MIXED SYMMETRIC DUALITY IN MULTIOBJECTIVE PROGRAMMING PROBLEMS

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Certificate

I hereby declare that the work which is being presented here in the dissertation entitled **"MIXED SYMMETRIC DUALITY IN MULTIOBJECTIVE PROGRAMMING PROBLEMS"** in partial fulfillment of the requirement for the award of degree of **Master of Science** in **Mathematics and Computing** submitted in School of Mathematics, Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of **Dr. Navdeep Kailey**, Lecturer, SOM and refer other researcher's work which are duly listed in the reference section.

The matter presented in this thesis has not been submitted to any other University/Institute for the award of my degree.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

The work being presented in the present thesis is devoted to the study of mixed symmetric duality in multiobjective programming problems under generalized convexity assumption.

In the first chapter of the dissertation, nonlinear and multiobjective programming problem is introduced. The brief description of basic concepts, definitions that are used throughout work and detailed review of duality in single and multiobjective programming problems and summary of the thesis has also been discussed in this chapter.

In chapter 2, we have reviewed a new pair of multiobjective second-order symmetric dual programs over arbitrary cones considered by Gupta et al. [20]and established weak, strong, converse and self duality theorems under K-η-bonvexity assumptions.

In chapter 3, we have reviewed a pair of second-order mixed symmetric nondifferentiable multiobjective dual programs over arbitrary cones considered by Gupta et al. [19] and established weak, strong and converse duality theorems under K-(F, ρ)-convexity assumptions.

In chapter 4, we have discussed higher order mixed multiobjective symmetric duality and prove weak duality theorem.

Contents

List of Figures

Chapter 1

Introduction

Mathematical programming arise naturally in various mathematical situation. They have played a large role in finding new connections between various branches of mathematics. They have also found application in physical sciences, biological sciences, engineering design, operation research, management science, computer science, financial engineering and economics etc. In optimization theory, we have calculate the maximum(or minimum) values of various functions. The function f is called, an objective function, a loss function or cost function (minimization), a utility function or fitness function (maximization), or, in certain fields, an energy function or energy functional. A feasible solution that minimizes (or maximizes, if that is the goal) the objective function is called an optimal solution. Number of problems in the world of science and engineering deal with optimization. An act, process, or methodology of making design, system, or decision as fully perfect, functional, or effective as possible; specifically : the mathematical procedures (as finding the maximum of a function) is called optimization.

Numerous algorithms are proposed for solving non-convex problems including the majority of commercially available solvers are not capable of making a distinction between local optimal solutions and strict optimal solutions, and will treat the former as actual solutions to the original problem. The branch of applied mathematics that is concerned with the development of deterministic algorithms that are capable of guaranteeing convergence in finite time to the actual optimal solution of a non-convex problem is called global optimization. An optimization problem consists of maximizing or minimizing a real function by systematically choosing input values from within an allowed set and computing the value of the function. The generalization of optimization theory and techniques to other formulations comprises a large area of applied mathematics. More generally, optimization includes finding "best available" values of some objective function given a defined domain (or a set of constraints), including a variety of different types of objective functions and different types of domains.

Linear programming plays a very important role in general optimization problems. LPP is applied to many-many real life problems like operation research, engineering and economics. Now days, business and economic situations are concerned with a problem of planning activity because there are limited resources at our disposal and our problem is to make use of these resources in such a way that gives the minimum cost of production or to maximize profit. These types of problems referred as constraint optimization. So one of the technique is linear programming with the help of which we determine an optimum schedule of independent activities of the available resources. Programming and planning are related to each other that is programming is just another word for planning and includes the process to find a particular plan of action from various alternatives resources. The word linear has deep meaning. Linear means for indicating that all relationships that involve particular problem are linear.

Optimization always tries to find the best and optimal solution to the given problem. For this purpose objectives of the organization are properly analyzed and defined in such a way that maximum profit take place. It also tries to find an optimal solutions with multiple variables. In most cases a large number of iterations are required to reach optimal solution. Written task becomes time consuming. With the help of computers, this has reduced written efforts and solutions can be obtained in a short period of time. No paper work is required. We can also transfer data from one place to another with the developments of computer. Errors also minimized. The reliability of solution is also high. Storage of data is faster and easy with the use of computers.

An extensive use of duality in mathematical programming has not only been made for many theoretical and computational developments in mathematical programming itself but also in economics, control theory, business problems and other diverse fields. It is well known that duality principles connect two programs one of which, called the primal problem, is a constrained minimization (or maximization) problem and the other, called the dual problem, is a constrained maximization (or minimization) problem, in such a way that the existence of an optimal solution to one of them guarantees an optimal solution to the other and optimal values of the two problems are equal. A pair of dual problems is called symmetric if the dual of the dual is the primal problem.

A mathematical programming problem with single objective function is called a scalar (or single objective) programming problem. A vector minimum (or maximum) problem is a mathematical optimization model with two or more objective functions. Such models are also called multiobjective programming problems. Often the several objectives are conflicting in nature. For example, it may be impossible to select an alternative to a problem which would maximize both profit and market share for a company. The existence of multiple objectives leads to many interesting questions, which do not arise in single objective models. It is difficult to obtain a unique solution since these problems rarely have feasible points that simultaneously minimize (or maximize) all the objectives. The concept of optimal solution in multiobjective optimization problems is clearly related to the preference attitude of the decision maker. A good decision is based on the principle that there is no other alternate that can be better in some aspect of consideration. Such a point is called an efficient point. An efficient solution is also known as noninferior or nondominated or Pareto optimal solution.

Convexity plays an important role in nonlinear programming. In inequality constrained problems the Karush-Kuhn-Tucker conditions are sufficient for optimality if the functions are convex. These conditions are used in many algorithms to solve nonlinear programming problems. However, the application of the Karush-Kuhn-Tucker sufficient optimality conditions cannot be restricted to convex problems as many mathematical models used in decision sciences, economics, management sciences, applied mathematics and engineering involve non convex functions. So, it is a necessity to generalize the notion of convexity and to explore the extent of the validity of results to larger classes of optimization problems. Thus, the literature contains various generalizations of convex functions.

The present chapter is divided into three sections. The first section gives important preliminaries. The second section contains a review of various developments in single and multiobjective mathematical programming which are relevant to the thesis and last one presents a summary of the thesis.

1.1 Preliminaries

1.1.1 Notations and definitions

Throughout the thesis the following notations are used. The n-dimensional Euclidean space is denoted by R^n and a vector $x \in R^n$ is denoted by $x = (x_1, x_2, \dots, x_n), x_i \in R, i = 1, 2, \dots, n$. Each vector x is a column vector and x^T denotes row vector where superscript T denotes the transpose of a vector. R^n_+ is the non-negative orthant of R^n and R_+ the set of nonnegative real numbers. For $x, y \in R^n$,

$$x \geqq y \Leftrightarrow x_i \geqq y_i, i = 1, 2, \dots, n,$$

$$x \geq y \Leftrightarrow x \geqq y \text{ and } x \neq y,$$

$$x > y \Leftrightarrow x_i > y_i, i = 1, 2, \dots, n.$$

$$x \nleq y \text{ is the negation of } x \leq y.$$

The vector $\nabla f(\bar{x})$ denotes the gradient of a scalar differentiable function $f : R^n \to R$ at $\bar{x}$, and is defined as

$$\nabla f(\bar{x}) = \left[\frac{\partial}{\partial x_1} \nabla f(\bar{x}), \frac{\partial}{\partial x_2} \nabla f(\bar{x}), \dots, \frac{\partial}{\partial x_n} \nabla f(\bar{x}) \right]^T$$

and for a vector valued differentiable function $f : R^n \to R^k$, the symbol $\nabla f(\bar{x})$ denotes $k \times n$ Jacobian matrix of f at $\bar{x}$, whose ith row is the vector $\nabla f_i(\bar{x})^T$. If the function $f : R^n \to R$ is twice differentiable at $\bar{x}$, in addition to the gradient vector there exists an $n \times n$ symmetric matrix $\nabla_{xx} f$ or $\nabla^2 f$, called the Hessian matrix of f at $\bar{x}$. The element in ith row and jth column of the Hessian matrix is the second-order partial derivative $\frac{\partial^2 f(\bar{x})}{\partial x_i \partial x_j}$. A vector valued function is differentiable if each of its components is differentiable and is twice differentiable if each of its components is twice differentiable.

Let $K : R^n \times R^m \to R$ be twice differentiable scalar function, $\nabla_x K(\bar{x}, \bar{y})$ and $\nabla_y K(\bar{x}, \bar{y})$ denote the gradient (column) vectors with respect to x and y at $(\bar{x}, \bar{y})$ respectively, and $\nabla_{xx} K(\bar{x}, \bar{y})$ and $\nabla_{yx} K(\bar{x}, \bar{y})$ denote respectively the $n \times n$ and $n \times m$ matrices of second-order partial derivatives evaluated at $(\bar{x}, \bar{y})$.

1.1.2 General mathematical programming problem

The general mathematical programming problem can be stated as follows :

$$\textbf{(NLP)} \quad \text{Minimize} f(x)$$
$$\text{subject to } x \in S = \{x \in X : g(x) \leq 0\}.$$

where X is an open subset of R^n, the functions $f : X \to R$ and $g : X \to R^m$ are differentiable on X.

This problem is called a **scalar (or single objective)** mathematical programming problem. The function f is known as the **objective function**, the components of g as the constraint functions and the corresponding inequalities as **constraints**. The set S is called the **feasible set** and any point $\bar{x} \in S$ is called a **feasible point** or simply **feasible**.

If $\bar{x} \in S$ and $f(x) \geq f(\bar{x})$ for each $x \in S$, then $\bar{x}$ is called **an optimal solution, a global optimal solution, or simply a solution** of Problem (NLP). If $\bar{x} \in S$ and if there exists an δ-neighborhood $N_\delta(\bar{x})$ around $\bar{x}$ such that $f(x) \geq f(\bar{x})$ for each $x \in S \cap N_\delta(\bar{x})$, then $\bar{x}$ is called a **local optimal solution**.

1.1.3 Duality in mathematical programming

Neumann [34] introduced the concept of duality in linear programming. He formulated the following dual pair and proved duality relations.

Primal Problem (PP)

$$\begin{aligned} \text{Minimize} \quad & f(x) = c^T x \\ \text{subject } to \quad & Ax \geq b, \\ & x \geq 0. \end{aligned}$$

Dual Problem (DP)

$$\begin{aligned} \text{Maximize} \quad & g(x) = b^T y \\ \text{subject } to \quad & A^T y \leq c, \\ & y \geq 0. \end{aligned}$$

The above pair shows that if the primal problem is a minimization of a linear function over a set of linear constraints, then the dual is a maximization of another linear function over a set of linear constraints. Moreover, dual of the dual is again the primal problem. In mathematical programming, a pair of primal and dual problems is called symmetric if the dual of the dual is the primal problem.

Duality in nonlinear programming has also been developed extensively. It originated with the duality results of quadratic programming given by Dennis [13]. Wolfe [41] and Mangasarian [25] gave duality results for convex primal and its dual program. Wolfe [41] formulated the following dual to (NLP):

$$\begin{aligned} \text{Maximize} \quad & f(y) + \mu^T g(y) \\ \text{subject to} \quad & \nabla f(y) + \nabla \mu^T g(y) = 0; \\ & y \in X, \mu \geq 0; \end{aligned}$$

and proved weak and strong duality theorems assuming f and g to be convex. Mangasarian [25] pointed out that these duality relations do not hold under weaker convexity assumptions. Mond and Weir[30] introduced the following dual to (NLP):

$$\begin{aligned} \text{Maximize} \quad & f(y) \\ \text{subject to} \quad & \nabla f(y) + \nabla \mu^T g(y) = 0; \\ & \mu^T g(y) \geq 0, \\ & y \in X, \mu \geq 0, \end{aligned}$$

and proved duality theorems by weakening the convexity assumptions of f and g to pseudoconvexity of f and quasiconvexity of $\mu^T g$. They also discussed duality results for the problems involving both equality and inequality constraints.

1.1.4 Mutliobjective programming

A mathematical programming problem with single objective function is called a scalar (or single objective) programming problem. A vector minimum (or maximum) problem is a mathematical optimization model with two or more objective functions. Such models are also called multiobjective programming problems. Multiobjective programming also known as multi-criteria or multi-attribute optimization, is the process of simultaneously optimizing two or more conflicting objectives subject to certain constraints.Consider an example of buying a house. A decision of buying a particular house is not based on a single criterion, of say cost alone, but many more criterion are equally important like land in acres, area, rooms, water supply, electricity, etc. Multiobjective optimization problems can be found in various fields:product and process design, finance, aircraft design, the oil and gas industry, automobile design, or wherever optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives. Maximizing profit and minimizing the cost of a product; maximizing performance and minimizing fuel consumption of a vehicle; and minimizing weight while maximizing the strength of a particular component are examples of multiobjective optimization problems.So, to solve this types of problem **multi-objective programming** is used. A general multi-objective nonlinear programming problem is in the form:

$$\text{(P)} \qquad \text{K-Minimize } f(x)$$
$$\text{subject to} \qquad x \in X^0 = \{x \in S : -g(x) \in Q\},$$

where $S \subseteq R^n, f : S \to R^k, g : S \to R^m$, K is a closed convex pointed cone in R^k with $\text{int}K \neq \phi$ and Q is a closed convex cone with a nonempty interior in R^m.

Definition 1.1[23, 38] A point $\bar{x} \in X^0$ is a weakly efficient solution of (P) if there exists no other $x \in X^0$ such that

$$f(\bar{x}) - f(x) \in \text{int}K.$$

Definition 1.2[23]A point $\bar{x} \in X^0$ is an efficient solution of (P) if there exists no other $x \in X^0$ such that

$$f(\bar{x}) - f(x) \in K \setminus \{0\}.$$

We can now define a positive polar cone and a K-η-bonvex function.

Let C_1 and C_2 be closed convex cones with nonempty interiors in R^n and R^m, respectively.

Definition 1.3[23, 38] The positive polar cone C_i^* of $C_i (i = 1, 2)$ is defined as

$$C_i^* = \{z : x^T z \geqq 0, \text{for all } x \in C_i\}.$$

Definition 1.4[29, 37] Let S be a compact convex set in R^n. The support function of S is defined by

$$s(x|S) = \max\{x^T y : y \in S\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists $z \in R^n$ such that

$$s(y|S) \geqq s(x|s) + z^T(y - x) \text{ for all } y \in S.$$

The subdifferential of $s(x|S)$ is given by

$$\partial s(x|S) = \{z \in S : z^T x = s(x|S)\}.$$

For any set $S \subset R^n$ the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leqq 0, \text{ for all } z \in S\}.$$

Also, y is in $N_S(x)$ if and only if $s(y|S) = x^T y$.

Let $F : X \times X \times R^n \to R$ (where $X \subseteq R^n$) be a sublinear functional.

1.1.5 Convex function and its generalization

Let X be an open convex set of R^n and the functions $f : X \to R$. Then at $x_2 \in X$,

(i) f is said to be **Convex** if for all $x_1 \in X$ and for all $0 \leqq \lambda \leqq 1$, we have

$$f[\lambda x_1 + (1 - \lambda)x_2] \leqq \lambda f(x_1) + (1 - \lambda)f(x_2),$$

or if f is differentiable at x_2, then we have

$$f(x_1) - f(x_2) \geqq \nabla f(x_2)^T(x_1 - x_2)$$

The function f is said to be strictly convex if the above conditions hold as strict inequalities for $x_1 \neq x_2, 0 < \lambda < 1$.

Convex function is shown in fig.1.1

Examples of convex function

(a) $f(x) = x^2, x \in R.$

(b) $f(x) = |x|, x \in R.$

(c) $f(x) = e^x, x \in R.$

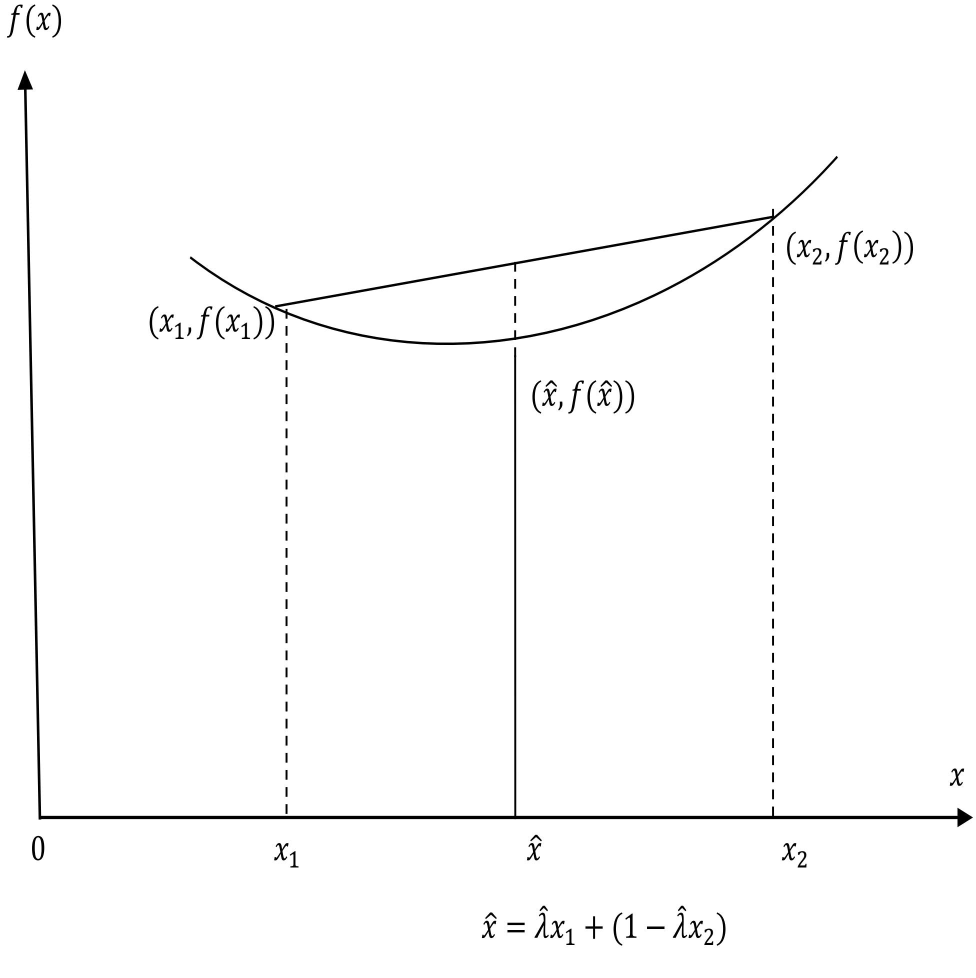

Figure 1.1: Convex function

(*ii*) f is said to be **Quasiconvex** if for all $x_1 \in X$ and for all $0 \leqq \lambda \leqq 1$, we have

$$f(x_1) \leqq f(x_2) \Rightarrow f[\lambda x_1 + (1-\lambda)x_2] \leqq f(x_2),$$

or if f is differentiable at x_2, then we have

$$f(x_1) \leqq f(x_2) \Rightarrow \nabla f(x_2)^T(x_1 - x_2) \leqq 0.$$

Remark Every convex function is quasiconvex, but the converse is not true. For example $f(x) = x^3$ is quasiconvex but not convex.

(*iii*) f is said to be **Pseudoconvex** if f is differentable at x_2 and for all $x_1 \in X$, we have

$$\nabla f(x_2)^T(x_1 - x_2) \geqq 0 \Rightarrow f(x_1) \geqq f(x_2),$$

or equivalently,if

$$f(x_1) < f(x_2) \Rightarrow \nabla f(x_2)^T(x_1 - x_2) < 0.$$

For example, $f(x) = x^3 + x$ is pseudoconvex.

(*iv*) **Strictly Pseudoconvex** if f is differentiable at x_2 and for all $x_1 \in X(x_1 \neq x_2)$

$$f(x_1) \leqq f(x_2) \Rightarrow \nabla f(x_2)^T(x_1 - x_2) < 0,$$

or equivalently, if

$$\nabla f(x_2)^T(x_1 - x_2) \geqq 0 \Rightarrow f(x_1) > f(x_2).$$

A new class of generalized convex functions, the so-called bonvex functions, introduced by Bector and Chandra [6] which was later on extended to η-bonvex by Pandey [35].

(*v*) **Second-order Convex (Bonvex)** if f is twice differentiable at x_2 and for all $x_1 \in X$, $p \in R^n$

$$f(x_1) - f(x_2) \geqq (\nabla f(x_2) + \nabla^2 f(x_2)p)^T(x_1 - x_2) - \frac{1}{2}p^T\nabla^2 f(x_2)p.$$

(*vi*) **Second-order Pseudoconvex (Pseudobonvex)** if f is twice differentiable at x_2 and for all $x_1 \in X, p \in R^n$

$$(\nabla f(x_2) + \nabla^2 f(x_2)p)^T(x_1 - x_2) \geqq 0 \Rightarrow f(x_1) \geqq f(x_2) - \frac{1}{2}p^T\nabla^2 f(x_2)p.$$

Further, f is said to be convex on X if f is convex at every point on X. A k-dimensional vector function $f = (f_1, f_2, \ldots, f_k)$ is said to be convex at x_2 (or on X) if for each $j \in K$; f_j is convex at x_2 (or on X). A function f is concave if and only if $-f$ is convex. Other definitions follow similarly.

A new class of generalized convex functions, the so-called invex functions, introduced by Hason [21] with the aim of extending the validity of the sufficiency of the Karush-Kuhn-Tucker conditions. It is obvious that the the particular case of (differentiable) convex function is obtained from by choosing $\eta(x_1, y) = x_1 - x_2$. The term invex was created by Craven [10].

(*vii*) f is said to be **invex** if there exists a function $\eta : X \times X \to R^n$ such that for all $x_1, x_2 \in X$.

$$f(x_1) - f(x_2) \geqq \eta(x_1, x_2)^T \nabla f(x_2).$$

(*viii*) f is said to be **pseudoinvex** if there exists a function $\eta : X \times X \to R^n$ such that for all $x_1, x_2 \in X$

$$\eta(x_1, x_2)^T \nabla f(x_2) \geqq 0 \Rightarrow f(x_1) \geqq f(x_2).$$

(*ix*) A functional $F : X \times X \times R^n \to R$ is sublinear in the third variable, if for all $x_1, x_2 \in X$

(*a*) $F(x_1, x_2; f_1 + f_2) \leqq F(x_1, x_2; f_1) + F(x_1, x_2, f_2)$, for all $f_1, f_2 \in R^n$,

(*b*) $F(x_1, x_2; \alpha a) = \alpha F(x_1, x_2; a)$,for all $\alpha \in R_+$ and $a \in R^n$.

Also we write it as

$$F_{x_1,x_2}(a) = F(x_1, x_2; a).$$

(*x*) f is said to be F-**convex** at x_2 if for all $x_1 \in X$,

$$f(x_1) - f(x_2) \geqq F(x_1, x_2; \nabla f(x_2)).$$

(xi) f is said to be F-**pseudoconvex** at x_2 if for all $x_1 \in X$,

$$F(x_1, x_2; \nabla f(x_2)) \geqq 0 \Rightarrow f(x_1) \geqq f(x_2).$$

(xii) f is said to be (F, ρ)-**convex** at x_2 if there exists $d : X \times X \to R$ and $\rho \in R$ then for all $x_1 \in X$

$$F(x_1, x_2; \nabla f(x_2)) + \rho d^2(x_1, x_2) \leqq f(x_1) - f(x_2).$$

$(xiii)$ f is said to be (F, ρ)-**pseudoconvex** at x_2, if for all $x_1 \in S$

$$F(x_1, x_2; \nabla f(x_2)) \geqq -\rho d^2(x_1, x_2) \Rightarrow f(x_1) \geqq f(x_2),$$

or equivalently

$$f(x_1) < f(x_2) \Rightarrow F(x_1, x_2; \nabla f(x_2)) < -\rho d^2(x_1, x_2).$$

(xiv) f is said to be (F, α, ρ, d)-**convex** at $x_2 \in X$ if there exists a function $\alpha : X \times X \to R_+ \setminus \{0\}, d : X \times X \to R$ and $\rho \in R$ then for all $x_1 \in X$

$$f(x_1) - f(x_2) \geqq F(x_1, x_2; \alpha(x_1, x_2)\nabla f(x_2)) + \rho d^2(x_1, x_2).$$

1.2 Review of related work

1.2.1 Symmetric Duality

In mathematical programming, a pair of primal and dual problems is called symmetric if the dual of the dual is the primal problem. Duality in linear programming is always symmetric but in non-linear programming duality is not always symmetric.

The notion of symmetric duality in which the dual of the dual is the primal, found its way in nonlinear programming in the excellent work of Dorn [14]. Dantzig et al. [11] formulated the following pair of symmetric dual programs and established weak and strong duality theorems.

$$\begin{aligned}
\text{(PS)}\quad \text{Minimize} \quad & F(x, y) = M(x, y) - y^T \nabla_y M(x, y) \\
\text{subject to} \quad & \nabla_y M(x, y) \leqq 0, \\
& x \geqq 0, \\
& y \geqq 0,
\end{aligned}$$

$$
\begin{array}{lll}
\text{(DS)} & \text{Maximize} & G(u,v) = M(u,v) - u^T \nabla_x M(u,v) \\
& \text{subject to} & \nabla_x M(u,v) \leqq 0, \\
& & u \geqq 0, \\
& & v \geqq 0,
\end{array}
$$

where $M : R^n \times R^m \to R$ is twice differentiable function.
Mond and Weir[32] considered the following pair of symmetric dual programs

$$
\begin{array}{lll}
\text{(PM)} & \text{Minimize} & M(x,y) \\
& \text{subject to} & \nabla_y M(x,y) \leqq 0, \\
& & y^T \nabla_y M(x,y) \geqq 0, \\
& & x \geqq 0.
\end{array}
$$

$$
\begin{array}{lll}
\text{(DM)} & \text{Maximize} & M(u,v) \\
& \text{subject to} & \nabla_x M(u,v) \geqq 0, \\
& & u^T \nabla_x M(u,v) \leqq 0, \\
& & v \geqq 0.
\end{array}
$$

Symmetric duality was generalized to multiobjective case by Weir and Mond [30] and as well as by Gulati et al.[17].

Bazaraa and Goode [5] generalized the results in [11] to arbitrary cones. They studied the Wolfe's type symmetric dual pair over arbitrary cones:

$$
\begin{array}{lll}
\text{(PC)} & \text{Minimize} & F(x,y) = M(x,y) - y^T \nabla_y M(x,y) \\
& \text{subject to} & \nabla_y M(x,y) \in C_2^*, \\
& & (x,y) \in C_1 \times C_2,
\end{array}
$$

$$
\begin{array}{lll}
\text{(DC)} & \text{Maximize} & G(u,v) = M(u,v) - u^T \nabla_x M(u,v) \\
& \text{subject to} & -\nabla_x M(u,v) \in C_1^*, \\
& & (u,v) \in C_1 \times C_2,
\end{array}
$$

where

(i) C_1 and C_2 are closed convex cones with non-empty interiors in R^n and R^m, respectively.

(ii) For $i = 1, 2, C_i^*$ is the polar of C_i.

(iii) $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$ and $M : S_1 \times S_2 \to R$ is a twice differentiable function.

Chandra and Kumar [7] formulated the following Mond-Weir type symmetric dual programs over arbitrary cones:

$$\begin{aligned}
\text{(PMC)} \quad \text{Minimize} \quad & M(x, y) \\
\text{subject to} \quad & \nabla_y M(x, y) \in C_2^*, \\
& y^T \nabla_y M(x, y) \geqq 0, \\
& x \in C_1.
\end{aligned}$$

$$\begin{aligned}
\text{(DMC)} \quad \text{Maximize} \quad & M(u, v) \\
\text{subject to} \quad & -\nabla_x M(u, v) \in C_1^*, \\
& u^T \nabla_x M(u, v) \leqq 0, \\
& v \in C_2.
\end{aligned}$$

and proved usual duality theorems under pseudoinvexity type assumptions.

Das and Nanda [12] studied symmetric duality in multiobjective programming with cone constraints. Subsequently, Kim et al.[24] derived symmetric duality results for multiobjective programs under pseudoinvex/strictly pseudoinvex functions and arbitrary cones.

1.2.2 Second order Duality

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used. Mond [28] considered the following second-order symmetric dual programs:

$$\begin{aligned}
\text{Minimize} \quad & M(x, y) - y^T \nabla_y M(x, y) - y^T \nabla_{yy} M(x, y)p - \frac{1}{2} p^T \nabla_{yy} M(x, y)p \\
\text{subject to} \quad & \nabla_y M(x, y) + \nabla_{yy} M(x, y)p \leqq 0, \\
& x \geqq 0.
\end{aligned}$$

$$\begin{aligned} \text{Maximize} \quad & M(x,y) - x^T\nabla_x M(x,y) - x^T\nabla_{xx}M(x,y)r - \frac{1}{2}r^T\nabla_{xx}M(x,y)r \\ \text{subject to} \quad & \nabla_x M(x,y) + \nabla_{xx}M(x,y)r \geqq 0, \\ & y \geqq 0. \end{aligned}$$

Ahmad and Husain [3] formulated the following-pair of Wolfe type multiobjective second order symmetric dual programs with cone constraints:

Primal Problem(WP)

$$\begin{aligned} \text{Minimize} \quad & f(x,y) - [y^T\nabla_y(\lambda^T f)(x,y)]e - [y^T\nabla_{yy}(\lambda^T f)(x,y)p]e - \frac{1}{2}[p^T\nabla_{yy}(\lambda^T f)(x,y)p]e \\ \text{subject to} \quad & \nabla_y(\lambda^T f)(x,y) + \nabla_{yy}(\lambda^T f)(x,y)p \in C_2^*, \\ & x \in C_1, \\ & \lambda > 0, \lambda^T e = 1. \end{aligned}$$

Dual Problem(WD)

$$\begin{aligned} \text{Maximize} \quad & f(u,v) - [u^T\nabla_x(\lambda^T f)(u,v)]e - [u^T\nabla_{xx}(\lambda^T f)(u,v)q]e - \frac{1}{2}[q^T\nabla_{xx}(\lambda^T f)(u,v)q]e \\ \text{subject to} \quad & -\nabla_x(\lambda^T f)(u,v) - \nabla_{xx}(\lambda^T f)(u,v)q \in C_1^*, \\ & v \in C_2, \\ & \lambda > 0, \lambda^T e = 1. \end{aligned}$$

where $f(x,y) : S_1 \times S_2 \to R^k (S_1 \subseteq R^n, S_2 \subseteq R^m), p \in R^m, q \in R^n, \lambda \in R^k$ and $e = (1,1,\ldots,1)^T \in R^k$, and usual duality results are established under second-order invexity assumptions.

1.2.3 Higher order Duality

Higher order duality in nonlinear programming has been studied by many researchers. By introducing two differentiable functions $h : R^n \times R^n \to R$ and $k : R^n \times R^n \to R^m$. Mangasarian [26] formulated the following higher order dual:

Primal problem(P5)

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{subject to} \quad & g(x) \leqq 0. \end{aligned}$$

Dual problem(D5)

$$\begin{aligned} \text{Maximize} \quad & f(x) + h(x,p) + y^T g(x) + y^T k(x,p) \\ \text{subject to} \quad & \nabla_p h(x,p) + \nabla_p y^T k(x,p) = 0, \\ & y \geqq 0. \end{aligned}$$

where $f : R^n \to R$ and $g : R^n \to R^m, \nabla_p h(u,p)$ denotes the gradient of h with respect to p and $\nabla_p(y^T k(u,p))$ denotes the gradient of $y^T k$ with respect to p

1.3 Summary of the thesis

The aim of the present thesis is to study the mixed symmetric duality in multiobjective programming problems for some dual mathematical programming problems under generalized convexity assumptions.

In Chapter 2, we have reviewed the following new pair of multiobjective second-order symmetric dual programs over arbitrary cones considered by Gupta et al[20]:

Primal problem (WP)

K-minimize

$$\begin{aligned} L(x,y,\lambda,p) \;=\; & \Bigg(f_1(x,y) - y^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \\ & -\frac{1}{2}\sum_{i=1}^{k} \lambda_i(p_i^T \nabla_{yy} f_i(x,y)p_i), \\ & \ldots, f_k(x,y) - y^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \\ & -\frac{1}{2}\sum_{i=1}^{k} \lambda_i(p_i^T \nabla_{yy} f_i(x,y)p_i)\Bigg) \end{aligned}$$

subject to

$$\begin{aligned} & -\sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \in C_2^*, \\ & \lambda^T e_k = 1, \\ & \lambda \in \text{int}K^*, x \in C_1. \end{aligned}$$

Dual problem (WD)

K-maximize

$$M(u,v,\lambda,q) = \Big(f_1(u,v) - u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{xx}f_i(u,v)q_i) - \frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx}f_i(u,v)q_i), \ldots, f_k(u,v) - u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{xx}f_i(u,v)q_i) - \frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx}f_i(u,v)q_i)\Big)$$

subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{xx}f_i(u,v)q_i) \in C_1^*,$$
$$\lambda^T e_k = 1,$$
$$\lambda \in \text{int}K^*, v \in C_2,$$

where

(i) $f_i : S_1 \times S_2 \to R, i = 1,2,\ldots,k$ is a differentiable function of x and y,$e_k = (1,\ldots,1)^T \in R^k$,

(ii) q_i and p_i are vectors in R^n and R^m, respectively, for $i = 1,2,\ldots,k$ and $\lambda \in R^k$.

and illustrate an example which is K-η-bonvex but not invex. We also study duality theorems weak, strong, converse and self duality under K-η-bonvexity assumptions.

In Chapter3, we have studied the following pair of multiobjective mixed second-order nondifferentiable symmetric dual programs over arbitrary cones considered by Gupta et al.[19]:

Primal Problem(P)

K-minimize

$$N(x^1,y^1,x^2,y^2,w^2,\lambda,p,r) = \{N_1(x^1,y^1,x^2,y^2,w_1^2,\lambda,p,r^1),\ldots, N_l(x^1,y^1,x^2,y^2,w_l^2,\lambda,p,r^l)\}$$

subject to

$$-\sum_{i=1}^{l}\lambda_i\big[\nabla_{y^1}f_i(x^1,y^1)-w^1+\nabla_{y^1y^1}f_i(x^1,y^1)p_i\big]\in C_3^*,$$
$$-\sum_{i=1}^{l}\lambda_i\big[\nabla_{y^2}g_i(x^2,y^2)-w_i^2+\nabla_{y^2y^2}g_i(x^2,y^2)r^i\big]\in C_4^*,$$
$$(y^2)^T\sum_{i=1}^{l}\lambda_i(\nabla_{y^2}g_i(x^2,y^2)-w_i^2+\nabla_{y^2y^2}g_i(x^2,y^2)r^i)\geqq 0,$$
$$w^1\in E,\ w_i^2\in B_i,\ i=1,2,\ldots,l$$
$$\lambda^Te_l=1,\ x^1\in C_1,\ x^2\in C_2,$$
$$\lambda\in \text{int}K^*.$$

Dual Problem(D)

K-maximize

$$M(u^1,v^1,u^2,v^2,z^2,\lambda,q,s)=\{M_1(u^1,v^1,u^2,v^2,z_1^2,\lambda,q,s^1),\ldots,$$
$$M_l(u^1,v^1,u^2,v^2,z_l^2,\lambda,q,s^l)\}$$

subject to

$$\sum_{i=1}^{l}\lambda_i\big[\nabla_{x^1}f_i(u^1,v^1)+z^1+\nabla_{x^1x^1}f_i(u^1,v^1)q_i\big]\in C_1^*,$$
$$\sum_{i=1}^{l}\lambda_i\big[\nabla_{x^2}g_i(u^2,v^2)+z_i^2+\nabla_{x^2x^2}g_i(u^2,v^2)s^i\big]\in C_2^*,$$
$$(u^2)^T\sum_{i=1}^{l}\lambda_i(\nabla_{x^2}g_i(u^2,v^2)+z_i^2+\nabla_{x^2x^2}g_i(u^2,v^2)s^i)\leqq 0,$$
$$z^1\in H,\ z_i^2\in D_i,\ i=1,2,\ldots,l$$
$$\lambda^Te_l=1,\ v^1\in C_3,\ v^2\in C_4,$$
$$\lambda\in \text{int}K^*.$$

where $\lambda\in R^l$, $e_l=(1,\ldots,1)^T\in R^l$, $w^1\in R^{|K_1|}$ and $z^1\in R^{|J_1|}$, H and E are compact and convex sets in $R^{|J_1|}$ and $R^{|K_1|}$, respectively and for $i=1,2,\ldots,l$

$$
\begin{aligned}
N_i(x^1, y^1, x^2, y^2, w_i^2, \lambda, p, r^i) \;=\;& f_i(x^1, y^1) + s(x^1|H) + g_i(x^2, y^2) + s(x^2|D_i) - (y^2)^T \\
& \times w_i^2 - (y^1)^T \sum_{i=1}^{l} \lambda_i(\nabla_{y^1} f_i(x^1, y^1) + \nabla_{y^1y^1} f_i(x^1, y^1) \\
& p_i) - \frac{1}{2}\sum_{i=1}^{l} \lambda_i\big[p_i^T \nabla_{y^1y^1} f_i(x^1, y^1)p_i\big] - \frac{1}{2}(r^i)^T \nabla_{y^2y^2} g_i(x^2, y^2)r^i,
\end{aligned}
$$

$$
\begin{aligned}
M_i(u^1, v^1, u^2, v^2, z_i^2, \lambda, q, s^i) \;=\;& f_i(u^1, v^1) - s(v^1|E) + g_i(u^2, v^2) - s(v^2|B_i) + (u^2)^T \\
& z_i^2 - (u^1)^T \sum_{i=1}^{l} \lambda_i(\nabla_{x^1} f_i(u^1, v^1) + \nabla_{x^1x^1} f_i(u^1, v^1)q_i) \\
& -\frac{1}{2}\sum_{i=1}^{l} \lambda_i\big[q_i^T \nabla_{x^1x^1} f_i(u^1, v^1)q_i\big] - \\
& \frac{1}{2}(s^i)^T \nabla_{x^2x^2} g_i(u^2, v^2)s^i,
\end{aligned}
$$

and

(*i*) $f_i : R^{|J_1|} \times R^{|K_1|} \to R$ and $g_i : R^{|J_2|} \times R^{|K_2|} \to R$ are differentiable functions,

(*ii*) B_i and D_i are compact and convex sets in $R^{|K_2|}$ and $R^{|J_2|}$, respectively,

(*iii*) $p_i \in R^{|K_1|}$, $q_i \in R^{|J_1|}$, r^i, $w_i^2 \in R^{|K_2|}$, and s^i, $z_i^2 \in R^{|J_2|}$.

Let $w^2 = (w_1^2, \ldots, w_l^2), p = (p^1, \ldots, p^l), q = (q^1, \ldots, q^l), r = (r^1, \ldots, r^l), s = (s^1, \ldots, s^l)$ and $z^2 = (z_1^2, \ldots, z_l^2)$.
and illustrate the example of second-order K-(F, ρ)-convex function which is not second-order F-convex. We also study duality theorems weak, strong and converse under the assumptions of second-order K-(F, ρ)-convexity.
In chapter4, we have discussed the following pair of multiobjective mixed higher order symmetric dual programs:

Primal problem (HMP)

K-minimize

$$\begin{aligned} L(x^1,x^2,y^1,y^2,p,q) &= \Big\{ f_1^1(x^1,y^1) + h_1^1(x^1,y^1,p_1) - p_1^T\nabla_{p_1}h_1^1(x^1,y^1,p_1) - (y^1)^T\sum_{i=1}^{l}\lambda_i \\ & \big[\nabla_{y^1}f_i^1(x^1,y^1) + \nabla_{p_i}h_i^1(x^1,y^1,p_i)\big] + f_1^2(x^2,y^2) + h_1^2(x^2,y^2,q_1) \\ & -q_1^T\nabla_{q_1}h_1^2(x^2,y^2,q_1), \ldots, f_l^1(x^1,y^1) + h_l^1(x^1,y^1,p_l) - \\ & p_l^T\nabla_{p_l}h_l^1(x^1,y^1,p_l) - (y^1)^T\sum_{i=1}^{l}\lambda_i\big[\nabla_{y^1}f_i^1(x^1,y^1) + \nabla_{p_i}h_i^1(x^1,y^1,p_i)\big] \\ & +f_l^2(x^2,y^2) + h_l^2(x^2,y^2,q_l) - q_l^T\nabla_{q_l}h_l^2(x^2,y^2,q_l)\Big\} \end{aligned}$$

subject to

$$-\Big(\sum_{i=1}^{l}\lambda_i\big[\nabla_{y^1}f_i^1(x^1,y^1) + \nabla_{p_i}h_i^1(x^1,y^1,p_i)\big]\Big) \in C_2^*$$

$$-\Big(\sum_{i=1}^{l}\lambda_i\big[\nabla_{y^2}f_i^2(x^2,y^2) + \nabla_{q_i}h_i^2(x^2,y^2,q_i)\big]\Big) \in C_4^*$$

$$(y^2)^T\sum_{i=1}^{l}\lambda_i\Big[\nabla_{y^2}f_i^2(x^2,y^2) + \nabla_{q_i}h_i^2(x^2,y^2,q_i)\Big] \geqq 0$$

$$x^1 \in C_1, x^2 \in C_3, \lambda > 0 \text{ and } \lambda^T e = 1$$

Dual problem(HMD)

K-maximize

$$\begin{aligned} M(u^1,u^2,v^1,v^2,r,s) &= \Big\{ f_1^1(u^1,v^1) + g_1^1(u^1,v^1,r_1) - r_1^T\nabla_{r_1}g_1^1(u^1,v^1,r_1) - (u^1)^T\sum_{i=1}^{l}\lambda_i \\ & \big[\nabla_{x^1}f_i^1(u^1,v^1) + \nabla_{r_i}g_i^1(u^1,v^1,r_i)\big] + f_1^2(u^2,v^2) + g_1^2(u^2,v^2,s_1) \\ & -s_1^T\nabla_{s_1}g_1^2(u^2,v^2,s_1), \ldots, f_l^1(u^1,v^1) + g_l^1(u^1,v^1,r_l) \\ & -r_l^T\nabla_{r_l}g_l^1(u^1,v^1,r_l) - (u^1)^T\sum_{i=1}^{l}\lambda_i\big[\nabla_{x^1}f_i^1(u^1,v^1) + \nabla_{r_i}g_i^1(u^1,v^1,r_i)\big] \\ & +f_l^2(u^2,v^2) + g_l^2(u^2,v^2,s_l) - s_l^T\nabla_{s_l}g_l^2(u^2,v^2,s_l)\Big\} \end{aligned}$$

subject to

$$\sum_{i=1}^{l} \lambda_i \big[\nabla_{x^1} f_i^1(u^1, v^1) + \nabla_{r_i} g_i^1(u^1, v^1, r_i)\big] \in C_1^*$$
$$\sum_{i=1}^{l} \lambda_i \big[\nabla_{x^2} f_i^2(u^2, v^2) + \nabla_{s_i} g_i^2(u^2, v^2, s_i)\big] \in C_3^*$$
$$(u^2)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{x^2} f_i^2(u^2, v^2) + \nabla_{s_i} g_i^2(u^2, v^2, s_i)\big] \leqq 0$$
$$v^1 \in C_2, v^2 \in C_4, \lambda > 0 \text{ and } \lambda^T e = 1.$$

where

(*i*) $f_i^1 : R^{|J_1|} \times R^{|K_1|} \to R$, $f_i^2 : R^{|J_2|} \times R^{|K_2|} \to R$,
$g_i^1 : R^{|J_1|} \times R^{|K_1|} \times R^{|K_1|} \to R$, $g_i^2 : R^{|J_2|} \times R^{|K_2|} \times R^{|K_2|} \to R$,
$h_i^1 : R^{|J_1|} \times R^{|K_1|} \times R^{|J_1|} \to R$, $h_i^2 : R^{|J_2|} \times R^{|K_2|} \times R^{|J_2|} \to R$,
$i = 1, ..., l$, are differentiable functions.

(*ii*) C_1, C_2, C_3, C_4 are closed convex cones in $R^{|J_1|}, R^{|K_1|}, R^{|J_2|}, R^{|K_2|}$, with non-empty interiors, respectively.

(*iii*) $C_1^*, C_2^*, C_3^*, C_4^*$ are polar cones of C_1, C_2, C_3, and C_4, respectively.

(*iv*) $p_i \in R^{|J_1|}, r_i \in R^{|K_1|}, q_i \in R^{|J_2|}, s_i \in R^{|K_2|}$.

(*v*) $e = (1, ..., 1)^T$ is a vector in R^l.

and prove weak duality theorem under the assumption of higher order (F, ρ, α, d)-convexity.

Chapter 2

Second-Order multiobjective symmetric duality involving cone-bonvex functions

2.1 Introduction

Duality in linear programming is symmetric, that is the dual of the dual is the primal problem. However, the majority of dual formulations in nonlinear programming do not possess this property. Symmetric dual programs have applications in the theory of games.The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used. Dantzig et al. [11] formulated a pair of symmetric dual nonlinear programs and discussed duality results involving convex/concave functions. The same results were later generalized by Bazaraa and Goode [5] to arbitrary cones. Mond and Weir [32] presented a distinct pair of symmetric dual nonlinear programs which admits the relaxation of the convexity/concavity assumption to pseudoconvexity/pseudoconcavity.

In this chapter, we studied duality theorems (weak, strong, converse and self duality) for a new pair of second-order multiobjective symmetric dual programs over arbitrary cones under the assumptions of K-η-bonvexity. This Chapter is divided into four sections. Section 2.1 is introductory, Section 2.2 contains notations and definitions. After that we described an example which is K-η-bonvex but not invex .In Section 2.3, we consider a new pair of multiobjective second order symmetric dual programs over arbitrary cones and prove weak, strong and converse duality theorems under K-η-bonvexity assumptions. Self duality is discussed in Section 2.4.

2.2 Notations and Definitions

Consider the following multiobjective programming problem:

$$(P) \qquad \text{K-minimize } \phi(x)$$
$$\text{subject to } x \in X^0 = \{x \in S : -g(x) \in Q\},$$

where $S \subseteq R^n, \phi : S \to R^k, g : S \to R^m$, K is a closed convex pointed cone in R^k with $\text{int}K \neq \phi$ and Q is a closed convex cone with a nonempty interior in R^m.

Multiobjective programming also known as multi-criteria or multi-attribute optimization, is the process of simultaneously optimizing two or more conflicting objectives subject to certain constraints. It has a large number of applications. As an example, it is generally used in goal programming, risk programming etc. Optimality conditions for multiobjective programming problems can be found in Miettinen [27] and Pardalos et al. [36]. Recently, Chinchuluun and Pardalos [9] discussed recent developments in multiobjective optimization. These include optimality conditions, applications, global optimization techniques, the new concept of epsilon pareto optimal solutions and heuristics.

Suppose that $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$.

Definition 2.1[18]A twice differentiable function $f : S_1 \times S_2 \mapsto R^k$ is said to be K-η_1-bonvex in the first variable at $u \in S_1$ for fixed $v \in S_2$, if there exists a function $\eta_1 : S_1 \times S_1 \mapsto R^n$ such that for $x \in S_1, q_i \in R^n, i = 1, 2, \ldots, k$,

$$\Big\{ f_1(x,v) - f_1(u,v) + \frac{1}{2} q_1^T \nabla_{xx} f_1(u,v) q_1 - \eta_1^T(x,u)[\nabla_x f_1(u,v) + \nabla_{xx} f_1(u,v) q_1], \ldots$$
$$f_k(x,v) - f_k(u,v) + \frac{1}{2} q_k^T \nabla_{xx} f_k(u,v) q_k - \eta_1^T(x,u)[\nabla_x f_k(u,v) + \nabla_{xx} f_k(u,v) q_k] \Big\} \in K,$$

and $f(x,y)$ is said to be K-η_2-bonvex in the second variable at $v \in S_2$ for fixed $u \in S_1$, if there exists a function $\eta_2 : S_2 \times S_2 \mapsto R^m$ such that for $y \in S_2, p_i \in R^m, 1, 2. \ldots, k$,

$$\left\{f_1(u,y) - f_1(u,v) + \frac{1}{2}p_1^T \nabla_{yy} f_1(u,v) p_1 - \eta_2^T(y,v)[\nabla_y f_1(u,v) + \nabla_{yy} f_1(u,v) p_1], \ldots, \right.$$

$$\left. f_k(u,y) - f_k(u,v) + \frac{1}{2}p_k^T \nabla_{yy} f_k(u,v) p_k - \eta_2^T(y,v)[\nabla_y f_k(u,v) + \nabla_{yy} f_k(u,v) p_k]\right\} \in K.$$

Remark 1

(*i*) If $K = R_+, p_1 = 0$ and $q_1 = 0$, then the above definition reduces to η-convexity/invexity in [22, 31].

(*ii*) If $K = R_+$, then the K-η-bonvexity becomes the η-bonvexity [16, 39].

(*iii*) Eliminating the second-order terms and substituting $\eta_1(x,u) = (x - u)$ and $\eta_2(v,y) = (v - y)$, the definition reduces to the K-convexity as in [38].

Example 2.1 Let $X = [2, 2.35] \subset R$, $n = m = 1$ and $k = 2$. Consider the function $f : X \to R^2$ be defined by $f(x) = (f_1(x), f_2(x))$, where $f_1(x) = x^3 \sin \frac{2}{x}$, $f_2(x) = \sin^2 x$. Let $K = \{(x,y) : x \geq 4y, x \geq 0\}$ and $\eta : X \times X \to R$ be defined by $\eta(x,u) = 12(1 - u)$. To show that f is K-η-bonvex, we have to prove that

$$\left(f_1(x) - f_1(u) + \frac{1}{2}q_1^T(\nabla_{xx} f_1(u) q_1) - \eta^T(x,u)[\nabla_x f_1(u) + \nabla_{xx} f_1(u) q_1], \right.$$

$$\left. f_2(x) - f_2(u) + \frac{1}{2}q_2^T(\nabla_{xx} f_2(u) q_2) - \eta^T(x,u)[\nabla_x f_2(u) + \nabla_{xx} f_2(u) q_2]\right) \in K,$$

or

$$\left(x^3 \sin\frac{2}{x} - u^3 \sin\frac{2}{u} + q_1^2\left[3u \sin\frac{2}{u} - \frac{2}{u}\sin\frac{2}{u} - 4\cos\frac{2}{u}\right] - 12(1-u)\right.$$

$$\times\left[-2u\cos\frac{2}{u} + 3u^2 \sin\frac{2}{u} + q_1\left(6u\sin\frac{2}{u} - \frac{4}{u}\sin\frac{2}{u} - 8\cos\frac{2}{u}\right)\right], \sin^2 x - \sin^2 u$$

$$\left. + q_2^2 \cos 2u - 12(1-u)[\sin 2u + 2q_2 \cos 2u]\right) \in K.$$

Let

$$\begin{aligned}
\phi &= x^3 \sin\frac{2}{x} - u^3 \sin\frac{2}{u} + q_1^2\left[3u\sin\frac{2}{u} - \frac{2}{u}\sin\frac{2}{u} - 4\cos\frac{2}{u}\right] \\
&\quad -12(1-u)\left[-2u\cos\frac{2}{u} + 3u^2\sin\frac{2}{u} + q_1\left(6u\sin\frac{2}{u} - \frac{4}{u}\sin\frac{2}{u} - 8\cos\frac{2}{u}\right)\right] \\
&= \phi_1 + \phi_2(\text{say}), \\
\psi &= \sin^2 x - \sin^2 u + q_2^2\cos 2u - 12(1-u)[\sin 2u + 2q_2\cos 2u]. \\
&= \psi_1 + \psi_2(\text{say}),
\end{aligned}$$

where

$$\begin{aligned}\phi_1 &= x^3 \sin\frac{2}{x} - u^3 \sin\frac{2}{u} - 12(1-u)\left[-2u\cos\frac{2}{u} + 3u^2 \sin\frac{2}{u}\right] \\ &\geq 0 \forall x, u \in X,\end{aligned}$$

as can be seen from Figure.2.1.

$$\begin{aligned}\phi_2 &= q_1^2\left[3u\sin\frac{2}{u} - \frac{2}{u}\sin\frac{2}{u} - 4\cos\frac{2}{u}\right] - 12(1-u)\left[q_1\left(6u\sin\frac{2}{u} - \frac{4}{u}\sin\frac{2}{u} - 8\cos\frac{2}{u}\right)\right] \\ &\geq 0 \ \forall \ u \in X \text{ and } q_1 \in (-10^{18}, 10^{18})\end{aligned}$$

as can be seen from Figure.2.2.

$$\begin{aligned}\psi_1 &= \sin^2 x - \sin^2 u - 12(1-u)\sin 2u \\ &< 0 \ \forall x, u \in X\end{aligned}$$

as can be seen from Figure.2.3 and

$$\begin{aligned}\psi_2 &= q_2^2 \cos 2u - 24(1-u) q_2 \cos 2u \\ &\leq 0 \forall u \in X \text{ and } q_2 \in (-10^{18}, 10^{18})\end{aligned}$$

as can be seen from Figure.2.4.

Hence $\phi \geq 0$ and $\psi < 0$. This implies $\phi - 4\psi \geq 0$ and $\phi \geq 0$. Thus $(\phi, \psi) \in K$ and hence f is K-η-bonvex function.

Next, we show that f is not invex. To prove this, either

$$f_1(x) - f_1(u) - \eta^T(x, u)\nabla_x f_1(u) \ngeq 0$$

or

$$f_2(x) - f_2(u) - \eta^T(x, u)\nabla_x f_2(u) \ngeq 0.$$

Since $f_2(x) - f_2(u) - \eta^T(x, u)\nabla_x f_2(u) = \sin^2 x - \sin^2 u - 12(1-u)\sin 2u < 0 \forall x, u \in X$, as can be seen from Fig.2.3.

Therefore, f is not invex. Hence f is K-η-bonvex but not invex.

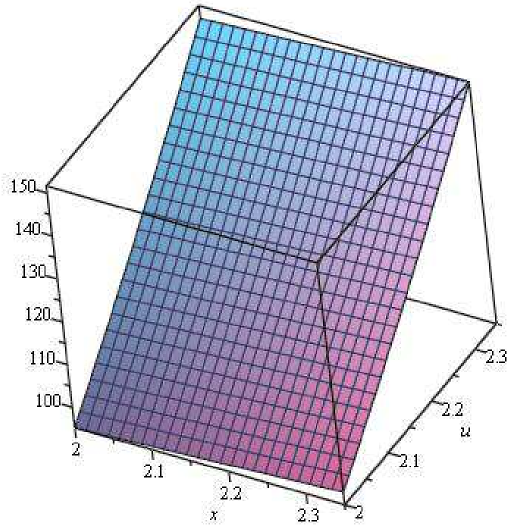

Figure 2.1: The graph of $\phi_1 = x^3 \sin \frac{2}{x} - u^3 \sin \frac{2}{u} - 12(1-u)[-2u \cos \frac{2}{u} + 3u^2 \sin \frac{2}{u}]$

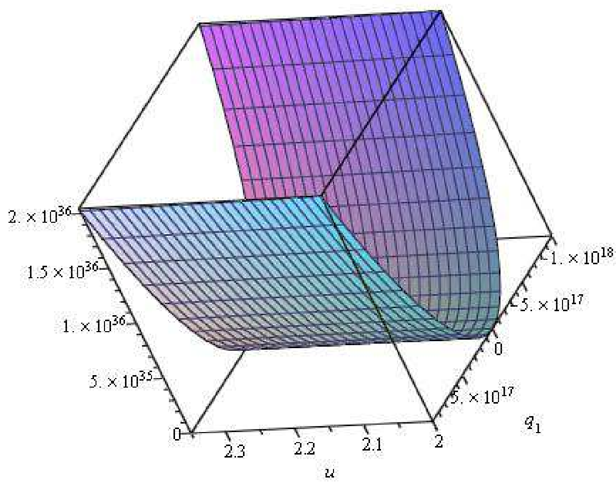

Figure 2.2: The graph of $\phi_2 = q_1^2[3u \sin \frac{2}{u} - \frac{2}{u} \sin \frac{2}{u} - 4 \cos \frac{2}{u}] - 12(1-u)q_1(6u \sin \frac{2}{u} - \frac{4}{u} \sin \frac{2}{u} - 8 \cos \frac{2}{u})$

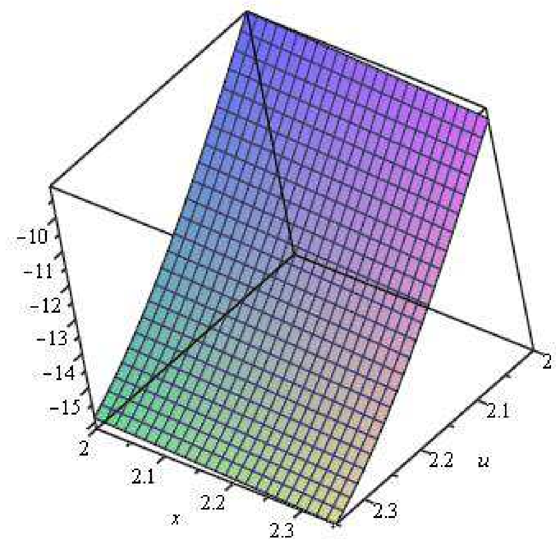

Figure 2.3: The graph of $\psi_1 = \sin^2 x - \sin^2 u - 12(1-u)\sin 2u$

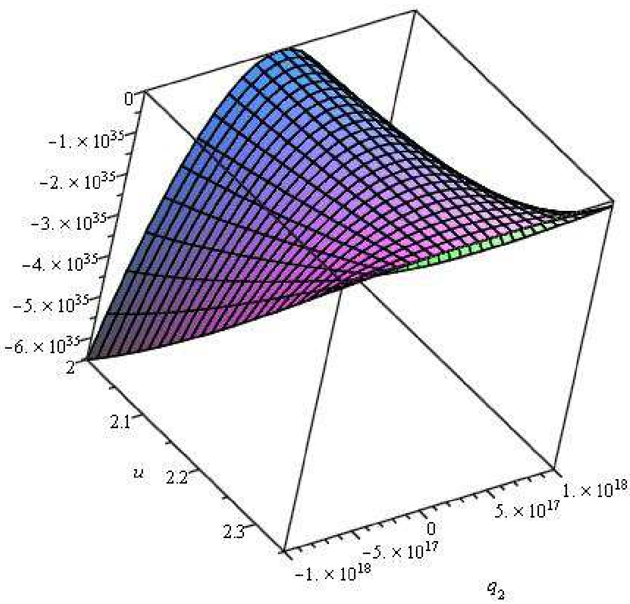

Figure 2.4: The graph of $\psi_2 = q_2^2 \cos 2u - 24(1-u)q_2 \cos 2u$

2.3 Multiobjective second-order symmetric dual programs

Consider the following pair of multiobjective second-order symmetric dual programs over arbitrary cones:

Primal problem (WP)

K-minimize

$$\begin{aligned} L(x,y,\lambda,p) \;=\; & \Big(f_1(x,y) - y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \\ & -\frac{1}{2}\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{yy} f_i(x,y)p_i), \\ & \ldots, f_k(x,y) - y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \\ & -\frac{1}{2}\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{yy} f_i(x,y)p_i)\Big) \end{aligned}$$

subject to

$$-\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \in C_2^*, \tag{2.1}$$

$$\lambda^T e_k = 1, \tag{2.2}$$

$$\lambda \in \operatorname{int}K^*, x \in C_1. \tag{2.3}$$

Dual problem (WD)

K-maximize

$$\begin{aligned} M(u,v,\lambda,q) \;=\; & \Big(f_1(u,v) - u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \\ & -\frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx} f_i(u,v)q_i), \\ & \ldots, f_k(u,v) - u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \\ & -\frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx} f_i(u,v)q_i)\Big) \end{aligned}$$

subject to

$$\sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) \in C_1^*, \tag{2.4}$$

$$\lambda^T e_k = 1, \tag{2.5}$$

$$\lambda \in \text{int}K^*, v \in C_2, \tag{2.6}$$

where

(i) $f_i : S_1 \times S_2 \to R, i = 1, 2, \ldots, k$ is a differentiable function of x and $y, e_k = (1, \ldots, 1)^T \in R^k$,

(ii) q_i and p_i are vectors in R^n and R^m, respectively, for $i = 1, 2, \ldots, k$ and $\lambda \in R^k$.

Theorem 2.1 (Weak duality) Let (x, y, λ, p) be feasible for (WP) and (u, v, λ, q) be feasible for (WD). Let

(i) $f(., v)$ be K-η_1-bonvex in the first variable at u,

(ii) $-f(x, .)$ be K-η_2-bonvex in the second variable at y,

(iii) $\eta_1(x, u) + u \in C_1$, $\forall\ x \in C_1$,

(iv) $\eta_2(v, y) + y \in C_2$, $\forall\ v \in C_2$,

Then

$$L(x, y, \lambda, p) - M(u, v, \lambda, q) \notin -K \setminus \{0\}.$$

Proof Suppose, to the contrary, that

$$L(x, y, \lambda, p) - M(u, v, \lambda, q) \in -K \setminus \{0\}.$$

that is,

$$\Bigg\{\Bigg[f_1(x,y) - y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - \frac{1}{2}\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{yy} f_i(x,y)p_i),$$

$$\ldots, f_k(x,y) - y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - \frac{1}{2}\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{yy} f_i(x,y)p_i)\Bigg]$$

$$-\Bigg[f_1(u,v) - u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - \frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx} f_i(u,v)q_i),$$

$$\ldots, f_k(u,v) - u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - \frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx} f_i(u,v)q_i)\Bigg]\Bigg\}$$

$$\in -K \setminus \{0\}.$$

Since $\lambda \in \text{int } K^*$ and $\lambda \neq 0$, we obtain

$$\sum_{i=1}^{k}\lambda_i\Bigg\{f_i(x,y) - y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - \frac{1}{2}\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{yy} f_i(x,y)p_i)$$

$$-(f_i(u,v) - u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i)$$

$$-\frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx} f_i(u,v)q_i))\Bigg\} < 0. \tag{2.7}$$

By K-η_1-bonvexity of $f(.,v)$, we have

$$\Bigg\{f_1(x,v) - f_1(u,v) + \frac{1}{2}q_1^T\nabla_{xx} f_1(u,v)q_1 - \eta_1^T(x,u)[\nabla_x f_1(u,v) + \nabla_{xx} f_1(u,v)q_1], \ldots,$$

$$f_k(x,v) - f_k(u,v) + \frac{1}{2}q_k^T\nabla_{xx} f_k(u,v)q_k - \eta_1^T(x,u)[\nabla_x f_k(u,v) + \nabla_{xx} f_k(u,v)q_k]\Bigg\} \in K.$$

Using $\lambda \in \text{int}K^*$, we get

$$\sum_{i=1}^{k}\lambda_i\Bigg\{f_i(x,v) - f_i(u,v) + \frac{1}{2}q_i^T\nabla_{xx} f_i(u,v)q_i$$

$$-\eta_1^T(x,u)[\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i]\Bigg\} \geqq 0. \tag{2.8}$$

Since (u, v, λ, q) is feasible for (WD), from the dual constraint (2.4) and hypothesis (iii), it follows that

$$[\eta_1(x,u) + u]^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \geqq 0,$$

which implies

$$\eta_1^T(x,u)\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i)$$
$$\geqq -u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i). \tag{2.9}$$

Using (2.8) and (2.9), we obtain

$$\sum_{i=1}^{k}\lambda_i(f_i(x,v)-f_i(u,v)+\frac{1}{2}q_i^T\nabla_{xx}f_i(u,v)q_i)$$
$$\geqq -u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i). \tag{2.10}$$

Similarly, by$K$-η_2-bonvexity of $-f(x,.)$, from(2.1) and hypothesis (iv), we get,

$$\sum_{i=1}^{k}\lambda_i(f_i(x,y)-f_i(x,v)-\frac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i)$$
$$\geqq y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)+\nabla_{yy}f_i(x,y)p_i). \tag{2.11}$$

It follows from (2.10),(2.11) and $\lambda^T e_k = 1$ that

$$\sum_{i=1}^{k}\lambda_i\{f_i(x,y)-y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)+\nabla_{yy}f_i(x,y)p_i)-\frac{1}{2}\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{yy}f_i(x,y)p_i)$$
$$-(f_i(u,v)-u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i-\frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx}f_i(u,v)q_i))\}\geqq 0,$$

which contradicts(2.7). Hence the result.

If the variable λ in the problems (WP) and (WD) is fixed to be $\bar{\lambda}$, we shall denote these problems by $(WP)_{\bar{\lambda}}$ and $(WD)_{\bar{\lambda}}$, respectively.

Theorem 2.2 (Strong duality) Let $f: S_1 \times S_2 \to R^k$ be thrice differentiable function and let $(\bar{x},\bar{y},\bar{\lambda},\bar{p_1},\bar{p_2},\ldots,\bar{p_k})$ be a weak efficient solution of (WP). Suppose that

(i) the matrix $\nabla_{yy}f_i(\bar{x},\bar{y})$ is non singular for $i = 1,2,\ldots,k$,

(ii) the vectors $\nabla_y f_1(\bar{x},\bar{y}),\ldots,\nabla_y f_k(\bar{x},\bar{y})$ are linearly independent,

(iii) $\sum_{i=1}^{k}\bar{\lambda_i}(\nabla_y(\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p_i}))\bar{p_i} \notin span\{\nabla_y f_1(\bar{x},\bar{y}),\ldots,\nabla_y f_k(\bar{x},\bar{y})\}\setminus\{0\}$,

(iv) $\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y(\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i))\bar{p}_i = 0$ implies $\bar{p}_i = 0 \forall\ i$, and

(v) K is closed convex pointed cone with $R^k_+ \subseteq K$.

Then $\bar{q}_i = 0, i = 1, 2, \ldots, k$ such that $(\bar{x}, \bar{y}, \bar{q}_1 = \bar{q}_2 = \ldots = \bar{q}_k = 0)$ is feasible for $(WD)_{\bar{\lambda}}$, and the objective function values of (WP) and $(WD)_{\bar{\lambda}}$ are equal. Furthermore, if the hypotheses of Theorem 2.1 are satisfied for all feasible solutions of (WP) and $(WD)_{\bar{\lambda}}$, then $(\bar{x}, \bar{y}, \bar{q}_1 = \bar{q}_2 = \ldots = \bar{q}_k = 0)$ is an efficient solution for $(WD)_{\bar{\lambda}}$.

Proof Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p}_1, \bar{p}_2, \ldots, \bar{p}_k)$ is a weakly efficient solution of (WP), there exists $\bar{\alpha} \in K^*, \bar{\beta} \in C_2, \bar{\eta} \in R$, such that the following by Fritz-John optimality conditions ([38], Lemma 1) are satisfied at $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p}_1, \bar{p}_2, \ldots, \bar{p}_k)$ (for simplicity, we write $\nabla_x f_i$, $\nabla_{xy} f_i$ instead of $\nabla_x f_i(\bar{x}, \bar{y})$, $\nabla_{xy} f_i(\bar{x}, \bar{y})$ etc.):

$$(x-\bar{x})^T\Bigg\{\sum_{i=1}^{k}\bar{\alpha}_i\nabla_x f_i + \sum_{i=1}^{k}\bar{\lambda}_i\nabla_{xy}f_i[\bar{\beta}-(\bar{\alpha}^T e_k)\bar{y}]$$
$$+\sum_{i=1}^{k}\bar{\lambda}_i[\nabla_x(\nabla_{yy}f_i\bar{p}_i)]\left[\bar{\beta}-(\bar{\alpha}^T e_k)\left(\bar{y}+\frac{1}{2}\bar{p}_i\right)\right]\Bigg\} \geqq 0, \forall\ x \in C_1, \tag{2.12}$$

$$(y-\bar{y})^T\Bigg\{\sum_{i=1}^{k}\bar{\alpha}_i\nabla_y f_i + \sum_{i=1}^{k}\bar{\lambda}_i\nabla_{yy}f_i[\bar{\beta}-(\bar{\alpha}^T e_k)\bar{y}] + \sum_{i=1}^{k}\bar{\lambda}_i[\nabla_y(\nabla_{yy}f_i\bar{p}_i)]$$
$$\times\left[\bar{\beta}-(\bar{\alpha}^T e_k)\left(\bar{y}+\frac{1}{2}\bar{p}_i\right)\right] - \sum_{i=1}^{k}\bar{\lambda}_i[\nabla_y f_i + \nabla_{yy}f_i\bar{p}_i](\bar{\alpha}^T e_k)\Bigg\} \geqq 0, \forall\ y \in R^m, \tag{2.13}$$

$$\nabla_y f(\bar{x},\bar{y})(\bar{\beta}-(\bar{\alpha}^T e_k)\bar{y}) + \bar{\eta}e_k + \Bigg\{\left(\bar{\beta}-(\bar{\alpha}^T e_k)\left(\bar{y}+\frac{1}{2}\bar{p}_1\right)\right)^T\nabla_{yy}f_1(\bar{x},\bar{y})\bar{p}_1,$$
$$\ldots,\left(\bar{\beta}-(\bar{\alpha}^T e_k)\left(\bar{y}+\frac{1}{2}\bar{p}_k\right)\right)^T\nabla_{yy}f_k(\bar{x},\bar{y})\bar{p}_k\Bigg\} = 0, \tag{2.14}$$

$$[(\bar{\beta}-(\bar{\alpha}^T e_k)(\bar{y}+\bar{p}_i))\bar{\lambda}_i]^T\nabla_{yy}f_i = 0, i = 1, 2, \ldots, k, \tag{2.15}$$

$$\bar{\beta}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i + \nabla_{yy}f_i\bar{p}_i) = 0, \tag{2.16}$$

$$\bar{\eta}^T[\bar{\lambda}^T e_k - 1] = 0 \tag{2.17}$$

$$(\bar{\alpha}, \bar{\beta}, \bar{\eta}) \neq 0. \tag{2.18}$$

Inequalities (2.13) and (2.14) are equivalent to

$$\sum_{i=1}^{k}\bar{\alpha}_i\nabla_y f_i+\sum_{i=1}^{k}\bar{\lambda}_i\nabla_{yy}f_i[\bar{\beta}-(\bar{\alpha}^Te_k)\bar{y}]$$
$$+\sum_{i=1}^{k}\bar{\lambda}_i[\nabla_y(\nabla_{yy}f_i\bar{p}_i)]\left[\bar{\beta}-(\bar{\alpha}^Te_k)\left(\bar{y}+\frac{1}{2}\bar{p}_i\right)\right]$$
$$-\sum_{i=1}^{k}\bar{\lambda}_i[\nabla_y f_i+\nabla_{yy}f_i\bar{p}_i](\bar{\alpha}^Te_k)=0, \tag{2.19}$$

$$\nabla_y f_i[\bar{\beta}-(\bar{\alpha}^Te_k)\bar{y}]+\nabla_{yy}f_i\bar{p}_i\left[\bar{\beta}-(\bar{\alpha}^Te_k)\left(\bar{y}+\frac{1}{2}\bar{p}_i\right)\right]$$
$$+\bar{\eta}=0, i=1,2,\ldots,k. \tag{2.20}$$

Since $R_+^k\subseteq K\Rightarrow K^*\subseteq R_+^k$ which implies $\text{int}(K^*)\subseteq\text{int}(R_+^k)$.
As $\bar{\lambda}\in\text{int}(K^*)$,therefore $\bar{\lambda}>0$.
As $\nabla_{yy}f_i$ is nonsingular and $\bar{\lambda}_i>0$ for $i=1,2\ldots,k$, (2.15)implies

$$\bar{\beta}=(\bar{\alpha}^Te_k)(\bar{y}+\bar{p}_i), i=1,2,\ldots,k. \tag{2.21}$$

If $\bar{\alpha}=0$ then (2.21) yields $\bar{\beta}=0$. Further, the Eq.2.20 gives $\bar{\eta}=0$. Consequently $(\bar{\alpha},\bar{\beta},\bar{\eta})=0$, contradicting (2.18). Hence $\bar{\alpha}\neq 0$. Further, $\bar{\alpha}\in K^*\subseteq R_+^k$ implies

$$\bar{\alpha}^Te_k>0. \tag{2.22}$$

Now, we claim that $\bar{p}_i=0$ for $i=1,2,\ldots,k$.
Using (2.21) and (2.22) in (2.19), we get

$$\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y(\nabla_{yy}f_i\bar{p}_i))\bar{p}_i=\frac{-2}{\bar{\alpha}^Te_k}\sum_{i=1}^{k}\nabla_y f_i[\bar{\alpha}_i-(\bar{\alpha}^Te_k)\bar{\lambda}_i], \tag{2.23}$$

which by hypotheses (iii) and (iv) implies

$$\bar{p}_i=0 \text{ for } i=1,2,\ldots,k, \tag{2.24}$$

and thus relation (2.21) gives

$$\bar{\beta}=(\bar{\alpha}^Te_k)\bar{y}. \tag{2.25}$$

Equation (2.23) and (2.24), yields

$$\sum_{i=1}^{k}(\bar{\alpha}_i - (\bar{\alpha}^T e_k)\bar{\lambda}_i)\nabla_y f_i = 0,$$

which on using hypotheses (ii) gives

$$\bar{\alpha}_i = (\bar{\alpha}^T e_k)\bar{\lambda}_i, i = 1, 2, \ldots, k. \tag{2.26}$$

Using (2.22), (2.24)-(2.26) in (2.12), we have

$$(x - \bar{x})^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i \geqq 0, \text{ for all } x \in C_1. \tag{2.27}$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$ and so (2.27) implies

$$x^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i \geqq 0, \text{ for all } x \in C_1.$$

Therefore,

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i \in C_1^*. \tag{2.28}$$

Also, from (2.25), we have

$$\bar{y} = \frac{\bar{\beta}}{\bar{\alpha}^T e_k} \in C_2.$$

Thus $(\bar{x}, \bar{y}, \bar{q}_1 = \bar{q}_2 = \ldots = \bar{q}_k = 0)$ satisfies the constraints of $(WD)_{\bar{\lambda}}$ and so it is a feasible solution for the dual problem $(WD)_{\bar{\lambda}}$.

Now, letting $x = 0$ and $x = 2\bar{x}$ in (2.27), we get

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i = 0. \tag{2.29}$$

Further, from (2.16),(2.22),(2.24), and (2.25), we obtain

$$\bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_y f_i = 0. \tag{2.30}$$

Therefore, using (2.24),(2.29) and (2.30), we get

$$\Big(f_1(\bar{x},\bar{y}) - \bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i + \nabla_{yy}f_i\bar{p}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{p}_i^T\nabla_{yy}f_i\bar{p}_i),$$
$$\ldots, f_k(\bar{x},\bar{y}) - \bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i + \nabla_{yy}f_i\bar{p}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{p}_i^T\nabla_{yy}f_i\bar{p}_i)\Big)$$
$$= \Big(f_1(\bar{x},\bar{y}) - \bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_x f_i + \nabla_{xx}f_i\bar{q}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{q}_i^T\nabla_{xx}f_i\bar{q}_i),$$
$$\ldots, f_k(\bar{x},\bar{y}) - \bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_x f_i + \nabla_{xx}f_i\bar{q}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{q}_i^T\nabla_{xx}f_i\bar{q}_i)\Big)$$

that is, the two objective function values are equal.

Now, let $(\bar{x},\bar{y},\bar{q_1} = \bar{q_2} = \ldots = \bar{q_k} = 0)$ is not an efficient solution of $(WD)_{\bar{\lambda}}$, then there exist $(\bar{u},\bar{v},\bar{q_1} = \bar{q_2} = \ldots = \bar{q_k} = 0)$ feasible for $(WD)_{\bar{\lambda}}$ such that,

$$\Big\{f_1(\bar{x},\bar{y}) - \bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \nabla_{xx}f_i(\bar{x},\bar{y})\bar{q}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{q}_i^T\nabla_{xx}f_i(\bar{x},\bar{y})\bar{q}_i),$$
$$\ldots, f_k(\bar{x},\bar{y}) - \bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \nabla_{xx}f_i(\bar{x},\bar{y})\bar{q}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{q}_i^T\nabla_{xx}f_i(\bar{x},\bar{y})\bar{q}_i)\Big\}$$
$$-\Big\{f_1(\bar{u},\bar{v}) - \bar{u}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{xx}f_i(\bar{u},\bar{v})\bar{q}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{q}_i^T\nabla_{xx}f_i(\bar{u},\bar{v})\bar{q}_i),$$
$$\ldots, f_k(\bar{u},\bar{v}) - \bar{u}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{xx}f_i(\bar{u},\bar{v})\bar{q}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{q}_i^T\nabla_{xx}f_i(\bar{u},\bar{v})\bar{q}_i)\Big\}$$
$$\in -K \setminus \{0\}.$$

As

$\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i\nabla_x f_i(\bar{x},\bar{y}) = \bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y f_i(\bar{x},\bar{y})$ and $\bar{p}_i = 0$, for $i = 1,2,\ldots,k$,

$$\Big\{f_1(\bar{x},\bar{y}) - \bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y}) + \nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{p}_i^T\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i),$$
$$\ldots, f_k(\bar{x},\bar{y}) - \bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y}) + \nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{p}_i^T\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i)\Big\}$$
$$\Big\{f_1(\bar{u},\bar{v}) - \bar{u}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{xx}f_i(\bar{u},\bar{v})\bar{q}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{q}_i^T\nabla_{xx}f_i(\bar{u},\bar{v})\bar{q}_i),$$
$$\ldots, f_k(\bar{u},\bar{v}) - \bar{u}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{xx}f_i(\bar{u},\bar{v})\bar{q}_i) - \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{q}_i^T\nabla_{xx}f_i(\bar{u},\bar{v})\bar{q}_i)\Big\}$$
$$\in -K \setminus \{0\}.$$

which contradicts weak duality theorem. Hence $(\bar{x}, \bar{y}, \bar{q_1} = \bar{q_2} = \ldots = \bar{q_k} = 0)$ is an efficient solution of $(WD)_{\bar{\lambda}}$.

Theorem 2.3(Converse duality) Let $f : S_1 \times S_2 \to R^K$ be thrice differentiable function and let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{q_1}, \bar{q_2}, \ldots, \bar{q_k})$ be a weak efficient solution of (WD). Suppose that

(*i*) the matrix $\nabla_{xx} f_i(\bar{u}, \bar{v})$ is non singular for $i = 1, 2, \ldots, k$,

(*ii*) the vectors $\nabla_x f_1(\bar{u}, \bar{v}), \ldots, \nabla_x f_k(\bar{u}, \bar{v})$ are linearly independent,

(*iii*) $\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x (\nabla_{xx} f_i(\bar{u}, \bar{v}) \bar{q_i})) \bar{q_i} \notin span\{\nabla_x f_1(\bar{u}, \bar{v}), \ldots, \nabla_x f_k(\bar{u}, \bar{v})\} \setminus \{0\}$,

(*iv*) $\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x (\nabla_{xx} f_i(\bar{u}, \bar{v}) \bar{q_i})) \bar{q_i} = 0$ implies $\bar{q_i} = 0 \forall$ i, and

(*v*) K is closed convex pointed cone with $R^k_+ \subseteq K$.

Then $\bar{p_i} = 0, i = 1, 2, \ldots, k$ such that $(\bar{u}, \bar{v}, \bar{p_1} = \bar{p_2} = \ldots = \bar{p_k} = 0)$ is feasible for $(WP)_{\bar{\lambda}}$, and the objective function values of $(WP)_{\bar{\lambda}}$ and (WD) are equal. Furthermore, if the hypotheses of Theorem 2.1 are satisfied for all feasible solutions of $(WP)_{\bar{\lambda}}$ and (WD), then $(\bar{u}, \bar{v}, \bar{p_1} = \bar{p_2} = \ldots = \bar{p_k} = 0)$ is an efficient solution for $(WP)_{\bar{\lambda}}$.

Proof It follows on the lines of Theorem 2.2.

2.4 Self-duality

A mathematical programming problem is said to be self dual if it is formally identical with its dual, that is, if the dual is recast in the form of primal, the new problem so obtained is the same as the primal. In general (WP) and (WD) are not self dual without an added restriction on f. The vectors function $f : R^n \times R^n \to R^k$ is said to be skew symmetric if for all $x, y \in R^n$,

$$f_i(x, y) = -f_i(y, x), \quad i \in \{1, 2, \ldots, k\}.$$

For the programs (WP) and (WD), self duality exists under the following assumptions:

(*i*) $m = n$,

(*ii*) $C_1 = C_2$,

(iii) the vector function $f(x,y)$ to be skew symmetric, i.e., $f(x,y) = -f(y,x)$.

Now recasting the dual problem (WD) as a minimization problem:

$(WD)'$

$$\text{K-minimize} \quad \Big(-f_1(u,v) + u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i)$$
$$+\frac{1}{2}\sum_{i=1}^{k} \lambda_i(q_i^T \nabla_{xx} f_i(u,v)q_i), \dots, -f_k(u,v) + u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i)$$
$$+\frac{1}{2}\sum_{i=1}^{k} \lambda_i(q_i^T \nabla_{xx} f_i(u,v)q_i)\Big)$$

$$\text{subject to} \quad \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \in C_1^*,$$
$$\lambda^T e_k = 1,$$
$$\lambda \in \text{int } K^*, v \in C_2.$$

Since f is skew symmetric,

$$\nabla_x f_i(u,v) = -\nabla_y f_i(v,u) \quad \text{and} \quad \nabla_{xx} f_i(u,v) = -\nabla_{yy} f_i(v,u) \quad \text{for } i = 1,2,\dots,k.$$

Therefore, the problem $(WD)'$ reduces to,

$$\text{K-minimize} \quad \Big(f_1(v,u) - u^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(v,u) + \nabla_{yy} f_i(v,u)q_i)$$
$$-\frac{1}{2}\sum_{i=1}^{k} \lambda_i(q_i^T \nabla_{yy} f_i(v,u)q_i), \dots, f_k(v,u) - u^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(v,u) + \nabla_{yy} f_i(v,u)q_i)$$
$$-\frac{1}{2}\sum_{i=1}^{k} \lambda_i(q_i^T \nabla_{yy} f_i(v,u)q_i)\Big)$$

$$\text{subject to} \quad -\sum_{i=1}^{k} \lambda_i(\nabla_y f_i(v,u) + \nabla_{yy} f_i(v,u)q_i) \in C_2^*,$$
$$\lambda^T e_k = 1,$$
$$\lambda \in \text{int } K^*, v \in C_1.$$

This shows that $(WD)'$ is formally identical to (WP), that is, the objective and constraint functions are identical. Hence(WP) is self dual. Consequently, the feasibility of $(x, y, \lambda, p_1, p_2, \dots, p_k)$ for (WP) implies the feasibility of $(y, x, \lambda, p_1, p_2, \dots, p_k)$ for (WD) and conversely.

Chapter 3

Second-Order Nondifferentiable Multiobjective Mixed Symmetric Dual Programs Over Cones

3.1 Introduction

Duality in mathematical programming has not only used in many theoretical and computational developments in mathematical programming itself but also used in economics, control theory and, business problems and other diverse fields. Multiobjective optimization problems can be found in various fields: product and process design, finance, aircraft design, the oil and gas industry, automobile design, or wherever optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives.The work of Dorn [14] was considerably developed and studied by Dantzig et al.[11], Mond [28] and Mond and Weir[32]. Bazaraa and Goode [5] extended the results in [11] to arbitrary cones. Weir and Mond [40] generalized the symmetric dual problem to multiobjective case. Recently, Ahmad and Husain [2] discussed a pair of multiobjective mixed symmetric dual programs over arbitrary cones and established duality under K-preinvexity/K-pseudoinvexity assumptions.

In this chapter, we studied duality theorems(weak, strong, and converse duality) for a pair of second-order mixed symmetric nondifferentiable multiobjective dual programs over arbitrary cones under the assumptions of second order K-(F,ρ)-convexity. This Chapter is divided into three sections. Section 3.1 is introductory, Section 3.2 contains notations and definitions. After that, we have described a nontrivial example of second-order K-(F, ρ)-convex function which is not second-order F-convex. In Section 3.3, we consider a pair of second order nondifferentiable multiobjective mixed symmetric dual programs over arbitrary cones and prove weak, strong and converse theorems under K-(F, ρ)-convexity assumptions.

3.2 Notations and definitions

Consider the following multiobjective programming problem:

$$\textbf{(P)} \qquad \text{K-Minimize } \xi(x)$$
$$\text{subject to } x \in X^0 = \{x \in S : -\phi(x) \in Q\},$$

where $S \subseteq R^n$ be open, $\xi : S \to R^k, \phi : S \to R^m$, K is closed convex pointed cone and Q is closed convex cone with nonempty interiors in R^k and R^m, respectively.

Definition 3.1 A real valued twice differential function $\xi : X \to R^k$ is said to be second-order K-(F, ρ)-convex at $u \in X$ with respect to $t_i \in R^n, i = 1, 2, \ldots, k$ if there exist a real valued function $d : X \times X \to R$ and a real number ρ such that $\forall x \in X$,

$$\Big(\xi_1(x) - \xi_1(u) + \frac{1}{2}t_1^T \nabla_{xx}\xi_1(u)t_1 - F_{x,u}(\nabla_x \xi_1(u) + \nabla_{xx}\xi_1(u)t_1)$$
$$-\rho d^2(x, u), \ldots, \xi_k(x) - \xi_k(u) + \frac{1}{2}t_k^T \nabla_{xx}\xi_k(u)t_k - F_{x,u}(\nabla_x \xi_k(u)$$
$$+\nabla_{xx}\xi_l(u)t_k) - \rho d^2(x, u)\Big) \in K.$$

3.1 A Nontrivial Example of Second-Order K-(F, ρ)-convex function

Example 3.1 Let $k = 1, K = R_+$ and $X = [-1.8, -1.5] \subset R$. Suppose that the function $\xi : X \to R$ be defined by

$$\xi(x) = x^3 + 8\sin^3 x$$

and the functional $F : X \times X \times R \to R$ be defined by

$$F_{x,u}(a) = a(1 - 2u).$$

Let $d : X \times X \to R$ be given by

$$d(x, u) = \sqrt{x^2 + u^2}.$$

For $\rho = -35$, we have

$$\begin{aligned} L &= \xi(x) - \xi(u) + \frac{1}{2}q^T\nabla_{xx}\xi(u)q - F_{x,u}(\nabla_x\xi(u) + \nabla_{xx}\xi(u)q) - \rho d^2(x,u) \\ &= x^3 - u^3 + 8(\sin^3 x - \sin^3 u) + q^2(3u + 9\sin 3u - 3\sin u) - (1-2u) \\ &\quad \times\big[3u^2 + 6(\cos u - \cos 3u) + q(6u + 18\sin 3u - 6\sin u)\big] - (-35)(\sqrt{x^2+u^2})^2 \\ &= x^3 - u^3 + 8(\sin^3 x - \sin^3 u) + (2u-1)\big[3u^2 + 6(\cos u - \cos 3u)\big] + 35x^2 \\ &\quad + q^2(3u + 9\sin 3u - 3\sin u) + (2u-1)(6uq + 18q\sin 3u - 6q\sin u) + 35u^2 \\ &= f_1 + f_2 (say) \end{aligned}$$

where

$$\begin{aligned} f_1 &= x^3 - u^3 + 8(\sin^3 x - \sin^3 u) + (2u-1)\big[3u^2 + 6(\cos u - \cos 3u)\big] + 35x^2 \\ &\geqq 0 \;\forall\, x, u, \in X \text{ as can be seen from Figure 3.1} \end{aligned}$$

and

$$\begin{aligned} f_2 &= q^2(3u + 9\sin 3u - 3\sin u) + (2u-1)(6uq + 18q\sin 3u - 6q\sin u) + 35u^2. \\ &\geqq 0 \;\forall\, u \in X \text{and } q \in (-10^{18}, 10^{18}) \text{ as can be seen from Figure 3.2.} \end{aligned}$$

Hence $L \geqq 0$. Therefore f is second-order K-(F,ρ)-convex. But f is not second-order F-convex since for $q = 1$, we have

$$\begin{aligned} M &= \xi(x) - \xi(u) + \frac{1}{2}q^T\nabla_{xx}\xi(u)q - F_{x,u}[\nabla_x\xi(u) + \nabla_{xx}\xi(u)q] \\ &= x^3 - u^3 + 8(\sin^3 x - \sin^3 u) + 3(u + 3\sin 3u - \sin u) - 3(1-2u) \\ &\quad \times[u^2 + 2\cos u - 2\cos 3u + 2u + 6\sin 3u - 2\sin u] \\ &< 0 \;\forall\, x, u \in X \text{ as be can seen from Figure 3.3.} \end{aligned}$$

Hence the function f is second-order K-(F,ρ)-convex but is not second-order F-convex.

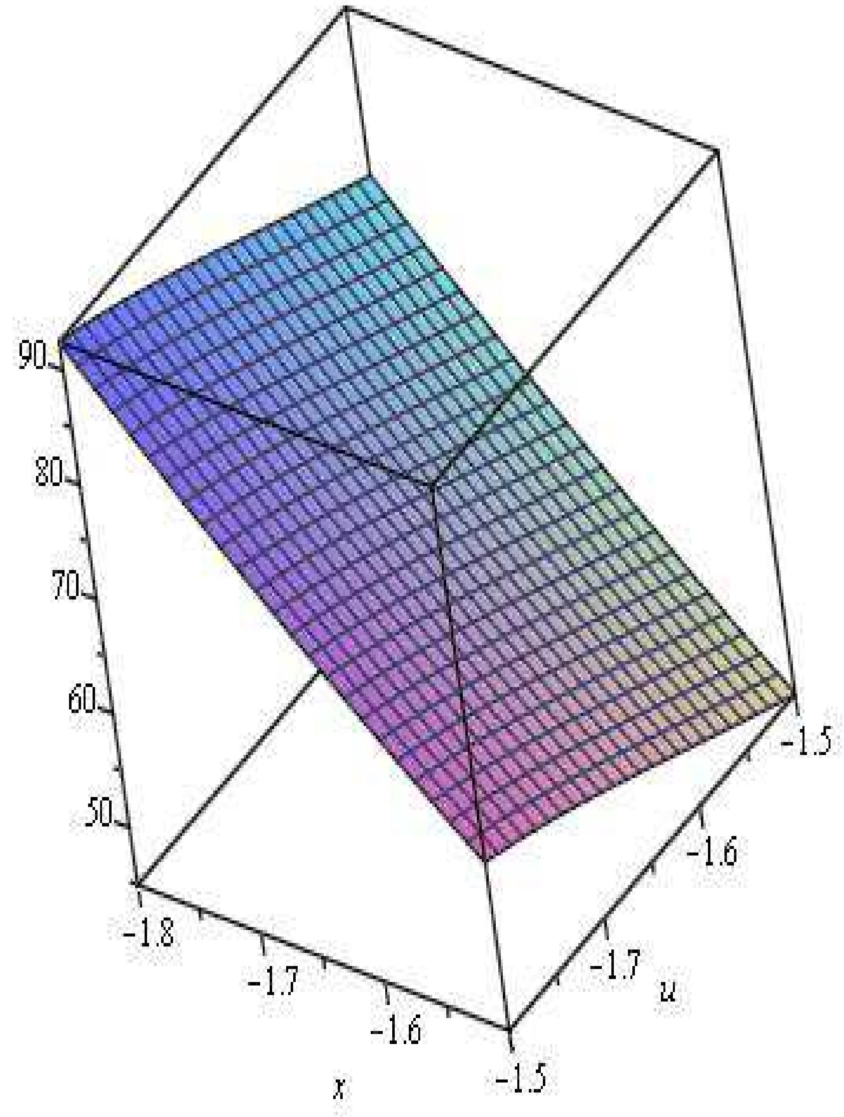

Figure 3.1: Graph of f_1 against u and x

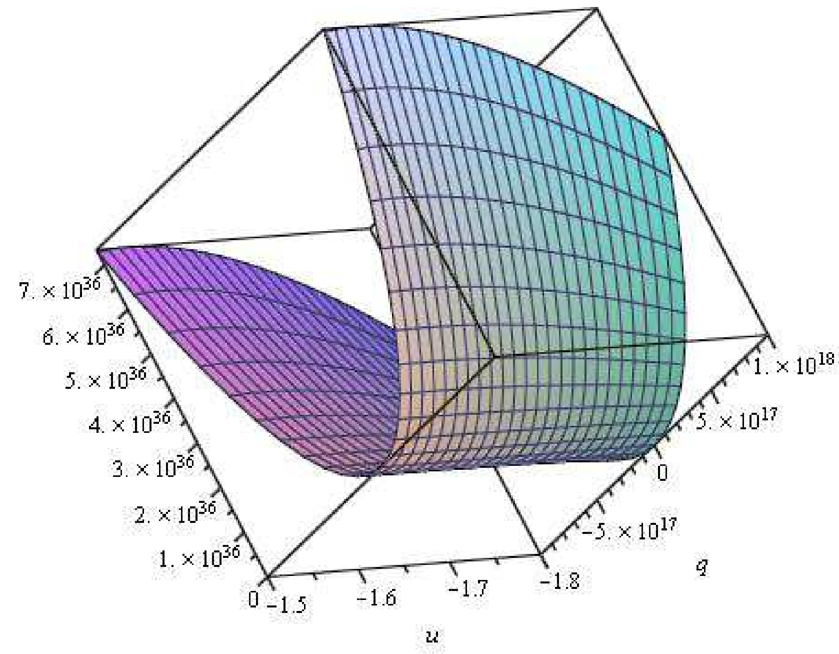

Figure 3.2: Graph of f_2 against u and x

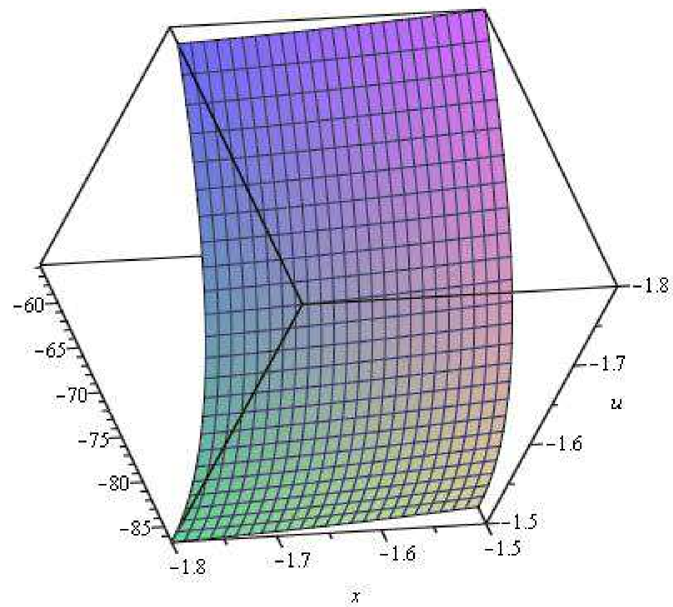

Figure 3.3: Graph of M against u and x

3.3 Multiobjective mixed second-order nondifferentiable symmetric dual programs

For $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$, let $J_1 \subseteq N, K_1 \subseteq M$ and $J_2 = N \setminus J_1$ and $K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in J_1. The other symbols $|J_2|, |K_1|$ and $|K_2|$ are defined similarly. Let $x^1 \in R^{|J_1|}$, $x^2 \in R^{|J_2|}$. Then any $x \in R^n$ can be written as (x^1, x^2). Similarly for $y^1 \in R^{|K_1|}, y^2 \in R^{|K_2|}, y \in R^m$ can be written as (y^1, y^2). It may be noted here that if $J_1 = \emptyset$, then $|J_1| = 0, J_2 = N$ and therefore $|J_2| = n$. In this case, $R^{|J_1|}, R^{|J_2|}$ and $R^{|J_1|} \times R^{|K_1|}$ will be zero-dimensional, n-dimensional and $|K_1|$-dimensional Euclidean spaces, respectively. Similarly, we can describe the cases $J_2 = \emptyset, K_1 = \emptyset$ or $K_2 = \emptyset$. Let C_1, C_2, C_3 and C_4 be closed convex cones with nonempty interiors in $R^{|J_1|}, R^{|J_2|}, R^{|K_1|}$ and $R^{|K_2|}$ respectively.

Consider the following pair of multiobjective mixed second-order nondifferentiable symmetric dual programs:

Primal Problem(P3.1)

K-minimize

$$N(x^1, y^1, x^2, y^2, w^2, \lambda, p, r) = \big\{N_1(x^1, y^1, x^2, y^2, w_1^2, \lambda, p, r^1), \ldots, N_l(x^1, y^1, x^2, y^2, w_l^2, \lambda, p, r^l)\big\}$$

subject to

$$-\sum_{i=1}^{l} \lambda_i \big[\nabla_{y^1} f_i(x^1, y^1) - w^1 + \nabla_{y^1 y^1} f_i(x^1, y^1) p_i\big] \in C_3^*, \tag{3.1}$$

$$-\sum_{i=1}^{l} \lambda_i \big[\nabla_{y^2} g_i(x^2, y^2) - w_i^2 + \nabla_{y^2 y^2} g_i(x^2, y^2) r^i\big] \in C_4^*, \tag{3.2}$$

$$(y^2)^T \sum_{i=1}^{l} \lambda_i (\nabla_{y^2} g_i(x^2, y^2) - w_i^2 + \nabla_{y^2 y^2} g_i(x^2, y^2) r^i) \geqq 0, \tag{3.3}$$

$$w^1 \in E, \; w_i^2 \in B_i, \; i = 1, 2, \ldots, l \tag{3.4}$$

$$\lambda^T e_l = 1, \; x^1 \in C_1, \; x^2 \in C_2, \tag{3.5}$$

$$\lambda \in \text{int} K^*. \tag{3.6}$$

Dual Problem(D3.1)

K-maximize

$$M(u^1,v^1,u^2,v^2,z^2,\lambda,q,s)=\big\{M_1(u^1,v^1,u^2,v^2,z_1^2,\lambda,q,s^1),\ldots,\\ M_l(u^1,v^1,u^2,v^2,z_l^2,\lambda,q,s^l)\big\}$$

subject to

$$\sum_{i=1}^{l}\lambda_i\big[\nabla_{x^1}f_i(u^1,v^1)+z^1+\nabla_{x^1x^1}f_i(u^1,v^1)q_i\big]\in C_1^*, \tag{3.7}$$

$$\sum_{i=1}^{l}\lambda_i\big[\nabla_{x^2}g_i(u^2,v^2)+z_i^2+\nabla_{x^2x^2}g_i(u^2,v^2)s^i\big]\in C_2^*, \tag{3.8}$$

$$(u^2)^T\sum_{i=1}^{l}\lambda_i(\nabla_{x^2}g_i(u^2,v^2)+z_i^2+\nabla_{x^2x^2}g_i(u^2,v^2)s^i)\leqq 0, \tag{3.9}$$

$$z^1\in H,\ z_i^2\in D_i,\ i=1,2,\ldots,l \tag{3.10}$$

$$\lambda^Te_l=1,\ v^1\in C_3,\ v^2\in C_4, \tag{3.11}$$

$$\lambda\in \text{int}K^*. \tag{3.12}$$

where $\lambda\in R^l$, $e_l=(1,\ldots,1)^T\in R^l$, $w^1\in R^{|K_1|}$ and $z^1\in R^{|J_1|}$, H and E are compact and convex sets in $R^{|J_1|}$ and $R^{|K_1|}$, respectively and for $i=1,2,\ldots,l$

$$\begin{aligned}N_i(x^1,y^1,x^2,y^2,w_i^2,\lambda,p,r^i) &= f_i(x^1,y^1)+s(x^1|H)+g_i(x^2,y^2)+s(x^2|D_i)-(y^2)^T\\ &\quad\times w_i^2-(y^1)^T\sum_{i=1}^{l}\lambda_i(\nabla_{y^1}f_i(x^1,y^1)+\nabla_{y^1y^1}f_i(x^1,y^1)p_i)\\ &\quad-\frac{1}{2}\sum_{i=1}^{l}\lambda_i\big[p_i^T\nabla_{y^1y^1}f_i(x^1,y^1)p_i\big]\\ &\quad-\frac{1}{2}(r^i)^T\nabla_{y^2y^2}g_i(x^2,y^2)r^i,\end{aligned}$$

$$\begin{aligned}M_i(u^1,v^1,u^2,v^2,z_i^2,\lambda,q,s^i) &= f_i(u^1,v^1)-s(v^1|E)+g_i(u^2,v^2)-s(v^2|B_i)+(u^2)^Tz_i^2\\ &\quad-(u^1)^T\sum_{i=1}^{l}\lambda_i(\nabla_{x^1}f_i(u^1,v^1)+\nabla_{x^1x^1}f_i(u^1,v^1)q_i)\\ &\quad-\frac{1}{2}\sum_{i=1}^{l}\lambda_i\big[q_i^T\nabla_{x^1x^1}f_i(u^1,v^1)q_i\big]\\ &\quad-\frac{1}{2}(s^i)^T\nabla_{x^2x^2}g_i(u^2,v^2)s^i,\end{aligned}$$

and

(i) $f_i : R^{|J_1|} \times R^{|K_1|} \to R$ and $g_i : R^{|J_2|} \times R^{|K_2|} \to R$ are differentiable functions,

(ii) B_i and D_i are compact and convex sets in $R^{|K_2|}$ and $R^{|J_2|}$, respectively,

(iii) $p_i \in R^{|K_1|}$, $q_i \in R^{|J_1|}$, r^i, $w_i^2 \in R^{|K_2|}$, and s^i, $z_i^2 \in R^{|J_2|}$.

Let w^2=$(w_1^2,\ldots,w_l^2)$, $p = (p^1, \ldots, p^l)$, $q = (q^1, \ldots, q^l)$, $r = (r^1, \ldots, r^l)$, $s = (s^1, \ldots, s^l)$, z^2=$(z_1^2,\ldots,z_l^2)$.

Theorem 3.1 (Weak duality) Let $(x^1, y^1, x^2, y^2, w^1, w^2\lambda, p, r)$ be feasible for (P3.1) and $(u^1, v^1, u^2, v^2, z^1, z^2, \lambda, q, s)$ be feasible for (D3.1). Let the sublinear functionals $F^1 : R^{|J_1|} \times R^{|J_1|} \times R^{|J_1|} \longmapsto R$, $F^2 : R^{|K_1|} \times R^{|K_1|} \times R^{|K_1|} \longmapsto R$, $G^1 : R^{|J_2|} \times R^{|J_2|} \times R^{|J_2|} \longmapsto R$ and $G^2 : R^{|K_2|} \times R^{|K_2|} \times R^{|K_2|} \longmapsto R$ satisfy the following conditions:

$$F^1_{x^1,u^1}(a^1) + (a^1)^T u^1 \geqq 0, forall\ a^1 \in C_1^*, \qquad (A)$$
$$F^2_{v^1,y^1}(a^2) + (a^2)^T y^1 \geqq 0, forall\ a^2 \in C_3^*, \qquad (B)$$
$$G^1_{x^2,u^2}(b^1) + (b^1)^T u^2 \geqq 0, forall\ b^1 \in C_2^*, \qquad (C)$$
$$G^2_{v^2,y^2}(b^2) + (b^2)^T y^2 \geqq 0, forall\ b^2 \in C_4^*. \qquad (D)$$

Suppose that

(i) $\{f_1(.,v^1) + (.)^T z^1,\ldots,f_l(.,v^1) + (.)^T z^1\}$ be second-order K-(F^1, ρ_1)-convex at u^1, and $\{-f_1(x^1, .) + (.)^T w^1,\ldots,-f_l(x^1, .) + (.)^T w^1\}$ be second-order K-(F^2, ρ_2)-convex at y^1,

(ii) $\{g_1(.,v^2) + (.)^T z_1^2,\ldots,g_l(.,v^2) + (.)^T z_l^2\}$ be second-order K-(G^1, σ_1)-convex at u^2, and $\{-g_1(x^2, .) + (.)^T w_1^2,\ldots,-g_l(x^2, .) + (.)^T w_l^2\}$ be second-order K-(G^2, σ_2)-convex at y^2.

(iii) either $\rho_1 d_1^2(x^1, u^1) + \rho_2 d_2^2(v^1, y^1) \geqq 0$ or $\rho_1, \rho_2 \geqq 0$ and

(iv) either $\sigma_1 d_1^2(x^2, u^2) + \sigma_2 d_2^2(v^2, y^2) \geqq 0$ or $\sigma_1, \sigma_2 \geqq 0$.

Then

$$N(x^1, y^1, x^2, y^2, w^2, \lambda, p, r) - M(u^1, v^1, u^2, v^2, z^2, \lambda, q, s) \notin -K \setminus \{0\}.$$

Proof Suppose, to the contrary, that

$$N(x^1, y^1, x^2, y^2, w^2, \lambda, p, r) - M(u^1, v^1, u^2, v^2, z^2, \lambda, q, s) \in -K \setminus \{0\}$$

that is,

$$\begin{aligned}
&\big\{N_1(x^1,y^1,x^2,y^2,w_1^2,\lambda,p,r^1),\ldots,N_l(x^1,y^1,x^2,y^2,w_l^2,\lambda,p,r^l)\big\}\\
&-\big\{M_1(u^1,v^1,u^2,v^2,z_1^2,\lambda,q,s^1),\ldots,M_l(u^1,v^1,u^2,v^2,z_l^2,\lambda,q,s^l)\big\}\in -K\setminus\{0\}.
\end{aligned}$$

Then, since $\lambda \in \text{int } K^*$, we have

$$\begin{aligned}
&\sum_{i=1}^{l}\lambda_i\bigg\{f_i(x^1,y^1)+s(x^1|H)+g_i(x^2,y^2)+s(x^2|D_i)-(y^2)^T w_i^2-(y^1)^T\\
&\qquad\times\sum_{i=1}^{l}\lambda_i(\nabla_{y^1}f_i(x^1,y^1)+\nabla_{y^1y^1}f_i(x^1,y^1)p_i)-\frac{1}{2}\sum_{i=1}^{l}\lambda_i\big[p_i^T\nabla_{y^1y^1}f_i(x^1,y^1)p_i\big]\\
&\qquad-\frac{1}{2}(r^i)^T\nabla_{y^2y^2}g_i(x^2,y^2)r^i\bigg\}\\
&<\sum_{i=1}^{l}\lambda_i\bigg\{f_i(u^1,v^1)-s(v^1|E)+g_i(u^2,v^2)-s(v^2|B_i)+(u^2)^T z_i^2-(u^1)^T\\
&\qquad\times\sum_{i=1}^{l}\lambda_i(\nabla_{x^1}f_i(u^1,v^1)+\nabla_{x^1x^1}f_i(u^1,v^1)q_i)-\frac{1}{2}\sum_{i=1}^{l}\lambda_i\big[q_i^T\nabla_{x^1x^1}f_i(u^1,v^1)q_i\big]\\
&\qquad-\frac{1}{2}(s^i)^T\nabla_{x^2x^2}g_i(u^2,v^2)s^i\bigg\}. \qquad (3.13)
\end{aligned}$$

By the hypothesis (i), we have

$$\begin{aligned}
&\bigg\{f_1(x^1,v^1)+(x^1)^Tz^1-f_1(u^1,v^1)-(u^1)^Tz^1+\frac{1}{2}q_1^T\nabla_{x^1x^1}f_1(u^1,v^1)q_1\\
&-F^1_{x^1,u^1}(\nabla_{x^1}f_1(u^1,v^1)+z^1+\nabla_{x^1x^1}f_1(u^1,v^1)q_1)-\rho_1 d_1^2(x^1,u^1),\ldots,\\
&\quad f_l(x^1,v^1)+(x^1)^Tz^1-f_l(u^1,v^1)-(u^1)^Tz^1+\frac{1}{2}q_l^T\nabla_{x^1x^1}f_l(u^1,v^1)q_l\\
&-F^1_{x^1,u^1}(\nabla_{x^1}f_l(u^1,v^1)+z^1+\nabla_{x^1x^1}f_l(u^1,v^1)q_l)-\rho_1 d_1^2(x^1,u^1)\bigg\}\in K \qquad (3.14)
\end{aligned}$$

and

$$\begin{aligned}
&\bigg\{f_1(x^1,y^1)-(y^1)^Tw^1-f_1(x^1,v^1)+(v^1)^Tw^1-\frac{1}{2}p_1^T\nabla_{y^1y^1}f_1(x^1,y^1)p_1\\
&-F^2_{v^1,y^1}(-\nabla_{y^1}f_1(x^1,y^1)+w^1-\nabla_{y^1y^1}f_1(x^1,y^1)p_1)-\rho_2 d_2^2(v^1,y^1),\ldots,\\
&f_l(x^1,y^1)-(y^1)^Tw^1-f_l(x^1,v^1)+(v^1)^Tw^1-\frac{1}{2}p_l^T\nabla_{y^1y^1}f_l(x^1,y^1)p_l\\
&-F^2_{v^1,y^1}(-\nabla_{y^1}f_l(x^1,y^1)+w^1-\nabla_{y^1y^1}f_l(x^1,y^1)p_l)-\rho_2 d_2^2(v^1,y^1)\bigg\}\in K. \qquad (3.15)
\end{aligned}$$

Using $\lambda \in \text{int } K^*$ and $\lambda^T e_l = 1$ in Eqs.3.14 and 3.15, we have

$$\sum_{i=1}^{l} \lambda_i \left\{ f_i(x^1, v^1) + (x^1)^T z^1 - f_i(u^1, v^1) - (u^1)^T z^1 + \frac{1}{2} q_i^T \nabla_{x^1 x^1} f_i(u^1, v^1) q_i \right\}$$
$$\geqq \sum_{i=1}^{l} \lambda_i F^1_{x^1, u^1}(\nabla_{x^1} f_i(u^1, v^1) + z^1 + \nabla_{x^1 x^1} f_i(u^1, v^1) q_i) + \rho_1 d_1^2(x^1, u^1) \tag{3.16}$$

$$\sum_{i=1}^{l} \lambda_i \left\{ f_i(x^1, y^1) - (y^1)^T w^1 - f_i(x^1, v^1) + (v^1)^T w^1 - \frac{1}{2} p_i^T \nabla_{y^1 y^1} f_i(x^1, y^1) p_i \right\}$$
$$\geqq \sum_{i=1}^{l} \lambda_i F^2_{v^1, y^1}(-\nabla_{y^1} f_i(x^1, y^1) + w^1 - \nabla_{y^1 y^1} f_i(x^1, y^1) p_i) + \rho_2 d_2^2(v^1, y^1) \tag{3.17}$$

As $R^l_+ \subset K \Rightarrow K^* \subset R^l_+$ and since $\lambda \in \text{int } K^*$, therefore $\lambda > 0$. This together with sublinearity of F^1 and F^2 using in Eqs.3.16 and 3.17, respectively yields

$$\sum_{i=1}^{l} \lambda_i \left\{ f_i(x^1, v^1) + (x^1)^T z^1 - f_i(u^1, v^1) - (u^1)^T z^1 + \frac{1}{2} q_i^T \nabla_{x^1 x^1} f_i(u^1, v^1) q_i \right\}$$
$$\geqq F^1_{x^1, u^1}\left(\sum_{i=1}^{l} \lambda_i \left[\nabla_{x^1} f_i(u^1, v^1) + z^1 + \nabla_{x^1 x^1} f_i(u^1, v^1) q_i \right] \right) + \rho_1 d_1^2(x^1, u^1) \tag{3.18}$$

and

$$\sum_{i=1}^{l} \lambda_i \left\{ f_i(x^1, y^1) - (y^1)^T w^1 - f_i(x^1, v^1) + (v^1)^T w^1 - \frac{1}{2} p_i^T \nabla_{y^1 y^1} f_i(x^1, y^1) p_i \right\}$$
$$\geqq F^2_{v^1, y^1}\left(-\sum_{i=1}^{l} \lambda_i [\nabla_{y^1} f_i(x^1, y^1) - w^1 + \nabla_{y^1 y^1} f_i(x^1, y^1) p_i] \right) + \rho_2 d_2^2(v^1, y^1). \tag{3.19}$$

Since $(x^1, y^1, x^2, y^2, w^1, w^2, \lambda, p, r)$ is feasible for primal problem (P3.1) and $(u^1, v^1, u^2, v^2, z^1, z^2, \lambda, q, s)$ is feasible for dual problem (D3.1), by the dual constraint(3.7), the vector $a^1 = \sum_{i=1}^{l} \lambda_i[\nabla_{x^1} f_i(u^1, v^1) + z^1 + \nabla_{x^1x^1} f_i(u^1, v^1)q_i] \in C_1^*$, and so from the hypothesis(A), we obtain

$$F^1_{x^1,u^1}(a^1) + (a^1)^T u^1 \geqq 0. \tag{3.20}$$

Similarly,

$$F^2_{v^1,y^1}(a^2) + (a^2)^T y^1 \geqq 0, \tag{3.21}$$

for the vector $a^2 = -\sum_{i=1}^{l} \lambda_i[\nabla_{y^1} f_i(x^1, y^1) - w^1 + \nabla_{y^1y^1} f_i(x^1, y^1)p_i] \in C_3^*$.
Using Eq.3.20 in Eqs.3.18 and 3.21 in Eq.3.19, we have

$$\begin{aligned}&\sum_{i=1}^{l} \lambda_i \Big\{ f_i(x^1, v^1) + (x^1)^T z^1 - f_i(u^1, v^1) - (u^1)^T z^1 + \frac{1}{2} q_i^T \nabla_{x^1x^1} f_i(u^1, v^1) q_i \Big\} \\ &\qquad \geqq -(u^1)^T a^1 + \rho_1 d_1^2(x^1, u^1)\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{l} \lambda_i \Big\{ f_i(x^1, y^1) - (y^1)^T w^1 - f_i(x^1, v^1) + (v^1)^T w^1 - \frac{1}{2} p_i^T \nabla_{y^1y^1} f_i(x^1, y^1) p_i \Big\} \\ &\qquad \geqq -(y^1)^T a^2 + \rho_2 d_2^2(v^1, y^1).\end{aligned}$$

Adding the above inequalities and hypothesis(iii), we obtain

$$\begin{aligned}&\sum_{i=1}^{l} \lambda_i \Big\{ f_i(x^1, y^1) + (x^1)^T z^1 - (y^1)^T w^1 + (y^1)^T a^2 - \frac{1}{2} p_i^T \nabla_{y^1y^1} f_i(x^1, y^1) p_i \Big\} \\ &\qquad \geqq \sum_{i=1}^{l} \lambda_i \Big\{ f_i(u^1, v^1) - (v^1)^T w^1 + (u^1)^T z^1 - (u^1)^T a^1 - \frac{1}{2} q_i^T \nabla_{x^1x^1} f_i(u^1, v^1) q_i \Big\}.\end{aligned} \tag{3.22}$$

Substituting the values of a^1 and a^2 in Eq.3.22, we get

$$\begin{aligned}&\sum_{i=1}^{l} \lambda_i \Big\{ f_i(x^1, y^1) + (x^1)^T z^1 - (y^1)^T \sum_{i=1}^{l} \lambda_i (\nabla_{y^1} f_i(x^1, y^1) + \nabla_{y^1y^1} f_i(x^1, y^1) p_i) \\ &\qquad - \frac{1}{2} \sum_{i=1}^{l} \lambda_i \big[p_i^T \nabla_{y^1y^1} f_i(x^1, y^1) p_i \big] \Big\} \\ &\geqq \sum_{i=1}^{l} \lambda_i \Big\{ f_i(u^1, v^1) - (v^1)^T w^1 - (u^1)^T \sum_{i=1}^{l} \lambda_i (\nabla_{x^1} f_i(u^1, v^1) \\ &\qquad + \nabla_{x^1x^1} f_i(u^1, v^1) q_i) - \frac{1}{2} \sum_{i=1}^{l} \lambda_i \big[q_i^T \nabla_{x^1x^1} f_i(u^1, v^1) q_i \big] \Big\}.\end{aligned}$$

Using $(x^1)^T z^1 \leqq s(x^1|H)$ and $(v^1)^T w^1 \leqq s(v^1|E)$, we have

$$\sum_{i=1}^{l} \lambda_i \Big\{ f_i(x^1, y^1) + s(x^1|H) - (y^1)^T \sum_{i=1}^{l} \lambda_i (\nabla_{y^1} f_i(x^1, y^1) + \nabla_{y^1 y^1} f_i(x^1, y^1) p_i)$$
$$- \frac{1}{2} \sum_{i=1}^{l} \lambda_i [p_i^T \nabla_{y^1 y^1} f_i(x^1, y^1) p_i] \Big\}$$
$$\geqq \sum_{i=1}^{l} \lambda_i \Big\{ f_i(u^1, v^1) - s(v^1|E) - (u^1)^T \sum_{i=1}^{l} \lambda_i (\nabla_{x^1} f_i(u^1, v^1)$$
$$+ \nabla_{x^1 x^1} f_i(u^1, v^1) q_i) - \frac{1}{2} \sum_{i=1}^{l} \lambda_i [q_i^T \nabla_{x^1 x^1} f_i(u^1, v^1) q_i] \Big\}. \quad (3.23)$$

The second-order K-(G^1, σ_1)-convexity of $\{g_1(., v^2) + (.)^T z_1^2, \ldots, g_l(., v^2) + (.)^T z_l^2\}$ at u^2 gives

$$\Big(g_1(x^2, v^2) + (x^2)^T z_1^2 - g_1(u^2, v^2) - (u^2)^T z_1^2 + \frac{1}{2}(s^1)^T \nabla_{x^2 x^2} g_1(u^2, v^2) s^1$$
$$- G^1_{x^2, u^2}(\nabla_{x^2} g_1(u^2, v^2) + z_1^2 + \nabla_{x^2 x^2} g_1(u^2, v^2) s^1) - \sigma_1 d_1^2(x^2, u^2), \ldots,$$
$$g_l(x^2, v^2) + (x^2)^T z_l^2 - g_l(u^2, v^2) - (u^2)^T z_l^2 + \frac{1}{2}(s^l)^T \nabla_{x^2 x^2} g_l(u^2, v^2) s^l$$
$$- G^1_{x^2, u^2}(\nabla_{x^2} g_l(u^2, v^2) + z_l^2 + \nabla_{x^2 x^2} g_l(u^2, v^2) s^l) - \sigma_1 d_1^2(x^2, u^2) \Big) \in K.$$

This follows from $\lambda \in \text{int } K^*$ and $\lambda^T e_l = 1$ that

$$\sum_{i=1}^{l} \lambda_i \big[g_i(x^2, v^2) + (x^2)^T z_i^2 \big] - \sum_{i=1}^{l} \lambda_i \Big[g_i(u^2, v^2) + (u^2)^T z_i^2 - \frac{1}{2}(s^i)^T \nabla_{x^2 x^2} g_i(u^2, v^2) s^i \Big]$$
$$\geqq \sum_{i=1}^{l} \lambda_i G^1_{x^2, u^2}(\nabla_{x^2} g_i(u^2, v^2) + z_i^2 + \nabla_{x^2 x^2} g_i(u^2, v^2) s^i) + \sigma_1 d_1^2(x^2, u^2).$$

Using sublinearity of G^1 and $\lambda > 0$ in the above inequality, we get

$$\sum_{i=1}^{l} \lambda_i \big[g_i(x^2, v^2) + (x^2)^T z_i^2 \big] - \sum_{i=1}^{l} \lambda_i \Big[g_i(u^2, v^2) + (u^2)^T z_i^2 - \frac{1}{2}(s^i)^T \nabla_{x^2 x^2} g_i(u^2, v^2) s^i \Big]$$
$$\geqq G^1_{x^2, u^2}\Big(\sum_{i=1}^{l} \lambda_i \big[\nabla_{x^2} g_i(u^2, v^2) + z_i^2 + \nabla_{x^2 x^2} g_i(u^2, v^2) s^i \big] \Big) + \sigma_1 d_1^2(x^2, u^2).$$

which further from hypothesis(C), and the dual constraint (3.8) and (3.9), we obtain

$$\sum_{i=1}^{l} \lambda_i \big[g_i(x^2, v^2) + (x^2)^T z_i^2 \big]$$
$$\geqq \sum_{i=1}^{l} \lambda_i \Big[g_i(u^2, v^2) + (u^2)^T z_i^2 - \frac{1}{2}(s^i)^T \nabla_{x^2 x^2} g_i(u^2, v^2) s^i \Big] + \sigma_1 d_1^2(x^2, u^2). \quad (3.24)$$

From the second-order K-(G^2, σ_2)-convexity of $\{-g_1(x^2, .) + (.)^T w_1^2, \ldots, -g_l(x^2, .) + (.)^T w_l^2\}$ at y^2, $\lambda \in \text{int } K^*$ and $\lambda^T e = 1$, we have

$$\sum_{i=1}^{l} \lambda_i \big[g_i(x^2, y^2) - (y^2)^T w_i^2\big] - \sum_{i=1}^{l} \lambda_i \left[g_i(x^2, v^2) - (v^2)^T w_i^2 + \frac{1}{2}(r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i\right]$$
$$\geqq \sum_{i=1}^{l} \lambda_i G^2_{v^2, y^2}\left[-(\nabla_{y^2} g_i(x^2, y^2) - w_i^2 + \nabla_{y^2 y^2} g_i(x^2, y^2) r^i)\right] + \sigma_2 d_2^2(v^2, y^2).$$

This together with sublinearity of G^2,$\lambda > 0$, primal constraints (3.2) and (3.3) and hypothesis (D) yield

$$\sum_{i=1}^{l} \lambda_i \big[g_i(x^2, y^2) - (y^2)^T w_i^2\big] \geqq \sum_{i=1}^{l} \lambda_i \left[g_i(x^2, v^2) - (v^2)^T w_i^2 + \frac{1}{2}(r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i\right] + \sigma_2 d_2^2(v^2, y^2). \tag{3.25}$$

Adding Eqs. 3.24 and 3.25 and using hypothesis (iv), we obtain,

$$\sum_{i=1}^{l} \lambda_i \left[g_i(x^2, y^2) + (x^2)^T z_i^2 - (y^2)^T w_i^2 - \frac{1}{2}(r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i\right]$$
$$\geqq \sum_{i=1}^{l} \lambda_i \left[g_i(u^2, v^2) + (u^2)^T z_i^2 - (v^2)^T w_i^2 - \frac{1}{2}(s^i)^T \nabla_{x^2 x^2} g_i(u^2, v^2) s^i\right].$$

Using $(x^2)^T z_i^2 \leqq s(x^2|D_i)$ and $(v^2)^T w_i^2 \leqq s(v^2|B_i)$, we obtain,

$$\sum_{i=1}^{l} \lambda_i \left[g_i(x^2, y^2) + s(x^2|D_i) - (y^2)^T w_i^2 - \frac{1}{2}(r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i\right]$$
$$\geqq \sum_{i=1}^{l} \lambda_i \left[g_i(u^2, v^2) - s(v^2|B_i) + (u^2)^T z_i^2 - \frac{1}{2}(s^i)^T \nabla_{x^2 x^2} g_i(u^2, v^2) s^i\right]. \tag{3.26}$$

Equations 3.23 and 3.26 together yield

$$\sum_{i=1}^{l} \lambda_i \Bigg\{ f_i(x^1, y^1) + s(x^1|H) + g_i(x^2, y^2) + s(x^2|D_i) - (y^2)^T w_i^2$$
$$- (y^1)^T \sum_{i=1}^{l} \lambda_i (\nabla_{y^1} f_i(x^1, y^1) + \nabla_{y^1 y^1} f_i(x^1, y^1) p_i)$$
$$- \frac{1}{2} \sum_{i=1}^{l} \lambda_i \big[p_i^T \nabla_{y^1 y^1} f_i(x^1, y^1) p_i\big] - \frac{1}{2}(r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i \Bigg\}$$

$$\geqq \sum_{i=1}^{l} \lambda_i \Bigg\{ f_i(u^1, v^1) - s(v^1|E) + g_i(u^2, v^2) - s(v^2|B_i) + (u^2)^T z_i^2$$
$$- (u^1)^T \sum_{i=1}^{l} \lambda_i (\nabla_{x^1} f_i(u^1, v^1) + \nabla_{x^1x^1} f_i(u^1, v^1) q_i)$$
$$- \frac{1}{2} \sum_{i=1}^{l} \lambda_i \big[q_i^T \nabla_{x^1x^1} f_i(u^1, v^1) q_i \big] - \frac{1}{2} (s^i)^T \nabla_{x^2x^2} g_i(u^2, v^2) s^i \Bigg\}$$

which contradicts Eq.3.13. Hence the results.

The notations $(P3.1)_{\overline{\lambda}}$ and $(D3.1)_{\overline{\lambda}}$ are used in the following theorems to denote (P3.1) and (D3.1), respectively when λ is fixed to be $\overline{\lambda}$.

Theorem 3.2 (Strong duality) Let $f : R^{|J_1|} \times R^{|K_1|} \to R^l$ and $g : R^{|J_2|} \times R^{|K_2|} \to R^l$ be differentiable functions and let ($\bar{x}^1$, $\bar{y}^1$, $\bar{x}^2$, $\bar{y}^2$, $\bar{w}^1$, $\bar{w}^2$,$\bar{\lambda}$, $\bar{p}$, $\bar{r}$) be a weak efficient solution of (P3.1). Suppose that

(*i*) the matrices $\nabla_{y^1y^1} f_i(\bar{x}^1, \bar{y}^1)$are non singular for $i = 1, 2, \ldots, l$,

(*ii*) the matrices $\nabla_{y^2y^2} g_i(\bar{x}^2, \bar{y}^2)$are non singular for $i = 1, 2, \ldots, l$,

(*iii*) the set $\{\nabla_{y^2} g_1(\bar{x}^2, \bar{y}^2) - \bar{w}^1 + \nabla_{y^2y^2} g_1(\bar{x}^2, \bar{y}^2)\bar{r}^1, \ldots, \nabla_{y^2} g_l(\bar{x}^2, \bar{y}^2) - \bar{w}^l + \nabla_{y^2y^2} g_l(\bar{x}^2, \bar{y}^2)\bar{r}^l\}$ is linearly independent,

(*iv*) for some $\varrho \in R^l_+ (\varrho > 0)$ and $\bar{r}^i \in R^{|K_2|}$, $\bar{r}^i \neq 0 (i = 1, 2, \ldots, l)$ implies that $\sum_{i=1}^{l} \varrho_i (\bar{r}^i)^T \big[\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) - \bar{w}_i^2 + \nabla_{y^2y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i\big] \neq 0$,

(*v*) $\sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{y^1}(\nabla_{y^1y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}_i))\bar{p}_i \notin span\{\nabla_{y^1} f_1(\bar{x}^1, \bar{y}^1), ..., \nabla_{y^1} f_l(\bar{x}^1, \bar{y}^1)\} \setminus \{0\}$,

(*vi*) $\sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{y^1}(\nabla_{y^1y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}_i))\bar{p}_i = 0$ implies $\bar{p}_i = 0 \forall\ i$, and

(*vii*) K is closed convex pointed cone with $R^l_+ \subseteq K$.

Then $\bar{q}_i = 0$ and $\bar{s}_i = 0, i = 1, 2, \ldots, l$, there exist $\bar{z}^1 \in R^{|J_1|}$ and $\bar{z}_i^2 \in R^{|J_2|}, i = 1, 2, \ldots, l$ such that $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{q}_1 = 0, \bar{q}_2 = 0, \ldots, \bar{q}_l = 0, \bar{s}_1 = 0, \bar{s}_2 = 0, \ldots, \bar{s}_l = 0)$ is feasible for $(D3.1)_{\overline{\lambda}}$ and the objective function values of (P3.1) and $(D3.1)_{\overline{\lambda}}$ are equal. Furthermore, if the assumptions of Theorem 3.1 are satisfied for all feasible solutions of (P3.1) and $(D3.1)_{\overline{\lambda}}$, then $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{\lambda}, \bar{q}_1 = 0, \bar{q}_2 = 0, \ldots, \bar{q}_l = 0, \bar{s}_1 = 0, \bar{s}_2 = 0, \ldots, \bar{s}_l = 0)$ is an efficient solution for $(D3.1)_{\overline{\lambda}}$.

Proof Since $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{w}^1, \bar{w}^2, \bar{\lambda}, \bar{p}, \bar{r})$ is a weak efficient solution of (P3.1), there exist $\alpha \in R^l_+, \beta \in C_3, \gamma \in C_4, \delta \in R_+$, and $\eta \in R$ such that the following by Fritz John optimality conditions [38] are satisfied at $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{w}^1, \bar{w}^2, \bar{\lambda}, \bar{p}, \bar{r})$:

$$\begin{aligned}&\big(x^1 - \bar{x}^1\big)\Big\{\sum_{i=1}^{l} \alpha_i(\nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1) + \bar{z}^1) + \sum_{i=1}^{l} \bar{\lambda}_i \nabla_{y^1 x^1} f_i(\bar{x}^1, \bar{y}^1)\big[\beta - (\alpha^T e_l)\bar{y}^1\big] \\ &\quad + \sum_{i=1}^{l} \bar{\lambda}_i\big(\nabla_{x^1}(\nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}_i)\big)\left[\beta - (\alpha^T e_l)\left(\bar{y}^1 + \frac{1}{2}\bar{p}_i\right)\right]\Big\} \\ &\geqq 0 \ \forall x^1 \in C_1,\end{aligned} \tag{3.27}$$

$$\begin{aligned}&(x^2 - \bar{x}^2)\Big\{\sum_{i=1}^{l} \alpha_i\left[\nabla_{x^2} g_i(\bar{x}^2, \bar{y}^2) + \bar{z}^2_i - \frac{1}{2}(\nabla_{x^2}(\nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i))\bar{r}^i\right] \\ &\quad + \sum_{i=1}^{l} \bar{\lambda}_i\big[\nabla_{y^2 x^2} g_i(\bar{x}^2, \bar{y}^2) + \nabla_{x^2}(\nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i)\big][\gamma - \delta\bar{y}^2]\Big\} \\ &\geqq 0 \ \forall x^2 \in C_2,\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\sum_{i=1}^{l} \alpha_i \nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1) + \sum_{i=1}^{l} \bar{\lambda}_i \nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\big[\beta - (\alpha^T e_l)\bar{y}^1\big] \\ &+ \sum_{i=1}^{l} \bar{\lambda}_i\big[\nabla_{y^1}(\nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}_i)\big]\left[\beta - (\alpha^T e_l)\left(\bar{y}^1 + \frac{1}{2}\bar{p}_i\right)\right] \\ &\quad - \sum_{i=1}^{l} \bar{\lambda}_i[\nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1) + \nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}_i](\alpha^T e_l) = 0,\end{aligned} \tag{3.29}$$

$$\begin{aligned}&\sum_{i=1}^{l} \alpha_i\big[\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) - \bar{w}^2_i - \frac{1}{2}(\nabla_{y^2}(\nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i)\bar{r}^i)\big] \\ &+ \sum_{i=1}^{l} \bar{\lambda}_i\big[\nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2) + \nabla_{y^2}(\nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i)\big][\gamma - \delta\bar{y}^2] \\ &\quad -\delta \sum_{i=1}^{l} \bar{\lambda}_i(\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) - \bar{w}^2_i + \nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i) = 0,\end{aligned} \tag{3.30}$$

$$\begin{aligned}&(\lambda_i - \bar{\lambda}_i)\bigg(\big[\beta - (\alpha^T e_l)\bar{y}^1\big]\nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1) + \big[\gamma - \delta\bar{y}^2\big]\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) \\ &\quad -\beta\bar{w}^1 + \eta + \left[\beta - (\alpha^T e_l)\left(\bar{y}^1 + \frac{1}{2}\bar{p}_i\right)\right]^T \nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}_i \\ &\quad +[\gamma - \delta\bar{y}^2]^T(\nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i - \bar{w}^2_i)\bigg) \geqq 0, i = 1, 2, \ldots, l\end{aligned} \tag{3.31}$$

$$\left[(\beta - (\alpha^T e_l)(\bar{y}^1 + \bar{p}_i))\bar{\lambda}_i\right]^T \nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1) = 0, i = 1, 2, \ldots, l \tag{3.32}$$

$$\left[(\gamma - \delta \bar{y}^2)\bar{\lambda}_i - \alpha_i \bar{r}^i\right]^T \nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2) = 0, i = 1, 2, \ldots, l, \tag{3.33}$$

$$\beta^T \sum_{i=1}^{l} \bar{\lambda}_i \left[\nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1) - \bar{w}^1 + \nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}_i\right] = 0, \tag{3.34}$$

$$\gamma^T \sum_{i=1}^{l} \bar{\lambda}_i \left[\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) - \bar{w_i^2} + \nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i\right] = 0, \tag{3.35}$$

$$\delta(\bar{y}^2)^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) - \bar{w_i^2} + \nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i) = 0, \tag{3.36}$$

$$\beta \in N_E(\bar{w}^1), \tag{3.37}$$

$$\alpha_i \bar{y}^2 + (\gamma - \delta \bar{y}^2)\bar{\lambda}_i \in N_{B_i}(\bar{w_i^2}), i = 1, 2, \ldots, l \tag{3.38}$$

$$\bar{z}^1 \in D, (\bar{z}^1)^T \bar{x}^1 = s(\bar{x}^1 | H) \tag{3.39}$$

$$\bar{z_i^2} \in D_i, (\bar{z_i^2})^T \bar{x}^2 = s(\bar{x}^2 | D_i), i = 1, 2, \ldots, l \tag{3.40}$$

$$\eta^T(\bar{\lambda}^T e_l - 1) = 0, \tag{3.41}$$

$$(\alpha, \beta, \gamma, \delta, \eta) \neq 0. \tag{3.42}$$

By hypothesis (i), Eq.3.32 gives

$$\beta = (\alpha^T e_l)(\bar{y}^1 + \bar{p}_i), i = 1, 2, \ldots, l. \tag{3.43}$$

Since the matrices $\nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2) for\ i = 1, 2, \ldots, l$ are non singular, Eq.3.33 yields

$$(\gamma - \delta \bar{y}^2)\bar{\lambda}_i = \alpha_i \bar{r}^i, i = 1, 2, \ldots, l. \tag{3.44}$$

Also, $R_+^l \subseteq K \Rightarrow K^* \subseteq R_+^l$ and since $\bar{\lambda} \in \text{int } K^*$, therefore $\bar{\lambda} > 0$.

Now, we claim that $\alpha_i \neq 0, i = 1, 2, \ldots, l$. Otherwise, if for some $k_0, \alpha_{k_0} = 0$, then it follows from $\lambda_{k_0} > 0$, and Eq.3.44 that $\gamma = \delta \bar{y}^2$.

From Eqs. 3.30 and 3.31, we get

$$\sum_{i=1}^{l} (\alpha_i - \delta \bar{\lambda}_i)(\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) - \bar{w_i^2}) + \sum_{i=1}^{l} \bar{\lambda}_i \nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)[\gamma - \delta \bar{y}^2 - \delta \bar{r}^i]$$
$$+ \sum_{i=1}^{l} \nabla_{y^2}(\nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i)\left[(\gamma - \delta \bar{y}^2)\bar{\lambda}_i - \frac{1}{2}\alpha_i \bar{r}^i\right] = 0 \tag{3.45}$$

and

$$\big[\beta-(\alpha^Te_l)\bar{y}^1\big]\nabla_{y^1}f_i(\bar{x}^1,\bar{y}^1)+\big[\gamma-\delta\bar{y}^2\big]\nabla_{y^2}g_i(\bar{x}^2,\bar{y}^2)-\beta\bar{w}^1+\eta$$
$$+\left[\beta-(\alpha^Te_l)\left(\bar{y}^1+\frac{1}{2}\bar{p}_i\right)\right]^T\nabla_{y^1y^1}f_i(\bar{x}^1,\bar{y}^1)\bar{p}_i+\big[\gamma-\delta\bar{y}^2\big]^T$$
$$\times(\nabla_{y^2y^2}g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i-\bar{w_i^2})=0, i-1,2,\dots,l. \tag{3.46}$$

By using Eq. 3.44, 3.45 follows that

$$\sum_{i=1}^{l}(\alpha_i-\delta\bar{\lambda}_i)\big[\nabla_{y^2}g_i(\bar{x}^2,\bar{y}^2)-\bar{w_i^2}+\nabla_{y^2y^2}g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i\big]$$
$$+\frac{1}{2}\sum_{i=1}^{l}\bar{\lambda}_i\big[\nabla_{y^2}(\nabla_{y^2y^2}g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i)\big]\big[\gamma-\delta\bar{y}^2\big]=0,$$

which on using hypothesis (iii) and $\gamma=\delta\bar{y}^2$ yields

$$\alpha_i=\delta\bar{\lambda}_i, i=1,2,\dots,l.$$

As $\bar{\lambda}_i>0, i=1,2,\dots,l$ and $\alpha_{k_0}=0$, for some k_0, the above equation shows $\delta=0$. Now, using $\delta=0$, the equations $\gamma=\delta\bar{y}^2$ and $\alpha_i=\delta\bar{\lambda}_i, i=1,2,\dots,l$ give $\gamma=0$ and $\alpha_i=0,\forall\ i$. Further, Eqs.3.43 and 3.46 implies $\beta=0$ and $\eta=0$, respectively.

Consequently,$(\alpha,\beta,\gamma,\delta,\eta)=0$, contradicting Eq. 3.42. Hence $\alpha_i>0,\forall\ i$.

Subtracting Eq.3.36 from 3.35 yields

$$\big[\gamma-\delta\bar{y}^2\big]^T\sum_{i=1}^{l}\bar{\lambda}_i\big[\nabla_{y^2}g_i(\bar{x}^2,\bar{y}^2)-\bar{w_i^2}+\nabla_{y^2y^2}g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i\big]=0,$$

Using Eq. 3.44, we get

$$\sum_{i=1}^{l}\alpha_i(\bar{r}^i)^T\big[\nabla_{y^2}g_i(\bar{x}^2,\bar{y}^2)-\bar{w_i^2}+\nabla_{y^2y^2}g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i\big]=0.$$

By the hypothesis (iv) with $\alpha_i>0, i=1,2,\dots,l$, we have

$$\bar{r}^i=0, i=1,2,\dots,l. \tag{3.47}$$

As $\bar{\lambda}_i>0, i=1,2,\dots,l$, Eq.3.44 yields

$$\gamma=\delta\bar{y}^2. \tag{3.48}$$

Using Eqs. 3.47 and 3.48 in Eq.3.30, we get

$$\sum_{i=1}^{l}(\alpha_i - \delta\bar{\lambda}_i)\big[\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) - \bar{w}_i^2\big] = 0,$$

which on using hypothesis (iii) and Eq.3.47 gives

$$\alpha_i = \delta\bar{\lambda}_i, i = 1, 2, \ldots, l. \tag{3.49}$$

From Eq.3.49 and $\bar{\lambda}^T e_l = 1$, it is clear that $\alpha^T e_l = \delta(\bar{\lambda}^T e_l) = \delta$. Since $\alpha_i > 0, i = 1, 2, \ldots, l$, therefore

$$\delta > 0. \tag{3.50}$$

Equation 3.48 yields

$$\bar{y}^2 = \frac{\gamma}{\delta} \in C_4.$$

Now using Eq. 3.43 and $\alpha_i > 0, \forall\, i$ in Eq. 3.29, we get

$$\sum_{i=1}^{l} \bar{\lambda}_i \big[\nabla_{y^1}(\nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}_i)\big]\bar{p}_i = \frac{-2}{\alpha^T e_l} \sum_{i=1}^{l} \nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1)\big[\alpha_i - (\alpha^T e_l)\bar{\lambda}_i\big]$$

which by hypothesis (v) and (vi) implies

$$\bar{p}_i = 0, i = 1, 2, \ldots, l. \tag{3.51}$$

From Eqs. 3.51 and 3.43, we obtain

$$\beta = (\alpha^T e_l)\bar{y}^1. \tag{3.52}$$

Also, from Eq. 3.52, we have $\bar{y}^1 = \frac{\beta}{(\alpha^T e_l)} \in C_3$.

Using Eqs. 3.49-3.52 in Eq.3.27, we have

$$(x^1 - \bar{x}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i(\nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1) + \bar{z}^1) \geqq 0 \forall\, x^1 \in C_1. \tag{3.53}$$

Let $x^1 \in C_1$. Then $x^1 + \bar{x}^1 \in C_1$, as C_1 is closed convex cone, and so Eq.3.53 implies

$$\sum_{i=1}^{l} \bar{\lambda}_i(\nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1) + \bar{z}^1) \in C_1^*.$$

Moreover, Eqs. 3.28 and 3.47-3.50 give

$$(x^2 - \bar{x}^2)^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^2} g_i(\bar{x}^2, \bar{y}^2) + \bar{z_i^2}) \geqq 0 \forall\ x^2 \in C_2. \tag{3.54}$$

Let $x^2 \in C_2$. Then $x^2 + \bar{x}^2 \in C_2$, as C_2 is closed convex cone, and so Eq.3.54 implies

$$\sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^2} g_i(\bar{x}^2, \bar{y}^2) + \bar{z_i^2}) \in C_2^*.$$

Let $x^2 = 0$ in Eq.3.54 Then $(\bar{x}^2)^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^2} g_i(\bar{x}^2, \bar{y}^2) + \bar{z_i^2}) \leqq 0$.

Thus$(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{q}_1 = 0, \bar{q}_2 = 0, \ldots, \bar{q}_l = 0, \bar{s}_1 = 0, \bar{s}_2 = 0, \ldots, \bar{s}_l = 0)$ satisfies the dual constraints from Eqs.3.7-3.12 and so it is a feasible solution for the dual problem $(D3.1)_{\bar{\lambda}}$.

Now, letting $x^1 = 0$ and $x^1 = 2\bar{x}^1$ in Eq.3.53, we get

$$(\bar{x}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1) + \bar{z}^1) = 0 \text{ or } (\bar{x}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1))$$
$$= -(\bar{x}^1)^T \bar{z}^1 = -s(\bar{x}^1 | H).$$

Using $\alpha > 0$, Eqs.3.51 and 3.52 in Eq.3.34, we get

$$(\bar{y}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i \nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1) = (\bar{y}^1)^T \bar{w}^1. \tag{3.55}$$

Moreover, since $\beta = (\alpha^T e_l)\bar{y}^1, \alpha > 0$ and $\beta \in N_E(\bar{w}^1)$ implies $\bar{y}^1 \in N_E(\bar{w}^1)$ so that

$$(\bar{y}^1)^T \bar{w}^1 = s(\bar{y}^1 | E). \tag{3.56}$$

From Eqs. 3.38,3.48-3.50, we have

$$\bar{y}^2 \in N_{B_i}(\bar{w_i^2}), i = 1, 2, \ldots, l \ as \bar{\lambda}_i > 0.$$

Since B_i, i=1,2,...,l are compact convex set in $R^{|k_2|}$, therefore, we get

$$(\bar{y}^2)^T \bar{w_i^2} = s(\bar{y}^2 | B_i), i = 1, 2, \ldots, l. \tag{3.57}$$

Therefore, using Eqs, 3.39, 3.40, 3.47, 3.51, 3.55-3.57, we obtain

$$\Big\{f_1(\bar{x}^1,\bar{y}^1)+s(\bar{x}^1|H)+g_1(\bar{x}^2,\bar{y}^2)+s(\bar{x}^2|D_1)-(\bar{y}^2)^T\bar{w}_1^2-(\bar{y}^1)^T\sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{y^1}f_i(\bar{x}^1,\bar{y}^1)$$
$$+\nabla_{y^1y^1}f_i(\bar{x}^1,\bar{y}^1)\bar{p}_i)-\frac{1}{2}\sum_{i=1}^{l}\bar{\lambda}_i(\bar{p}_i^T\nabla_{y^1y^1}f_i(\bar{x}^1,\bar{y}^1)\bar{p}_i)$$
$$-\frac{1}{2}(\bar{r}^1)^T\nabla_{y^2y^2}g_1(\bar{x}^2,\bar{y}^2)\bar{r}^1,\ldots,f_l(\bar{x}^1,\bar{y}^1)$$
$$+s(\bar{x}^1|H)+g_l(\bar{x}^2,\bar{y}^2)+s(\bar{x}^2|D_l)-(\bar{y}^2)^T\bar{w}_l^2-(\bar{y}^1)^T\sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{y^1}f_i(\bar{x}^1,\bar{y}^1)$$
$$+\nabla_{y^1y^1}f_i(\bar{x}^1,\bar{y}^1)\bar{p}_i)-\frac{1}{2}\sum_{i=1}^{l}\bar{\lambda}_i(\bar{p}_i^T\nabla_{y^1y^1}f_i(\bar{x}^1,\bar{y}^1)\bar{p}_i)-\frac{1}{2}(\bar{r}^l)^T\nabla_{y^2y^2}g_l(\bar{x}^2,\bar{y}^2)\bar{r}^l\Big\}$$

$$=\Big\{f_1(\bar{x}^1,\bar{y}^1)-s(\bar{y}^1|E)+g_1(\bar{x}^2,\bar{y}^2)-s(\bar{y}^2|B_1)+(\bar{x}^2)^T\bar{z}_1^2-(\bar{x}^1)^T\sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{x^1}f_i(\bar{x}^1,\bar{y}^1)$$
$$+\nabla_{x^1x^1}f_i(\bar{x}^1,\bar{y}^1)\bar{q}_i)-\frac{1}{2}\sum_{i=1}^{l}\bar{\lambda}_i(\bar{q}_i^T\nabla_{x^1x^1}f_i(\bar{x}^1,\bar{y}^1)\bar{q}_i)$$
$$-\frac{1}{2}(\bar{s}^1)^T\nabla_{x^2x^2}g_1(\bar{x}^2,\bar{y}^2)\bar{s}^1,\ldots,f_l(\bar{x}^1,\bar{y}^1)$$
$$-s(\bar{y}^1|E)+g_l(\bar{x}^2,\bar{y}^2)-s(\bar{y}^2|B_l)+(\bar{x}^2)^T\bar{z}_l^2-(\bar{x}^1)^T\sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{x^1}f_i(\bar{x}^1,\bar{y}^1)$$
$$+\nabla_{x^1x^1}f_i(\bar{x}^1,\bar{y}^1)\bar{q}_i)-\frac{1}{2}\sum_{i=1}^{l}\bar{\lambda}_i(\bar{q}_i^T\nabla_{x^1x^1}f_i(\bar{x}^1,\bar{y}^1)\bar{q}_i)-\frac{1}{2}(\bar{s}^l)^T\nabla_{x^2x^2}g_l(\bar{x}^2,\bar{y}^2)\bar{s}^l\Big\}$$

that is, the two objective function values are equal.

Now, let $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{z}^1,\bar{z}^2,\bar{q}_1=0,\bar{q}_2=0,\ldots,\bar{q}_l=0,\bar{s}_1=0,\bar{s}_2=0,\ldots,\bar{s}_l=0)$ is not an efficient solution of $(D3.1)_{\bar{\lambda}}$, then there exist $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{z}^1,\bar{z}^2,\bar{q}_1=0,\bar{q}_2=0,\ldots,\bar{q}_l=0,\bar{s}_1=0,\bar{s}_2=0,\ldots,\bar{s}_l=0)$ feasible for $(D3.1)_{\bar{\lambda}}$, such that

$$\{N_1(x^1,y^1,x^2,y^2,w_1^2,p,r^1),\ldots,N_l(x^1,y^1,x^2,y^2,w_l^2,p,r^l)\}$$
$$-\{M_1(u^1,v^1,u^2,v^2,z_1^2,q,s^1),\ldots,M_l(u^1,v^1,u^2,v^2,z_l^2,q,s^l)\}\in -K\setminus\{0\},$$

which contradicts Theorem 3.1. Hence $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{z}^1,\bar{z}^2,\bar{q}_1=0,\bar{q}_2=0,\ldots,\bar{q}_l=0,\bar{s}_1=0,\bar{s}_2=0,\ldots,\bar{s}_l=0)$ is an efficient solution of $(D3.1)_{\bar{\lambda}}$.

Theorem 3.3(Converse duality) Let $f: R^{|J_1|}\times R^{|K_1|}\to R^l$ and $g: R^{|J_2|}\times R^{|K_2|}\to R^l$ be differentiable functions and let $(\bar{u}^1,\ \bar{v}^1,\ \bar{u}^2,\ \bar{v}^2,\ \bar{z}^1,\ \bar{z}^2,\bar{\lambda},\ \bar{q},\ \bar{s})$ be a weak efficient solution of (D3.1). Suppose that

(i) the matrices $\nabla_{x^1x^1} f_i(\bar{u}^1,\ \bar{v}^1)$are non singular for $i = 1, 2, \ldots, l$,

(ii) the matrices $\nabla_{x^2x^2} g_i(\bar{u}^2,\ \bar{v}^2)$are non singular for $i = 1, 2, \ldots, l$,

(iii) the set $\{\nabla_{x^2} g_1(\bar{u}^2,\ \bar{v}^2) + \bar{z}^1 + \nabla_{x^2x^2} g_1(\bar{u}^2, \bar{v}^2)\bar{s}^1, \ldots, \nabla_{x^2} g_l(\bar{u}^2,\ \bar{v}^2) + \bar{z}^l + \nabla_{x^2x^2} g_l(\bar{u}^2, \bar{v}^2)\bar{s}^l\}$ is linearly independent,

(iv) for some $\varrho \in R^l_+ (\varrho > 0)$ and $\bar{s}^i \in R^{|J_2|}$, $\bar{s}^i \neq 0 (i = 1, 2, \ldots, l)$ implies that $\sum\limits_{i=1}^{l} \varrho_i (\bar{s}^i)^T [\nabla_{x^2} g_i(\bar{u}^2, \bar{v}^2) + \bar{z}_i^2 + \nabla_{x^2x^2} g_i(\bar{u}^2, \bar{v}^2)\bar{s}^i] \neq 0$,

(v) $\sum\limits_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^1}(\nabla_{x^1x^1} f_i(\bar{u}^1, \bar{v}^1)\bar{q}_i))\bar{q}_i \notin span\{\nabla_{x^1} f_1(\bar{u}^1, \bar{v}^1), \ldots, \nabla_{x^1} f_l(\bar{u}^1, \bar{v}^1)\} \setminus \{0\}$,

(vi) $\sum\limits_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^1}(\nabla_{x^1x^1} f_i(\bar{u}^1, \bar{v}^1)\bar{q}_i))\bar{q}_i = 0$ implies $\bar{q}_i = 0\ \forall\ i$, and

(vii) K is closed convex pointed cone with $R^l_+ \subseteq K$.

Then $\bar{p}_i = 0$ and $\bar{r}_i = 0, i = 1, 2, \ldots, l$, there exist $\bar{w}^1 \in R^{|K_1|}$ and $\bar{w}_i^2 \in R^{|K_2|}, i = 1, 2, \ldots, l$ such that $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{w}^1, \bar{w}^2, \bar{p}_1 = 0, \bar{p}_2 = 0, \ldots, \bar{p}_l = 0, \bar{r}_1 = 0, \bar{r}_2 = 0, \ldots, \bar{r}_l = 0)$ is feasible for $(P3.1)_{\overline{\lambda}}$ and the objective function values of $(P3.1)_{\overline{\lambda}}$ and (D3.1) are equal. Furthermore, if the assumptions of Theorem 3.1 are satisfied for all feasible solutions of $(P3.1)_{\overline{\lambda}}$ and (D3.1), then $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{w}^1, \bar{w}^2, \bar{p}_1 = 0, \bar{p}_2 = 0, \ldots, \bar{p}_l = 0, \bar{r}_1 = 0, \bar{r}_2 = 0, \ldots, \bar{r}_l = 0)$ is an efficient solution for $(P3.1)_{\overline{\lambda}}$.

Proof It follows on the lines of Theorem 3.2.

Chapter 4

Higher Order Mixed Multiobjective Symmetric Duality

4.1 Introduction

The aim of this chapter is to make some investigations concerning duality for higher order mixed multiobjective symmetric duality. One practical example of higher-order duality is that it provides tighter bounds for the value of objective function of the primal problem when approximations are used because there are more parameters involved. Higher-order duality in nonlinear programming has been studied in the last few years by many researchers [1, 4, 8, 15, 26, 33]. A class of higher-order dual problems for nonlinear programming problems first formulated by Mangasarian [26].

In this chapter, we studied weak duality theorem for a higher order mixed multiobjective symmetric duality under the assumptions of higher order (F, ρ, α, d)-convexity. This chapter is divided into three sections. Section 4.1 is introductory, Section 4.2 contains notations and definitions. In Section 4.3, we consider a pair of higher order mixed multiobjective symmetric duality and prove weak duality theorem under (F, ρ, α, d)-convexity assumptions.

4.2 Notations and Definitions

Consider the following multiobjective programming problem:

$$\textbf{(P)} \qquad \text{K-Minimize } f(x)$$
$$\text{subject to } x \in X^0 = \{x \in S : -h(x) \in Q\},$$

where $S \subseteq R^n$ be open, $f : S \to R^k, h : S \to R^m, K$ is closed convex pointed cone and Q is closed convex cone with nonempty interiors in R^k and R^m, respectively.

Definition 4.1 A differentiable function $f : R^n \times R^m \to R$ is said to be higher order (F, ρ, α, d) convex in the first variable at $u \in R^n$ for fixed $v \in R^m$, if there exists a sublinear functional $F : R^n \times R^n \times R^n \to R$ and $\alpha : R^n \times R^n \to R_+ \setminus \{0\}, \rho \in R$ and $d : R^n \times R^n \to R$ be a metric on R such that for $x \in R^n, p_i \in R^n, i = 1, 2, \ldots, l$.

$$f_i(x, v) - f_i(u, v) \geqq F(x, u; \nabla_x f(u, v) + \nabla_{p_i} h(u, v, p_i)) + h(u, v, p_i) - p_i^T \nabla_{p_i} h(u, v, p_i).$$

4.3 Higher order mixed multiobjective symmetric duals

Consider the following pair of multiobjective higher order mixed symmetric dual programs:

Primal problem (HMP)

K-minimize

$$\begin{aligned} L(x^1, x^2, y^1, y^2, p, q) \quad = \quad & \Big\{ f_1^1(x^1, y^1) + h_1^1(x^1, y^1, p_1) - p_1^T \nabla_{p_1} h_1^1(x^1, y^1, p_1) - (y^1)^T \sum_{i=1}^{l} \lambda_i \\ & \big[\nabla_{y^1} f_i^1(x^1, y^1) + \nabla_{p_i} h_i^1(x^1, y^1, p_i)\big] + f_1^2(x^2, y^2) + h_1^2(x^2, y^2, q_1) \\ & -q_1^T \nabla_{q_1} h_1^2(x^2, y^2, q_1), \ldots, f_l^1(x^1, y^1) + h_l^1(x^1, y^1, p_l) - \\ & p_l^T \nabla_{p_l} h_l^1(x^1, y^1, p_l) - (y^1)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{y^1} f_i^1(x^1, y^1) + \nabla_{p_i} h_i^1(x^1, y^1, p_i)\big] \\ & + f_l^2(x^2, y^2) + h_l^2(x^2, y^2, q_l) - q_l^T \nabla_{q_l} h_l^2(x^2, y^2, q_l) \Big\} \end{aligned}$$

subject to

$$-\left(\sum_{i=1}^{l}\lambda_i\big[\nabla_{y^1}f_i^1(x^1,y^1)+\nabla_{p_i}h_i^1(x^1,y^1,p_i)\big]\right)\in C_2^* \tag{4.1}$$

$$-\left(\sum_{i=1}^{l}\lambda_i\big[\nabla_{y^2}f_i^2(x^2,y^2)+\nabla_{q_i}h_i^2(x^2,y^2,q_i)\big]\right)\in C_4^* \tag{4.2}$$

$$(y^2)^T\sum_{i=1}^{l}\lambda_i\Big[\nabla_{y^2}f_i^2(x^2,y^2)+\nabla_{q_i}h_i^2(x^2,y^2,q_i)\Big]\geqq 0 \tag{4.3}$$

$$x^1\in C_1, x^2\in C_3, \lambda>0 \text{ and } \lambda^T e=1$$

Dual problem(HMD)

K-maximize

$$\begin{aligned}M(u^1,u^2,v^1,v^2,r,s) \;=\; &\Big\{f_1^1(u^1,v^1)+g_1^1(u^1,v^1,r_1)-r_1^T\nabla_{r_1}g_1^1(u^1,v^1,r_1)-(u^1)^T\sum_{i=1}^{l}\lambda_i\\ &\big[\nabla_{x^1}f_i^1(u^1,v^1)+\nabla_{r_i}g_i^1(u^1,v^1,r_i)\big]+f_1^2(u^2,v^2)+g_1^2(u^2,v^2,s_1)\\ &-s_1^T\nabla_{s_1}g_1^2(u^2,v^2,s_1),\ldots,f_l^1(u^1,v^1)+g_l^1(u^1,v^1,r_l)\\ &-r_l^T\nabla_{r_l}g_l^1(u^1,v^1,r_l)-(u^1)^T\sum_{i=1}^{l}\lambda_i\big[\nabla_{x^1}f_i^1(u^1,v^1)+\nabla_{r_i}g_i^1(u^1,v^1,r_i)\big]\\ &+f_l^2(u^2,v^2)+g_l^2(u^2,v^2,s_l)-s_l^T\nabla_{s_l}g_l^2(u^2,v^2,s_l)\Big\}\end{aligned}$$

subject to

$$\sum_{i=1}^{l}\lambda_i\big[\nabla_{x^1}f_i^1(u^1,v^1)+\nabla_{r_i}g_i^1(u^1,v^1,r_i)\big]\in C_1^* \tag{4.4}$$

$$\sum_{i=1}^{l}\lambda_i\big[\nabla_{x^2}f_i^2(u^2,v^2)+\nabla_{s_i}g_i^2(u^2,v^2,s_i)\big]\in C_3^* \tag{4.5}$$

$$(u^2)^T\sum_{i=1}^{l}\lambda_i\big[\nabla_{x^2}f_i^2(u^2,v^2)+\nabla_{s_i}g_i^2(u^2,v^2,s_i)\big]\leqq 0 \tag{4.6}$$

$$v^1\in C_2, v^2\in C_4, \lambda>0 \text{ and } \lambda^T e=1.$$

where

(i) $f_i^1 : R^{|J_1|}\times R^{|K_1|}\to R$, $f_i^2 : R^{|J_2|}\times R^{|K_2|}\to R$,
$g_i^1 : R^{|J_1|}\times R^{|K_1|}\times R^{|K_1|}\to R$, $g_i^2 : R^{|J_2|}\times R^{|K_2|}\times R^{|K_2|}\to R$,
$h_i^1 : R^{|J_1|}\times R^{|K_1|}\times R^{|J_1|}\to R$, $h_i^2 : R^{|J_2|}\times R^{|K_2|}\times R^{|J_2|}\to R$,
$i=1,\ldots,l$, are differentiable functions.

(ii) C_1, C_2, C_3, C_4 are closed convex cones in $R^{|J_1|}, R^{|K_1|}, R^{|J_2|}, R^{|K_2|}$ with non-empty interiors, respectively.

(iii) $C_1^*, C_2^*, C_3^*, C_4^*$ are polar cones of C_1, C_2, C_3, and C_4 respectively.

(iv) $p_i \in R^{|J_1|}, r_i \in R^{|K_1|}, q_i \in R^{|J_2|}, s_i \in R^{|K_2|}$.

(v) $e = (1, \ldots, 1)^T$ is a vector in R^l.

Theorem 4.1(Weak duality) Let $(x^1, x^2, y^1, y^2, p, q)$ and $(u^1, u^2, v^1, v^2, r, s)$ be feasible solutions of (HMP) and (HMD), respectively. Let for $i = 1, 2, \ldots, l$

(i) $f_i^1(\cdot, v^1)$ is higher order $(F, \alpha_1, \rho_i^1, d_i^1)$-convex at u^1 with respect to $g_i^1(u^1, v^1, r_i)$ and $f_i^2(\cdot, v^2)$ is higher order $(F, \alpha_2, \rho_i^2, d_i^2)$-convex at u^2 with respect to $g_i^2(u^2, v^2, s_i)$.

(ii) -$f_i^1(x^1, \cdot)$ is higher order $(G, \alpha_2, \rho_i^1, d_i^1)$-convex at y^1 with respect to $-h_i^1(x^1, y^1, p_i)$ and $-f_i^2(x^2, \cdot)$ is higher order $(G, \alpha_2, \rho_i^2, d_i^2)$-convex at y^2 with respect to $-h_i^2(x^2, y^2, q_i)$.

(iii) either

(a) $\sum_{i=1}^{l} \lambda_i \left[\rho_i^1 (d_i^1(x, u))^2 + \rho_i^2 (d_i^2(x, u))^2\right] \geqq 0$ or

(b) $\sum_{i=1}^{l} \lambda_i \left[\rho_i^1 (d_i^1(v, y))^2 + \rho_i^2 (d_i^2(v, y))^2\right] \geqq 0$ or

(c) $\rho_i^1 \geqq 0$ and $\rho_i^2 \geqq 0 \ \forall \ i$

where the sublinear functional $F : R^n \times R^n \times R^n \to R$ and $G : R^m \times R^m \times R^m \to R$ satisfy the following conditions:

(iv) $F(x, u; a) + \alpha_1^{-1} a^T u^1 \geqq 0, \forall \ a \in C_1^*$.

(v) $F(x, u; b) + \alpha_2^{-1} b^T u^2 \geqq 0, \forall \ b \in C_3^*$.

(vi) $G(v, y; c) + \alpha_1^{-1} c^T y^1 \geqq 0, \forall \ c \in C_2^*$.

(vii) $G(v, y; d) + \alpha_2^{-1} d^T y^2 \geqq 0, \forall \ d \in C_4^*$.

Then

$$L(x^1, x^2, y^1, y^2) \nleq M(u^1, u^2, v^1, v^2, r, s).$$

Proof Assume to the contrary that

$$L(x^1, x^2, y^1, y^2, p, q) \leq M(u^1, u^2, v^1, v^2, r, s)$$

or

$$\begin{aligned}
&f_i^1(x^1, y^1) + h_i^1(x^1, y^1, p_i) - (p_i)^T \nabla_{p_i} h_i^1(x^1, y^1, p_i) - (y^1)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{y^1} f_i^1(x^1, y^1) \\
&+\nabla_{p_i} h_i^1(x^1, y^1, p_i)\big] + f_i^2(x^2, y^2) + h_i^2(x^2, y^2, q_i) - q_i^T \nabla_{q_i} h_i^2(x^2, y^2, q_i) \\
&\leq f_i^1(u^1, v^1) + g_i^1(u^1, v^1, r_i) - (r_i)^T \nabla_{r_i} g_i^1(u^1, v^1, r_i) - (u^1)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{x^1} f_i^1(u^1, v^1) \\
&+\nabla_{r_i} g_i^1(u^1, v^1, r_i)\big] + f_i^2(u^2, v^2) + g_i^2(u^2, v^2, s_i) - (s_i)^T \nabla_{s_i} g_i^2(u^2, v^2, s_i), i = 1, 2, \ldots, l.
\end{aligned}$$

Since $\lambda > 0$ and $\lambda^T e_l = 1$

$$\begin{aligned}
&\sum_{i=1}^{l} \lambda_i \Bigg[f_i^1(x^1, y^1) + h_i^1(x^1, y^1, p_i) - (p_i)^T \nabla_{p_i} h_i^1(x^1, y^1, p_i) - (y^1)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{y^1} f_i^1(x^1, y^1) \\
&+\nabla_{p_i} h_i^1(x^1, y^1, p_i)\big] + f_i^2(x^2, y^2) + h_i^2(x^2, y^2, q_i) - q_i^T \nabla_{q_i} h_i^2(x^2, y^2, q_i) \Bigg] \\
&< \sum_{i=1}^{l} \lambda_i \Bigg[f_i^1(u^1, v^1) + g_i^1(u^1, v^1, r_i) - (r_i)^T \nabla_{r_i} g_i^1(u^1, v^1, r_i) - (u^1)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{x^1} f_i^1(u^1, v^1) \\
&+\nabla_{r_i} g_i^1(u^1, v^1, r_i)\big] + f_i^2(u^2, v^2) + g_i^2(u^2, v^2, s_i) - (s_i)^T \nabla_{s_i} g_i^2(u^2, v^2, s_i) \Bigg] \qquad (4.7)
\end{aligned}$$

Since $(x^1, y^1, x^2, y^2, \lambda, p, q)$ and $(u^1, v^1, u^2, v^2, r, s)$ be the feasible for (HMP) and (HMD), $\alpha_1(x, u) > 0$, by the dual constraint (4.4), the vector

$$a = \alpha_1(x, u) \Bigg[\sum_{i=1}^{l} \lambda_i \big[\nabla_{x^1} f_i^1(u^1, v^1) + \nabla_{r_i} g_i^1(u^1, v^1, r_i)\big] \Bigg] \in C_1^*$$

and so from the hypothesis (iv), we obtain

$$F_{x,u}(a) + \alpha_1^{-1} a^T u^1 \geqq 0$$

Similarly

$$F_{x,u}(b) + \alpha_2^{-1} b^T u^2 \geqq 0$$

for the vector

$$b = \alpha_2(x, u) \Bigg[\sum_{i=1}^{l} \lambda_i \big[\nabla_{x^2} f_i^2(u^2, v^2) + \nabla_{s_i} g_i^2(u^2, v^2, s_i)\big] \Bigg] \in C_3^*$$

Similarly

$$G_{v,y}(c) + \alpha_1^{-1} c^T y^1 \geqq 0$$

for the vector

$$c = -\alpha_1(v,y)\Bigg[\sum_{i=1}^{l} \lambda_i \big[\nabla_{y^1} f_i^1(x^1, y^1) + \nabla_{p_i} h_i^1(x^1, y^1, p_i)\big]\Bigg] \in C_2^*$$

Similarly

$$G_{v,y}(d) + \alpha_2^{-1} d^T y^2 \geqq 0$$

for the vector

$$d = -\alpha_2(v,y)\Bigg[\sum_{i=1}^{l} \lambda_i \big[\nabla_{y^2} f_i^2(x^2, y^2) + \nabla_{q_i} h_i^2(x^2, y^2, q_i)\big]\Bigg] \in C_4^*$$

By higher order $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$-convexity of $f_i^1(\cdot, v)$, $i = 1, \dots, l$ w.r.t $g_i^1(u^1, v^1, r_i)$, we have

$$\begin{aligned} &f_i^1(x^1, v^1) - f_i^1(u^1, v^1) - g_i^1(u^1, v^1, r_i) + (r_i)^T \nabla_{r_i} g_i^1(u^1, v^1, r_i) \\ &\geqq F_{x,u}\Bigg[\alpha_1(x,u)\big[\nabla_{x^1} f_i^1(u^1, v^1) + \nabla_{r_i} g_i^1(u^1, v^1, r_i)\big]\Bigg] + \rho_i^1 (d_i^1(x,u))^2 \end{aligned}$$

It follows from $\lambda > 0$ and sublinearity of F that

$$\begin{aligned} &\sum_{i=1}^{l} \lambda_i \Bigg[f_i^1(x^1, v^1) - f_i^1(u^1, v^1) - g_i^1(u^1, v^1, r_i) + (r_i)^T \nabla_{r_i} g_i^1(u^1, v^1, r_i)\Bigg] \\ &\geqq F_{x,u}\Bigg[\alpha_1(x,u)\Bigg(\sum_{i=1}^{l} \lambda_i \big[\nabla_{x^1} f_i^1(u^1, v^1) + \nabla_{r_i} g_i^1(u^1, v^1, r_i)\big]\Bigg)\Bigg] + \rho_i^1 (d_i^1(x,u))^2 \end{aligned} \tag{4.8}$$

By higher order $(F, \alpha_2, \rho_i^{(2)}, d_i^{(2)})$-convexity of $f_i^2(\cdot, v)$, $i = 1, \dots, l$ w.r.t $g_i^2(u^2, v^2, s_i)$, we have

$$\begin{aligned} &f_i^2(x^2, v^2) - f_i^2(u^2, v^2) - g_i^2(u^2, v^2, s_i) + (s_i)^T \nabla_{s_i} g_i^2(u^2, v^2, s_i) \\ &\geqq F_{x,u}\Bigg[\alpha_2(x,u)\big[\nabla_{x^2} f_i^2(u^2, v^2) + \nabla_{s_i} g_i^2(u^2, v^2, s_i)\big]\Bigg] + \sum_{i=1}^{l} \lambda_i \rho_i^2 (d_i^2(x,u))^2 \end{aligned}$$

Since $\lambda > 0$ and sublinearity of F that

$$\begin{aligned} &\sum_{i=1}^{l} \lambda_i \big[f_i^2(x^2, v^2) - f_i^2(u^2, v^2) - g_i^2(u^2, v^2, s_i) + s_i^T g_i^2(u^2, v^2, s_i)\big] \\ &\geqq F_{x,u}\Bigg[\alpha_2(x,u)\Bigg(\sum_{i=1}^{l} \lambda_i \big[\nabla_{x^2} f_i^2(u^2, v^2) + \nabla_{s_i} g_i^2(u^2, v^2, s_i)\big]\Bigg)\Bigg] \\ &+ \sum_{i=1}^{l} \lambda_i \rho_i^2 (d_i^2(x,u))^2 \end{aligned} \tag{4.9}$$

Add (4.8) and (4.9), we get

$$\sum_{i=1}^{l}\lambda_i\big[f_i^1(x^1,v^1)+f_i^2(x^2,v^2)-f_i^1(u^1,v^1)-f_i^2(u^2,v^2)-g_i^1(u^1,v^1,r_i)+r_i^T$$
$$\nabla_{r_i}g_i^1(u^1,v^1,r_i)-g_i^2(u^2,v^2,s_i)+s_i^T\nabla_{s_i}g_i^2(u^2,v^2,s_i)\big]$$
$$\geqq F_{x,u}\left[\alpha_1(x,u)\left(\sum_{i=1}^{l}\lambda_i(\nabla_{x^1}f_i^1(u^1,v^1)+\nabla_{r_i}g_i^1(u^1,v^1,s_i)\right)\right]+F_{x,u}\left[\alpha_2(x,u)\left(\sum_{i=1}^{l}\lambda_i\right.\right.$$
$$\left.\left.\big[\nabla_{x^2}f_i^2(u^2,v^2)+\nabla_{s_i}g_i^2(u^2,v^2,s_i)\big]\right)\right]+\sum_{i=1}^{l}\lambda_i\big[\rho_i^1(d_i^1(x,u))^2+\rho_i^2(d_i^2(x,u))^2\big]$$

By hypothesis (iv), (v), and (iii), we get

$$\sum_{i=1}^{l}\lambda_i\big[f_i^1(x^1,v^1)+f_i^2(x^2,v^2)-f_i^1(u^1,v^1)-f_i^2(u^2,v^2)-g_i^1(u^1,v^1,r_i)$$
$$+r_i^T\nabla_{r_i}g_i^1(u^1,v^1,r_i)-g_i^2(u^2,v^2,s_i)+s_i^T\nabla_{s_i}g_i^2(u^2,v^2,s_i)\big]$$
$$\geqq-\alpha_1^{-1}a^Tu^1-\alpha_2^{-1}b^Tu^2 \tag{4.10}$$

Similarly, using hypothesis (ii), (v), (vi) and (iii) and $\lambda>0$, $\alpha_2(v,y)>0$ and sublinearity of G, we get

$$\sum_{i=1}^{l}\lambda_i\big[f_i^1(x^1,y^1)+f_i^2(x^2,y^2)-f_i^1(x^1,v^1)-f_i^2(x^2,v^2)+h_i^1(x^1,y^1,p_i)$$
$$+p_i^T\nabla_{p_i}h_i^1(x^1,y^1,p_i)+h_i^2(x^2,y^2,q_i)-q_i^T\nabla_{q_i}h_i^2(x^2,y^2,q_i)\big]$$
$$\geqq-\alpha_1^{-1}c^Ty^1-\alpha_2^{-1}d^Ty^2 \tag{4.11}$$

Adding (4.10) and (4.11), we get

$$\sum_{i=1}^{l}\lambda_i\big[f_i^1(x^1,y^1)+f_i^2(x^2,y^2)-f_i^1(u^1,v^1)-f_i^2(u^2,v^2)+h_i^1(x^1,y^1,p_i)+p_i^T\nabla_{p_i}h_i^1(x^1,y^1,p_i)$$
$$+h_i^2(x^2,y^2,q_i)-q_i^T\nabla_{q_i}h_i^2(x^2,y^2,q_i)-g_i^1(u^1,v^1,r_i)+r_i^T\nabla_{r_i}g_i^1(u^1,v^1,r_i)-g_i^2(u^2,v^2,s_i)$$
$$+s_i^T\nabla_{s_i}g_i^2(u^2,v^2,s_i)\big]\geqq-\alpha_1^{-1}a^Tu^1-\alpha_2^{-1}b^Tu^2-\alpha_1^{-1}c^Ty^1-\alpha_2^{-1}d^Ty^2$$

Put the value of (a),(b),(c) and (d), we get

$$\sum_{i=1}^{l}\lambda_i\big[f_i^1(x^1,y^1)+h_i^1(x^1,y^1,p_i)-p_i^T\nabla_{p_i}h_i^1(x^1,y^1,p_i)+f_i^2(x^2,y^2)+h_i^2(x^2,y^2,q_i)$$
$$-q_i^T\nabla_{q_i}h_i^2(x^2,y^2,q_i)-f_i^1(u^1,v^1)-f_i^2(u^2,v^2)-g_i^1(u^1,v^1,r_i)+r_i^T\nabla_{r_i}g_i^1(u^1,v^1,r_i)$$
$$-g_i^2(u^2,v^2,s_i)+s_i^T\nabla_{s_i}g_i^2(u^2,v^2,s_i)\big]$$

$$\geqq -(u^1)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{x^1} f_i^1(u^1, v^1) + \nabla_{r_i} g_i^1(u^1, v^1, r_i)\big] - (u^2)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{x^2} f_i^2(u^2, v^2) +$$

$$\nabla_{s_i} g_i^2(u^2, v^2, s_i)\big] + (y^1)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{y^1} f_i^1(x^1, y^1) + \nabla_{p_i} h_i^1(x^1, y^1, p_i)\big] +$$

$$(y^2)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{y^2} f_i^2(x^2, y^2) + \nabla_{q_i} h_i^2(x^2, y^2, q_i)\big]$$

which implies

$$\sum_{i=1}^{l} \lambda_i \Bigg[f_i^1(x^1, y^1) + h_i^1(x^1, y^1, p_i) - (p_i)^T \nabla_{p_i} h_i^1(x^1, y^1, p_i) - (y^1)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{y^1} f_i^1(x^1, y^1)$$

$$+\nabla_{p_i} h_i^1(x^1, y^1, p_i)\big] + f_i^2(x^2, y^2) + h_i^2(x^2, y^2, q_i) - q_i^T \nabla_{q_i} h_i^2(x^2, y^2, q_i) \Bigg]$$

$$\geqq \sum_{i=1}^{l} \lambda_i \Bigg[f_i^1(u^1, v^1) + g_i^1(u^1, v^1, r_i) - r_i^T \nabla_{r_i} g_i^1(u^1, v^1, r_i) - (u^1)^T \sum_{i=1}^{l} \lambda_i \big[\nabla_{x^1} f_i^1(u^1, v^1) +$$

$$\nabla_{r_i} g_i^1(u^1, v^1, r_i)\big] + f_i^2(u^2, v^2) + g_i^2(u^2, v^2, s_i) - s_i^T \nabla_{s_i} g_i^2(u^2, v^2, s_i) \Bigg]$$

$$\Bigg[\because (u^2)^T \sum_{i=1}^{l} \lambda_i (\nabla_{x^2} f_i^2(u^2, v^2) + \nabla_{s_i} g_i^2(u^2, v^2, s_i)) \geqq 0$$

$$\text{and } (y^2)^T \sum_{i=1}^{l} \lambda_i (\nabla_{y^2} f_i^2(x^2, y^2) + \nabla_{q_i} h_i^2(x^2, y^2, q_i)) \geqq 0 \Bigg]$$

which contradicts (4.7)

Hence

$$L(x^1, y^1, x^2, y^2, \lambda, p, q) \nleq M(u^1, v^1, u^2, v^2, \lambda, r, s)$$

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