EXACT SOLUTIONS OF NONLINEAR PARTIAL DIFFRENTIAL EQUATIONS

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under
the guidance of
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to the

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CHAPTER

CERTIFICATE

I hereby certify that the work presented in the thesis entitled " Exact Solutions of Nonlinear Partial Differential Equations*" which is being presented for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Rajesh Kumar Gupta.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

Shivali Singla

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

Exact solutions to nonlinear partial differential equations play an important role for understanding of qualitative as well as quantitative features of many phenomena and processes. Exact solutions visually demonstrate and make it possible to understand the mechanism of complex nonlinear effects.

The thesis entitled “ EXACT SOLUTIONS OF NONLINEAR PARTIAL DIFFRENTIAL EQUATIONS ” is an attempt to obtain the exact solutions of some nonlinear partial differential equations.

The thesis has been divided into six chapters. The brief outline of the research work presented chapter wise in the thesis is as follows:

First chapter is introductory in nature, in this chapter, definition of nonlinear differential equations and basic concepts are discussed. A brief summary of literature available on the subject and summary of the work presented in the thesis also appears in this chapter.

In the second chapter, methodology of hyperbolic functions expansion method and jacobi elliptic functions expansion method have been presented.

In the third chapter, Dreinfeld’s- Sokolov - Wilson equation has been solved by using both the methods: Hyperbolic functions expansion method and Jacobi elliptic functions expansion method. Jacobi elliptic functions expansion method, which is more general than the hyperbolic functions expansion method, is proposed to construct the exact solutions. The main idea of this method is to take full advantage of the elliptic equations that Jacobian elliptic functions satisfy and use its solutions to replace Jacobian elliptic functions in Jacobi elliptic functions method.

The fourth chapter comprises Gardener’s equation with both the methods: hyperbolic functions expansion method (tanh method) and jacobi elliptic functions expansion method.

The fifth chapter contains Whitham- broer- kaup equation, which is solved by hyperbolic functions expansion method, and in the sixth chapter Benjamin- Bona- Mahony equation has been solved with the hyperbolic functions expansion method.

It is worth to mention that all the solutions reported in this thesis are checked by Maple software.

CHAPTER – 1

INTRODUCTION

1.1 Basic Definitions

Differential equation is an equation $f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots\ldots, \frac{d^ny}{dx^n}) = 0$, which expresses a relation between independent and dependent variables and their derivatives of any order.

There are two types of differential equations:

1. Ordinary differential equations.
2. Partial differential equations.

Ordinary differential equation is an equation involving only one independent variable.

Example:

$$\frac{dy}{dx} = 2x + 1.$$

Order of an ordinary differential equation is the highest order derivative present in the equation.

Example:

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + 1, \text{ Order} = 2.$$

Degree of an ordinary differential equation is the exponent of the derivative of the highest order in the differential equation after the equation is free from radicals and fractions in its derivatives.

Example:

$$\left(\frac{dy}{dx}\right)^{7/2} = \frac{dy}{dx} + 2, \text{ Degree} = 7.$$

Linear differential equation is an equation if the unknown function and its derivative, which occur in the equation, occur only in the first degree and are not multiplied together. Otherwise, the differential equation is said to be non-linear.

Example:

$$(1).\ \frac{d^3y}{dx^3}+\frac{dy}{dx}+y=0,\ \text{is linear.}$$

$$(2).\ \frac{dy}{dx}+\left[1+\left(\frac{d^2y}{dx^2}\right)^3\right]^{5/2}+y=0,\ \text{is nonlinear.}$$

$$(3).\ \frac{d^2y}{dx^2}+\sin y=0,\ \text{is nonlinear.}$$

Partial differential equation is an equation involving one dependent variable and one or more independent variables.

Examples:

$$(1).\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u.$$

$$(2).\ \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=z.$$

Order of the partial differential equation is the order of the highest partial derivatives in the equation.

Example:

$$\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{\partial^2 u}{\partial z^2}\right)^3,\ \text{Order} = 2.$$

Degree of a partial differential equation is the degree of the highest order derivative occurring in the equation.

Example:

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial^2 u}{\partial z^2} \right)^3, \text{ Degree} = 3.$$

1.2 Nonlinear equations and exact solutions

An equation in which one or more terms have a degree 2 or higher is called nonlinear equation. A nonlinear system of equations contains at least one nonlinear equation. In maths linear generally means "simple" and nonlinear means "complicated". The theory for solving linear equations is very well developed because linear equations are simple enough to be solveable. Non-linear equations can usually not be solved exactly and are the subject of much on-going research.

Nonlinear equations are of the interest to the engineers, physicists and mathematicians because most physical systems are inherently nonlinear in nature. Nonlinear equations are difficult to solve and give rise to interesting phenomena such as chaos. The weather is famously nonlinear, where simple changes in one part of the system produce complex effects throughout.

The study of nonlinear differential equations has not only provided information about the phenomenon but has, in fact, helped in making more precise some of the concepts and theories developed in the last century mathematics. The standard strategies adopted to get the solutions of nonlinear partial differential equations (PDEs) to date are following:

(i). Linearize the given set of nonlinear equations by invoking certain physical assumptions.
(ii). Numerical integration of the equations under appropriate boundary conditions.
(iii). To derive exact solutions of nonlinear equations.

Exact solutions of differential equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. These solutions can be used to verify the consistencies and estimate errors of various numerical, asymptotic and approximate analytical methods.

1.3 Literature review

Nonlinear complex physics phenomena are related to nonlinear partial differential equations (NLPDEs) which are involved in many fields from physics to biology, chemistry, mechanics, etc. As mathematical model of the phenomena, the investigation of the exact solutions of nonlinear partial differential equations (NLPDEs) will help us to understand these phenomena better.

Exact analytical solutions to nonlinear partial differential equations (NLPDEs) play a very important role in nonlinear science, especially in nonlinear physical science, since they can provide much physical informations and more insight into the physical aspects of the problems and thus lead to further applications. Much work has been done over the years on the subject of obtaining the analytical solutions to the partial differential equations. Especially with the development of computer algebra, the simple and direct methods such as tanh-method [15, 18, 19, 21, 33] for obtaining exact analytical solutions of partial differential equations have drawn a lot of interest.

The search for exact solutions of nonlinear equations has been in more interest in the recent years because of the availability of the symbolic Computation Mathematica or Maple. These programs allow us to perform some complicated and differential calculations on a computer. Physical structures of the exact solutions are important to give more insight into the physical aspects of the nonlinear equations applicable for a considerably number of nonlinear systems. One of the most exciting advances of the nonlinear and theoretical physics has been development of methods to look for exact solutions for nonlinear partial differential equations.

Due to the increasing interest for obtaining the exact solutions of the nonlinear partial differential equations (NLPDEs), a wide range of methods are now available for obtaining the exact solutions such as:

(1). Backlund transformation [17].
(2). Hyperbolic functions expansion method [18, 19].
(3). Sine-cosine method [37].
(4). Jacobi elliptic functions expansion method [20].
(5). Homotopy perturbation method [10].
(6). Inverse scattering method [1].
(7). F-expansion method [38], etc.

Tanh method is one of these methods for obtaining the exact solution of nonlinear partial differential equations (NLPDEs). Further this is modified and named as Extended Tanh method. This extended tanh method is used to construct exact solutions of the (2+1) – dimensional dispersive long wave equations [36]. As a result, many new and more general solutions are obtained including soliton-like solutions, periodic formal solutions and rational function solutions.

All these solutions are compared with most existing tanh function methods, the proposed method give new and more general exact solutions. More importantly, with the aid of symbolic computation, this method provides a powerful mathematical tool for solving many nonlinear partial differential equations in mathematical physics.

Tanh method is more straightforward than the real exponential method. However, for other than simple equations it is still tedious to use by hand. If we take the full advantage of modern computer algebra system such as Maple and Mathematica, the limitation of tanh function expansion method would be eliminated.

Many equations have already been solved by tanh and extended tanh method, which are as follows:
(1). (2+1)- dimensional Dispersive Long Wave equations [36].

$$u_{yt} + H_{xx} + \frac{1}{2}\left(u^2\right)_{xy} = 0$$

$$H_t + (uH + u + u_{xy})_x = 0.$$

(2). MBBM equation (Modified Benjamin- Bona- Mahony equation) [34].

$u_t + u_x + a u^2 u_x + b u_{xxt} = 0$, where a, b > 0.

(3). Hirota-Satsuma Coupled kdV equation [35].

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} - 3uu_x + 3\left(vw\right)_x \\ v_t &= - v_{xxx} + 3uv_x \\ w_t &= - w_{xxx} + 3uw_x. \end{aligned}$$

(4). Double Sin-Gordon equation [8].

$$u_{xt} = \sin u + \sin 2u.$$

(5). Kdv (Kortweg- de Vries) Burger equation [34].

$$u_t + auu_x + bu_{xx} + cu_{xxx} = 0.$$

(6). Klein-Gordon equation [24].

$$u_{xx} - u_{tt} - u + 2u^3 + 2uv = 0$$

$$u_x - v_t - 4uu_t = 0.$$

(7). The Korteweg- de Vries equation (KdV equation) [29].

$$u_t + auu_x + bu_{xxx} = 0.$$

Jacobi elliptic function method is also straight forward and easy to implement. The solution to the equations can also be obtained with the help of Maple software. Many equations have already been solved by Jacobi elliptic functions expansion method, which are as follows:

(1). The Nizhnik- Novikov- Veselov equation [23].

$$\begin{aligned} &u_t + au_{xxx} + bu_{yyy} - 3a(uv)_x - 3b(uw)_y = 0 \\ &u_x = v_y \\ &u_y = w_{x.} \end{aligned}$$

(2). The Schrodinger equation [12].

$$\frac{\partial \psi}{\partial z} = i\alpha_1 \frac{\partial^2 \psi}{\partial t^2} + i\alpha_2 \psi |\psi|^2 + \alpha_3 \frac{\partial^3 \psi}{\partial t^3} + \alpha_4 \frac{\partial \psi |\psi|^2}{\partial t} + \alpha_5 \psi \frac{\partial |\psi|^2}{\partial t}.$$

CHAPTER - 2

METHODOLOGY

2.1 Hyperbolic functions expansion method

The method was originally proposed by W. Malfliet and W. Hereman [18, 19]. But they considered only tan hyperbolic. Only tanh type traveling wave solutions can be obtained by tanh-function expansion method. If a given PDE has other types of traveling wave solutions for example, tan type solutions, we have to repeat the similar but tedious calculations. With this consideration in mind, Fan [6, 7] proposed an extended tanh-function expansion method by which one can simultaneously obtain three types of traveling wave solutions to a given PDE.

Later the extended tanh method, developed by Wazwaz [28] is a direct and effective algebraic method for handling nonlinear equations. Various extensions of the method were developed as well. Also R. X. Yao, et al. [33] considered both the tan and sec hyperbolic.

This method mainly consists of following steps:

Step 1: Transform the PDEs into ODEs.

For a given system of partial differential equations, say, in two dependent variables ψ *and* ϕ :

$$P(\psi, \phi, \psi_x, \phi_x, \psi_t, \phi_t, \psi_{xx}, \phi_{xx}, \ldots) = 0$$
$$Q(\psi, \phi, \psi_x, \phi_x, \psi_t, \phi_t, \psi_{xx}, \phi_{xx}, \ldots) = 0,$$

(2.1.1)

where $\psi(x,t)$, $\phi(x,t)$ are the functions of the variables x and t. P and Q are polynomials about ψ, ϕ and their derivatives.

Now by using the simple transformation $\xi = x - ct$, where ξ is the traveling wave variable and c is arbitrary constant.

The system (2.1.1) becomes a system of ordinary differential equations:

$$P(\psi, \phi', \psi', \phi', \psi'', \phi'', \ldots) = 0$$
$$Q(\psi, \phi, \psi', \phi', \psi'', \phi'', \ldots) = 0,$$

(2.1.2)

where ' denotes $d/d\xi$.

The method is originally for the system of ordinary differential equations with constant coefficients.

Now we assume equation (2.1.2) admits the solution of the form:

$$\psi(\xi) = \sum_{i=0}^{m} a_i u^i + \sum_{i=1}^{m} b_i v u^{i-1}$$

$$\phi(\xi) = \sum_{j=0}^{n} c_j u^j + \sum_{j=1}^{n} d_j v u^{j-1},$$

(2.1.3)

where a_i, b_i, c_j, d_j are the constants which are to be determined.

Here $u = \operatorname{sec}h(k\xi)$ and $v = \tanh(k\xi)$.

(2.1.4)

u and v satisfies the relation: $v^2 = 1 - u^2$. If we take the first order derivatives we will get the relations as:

$u' = -kuv$ and $v' = k(1 - v^2)$, and so on… (2.1.5)

As further higher derivatives are also polynomial expressions in u and v.

Step 2: Determine the degree of the polynomial solution.

The values of m and n are the polynomial degree of the solution in (ψ ,ϕ) respectively, which can be determined by using the Homogenous balancing method, i.e. by balancing the highest nonlinear terms with the highest- order derivative terms in (2.1.2).

Step 3: Derive the algebraic system for the coefficients.

Substituting (2.1.3) into (2.1.2), using (2.1.5) repeatedly, all higher derivatives of u and v can be expressed as polynomials in u and v. To be precise, all derivatives of v are polynomials in v. Odd derivatives of u are polynomials in u and v (the latter appears only up to the first power). Even derivatives of u are polynomial in u, again after using the relation $v^2 = 1 - u^2$, we remove all even powers of v. Because the coefficients of u^i and vu^i have to vanish, we get two sets of algebraic equations comprising the nonlinear equations in a_i, b_i, c_j and d_j's involving k and c. The solutions to the algebraic equations give us various relations among the physical parameters and the undetermined constants in the form (2.1.3).

Step 4: Solve the nonlinear algebraic system.

Now we will solve the algebraic system of the nonlinear equations.

Step 5: Build and test the solutions

The outputs of solving the algebraic system comprise a list of the for a_i, b_i, c_j and d_j 's with *k* and *c*. Using these constants in (2.1.3), solutions of the system (2.1.1) can be obtained.

2.2 Jacobi elliptic functions method

The jacobi elliptic functions were introduced by Carl Gustav Jakob Jacobi, around 1830. They also have useful analogies to the functions of trigonometry, as indicated by the matching notation sn for sin etc. An elliptic function is a doubly periodic analytic function whose only possible singular points, in the finite part of the z-plane, are the poles. Porubov et al. [25-27] have obtained some exact periodic solutions to some nonlinear wave equations; they use the Weierstrass elliptic function. Liu et al. [16] recently proposed Jacobi elliptic functions expansion method for finding the periodic wave solutions to nonlinear evolution equations; it involves seeking a solution in the form of a polynomial in Jacobi elliptic sine function. Shortly afterwards, Fu et al. [9] showed that the method also works with other Jacobi elliptic functions. The Jacobi sine-function and cosine-function methods are natural generalization of the tanh-function and sech-function methods, respectively, for finding solitary wave solutions. Since it is algorithmic procedure, with the advent of computer software this method is easy to implement. Software Mathematica and Maple are used successfully by different authors [9, 22] to obtain variety of periodic solutions including some shock wave solutions and solitary wave solutions.

Consider a constant coefficients partial differential equation for a function $u(x,t)$. The Jacobi sine function method for solving this equation consists of the following steps:

(i). Taking the transformation $u(x,t)=U(\eta)$, where $\eta = x - ct - x_0$ with c a real nonzero constant Substitution in the partial differential equation yields an ordinary differential equation for $U(\eta)$.

(ii). Now we assume that the ordinary differential equation has the solution of the form:

$$U(\eta)=\sum_{i}^{M} a_i\, sn^i(k\eta\ /m). \qquad (2.2.1)$$

where $sn(\xi\ |m)$ is the Jacobi elliptic sine function with argument ξ and modulus m, M is a positive integer to be determined. The $a_i(i = 0,1,K,M)$ are real constants with $a_M \neq 0$, and k is a real nonzero constant.

The Jacobi elliptic cosine function $cn(\xi)$, and the Jacobi elliptic function of third kind $dn(\xi)$, satisfy the following:

$$cn^2(\xi)=1-sn^2(\xi),\quad dn^2(\xi)=1-m^2sn^2(\xi)$$

$$\frac{d}{d\xi}sn(\xi)=cn(\xi)dn(\xi),\quad \frac{d}{d\xi}cn(\xi)=-sn(\xi)dn(\xi)$$

$$\frac{d}{d\xi}dn(\xi)=-m^2sn(\xi)cn(\xi),\quad \text{with modulus } m\ (0<m<1).$$

(2.2.2)

Substitution of (2.2.1) and (2.2.2) into the ordinary differential equation from step (i) yields an algebraic expression involving $cn(\xi\ |m)dn(\xi\ |m)$ and powers of $sn(\xi\ |m)$.

(iii). Determine M (if possible); usually this involves balancing the linear term(s) of highest order with the highest order nonlinear term(s). In order to accomplish this it is useful to note that:

(a) the highest power of $sn(\xi \mid m)$ in $U^n\ sn^{nM}(\xi)$.

(b) when n is even, $\dfrac{d^n U}{d\eta^n}$ is a polynomial in $sn(\xi \mid m)$ of degree $n + M$.

(c) when n is odd, $\dfrac{d^n U}{d\eta^n}$ is of the form $cn(\xi \mid m)\, dn(\xi \mid m)$ times a polynomial in $sn(\xi \mid m)$ of degree $n + M - 2$.

(iv). The equations from step (ii) may be written in the form:

$$P_1(sn(\xi)) + cn(\xi)dn(\xi)P_2(sn(\xi)) = 0,$$

where $P_1(sn(\xi))$ and $P_2(sn(\xi))$ are polynomials in $sn(\xi \mid m)$. With M, as determined in step (iii), equate the coefficients of each power of $sn(\xi \mid m)$ in $P_1(sn(\xi))$ and $P_2(sn(\xi))$ to zero. This yields a system of algebraic equations involving the $a_i (i = 0, 1, K, M), k, c, m$.

(v). To get the solutions of algebraic equations obtained in step (iv), one can attempt on similar lines as in the hyperbolic functions expansion method.

(vi). If there is a real nontrivial solution of these equations, then we have a solution of the form (2.2.1) for the differential equation.

CHAPTER – 3

DRINFEL'D- SOKOLOV- WILSON EQUATIONS

3.1 Introduction

Drinfel'd- sokolov- wilson (DSW) equations is a nonlinear partial differential equation with (1+1) dimension

$$\begin{aligned} u_t &= 2vv_x \\ v_t &= 2v_{xxx} + 2uu_x + \alpha\, u_x v = 0. \end{aligned}$$

The system with $\alpha = 1$ was proposed by Drinfel'd [4], Sokolov [5], and Wilson [31]. It can be obtained [11] as a reduction of the Kadomtsev- Petviashvili equation (i.e. a two- dimensional equation) and is completely integrable system. In [32] Yao and Lie computed conservation laws of the Drinfel'd- Sokolov- Wilson equations, where they had introduced four arbitrary coefficients. Using scales on x, t, u and v, all but one coefficients in the above equation can be scaled to any real number. Therefore, to cover the entire family of DSW equation, it suffices to leave one coefficient arbitrary i.e. α in front of $u_x v$.

3.2 Hyperbolic functions expansion method

$$\begin{aligned} u_t &= 2ww_x \\ w_t &= 2w_{xxx} + 2uu_x + u_x w = 0. \end{aligned} \qquad (3.2.1)$$

To find the solution of equation (3.2.1), we use the transformations:

$$\begin{aligned} u(x,t) &= u(\xi) \\ w(x,t) &= w(\xi) \end{aligned} \qquad \text{where } \xi = x - ct$$

$$\frac{\partial u}{\partial t} = u_t = \frac{\partial u}{\partial \xi}\,\frac{\partial \xi}{\partial t} = -c\frac{\partial u}{\partial \xi} = -cu'$$

$$\frac{\partial w}{\partial t} = w_t = \frac{\partial w}{\partial \xi}\,\frac{\partial \xi}{\partial t} = -c\frac{\partial w}{\partial \xi} = -cw'$$

$$\frac{\partial w}{\partial x} = w_x = \frac{\partial w}{\partial \xi}\,\frac{\partial \xi}{\partial x} = \frac{\partial w}{\partial \xi}\,.1 = w'$$

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial \xi}\,\frac{\partial \xi}{\partial x} = \frac{\partial u}{\partial \xi}\,.1 = u'$$

$$\frac{\partial^3 w}{\partial x^3} = w_{xxx} = \frac{\partial^3 w}{\partial \xi\;\partial \xi\;\partial \xi}.1 = w'''$$

Then equation (3.2.1) will become the ordinary differential equation of the form

$$-cu' = 3ww' \tag{3.2.2}$$

$$-cw' = 2w''' + 2uw' + u'w. \tag{3.2.3}$$

Now suppose that equation (3.2.2) and (3.2.3) have the solutions of the form

$$u(\xi) = \sum_{i=0}^{m} a_i r^i + \sum_{i=1}^{m} b_i s r^{i-1} \tag{3.2.4}$$

and

$$w(\xi) = \sum_{j=0}^{n} c_j r^j + \sum_{j=1}^{n} d_j s r^{i-1} \tag{3.2.5}$$

where order of u is *m* and order of w is *n*.

Now by balancing highest order derivative term w''' with nonlinear term uw', we get

$$n + 3 = n + m + 1$$
$$\Rightarrow \quad m = 2.$$

Now balancing u' with ww'

That is:

$$m + 1 = n + n + 1$$
$$\Rightarrow\ 2 + 1 = 2n + 1$$
$$\Rightarrow\ n = 1.$$

Therefore equations (3.2.4) and (3.2.5) becomes

$$u(\xi) = \sum_{i=0}^{2} a_i r^i + \sum_{i=1}^{2} b_i s r^{i-1}$$

$$= a_0 + a_1 r + a_2 r^2 + b_1 s + b_2 s\, r.$$

$$w(\xi) = \sum_{j=0}^{1} c_j r^j + \sum_{j=1}^{1} d_j s r^{i-1}$$

$$= c_0 + c_1 r + d_1 s,$$

where $r = \sec h(k\xi)$ and $s = \tanh(k\xi)$

Hence

$$u(\xi) = a_0 + a_1 \sec h(k\xi) + a_2 \sec h^2(k\xi) + b_1 \tanh(k\xi) + b_2 \tanh(k\xi) \sec h(k\xi). \qquad (3.2.6)$$
a

nd

$$w(\xi) = c_0 + c_1 \sec h(k\xi) + d_1 \tanh(k\xi). \qquad (3.2.7)$$

$$u' = -ka_1 \sec h(k\xi)\tanh(k\xi) - 2a_2 k \sec h^2(k\xi)\tanh(k\xi) + b_1 k \sec h^2(k\xi) + b_2 k$$
$$\sec h^3(k\xi) - b_2 k \sec h(k\xi)\tanh^2(k\xi).$$

$$u' = -ka_1 \sec h(k\xi)\tanh(k\xi) - 2a_2 k \sec h^2(k\xi)\tanh(k\xi) + b_1 k \sec h^2(k\xi) + b_2 k$$
$$\sec h^3(k\xi) - b_2 k \sec h(k\xi) + b_2 k \sec h^3(k\xi).$$

$$w' = -c_1 k \sec h(k\xi)\tanh(k\xi) + d_1 k \sec h^2(k\xi).$$

$$ww' = [c_0 + c_1 \sec h(k\xi) + d_1 \tanh(k\xi)]\,[-c_1 k \sec h(k\xi)\tanh(k\xi) + d_1 k \sec h^2(k\xi)].$$

$$\begin{aligned}&= -c_0c_1k\sec h(k\xi)\tanh(k\xi) + c_0d_1k\sec h^2(k\xi) - c_1{}^2k\sec h^2(k\xi)\tanh(k\xi) + c_1d_1k\\ &\quad \sec h^3(k\xi) - c_1d_1k\sec h(k\xi)\tanh^2(k\xi) + d_1{}^2k\sec h^2(k\xi)\tanh(k\xi).\end{aligned}$$

$$\begin{aligned}&= -c_0c_1k\sec h(k\xi)\tanh(k\xi) + c_0d_1k\sec h^2(k\xi) - c_1{}^2k\sec h^2(k\xi)\tanh(k\xi) + c_1d_1k\\ &\quad \sec h^3(k\xi) - c_1d_1k\sec h(k\xi) + c_1d_1k\sec h^3(k\xi) + d_1{}^2k\sec h^2(k\xi)\tanh(k\xi).\end{aligned}$$

$$\begin{aligned}&= -c_0c_1k\sec h(k\xi)\tanh(k\xi) + c_0d_1k\sec h^2(k\xi) - c_1{}^2k\sec h^2(k\xi)\tanh(k\xi) + 2c_1d_1k\\ &\quad \sec h^3(k\xi) - c_1d_1k\sec h(k\xi) + d_1{}^2k\sec h^2(k\xi)\tanh(k\xi).\end{aligned}$$

$$\begin{aligned}u\,w' = {}& [a_0 + a_1\sec h(k\xi) + a_2\sec h^2(k\xi)\tanh(k\xi) + b_1\tanh(k\xi) + b_2\tanh(k\xi)\sec h(k\xi)]\\ & [-c_1k\sec h(k\xi)\tanh(k\xi) + d_1k\sec h^2(k\xi)].\end{aligned}$$

$$\begin{aligned}&= -a_0c_1k\sec h(k\xi)\tanh(k\xi) + a_0d_1k\sec h^2(k\xi) - a_1c_1k\sec h^2(k\xi)\tanh(k\xi) + a_1d_1k\\ &\quad \sec h^3(k\xi) - a_2c_1k\sec h^3(k\xi)\tanh(k\xi) + a_2d_1k\sec h^4(k\xi) - b_1c_1k\sec h(k\xi) + b_1c_1\\ &\quad k\sec h^3(k\xi) + b_1d_1k\sec h^2(k\xi)\tanh(k\xi) - b_2c_1k\sec h^2(k\xi) + b_2c_1k\sec h^4(k\xi) +\\ &\quad b_2d_1k\sec h^3(k\xi)\tanh(k\xi).\end{aligned}$$

$$\begin{aligned}&= -a_0c_1k\sec h(k\xi)\tanh(k\xi) + (a_0d_1k - b_2c_1k)\sec h^2(k\xi) - (a_1c_1k + b_1d_1k)\sec h^2(k\xi)\\ &\quad \tanh(k\xi) + (a_1d_1k + b_1c_1k)\sec h^3(k\xi) + (-a_2c_1k + b_2d_1k)\sec h^3(k\xi)\tanh(k\xi)\\ &\quad + (a_2d_1k + b_2c_1k)\sec h^4(k\xi) + (-b_1c_1k)\sec h(k\xi).\end{aligned}$$

$$\begin{aligned}u'\,w = {}& [-ka_1\sec h(k\xi)\tanh(k\xi) - 2a_2k\sec h^2(k\xi)\tanh(k\xi) + b_1k\sec h^2(k\xi) + b_2k\sec h^3(k\xi)\\ & - b_2k\sec h(k\xi) + b_2k\sec h^3(k\xi)]\,[c_0 + c_1\sec h(k\xi) + d_1\tanh(k\xi)].\end{aligned}$$

$$
\begin{aligned}
= & - c_0 k a_1 \sec h(k\xi)\tanh(k\xi) - 2a_2 c_0 k \sec h^2(k\xi)\tanh(k\xi) + c_0 b_1 k \sec h^2(k\xi) + c_0 b_2 k \\
& \sec h^3(k\xi) - c_0 b_2 k \sec h(k\xi) + c_0 b_2 k \sec h^3(k\xi) - a_1 c_1 k \sec h^2(k\xi)\tanh(k\xi) - 2a_2 c_1 k \\
& \sec h^3(k\xi)\tanh(k\xi) + c_1 b_1 k \sec h^3(k\xi) + c_1 b_2 k \sec h^4(k\xi) - c_1 b_2 k \sec h^2(k\xi) + c_1 b_2 k \\
& \sec h^4(k\xi) - a_1 d_1 k \sec h(k\xi)\tanh^2(k\xi) - 2a_2 d_1 k \sec h^2(k\xi)\tanh^2(k\xi) + b_1 d_1 k \\
& \sec h^2(k\xi)\tanh(k\xi) + d_1 b_2 k \sec h^3(k\xi)\tanh(k\xi) - d_1 b_2 k \sec h(k\xi)\tanh(k\xi) + d_1 b_2 k \\
& \sec h^3(k\xi)\tanh(k\xi).
\end{aligned}
$$

$$
\begin{aligned}
= & - c_0 k a_1 \sec h(k\xi)\tanh(k\xi) - 2a_2 c_0 k \sec h^2(k\xi)\tanh(k\xi) + c_0 b_1 k \sec h^2(k\xi) + c_0 b_2 k \\
& \sec h^3(k\xi) - c_0 b_2 k \sec h(k\xi) + c_0 b_2 k \sec h^3(k\xi) - a_1 c_1 k \sec h^2(k\xi)\tanh(k\xi) - 2a_2 c_1 k \\
& \sec h^3(k\xi)\tanh(k\xi) + c_1 b_1 k \sec h^3(k\xi) + 2c_1 b_2 k \sec h^4(k\xi) - c_1 b_2 k \sec h^2(k\xi) - a_1 d_1 k \\
& \sec h(k\xi) + a_1 d_1 k \sec h^3(k\xi) - 2a_2 d_1 k \sec h^2(k\xi) + 2a_2 d_1 k \sec h^4(k\xi) + b_1 d_1 k \sec h^2(k\xi) \\
& \tanh(k\xi) + d_1 b_2 k \sec h^3(k\xi)\tanh(k\xi) - d_1 b_2 k \sec h(k\xi)\tanh(k\xi) + d_1 b_2 k \sec h^3(k\xi) \\
& \tanh(k\xi).
\end{aligned}
$$

$$
w'' = - c_1 k^2 \sec h(k\xi)\tanh^2(k\xi) - c_1 k^2 \sec h^3(k\xi) - 2d_1 k^2 \sec h^2(k\xi)\tanh(k\xi).
$$

$$
= c_1 k^2 \sec h(k\xi) - c_1 k^2 \sec h^3(k\xi) - c_1 k^2 \sec h^3(k\xi) - 2d_1 k^2 \sec h^2(k\xi)\tanh(k\xi).
$$

$$
= c_1 k^2 \sec h(k\xi) - 2c_1 k^2 \sec h^3(k\xi) - 2d_1 k^2 \sec h^2(k\xi)\tanh(k\xi).
$$

$$
\begin{aligned}
w''' = & - c_1 k^3 \sec h(k\xi)\tanh(k\xi) + 6c_1 k^3 \sec h^3(k\xi)\tanh(k\xi) + 4d_1 k^3 \sec h^2(k\xi)\tanh^2(k\xi) \\
& - 2d_1 k^3 \sec h^4(k\xi).
\end{aligned}
$$

$$
\begin{aligned}
= & - c_1 k^3 \sec h(k\xi)\tanh(k\xi) + 6c_1 k^3 \sec h^3(k\xi)\tanh(k\xi) + 4d_1 k^3 \sec h^2(k\xi) - 4d_1 k^3 \\
& \sec h^4(k\xi) - 2d_1 k^3 \sec h^4(k\xi).
\end{aligned}
$$

$$
\begin{aligned}
w''' = & - c_1 k^3 \sec h(k\xi)\tanh(k\xi) + 6c_1 k^3 \sec h^3(k\xi)\tanh(k\xi) + 4d_1 k^3 \sec h^2(k\xi) - 6d_1 k^3 \\
& \sec h^4(k\xi).
\end{aligned}
$$

Now substituting the values of u' and ww' in equation (3.2.2).

$-cu' = 3ww'.$

$-cu' - 3ww' = 0.$

$$-c[-a_1k\operatorname{sec}h(k\xi)\tanh(k\xi) - 2a_2k\operatorname{sec}h^2(k\xi)\tanh(k\xi) + b_1k\operatorname{sec}h^2(k\xi) + 2b_2k \operatorname{sec}h^3(k\xi) - b_2k\operatorname{sec}h(k\xi)] - 3[-c_0c_1k\operatorname{sec}h(k\xi)\tanh(k\xi) + c_0d_1k\operatorname{sec}h^2(k\xi) - c_1^{\,2}k\operatorname{sec}h^2(k\xi)\tanh(k\xi) + 2c_1d_1k\operatorname{sec}h^3(k\xi) - c_1d_1k\operatorname{sec}h(k\xi) + d_1^{\,2}k\tanh(k\xi) \operatorname{sec}h^2(k\xi)] = 0.$$

Now by comparing the coefficients, we will get the one set of algebraic equations:

Coefficient of $\operatorname{sec}h(k\xi)\ \tanh(k\xi)$

$a_1c + 3c_0c_1 = 0.$

Coefficient of $\operatorname{sec}h^2(k\xi)\ \tanh(k\xi)$

$2ca_2 + 3c_1^{\,2} = 0.$

Coefficient of $\operatorname{sec}h^2(k\xi)$

$cb_1 + 3c_0d_1 = 0.$

Coefficient of $\operatorname{sec}h^3(k\xi)$

$cb_2 + 3c_1d_1 = 0.$

Coefficient of $\operatorname{sec}h(k\xi)$

$cb_2 + 3c_1d_1 = 0.$

Now substituting the values of $w', w''', uw', u'w$ in equation (3.2.3)

Thus we have

$$-cw' - 2w''' - 2uw' - u'w = 0$$
$$cw' + 2w''' + 2uw' + u'w = 0.$$

$$c[-c_1 k \sec h(k\xi)\ \tanh(k\xi) + d_1 k \sec h^2(k\xi)] + 2[-c_1 k^3 \sec h(k\xi)\ \tanh(k\xi)$$
$$+6c_1 k^3 \sec h^3(k\xi)\tanh(k\xi) + 4d_1 k^3 \sec h^2(k\xi) - 6d_1 k^3 \sec h^4(k\xi)] + 2[-a_0 c_1$$
$$k \sec h(k\xi)\tanh(k\xi) + (a_0 d_1 k - b_2 c_1 k)\sec h^2(k\xi) + (-a_1 c_1 k + b_1 d_1 k)\sec h^2(k\xi)$$
$$\tanh(k\xi) + (a_1 d_1 k + b_1 c_1 k)\sec h^3(k\xi) + (-a_2 c_1 k + b_2 d_1 k)\sec h^3(k\xi)\tanh(k\xi) +$$
$$(a_2 d_1 k + b_2 c_1 k)\sec h^4(k\xi) + (-b_1 c_1 k)\sec h(k\xi)] + [-a_1 c_0 k \sec h(k\xi)\tanh(k\xi)$$
$$-2a_2 c_0 k\ \sec h^2(k\xi)\tanh(k\xi) + b_1 c_0 k \sec h^2(k\xi) + b_2 c_0 k \sec h^3(k\xi) - b_2 c_0 k$$
$$\sec h^3(k\xi) - a_1 c_1 k \sec h^2(k\xi)\ \tanh(k\xi) - 2a_2 c_1 k \sec h^3(k\xi)\tanh(k\xi) + b_1 c_1 k$$
$$\sec h^3(k\xi) + 2b_2 k c_1 \sec h^4(k\xi) - b_2 k c_1 \sec h^2(k\xi) - a_1 d_1 k \sec h(k\xi) + a_1 d_1 k$$
$$\sec h^3(k\xi) - 2a_2 d_1 k \sec h^2(k\xi) + 2a_2 d_1 k \sec h^4(k\xi) + b_1 d_1 k \sec h^2(k\xi)\tanh(k\xi)$$
$$+b_2 k d_1 \sec h^3(k\xi)\tanh(k\xi) - b_2 k d_1 \sec h(k\xi)\tanh(k\xi) + b_2 k d_1 \sec h^3(k\xi)\tanh(k\xi)] = 0.$$

Now by comparing the coefficients, we will get the another set of algebraic equations:

Coefficient of $\sec h(k\xi)\tanh(k\xi)$

$$-cc_1 - 2c_1 k^2 - 2a_0 c_1 - a_1 c_0 - b_2 d_1 = 0.$$

Coefficient of $\sec h^2(k\xi)$

$$cd_1 + 8d_1 k^2 + 2a_0 d_1 + b_1 c_0 - 3b_2 c_1 = 0.$$

Coefficient of $\sec h^3(k\xi)\tanh(k\xi)$

$$3c_1 k^2 - a_2 c_1 + b_2 d_1 = 0.$$

Coefficient of $\sec h^4(k\xi)$

$$-3d_1 k^2 + b_2 c_1 + a_2 d_1 = 0$$

Coefficient of $\sec h^2(k\xi)\tanh(k\xi)$

$$-3a_1 c_1 + 3b_1 d_1 - 2a_2 c_0 = 0.$$

Coefficient of $\sec h^3(k\xi)$

$$3a_1d_1 + 3b_1c_1 + 2b_2c_0k = 0.$$

Coefficient of $\sec h(k\xi)$

$$-2b_1c_1 - b_2c_0 - a_1d_1 = 0.$$

Thus all the equations are:

$$a_1c + 3c_0c_1 = 0 \qquad (3.2.8)$$
$$2ca_2 + 3c_1^2 - d_1^2 = 0 \qquad (3.2.9)$$
$$cb_1 + 3c_0d_1 = 0 \qquad (3.2.10)$$
$$cb_2 + 3c_1d_1 = 0 \qquad (3.2.11)$$
$$-cc_1 - 2c_1k^2 - 2a_0c_1 - a_1c_0 - b_2d_1 = 0 \qquad (3.2.12)$$
$$cd_1 + 8d_1k^2 + 2a_0d_1 + b_1c_0 - 3b_2c_1 = 0 \qquad (3.2.13)$$
$$3c_1k^2 - a_2c_1 + b_2d_1 = 0 \qquad (3.2.14)$$
$$-3d_1k^2 + b_2c_1 + a_2d_1 = 0 \qquad (3.2.15)$$
$$-3a_1c_1 + 3b_1d_1 - 2a_2c_0 = 0 \qquad (3.2.16)$$
$$3a_1d_1 + 3b_1c_1 + 2b_2c_0 = 0 \qquad (3.2.17)$$
$$-2b_1c_1 - b_2c_0 - a_1d_1 = 0 \qquad (3.2.18)$$

Multiplying equation (3.2.18) by 3 and then adding equation (3.2.17) and equation (3.2.18), we get:

$$3b_1c_1 + 2b_2c_0 - 6b_1c_1 - 3b_2c_0 = 0$$
$$-3b_1c_1 - b_2c_0 = 0$$
$$3b_1c_1 + b_2c_0 = 0$$

Thus all the equations are:

$$a_1c + 3c_0c_1 = 0 \qquad (3.2.8)$$
$$2ca_2 + 3c_1^2 - d_1^2 = 0 \qquad (3.2.9)$$

$$cb_1 + 3c_0d_1 = 0 \tag{3.2.10}$$
$$cb_2 + 3c_1d_1 = 0 \tag{3.2.11}$$
$$-cc_1 - 2c_1k^2 - 2a_0c_1 - a_1c_0 - b_2d_1 = 0 \tag{3.2.12}$$
$$cd_1 + 8d_1k^2 + 2a_0d_1 + b_1c_0 - 3b_2c_1 = 0 \tag{3.2.13}$$
$$3c_1k^2 - a_2c_1 + b_2d_1 = 0 \tag{3.2.14}$$
$$-3d_1k^2 + b_2c_1 + a_2d_1 = 0 \tag{3.2.15}$$
$$-3a_1c_1 + 3b_1d_1 - 2a_2c_0 = 0 \tag{3.2.16}$$
$$-2b_1c_1 - b_2c_0 - a_1d_1 = 0 \tag{3.2.18}$$
$$3b_1c_1 + b_2c_0 = 0 \tag{3.2.19}$$

Now multiplying equation (3.2.8) by b_1 and equation (3.2.10) by c_1

$$3b_1c_0c_1 - 3c_0c_1d_1 = 0$$
$$b_1c_0c_1 - c_0c_1d_1 = 0$$
$$\Rightarrow\ c_0c_1(b_1 - d_1) = 0$$

$$\Rightarrow \text{ either } \ c_0c_1 = 0 \quad or \quad (b_1 - d_1) = 0$$

$$\text{let } \ (b_1 - d_1) = 0$$
$$\Rightarrow\ b_1 = d_1$$

Now substituting $b_1 = d_1$ in all the above equations.

$$a_1c + 3c_0c_1 = 0 \tag{3.2.8}$$
$$2ca_2 + 3c_1^2 - d_1^2 = 0 \tag{3.2.9}$$
$$cb_2 + 3c_1d_1 = 0 \tag{3.2.11}$$
$$-cc_1 - 2c_1k^2 - 2a_0c_1 - a_1c_0 - b_2d_1 = 0 \tag{3.2.12}$$
$$cd_1 + 8d_1k^2 + 2a_0d_1 + d_1c_0 - 3b_2c_1 = 0 \tag{3.2.13}$$
$$3c_1k^2 - a_2c_1 + b_2d_1 = 0 \tag{3.2.14}$$
$$-3d_1k^2 + b_2c_1 + a_2d_1 = 0 \tag{3.2.15}$$
$$-3a_1c_1 + 3d_1^2 - 2a_2c_0 = 0 \tag{3.2.16}$$
$$-2d_1c_1 - b_2c_0 - a_1d_1 = 0 \tag{3.2.18}$$
$$3d_1c_1 + b_2c_0 = 0 \tag{3.2.19}$$

From equation (3.2.11)

$$cb_2 + 3c_1d_1 = 0$$
$$\Rightarrow 3c_1d_1 = -cb_2$$

Substituting this value in $3d_1c_1 + b_2c_0 = 0$, we get:

$$-cb_2 + b_2c_0 = 0$$
$$\Rightarrow b_2(-c + c_0) = 0$$

$$\Rightarrow \text{either} \quad b_2 = 0 \quad \text{or} \quad c_0 = c$$

Let $b_2 = 0$

$$a_1c + 3c_0c_1 = 0 \qquad (3.1.8)$$
$$2ca_2 + 3c_1^2 - d_1^2 = 0 \qquad (3.1.9)$$
$$3c_1d_1 = 0 \qquad (3.1.11)$$
$$-cc_1 - 2c_1k^2 - 2a_0c_1 - a_1c_0 = 0 \qquad (3.1.12)$$
$$cd_1 + 8d_1k^2 + 2a_0d_1 + d_1c_0 = 0 \qquad (3.1.13)$$
$$3c_1k^2 - a_2c_1 = 0 \qquad (3.1.14)$$
$$-3d_1k^2 + a_2d_1 = 0 \qquad (3.1.15)$$
$$-3a_1c_1 + 3d_1^2 - 2a_2c_0 = 0 \qquad (3.1.16)$$
$$-2d_1c_1 - a_1d_1 = 0 \qquad (3.1.18)$$

From equation (3.2.11) i.e.

$$3c_1d_1 = 0$$
$$\Rightarrow \text{either} \quad c_1 = 0 \quad \text{or} \quad d_1 = 0$$

Now taking $d_1 = 0$

$$a_1c + 3c_0c_1 = 0 \qquad (3.2.8)$$
$$2ca_2 + 3c_1^2 - d_1^2 = 0 \qquad (3.2.9)$$
$$-cc_1 - 2c_1k^2 - 2a_0c_1 - a_1c_0 = 0 \qquad (3.2.12)$$
$$3c_1k^2 - a_2c_1 = 0 \qquad (3.2.14)$$
$$-3a_1c_1 - 2a_2c_0 = 0 \qquad (3.2.16)$$

Multiplying equation (3.2.8) by $3c_1$ and equation (3.2.16) by c we get:

$$9c_0c_1^2 - 2a_2c_0 = 0$$
$$\Rightarrow c_0(9c_1^2 - 2a_2) = 0$$
$$\Rightarrow \text{either } c_0 = 0 \quad \text{or} \quad 9c_1^2 - 2a_2 = 0$$

let $c_0 = 0$

$$a_1c = 0 \tag{3.2.8}$$
$$2ca_2 + 3c_1^2 = 0 \tag{3.2.9}$$
$$-cc_1 - 2c_1k^2 - 2a_0c_1 = 0 \tag{3.2.12}$$
$$3c_1k^2 - a_2c_1 = 0 \tag{3.2.14}$$
$$-3a_1c_1 = 0 \tag{3.2.16}$$

From equation (3.2.8) we get:

$a_1 = 0$

$\therefore$ the equations becomes

$$2ca_2 + 3c_1^2 = 0 \tag{3.2.9}$$
$$-cc_1 - 2c_1k^2 - 2a_0c_1 = 0 \tag{3.2.12}$$
$$3c_1k^2 - a_2c_1 = 0 \tag{3.2.14}$$

Now from equation (3.2.12) i.e. $-cc_1 - 2c_1k^2 - 2a_0c_1 = 0$

$$-cc_1 - 2c_1k^2 - 2a_0c_1 = 0$$
$$\Rightarrow c_1(-c - 2k^2 - 2a_0) = 0$$

$$\Rightarrow \text{either } c_1 = 0 \quad \text{or} \quad -c - 2k^2 - 2a_0 = 0$$

let $-c - 2k^2 - 2a_0 = 0$

$$-c - 2k^2 - 2a_0 = 0$$
$$\Rightarrow 2a_0 = -c - 2k^2$$
$$\Rightarrow a_0 = \frac{-c - 2k^2}{2}$$

$$2ca_2 + 3c_1^{\ 2} = 0 \tag{3.2.9}$$
$$3c_1k^2 - a_2c_1 = 0 \tag{3.2.14}$$

From equation (3.2.14) i.e. $3c_1k^2 - a_2c_1 = 0$

$$3c_1k^2 - a_2c_1 = 0$$
$$\Rightarrow c_1(3k^2 - a_2) = 0$$
$$\Rightarrow \text{either } c_1 = 0 \quad \text{or} \quad 3k^2 - a_2 = 0$$

let $3k^2 - a_2 = 0 \quad \Rightarrow \quad a_2 = 3k^2$

By $a_2 = 3k^2$ we get:

Thus all the values of the first solution set are:

$b_2 = 0$
$d_1 = 0$
$b_1 = 0$
$c_0 = 0$
$a_1 = 0$
$a_0 = \dfrac{-c - 2k^2}{2}$

$a_2 = 3k^2$

with $6ck^2 + 3c_1^{\ 2} = 0$, c_1 is arbitrary constant.

Thus we have

$$u(x,t) = \frac{-c-2k^2}{2} + 3k^2 \sec h^2[k(x-ct)]$$

$$w(x,t) = 0 \tag{3.2.20}$$

with $6ck^2 + 3c_1^2 = 0$, k, c, c_1 are arbitrary constants.

The values of second solution set are:

$a_0 = 2 + k_1 - \frac{1}{2}c$
$b_1 = -3k_1$
$b_2 = 0$
$d_1 = \left(\sqrt{2c}\right)\left(k_1\right)$
$c_0 = 0$
$a_1 = 0$
$c_1 = 0$

Thus we have

$$u(x,t) = 2 + k_1 - \frac{1}{2}c - 3k_1 \tanh[k(x-ct)]$$

$$w(x,t) = \left(\sqrt{2c}\right)\left(k_1\right) \tanh[k(x-ct)],$$

where k_1, c, k are arbitrary constants.

The values of third solution set are:

$b_2 = 0$
$d_1 = \left(-\sqrt{2c}\right)\left(k_1\right)$
$c_0 = 0$
$a_1 = 0$
$c_1 = 0$
$a_0 = 2 + k_1 - \frac{1}{2}c$
$b_1 = -3k_1$

Hence

$$u(x,t) = 2 + k_1 - \frac{1}{2}c - 3k_1 \tanh[k(x - ct)]$$

$$w(x,t) = \left(-\sqrt{2c}\right)\left(k_1\right) \tanh[k(x - ct)],$$

where $k_1,\ c,\ k$ are arbitrary constants.

3.3 Jacobi elliptic functions expansion method

$$\begin{aligned} u_t &= 2ww_x \\ w_t &= 2w_{xxx} + 2uu_x + u_x w = 0. \end{aligned} \tag{3.3.1}$$

Now we use the transformations:

$u(x,t) = U(\eta)$

$w(x,t) = W(\eta)$ where $\eta = x - ct - x_0$

$$\frac{\partial u}{\partial t} = u_t = \frac{\partial U}{\partial \eta}\ \frac{\partial \eta}{\partial t} = -c\frac{\partial U}{\partial \eta} = -cU'$$

$$\frac{\partial w}{\partial t} = w_t = \frac{\partial W}{\partial \eta}\ \frac{\partial \eta}{\partial t} = -c\frac{\partial W}{\partial \eta} = -cW'$$

$$\frac{\partial w}{\partial x} = w_x = \frac{\partial W}{\partial \eta}\ \frac{\partial \eta}{\partial x} = \frac{\partial W}{\partial \eta}\ .1 = W'$$

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial U}{\partial \eta}\ \frac{\partial \eta}{\partial x} = \frac{\partial U}{\partial \eta}\ .1 = U'$$

$$\frac{\partial^3 w}{\partial x^3} = w_{xxx} = \frac{\partial^3 W}{\partial \eta\ \partial \eta\ \partial \eta}.1 = W'''$$

Then equation (3.3.1) will become the ordinary differential equation of the form:

$$-cU' = 3WW' \tag{3.3.2}$$

$$-cW' = 2W''' + 2UW' + U'W. \tag{3.3.3}$$

Now suppose that equation (3.3.2) and (3.3.3) has the solution of the form

$$U(\eta) = \sum_{i}^{M} a_i sn^i(k\eta \,/\, m) \tag{3.3.4}$$

$$W(\eta) = \sum_{j}^{N} b_j sn^i(k\eta \,/\, m), \tag{3.3.5}$$

where *M* is the order of *U* and *N* is the order of *W*.

Now by balancing W''' with UW'.

That is:

$$N + 3 = N + M + 1$$
$$\Rightarrow \quad M = 2$$

Now balancing U' with WW'.

That is:

$$M + 1 = N + N + 1$$
$$\Rightarrow 2 + 1 = 2N + 1$$
$$\Rightarrow N = 1$$

Therefore equation (3.3.4) and (3.3.5) becomes:

$$U(\eta) = \sum_{i=0}^{2} a_i sn^i(k\eta \mid m)$$

$$= a_0 + a_1 sn(k\eta \mid m) + a_2 sn^2(k\eta \mid m).$$

$$W(\eta) = \sum_{j=0}^{1} b_i sn^i(k\eta \mid m)$$

$$= b_0 + b_1\, sn(k\eta \mid m).$$

$$W(\eta) = b_0 + b_1\, sn(k\eta \mid m).$$

$$W'(\eta) = \frac{b_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m).$$

$$WW' = \frac{b_0 b_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{b_1^2 k}{m} sn(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m).$$

$$W'' = -\frac{b_1 k^2}{m^2} sn(k\eta \mid m)\, dn^2(k\eta \mid m) + \frac{b_1 k^2}{m^2}\left[-m^2 sn(k\eta \mid m)\, cn^2(k\eta \mid m)\right].$$

$$= -\frac{b_1 k^2}{m^2} sn(k\eta \mid m)\left(1 - m^2 sn^2(k\eta \mid m)\right) - b_1 k^2\, sn(k\eta \mid m)\left(1 - sn^2(k\eta \mid m)\right).$$

$$W'' = -\frac{b_1 k^2}{m^2} sn(k\eta \mid m) + b_1 k^2\, sn^3(k\eta \mid m) - b_1 k^2\, sn(k\eta \mid m) + b_1 k^2\, sn^3(k\eta \mid m).$$

$$W'' = -\frac{b_1 k^2}{m^2} sn(k\eta \mid m) + 2 b_1 k^2\, sn^3(k\eta \mid m) - b_1 k^2\, sn(k\eta \mid m).$$

$$W''' = -\frac{b_1 k^3}{m^3} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{6 b_1 k^3}{m} sn^2(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m) - \frac{b_1 k^3}{m}$$
$$cn(k\eta \mid m)\, dn(k\eta \mid m).$$

$$U(\eta) = a_0 + a_1 sn(k\eta \mid m) + a_2 sn^2(k\eta \mid m).$$

$$U'(\eta) = \frac{a_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{2 a_2 k}{m} sn(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m).$$

$$2UW' = 2\left[a_0 + a_1 sn(k\eta \mid m) + a_2 sn^2(k\eta \mid m)\right]\left[\frac{b_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m)\right].$$

$$= \frac{2b_1 k a_0}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{2a_1 b_1 k}{m} sn(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{2a_2 b_1 k}{m}$$
$$sn^2(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m).$$

$$U'W = \left[\frac{a_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{2a_2 k}{m} sn(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m)\right]$$
$$\left[b_0 + b_1 sn(k\eta \mid m)\right].$$

$$= \frac{a_1 b_0 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{a_1 b_1 k}{m}\frac{a_1 b_0 k}{m} sn(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{2a_2 b_0 k}{m}$$
$$sn(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{2a_2 b_1 k}{m} sn^2(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m).$$

Now substituting the values of U' and WW' in equation (3.3.2) we get:

$$-c\left[\frac{a_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{2a_2 k}{m} sn(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m)\right]$$
$$-3\left[\frac{b_1 b_0 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{b_1^{\,2} k}{m} sn(k\eta \mid m) cn(k\eta \mid m)\, dn(k\eta \mid m)\right] = 0.$$

Now by comparing the coefficients, we will get a set of algebraic equations:

Coefficient of $cn(k\eta \mid m)\, dn(k\eta \mid m)$

$$-\frac{ca_1 k}{m} - \frac{3b_0 b_1 k}{m} = 0$$
$$-ca_1 - 3b_0 b_1 = 0.$$

Coefficient of $sn(k\eta \mid m) cn(k\eta \mid m)\, dn(k\eta \mid m)$

$$-\frac{2ca_2k}{m}-\frac{3b_1{}^2k}{m}=0$$

$$-2ca_2-3b_1^2=0.$$

Now substituting the values of $W', W''', UW', U'W$ in equation (3.2.3)

$$-c\left[\frac{b_1k}{m}cn(k\eta\mid m)\,dn(k\eta\mid m)\right]-2\left[\begin{array}{l}-\frac{b_1k^3}{m^3}cn(k\eta\mid m)\,dn(k\eta\mid m)+\frac{6b_1k^3}{m}sn^2(k\eta\mid m)\\ cn(k\eta\mid m)\,dn(k\eta\mid m)-\frac{b_1k^3}{m}cn(k\eta\mid m)\,dn(k\eta\mid m)\end{array}\right]-$$

$$\left[\begin{array}{l}\frac{2b_1ka_0}{m}cn(k\eta\mid m)\,dn(k\eta\mid m)+\frac{2\,a_1b_1k}{m}sn(k\eta\mid m)\,cn(k\eta\mid m)\,dn(k\eta\mid m)+\frac{2\,a_2b_1k}{m}\\ \qquad sn^2(k\eta\mid m)\,cn(k\eta\mid m)\,dn(k\eta\mid m)\end{array}\right]-$$

$$\left[\begin{array}{l}\frac{a_1b_0k}{m}cn(k\eta\mid m)\,dn(k\eta\mid m)+\frac{a_1b_1k}{m}\frac{a_1b_0k}{m}sn(k\eta\mid m)\,cn(k\eta\mid m)\,dn(k\eta\mid m)+\frac{2a_2b_0k}{m}\\ sn(k\eta\mid m)\,cn(k\eta\mid m)\,dn(k\eta\mid m)+\frac{2a_2b_1k}{m}sn^2(k\eta\mid m)\,cn(k\eta\mid m)\,dn(k\eta\mid m)\end{array}\right]=0.$$

Now by comparing the coefficients, we will get the another set of algebraic equations:

Coefficient of $cn(k\eta\mid m)\,dn(k\eta\mid m)$

$$-\frac{cb_1k}{m}+\frac{2b_1k^3}{m^3}+\frac{2b_1k^3}{m}-\frac{2a_0b_1k}{m}-\frac{a_1b_0k}{m}=0.$$

Coefficient of $sn^2(k\eta\mid m)cn(k\eta\mid m)\,dn(k\eta\mid m)$

$$-\frac{12b_1k^3}{m}-\frac{2a_2b_1k}{m}-\frac{2a_2b_1k}{m}=0.$$

$$-6b_1k^2-2a_2b_1=0$$

$$-3b_1k^2-a_2b_1=0.$$

Coefficient of $sn(k\eta \mid m)cn(k\eta \mid m)\,dn(k\eta \mid m)$

$$-\frac{2a_1b_1k}{m}-\frac{a_1b_1k}{m}-\frac{2a_2b_0k}{m}=0$$

$$-3a_1b_1-2a_2b_0=0.$$

Thus all the equations are:

$$-ca_1-3b_0b_1=0 \tag{3.3.5}$$
$$-2ca_2-3b_1^2=0 \tag{3.3.6}$$
$$-\frac{cb_1k}{m}+\frac{2b_1k^3}{m^3}+\frac{2b_1k^3}{m}-\frac{2a_0b_1k}{m}-\frac{a_1b_0k}{m}=0 \tag{3.3.7}$$
$$-3b_1k^2-a_2b_1=0 \tag{3.3.8}$$
$$-3a_1b_1-2a_2b_0=0 \tag{3.3.9}$$

Now from equation (3.3.8) we have:

$$-3b_1k^2-a_2b_1=0$$
$$\Rightarrow b_1(-3k^2-a_2)=0$$
$$\Rightarrow \text{ either } b_1=0 \text{ or } a_2=-3k^2$$

let $a_2=-3k^2$

Now substituting $a_2=-3k^2$ in all the above equations, we have

$$-ca_1-3b_0b_1=0 \tag{3.3.5}$$
$$-2c(-3k^2)-3b_1^2=0 \tag{3.3.6}$$
$$-\frac{cb_1k}{m}+\frac{2b_1k^3}{m^3}+\frac{2b_1k^3}{m}-\frac{2a_0b_1k}{m}-\frac{a_1b_0k}{m}=0 \tag{3.3.7}$$
$$-3a_1b_1-2b_0(-3k^2)=0 \tag{3.3.8}$$

Now from equation (3.3.6) we have

$$-2c(-3k^2) - 3b_1^2 = 0$$
$$\Rightarrow b_1^2 = 2ck^2$$
$$\Rightarrow b_1 = \pm k\sqrt{2c}$$

Now substituting the value of $b_1 = \pm k\sqrt{2c}$ in equations (3.3.5), (3.3.7) and (3.3.8)

$$-ca_1 - 3b_0\left(k\sqrt{2c}\right) = 0 \tag{3.3.5}$$
$$-\frac{ck\sqrt{2c}}{m} + \frac{2k\sqrt{2c}k^2}{m^3} + \frac{2k\sqrt{2c}k}{m} - \frac{2a_0k\sqrt{2c}}{m} - \frac{a_1b_0}{m} = 0 \tag{3.3.7}$$
$$-3a_1k\sqrt{2c} - 2b_0(-3k^2) = 0 \tag{3.3.8}$$

Now multiplying equation (3.3.5) by $2k$ and equation (3.3.8) by $\sqrt{2c}$ and adding these we get:

$$-2ca_1k - 6ca_1k = 0$$
$$\Rightarrow a_1 = 0$$

Now substituting $a_1 = 0$ in equation (3.3.7)

$$-\frac{ck\sqrt{2c}}{m} + \frac{2k\sqrt{2c}k^2}{m^3} + \frac{2k\sqrt{2c}k}{m} - \frac{2a_0k\sqrt{2c}}{m} = 0$$

$$\Rightarrow 2a_0km^2\sqrt{2c} = ckm^2\sqrt{2c} + 2\sqrt{2c}k^3 + 2k^3m^2\sqrt{2c}$$

$$\Rightarrow 2a_0m^2\sqrt{2c} = cm^2\sqrt{2c} + 2\sqrt{2c}k^2 + 2k^2m^2\sqrt{2c}$$

$$\Rightarrow a_0 = \frac{cm^2 + 4k^4m^2}{2}$$

Thus all the values are:

$$a_1 = 0$$
$$b_1 = \pm k\sqrt{2c}$$
$$a_2 = -3k^2$$

$$b_0 = 0$$

$$a_0 = \frac{cm^2 + 4k^4m^2}{2}$$

Thus

$$u(x,t) = \frac{cm^2 + 2k^2 2k^2 m^2}{2} - 3k^2 sn^2(k(x - ct - x_0) \mid m)$$

$$w(x,t) = \pm\, k\sqrt{2c}\, sn(k(x - ct - x_0) \mid m),$$

where *k*, *c* are arbitrary constants.

CHAPTER – 4

GARDNER'S EQUATION

4.1 Introduction

In some physical situations, it becomes necessary to complement the KdV equation i.e.

$$A_t + cA_x + \mu AA_x + \lambda A_{xxx} = 0. \tag{4.1.1}$$

wiith a higher order cubic nonlinear term of the form $\nu A^2 A_x$. After transformation and rescaling, the amended equation (4.1.1) can be transformed to the so- called Gardner equation i.e.

$$u_t = 6(u + \delta^2 u^2) u_x + u_{xxx}.$$

The Gardner's equation is the nonlinear partial differential equation and is of (1+1) dimension. Here in above equation (4.1.1) c is the relevant linear long wave speed for the mode whose Amplitude is $A(x,t)$, while λ and μ , are the coefficients of the quadratic nonlinear and linear dispersive terms respectively.

In the Gardner's equation, δ can either be positive or negative, and the structure of the solutions depend crucially on which sign is appropriate. Thus the Gardner's equation differs from the kdV- equation by the presence of an additional term of cubic nonlinearity may be used as the generalized model.

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4.2 Hyperbolic functions expansion method

$$u_t = 6(u + \varepsilon^2 u^2) u_x + u_{xxx}, \tag{4.2.1}$$

where ε is arbitrary parameter.

To have the solution of (4.2.1) we use the transformation $u(x,t) = u(\xi)$, where $\xi = x - ct$, with c as a constant.

$$\frac{\partial u}{\partial t} = u_t = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} = -c\frac{\partial u}{\partial \xi} = -cu'$$

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial u}{\partial \xi}.1 = u'$$

$$\frac{\partial^3 u}{\partial x^3} = u_{xxx} = \frac{\partial^3 u}{\partial \xi\ \partial \xi\ \partial \xi}.1 = u'''$$

Then the equation (4.2.1) will become the ordinary differential equation:

$$-cu' - 6uu' - 6\varepsilon^2 u^2 u' - u''' = 0 \tag{4.2.2}$$

Now suppose that the equation (4.2.2) has the solution of the form

$$u(\xi) = \sum_{i=0}^{m} a_i r^i + \sum_{i=1}^{m} b_i s r^{i-1}. \tag{4.2.3}$$

and the order of *u* in (4.2.3) is *m*.

Now we can show that m = 1 if balancing u''' with u^2u'.

That is degree of u''' is *m+3* and degree of u^2u' is *2m + m + 1*.

By equating we get:

$$m + 3 = 2m + m + 1$$
$$\Rightarrow m = 1$$

Thus equation (4.2.3) becomes:

$$u(\xi) = \sum_{i=0}^{1} a_i r^i + \sum_{i=1}^{1} b_i s r^{i-1}$$

$$=a_0 r^0 + a_1 r + b_1 s.$$

where $r = \sec h(k\xi)$ and $s = \tanh(k\xi)$

Therefore

$$u(\xi) = a_0 + a_1 \sec h(k\xi) + b_1 \tanh(k\xi). \tag{4.2.4}$$

$$u' = -a_1 k \sec h(k\xi) \tanh(k\xi) + b_1 k \sec h^2(k\xi).$$

$$u u' = [a_0 + a_1 \sec h(k\xi) + b_1 \tanh(k\xi)]\ [-k a_1 \sec h(k\xi) \tanh(k\xi) + b_1 k \sec h^2(k\xi)].$$

$$= -k a_0 a_1 \sec h(k\xi) \tanh(k\xi) + a_0 b_1 k \sec h^2(k\xi) - k a_1^2 \sec h^2(k\xi) \tanh(k\xi) + a_1 b_1 k$$
$$\sec h^3(k\xi) - a_1 b_1 k \sec h(k\xi) + a_1 b_1 k \sec h^3(k\xi) + b_1^2 k \sec h^2(k\xi) \tanh(k\xi).$$

$$u u' = -k a_0 a_1 \sec h(k\xi) \tanh(k\xi) + a_0 b_1 k \sec h^2(k\xi) - k a_1^2 \sec h^2(k\xi) \tanh(k\xi) + 2 a_1 b_1 k$$
$$\sec h^3(k\xi) - a_1 b_1 k \sec h(k\xi) + b_1^2 k \sec h^2(k\xi) \tanh(k\xi).$$

$$u^2 = [a_0 + a_1 \sec h(k\xi) + b_1 \tanh(k\xi)]\,[a_0 + a_1 \sec h(k\xi) + b_1 \tanh(k\xi)].$$

$$u^2 = a_0^2 + a_0 a_1 \sec h(k\xi) + a_0 b_1 \tanh(k\xi) + a_0 a_1 \sec h(k\xi) + a_1^2 \sec h^2(k\xi) + a_1 b_1 \tanh(k\xi)$$
$$\sec h(k\xi) + a_0 b_1 \tanh(k\xi) + a_1 b_1 \sec h(k\xi) \tanh(k\xi) + b_1^2 - b_1^2 \sec h^2(k\xi).$$

$$u^2 = a_0^2 + 2a_0 a_1 \sec h(k\xi) + 2a_0 b_1 \tanh(k\xi) + (a_1^2 - b_1^2) \sec h(k\xi)^2 + 2 a_1 b_1 \tanh(k\xi)$$
$$\sec h(k\xi) + b_1^2.$$

$$u^2 u' = [a_0^2 + b_1^2 + 2a_0 a_1 \sec h(k\xi) + 2a_0 b_1 \tanh(k\xi) + (a_1^2 - b_1^2) \sec h^2(k\xi) + 2 a_1 b_1$$
$$\tanh(k\xi) \sec h(k\xi)]\,[-k a_1 \sec h(k\xi) \tanh(k\xi) + b_1 k \sec h^2(k\xi)].$$

$$\begin{aligned}u^2 u' = &-ka_1a_0^2 \sec h(k\xi)\tanh(k\xi) - a_1 b_1^2 k \sec h(k\xi)\tanh(k\xi) - 2ka_0a_1b_1 \tanh(k\xi) \\ &\sec h^2(k\xi) - 2a_0a_1b_1k \sec h(k\xi) + 2a_0a_1b_1k \sec h^3(k\xi) + (a_1^2 - b_1^2)(-a_1k) \\ &\sec h^3(k\xi)\tanh(k\xi) - 2a_1^2b_1k \sec h^2(k\xi) + 2a_1^2b_1k \sec h^4(k\xi) + a_0^2b_1k \sec h^2(k\xi) \\ &+ b_1^3k \sec h^2(k\xi) + 2ka_1a_0b_1 \sec h^3(k\xi) + 2ka_0b_1^2 \sec h^2(k\xi)\tanh(k\xi) + (a_1^2 - b_1^2) \\ &(b_1k)\sec h^4(k\xi) + 2ka_1b_1^2 \sec h^3(k\xi)\tanh(k\xi).\end{aligned}$$

$$u = a_0 + a_1 \sec h(k\xi) + b_1 \tanh(k\xi).$$

$$u' = -a_1 k \sec h(k\xi)\tanh(k\xi) + b_1k \sec h^2(k\xi).$$

$$u'' = k^2a_1 \sec h(k\xi) - 2k^2a_1 \sec h^3(k\xi) - 2b_1k^2 \sec h^2(k\xi)\tanh(k\xi).$$

$$\begin{aligned}u''' = &-k^3a_1 \sec h(k\xi)\tanh(k\xi) + 6k^3a_1 \sec h^3(k\xi)\tanh(k\xi) + 4b_1k^3 \sec h^2(k\xi) \\ &\tanh^2(k\xi) - 2b_1k^3 \sec h^3(k\xi).\end{aligned}$$

$$\begin{aligned}u''' = &-k^3a_1 \sec h(k\xi)\tanh(k\xi) + 6a_1k^3 \sec h^3(k\xi)\tanh(k\xi) + 4b_1k^3 \sec h^2(k\xi) - 6 \\ &b_1k^3 \sec h^4(k\xi).\end{aligned}$$

Now substituting the values of u', uu', u^2u', u''' in equation (4.2.2)

Thus we have:

$$\begin{aligned}&-c[-ka_1 \sec h(k\xi)\tanh(k\xi) - b_1k \sec h^2(k\xi)] - 6[-ka_0a_1 \sec h(k\xi)\tanh(k\xi) + a_0b_1k \\ &\sec h^2(k\xi) - ka_1^2 \sec h^2(k\xi)\tanh(k\xi) + 2a_1b_1k \sec h^3(k\xi) - a_1b_1k \sec h(k\xi) + b_1^2k \\ &\sec h^2(k\xi)\tanh(k\xi)] - 6\varepsilon^2[-ka_1a_0^2 \sec h(k\xi)\tanh(k\xi) - a_1 b_1^2k \sec h(k\xi)\tanh(k\xi) \\ &- 2ka_0a_1b_1 \tanh(k\xi) \sec h^2(k\xi) - 2a_0a_1b_1k \sec h(k\xi) + 2a_0a_1b_1k \sec h^3(k\xi) + (a_1^2 - b_1^2) \\ &(-a_1k)\sec h^3(k\xi)\tanh(k\xi) - 2a_1^2b_1k \sec h^2(k\xi) + 2a_1^2b_1k \sec h^4(k\xi) + a_0^2b_1k \sec h^2(k\xi) \\ &+ b_1^3k \sec h^2(k\xi) + 2ka_1a_0b_1 \sec h^3(k\xi) + 2ka_0b_1^2 \sec h^2(k\xi)\tanh(k\xi) + (a_1^2 - b_1^2)(b_1k) \\ &\sec h^4(k\xi) + 2ka_1b_1^2 \sec h^3(k\xi)\tanh(k\xi) - [k^3a_1 \sec h(k\xi)\tanh(k\xi) + 6a_1k^3 \sec h^3(k\xi) \\ &\tanh(k\xi) + 4b_1k^3 \sec h^2(k\xi) - 6b_1k^3 \sec h^4(k\xi)] = 0.\end{aligned}$$

Now by comparing the coefficients, we will get a set of algebraic equations:

Coefficient of $\sec h^2(k\xi)$

$$-cb_1 - 6a_0b_1 + 12\varepsilon^2 a_1^2 b_1 - 6a_0^2 b_1 \varepsilon^2 - 4b_1 k^2 = 0$$

Coefficient of $\sec h(k\xi)\tanh(k\xi)$

$$ca_1 + 6a_0a_1 + 6\varepsilon^2 a_0^2 a_1 + 6a_1 b_1^2 \varepsilon^2 + a_1 k^2 = 0$$

Coefficient of $\sec h^2(k\xi)\tanh(k\xi)$

$$a_1^2 - b_1^2 + 2\varepsilon^2 a_0 a_1 b_1 - 2a_0 b_1^2 = 0$$

Coefficient of $\sec h(k\xi)$

$$1 + 2\varepsilon^2 a_0 = 0$$

Coefficient of $\sec h^3(k\xi)$

$$-a_1 b_1 - 2\varepsilon^2 a_0 a_1 b_1 = 0$$

Coefficient of $\sec h^4(k\xi)$

$$-3\varepsilon^2 a_1^2 b_1 + \varepsilon^2 b_1^3 + b_1 k^2 = 0$$

Coefficient of $\sec h^3(k\xi)\tanh(k\xi)$

$$\varepsilon^2 a_1^3 - 3a_1 b_1^2 k \varepsilon^2 - k^2 a_1 = 0$$

Thus the algebraic set of equations are:

$$ca_1 + 6a_0a_1 + 6\varepsilon^2 a_1 a_0^2 + 6a_1 b_1^2 \varepsilon + a_1 k^2 = 0 \qquad (4.2.5)$$

$$-cb_1 + 6a_0b_1 + 6\varepsilon^2 a_0^2 b_1 + 6a_1^2 b_1 \varepsilon^2 = 0 \qquad (4.2.6)$$

$$a_1^2 - b_1^2 + 2\varepsilon^2 a_0 a_1 b_1 - 2a_0 b_1^2 = 0 \tag{4.2.7}$$
$$1 + 2\varepsilon^2 a_0 = 0 \tag{4.2.8}$$
$$\varepsilon^2 a_1^3 - 3\varepsilon^2 a_1 b_1^2 - a_1 k^2 = 0 \tag{4.2.9}$$
$$-3b_1^2 a_1^2 \varepsilon^2 + \varepsilon^2 b_1^3 + b_1 k^2 = 0 \tag{4.2.10}$$

From equation (4.2.8) we have:

$$1 + 2\varepsilon^2 a_0 = 0$$

$$a_0 = \frac{-1}{2\varepsilon^2}$$

Substituting the value of a_0 in all the whole equation set, we get:

$$ca_1 - 3\frac{1}{\varepsilon^2}a_1 + \left(\frac{3}{2\varepsilon^2}\right)a_1 + 6a_1 b_1^2 \varepsilon^2 + a_1 k^2 = 0 \tag{4.2.5}$$
$$-cb_1 + 3\frac{1}{\varepsilon^2}b_1 + 12a_1^2 b_1 \varepsilon^2 - \frac{3}{2\varepsilon^2}b_1 - 4k^2 b_1 = 0 \tag{4.2.6}$$
$$a_1^2 - b_1^2 - a_1 b_1 + \frac{b_1^2}{\varepsilon^2} = 0 \tag{4.2.7}$$
$$\varepsilon^2 a_1^3 - 3\varepsilon^2 a_1 b_1^2 - a_1 k^2 = 0 \tag{4.2.9}$$
$$-3b_1^2 a_1^2 \varepsilon^2 + \varepsilon^2 b_1^3 + b_1 k^2 = 0 \tag{4.2.10}$$

Now solving these equations we get

$$a_1 = -\frac{1}{2\varepsilon^2}\sqrt{-4c\varepsilon^2 + k}$$

Thus

$$u(x,t) = \frac{-1}{2\varepsilon^2} - \frac{1}{2\varepsilon^2}\sqrt{-4c\varepsilon^2 + k}\ \sec h[k(x - ct)], \tag{4.2.11}$$

where *k* and *c* are arbitrary constants.

4.3 Jacobi elliptic functions method

$$u_t = 6(u + \varepsilon^2 u^2)u_x + u_{xxx} \tag{4.3.1}$$

where ε is arbitrary parameter

Now we use the transformation $u(x,t) = U(\eta)$, where $\eta = x - ct - x_0$, with c as a constant.

$$\frac{\partial u}{\partial t} = u_t = \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial U}{\partial \eta} = -cU'$$

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial U}{\partial \eta}.1 = U'$$

$$\frac{\partial^3 u}{\partial x^3} = u_{xxx} = \frac{\partial^3 U}{\partial \eta\, \partial \eta\, \partial \eta}.1 = U'''$$

Then the equation (4.3.1) will become the ordinary differential equation:

$$-cU' - 6UU' - 6\varepsilon^2 U^2 U' - U''' = 0 \tag{4.3.2}$$

Now suppose that the equation (4.3.2) has the solution of the form

$$U(\eta) = \sum_{i}^{M} a_i\, sn^i(k\eta / m) \tag{4.3.3}$$

where M is the order of u.

Now we can show that $M = 1$ if balancing U''' with U^2U'

That is:

$$M + 3 = 2M + M + 1$$
$$\Rightarrow M = 1$$

Thus equation (4.3.3) becomes:

$$U(\eta) = \sum_{i=0}^{1} a_i sn^i(k\eta \mid m)$$

$$U(\eta) = a_0 + a_1\, sn(k\eta \mid m)$$

$$U'(\eta) = \frac{a_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m)$$

$$UU' = \frac{a_0 a_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{a_1^2 k}{m} sn(k\eta \mid m)\, cn(k\eta \mid m)\, dn(k\eta \mid m)$$

$$U'' = -\frac{a_1 k^2}{m^2} sn(k\eta \mid m)\, dn^2(k\eta \mid m) + \frac{a_1 k^2}{m^2}\left[-m^2 cn^2(k\eta \mid m)\, sn(k\eta \mid m)\right]$$

$$= -\frac{a_1 k^2}{m^2} sn(k\eta \mid m)\left(1 - m^2 sn^2(k\eta \mid m)\right) - a_1 k^2\, sn(k\eta \mid m)\left(1 - sn^2(k\eta \mid m)\right)$$

$$= -\frac{a_1 k^2}{m^2} sn(k\eta \mid m) + a_1 k^2\, sn^3(k\eta \mid m) - a_1 k^2\, sn(k\eta \mid m) + a_1 k^2\, sn^3(k\eta \mid m)$$

$$= -\frac{a_1 k^2}{m^2} sn(k\eta \mid m) + 2 a_1 k^2\, sn^3(k\eta \mid m) - a_1 k^2\, sn(k\eta \mid m)$$

$$U''' = -\frac{a_1 k^3}{m^3} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{6 a_1 k^3}{m} sn^2(k\eta \mid m) cn(k\eta \mid m)\, dn(k\eta \mid m) - a_1 k^2 \\ cn(k\eta \mid m)\, dn(k\eta \mid m)\left(\frac{k}{m}\right)$$

$$U''' = -\frac{a_1 k^3}{m^3} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{6 a_1 k^3}{m} sn^2(k\eta \mid m) cn(k\eta \mid m)\, dn(k\eta \mid m) - \frac{a_1 k^3}{m} \\ cn(k\eta \mid m)\, dn(k\eta \mid m)$$

$$U^2 = a_0^2 + a_1^2\, sn^2(k\eta \mid m) + 2a_0 a_1 sn(k\eta \mid m)$$

$$U^2U' = \frac{a_0^2 a_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \frac{a_1^3 k}{m} sn^2(k\eta \mid m)\; cn(k\eta \mid m)\; dn(k\eta \mid m) + \frac{2a_0 a_1{}^2 k}{m}$$
$$sn(k\eta \mid m)\; cn(k\eta \mid m)\; dn(k\eta \mid m)$$

Now substituting the values of U', UU', U^2U', U''' in equation (4.3.2), we have

$$\frac{-ca_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) - 6\left[\begin{array}{l} \dfrac{a_0 a_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \dfrac{a_1{}^2 k}{m} sn(k\eta \mid m) \\ cn(k\eta \mid m) dn(k\eta \mid m) \end{array}\right]$$
$$-6\varepsilon^2 \left[\begin{array}{l} \dfrac{a_0^2 a_1 k}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \dfrac{a_1^3 k}{m} sn^2(k\eta \mid m)\; cn(k\eta \mid m)\; dn(k\eta \mid m) + \\ \dfrac{2a_0 a_1{}^2 k}{m} sn(k\eta \mid m)\; cn(k\eta \mid m)\; dn(k\eta \mid m) \end{array}\right]$$
$$-\left[\begin{array}{l} \dfrac{a_1 k^3}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) + \dfrac{6a_1 k^3}{m} sn^2(k\eta \mid m)\; cn(k\eta \mid m)\; dn(k\eta \mid m) - \\ \dfrac{a_1 k^3}{m} cn(k\eta \mid m)\, dn(k\eta \mid m) \end{array}\right] = 0$$

Now by comparing the coefficients, we will get a set of algebraic equations:

Coefficient of $cn(k\eta \mid m)\, dn(k\eta \mid m)$

$$\frac{-ca_1}{m} - \frac{6a_0 a_1}{m} - \frac{6a_0{}^2 a_1 \varepsilon^2}{m} + \frac{a_1 k^2}{m^3} + \frac{a_1 k^2}{m} = 0.$$

Coefficient of $sn(k\eta \mid m) cn(k\eta \mid m)\, dn(k\eta \mid m)$

$$-6a_1^{\,2} - 12\varepsilon^2 a_0 a_1^{\,2} = 0.$$

Coefficient of $sn^2(k\eta \mid m)cn(k\eta \mid m)\,dn(k\eta \mid m)$

$$-6\varepsilon^2 a_1^{\,3} - 6a_1 k^2 = 0.$$

Thus the equations are:

$$\frac{-ca_1}{m} - \frac{6a_0 a_1}{m} - \frac{6a_0^{\,2} a_1 \varepsilon^2}{m} + \frac{a_1 k^2}{m^3} + \frac{a_1 k^2}{m} = 0 \qquad (4.3.4)$$

$$-6a_1^{\,2} - 12\varepsilon^2 a_0 a_1^{\,2} = 0 \qquad (4.3.5)$$

$$-6\varepsilon^2 a_1^{\,3} - 6a_1 k^2 = 0 \qquad (4.3.6)$$

From equation (4.3.5) we have:

$$-6a_1^{\,2} - 12\varepsilon^2 a_0 a_1^{\,2} = 0$$
$$\Rightarrow\; a_1^{\,2}(6 + 12\varepsilon^2 a_0) = 0$$
$$\Rightarrow\; \text{either } a_1 = 0 \text{ or } 6 + 12\varepsilon^2 a_0 = 0$$

Let $6 + 12\varepsilon^2 a_0 = 0$

$$a_0 = -\frac{1}{2\varepsilon^2}$$

Now from equation (4.3.6) we have:

$$-6\varepsilon^2 a_1^{\,3} - 6a_1 k^2 = 0$$
$$\Rightarrow\; \varepsilon^2 a_1^{\,2} = -k^2$$
$$\Rightarrow\; a_1 = \frac{\pm\, ki}{\varepsilon}$$

Thus we have

$$u(x,t) = -\frac{1}{2\varepsilon^2} \pm i\frac{k}{\varepsilon} sn(k(x - ct - x_0)\,|\,m),$$

with

$$\frac{-ic\left(-\frac{1}{2\varepsilon^2}\right)\left(\frac{\pm k}{\varepsilon}\right)}{m} - \frac{6i\left(-\frac{1}{2\varepsilon^2}\right)\left(\frac{\pm k}{\varepsilon}\right)}{m} - \frac{6i\left(-\frac{1}{2\varepsilon^2}\right)^2\left(\frac{\pm k}{\varepsilon}\right)\varepsilon^2}{m} + \frac{\left(\frac{\pm ki}{\varepsilon}\right)k^2}{m^3} + \frac{i\left(\frac{\pm k}{\varepsilon}\right)k^2}{m} = 0,$$

and k and ε are arbitrary constants.

CHAPTER – 5

WHITHAM- BROER- KAUP EQUATIONS

5.1 Introduction

The coupled Whitham- Broer- Kaup (WBK) equations which have been studied by Whitham [30], Broer [3], and Kaup [13], describes the propagation of shallow water waves, with different dispersion relations. The WBK equations are as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \beta\,\frac{\partial^2 u}{\partial x^2} = 0$$
$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \alpha\,\frac{\partial^3 u}{\partial x^3} - \beta\,\frac{\partial^2 v}{\partial x^2} = 0,$$

where $u = u(x,t)$ is the horizontal velocity, $v = v(x,t)$ is the height that deviates from equilibrium position of the liquid, and $\alpha\ ,\beta$ are the constants which are represented in different diffusion powers [14].

By using a new method and Mathematica, the Backlund transformations for Whitham- Broer- Kaup equations (WBK) are derived. The connections between WBK equation, heat equation and Burgers equation are found, which are used to obtain three families of solutions for WBK equations, one of which is the family of solitary wave solutions.

5.2 Hyperbolic functions expansion method.

$$\begin{aligned} &u_t + uu_x + H_x + qu_{xx} = 0 \\ &H_t + (Hu)_x + pu_{xxx} - qH_{xx} = 0, \end{aligned} \tag{5.2.1}$$

where p, q are arbitrary constants.

To find the solution of equation (5.2.1) we use the transformations:

$$\begin{aligned} u(x,t) &= u(\xi) \\ w(x,t) &= w(\xi) \end{aligned} \qquad \text{where } \xi = x - ct$$

$$\frac{\partial u}{\partial t} = u_t = \frac{\partial u}{\partial \xi}\,\frac{\partial \xi}{\partial t} = -c\frac{\partial u}{\partial \xi} = -cu'$$

$$\frac{\partial H}{\partial t} = H_t = \frac{\partial H}{\partial \xi}\,\frac{\partial \xi}{\partial t} = -c\frac{\partial H}{\partial \xi} = -cH'$$

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial \xi}\,\frac{\partial \xi}{\partial x} = \frac{\partial u}{\partial \xi}\,.1 = u'$$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = \frac{\partial^2 u}{\partial \xi^2} = u''$$

$$\frac{\partial^3 u}{\partial x^3} = u_{xxx} = 1.\frac{\partial^3 u}{\partial \xi^3} = u'''$$

$$\frac{\partial H}{\partial x} = H_x = \frac{\partial H}{\partial \xi}\,\frac{\partial \xi}{\partial x} = \frac{\partial H}{\partial \xi} = H'$$

$$\frac{\partial^2 H}{\partial x^2} = H_{xx} = \frac{\partial^2 H}{\partial \xi^2}\,\frac{\partial \xi^{2}}{\partial x^2} = \frac{\partial^2 H}{\partial \xi^2} = H''$$

$$\frac{\partial\left(Hu\right)}{\partial x} = \left(Hu\right)_x = H\frac{\partial u}{\partial x} + u\frac{\partial H}{\partial x} = H\frac{\partial u}{\partial \xi}\,\frac{\partial \xi}{\partial x} + u\frac{\partial H}{\partial \xi}\,\frac{\partial \xi}{\partial x} = H\frac{\partial u}{\partial \xi} + u\frac{\partial H}{\partial \xi} = Hu' + uH'$$

Then equation (5.2.1) will become the ordinary differential equation of the form:

$$u' + uu' + H' + qu'' = 0 \tag{5.2.2}$$

$$H' + (Hu)' + pu''' - qH'' = 0. \tag{5.2.3}$$

Now suppose that equations (5.2.2) and (5.2.3) has the solution of the form

$$u(\xi) = \sum_{i=0}^{m} a_i r^i + \sum_{i=1}^{m} b_i s r^{i-1} \tag{5.2.4}$$

and

$$H(\xi) = \sum_{j=0}^{n} c_j r^j + \sum_{j=1}^{n} d_j s r^{i-1} \tag{5.2.5}$$

where order of u is m and order of H is n.

Now balancing uu' with u''

That is:

$$m + m + 1 = m + 2$$
$$\Rightarrow \quad m = 1$$

Now balancing u''' with H''

That is:

$$m + 3 = n + 2$$
$$\Rightarrow \; 1 + 3 = n + 2$$
$$\Rightarrow \quad n = 2$$

Therefore equation (5.2.4) and (5.2.5) becomes:

$$u(\xi) = \sum_{i=0}^{1} a_i r^i + \sum_{i=1}^{1} b_i s r^{i-1}$$

$$= a_0 + a_1 r + b_1 s$$

$$H(\xi) = \sum_{j=0}^{2} c_j r^j + \sum_{j=1}^{2} d_j s r^{i-1}$$

$$= c_0 + c_1 r + c_2 r^2 + d_1 s + d_2 s r,$$

where $s = \tanh(k\xi)$ and $r = \sec h(k\xi)$

Therefore

$$u(\xi) = a_0 + a_1 \sec h(k\xi) + b_1 \tanh(k\xi)$$

and

$$H(\xi) = c_0 + c_1 \sec h(k\xi) + c_2 \sec h^2(k\xi) + d_1 \tanh(k\xi) + d_2 \tanh(k\xi) \sec h(k\xi).$$

$$u'(\xi) = -a_1 k \sec h(k\xi) \tanh(k\xi) + b_1 k \sec h^2(k\xi).$$

$$u''(\xi) = a_1 k^2 \sec h(k\xi) - 2 a_1 k^2 \sec h^3(k\xi) - 2 b_1 k^2 \sec h^2(k\xi) \tanh(k\xi).$$

$$u'''(\xi) = -a_1 k^3 \sec h(k\xi) \tanh(k\xi) + 6 a_1 k^3 \sec h^3(k\xi) \tanh(k\xi) + 4 b_1 k^3 \sec h^2(k\xi) \\ - 6 b_1 k^3 \sec h^4(k\xi).$$

$$uu' = [a_0 + a_1 \sec h(k\xi) + b_1 \tanh(k\xi)][-a_1 k \sec h(k\xi) \tanh(k\xi) + b_1 k \sec h^2(k\xi)].$$

$$= -a_0 a_1 k \sec h(k\xi) \tanh(k\xi) + a_0 b_1 k \sec h^2(k\xi) - a_1^2 k \sec h^2(k\xi) \tanh(k\xi) + b_1 a_1 k \\ \sec h^3(k\xi) - b_1 a_1 k \sec h(k\xi) \tanh^2(k\xi) + b_1^2 k \sec h^2(k\xi) \tanh(k\xi).$$

$$= -a_0 a_1 k \sec h(k\xi) \tanh(k\xi) + a_0 b_1 k \sec h^2(k\xi) - a_1^2 k \sec h^2(k\xi) \tanh(k\xi) + b_1 a_1 k \\ \sec h^3(k\xi) - b_1 a_1 k \sec h(k\xi) + b_1 a_1 k \sec h^3(k\xi) + b_1^2 k \sec h^2(k\xi) \tanh(k\xi).$$

$$uu' = -a_0 a_1 k \sec h(k\xi) \tanh(k\xi) + a_0 b_1 k \sec h^2(k\xi) - a_1^2 k \sec h^2(k\xi) \tanh(k\xi) + 2 b_1 a_1 k \\ \sec h^3(k\xi) - b_1 a_1 k \sec h(k\xi) + b_1^2 k \sec h^2(k\xi) \tanh(k\xi).$$

$$H(\xi) = c_0 + c_1 \sec h(k\xi) + c_2 \sec h^2(k\xi) + d_1 \tanh(k\xi) + d_2 \tanh(k\xi) \sec h(k\xi).$$

$$H'(\xi) = -c_1 k \sec h(k\xi) \tanh(k\xi) - 2 c_2 k \sec h^2(k\xi) \tanh(k\xi) + d_1 k \sec h^2(k\xi) + d_2 k \\ \sec h^3(k\xi) - d_2 k \sec h(k\xi) \tanh^2(k\xi).$$

.

$$= -c_1k\,\sec h(k\xi)\tanh(k\xi) - 2c_2k\,\sec h^2(k\xi)\tanh(k\xi) + d_1k\,\sec h^2(k\xi) + d_2k\,\sec h^3(k\xi) - d_2k\,\sec h(k\xi) + d_2k\,\sec h^3(k\xi).$$

$$H'(\xi) = -c_1k\,\sec h(k\xi)\tanh(k\xi) - 2c_2k\,\sec h^2(k\xi)\tanh(k\xi) + d_1k\,\sec h^2(k\xi) + 2d_2k\,\sec h^3(k\xi) - d_2k\,\sec h(k\xi).$$

$$H''(\xi) = c_1k^2\sec h(k\xi)\tanh^2(k\xi) - c_1k^2\sec h^3(k\xi) + 4c_2k^2\sec h^2(k\xi)\tanh^2(k\xi) - 2c_2k^2\sec h^3(k\xi) - d_1k^2\sec h^2(k\xi)\tanh(k\xi) - 6d_2k^2\sec h^3(k\xi)\tanh(k\xi) + d_2k^2\sec h(k\xi)\tanh(k\xi).$$

$$Hu' = [c_0 + c_1\sec h(k\xi) + c_2\sec h^2(k\xi) + d_1\tanh(k\xi) + d_2\tanh(k\xi)\sec h(k\xi)]\,[-a_1k\,\sec h(k\xi)\tanh(k\xi) + b_1k\,\sec h^2(k\xi)].$$

$$= -c_0a_1k\,\sec h(k\xi)\tanh(k\xi) + c_0b_1k\,\sec h^2(k\xi) - a_1c_1k\,\sec h^2(k\xi)\tanh(k\xi) + b_1c_1k\,\sec h^3(k\xi) - c_2a_1k\,\sec h^3(k\xi)\tanh(k\xi) + b_1c_2k\,\sec h^3(k\xi) - a_1d_1k\,\sec h(k\xi) + a_1d_1k\,\sec h^3(k\xi) + b_1d_1k\,\sec h^2(k\xi)\tanh(k\xi) - d_2a_1k\,\sec h^2(k\xi) + d_2a_1k\,\sec h^4(k\xi) + b_1d_2k\,\sec h^3(k\xi)\tanh(k\xi).$$

$$u\,H' = [a_0 + a_1\sec h(k\xi) + b_1\tanh(k\xi)]\,[-c_1k\,\sec h(k\xi)\tanh(k\xi) - 2c_2k\,\sec h^2(k\xi)\tanh(k\xi) + d_1k\,\sec h^2(k\xi) + 2d_2k\,\sec h^3(k\xi) - d_2k\,\sec h(k\xi)]$$

$$= -a_0c_1k\,\sec h(k\xi)\tanh(k\xi) - 2a_0c_2k\,\sec h^2(k\xi)\tanh(k\xi) + a_0d_1k\,\sec h^2(k\xi) + 2a_0d_2k\,\sec h^3(k\xi) - a_0d_2k\,\sec h(k\xi) - a_1c_1k\,\sec h^2(k\xi)\tanh(k\xi) - 2a_1c_2k\,\sec h^3(k\xi)\tanh(k\xi) + a_1d_1k\,\sec h^3(k\xi) + 2a_1d_2k\,\sec h^4(k\xi) - a_1d_2k\,\sec h^2(k\xi) - b_1c_1k\,\sec h(k\xi) + b_1c_1k\,\sec h^3(k\xi) - 2b_1c_2k\,\sec h^2(k\xi) + 2b_1c_2k\,\sec h^4(k\xi) + b_1d_1k\,\sec h^2(k\xi)\tanh(k\xi) + 2b_1d_2k\,\sec h^3(k\xi)\tanh(k\xi) - b_1d_2k\,\sec h(k\xi)\tanh(k\xi).$$

$$(Hu)' = H\,u' + u\,H'$$

$$= (-c_0a_1k - a_0c_1k - b_1d_2k)\operatorname{sec}h(k\xi)\tanh(k\xi) + (c_0b_1k - a_1d_2k + a_0d_1k - d_2a_1k - 2c_2 b_1k)\operatorname{sec}h^2(k\xi) + (-a_1c_1k + b_1d_1k - 2a_0c_2k - a_1c_1k + b_1d_1k)\operatorname{sec}h^2(k\xi)\tanh(k\xi) + (b_1c_1k + a_1d_1k + 2a_0d_2k + a_1d_1k + b_1c_1k)\operatorname{sec}h^3(k\xi) + (-c_2a_1k + 3b_1d_2k - 2a_1c_2k)\operatorname{sec}h^3(k\xi)\tanh(k\xi) + (-a_1d_1k - a_0d_2k - b_1c_1k)\operatorname{sec}h(k\xi) + (c_2b_1k + a_1d_2k + 2a_1d_2k\ 2c_2b_1k)\operatorname{sec}h^4(k\xi).$$

$$= (-c_0a_1k - a_0c_1k - b_1d_2k)\operatorname{sec}h(k\xi)\tanh(k\xi) + (c_0b_1k - a_1d_2k + a_0d_1k - d_2a_1k - 2c_2 b_1k)\operatorname{sec}h^2(k\xi) + (-2a_1c_1k + 2b_1d_1k - 2a_0c_2k)\operatorname{sec}h^2(k\xi)\tanh(k\xi) + (2b_1c_1k + 2a_1d_1k + 2a_0d_2k)\operatorname{sec}h^3(k\xi) + (-3c_2a_1k + 3b_1d_2k)\operatorname{sec}h^3(k\xi)\tanh(k\xi) + (-a_1d_1k - a_0d_2k - b_1c_1k)\operatorname{sec}h(k\xi) + (3c_2b_1k + 3a_1d_2k)\operatorname{sec}h^4(k\xi).$$

Now substituting the values of u', u'' and uu' in equation (5.2.2):

$$-c[-a_1k\operatorname{sec}h(k\xi)\tanh(k\xi) + b_1k\operatorname{sec}h^2(k\xi)] + [-a_0a_1k\operatorname{sec}h(k\xi)\tanh(k\xi) + a_0b_1k\operatorname{sec}h^2(k\xi) - a_1^2k\operatorname{sec}h^2(k\xi)\tanh(k\xi) + 2b_1a_1k\operatorname{sec}h^3(k\xi) - b_1a_1k\operatorname{sec}h(k\xi) + b_1^2k\operatorname{sec}h^2(k\xi)\tanh(k\xi)] + [-c_1k\operatorname{sec}h(k\xi)\tanh(k\xi) - 2c_2k\operatorname{sec}h^2(k\xi)\tanh(k\xi) + d_1k\operatorname{sec}h^2(k\xi) + 2d_2k\operatorname{sec}h^3(k\xi) - d_2k\operatorname{sec}h(k\xi)] + q[a_1k^2\operatorname{sec}h(k\xi) - 2a_1k^2\operatorname{sec}h^3(k\xi) - 2b_1k^2\operatorname{sec}h^2(k\xi)\tanh(k\xi)] = 0.$$

Now by comparing the coefficients, we will get the one set of algebraic equations

Coefficient of $\operatorname{sec}h(k\xi)\ \tanh(k\xi)$

$$ca_1 - a_0a_1 - c_1 = 0.$$

Coefficient of $\operatorname{sec}h^2(k\xi)$

$$-cb_1 - a_0b_1 - d_1 = 0.$$

Coefficient of $\sec h^2(k\xi)\ \tanh(k\xi)$

$$-a_1^2 + b_1^2 - 2c_2 - 2b_1qk = 0.$$

Coefficient of $\sec h^3(k\xi)$

$$2a_1b_1 + 2d_2 - 2a_1qk = 0.$$

Coefficient of $\sec h(k\xi)$

$$-a_1b_1 - d_2 + a_1qk = 0.$$

Now substituting the values of $H', u''', H'', (Hu)'$ in equation (5.2.3).

$$\begin{gathered}
-c[-c_1k\sec h(k\xi)\tanh(k\xi) - 2c_2k\sec h^2(k\xi)\tanh(k\xi) + d_1k\sec h^2(k\xi) + 2d_2k \\
\sec h^3(k\xi) - d_2k\sec h(k\xi) + [(-c_0a_1k - a_0c_1k - b_1d_2k)\sec h(k\xi)\tanh(k\xi) + (c_0b_1k \\
-a_1d_2k + a_0d_1k - d_2a_1k - 2c_2b_1k)\sec h^2(k\xi) + (-2a_1c_1k + 2b_1d_1k - 2a_0c_2k)\sec h^2(k\xi) \\
\tanh(k\xi) + (2b_1c_1k + 2a_1d_1k + 2a_0d_2k)\sec h^3(k\xi) + (-3c_2a_1k + 3b_1d_2k)\sec h^3(k\xi) \\
\tanh(k\xi) + (-a_1d_1k - a_0d_2k - b_1c_1k)\sec h(k\xi) + (3c_2b_1k + 3a_1d_2k)\sec h^4(k\xi)] + p[-a_1 \\
k^3\sec h(k\xi)\tanh(k\xi) + 6a_1k^3\sec h^3(k\xi)\tanh(k\xi) + 4b_1k^3\sec h^2(k\xi) - 6b_1k^3 \\
\sec h^4(k\xi)] - q[c_1k^2\sec h(k\xi)\tanh^2(k\xi) - 2c_1k^2\sec h^3(k\xi) + 4c_2k^2\sec h^2(k\xi) \\
-4c_2k^2\sec h^4(k\xi) - 2c_2k^2\sec h^3(k\xi) - 2d_1k^2\sec h^2(k\xi)\tanh(k\xi) - 6d_2k^2\sec h^3(k\xi) \\
\tanh(k\xi) + d_2k^2\sec h(k\xi)\tanh(k\xi)] = 0.
\end{gathered}$$

Now by comparing the coefficients, we will get the another set of algebraic equations:

Coefficient of $\sec h(k\xi)\tanh(k\xi)$

$$cc_1 - c_0a_1 - a_0c_1 - b_1d_1 - pa_1k^2 + d_2qk = 0.$$

Coefficient of $\sec h^3(k\xi)\tanh(k\xi)$

$$cc_2 - c_1a_1 - a_0c_2 + b_1d_1 + d_1qk = 0.$$

Coefficient of $\sec h^2(k\xi)$

$$-cd_1 + c_0b_1 - 2a_1d_2 + a_0d_1 - 2c_2b_1 + 4b_1pk^2 - 4qc_2k = 0.$$

Coefficient of $\sec h^3(k\xi)$

$$-cd_2 + c_1b_1 + a_1d_1 + a_0d_2 + c_1qk + qc_2k = 0.$$

Coefficient of $\sec h(k\xi)$

$$cd_2 - c_1b_1 - a_1d_1 - a_0d_2 - qc_1k = 0.$$

Coefficient of $\sec h^3(k\xi)\tanh(k\xi)$

$$-3c_2a_1 + 3d_2b_1 + 6pa_1k^2 + 6d_2qk = 0.$$

Coefficient of $\sec h^4(k\xi)$

$$3d_2a_1 + 3c_2b_1 - 6pb_1k^2 + 4c_2qk = 0.$$

Thus all the equations are:

$$ca_1 - a_0a_1 - c_1 = 0 \quad (5.2.6)$$
$$-cb_1 - a_0b_1 - d_1 = 0 \quad (5.2.7)$$
$$-a_1^2 + b_1^2 - 2c_2 - 2b_1qk = 0 \quad (5.2.8)$$
$$-a_1b_1 - d_2 + a_1qk = 0 \quad (5.2.9)$$
$$cc_1 - c_0a_1 - a_0c_1 - b_1d_2 - pa_1k^2 + d_2qk = 0 \quad (5.2.10)$$
$$cc_2 - c_1a_1 - a_0c_2 + b_1d_1 + d_1qk = 0 \quad (5.2.11)$$
$$-cd_1 + c_0b_1 - 2a_1d_2 + a_0d_1 - 2c_2b_1 + 4b_1pk^2 - 4qc_2k = 0 \quad (5.2.12)$$
$$-cd_2 + c_1b_1 + a_1d_1 + a_0d_2 + c_1qk + qc_2k = 0 \quad (5.2.13)$$
$$cd_2 - c_1b_1 - a_1d_1 - a_0d_2 - qc_1k = 0 \quad (5.2.14)$$
$$-3c_2a_1 + 3d_2b_1 + 6pa_1k^2 + 6d_2qk = 0 \quad (5.2.15)$$

$$3d_2a_1 + 3c_2b_1 - 6pb_1k^2 + 4c_2qk = 0 \qquad (5.2.16)$$

By adding equation (5.2.13) and equation (5.2.14) we get:

$$c_2qk = 0$$
$$\Rightarrow\ c_2 = 0$$

Now substituting $c_2 = 0$ in all the equations we get:

$$ca_1 - a_0a_1 - c_1 = 0 \qquad (5.2.6)$$
$$-cb_1 - a_0b_1 - d_1 = 0 \qquad (5.2.7)$$
$$-a_1^2 + b_1^2 - 2c_2 - 2b_1qk = 0 \qquad (5.2.8)$$
$$-a_1b_1 - d_2 + a_1qk = 0 \qquad (5.2.9)$$
$$cc_1 - c_0a_1 - a_0c_1 - b_1d_2 - pa_1k^2 + d_2qk = 0 \qquad (5.2.10)$$
$$-c_1a_1 + b_1d_1 + d_1qk = 0 \qquad (5.2.11)$$
$$-cd_1 + c_0b_1 - 2a_1d_2 + a_0d_1 + 4b_1pk^2 = 0 \qquad (5.2.12)$$
$$cd_2 - c_1b_1 - a_1d_1 - a_0d_2 - qc_1k = 0 \qquad (5.2.14)$$
$$3d_2b_1 + 6pa_1k^2 + 6d_2qk = 0 \qquad (5.2.15)$$
$$3d_2a_1 - 6pb_1k^2 = 0 \qquad (5.2.16)$$

By solving these equations we get:

$$c_0 = -\left(-2p - 2q^2 - 2\sqrt{q^2p + q^4}\right)k^2$$
$$d_1 = \left(-2p - 2q^2 - 2\sqrt{q^2p + q^4}\right)k^2$$
$$a_0 = c$$
$$a_1 = 0$$
$$c_1 = 0$$
$$c_2 = 0$$
$$d_2 = 0$$
$$b_1 = \frac{-2qk\left(-2p - 2q^2 - 2\sqrt{q^2p + q^4}\right)}{-2q^2 - 2\sqrt{q^2p + q^4}}$$

Thus we have:

$$u(\xi)=c-\frac{2qk\left(-2p-2q^2-2\sqrt{q^2p+q^4}\right)}{-2q^2-2\sqrt{q^2p+q^4}}\tanh[k(x-ct)]$$

$$h(\xi)=-\left(-2p-2q^2-2\sqrt{q^2p+q^4}\right)k^2+\left(-2p-2q^2-2\sqrt{q^2p+q^4}\right)k^2\tanh[k(x-ct)],$$

where k,c are arbitrary constants.

The values of the second set are:

$$c_0=-\left(-2p-2q^2+2\sqrt{q^2p+q^4}\right)k^2$$
$$d_1=\left(-2p-2q^2+2\sqrt{q^2p+q^4}\right)k^2$$
$$a_0=c$$
$$a_1=0$$
$$c_1=0$$
$$c_2=0$$
$$d_2=0$$
$$b_1=\frac{-2qk\left(-2p-2q^2+2\sqrt{q^2p+q^4}\right)}{-2q^2+2\sqrt{q^2p+q^4}}$$

Thus we have:

$$u(x,t)=c-\frac{2qk\left(-2p-2q^2+2\sqrt{q^2p+q^4}\right)}{-2q^2+2\sqrt{q^2p+q^4}}\tanh[k(x-ct)]$$

$$h(x,t)=-\left(-2p-2q^2+2\sqrt{q^2p+q^4}\right)k^2+\left(-2p-2q^2+2\sqrt{q^2p+q^4}\right)k^2\tanh[k(x-ct)], \tag{5.2.17}$$

where k,c are arbitrary constants.

CHAPTER – 6

BENJAMIN-BONA-MAHONY EQUATION

6.1 Introduction

The Benjamin- Bona- Mahony equation (or BBM equation) [2] is the partial differential equation

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

where $u = u(x,t)$.

This equation was introduced (T. B. Benjamin, J. L. Bona & J. J. Mahony 1972) as an improvement of the Korteweg–de Vries equation (KdV equation) for modeling long waves of small amplitude in 1+1 dimensions.

The Benjamin- Bona- Mahony equation has improved short- wavelength behaviour, as compared to the Korteweg- de Vries equation, and is another uni-directional wave equation with cnoidal wave solutions.(In fluid dynamics, a cnoidal wave is a nonlinear and exact periodic wave solution of the Korteweg- de Vries equation. These solutions are in terms of the Jacobi elliptic function *cn*, which is why they are coined *cn*oidal waves. They are used to describe surface gravity waves of fairly-long wavelength, as compared to the water depth.) Further, since the Korteweg- de Vries equation is an approximation to the Boussinesq equations for the case of one-way wave propagation, cnoidal waves are approximate solutions to the Boussinesq equations.

For practical applications, the Benjamin- Bona- Mahony equation (BBM equation) is preferable over the KdV equation, a forward-propagating model similar to KdV but with much better frequency- dispersion behaviour at shorter wavelengths. Further improvements in short-wave performance can be obtained by starting to derive a one-way wave equation from a modern improved Boussinesq model, valid for even shorter wavelengths.

6.2 Hyperbolic functions expansion method

$$u_t + u_x + uu_x - u_{xxt} = 0. \qquad (6.2.1)$$

To find the solution of equation (6.2.1) we use the transformations:

$$\begin{aligned} u(x,t) &= u(\xi) \\ w(x,t) &= w(\xi) \end{aligned} \qquad \text{where} \quad \xi = x - ct$$

$$\frac{\partial u}{\partial t} = u_t = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} = -c\frac{\partial u}{\partial \xi} = -cu'$$

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial u}{\partial \xi}.1 = u'$$

$$\frac{\partial^3 u}{\partial xxt} = u_{xxt} = -cu'''$$

Then equation (6.2.1) will become the ordinary differential equation of the form:

$$-cu' + u' + uu' + cu''' = 0. \qquad (6.2.2)$$

Now suppose that equation (6.2.2) has the solution of the form

$$u(\xi) = \sum_{i=0}^{m} a_i r^i + \sum_{i=1}^{m} b_i s r^{i-1}, \qquad (6.2.3)$$

where order of u is m.

Now balancing u''' with uu'

That is:

$$\begin{aligned} &m + 3 = m + m + 1 \\ \Rightarrow \quad &m = 2m + 2 \\ \Rightarrow \quad &m = 2 \end{aligned}$$

Therefore equation (6.2.3) becomes:

$$u(\xi) = \sum_{i=0}^{2} a_i r^i + \sum_{i=1}^{2} b_i s r^{i-1}$$

$$= a_0 + a_1 r + a_2 r^2 + b_1 s + b_2 s r,$$

where $s = \tanh(k\xi)$ and $r = \sec h(k\xi)$

Therefore

$$u(\xi) = a_0 + a_1 \sec h(k\xi) + a_2 \sec h^2(k\xi) + b_1 \tanh(k\xi) + b_2 \sec h(k\xi)\tanh(k\xi).$$

$$u'(\xi) = -a_1 k \sec h(k\xi)\tanh(k\xi) - 2a_2 k \sec h^2(k\xi)\tanh(k\xi) + b_1 k \sec h^2(k\xi) - b_2 k \sec h(k\xi) + b_2 k \sec^3(k\xi) + b_2 k \sec^3(k\xi).$$

$$u'(\xi) = -a_1 k \sec h(k\xi)\tanh(k\xi) - 2a_2 k \sec h^2(k\xi)\tanh(k\xi) + b_1 k \sec h^2(k\xi) - b_2 k \sec h(k\xi) + 2b_2 k \sec^3(k\xi).$$

$$u''(\xi) = a_1 k^2 \sec h(k\xi)\tanh^2(k\xi) - a_1 k^2 \sec h^3(k\xi) + 4a_2 k^2 \sec h^2(k\xi)\tanh^2(k\xi) - 2a_2 k^2 \sec h^4(k\xi) - 2b_1 k^2 \sec h^2(k\xi)\tanh(k\xi) + b_2 k^2 \sec h(k\xi)\tanh(k\xi) - 3b_2 k^2 \sec h^3(k\xi)\tanh(k\xi) - 3b_2 k^2 \sec h^3(k\xi)\tanh(k\xi).$$

$$u'' = a_1 k^2 \sec h(k\xi)\tanh^2(k\xi) - a_1 k^2 \sec h^3(k\xi) - a_1 k^2 \sec h^3(k\xi) + 4a_2 k^2 \sec h^2(k\xi) - 4a_2 k^2 \sec h^4(k\xi) - 6a_2 k^2 \sec h^4(k\xi) - 2b_1 k^2 \sec h^2(k\xi)\tanh(k\xi) + b_2 k^2 \sec h(k\xi) \tanh(k\xi) - 3b_2 k^2 \sec h^3(k\xi)\tanh(k\xi) - 3b_2 k^2 \sec h^3(k\xi)\tanh(k\xi).$$

$$u'' = a_1 k^2 \sec h(k\xi)\tanh^2(k\xi) - 2a_1 k^2 \sec h^3(k\xi) + 4a_2 k^2 \sec h^2(k\xi)\tanh^2(k\xi) - 6a_2 k^2 \sec h^4(k\xi) - 2b_1 k^2 \sec h^2(k\xi)\tanh(k\xi) + b_2 k^2 \sec h(k\xi)\tanh(k\xi) - 6b_2 k^2 \sec h^3(k\xi)\tanh(k\xi).$$

$$\begin{aligned}u''' =& -a_1k^3\sec h(k\xi)\tanh(k\xi)+6a_1k^3\sec h^3(k\xi)\tanh(k\xi)-8a_2k^3\sec h^2(k\xi)\tanh(k\xi)\\ &-24a_2k^3\sec h^4(k\xi)\tanh(k\xi)+4b_1k^3\sec h^2(k\xi)\tanh(k\xi)-2b_1k^3\sec h^4(k\xi)-b_2k^3\\ &\sec h(k\xi)\tanh^2(k\xi)+b_2k^3\sec h^3(k\xi)+18b_2k^3\sec h^3(k\xi)\tanh^2(k\xi)-6b_2k^3\sec h^5(k\xi).\end{aligned}$$

$$\begin{aligned}u''' =& -a_1k^3\sec h(k\xi)\tanh(k\xi)+6a_1k^3\sec h^3(k\xi)\tanh(k\xi)-8a_2k^3\sec h^2(k\xi)\tanh(k\xi)\\ &-24a_2k^3\sec h^4(k\xi)\tanh(k\xi)+4b_1k^3\sec h^2(k\xi)\tanh(k\xi)-2b_1k^3\sec h^4(k\xi)-b_2\\ &k^3\sec h(k\xi)+b_2\,k^3\sec h^3(k\xi)+b_2k^3\sec h^3(k\xi)+18b_2k^3\sec h^3(k\xi)+18b_2k^3\\ &\sec h^5(k\xi)-6b_2k^3\sec h^5(k\xi).\end{aligned}$$

$$\begin{aligned}u''' =& -a_1k^3\sec h(k\xi)\tanh(k\xi)+6a_1k^3\sec h^3(k\xi)\tanh(k\xi)-8a_2k^3\sec h^2(k\xi)\tanh(k\xi)\\ &-24a_2k^3\sec h^4(k\xi)\tanh(k\xi)+4b_1k^3\sec h^2(k\xi)\tanh(k\xi)-2b_1k^3\sec h^4(k\xi)-b_2\\ &k^3\sec h(k\xi)+20b_2k^3\sec h^3(k\xi)-24b_2k^3\sec h^5(k\xi).\end{aligned}$$

$$\begin{aligned}uu' =& [a_0+a_1\sec h(k\xi)+a_2\sec h^2(k\xi)+b_1\tanh(k\xi)+b_2\tanh(k\xi)\sec h(k\xi)]\,[-a_1k\sec h(k\xi)\\ &\tanh(k\xi)-2a_2k\sec h^2(k\xi)\tanh(k\xi)+b_1k\sec h^2(k\xi)-b_2k\sec h(k\xi)+2b_2k\sec h^3(k\xi)].\end{aligned}$$

$$\begin{aligned}uu' =& -a_0a_1k\sec h(k\xi)\tanh(k\xi)-2a_0a_2k\sec h^2(k\xi)\tanh(k\xi)+a_0b_1k\sec h^2(k\xi)-a_0b_2k\\ &\sec h(k\xi)+2a_0b_2k\sec h^3(k\xi)-a_1^2k\sec h^2(k\xi)\tanh(k\xi)-2a_1a_2k\sec h^3(k\xi)\tanh(k\xi)\\ &a_1b_1k\sec h^3(k\xi)-a_1b_2k\sec h^2(k\xi)+2a_1b_2k\sec h^4(k\xi)-a_1a_2k\sec h^3(k\xi)\tanh(k\xi)\\ &-2a_2^2k\sec h^4(k\xi)\tanh(k\xi)+b_1a_2k\sec h^4(k\xi)-b_2a_2k\sec h^3(k\xi)+2b_2a_2k\sec h^5(k\xi)\\ &-a_1b_1k\sec h(k\xi)+a_1b_1k\sec h^3(k\xi)-2a_2b_1k\sec h^2(k\xi)+2a_2b_1k\sec h^4(k\xi)+b_1^2k\\ &\sec h^2(k\xi)\tanh(k\xi)-b_1b_2k\sec h(k\xi)\tanh(k\xi)+2b_1b_2k\sec h^3(k\xi)\tanh(k\xi)-a_1b_2\\ &k\sec h^2(k\xi)+a_2b_1k\sec h^4(k\xi)-2b_2a_2k\sec h^3(k\xi)+2b_2a_2k\sec h^5(k\xi)+b_1b_2k\\ &\sec h^3(k\xi)\tanh(k\xi)-b_2^2k\sec h^2(k\xi)\tanh(k\xi)+2b_2^2k\sec h^4(k\xi)\tanh(k\xi).\end{aligned}$$

Now substituting the values of u', uu', u''' in equation (6.2.2):

$$\begin{aligned}
&(1-c)[-a_1k\sec h(k\xi)\tanh(k\xi)-2a_2k\sec h^2(k\xi)\tanh(k\xi)+b_1k\sec h^2(k)-b_2k\sec h(k\xi)\\
&+2b_2k\sec h^3(k\xi)]+[-a_0a_1k\sec h(k\xi)\tanh(k\xi)-2a_0a_2k\sec h^2(k\xi)\tanh(k\xi)+a_0b_1k\\
&\sec h^2(k\xi)-a_0b_2k\sec h(k\xi)+2a_0b_2k\sec h^3(k\xi)-a_1^2k\sec h^2(k\xi)\tanh(k\xi)-2a_1a_2k\\
&\sec h^3(k\xi)\tanh(k\xi)+a_1b_1k\sec h^3(k\xi)-a_1b_2k\sec h^2(k\xi)+2a_1b_2k\sec h^4(k\xi)-a_1a_2k\\
&\sec h^3(k\xi)\tanh(k\xi)-2a_2^2k\sec h^4(k\xi)\tanh(k\xi)+b_1a_2k\sec h^4(k\xi)-b_2a_2k\sec h^3(k\xi)+\\
&2b_2a_2k\sec h^5(k\xi)-a_1b_1k\sec h(k\xi)+a_1b_1k\sec h^3(k\xi)-2a_2b_1k\sec h^2(k\xi)+2a_2b_1k\sec h^4(k\xi)\\
&+b_1^2k\sec h^2(k\xi)\tanh(k\xi)-b_1b_2k\sec h(k\xi)\tanh(k\xi)+2b_1b_2k\sec h^3(k\xi)\tanh(k\xi)-a_1b_2k\\
&\sec h^2(k\xi)+a_2b_1k\sec h^4(k\xi)-2b_2a_2k\sec h^3(k\xi)+2b_2a_2k\sec h^5(k\xi)+b_1b_2k\sec h^3(k\xi)\\
&\tanh(k\xi)-b_2^2k\sec h^2(k\xi)\tanh(k\xi)+2b_2^2k\sec h^4(k\xi)\tanh(k\xi)]+c[-a_1k^3\sec h(k\xi)\\
&\tanh(k\xi)+6a_1k^3\sec h^3(k\xi)\tanh(k\xi)-8a_2k^3\sec h^2(k\xi)\tanh(k\xi)+24a_2k^3\sec h^4(k\xi)\\
&\tanh(k\xi)+4b_1k^3\sec h^2(k\xi)\tanh(k\xi)-2b_1k^3\sec h^4(k\xi)-b_2k^3\sec h(k\xi)+20b_2k^3\sec h^3(k\xi)\\
&-24b_2k^3\sec h^5(k\xi)]=0.
\end{aligned}$$

Now by comparing the coefficients, we will get a set of algebraic equations

Coefficient of $\sec h(k\xi)\ \tanh(k\xi)$

$$-(1-c)a_1-a_0a_1-b_1b_2-ca_1k^2=0.$$

Coefficient of $\sec h^2(k\xi)\ \tanh(k\xi)$

$$-(1-c)2a_2-2a_0a_2-a_1^2+b_1^2-8ca_2k^2+4cb_1k^2=0.$$

Coefficient of $\sec h^2(k\xi)$

$$(1-c)b_1+a_0b_1-2a_1b_2-2a_2b_1=0.$$

Coefficient of $\sec h(k\xi)$

$$-(1-c)b_2-a_0b_2-a_1b_1-cb_2k^2=0.$$

Coefficient of $\sec h^3(k\xi)$

$$(1-c)2b_2+2a_0b_2+2a_1b_1-3a_2b_2+20cb_2k^2=0.$$

Coefficient of $\sec h^3(k\xi)\tanh(k\xi)$

$$-a_1a_2 + b_1b_2 + 2ca_1k^2 = 0.$$

Coefficient of $\sec h^4(k\xi)\tanh(k\xi)$

$$-a_2^2 + b_2^2 - 12ca_2k^2 = 0.$$

Coefficient of $\sec h^4(k\xi)$

$$3a_1b_2 + 3a_2b_1 - 2cb_1k^2 = 0.$$

Coefficient of $\sec h^5(k\xi)$

$$a_2b_2 - 6cb_2k^2 = 0.$$

Thus all the equations are:

$$-(1-c)a_1 - a_0a_1 - b_1b_2 - ca_1k^2 = 0 \qquad (6.2.4)$$
$$-(1-c)2a_2 - 2a_0a_2 - a_1^2 + b_1^2 - 8ca_2k^2 + 4cb_1k^2 = 0 \qquad (6.2.5)$$
$$(1-c)b_1 + a_0b_1 - 2a_1b_2 - 2a_2b_1 = 0 \qquad (6.2.6)$$
$$-(1-c)b_2 - a_0b_2 - a_1b_1 - cb_2k^2 = 0 \qquad (6.2.7)$$
$$(1-c)2b_2 + 2a_0b_2 + 2a_1b_1 - 3a_2b_2 + 20cb_2k^2 = 0 \qquad (6.2.8)$$
$$-a_1a_2 + b_1b_2 + 2ca_1k^2 = 0 \qquad (6.2.9)$$
$$-a_2^2 + b_2^2 - 12ca_2k^2 = 0 \qquad (6.2.10)$$
$$3a_1b_2 + 3a_2b_1 - 2cb_1k^2 = 0 \qquad (6.2.11)$$
$$a_2b_2 - 6cb_2k^2 = 0 \qquad (6.2.12)$$

Now multiplying equation (6.2.7) by 2 and adding equation (6.2.7) and equation (6.2.8)

$$-2cb_2k^2 - 3a_2b_2 + 20cb_2k^2 = 0$$
$$-3a_2b_2 + 18cb_2k^2 = 0$$
$$\Rightarrow\ -a_2b_2 + 6cb_2k^2 = 0$$
$$\Rightarrow\ a_2b_2 - 6cb_2k^2 = 0$$
$$\Rightarrow\ b_2(a_2 - 6ck^2) = 0$$
$$\Rightarrow\ \text{either}\ b_2 = 0\ \text{or}\ a_2 - 6ck^2 = 0$$

Let $b_2 = 0$

Now substituting $b_2 = 0$ in all the above equations:

$$-(1-c)a_1 - a_0a_1 - ca_1k^2 = 0 \tag{6.2.4}$$

$$-(1-c)2a_2 - 2a_0a_2 - a_1^2 + b_1^2 - 8ca_2k^2 + 4cb_1k^2 = 0 \tag{6.2.5}$$

$$(1-c)b_1 + a_0b_1 - 2a_2b_1 = 0 \tag{6.2.6}$$

$$a_1b_1 = 0 \tag{6.2.7}$$

$$-a_1a_2 + 2ca_1k^2 = 0 \tag{6.2.9}$$

$$-a_2^2 - 12ca_2k^2 = 0 \tag{6.2.10}$$

$$3a_2b_1 - 2cb_1k^2 = 0 \tag{6.2.11}$$

Now from equation (6.2.10) we have:

$$-a_2^2 - 12ca_2k^2 = 0$$

$$\Rightarrow \; -a_2(a_2 - 12ck^2) = 0$$

$$\Rightarrow \text{ either } a_2 = 0 \text{ or } a_2 = -12ck^2$$

Let $a_2 = -12ck^2$

Now substituting $a_2 = -12ck^2$ in the above equations we get:

$$-(1-c)a_1 - a_0a_1 - ca_1k^2 = 0 \tag{6.2.4}$$

$$(1-c)24ck^2 - 2a_0a_2 - a_1^2 + b_1^2 + 96c^2k^3 + 4cb_1k^2 = 0 \tag{6.2.5}$$

$$(1-c)b_1 + a_0b_1 + 24ck^2b_1 = 0 \tag{6.2.6}$$

$$12ck^2a_1 + 2ca_1k^2 = 0 \tag{6.2.9}$$

$$3(-12ck^2)b_1 - 2cb_1k^2 = 0 \tag{6.2.11}$$

From equation (6.2.11) we have:

$$3(-12ck^2)b_1 - 2cb_1k^2 = 0$$

$$\Rightarrow \; -36ck^2b_1 - 2cb_1k^2 = 0$$

$$\Rightarrow \; b_1 = 0$$

From equation (6.2.4)

$-(1-c)a_1 - a_0a_1 - ca_1k^2 = 0$

$\Rightarrow a_1(-1+c-a_0-ck^2) = 0$

$\Rightarrow$ either $a_1 = 0$ or $a_0 = -1+c-ck^2$

Let $a_0 = -1+c-ck^2$

From equation (6.2.9) we get, $a_1 = 0$

Hence all the values are as follows:

$b_2 = 0$

$a_0 = -1+c-ck^2$

$b_1 = 0$

$a_2 = -12ck^2$

$a_1 = 0$

Thus

$$u(x,t) = (-1+c-ck^2) + (-12ck^2)\sec h^2[k(x-ct)], \qquad (6.2.13)$$

where k is arbitrary constant.

CONCLUSION

The Hyperbolic functions expansion method (tanh method) and Jacobi elliptic functions expansion method are the straightforward and efficient methods for finding the exact solutions of the nonlinear partial differential equations. These methods are a great tool for finding the exact.

A wide range of solutions are obtained which will be helpful for the understanding of various physical phenomena described by these equations. But here we restrict ourself for obtaining the exact solutions only. For better understanding of the nature of the solutions obtained, here we are presenting plots of some of these solutions:

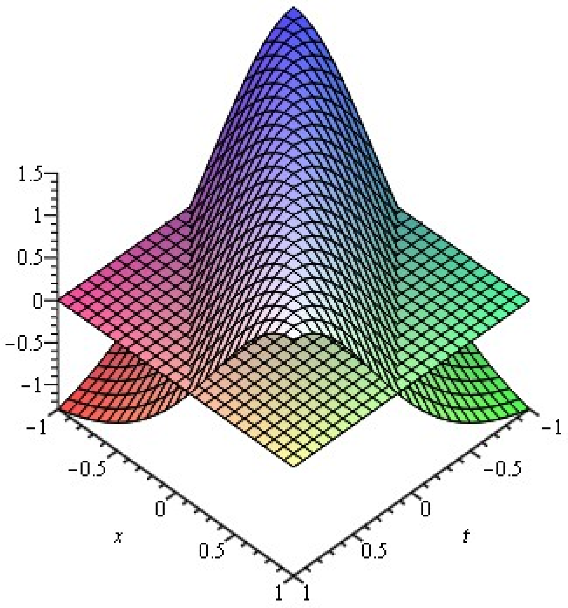

Figure- 1 in section (3.2), solution (3.2.20) with c = 1 and k = 1.

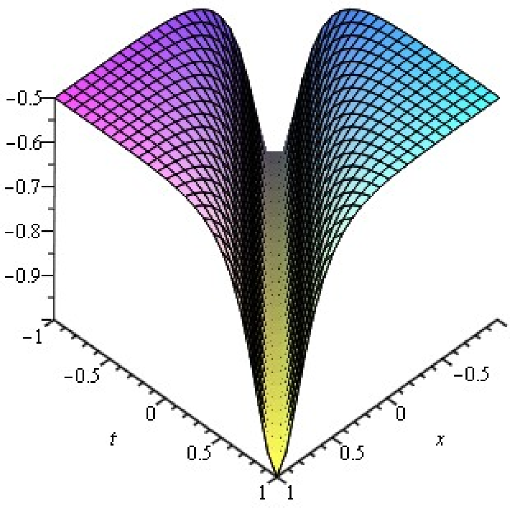

Figure- 2 in section (4.2), solution (4.2.11) with c = 1, k = 5 and ε = 1

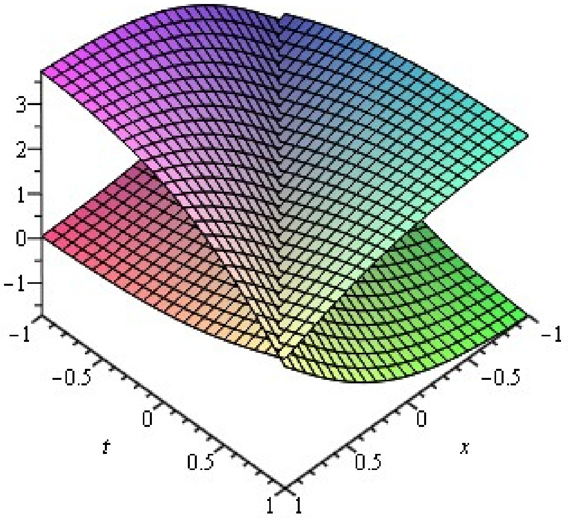

Figure- 3 in section (5.2), solution (5.2.17) with c = 1, k = 1, p = 1 and q =1.

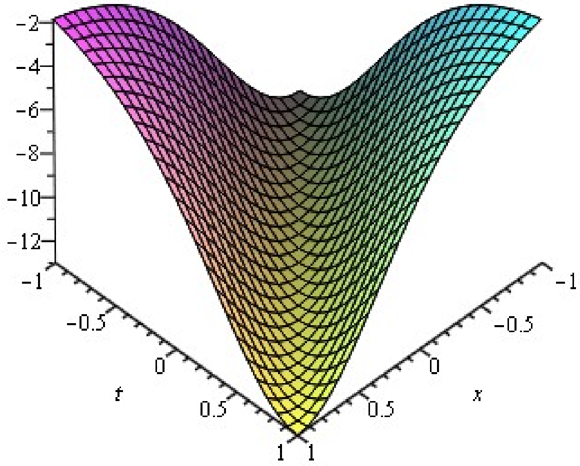

Figure- 4 in section (6.2), solution (6.2.13) with c = 1, k = 1.

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