Daubechies wavelets: Theory and Applications

A Thesis

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by

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Certificate

This is to certify that the dissertation titled **'Daubechies wavelets: Theory and applications'**, submitted to School of mathematics, TIET, Patiala in partial fulfilment for the award of masters of science in mathematics and computing, is a record of bonafide work carried out by Ms. Kanika Gupta, roll no. 301703016, under the supervision and guidance of Dr. Kavita, Assistant Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

All help received by her from various sources have been duly acknowledged.

No part of this dissertation has been submitted elsewhere for award of any other degree.

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It is certified that the above statement made by the candidate is correct to the best of my knowledge.

Dr. Kavita
Supervisor

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Abstract

The very first tool that strikes in mind for signal processing is fourier transform but its incapability to detect the discontinuity and being localized in time only. Short term fourier transform (STFT) was a improvement to fourier transform. According, to Heisenberg uncertainty principle both frequency and time cannot be measured simultaneously. Therefore, wavelets have proved to be useful enough to analyse the non-stationary signal more accurately incomparison to STFT and fourier transform. Wavelets are useful discovery as it has wide number of applications to name a few are medicine, fingerprint verification.

The thesis is a review of Daubechies wavelets with the application in the field of signal processing, partial differential equations (PDE) and applications of wavelets.

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Chapter 1

Introduction

Initially, Fourier transform was used to analyze the given signal. Fourier transform uses the complex exponentials which are periodic in nature. One of the major drawback is that it does not approximate a function at a point of discontinuity. The first modification that was made to Fourier transform was Short Time Fourier Transform(STFT). Since, Fourier transform can only provide information in time. Therefore, it is not suitable to approximate a non-stationary signal which has spectral content changes both in time and frequency domain.

Short term Fourier Transform basically captures the information of longer sinusoidal frequency and phase content into shorter time signal of equal length. Further, Fourier Transform can be applied on each of these segments. The study of wavelets is the combination of pure and applied mathematics, engineering , computer science and also physics. This field is collective work of researchers from various fields. This is one of the reason why they have wide range of applicability in diverse fields. To name a few are: medicine, meteorology, signal processing. Wavelets are serving as standardized tool and are producing better results with few amendments to the previously existing techniques. The analyzing signal is first windowed. Then, its fourier coefficients are calculated which is equivalent to take inner product of the given signal with the windowed fourier function. This window further has dependance on 2 deciding factors:

1. Window localization.
2. Different coefficient representing frequency.

This is the standardized procedure for the signal analysis. Jean Morlet, a geophysical engineer at French Oil company Elf Acquitaine in late 1970'swas studying a signal that had different characteristic in time and frequency which he wanted to extract [1]. Typically, using frequency would corresponds to short time span and vice-versa. This is also known the Heinsberg uncertainty principle. But, here demand was to analyze a given signal which contains various different features at different times. To have good time resolution at higher frequency one uses wide band consisting of short time transform. Similarly, if one wishes to achieve good frequency for lower frequency then, narrower band short time fourier transform is applied. But, aim was to obtain both by single transform. To resolve this problem Morlet experimented with windowed cosine wave (Gaussian Window) and compressing this in time yields high frequency function and widening in time yields lower frequency function. Gaussian window was used in which its width was a variable. This function further depended upon compression degree, scale. The remaining methodology was similar i.e. taking the inner product of the signal with transformed function. The major difference was functions with high frequency were narrow whereas functions with lower frequency were not narrow. Morlet name it as Wavelets of constant shape. But other researchers soon named it as wavelets. Morlet faced much opposition from his geophysics colleagues when he tried to explain the worth of this tool stating that if it was a tool then, mathematics books would contain it. But Morlet believed in his work and to find the recognition he connected with A. Grossmann, was a researcher in field of quantum mechanics. In quantum mechanics one encounters similar problem when one tries to find out local features of signal with certainty. He found the similarity of Morlet's work with coherent state formalism (technique in quantum mechanisms). Grossmann then, build the inversion formula of Morlet's transform. Both Grossmann and Morlet worked together on applicability in different fields. E.Aslaken and J. Klauder had formulated the inversion formula earlier [1]. Ingrid Daubechies was a student and during her PhD dissertation, she worked with A. Grossmann. In spring 1985 when Daubechies visited Marseille where she learned about wavelets. Grossmann guided Daubechies to collectively work learn about the concept of frames given by A. Cohen and J. Kovacevic, which played an important part.

While waiting in line for photocopying machine Vyes Meyer, a pure mathematician then working at Ecole Polytechnique heard about the work of Grossman and Morlet in year 1985 [2]. He further looked up on the work of Grossman and Morlet and found it to be similar with the work of A. caledron, who was working in harmonic analysis in 1960's. Harmonic analysis is a field which has its origins from Fourier analysis. Meyer created orthonormal wavelets basis which had good time and frequency localisation. He established correspondence between smoothness and decay properties of function , approximated its wavelet expansion in various norm.

Wavelets series fascinated Meyer because they used redundant families of wavelet series to give time frequency localisation but concept of redundancy was always thought as unavoidable.

Mallat, a graduate student at Penn was on a vacation. There he met an old friend, who was graduate student of Meyer where he referred wavelet basis to him. This captured the interest of Mallat as he found similarity with his field. Mallat was specialized in computer vision and image analysis. Coarse features of image includes large scale objects where fine features needed detailed study. Meyer and Mallat shaped the mathematical details of multiresolution analysis for discrete wavelet transform. The first and simplest orthonormal wavelet basis was discovered by Alfred Haar in year 1909.

Singularity is an important aspect that needs to be covered. Littlewood Paley in year 1930, created dyadic rearrangement of the fourier transform to tackle singularities more effectively. Since, singularity is the local phenomenon. Vyes Meyer was best in field, he could easily see the similarity with harmonic analysis. Meyer was excited to work in this new arena of application [1]. He tried to bring together harmonic analysts and applied researchers to link up the fields. Meyer was curious since, it was believed that the redundancy is not avoidable to get good time frequency localization. M.Frazier and B. Jawerth, harmonic analysts had developed similar series to wavelet. Meyer worked on this relation and wanted to prove it After few weeks of brainstorming, he came up with the orthornormal wavelet which had great time frequency localization. With help of P.G. Lemarie they extended it to N dimensions. Later Lemarie and G. Battle developed spline functions as wavelets bases. These decayed better than Meyer wavelets but regularity was affected. G. Battle was a mathematical physicist and quantum mechanics interested him. P. Federbush has developed a complex machinery along with Battle which became simpler after that incoporated wavelet bases in this model. The idea behind normalization group techniques, a tool to study different scales.

For an image, it can be divided into 2 parts large scale and finer scale features. A Witkin gave the concept of scale-space. The idea that narrow scale represents the fine scale features and coarse scale is represented by wider scale. P. Burt and E. Adelson gave the laplacian pyramid. In laplacian pyramid method, the original image and blurred images are subtracted which eventually yields the details or fine scale features. These can be further broken down into elementary pieces. Mallat designed the similar layer structure inspired from their concept. Also, he gave the concept to decompose the function into one basic wavelet. When it came to the knowledge of Mallat that Meyer would be paying a visit to Chicago, Mallat planned a meet with him. Both of them worked on it for few days together which led to the fabrication of Multiresolution analysis. Stomberg came up with the basis that correlated to IIR filters with z-transform [2]. The question that now arised was how to have wavelet basis in which truncation was not required. The solution was simple to define the filters and check that they corresponds to orthonormal wavelet basis. Electrical engineers used the idea to group frequencies into bands such that width corresponds to average frequency bands. This is called as constant-Q filtering. The method to achieve splitting is that range of frequency is divided into two halves having different bandwidths by using two filters high pass and low pass filters. The method is iteratively applied on lower frequency half. Nqyuist sampling is applied at end of each step to keep a check if the result is accurate. If signal obtained is distorted then, it is known as aliasing. This effect is due to not using proper filter. A.Croiser, D Esteban, C. Gallad designed a method that would not let aliasing take place. Quadrature mirror filters(QMF) was the solution to stop aliasing effect. After, 10 years T. Barnwell, F. Mintzer gave QMC like filter that gave exact reconstruction. These are better known as Conjugated quadrature filter(CQF) [1]. W. Dahmen showed that wavelets serve as unconditional bases for large part of the functional spaces and this proved interesting concept for harmonic analysts. Wavelets does not necessarily implies orthogonal wavelets, but few similar ideas led to formation of Coifman wavelets.

Ingrid Daubechies, graduate student of Grossman developed a Wavelet transform which allowed to choose basis function but with some redundancy. Daubechies and Mallat together developed progressed from continuous to discrete signal analysis. In later 1983, M Smith and T. Barnwell discovered quadrature mirror filters which gave exact resolution. They had designed filters which had many practical applications in electrical engineering. Daubechies gave orthonormal bases of compactly supported wavelets in 1988.

1.1 Advantages of wavelets over fourier transform

(1.) Wavelet transform can be applied to both stationary and non-stationary signals where Fourier Transform can only be applied to stationary signals.

(2.) The features of the signal obtained by wavelets are more accurate in comparison to the Fourier transform.

(3.) Better compression and de-noising is performed by the wavelets without breaking down components into various levels of details.

(4.) As, less number of coefficients are required by the wavelet transform to represent the given signal. Therefore,result obtained by using wavelets is fast and efficient [3].

(5.) **Gibbs phenomenon** It is a phenomenon exhibited by Fourier Transform in which discontinuity spreads to the surrounding region instead of a point. Therefore, while approximating a function using Fourier Transform the error is not restricted to only single point but to surrounding area also.

Consider an example, **Sawtooth function**

$$f(x) = \begin{cases} x & 0 \leq x < 1/2 \\ x-1 & 1/2 \leq x < 1 \\ 0 & otherwise \end{cases} \tag{1.1.1}$$

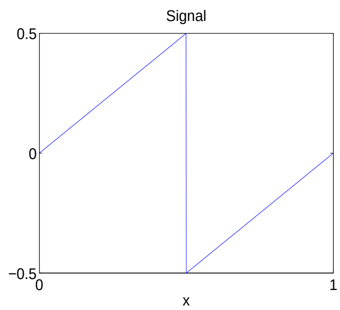

Figure 1.1: Sawtooth Function.

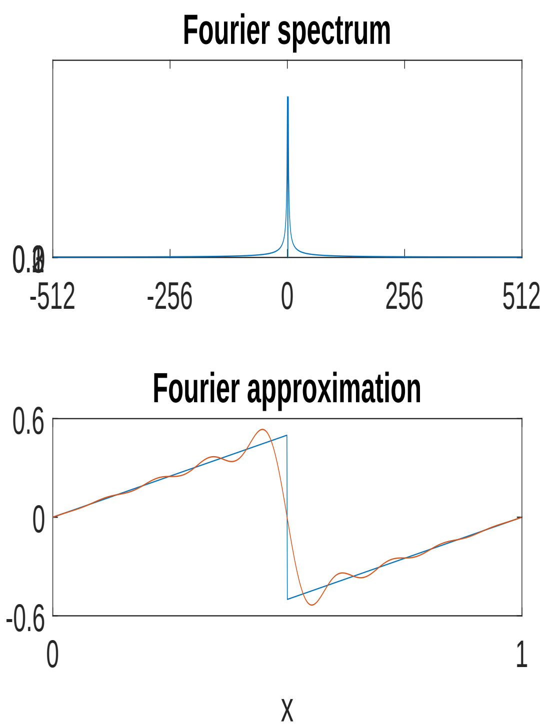

Figure 1.2: The function and Fourier representation.

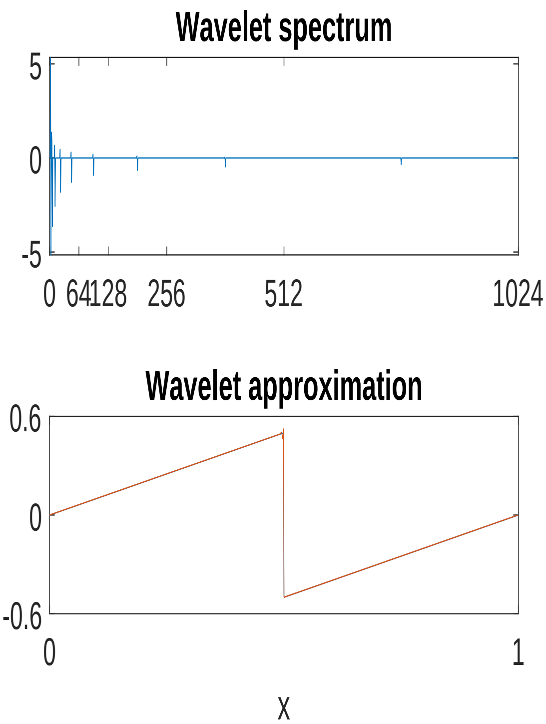

Figure 1.3: The wavelet approximation.

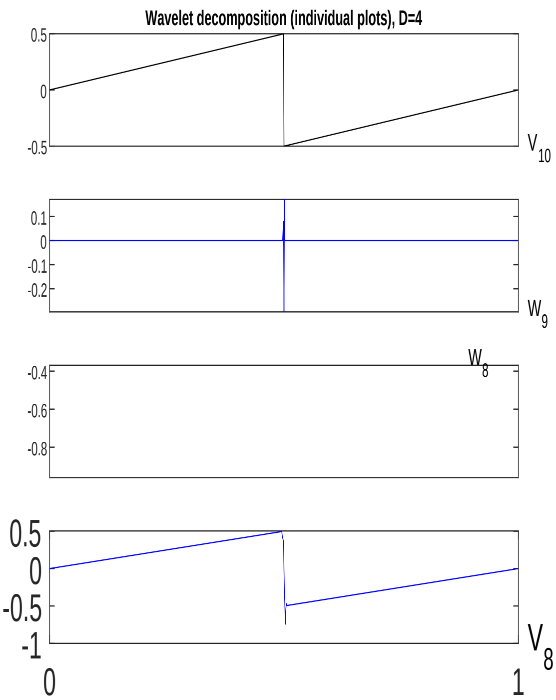

Figure 1.4: The wavelet decomposition representation.

If we approximate upto 17 largest coefficients. The function has discontinuity at $x = 0.5$ and Fourier Transform fails to detect the discontinuity at the given point. The primary reason for this failure is that because it contains complex exponentials which spreads all over the interval.

1.2 Types of wavelets

1.2.1 Haar wavelet

(1.) Alfred Haar discovered the Haar Wavelet in 1909.

(2.) It is the most simplest form of the wavelet.

(3.) It is not continuous and differentiable.

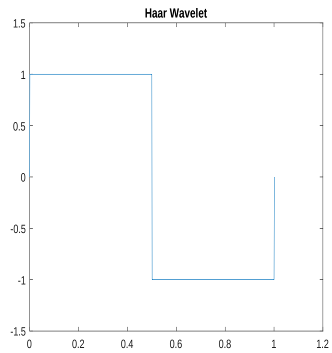

Figure 1.5: Haar wavelet.

$$\psi(t) = \begin{cases} 1 & 0 \leq t < 1/2 \\ -1 & 1/2 \leq t < 1 \\ 0 & otherwise \end{cases}$$

$$\phi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & otherwise \end{cases}$$

1.2.2 Mexican hat wavelet

(1.) It is also known as Ricker Wavelet.

(2.) It is a special case of continuous wavelet.

(3.) It is frequently used to study the seismic data.

$$mexh(x) = c * exp(-x^2/2) * (1 - x^2)$$

where

$$c = 2/(\sqrt{3} * \pi^{1/4}).$$

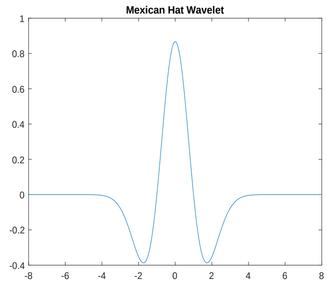

Figure 1.6: Mexican hat wavelet

1.2.3 Meyer wavelet

(1.) It was proposed by Yves Meyer.

(2.) It is an orthogonal wavelet.

(3.) It a Continuous wavelet.

(4.) It is infinitely differentiable with infinite support.

$$\psi(\omega) = \begin{cases} 1/\sqrt{2\pi} sin(\pi/2v(3|\omega|/2\pi - 1)) \exp(j\omega/2) & 2\pi/3 \leq |\omega| < 4\pi/3 \\ 1/\sqrt{2\pi} cos(\pi/2v(3|\omega|/2\pi - 1)) \exp(j\omega/2) & 4\pi/3 \leq |\omega| < 8\pi/3 \\ 0 & otherwise \end{cases}$$

where

$$\phi(t) = \begin{cases} 0 & 0 < x \\ x & o < x < 1 \\ 1 & x > 1 \end{cases}$$

$$\phi(t) = \begin{cases} 1/2\pi & \omega < 2\pi/3 \\ 1/\sqrt{2\pi} cos(\pi/2v(3|\omega|/2\pi - 1)) \exp(j\omega/2) & 2\pi/3 \leq |\omega| < 4\pi/3 \\ 0 & otherwise \end{cases}$$

1.2.4 Daubechies wavelet

(1.) It was proposed by Ingrid Daubechies. Compactly supported wavelet with degree of inclusiveness (maximum or minimum value).

(2.) Highest number of vanishing moments for given support interval.

(3.) The scaling functions associated with it are minimum-phase filters.

(4.) It is both orthogonal and biorthogonal.

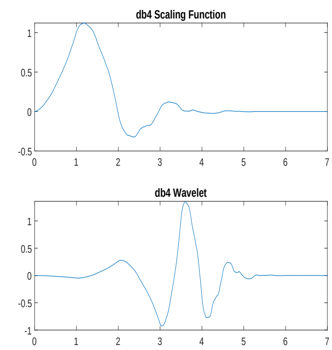

Figure 1.7: Scaling function and Wavelet function for D=4

1.2.5 Coiflets

(1.) They were formulated collectively by Ingrid Daubechies and Ronald Coifman.

(2.) They do not have any specific formula.

(3.) They are orthogonal, symmetric, bi-orthogonal.

(4.) Coiflets have five types coiflet1, coiflet2, coiflet3, coiflet4, coiflet5.

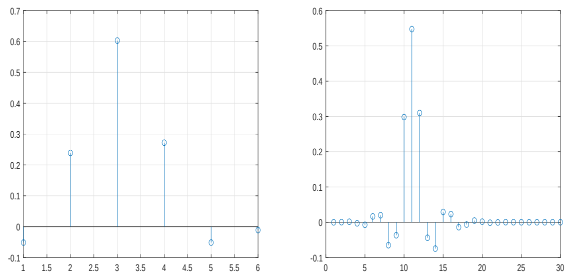

Figure 1.8: Coiflet1 scaling function, Coiflet5 scaling function

1.2.6 Shannon Wavelets

(1.) They are orthogonal wavelet.

(2.) They are analytic functions and are infinitely times differentiable.

(3.) They forms a base for definition of connection coefficients.

(4.) The Shannon wavelet originates from the shannon scaling function.

The Shannon function is defined as:

$$\psi(x) = \frac{\exp(\pi i x) - \exp(-\pi i x)}{2\pi i x}.$$

The translation and dilation equation is given by:

$$\begin{aligned}\psi(x)_l^m &= 2^{\frac{m}{2}}\psi(2^m x - l) \\ &= 2^{\frac{m}{2}}\frac{sin\pi(2^m x - l)}{\pi(2^m x - l)}.\end{aligned}$$

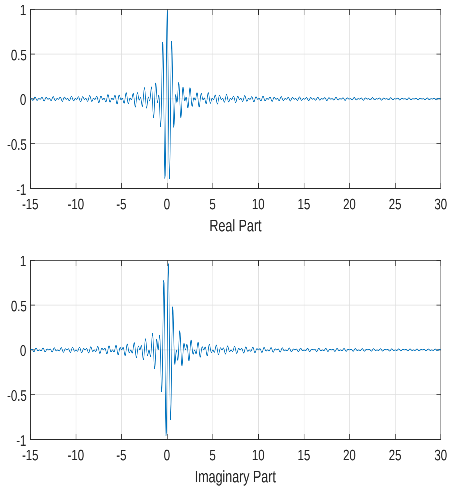

Figure 1.9: Shannon wavelets

1.2.7 Biorthogonal wavelets

(1.) Ph. Tchamitician formulated Biorthogonal wavelet.

(2.) They have symmetric, compact support.

(3.) $\psi(x)$ and $\tilde{\psi}(x)$ represents the mother wavelet.

(4.) They are represented as :

$$g(x) = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} (g, \psi_{k,l})\tilde{\psi}_{k,l}(x) = \sum_{k\in\mathbb{Z} \sum_{l\in\mathbb{Z}}} (g, \tilde{\psi}_{k,l})\psi_{k,l}(x)$$

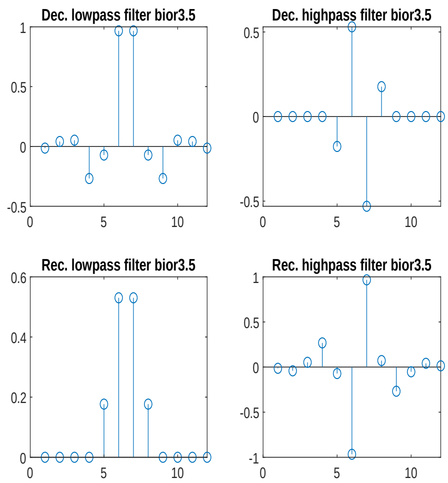

Figure 1.10: Biorthogonal Wavelet

1.2.8 Curvelets

(1.) They were formulated by Candes and Donoho.

(2.) The singularities on a given curve can be represented by curvelet effectively but wavelets are suitable for point singularity.

(3.) Curvelet are designed on the basis of scale, location and orientation.

(4.) This originates from ridgelet transform [24].

(5.) The ridgelet transform is defined as :

$$R_g(m, n, \theta) = \int\int \psi_{m,n,\theta}(y, z)g(y, z)dydz, \theta \in [0, 2\pi].$$

where m represents the scale, n represents the translation, $\theta \in [0, 2\pi]$ represents the orientation. Ridgelet is defined as:

$$\psi_{m,n,\theta} = m^{\frac{1}{2}}\psi\left(\frac{mcos\theta + zsin\theta - n}{m}\right).$$

By changing the orientation ridgelet becomes curvelet.

1.3 Applications

1.3.1 Analysing sound of inner ear

Wavelet transform is used for examining the sound of the human ear. The oscillations are further carried from the basilar membrane. This membrane covers the whole length of the cochlea. The cochlea is present inside the inner ear. Imagine, basilar segment being extended outwards. Assume, this membrane as y co-ordinate. Experimentally, the pressure wave which represents the pure tone

$$f_w(t) = exp(iwt)$$

gives the simulation along the basilar membrane. It has same frequency in time along with the envelope in y.

$$F_w(t, y) = exp(iwt)\phi_w(y)$$

Above, 500 hertz the approximation yields good results. The w of $\psi_w(y)$ is consistent with log$w$. It means that there exists a function such that $\phi_w(y)$ is very near to $\phi(y - log\text{w})$. Excitation function is represented by:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int dw\hat{f}(w)exp(iwt)$$

and response function is given by:

$$F(t, y) = \frac{1}{\sqrt{2\pi}} \int dw\hat{f}(w)F_w(t, y)$$

$$= \left(\frac{1}{2}\right)\pi \int dw\hat{f}(w)exp(iwt)\phi(y - logw)$$

by change of parameters we have,

$$\hat{\psi}(exp(-x)) = (2\pi)^{(-1/2)}\phi(x).$$

$$G(a, t) = F(t, loga).$$

then, we have

$$G(a, t) = \int dtf(t^{'})\psi(a(t - t^{'})).$$

which represents wavelet transform. This means that the wavelet method can study sound better over other methods.

1.3.2 Facial recognition

The wave packet analyses can be used for facial recognition. As, in wave packet analyzes first the decorrelation between spatial and frequency domain is performed. In wave packet analyses we can breakdown both approximate subspaces as well as detailed subspaces. As, when studying a signal we should not miss upon any part of the information. The face recognition can be changed smoothly by small gradual steps like faking a moustache, sunglasses ectera. There are two methods in which facial recognition is carried out:

(1.) Geometric approach.

(2.) Feature based approach.

In geometric approach, basic features such as nose, chin, eyes are used and face is generated using them. Since, features are bit difficult to extract, therefore feature based approach is followed. In feature based approach, the face is extracted and then sub-band decomposition is applied. On, which further wavepacket transform is used. Since, some learnt prototype are already stored then, image matching is performed.

1.3.3 Electrocardiography using wavelet transform

Electrocardiography (ECG) is used to ensure proper functioning of the heart. Electric potential is calculated on the surface of the given living tissue. Then, it is measured by the galvanometer. The diagnostic information is given by the various waves and vectors of polarisation and depolarisation. The ability of the wavelet transform to represent the temporal domain and localisation in frequency. Hence, this gives the better examination of the signal. The ECG signal has cyclic occurring patterns at various frequencies. Thereby, making it easy to separate out the ECG signal from the noise and baseline drift [4].

1.3.4 Wavelet analysis in hydrology

Hydrology is the branch of science that deals with the study of distribution, natural occurrence of water and its various properties. The hydrological cycle is basically the recharge of the ground water and its evaporation leads to the purification of the water, which further reaches down through rain. Earlier, artificial neural network (ANN) were employed in the study of hydrology [10]. But accurate results were not obtained. This was further modified with help of wavelets and method was known as wavelet artificial neural network. In town planning water management plays an important role. Since, hydrology consists of number of factors such as vegetation area, evaporation etc. These add upto non-linear and multi-scale parameters which cannot be neglected in the model. The time series in this model is represented in form of wavelets.

1.3.5 Wavelets in speech recognition

Speech recognition is a part of computational linguistics that deals with identifying the phrases or words and converting them as a input for computer or device which performs the desired action. Playing music on a given device just by verbal command is an example of speech recognition. Automatic speech recognition has some challenges which can be remove with help of the wavelets. The challenges are:

(1.) The signal distorts due to background noise or the noise in the channel.

(2.) The incapability to judge the redundancy in speech [5].

(3.) The combination of more than one word spoken together which dominates the desired speech(signal).

The speech are mainly of 2 types namely:

(1.) Isolated

(2.) Continuous

Isolated: In isolated speech signal there are short pauses. This kind of speech is to recognize by the system.
Continuous: In continuous speech signal there are chances that the co-articulation effect takes place. In speech command the efficient method of recognizing speech is to coach the system about the phonemes rather than the words [6].
If the system is coached for words then, the execution and complexity of the signal increases thereby delaying the response time. wavelets are employed better producing better results of the speech recognition. The speech signal is expressed in the form of the translated and scaling wavelet. Discrete scaling function or fast wavelet transform are applied for applied to the signal then, inverse discrete wavelet transform is applied to retrieve the original signal. Daubechies - 8 wavelet is preferred since it has least support for the given vanishing moments.

1.4 Literature review

Wavelets serve as basis for data sets or functions. First generation wavelets are defined in [7, 8] are infinite or in periodic domain. There is fixed mother wavelet which can translate and dilate. If one is dealing with the arbitrary weight functions, boundaries. Then, these cannot be expressed in wavelet form having translate and dilate of one set function. Therefore, second generation wavelets [9] are put to use. They are formulated in spatial domain and can be modeled in accordance to complex geometry.

In modern times, where any piece of information can be sent from one part to another part of the world within just few seconds. The science behind it is the effective data compression and transmission. Data compression can be defined as scaling down the data size to reduce the memory and time of transmission of the signal [5]. If data compression is not used then, it would become time consuming task to transmit data. Consider, an example if data of size 20Gb without using any data compression technique would take up to 3 hours for the transmission. Wavelet transform is a procedure to breakdown the signal and reduce its storage space by effective data compression of the given signal [5]. Compactly supported orthonormal wavelets with fast wavelet transform was introduced by Daubechies and Mallat [25].

In [5] author considers the wavelet transform for speech compression. The author has used 8Khz 8 bit speech signal. Peak signal noise ratio(PSNR), signal noise ratio(SNR), normalized root mean square error(NRMSE) and compression ratio are the parameters considered to find out which provides better results for signal compression. The author concluded that D10 wavelet filter gives the better speech standard, the rate is lowered from 64Kbps to 13 Kbps, compression ratio being 4.31 times. The author in [12], have used wavelets for speech signal compression. They have used D20 wavelet filter and concluded that with increase in the compression ratio the SNR decreases. In [19], they have used D4, D8, D10, D20 to study arabic speech signal and found that the best result was obtained with D10 wavelet filter that gave best result with less number of truncated coefficients and also better SNR.

The wavelet bases for solving partial differential equations(pde) was used in 1990s when the adaptive wavelet method emerged [20]. In 1984, Grossmann and Morlet developed a continuous wavelet transform [?]. In 1988, Farge and Rabreau used wavelets for the very first time in fluid mechanics. After that, wavelets have been used to solve both linear and non-linear pde, to consider an example poisson equation [10], heat equation and transport equation [13, 14], Burgers equation [15, 16], non-linear schrodinger equation [17].

Vasilyer [18] in year 1995 formulated multilevel wavelet collocation method for solving pdes. In this method, the collocation methodology is used with wavelet approximation. Liandrat and Tchiamichian [15] Maday et al [20] have proved multiresolution construction of the wavelet bases is simple and productive structure for spatial adaptive algorithm. In 1996, Vasilyev and Paolucci [18] formulated adaptive multilevel wavelet collocation method for finding the solution of pde's in spatial domain. In 1997, Vasilyev and Paolucci formed a method for multidimensional Pde known as fast adaptive wavelet collocation algorithm. The pde's have multiscale solution on manifold and are systematically solved by wavelets [8]. Despite vast literature availability on wavelet method for finding the solution of pde's is in the budding stage. In 2008, M.Mehra and N.K.R. Kevlahan have used second generation wavelet.

Uncertainty Quantification (UQ) is the field of science which studies about the quantitative characterization and reducing the uncertainty in execution. UQ in fluid mechanics is solve using by polynomial chaos (pc) method [6]. The pc method fails when data variation is instantenous. This is called as Weiner hermite represenation (Whe). In [11] research performed by the chorin states that Whe is more practical approach to work with in complex situations that involve shock generation. In year 2007, O.P. Le Maitro and O.M. Knio formulated wavelet based pc. They even used haar and multiwavelet concept [30,32].

Chapter 2

Multiresolution analysis

Multiresolution analysis(MRA) was introduced by Stephane Mallat and Yves Meyer in year 1988. The theory Of MRA has its origins from Microlocal analysis. Microlocal analysis consists of techniques that are based on the Fourier Transform. It consists of variable-coefficient linear and non-linear partial differential equations. **The natural question arises why do we need MRA?**
To study different levels of details of a given function $f \in L^2(\mathbb{R})$ [28].

The following axioms are satisfied by the MRA:

$$0 \subset ... \subset \mathbb{V}_{-1} \subset \mathbb{V}_0 \subset ...L^2(R). \tag{2.0.1}$$

$$\cup_{k=-\infty}^{\infty} V_k = L^2(\mathbb{R}). \tag{2.0.2}$$

$$\{\phi(x-l)_{l \in Z}\}\text{is an orthonormal basis for } \mathbb{V}_0. \tag{2.0.3}$$

$$f(1.) \in \mathbb{V}_k \Leftrightarrow f(2(.)) \in \mathbb{V}_{k+1}. \tag{2.0.4}$$

$$\cap_{k=-\infty}^{\infty} \mathbb{V}_k = 0. \tag{2.0.5}$$

2.1 Detail spaces:

W_k represents the detail space which is orthogonal complement of V_k in V_{k+1} given by:

$$V_{k+1} = V_k \bigoplus W_k.$$

If V_{k_0} and V_k are the two spaces then, for $k > k_0$ applying recursive relation yields

$$V_k = V_{k_0} \bigoplus (\bigoplus_{k=k_0}^{k-1} W_k).$$

This allows to examine the function at different level of scales. Iteratively, it can be expressed in the form:

$$L^2(\mathbb{R}) = \bigoplus_{k=-\infty}^{\infty} W_k.$$

2.2 Scaling and wavelet expansion:

$\{\phi(x-l)\}$ represents orthonormal basis for $\mathbb{V}_0$. Repeatedly, using property (1.5) would yield basis of $\mathbb{V}_k$ which is given by: $\{\phi(2^k x-l)\}, l \in \mathbb{Z}$. $2^{k/2}\phi(2^k x-l)dx$ is orthonormal basis for $\mathbb{V}_k$. Similarly, $2^{k/2}\psi(2^k x-l)dx$ is orthonormal basis for $\mathbb{W}_k$. Where *phi* is the basic scaling function and *psi* is the basic wavelet.

2.3 Expanding function in V_k:

In terms of scaling function:

$$g(x) = \sum_{m=-\infty}^{\infty} c_{k,m}\psi_{k,m}(x), x \in \mathbb{R}. \tag{2.3.1}$$

The wavelet expansion is given by:

$$\boxed{g(x) = \sum_{m=-\infty}^{\infty} c_{k_0,m}\phi_{k_0,m}(x) + \sum_{k=k_0}^{k-1}\sum_{m=-\infty}^{\infty} d_{k,m}\psi_{k,m}(x), \quad x \in R.} \tag{2.3.2}$$

where

$$c_{k_0,l} = \int_{-\infty}^{\infty} g(x)\phi_{k_0,m}(x)dx. \tag{2.3.3}$$

$$d_{l,m} = \int_{-\infty}^{\infty} g(x)\psi_{k,m}(x)dx. \tag{2.3.4}$$

$$\tag{2.3.5}$$

$\lambda = k - k_0$ denotes the depth of wavelet expansion.

2.3.1 Dilation equation

For the given function can be written in form of basis functions as follows:

$$\phi(x) = \sum_{l=-\infty}^{\infty} a_l\phi_{1,l}(x) = \sqrt{2}\sum_{l=-\infty}^{\infty} a_l\phi(2x-l).$$

where

$$a_l = \int_{-\infty}^{\infty} \phi(x)\phi_{1,l}(x)dx.$$

The **Dilation equation** is given:

$$\boxed{\phi(x) = \sqrt{2}\sum_{l=0}^{D-1} a_l \phi(2x-l).} \tag{2.3.6}$$

$a_0, a_1, ..., a_{D-1}$ represents filter coefficients

Similarly, **Wavelet equation** is given by:

$$\boxed{\psi(x) = \sqrt{2}\sum_{l=0}^{D-1} b_l \phi(2x-l).} \tag{2.3.7}$$

D represents wavelet genus and it must be even positive integer. where

$$b_l = \int_{-\infty}^{\infty} \psi(x)\phi_{1,l}(x)dx.$$

$b_0, b_1, ..., b_{D-1}$ represents filter coefficients. ϕ and ψ being implicit function. Hence, they are computed using their properties.

Theorem 2.3.1 $b_l = (-1)^l a_{D-1-l}$, $l = 0, 1, ..., D-1$

Proof We use orthogonality property. As,

$$\int_{-\infty}^{\infty} \phi(x)\psi(x)dx = 0.$$

Substituting the dilation and wavelet equation we get,

$$\int_{-\infty}^{\infty} \phi(x)\psi(x)dx = 2\int_{-\infty}^{\infty} \sum_{l=0}^{D-1} a_l \phi(2x-l) \sum_{m=0}^{D-1} b_m \phi(2x-m)dx.$$

Substitute $2x$ as y:

$$= \sum_{l=0}^{D-1}\sum_{m=0}^{D-1} a_l b_m \int_{-\infty}^{\infty} \phi(y-l)\psi(y-m)dy$$

$$= \sum_{l=0}^{D-1}\sum_{m=0}^{D-1} a_l b_m \int_{-\infty}^{\infty} \phi_l(y)\psi_m(y)dy$$

$$= \sum_{l=0}^{D-1}\sum_{m=0}^{D-1} a_l b_m \int_{-\infty}^{\infty} \delta_{l,m}dy$$

Since, integral is always a positive quantity,

$$= \sum_{l=0}^{D-1} a_l b_m = 0.$$

Trivially, this holds if $a_l = 0$ or $b_m = 0$ Non-trivial solution is:

$$b_l = (-1)^l a_{t-l},$$

t being the odd integer and set $a_{t-l} = 0$ for (t-l) $\notin$ [0,D-1].

2.3.2 Some properties of wavelet functions:

(1.) **Orthonormality property:**

Orthornormality means that the norm of given function is one and is also orthogonal.

$$\delta_{0,r} = \int_{-\infty}^{\infty} \phi(x)\phi(x-r)dx.$$

Using, dilation equation,

$$\begin{aligned}
&= \int_{-\infty}^{\infty} (\sqrt{2}\sum_{l=0}^{D-1} a_l\phi(2x-l))(\sqrt{2}\sum_{l=0}^{D-1} a_l\phi(2x-2r-m))dx. \\
&= \sum_{l=0}^{D-1}\sum_{k=0}^{D-1} a_l a_m \int_{-\infty}^{\infty} \phi(y)\phi(y+k-2r-m)dx.
\end{aligned}$$

where we have substituted $y = 2x - l$

$$= \sum_{l=0}^{D-1}\sum_{m=0}^{D-1} a_l a_m \delta_{l-2n,m}$$

$$= \sum_{l=l1(r)}^{l_2(r)} a_l a_{l-2r}, r \in \mathbb{Z}$$

$l_1(n) = max(0, 2r) \quad and \quad l_2(r) = min(D-1, D-1+2r).$

This gives distinct $D/2$ distinct equations for $n = 0, 1, ..., D/2-1$. Finally,

$$\boxed{\sum_{l=l1(r)}^{l_2(r)} a_l a_{l-2r} = \delta_{0,r}, r = 0, 1, ..., D/2-1.} \tag{2.3.8}$$

Similarly, using above theorem,

$$\boxed{\sum_{l=l1(r)}^{l_2(r)} b_l b_{l-2r} = \delta_{0,r}, r = 0, 1, ..., D/2-1} \tag{2.3.9}$$

(2.) **Area conservation:**

We know that, $\int_{-\infty}^{\infty} \phi(x)dx = 1$. Integrating on both sides of the dilation equation,

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(x)dx &= \sqrt{2}\sum_{l=0}^{D-1} a_l \int_{-\infty}^{\infty} \phi(2x-l)dx \\ &= \frac{1}{\sqrt{2}}\sum_{l=0}^{D-1} a_l \int_{-\infty}^{\infty} \phi(y)dy \end{aligned}$$

$$\boxed{\sum_{l=0}^{D-1} a_l = \sqrt{2}.} \tag{2.3.10}$$

(3.) **Vanishing moments:** Vanishing moments give the accurate representation of the polynomial exactly to Q-1 for given wavelet genus D [28].

$$x^q = \sum_{l=-\infty}^{\infty} M_l^q \int_{-\infty}^{\infty} \phi(x-l)\psi(x)dx = 0, \quad q = 0, 1, ..., Q-1.$$

and moments are defined as follows:

$$M_l^q = \int_{-\infty}^{\infty} x^q \phi(x-l)dx, l \in \mathbb{Z}, \quad q = 0, 1, ..., Q-1.$$

Consider, inner product of above equation with $\psi(x)$ gives:

$$\int_{-\infty}^{\infty} x^q \psi(x)dx = \sum_{l=-\infty}^{\infty} M_l^q \int_{l=-\infty}^{\infty} \phi(x-l)\phi(x)dx = 0.$$

As,ϕ and ψ are orthonormal. The property of Q vanishing moments gives:

$$\boxed{\int_{-\infty}^{\infty} x^q \psi(x) dx = 0, \quad x \in \mathbb{R}, \quad q = 0, 1, 2, ..., Q-1} \tag{2.3.11}$$

In terms of filter coefficients, we can possible express it as:

$$\int_{-\infty}^{\infty} x^q \psi(x) dx = 0$$

Substitute the wavelet equation:

$$= \sqrt{2} \sum_{l=0}^{D-1} b_l \int_{-\infty}^{\infty} \phi(2x - l) dx.$$

$$= \frac{\sqrt{2}}{2^{q+1}} \sum b_l \int_{-\infty}^{\infty} (y + l)^q \phi(y) dy, y = 2x - l$$

$$= \sqrt{2} \sum_{l=0}^{D-1} b_l \sum_{r=0}^{q} \binom{q}{r} l^r \int_{-\infty}^{\infty} y^{q-r} \psi(y).$$

$$= \frac{\sqrt{2}}{2^{q+1}} \sum_{r=0}^{q} \binom{q}{r} M_0^{q-r} \sum_{l=0}^{D-1} b_l l^r.$$

the binomial formula

$$(y + l)^q = \sum_{r=0}^{q} \binom{q}{r} y^{q-r} l^r.$$

for $q = 0$, we have

$$\sum_{l=0}^{D-1} b_l = 0.$$

$$\sum_{l=0}^{D-1} (-1)^l a_{D-1-l} l^q = 0, q = 0, 1, 2, ..., Q-1.$$

Put $m = D - 1 - l$

$$\sum_{m=0}^{D-1} (-1)^{D-1-m} a_m (D - 1 - m)^q.$$

Using, binomial expression we get,

$$\boxed{\sum_{l=0}^{D-1}(-1)^m a_m m^q = 0, \quad q = 0,1,2,...,Q-1.} \tag{2.3.12}$$

We can calculate the qth moment by following method:

$$M_l^q = \int_{-\infty}^{\infty} x^q \phi(x-m)dx, m, q \in \mathbb{Z}.$$

Substitute $q = 0$ in above equation , we get,

$$M_m = \int_{-\infty}^{\infty} \phi(x-m)dx = 1.$$

As, unit area of ϕ is:

$$\int_{-\infty}^{\infty} \phi(x)dx = 1.$$

$$\Rightarrow \boxed{M_m^0 = 1.} \tag{2.3.13}$$

Consider,

$$M_0^q = \int_{-\infty}^{\infty} x^q \phi(x)dx.$$

Substitute value of dilation equation we get,

$$= \sqrt{2}\sum_{l=0}^{D-1} a_l \int_{-\infty}^{\infty} x^q \phi(2x-l)dx.$$

$$= \sqrt{2}\sum_{l=0}^{D-1} a_l \int_{-\infty}^{\infty} (y/2)^q \phi(y-l)dy/2.$$

$$= \sqrt{2}/2^{q+1}\sum_{l=0}^{D-1} a_l \int_{-\infty}^{\infty} y^q \phi(y-l)dy.$$

$$M_0^q = \sqrt{2}/2^{q+1}\sum_{l=0}^{D-1} a_l M_l^q.$$

To decrease the number of equations put $y = x - m$ in M_m^q we get,

$$M_m^q = \int_{-\infty}^{\infty} (y+m)^q \phi(y)dy.$$

Expand using binomial expansion we have,

$$= \sum_{r=0}^{q} \binom{q}{r} m^{q-r} \int_{-\infty}^{\infty} y^r \phi(y) dy.$$

$$M_m^q = \sum_{r=0}^{q} \binom{q}{r} m^{q-r} M_0^r.$$

$$M_0^q = \sqrt{2}/2^{q+1} \sum l = 0^{D-1} a_l M_l^q.$$

Substitute the value of M_l^q we get,

$$M_0^q = \sqrt{2}/2^{q+1} \sum_{l=0}^{D-1} a_l \sum_{l=0}^{q} \binom{q}{r} l^{q-r} M_0^r.$$

$$= \sqrt{2}/2^{q+1} \sum_{l=0}^{q} \binom{q}{r} l^{q-r} M_0^r \sum_{l=0}^{D-1} a_l k^{q-r} + \sqrt{2}/2^{q+1} M_0^q \sum_{l=0}^{D-1} a_l.$$

$$\text{By properties:} \quad \sum_{l=0}^{D-1} a_l = \sqrt{2}.$$

$$\left(\frac{2^{q+1} - 2}{2^{q+1}} \right) M_0^q = \frac{\sqrt{2}}{2^{q+1}} \sum_{r=0}^{q-1} \binom{q}{r} M_0^r \sum_{l=0}^{D-1} a_l l^{q-r}.$$

$$\boxed{M_0^q = \frac{\sqrt{2}}{2(2^q - 1)} \sum_{r=0}^{q-1} \binom{q}{r} M_0^r \sum_{l=0}^{D-1} a_l l^{q-r}.} \tag{2.3.14}$$

2.4 Discrete scaling function and inverse discrete scaling functions:

If ϕ is the basic scaling function and if we want to compute the function given by:

$$g(x) = \sum_{-\infty}^{\infty} c_{k,m} \phi_{k,l}(x) \tag{2.4.1}$$

and we assume that the ϕ is known at the dyadic rationals $\frac{n}{2^s}$,$n = 0, 1, ..., (D-1)2^s$ At the grid points we have,

$$\phi_{k,m}\left(\frac{l}{2^c}\right) = 2^{k/2}\phi(2^k\left(\frac{l}{2^c}\right) - m).$$

$$= 2^{k/2}(\phi^{k-c}l - m).$$

$$= 2^{k/2}\phi((2^{k+s-c}l - 2^s m)/2^s).$$

$$= 2^{k/2}\phi(m(l, m)/2^s).$$

where:

$$m(l, m) = l2^{k+s-c} - m2^s.$$

$m(l, m)$ is the index for already computed values ϕ. $m(l, m)$ must be integer, therefore:

$$k + s - c \geq 0$$

Since, $D - 1$ terms are non- zero

$$0 < \frac{m(l, m)}{2^s} < D - 1.$$

At grid values it can be written as:

$$g\left(\frac{l}{2^c}\right) = 2^{\frac{k}{2}} \sum_{m=m_0}^{m_0(l)+D-2} c_{k,m}\phi\left(\frac{m(l, m)}{2^c}\right), l \in \mathbb{Z}. \tag{2.4.2}$$

The above equation represents the map from scaling function coefficients 2^k to function's samples 2^c [28]. Mapping is denoted by:

$$g_c = T_{c,k}c_k.$$

at $c = k$

$$g_k = T_{k,k}c_k.$$

where $c_k = [c_{k,0}, c_{k,1}, ..., c_{k,2^k-1}]^T$ and $g_c = [g(0), g(1/2)^c, ..., g((2^c - 1)/2^c)]^T$

$$\boxed{g = Tc} \tag{2.4.3}$$

$$c = T^{-1}g \tag{2.4.4}$$

(2.4.3) represents **Indiscrete scaling transform** and eqrefa3 represents **Discrete scaling function**

2.5 Fast wavelet transform:

A relation between the coefficients of scaling function coefficients and wavelet coefficients can be obtained by using the orthogonality relation, with help of dyadic coupling between MRA can be used to find coefficients on different scales [28]. This is called fast wavelet transform. Consider, $g \in L^2(\mathbb{R})$, then , the projection

$$(p_{v_k})(x) = \sum_{m=-\infty}^{m=\infty} c_{k,m}\phi_{k,m}(x). \tag{2.5.1}$$

Since, $P_{V_k} = P_{v_{k-1}}g + P_{w_{k-1}}g$. In terms of scaling function and wavelets we have:

$$(P_{v_k})(x) = \sum_{-\infty}^{\infty} c_{k-1,m}\phi k-1,m(x) + \sum_{-\infty}^{\infty} d_{k-1,m}\psi_{k-1,m}(x). \tag{2.5.2}$$

$$\begin{aligned}\psi_{k-1,m}(x) &= 2^{(k-1)/2}\psi(2^{k-1}x-m).\\ &= 2^{k/2}\sum_{l=0}^{D-1} a_l\psi(2^k x-2m-l).\end{aligned}$$

$$= \sum_{l=0}^{D-1} a_l\psi_{k,2m+l}(x). \tag{2.5.3}$$

and similarly we can write,

$$
\begin{aligned}
\psi_{k-1,m}(x) &= \sum_{l=0}^{D-1} b_l \psi_{k,2m+l}(x) \\
\text{We know that,} & \\
g(x) &= \sum_{m=-\infty}^{m=\infty} c_{k,m}\psi_{k,m}(x), x \in \mathbb{R} \\
\text{and also,} & \\
c_{k,m} &= \int_{-\infty}^{\infty} g(x)\psi_{k,m}(x)dx.
\end{aligned}
$$

substitute the value we get,

$$
\begin{aligned}
c_{k-1,m} &= \int_{-\infty}^{\infty} g(x) \sum_{l=0}^{D-1} a_l \phi_{k,2m+l}(x)dx \\
&= \sum_{l=0}^{D-1} a_l \int_{-\infty}^{\infty} g(x)\phi_{k,2m+l}(x)dx \\
&= \sum_{l=0}^{D-1} a_l c_{k,2m+l}.
\end{aligned}
$$

and similarly, we get,

$$
\begin{aligned}
c_{k-1,m} &= \sum_{l=0}^{D-1} a_l c_{k,2m+l} \\
d_{k-1,m} &= \sum_{l=0}^{D-1} b_l c_{k,2m+l}.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{m=-\infty}^{\infty} c_{k,m}\phi_{k,m}(x) &= \sum_{r=\infty}^{\infty} c_{k-1,r} \sum_{l=0}^{D-1} \phi_{k-1,r}(x) + \sum_{r=-\infty}^{\infty} d_{k-1,r} \sum_{l=0}^{D-1} b_l \phi_{k,2r+l}(x) \\
&= \sum_{l=0}^{D-1} \sum_{r=-\infty}^{\infty} [c_{k-1,n} a_l + d_{k-1,r} b_l]\phi_{k,2r+l}(x).
\end{aligned}
$$

Put $2r + l = h$. Therefore, $l = h - 2r$ and $l \in [0, D-1]$. The bounds are given on the interval by:

$$\left\lceil \frac{m-D+1}{2} \right\rceil \equiv t_1(k) \leq t \leq t_2(k) \equiv \left\lfloor \frac{k}{2} \right\rfloor \tag{2.5.4}$$

so, final equation becomes

$$\sum_{m=-\infty}^{\infty} c_{k,m}\phi_{k,m}(x) = \sum_{k=-\infty}^{\infty} \sum_{t=t_1(m)}^{t_2(m)} [c_{k-1,t}a_{m-2t} + d_{k-1,t}b_{k-2t}]\phi_{k,m}(x).$$

Chapter 3

Application of discrete wavelet to differential equation

Various approaches are put into consideration to solve partial differential equation and one such approach is Galerkin method approach.

Galerkin method is a projection method approach. In projection method we approximate the solution using some functions. Galerkin method was proposed was by the Russian engineer V.I. Galerkin in year 1917.

The difference that wavelets were expected to create in finding the solution of the differential equations is not proven upto the mark. The primary reason being computational help by applying wavelet compression does not reduce the workload [28].

Few ways of solving the partial differential equations using wavelets as basis has been discussed. Methods are distinguished into classes as follows:

CLASS 1: ***Expanding in terms of scaling functions:*** The approximated solution is expressed in the form of scaling function at some level of J and is further solved using Galerkin approach. However, this approach fails to use wavelet compression extensively. This method has super convergence at the grid points and order of approximation is two times as that approximated using projection method approach.

CLASS 2: ***Expanding in terms of wavelet functions:*** The partial differential equation is solved by expressing the approximated solution in form of the wavelets. Then, wavelet compression is applied either to the solution or to the differential equation. One approach to use the sparsity of the wavelet representation is to make operators sparse. Ratio of the significant coefficients in the solution to the dimension of the problem should be very small. This is successful approach for non-linear operators. In other approach, linear operations are performed in the physical domain and non-linear operations are performed in the wavelet domain, where wavelet compression must be large enough.

CLASS 3: ***Finite difference using wavelets approach:*** In this approach, we work with the point values in the physical representation . Wavelet transform is used with the finite difference method and is used to find where grid must be coarsened or refined.

3.1 Connection coefficients

Connection coefficients are second kind of the Christoffel symbol. The connection coefficients are defined as:

$$\Gamma^{d1,d2}_{j,l,m} = \int_{-\infty}^{\infty} \phi^{d1}_{j,l}(x)\phi_{j,m}(x)dx, j, l, m \in \mathbb{Z} \tag{3.1.1}$$

d1, d2 represents order of differentiation. By scaling function, we have,

$$\phi_{j,l} = 2^{j/2}(\phi 2^j x - l)$$

we subsitute $2^j x - l$ as x

$$\phi_{j,m} = 2^{j/2}(\phi 2^j x - m)$$

Now, (3.1.1) becomes:

$$\Gamma^{d1,d2}_{j,l,m} = 2^{jd}\int_{-\infty}^{\infty} \phi^{(d1)}(x)\phi^{(d2)}(x - m + l)dx.$$

[26]

$$\Gamma_{j,l,m} = 2^{jd}\Gamma^{d1,d2}_{0,0,m-l}$$

where d = d1 + d2

$$\Gamma^d_l = \int_{\infty}^{\infty} \phi(x)\phi^{(d)}_l(x)dx, l \in \mathbb{Z} \tag{3.1.2}$$

and similarly we can write,

$$\Gamma^{d1,d2}_{j,l,m} = (-1)^{d1}2^j d\Gamma^d_{m-l} \tag{3.1.3}$$

Both ϕ and $\phi^{(d)}_l$ share common range in $-(D-2) \leq l \leq (D-2)$ and (2D-3) connection coefficients which are to be found. Then, we can write:

$$\Gamma^d = \Gamma_l^d {}_{l=2-D}^{D-2} \tag{3.1.4}$$

Consider, the scaling equation:

$$\phi_{j-1,l}(x) = \sum_{k=0}^{D-1} a_k \phi_{j,2l+k}(x)$$

which can alternatively be represented as

$$\phi_{j-1,l}(x) = \sum_{k=0}^{D-1} a_k \phi(2^j x - 2l - k) \tag{3.1.5}$$

Put j=1 in the above equation, we get,

$$\phi_l^{(d)}(x) = \sqrt{2} \sum_{k=0}^{D-1} a_k \phi_{1,2l+k}^{(d)}(2x) \tag{3.1.6}$$

Differentiating it d times will yield

$$\phi_l^{(d)}(x) = 2^d \sqrt{2} \sum_{k=0}^{D-1} a_k \phi_{2l+k}^{(d)}(2x). \tag{3.1.7}$$

Put dilation equation and (3.1.7) in (3.1.2),

$$\Gamma_l^d = \int_{-\infty}^{\infty} [\sqrt{2} \sum_{s=0}^{D-1} a_s \phi_s(2x)][2^d \sqrt{2} \sum_{r=0}^{D-1} a_r \phi_{2l+r}(2x)] dx$$

$$= 2^{(d+1)} \sum_{s=0}^{D-1} \sum_{r=0}^{D-1} a_s a_r \int_{-\infty}^{\infty} \phi_s(2x) \phi_{2l+r}^{(d)}(2x) dx$$

Put $2x$ as x

$$= 2^d \sum_{s=0}^{D-1} \sum_{r=0}^{D-1} a_s a_r \int_{-\infty}^{\infty} \phi_s(x) \phi_{2l+r}^{(d)}(x) dx$$

and now put $x - s$ as x

$$= 2^d \sum_{s=0}^{D-1} \sum_{r=0}^{D-1} a_s a_r \int_{-\infty}^{\infty} \phi_s(x) \phi_{2l+r-s}^{(d)}(x) dx$$

$$\sum_{s=0}^{D-1}\sum_{r=0}^{D-1} a_s a_r \Gamma^d_{2l+r-s} = (1/2)^d \Gamma^d_l, l \in [2-D, D-2] \tag{3.1.8}$$

[21] which is obvious. Substituting $n = 2l + r - s$. s must be restricted to [0,D-1]. This condition is satisfied if $max(0, 2l - n) \leq s \leq min(D - 2, D - 2 + 2l - n)$. Also, put $t = 2l - n$

$$\bar{a}_t = \sum_{s=s_1(t)}^{s_2(t)} a_s a_{s-t}$$

where $s_1(t) = max(0, t)$ and $s_2(t) = min(D - 1, D - 1 + t)$. Therefore, equation takes the form:

$$\sum_{2-D}^{D-2} \bar{a}_{2l-n} \Gamma^d_n = (1/2)^d \Gamma^d_l, l \in [2-D, D-2].$$

This has matrix representation as:

$$(A - 2^{-d} I)\Gamma^d = 0 \tag{3.1.9}$$

A has order (2D-3)*(2D-3) and elements are represented as:

$$[A]_{l,n} = \bar{a}_{2l-n}, l, n \in [2-D, D-2].$$

$\bar{a}$ has following properties :

1. According to the orthogonality property

$$\bar{a_p} = \begin{cases} 1 & for \quad t = 0 \\ 0 & for \quad t = 0, \pm 2, \pm 4, ... \end{cases}$$

2. $$\overline{a}_t = \overline{a}_{-t} \text{we calculate only } \overline{a}_p \text{ for } p > 0$$

3. $$\sum_{podd} \overline{a}_p = 1$$

For $D = 6$ we find that matrix A has entries as:

From system of equations we find that 2^{-d} is the eigen value of A matrix. This forms homogeneous system of equations. Therefore, we use vanishing moments to find the solutions.

Since,

$$x^d = \sum_{l=-\infty}^{\infty} M_l^d \phi(x-l)$$

$$d! = \sum_{l=\infty}^{l=\infty} M_l^d \int_{-\infty}^{\infty} \phi^{(d)}(x-l)dx$$

Pre-multiply both sides by $\phi(x)$ then, integrating, we get

$$d! \int_{-\infty}^{\infty} \phi(x)dx = \sum_{l=\infty}^{l=\infty} M_l^d \int_{-\infty}^{\infty} \phi(x)\phi^{(d)}(x-l)dx$$

$$\sum_{l=2-D}^{D-2} M_l^d \Gamma_l^d = d! \tag{3.1.10}$$

3.2 Obtaining differentiation matrix with respect to scaling function

Let a function g(x) $\in V_j \cap c^d(\mathbb{R}), J \in N_0$. We use connection coefficients to find the derivative of the function in terms of scaling function coefficients. We know that scaling function has following expansion if a given function has to be expressed in V_j space:

$$f(x) = \sum_{l=-\infty}^{\infty} c_{j,l}\phi_{j,l}(x), x \in \mathbb{R}$$

Differentiating d times above equation we get,

$$f^d(x) = \sum_{l=\infty}^{\infty} c_{j,l}\phi_{j,l}^{(d)}(x), x \in \mathbb{R} \tag{3.2.1}$$

since, $f^{(d)}$ does not belongs to space V_j. Therefore, we project it using the formula :

$$(P_{v_j} f^{(d)})(x) = \sum_{k=-\infty}^{\infty} c_{j,k}^{(d)}\phi_{j,k}(x), x \in \mathbb{R} \tag{3.2.2}$$

and we know that scaling function coefficients are given by:

$$c_{j,k} = \int_{-\infty}^{\infty} f(x)\phi_{j,k}(x)dx$$

Again, differentiating the above equation we have,

$$c_{j,k}^{(d)} = \int_{-\infty}^{\infty} f^{(d)}(x)\phi_{j,k}(x)dx \tag{3.2.3}$$

Substitute the value of $f^{(d)}(x)$ in above we get,

$$c_{j,k}^{(d)} = \int_{-\infty}^{\infty} \sum_{l=\infty}^{\infty} c_{j,l}\phi_{j,l}^{(d)}(x)\phi_{j,k}(x)dx \tag{3.2.4}$$

By definition of connection coefficients we have,

$$= \sum_{l=\infty}^{\infty} c_{j,l}\Gamma_{j,k,l}^{0,d}$$

$$= \sum_{l=-\infty}^{\infty} c_{j,l}2^{jd}\Gamma_{k-l}^{d}$$

$$= \sum_{l=-\infty}^{\infty} c_{j,n+k}2^{jd}\Gamma_{n}^{d}, j, k \in \mathbb{Z},$$

by definition of one periodic function we have,

$$c_{j,l} = c_{j,l+p2^j}, l, p \in \mathbb{Z}$$

$$c_{j,k}^{(d)} = c_{j,k+p2^j}, k, p \in \mathbb{Z}$$

$$c_{j,k}^{(d)} = \sum_{n=2-D}^{D-2} c_{j,\langle n+k\rangle_{2^j}} 2^{jd}\Gamma_{n}^{d}, k = 0, 1, ..., 2^j - 1 \tag{3.2.5}$$

In matrix form the system can be represented as:

$$c^{(d)} = D^{(d)}c \tag{3.2.6}$$

and we have:

$$[D^{(d)}]_{k,\langle n+k\rangle_{2^j}} = 2^{jd}\Gamma_n^d$$

$$k = 0, 1, 2, ..., 2^j - 1n = 2 - D, 3 - D, ..., D - 2c^{(d)} = [c_{j,0}^{(d)}, c_{j,1}^{(d)}, ..., c_{2^j-1}^{(d)}] \qquad (3.2.7)$$

3.3 Obtaining differentiation matrix with respect to physical space

We consider periodic case. Since,

$$f = Tc \qquad (3.3.1)$$

$$c = T^{-1}f$$

Differentiating the above equation and applying projection of $f^{(d)}$ into space $\tilde{v}_j$ we obtain:

$$f^{(d)} = Tc^{(d)}$$

Put value of $c^{(d)}$ we get,

$$f^{(d)} = T(D^{(d)}c)$$

$$= T(D^{(d)}T^{-1}f)$$

here, $D^{(d)}$ is the differentiation matrix which in respect to coefficient space and with respect to physical space it is given by TDT^{-1}.

Theorem 3.3.1 *Circulant matrices of same dimensions always commute.*

By use of above theorem, the following is possible as both the matrices T and $D^{(d)}$ have same dimension and are shift circulant.

$$TD^{(d)}T^{-1} = D^{(d)}TT^{-1} = D^{(d)}$$

Therefore,

$$f^{(d)} = D^{(d)}f \qquad (3.3.2)$$

where $D^{(d)}$ now represents differentiation matrix wrt. physical and coefficient space.

Calculating the approximation errors we have,

$$E^{(d)}(f,j) = max_{k=0,1,\ldots,2^j-1}|\left[f^{(d)}\right]_k - f^{(d)}(k/2^j)|$$

The rate of convergence for the differentiation matrix $D^{(1)}$ is given by:

$$f \in C^D(\mathbb{R}) \Rightarrow E^{(1)}(f,j) \leq C2^{-jD} \tag{3.3.3}$$

where C is a constant. For higher order of differentiation assume that results holds true, which is given by:

$$E^{(d)}(f,j) = C2^{-jr}, r \in \mathbb{R}, d \geq 1 \tag{3.3.4}$$

here, r depends upon d and D. Take log on both sides of the equation we have,

$$\log_2(E^{(d)}(f,J)) = \log_2(C) - Jr \tag{3.3.5}$$

Put, $J = J+1$ in above equation and subtracting the two equation we get,:

$$log_2(E^{(d)}(f,J)) - log_2(E^{(d)}(f,J+1)) = log_2(C) - Jr - log_2(C) - (J+1)r$$

$$log_2(E^{(d)}(f,J)) - log_2(E^{(d)}(f,J+1)) = r$$

Also, r is given by:

$$r = D - 2\lfloor d/2 \rfloor \tag{3.3.6}$$

3.4 Solving examples of pde using different approaches

3.5 Galerkin method

We now discuss an example based on the Galerkin method using approach mentioned in class 1. Consider, 1 Dimensional Helmholtz equation

$$\left\{ \begin{array}{l} -v'' + \alpha v = g(x) \\ v(x) = v(x+1) \end{array} \right\} x \in \mathbb{R} \tag{3.5.1}$$

Firstly, we approximate the solution in terms of scaling functions.

$$v_k(x) = \sum_{l=0}^{2^{k-1}} (c_v)_{k,l}\tilde{\phi}_{k,l}(x), k \in N_0 \tag{3.5.2}$$

$$v_k''(x) = \sum_{l=0}^{2^{k-1}} (c_v^{(2)})_{k,l}\tilde{\phi}_{k,l}(x) \tag{3.5.3}$$

We have the relation:

$$c^{(d)} = D^{(d)}c$$

Similarly, it can be written as:

$$(c_v^{(2)})_{k,l} = [D^{(2)}c_v]_l$$

Consider, the relation:

$$c_{k,l}^{(2)} = \sum_{n=D-2}^{D-2} c_k, \langle n+l\rangle_{2^l} 2^{ld}\Gamma_n^d$$

Coefficients $(c_v)_{k,l}$ is found using Galerkin approach. Multiply both sides of (3.5.3) by $\phi_{l,m}(x)$ and integrating it over [0,1].

$$\int_0^1 -v_k''(x)\tilde{\phi}_{k,m}(x)dx + \alpha\int_0^1 v\tilde{\phi}_{l,m}(x)dx = \int_0^1 g(x)\tilde{\phi}_{l,m}(x)dx$$

$$-\int_0^1 \sum_{l=0}^{2^k-1} (c_v^{(2)})_{k,l}\phi_{k,l}(x)\tilde{\phi}_{l,m}dx + \alpha\int_0^1 \sum_{l=0}^{2^k-1} (c_v)_{k,l}\tilde{\phi}_{l,m}(x)dx = \int_0^1 g(x)\tilde{\phi}_{k,m}dx$$

By orthogonality condition we have,

$$\int_0^1 \tilde{\phi}_{k,l}(x)\tilde{\phi}_{k,m}(x)dx = \delta_{l,m}, k \geq 0$$

$$-\sum_{l=0}^{2^k-1} (c_v^{(2)})_{k,l}\delta_{l,m} + \alpha\sum_{l=0}^{2^k-1} (c_v)_{k,l}\delta_{l,m} = \int_0^1 g(x)\tilde{\phi}_{l,m}(x)dx$$

$$-\sum_{l=0}^{2^k-1} (c_v^{(2)})_{k,l} + \alpha\sum_{l=0}^{2^k-1} (c_v)_{k,l} = \int_0^1 g(x)\tilde{\phi}_{k,m}$$

substitute

$$(c_g)_{k,m} = \int_0^1 g(x)\tilde{\phi}_{k,m}(x)dx$$

Therefore, the system becomes:

$$-c_v^{(2)} + \alpha c_v = c_g \tag{3.5.4}$$

Using the relation we have,

$$c^{(d)} = D^{(d)}c$$

$$c_v^{(2)} = D^{(2)}c_v$$

$$-D^{(2)}c_v + \alpha c_v = c_g$$

$$(-D^{(2)} + \alpha I)c_v = c_g$$

$$Ac_v = c_g$$

where A is

$$A = (-D^{(2)} + \alpha I)$$

The system of equations can be represented as :

$$Ac_v = c_g \tag{3.5.5}$$

3.6 Collocation method:

In wavelet collocation method, choose a wavelet and a grid, which can be modified numerically. Then, $g(x)$ is the function which has to be approximated in space V_k, by using the following formula:

$$P_{v_k}g(x) = \sum_{l=0}^{2^k-1} c_{k,l}\tilde{\phi}_{k,l}(x)$$

where $c_{k,l}$ represents the coefficients of scaling functions. Differentiate both sides 'd' times, we get,

$$g^{(d)}(x) = 2^{kd}\sum_{l=0}^{2^k-1} c_{k,l}\tilde{\phi}_{k,l}(x)$$

Point of nodes at level k, exact function and calculated function is

$$g^{(d)}\left(\frac{m}{2^k}\right) = 2^{jd}\sum_{l=0}^{2^k-1} c_{k,l}\tilde{\phi}_{k,l}\left(\frac{m}{2^k}\right)$$

and $m = 0, 1, ..., 2^k - 1$

$$g^{jd} = D^{(d)} c_k$$

and $g^{j(d)} = (g^(d)(0), ..., g^{(d)}(\frac{2^k-1}{2^k}))$ and $c^k = (c_{k_0}, ..., c_{k_{\frac{2^k-1}{2^k}}})$ The matrix $D^{(d)}$ is obtained in terms of the scaling functions.

Wavelet alogrithms:

Calculating ϕ at integers:

Since, ψ and *phi* are implicit functions. Therefore, filter coefficients are used to calculate ψ and ϕ [28]. Algorithm comes handy when one wants to compute these values. The compact support of the scaling function is on the interval$[0, D-1]$ and it satisfies the following conditions:

$$\phi(0) = 0 \quad and \quad \phi(D-1) = 0.$$

But only $\phi(0) = 0$ will be used in the computations.

We put $x = 0, 1, ..., D-2$ in the dilation equation yields a homogeneous linear systems of equations, consider for $D = 6$.

$$\begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \\ \phi(4) \end{bmatrix} = \sqrt{2} \begin{bmatrix} a_0 & & & & \\ a_2 & a_1 & a_0 & & \\ a_4 & a_3 & a_2 & a_1 & a_0 \\ & a_5 & a_4 & a_3 & a_2 \\ & & & a_5 & a_4 \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \\ \phi(4) \end{bmatrix}$$

.

Consider,the eigen value problem for A^0

$$A^0 \phi(0) = \lambda \phi(0) \tag{3.6.1}$$

Now, we have to search for the solution which satisfy the above equation. Since, $\lambda = 1$ satisfy the above system. Also, $\lambda = 2^{-m}$,$m = 0, 1, 2, ..., D/2 - 1$ also satisfy the system of solution.

Also, the solution must be such that it satisfy the following property:

$$\sum_{k=0}^{D-1} \phi(k) = 1.$$

3.7 Calculating ϕ at dyadic rationals:

In dilation equation, if we substitute $x = \frac{1}{2}, \frac{3}{2}, ...$. The following matrix is obtained:

The above can be represented as

$$\Phi\left(\frac{1}{2}\right) = A^{(1)}\Phi(0).$$

For the integers of the form, the pattern repeats $\frac{l}{4}$, l is odd. The system thus obtained:

$$\begin{bmatrix} \phi(\frac{1}{4}) \\ \phi(\frac{3}{4}) \\ \phi(\frac{5}{4}) \\ \phi(\frac{7}{4}) \\ \phi(\frac{9}{4}) \\ \phi(\frac{11}{4}) \\ \phi(\frac{13}{4}) \\ \phi(\frac{15}{4}) \\ \phi(\frac{17}{4}) \\ \phi(\frac{19}{4}) \end{bmatrix} = \sqrt{2} \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ a_4 & a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 & a_1 \\ & a_5 & a_4 & a_3 & a_2 \\ & & a_5 & a_4 & a_3 \\ & & & a_5 & a_4 \\ & & & & a_5 \end{bmatrix} \begin{bmatrix} \phi(\frac{1}{2}) \\ \phi(\frac{3}{2}) \\ \phi(\frac{5}{2}) \\ \phi(\frac{7}{2}) \\ \phi(\frac{9}{2}) \end{bmatrix} \tag{3.7.1}$$

We obtain the recursive relation such that, which can be put in the form:

$$\phi\left(\frac{1}{4}\right) = A^{(0)}\phi\left(\frac{1}{2}\right).$$

$$\phi\left(\frac{3}{4}\right) = A^{(1)}\phi\left(\frac{1}{2}\right).$$

$$\begin{aligned}
\phi\left(\frac{1}{8}\right) &= A^{(0)}\phi\left(\frac{1}{4}\right) \qquad \phi\left(\frac{5}{8}\right) = A^{(1)}\phi\left(\frac{1}{4}\right) \\
\phi\left(\frac{3}{8}\right) &= A^{(0)}\phi\left(\frac{3}{4}\right) \qquad \phi\left(\frac{7}{8}\right) = A^{(1)}\phi\left(\frac{3}{4}\right) \\
\phi\left(\frac{1}{16}\right) &= A^{(0)}\phi\left(\frac{1}{8}\right) \qquad \phi\left(\frac{9}{16}\right) = A^{(1)}\phi\left(\frac{1}{8}\right) \\
\phi\left(\frac{3}{16}\right) &= A^{(0)}\phi\left(\frac{3}{8}\right) \qquad \phi\left(\frac{11}{16}\right) = A^{(1)}\phi\left(\frac{3}{8}\right) \\
\phi\left(\frac{5}{16}\right) &= A^{(0)}\phi\left(\frac{5}{8}\right) \qquad \phi\left(\frac{13}{16}\right) = A^{(1)}\phi\left(\frac{5}{8}\right) \\
\phi\left(\frac{7}{16}\right) &= A^{(0)}\phi\left(\frac{7}{8}\right) \qquad \phi\left(\frac{15}{16}\right) = A^{(1)}\phi\left(\frac{7}{8}\right)
\end{aligned}$$

The series can be obtained at desired resolution :

$$\Phi\left(\frac{l}{2^k}\right) = A^0\Phi\left(\frac{l}{2^k-1}\right).$$

$$\Phi\left(\frac{l}{2^k}+\frac{1}{2}\right) = A^1\Phi\left(\frac{l}{2^k-1}\right).$$

$Cascade(D, q)$ matlab function gives the value of ψ and ϕ at dyadic rationals.

To compute the value of ψ the wavelet equation is used.

$$\psi(x) = \sqrt{2}\sum_{l=0}^{D-1} a_l\psi(2x-l)$$

$$\psi\left(\frac{m}{2^k}\right) = \sqrt{2}\sum_{l=0}^{D-1} a_l\psi\left(\frac{2m}{2^k}-l\right)$$

Differentiate dilation equation (2.3.6) d times,

$$\psi^d(x) = 2^d\sqrt{(2)}\sum_{l=0}^{D-1} a_l\psi^{(d)}(2x-l) \tag{3.7.2}$$

put $x = 0, 1, ..., D-1$ in above equation yields the system given by:

$$2^{-d}\psi^{(d)}(0) = A^0\psi^{(d)}(0) \tag{3.7.3}$$

and $x = \frac{1}{2}, \frac{3}{2}, ...$

$$\psi^{(d)}\left(\frac{1}{2}\right) = 2^d A^1 \psi^{(d)}\left(\frac{1}{2}\right) \tag{3.7.4}$$

Differentiate wavelet equation (2.3.7) gives:

$$\psi^{(d)}(x) = 2^d \sqrt{2} \sum_{l=0}^{D-1} b_l \psi^{(d)}(2x - l). \tag{3.7.5}$$

The values of $\psi^{(d)}$ is computed from $\phi^{(d)}$.

3.8 WOFD:

WOFD is wavelet optimized finite difference method. It was given by Leland Jameson. In WOFD, a grid is considered at different scales and finite difference is applied. Consider, a function $g(x)$ where $x \in J$, J represents the interval. This function can be expressed in the form of wavelet coefficients. As, wavelets depend upon two parameters namely scale and shift function $d_{k,l}$. A threshold value is considered to filter the wavelet coefficients, the coefficients above threshold remains in the system and those below it are neglected. Let the threshold value be K. Then, $|d_{k,l} > K|$ gives the desired wavelet coefficient that can be defined by manually. The grid points are added at shift l and density of grid at the scale k. The grid is employed which will resolve the function on the domain. The function is not resolved in those part of the interval where it is smooth. For fourth order finite differencing it is equivalent to the D4 wavelet decomposition.

3.9 Decay of wavelet coefficients:

For smooth function the wavelet coefficient decrease very fast. But, they decrease slowly around the point of discontinuity. Following theorem represents wavelet coefficient's decay. The motive of the theorem is with increase in the smoothness of the curve, less number of wavelet coefficients are required to represent it. The wavelet grid is generated by using the following theorem.

Theorem 3.9.1 *If $Q = D/2$ gives the number of vanishing moments for wavelet $\psi_{k,l}$ and $g \in C^q(\mathbb{R})$. Then, wavelet coefficient decay is given by:*

$$\boxed{|d_{k,l}| \leq C_q 2^{-k(Q+\frac{1}{2})} max_{\xi \in I_{k,l}} |f^{(Q)} \xi|.} \tag{3.9.1}$$

where C_q represents a constant independent of k, l, g and $I_{k,l} = supp\psi_{k,l} = \left[\frac{l}{2^k}, \frac{(l+D-1)}{2^k}\right]$

3.10 Algorithm for solving PDE

(1.) Differential equation domain is discretized.

(2.) Operators of the differential equation is discretized.

(3.) Let the time ti's be decided for the refinement of the grid. The initial condition is used to find the solution at t_1, which is given by $u(t_1)$.

(4.) Using $u(t_1)$ in accordance with the grid X_{t_1} to get $X_{t1+\delta t}$.

(5.) On $X_{t_1+\delta t}$ find the differential operator of differential equation.

(6.) Integrate the resultant of ordinary differential equation gives the solution $t = \delta + t_1$.

(7.) Repeat the above step to get $u(t_2)$ after that follow step 4.

Numerical example

Consider, the burger's equation given by:

$$\begin{aligned} u_t &= vu_{xx} - (u+p)u_x \\ u(x,0) &= h(x) \\ u(x,t) &= u(x+1,t) \end{aligned}$$

where $x \in \mathbb{R}$, v is positive constant, $p \in \mathbb{R}$ and $h(x) = h(x+1)$. This is solved using above mentioned WOFD technique. The graphical results are provided as follows.

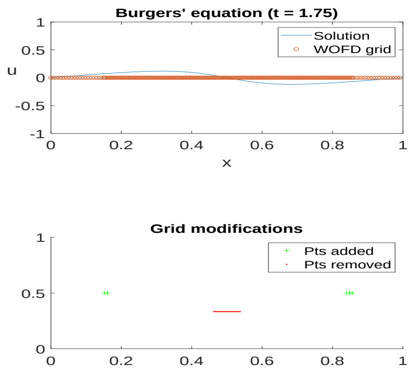

Figure 3.1: Burger's equation

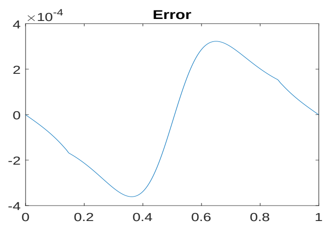

Figure 3.2: Error estimate

Chapter 4

Application of wavelets to other fields:

4.1 Inverse problem

What are inverse problems?

Inverse problems are subset of the ill-posed problems. Ill posed problems are inverse of the well posed problems. Jacques Salmon Hadamard (1865-1963) introduced the concept of well posed problems. A problem is said to be well posed problem if satisfies following axioms:

(1.) **Existence**: Solution of a mathematical model of physical problem must exist.

(2.) **Uniqueness**: Solution of mathematical model of physical problem must possess a unique solution.

(3.) **Stability**: Solution of mathematical model of physical model must depend continuously on the data in some reasonable space.

If any of the above mentioned axiom is not satisfied then, the problem is called **ill posed**. Inverse problem can be thought as something where we have to find the cause of the consequence with maximum accuracy. The applicability of inverse problem is diverse in real life situations. Consider, a glass of water with droplet of ink dropped just at the center of the glass. After, 24 hours it would be extremely difficult to know the exact position of the droplet as diffusion takes place, given the volume and dimensions of the glass. To determine the position of the droplet initially when it was introduced in the water is the Inverse problem.

There are various applications of inverse problems to list a few:

(1.) **Image deblurring:** In image deblurring, we use mathematical model for obtaining the original and sharp image. In blurred image the details are present in it but the fact is they are only hidden. Once, details of blurring process are retrieved then, hidden information can be easily extracted. It is difficult to get back actual original image. Since, approximating anything will definitely consist of alot of errors. Fluctuations

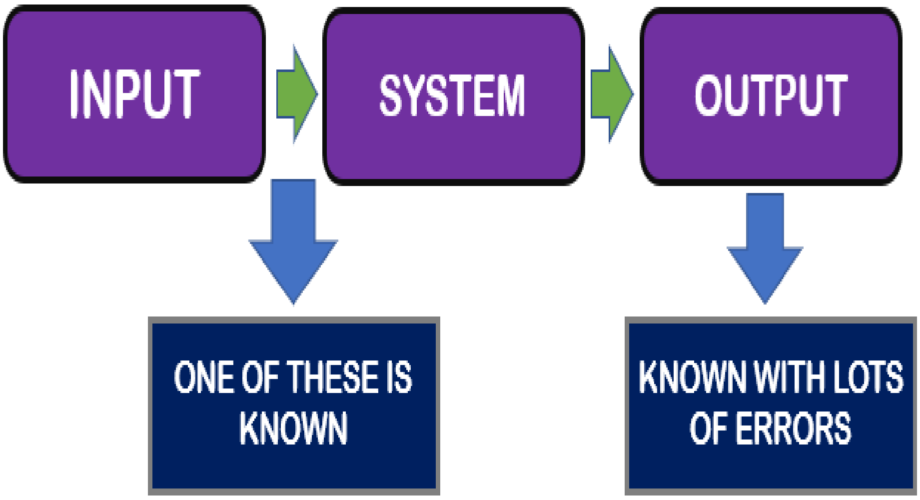

Figure 4.1: Outline of inverse problem

errors while recording the signal. Image has to be represented with finite number of digits the approximation error comes into play.

(2.) **Tomography:** Tomography is obtained from the greek word "Tomos" meaning slice and "Graphos" meaning to write. Tomography reconstruction is the mathematical procedure on which production of these images takes place. Optical tomography uses low energy / infrared light is used to reconstruct the images of the given medium. Since, infrared light can penetrate and interact with the tissue thereby, results in absorption and scattering. The image thus obtained from the boundary measurements is called the Inverse problem.

(3.) **Seismology:** It is the study of the Earthquake and seismic wave that travel through the Earth's layers. The inverse problem would be to determine the structure of the Earth or the position of the boundaries by defining the different material properties.

Mathematical representation of inverse problem:

Inverse problem can be represented mathematically as:

$$\boxed{d = Hm} \tag{4.1.1}$$

where

(1.) d represents data.

(2.) m represents model parameter.

(3.) H is compact linear or non-linear operator s.t. $H : X \to Y$.

4.1.1 Concept of regularization:

The concept of regularization was required because when third Hamadard third condition fails, even if the deviation from the true solution is small then large error occurs. Regularization considers operator family maybe linear or non-linear operators such that:

$$R_\beta : Y \to X, \beta > 0$$

where

$$\lim_{\beta \to 0} R_\beta(H(x)) = x, \forall x \in X \tag{4.1.2}$$

where β is called as regularization parameter.

(1.) If β is small then, noise is not properly filtered which makes m_β oscillatory in nature.

(2.) If β is large then, m_β obtained is overly smooth and noise is completely filtered out.

If eqn.(3.2) is satisfied then, R_β is known as regularisation strategy. Data has error of size δ, then inverse problem can re-written as:

$$m_\beta = R_{\beta(\delta)} d^\delta$$

Error level δ is known as regular if $d \in H(X)$ and $\forall \quad Y$ with $||d^\delta - d|| \leq \delta$ and following conditions holds:

$$R_\beta d^\delta \to H^{-1} d, \delta \to 0$$

Given below are few methods for regularization:

4.1.2 Regularisation by filtering:

In this approach, assume H is an invertible and real valued matrix. Then, by singular value decomposition it can be expressed as:

$$H = V diag(\omega_i) W^T$$

ω takes decreasing positive values only. Column vector v_i and w_i satisfies:

$$v_j^T v_k = \delta_{j,k}, w_j^T w_k = \delta_{j,k}, Hv_j = \omega_i v_i, H^T v_j = \omega_j w_j$$

$V^T = V^{-1}$ and similarly, $W^T = W^{-1}$. Then,

$$H^{-1}d = Wdiag(\omega^{-1})V^T d = m + \sum_{j=1}^{n} \omega_j^{-1}(v_j^T \nu)w_i \tag{4.1.3}$$

where ν represents noise. Since, here division is done by small singular values and it is the reason behind instability. Therefore, modifying the value of ω_j will yield better results.

Multiply, the above (4.1.3) by $\sigma_\beta(\omega_j^2)(\omega^{-1}) \to 0$, then

$$m_\beta = Wdiag(\sigma_\beta(\omega_j^2)\omega^{-1})V^T d = \sum_{j=1}^{n} \sigma_\beta(\omega_j^2)\omega_j^{-1}(v_j^T d)w_j \tag{4.1.4}$$

Choosing, different values for $\sigma_\beta(\omega^2)$ will give different methods.

4.1.2.1 TSVD:

TSVD is also known as 'Truncated singular value decomposition'

if $\sigma(\omega^2)$ has value

$$\sigma_\beta(\omega^2) = \begin{cases} 1 & if \quad \omega^2 > \beta \\ 0 & if \quad \omega^2 \leq \beta \end{cases}$$

and m_β is modified as:

$$m_\beta = \sum_{\omega > \beta} \omega_j^{-1}(v_j^T d)w_j$$

4.1.3 Tikhonov regularization:

If we choose,

$$\sigma_\beta(\omega^2) = \frac{\omega^2}{(\omega^2 + \beta)}$$

then, m_β becomes:

$$\begin{aligned} m_\beta &= \sum_{j=1}^{n} \frac{\omega_j (v_j^T d)}{\omega^2 + \beta} w_j \\ &= (H^T H + \beta I)^{-1} H^T d \end{aligned}$$

This is called Tikhonov regularization.

4.1.4 Solving inverse problem using wavelets:

Desired results are not produced by singular value decomposition technique when original function is spatially inhomogeneous. Wavelets possess the property of localisation in both time and frequency. Obviously, wavelets will give much precise results for the same. The implementation process is given by:

(1.) On spatially inhomogeneous functions consisting of noise apply wavelet transform to it. This gives function which has the same length. Then, large wavelet coefficient will represent the function and smaller one's corresponds to the noise.

(2.) Apply appropriate threshold to the obtained so as to remove noise and then, apply inverse wavelet transform. Further, setting the level of threshold and threshold function will lead to a smooth function which was to reconstructed.

There are several methods to name a few :

(1.) Wavelet-vaguelette decomposition [41].

(2.) Generalized Wavelet-Galerkin method.

(3.) Wavelet Domain linear inversion.

4.2 Wavelet application in medicine:

Wavelets have proved to be useful in medical application because they can detect diagnostic features and in medical image compression. To study non-stationary signals, short

-time fourier transform can be studied by partitioning it into blocks of short, pseudo stationary sections, whose sections remain unchanged for the required duration. But, due to Heisenberg uncertainty principle if time duration is large enough, then , the frequency will be small. This further disproves the supposition of stationary signal within considered window.

Time resolution at high frequency, frequency resolution at low frequency is given by wavelet transform. Early, detection of heart disease, breast cancer , separating speech from the noise in hearing aid machine are some examples of wavelet transform applications.

Wavelets and heart disease: Wavelets can be applied to study heart sounds. The arteries produces the sound which can detect the severness and kind of blockage that may take place. Also, higher frequency content greater the degree of stenosis. The sounds that are produced by opening and closing of the heart valve are similarity that are produced by stenosis coronary artery sounds. Diastole is perfect condition to study the sounds that are produced by turbulent blood flow.

At this time blood flow is at maximum and therefore, sounds produced are also the loudest [4]. The heart cycle is separated from the signal by time window. This is further studied by wavelet based fractal analysis.

Fractal analyzes can analyze the turbulence due to diastole heart sound and rescaling it properly would give the original signal nearly. Turbulent flow is a fractal process as on small scales it is not uniform in space and energy distribution is also irregular. H denotes the hurst exponent and D represents the fractal dimension. Hurst exponent represents scaling behaviour. The relation between hurst exponent and fractal dimension is given by $D = 2 - H$. It is assumed that the details of the signals are stationary. The difference in information of the given two successive scales represents the wavelet coefficients. Hurst exponent can be calculated from the variance of the scale and wavelet coefficients gives the relation for the variance.

The log-log plot for the given variance of the wavelet coefficients versus scales. Then, the hurst exponent can be calculated as follows:

$$H = (\gamma - 1)$$

where γ represents the regression line.
The plot was plotted for normal subject and abnormal subject. For normal subject the $\gamma = 2.1$ and for abnormal subject $\gamma = 1.1$. And obviously, for the normal subject the heart signal was smooth in comparison to the abnormal subject.

WAVELETS FOR HEARING AIDS: Hearing aids are used to correct the hearing loss. Hearing loss is generally affects the old age people. The reason is cochlea's ability to analyse the sound and speech decreases. Experiments have shown that the speech recognising test, the hearing impaired patients score less in comparison to normal listeners. To improve the speech signal single microphone method was developed to reduce the components of unwanted sound. The method did not produce the required results.

Wavelet methods were adopted for removing noise from the background. The improvement was seen in speech-noise ratio. Wavelet transform produced much better results. It used local discriminant basis to separate out the signal required and noise.

4.3 Fingerprint storage using wavelets:

Wavelets are used by FBI to store fingerprints. Approximately, $30,000$ fingerprints are added daily to the dataset. Earlier, FBI was use to store fingerprints in paper form and accumulation of such enormous data lead to utilizing space equivalent to the football field. Sharing huge information was also troublesome. Dividing each square-inch into 256*256 pixels and 0 represents completely white and 256 represents completely black picture [31]. Fingerprint image has certain details to it, where gray tone represents its relationship with the pixels.This also enables to store the data in form sequence of numbers.

Since, wavelets have unique capability to represent large amount of data using few coefficients accurately. Therefore, bi-orthogonal spline wavelets were put to use. ψ

$$g(t) = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} d_{k,l}\psi_{k,l}(t)$$

Wavelet scalar quantization gray scale fingerprint image compression algorithm was used, which is also famously known as WSQ. Only, eight percent of the original information was required to reconstruct the fingerprint accurately. This further also makes data compression easy.

4.4 Compression of data:

Assume, that following sequence of numbers represents a signal.

60	44	12	28	52	52	44	20

These eight numbers represents the input signal. The transformation applied here is taking average of the pairs and second transformation is subtracting the first term and the averaged term [3]. Mathematically, it is represented by:

$$(x, y) \rightarrow (w, z)$$

where:

$$w = \frac{x+y}{2}$$

$$z = \frac{x-y}{2}$$

52	8	20	-8	52	0	32	12

The first term of the sequence is given by $\frac{(60+44)}{2} = 52$ and similarly, the second term of the sequence is calculated by $(60 - 52) = 8$. As, four pairs are formed and first term of each pair is the average of the two numbers and second term of the pair is the difference of first term and the averaged term. Calculating, by using the mentioned scheme yields the above table [17].

36	8	16	-8	42	0	10	12

In above table, we transform only the averaged terms and remaining terms remain the same. Further, average of averaged terms and difference of the averaged terms are replaced by them i.e. only 36 and 42 are considered.

39	8	16	-8	-3	0	10	12

Now, delete terms that are less than 4. This is also known as threshold condition. Threshold here is set to 4. Final table is given by:

39	8	16	-8	0	0	10	12

Now, the signal is transformed using the above transformation and then, signal obtained is reconstructed using the formula:

$$x = w + z$$

$$y = w - z$$

so, using the above formula on position 1 and on position 5 will yield:

39	8	16	-8	39	0	10	12

Considering, the average terms again i.e. $39, 16, 39, 10$ and performing the reconstruction yields the following table:

55	8	23	-8	49	0	29	12

Now, the average and difference between the average term is calculated pairwise. This, gives the final reconstruction table: The difference between the original signal and recon-

63	47	15	31	49	49	41	17

structed signal is shown by following table. It also shows that the reconstructed signal has small variation in comparison to the original signal. This means that approximately same signal can be retrieved after compression.

63	47	15	31	49	49	41	17
60	44	12	28	52	52	44	20

Conclusion

The dissertation discusses studies different types of wavelets and advantages of Wavelets over fourier transform. Some properties that are necessary for calculation of wavelets have been discussed.

Various methods that are required to solve partial differential equations are reviewed. They are further elaborated with the help of an example.

Applicability in different fields of wavelets is studied.

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