STATIC ANALYSIS OF LAMINATED CIRCULAR PLATE USING DIFFERENTIAL QUADRATURE METHOD

A Thesis

Submitted in partial fulfillment of the requirements for the award of degree

of

MASTER OF ENGINEERING

IN

CAD/CAM & ROBOTICS ENGINEERING

Submitted By

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June 2005

CERTIFICATE

This is to certify that the Thesis titled, "**Static Analysis of Laminated Circular Plate using Differential Quadrature Method",** submitted by **Ms. Shifali Bansal,** in partial fulfillment of the requirements for the award of degree of **Master of Engineering (CAD/CAM & Robotics Engineering) at Thapar Institute of Engineering and Technology (Deemed University), Patiala** is a bonafide work carried out by her under our guidance and supervision and that no part of this thesis has been submitted for the award of any other degree.

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ACKNOWLEDGEMENT

I express my deep sense of respect and sincere gratitude to my guide Mr. Ashish Sharma, Lecturer, Mechanical Engineering Department, Thapar Institute of Engineering and Technology, Patiala, for his individual guidance, sympathetic treatment and much needed encouragement while doing my thesis work.

I am also thankful to our PG coordinator Dr. Vijay Jadon, Asst. Prof, Mechanical Engineering Department, Thapar Institute of Engineering and Technology, Patiala for his moral support during my thesis work. I do not find enough words with which I can express my feeling of thanks to entire faculty and staff of M.E.D., T.I.E.T., Patiala, for their help and inspiration which went a long way in successful completion of my thesis. I thank all those who have contributed directly or indirectly to my thesis.

I am also thankful to the authors whose works I have consulted and quoted in this work. Last, but not the least, very special thanks to all my friends, who devoted their valuable time and helped me in all possible ways.

(Shifali Bansal)

ABSTRACT

The aim of this study is the static analysis of laminated circular plate using differential quadrature method. The governing equations for cylindrical coordinates have been developed using first order shear deformation theory and solved using differential quadrature method. The axisymmetric case is considered in the present study. Clamped and simply supported boundary conditions have been considered in the present study.

The convergence study has been carried out. The results obtained are verified with the available results from literature for the isotropic plate. Parametric study ahs been done to analyze the behavior of laminate with different radius to thickness ratio and lamination schemes.

NOMENCLATURE

(u_0, v_0, w_0)	displacements in (x, y, z) direction at midplane
(u, v, w)	displacements at any point in (x, y, z) direction
(u^0, v^0, w^0)	radial, circumferential and axial displacements in (r, θ, z) direction
ψ	rotation of the normal
(ϕ_x, ϕ_y)	rotations of a transverse normal about y- and x-axis.
a	radius of the plate
h	thickness of the plate
(N_x, N_y)	normal force per unit length in x and y direction
N_{xy}	shear force per unit length
(M_x, M_y)	bending moments per unit length in x and y direction
Q_{ij}	plane stress reduced stiffnesses
E	Young’s modulus
υ	Poisson’s ratio
G	Shear modulus
A	extensional stiffness
D	bending stress
B	bending extensional coupling stiffness
N	number of grid points
□	shear correction factor
[S]	matrix of elastic compliance
$C_{ij}^{(n)}$	weighting coefficients associated with nth-order derivative of *f(r)* with respect to *r* at discrete point *ri*
R	radial orientation
C	circumferential orientation

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CHPATER 1
INTRODUCTION

Composite materials are formed by combining two or more materials on a macroscopic scale which together produce better engineering properties that cannot be achieved with any of the constituents alone[1]. Some of the properties that can be improved by forming a composite material are stiffness, strength, weight reduction, corrosion resistance, thermal properties, fatigue life and wear resistance. Most man made composite materials are made from two materials: a reinforcement material called fiber and a base material, called matrix material.

Composite materials are commonly of three different types:

(1) Fibrous composite, which consist of fibers of one material in a matrix material of another.
(2) Particulate composites, which are composed of macro size particles of one material in a matrix of another.
(3) Laminated composites, which are made of layers of different materials, including composites of the first two types.

A lamina or ply is a single flat layer of unidirectional fibers or woven fibers arranged in a matrix. A laminate is a collection of laminae stacked in predetermined directions and thicknesses to achieve the desired stiffness and strength properties.

NEED OF LAMINATES [2]

- The lamina thickness is of the order of 0.125 mm which implies that several laminae will be required to take the realistic loads.
- The mechanical properties of unidirectional laminae are severely limited in the transverse direction. Therefore stacking of several unidirectional laminae makes them optimum for unidirectional loads.

A laminate may have any number of layers and the fiber orientation changes from layer to layer in a regular manner through the thickness of the laminate. Each layer can be identified by its location in the laminate, its material, and its angle of orientation with a reference axis as shown in Figure 1.1.

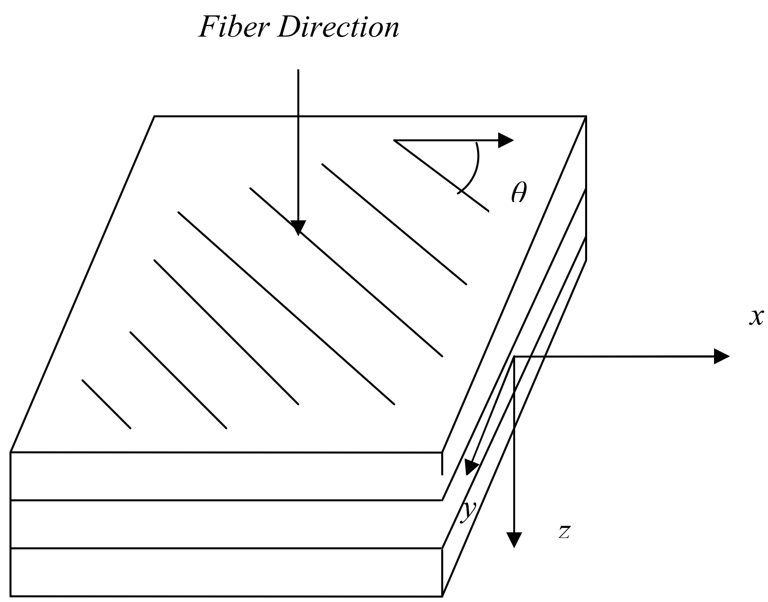

Figure 1.1: Schematic of a laminate [2]

Fiber orientation in each lamina and stacking sequence of the layers can be chosen to achieve desired strength and stiffness for a specific application. For complex loading and stiffness requirements, the layers of laminate are stacked at different angles. Knowing the mechanics of a single lamina, one can develop the mechanics of a laminate. Structural elements, such as bars, beams or plates are also formed by stacking the layers to achieve desired strength and stiffness. Laminated plates are widely used as machine elements and structural components in civil, marine, aeronautical and mechanical engineering applications. The skins of aeroplane wings and tails, the hull sides and decking of ships, the sides and bottom of water tanks are typical examples. Cylindrical components, such as filament wound tanks, can be treated as laminates, provided the radius to thickness ratio is sufficiently large(>50).

Several steps are involved in the analysis of structural elements made of laminated composite materials [2]. The analysis requires knowledge of anisotropic elasticity, structural theories of laminates and analytical or computational methods to determine solutions of the governing equations and failure theories to predict mode of failures and to determine failure loads.

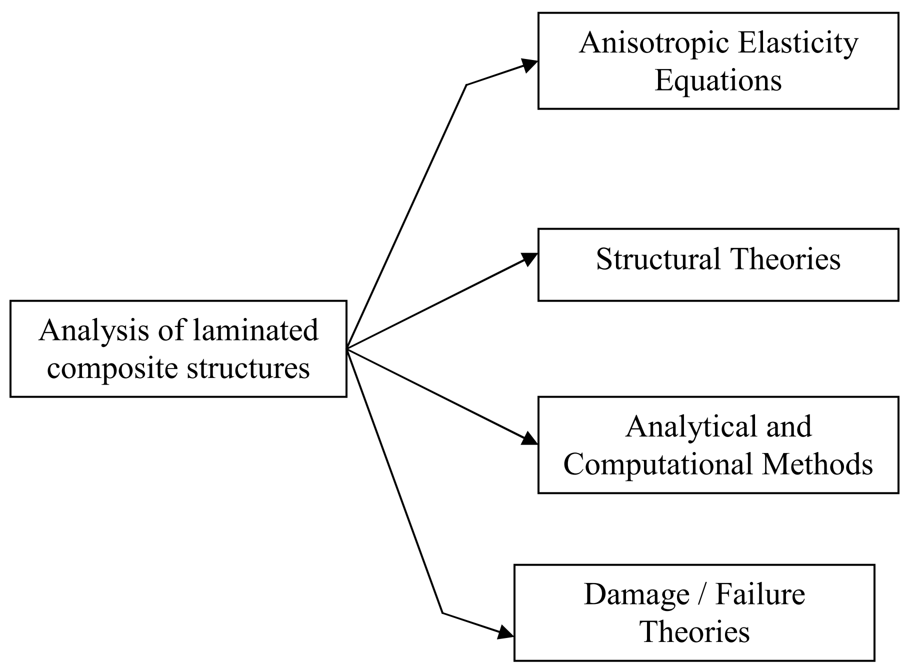

Figure 1.2: Basic steps in the analysis of laminated structures [1]

Laminated composite structures also possess some disadvantages. The shear stress is produced between the layers, especially at the edges of laminates due to the mismatch of material properties between layers, which causes delamination. Material defects such as interlaminar voids, delamination, incorrect orientation, damaged fibers, and variation in thickness may be introduced during the manufacturing of laminates.

Several theories are available for the analysis of laminates i.e. classical laminated plate theory [5], first order laminated plate theory [19] and higher order theory [6]. These theories differ in the terms of assumed displacement field and computational effort required. The first order laminated plate theory extends the kinematics of classical laminated plate theory by including effects of transverse shear strain which is neglected in case of classical laminated plate theory. The third order theory provides a slight increase in accuracy relative to first order theory but there is increase in computational effort required. Most of the research in this field is based on rectangular coordinates. The research in the field of cylindrical coordinates is limited.

Various methods are available for solving governing equations obtained from the above mentioned methods. Traditionally, there are three numerical methods i.e. the finite differences, the finite elements [3] and the boundary elements that are used to solve linear and nonlinear differential equations. Differential quadrature method is a relatively new method. It is efficient method for solving initial and boundary value problems and thus can serve as alternative to the existing methods such as the finite elements or finite differences.

In the present work, the static analysis of the laminated circular plate is done. The plates are assumed to be made up of cylindrically orthotropic layers. The laminate stiffnesses have been calculated and further used to calculate various parameters of the laminated circular plate. The governing equations are solved using Differential Quadrature method which is explained in Chapter 3. On the basis of results obtained, convergence study and parametric study is done. The results for isotropic plate have been verified.

CHPATER 2
MECHANICS OF LAMINATED PLATES

2.1 INTRODUCTION

Laminates are used in applications that require axial and bending strengths. Therefore composite laminates are treated as plate elements. The objective of this chapter is to present the most commonly used laminate plate theories, namely [1]:

- Classical laminated plate theory
- First order shear deformation theory (FSDT)
- Third order plate theory

These theories differ in terms of their assumed displacement fields. In formulating the various plate theories following assumptions are made [1]:

(1) The layers are perfectly bonded together.
(2) The material of each layer has two planes of material symmetry (orthotropic).
(3) Each layer has uniform thickness.
(4) The strains and displacements are small.

Consider a plate of total thickness *h* composed of N orthotropic layers with the principal material coordinates $\left(x_1^k, x_2^k, x_3^k\right)$ of the kth lamina oriented at an angle θ_k to the laminate coordinate *x*. The *xy*- plane is taken in the undeformed midplane of the laminate. The *z*-axis is taken positive upward from the midplane. The k^{th} layer is located between the points $z = z_k$ and $z = z_{k+1}$ in the thickness direction as shown in Figure 2.1. The boundary of laminate consists of top surface at $z = h/2$ and bottom surface at $z = -h/2$.

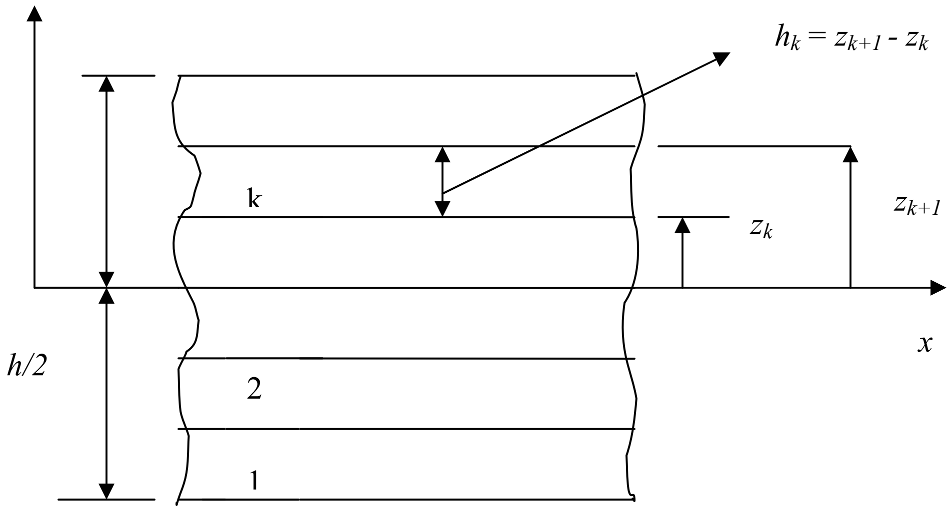

Figure2.1: Coordinate system and layer numbering for a typical laminated plate

Now assume that (u_0, v_0, w_0) to be displacements in the x, y and z directions, respectively and (u, *v, w)* are the displacements at any point in the x, y, and z directions and $\alpha = \left(\frac{\partial w_0}{\partial x}\right)$. At any point other than the midplane, the two displacements in the x-y plane will depend on the axial location of the point and the slope of laminate midplane with the x and y direction as shown in Figure 2.2.

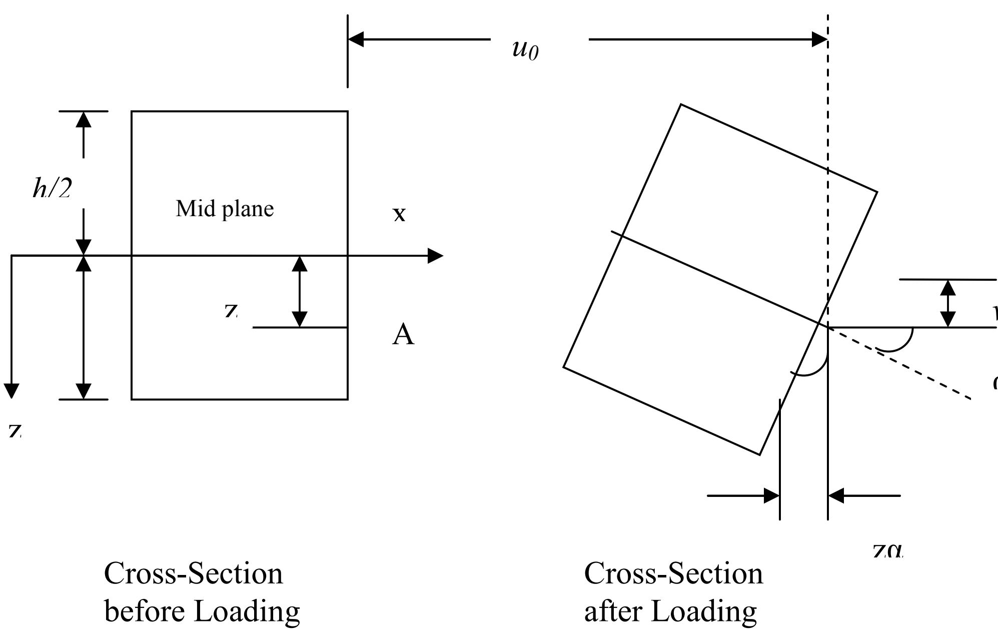

Figure 2.2: Relationship between displacements through the thickness of a plate to midplane displacements [2]

If (N_x, N_y) are normal forces per unit length, N_{xy} is the shear force per unit length and (M_x, M_y) are bending moments per unit length, then these can be shown on a laminate as in Figure 2.3 and Figure 2.4.

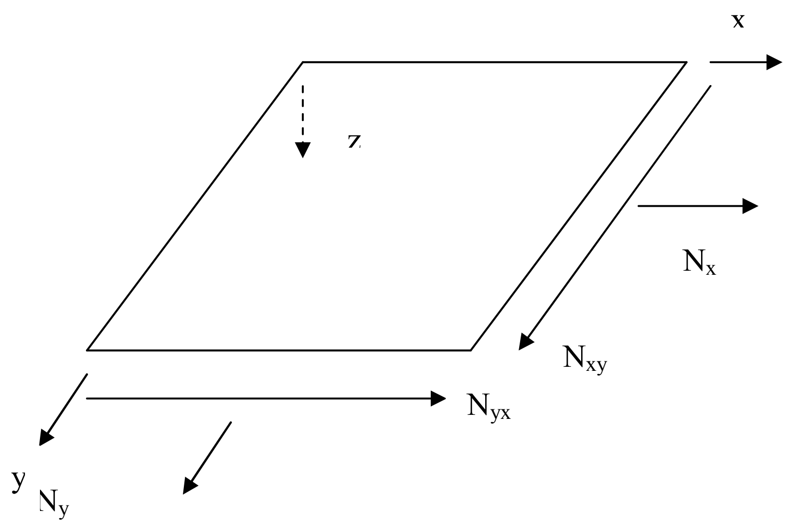

Figure 2.3: Resultant forces on a laminate [2]

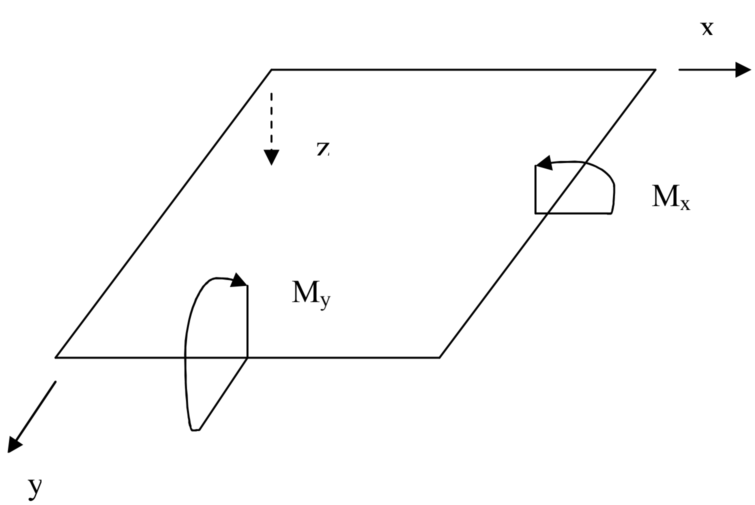

Figure2.4: Resultant moments on a laminate [2]

2.2 LAMINA CONSTITUTIVE RELATIONS

The linear constitutive relations for the *k*th orthotropic lamina in the principal material coordinates of lamina are [1],

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2.1}$$

Where $Q_{ij}^{(k)}$ are the plane stress reduced stiffnesses and are known in terms of the engineering constants of the *k*th layer [1],

$$Q_{11} = \frac{E_x}{1-\upsilon_{xy}\upsilon_{yx}}, \qquad Q_{12} = \frac{\upsilon_{xy}E_y}{1-\upsilon_{xy}\upsilon_{yx}} = \frac{\upsilon_{yx}E_x}{1-\upsilon_{xy}\upsilon_{yx}}$$

$$Q_{22} = \frac{E_y}{1-\upsilon_{xy}\upsilon_{yx}}, \qquad \mathrm{Q}_{66} = G_{xy} \tag{2.2}$$

Since the laminate is made of several orthotropic layers, with their material axes oriented arbitrarily with respect to the laminate coordinates, therefore constitutive equation of each layer is transformed to the laminate coordinates. The stress-strain relation when transformed to the laminate coordinates is given by [1],

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2.3}$$

Where

$$\overline{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$
$$\overline{Q}_{12} = (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta+\cos^4\theta)$$
$$\overline{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta$$
$$\overline{Q}_{16} = (Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta + (Q_{12}-Q_{22}-2Q_{66})\sin^3\theta\cos\theta$$
$$\overline{Q}_{26} = (Q_{11}-Q_{12}-2Q_{66})\sin^3\theta\cos\theta + (Q_{12}-Q_{22}-2Q_{66})\sin\theta\cos^3\theta$$
$$\overline{Q}_{66} = (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta+\cos^4\theta)$$

Here θ is the angle measured anticlockwise from the x-coordinate to the $x1$-coordinate.

2.3 THE CLASSICAL LAMINATED PLATE THEORY

The classical laminate plate theory is the extension of classical plate theory to composite laminates. In classical laminated plate theory, it is assumed that straight lines perpendicular to the midsurface before deformation remain straight after deformation. The transverse normals rotate such that they remain perpendicular to the midsurface after deformation. Undeformed and deformed geometries of the edge of the plate are shown in Figure 2.5.

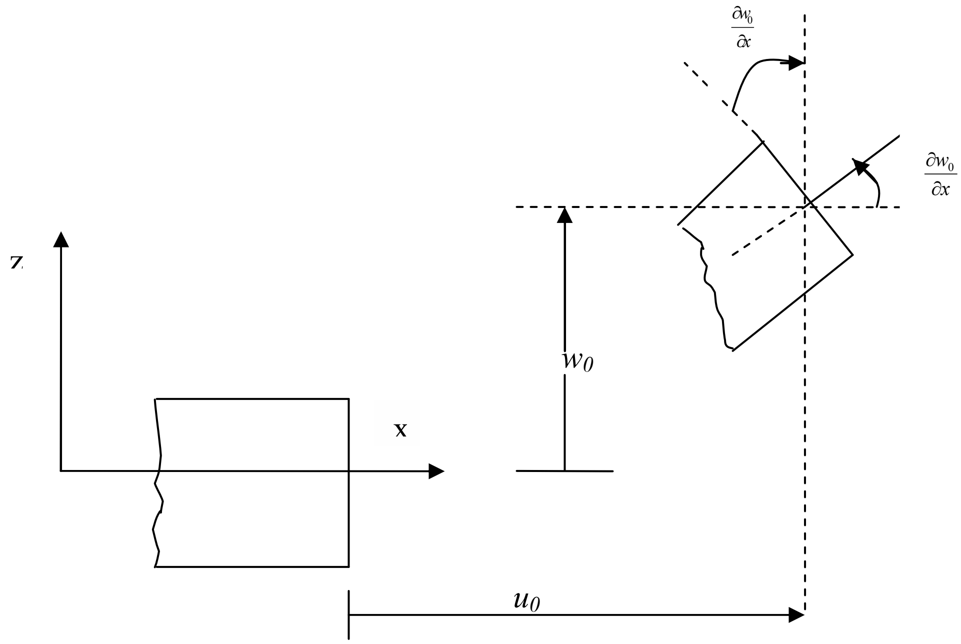

Figure2.5: Undeformed and deformed geometries of an edge of plate under classical laminated plate theory [1]

2.3.1 DISPLACEMENTS AND STRAINS

The displacement field of classical laminated plate theory is given by [1],

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) - z\frac{\partial w_0}{\partial x} \\
v(x,y,z) &= v_0(x,y) - z\frac{\partial w_0}{\partial y} \\
w(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{2.4}
$$

Where (u_0, v_0, w_0) denote the displacements of a point on the plane z = 0.

The strains are given by [1],

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} \\
\varepsilon_y &= \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} \\
\gamma_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - 2z\frac{\partial^2 w_0}{\partial x \partial y}, \qquad \varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0.
\end{aligned}
\tag{2.5}
$$

Once the displacements (u_0, v_0, w_0) of the midplane are known, strains at any point (x, y, z) in the plate can be calculated. All strains components vary linearly through the laminate thickness, and they are independent of the material variations through the laminate thickness. Lamina constitutive relations for classical laminated plate theory are given by equation (2.3).

If (N_x, N_y) are normal forces per unit length, (M_x, M_y) are bending moments per unit length, Q_x,Q_y are the shear forces per unit length, then for the midplane, they are defined as,

$$
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz, \tag{2.6a}
$$

$$
\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} z \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz, \tag{2.6b}
$$

Using equations (2.1), (2.5), (2.6a) and (2.6b), the force and moment resultants are given by,

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^1 \\ \varepsilon_y^1 \\ \gamma_{xy}^1 \end{bmatrix} \quad (2.7a)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^1 \\ \varepsilon_y^1 \\ \gamma_{xy}^1 \end{bmatrix} \quad (2.7b)$$

Where

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{bmatrix}, \qquad \begin{bmatrix} \varepsilon_x^1 \\ \varepsilon_y^1 \\ \gamma_{xy}^1 \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix} \quad (2.8)$$

$$\begin{bmatrix} A_{ij} & B_{ij} & D_{ij} \end{bmatrix} = \int_{-h/2}^{h/2} Q_{ij} \begin{bmatrix} 1 & z & z^2 \end{bmatrix} dz, \quad (2.9)$$

Where A_{ij} are called extensional stiffnesses, D_{ij} the bending stresses, and B_{ij} are bending extensional coupling stiffnesses.

Following associations of A, B and D are made [2]:

(1) A_{16} and A_{26} relate in-plane direct forces to in-plane shear strain, or in-plane shear force to in-plane direct strains.

(2) B_{11}, B_{12} and B_{22} relate in-plane direct forces to plate curvatures, or bending moments to in-plane direct strains.

(3) B_{16} and B_{26} relate in-plane direct forces to plate twisting, or torque to in-plane direct strains.

(4) B_{66} relates in-plane shear force to plate twisting, or torque to in-plane shear strain.

(5) D_{16} and D_{26} relate bending moments to plate twisting, or torque to plate curvatures.

Therefore if $A_{16} = A_{26} = 0$ then there will be no coupling between direct stresses and shear strains. The examination of stiffnesses reveals that their values depend on the material stiffnesses, layer thicknesses, and the lamination scheme. Symmetry or antisymmetry of the lamination scheme and material properties about the midplane of the laminate reduce some of the laminate stiffness to zero. When ply stacking sequence, material, and geometry (i.e., ply thicknesses) are symmetric about the midplane of the laminate, the laminate is called a symmetric laminate. For a symmetric laminate, the upper half through the laminate thickness is a mirror image of the lower half. The laminate (0/90/90/0), with all layers having the same thickness and material is example of a symmetric cross ply laminate. For layers with 0 or 90 orientations, the layer stiffnesses $\overline{Q}_{16}, \overline{Q}_{26}, \overline{Q}_{45}$ are zero. Hence, $A_{16} = A_{26} = A_{45} = D_{16} = D_{26} = 0$.

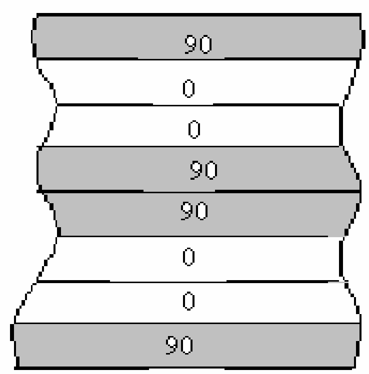

Figure 2.6: Symmetrical laminate

An unsymmetrical or asymmetric laminate is a laminate that is not symmetric. An antisymmetric laminate is one whose lamination scheme is antisymmetric and material and thicknesses are symmetric about the midplane. The example of antisymmetric cross ply laminate is (0/90/0/90/0/90).

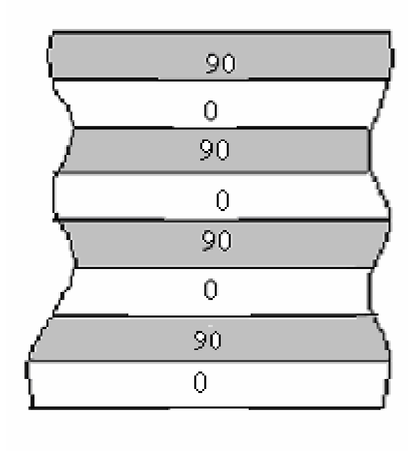

Figure 2.7: Unsymmetrical laminate

Laminate stiffnesses A_{ij}, B_{ij}, and D_{ij} can be calculated for various lamination schemes. Laminate stiffnesses A_{ij} depend only on the thicknesses and the stiffnesses of the layers but on their placement through the thickness. On the other hand, laminate stiffnesses D_{ij} depend not only on the layer thickness and stiffnesses but also on their location relative to the midplane. For example, both $(0/90)_s$ and $(90/0)_s$ laminates will have same in plane stiffnesses A_{ij}. But $(0/90)_s$ laminate will have larger bending stiffnesses D_{ij} about an axis perpendicular to the fiber direction than the $(90/0)_s$ laminate. Both A_{ij} and D_{ij} are always positive. Laminate stiffnesses B_{ij} can be negative, depending on the lamination scheme and on the number of layers. Bending membrane coupling can be avoided if the B matrix is zero. This is achieved by making the laminate symmetric about its midplane. The bending twisting coupling is eliminated if $D_{16} = D_{26} = 0$. This is achieved with unidirectional or cross ply laminates i.e. for every layer at $+\theta$ orientation and a given distance above the midplane there is a layer with identical thickness and properties oriented at $-\theta$ and the same distance below the midplane. For symmetrical laminates D_{16} and D_{26} is not zero but these terms tend to zero for thick multilayer symmetric laminates. The laminate stiffnesses A_{ij}, B_{ij}, and D_{ij} for any arbitrary laminate can be calculated by using equation (2.9).

2.4 FIRST ORDER LAMINATED PLATE THEORY

In FSDT, the transverse normal do not remain perpendicular to the midsurface after deformation as shown in Figure 2.8. Therefore transverse shear strains are also included in FSDT.

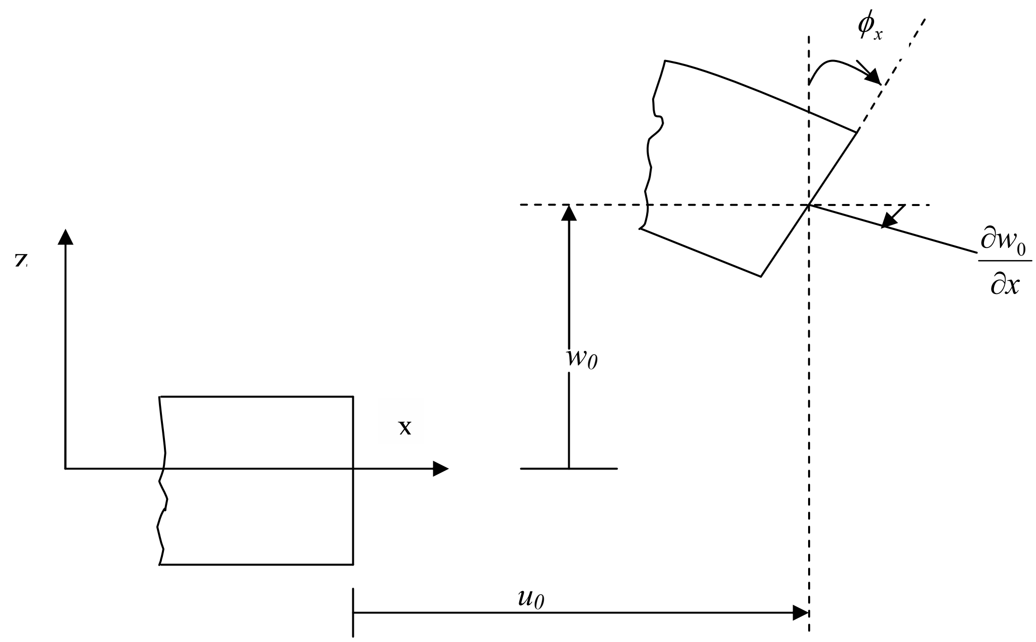

Figure 2.8: Undeformed and deformed geometries of an edge of plate under first order laminated plate theory [1]

2.4.1 DISPLACEMENTS AND STRAINS

The displacement field of the first order laminate plate theory is of the form [1]:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\phi_x(x,y) \\ v(x,y,z) &= v_0(x,y) + z\phi_y(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{2.10}$$

Where (u_0, v_0, w_0) denote the displacements of a point on the plane z = 0. ϕ_x and ϕ_y are the rotations of a transverse normal about y- and x- axis respectively given by,

$$\frac{\partial u}{\partial z} = \phi_x, \qquad \frac{\partial v}{\partial z} = \phi_y \tag{2.11}$$

The strains associated with the displacement field (2.10) are given by [1],

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + z\frac{\partial \phi_x}{\partial x}, \qquad \varepsilon_y = \frac{\partial v_0}{\partial y} + z\frac{\partial \phi_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right), \qquad \gamma_{xz} = \frac{\partial w_0}{\partial x} + \phi_x, \qquad \gamma_{yz} = \frac{\partial w_0}{\partial y} + \phi_y, \tag{2.12}$$

The strains $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$ are linear through the laminate thickness, while the transverse shear strain $(\gamma_{xz}, \gamma_{yz})$ are constant in first order laminated plate theory.

For laminated plates, the stress and strains are related by,

$$\varepsilon = S\sigma \tag{2.13}$$

S is matrix of elastic compliance. The lamina constitutive relations are given by equation (2.1) and following equation,

$$\begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} Q_{44}\gamma_{yz} \\ Q_{55}\gamma_{zx} \end{bmatrix} \tag{2.14}$$

$$Q_{44} = G_{yz}, \qquad Q_{55} = G_{zx} \tag{2.15}$$

Where G_{ij} are shear modulli.

The equations for equilibrium for the rectangular laminated plate are given by,

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \qquad \frac{\partial N_{xy}}{\partial y} + \frac{\partial N_y}{\partial y} = 0, \qquad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \qquad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{2.16}$$

Where N_x, N_y are in-plane force resultants, M_x, M_y are in-plane moment resultants, Q_x, Q_y are the shear forces, p_z is force.

For the midplane, the force resultants are given by equations (2.7a) and (2.7b) and following equation [1],

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \kappa \int_{-h/2}^{h/2} \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} \mathrm{dz} \tag{2.17}$$

Where κ is shear correction factor.

Equation (2.17) when solved results in,

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \begin{bmatrix} A_{55}\left(\dfrac{dw^0}{dx} + \phi_x\right) \\ A_{44}\left(\dfrac{dw^0}{dy} + \phi_y\right) \end{bmatrix}, \tag{2.18}$$

For first order laminated plate theory [1],

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \gamma_{yz}^0 \\ \gamma_{zx}^0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \\ \dfrac{\partial w_0}{\partial y} + \phi_y \\ \dfrac{\partial w_0}{\partial x} + \phi_x \end{bmatrix}, \qquad \begin{bmatrix} \varepsilon_x^1 \\ \varepsilon_y^1 \\ \gamma_{xy}^1 \\ \gamma_{yz}^1 \\ \gamma_{zx}^1 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \\ 0 \\ 0 \end{bmatrix} \tag{2.19}$$

2.5 THIRD ORDER LAMINATE PLATE THEORY

Third order laminate plate theory provides increase in accuracy than the first order laminate plate theory but the computational effort in case of third order laminate plate theory is more.

2.5.1 DISPALCEMENT AND STRAINS

The displacement field of the third order laminate plate theory is of the form [1]:

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) + z\phi_x(x,y) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \\
v(x,y,z) &= v_0(x,y) + z\phi_y(x,y) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \qquad (2.20) \\
w(x,y,z) &= w_0(x,y)
\end{aligned}
$$

Where (u_0, v_0, w_0) denote the displacements of a point on the plane z = 0. ϕ_x and ϕ_y are the rotations of a transverse normal about y- and x- axis respectively and *h* being the total thickness of plate. The transverse shear stresses on the top and bottom of a general laminate vanishes and hence there is no need of shear correction factor in case of third order laminate plate theory.

The strains are given by [1]:

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u_0}{\partial x} + z\frac{\partial \phi_x}{\partial x} + z^3\left(-\frac{4}{3h^2}\right)\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) \\
\varepsilon_y &= \frac{\partial v_0}{\partial y} + z\frac{\partial \phi_y}{\partial y} + z^3\left(-\frac{4}{3h^2}\right)\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right) \\
\gamma_{xy} &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + z^3\left(-\frac{4}{3h^2}\right)\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \qquad (2.21) \\
\gamma_{xz} &= \frac{\partial w_0}{\partial x} + \phi_x + z^2\left(-\frac{4}{3h^2}\right)\left(\frac{\partial w_0}{\partial x} + \phi x\right), \\
\gamma_{yz} &= \frac{\partial w_0}{\partial y} + \phi_y + z^2\left(-\frac{4}{3h^2}\right)\left(\frac{\partial w_0}{\partial y} + \phi_y\right),
\end{aligned}
$$

CHPATER 3
DIFFERENTIAL QUADRATURE METHOD

3.1 INTRODUCTION

Bellman and Casti in the early 1970s originated Differential Quadrature method. The Differential Quadrature method is a distinct numerical solution technique for the initial and boundary value problems of physical and engineering sciences. The main difference between initial value problem and boundary value problem is that the known boundary conditions are all given at the beginning of the domain and the results at the other end of the time interval are unknown and are to be determined. The DQ method is first employed to convert the differential equations into a set of linear algebraic equations. Then by solving the algebraic equations, solutions to the problem are obtained. The essence of the method relies on the idea that a derivative of a function with respect to variable at any discrete point can be approximated as a weighted linear sum of the function values at all the discrete points chosen in the overall domain of that variable. The key point in the method is to determine the weighted coefficients for the partial derivative approximation.

3.2 DIFFRENTIAL QUADRATURE PROCEDURE

The differential Quadrature procedure for circular domain is presented here.

Suppose that there are N grid points in the r-direction with r_1, r_2, r_3 . . . r_N as the coordinates. The nth-order derivative of the *f*(r) with respect to r is given discretely at the point r_i as

$$f_r^{(n)}(r_i) = \sum_{k=1}^{N} C_{ik}^{(n)} f(r_k); \qquad n = 1,2,3...............N-1$$

Where $C_{ij}^{(n)}$ = weighting coefficients associated with nth-order derivative of *f*(r) with respect to r at discrete point r_i, which can be calculated as follows:

$$C_{ij}^{(1)} = \frac{M^{(1)}(r_i)}{(r_i - r_j)M^{(1)}(r_j)}; \qquad i, j = 1,2,3................N, but\ j \neq i$$

where

$$M^{(1)}(r_i) = \prod_{j=1}^{N}(r_i - r_j) \qquad j \neq i$$

And

$$C_{ij}^{(n)} = n\left[C_{ii}^{(n-1)}C_{ij}^{(n)} - \frac{C_{ij}^{(n-1)}}{r_i - r_j}\right]; \qquad for\ i, j = 1,2,3..........N,$$

$$but \quad j \neq i; \qquad and\ n = 2,3,...................,N-1$$

$$C_{ii}^{(n)} = -\sum_{j=1, j\neq i}^{N} C_{ij}^{(n)}; \qquad i = 1,2,............,N, and\ n = 1,2,.........,N-1$$

Once the functional values at all the grid points are calculated, the value at any point could be readily obtained.

3.3 CHOICE OF GRID POINTS

A decisive factor for the accuracy of the Differential Quadrature solution is the choice of the sampling or grid points. Three different types for sampling grid points are given. A natural and convenient, choice for sampling points is that often equally spaced points. These points are given by

Type 1 $\qquad r_i = \frac{i-1}{N-1}; \qquad i = 1,2,........,N$

Sometimes, the Differential Quadrature solutions deliver more accurate results with unequally spaced sampling point given by

Type 2 $\qquad r_i = \frac{1}{2}\left[1 - \cos\left(\frac{2i-1}{N-1}\right)\pi\right]; \qquad i = 1,2,.........N$

Equally spaced sampling points with adjacent-δ points method employs points at a very small distance (δ = 0:00001) from the boundary points. The sampling point are given by,

Type 3

$$r_1 = 0 \qquad r_2 = \delta \qquad r_{N-1} = 1-\delta$$
$$r_N = 0 \qquad \textit{for adjacent points}$$
$$r_i = \frac{i-1}{N-3}; \qquad i = 3,4,\dots\dots\dots,(N-2)$$

3.4 LITERATURE REVIEW

C. W. Bert, S. K. Jang and A. G. Striz [7] were the first to apply DQM to the structural problems. DQM was applied to various bar, beam, membrane and plate vibration problems in this paper. A differential quadrature approximation for *i*th discreet point is given by,

$$L\{f(x)\}_i \cong \sum_{j=1}^{N} A_{ij} f(x_j)$$

Where

L is linear differential operator applied to a function *f*(x).

x_i are the discrete points in the variable domain and $f\,(x_j)$ are function value at these points.

A_{ij} are weighting coefficients attached to these function values given by,

$$\sum_{j=1}^{N} A_{ij} x_j^{k-1} = (k-1)(k-2)\dots(k-n)x_i^{(k-n-1)} \qquad i,k = 1,2,\dots\dots,N$$

Where N is the number of grid points.

It is concluded that DQM requires less computational effort than the finite element and finite differences method to achieve comparable accuracy.

J.B. Han and K.M. Liew [8] applied the DQM method to analyze the axisymmetric bending behavior of moderately thick circular plate. Linear shear deformation plate theory proposed by Reissner and Mindlin is used to model the plate. Reissner-Mindlin circular plate of radius *a* and thickness *h* as shown in Figure 3.1 is considered.

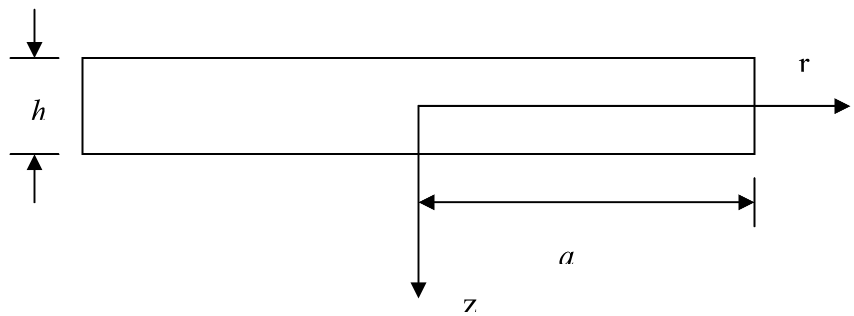

Figure 3.1: Dimensions of thick circular plate

The equilibrium equations for the plate for axisymmetric bending is given by,

$$D\left(\frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr}-\frac{\psi}{r^2}\right)-\kappa Gh\left(\frac{dw}{dr}+\psi\right)=0$$
$$\kappa Gh\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}+\frac{d\psi}{dr}+\frac{\psi}{r}\right)+q=0 \qquad (3.1)$$

where

w = transverse deflection;

ψ = rotation of normal about r-axis;

$D=\frac{Eh^3}{\left[12\left(1-\upsilon^2\right)\right]}$ = plate flexural rigidity;

E, G, and υ = Young's modulus, shear modulus, and Poisson's ratio respectively;

q = shear load intensity;

□ = shear correction factor taken to be 5/6.

Non dimensional parameters used to normalize the equations are given by,

$$R=\frac{r}{a};\qquad W=\frac{w}{a};\qquad \delta=\frac{h}{a};\qquad \psi=\psi \qquad (3.2)$$

The governing equation (3.1) is normalized as,

$$\delta^2\left(R^2\frac{d^2\psi}{dR^2}+R\frac{d\psi}{dR}-\psi\right)-6\kappa\left(1-\upsilon\right)R^2\left(\frac{dW}{dR}+\psi\right)=0 \qquad (3.3a)$$

$$R^2\frac{d^2W}{dR^2}+\frac{dW}{dR}+R\frac{d\psi}{dR}+\psi=-\frac{R}{\kappa G\delta}q \qquad (3.3b)$$

After applying DQM and setting R_1=0 and R_N = 1, the equations (3.3a) and (3.3b) results in following form,

$$\delta^2 \sum_{k=1}^{N} \left[C_{ik}^{(2)} R_i^2 + C_{ik}^{(1)} R_i \right] \psi_k - \left[\delta^2 + 6\kappa(1-\nu) R_i^2 \right] \psi_i - 6\kappa(1-\nu) R_i^2 \sum_{k=1}^{N} C_{ik}^{(1)} W_k = 0 \qquad (3.4a)$$

$$\sum_{k=1}^{N} \left[C_{ik}^{(2)} R_i + C_{ik}^{(1)} \right] W_k + \psi_i + R_i \sum_{k=1}^{N} C_{ik}^{(1)} \psi_k = -\frac{R_i}{\kappa G \delta} q_i \qquad (3.4b)$$

Where N= number of grid points, i = 1, 2…….N; $C_{rs}^{(n)}$ = weighting coefficients for nth order derivatives of W and ψ with respect to R.

The algebraic equations are solved to get the solutions to the problem. In this paper three types of plates are considered:

(1) A clamped uniformly loaded plate
(2) A simply supported plate subject to different loading conditions
(3) A clamped uniformly loaded plate with a central rigid support.

DQ results are compared with exact results to check the accuracy of the method. It is concluded that the convergence characteristics and accuracy of the method depend strongly on the system of governing equations and boundary equations of a problem.

K. M. Liew, J. B. Han and Z. M. Xiao [9] analyzed the free vibrations of moderately thick Mindlin circular plates with free, simply supported and clamped edges. The first fifteen natural frequencies of the plate were calculated. Circular plate of radius a and thickness h was considered. The normalized governing equations were given by,

$$\delta^2 \left(R^2 \frac{d^2\psi}{dR^2} + R\frac{d\psi}{dR} - \psi \right) - 6\kappa(1-\upsilon) R^2 \left(\frac{dW}{dR} + \psi \right) - \delta^2 R^2 \Omega^2 \psi = 0\,, \qquad (3.5a)$$

$$R^2 \frac{d^2 W}{dR^2} + \frac{dW}{dR} + R\frac{d\psi}{dR} + \psi + \frac{2R\Omega^2}{(1-\upsilon)\kappa} \mathrm{W} = 0, \qquad (3.5b)$$

where $R = \frac{r}{a}$; $\delta = \frac{h}{a}$; $W = \frac{w}{a}$; $T = t\sqrt{\frac{E}{\rho a^2 (1-\upsilon^2)}}$ (3.5c)

w = transverse deflection;

ψ = rotation of normal about the r-axis;

$D = \frac{Eh^3}{\left[12\left(1-\upsilon^2\right)\right]}$ = plate flexural rigidity;

E, G, and υ = Young's modulus, shear modulus, and Poisson's ratio respectively;

ρ = density of the plate material;

□ = shear correction factor.

The plate is discretized into N grid points given by,

$$R_i = \frac{1}{2}\left[1-\cos\left(\frac{(i-1)\pi}{N-1}\right)\right], \qquad i = 1,2,............, N \tag{3.6}$$

Setting $R_1 = 0$ and $R_N = 1$, applying DQ procedure, equations (3.5a) and (3.5b) takes the following discrete form,

$$\delta^2 \sum_{k=1}^{N}\left(C_{ik}^{(2)} R_i^2 + C_{ik}^{(1)} R_i\right)\psi_k - \left[\delta^2 + 6\kappa(1-\upsilon)R_i^2\right]\psi_i - 6\kappa(1-\upsilon)R_i^2 \sum_{k=1}^{N} C_{ik}^{(1)} W_k + \delta^2 R_i^2 \Omega^2 W_i = 0,$$

$$\sum_{k=1}^{N}\left(C_{ik}^{(2)} R_i + C_{ik}^{(1)}\right)W_k + \psi_i + R_i \sum_{k=1}^{N} C_{ik}^{(1)} \psi_k + \frac{2R_i}{(1-\upsilon)\kappa}\Omega^2 W_i = 0,$$

Where i = 1, 2…….N; $C_{rs}^{(n)}$ = weighting coefficients for *n*th order derivatives of W and ψ with respect to R.

The solutions of the problems were obtained by solving set of equations which consists of 2 x (N-2) governing equations at all non boundary points and 2 x 2 constraint conditions at boundary points. Following conclusions were made from the solutions obtained:

(1) DQ results converge to the corresponding exact solutions by increasing grid points.
(2) More accurate results were obtained by using the same number of grid points but increasing the thickness.
(3) For various boundary conditions and relative thicknesses, more grid points are required to get result for a higher frequency than for a lower one.

X. Wang, Y. Wang and R. B. Chen [10] proposed the DQ element method for the analysis of two dimensional rectangular plate problems. The weighting coefficient matrix is obtained directly from the differential equation rather than using the integral weak form. The weighting coefficient matrix is so obtained was symmetrical. Plate problems, including deflection and free vibrational problems, can be solved successfully by the proposed method. DQ provided a new way to apply for the boundary conditions and therefore can be used to solve structural problems. Hence the paper emphasized on the practical use of DQM.

H. Zeng and C. W. Bert [11] presented the differential quadrature analysis for the free vibration of eccentrically stiffened plate. The plate and stiffeners are separated at the interface with equilibrium and continuity conditions satisfied. Differential Quadrature equivalents are used to represent the governing equations and boundary conditions for each plate segment or stiffener. The plate and stiffeners have displacements in three dimensions. Three examples are considered:

(1) A simply supported plate with central eccentric stiffener.

(2) A clamped square plate with central eccentric stiffener.

(3) A double-ribbed plate with all edges clamped.

It is concluded in the paper that DQ method decreases computational efforts significantly.

H. Zhong and Q. Guo [12] presented the DQ method for the analysis of simply supported Timoshenko beams with immovable ends. Various nonlinear effects are considered and governing equations were obtained by using Hamilton's principle. DQ method is used to solve the nonlinear differential equations. The formulation is validated by comparing the results obtained with exact solutions. Excellent agreement is obtained between the two. It is concluded that the nonlinear term of the axial force is dominant factor in nonlinear vibration of Timoshenko beam.

S. Moradi and F. Taheri [13] focused on the application of DQM to the delamination buckling of laminated composite plates. The composite modeled as a one dimensional beam plate with through the width delamination that can be at any depth, through the thickness of the geometry. The delamination is one of the most important modes of failure in composites. The buckling strength of the composite is reduced by delamination. Delamination leads to stiffness loss and premature failure of the laminate. The DQM is applied to each region and by imposing suitable boundary conditions; the problem is transformed into a system of linear algebraic equations which can be solved easily. It is concluded that for obtaining accurate results each section can be discretized by only 9-11 sampling points. DQ method saves the modeling time as compared to finite element or finite difference method. Effect of various grid spacing scheme is also studied and concluded that the use of unequally grid spacing schemes can produce the fastest convergence. This method can also be used efficiently for more complicated cases of delamination buckling in composites.

T. C. Fung [14] used the differential quadrature method to solve the first order initial value problems. The initial condition is given at the beginning of a time interval. The order of accuracy and the stability property of the DQ method depend on the location of sampling grid points. The technique for choice of sampling grid point by the roots of the modified shifted Legendre polynomials is presented. If n sampling grid points are used, the solutions are in general at least nth-order accurate. But by using the proposed sampling grid points, the order of accuracy can be increased up to $2n$ -1 or $2n$ for the solutions. The results were verified by using numerical examples and computational efficiency of the method was verified.

T. Y. Wu and G. R. Liu [15] propose differential quadrature method to solve boundary value and initial value problem. In the proposed DQ, the function values and derivatives can be independent variables wherever necessary. The total number of independent variables at a point must be equal to total number of equations from that point. DQ has only the function value as the independent variables. Therefore at any point only one differential quadratue can be implemented but in proposed DQ one has the function value and its derivative wherever necessary as the independent variable. Thus at one point, more than one differential quadratue can be implemented in proposed method. The resulting weighting coefficients of the classical DQM is a matrix of N x N. But the resulting weighting coefficients of the proposed DQM is a matrix of N x M, where M is total independent variables and N is number of grid points. Any finite boundary differential equation with finite function values, their derivatives and their combinations within its domain can be solved by using the proposed method.

X. Wang, C. W. Bert and A. G. Striz [16] presented the new approach to apply the differential quadrature method to the deflection, buckling, and free vibration analysis of beams and plates with various boundary conditions. The first derivative of a deflection function $w(x)$ at a given discrete point $i, w_i'(x)$, is given by,

$$w_i' = A_{ij} w_j, \qquad i = 1,2,..............., N \qquad (3.7)$$

Where A_{ij} are known as weighting coefficients of first derivative and N being the number of grid points. Expanding equation (3.7) in matrix form one obtains an equivalent equation for the first derivative as,

$$\begin{bmatrix} A_{11} & A_{12} & . & . & A_{1,N-1} & A_{1,N} \\ A_{21} & A_{22} & . & . & A_{2,N-1} & A_{2,N} \\ . & . & & & . & . \\ . & . & & & . & . \\ . & . & & & . & . \\ A_{N1} & A_{N2} & . & . & A_{N,N-1} & A_{N,N} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_N \end{Bmatrix} = \begin{Bmatrix} w_1' \\ w_2' \\ . \\ . \\ . \\ w_N' \end{Bmatrix} \qquad (3.8)$$

The essence of the new approach is that the boundary conditions are applied during formulation of the weighting coefficients for the inner grid points. Now if one wants to apply boundary condition at first and last grid point then, equation (3.8) would be modified as,

$$\begin{bmatrix} 0 & A_{12} & . & . & A_{1,N-1} & 0 \\ 0 & A_{22} & . & . & A_{2,N-1} & 0 \\ . & . & & & . & . \\ . & . & & & . & . \\ . & . & & & . & . \\ 0 & A_{N2} & . & . & A_{N,N-1} & 0 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_N \end{Bmatrix} = \begin{Bmatrix} w_1' \\ w_2' \\ . \\ . \\ . \\ w_N' \end{Bmatrix} \qquad (3.9)$$

All the elements in the columns corresponding to coordinates of boundary grid points in the 1^{st} order A matrix are set to zero.

It is concluded that the new approach of applying the differential quadrature method is more compact and convenient procedure for static and free vibrational analyses of structural components.

F. L. Liu [17] developed the two-dimensional differential quadrature element method (DQEM) for the static analysis of symmetric cross-ply laminates. The first-order shear deformation plate theory is used. In this study, the laminated plate is first divided into several simple plate elements and then the differential quadrature method (DQM) is applied to each simple element.

Following conclusions were made:-

(1) Both methods, i.e. increasing the number of elements on the whole plate or increasing the number of grid points in each element will yield convergent numerical results.

(2) It is much more effective to increase the number of grid points in each element with a fixed number of elements than to increase the number of elements on a plate with a fixed number of grid points in each element in order to obtain a faster convergent rate and a higher accuracy.

J. Z. Zhang, T. Y. Ng and K. M. Liew [18] analyze the free vibration of simply-supported and clamped composite laminates, especially thick laminates. The three-dimensional theory of elasticity is integrated into a layerwise model via differential quadrature discretization. Effects of plate aspect and thickness ratios on the free vibration of these laminates are examined. Distinguishable from existing methods, relative displacement fields are presented, which are kinematically compatible with continuity and discontinuity conditions along interfaces of laminates.

K. M. Liew and Y. Q. Huang [19] proposed the moving least squares differential Quadrature method and applied this method to the bending and buckling analysis of moderately thick symmetric laminates based on first order shear deformation theory. The transverse deflection and two rotations of the laminate are independently assumed with the moving least squares approximations. The weighting coefficients used in this method are obtained through the fast computation of the moving least squares shape functions and their derivatives. The weighting coefficients are calculated directly from the partial derivatives of shape functions. The displacements, stresses and critical buckling loads of various laminated plates were presented and compared with analytical results. This method is capable of solving problems with irregular discretizations. It is concluded that the present method provides rapid and convergent solutions of high accuracy for displacements, stresses and critical buckling loads.

CHAPTER 4
PROBLEM FORMULATION

The static analysis of laminated composite circular plate is done. The first order laminated plate theory is used to model the plate.

4.1 GOVERNING EQUATIONS FOR THE PROBLEM

Consider a laminated elastic circular plate of thickness h having polar orthotropic layers (Figure 4.1). The radius of the hybrid plate is R with midplane at z = 0. The plate is supported at its circular boundary. The analysis is done for the axissymmetric case.

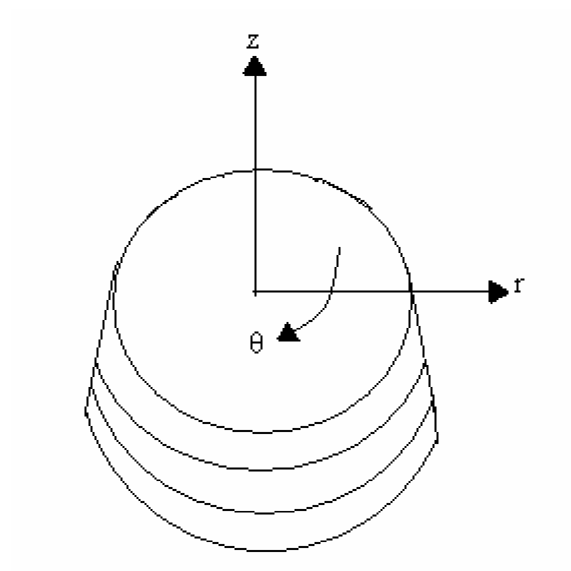

Figure 4.1: Circular laminated plate

Let $u^0, v^0(=0), w^0$ be the radial, circumferential and axial displacements of the midplane and ψ be the rotation of the normal. The displacement field is given by [20],

$$\begin{aligned} u(r,z) &= u^0(r) + z\psi(r), \\ v(r,z) &= 0, \\ w(r,z) &= w^0(r) \end{aligned} \tag{4.1}$$

The strains are given by [20],

$$
\begin{aligned}
\varepsilon_r &= \frac{du^0}{dr} + z\frac{d\psi}{dr}, \\
\varepsilon_\theta &= \frac{u^0 + z\psi}{r}, \\
\gamma_{zr} &= \psi + \frac{dw^0}{dr}, \\
\varepsilon_z &= \gamma_{r\theta} = \gamma_{\theta z} = 0.
\end{aligned}
\tag{4.2}
$$

For linear material, the constitutive equation relating stress σ and strain ε is given by,

$$
\varepsilon = S\sigma \tag{4.3}
$$

Where S is the matrix of elastic compliance. Only elastic effects are considered in present study. Piezoelectric and thermal effects are neglected.

For a polar orthotropic material with polarization in the *z* direction S is given by,

$$
S = \begin{bmatrix}
\frac{1}{E_r} & -\frac{\upsilon_{\theta r}}{E_\theta} & -\frac{\upsilon_{zr}}{E_z} & 0 & 0 & 0 \\
-\frac{\upsilon_{r\theta}}{E_r} & \frac{1}{E_\theta} & -\frac{\upsilon_{z\theta}}{E_z} & 0 & 0 & 0 \\
-\frac{\upsilon_{rz}}{E_r} & -\frac{\upsilon_{\theta z}}{E_\theta} & \frac{1}{E_z} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{\theta z}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{zr}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{r\theta}}
\end{bmatrix}
\tag{4.4}
$$

Where E_i are Young's modulli, υ_{ij} are Poisson's ratios, G_{ij} are shear modulli.

Using equation (4.2), (4.3) and assumption that $\sigma_z = 0$, the governing equation is,

$$\sigma_r = Q_{11}\left(\frac{du^0}{dr} + z\frac{d\psi}{dr}\right) + Q_{12}\left(\frac{u^0}{r} + \frac{z\psi}{r}\right) \tag{4.5}$$

$$\sigma_\theta = Q_{12}\left(\frac{du^0}{dr} + z\frac{d\psi}{dr}\right) + Q_{22}\left(\frac{u^0}{r} + \frac{z\psi}{r}\right) \tag{4.6}$$

$$\tau_{zr} = Q_{55}\left(\psi + \frac{dw^0}{dr}\right) \tag{4.7}$$

Where

$$\begin{aligned} Q_{11} &= \frac{E_r}{(1-\upsilon_{r\theta}\upsilon_{\theta r})}, \\ Q_{12} &= \frac{\upsilon_{12}E_2}{(1-\upsilon_{r\theta}\upsilon_{\theta r})}, \\ Q_{22} &= \frac{E_2}{(1-\upsilon_{r\theta}\upsilon_{\theta r})}, \\ Q_{55} &= G_{31} = G_{zr} \end{aligned} \tag{4.8}$$

Equations of equilibrium for axissymmetric case are,

$$\begin{aligned} &\frac{dN_r}{dr} + \frac{(N_r - N_\theta)}{r} = 0, \\ &\frac{dM_r}{dr} + \frac{(M_r - M_\theta)}{r} - Q_r = 0, \\ &\frac{dQ_r}{dr} + \frac{Q_r}{r} + p_z = 0, \end{aligned} \tag{4.9}$$

$$Q_r = -\frac{1}{r}\int_0^r rp_z dr \tag{4.10}$$

Where

p_z = the load in z-direction.

Q_r = shear force.

N_r, N_θ = in plane force resultants (Figure 4.2) given by,

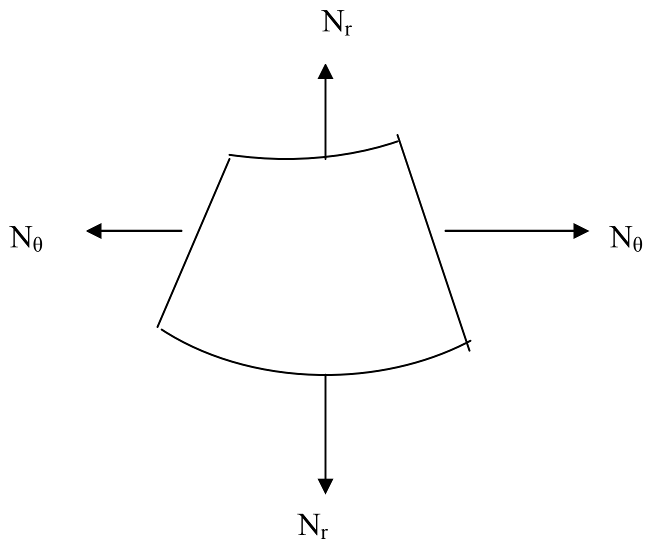

Figure 4.2: Resultant forces on circular laminated plate

$$\begin{bmatrix} N_r \\ N_\theta \\ Q_r \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{zr} \end{bmatrix} dz \tag{4.11a}$$

M_r, M_θ = in plane moment resultants (Figure 4.3) given by,

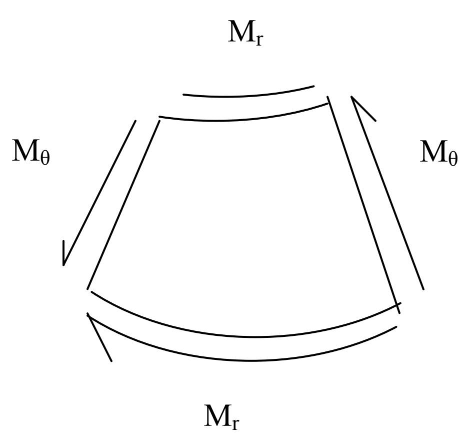

Figure 4.3: Resultant moments on circular laminated plate

$$\begin{bmatrix} M_r \\ M_\theta \end{bmatrix} = \int_{-h/2}^{h/2} z \begin{bmatrix} \sigma_r \\ \sigma_\theta \end{bmatrix} dz \tag{4.11b}$$

Using equations (4.5), (4.6) and (4.7) in (4.11a) and (4.11b), the following equations were obtained,

$$\begin{aligned}
N_r &= A_{11}\frac{du^0}{dr} + A_{12}\frac{u^0}{r} + B_{11}\frac{d\psi}{dr} + B_{12}\frac{\psi}{r} \\
N_\varsigma &= A_{12}\frac{du^0}{dr} + A_{22}\frac{u^0}{r} + B_{12}\frac{d\psi}{dr} + B_{22}\frac{\psi}{r} \\
M_r &= B_{11}\frac{du^0}{dr} + B_{12}\frac{u^0}{r} + D_{11}\frac{d\psi}{dr} + D_{12}\frac{\psi}{r} \\
M_\theta &= B_{12}\frac{du^0}{dr} + B_{22}\frac{u^0}{r} + D_{12}\frac{d\psi}{dr} + D_{22}\frac{\psi}{r} \\
Q_r &= A_{55}\left(\psi + \frac{d\psi^0}{dr}\right)
\end{aligned} \tag{4.12}$$

where $\begin{bmatrix} A_{ij} \\ B_{ij} \\ D_{ij} \end{bmatrix} = \int_{-h/2}^{h/2} Q_{ij} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix} dz$ except for A_{55} which is given by $A_{55} = \int_{-h/2}^{h/2} k_5^2 Q_{55} dz$,

where k_{55}^2 is the shear correction factor taken as 5/6.

Using equation (4.2), the equation (4.9) can be reduced as,

$$A_{11}\left[\frac{d^2u^0}{dr^2} + \left(\frac{\frac{du^0}{dr}}{r}\right)\right] - A_{22}\frac{u^0}{r^2} + B_{11}\left[\frac{d^2\psi}{dr^2} + \left(\frac{\frac{d\psi}{dr}}{r}\right)\right] - B_{22}\frac{\psi}{r^2} = 0 \tag{4.13}$$

$$B_{11}\left[\frac{d^2u^0}{dr^2} + \left(\frac{\frac{du^0}{dr}}{r}\right)\right] - B_{22}\frac{u^0}{r^2} + D_{11}\left[\frac{d^2\psi}{dr^2} + \left(\frac{\frac{d\psi}{dr}}{r}\right)\right] - D_{22}\frac{\psi}{r^2} = -\frac{p_z}{2}r \tag{4.14}$$

$$\mathrm{A}_{55}\left(\psi + \frac{\mathrm{d}w^0}{\mathrm{dr}}\right) = -\frac{p_z}{2}r \tag{4.15}$$

Using the non dimensional parameter $r = a * R$, where a is radius of plate, the above equations are normalized as,

$$\frac{A_{11}}{a}\left[R^2\frac{d^2u^0}{dR^2}+R\frac{du^0}{dR}\right]-\frac{A_{22}}{a}u^0+\frac{B_{11}}{a}\left[R^2\frac{d^2\psi}{dR^2}+R\frac{d\psi}{dR}\right]-\frac{B_{22}}{a}\psi=0 \tag{4.16}$$

$$\frac{B_{11}}{a}\left[R^2\frac{d^2u^0}{dR^2}+R\frac{du^0}{dR}\right]-\frac{B_{22}}{a}u^0+\frac{D_{11}}{a}\left[R^2\frac{d^2\psi}{dR^2}+R\frac{d\psi}{dR}\right]-\frac{D_{22}}{a}\psi=-\frac{p_z}{2}\,a^2R^3 \tag{4.17}$$

$$A_{55}\left(\psi+\frac{1}{a}\frac{dw^0}{dR}\right)=-\frac{p_z}{2}aR \tag{4.18}$$

Boundary conditions for movable simply supported (SM), immovable simply supported (SI) and immovable clamped plate (CI) are given by,

For SI $\quad w^0(R)=0, \quad M_r(R)=0, \quad u^0(R)=0;$ (4.19)

For SM $\quad w^0(R)=0, \quad M_r(R)=0, \quad N_r(R)=0;$ (4.20)

For CI $\quad w^0(R)=0, \quad \psi(R)=0, \quad u^0(R)=0;$ (4.21)

The symmetry conditions at the centre are,

$$u^0(0)=0, \qquad \psi(0)=0; \qquad Q_r(0)=0 \tag{4.22}$$

4.2 DQM APPLIED TO THE PROBLEM

Now discretizing plate into N number of grid points, setting $R_0 = 0$ and $R_N = 1$ and applying DQM to equations (4.16), (4.17) and (4.18), normalized equations take the discretized form as,

$$\frac{A_{11}}{a}\left[\sum_{k=1}^{N}\left(C_{ik}^{(2)}R_i^2+C_{ik}^{(1)}R_i\right)u_k\right]-\frac{A_{22}}{a}u_i+\frac{B_{11}}{a}\left[\sum_{k=1}^{N}\left(C_{ik}^{(2)}R_i^2+C_{ik}^{(1)}R_i\right)\psi_k\right]-\frac{B_{22}}{a}\psi_i=0 \tag{4.23}$$

$$\frac{B_{11}}{a}\left[\sum_{k=1}^{N}\left(C_{ik}^{(2)}R_i^2+C_{ik}^{(1)}R_i\right)u_k\right]-\frac{B_{22}}{a}u_i+\frac{D_{11}}{a}\left[\sum_{k=1}^{N}\left(C_{ik}^{(2)}R_i^2+C_{ik}^{(1)}R_i\right)\psi_k\right]-\frac{D_{22}}{a}\psi_i=-\frac{p_z}{2}a^2R_i^3 \tag{4.24}$$

$$A_{55}\psi_i+\frac{A_{55}}{a}\left(\sum_{k=1}^{N}C_{ik}^{(1)}w_k\right)=-\frac{p_z}{2}aR_i \tag{4.25}$$

Where i = 1, 2, 3… N.

Similarly, in plane force and moment resultants are normalized as,

$$
\begin{aligned}
N_r &= \frac{A_{11}}{a}\left(\frac{du^0}{dR}\right)+\frac{A_{12}}{a}\left(\frac{u^0}{R}\right)+\frac{B_{11}}{a}\left(\frac{d\psi}{dR}\right)+\frac{B_{12}}{a}\left(\frac{\psi}{R}\right)\\
N_\theta &= \frac{A_{12}}{a}\left(\frac{du^0}{dR}\right)+\frac{A_{22}}{a}\left(\frac{u^0}{R}\right)+\frac{B_{12}}{a}\left(\frac{d\psi}{dR}\right)+\frac{B_{22}}{a}\left(\frac{\psi}{R}\right)\\
M_r &= \frac{B_{11}}{a}\left(\frac{du^0}{dR}\right)+\frac{B_{12}}{a}\left(\frac{u^0}{R}\right)+\frac{D_{11}}{a}\left(\frac{d\psi}{dR}\right)+\frac{D_{12}}{a}\left(\frac{\psi}{R}\right)\\
M_\theta &= \frac{B_{12}}{a}\left(\frac{du^0}{dR}\right)+\frac{B_{22}}{a}\left(\frac{u^0}{R}\right)+\frac{D_{12}}{a}\left(\frac{d\psi}{dR}\right)+\frac{D_{22}}{a}\left(\frac{\psi}{R}\right)\\
Q_r &= A_{55}\psi+\frac{A_{55}}{a}\left(\frac{dw^0}{dr}\right)
\end{aligned}
\tag{4.26}
$$

After applying DQM, equation (4.26) results in discrete form as,

$$
\begin{aligned}
(N_r)_i &= \frac{A_{11}}{a}\left(\sum_{k=1}^{N} C_{ik}^{(1)}u_k\right)+\frac{A_{12}}{a}\left(\frac{u_i}{R_i}\right)+\frac{B_{11}}{a}\left(\sum_{k=1}^{N} C_{ik}^{(1)}\psi_k\right)+\frac{B_{12}}{a}\left(\frac{\psi_i}{R_i}\right)\\
(N_\theta)_i &= \frac{A_{12}}{a}\left(\sum_{k=1}^{N} C_{ik}^{(1)}u_k\right)+\frac{A_{22}}{a}\left(\frac{u_i}{R_i}\right)+\frac{B_{12}}{a}\left(\sum_{k=1}^{N} C_{ik}^{(1)}\psi_k\right)+\frac{B_{22}}{a}\left(\frac{\psi_i}{R_i}\right)
\end{aligned}
$$

$$
\begin{aligned}
(M_r)_i &= \frac{B_{11}}{a}\left(\sum_{k=1}^{N} C_{ik}^{(1)}u_k\right)+\frac{B_{12}}{a}\left(\frac{u_i}{R_i}\right)+\frac{D_{11}}{a}\left(\sum_{k=1}^{N} C_{ik}^{(1)}\psi_k\right)+\frac{D_{12}}{a}\left(\frac{\psi_i}{R_i}\right)\\
(M_\theta)_i &= \frac{B_{12}}{a}\left(\sum_{k=1}^{N} C_{ik}^{(1)}u_k\right)+\frac{B_{22}}{a}\left(\frac{u_i}{R_i}\right)+\frac{D_{12}}{a}\left(\sum_{k=1}^{N} C_{ik}^{(1)}\psi_k\right)+\frac{D_{22}}{a}\left(\frac{\psi_i}{R_i}\right)\\
(Q_r)_i &= A_{55}\psi_i+\frac{A_{55}}{a}\sum_{k=1}^{N} C_{ik}^{(1)}w_k
\end{aligned}
\tag{4.27}
$$

Similarly boundary conditions given in equations (4.19), (4.20), (4.21) and (4.22) are also normalized and discretized using DQM and results in,

For movable simply supported beam,

$$
\begin{aligned}
& w_N = 0\\
& \frac{B_{11}}{a}\left(\sum_{k=1}^{N} C_{Nk}^{(1)}u_k\right)+\frac{B_{12}}{a}\left(\frac{u_N}{R_N}\right)+\frac{D_{11}}{a}\left(\sum_{k=1}^{N} C_{Nk}^{(1)}\psi_k\right)+\frac{D_{12}}{a}\left(\frac{\psi_N}{R_N}\right)=0\\
& \frac{A_{11}}{a}\left(\sum_{k=1}^{N} C_{Nk}^{(1)}u_k\right)+\frac{A_{12}}{a}\left(\frac{u_N}{R_N}\right)+\frac{B_{11}}{a}\left(\sum_{k=1}^{N} C_{Nk}^{(1)}\psi_k\right)+\frac{B_{12}}{a}\left(\frac{\psi_N}{R_N}\right)=0
\end{aligned}
\tag{4.28}
$$

For immovable simply supported beam,

$$w_N = 0;$$

$$\frac{B_{11}}{a}\left(\sum_{k=1}^{N} C_{Nk}^{(1)} u_k\right) + \frac{B_{12}}{a}\left(\frac{u_N}{R_N}\right) + \frac{D_{11}}{a}\left(\sum_{k=1}^{N} C_{Nk}^{(1)} \psi_k\right) + \frac{D_{12}}{a}\left(\frac{\psi_N}{R_N}\right) = 0; \tag{4.29}$$

$$u_N = 0;$$

For clamped plate,

$$\begin{aligned} w_N &= 0 \\ \psi_N &= 0 \\ u_N &= 0 \end{aligned} \tag{4.30}$$

Symmetry conditions at the centre are,

$$\begin{aligned} u_1 &= 0 \\ \psi_1 &= 0 \\ A_{55}\psi_1 + \frac{A_{55}}{a}\left(\sum_{k=1}^{N} C_{1k}^{(1)} \psi_k\right) &= 0 \end{aligned} \tag{4.31}$$

The cross ply laminates are considered in present study. The circular plate can have only two orientations, i.e. 0^0 known as radial represented by R and 90^0 known as circumferential orientation represented by C. So, if one want to represent lamination scheme (0/90/0/90), it will be represented as (R/C/R/C) for circular plates.

CHAPTER 5
RESULTS AND DISCUSSION

5.1 PROCEDURE

The governing, boundary and symmetry equations are formulated for the problem in the previous chapter. These discretized linear algebraic equations obtained by applying DQM are further solved to obtain the results to the problem. Both boundary and symmetry conditions are used to obtain the results. The solution to the problem is obtained by solving linear algebraic equations, consisting of $3\times(N-2)$ governing equations at all the interior points (where N is the number of grid points) and 3×2 boundary and symmetry conditions at both the edge and central points. Thus total of 3 x N equations are solved to obtain the results.

Computer program in VC^{++} environment based on the formulation presented in the previous chapter is developed and numerical results for laminated circular plate made up of a substrate of layered graphite/epoxy laminae were obtained. Uniform thickness of the plate is assumed.

The properties of the graphite/epoxy laminae are [21]:

$E_1 = 181GPa$, $E_2 = 10.3GPa$,

$G_{23} = 2.87GPa$, $G_{31} = 7.17GPa$, $G_{12} = 7.17GPa$,

$\nu_{12} = 0.28$

The properties of the isotropic plate are [21]:

$E_1 = 181GPa$, $\nu_{12} = 0.28$, $G_{12} = 7.17GPa$

Mechanical load $P_z = 160Nm^{-2}$ is considered for the problem.

Numerical results for simply supported and clamped laminated beam are obtained. Both symmetrical and unsymmetrical laminates are considered. Various lamination schemes of the laminates are also considered. Various studies are done on the basis of results obtained that are explained in the following pages. For the comparison purpose, the exact analytical solution to the problem is obtained by following equation,

$$
\begin{aligned}
\psi &= -\frac{p_z r^3}{2d_3} - C_1 r^{n1} \\
u &= 0 \\
w &= \frac{p_z r^4}{8d_3} - \frac{C_1 r^{n1+1}}{(n1+1)} - \frac{p_z r^2}{4A_{55}} + \frac{C_1 a^{n1+1}}{(n1+1)} - a^4\left(\frac{p_z}{8d_3}\right) + \left(\frac{p_z}{4A_{55}}\right)
\end{aligned}
\tag{5.1}
$$

Where

For clamped laminated plate,

$$
\begin{aligned}
d_3 &= 9D_{11} - D_{22}, \\
C_1 &= \left(\frac{p_z}{2d_3}\right) a^{3-n1}, \\
n1 &= \left(\frac{D_{22}}{D_{11}}\right)^{\frac{1}{2}}
\end{aligned}
\tag{5.2}
$$

Where D_{11} and D_{22} are the bending stiffnesses and A_{55} is the extensional stiffness.

For simply supported laminated plate,

$$
C_1 = \frac{\left[(3D_{11} + D_{22})\left(\frac{p_z}{2d_3}\right)\right] a^{3-n1}}{(n1 D_{11} + D_{22})}
$$

n1 and d_3 is same as for clamped laminated plate.

5.2 COMPARISON WITH ISOTROPIC PLATE

In this section, results for the isotropic plate obtained from the program are compared with the results available in literature [8]. Comparisons of both the results are shown in Table 1. Comparison study is done for clamped circular plate. Comparison study for w and ψ is shown in Table 1 and comparison study for M_{rmax} and Q_{rmax} is shown in Table 2. Close agreement is found between two results.

Table 1: Comparison study of w and ψ for DQ results of clamped isotropic plate under uniformly distributed load

h/a	Grid Points	Results from [8]		Present results	
		W_{max}	$\Psi_{(R=0.5}$ *(radian)*	W_{max}	$\Psi_{(R=0.5)}$ *(radians)*
.001	3	0.91428 e-5	0.45714 e-5	0.91428 e-5	0.45714 e-5
.001	4	1.11852	0.88889	1.1187	0.78716
.001	5	1.00000	0.75000	1.0000	0.75000
.001	6	1.00000	0.75000	0.9997	0.75600
.001	Exact	1.00000	0.75000	1.00000	0.75000
.100	3	0.088495	0.042781	0.088495	0.042781
.100	4	1.2183	0.86519	1.21826	0.76905
.100	5	1.0457	0.75000	1.04571	0.75000
.100	6	1.0457	0.75000	1.04571	0.74900
.100	Exact	1.0457	0.75000	1.04571	0.75000

Table 2: Comparison study of M_{rmax} and Q_{rmax} for DQ results of clamped isotropic plate under uniformly distributed load

h/a	Grid Points	Results from [8]		Present results	
		M_{rmax} *(Nm/m)*	Q_{rmax} *(N/m)*	M_{rmax} *(Nm/m)*	Q_{rmax} *(N/m)*
.001	3	-0.91428 e-5	-2.0000	-0.91428 e-5	-1.9999
.001	4	-1.7778	-2.0000	-1.77118	-2.00495
.001	5	-2.0000	-1.0000	-2.0000	-1.0000
.001	6	-2.0000	-1.0000	-1.9999	-1.0013
.001	Exact	-2.0000	-1.0000	-2.0000	-1.0000
.100	3	-0.085561	-1.9358	-0.085561	-1.9358
.100	4	-1.8157	-1.8294	-1.81569	-1.82938
.100	5	-2.0000	-1.0000	-2.0000	-1.0000
.100	6	-2.0000	-1.0000	-2.0000	-1.0000

Table 3: Comparison study of *w* for DQ results of simply supported isotropic plate under uniformly distributed load

h/a	Grid Points	Results from [8]	Present results
		W_{max}	W_{max}
.001	3	0.23209×10^{-4}	0.23209×10^{-4}
.001	5	0.40769×10^{1}	0.40769×10^{1}
.001	7	0.40769×10^{1}	0.39915×10^{1}
.001	9	0.40769×10^{1}	0.40769×10^{1}
.001	Exact	0.40769×10^{1}	0.40769×10^{1}
.100	3	0.22013×10^{0}	0.22013×10^{0}
.100	5	0.41226×10^{1}	0.41226×10^{1}
.100	7	0.41226×10^{1}	0.41227×10^{1}
.100	9	0.41226×10^{1}	0.41226×10^{1}
.100	Exact	0.41226×10^{1}	0.41226×10^{1}

5.3 CONVERGENCE STUDY

In this section convergence study is carried out by changing number of grid points. Convergence study is done for two cases i.e. clamped laminated plate and movable simply supported laminated plate is carried out. Two *h/a* ratios are considered for the convergence study. The results obtained for the clamped plate are shown in Table 4 and for simply supported case is shown in Table 5. The two plates are considered symmetrically laminated for this study.

Table 4: Convergence study of DQM results of clamped uniformly loaded symmetrically laminated(R/C/R/R/C/R) circular plate (6 layers & $p_z = 160\ N/m^{-2}$)

h/a	Grid Points	W_{max}	$\Psi_{(R=0.5)}$ *(radians)*
.001	3	1.26201	1.26198
.001	4	1.02977	1.26117
.001	5	0.95550	1.42128
.001	6	0.948784	1.36385
.001	7	0.954747	1.42198
.001	9	0.954902	1.42226
.001	11	0.955087	1.42239
.001	21	0.93918	1.41935
.001	27	1.48217	2.12385
.001	33	43.0379	2.74485
.001	Exact	0.955681	1.42661
.100	3	1.53721×10^{-6}	1.26198×10^{-6}
.100	4	1.30497×10^{-6}	1.26117×10^{-6}
.100	5	1.23070×10^{-6}	1.42128×10^{-6}
.100	6	1.22399×10^{-6}	1.36385×10^{-6}
.100	7	1.22995×10^{-6}	1.42198×10^{-6}
.100	9	1.23010×10^{-6}	1.42226×10^{-6}
.100	11	1.23029×10^{-6}	1.42239×10^{-6}
.100	21	1.21242×10^{-6}	1.41935×10^{-6}
.100	27	1.43253×10^{-6}	1.48214×10^{-6}
.100	33	4.76764×10^{-5}	2.44124×10^{-6}
.100	Exact	1.23088×10^{-6}	1.42261×10^{-6}

It is observed from the Table 4 that the parameter *w* converges to the exact solutions with the increase in grid points but on the other hand, too many grid points also results in divergent results and too few grid points may give poor results. Therefore exact results are obtained by choosing optimum number of grid points neither too few grid points nor too many grid points. Same conclusion is made for the parameter ψ. For ψ also the exact results are obtained by keeping optimum number of grid points.

Table 5: Convergence study of DQM results of simply supported uniformly loaded symmetrically laminated(R/C/R/R/C/R) circular plate (6 layers & $p_z = 160$ N/m^{-2})

h/a	Grid Points	W_{max}	$\Psi_{(R=0.5)}$ *(radians)*
.001	3	2.82646	2.82643
.001	4	3.14001	3.43081
.001	5	3.18769	3.68840
.001	6	3.10324	3.55422
.001	7	3.17333	3.67707
.001	9	3.16730	3.67119
.001	11	3.16395	3.66765
.001	21	3.18429	3.72348
.001	27	4.91738	4.91738
.001	33	3.00003	51.4713
.001	Exact	3.15096	3.65362
.100	3	3.10166×10^{-6}	2.82643×10^{-6}
.100	4	3.41521×10^{-6}	3.43080×10^{-6}
.100	5	3.46289×10^{-6}	3.68840×10^{-6}
.100	6	3.37844×10^{-6}	3.55422×10^{-6}
.100	7	3.44853×10^{-6}	3.67707×10^{-6}
.100	9	3.44250×10^{-6}	3.67119×10^{-6}
.100	11	3.43915×10^{-6}	3.66765×10^{-6}
.100	21	3.45679×10^{-6}	3.95635×10^{-6}
.100	27	5.21570×10^{-6}	4.91738×10^{-6}
.100	33	2.86332×10^{-6}	5.61097×10^{-5}
.100	Exact	3.42616×10^{-6}	3.65362×10^{-6}

For simply supported circular symmetrically laminated plate similar convergence for all the parameters is observed. Here again optimum number of grid points is needed to get the exact result. Too high grid points result in divergent results.

5.4 PARAMETERIC STUDY

Parameter study is done by varying *a/h* ratios and results are obtained for both the cases. The results for various *a/h* ratios for clamped circular symmetrically laminated plate are shown in Table 6 and for simply supported circular symmetrically laminated plate are shown in Table 7. The normalized variables taken for parametric study are:

$$\overline{p}_z = \frac{p_z}{\left(\frac{E_1 h^4}{a^4}\right)}, \qquad \overline{M}_{r\max} = \frac{M_{r\max}}{\left(\frac{E_1 h^4}{a^2}\right)}, \qquad \overline{Q}_{r\max} = \frac{Q_{r\max}}{\left(\frac{E_1 h^4}{a^3}\right)}$$

Table 6: DQM results of clamped circular symmetrically laminated(R/C/R/R/C/R) plate having 6 layers and $\overline{p}_z$ = 5 for various *a/h* ratios

a/h	Grid Points	W_{max}	$\Psi_{(R=0.5)}$	$\overline{M}_{r\max}$	$\overline{Q}_{r\max}$
100	7	0.01733	0.02574	-0.64243	-12.2807
50	7	0.03496	0.05147	-0.64240	-4.95200
20	7	0.09269	0.12878	-0.83387	-2.89423
10	7	0.22262	0.25738	-0.64243	-2.59780
5	7	0.74414	0.51476	-0.64240	-2.52448

Table 7: DQM results of movable simply supported circular symmetrically laminated (R/C/R/R/C/R) plate having 6 layers and $\overline{p}_z$ = 5 for various *a/h* ratios

a/h	Grid Points	W_{max}	$\Psi_{(R=0.5)}$	$\overline{M}_{r\max}$	$\overline{Q}_{r\max}$
100	7	0.057487	0.066555	-0.29790	-24.1989
50	7	0.115272	0.133110	-0.29799	-7.92472
20	7	0.293606	0.332995	-0.29807	-3.37017
10	7	0.624184	0.665549	-0.29790	-2.71702
5	7	1.54727	1.331100	-0.29789	-2.55425

It is observed from the Table 6 that as the *a/h* ratio increases the value of parameters *w* and ψ decreases. The value of M_{rmax} and Q_{rmax} also decreases with the increase in *a/h* ratios. The above observation holds for both the plates i.e. clamped as well as simply supported plate.

5.5 EFFECT OF LAMINATION SCHEMES

Symmetrical as well as unsymmetrical lamination schemes are considered. Symmetrical laminates having all layers of equal thickness as well as varying thickness (but symmetrical about the midplane) are considered. Plate with total thickness 0.001 discretized into seven grid point is considered for the problem. Here *n*R and *n*C means that *n* consecutive layers are having same orientation. Results of various lamination schemes are shown in Table 8.

Table 8: DQM results of clamped circular laminated plate for various lamination schemes p_z = 160 Nm^{-2} (grid points = 7)

Lamination Scheme	W_{max}	$\Psi_{(R=0.5)}$	M_{rmax}
Isotropic layer	1.00000	0.75000	-2.0000
Orthotropic layer	9.8219×10^{-7}	8.8724×10^{-7}	-62.4999
Symmetrical lamination schemes			
(R/C/2R /C/R) Equal thickness	0.954746	1.42198	-64.2416
(R/C/2R/C/R) Varying thickness	0.954746	1.42198	-64.2416
(C/R/2C/R/C) Equal thickness	1.00128	1.5115	-60.7707
(C/R/2C/R/C) Varying thickness	1.00128	1.5115	-60.7707
(R/4C/R) Equal thickness	0.981254	1.4741	-62.1002

(R/4C/R) Varying thickness	0.981254	1.4741	-62.1002
Unsymmetrical lamination schemes			
(R/2C/2R/C) Equal thickness	0.98815	1.48069	-62.528
(R/2C/2R/C) Varying thickness	0.98815	1.48069	-62.528
(R/C/R/C/R/C) Equal thickness	0.97603	1.46399	-62.5006
(R/C/R/C/R/C) Varying thickness	0.97603	1.46399	-62.5006
(C/2R/2C/R) Equal thickness	0.98815	1.48069	-62.528
(C/2R/2C/R) Varying thickness	0.98815	1.48069	-62.528
(C/R/C/R/C/R) Equal thickness	0.97603	1.46399	-62.5006
(C/R/C/R/C/R) Varying thickness	0.97603	1.46399	-62.5006

It is observed from the Table 8 that the laminates having all layers of equal thickness or having different thicknesses but symmetrical to the midplane gives the same results if they have same orientation of layers. In both the cases i.e. symmetrical and unsymmetrical laminates difference in layers orientation gives different results. In case of unsymmetrical laminates, if we replace R's of one orientation with C's and C's with R's then results of both the orientation schemes matches as shown in the Table 8.

CHAPTER 6
CONCLUSION

The static analysis of the laminated circular plate has been done. The plates are assumed to be made of cylindrically orthotropic layers. The axisymmetric case is considered in the present study. The governing equation for the cylindrical coordinates is developed and solved by using Differential Quadrature method. Grid points with equal spacing have been considered in the present study. Results have been verified by the comparison study. It is seen that differential quadrature method is computationally efficient method. The key point in the method is to calculate the weighing coefficients. Once they are calculated, they can be used for solving any kind of problem. The analysis shows that the optimum number of grid points for accurate results are 5 to 7. The grid points beyond 20 give divergent results.

In the present study, two types of boundary conditions are analyzed, clamped and simply supported. Future work can be done by considering complex situations using same weighing coefficients and with non uniform spacing. Work on non axisymmetric case can also be done. In the present study only elastic effects are considered, piezoelectric and thermal effects are neglected. These effects can also be considered for future work.

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