ON L^1 - CONVERGENCE OF MODIFIED TRIGONOMETRIC SUMS

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CERTIFICATE

I hereby certify that the dissertation entitled **"ON L^1 - CONVERGENCE OF MODIFIED TRIGONOMETRIC SUMS "**, which is being submitted by **Ms. Hardeep Kaur** (Roll no. 301103005), in the fulfillment of the requirements for the award of degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision of **Dr. S.S. Bhatia,** Professor, SMCA, Thapar University, Patiala, during the period from January 2013 to June 2013.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other university for the award of any degree.

(Hardeep Kaur)

This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.

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ABSTRACT

The present dissertation entitled, "**ON L^1 - CONVERGENCE OF MODIFIED TRIGONOMETRIC SUMS**", contains a brief account of the study carried out by me on L^1-convergence of trigonometric series under the supervision of **Dr. S.S. Bhatia,** Professor, School of Mathematics and computer Applications, Thapar University, Patiala.

The whole work presented in this dissertation is divided into four chapters. Chapter-I is introductory. In this chapter, apart from setting up the notations and terminology to be used in subsequent chapters, I have presented some known results related with our results along with a brief plan of the results presented in chapters to follow. The aim of chapter II is to study the L^1-convergence of modified cosine sums of Kumari and Ram given in 1988 for the class S of Sidon. Chapter III is devoted to the study of L^1- convergence of r^{th} derivatives of the modified cosine sums of Kumari and Ram under class S_r of coefficient sequences introduced by Tomovski.

In chapter IV, the L^1- convergence of modified cosine sums introduced by Kumari and Ram for the class of the coefficient sequence given by Telijakovski have been studied.

References of various publications cited in the present dissertation have been reported towards the end of the dissertation.

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CHAPTER I

INTRODUCTION

1.1 Introduction.

The present dissertation comprises certain investigations carried out by the author **"On L^1-Convergence of Modified Trigonometric Sums".** It is well known that if trigonometric series converges in L^1-metric to a function $f \in L^1(T)$, then it is the Fourier series of the function f. Riesz [1, Vol.II,ch.VIII, p.144] gave a counter example showing that in a metric space L^1, we can not expect the converse of the above said result to hold true. This motivated various authors to study the L^1-convergence of trigonometric series with special coefficient. The work on this topic was introduced in 1913 by W. H. Young [43] by taking the classes of convex sequence $(\Delta^2 a_n \geq 0)$ and A. N. Kolmogorov [20] in the year 1923 by taking the classes of quasi-convex sequeuence $(\sum_{n=1}^{\infty} n \mid \Delta^2 a_n \mid < \infty)$.Teljakovaskii [42] in 1973 studied the L^1-convergence of certain modified cosine sums under the class S introduced by Sidon [31]. The result generalized by these authors were further generalized and extended by G. H. Hardy and J. E. Littelewood [12], T. Kano [14], J. W. Garret and C. V.Stanojevic ([8], [9]), B. Ram ([25], [26], [27]), N. Singh and K. M. Sharma ([32], [33], [34]), R. Bojanic and C. V. Stanojevic [4], C. P. Chen [5], R. Bala and B. Ram [2], F. Moricz [23], S. S. Bhatia and B. Ram [3], Z. Tomovski ([37], [38], [39], [40]), N. Hooda and B. Ram [13], K. Kaur, S. S. Bhatia and B. Ram [15], J. Kaur and S. S. Bhatia ([17], [18], [19]) and others by considering various generalizations of classes of sequences mentioned above for one-dimensional trigonometric cosine and sine sums.

During their investigations, some authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that they converge in L^1-metric to the sum of the trigonometric series, whereas the classical series itself may not. Garrett and Stanojevic ([8], [9], [10]), Rees and Stanojevic [29], Kumari and Ram [21], Chen ([6], [7]), Bhatia and Ram [3], Hooda and Ram [13] and others, introduced new modified trigonometric sums and studied their integrability and L^1-convergence under various classes of coefficient sequences. Bhatia and Ram [3] generalized the results of Kumari and Ram [21] and obtained their results as corollary.

In the present dissertation, number of results have been studied by the author, most of which are directly associated with the works of above mentioned authors.

To provide sufficient background for later chapters, we in this introductory chapter give a summary of basic definitions and a brief chapter wise resume of the results contained in the dissertation. However, some of the definitions and notations will be repeated in various chapters for the convenience.

1.2 Definition and Notations.

Let $\{a_n\}$ be a sequence. Then we write

$$\Delta a_n = a_n - a_{n+1}$$

$$\Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}$$

Abel's transformation ([1], Vol.I, p.1): Let $a_0, a_1, a_2, ..., v_0, v_1, ..., v_n, ...$ be any real numbers and also let us assume that

$$V_n = v_0 + v_1 + ... + v_n ,$$

then for any values of m and n, we have

$$\sum_{k=m}^{n} a_k v_k = \sum_{k=m}^{n-1} \Delta a_k V_k + a_n V_n - a_m V_{m-1}$$

(under the condition that if $m = 0$, $V_{-1} = 0$).

Null sequence: The sequence $\{a_n\}$ is null sequence if $a_n \to 0 \; as \; n \to \infty$.

Convex sequence: A sequence $\{a_n\}$ is said to be convex if $\Delta^2 a_n \geq 0$.

Quasi-Convex sequence ([1], Vol.II, p.202): A sequence $\{a_n\}$ is said to be quasi-convex if

$$\sum_{n=1}^{\infty} n \, | \Delta^2 a_n | < \infty$$

Semi-convex sequence [14]: A null sequence $\{a_n\}$ is said to be semi-convex if

$$\sum_{n=1}^{\infty} n\,|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \qquad (a_0 = 0)$$

Generalized semi-convex sequence [16]: A null sequence $\{a_n\}$ is said to be generalized semi-convex, if

$$\sum_{n=1}^{\infty} n^{\alpha}\,|\Delta^{\alpha+1} a_{n-1} + \Delta^{\alpha+1} a_n| < \infty, \qquad (a_0 = 0)$$

If we take $\alpha = 1,$ then this sequence reduce to the semi-convex sequence as in [14].

Sequence of Bounded Variation ([1], Vol.I, p.3): A sequence $\{a_n\}$ is said to be of bounded variation, if

$$\sum_{n=1}^{\infty} |\Delta a_n| < \infty.$$

The class of all null sequences of bounded variation are denoted by BV.

Sequence of Bounded Variation of order m [11]: A sequence $\{a_n\}$ is said to be of bounded variation of order $m,$ if

$$\sum_{n=1}^{\infty} |\Delta^m a_n| < \infty,$$

where $\quad \Delta^m a_n = \Delta(\Delta^{m-1} a_n) = \Delta^{m-1} a_n - \Delta^{m-1} a_{n+1}.$

The class of all null-sequences of bounded variation of order m are denoted by $(BV)^m$. Clearly for $m = 1$, the class $(BV)^m$ reduces to the class BV.

Quasi-monotone sequence ([29], [35]): A sequence of non-negative numbers is said to be quasi-monotone, if $a_{n+1} \le a_n\left(1 + \frac{\alpha}{n}\right)$ for some $\alpha > 0$ and all $n > n_0(\alpha)$. An equivalent definition is that $n^{-\beta}\alpha_n \downarrow 0$ for some $\beta > 0$.

Trigonometric Series: A series of the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

is called trigonometric series, where $a_0, a_1, a_2, ..., b_1, b_2, ...$ are the coefficients. These coefficients may be real or complex.

Fourier Series: A trigonometric series

$$f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

is called the Fourier Series, if $f(x)$ is periodic and intergrable function over the interval $(-\pi, \pi)$ and the Fourier formulae

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x),$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.$$

Dirichlet kernel [1, Vol.I, p.85]: It can be defined as

$$D_n(x)=\frac{1}{2}+\cos x+\cos 2x+...+\cos nx.$$

This implies that

$$2\sin\frac{x}{2}D_n(x)=\sin\frac{x}{2}+2\sin\frac{x}{2}\cos x+...+2\sin\frac{x}{2}\cos nx$$

$$=\sin\left(n+\frac{1}{2}\right)x$$

Hence,

$$D_n(x)=\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

This expression is known as Dirichlet's kernel.

Also,

$$\tilde{D}_n(x) = \sin x + \sin 2x + ... + \sin nx$$

$$= \frac{\cos\frac{x}{2} - \cos\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

is called the conjugate Dirichlet kernel.

If $x \neq 0 \pmod{2\pi}$, then

$$|D_n(x)| \leq \frac{\pi}{2x}, \qquad \textit{for } 0 < |x| \leq \pi$$

and

$$\left|\tilde{D}_n(x)\right| \leq \frac{\pi}{2}, \qquad \textit{for } 0 < |x| \leq \pi$$

We also note that the uniform estimate of Dirichlet kernel is given as

$$|D_n(x)| \leq n + \frac{1}{2}, \qquad \textit{for any } x$$

Fejer kernel ([1], [66]): The Fejer kernel $K_n(x)$ is defined as

$$K_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(x)$$

$$= \frac{1}{n+1}\sum_{j=0}^{n} \frac{\sin\left(j + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

on using the estimate $|D_n(x)| \leq n+1$, we find that $K_n(x) \leq n+1$.

The Fejer kernel $K_n(x)$ has the following properties:

(i) $K_n(x) \geq 0, \quad \forall x$

(ii) $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)\, dx = 1.$

The conjugate Fejer kernel $\tilde{K}_n(x)$ is defined as

$$\tilde{K}_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} \tilde{D}_n(x)$$

We have

$$\tilde{K}_n(x) > 0 \quad for \;\; 0 < x < \pi, \quad n = 1,2,3,...$$

and the estimate:

$$|\tilde{K}_n(x)| < \frac{1}{2}n \;.$$

In 1939, Sidon [31] introduced a new class S of coefficient sequence as:

The Class S ([31],[42]): A null sequence $\{a_n\}$ belongs to class S, if $a_n \to 0 \; as \; n \to \infty$ and there exists sequence $\{A_n\}$ such that

(i) $A_n \downarrow 0,\, n \to \infty.$

(ii) $\sum_{n=0}^{\infty} A_n < \infty.$

(iii) $|\Delta a_n| \leq A_n, \quad for \;\; all \;\; n.$

Moreover, if we let $A_n = \sum_{k=n}^{\infty} |\Delta^2 a_k|$, then we find that every quasi-convex null sequence satisfies the class S.

In 2001, Tomovski [37] introduced a new class $S_r; r = 0,1,2,...$ of coefficient sequence which is an extention of class S, defined as follows:

The class S_r [37]: A null sequence $\{a_n\}$ belongs to class S_r, $r = 0,1,2,...,$ if there exists a sequence $\{A_n\}$ such that

(i) $A_n \downarrow 0,\ n \to \infty$,

(ii) $\sum_{n=0}^{\infty} n^r A_n < \infty,\quad for\ r = 0,1,2,...$

(iii) $|\Delta a_n| \le A_n,\quad for\ all\ n.$

For $r = 0$, this class S_r reduces to class S of Sidon.

Singh and Sharma ([32], [33]) gave a definition of one more class namely S^* in the following manner :

The Class S^* ([32],[33]): A sequence $\{a_n\}$ of numbers is said to belong class S^*, if $a_n \to 0\ as\ n \to \infty$ and there exists a sequence $\{A_n\}$ such that

(i) $\{A_n\}$ is quasi monotone sequence,

(ii) $\sum_{n=0}^{\infty} A_n < \infty$,

(iii) $|\Delta a_n| \le A_n,\ for\ all\ n.$

However, Leindler [22] in 2000 proved that class S and the class S^* are equivalent.

1.3 The following results about the behaviour of cosine and sine series are known:

Theorem I([1], [20], [43]). If $\{a_n\}$ is a quasi-convex null sequence, then

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L^1[0,\pi]. \tag{1.3.1}$$

Theorem II([1], [40]). If $\{a_n\}$ is a quasi-convex null sequence, then

$$\sum_{n=1}^{\infty} a_n \sin nx, \tag{1.3.2}$$

is a Fourier series if and only if $\sum_{k=0}^{\infty} \frac{|a_k|}{k} < \infty$.

In 1968, Theorem I and Theorem II were generalized by Kano [14] as given below:

Theorem III. If $\{a_n\}$ is null sequence satisfying

$$\sum_{n=1}^{\infty} n^2 \,|\, \Delta^2 \left(\frac{a_n}{n}\right) | < \infty, \tag{1.3.3}$$

then (1.3.1) and (1.3.2) are the Fourier series or equivalenly they represent integrable functions.

Teljakovaskii [42] has given the following results concerning the integrability of trigonometric series belonging to class S ([31], [42]).

Theorem IV. Let the coefficient sequence $\{a_n\}$ of the cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.3.4}$$

belongs to class S, then this represents a Fourier series and the following relation hold good:

$$\int_0^{\pi} |\, \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \,| dx \le C \sum_{k=0}^{\infty} A_k$$

where C is absolute positive constant.

Theorem V. Consider $\sum_{n=1}^{\infty} a_n \sin nx$ be a sine series belonging to class S. The following result holds for $p=1,2,3,\ldots,$

$$\int_{\pi/(p+1)}^{\pi} |\sum_{n=1}^{\infty} a_n \sin nx \,|\, dx = \sum_{k=1}^{p} \frac{|\, a_k \,|}{k} + O\left(\sum_{k=1}^{\infty} A_k\right).$$

Theorem I and III give only sufficient conditions for the integrability of cosine series. Rees and Stanojevic [28] in 1973 have shown that the condition $\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty$ is a necessary and sufficient condition for L^1-convergence and integrability for a different type of cosine sums. The results proved by them are as follows:

Theorem VI. Consider $b_k = \frac{a_k}{k} \downarrow 0$. Then

$$g(x) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \left[\frac{b_k}{k} + \left(\sum_{j=k}^{n} b_j \right) \cos kx \right],$$

exists for $x \in (0, \pi]$ and $g \in L^1[0, \pi]$ if and only if $\sum_{k=1}^{\infty} b_k < \infty$.

Theorem VII. Let $(k+1)\,|\Delta^2 a_k| \downarrow 0$. Then

$$h(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{2}\,(k+1)\,|\Delta^2 a_k| + \left(\sum_{j=k}^{n} (j+1)\,|\Delta^2 a_j| \right) \cos kx \right],$$

exists for $x \in (0, \pi]$ and $h \in L^1[0, \pi]$ if and only if $\{a_k\}$ is quasi-convex sequence.

Ram [26] in 1979 have shown that the conditions of the class S are sufficient for the integrability of Rees-Stanojevic [28] cosine sums given by

$$f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_k \cos kx \qquad (1.3.5)$$

Ram [26] proved the following result:

Theorem VIII. Let the sequence $\{a_k\}$ satisfy the condition S. Then $f(x) = \lim_{n\to\infty} f_n(x)$ exist for $x \in (0, \pi]$ and $\int_0^{\pi} |g(x)| \le C \sum_{k=0}^{\infty} A_k$.

In 1977, Ram [25] has proved the following result on L^1-convergence of cosine trigonometric sum (1.3.5) introduced by Rees-Stanojevic [28]:

Theorem IX. If (1.3.5) belongs to class S, then $\| f - f_n \|_{L^1} = o(1)$, $n \to \infty$.

Further, Kumari and Ram [21] in 1988 introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \qquad (1.3.6)$$

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \sin kx \qquad (1.3.7)$$

and studied their L^1-convergence under the condition that the cosine series and sine series belong to the class S .

Kolomogorov [20] have proved the following result regarding the L^1-convergence of cosine trigonometric series.

Theorem X. If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (1.3.5) in metric space L^1. It is necessary and sufficient that $a_k \log k = o(1), k \to \infty$.

In 1973, Teljakovaskii [42] generalized the Theorem X as:

Theorem XI. If the coefficient sequence $\{a_k\}$ of the cosine series (1.3.5) belongs the class S,

then a necessary and sufficient conditions for L^1-convergence of (1.3.5) is $a_k \log k = o(1), k \to \infty$.

In 1967, Teljakovskii [41] generalized Theorem X for the cosine series (1.3.5) with coefficient $\{a_k\}$ satisfying conditions of quasi-convexity and established the following result:

Theorem XII. Let the coefficient $\{a_k\}$ of the series (1.3.5) satisfy the conditions of quasi-convexity If $\lim_{n\to\infty} a_k = 0,$ then the series (1.3.5) converges in the L^1-metric space.

Also Teljakovaskii [40] has shown that the series (1.3.5) is Fourier series under the conditions of quasi-condition and

$$\int_0^{\pi} \left| \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \right| dx \le C(T_1 + T_2),$$

where C is a positive absolute constant.

In chapter II, we have studied the L^1-convergence of new modified cosine sums (1.3.6) (introduced by Kumari and Ram) under the class S and have also studied the L^1-convergence of cosine trigonometric series as corollary.

In chapter III, we have studied the L^1-convergence of r^{th} derivative of modified cosine sums (1.3.6) of Kumari and Ram under the class S_r of coefficient sequence.

The aim of chapter IV is to study the L^1-convergence of the cosine sums (1.3.6) under a new class of coefficient sequence which generalizes the idea of quasi-convexity and the result of Taljakovaskii [41] has been studied as a corollary.

CHAPTER II

L^1-CONVERGENCE OF A MODIFIED COSINE SUM

2.1 Introduction.

Consider the cosine trigonometric series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx \tag{2.1.1}$$

satisfying $a_k = o(1), k \to \infty$ and if there exists a sequence $\{A_k\}$ such that

(i) $A_k \downarrow 0, k \to \infty$ (2.1.2)

(ii) $\sum_{k=0}^{\infty} A_k < \infty$ (2.1.3)

(iii) $|\Delta a_k| \leq A_k \quad \forall k = 1,2,3,\ldots,$ (2.1.4)

then, we say that the coefficient of sequence $\{a_k\}$ belongs to the class S introduced by Sidon [31].

Let $S_n(x)$ denotes the n^{th} partial sum of (2.1.1) and let $f(x)=\lim_{n\to\infty} S_n(x)$.

In 1973, Rees-Stanojevic [28] have introduced modified cosine sums as

$$f_n(x)=\frac{1}{2}\sum_{k=0}^{\infty}\Delta a_k+\sum_{k=1}^{n}\sum_{j=k}^{n}a_k\cos kx \tag{2.1.5}$$

and studied their L^1-convergence.

Further in 1977, Ram [25] obtained L^1- convergence of the modified sums (2.1.5) and proved the following result:

Theorem I. If the coefficient $\{a_k\}$ of the cosine trigonometric series (2.1.1) belongs to the class S, then

$$\| f - f_n \|_{L^1} = o(1),\ n \to \infty.$$

Later, in 1988, Kumari and Ram [21] introduced new modified cosine sums.

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

and studied their L^1-convergence. Also, they deduced the L^1-convergence of the cosine trigonometric series as corollary.

In this chapter we have studied the L^1-convergence of modified cosine sum (introduced by Kumari and Ram [21]) under the class S and shall also study the L^1-convergence of cosine series as well.

2.2 Lemmas.

The following lemmas will be required for the proof of the main result.

Lemma 2.2.1[7]. If $|c_k| \leq 1,$ then

$$\int_0^{\pi} \left| \sum_{k=0}^{n} c_k \frac{\sin\left(k + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| \leq C(n+1)$$

where C is a positive absolute constant.

Proof: We know that

$$D_k(x) = \frac{\sin\left(k + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

Since Dirichlet kernel is bounded.

$$\therefore \qquad \left| \frac{\sin\left(k + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| \leq B$$

$$\Rightarrow \int_0^{\pi} \left| \sum_{k=0}^{n} c_k \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx \le \int_0^{\pi} \left| B \sum_{k=0}^{n} 1 \right| dx$$

$$\le B(n+1)\pi = C(n+1)$$

$$\Rightarrow \int_0^{\pi} \left| \sum_{k=0}^{n} c_k \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx \le C(n+1)$$

Lemma 2.2.2[30]. Let $\tilde{D}_n(x) = \dfrac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$ be the conjugate Dirichlet kernel. Then

$$\int_{-\pi}^{\pi} |\ \tilde{D}_n{}^r(x)\ | = O(n^r \log n);\ n \to \infty,\ r = 0,1,2,...$$

where $\tilde{D}_n^r(x)$ represents the r^{th} derivative of conjugate Dirichlet kernel.

In particular if $r = 1$, then $\int_{-\pi}^{\pi} |\ \tilde{D}_n'(x)\ | = o(n \log n);\ n \to \infty.$

Proof. Let r be any non-negative integer. Use of Bernstein inequality ([1], Vol.I, p.35) gives

$$\int_0^{\pi} \left| \tilde{D}_n^r(x) \right| dx \le n^r \int_0^{\pi} \left| \tilde{D}_n(x) \right| dx \qquad (2.2.1)$$

Now,

$$\lim_{n\to\infty} \int_{\frac{\pi}{n}}^{\pi} \left| \tilde{D}_n(x) \right| dx \le \lim_{n\to\infty} \int_{\frac{\pi}{n}}^{\pi} \frac{\pi}{x}\, dx$$

$$= \lim_{n\to\infty} \pi \int_{\frac{\pi}{n}}^{\pi} \frac{1}{x}\, dx$$

$$= \lim_{n\to\infty} \pi \left[\log x\right]_{\frac{\pi}{n}}^{\pi}$$

$$= \pi \left[\log \pi - \log \frac{\pi}{n} \right]$$

$$= \pi \left[\log \pi - \log \pi + \log n \right]$$

$$= \pi \log n = O(\log n)$$

Therefore,

$$\int_0^{\pi} \left| \tilde{D}_n(x) \right| = O(\log n)$$

and hence from (2.2.1), we have

$$\int_0^{\pi} \left| \tilde{D}_n^r(x) \right| = O(n^r \log n), \qquad r = 0,1,2,...$$

2.3 Main Result.

Theorem 2.3.1. Let (2.1.1) belongs to the class S. If $\lim_{n\to\infty} |a_{n+1}| \log n = 0$, then $\| f - g_n \|_{L^1} = o(1)$, $n \to \infty$.

Proof : Consider,

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + ... + \Delta\left(\frac{a_n}{n}\right) \right)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1}\right) + \left(\frac{a_{k+1}}{k+1} - \frac{a_{k+2}}{k+2}\right) + ... + \left(\frac{a_n}{n} - \frac{a_{n+1}}{n+1}\right) \right)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1} \right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx$$

$$= S_n(x) - \frac{a_{n+1}}{n+1}\ \tilde{D}'_n(x),$$

here $\tilde{D}'_n(x)$ denotes the derivative of conjugate Dirichlet kernel.

$$f(x) - g_n(x) = \sum_{k=1}^{\infty} a_k \cos kx - S_n(x) + \frac{a_{n+1}}{n+1}\ \tilde{D}'_n(x)$$

$$= S_n(x) + \sum_{k=n+1}^{\infty} a_k \cos kx - S_n(x) + \frac{a_{n+1}}{n+1}\ \tilde{D}'_n(x)$$

$$= \sum_{k=n+1}^{\infty} a_k \cos kx + \frac{a_{n+1}}{n+1}\ \tilde{D}'_n(x)$$

Now, using Abel's transformation, we obtain

$$f(x) - g_n(x) = \sum_{k=n+1}^{\infty} \Delta a_k\ D_k(x) - a_{n+1} D_n(x) + \frac{a_{n+1}}{n+1}\ \tilde{D}'_n(x)$$

Now consider,

$$\int_0^{\pi} | f(x) - g_n(x) |\, dx = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1} D_n(x) + \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx$$

$$\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^{\pi} | a_{n+1} D_n(x) |\, dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1}\ \tilde{D}'_n(x) \right| dx$$

$$= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k}\ D_k(x) \right| dx + \int_0^{\pi} | a_{n+1} D_n(x) |\, dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1}\ \tilde{D}'_n(x) \right| dx$$

Applying Abel's transformation and making use of Lemma 2.2.1, we get

$$\int_0^{\pi} | f(x) - g_n(x) |\, dx = \left| \int_0^{\pi} \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx + \int_0^{\pi} | a_{n+1} D_n(x) |\, dx$$

$$+ \int_0^{\pi} \left| \frac{a_{n+1}}{n+1}\ \tilde{D}'_n(x) \right| dx$$

$$\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx + \int_0^{\pi} | a_{n+1} D_n(x) |\, dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1}\ \tilde{D}'_n(x) \right| dx$$

$$\leq C\sum_{k=n+1}^{\infty}(k+1)\Delta A_k+\int_0^{\pi}|a_{n+1}D_n(x)|\,dx+\int_0^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\right|dx \tag{2.3.1}$$

Applying Abel's transformation on $\sum A_k$, we have $\sum_{k=1}^{n}A_k=\sum_{k=1}^{n-1}(k+1)\Delta A_k-nA_n$. By use of (2.1.2) and (2.1.3), the series $\sum_{k=1}^{\infty}(k+1)\Delta A_k$ is convergent and therefore the first term in (2.3.1) goes to zero as $n\to\infty$.

Also, by using lemma (2.2.2), we obtain

$$\int_0^{\pi}\left|\frac{a_{n+1}}{n+1}\right|\tilde{D}_n'(x)\leq\int_{-\pi}^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\right|dx$$

$$=\frac{|a_{n+1}|}{n+1}\int_{-\pi}^{\pi}|\tilde{D}_n'(x)|\,dx$$

$$=\frac{|a_{n+1}|}{n+1}\int_{-\pi}^{\pi}|\tilde{D}_n'(x)|\,dx$$

$$\sim|a_{n+1}|\log n \tag{2.3.2}$$

Since $\int_{-\pi}^{\pi}|\tilde{D}_n'(x)|\,dx$ behave like $n\log n$.

This implies,

$$\int_0^{\pi}|f(x)-g_n(x)|\,dx=o(1)+O(|a_{n+1}|\log n)+O(|a_{n+1}|\log n)$$

Hence, under the given assumption that $|a_{n+1}|\log n\to 0\ as\ n\to\infty,$ we have

$$\|f-g_n\|=o(1)\ as\ n\to\infty.$$

Thus, the conclusion of main result holds.

Corollary. If (2.1.1) belongs to the class S and $\lim_{n\to\infty}|a_{n+1}|\log n=0,$ then $\|f-S_n\|=o(1),\ n\to\infty$.

Proof : Consider,

$$\int_{-\pi}^{\pi} | f(x) - S_n(x) | dx = \int_{-\pi}^{\pi} | f(x) - g_n(x) + g_n(x) - S_n(x) | dx$$

$$\leq \int_{-\pi}^{\pi} | f(x) - g_n(x) | dx + \int_{-\pi}^{\pi} | g_n(x) - S_n(x) | dx$$

$$\leq \int_{-\pi}^{\pi} | f(x) - g_n(x) | dx + \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx$$

Now, by the use of our Theorem 2.3.1 and Lemma 2.2.2, the conclusion of the corollary follows; i.e.,

$$\int_{-\pi}^{\pi} | f(x) - S_n(x) | dx \to 0, \text{ as } n \to \infty.$$

CHAPTER III

ON L^1- CONVERGENCE OF r^{th} DERIVATIVE OF KUMARI AND RAM'S MODIFIED COSINE TRIGONOMETRIC SUMS

3.1 Introduction.

Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{3.1.1}$$

be a cosine series. Let $S_n(x)$ be the n^{th} partial sum of the series (3.1.1) and $\lim_{n\to\infty} S_n(x) = f(x)$.

Further, let $\lim_{n\to\infty} S_n^r(x) = f^r(x)$, $r \in \{0,1,2,...\}$, where $f^r(x)$ denotes r^{th} derivative of $f(x)$ and $S_n^r(x)$ denotes r^{th} derivative of n^{th} partial sum $S_n(x)$.

In 1988, Kumari and Ram [21] introduced a new modified cosine sum as

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \tag{3.1.2}$$

and studied the L^1-convergence of cosine series (3.1.1) by using modified cosine sum (3.1.2).

In 1939, Sidon [31] introduced class S of coefficient sequence defined as follows :

Class S of coefficients Sequence ([31], [42]): A null sequence $\{a_n\}$ belongs to class S, if there exists a sequence $\{A_n\}$ such that

(i) $A_n \downarrow 0 \text{ as } n \to \infty,$

(ii) $\sum_{n=1}^{\infty} A_n < \infty,$

(iii) $|\Delta a_n| < A_n \quad \forall n.$

Further, Tomovskii [37] in 2000 extended the class S to a new class $S_r; r = 0,1,2,...$ of coefficient sequence defined as follows:

Class S_r of coefficient sequence [37]: A null sequence $\{a_n\}$ belongs to class $S_r; r = 0,1,2,...$, if there exists a sequence $\{A_n\}$ such that

(i) $A_n \downarrow 0 \ as \ n \to \infty,$

(ii) $\sum_{n=1}^{\infty} n^r A_n < \infty,$

(iii) $\left|\Delta a_n\right| < A_n \quad \forall n.$

For $r = 0$, this class reduces to class S.

In this chapter, we have studied the L^1-convergence of the r^{th} derivative of modified cosine sums (3.1.2) under a new class S_r of coefficient sequence.

3.2 Lemmas.

The following lemmas will be required for the proof of our main result:

Lemma 3.2.1 [7]. If $|a_k| \leq 1,$ then

$$\int_0^{\pi} \left| \sum a_k \frac{\sin\left(k + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx \leq C\,(n+1)$$

where C is a positive absolute constant.

Moreover, by Bernstein's inequality ([1],Vol.I, p.35), for $r = 0,1,2,...$, we have

$$\int_{\frac{\pi}{n+1}}^{\pi} \left| \sum_{k=0}^{n} a_k D_k^r(x) \right| dx \leq C\,(n+1)^{r+1}$$

where $D_n^r(x)$ represents the r^{th} derivative of Dirichlet kernel.

Lemma 3.2.2.[30]. $\|\widetilde{D}_n^r(x)\|_{L^1} = O(n^r \log n)$, $r = 0,1,2,...$ where $\widetilde{D}_n^r(x)$ denotes the r^{th} derivative of conjugate Dirichlet kernel.

Note: The proof of above two lemmas have already been given in chapter II.

Lemma 3.2.3. Let $a_n \downarrow 0 \ as \ n \to \infty$, if $\sum n^r a_n < \infty$ then $\sum (n+1)^{r+1} \Delta a_n < \infty$.

Proof. In order to prove the Lemma, we shall first prove the following inequality:

$$\sum_{n=0}^{m} n^r a_n \leq \sum_{n=0}^{m-1} (n+1)^{r+1} \Delta a_n + m^{r+1} a_m \tag{3.2.1}$$

We shall make use of Method of Induction to prove this inequality.

Clearly, for $r = 0$, the inequality (3.2.1) holds.

Now, Assume that (3.2.1) hold for $r = k$, i.e.,

$$\sum_{n=0}^{m} n^k a_n \leq \sum_{n=0}^{m-1} (n+1)^{k+1} \Delta a_n + m^{k+1} a_m \tag{3.2.2}$$

We now prove it for $r = k+1$.

Multiply (3.2.2) by n both sides, we get

$$\sum_{n=0}^{m} n^{k+1} a_n \leq \sum_{n=0}^{m-1} n(n+1)^{k+1} \Delta a_n + m^{k+1} n a_m$$

$$\leq \sum_{n=0}^{m-1} (n+1)^{k+2} \Delta a_n + m^{k+2} a_m$$

Hence, (3.2.1) is true for $r = k+1$.

Further, in the proof of Lemma 3.2.3, we require Oliver's theorem. The theorem says that if the series $\sum_{n=1}^{\infty} a_n < \infty$ and $a_n \downarrow 0 \ as \ n \to \infty$, then $na_n \to 0 \ as \ n \to \infty$.

Thus, by inequality (3.2.1) and by use of Oliver's theorem, the series $\sum_{n=1}^{\infty} (n+1)^{r+1} \Delta a_n$ is convergent.

This proves the lemma 3.2.3.

3.3 Main result.

Theorem 3.3.1. If the $\{a_k\}$ belongs to class S_r and $n^r|a_{n+1}|\log n = o(1),\ n \to \infty,$ then

$$\|f^r - g_n^r\| = o(1),\ n \to \infty.$$

Proof. Let

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx\left(\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \dots\dots + \Delta\left(\frac{a_n}{n}\right)\right)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx\left(\left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1}\right) + \left(\frac{a_{k+1}}{k+1} - \frac{a_{k+2}}{k+2}\right) + \dots\dots + \left(\frac{a_n}{n} - \frac{a_{n+1}}{n+1}\right)\right)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx\left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k\cos kx$$

$$= S_n(x) - \frac{a_{n+1}}{n+1} \tilde{D}_n'(x)$$

Here, $\tilde{D}_n'(x)$ denotes the first derivative of the conjugate Dirichlet kernel.

Taking r^{th} derivative both sides, we have

$$g_n^r(x) = S_n^r(x) - \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \tag{3.3.1}$$

Here, $g_n^r(x)$ denotes the r^{th} derivative of $g_n(x)$ and $\tilde{D}_n^r(x)$ denotes the r^{th} derivative of conjugate Dirichlet kernel.

Since $\{a_n\} \in S_r$, $r \in \{0,1,2,...\}$ is a null sequence and $\tilde{D}_n^r(x)$ is bounded in $(0,\pi]$,

Therefore,

$$\lim_{n\to\infty} g_n^r(x) = \lim_{n\to\infty} S_n^r(x) = f^r(x).$$

Thus the limit exists in $(0,\pi]$.

Now, it follows from (3.3.1) that

$$f^r(x) - g_n^r(x) = \sum_{k=1}^{\infty} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) - S_n^r(x) + \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)$$

$$= \sum_{k=1}^{\infty} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) - \sum_{k=1}^{n} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) + \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)$$

$$= \sum_{k=n+1}^{\infty} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) + \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)$$

By the use of Abel's transformation, we get

$$f^r(x) - g_n^r(x) = \sum_{k=1}^{\infty} \Delta a_k D_k^r(x) - a_{n+1} D_n^r(x) + \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)$$

Here, $D_n^r(x)$ denotes the r^{th} derivative of Dirichlet kernel.

Further, we consider

$$\left\| f^r(x) - g_n^r(x) \right\| = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) - a_{n+1} D_n^r(x) + \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) \right| dx$$

$$\le \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) \right| dx + \int_0^{\pi} \left| a_{n+1} D_n^r(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) \right| dx$$

$$= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k^r(x) \right| dx + \int_0^{\pi} \left| a_{n+1} D_n^r(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) \right| dx$$

Again, applying Abel's transformation, we get

$$\left\| f^r - g_n^r \right\| = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \sum_{j=0}^{k} \frac{\Delta a_j}{j} D_j^r(x) \right| dx + \int_0^{\pi} \left| a_{n+1} D_n^r(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx$$

$$\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i^r(x) \right| dx + \int_0^{\pi} \left| a_{n+1} D_n^r(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx$$

Using Lemmas 3.2.1 and 3.2.2, we have

$$\left\| f^r - g_n^r \right\| \leq C \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} (k+1)^{r+1} \Delta A_k \right| dx + O\left(n^r \left| a_{n+1} \right| \log n\right) + O\left(n^r \left| a_{n+1} \right| \log n\right) \quad (3.3.2)$$

Therefore by the use Lemma 3.2.3, $\sum_{k=1}^{\infty} (k+1)^{r+1} \Delta A_k$ is convergent and therefore the first term of (3.3.2) is zero.

Hence, this implies that

$$\left\| f^r - g_n^r \right\| = o(1) + O\left(n^r \left| a_{n+1} \right| \log n\right)$$

Therefore, by the given hypothesis, the conclusion the result follows.

Corollary. If $\{a_n\}$ belongs to class S_r, $r = 0,1,2,...$ and if $\left| a_{n+1} \right| n^r \log n = o(1)$, $n \to \infty$. Then $\left\| f^r - S_n^r \right\|_{L^1} = o(1)$, $n \to \infty$.

Proof. Consider

$$\left\| f^r - S_n^r \right\| = \left\| f^r - g_n^r + g_n^r - S_n^r \right\|$$

$$\leq \left\| f^r - g_n^r \right\| + \left\| g_n^r - S_n^r \right\|$$

$$= \left\| f^r - g_n^r \right\| + \left\| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right\|$$

$$= \left\| f^r - g_n^r \right\| + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx$$

Since $\left\| f^r - g_n^r \right\|_{L^1} = o(1),\, n \to \infty$ by our theorem and $\int_0^\pi \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx$ behaves like $n^r \left| a_{n+1} \right| \log n$ (by Lemma 3.2.2), the conclusion of corollary follows.

CHAPTER IV

ON CONVERGENCE OF CERTAIN TRIGONOMETRIC SUMS

IN THE METRIC SPACE L

4.1 Introduction.

Let

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \qquad (4.1.1)$$

be the cosine series and $f(x)=\lim_{n\to\infty}S_n(x)$, where $S_n(x)$ is the n^{th} partial sum of the series (4.1.1).

A sequence $\{a_k\}$ is said to be convex if $\Delta^2 a_k \geq 0$ and quasi-convex if $\sum_{k=1}^{\infty}(k+1)\,|\,\Delta^2\,a_k\,|<\infty$.

where $$\Delta^2 a_k = \Delta(\Delta\, a_k)$$

$$= \Delta(a_k - a_{k+1})$$

$$= a_k - a_{k+1} - a_{k+1} + a_{k+2}$$

$$= a_k - 2a_{k+1} + a_{k+2}$$

The concept of quasi-convexity was further generalized by Sidon [31] in 1939 by defining a new class S of coefficient sequences as follows:

The class S of coefficient sequence ([31], [42]): A sequence $\{a_k\}$ is said to belong to the class S if $a_k = o(1)$, $k\to\infty$ and there exists a sequence of numbers $\{A_k\}$ such that

(i) $A_k \downarrow 0,\ k\to\infty,$

(ii) $\sum_{k=1}^{\infty}A_k<\infty,$

(iii) $\left|\Delta a_k\right| \leq A_k,\ for\ all\ k.$

Remark: If we choose $A_n = \sum_{m=n}^{\infty} |\Delta^2 a_m|$, then the quasi-convex null sequence satisfies the condition of the class S.

Later in 1964, Teljakovaskii [40] generalized the idea of quasi-convexity in the following way :

Let $\{a_k\}$ be a sequence satisfying the following condition:

$$a_k = o(1),\ k \to \infty, \tag{4.1.2}$$

$$T_1 = \sum_{k=0}^{\infty} |\Delta a_k| < \infty, \tag{4.1.3}$$

$$T_2 = \sum_{m=2}^{\infty} \left| \sum_{k=1}^{[m/2]} \frac{\Delta a_{m-k} - \Delta a_{m+k}}{k} \right| < \infty. \tag{4.1.4}$$

It can be easily noted that a quasi-convex null sequence satisfies the (4.1.3)-(4.1.4).

Concerning the L^1-convergence of the cosine series (4.1.1), Kolmogorov [20] proved the following result:

Theorem I. *If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (4.1.1) in the metric space L, it is necessary and sufficient that $a_k \log k = o(1), k \to \infty$.*

In 1973, Teljakovaskii [42] proved the L^1-convergence of cosine series under class S. The result reads as follows:

Theorem II. *If the coefficient sequence $\{a_k\}$ of the cosine series (4.1.1) belongs the class S, then a necessary and sufficient conditions for L^1-convergence of (4.1.1) is $a_k \log k = o(1),\ k \to \infty$.*

In 1967, Teljakovaskii [41] generalized Theorem I for the cosine series (4.1.1) with the coefficient $\{a_k\}$ satisfying conditions (4.1.2)-(4.1.4) and obtained the following result:

Theorem III. *Let the coefficient* $\{a_k\}$ *of the series (4.1.1) satisfy the conditions (4.1.2)-(4.1.4). If* $\lim_{k\to\infty} a_k = 0,$ *then the series (4.1.1) converges in the metric space* L.

Also, Teljakovaskii [40] proved that the series (4.1.1) is a Fourier series under the conditions (4.1.2)-(4.1.4) and

$$\int_0^{\pi} \left|\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx\right| dx \leq C(T_1 + T_2) \tag{4.1.5}$$

where C is a positive absolute constant.

In 1977, Ram [25] studied the L^1-convergence of the Rees-Stanojevic sums as

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx \tag{4.1.6}$$

and obtained the Theorem II as a corollary of his result.

In 1988, Kumari and Ram [21] considered modified sums

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \tag{4.1.7}$$

The aim of this chapter is to study the L^1-convergence of modified sums (4.1.7) under a new class of coefficient sequences which generalizes the concept of quasi-convexity and to get the Theorem III of Teljakovaskii [41] as a corollary.

4.2 Lemmas.

The following lemmas will be required in the proof of the our main result.

Lemma 4.2.1[41]. Let $\{a_k\}$ be a sequence of numbers satisfying the condition (4.1.3), and let n be a natural number. If

$$b_k = \begin{cases} 0 & \text{for } k \leq n \\ a_k & \text{for } k > n, \end{cases}$$

then for all $n \geq 2$

$$\sum_{m=2}^{\infty} \left| \sum_{k=1}^{[m/2]} \frac{\Delta b_{m-k} - \Delta b_{m+k}}{k} \right| = \sum_{m=n}^{\infty} \left| \sum_{k=1}^{[m/2]} \frac{\Delta a_{m-k} - \Delta a_{m+k}}{k} \right| + o\left(\max_{n/2 \le k < 3n/2} |a_k| \log n\right).$$

Lemma 4.2.2[30]. If $\tilde{D}_n(x) = \dfrac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)}{2\sin\frac{x}{2}}$ be the conjugate Dirichlet kernel, then

$$\int_{-\pi}^{\pi} |\tilde{D}_n'(x)|\, dx = O(n \log n).$$

The proof of this lemma has been reported in chapter II.

4.3 Main result.

Theorem 4.3.1. Let $\{a_k\}$ be a sequence of numbers satisfying the conditions (4.1.2)-(4.1.4). If $\lim_{n\to\infty} a_n \log n = 0$, then $\| f - g_n \|_{L^1} = o(1),\ n \to \infty.$

Proof. Let b_k be the numbers as defined in Lemma (4.2.1). Then the cosine series (4.1.1) can be written as

$$\frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \cos kx = \sum_{k=n+1}^{\infty} a_k \cos kx \tag{4.3.1}$$

Furthermore, we have

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \ldots + \Delta\left(\frac{a_n}{n}\right) \right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[\left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1}\right) + \left(\frac{a_{k+1}}{k+1} - \frac{a_{k+2}}{k+2}\right) + \ldots + \left(\frac{a_n}{n} - \frac{a_{n+1}}{n+1}\right) \right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1} \right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx$$

$$= S_n(x) - \frac{a_{n+1}}{n+1}\ \tilde{D}_n'(x)\ .$$

Consider,

$$\int_0^{\pi} | f(x) - g_n(x) | dx = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} a_k \cos kx + \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx$$

By using (4.3.1), we have

$$= \int_0^{\pi} \left| \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \cos kx + \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx$$

$$\leq \int_0^{\pi} \left| \frac{b_o}{2} + \sum_{k=1}^{\infty} b_k \cos kx \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx$$

By using the inequality (4.1.5), we get

$$\leq C \left(\sum_{k=0}^{\infty} |\Delta b_k| + \sum_{m=2}^{\infty} \left| \sum_{k=1}^{[m/2]} \frac{\Delta b_{m-k} - \Delta b_{m+k}}{k} \right| \right) + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx$$

Now, making use of Lemma (4.2.1), we have

$$\| f(x) - g_n(x) \| \leq C \left(\sum_{k=n+1}^{\infty} |\Delta a_k| + \sum_{m=n}^{\infty} \left| \sum_{k=1}^{[m/2]} \frac{\Delta a_{m-k} - \Delta a_{m+k}}{k} \right| + \max_{n/2 \leq k < 3n/2} |a_k| \log n \right)$$

$$+ \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx \qquad (4.3.2)$$

Use of Lemma (4.2.2) implies that

$$\int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx \leq \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx$$

$$= \frac{|a_{n+1}|}{n+1} \int_{-\pi}^{\pi} \left| \tilde{D}'_n(x) \right| dx$$

$$\sim |a_{n+1}| \log n, \qquad (4.3.3)$$

Since $\int_{-\pi}^{\pi} \left| \tilde{D}'_n(x) \right| dx$ behaves like $n \log n$.

Now, the use of given hypothesis both the terms of (4.3.2) tends to zero as $n \to \infty$.

Therefore, the conclusion of the result follows.

4.4 Corollary.

Corollary. Let $\{a_k\}$ be a sequence of numbers satisfying the conditions (4.1.2)-(4.1.4) and if $\lim_{n\to\infty} a_n \log n = 0$, then $\|f - S_n\| = o(1)$, $n \to \infty$.

Proof. Consider,

$$\int_0^{\pi} |\, f(x) - S_n(x) \,| = \int_{-\pi}^{\pi} |\, f(x) - g_n(x) + g_n(x) - S_n(x) \,|\, dx$$

$$\leq \int_{-\pi}^{\pi} |\, f(x) - g_n(x) | dx + \int_{-\pi}^{\pi} | g_n(x) - S_n(x) | dx$$

$$\leq \int_{-\pi}^{\pi} |\, f(x) - g_n(x) \,|\, dx + \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx.$$

Since, $\lim_{n\to\infty} \int_{-\pi}^{\pi} |\, f(x) - g_n(x) |\, dx = 0$ by Theorem 4.3.1 and using Lemma 4.2.2, the conclusion of corollary follows.

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