Study of wave propagation in elastic and thermoelastic solids

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under the guidance of

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DEDICATED

TO

MY FATHER LATE S. KARAM SINGH AND MY TEACHERS

Certificate

This is to certify that the thesis entitled "**Study of wave propagation in elastic and thermoelastic solids**", which is being submitted for the award of degree of Master of science, School of Mathematics and computer Applications, Thapar university Patiala, is an authentic record of the work carried by me under the supervision of **Dr. Satish Kumar Sharma**.

The contents of this thesis has not been submitted for the award of any degree to any other institute or university.

Gurpreet Singh

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

Lamb wave (1917), which propagate in solid plates with free boundaries, are elastic perturbations for which displacement occurs both in the direction of wave propagation and perpendicular to the plane of plate. These waves are commonly used in ultrasonic non-destructive testing applications. Rayleigh surface waves have been well recognized in the study of seismology, geophysics and geodynamics. These types of surface waves propagate in half space.

In the present work, a mathematical analysis has been done to study Rayleigh and Lamb waves in Thermoelastic solids in the context of Generalized theories of Thermoelasticity. The thesis has been divided into four chapters.

Chapter 1 includes the discussion of classical theory of elasticity and Generalized Hooke's law. This chapter also includes a brief theory of Thermoelasticity and Generalized Thermoelasticity.

In Chapter 2, we have studied Rayleigh waves in homogeneous isotropic elastic half space and Lamb waves in homogeneous isotropic elastic plate. Formulation of the problem is done and equations are solved by assuming the boundary conditions.

Chapter 3 includes the discussion of Generalized Thermoelastic waves in homogeneous isotropic plates. The secular equations are derived and discussed in this chapter. The variations of phase velocity with wave number are shown graphically.

Chapter 4 includes the discussion of Rayleigh waves in Generalized Thermoelastic half-space. The secular equations are derived and discussed in this chapter. The variations of phase velocity with wave number are also shown graphically.

Contents

List of symbols

λ, μ	Elastic constants
ρ	Density
σ_{ij}	Stress tensor
ε_{ij}	Strain tensor
q, ψ	Potential function
$\vec{u}$	Displacement vector
β	Thermal expansion
C_e	Specific heat
K	Thermal conductivity
e	Dilatation
T	Temperature change
δ_{ij}	Kronecker's delta
t_0, t_1	Relaxation time

Chapter 1

Introduction

1.1 Theory of elasticity

Solid bodies can be divided into namely two categories; elastic body and plastic body. A body is called elastic if it returns to its original shape upon the removal of applied forces. All bodies exhibit elastic behavior under sufficiently small loads. The mathematical analysis of elastic behavior of solid body is called theory of elasticity. A body that does not return to its original shape or size, upon the removal of deforming force is called plastic body.

When an elastic material is deformed due to an external force, it experiences internal forces that oppose the deformation and restore it to its original state, if the external force is no longer applied. There are various elastic moduli, such as Young's modulus, the shear modulus, and the bulk modulus, all of which are measures of the inherent stiffness of a material as a resistance to deformation under an applied load. The various moduli apply to different kinds of deformation. For instance, Young's modulus applies to uniform extension, whereas the shear modulus applies to shearing.

The elasticity of materials is described by a stress-strain curve, which shows the relation between stress (the average restorative internal force per unit area) and strain (the relative deformation). For most metals or crystaline materials, the curve is linear for small deformations, and hence the stress-strain relationship can adequately be described by Hooke's law and higher-order terms can be ignored. However, for larger stresses beyond the elastic limit the relation is no longer linear. For even higher stresses, materials exhibit plastic behavior, that is, they deform irreversibly and do not return to their original shape after stress is no longer applied. For rubber-like materials such as elastomers, the gradient of the stress-strain curve increases with stress, meaning that rubbers progressively become more difficult to stretch, while for most metals, the gradient decreases at very high stresses, meaning that they progressively become easier to stretch.

The theory of elasticity is concerned with the displacements and angular movements of the volume element with respect to their neighborhood. If the volume elements were small enough , their movements can be described completely in terms of three positional coordinates of each element. Likewise, when the volume elements are small enough, the interactions between each element and its neighborhood can be expressed in terms of tractive force alone. Now from these ideas the state of strain in solid at any given point can be expressed by resolving the displacement of elementary volume , originally located at such points , along three mutually perpendicular directions and differentiating three component of displacement again along each of three axis in turn. Thus, we obtain nine component of strain tensor. We consider the tractive force acting on an infinitesimal area drawn normal to three coordinate planes in turn at the position and again resolving these tractive force along each of the three coordinate axes. We thus obtain the nine components of stress tensor.

1.2 Generalized Hooke's law

In stress strain relationship Hooke's law states that the stress is a quantity that is proportional to the force causing deformation and strain is a measure of the degree of deformation. Stress can be divided into two components; normal and shearing stress. In 3 dimensional axes of the cubic body, stress can be resolved into three parts, one normal stress and shearing stress which itself can be resolved into two components parallel to the direction of two coordinates.

The Generalized Hooke's law states that nine component of stress acts on the face of the cube. These are $(\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yx}, \sigma_{yy}, \sigma_{yz}, \sigma_{zx}, \sigma_{zy}, \sigma_{zz})$ where the first suffix refers to the normal to the plane on which shear stress acts and second suffix refers to the direction of the shear on the plane. Similarly strain can also be resolved into two components; longitudinal and shearing strain. There are nine component of strain. These are $(\varepsilon_{\mathrm{xx}}, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{\mathrm{yx}}, \varepsilon_{yy}, \varepsilon_{yz}, \varepsilon_{\mathrm{zx}}, \varepsilon_{zy}, \varepsilon_{zz})$.

In general σ_{ij} and ε_{kl} will have nine components and c_{ijkl} will have 81 components. Considering the symmetry of stress and strain components, the number of independent component reduces from 81 to 36. We write Hooke's law in the form

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & c_{1113} & c_{1123} \\ c_{2211} & c_{2222} & c_{2233} & c_{2212} & c_{2213} & c_{2223} \\ c_{3311} & c_{3322} & c_{3333} & c_{3312} & c_{3313} & c_{3323} \\ c_{1211} & c_{1222} & c_{1233} & c_{1212} & c_{1213} & c_{1223} \\ c_{1311} & c_{1312} & c_{1333} & c_{1312} & c_{1313} & c_{1323} \\ c_{2311} & c_{2332} & c_{2333} & c_{2312} & c_{2313} & c_{2323} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}.$$

Here $\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$, where I, j, .k, l=1,2,3.

For other crystal the number of independent constants is diminished by reason of their symmetry property. For monoclinic materials there are only 13 independent elastic constants. In case of orthotropic materials the number of elastic constant is further reduced by 4 from thirteen to nine. For transversely isotropic material there are only five independent constants. Finally, there are only two independent elastic constants known as Lame's parameter for isotropic elastic material.

The constitutive equation for classical isotropic elasticity is

$\sigma_{ij} = \lambda u_{r,r}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$ where δ_{ij} and σ_{ij} are the Kronecker's delta and stress tensor respectively.

1.3 Wave propagation in elastic solids

Depending upon the nature of medium and the boundary conditions the phenomenon of wave propagation can be divided into following categories.

1. Waves in infinite media
2. Waves in semi infinite media
3. Waves in plate.

1.4 Waves in infinite media

The equation of motion for a homogeneous isotropic medium without body force is given by

$$(\lambda+\mu)\nabla(\nabla.\vec{u})+\mu\nabla^2\vec{u}=\rho\frac{\partial^2\vec{u}}{\partial t^2}$$

Where $\vec{u}=(u_1,u_2,u_3)$ is the displacement vector. λ,μ are Lami's constant and ρ is density.

We take $\vec{u}=(u_1,u_2,u_3)$ and $\vec{\psi}=(0,-\psi,0)$ for two- dimensional problems.

Using Helmholtz decomposition, this vector can be decomposed in terms of scalars and vector potentials,

$$\vec{u}=\nabla q+\nabla\times\vec{\psi}$$

q and ψ are the scalars and vector potentials respectively. The resolution of a vector field into the gradient of a scalar and the curl of a zero- divergence vector is due to a theorem by Helmholtz. The condition $\nabla.\vec{\psi}=0$ provides the necessary additional condition to uniquely determine the three components of u.

Now Inserting $\vec{u}$ in basic equation and interchanging the order of some operations, we obtain

$$\nabla^2\psi=\frac{\rho}{\mu}\frac{\partial^2\psi}{\partial t^2}$$

$$\nabla^2 q=\frac{\rho}{\lambda+2\mu}\frac{\partial^2 q}{\partial t^2}$$

we take

$$c_1^2=\frac{\lambda+2\mu}{\rho}\ \textit{and}\ c_2^2=\frac{\mu}{\rho}.$$

Therefore waves may propagate in the interior of elastic solid at two different velocities. c_1 is the wave velocity of longitudinal and c_2 is the velocity of shear waves. In longitudinal waves, the oscillations occur in the direction of wave propagation. Since compressional and dilatational forces are active in these waves. They are also called density waves because their particle density fluctuates as they move. In the transverse or shear wave, the particles oscillates at right angle. Shear waves require an acoustically solid material for effective propagation and therefore, are not effectively propagated in materials such as liquids or gasses. Shear waves are weaker than longitudinal waves. Shear waves are usually generated in materials using some of the energy from longitudinal waves. A variety of terminology exists for the two wave- types. Longitudinal waves are also called irrotational, Dilatational and primary waves(P). The Shear waves are also called equi-voluminal, distortional, rotational and secondary waves (S). The P and S waves designations have arisen in seismology, where they are also occasionally picturesquely designated as the ‘push’ and ‘shake’ waves.

1.5 Waves in semi infinite media

There are different types of waves in semi infinite media. But in the present work we are considering only the types of waves whose effects are closely confined to the surface. These types of waves were first investigated by Lord Rayleigh who showed that there is a wave type that could propagate along surfaces, such that the motion associated with the wave decayed exponentially with the distance into material from the surface. This type of surface wave is called Rayleigh wave. Longitudinal and transverse both motions may be found in solids as Rayleigh surface wave. When Rayleigh surface wave passes through a solid, the particles in a solid moves in elliptical paths, with the major axis of the ellipse perpendicular to the surface of solid. The simplest medium in which Rayleigh wave propagate is homogeneous isotropic half space. The transverse velocity is slightly greater than the velocity of Rayleigh waves.

1.6 Waves in plates

Here we consider the propagation of waves in a plate having traction- free boundary. This is the case of greatest practical interest and is the classical case first studied by Rayleigh and Lamb (Graff 1991) and in general are known as Lamb waves. These are the elastic waves whose particle motion lies in the plane that contains the direction of wave propagation. These type of waves can travel long distances with little attenuation.

Rayleigh –Lamb theory applies to the propagation of continuous, straight crested waves in an infinite plate with free surface. The Rayleigh-Lamb frequency equation gives the relationship between phase velocity and wave number. The displacement and stress distribution functions can be obtained after the Rayleigh-Lamb frequency are solved. As in Rayleigh waves which propagates along single free surfaces, the particle motion in Lamb wave is elliptical with its x and z components depending on the depth within the plate. In one family of modes, the motion is symmetrical about the mid thickness plate. In the other family it is antisymmetric.

When Lamb waves are transmitted, particles move in one of two different ways. If the particle motion is symmetric to the mid surface, it is called symmetric Lamb wave. If the particle motion is antisymmetric with respect to the mid surface, it is called anti symmetric Lamb waves. Furthermore, a number of modes exist for each type of the Lamb waves. Lamb wave can be generated in a plate with free boundaries with an infinite number of modes for both symmetric and antisymmetric displacements within the layer. The symmetric modes are also called longitudinal modes because the average displacement over the thickness of the plate or layer in the longitudinal direction. The anti-symmetric modes are observed to exhibit average displacement in the transverse direction and these modes are also called flexural modes. Symmetric and anti-symmetric Lamb waves have different phase and group velocity, as well as distribution of particle displacement and stress through the plate thickness.

1.7 Thermoelasticity

Thermoelasticity deals with the study of thermodynamical system of bodies in equilibrium, whose interactions with surrounding are limited to mechanical work, heat exchange and external work. We know from experiment that deformation of a body is associated with a change of heat. The time varying loading of a body causes in it not only displacements but also temperature distribution changing in time. Conversely, the heating of a body produces in it deformation and temperature change. The motion of a body is characterized by mutual interaction between deformation and temperature fields. The domain of science dealing with the mutual interaction of these fields is called Thermoelasticity.

The theory of coupling of thermal and the strain fields give rise to coupled theory of Thermoelasticity and was first postulated by Duhamel [1837] shortly after the formulation of elasticity. He derived the equations for the distributions of strains in an elastic medium subjected to temperature gradient and introduced the dilatation term in heat conduction equation.

However, this equation was not well grounded in the thermo dynamical sense. Next, an attempt at thermo dynamical Justification of this equation was undertaken by Voigt and Jefreys [1930]. Biot[1956] gave the full justification of the thermal conductivity equation on the basis of thermodynamics of irreversible processes. Biot also presented the fundamental methods for solving the thermo elasticity equation as well also variational theorem.

Thermoelasticity describes a broad range of phenomena; it is the generalization of the classical theory of elasticity and of the theory of thermal conductivity. It is known, that research in the field of thermo elasticity was preceded by broad-scale investigation within the framework of what is called the theory of thermal stresses. By this term we mean the investigation of strains and stresses produced by heating a body, with the simplifying assumption that the deformation of an elastic body does not affect the thermal conductivity.

Coupled theory of Thermoelasticity

The coupling between thermal and strain fields give rise to coupled theory of Thermoelasticity. For static problems this coupling vanishes and two fields becomes independent of each other .The coupling effects were considered by Weiner [1957] and Lesson[1957] in their works. Nowacki [1962] discussed the propagation of longitudinal waves in an unbounded Thermoelastic medium.

1.8 Generalized theories of Thermoelasticity

The basic governing equations of Thermoelasticity in the usual framework of linear coupled Thermoelasticity consist of wave type (hyperbolic) equations of motion and diffusion type (parabolic) equation of heat conduction. It is observed that a part of solution of the energy tends to infinity. This implies that if an isotropic homogeneous elastic medium is subjected to thermal or mechanical disturbances, the effects in the temperature and displacement fields are felt at an infinite distance from the source of disturbance instantaneously. This implies that a part of solution has infinite velocity of Researchers such as kaliski[1965] , Lord and Shulman[1967]. Fox[1969], Gurtin and Pipkkin[1969] have tried to modify the Fourier law of heat conduction. These works include the time needed for acceleration of the hest flow, in the heat conduction equation along with the coupling between temperature and stain fields. This paradox in existing coupled theory of Thermoelasticity has also been discussed by Boley[1956].

Thermoelasticity covers a wide range of extensions of classical dynamical coupled Thermoelasticity. This theory eliminates the paradox of an infinite velocity of propagation and is based upon the more general linear functional relationship between the heat flow and temperature gradients. The term generalized Thermoelasticity stands for “Hyperbolic Thermoelasticity” in which thermo mechanical load applied to the body is transmitted in a wave like manner throughout the body. In the present work we have applied the following two models of Thermoelasticity namely, Lord-Shulman(LS) and Green-Lindsay(GL).

1.8.1 Lord Shulman theory

Lord and Shulman[1967] have formulated a generalized dynamical theory of Thermoelasticity by using Maxwell-Cattano law that generalizes the Fourier’s heat conduction equation by introducing a single relaxation time needed for acceleration of heat flow.

1.8.2 Green –Lindsay (GL)

Green and Lindsay[1972] have also obtained a generalization of coupled theory of Thermoelasticity which proved that the second sound effects are short lived. Their analysis is based on modified form of entropy production inequality. In this theory the constitutive relations for stress and entropy are generalized by introducing two different relaxation times into consideration constrained by inequality $t_1 \geq t_0 \geq 0$.

Basic differences between GL and LS theory

(1) GL theory modifies both constitutive equation and energy equation. But LS theory modifies only the energy equation.

(2) GL involves two relaxation times while LS theory involves one relaxation time.

(3) The energy equation of LS theory depends on strain velocity and strain acceleration but corresponding equation of GL theory depends only on the strain constraints.

(4) In GL theory the heat cannot propagate with a finite speed unless the stresses depend on temperature velocity. But according to LS theory the heat can propagate with a finite speed even though the stresses are independent of temperature velocity.

Chapter -2

Rayleigh wave in a homogeneous isotropic elastic half space and Rayleigh Lamb wave Propagation in homogeneous isotropic elastic plate.

2.1 Introduction

Rayleigh or surface waves travel along the surface of a relative thick solid material penetrating to a depth of one wavelength. Rayleigh waves are useful because they are very sensitive to surface defects and since they will follow the surface around, so these can be used to assess the surface, the other waves might have difficulty in reaching.

Lamb wave can propagates in thin metals. Lamb wave are complex vibrational waves that travel through the entire thickness of a material. These provide a means for inspection of very thin materials. The propagation of lamb waves depend on density, elastic, and material properties of a component. With Lamb waves, a number of modes of a particles vibrations are possible, but most two common are symmetrical and asymmetrical. In this chapter, we have derived the secular equations for Rayleigh and Rayleigh –Lamb type wave propagation.

2.2 Rayleigh wave

Consider a homogeneous, isotropic elastic half-space. We introduce a Cartesian coordinate system whose (x-plane coincides with the surface of the medium, and z- axis is positive downwards) . the z–axis is pointing vertically downward into the semi space. The x-axis is taken along the direction of wave propagation so that all particles on a line parallel to y- axis are equally displaced and hence all the quantities are independent of y- coordinate. The surfaces are assumed to be stress free.

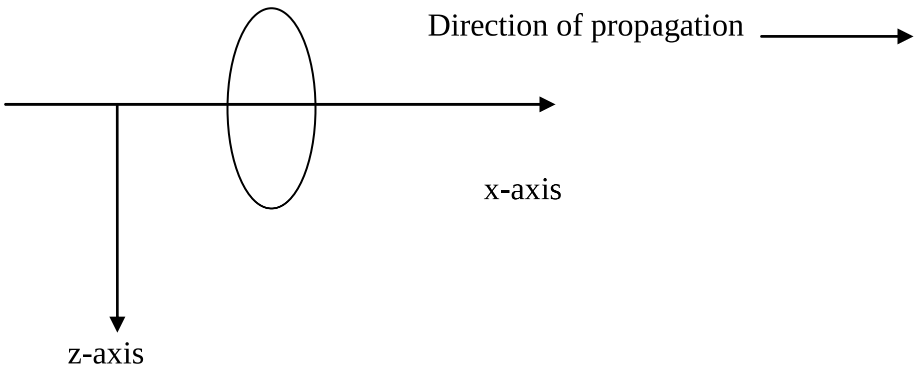


Fig 2.1 (geo.mff.cuni.cz/vyuka/Novotony-SeismicSurfaceWaves)

The basic governing equations of generalized elasticity in the absence of body forces and heat sources are

$$(\lambda+\mu)\nabla\left(\nabla.\vec{u}\right)+\mu\nabla^2\vec{u}=\rho\frac{\partial^2\vec{u}}{\partial t^2} \tag{2.2.1}$$

The constitutive relation is given by

$$\sigma_{ij}=\lambda u_{r,r}\delta_{ij}+\mu(u_{i,j}+u_{j,i}) \tag{2.2.2}$$

Where $\vec{u}=(u_1,u_2,u_3)$ is the displacement vector, ρ is the density.

We define the quantities

$$\Delta^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2},\ x^{'}=\frac{\omega^* x}{c_1},\ z^{'}=\frac{\omega^* z}{c_1},\ t^{'}=\omega^* t,\ u^{'}=\frac{\rho\omega^* c_1}{\nu T_0}u,$$
$$w^{'}=\frac{\rho\omega^* c_1}{\nu T_0}w,\ \delta^2=\frac{c_2^2}{c_1^2},\ \sigma_{ij}^{'}=\frac{\sigma_{ij}}{\nu T_0},\ c_1^2=\frac{\lambda+2\mu}{\rho},\ c_2^2=\frac{\mu}{\rho}. \tag{2.2.3}$$

c_1 and c_2 are longitudinal and shear velocities.w^* is characteristic frequency of medium.

For two dimensional problem, we take

$$\vec{u} = \left(u_{1,} 0, u_3\right)$$

The equations in the non dimensional form for Rayleigh wave can be written as

$$\left(1-\delta^2\right)\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_3}{\partial x \partial z}\right)+\delta^2 \frac{\partial^2 u_1}{\partial x^2}=\frac{\partial^2 u_1}{\partial t^2} \tag{2.2.4}$$

$$\left(1-\delta^2\right)\left(\frac{\partial^2 u_3}{\partial z^2}+\frac{\partial^2 u_1}{\partial x \partial z}\right)+\delta^2 \frac{\partial^2 u_3}{\partial z^2}=\frac{\partial^2 u_3}{\partial t^2} \tag{2.2.5}$$

In order to solve above equations, we introduce potential function q and ψ in the solid through the relations

$$u_1 = \frac{\partial q}{\partial x}+\frac{\partial \psi}{\partial z} \tag{2.2.6}$$

$$u_3 = \frac{\partial q}{\partial z}-\frac{\partial \psi}{\partial x} \tag{2.2.7}$$

Substituting (2.2.6) and (2.2.7) in equations (2.2.4) and (2.2.5) we obtained

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) q = 0 \tag{2.2.8}$$

$$\left(\nabla^2 - \frac{1}{\delta^2}\frac{\partial^2}{\partial t^2}\right) \psi = 0 \tag{2.2.9}$$

Now on Substituting (2.2.6) and (2.2.7) in constitutive relation, we obtain

$$\sigma_{zz} = \nabla^2 q - 2\delta^2 (q_{,xx} + \psi_{,xz}) \tag{2.2.10}$$

$$\sigma_{zx} = \delta^2 (2q_{,xz} - \psi_{,xx} + \psi_{,xz}) \tag{2.2.11}$$

Boundary conditions

The surface of the solid is assumed to be free of stresses, couples so that,

$$\sigma_{zz} = 0, \sigma_{zx} = 0$$

The boundary condition is to be satisfied at the surface (z=0) of the solid half space.

2.2.1 Solution of the problem

We assume the solution of the form,

$$q = q(z)e^{\iota\xi(x-ct)} \tag{2.2.12}$$

$$\psi = \psi(z)e^{\iota\xi(x-ct)} \tag{2.2.13}$$

Where c is the non dimensional phase velocity, ω is the circular frequency, ξ is wave number

Upon using solutions in (2.2.8) and (2.2.9), we get

$$q = (A_1 e^{\alpha^* z} + B_1 e^{-\alpha^* z})e^{i\xi(x-ct)} \tag{2.2.14}$$

$$\psi = (A_2 e^{\beta^* z} + B_2 e^{-\beta^* z})e^{\iota\xi(x-ct)} \tag{2.2.15}$$

Where $\alpha^{*2} = \xi^2(1-c^2)$ and $\beta^{*2} = \xi^2(1-\frac{c^2}{\delta^2})$

Since we are primarily interested in surface wave propagation along the free surface of half space. So choose the partial wave that satisfied the radial condition. That is all field variables should be bounded for location deep within the solid. Therefore, we choose the solution for q and as ψ

$$q = B_1 e^{-\alpha^* z} e^{i\xi(x-ct)} \tag{2.2.16}$$

$$\psi = B_2 e^{-\beta^* z} e^{\iota\xi(x-ct)} \tag{2.2.17}$$

Now applying q and ψ in σ_{zz} *and* σ_{zx}

$$\sigma_{zz} = B_1\left[(\alpha^{*2} - \xi^2) + 2\delta^2\xi^2\right] + 2\iota\xi\delta^2\beta^* B_2$$

$$\sigma_{ZX} = \delta^2\left[B_2(\beta^{*2} + \xi^2) - 2B_1\iota\xi\alpha^*\right]$$

Now invoking boundary condition, $\sigma_{zz} = \sigma_{xz} = 0$

$$B_2(\beta^{*2} + \xi^2) - 2B_1 \iota\xi\alpha^* = 0 \tag{2.2.18}$$

$$B_1(\beta^{*2} + \xi^2) + 2\iota\xi\beta^* B_2 = 0 \tag{2.2.19}$$

This is a homogeneous system of equation for the unknown amplitude B_1 and B_2

The system of equations (2.2.15) and (2.2.16) has a non trivial solution if corresponding determinant equals to zero.

Now solving determinant, we get

$$\begin{vmatrix} \left(\beta^{*2} + \xi^2\right) & 2i\xi\beta^* \\ -2i\xi\alpha^* & \left(\beta^{*2} + \xi^2\right) \end{vmatrix} = 0$$

$$\left(\beta^{*2} + \xi^2\right)^2 - 4\alpha^*\beta^*\xi^2 = 0 \tag{2.2.20}$$

This is the Rayleigh equation which is same as is obtained and discussed in by Graff (1991)

2.3 Rayleigh lamb wave

We consider an infinite homogeneous isotropic elastic plate of thickness 2d. We take the origin of the coordinate system (x, y, z) on the middle surface of the plate. The x-y plane is chosen to coincide with the middle surface and the z- axis normal to it along the thickness. The surface are assumed to be stress free .The basic governing equation, constitutive relation and partial differential equations for potential function of Rayleigh-lamb wave are same as that of Rayleigh wave equations (2.2.1 to 2.2.9) .

Boundary conditions

The boundaries of the plate are assumed to be stress free. Therefore, the non dimensional mechanical boundary conditions are given by

$\sigma_{zz} = \sigma_{xz} = 0$ at z= $\pm$ d

For Rayleigh lamb type waves, solution for potential functions are obtained as

$$q = (A\sin\alpha' z + B\cos\alpha' z)e^{i\xi(x--ct)}$$

where $\alpha'^2 = \xi^2(c^2 - 1)$

and

$$\psi = \left(E\sin\beta' z + F\cos\beta' z\right)e^{i\xi(x-ct)}$$

where $\beta'^2 = \xi^2(\frac{c^2}{\delta^2} - 1)$

On using (2.2.6) and (2.2.7) in constitutive relation, we get

$$\sigma_{zz} = \left[-(\lambda+2\mu)\alpha'^2 - \lambda\xi^2\right](A\sin\alpha' z + B\cos\alpha' z) - 2\mu i\xi\beta'(E\cos\beta' z - F\sin\beta' z) \quad (2.2.21)$$

$$\sigma_{xz} = 2i\xi\alpha'\mu\left(A\cos\alpha' z - B\sin\alpha' z\right) + \mu\left(\xi^2 - \beta'^2\right)\left(E\sin\beta' z + F\cos\beta' z\right) \quad (2.2.22)$$

The displacement are obtained from equation (2.2.6) and (2.2.7) as

$$u_1 = \left[i\xi(A\sin\alpha' z + B\cos\alpha' z) + \beta'(E\cos\beta' z - F\sin\alpha' z)\right]e^{i\xi(x-ct)}$$

$$u_3 = \alpha'((A\cos\alpha' z - B\sin\alpha' z) - i\xi\beta'(E\sin\beta' z + F\cos\beta' z))e^{i\xi(x-ct)}$$

2.3.1 Derivation of Secular Equations

Invoking the boundary conditions at the surface $z = \pm d$ of the plate and using equations of q and ψ, we obtain a system of four simultaneous linear equation as below:

$$p(-As_1 + Bc_1) + q(Ec_2 + Fs_2) = 0$$

$$p(As_1 + Bc_1) + q(Ec_2 - Fs_2) = 0$$

$$f(Ac_1 + Bs_1) + r(-Es_2 + Fc_2) = 0$$

$$f(Ac_1 - Bs_1) + r(Es_2 + Fc_2) = 0$$

Where

$$p=(\lambda+2\mu)\alpha'^2+\lambda\xi^2, q=2i\xi\beta'\mu, c_1=\cos\alpha' z, c_2=\cos\beta' z, s_1=\sin\alpha' z, s_2=\sin\beta' z, f=2i\xi\alpha', r=(\xi^2-\beta'^2)$$

For symmetric modes, the reduced system of equations can be written in the matrix form as

$$\begin{bmatrix} p\cos\alpha' d & q\cos\beta' d \\ -f\sin\alpha' d & r\sin\beta' d \end{bmatrix}\begin{bmatrix} B \\ E \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the system of equation is homogeneous, the determinant of coefficient has to vanish for non trivial solution, which results in the frequency equation.

$$pr\cos\alpha' d\sin\beta' d+qf\cos\beta' d\sin\alpha' d=0$$

it can be written as

$$\frac{\tan\beta' d}{\tan\alpha' d}=\frac{-4\xi^2\alpha'\beta'}{(\xi^2-\beta'^2)^2} \quad \text{where} \quad \alpha'^2=\xi^2(c^2-1), \beta'^2=\xi^2(\frac{c^2}{\delta^2}-1) \qquad (2.2.23)$$

Similarly for anti-symmetric modes, the reduced system of equations can be written in the matrix form as

$$\begin{bmatrix} p\sin\alpha' d & -q\sin\beta' d \\ f\cos\alpha' d & r\cos\beta' d \end{bmatrix}\begin{bmatrix} A \\ F \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives the Rayleigh –Lamb frequency equation for the propagation of anti symmetric waves in a plate

$$\frac{\tan\alpha' d}{\tan\beta' d}=\frac{-4\xi^2\alpha'\beta'}{\left(\xi^2-\beta'^2\right)^2}. \qquad (2.2.24)$$

Combining (2.2.23) and (2.2.24) ,

$$\left(\frac{\tan\alpha' d}{\tan\beta' d}\right)^{\pm 1}=\frac{-4\xi^2\alpha'\beta'}{\left(\xi^2-\beta'^2\right)^2}$$

Here +1 is for anti symmetric mode and – 1 for symmetric mode.

The equations are same as obtained and discussed by Graff(1991).

CHAPTER – 3

Propagation of Thermoelastic waves in isotropic plates.

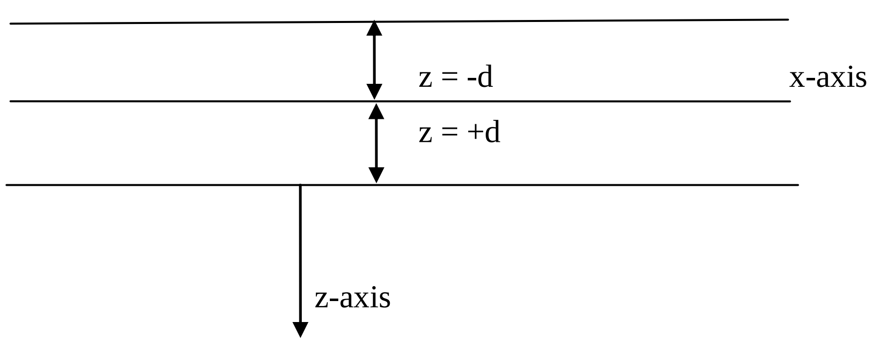
3.1 Introduction

The concept of generalized Thermoelasticity covers a wide range of extensions of classical dynamical coupled Thermoelasticity. Thermoelasticity proposed by Lord and Shulman in1967 (L–S model) in which, in comparison to the classical theory, the Fourier law of heat conduction is modified by taking into consideration a single relaxation time. To explain and remove the paradox of infinite velocity of heat propagation, some researchers derived and formulated generalized theories of thermo elasticity. The propagation of plane harmonic waves in homogeneous transversely isotropic materials has also been studied in generalized theories of thermo elasticity. In the present work we have considered the propagation of plane waves in an infinite homogeneous, isotropic plate of thickness 2d. In the context of generalized theories of Thermoelasticity. Here we have reviewed the work carried out by Sharma etal. [2000].

(Generalized Thermoelastic waves in homogeneous isotropic plates", J. Acoust. Soc. Am. 108(2), August 2000)

3.2 Formulation of the problem and its solution

We consider an infinite homogeneous isotropic thermally conducting elastic plate of thickness $2d$ initially at uniform temperature T_0 . We take origin of the coordinate system (x ,y ,z) on the middle surface of the plate. The $x– y$ plane is chosen to coincide with the middle surface and the z axis normal to it along the thickness. The surfaces $z=\pm d$ are assumed to be (i) stress free insulated and (ii) rigid fixed insulated.



The basic governing equations of generalized Thermoelasticity in the absence of heat sources and body forces are

$$(\lambda+\mu)\nabla(\nabla.\vec{u})+\mu\nabla^2 u-\beta\nabla(T+\delta_{2k}t_1\dot{T})=\rho\ddot{u} \tag{3.1}$$

$$K\nabla^2 T-\rho C_e(\dot{T}+t_0\ddot{T})=\beta T_0(\frac{\partial}{\partial t}+\delta_{1k}t_0\frac{\partial^2}{\partial t^2})\nabla.\vec{u} \tag{3.2}$$

where $\vec{u}=(u_1,u_2,u_3)$ is the displacement vector,

$T(x,y,z,t)$ is the temperature change λ *and* μ are Lame's parameters;

K is thermal conductivity; ρ and c_e are the density and specific heat at constant strain respectively. $\beta=(3\lambda+2\mu)\alpha_t$, α_t is the linear thermal expansion and e is the dilatation.

The dot notation denotes time differentiation and δ_{ij} is the Kronecker's delta. Here K=1 for Lord Shulman (LS) theory and K=2 for Green- Lindsay (GL) theory. The thermal relaxation times t_0 and t_1 satisfies the inequalities $t_0\ge t_1\ge 0$ for GL theory only.

We take the x– z plane as the plane of incidence and we assume that the solutions are explicitly Independent of y, but implicit dependence is there so that the transverse component u_2 of displacement is non vanishing.

The governing equations in non-dimensional form can be written as:

$$\frac{\partial^2 u_1}{\partial x^2}+(1-\delta^2)\frac{\partial^2 u_3}{\partial x\partial z}+\delta^2\frac{\partial^2 u_1}{\partial z^2}-\frac{\partial}{\partial x}(T+\delta_{2k}t_1\dot{T})=\frac{\partial^2 u_1}{\partial t^2} \tag{3.3}$$

$$\frac{\partial^2 u_3}{\partial z^2}+(1-\delta^2)\frac{\partial^2 u_1}{\partial x\partial z}+\delta^2\frac{\partial^2 u_3}{\partial x^2}-\frac{\partial}{\partial z}(T+\delta_{2k}t_1\dot{T})=\frac{\partial^2 u_3}{\partial t^2} \tag{3.4}$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}-(\dot{T}+t_0\ddot{T})=\varepsilon\left[\frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial x}\right)+\frac{\partial}{\partial t}\left(\frac{\partial u_3}{\partial z}\right)+\delta_{1k}t_0\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_1}{\partial x}\right)+\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_3}{\partial z}\right)\right] \tag{3.5}$$

Here we have defined the quantities,

$$x'_i = \omega^* x_i / c_i, \quad t' = \omega^* t$$
$$u'_i = \rho \omega^* c_1 u_i / \beta T_0$$
$$T' = T / T_0, \quad t_1' = \omega^* t_1$$
$$t_0' = \omega^* t_0, \quad \omega^* = c_e(\lambda + 2\mu)/k$$
$$\varepsilon = T_0 \beta^2 / \rho c_e(\lambda + 2\mu)$$
$$\delta^2 = \mu / (\lambda + 2\mu)$$
$$c_1{}^2 = (\lambda + 2\mu)/\rho$$
$$c_2{}^2 = \mu / \rho \tag{3.6}$$

The constitutive relation is given as :

$$\sigma_{ij} = \lambda u_{r,r} \delta_{ij} + \mu(u_{i,j} + u_{j,j}) - \beta(T + t_1 \delta_{2k} \frac{\partial T}{\partial t}) \delta_{ij} \tag{3.7}$$

Boundary conditions

(a) Mechanical boundary conditions

The non-dimensional mechanical boundary condition at $z = \pm d$ are given by:

(1)For stress free boundary conditions-

$$\sigma_{33} = u_{3,3} + \left(1 - 2\delta^2\right) u_{1,1} - \left(T + t_1 \delta_{2k} \dot{T}\right) = 0 \tag{3.8}$$

$$\sigma_{13} = u_{1,3} + u_{3,1} = 0 \tag{3.9}$$

(2)For rigidly fixed boundary conditions-

$$u_1 = u_2 = u_3 = 0$$

(b)Thermal condition

The non dimensional thermal boundary conditions at $z = \pm d$ are given by:

$$\frac{\partial T}{\partial z} + hT = 0$$

Where h is the surface heat transfer coefficient

$h \to 0$ Corresponds to Thermally Insulated Boundary

Using Helmholtz decomposition,

Now, $\vec{u} = \nabla q + \nabla \times \vec{\psi}$

$$\nabla^2 q - \left(1 + t_1 \delta_{2k} \frac{\partial}{\partial t}\right) T = \ddot{q} \tag{3.10}$$

and

$$\nabla^2 \psi - \frac{1}{\delta^2} \ddot{\psi} = 0 \tag{3.11}$$

and

$$\nabla^2 T - \frac{\partial T}{\partial t} + t_0 \frac{\partial^2 T}{\partial t^2} = \in \left(\frac{\partial}{\partial t} + \delta_{1k} t_0 \frac{\partial^2}{\partial t^2} \right) q \tag{3.12}$$

Now from equation (3.10) , we get

$$\psi = \left(E \sin \beta z + F \cos \beta z\right) e^{i\xi(x-ct)} \tag{3.13}$$

And from equation (3.10) and (3.12) we get

$$q = \left(A \sin m_1 z + B \cos m_1 z\right) e^{i\xi(x-ct)} + \left(C \sin m_2 z + D \cos m_2 z\right) e^{i\xi(x-ct)} \tag{3.14}$$

$$T = i\tau_1^{-1}\omega^{-1}\left\{\left(\alpha^2 - m_1^{\,2}\right)\left(Ar_1 + Bc_1\right) + \left(\alpha^2 - m_2^{\,2}\right)\left(Cr_3 + Dc_3\right)\right\} e^{i\xi(x-ct)} \tag{3.15}$$

Displacement u_1, u_3 are given by,

$$u_1 = \frac{\partial q}{\partial x} + \frac{\partial \psi}{\partial z},$$

$$u_3 = \frac{\partial q}{\partial z} - \frac{\partial \psi}{\partial x}$$

Putting the values of q and ψ , we get

$$u_1 = \left\{ i\xi \left(Ar_1 + Bc_1 + Cr_2 + Dc_2 \right) + (\beta Ec - Fr) \right\} e^{i\xi(x-ct)}, \tag{3.16}$$

$$u_3 = \left\{ m_1 \left[-Br_1 + Ac_1 \right] + m_2 \left[-Dr_2 + Cc_2 \right] - i\xi \left[Fc + Cr \right] \right\} e^{i\xi(x-ct)} \tag{3.17}$$

Here ,

$$\tau_0 = t_0 + i\omega^{-1},$$
$$\tau_0` = t_0 \delta_{1k} + i\omega^{-1},$$
$$\tau_1` = t_1 \delta_{2k} + i\omega^{-1}, r = \sin \beta z, c = \cos \beta z, r_i = \sin m_i z, c_i = \cos m_i z, i = 1, 2$$
$$\alpha^2 = \xi^2 \left(c^2 - 1 \right),$$
$$\beta^2 = \xi^2 \left(\frac{c^2}{\delta^2} - 1 \right),$$
$$m_1^{\ 2} = \xi^2 \left(a_1^{\ 2} c^2 - 1 \right),$$
$$m_2^{\ 2} = \xi^2 \left(a_2^{\ 2} c^2 - 1 \right),$$
$$a_1^{\ 2}, a_2^{\ 2} = \frac{1}{2} \left\{ \left(1 + \tau_0 - i\omega \in \tau_0 ` \tau_1 \right) \pm \left[ \left( 1 - \tau_0 - i\omega \in \tau_0 ` \tau_1 \right)^2 - 4i\omega \in \tau_0 ` \tau_1 \right]^{\frac{1}{2}} \right\}$$

3.3 Secular equations

Invoking the boundary conditions at the surfaces $z = \pm d$ of the plate. We obtain a system of six simultaneous linear equations for stress free conditions as

$$p(-As_1 - Bc_1 - Cs_2 - Dc_2) + q(-Ec + Fr) = 0$$
$$\mathrm{f}(Ac_1 - Bs_1) + g(Cc_2 - Ds_2) + h(Er + Fc) = 0$$
$$l(Ac_1 - Bs_1) + n(Cc_2 - Ds_2) = 0$$
$$p(As_1 - Bc_1 + Cs_2 - Dc_2) + q(-Ec - Fr) = 0$$
$$\mathrm{f}(Ac_1 + Bs_1) + g(Cc_2 + Ds_2) + h(-Er + Fc) = 0$$
$$l(Ac_1 + Bs_1) + n(Cc_2 + Ds_2) = 0$$

Where

$p=\left((1-2\delta^2)\xi^2+\alpha^2\right)$, q=2i$\xi\delta^2\beta$, f = 2m$_1 i\xi$, g = 2m$_2 i\xi$, h=$(\xi^2-\beta^2)$

$l=(\alpha^2-m_1^{\,2})m_1$, n=$(\alpha^2-m_2^{\,2})m_2$

$r=\sin\beta z, c=\cos\beta z, s_i=\sin m_i z, c_i=\cos m_i z, i=1,2$

Equations possess nontrivial solution if the determinant of the coefficients of amplitudes *[A,B,C,D,E,F]*^{T} vanishes.

For skew symmetric mode , under stress free conditions we obtained the matrix as

$$\begin{vmatrix} -((-1+2\delta^2)\xi^2+\alpha^2)T_1 & -((-1+2\delta^2)\xi^2+\alpha^2)T_1 & 2\delta^2 i\xi\beta T_4 \\ 2m_1 i\xi & 2m_2 i\xi & \xi^2-\beta^2 \\ (\alpha^2-m_1^{\,2})m_1 & (\alpha^2-m_2^{\,2})m_2 & 0 \end{vmatrix}=0 \tag{3.18}$$

Equations for the symmetric mode can also be obtained in the similar way.

After applying lengthy algebraic reductions and manipulations, we obtain the equations for a plate with thermally insulated boundaries as:

$$\left[\frac{\tan \mathrm{m}_1 d}{\tan\beta \mathrm{d}}\right]^{\pm 1}-\frac{\mathrm{m}_1(\alpha^2-\mathrm{m}_1^{\,2})}{\mathrm{m}_2(\alpha^2-\mathrm{m}_2^{\,2})}\left[\frac{\tan \mathrm{m}_2 d}{\tan\beta \mathrm{d}}\right]^{\pm 1}=\left\{\frac{4\xi^2\beta \mathrm{m}_1(m_2^{\,2}-\mathrm{m}_1^{\,2})}{\left(\xi^2-\beta^2\right)^2\left(\alpha^2-m_2^{\,2}\right)}\right\}$$

Where +1 corresponds to skew symmetric and -1 corresponds to symmetric modes.

Similarly, Invoking the boundary conditions at the surfaces $z=\pm d$ of the plate. We obtain a system of six simultaneous linear equations for rigidly fixed case as

$$i\xi(As_1+Bc_1+Cs_2+Dc_2)+\beta(c-r)=0$$

$$m_1(Ac_1-Bs_1)-m_2(Cc_2-Ds_2)-i\xi(Er+Fc)=0$$

$$l(Ac_1-Bs_1)+n(Cc_2+Ds_2)=0$$

$$i\xi(-As_1+Bc_1-Cs_2+Dc_2)+\beta(c+r)=0$$

$$m_1(Ac_1+Bs_1)-m_2(Cc_2+Ds_2)-i\xi(-Er+Fc)=0$$

$$l(Ac_1+Bs_1)+n(Cc_2-Ds_2)=0$$

which has nontrivial solution if the determinant of the coefficients of amplitudes *[A,B,C,D,E,F]*^{T} vanishes.

For skew symmetric mode , under rigidly fixed conditions we obtained the matrix as

$$\begin{vmatrix} i\xi T_1 & i\xi T_2 & -\beta T_4 \\ m_1 & m_2 & -i\xi \\ (\alpha^2 - m_1^2)m_1 & (\alpha^2 - m_2^2)m_2 & 0 \end{vmatrix} = 0 \qquad (3.19)$$

Similarly, after obtaining the corresponding equations for symmetric modes we can write the frequency equation for both symmetric and skew symmetric modes as:

$$\left[\frac{\tan m_1 d}{\tan\beta d}\right]^{\pm 1} - \frac{m_1(\alpha^2 - m_1^2)}{m_2(\alpha^2 - m_2^2)}\left[\frac{\tan m_2 d}{\tan\beta d}\right]^{\pm 1} = \left\{\frac{\beta m_1(m_2^2 - m_1^2)}{\xi^2\left(\alpha^2 - m_2^2\right)}\right\}$$

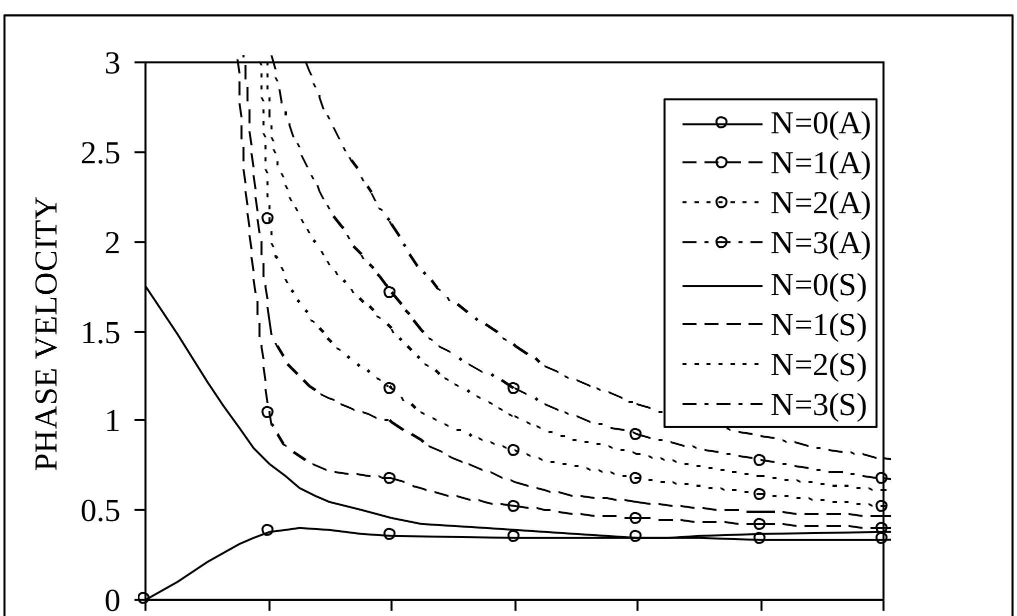
Here the subscript +1 corresponds to skew symmetric and -1 refers to symmetric modes

3.4 Numerical results and discussion

To illustrate the theoretical results obtained in the preceding sections, we now present some numerical results. The material for this purpose is aluminum-epoxy composite, the physical data for which is given as

$$\varepsilon = 0.073,\ \lambda = 7.59\times10^{10} Nm^{-2},\ \mu = 1.89\times10^{-10} Nm^{-2},$$
$$\rho = 2.19\times10^3 kg\ m^{-3},\ \mathrm{K} = 2.508\ \mathrm{J/ms}^{\circ}C,\ \mathrm{C_e} = 961.4\ \mathrm{Jkg^{-1}}/{}^{\circ}C,$$
$$T_0 = 23^{\circ}C,\ t_0 = 6.131\times10^{-13}s,\ t_1 = 8.765s,\ d = 1.0.$$

The phase velocity of symmetric and skew symmetric modes of wave propagation have been computed for various value of wave number from the dispersion relations for stress–free insulated boundary conditions. The corresponding classical and modified dispersion curves for Rayleigh –Lamb types are presented in Fig. 1 in the context of LS theories of Thermoelasticity.



The phase velocity of the lowest skew symmetric mode is observed to increase from zero value at vanishing wave number to become closer the Rayleigh wave velocity at higher values of the wave number, where as in the case of lowest symmetric mode it decrease from a value greater than unity towards the Rayleigh velocity asymptotically with an increase in wave number. The phase velocity of higher modes of propagation , symmetric and skew symmetric, attain quite large values at vanishing wave number, which sharply slash down to become steady with increasing values of wave number.

Chapter 4

Propagation of Rayleigh wave in Thermoelastic solid.

4.1 Introduction

In this chapter the work carried out in chapter 3 is being extended to study the Rayleigh wave propagation in a Thermoelastic half space . Here I have reviewed the work done by Sharma etal. in the absence of microstretch and micropolarity (Effect of micropolarity, microstretch and relaxation times on Rayleigh Surface waves in Thermoelastic Solids , Int J. of Appl. Math and Mech. 5(2):17-38,2009.)

4.2 Formulation of the problem and its solution

Consider a homogeneous, isotropic and elastic half space for Rayleigh wave. We introduce a Cartesian coordinate system whose (x-plane coincides with the surface of the medium, and z-axis is positive downwards) . the z – axis is pointing vertically downward into the half-space. The x- axis is taken along the direction of wave propagation so that all particles on a line parallel to y- axis are equally displaced and hence all the quantities are independent of y- coordinate. The surfaces are assumed to be stress free.

Equations 3.1 to 3.13 are same as obtained and discussed in chapter 3.

Boundary conditions

(a)Mechanical boundary condition

$$\sigma_{33} = u_{3,3} + \left(1 - 2\delta^2\right) u_{1,1} - \left(T + t_1 \delta_{2k} \dot{T}\right) = 0 \tag{4.1}$$

$$\sigma_{13} = u_{1,3} + u_{3,1} = 0 \tag{4.2}$$

(b)Thermal boundary condition

$$\frac{\partial T}{\partial z} + hT = 0$$

Where h is the surface heat transfer coefficient

$h \to 0$ Corresponds to Thermally Insulated Boundaries.

Here, we obtain the solutions

$$\psi = A_4 e^{m_4 z} + B_4 e^{-m_4 z} (e^{i\xi(x-ct)}) \tag{4.3}$$

$$q = \left(A_1 e^{m_1 z} + B_1 e^{-m_1 z}\right) e^{i\xi(x-ct)} + \left(A_2 e^{m_2 z} + B_2 e^{-m_2 z}\right) e^{i\xi(x-ct)}$$

and

$$T = i\tau_1 \omega^{-1} \left\{ \left(\alpha^2 + m_1^{\ 2}\right)\left(A_1 e^{m_1 z} + B_1 e^{-m_1 z}\right) + \left(\alpha^2 + m_2^{\ 2}\right)\left(\left(A_2 e^{m_2 z} + B_2 e^{-m_2 z}\right)\right)\right\} e^{i\xi(x-ct)} = 0 \tag{4.4}$$

Where

$$m_1^{\ 2} = \xi^2 (1 - a_1^{\ 2} c^2)$$
$$m_2^{\ 2} = \xi^2 (1 - a_2^{\ 2} c^2)$$

Since displacement u_1, u_3 can be obtained by using the relations

$$u_1 = \frac{\partial q}{\partial x} + \frac{\partial \psi}{\partial z},$$

$$u_3 = \frac{\partial q}{\partial z} - \frac{\partial \psi}{\partial x}$$

Since we are primarily interested in surface wave propagation along the free surface of half space. So choose the partial wave that satisfied the radial condition. That is all field variables should be bounded for location deep within the solid. Therefore, we choose the solution as

$$\psi = B_4 e^{-m_4 z} (e^{i\xi(x-ct)})$$

$$q = B_1 e^{-m_1 z} e^{i\xi(x-ct)} + B_2 e^{-m_2 z} e^{i\xi(x-ct)}$$

$$and \qquad (4.5)$$

$$T = i\tau_1 \omega^{-1} \left\{ \left(\alpha^2 + m_1^{\;2}\right) B_1 e^{-m_1 z} + \left(\alpha^2 + m_2^{\;2}\right) B_2 e^{-m_2 z} \right\} e^{i\xi(x-ct)}$$

4.3 Secular equations

Invoking the boundary conditions at the surfaces $z = 0$. We obtain a system of six simultaneous linear equations

$$p(-B_1 - B_2) + qB_4 = 0$$

$$-2m_1 i\xi B_1 - 2m_2 i\xi B_2 + (m_4^{\;2} + \xi^2)B_4 = 0$$

$$-\left(\alpha^2 + m_1^{\;2}\right) m_1 B_1 - \left(\alpha^2 + m_2^{\;2}\right) m_2 B_2 e^{-m_2 z} = 0$$

$$p(-A_1 - A_2) - qA_4 = 0$$

$$2m_1 i\xi A_1 + 2m_2 i\xi A_2 + (m_4^{\;2} + \xi^2)A_4$$

$$\left(\alpha^2 + m_1^{\;2}\right) m_1 A_1 + \left(\alpha^2 + m_2^{\;2}\right) m_2 A_2 = 0$$

Where

$$p = \left(1 - 2\delta^2\right)\xi^2 + \alpha^2, \ \text{q}=2\delta^2 i\xi m_4$$

In each case which has nontrivial solution if the determinant of the coefficients of amplitudes $\left[A_1, B_1, A_2, B_2, A_4, B_4\right]^T$ vanishes.

Under stress free conditions, we obtained the matrix as

$$\begin{vmatrix} -((1-2\delta^2)\xi^2 + \alpha^2) & -((1-2\delta^2)\xi^2 + \alpha^2) & 2\delta^2 i\xi m_4 \\ 2m_1 i\xi & 2m_2 i\xi & \xi^2 + m_4^{\;2} \\ (\alpha^2 + m_1^{\;2})m_1 & (\alpha^2 + m_2^{\;2})m_2 & 0 \end{vmatrix} = 0$$

And solving determinant for stress free conditions, we get

$$\xi^2 \left(2 - \frac{c^2}{\delta^2}\right)^2 \left(\alpha^2 + m_1^{\;2} + m_2^{\;2} + m_1 m_2\right) + 4m_1 m_2 m_4 (m_1 + m_2) = 0 \qquad (4.6)$$

Now put the value of m_1, m_2, m_4

if we take

$\alpha_1^2 = 1 - a_1^2 c^2$

and

$\alpha_2^2 = 1 - a_2^2 c^2$

and

$m_4^2 = \xi^2 \alpha_4^2$

we get,

$$\left(2 - \frac{c^2}{\delta^2}\right)^2 \left(\alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2 + c^2 - 1\right) + 4\alpha_1\alpha_2\alpha_4\left(\alpha_1 + \alpha_2\right) = 0$$

This equation is the secular equations for Thermoelastic Rayleigh waves and is same as discussed in detail by many authors such as Sharma etal[2000], Lockett[1958], Chadwick and Windle[1964], Atkin and Chadwick[1981] , Nayfeh and Nasser[1971].

4.4 Discussion

Variations of phase velocity with respect to wave number in the case of elasticity have been studied in Figure 2.

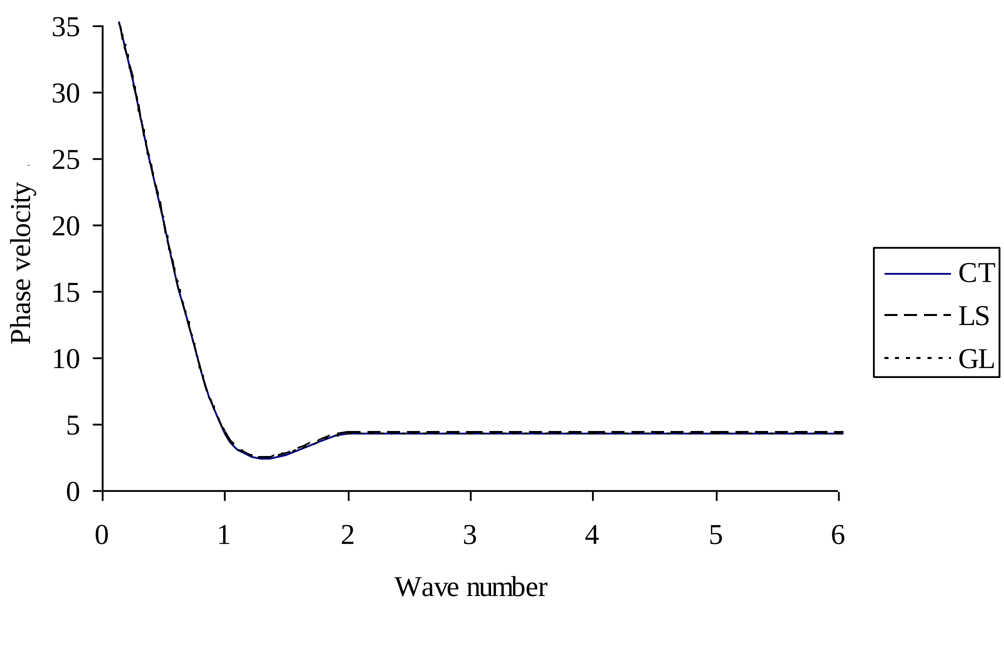


FIG 2: Variation of phase velocity with wave number

It is observed that the phase velocity of Rayleigh waves in half space decreases sharply from its peak values in the wave number range 0 to 1, increases moderately in the wave number range 1 to 2 and becomes dispersion less for wave number range greater than 2. The phase velocity profiles are significantly dispersive at small values of wave number and become dispersion less with increasing wave number.

Conclusion

A Thermoelastic model for studying Rayleigh and Rayleigh Lamb type waves has been presented for insulated boundary conditions. Secular equations are obtained for both Rayleigh and Rayleigh Lamb waves and variations of phase velocity with the wave number have been explained graphically also. Rayleigh waves are widely used for material characterization, to discover the mechanical and structural properties of the objects. Guided waves like Rayleigh and Lamb wave have great potential for non destructive evaluation. In the present work, modified secular equations have been derived considering Thermoelastic conditions, application of our model may be useful for further applications of Rayleigh and Lamb waves.

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