^*, \zeta \rangle_{s^*} = 0, \tag{4.33}$$

that is, the vector $\langle a \rangle$ is a zero vector, where

$$\langle t^*, \xi \rangle_{s^*} \equiv \sum_{i=1}^{s^*} t_i^* \xi_i, \quad \langle t^*, \zeta \rangle_{s^*} \equiv \sum_{i=1}^{s^*} t_i^* \zeta_i, \quad \langle \mu^*, \rho \rangle_p \equiv \sum_{j=1}^{p} \mu_j^* \rho_j,$$

$$\xi = (\xi_1, \ldots, \xi_{s^*}),\ \zeta = (\zeta_1, \ldots, \zeta_{s^*}), \text{ and } \rho = (\rho_1, \ldots, \rho_p).$$

This implies that

$$\frac{1}{p} \big\langle < a >, (e^{p\eta(x^*, u^*)} - \mathbf{1}) \big\rangle = 0. \tag{4.34}$$

If E is strictly Exp B-$(p, r)-$invexity w.r.t η at u^* in L_4, we get

$\frac{1}{r} b(x^*, u^*)(e^{r(E(x^*) - E(u^*))} - 1) > \frac{1}{p} \big\langle < a >, (e^{p\eta(x^*, u^*)} - \mathbf{1}) \big\rangle.$

From above relation and (4.34), we have

$$\frac{1}{r} b(x^*, u^*)(e^{r(E(x^*) - E(u^*))} - 1) > 0.$$

By the exponential function, we obtain

$$E(x^*) > E(u^*) = 0 \text{ for } r \neq 0. \tag{4.35}$$

From Theorem 4.5.2, we see that

$$\max_{y \in Y} \frac{f(x^*, y)}{g(x^*, y)} = \frac{\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^*, h(u^*) \rangle_p}{\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)}$$

and so

$$\frac{f(x^*, y)}{g(x^*, y)} \leq \frac{\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^*, h(u^*) \rangle_p}{\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)}. \text{ for all } y \in Y,$$

Or equivalently,

$$f(x^*, y) \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) - g(x^*, y) \left[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \langle \mu^*, h(u^*) \rangle_p \right] \leq 0 \text{ for all } y \in Y.$$

Multiplying by $t_i^* \geq 0, i = 1, 2, \ldots, s^*$ and summing up, it yields

$$\sum_{i=1}^{s^*} t_i^* f(x^*, y_i^*) \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)$$
$$- \sum_{i=1}^{s^*} t_i^* g(x^*, y_i^*) \left[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \left\langle \mu^*, h(u^*) \right\rangle_p \right] \leq 0, \tag{4.36}$$

it reduces to

$$\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) \left[\sum_{i=1}^{s^*} t_i^* f(x^*, y_i^*) + \left\langle \mu^*, h(x^*) \right\rangle_p \right]$$
$$- \sum_{i=1}^{s^*} t_i^* g(x^*, y_i^*) \left[\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \left\langle \mu^*, h(u^*) \right\rangle_p \right]$$
$$= E(x^*) \leq \sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) \left\langle \mu^*, h(x^*) \right\rangle_p. \tag{4.37}$$

Since the relations (4.29), $g(u^*, y_i^*) > 0, i = 1, 2, \ldots, s$, and $h_j(x^*) \leq 0$, we get

$$\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*) \left\langle \mu^*, h(x^*) \right\rangle_p \leq 0.$$

Therefore, from (4.37), we obtain

$$E(x^*) \leq 0 = E(u^*).$$

Consequently, the above expression contradicts the inequality (4.35). Hence u^* is an optimal solution to (P), and $E(x^*) = E(u^*)$ deduces $u^* = x^*$. Therefore

$$\max_{y \in Y} \frac{f(u^*, y)}{g(u^*, y)} = \frac{\sum_{i=1}^{s^*} t_i^* f(u^*, y_i^*) + \left\langle \mu^*, h(u^*) \right\rangle_p}{\sum_{i=1}^{s^*} t_i^* g(u^*, y_i^*)}.$$

This proves the optimal values of the dual problem (WD) and the primal problem (P) are equal.

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