CONSTRUCTION OF SOME NEW ITERATIVE FAMILIES FOR SOLVING SYSTEM OF NONLINEAR EQUATIONS

Thesis Submitted in partial fulfillment of the requirements for

The award of the degree of

Masters of Science

In

Mathematics and Computing

Submitted by

Dilpreet Kaur Bhatia

Roll no.- 301203003

Under

the guidance of

Dr. Sanjeev Kumar

JULY, 2014

School of Mathematics and Computer Applications

Thapar University

Patiala- 147004(PUNJAB)

INDIA

DEDICATED

TO

GOD, MY PARENTS

AND

SUPERVISOR

CERTIFICATE

I hereby certify that the work which is bring presented in the thesis entitled " **Construction of some new iterative families for solving system of nonlinear equations**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Sanjeev Kumar.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

Dilpreet Kaur

(Dilpreet Kaur Bhatia)

Reg. No. 301203003

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

Sanjeev Kumar 16/7/14

(Dr. Sanjeev Kumar)

Assistant Professor

SMCA, Thapar University

Patiala.

Countersigned by:

Dr. Rajesh Kumar

(Associate Professor & Head)

School of Mathematics & Computer Applications

Thapar University, Patiala.

Dr. S.K. Mohapatra

Dean of Academic Affairs

Thapar University

Patiala.

ACKNOWLEDGEMENTS

The writing of this dissertation has been one of the most significant academic challenges I have ever had to face. Without the support, patience and guidance of the following people, this study would not have been completed. It is to them that I owe my deepest gratitude.

First and foremost, I would like to express my special appreciation and thanks to my supervisor, ***Dr. Sanjeev Kumar*** *, Assistant Professor, School of Mathematics and Computer Application, Thapar University, Patiala, you have been a tremendous mentor for me. He has supported me throughout my thesis with his patience and knowledge. I would like to thank him for the guidance, encouragement and advice he has provided my time as his student. I have been extremely lucky to have a supervisor who cared so much about my work, and who responded to my questions and queries so promptly and his challenges brought this work towards a completion. He also helped me get on the road to LaTeX and provided an experienced ear for my doubts about writing a thesis. It is with his supervision that this work came into existence.*

I would like to express my sincere thanks to ***Dr. Rajesh Kumar****, Head SMCA, Thapar University, Patiala, for providing the necessary facilities needed for this work and directly or indirectly encouraged me to do the work.*

I wish to thank my parents for their undivided support and interest who inspired me and encouraged me to go on my own way, without whom I would be unable to complete my work.

I would also thank all academic and administrative staff of School of Mathematics and Computer Applications, Thapar University, Patiala.

At last but not the least I want to thank my friends who appreciated me for my work and motivated me and finally to God who made all the things possible

I am very thankful to everyone who all supported me , for I have completed my thesis effectively and moreover, on time.

(DILPREET KAUR BHATIA)

ABSTRACT

Nonlinear systems of equations appear in many disciplines such as engineering, mathematics, robotics and computer sciences because majority of physical systems are nonlinear in nature. One of the most basic numerical problems encountered in computational economics is to find the solution to a nonlinear equation or a whole system of nonlinear equations. In this thesis we introduce a technique for solving nonlinear system of equations that improve the order of convergence for any given iterative method. The family of new iterative methods are built up and analyzed. A development of an inverse first-order divided difference operator for functions of several variables is presented. The main advantage of these methods is that, they have higher - order of convergence and they do not require the evaluation of any second or higher order Fréchet derivatives. This thesis consists of three chapters

CHAPTER 1 gives a brief explanation about the need of iterative methods in scientific and engineering problems. Some basic concepts and definitions regarding system of nonlinear equations are introduced. Some fundamental concepts are also explained in this chapter.

CHAPTER 2 gives a brief survey of literature. It gives a review of methods with cubic as well higher- order convergent iterative methods.

In **CHAPTER 3** the family of iterative methods for computing the solution of system of nonlinear equations has been introduced. The proposed methods do not require the evaluation of second or higher order Fréchet derivatives per iteration to proceed and reach fourth and sixth order of convergence. An improvement of the local order of convergence is presented to increase the efficiency of the iterative methods with an appropriate number of evaluations of the function and its derivative. First-order divided difference operator for functions of several variables are used to prove the local convergence order. A discussion on the efficiency index of the contribution with comparison to the other iterative methods is also given. Finally, numerical tests illustrate the theoretical aspects using the programming package *Mathematica* 7.1.

CONTENTS

CERTIFICATE i

ACKNOWLEDGEMENTS ii

ABSTRACT iii

GLOSSARY OF SYMBOLS vi

1 INTRODUCTION 1

1.1 FUNDAMENTAL CONCEPTS 5

1.1.1 Root or solution 5

1.1.2 Approximated Computational Order of Convergence, ACOC 5

1.1.3 Convergence analysis 5

1.1.4 Operational count for Gauss elimination 6

1.1.5 Operational count for LU decomposition 6

1.1.6 Operational count for Resolution of two triangular linear systems 7

1.1.7 Fréchet Derivative 7

2 LITERATURE SURVEY 8

2.1 SURVEY OF SOME ITERATIVE METHODS FOR SYSTEM OF NONLINEAR EQUATIONS 9

2.1.1 Methods having cubic-order of convergence 10

2.1.2 Higher-order iterative methods 14

3 FAMILIES OF SOME NEW ITERATIVE METHODS **18**

3.1 BRIEF INTRODUCTION 18

3.2 MAIN RESULTS 19

3.3 SPECIAL CASES 20

3.4 CONVERGENCE THEOREM 21

3.5 COMPUTATIONAL EFFICIENCY AND COMPARISON AMONG PROPOSED METHODS 22

Case (i) ψ_1 *versus* ψ_2 24

Case (ii) ψ_3 *versus* ψ_4 24

Case (iii) ψ_5 *versus* ψ_6 25

Case (iv) ψ_1 *versus* ψ_4 25

Case (v) ψ_1 *versus* ψ_6 26

Case (vi) ψ_3 *versus* ψ_6 26

Case (vii) ψ_2 *versus* ψ_3 29

Case (viii) ψ_2 *versus* ψ_5 29

Case (ix) ψ_4 *versus* ψ_5 30

3.6 NUMERICAL RESULTS 32

3.7 CONCLUSIONS AND DISCUSSIONS 37

REFERENCES **38**

GLOSSARY OF SYMBOLS

$\mathbb{R}^t$	the real t – dimensional space of column vectors
$\{X^{(k)}\}_{k\geq 0}$	a sequence
ρ	order of convergence
$e^{(k)}$	an error at k^{th} iteration
r	solution of system of nonlinear equations
$F : D \subset \mathbb{R}^t \to \mathbb{R}^t$	a mapping with domain D in $\mathbb{R}^t$ and range in $\mathbb{R}^t$
α_1, α_2	real numbers
$\hat{\rho}_c$	approximated computational order of convergence (ACOC)

CHAPTER

ONE

INTRODUCTION

Numerical analysis is the study of algorithms that use numerical approximation for the problems of mathematical analysis (as distinguished from discrete mathematics). It is a wide-ranging discipline having close connections with medical science, engineering and the applied sciences. It has numerous applications in all field of science and some fields of engineering, and essentially any type of work that requires calculations to give very precise solutions. It involves the study of methods of computing numerical data. In many problems this implies producing a sequence of approximations by repeating the procedure again and again.

Numerical methods greatly expand the types of problems we can address. They are capable of handling large systems of equations nonlinearities and complicated geometries that are not uncommon in engineering and science and that are often impossible to solve analytically with standard calculus. Numerical methods are an efficient vehicle for learning to use computers. Because numerical methods are expressly designed for computer implementation they are ideal for illustrating the computers powers and limitations. When we successfully implement numerical methods on a computer, and then apply them to solve otherwise intractable problems, we will be provided with a dramatic demonstration of how computers can serve our professional development. At the same time, we will also learn to acknowledge and control the errors of approximation that are part and parcel of large-scale numerical calculations. Therefore with the advancement of computer, numerical analysis gained more importance than before.

Generally, such types of problems originate from real world applications of the four major disciplines of engineering : chemical, electrical, civil and mechanical and can be modeled by different mathematical equations. For example, the problems of solving integral equation for fluid pair distribution function in fluid mechanics, the problem of investigating coarse-grained dynamical properties of neuronal networks in kinetic theory many problems in control and optimization theory and some other mathematical problems namely, the problem of finding the eigen values of a square matrix, the problem of finding the roots of an auxiliary equation of higher-order homogeneous differential equations with constant coefficients, the solution of integral and differential equations using finite difference methods etc. can be formulated as a nonlinear equation or a system of nonlinear equations. These equations may be higher order algebraic equations and possibly they may involve exponential, trigonometric and hyperbolic terms or completely be transcendental equations.

The overall goal of field of numerical analysis is the design and analysis of techniques to give approximate but accurate solutions to hard problems, the variety of which is suggested by the following:

- Advanced numerical methods are essential in making numerical weather prediction feasible.
- Computing the trajectory of a spacecraft requires the accurate numerical solution of a system of ordinary differential equations.
- Car companies can improve the crash safety of their vehicles by using computer simulations of car crashes. Such simulations essentially consist of solving partial differential equations numerically.
- Hedge funds (private investment funds) use tools from all fields of numerical analysis to attempt to calculate the value of stocks and derivatives more precisely than other market participants.
- Airlines use sophisticated optimization algorithms to decide ticket prices, airplane and crew assignments and fuel needs. Historically, such algorithms were developed within the overlapping field of operations research.
- Insurance companies use numerical programs for actuarial analysis.

There are two types of methods for finding the roots of such problems, first one is the direct methods (analytical methods) and second one is iterative methods. Direct methods give the exact value of roots in finite number of steps. Further, these methods give all the roots at a time, but many times these methods are not able to find the roots of many nonlinear equations (e.g. there is no direct method that can solve every polynomial equation of degree greater than four) and then we turn towards iterative methods . These methods are based on the idea of successive approximation, i.e., starting from an arbitrary point (initial approximation) - the closest possible point to the solution sought - and involves arriving at solution gradually through successive tests by substituting the initial approximations into some formula involving the equation to obtain a sequence of successive approximations which in the limit converges to the root. A large number of methods have been proposed by researchers for their solutions.

So analytical methods and numerical methods both serve for different purposes. In analytical analysis we should use related formulas to obtain the analysis results. However many systems possess complex functionality that it is hard to track the system behavior by formulas, in such cases we should simulate the system and analyze it in some well defined situation to estimate its behavior. Numerical methods provide approximations to the problem in question. No matter how accurate they are they do not, in most cases, provide the exact answer. In some instances working out the exact answer by a different approach may not possible or may be too time consuming and it is in these cases where numerical methods are more often used.

The point of numerical analysis is to analyze method that are used to give approximate number solutions to situations where it is unlikely to find the real solution quickly and to try and improve upon these methods so as to reduce the amount of error generated by computer calculation. It is essential in work that requires precise numbers to get very good approximations with very little error in them, if approximations with just even 1 or 2 % error are used in another calculation, and the answer of that calculation used in another and so on, the error will build up and we end up with very unreliable numbers. That is why it is a good idea to study numerical analysis.

The two criteria to take into account when choosing a method for solving nonlinear equations are

1. Method convergence (conditions of convergence, speed of convergence etc.).

2. The computational cost of method.

Generally these iterative methods do not strive for the exactness. Therefore, researchers attempted to devise a method which will yield an approximate root differ from the exact root within a specific tolerance limit, or by an amount which has less than a specified probability of exceeding that tolerance. In our point of view, this is a first main reason for huge variety of research papers on iterative methods and the second one is that no iterative method apply to all types of problems. **Therefore, construction and analysis of some higher-order numerical methods for finding the solutions of system of nonlinear equations for the present context is the main focus of present thesis.**

Solving system of nonlinear equations is much more difficult than scalar case because

- Wide variety of behavior is possible, so determining existence and number of solutions or good starting guess is much more complex.
- In general, there is no simple way to guarantee convergence to desired solution or to bracket solution to produce absolutely safe methods.
- Computational overhead increases rapidly with dimension of problem.

One of the most basic and earliest problems of numerical analysis concerns with finding efficiently and accurately the approximate solution to nonlinear systems:

$$F(X) = 0. \tag{1.1}$$

where $F(X) = (f_1(X), f_2(X), ..., f_t(X))^T$, $X = (x_1, x_2, ..., x_t)^T$ and $F : \mathbb{R}^t \to \mathbb{R}^t$ is a sufficiently differentiable vector function.

The construction of iterative methods for solving system of nonlinear equations is an interesting task in numerical analysis and applied scientific branches, which has attracted so much attention recently. In the last years, iterative techniques have been applied in many diverse fields as economics, engineering, physics, dynamical models, and so on.

In this thesis, the authors have presented many new computationally effective root-finding methods for solving system of nonlinear equations, those are equally competitive with their existing counter parts. In comparison of iterative methods, one should take into account their convergence order, the numerical stability, computational costs, asymptotic error constants, the dependence of convergence on the choice of initial guesses, the simple body structures etc. The study of some of the mentioned features is often complicated so the comparison procedure is usually reduced to

(i) Number of iterations,

(ii) Approximated computational order of convergence (ACOC),

In essence, one can apply more simple procedure for checking which method is superior than others in the following way: method A is superior than method B if A attains more accuracy in terms of significant figures gained by each method by utilizing same total numbers of function evaluation with same initial approximations. Many computer algebra software systems are available such as Mathematica 7.1, Matlab 7.12 and Mapple 18 etc. We use Wolfram Mathematica 7.1 (in multiple precision) for the computation work throughout the thesis.

1.1 FUNDAMENTAL CONCEPTS

1.1.1 Root or solution

The value $r = (r_1, r_2, ..., r_t)^T$, which satisfies

$$f_1(r) = 0, f_2(r) = 0, ..., f_t(r) = 0,$$

is called a root or solution of $F(X) = 0$

1.1.2 Approximated Computational Order of Convergence, ACOC

The approximated computational order of convergence (ACOC)[1] is defined by

$$\hat{\rho}_c \approx \frac{ln(||\hat{e}^{(k+1)}||/||\hat{e}^{(k)}||)}{ln(||\hat{e}^{(k)}||/||\hat{e}^{(k-1)}||)},$$

where $\hat{e}^{(k+1)} = X^{(k)} - X^{(k-1)}$ and $X^{(k)}$, $X^{(k-1)}$, $X^{(k-2)}$ are three consecutive iterations.

1.1.3 Convergence analysis

In order to explore the convergence properties, we recall the following result of Taylor's series expansion on vector function (see [2]).

Lemma :

Let $F : V \subset \mathbb{R}^t \to \mathbb{R}^t$ be p-times Fréchet differentiable in a convex set $V \subseteq \mathbb{R}^t$ then for any $X, h \in \mathbb{R}^t$, the following expression holds:

$$F(X+h) = F(X) + F'(X)h + \frac{1}{2!}F''(X)h^2 + \frac{1}{3!}F'''(X)h^3 + ... + \frac{1}{(p-1)!}F^{(p-1)}(X)h^{(p-1)} + \mathbb{R}_p,$$

where

$$||\mathbb{R}_p|| \leq \sup_{0 \leq u \leq 1} \frac{1}{p!}||F^{(p)}(X+uh)||||h||^p \text{ and } h^p = (h, h, ...^p...., h).$$

1.1.4 Operational count for Gauss elimination

$$\text{Total number of quotients} = \sum t = \frac{t}{2}(t+1).$$

$$\text{Total products in the forward elimination} = \sum (t-1)t = \frac{t}{3}(t+1)(t-1).$$

$$\text{Total products in the back substitutions} = \sum (t-1) = \frac{t}{2}(t-1).$$

$$\begin{aligned}\text{Total products} &= \frac{t}{3}(t+1)(t-1)+\frac{t}{2}(t-1).\\ &= \frac{t}{6}(t-1)(2t+5).\end{aligned}$$

$$\begin{aligned}\text{Operational count} &= \text{Total number of products and quotients}\\ &= \frac{t}{2}(t+1)+\frac{t}{6}(t-1)(2t+5).\\ &= \frac{t}{3}(t^2+3t-1).\end{aligned}$$

$$\text{For large t , operation count} \approx \frac{t^3}{3}.$$

$$\text{Total additions and subtractions} = \frac{t}{6}(t-1)(2t+5).$$

1.1.5 Operational count for LU decomposition

$$\text{Total number of products} = \frac{t}{6}(t-1)(2t-1).$$

$$\text{Total number of quotients} = \frac{t}{2}(t-1).$$

$$\text{Total number of products and quotients} = \frac{t^3-t}{3}.$$

$$\text{Total number of addition and subtraction} = \frac{t}{6}(t-1)(2t-1).$$

1.1.6 Operational count for Resolution of two triangular linear systems

$$\text{Total number of products} = t(t-1).$$

$$\text{Total number of quotients} = t.$$

$$\text{Total number of products and quotients} = t^2.$$

$$\text{Total number of addition and subtraction} = t(t-1).$$

1.1.7 Fréchet Derivative

Let X , Y be normed linear spaces. The Fréchet derivative of an operator $F : X \to Y$ is the bounded linear operator $DF(a) : X \to Y$ which satisfies the following relation,

$$\lim_{h \to 0} \frac{||F(a+h) - F(a) - DF(a)h||}{||h||} = 0.$$

It is a generalization of the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ encountered in first-year calculus and the Jacobian of a function $f : \mathbb{R}^t \to \mathbb{R}^t$ studied in advanced calculus. For functions $F : \mathbb{R}^t \to \mathbb{R}^t$ the Fréchet derivative $DF(a)$ is the Jacobian of F, a linear operator which is represented by an $t \times t$ matrix.

CHAPTER

TWO

LITERATURE SURVEY

This section highlights the brief review of research work of iterative methods carried out in solving the systems of nonlinear equations. The paradigm involving these equations by iterative methods for finding solutions is one of the most important and challenging areas of applied mathematics. A large number of applications lead us to thousands of such equations depending on one or more parameters which are to be solved effectively. A large number of research workers have done extensive research and developed many methods of different orders to solve these equations. Solving these equations is a difficult task in general. Unless the solutions can be expressed in analytical forms, the equations are solved numerically. Even if the solution is given as an expression (e.g. the roots of quadratic equation), it contains nonlinear functions that are not readily available on standard micro-processors and micro-controllers. Consider a simple example $Ax^2+Bx+C=0$, involving a quadratic polynomial with real coefficients A,B,C and $A\neq 0$. The two solutions of this equation, labeled x_1 and x_2, are found in terms of the coefficients of the polynomial form the familiar formulae

$$x_1=\frac{-B+\sqrt{B^2-4AC}}{2A} \text{ and } x_2=\frac{-B-\sqrt{B^2-4AC}}{2A}. \tag{2.1}$$

Although the solution is given by the formula (2.1), it contains a square root operation that must be implemented in software. Therefore, one can hope to obtain only approximate solutions by relying on iteration methods.

2.1 SURVEY OF SOME ITERATIVE METHODS FOR SYSTEM OF NONLINEAR EQUATIONS

Consider a system of 't' nonlinear equations in 't'unknowns of the form

$$\left.\begin{aligned} f_1(x_1,x_2,......,x_t) &= 0,\\ f_2(x_1,x_2,......,x_t) &= 0,\\ \vdots \quad \vdots \qquad &\vdots\\ f_t(x_1,x_2,......,x_t) &= 0. \end{aligned}\right\}. \tag{2.2}$$

We denote $\mathbb{R}^t$ the real t-dimensional space of column vectors with components $x_1,x_2,......,x_t$. Each function f_i(where $i = 1,2,....,t$) can be thought of as mapping a vector $(x_1,x_2,......,x_t)^T$ of the t-dimensional space $\mathbb{R}^t$ into the real line $\mathbb{R}$. The elements of $\mathbb{R}^t$ are called vectors. We use the following compact notations to denote vectors:

$$F = (f_1, f_2,.....,f_t)^T, X = (x_1,x_2,.....,x_t)^T.$$

This system of 't' nonlinear equations in t-unknowns can be represented by defining a function F mapping $\mathbb{R}^t$ into $\mathbb{R}^t$ as

$$F = (f_1(x_1,x_2,......,x_t), f_2(x_1,x_2,......,x_t),....,f_t(x_1,x_2,......,x_t))^T. \tag{2.3}$$

Thus the system of equations (2.2) reduces to

$$F(X) = 0. \tag{2.4}$$

The functions $f_1(X), f_2(X),.....,f_t(X)$ are the coordinate functions of $F(X)$. The derivative $F'(X)$ of $F(X)$ is called Fréchet derivative. The matrix representation of $F'(X)$ is given by the Jacobian matrix denoted by $J_F(X)$ and is given by

$$J_F(X) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(X) & \frac{\partial f_1}{\partial x_2}(X) & \cdots & \frac{\partial f_1}{\partial x_t}(X)\\ \frac{\partial f_2}{\partial x_1}(X) & \frac{\partial f_2}{\partial x_2}(X) & \cdots & \frac{\partial f_2}{\partial x_t}(X)\\ \vdots & \vdots & & \vdots\\ \frac{\partial f_t}{\partial x_1}(X) & \frac{\partial f_t}{\partial x_2}(X) & \cdots & \frac{\partial f_t}{\partial x_t}(X) \end{pmatrix}. \tag{2.5}$$

There is no doubt that Newton's method is one of the best root-finding method for solving nonlinear equations $F(X) = 0$ and is given by

$$Y^{(k)} = X^{(k)} - \left\{F'(X^{(k)})\right\}^{-1} F(X^{(k)}), \tag{2.6}$$

where $\left\{F'(X^{(k)})\right\}^{-1}$ is the inverse of first Fréchet derivative $F'(X^{(k)})$ of the function of $F(X)$.

Despite of the fact Newton's method is very effective for finding the solution of system of nonlinear equations. There are well known two drawbacks of Newton's method. Firstly, Newton's method is very sensitive to the choice of initial guess and the Newton's method fails miserably if at any stage of the computation Jacobian is zero. Kanwar et al. [3] and Kou et al. [4] have given an alternative by proposing a modified Newton's method that works even if Jacobian is zero. Time to time a lot of work have been done for increasing the order of convergence of Newton method.

2.1.1 Methods having cubic-order of convergence

Some third order iterative methods for solving the system of nonlinear equations (2.4) are described as follows :

- Chebyshev's method[5, 6],

$$X^{(k+1)} = X^{(k)} - \left[I + \frac{1}{2}L_F(X^{(k)})\right] \{F'(X^{(k)})\}^{-1} F(X^{(k)}). \tag{2.7}$$

- Halley's method(or method of tangent hyperbolas)[5, 6],

$$X^{(k+1)} = X^{(k)} - \left[I + \frac{1}{2}L_F(X^{(k)}) \left\{I - \frac{1}{2}L_F(X^{(k)})\right\}^{-1}\right] \{F'(X^{(k)})\}^{-1} F(X^{(k)}), \tag{2.8}$$

where I is the identity operator on a Banach space and $L_F(X)$ is the linear operator defined by

$$L_F(X) = \{F'(X)\}^{-1} F''(X) \{F'(X)\}^{-1} F(X),$$

provided that $\{F'(X)\}^{-1}$ exists. The main practical difficulty related to the class of methods is the evaluation of the second-order Fréchet derivative. For a nonlinear system of t equations and t unknowns the first Fréchet derivative is a matrix with t^2 values, while the second Fréchet derivative has t^3 values. This implies a huge amount of operations in order to evaluate every iteration. To overcome these difficulties many authors have considered methods that do not use the second derivative. The authors have derived many third order two-point iterative methods free from second derivatives by using quadrature formulae, approximation to second derivatives, the Adomian Decomposition method etc. Some of them are as follows:

In 2000, Hernández [7] has introduced a numerical method for nonlinear equations, based on the Chebyshev third-order method [8], in which the second-derivative operator is replaced by a finite difference between first derivatives and this method is given by

$$\left.\begin{aligned} Y^{(k)} =& X^{(k)} - \{F'(X^{(k)})\}^{-1} F(X^{(k)}), \\ Z^{(k)} =& X^{(k)} + \frac{1}{2}(Y^{(k)} - X^{(k)}), \\ X^{(k+1)} =& Y^{(k)} - \{F'(X^{(k)})\}^{-1} \left[F'(Z^{(k)}) - F'(X^{(k)})\right](Y^{(k)} - X^{(k)}). \end{aligned}\right\}. \tag{2.9}$$

In 2003, Ezquerro and Hernández [9] have proposed Halley's type method given as

$$\left.\begin{aligned} &Y^{(k)} = X^{(k)} - \left\{F'(X^{(k)})\right\}^{-1} F(X^{(k)}), \\ &H(X^{(k)}, Y^{(k)}) = \left\{F'(X^{(k)})\right\}^{-1} F''(X^{(k)})(Y^{(k)} - X^{(k)}), \\ &X^{(k+1)} = Y^{(k)} - \frac{1}{2}H(X^{(k)}, Y^{(k)})\left\{I + \frac{1}{2}H(X^{(k)}, Y^{(k)})\right\}^{-1}(Y^{(k)} - X^{(k)}). \end{aligned}\right\}. \tag{2.10}$$

Further in 2004, Ezquerro and Hernández [10] have relaxed the convergence criterium required in [9] and proposed the following algorithm :

$$\left.\begin{aligned} &Y^{(k)} = X^{(k)} - \left\{F'(X^{(k)})\right\}^{-1} F(X^{(k)}), \\ &Z^{(k)} = X^{(k)} + \beta(Y^{(k)} - X^{(k)}),\ \beta \in (0,1], \\ &H(X^{(k)}, Y^{(k)}) = \frac{1}{\beta}\left\{F'(X^{(k)})\right\}^{-1}(F'(Z^{(k)}) - F'(Y^{(k)})), \\ &X^{(k+1)} = Y^{(k)} - \frac{1}{2}H(X^{(k)}, Y^{(k)})\left\{I + \frac{1}{2}H(X^{(k)}, Y^{(k)})\right\}^{-1}(Y^{(k)} - X^{(k)}). \end{aligned}\right\}, \tag{2.11}$$

to solve nonlinear equations $F(X) = 0$ in Banach spaces. In this case, $F : \Omega \subseteq X \to Y$ is an operator defined in an open convex subset Ω of the Banach space X into the Banach space Y. Although the above algorithm is free from second Fréchet derivative of the operator F, the order three of convergence of Halley's method is maintained.

In 2006, Cordero and Torregrosa [11] have introduced some variants of Newton's Method in order to solve systems of nonlinear equations, based in trapezoidal and midpoint rules of quadrature which are given as

Trapezoidal Newton's method (TN):

$$X^{(k+1)} = X^{(k)} - 2\left\{J_F(X^{(k)}) + J_F(X^{(k+1)})\right\}^{-1} F(X^{(k)}), \tag{2.12}$$

where $J_F(X^{(k)})$ is jacobian matrix of function F evaluated in $X^{(k)}$. In order to avoid the implicit problem that the equation (2.12) involves, they use the $(k+1)^{th}$ iteration of Newton method in the right side,

$$\left.\begin{array}{r} X^{(k+1)} = X^{(k)} - 2\left\{J_F(X^{(k)}) + J_F(Z^{(k)})\right\}^{-1} F(X^{(k)}), \\ \text{where } Z^{(k)} = X^{(k)} - \left\{J_F(X^{(k)})\right\}^{-1} F(X^{(k)}). \end{array}\right\}. \tag{2.13}$$

This variant of Newton's method is called Trapezoidal Newton's method (TN)

Midpoint Newton's method (MN):

$$X^{(k+1)} = X^{(k)} - \left\{J_F\left(\frac{X^{(k)} + X^{(k+1)}}{2}\right)\right\}^{-1} F(X^{(k)}), \tag{2.14}$$

Using again the $(k+1)^{th}$ iteration of Newton's method in the right side of (2.14), the implicit problem is avoided. Then,

$$\left.\begin{array}{r} X^{(k+1)} = X^{(k)} - \left\{J_F\left(\frac{X^{(k)} + Z^{(k)}}{2}\right)\right\}^{-1} F(X^{(k)}), \\ \text{where } Z^{(k)} = X^{(k)} - \left\{J_F(X^{(k)})\right\}^{-1} F(X^{(k)}). \end{array}\right\}. \tag{2.15}$$

This iterative process is called Midpoint Newton's method (MN)

Since the variants of Newton's method with cubic-order of convergence have become popular iterative methods to find approximate solutions to the solution of nonlinear equations. These methods both enjoy cubic convergence at simple roots and do not require the evaluation of second order derivatives. In 2006, Babajee and Dauhoo [12] have investigated about the relationship between these methods which are in fact based on the approximation of the second order derivative present in the third-order limited Taylor expansion. They have also proved that these methods are different forms of Halley's method and are all contractive iterative methods in a common neighbourhood. They have extended three variants namely, the Mid-point method, VS variant (Hasanov) and VI variant (Nedzhibov) to multidimension. The linear forms of these variants are given by

$$F_L^{(k)} = F(X^{(k)}) + D_m(X^{(k)})(X - X^{(k)}), \;\; m = 3,4,5,$$

where the matrix D_m at the $X^{(k)}$ defined as

Mid-point method:

$$D_3(X^{(k)}) = J\left(\frac{1}{2}\left(X^{(k)} + X_N^{(k+1)}\right)\right). \tag{2.16}$$

***VS* variant (Hasanov):**

$$D_4(X^{(k)}) = \frac{1}{6}\left[J(X_N^{(k+1)}) + 4J\left(\frac{1}{2}\left(X^{(k)} + X_N^{(k+1)}\right)\right) + J(X^{(k)})\right]. \tag{2.17}$$

***VI* variant (Nedzhibov):**

$$D_5(X^{(k)}) = \frac{1}{4}\left[J(X_N^{(k+1)}) + 2J\left(\frac{1}{2}\left(X^{(k)} + X_N^{(k+1)}\right)\right) + J(X^{(k)})\right]. \tag{2.18}$$

O Varmann [13] in 2006 has introduced numerical methods in which procedural and rounding errors are unavoidable, for example, those arising in mathematical modelling and simulation. For the solution of involving decomposition-coordination problems some rapidly convergent iterative methods are developed based on the classical cubically convergent method of tangent hyperbolas (Chebyshev-Halley Method) and the method of tangent parabolas (Euler-Chebyshev method) which is given by

$$\left.\begin{aligned} X^{(k+1)} &= V^{(k)} + \left(A^{(k)}F'(V^{(k)}) - 2I\right)A^{(k)}F(V^{(k)}), \\ \text{where } V^{(k)} &= X^{(k)} - A^{(k)}F(X^{(k)}) \text{ and } A^{(k)} = \{F'(X^{(k)})\}^{-1}. \end{aligned}\right\}. \tag{2.19}$$

In 2007, Darvishi and Barati [14] have presented a third-order Newton type method to solve system of nonlinear equations which is given as follows

$$\left.\begin{aligned} X^{(k+1)} &= X^{(k)} - \left\{F'(X^{(k)})\right\}^{-1}\left(F(X^{(k)} + F(X^{(k+1)^*})\right), \\ \text{where } X^{(k+1)^*} &= X^{(k)} - \left\{F'(X^{(k)})\right\}^{-1}F(X^{(k)}). \end{aligned}\right\}. \tag{2.20}$$

In 2008, Wang Haijun [15] has introduced a cubic order method that does not require the evaluation of any second or higher order Fréchet derivative and this method is given as follows:

$$\left.\begin{array}{l} X^{(k+1)} = X^{(k)} + \left\{D(X^{(k)}) - F'(X^{(k)})\right\}^{-1} (F(X^{(k)}) + F(Y^{(k)})), \\ \text{where} \quad Y^{(k)} = X^{(k)} + \left\{D(X^{(k)}) - F'(X^{(k)})\right\}^{-1} F(X^{(k)}), \\ \text{and} \quad D(X^{(k)}) = \operatorname{diag}(\gamma_1 f_1(X^{(k)}), \gamma_2 f_2(X^{(k)}), ..., \gamma_t f_t(X^{(k)})), \end{array}\right\}, \tag{2.21}$$

where $\gamma_k \in [-1,1]$. The above iterative method works well even if the Jacobian matrix is singular.

2.1.2 Higher-order iterative methods

Some higher-order iterative methods have also been proposed by various researchers for solving systems of nonlinear equations which are described as follows:

In 2007, Darvishi and Barati [17] have introduced a fourth-order convergence method based on a quadrature formulae to solve systems of nonlinear equations, this method is given by

$$X^{(k+1)} = X^{(k)} - \left\{\frac{1}{6}F'(X^{(k)}) + \frac{2}{3}F'\left(\frac{X^{(k)} + g(X^{(k)})}{2}\right) + \frac{1}{6}F'(g(X^{(k)})\right\}^{-1} F(X^{(k)}), \tag{2.22}$$

and its error equation is given by

$$\left[\frac{1}{6}F'(X^{(k)}) + \frac{2}{3}F'\left(\frac{X^{(k)} + g(X^{(k)})}{2}\right) + \frac{1}{6}F'(g(X^{(k)}))\right] e^{(k+1)} = O(e^{(k)})^4. \tag{2.23}$$

In 2007, Golbabai and Javidi [18] have applied Homotopy perturbation method (HPM) to construct an iterative method for solving system of nonlinear algebraic equations. They have introduced the following iterative methods for solving system of nonlinear equation

Algorithm 1. For a given $\mathbf{z^{(k)}} = \begin{bmatrix} z^{(k)} \\ w^{(k)} \end{bmatrix}$ calculate the approximation solution $\mathbf{z^{(k+1)}} = \begin{bmatrix} z^{(k+1)} \\ w^{(k+1)} \end{bmatrix}$

by the iterative scheme

$$\begin{bmatrix} z^{(k+1)} \\ w^{(k+1)} \end{bmatrix} = \begin{bmatrix} z^{(k)} \\ w^{(k)} \end{bmatrix} - \begin{bmatrix} f_x(z^{(k)}) & f_y(z^{(k)}) \\ g_x(z^{(k)}) & g_y(z^{(k)}) \end{bmatrix}^{-1} \begin{bmatrix} f(z^{(k)}) \\ g(z^{(k)}) \end{bmatrix}, \; t = 0,1,2,.... \tag{2.24}$$

Algorithm 2. For a given $\mathbf{z^{(k)}} = \begin{bmatrix} z^{(k)} \\ w^{(k)} \end{bmatrix}$ calculate the approximation solution $\mathbf{z^{(k+1)}} = \begin{bmatrix} z^{(k+1)} \\ w^{(k+1)} \end{bmatrix}$

by the iterative scheme

$$\left.\begin{aligned} \begin{bmatrix} l^{(k)} \\ m^{(k)} \end{bmatrix} &= -\{A_1\}^{-1} \begin{bmatrix} f(z^{(k)}) \\ g(z^{(k)}) \end{bmatrix}, \\ \begin{bmatrix} z^{(k+1)} \\ w^{(k+1)} \end{bmatrix} &= \begin{bmatrix} z^{(k)} \\ w^{(k)} \end{bmatrix} + \begin{bmatrix} l^{(k)} \\ m^{(k)} \end{bmatrix} - \{A_1\}^{-1} \begin{bmatrix} \frac{1}{2!}\left[{l^{(k)}}^2 f_{xx}(z^{(k)}) + 2l^{(k)}m^{(k)} f_{xy}(z^{(k)}) {m^{(k)}}^2 f_{yy}(z^{(k)}) \right] \\ \frac{1}{2!}\left[{l^{(k)}}^2 g_{xx}(z^{(k)}) + 2l^{(k)}m^{(k)} g_{xy}(z^{(k)}) {m^{(k)}}^2 g_{yy}(z^{k}) \right] \end{bmatrix}, \\ \text{where } A_1 &= \begin{bmatrix} f_x(z^{(k)}) & f_y(z^{(k)}) \\ g_x(z^{(k)}) & g_y(z^{(k)}) \end{bmatrix}. \end{aligned}\right\}. \tag{2.25}$$

In 2011, Cordero et al. [19] have derived iterative methods with order of convergence four and higher, for solving nonlinear systems by composing iteratively golden ratio methods with a modified Newton's method, this method is given by

$$\left.\begin{aligned} X^{(k)} &= X^{(k)} - \kappa_1 \left\{F'(X^{(k)})\right\}^{-1} F(X^{(k)}), \\ X^{(k+1)} &= X^{(k)} - A_i \left\{F'(X^{(k)})\right\}^{-1} F(Y^{(k)}). \end{aligned}\right\}, \tag{2.26}$$

for $i = 1,2$, where $\kappa_1 = \frac{1}{\phi}$, $\kappa_2 = -\phi$, $A_1 = 1+\phi$ and $A_2 = 2-\phi$, where $\phi = \frac{1+\sqrt{5}}{2}$. The equation

$$e^{(k+1)} = L(e^{(k)})^{\rho} + O(e^{(k)})^{\rho+1},$$

where L is a ρ-linear function, is called the error equation and ρ is called the order of convergence.

In 2012, Montazeri et al. [20] have introduced a sixth-order iterative method which is given by

$$\left.\begin{aligned} Y^{(k)} &= X^{(k)} - \frac{2}{3}V^{(k)}, \\ Z^{(k)} &= X^{(k)} - \left[\frac{23}{8}I - 3M^{(k)} + \frac{9}{8}\left(M^{(k)}\right)^2\right] V^{(k)}, \\ X^{(k+1)} &= Z^{(k)} - \left[\frac{5}{2}I - \frac{3}{2}M^{(k)}\right] W^{(k)}, \end{aligned}\right\}, \tag{2.27}$$

where $F'(X^{(k)})V^{(k)} = F(X^{(k)})$, $F'(X^{(k)})M^{(k)} = F'(Y^{(k)})$ and $F'(X^{(k)})W^{(k)} = F(Z^{(k)})$.

The sequence $\{X^{(k)}\}_{k\geq 0}$ obtained using the iterative method (2.27) converges to r with convergence rate 6, and the error equation reads as

$$e^{(k+1)} = \frac{1}{9}\left(6C_2^2 - C_3\right)\left(45C_2^3 - 9C_3C_2 + C_4\right)(e^{(k)})^6 + O(e^{(k)})^7, \tag{2.28}$$

where $C_m = \frac{1}{m!}\{F'(r)\}^{-1}F^{(k)}(r), m = 2,3,....$

Further more in 2012, Cordero et al.[21] have introduced a new set of predictor-corrector iterative methods with increasing order of convergence in order to estimate the solution of nonlinear equations which is given by

$$\left.\begin{aligned} Y^{(k)} &= X^{(k)} - \frac{2}{3}\left\{F'(X^{(k)})\right\}^{-1} F(X^{(k)}),\\ Z^{(k)} &= Y^{(k)} + \frac{1}{6}\left\{F'(X^{(k)})\right\}^{-1} F(X^{(k)}),\\ U^{(k)} &= Z^{(k)} + \left\{F'(X^{(k)}) - 3F'(Y^{(k)})\right\}^{-1} F(X^{(k)}),\\ V^{(k)} &= Z^{(k)} + \left\{F'(X^{(k)}) - 3F'(Y^{(k)})\right\}^{-1}\left[F(X^{(k)}) + 2F(U^{(k)})\right],\\ X^{(k+1)} &= V^{(k)} - \frac{1}{2}\left\{F'(X^{(k)})\right\}^{-1}\left[5F'(X^{(k)}) - 3F'(Y^{(k)})\right]\left\{F'(X^{(k)})\right\}^{-1} F(V^{(k)}). \end{aligned}\right\}. \tag{2.29}$$

The sequence $\{X^{(k)}\}_{k\geq 0}$ obtained using the iterative method converges to r with convergence rate 8, and the error equation reads as

$$e^{(k+1)} = \left(C_2^2 - \frac{1}{2}C_3\right)\left(2C_2^3 + 2C_3C_2 - 2C_2C_3 - \frac{20}{9}C_4\right)(e^{(k)})^8 + O(e^{(k)})^9, \tag{2.30}$$

where $C_m = \frac{1}{m!}\{F'(r)\}^{-1}F^{(k)}(r), m = 2,3,....$

Further in 2012, Babajee et al.[22] have proposed an efficient iterative scheme including two steps and fourth order of convergence. The fourth-order Arithmetic Mean Newton method (4th AM) for system of nonlinear equations can be suggested by the authors are as follows:

$$\left.\begin{aligned} X^{(k+1)} &= G_{4th\ AM}(X^{(k)})\\ &= X^{(k)} - H_1(X^{(k)})\left\{A_1(X^{(k)})\right\}^{-1} F(X), \end{aligned}\right\}, \tag{2.31}$$

where $H_1 = (I - (1/4)(\tau(X) - I) + (3/4)(\tau(X) - I)^2)$, $\tau(X) = \{F'(X)\}^{-1}F'(X - (2/3)U(X))$, and I is the $t \times t$ identity matrix and $U(x) = \{F'(X)\}^{-1}F(X)$. The sequence $\{X^{(k)}\}_{k\geq 0}$ obtained using the iterative expression (2.31) converges to r with order 4, and the error equation reads as

$$e^{(k+1)} = \left(\frac{1}{9}C_4 + \frac{11}{3}C_2^3 - C_3C_2\right)(e^{(k)})^4 + O(e^{(k)})^5, \tag{2.32}$$

where $C_m = \frac{1}{m!}\{F'(r)\}^{-1}F^{(k)}(r), m = 2,3,....$

Recently in 2013, Arygros and Ren [23] have introduced a steffensen-type method (STTM) for nonlinear equations in a Banach space which is given by

$x_0 \in \Omega$ where Ω is an open subset of a Banach space.

$$\left.\begin{aligned} Y^{(k)} =& X^{(k)} - \{A^{(k)}\}^{-1}F(X^{(k)}), \\ A^{(k)} =& [X^{(k)}, G(X^{(k)}) : F], \\ Z^{(k)} =& X^{(k)} + a(Y^{(k)} - X^{(k)}), \\ X^{(k+1)} =& X^{(k)} - \left\{A^{(k)}\right\}^{-1}(bF(X^{(k)}) + cF(Z^{(k)})), \end{aligned}\right\}, \tag{2.33}$$

where $G : X \to X$ and a, b are non-negative parameters.

More recently in 2014, Soleymani et al. [24] have presented the class of iterative methods without restriction on the computation of Fréchet derivatives including multisteps for solving systems of nonlinear equations, they consider a three-step structure with the same correcting factor as follows:

$$\left.\begin{aligned} Y^{(k)} =& X^{(k)} - \left\{M^{(k)}\right\}^{-1}F(X^{(k)}), \\ Z^{(k)} =& Y^{(k)} - \left\{M^{(k)}\right\}^{-1}F(Y^{(k)}), \\ X^{(k+1)} =& Z^{(k)} - \left\{M^{(k)}\right\}^{-1}F(Z^{(k)}), \end{aligned}\right\}, \tag{2.34}$$

$$\text{where } M^{(k)} = \left[Y^{(k)}, X^{(k)}; F\right]. \tag{2.35}$$

The sequence $\{X^{(k)}\}_{k\geq 0}$ obtained using the iterative method (2.34) using (2.35) converges to r with convergence rate 4, and the error equation reads as

$$e^{(k+1)} = C_2^3\left(I + F'(r)\right)\left(2I + F'(r)\right)^2(e^{(k)})^4 + O(e^{(k)})^5, \tag{2.36}$$

where $C_2 = \frac{1}{2!}\{F'(r)\}^{-1}F^{(k)}(r)$.

Some more iterative methods in this field have also been reported in Literature $[25-27]$. The main goal of this thesis is to construct the new higher order iterative methods to solve system of nonlinear equations which are equally competent as the existing methods are. In this study , the authors have made an attempt to provide an innovative iterative schemes to solve system of nonlinear equations.

CHAPTER

THREE

FAMILIES OF SOME NEW ITERATIVE METHODS

3.1 BRIEF INTRODUCTION

Recently for $t = 1$ many robust and efficient methods have been proposed with higher convergence order $[28-35]$. But most of the cases the methods have not been extended to several variables. However during last decade, few of the researchers have carried out the work for multidimensional case to introduce methods of higher order of convergence $[14,17,36-45]$ by using different techniques eg. quadrature formula, Adomian decomposition, approximation of second order derivative etc. Many of these methods require the function or its first derivative evaluated at two different points, usually using the Newton's method as the first step.

Recently, Behl et al. [46] have proposed new optimal families of Ostrowski's method for solving scalar equations having cubic scaling factor of functions in the correction factor and is given by

$$\left.\begin{aligned} y_t &= x_t - \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} &= x_t - \frac{f(x_t)}{f'(x_t)}\left[\frac{(\alpha_1^2+\alpha_1\alpha_2-\alpha_2^2)f(x_t)f(y_t)-\alpha_1(\alpha_1-\alpha_2)\{f(x_t)\}^2}{(\alpha_1 f(x_t)-\alpha_2 f(y_t))((2\alpha_1-\alpha_2)f(y_t)-(\alpha_1-\alpha_2)f(x_t))}\right]. \end{aligned}\right\}, \tag{3.1}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ but choose α_1 and α_2 such that neither $\alpha_1 = 0$ nor $\alpha_1 = \alpha_2$.

More recently, in [47, 48] authors have approximated the derivatives by using divided difference operators preserving the local convergence order of iterative methods. However there is some more work on nonlinear systems of equations using the divided difference operator has been reported in Literature$[17,38-41]$. In this study, we also construct a generalization for several variables of given families of ostrowski's methods (3.1) by using divided difference operator and provide several methods of fourth and sixth order convergence.

3.2 MAIN RESULTS

Let $F : D \subset \mathbb{R}^t \to \mathbb{R}^t$ and assume that F has at least, third-order Fréchet derivatives with continuity on a convex set D. Suppose that the equation $F(X) = 0$ has a solution $r \in D$ at which $F'(r)$ is nonsingular.

Let us consider the first divided difference operator of F on $\mathbb{R}^t$ as a mapping $[\cdot,\cdot : F] : D \times D \subset \mathbb{R}^t \times \mathbb{R}^t \to L(\mathbb{R}^t)$ which is defined by (see[40, 50] and the references therein)

$$\left[X+h, X; F\right] = \int_0^1 F'(X+uh)du, \quad \forall (X,h) \in \mathbb{R}^t \times \mathbb{R}^t. \tag{3.2}$$

Developing $F'(X+uh)$ in Taylor's series at X and integrating one can obtain

$$\int_0^1 F'(X+uh)du = F'(X) + \frac{1}{2}F'''(X)h + \frac{1}{6}F'''(X)h^2 + O(h^3). \tag{3.3}$$

Taking into account $e^{(k)} = X^{(k)} - r$ we develop $F(X^{(k)})$ and its derivatives in a neighborhood of r. Namely, assuming that $\Gamma = \left\{F'(r)\right\}^{-1}$ exists, we have

$$F(X^{(k)}) = F'(r)\Big(e^{(k)} + A_2\big(e^{(k)}\big)^2 + A_3\big(e^{(k)}\big)^3 + O(e^{(k)})^4\Big), \tag{3.4}$$

where $A_p = \frac{1}{p!}\Gamma F^{(p)}(r) \in L_p(\mathbb{R}^t, \mathbb{R}^t), p = 2,3$. From (3.4) the derivative of $F(X^{(k)})$ can be written as

$$\left.\begin{aligned} F'(X^{(k)}) =& F'(r)\Big(I + 2A_2\big(e^{(k)}\big) + 3A_3\big(e^{(k)}\big)^2 + O(e^{(k)})^3\Big),\\ F''(X^{(k)}) =& F'(r)\Big(2A_2 + 6A_3\big(e^{(k)}\big) + O(e^{(k)})^2\Big),\\ F'''(X^{(k)}) =& F'(r)\Big(6A_3 + O(e^{(k)})\Big). \end{aligned}\right\}. \tag{3.5}$$

Setting $Y^{(k)} = X^{(k)} + h$ & $E^{(k)} = Y^{(k)} - r$, we have $h = Y^{(k)} - X^{(k)} = E^{(k)} - e^{(k)}$. Using equations (3.5) into (3.3), we get

$$\left[Y^{(k)}, X^{(k)}; F\right] = F'(r)\Big(I + A_2\big(E^{(k)} + e^{(k)}\big) + A_3\big(e^{(k)}\big)^2 + O(e^{(k)})^3\Big), \tag{3.6}$$

where $E^{(k)}$ is the local error of Newton's method: $E^{(k)} = A_2\big(e^{(k)}\big)^2 + O(e^{(k)})^3$.
Let us introduce the following modification over family (3.1)

$$\left.\begin{aligned} Y^{(k)} =& X^{(k)} - \left\{F'(X^{(k)})\right\}^{-1} F(X^{(k)}),\\ X^{(k+1)} =& \phi_1\big(X^{(k)}\big) = Y^{(k)} - \eta_1 F(Y^{(k)}). \end{aligned}\right\}, \tag{3.7}$$

where $\eta_1 = \tau^{-1}\left[-\left(\alpha_2^2 - 2\alpha_1\alpha_2\right)\left[Y^{(k)}, X^{(k)}; F\right] + \left(\alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2\right)F'(X^{(k)})\right]$,

$$\textit{and}\quad \tau = \left(2\alpha_1\alpha_2 - \alpha_2^2\right)\left[Y^{(k)}, X^{(k)}; F\right]\left[Y^{(k)}, X^{(k)}; F\right] - \alpha_1\alpha_2\left[Y^{(k)}, X^{(k)}; F\right]F'(X^{(k)})$$
$$+ \left(2\alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2\right)F'(X^{(k)})\left[Y^{(k)}, X^{(k)}; F\right] - \left(\alpha_1^2 - \alpha_1\alpha_2\right)F'(X^{(k)})F'(X^{(k)}).$$

This is two step family with order of convergence four. Further, a sixth order three step family consisting an additional evaluation of the function at $Z^{(k)} = \phi_1\left(X^{(k)}\right)$ is introduced as follows:

$$Z^{(k+1)} = \phi_2\left(X^{(k)}\right) = Z^{(k)} - \eta_1 F(Z^{(k)}). \tag{3.8}$$

3.3 SPECIAL CASES

For different values of α_1 and α_2, we get the various methods from the families (3.7) & (3.8) as follows:

(i) For $\alpha_1 \in \mathbb{R} - \{0\}$ and $\alpha_2 = 0$, family (3.7) reads as

$$\left.\begin{aligned} Y^{(k)} =& X^{(k)} - \{F'(X^{(k)})\}^{-1}F(X^{(k)}),\\ X^{(k+1)} =& \psi_1\left(X^{(k)}\right) = Y^{(k)} - \left\{2\left[Y^{(k)}, X^{(k)}; F\right] - F'(X^{(k)})\right\}^{-1}F(Y^{(k)}). \end{aligned}\right\}. \tag{3.9}$$

(ii) For $\alpha_1 \in \mathbb{R} - \{0\}$ and $\alpha_2 = 0$, if we recall $Z^{(k)} = \psi_1\left(X^{(k)}\right)$ then family (3.8) reduces to

$$Z^{(k+1)} = \psi_2\left(X^{(k)}\right) = Z^{(k)} - \left\{2\left[Y^{(k)}, X^{(k)}; F\right] - F'(X^{(k)})\right\}^{-1}F(Z^{(k)}). \tag{3.10}$$

(iii) For $\alpha_1 = \pm\frac{1}{\sqrt{3}}$ and $\alpha_2 = \pm\sqrt{3}$, family (3.7) reads as

$$\left.\begin{aligned} Y^{(k)} =& X^{(k)} - \{F'(X^{(k)})\}^{-1}F(X^{(k)}),\\ X^{(k+1)} =& \psi_3\left(X^{(k)}\right) = Y^{(k)} - \tau_1^{-1}\left\{3[Y^{(k)}, X^{(k)}; F] - F'(X^{(k)})\right\}F(Y^{(k)}). \end{aligned}\right\}. \tag{3.11}$$

(iv) For $\alpha_1 = \pm\frac{1}{\sqrt{3}}$ and $\alpha_2 = \pm\sqrt{3}$, if we recall $U^{(k)} = \psi_3\left(X^{(k)}\right)$ then family (3.8) reduces to

$$Z^{(k+1)} = \psi_4\left(X^{(k)}\right) = U^{(k)} - \tau_1^{-1}\left\{3[Y^{(k)}, X^{(k)}; F] - F'(X^{(k)})\right\}F(U^{(k)}), \tag{3.12}$$

$$\begin{aligned} \text{where}\quad \tau_1 =& 3\left[Y^{(k)}, X^{(k)}; F\right]\left[Y^{(k)}, X^{(k)}; F\right] + 3\left[Y^{(k)}, X^{(k)}; F\right]F'(X^{(k)})\\ & - 2F'(X^{(k)})\left[Y^{(k)}, X^{(k)}; F\right] - 2F'(X^{(k)})F'(X^{(k)}). \end{aligned} \tag{3.13}$$

(v) For $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2}$, family (3.7) reads as

$$\left.\begin{aligned} Y^{(k)} =& X^{(k)} - \{F'(X^{(k)})\}^{-1} F(X^{(k)}), \\ X^{(k+1)} =& \psi_5\big(X^{(k)}\big) = Y^{(k)} - \tau_2^{-1}\Big\{3\big[Y^{(k)}, X^{(k)}; F\big] - F'(X^{(k)})\Big\} F(Y^{(k)}). \end{aligned}\right\}. \tag{3.14}$$

(vi) For $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2}$, if we recall $W^{(k)} = \psi_5\big(X^{(k)}\big)$ then family (3.8) reduces to

$$Z^{(k+1)} = \psi_6\big(X^{(k)}\big) = W^{(k)} - \tau_2^{-1}\Big\{3\big[Y^{(k)}, X^{(k)}; F\big] - F'(X^{(k)})\Big\} F(W^{(k)}), \tag{3.15}$$

$$\begin{aligned} \text{where } \ \tau_2 =& (3\big[Y^{(k)}, X^{(k)}; F\big]\big[Y^{(k)}, X^{(k)}; F\big] - 2\big[Y^{(k)}, X^{(k)}; F\big] F'(X^{(k)}) \\ & + 3F'(X^{(k)})\big[Y^{(k)}, X^{(k)}; F\big] - 2F'(X^{(k)})F'(X^{(k)}). \end{aligned}$$

By taking different real values of α_1 and α_2 one can deduce many more higher-order iterative methods.

3.4 CONVERGENCE THEOREM

The following theorem establishes the prove of convergence of the proposed iterative methods (3.7) & (3.8)

Theorem 3.4.1 *The iterative method* ϕ_1(3.7) *has local order of convergence at least four with the following error equation :*

$$\begin{aligned} \varepsilon = Z^{(k)} - r = \frac{A_2}{\left(\alpha_1^2 - \alpha_1\alpha_2\right)^2} \Big[& \left(5\alpha_1^4 - 20\alpha_1^3\alpha_2 + 24\alpha_1^2\alpha_2^2 - 9\alpha_1\alpha_2^3 + \alpha_2^4\right) A_2{}^2 - \big(\alpha_1^4 - 2\alpha_1^3\alpha_2 \\ & + \alpha_1^2\alpha_2^2\big) A_3 \Big] \left(e^{(k)}\right)^4 + O(e^{(k)})^5, \end{aligned}$$

and iterative method ϕ_2(3.8) *has at least order six with the following error equation:*

$$\begin{aligned} e^{(k+1)} = Z^{(k+1)} - r = \frac{1}{\left(\alpha_1^2 - \alpha_1\alpha_2\right)^4} \Big[& \big(30\alpha_1^8 - 230\alpha_1^7\alpha_2 + 709\alpha_1^6\alpha_2^2 - 1127\alpha_1^5\alpha_2^3 + 989\alpha_1^4\alpha_2^4 \\ & - 483\alpha_1^3\alpha_2^5 + 130\alpha_1^2\alpha_2^6 - 18\alpha_1\alpha_2^7 + \alpha_2^8\big) A_2^5 + \big(-11\alpha_1^8 \\ & + 64\alpha_1^7\alpha_2 - 144\alpha_1^6\alpha_2^2 + 158\alpha_1^5\alpha_2^3 - 87\alpha_1^4\alpha_2^4 + 22\alpha_1^3\alpha_2^5 \\ & - 2\alpha_1^2\alpha_2^6\big) A_2^3 A_3 + \big(\alpha_1^8 - 4\alpha_1^7\alpha_2 + 6\alpha_1^6\alpha_2^2 - 4\alpha_1^5\alpha_2^3 \\ & + \alpha_1^4\alpha_2^4\big) A_2 A_3 \Big] \left(e^k\right)^6 + O(e^{(k)})^7. \end{aligned}$$

Proof : Taking into account (3.5) and (3.6) one gets,

$$\eta_1 = \Gamma\Big[I + \frac{\big(-2\alpha_1^4+4\alpha_1^3\alpha_2-2\alpha_1^2\alpha_2^2\big)}{\big(\alpha_1^2-\alpha_1\alpha_2\big)^2}A_2E^{(k)} + \frac{\big(-4\alpha_1^4+18\alpha_1^3\alpha_2-23\alpha_1^2\alpha_2^2+9\alpha_1\alpha_2^3-\alpha_2^4\big)}{\big(\alpha_1^2-\alpha_1\alpha_2\big)^2}A_2^2\big(e^{(k)}\big)^2$$
$$+\frac{\big(\alpha_1^4-2\alpha_1^3\alpha_2+\alpha_1^2\alpha_2^2\big)}{\big(\alpha_1^2-\alpha_1\alpha_2\big)^2}A_3\big(e^{(k)}\big)^2\Big] + O(e^{(k)})^3. \tag{3.16}$$

Considering (3.7), applying (3.4) to $F(Y^{(k)})$ and setting $Z^{(k)} = X^{(k+1)}$, one can obtain

$$\varepsilon = Z^{(k)} - r = E^{(k)} - \eta_1 F'(r)\Big[E^{(k)} + A_2\big(E^{(k)}\big)^2 + A_3\big(E^{(k)}\big)^3 + O(E^{(k)})^4\Big]. \tag{3.17}$$

Substituting the value of η_1 from equation (3.16) in (3.17), one can have

$$\varepsilon = \frac{A_2}{\big(\alpha_1^2-\alpha_1\alpha_2\big)^2}\Big[(5\alpha_1^4-20\alpha_1^3\alpha_2+24\alpha_1^2\alpha_2^2-9\alpha_1\alpha_2^3+\alpha_2^4)A_2{}^2 - \big(\alpha_1^4-2\alpha_1^3\alpha_2+\alpha_1^2\alpha_2^2\big)A_3\Big]\big(e^{(k)}\big)^4$$
$$+O(e^{(k)})^5. \tag{3.18}$$

Moreover, subtracting the root r from both sides of (3.8) we have

$$e^{(k+1)} = Z^{(k+1)} - r = \varepsilon - \eta_1 F'(r)\Big[\varepsilon + O(\varepsilon^2)\Big]$$
$$= \Big[\frac{\big(6\alpha_1^4-22\alpha_1^3\alpha_2+25\alpha_1^2\alpha_2^2-9\alpha_1\alpha_2^3+\alpha_2^4\big)}{\big(\alpha_1^2-\alpha_1\alpha_2\big)^2}A_2^2 + \frac{\big(-\alpha_1^4+2\alpha_1^3\alpha_2-\alpha_1^2\alpha_2^2\big)}{\big(\alpha_1^2-\alpha_1\alpha_2\big)^2}A_3\Big]\big(e^{(k)}\big)^2\varepsilon + O(e^{(k)})^7$$
$$= \frac{1}{\big(\alpha_1^2-\alpha_1\alpha_2\big)^4}\Big[\big(30\alpha_1^8-230\alpha_1^7\alpha_2+709\alpha_1^6\alpha_2^2-1127\alpha_1^5\alpha_2^3+989\alpha_1^4\alpha_2^4-483\alpha_1^3\alpha_2^5$$
$$+130\alpha_1^2\alpha_2^6-18\alpha_1\alpha_2^7+\alpha_2^8)A_2^5 + \big(-11\alpha_1^8+64\alpha_1^7\alpha_2-144\alpha_1^6\alpha_2^2+158\alpha_1^5\alpha_2^3-87\alpha_1^4\alpha_2^4$$
$$+22\alpha_1^3\alpha_2^5-2\alpha_1^2\alpha_2^6)A_2^3A_3 + (\alpha_1^8-4\alpha_1^7\alpha_2+6\alpha_1^6\alpha_2^2-4\alpha_1^5\alpha_2^3+\alpha_1^4\alpha_2^4)A_2A_3\Big]\big(e^k\big)^6 + O(e^k)^7. \tag{3.19}$$

This establishes the proof of the theorem.

3.5 COMPUTATIONAL EFFICIENCY AND COMPARISON AMONG PROPOSED METHODS

For the systems with 't' nonlinear equations and 't' unknowns, author suggests the following definition of the computational efficiency index *(CEI)* of iterative methods of order of convergence ρ

$$CEI(\theta_0,\theta_1,t) = \rho^{\frac{1}{C(\theta_0,\theta_1,t)}}, \tag{3.20}$$

where $C(\theta_0,\theta_1,t)$ is the computational cost per iteration given by

$$C(\theta_0,\theta_1,t) = A_0(t)\theta_0 + A_1(t)\theta_1 + P(t). \tag{3.21}$$

In (3.21) $A_0(t)$ represents the number of evaluations of scalar functions $(F_1, F_2, \ldots\ldots\ldots, F_t)$ used in the evaluation of F and $[Y,X;F]$ where

$$[Y,X;F]_{ij} = \frac{F_i(y_1,\ldots\ldots., y_{j-1}, y_j, x_{j+1},\ldots\ldots x_t) - F_i(y_1,\ldots\ldots., y_{j-1}, x_j, x_{j+1},\ldots\ldots x_t)}{(y_j - x_j)}, 1 \leq i,j \leq t.$$

The number of evaluations of scalar functions of F' , say $\frac{\partial F_i}{\partial x_j}, 1 \leq i,j \leq t$, is $A_1(t)$, $P(t)$ is the number of products per iteration and θ_0 and θ_1 are the ratios between products and evaluations required to express the value of $C(\theta_0, \theta_1, t)$ in terms of products.

When we evaluate F in any iterative function we calculate t component functions and if we compute a divided difference then we evaluate $t(t-1)$ scalar functions, where $F(X)$ and $F(Y)$ are computed separately. The number of scalar evaluations is t^2 evaluations for any new derivative F'. We must add t^2 quotients from any divided difference. In order to compute an inverse linear operator we have $(t^3-t)/3$ products or quotients in the decomposition LU and t^2 products or quotients in the resolution of two triangular linear systems. Taking into account the previous considerations, and considering $A_0(t) = t^2+t$, $A_1(t) = t^2$, $P(t) = 2(t^3-t)/3 + 4t^2$ for the first iterative function ψ_1, we obtain

$$C_1 = (t^2+t)\theta_0 + t^2\theta_1 + 4t^2 + \frac{2(t^3-t)}{3} \quad \text{and} \quad CEI_1 = 4^{1/C_1}. \tag{3.22}$$

For ψ_2 we have

$$C_2 = (t^2+2t)\theta_0 + t^2\theta_1 + 5t^2 + \frac{2(t^3-t)}{3} \quad \text{and} \quad CEI_2 = 6^{1/C_2}. \tag{3.23}$$

For ψ_3 we have

$$C_3 = (t^2+t)\theta_0 + t^2\theta_1 + 10t^2 + \frac{2(t^3-t)}{3} \quad \text{and} \quad CEI_3 = 4^{1/C_3}. \tag{3.24}$$

For ψ_4 we have

$$C_4 = (t^2+2t)\theta_0 + t^2\theta_1 + 11t^2 + \frac{2(t^3-t)}{3} \quad \text{and} \quad CEI_4 = 6^{1/C_4}. \tag{3.25}$$

For ψ_5 we have

$$C_5 = (t^2+t)\theta_0 + t^2\theta_1 + 12t^2 + \frac{2(t^3-t)}{3} \quad \text{and} \quad CEI_5 = 4^{1/C_5}. \tag{3.26}$$

For ψ_6 we have

$$C_6 = (t^2+2t)\theta_0 + t^2\theta_1 + 13t^2 + \frac{2(t^3-t)}{3} \quad \text{and} \quad CEI_6 = 6^{1/C_6}. \tag{3.27}$$

In order to compare the iterative methods $\psi_i,\ 1 \le i \le 6$, we define the ratio

$$R_{i,j} = \frac{log\ CEI_i(\theta_0,\theta_1,t)}{log\ CEI_j(\theta_0,\theta_1,t)} = \frac{log(\rho_i)C_j(\theta_0,\theta_1,t)}{log(\rho_j)C_i(\theta_0,\theta_1,t)}. \tag{3.28}$$

It is clear that if $R_{i,j} > 1$, the iterative method ψ_i is more efficient than ψ_j. Taking into account that the border between two computational efficiencies is given by $R_{i,j} = 1$, this boundary is given by the equation of θ_0 written as a function of θ_1 and t, that is $\theta_0 = M_{i,j}(\theta_1,t)$. Here $\theta_0 > 0, \theta_1 > 0$ and t is a positive integer $t \ge 2$.

Case (i) ψ_1 ***versus*** ψ_2

The boundary $R_{2,1} = 1$ expressed by θ_0 written as a function of θ_1 and t is

$$M_{2,1} = \frac{1}{3}\frac{2at^2 + (3a\theta_1 + 12a - 3b)t - 2a}{b - at - a}, \tag{3.29}$$

where $a = ln(3/2)$ and $b = 2ln(2)$.

This function has the vertical asymptote for $t = (b-a)/a = 2.419...$.

Note that the numerator of (3.29) is positive for $t \ge 1$, since for $t = 1$ yields $3a\theta_1 + 12a - 3b > 0$ and the denominator of (3.29) is negative for $t > 2.419...$. Consequently we obtain that $\mu_0 = M_{2,1}(\theta_1,t)$ is always negative for $t \ge 3$. That is the boundary is out of the admissible region for $t \ge 3$ and we have $\forall\ \theta_0 > 0\ ,\ \theta_1 > 0$ and $CEI_2 > CEI_1$.

In particular for $t = 2$ the boundary (3.29) is straight line with positive slope

$$\theta_0 = \frac{2(a\theta_1 + 5a - b)}{b - 3a}, \tag{3.30}$$

where $CEI_1 > CEI_2$ over it and $CEI_2 > CEI_1$ under it(see Fig. 1).

Case (ii) ψ_3 ***versus*** ψ_4

The boundary $R_{4,3} = 1$ expressed by θ_0 written as a function of θ_1 and t is

$$M_{4,3} = \frac{1}{3}\frac{2at^2 + (3a\theta_1 + 30a - 3b)t - 2a}{b - at - a}, \tag{3.31}$$

where $M_{4,3}$ function has the vertical asymptote for $t = 2.419...$.

Observe that the numerator of (3.31) is positive for $t \ge 1$ and the denominator is negative for $t > 2.419...$. Consequently we obtain that $\mu_0 = M_{4,3}(\theta_1,t)$ is always negative for $t \ge 3$. That is the boundary is out of the admissible region for $t \ge 3$ and we have $\forall\ \theta_0 > 0\ ,\ \theta_1 > 0$ and $CEI_4 > CEI_3$.

For $t = 2$ the boundary (3.31) is straight line with positive slope

$$\theta_0 = \frac{2(a\theta_1 + 11a - b)}{b - 3a}, \tag{3.32}$$

where $CEI_3 > CEI_4$ over it and $CEI_4 > CEI_3$ under it(see Fig.1).

Case (iii) ψ_5 ***versus*** ψ_6

The boundary $R_{6,5} = 1$ expressed by θ_0 written as a function of θ_1 and t is

$$M_{6,5} = \frac{1}{3}\frac{2at^2 + (3a\theta_1 + 36a - 3b)t - 2a}{b - at - a}, \tag{3.33}$$

where $M_{6,5}$ function has the vertical asymptote for $t = 2.419....$

Observe that the numerator of (3.33) is positive for $t \geq 1$ and the denominator is negative for $t > 2.419....$ Consequently we obtain that $\mu_0 = M_{6,5}(\theta_1, t)$ is always negative for $t \geq 3$. That is the boundary is out of the admissible region for $t \geq 3$ and we have $\forall\ \theta_0 > 0$, $\theta_1 > 0$ and $CEI_6 > CEI_5$.

For $t = 2$ the boundary (3.33) is straight line with positive slope

$$\theta_0 = \frac{2(a\theta_1 + 13a - b)}{b - 3a}, \tag{3.34}$$

where $CEI_5 > CEI_6$ over it and $CEI_6 > CEI_5$ under it(see Fig.1).

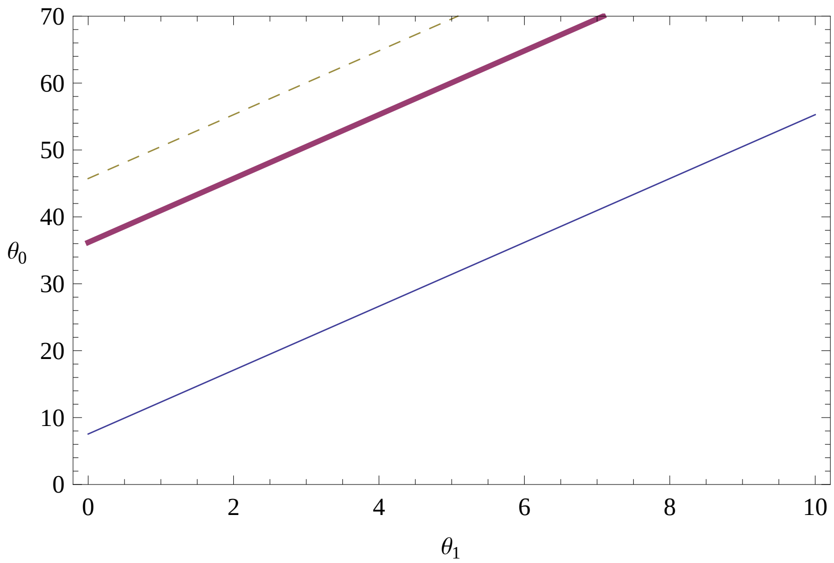

Fig. 1 CEI_1 versus CEI_2(solid line), CEI_3 versus CEI_4(thick line), CEI_5 versus CEI_6(dashed line) for $t = 2$

Case (iv) ψ_1 *versus* ψ_4 *case*

The boundary $R_{4,1} = 1$ expressed by θ_0 written as a function of θ_1 and t is

$$M_{4,1} = \frac{1}{3}\frac{2at^2 + (3a\theta_1 + 12a - 21b)t - 2a}{b - at - a}, \tag{3.35}$$

where $M_{4,1}$ has the vertical asymptote for $t = (b-a)/a = 2.419...$.

Note that the numerator of (3.35) is positive for $t \geq 3$ and the denominator is negative for $t > 2.419...$, we obtain $CEI_4 > CEI_1$ for $t \geq 3$.

For $t = 2$ the boundary (3.35) is the straight line with positive slope

$$\theta_0 = \frac{2(a\theta_1 + 5a - 7b)}{b - 3a}, \tag{3.36}$$

where $CEI_1 > CEI_4$ over it and $CEI_4 > CEI_1$ under it (see Fig. 2).

Case(v) ψ_1 *versus* ψ_6

The boundary $R_{6,1} = 1$ expressed by θ_0 written as a function of θ_1 and t is

$$M_{6,1} = \frac{1}{3}\frac{2at^2 + (3a\theta_1 + 12a - 27b)t - 2a}{b - at - a}, \tag{3.37}$$

where $M_{6,1}$ has the vertical asymptote for $t = (b-a)/a = 2.419...$.

Note that the numerator of (3.37) is positive for $t \geq 3$ and the denominator is negative for $t > 2.419...$, we obtain $CEI_6 > CEI_1$ for $t \geq 3$.

For $t = 2$ the boundary (3.37) is the straight line with positive slope

$$\theta_0 = \frac{2(a\theta_1 + 5a - 9b)}{b - 3a}, \tag{3.38}$$

where $CEI_1 > CEI_6$ over it and $CEI_6 > CEI_1$ under it (see Fig. 2).

Case (vi) ψ_3 *versus* ψ_6

The boundary $R_{6,3} = 1$ expressed by θ_0 written as a function of θ_1 and t is

$$M_{6,3} = \frac{1}{3}\frac{2at^2 + (3a\theta_1 + 30a - 9b)t - 2a}{b - at - a}, \tag{3.39}$$

where $M_{6,1}$ has the vertical asymptote for $t = (b-a)/a = 2.419...$.

Note that the numerator of (3.39) is positive for $t \geq 3$ and the denominator is negative for $t > 2.419...$, we obtain $CEI_6 > CEI_3$ for $t \geq 3$.

For $t = 2$ the boundary (3.39) is the straight line with positive slope

$$\theta_0 = \frac{2(a\theta_1 + 11a - 3b)}{b - 3a}, \tag{3.40}$$

where $CEI_3 > CEI_6$ over it and $CEI_6 > CEI_3$ under it (see Fig. 2).

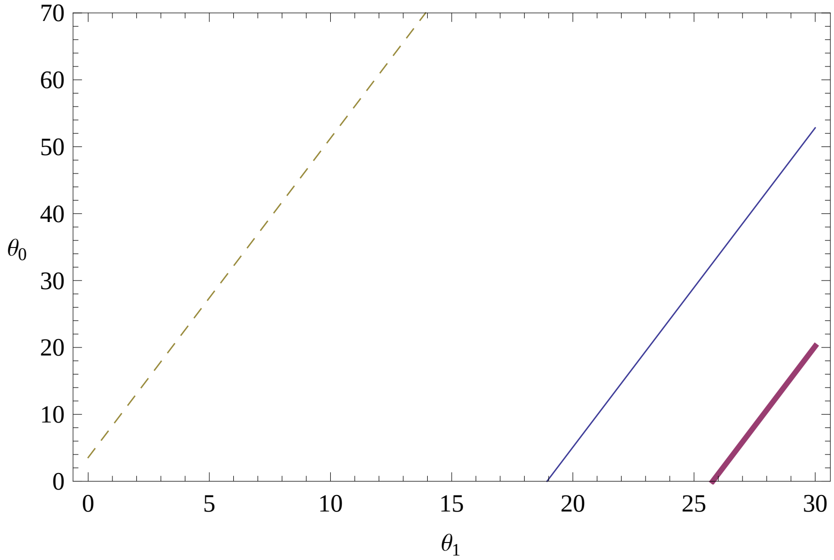

Fig. 2 CEI_1 versus CEI_4(solid line), CEI_1 versus CEI_6(thick line), CEI_3 versus CEI_6(dashed line) for $t = 2$

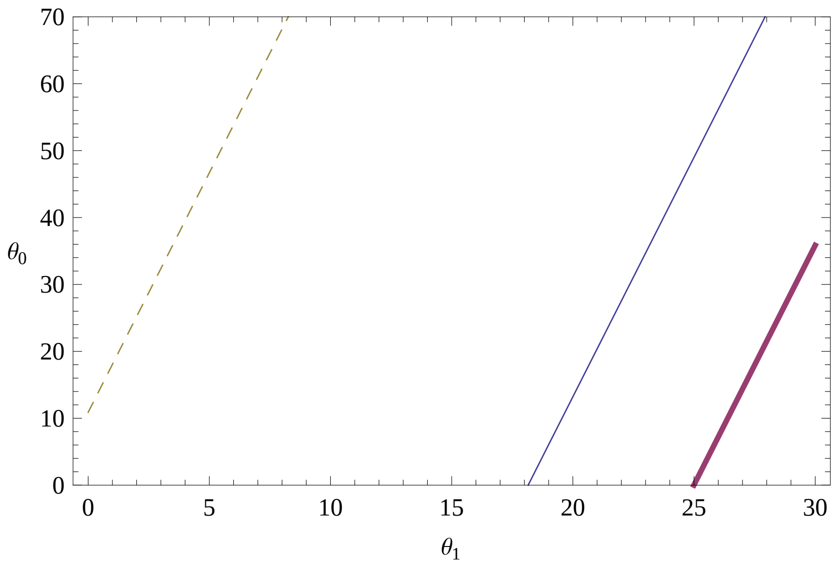

Fig. 3 CEI_1 versus CEI_4(solid line), CEI_1 versus CEI_6(thick line), CEI_3 versus CEI_6(dashed line) for $t = 3$

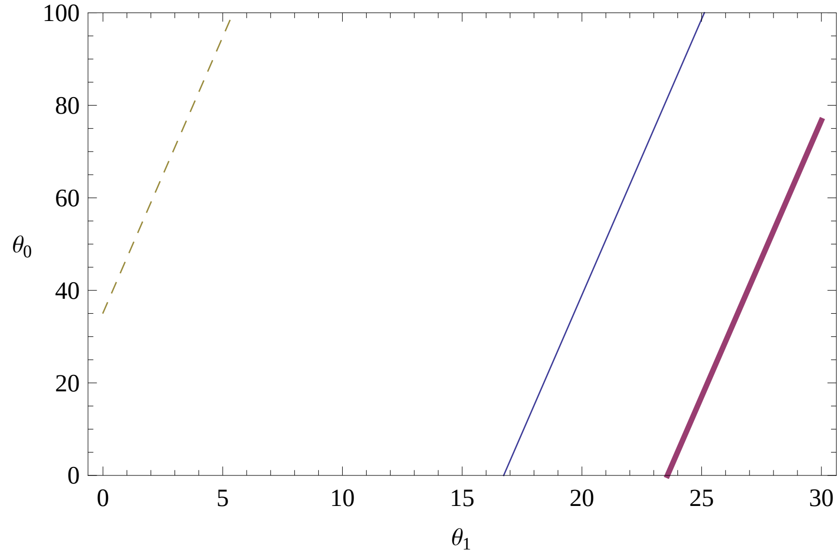

Fig. 4 CEI_1 versus CEI_4(solid line), CEI_1 versus CEI_6(thick line), CEI_3 versus CEI_6(dashed line) for $t = 5$

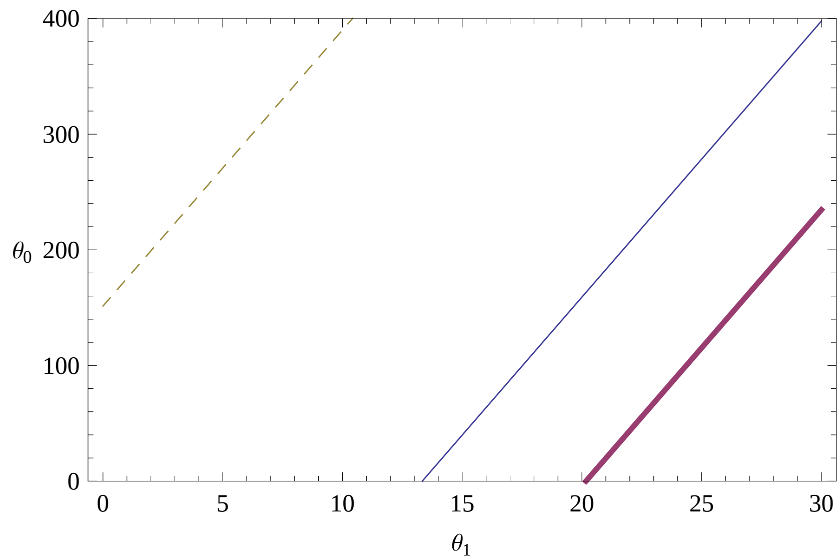

Fig. 5 CEI_1 versus CEI_4(solid line), CEI_1 versus CEI_6(thick line), CEI_3 versus CEI_6(dashed line) for $t = 10$

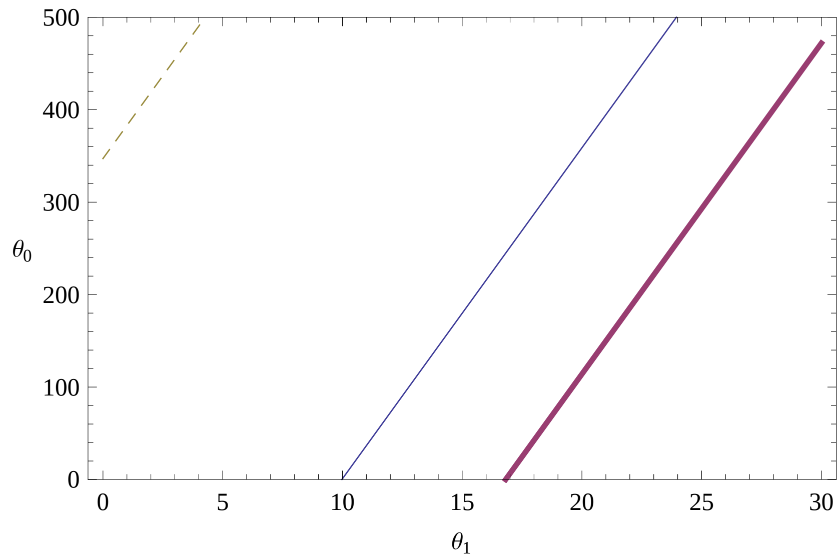

Fig. 6 CEI_1 versus CEI_4(solid line), CEI_1 versus CEI_6(thick line), CEI_3 versus CEI_6(dashed line) for $t = 15$

Many more figures have been drawn by taking different values of t.

Case (vii) ψ_2 ***versus*** ψ_3

The boundary $R_{2,3} = 1$ expressed by θ_0 written as a function of θ_1 and t is

$$M_{2,3} = \frac{1}{3}\frac{2at^2 + (3a\theta_1 + 30a + 15b)t - 2a}{b - at - a}, \tag{3.41}$$

where $a = ln(3/2)$ and $b = 2ln(2)$.

This function has the vertical asymptote for $t = (b-a)/a = 2.419....$

Note that the numerator of (3.41) is positive for $t \geq 1$, since for $t = 1$ yields $3a\theta_1 + 30a + 15b > 0$ and the denominator of (3.41) is negative for $t > 2.419....$ Consequently we obtain that, $CEI_2 > CEI_3$.

In particular for $t = 2$ the boundary (3.41) is straight line with positive slope

$$\theta_0 = \frac{2(a\theta_1 + 11a + 5b)}{b - 3a}. \tag{3.42}$$

Case (viii) ψ_2 ***versus*** ψ_5

The boundary $R_{2,5} = 1$ expressed by θ_0 written as a function of θ_1 and t is

$$M_{2,5} = \frac{1}{3}\frac{2at^2 + (3a\theta_1 + 36a + 21b)t - 2a}{b - at - a}, \tag{3.43}$$

where $M_{2,5}$ function has the vertical asymptote for $t = 2.419....$

Observe that the numerator of (3.43) is positive for $t \geq 1$ and the denominator is negative for $t > 2.419....$

Consequently we obtain that $CEI_2 > CEI_5$ for all $t \geq 1$.

For $t = 2$ the boundary (3.43) is straight line with positive slope

$$\theta_0 = \frac{2(a\theta_1 + 13a + 7b)}{b - 3a}. \tag{3.44}$$

<u>Case (ix) ψ_4 *versus* ψ_5</u>

The boundary $R_{4,5} = 1$ expressed by θ_0 written as a function of θ_1 and t is

$$M_{4,5} = \frac{1}{3}\frac{2at^2 + (3a\theta_1 + 36a + 3b)t - 2a}{b - at - a}, \tag{3.45}$$

where $M_{4,5}$ function has the vertical asymptote for $t = 2.419...$.

Observe that the numerator of (3.45) is positive for $t \geq 1$ and the denominator is negative for $t > 2.419...$. Consequently we obtain that $CEI_4 > CEI_5$.

For $t = 2$ the boundary (3.45) is straight line with positive slope

$$\theta_0 = \frac{2(a\theta_1 + 13a + b)}{b - 3a}. \tag{3.46}$$

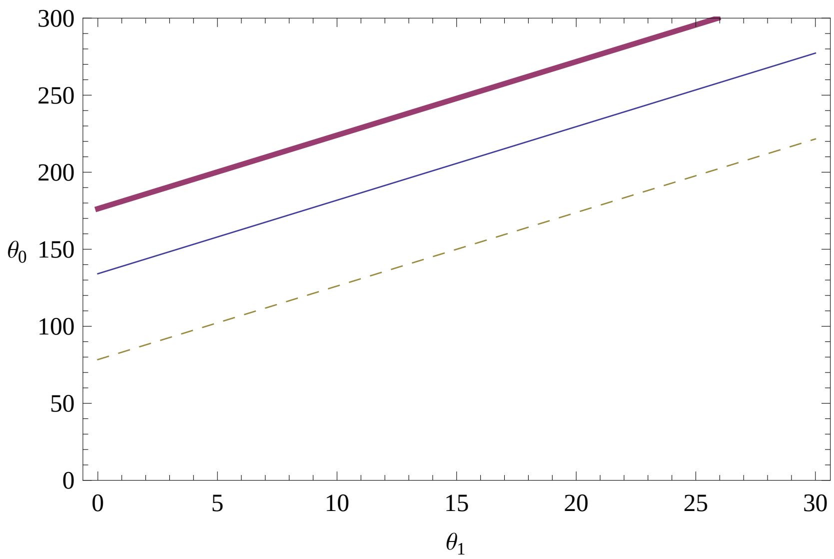

Fig. 7 CEI_2 versus CEI_3(solid line), CEI_2 versus CEI_5(thick line), CEI_4 versus CEI_5(dashed line) for $t = 2$

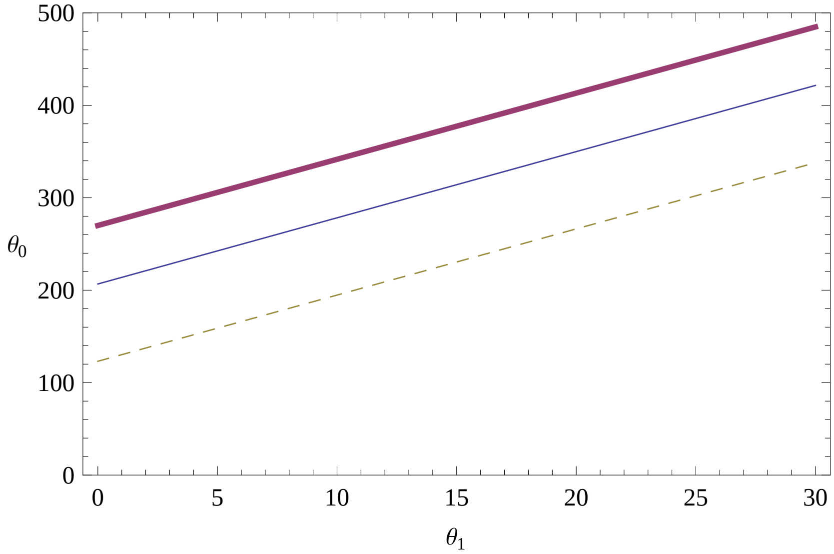

Fig. 8 CEI_2 versus CEI_3(solid line), CEI_2 versus CEI_5(thick line), CEI_4 versus CEI_5(dashed line) for $t = 3$

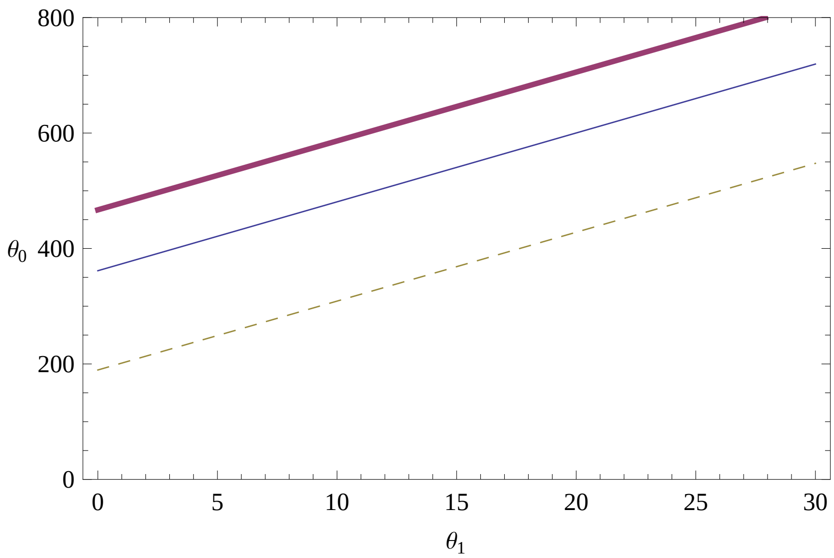

Fig. 9 CEI_2 versus CEI_3(solid line), CEI_2 versus CEI_5(thick line), CEI_4 versus CEI_5(dashed line) for $t = 5$

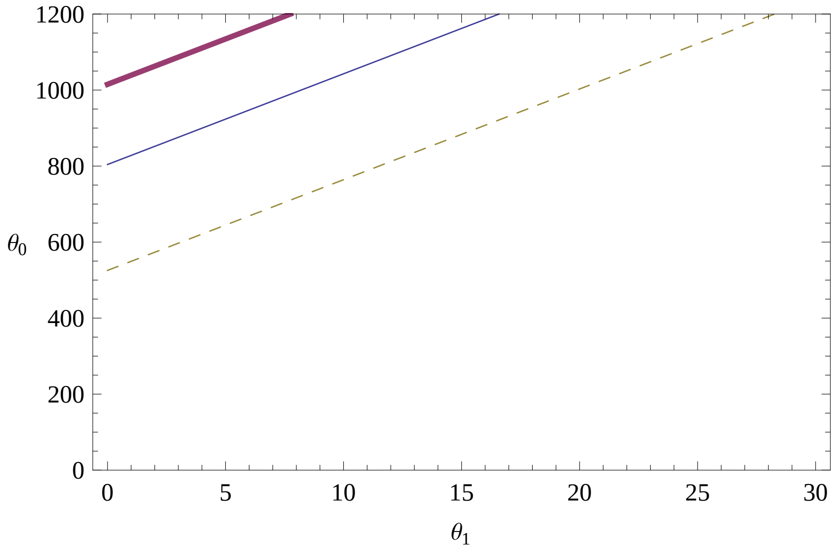

Fig. 10 CEI_2 versus CEI_3(solid line), CEI_2 versus CEI_5(thick line), CEI_4 versus CEI_5(dashed line) for $t = 10$

Many more figures have been drawn by taking different values of t.

Theorem 3.5.1 *For all* $\theta_0 > 0$ *and* $\theta_1 > 0$ *we have:*

(a) *$CEI_1 > CEI_3$, $CEI_3 > CEI_5$ and $CEI_1 > CEI_5$ for all $t \geq 2$.*

(b) *$CEI_2 > CEI_4$, $CEI_4 > CEI_6$ and $CEI_2 > CEI_6$ for all $t \geq 2$.*

(c) *$CEI_2 > CEI_3$, $CEI_2 > CEI_5$ and $CEI_4 > CEI_5$ for all $t \geq 2$.*

(d) *$CEI_2 > CEI_1$, $CEI_4 > CEI_3$ and $CEI_6 > CEI_5$ for all $t \geq 3$.*

(e) *$CEI_4 > CEI_1$, $CEI_6 > CEI_1$ and $CEI_6 > CEI_3$ for all $t \geq 3$.*

3.6 NUMERICAL RESULTS

In this section, some numerical problems are considered to illustrate the convergence behavior and computational efficiency of the proposed method. The numerical computations are performed in *Mathematica* 7.1 [49] using multiple-precision arithmetic with 2048 digits. The classical stopping criterium

$||e^{(l+1)}|| = ||X^{(l+1)} - r|| < 0.5 \cdot 10^{-\zeta}$,

where $\zeta = 2048$ is replaced by $E^{(l+1)} = \frac{||\hat{e}^{(l+1)}||}{||\hat{e}^{(l)}||} < 0.5 \cdot 10^{-\xi}$, where $\hat{e}^{(l)} = X^{(l)} - X^{(l-1)}$ and $\xi = \frac{\rho-1}{\rho^2}\zeta$. Note that this criterion is independent of knowledge of the root (see [1]).

According to the definition of the computational cost (3.20), (3.21), an estimation of the factors $\theta_{0,1}$ is claimed. To do this, we express the cost of the evaluation of the elementary functions in terms of products which depends on the machine, the software and the arithmetics used. In Table 1 an estimation of the cost of the elementary functions in number of equivalent products is shown, where running time of one product is measured in milliseconds.

Table 1:Estimation of computational cost of elementary functions computed with Mathematica 7.1 in a processor Intel(R) Core (TM) i5-2430M CPU @ 2.40 GHz (32-bit machine) Microsoft Windows 7 Ultimate 2009, where $x = \sqrt{3} - 1$ and $y = \sqrt{5}$.

$Digits$	$x * y$	x/y	$\sqrt{x}$	$exp(x)$	$ln(x)$	$sin(x)$	$cos(x)$	$arccos(x)$	$arctan(x)$
2048	$0.0301ms$	3	1.5	77	78	78	77	119	118

We present four examples corresponding to the definition 3.5.1. Table 2-5 show the results obtained for the iterative methods $\psi_k, 1 \leq k \leq 6$. In each table we can read the necessary iterations l, the computational cost C_k in terms of products, the computational efficiency index CEI_k (3.22)-(3.27), the measure time factor TF defined as $TF = 1/log_{10}CEI_k$, the computed value of the approximated computational order of convergence (ACOC) with its higher bound $\Delta\widehat{\rho_k}l$, where the local order is computed by $\rho = \widehat{\rho_k}l \pm \Delta\widehat{\rho_k}l$ and ACOC is

$$\hat{\rho}_l = \frac{lnE^{(l)}}{lnE^{(l-1)}}.$$

as it is defined in [1].

Table 2: Numerical results for the nonlinear system (3.47)

	l	C	CEI	TF	$\hat{\rho} \pm \Delta\hat{\rho}$
ψ_1	6	773	1.0017950	1283.93	$4 \pm 8.2 \times 10^{-1}$
ψ_2	5	1009	1.0017773	1296.66	$6 \pm 6.6 \times 10^{-1}$
ψ_3	10	797	1.0017409	1323.79	$4 \pm 9.15 \times 10^{-3}$
ψ_4	8	1033	1.0017360	1327.51	$6 \pm 8.9 \times 10^{-2}$
ψ_5	10	805	1.0017235	1337.14	$4 \pm 5.37 \times 10^{-1}$
ψ_6	9	1041	1.00172267	1337.78	$6 \pm 2.89 \times 10^{-1}$

Table 3: Numerical results for the nonlinear system (3.48)

	l	C	CEI	TF	$\hat{\rho} \pm \Delta\hat{\rho}$
ψ_1	4	504	1.0027543	837.13	$4 \pm 7.7 \times 10^{-2}$
ψ_2	3	667	1.0026899	857.16	$6 \pm 1.0 \times 10^{-1}$
ψ_3	4	528	1.0026290	876.99	$4 \pm 7.9 \times 10^{-1}$
ψ_4	3	691	1.0025964	888	$6 \pm 9.6 \times 10^{-1}$
ψ_5	5	536	1.0025897	890.28	$4 \pm 2.7 \times 10^{-2}$
ψ_6	3	699	1.0025661	898.459	$6 \pm 6.3 \times 10^{-2}$

Table 4: Numerical results for the nonlinear system (3.49)

	l	C	CEI	TF	$\hat{\rho} \pm \Delta\hat{\rho}$
ψ_1	4	1207	1.0011492	2004.79	$4 \pm 7.5 \times 10^{-2}$
ψ_2	3	1447	1.0012390	1859.57	$6 \pm 1.7 \times 10^{-2}$
ψ_3	4	1261	1.0010999	2094.60	$4 \pm 2.7 \times 10^{-2}$
ψ_4	3	1501	1.0011944	1928.96	$6 \pm 7.8 \times 10^{-1}$
ψ_5	5	1279	1.0010845	2124.33	$4 \pm 2.7 \times 10^{-2}$
ψ_6	3	1519	1.0011803	1952.06	$6 \pm 3.2 \times 10^{-1}$

Table 5: Numerical results for the nonlinear system (3.50)

	l	C	CEI	TF	$\hat{\rho} \pm \Delta\hat{\rho}$
ψ_1	2	2875	1.000482306	4774.72	4
ψ_2	1	3285	1.000545585	4221.54	6
ψ_3	3	3025	1.000458384	5024.41	$4 \pm 9.2 \times 10^{-1}$
ψ_4	2	3435	1.000521755	4414.30	6
ψ_5	3	3075	1.000450929	5107.46	$4 \pm 9.15 \times 10^{-1}$
ψ_6	2	3485	1.000514267	4478.56	6

Example 1 We begin with the system $F(x_1, x_2) = 0$ defined by

$$F(x_1, x_2) = \begin{cases} 2 - e^{x_1} + arctan(x_2), \\ arctan(x_1^2 + x_2^2 - 5.0), \end{cases} \tag{3.47}$$

where $(t, \theta_0, \theta_1) = (2, 116, 14.25)$ are the values used in (3.22)-(3.27).

We test the convergence of the methods towards the root $r = (1.12906503916019.........., 1.930080862903468.......)^t$, where the initial value is $x_0 = (1.0, 2.0)^t$. The results shown in Table 2 confirm the first, second and third assertion of Theorem 3.5.1 for $t = 2$. Namely , CEI attains its maximum when we use ψ_1 and in consequence , this method spends minimum time in the computation of the numerical solution. Moreover, we have $CEI_1 > CEI_2$, $CEI_3 > CEI_4$ and $CEI_5 > CEI_6$ (see Fig. 1).

Example 2 We begin with the system $F(x_1, x_2) = 0$ defined by

$$F(x_1, x_2) = \begin{cases} ln(x_2) - x_1^2 + x_1 x_2, \\ ln(x_1) - x_2^2 + x_1 x_2, \end{cases} \tag{3.48}$$

where $(t, \theta_0, \theta_1) = (2, 79.5, 1.8)$ are the values used in (3.22)-(3.27).

We test the convergence of the methods towards the root $r = (1.0, 1.0)^t$, where the initial value is

$x_0 = (0.5, 1.5)^t$. The results shown in Table 3 confirm the first, second and third assertion of Theorem 3.5.1 for $t = 2$. Namely , CEI attains its maximum when we use ψ_1 and in consequence , this method spends minimum time in the computation of the numerical solution. Moreover, we have $CEI_1 > CEI_2$, $CEI_3 > CEI_4$ and $CEI_5 > CEI_6$ (see Fig. 1).

Example 3 We begin with the system $F(x_1, x_2, x_3) = 0$ defined by

$$F(x_1,x_2,x_3) = \begin{cases} x_2 + x_3 - e^{-x_1}, \\ x_1 + x_3 - e^{-x_3}, \\ x_1 + x_2 - e^{-x_3}, \end{cases} \tag{3.49}$$

where $(t, \theta_0, \theta_1) = (3, 77, 25.67)$ are the values used in (3.22)-(3.27).

We test the convergence of the methods towards the root $r = (-0.83202503981029.........,$ $1.14898375369576......, 1.14898375369576......)^t$, where the initial value is $x_0 = (-0.8, 1.1, 1.1)^t$. The results shown in Table 4 confirm the fourth and fifth assertion of Theorem 3.5.1 for $t = 3$. Namely , CEI attains its maximum when we use ψ_2 and in consequence , this method spends minimum time in the computation of the numerical solution.

Example 4 Now considering the system of five equations:

$$F(x_1,x_2,x_3,x_4,x_5) = \begin{cases} x_2 + x_3 + x_4 + x_5 - e^{-x_1}, \\ x_1 + x_3 + x_4 + x_5 - e^{-x_2}, \\ x_1 + x_2 + x_4 + x_5 - e^{-x_3}, \\ x_1 + x_2 + x_3 + x_5 - e^{-x_4}, \\ x_1 + x_2 + x_3 + x_4 - e^{-x_5}, \end{cases} \tag{3.50}$$

where $(t, \theta_0, \theta_1) = (5, 77, 15.4)$ are the values used in (3.22)-(3.27).

We test the convergence of the methods towards the root $r = (0.20388835470222402..,$ $0.20388835470222402..., 0.20388835470222402..., 0.20388835470222402..., 0.20388835470222402...)^t$, where the initial value is $x_0 = (1, 1, 1, 1, 1)^t$. The results shown in Table 5, CEI attains its maximum when we use ψ_2 and in consequence, this method spends minimum time in the computation of the numerical solution.

3.7 CONCLUSIONS AND DISCUSSIONS

Authors have considered two new families for solving system of nonlinear equations. Many iterative methods of order four and six have been developed to solve system of nonlinear equations. Their simplicity lies in the fact that the second-order Fréchet derivative has not been used. A development of an inverse first-order divided difference operator for functions of several variables is analyzed. Moreover researcher have studied the computational efficiency index and a comparison for these iterative methods is given.

Finally, authors would like to mention that the analysis presented is applicable directly to any iterative method that uses divided differences instead of derivatives.

REFERENCES

[1] Miquel Grau-Sánchez, Miquel Noguera, José Manuel Gutiérrez, On some computational orders of convergence, Appl. Math. Lett. 23(2010) 472-478.

[2] J.M.Örtega, W.C. Rheinboldt, Iterative solutions of Nonlinear equations in several variables, Academic press, New York 1970.

[3] V. Kanwar, Sanjeev Kumar , Ramandeep Behl, Several New Families of Jarratt's method for Solving Systems of Nonlinear Equations, Applications and App. Math. Vol. 8, Issue 2 (December 2013), pp. 701 -716

[4] Kou Jisheng, Li Yitian, Wang Xiuhua, Efficient continuation Newton-like method for solving systems of non-linear equations, Appl. Math. Comput. 174(2006)846-853.

[5] J.M. Gutiérrez, M.A. Hernández, A family of Chebyshev-Halley type methods in Banach spaces, Bull. Austral. Math. Soc. 55(1997)113-130.

[6] S.Amat, S.Busquier, J.M. Gutiérrez, Geometric constructions of iterative methods to solve nonlinear equations, Comput. Appl. Math. 157(2003)197-205.

[7] M.A. Hernández, Second-derivative free variant of the Chebyshev method for nonlinear equations, J. Optim. Theory Appl. 104(2000)501-515.

[8] I.K. Argyros and D. Chen, Results on the Chebyshev Method in Banach spaces, Proyecciones, Vol. 12, pp. 119-128, 1993.

[9] J.A. Ezquerro, M.A. Hernández, A uniparametric halley-type iteration with free second derivative, Int. J. Pure Appl. Math.6(2003)104-114.

[10] J.A. Ezquerro, M.A. Hernández, On Halley-type iterations with free second derivative, J. Comput. Appl. Math. 170(2004)455-459.

[11] A. Cordero, J.R. Torregrosa, Variants of Newton's method for functions for several functions, Appl. Math. Comput. 183(2006)199-208.

[12] D.K.R. Babajee, M.Z. Dauhoo, An Analysis of the properties of the variants of Newton's method with third order convergence, Appl. Math. Comput. 183(2006)659-684.

[13] O Varmann, High order iterative methods for decomposition-coordination problems, Okio Technol. IR Ekon. Vystymas Technol. econ. Dev. Econ. 7(2006)56-61

[14] M.T. Darvishi , A. Barati, A third-order Newton-type method o solve systems of nonlinear equations, Appl. Math. and Comput 187 (2007) 630-635.

[15] Wang Haijun, New third-order method for solving systems of nonlinear equations, Numer Algor(2009) 50:271-282, DOI 10.1007/s11075-008-9227-2.

[16] D.K.R. Babajee, M.Z. Dauhoo, M.t.Darvishi, A. Karami, A. Barati, Analysis of two Chebyshev-like third order methods free from second derivatives for solving systems of nonlinear equations, J. Comput. Appl. Math. 233(2010)2002-2012.

[17] Darvishi, M. T. and Barati , A., A fourth-order method from quadrature formulae to solve systems of nonlinear equations, Applied Mathematics and Computation, 188 (1) (2007), 257-261.

[18] A. Golbabai, M. Javidi, A new family of iterative methods for solving system of nonlinear algebraic equations, Appl. Math. Comput. 190(2007)1717-1722.

[19] Alicia Cordero, José L. Hueso, Eulalia Martínez, Juan R. Torregrosa, Efficient high-order methods based on golden ratio for nonlinear systems, Appl. Math. Comput.217(2011)4548-4556.

[20] H. Montazeri, F. Soleymani, S. Shateyi and S.S. Motsa, On a New Method for Computing the Numerical Solution of Systems of Nonlinear Equations, J. Appl. Math. Volume 2012, Article ID 751975, 15 pages DOI:10.1155/2012/751975.

[21] Alicia Cordero, Juan R. Torregrosa, María P.Vassileva, New Predictor-Corrector Methods with High Efficiency for Solving Nonlinear Systems, J. Appl. Math Vol. 2012, DOI:10.1155/2012/709843.

[22] Diyashvir K.R. Babajee, Alicia Cordero, Fazlollah Soleymani, Juan R. Torregrosa, On a Novel Fourth-Order Algorithm for Solving Systems of Nonlinear Equations J. Appl. Math. DOI:10.1155/2012/165452.

[23] I.K.Arygros, H.Ren, Efficient Steffensen-type algorithms for solving nonlinear equations, Int. J. Comput. Math. 90(2013)691-704.

[24] F. Soleymani, M. Sharifi, S. Shateyi, and F. Khaksar Haghani, A Class of Steffensen-Type Iterative Methods for Nonlinear Systems,http://dx.doi.org/10.1155/2014/705375

[25] M. Grau-S*á*nchez, M. Noguera, J.L.*Ď*iaz-Barrero, On the local convergence of a family of two-step iterative methods for solving nonlinear equations, J. Comput.Appl. Math. 255(2014)753-764.

[26] Alicia Cordero, José L. Hueso, Eulalia Martínez, Juan R. Torregrosa, Increasing the convergence order of an iterative method for nonlinear systems, Appl. math. Lett. 25(2012)2369-2374.

[27] M. Grau-S*á*nchez, Ángela Grau, M. Noguera, On the computational efficiency index and some iterative methods for solving systems of nonlinear equations, J. Comput.Appl. Math. 236(2011)1259-1266.

[28] W. Gaustchi, Alexander M. Ostrowski, (1893-1986): His life, work, and students (2010). <http://www.cs.purdue.edu/homes/wxg/AMOengl.pdf>.

[29] A. M. Ostrowski, Solutions of Equations and System of Equations, Academic Press, New York 1960.

[30] M. Grau, J.L.*Ď*iaz-Barrero, An improvement to Ostrowski root-finding method, Appl. Math. Comput. 173 (2006) 450-456.

[31] B. Neta, A sixth-order family of methods for nonlinear equations, Int. J. Comput. Math. 7 (1979) 157-161.

[32] B. Neta, A New Iterative Method for the solution of Systems of Nonlinear Equations, in: Z. Ziegler (Ed.), Approx. Theor. Appl., Academic Press, 1981, pp.249-263.

[33] J. R. Sharma, R.K. Guha, A family of modified Ostrowski methods with accelerated sixth order convergence, Appl. Math. Comput. 190 (2007) 111-115.

[34] C. Chun, Y. Ham, Some sixth-order variants of Ostrowski root-finding methods, Appl. Math. Comput. 190 (2007) 389-394.

[35] J. Kou, Y. Li, X. Wang, Some variants of Ostrowski's method with seventh-order convergence, J. Comput.Appl. Math. 209 (2007) 153-159.

[36] M. Frontini, E. Sormani, Third-order methods from quadrature formulae for solving systems of nonlinear equations , appl. math. comput 149 (2004) 771-782.

[37] W. Bi, H. Ren, Q. Wu, Three-step iterative methods with eighth-order convergence for solving nonlinear equations, J. Comput. Appl. Math. 225 (2009) 105-112.

[38] Nedzhibov, G.H., A family of multi-point iterative methods for solving systems of nonlinear equations, Journal of Computational and Applied Mathematics 222 (2008) 244-250.

[39] Cordero, A., Martínez, E. and Torregrosa, J.R., Iterative methods of order four and five for systems of nonlinear equations 231, 541-551 (2009).

[40] J.M. Örtega, W.C. Rheinboldt, Iterative Solution of Non-linear Equations in Several Variables, Academic Press, New York, 1970. .

[41] J.R. Sharma, R.K. Guha, R. Sharma, An efficient fourth-order weighted-Newton method for systems of non-linear equations, Numerical Algoriths, *DOI 10.1007/s11075-012-9585-7.*

[42] H. H. H. Homeier, A modified Newton method with cubic convergence: the multivariate case, Journal of Computational and Applied Mathematics 169, (2004), 161-169

[43] M. T. Darvishi, A. Barati, Super cubic iterative methods to solve systems of nonlinear equations.Appl. Math. and Comput. 188 (2007) 1678-1685.

[44] J.R. Sharma, Himani Arora, An efficient Derivative free iterative method for solving systems of nonlinear equations, Appl. Anal. Discrete Math. 7 (2013), 390-403.

[45] M. Grau-S*á*nchez, A. Grau, M. Noguera, Ostrowski type methods for solving systems of nonlinear equations, Appl. Math. and Comput. 218 (2011) 2377-2385.

[46] Ramandeep Behl, V. kanwar, Kapil K. Sharma (2012): Optimal equi-scaled families of Jarrat's method, International journel of Computer Mathematics, *DOI: 10.1080/00207160.2012.719078.*

[47] M. Grau-S*á*nchez, M. Noguera, S. Amat, On the approximation of derivatives using divided diiference operators preserving the local convergence order of iterative methods, J. Comput.Appl. Math. 237 (2013) 363-372.

[48] M. Grau-S*á*nchez, M. Noguera, J.L.*Ď*iaz-Barrero, On the local convergence of a family of two-step iterative methods for solving nonlinear equations, J. Comput.Appl. Math. 255 (2014) 753-764.

[49] S.Wolfram : The Mathematica Book, fifth ed. Wolfram Media , 2003.

[50] M. Grau-S*á*nchez, A. Grau, M. Noguera, Frozen divided difference scheme for solving systems of nonlinear equations, J. Comput.Appl. Math. 235 (2011) 1739-1743.