Design and Analysis of Variable Digital Filters using Fractional Fourier Transform

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CERTIFICATE

This is to certify that the thesis entitled **"Design and Analysis of Variable Digital Filters using Fractional Fourier Transform"** which is being submitted to the **Department of Electronics and Communication Engineering, Thapar University**, Patiala in fulfilment of the requirements for the award of degree of **"Doctor of Philosophy"** is a bonafide research work carried out under the supervision and guidance of Dr. Rajesh Khanna and Dr. S. S. Bhatia. The requirements for the submission of this thesis are fulfilled. The material previously published or written by any other person is incorporated in this thesis with the due acknowledgements and with the references made in the text.

The results contained in this thesis have not been submitted in part or full to any other institute or university for the award of any diploma or degree.

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ABSTRACT

The present thesis comprises research work carried out on the design and analysis of variable digital filters using Fractional Fourier Transform (FrFT). Variable Digital Filters (VDFs) are digital filters with controllable spectral characteristics. In many practical applications of digital signal processing, there is a need for real-time tuning of the frequency-domain characteristics of digital filters in order to meet the new desired specifications. Such tunable frequency selective digital filters are called VDFs. The frequency responses of VDFs can be varied instantaneously according to some tuning parameter. Different responses or delays can be immediately obtained by tuning the variable parameters without the need to design a new filter. Variable digital filters (also called tunable digital filters) have applications in different areas of signal processing and communications, e.g., digital tuners, audio tone control, software radio, digital hearing aids, sigma-delta modulators, multimedia signal enhancement and correction, in multi-standard wireless communication receiver systems, channel equalization, matched filtering and pulse shaping etc. VDFs are needed to perform a slightly modified task in accordance with the need that arises in real time processing.

Fractional Fourier transform is applied in the design of digital filters so that the frequency response characteristics of window-based fixed length FIR filter can be modified during operation simply by adjusting the fractional order parameter a, while keeping the coefficients of the fixed length FIR filter unchanged. FrFT is having distinctive properties as one extra degree of freedom is possessed by the fractional order parameter a (or rotation angle parameter α, where $\alpha = a\pi/2$) of the transform. The adjustable parameter denoted by notation a (or α) can be adjusted in different applications so that enhanced results are obtained in comparison to the other existing methods.

A design methodology based on FrFT is presented for designing window-based Low-Pass (LP) FIR filters with adjustable transition-width while keeping the length of the filter same. Sharp transition-width FIR filter is obtained by using fractional order of the transform as a tuning parameter. The transition width of the filter is controlled by FrFT order parameter with low computational cost.

A new closed-form expression of FrFT based frequency response function of a Rectangular window-based fixed length LP FIR filter is derived. The derived expression is analyzed for a range of fractional angle and on the basis of the analysis it is observed that transition width and pass-band width of FIR filter can be reduced, whereas, improved stop-band attenuation can be achieved by reducing the fractional angle α from $\pi/2 \; to \; 0$. Because $\alpha = a\pi/2$, reducing the fractional angle α from $\pi/2 \; to \; 0$ is analogous to reducing the fractional order parameter a from $1 \; to \; 0$. The performance comparison of the proposed FrFT based method over the ordinary method based on FFT (by varying the window length) in tuning FIR filters is also done.

Further, a new FrFT based design technique for obtaining the variable magnitude as well as the phase characteristics of fixed length window-based LP FIR filters is proposed. In the proposed design, the pass-band width and the phase response characteristics of a filter are modified simultaneously by using the fractional order parameter a as a tuning parameter. The behaviour of desired impulse response of FIR filter is analyzed and it is observed that the pass-band width of a digital filter can be controlled using FrFT. An increase in pass-band width of FIR filter is experienced by reducing the FrFT order a from 1 to 0. Also, variations in phase response are observed for a range of fractional order a of the transform.

Finally, a finite duration Chirp function is analyzed as windowing function by using fractional Fourier transform. A new mathematical model for obtaining the FrFT of Chirp function is developed and analyzed in order to add to the advantage of using FrFT in adjustable FIR filter design and in spectral analysis. It is put forward through the derived closed-form expression that FrFT of finite duration Chirp depends on the order of the transform and the Chirp parameter. The variations in the parameters such as Half Main-Lobe Width (HMLW), Maximum Side-Lobe Level (MSLL) & Side-Lobe Fall-Off Rate (SLFOR) of Chirp window are observed for a range of fractional orders. The performance of Chirp window under FrFT is compared with some of the existing windows and improvement in terms of main lobe and side ripples is observed for a range of fractional angle. It is found that the Chirp window is superior and provides better spectral parameters for some particular values of fractional angle α. Also, the Chirp window-based LP FIR filter using FrFT is designed with modified transition-width characteristics.

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ACRONYMS AND ABBREVIATIONS

BP	Band-Pass
BS	Band-Stop
BW	Bandwidth
CFrFT	Continuous Fractional Fourier Transform
CFT	Continuous Fourier Transform
DFT	Discrete Fourier Transform
DFrFT	Discrete Fractional Fourier Transform
dB	Decibel
FT	Fourier Transform
FFT	Fast Fourier transform
FrFT	Fractional Fourier Transform $(F^a = F^\alpha; \alpha = a\pi/2)$
FIR	Finite Impulse Response
FRM	Frequency Response Masking
HMLW	Half Main Lobe Width
HP	High-Pass
IDFT	Inverse Discrete Fourier Transform
IIR	Infinite Impulse Response
LTI	Linear Time Invariant
LP	Low-Pass
LS	Least Square
MLW	Main Lobe Width
MSLL	Maximum Side Lobe Level

SLFOR	Side Lobe Fall Off Rate
SDR	Software Defined Radio
TF	Time-Frequency
VLSI	Very Large Scale Integrated
VDFs	Variable Digital Filters
WLS	Weighted Least Square

GLOSSARY OF SYMBOLS

$\lvert x(n) \rvert$	Absolute function of $x(n)$
ω	Angular frequency
$T(z)$	All pass transfer function
k	An integer variable
Δ	Attenuation in dB
z	A complex variable
$*$	Complex conjugation
$\otimes$	Convolution operator for the ordinary FT
t or u'	Continuous time domain variable
i	Complex number $i = \sqrt{-1}$
β	Chirp parameter
n	Discrete time domain variable
F	Fourier operator
F^{a}	FrFT operator of order a
F^{α}	FrFT operator with rotation angle α
f	Frequency variable
α	Fractional angle parameter
a	Fractional order parameter (in terms of rotation angle, $a = 2\alpha / \pi$)
$H(\omega)$	Frequency response of filter (or Fourier transform of $h(n)$)
$x(t)$	Function $x(.)$ of independent variable t
$X(\omega)$	Fourier transform of $x(t)$

$x(n)$	Function $x(.)$ of independent variable n
$H_d(e^{i\omega})$	Frequency response of ideal filter
$H_d(\omega)$	Fourier transform of desired impulse response $h_d(n)$
u	Hybrid time/frequency variable
F^{-a}	Inverse FrFT operator of order a
$\hat{h}_{LP}[n]$	Impulse response coefficients of a variable filter
$h(n)$	Impulse response of filter
$h_d[n]$	Impulse response of ideal LP FIR filter
$h_{LP}[n]$	Impulse response of practical LP FIR filter
$K_{-\alpha}$	Inverse FrFT Kernel
F^{-1}	Inverse Fourier operator
$\delta(t-t')$	Impulse function
A_s	Minimum stop-band attenuation
$\lvert H(\omega)\rvert$	Magnitude response function of filter
N	Number of points of a variable in discrete domain
π	Pi
R_p	Pass-band ripple
R^{α}	Rotation operator
v	Separate frequency variable
ρ	Shape parameter of Gaussian window
σ	Standard deviation of Gaussion function

γ	Shape parameter of Kaiser window
$G(z)$	Transfer function of transformed filter
η	Transformation parameter or tuning parameter
$H(z)$	Transfer function of original filter
K_α	Transformation kernel of FrFT
δ_k	Tolerance or ripple
z^{-1}	Unit delay element
$w(u,v)$	Weighting function
C_α	$\sqrt{\frac{1-i\cot\alpha}{2\pi}}$

CHAPTER 1
INTRODUCTION

1.1 Introduction

Digital filter is a Linear Time Invariant (LTI) discrete-time system which is used to pass the certain desired frequency components in an input signal without distortion and to block/attenuate the other frequency components [69, 74]. Digital filters are widely employed in diverse fields of signal processing applications including multimedia systems, communications, biomedical systems, consumer electronics, automobile systems, industrial control systems, instrumentations etc [50, 78, 89, 98]. With the advent of modern high-speed digital computing devices, digital filters are also used in areas such as digital audio, sonar and radar processing, speech and data communication and in many more applications, which require high speed calculations and results. Digital filters are used for various purposes such as removal of undesirable noise from desired signals, separating two or more signals which are formerly combined, signal detection in radar, sonar and communications, minimizing inter-symbol interference in communication architectures, for performing spectral analysis of signals, and so on [30, 42, 51, 68].

Digital filter can be uniquely defined either in the time domain by its impulse response $h(n)$ or alternatively in the frequency domain by its frequency response $H(\omega)$. The notation ω corresponds to angular frequency in radians. The frequency response of a filter describes how the filter alters the magnitude and phase of the input signal frequencies. The frequency response of a digital filter can be obtained by taking the Fourier transform of impulse response of the filter. Digital filters are termed as Finite Impulse Response (FIR) or Infinite Impulse Response (IIR) in terms of impulse response. FIR filter has an impulse response that is zero outside of some finite time interval, whereas, an IIR filter has an infinite duration impulse response. The output of FIR filter depends only on the present and past inputs. In contrast, output of IIR filter depends on the present and past inputs as well as previous outputs.

The FIR or IIR digital filters are classified as low-pass, high-pass, band-pass, or band-stop filter on the basis of the frequency characteristics. A Low-Pass (LP) filter passes low-frequency components to the output, while eliminating high-frequency components, whereas, the High-Pass (HP) filter passes all high-frequency components and rejects all low-frequency components. The Band-Pass (BP) filter blocks both low and high frequency components while passing the intermediate range. The Band-Stop (BS) eliminates the intermediate band of frequencies while passing both low and high frequency components.

The ideal filters have a constant magnitude characteristic within their pass/stop bands and are impractical and physically unrealizable for real-time applications. The frequency response characteristics of ideal filter can be approximated very closely by practical, physically realizable filters. The filter design problem consists of a realizable filter whose order is low and whose frequency response best approximates the ideal frequency response to meet the certain desired design specifications [51]. The specifications of magnitude response (or, equivalently, a gain function) $|H(\omega)|$ of practical filters are usually expressed in terms of the desired pass-band and stop-band edge angular frequencies ω_p and ω_s respectively, the permitted deviations in the pass-band R_p (pass-band ripple) in dB, and the desired minimum stop-band attenuation A_s in dB as shown in Fig. 1.1.

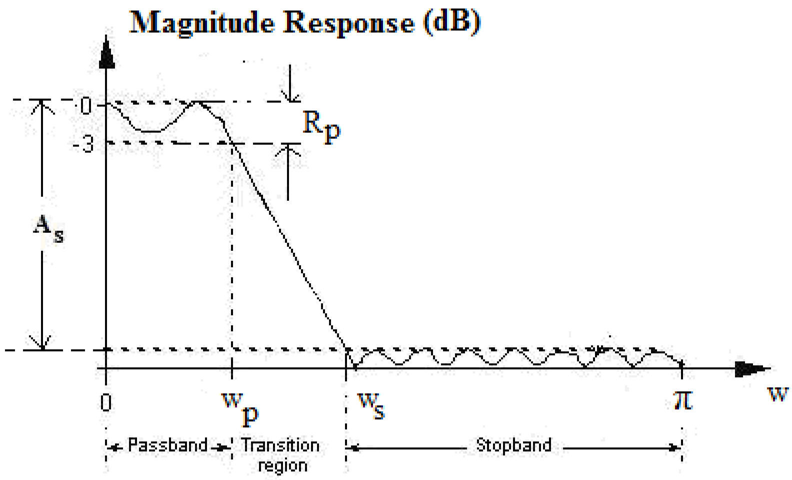

Fig. 1.1: Magnitude characteristics of physically realizable LP filter.

The pass-band defined by [0, ω_p] is the frequency range allowed to pass through the filter. It is the range of frequencies where the filter's frequency response is greater than or equal to -3dB, as depicted in Fig. 1.1. The pass-band edge frequency ω_p is the frequency beyond which attenuation can start. The stop-band defined by $[\omega_s, \pi]$ is that band of frequencies which are attenuated or blocked by a digital filter. It is the region in which magnitude response is near zero. The stop-band edge frequency ω_s denotes the beginning of the stop-band. It is the frequency beyond which required stop-band attenuation is met. The stop-band attenuation denoted by A_s, is the minimum amount by which frequency components in the stop-band are attenuated. It is formally specified as the attenuation to the top of the first side-lobe of the filter's frequency response. The pass-band ripple R_p specifies maximum variation of gain in the pass-band. Ideally the gain should be constant so that frequency components of signal will not get distorted and hence pass-band ripple must be as small as possible. To permit the magnitude response to decay more gradually from its maximum value in the pass-band to zero value in the stop-band, the specification includes a transition band $(\omega_s - \omega_p)$ between the pass-band and the stop-band. The transition-width $[\omega_s - \omega_p]$ is the frequency range between the pass-band and the stop-band of a digital filter to permit the magnitude to drop off smoothly.

The magnitude response is allowed to vary by a small amount both in the pass-band and in the stop-band but the variation of $|H(\omega)|$ within the pass band and within the stop band should be as small as possible. Applications that require fine frequency discrimination use filters with correspondingly sharp (narrow) transition bands. The measure of sharpness is the average slope of $|H(\omega)|$ in the transition band [3]. The best filter will have narrowest transition band, smallest pass-band ripple and greatest minimum stop-band attenuation. Improving one performance constraint in filter design usually worsens others. Increasing filter order (i.e. cost) can improve all the three measures.

With the advancement of Very Large Scale Integrated (VLSI) technology, most of the modern tasks in signal processing are performed in the digital domain so that complex operations can also be performed in real-time applications without the well-known shortcomings of analog implementations [31, 91]. In many filtering applications, digital filters are required to change their frequency-domain characteristics during the course of signal processing. For example, after designing a LP filter with a pass-band edge at 2 kHz, it may be required to move the pass-band edge to 2.1 kHz or 1.9 kHz. It is possible to design digital filters with adjustable frequency characteristics so that the parameters of digital filters can be changed instantaneously as and when required in many signal processing applications [34, 95]. The use of digital signal processing techniques in wireless communication, digital televisions, and multimedia requires an efficient method which can quickly obtain the new desired frequency-domain characteristics for digital filters.

The wide-ranging term used for digital filters with changeable parameters is Variable Digital Filters (VDFs). Digital filters with changeable parameters are used to tune the frequency/magnitude response immediately on-line without redesigning a new filter off-line. The most straightforward approach or the direct method for the design of VDF is to change the frequency response of a digital filter by changing the length of the filter. The length of the filter needs to be increased in order to reduce the bandwidth or the transition width. Whereas, it is required to decrease the same in order to increase the bandwidth. But, the direct method involves re-computation of filter coefficients and requires redesigning of new filter from the existing one. The design of a new filter in hardware from the existing old one is cumbersome and increases the computational burden. Different computational resources such as different number of multipliers and adders are needed to compute the new coefficients and more memory is required to store large number of filter coefficients in order to meet the modified specifications by using the direct approach. The problem of designing a variable digital filter received a considerable attention in the past few years and several techniques are extensively investigated for the design of VDFs.

An introduction to digital filters with variable characteristics is presented next.

1.2 Variable Digital Filters

Definition: *Digital filters with variable frequency responses or whose characteristics can be varied instantaneously are called Variable Digital Filters*.

A VDF is required:

(i) For performing a slightly modified task during the course of signal processing in order to meet the desired requirement.

(ii) To reduce the amount of design time.

(iii) For obtaining the new desired frequency-domain characteristics quickly by simply changing the coefficients of variable filter on-line without redesigning a new filter off-line.

(iv) To have narrow transition bandwidth filter for the professional digital audio applications.

(v) To increase or decrease the pass-band width of anti-aliasing and anti-imaging filters accordingly as the sample rate changes in sample rate converters and to have sharp transient response of both the filters in order to achieve best results.

In most general case of a variable digital filter, parameters related to magnitude and frequency can be subjected to change and such filters are called variable magnitude response filters. The magnitude related parameters include pass-band deviations and stop-band attenuation; whereas the frequency related parameters include variable cut-off frequency, adjustable pass-band and stop-band width, variable transition bandwidth, controllable center frequency and bandwidth of band-pass or band-stop filters. This variable feature is useful in implementing digital filters with tunable magnitude responses for signal processing. In the design of the VDF which deals with the variable magnitude response filters, the possible variable magnitude response parameters for a LP filter and a BP filter are shown in Fig. 1.2(a) and Fig. 1.2(b) respectively. The absolute magnitude specifications are taken, where the notation $|H(\omega)|$ denotes the absolute value of the magnitude response function $H(\omega)$.

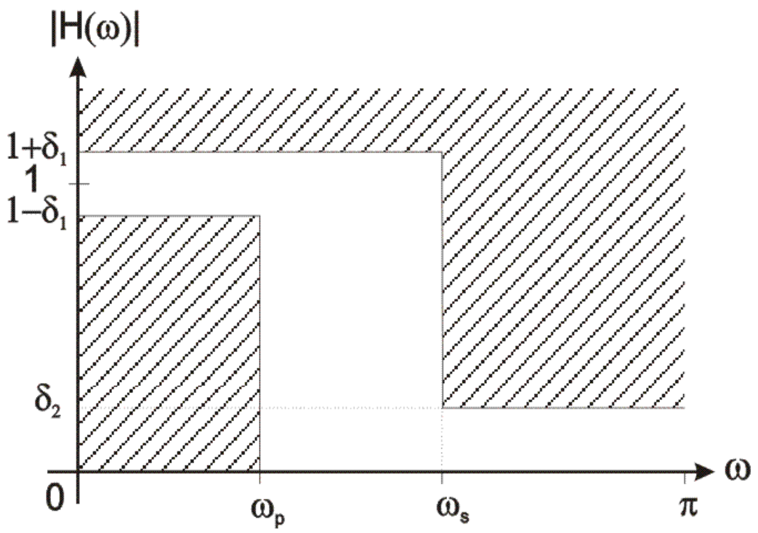

Fig. 1.2(a): Illustration of possible variable magnitude response parameters for LP filter.

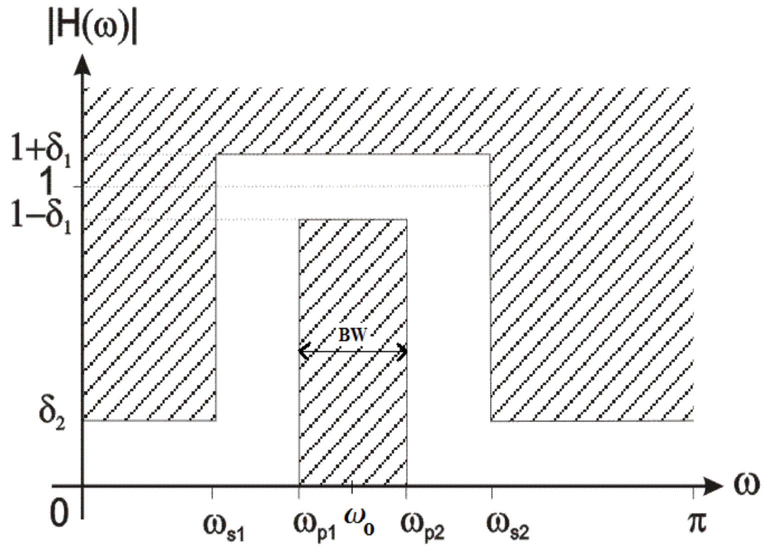

Fig. 1.2(b): Illustration of possible variable magnitude response parameters for BP filter.

There are two possible approaches to represent the magnitude response specifications of digital filters graphically [108]:

(i) Absolute specifications: The first approach is called absolute specifications, which provide a set of requirements on the magnitude response function $|H(\omega)|$ and are generally used for FIR filters.

(ii) Relative specifications: The second approach is called relative specifications, which provide requirements in decibels (dB), where

$$\text{dB scale} = -20\log_{10}\frac{|H(\omega)|}{|H(\omega)|_{\max}} \geq 0 \tag{1.1}$$

The relative approach is more popular and is used for both FIR and IIR filters.

The typical magnitude as well as relative specifications for a low-pass filter are shown in Fig. 1.3(a) and Fig. 1.3(b), respectively.

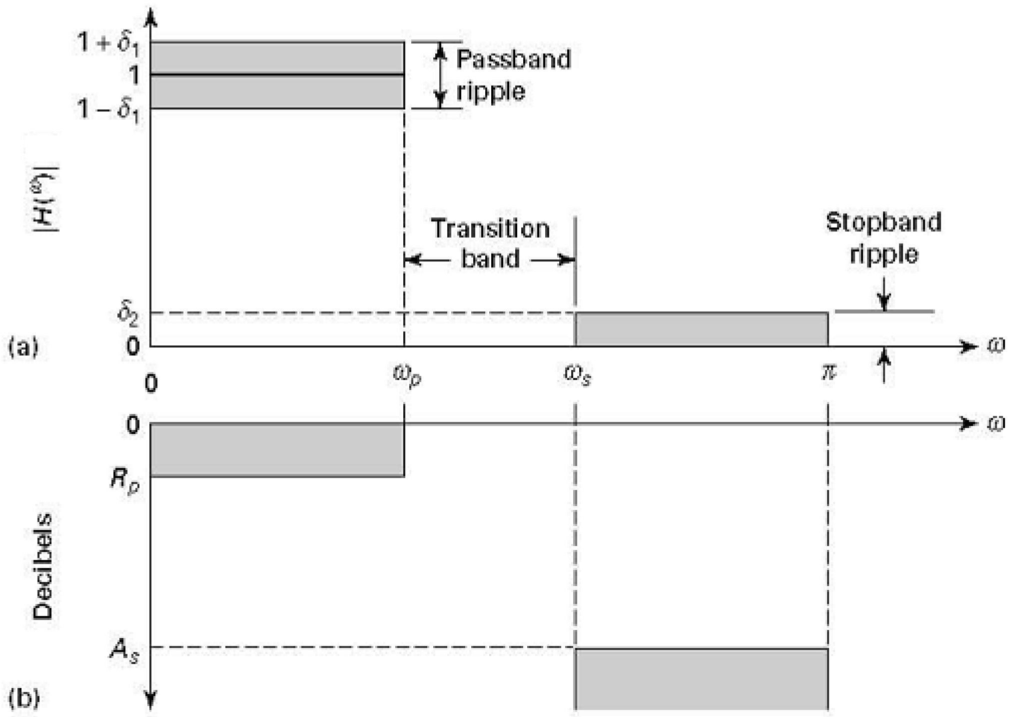

Fig. 1.3: Low-pass digital filter specifications, (a) Absolute, (b) Relative (dB) specifications.

The absolute magnitude specifications for a low-pass filter as shown in Fig. 1.3(a) are defined as:

(a) Band $[0,\omega_p]$ is called the pass-band where ω_p is normalized cut-off frequency in the pass-band, and δ_1 is the tolerance (ripple) that is acceptable in the ideal pass-band response such that $|H(\omega)| \geq 1-\delta_1$ in the pass-band.

(b) Band $[\omega_s,\pi]$ is called the stop-band, and δ_s is the tolerance (ripple) in the stop-band such that $|H(\omega)| \leq \delta_2$ in the stop band.

(c) Band $[\omega_p,\omega_s]$ is called the transition band. There are no restrictions on the magnitude response in this band.

Whereas, the relative (dB) specifications for a low pass filter as shown in Fig. 1.3(b) are defined as:

(d) R_p is the allowable pass-band ripple in dB

(e) A_s is the stop-band attenuation in dB

The relationships between absolution specifications and relative specifications are given by:

$$R_p = -20\log_{10}\frac{1-\delta_1}{1+\delta_1} > 0(\approx 0) \quad \textit{for passband} \tag{1.2}$$

$$A_s = -20\log_{10}\frac{\delta_2}{1+\delta_1} > 0(>>1) \quad for\,stopband \tag{1.3}$$

Variable magnitude deviations δ_2 (in the stop-band) and δ_1 (in the pass-band) are not easily achievable. The complicated recalculations of the entire transfer function are required and usually many circuit elements must be changed in order to obtain the new δ_2 and δ_1. The rarest case in which only magnitude parameters are varied is called magnitude equalizer. More often tuning of only frequency parameters e.g. pass-band cut-off frequency ω_p and stop-band edge frequency ω_s of LP (Fig. 1.2(a)) or ω_{p1}, ω_{p2}, ω_{s1} and ω_{s2} , center frequency ω_0 and Bandwidth (BW) of BP filters (Fig. 1.2(b)) is required. This is an easier task compared with all-magnitude parameters tuning. Hence VDFs are often named as 'tunable' filters when only frequency related parameters are modified. VDFs are called as 'adjustable' filters when changes occur in some narrow range of values of a given parameter or 'programmable' filters when parameters can be reprogrammed or are controlled by a computer [95].

The variable filter whose frequency response changes by adjusting its coefficients possess higher flexibility than the normal fixed-coefficient digital filter and all the above mentioned tasks can be easily achieved using VDFs. VDFs are different from time-varying filters in a way that while the characteristics of time-varying filters vary in time in a prescribed manner, the characteristics of variable filters vary in accordance with the need that arises in real time processing.

A variable digital filter contains number of parameters in transfer function that can be used to tune the frequency response easily without redesigning a new filter. The main objective in the design of a VDF is to find a parameterized transfer function which in a certain sense best approximates a given set of frequency response characteristics that vary with the parameters in a desired manner.

VDFs can be constructed using either finite impulse response or infinite impulse response (recursive) filters. In comparison to FIR filter, recursive filter design is much more difficult and complicated because the stability of filter must be considered. The recursive filter design requires lower order and less computational complexity to satisfy the same desired variable magnitude responses.

The design of variable cut-off/band-edge FIR filters is broadly classified into two categories: (i) Frequency transformation method and (ii) Spectral parameter approximation method. Most of the design methods for designing digital filters with variable cut-off frequencies are based on different frequency transformations applied to a prototype (usually LP) filter. The frequency transformation method is used to design a digital filter with modified characteristics by transforming a given digital filter. A prototype filter with certain desirable frequency characteristics is first designed. Certain transformation such as the all-pass transformation method is then applied to the prototype filter to obtain the final VDF.

One of the most popular frequency transformations is transformation proposed by Constantinides [17] which is based on the idea of replacing each delay element of an existing FIR or IIR filter transfer function by the same all-pass filter. This technique is briefly discussed below:

Let $h(n)$ be the impulse response of original digital filter and $H(z)$ be its transfer function. The new transfer function $G(z)$ of a variable digital filter can be written as:

$$G(z) = H(z)\big|_{z^{-1}=T(z)} \quad , \tag{1.4}$$

where $T(z)$ is an appropriate stable all-pass function. In the case of design of variable low-pass filters, $T(z)$ is given by:

$$T(z) = \left(\frac{-\eta + z^{-1}}{1 - \eta z^{-1}} \right) \quad \text{for } |\eta| < 1 \ , \tag{1.5}$$

where the parameter of the transformation denoted by symbol η is called as the tuning parameter.

The relation between the original cut-off frequency ω_{c1} of $H(z)$ with the transformed cut-off frequency ω_{c2} of $G(z)$ is given by:

$$\tan\left(\frac{\omega_{c1}}{2}\right) = \left(\frac{1-\eta}{1+\eta}\right)\tan\left(\frac{\omega_{c2}}{2}\right), \tag{1.6}$$

which can be solved for η yielding:

$$\eta = \frac{\tan(\omega_{c1}/2) - \tan(\omega_{c2}/2)}{\tan(\omega_{c1}/2) + \tan(\omega_{c2}/2)} = \frac{\sin\left(\frac{\omega_{c1} - \omega_{c2}}{2}\right)}{\sin\left(\frac{\omega_{c1} + \omega_{c2}}{2}\right)} \tag{1.7}$$

For the range $-1 < \eta < 0$, the transformation is backward which means the transformed cut-off frequency is smaller than the original cut-off frequency. For the range $0 < \eta < 1$, the effect is reverse i.e. the transformation is forward which means the transformed frequency is higher than the original cut-off frequency. Thus, spectral transformation can be used to transform a given digital LP filter to another digital transfer function. But with spectral transformation approach, the resulting structure is no longer FIR because the delays in FIR filters are replaced by all-pass sections making it an IIR filter.

In the second approach, i.e. spectral parameter approximation technique, the coefficients of the variable filter are expressed as functions (trigonometric or polynomial functions) of the adjustable parameters which determine the desired frequency response. The adjustable parameters defining the desired filter characteristics can be cut-off frequency, center frequency of the pass-band, transition bandwidth, pass-band width etc. A simple method based on the approximation technique is proposed by Jarske et al. [38-39] for designing tunable LP FIR filters. In the method given by the researchers, the impulse response coefficients of the transfer function of variable filter are expressed as simple functions of the cut-off frequency. The idea behind the tuning procedure is to meet proposed specification and to change desired magnitude response of filter during operation by varying a single spectral parameter. The method preserves the FIR structure and permits easy tuning.

The approximation method suggested by the Jarske et al. for designing FIR digital filter with adjustable cut-off frequency is based on the observation that for an ideal LP FIR filter with a zero phase response given by:

$$H_d(e^{i\omega}) = \begin{cases} 1, & \text{for } 0 \le |\omega| \le \omega_c \\ 0, & \text{for } \omega_c < |\omega| \le \pi \end{cases} \tag{1.8}$$

The impulse response coefficients of ideal LP FIR are given by:

$$h_d[n] = \begin{cases} \dfrac{\sin(\omega_c n)}{\pi n}, & n \neq 0 \quad -\infty \leq n \leq \infty \\ \dfrac{\omega_c}{\pi}, & n = 0 \end{cases} \tag{1.9}$$

The above expression can be truncated to obtain the coefficients of a realizable approximation given by:

$$h_{LP}[n] = \begin{cases} c[n]\omega_c, & for\, n = 0, \\ c[n]\sin(\omega_c n), & for\, 1 \leq |n| \leq N \\ 0, & otherwise \end{cases} \tag{1.10}$$

where ω_c is the 6-dB cut-off frequency, and

$$c[n] = \begin{cases} 1/\pi, & for\, n = 0, \\ 1/\pi n, & for\, 1 \leq |n| \leq N \end{cases} \tag{1.11}$$

It follows from the above that FIR LP filter can be designed for a given cut-off frequency as per the initial filtering requirements.

Now, solving for the constants $c[n]$ in terms of coefficients of initially designed filter for a cut-off frequency ω_c using (1.10), one gets:

$$c[0] = \frac{h_{LP}[0]}{\omega_c}, \tag{1.12}$$

$$c[n] = \frac{h_{LP}[n]}{\sin(\omega_c n)}, \quad 1 \leq |n| \leq N \tag{1.13}$$

Tunable FIR LP filter can be designed where the coefficients of the transformed FIR filter with a new desired cut-off frequency $\hat{\omega}_c$ are given by:

$$\hat{h}_{LP}[0] = c[0]\hat{\omega}_c = \left(\frac{\hat{\omega}_c}{\omega_c}\right) h_{LP}[0], \tag{1.14}$$

$$\hat{h}_{LP}[n] = c[n]\sin(\hat{\omega}_c n) = \left(\frac{\sin(\hat{\omega}_c n)}{\sin(\omega_c n)}\right) h_{LP}[n], \quad 1 \le |n| \le N \tag{1.15}$$

Although, the filter coefficients are a simple function of the cut-off frequency as shown in (1.10), the $\sin e$-function is difficult to calculate in real-time implementations. This can be solved using pre-calculated look-up tables, series expansions of the $\sin x$-function or recursive calculation with a digital $\sin e$-wave generator.

Also, digital filters can be designed to have adjustable transition width. The simplest method to design variable transition bandwidth filters is discussed by Huang and Shyu [34] which is based on applying variable windows in FIR filter design.

There is another group of variable response circuits in which only phase parameters such as controllable fractional delay and group delay are varied. These are called as variable delay response filters.

Below, a literature survey is presented on the design of variable digital filters. The work done in the area of design of digital filters with adjustable characteristics using FrFT is also included.

1.3 Literature Review on Variable Digital Filters

The design of digital filters with variable characteristics is a constant area of interest for many researchers. Several methods are proposed in the literature for designing adjustable filters. Most of the design methods for digital filters with variable cut-off and center frequencies are based on different transformations applied to a prototype filter.

Schuessler and Winkelnkemper [85] were the first to propose a transform approach in 1970 in which each delay element of a prototype filter is replaced by a first order all pass network to transform the frequency. The resulting filter has an identical frequency response as that of prototype filter but on a distorted frequency scale.

The spectral transformations proposed by Constantinides [17] and the transformations suggested by Oppenheim et al. [59] are the most popular transformations. The authors proposed a new class of transformation in [59] so that the resulting impulse response of the filter is finite and the phase of the filter is linear.

Stoyanov and Kawamata [94-95] developed a new method for the design of variable bandpass/bandstop real coefficient IIR digital filters with canonic number of multipliers and delays (equal to the order of the transfer function) and with high accuracy independent tuning of their center frequency and bandwidth. The method proposed by them is based on cascade realization, spectral transformations and on development of prototype and all-pass circuits with very low sensitivity.

Jarske and Mitra [37] implemented the proposed variable digital filter schemes for FIR and IIR using a TMS320-series digital signal processor. The FIR filter is implemented by updating the coefficients of finite-impulse response filter in a simple manner such that the cutoff frequencies can be controlled through a single parameter. The IIR is implemented by using a series expansion of the low-pass-low-pass frequency transformation.

Mitra et al. [52] proposed a new method for changing the cut-off frequency of infinite impulse response digital filters with a single parameter. This method is based on the use of the Taylor series expansion of the lowpass-to-lowpass frequency transformation and the resulting filter structure is a parallel connection of real or complex allpass sections. the tuning range is several octaves for narrowband filters. Also, a special lowpass-to-bandpass transformation is incorporated into the structure to obtain tunable centre frequency and bandwidth. The authors also included a description of a signal processor implementation.

Sia and Fahmy [90] proposed a more general and flexible approach where the coefficients of the variable filter are expressed as functions of the spectral parameters defining the desired filter characteristics.

Jarske et al. [38-39] developed direct-form linear-phase FIR variable filters by using an approach of having a simple relationship between filter coefficients and the cut-off frequency. The authors approximated the variable filter coefficients by simple sine functions of the cut-off frequency.

Zarour and Fahmy [105] proposed the spectral parameter approximation method in which the poles and zeros of an infinite impulse response filter are assumed to be polynomials of the spectral or tuning parameters. The authors in their work represented filter coefficients by explicit analytic functions and assumed the cascade realization for the transfer function. The new designed algorithm is much faster and gives better results and allows on-line variation of coefficients to meet the changing specifications.

Pun et al. [75] proposed the design and implementation technique of FIR and IIR digital filters. They suggested Least Squares (LS) approach for the design of FIR variable digital filters and approximated the impulse response of the filter as a linear combination of basis functions. The basis functions taken as piecewise polynomials give larger tuning range than the ordinary polynomial based approach, and it also reduces the number of multiplications. The resulting VDFs can be efficiently implemented using familiar Farrow structure. The model reduction approach is proposed for the design of IIR filters which is simple, guarantees stability and free from undesirable transients during parameter tuning. Chan et al. [14] proposed a method for designing FIR filters with variable characteristics. In the method, the impulse response of the VDF is parameterized as a linear combination of functions in the spectral or tuning parameters.
Johansson and Lowenborg [40] implemented variable band-edge FIR filters by using over-designed fixed filters, each having several times sharper transition band than that required by the variable filter. Thus, each filter is taking care of only part of the variable frequency regions and only one of the filters is used at any moment of operation.

Deng [20-22] proposed a closed-form solution for designing variable FIR digital filters whose magnitude and fractional phase-delay responses can be tuned simultaneously [22-24]. The authors showed that the designed variable FIR filter has independently tunable magnitude and fractional delay and is much more flexible than the existing filters that only have variable magnitude or variable fractional delay.

Yu et al. [104] proposed a method to design variable band-edge FIR filters with sharp transition band. The variation of the filter is realized by shifting the input signal frequency spectrum. The frequency-shifted signal is shaped by a filter with a fixed band-edge, and then shifted back to its original frequency region to achieve the same effect of varying the band-edge by shifting along the frequency axis. The variable filter is constructed from a fixed FFB and a fixed half band filter.

Harris [33] presented a technique to obtain continuously variable bandwidth filters as a cascade of input and output arbitrary interpolators. The input interpolator lowers the input sample rate to increase the interval between samples so that bandwidth can be decreased without increasing the number of taps in a filter. The authors proposed in their work that there is no need to compute different filter coefficients for the range of variable bandwidth filters.

Toma and Naoyuki [96] presented a method for designing a variable linear-phase FIR filters with multiple variable factors and a reduction method for the number of polynomial coefficients. The obtained filter has a high piecewise attenuation in the stop-band. The proposed VDF is implemented using the Farrow structure and is suitable for real time signal processing.

George and Elias [28] proposed the technique of obtaining a continuously variable bandwidth filter using re-sampling in such a way that a bandwidth increase as well as decrease can be achieved with low distortion and low complexity. A continuously variable bandwidth filter is obtained as a combination of two arbitrary sample rate converters and a fixed length, sharp transition band filter with low complexity. The first sample rate converter either increases or decreases the sample rate according to the proportional requirement of the effective bandwidth. This signal is processed with the fixed length, fixed bandwidth filter and the output of the filter is given to a second sample rate converter for converting back to the original input sample rate. This effectively changes the band-width of the filter. Arbitrary sampling rate is achieved by using a poly-phase interpolator and the fixed length filter is implemented by Frequency Response Masking (FRM) technique for reducing the complexity. Sharp transition band is achieved which helps in reducing the inter-channel interference.

Darak et al. [18] presented the design of a variable linear phase FIR filter based on second order frequency transformations and coefficient decimation. The VDF using second order transformation has better cut-off slope characteristics compared to the VDF using first order transformation. The design example illustrated in the work shows that cut-off frequency can be varied over a wider range using the proposed technique.

Huang and Shyu [34] proposed the design of variable transition bandwidth FIR filters. They first presented the technique for generating the adjustable Kaiser window. By exploiting this variable window, the low-pass variable transition bandwidth FIR filter is designed.

Different researchers applied fractional Fourier transform in the design of digital filters in order to take the advantage of one extra degree of freedom possessed by the fractional order a.

Saxena and Singh [84] presented that importance of FrFT in the area of signal analysis and filter design. The authors presented the analysis of the Rectangular and Cos window with FrFT and discussed the application of FrFT in filtering using window functions.

Sharma et al. [87] proposed a methodology to sharpen the transition band of Kaiser window-based FIR filter using fractional Fourier transform. The authors show that apart from the traditional usage of window shape parameter and window length, the parameter a of the transform can also be used to design variable transition bandwidth FIR filter. The merit of using FrFT in digital filter design is that there is no need to re-compute the impulse response coefficients to change the frequency response of a window-based digital filter.

Sharma et al. [88] presented a method based on FrFT in order to tune the width of the transiton band of FIR filter realized with Kaiser as well as Parzen-$\cos^6(\pi t)$ window. The authors also compared the performance of the method proposed by them with the direct tuning scheme (with filter order as the tuning parameter).

Singh [93] analyzed the Triangular, Hanning, Blackman and Kaiser window functions with FrFT and observed the change in the behavior of these windows with change in order parameter a of the transform. The author evaluated the window parameters such as Maximum Side Lobe Level (MSLL), Half Main Lobe Width (HMLW) and Side Lobe Fall-off Rate (SLFOR) for different values of a.

Kumar et al. [43] developed a new mathematical model for obtaining the fractional Fourier transform of Dirichlet and Generalized Hamming window functions. The authors presented that FrFT of these windows is directly dependent on the FrFT angle parameter α. This variable angle parameter can control the main-lobe width and correspondingly the minimum stop-band attenuation of these windows.

Goel and Singh [29] developed the mathematical model of Dirichlet, Generalized “Hamming”, and Triangular window functions using linear canonical transform. The authors discussed the two special cases when linear canonical transform reduces to FrFT and ordinary Fourier transform. They also derived the closed-form expression for obtaining the FrFT of these window functions. The application of linear canonical transform and FrFT (a special case of linear canonical transform) in tuning of window-based FIR filter transition bandwidth as well as in the area of filtering using window functions is also discussed.

Singh and Saxena [83] presented the growth of FrFT based on the recent patents and publications. The development of mathematical properties of FrFT is discussed and a comparative analysis is made to establish the best one amongst various algorithms available for evaluating the discrete FrFT. Also, the application of FrFT in filtering, beam-forming, encryption and watermarking is presented.

Muralidhar et al. [55] evaluated different types of window functions such as Blackman, Hanning and Raised Cosine windows with FrFT. The parameters of windows which are used for the evaluation are MSLL and HMLW. The results of Fourier transform and fractional Fourier transform are compared by applying each of the transform on the windows.

Muralidhar et al. [54] introduced HP linear phase FIR filter with the help of the Kaiser window and FrFT. The high-pass FIR filter is tuned for the transition band by changing the FrFT angle. The authors implemented Rectangular, Bartlett, Hanning, Hamming and Blackman window from Kaiser window by adjusting the variable shape parameter of Kaiser to different values. Chaturvedi et al. [16] implemented FIR filter using the Kaiser window and FrFT in order to tune the response of FIR filter.

1.4 Gaps in Existing Work on Variable Digital Filters

Following gaps are found in the work done on the design of digital filters with variable characteristics:

1.4.1 Research Motivation in Variable Cut-off/Band-edge Frequency Digital Filters

A transformation approach is proposed in [17, 85], which generally increases the filter length, thereby requiring a special filter structure. The edge frequency of various bands can also not be independently controlled. It is also not applicable to variable digital filters with variable fractional delay. The frequency transformation method projected by the Constantinides in [17] is very simple, but this method can tune only the cut-off frequencies of filters.

The all known variable filters are realized as transformed Taylor's structures in [59], which are too complicated for higher-order transformation. Hardware implementation is also difficult and impractical because there is no simple relation between the filter coefficients and the cut-off frequency. The frequency-transformation-based methods are not applicable for the design when the given variable specifications of a digital filter are more complicated.

In [38-39, 90, 105], a spectral parameter approximation method is proposed where the direct approximation of coefficient affects coefficient precision and leads to deterioration of magnitude response. In [38-39], the greatest stop-band attenuation achievable is only 40dB. The method is not based on any clear or strong background and some functions used in the method are the results of experiments collected after designing a large number of filters. The pass-band and stop-band edges cannot be set to the desired frequencies and the pass-band and stop-band deviations cannot be predicted in the design process.

In [90], the disadvantage of the proposed design technique is huge amount of design time required. In addition, it is based on direct form realization which is known to be highly sensitive to quantization errors. Recently Farrow structures [75] are used in realizing variable band-edge filters but their computational complexity is very high. Over-designed set of filters [40] are also used as an alternative but their computational complexity is also high and large memory size is required to store the coefficient values.

The technique proposed in [28, 33] requires additional hardware such as sample rate converters to implement filters with variable characteristics.

In [104], a technique is proposed is to vary the band-edge by shifting the signal along the frequency axis but it uses more delay elements and additional hardware such as digital comparators. Also, this method results in increased group delay.

The method proposed in [18] is suitable only for designing tunable filters for a finite set of cut-off frequencies.

1.4.2 Research Motivation in Variable Transition Bandwidth FIR Filters

The Farrow structures having several fixed length subfilters and variable multipliers are used to implement the designed variable filter in [34]. The Farrow structure has high arithmetic complexity.

The variable transition bandwidth FIR filter is designed using fractional Fourier transform and significant computational saving is achieved in [87-88, 16, 54-55], but the range of adjustability in transition width is lesser than the range obtained using direct approach. The tuning of transition bandwidth is considered only. The pass-band performance is also not evaluated for different values of fractional order. No analytical relationship is found to formulate the transfer function of a variable digital filter. The motivation behind this research is to add to the advantage of using FrFT in the area of design of variable digital FIR filters and to overcome the shortcomings, if present, in the existing work. Table 1.1 shows the comparison of existing literature on the design of variable digital filters in a tabular form.

Table 1.1:

Comparative Table of Literature Review

Authors	Proposed Technique	Advantages	Disadvantages
Schuessler and Winkelnkemper	Transform approach	Simple	The edge frequency of various bands can not be independently controlled, only the cut-off frequencies of filters can be tuned, no simple relation between the filter coefficients and the cut-off frequency
Constantinides and Oppenheim	Transform approach	Simple,The resulting impulse response of the filter is finite and phase is linear	
Stoyanov and Kawamata, Jarske and Mitra, Mitra et al.	Transform approach	High accuracy independent tuning of center frequency and bandwidth	
Jarske et al. , Zarour and Fahmy, Sia and Fahmy	Spectral parameter approximation	Simple relation between the filter coefficients and the cut-off frequency	Huge amount of design time, Deterioration of magnitude response , pass-band and stop-band edges cannot be set to the desired frequencies
Pun et al.	Least Square approach	Filter coefficients are approximated as a linear combination of basis functions	Implementation cost is high because Farrow structures are used
Chan et al., Johansson and	Over-designed fixed filters are		Computational complexity is high and large memory is

Lowenberg	used		required to store the coefficient values
Yu et al.	Filter is realized by shifting the input signal spectrum	Filter is realized using fixed FFB	Requires more delay elements and additional hardware such as digital comparators, increases group delay
Harris, George and Elias	Variable bandwidth filters as a cascade of input and output arbitrary interpolators	No need to compute different filter coefficients	Requires additional hardware such as sample rate converters
Huang and Shyu	Designed variable transition width filter	Adjustable Kaiser window is exploited	Farrow structures are employed which has high arithmetic complexity
Sharma et al., Sharma et al.	FrFT is applied to obtain VDFs	No need to compute coefficients of new desired filter, save design time	Range of adjustability is less, No analytical relationship is found to formulate the transfer function of filter
Saxena and Singh, Singh, Muralidhar et al., Chaturvedi et al.	Evaluated different types of window functions with FrFT	Better spectral parameters can be obtained	No mathematical model is developed
Kumar et al.	Evaluated different types of window functions with FrFT	Better spectral parameters can be obtained, Closed-form expression is derived to obtain FrFT of different window functions	Study is limited to only application of FrFT to window functions
Goel and Singh	Evaluated different types of window functions with LCT	Better spectral parameters can be obtained, Closed-form expression is derived to obtain LCT of different window functions	No analytical relationship is found to formulate the transfer function of filter

To sum up, the design of VDFs whose response changes according to some tuning parameter is taken up as a major research issue in this thesis. To start with, an overview of variable digital filters is presented. It is shown that the frequency response characteristics of digital filters can be varied in order to meet the new desired specifications immediately. The thesis concentrates on designing VDFs by using fractional Fourier transform based technique. The overview of fractional Fourier transform is also presented. A comprehensive literature survey on different methods for the design of VDFs is further presented in this chapter. This is followed by research gaps found in the existing work on VDFs. The objectives of the present research along with the organization of the thesis are given towards the end of this chapter.

1.5 Research Objectives

The work presents the variable digital filter design using FrFT. The demand for VDFs is increased considerably because they are widely used in communication systems, signal processing and image processing.

The research objectives of the thesis include:

(1) To study the present techniques for the design of variable digital filters,

(2) To develop a novel algorithm for the design of variable transition-width FIR filters using fractional Fourier transform,

(3) To develop a novel algorithm for the design of variable cut-off/ band-edge FIR filters using fractional Fourier transform and

(4) Performance evaluation and analysis of the proposed algorithms by comparing with the existing techniques.

The aim of this research is to use the rotation angle α (or fractional order a) of FrFT as a design parameter for variable digital filters. The thesis presents analytical as well as simulation results in order to tune the spectral characteristics of window-based LP FIR filter using the proposed methods. It is concluded that variable digital filters can be designed easily and quickly using FrFT by changing only one adjustable parameter of the transform, while keeping the length of the filter same.

1.6 Thesis Organization

The rest of this thesis is organized as follows:

Chapter 2 presents a detailed introduction to the design of window-based FIR digital filter. Further in this chapter a review on the design of variable transition band window-based LP FIR filters with fractional order a of FrFT as a tuning parameter is discussed. Then, the methodology for designing sharp window-based FIR filters using FrFT is illustrated..The Triangular window, the Gaussian window and the Kaiser window are analyzed with FrFT to establish the dependence of the main-lobe width of the transform of these windows on the fractional order parameter a of the transform. Finally, some examples of LP FIR filters are illustrated to show the variation in transition width with change in the fractional order parameter a (or the rotation angle α) of the transform. The comparison of FrFT based technique is done with the direct methods of tuning digital filters. The summary of the chapter is given towards the end.

In **Chapter 3,** a mathematical model for computing the variable Rectangular windowed low-pass FIR filter transfer function with rotation angle of fractional Fourier transform is developed. The variable magnitude response of Rectangular windowed LP FIR filter is observed for a range of fractional angle α. By changing the rotation angle in the range from $0 \text{ to } \pi/2$, the transition bandwidth, pass-band width and the stop-band attenuation of the fixed length FIR filter are tuned easily. The performance comparison of the proposed FrFT based method over the ordinary method based on FFT (by varying the window length) in tuning FIR filter is also shown.

In **Chapter 4,** a FrFT based design method for simultaneously varying magnitude and phase characteristics of a fixed length window-based LP FIR filter is proposed. It is shown that on changing the fractional order parameter a in the range from 0 to 1 (or by changing the rotation angle α in the range from $0 \text{ to } \pi/2$), the filter characteristics can be tuned easily. FrFT based tuning methodology for tuning the pass-band width and phase characteristics of FIR filter is also discussed. The tuning process involves convolution of FrFT of the filter's desired frequency response (instead of convolving the ordinary FT of the desired filter's impulse response) with the spectra of the suitable window function. The proposed technique is illustrated with some examples to show the variation in the magnitude and phase characteristics of a LP FIR filter with change in fractional order parameter (or the rotation angle parameter) of the transform. It is observed that pass-band width can be increased by reducing the fractional order parameter a of the transform from $1\ to\ 0$. Also, the proposed FrFT based technique is compared with the traditional methods of designing VDFs.

Chapter 5 presents the analysis of finite duration Chirp signal as windowing function using FrFT. Based on the results obtained, it is shown that Chirp under FrFT can prove to be an alternative window function to the existing Rectangular window in achieving better spectral characteristics. A mathematical model for obtaining the closed-form expression of fractional Fourier transform of Chirp window function is developed. The variations in the parameters such as Half Main-Lobe Width (HMLW), Maximum Side-Lobe Level (MSLL) & Side-Lobe Fall-Off Rate (SLFOR) of Chirp with the variation of fractional order parameter a are illustrated through simulation results. The performance of Chirp window with FrFT is compared with some of the existing windows. Finally, the Chirp window-based LP FIR filter using FrFT is designed with modified transition-width characteristics in order to add to the advantage of using FrFT in adjustable FIR filter design.

Finally, the conclusions and future scope is given in the **Chapter 6**.

CHAPTER 2

STUDY OF WINDOW BASED LP FIR FILTER DESIGN WITH VARIABLE TRANSITION BAND USING FrFT

This chapter presents a review on fractional Fourier transformation based technique to obtain FIR filter whose transition width can be controlled with fractional order a of the transform as a tuning parameter while keeping the filter length fixed.

2.1 Introduction

The transition width of window-based digital FIR filters should be narrow for various filtering applications that require fine frequency discrimination. For example, the width of the transition band of window-based FIR filters must be narrow in professional digital audio applications so that signal components just below half the source frequency must pass unaltered, whereas those just above must be sufficiently suppressed. The transition-width is inversely proportional to the filter length. In conventional design, the transition bandwidth of FIR filters can be directly modified either by varying the length (or order) of the filter or by applying variable windows in FIR filter design so that the window shape parameter can be adjusted to control the characteristics of the filter. The direct tuning involves re-computation of the impulse response coefficients to design of a new filter from the existing one which adds to more computations (in terms of arithmetic operations like multiplications, additions and divisions). Also, more memory is needed to store large number of filter coefficients and data registers.

The advantage of tuning FIR filter using FrFT is that impulse response need not be recomputed and filter characteristics can be modified on-line without the need to design a new filter from the existing one. In the present work, the narrow transition width of window-based fixed length LP FIR filter is obtained simply by adjusting the value of fractional order parameter a of the transform. The window method is preferable for the design of FIR filter because of its simplicity as compared to other methods and ease of use. The well defined equations are often available for calculating the window coefficients. The window method is used when desired frequency response specification of a filter can be easily put into a closed form mathematical expression so that desired impulse response coefficients can be evaluated easily. This method is basically used for design of prototype filters like low pass, high pass, band pass etc [51]. The window-based LP FIR filter design is first introduced as below.

2.1.1 FIR Design Based on Window Functions

The FIR filter design by window method uses the idea that the desired frequency response of an ideal LP filter is equal to one for all the pass-band frequencies and equal to zero for all the stop-band frequencies. The desired impulse response of an ideal LP filter, which can be obtained by taking the inverse Fourier transform of the ideal frequency response, has the form of a sinc function of infinite length. A finite impulse response filter is obtained by multiplying the time domain filter coefficients with a window function of a finite width. Truncating and windowing this sinc gives a response that is close to the desired response.

The desired frequency response $H_d(w)$ of an ideal LP digital filter is given by:

$$H_d(\omega) = \begin{cases} 1 & |\omega| \le \omega_c, \\ 0 & otherwise \end{cases} \tag{2.1}$$

where ω_c is the cut-off frequency corresponding to the location of sharp cut-off edge, as shown in Fig. 2.1.

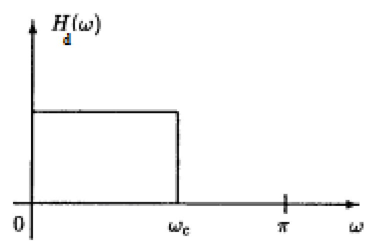

Fig. 2.1 Ideal characteristics of a digital LP filter.

The desired impulse response function of an ideal low-pass system is the inverse Fourier transform of Eq. (2.1) and is given by:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{i\omega n}d\omega,$$

$$= \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{iwn}dw,$$

$$= \begin{cases} \dfrac{\sin(w_c n)}{\pi n}, & n \neq 0 \quad -\infty \leq n \leq \infty \\ \dfrac{w_c}{\pi}, & n = 0 \end{cases} \tag{2.2}$$

The impulse response $h_d(n)$ is infinite in duration. It must be truncated at some point say $n = N-1$ to yield an FIR filter of length N (i.e. 0 *to* $N-1$). The truncation of $h_d(n)$ to length $N-1$ is same as multiplying $h_d(n)$ by a finite length uniform window function called as Rectangular window. The Rectangular window is defined as:

$$\begin{aligned} w(n) &= 1 \qquad 0 \leq n \leq N-1 \\ &\quad 0 \qquad otherwise \end{aligned} \tag{2.3}$$

So that the modified impulse response $h(n)$ of the FIR filter becomes:

$$\begin{aligned} h(n) &= h_d(n) w(n) \\ &= \begin{cases} h_d(n), & 0 \leq n \leq N-1 \\ 0 & otherwise \end{cases} \end{aligned} \tag{2.4}$$

Now, the multiplication of the window function $w(n)$ with $h_d(n)$ in time-domain (also known as windowing technique) is equivalent to convolution of $H_d(\omega)$ with $W(\omega)$ in the frequency-domain i.e.:

$$H(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(v) W(\omega - v) dv \tag{2.5}$$

where $W(\omega)$ is the frequency domain representation of the window function and is given by:

$$W(\omega) = \sum_{n=0}^{N-1} w(n) e^{-j\omega n} \quad , \tag{2.6}$$

and $H(\omega)$ is the modified or actual frequency response of the truncated FIR filter. The ideal frequency response is smoothed by the Fourier transform $W(\omega)$ of the window. From Eq. (2.5), it is clear that if $W(\omega)$ ideally approximates a delta function, then $H(\omega)$ appears very much similar to $H_d(\omega)$ except where $H_d(\omega)$ changes very abruptly. The window function is selected so that the length of the window $w(n)$ should be as small as possible in duration in order to minimize the computational complexity, while having $W(\omega)$ approximate an impulse. Clearly, these two requirements are conflicting as no signal can be time-limited and band-limited simultaneously [74].

The frequency response $W(\omega)$ of a window function is continuous spectrum characterized by a main-lobe in the middle of the spectrum along with a series of side lobes on both sides of main-lobe with decreasing amplitudes [13]. These side lobes decay with asymptotic attenuation of f^{-n} as $f \to \infty$ where n is an integer. The two parameters that predict the performance of a window in FIR filter design are its main-lobe width and the relative side-lobe level. The width of the main lobe can be defined in many ways. For example, it may be taken to be the distance (or half the distance) between the nearest zero crossings on both sides of the main-lobe in units of radians/sample or Hz. It can be taken to be 3-dB width i.e. where the squared magnitude response of window function $|W(\omega)|^2$ drops by $1/2$. The relative side-lobe level is the difference in dB between the amplitudes of the largest side-lobe and the main-lobe.

The application of Rectangular window in FIR filter design corresponds to direct truncation of the infinite-duration impulse response. The direct truncation of $h_d(n)$ to N terms to obtain $h(n)$ introduces transition band in $H(\omega)$ at frequency points where the frequency response of the ideal filter has discontinuities, i.e., at pass band edges. The width of the transition bands depend on the width of the main lobe of the Fourier transform of the window function. Also, the side ripples present in the window transform leads to having large ripples (oscillations) in the frequency response of the actual filter. This is the well-known Gibbs phenomenon effect.

As the length N of the window function is increased, the width of each lobe present in the Fourier transform $W(\omega)$ of the window function decreases, however, the area under each lobe remains constant so that height of the side-lobes also increases relative to increase in the main-lobe height [60]. The oscillations of the resulting filter response occur more rapidly in $H(\omega)$ around the point of discontinuity as N increases but do not decrease in amplitude. To reduce the oscillations in the resulting filter response, the area under the side-lobes should be very small. The Gibbs phenomenon can be reduced by multiplying $h_d(n)$ with a smoother window function that decays gradually near its endpoints and cuts-off to zero less sharply than the Rectangular window.

There exist dozens of possible shapes of windows in literature [6, 32, 44, 58, 80]. On the basis of shapes of window functions, the windows can be divided into two categories (i) **Fixed windows** and (ii) **Adjustable windows**. The windows which have only one adjustable parameter i.e. the window length N are called as fixed windows and have fixed values of side-lobe suppression. The examples of fixed windows are Rectangular, Hanning, Hamming, and Triangular window etc. The height of the ripples can be controlled by the type of window as the side-lobe suppression depends on the particular window shape and remains relatively independent of window length. The window which has two parameters, one is the window length and the other one is the adjustable shape parameter, is called an adjustable window. The merit of adjustable window is that it can provide any desired value of side-lobe suppression. The examples of adjustable windows are Kaiser and Gaussian window etc. Some general properties of windows are given in [10].

By using a smoother window, the peak side-lobe decreases but the main-lobe width increases. This will increase the width of the transition band as it is proportional to the main-lobe of the window transform. In order to achieve the same transition bandwidth with a smoother window, the length of window needs to be increased.

The objective of this research is to apply Fractional Fourier Transform (FrFT) in filter design so that the frequency response characteristics of window-based fixed length FIR filter can be tuned simply by adjusting the parameter a of the transform. The notation a denotes the adjustable fractional order parameter of FrFT. The advantage of using FrFT in the design of variable digital filters is that the computational resources remain fixed and there is no need to operate a filter design algorithm in the background to compute new coefficients for each desired filter characteristic. An introduction to fractional Fourier transform is given below.

2.2 Fractional Fourier Transform

The Fourier transform is a widely used tool for representing the relation of signal between the time and frequency domain in digital signal processing [9, 15, 46]. The fractional Fourier transform is a generalization of the ordinary Fourier Transform and was originally introduced in

1980 by Namias [56]. The author defined FrFT as a mathematical tool for solving quantum mechanics problems. In 1994, Almeida [5] introduced FrFT to the signal processing community. Ozaktas et al. [67] defined fractional Fourier transform as an important mathematical tool which can be used in signal processing and optical information processing with enhanced results.

Almeida [5] suggested that fractional Fourier transform of a signal is interpreted as a rotation of signal by an arbitrary angle α in the Time-Frequency (TF) plane. The angle of rotation α is a non-integer multiple of $\pi/2$. An FrFT with $\alpha = \pi/2$ corresponds to the ordinary Fourier transform, and an FrFT with $\alpha = 0$ corresponds to the identity operator. The successive application of FrFT of various angles is equal to a single application of the FrFT whose angle is equal to the sum of individual angles. The authors defined FrFT by taking two orthogonal axes corresponding to time and frequency in the time-frequency plane as shown in Fig. 2.2.

Let $x(t)$ be a signal represented along the time axis. Its Fourier transform $X(\omega)$ is represented along frequency axis, so the Fourier transform operator designated by F can be viewed as a change in representation of the signal corresponding to counterclockwise rotation of $\pi/2$ radians.

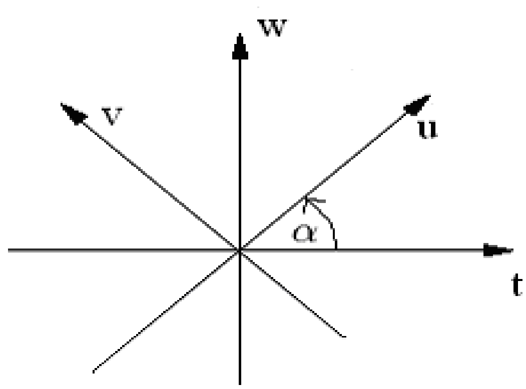

Fig. 2.2: Time-frequency plane and set of coordinates (u,v) rotated by an angle α relative to the original coordinates (t,ω).

This is consistent with the result of the repeated application of the Fourier transform operator F. If the operator F is applied twice (which corresponds to two successive rotations through angle $\pi/2$ of the t axis), the result becomes inversion of time axis i.e. $F^2 x(t) = x(-t)$. Similarly, rotation through angle $3\pi/2$ results in an axis directed along $-\omega$ i.e.

$F^3 x(t) = X(-\omega)$. And four successive rotations through angle $\pi/2$ means doing no transform i.e. $F^4 x(t) = x(t)$, since rotation through 2π should leave the signal unaltered.

Let the linear operator, which corresponds to a rotation by an angle α that is not a multiple of $\pi/2$, is represented by R^{α}. This operator should have the following properties as listed in the Table 2.1.

Table 2.1:
Basic properties of fractional Fourier transform

$R^0 = I$ (Identity operator): zero rotation
$R^{\pi/2}$: FT operator
R^{π} : Time-reverse operator
$R^{3\pi/2}$: Inverse FT operator
$R^{2\pi} = I$: 2π rotation
$R^{\alpha 1}\, R^{\alpha 2} = R^{\alpha 1 + \alpha 2}$: Additivity of rotations

The FrFT which satisfies these properties is defined by means of the transformation kernel:

$$K_{\alpha}(t,u) = \begin{cases} \sqrt{\dfrac{1 - i\cot\alpha}{2\pi}} \exp\left\{ i\dfrac{(t^2+u^2)}{2}\cot\alpha - iut\csc\alpha \right\}, & \text{if } \alpha \text{ is not a multiple of } \pi \\ \delta(t-u), & \text{if } \alpha \text{ is a multiple of } 2\pi \\ \delta(t+u), & \text{if } \alpha + \pi \text{ is a multiple of } 2\pi, \end{cases} \quad (2.7)$$

where the imaginary unit $i = \sqrt{-1}$.

The fractional Fourier transform of a function x, with an angle α, is defined by $R^{\alpha} x = X_{\alpha}$. The corresponding FrFT is given by:

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,u)\, dt \tag{2.8}$$

$$= \begin{cases} \sqrt{\dfrac{1 - i\cot\alpha}{2\pi}} \exp\left(i\dfrac{u^2}{2}\cot\alpha \right) \displaystyle\int_{-\infty}^{\infty} x(t) \exp\left\{ i\dfrac{t^2}{2}\cot\alpha - iut\csc\alpha \right\} dt, & \text{if } \alpha \text{ is not a multiple of } \pi \\ x(t), & \text{if } \alpha \text{ is a multiple of } 2\pi \\ x(-t), & \text{if } \alpha + \pi \text{ is a multiple of } 2\pi \end{cases}$$

(2.9)

The original function $x(t)$ can also be recovered from a FrFT operation with angular parameter $(-\alpha)$ by using the relation:

$$x(t) = \int_{-\infty}^{\infty} X_\alpha(u) K_{-\alpha}(t,u)\, du\,, \tag{2.10}$$

where $K_{-\alpha}(t,u) = K_\alpha{}^*(t,u)$ (superscript $*$ denotes complex conjugation). FrFT operator corresponds to a counterclockwise rotation by an angle α along an axis u (Fig. 2.2). The Eq. (2.9) shows that for angles that are not multiples of π, the fractional Fourier transform can be computed with the following four steps:

1. Multiplication of function with a chirp in one domain,
2. Taking the Fourier transform of this product (with its argument scaled by $\csc\alpha$),
3. Again multiplying with a chirp in the transform domain, and
4. Then finally multiplication with a complex amplitude factor.

Three different cases are discussed by Saxena and Singh [84] in order to show that the FrFT defined by Eq. (2.9) satisfy the basic properties given in Table 2.1:

Case (I): When $\alpha = \pi/2$,

The Eq. 2.9 corresponds to:

$$X_{\pi/2}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) \exp(-iut)\, dt \tag{2.11}$$

This shows that FrFT corresponds to the ordinary Fourier transform for $\alpha = \pi/2$.

Case (II): When $\alpha = 0$,

As α approaches 0, $\sin\alpha$ approaches α, $\cot\alpha$ approaches $1/\alpha$. Using the fact that in sense of generalized functions:

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{i\pi\varepsilon}} e^{-x^2 / i\varepsilon} = \delta(x) \tag{2.12}$$

So that Eq. (2.9) reduces to:

$$X_0(u) = \int_{-\infty}^{\infty} \delta(u-t)x(t)dt = x(t) \tag{2.13}$$

This implies that the transform kernel reduces to identity transform and FrFT corresponds to identity operator.

Case (III): When $\alpha = \pi$,

Using the same procedure as used in case (II), the result of FrFT is given by:

$$X_\pi(u) = \int_{-\infty}^{\infty} \delta(u+t)x(t)dt = x(-t)\,. \tag{2.14}$$

The FrFT corresponds to the time-reverse operation. Similarly, the other remaining properties can also be derived easily.

Ozaktas et al. [67] defined fractional Fourier transform as a generalization of the ordinary Fourier transform with an order (or power) parameter denoted by a. Let F^a represents taking Fourier transform a times where a is called as order (or power) parameter of the Fourier transform operation. FT allows a having only integer values. Integer values of a correspond to repeated application of the Fourier transform operator. Fourier transform which is generalized so that parameter a can take any fractional value is called fractional Fourier transform.

Thus, the *ath* order fractional Fourier transform operator is the *ath* power of the ordinary Fourier transform operator or applying Fourier operator a times where a need not be an integer. If F denotes the ordinary Fourier transform operator, and F^a will represent the *ath* order fractional Fourier transform operator. The authors defined the meaning of fractional powers of the Fourier transform operator by taking a discrete case where the discrete ordinary and fractional Fourier transform operators are represented by matrices.

Let the $N \times 1$ column vector x denote a discrete signal, the $N \times 1$ column vector x_1 its discrete Fourier transform, and F the $N \times N$ discrete Fourier transform matrix with elements $F_{kl} = N^{-1/2} \exp[-i2\pi(k-1)(l-1)/N]$ so that:

$$x_1 = Ff \tag{2.15}$$

The notation k and l are integers which denote the row and column index respectively.

The ath order fractional Fourier transform of x denoted as x_a can be defined by multiplying f with the ath power of discrete Fourier transform matrix:

$$x_a = F^a x \tag{2.16}$$

When $a=1$, $F^a = F^1 = F$ and $x_a = x_1$ which implies that the first-order fractional Fourier transform operator is the ordinary FT. When $a = 0$, the zeroth-order fractional Fourier transform operator F^0 is equal to the identity operator I, so that $x_0 = x$. The meaning of F^a can be easily interpreted when a is a rational number (ratio of two integers) i.e. $a = m/n$. $F^{m/n}$ is simply the mth power of $F^{1/n}$, where the latter is the nth root of F. It can be stated that $(m/n)th$ fractional Fourier transform operation is that operation is simply the $(1/n)th$ transform applied m times in succession.

Further, the authors used the notation $x_a(u)$ or equivalently $F^a x(u)$ to denote the ath order fractional Fourier transform of the signal $x(u)$. They defined the fractional Fourier transform in a most direct and concrete way by interpreting it as the operator F^a acting on the function $x(u)$ with the result being expressed in the u domain:

$$x_a(u) \equiv F^a x(u) \equiv F^a[x(u)](\mathrm{u}) \equiv \left(F^a[x(u)]\right)(\mathrm{u}) \tag{2.17}$$

This interpretation is appropriate regardless of whether the operator F^a denotes a system or a transformation. The same dummy variable u is used both for the original function in the time domain, and its fractional Fourier transform. When the fractional Fourier transform operator is interpreted as system acting on an input signal x, it is possible to suppress both dummy variables and write:

$$x_a \equiv F^a x \equiv F^a[x] \tag{2.18}$$

The notation $F^a[\cdot]$ or simply F^a is used as the ath order fractional Fourier transform operator which transforms a signal into its fractional Fourier transform. A system is usually a rule or a mathematical abstraction which alters the characteristics of input signal in a certain way to produce output signal. A system is an abstract entity whose existence is independent of which coordinate system is used. The output of a system may be represented in either the same representation as the input, or a different one, without affecting the nature of the system.

Also sometimes, especially when the fractional Fourier transform is interpreted as a transformation from one representation of a signal to another, it is useful to distinguish the argument of the transformed signal from that of original signal. In this case u_a denote the argument of the ath order fractional Fourier transform and

$$x_a(u_a) = (F^a[x(u)])(u_a) \tag{2.19}$$

With this convention, u_0 corresponds to u, the time coordinate, and u_1 corresponds to the frequency coordinate. The order parameter a is restricted to a real number. The ath order fractional Fourier transform is a linear operation defined by the integral

$$x_a(u) = \int_{-\infty}^{\infty} K_a(u,u')x(u')du' \tag{2.20}$$

$$K_a(u,u') = \sqrt{1-i\cot\alpha}\exp\left[i\pi\left(u^2\cot\alpha - 2uu'\csc\alpha + u'^2\cot\alpha\right)\right], \quad \alpha = a\pi/2 \tag{2.21}$$

where α is a continuous transformation parameter interpreted as a rotation angle in the phase plane.

The relation between FrFT order a and α is given by $\alpha = a\pi/2$. Abushagur and Almanasrah [2] defined the variable u as an arbitrary frequency scaling representing the FrFT plane. It is a time variable when $\alpha = 0$, and a frequency variable when $\alpha = \pi/2$. The variable u can be interpreted as some hybrid time/frequency variable. The interpretation of u changes gradually from time to frequency, reflecting temporal changes in the frequency content of the transformed signal as α takes value from 0 to $\pi/2$. The changed frequency characteristics of the transformed signal are revealed for different values of α.

In summary, the FrFT is a linear transform, continuous in the angle α and satisfies the basic conditions for being interpretable as a rotation in the time-frequency plane. As the rotation operation can be generalized for intermediate angles also, a wider class of signals can be analyzed using FrFT. In other words, fractional Fourier transform is a parameterized transform with an adjustable transform parameter $a = 2\alpha/\pi$, where a is a real number. For α lying in the range 0 to $\pi/2$, the parameter a varies from 0 to 1. Both the parameters α and a of the transform can be used interchangeably.

FrFT takes advantage of an extra degree of freedom possessed by the order parameter a in achieving better performance over ordinary Fourier transform [19, 23-25, 65-66, 81, 107]. Also, the FrFT has the same order of complexity (in terms of multiplications, additions and divisions) as that of ordinary FT and can be used successfully in the applications where FT fails to work. Every property and application of the FT becomes a special case of the fractional Fourier transform and there exists a potential for generalization and improvement by using FrFT in every area where Fourier transforms are used [77, 86, 102]. The FrFT is a generalization of the FT and most of the properties of the FT have their corresponding generalization versions of the FrFT. The well established properties of FrFT which are extensions of the corresponding properties of the FT are listed in Table 2.2.

Table 2.2:

Properties of Factional Fourier Transform

Operation	Signal $x(t)$	Fractional Fourier Transform $X_\alpha(u)$

Time shift	$x(t-\tau)$	$e^{-\frac{1}{2}i\tau^2 \sin\alpha\cos\alpha + iu\tau\sin\alpha} X_\alpha(u-\tau\cos\alpha)$
Modulation	$x(t)e^{ivt}$	$e^{-\frac{1}{2}iv^2 \sin\alpha\cos\alpha + iuv\cos\alpha} X_\alpha(u-v\sin\alpha)$
Inversion of time axis	$x(-t)$	$X_\alpha(-u)$
Scaling of time axis	$\lvert M\rvert^{-1/2}x(t/M)$	$\sqrt{\frac{1-i\cot\alpha}{1-iM^2\cot\alpha}}\exp\left(i\pi u^2\cot\alpha\left(1-(\cos^2\alpha')/(\cos^2\alpha)\right)\right)$ $\times X_{\alpha'}(\frac{Mu\sin\alpha'}{\sin\alpha})$, where $\alpha' = \arctan(M^{-2}\tan\alpha)$
Differentiation	$\frac{d^m x(t)}{dt^m}$	$\left(-iu\sin\alpha + \cos\alpha\frac{d}{du}\right)^m X_\alpha(u)$
Integration	$\int_b^t x(t')dt'$	$\sec\alpha\exp\left(-i(u^2/2)\tan\alpha\right)\int_b^u X_\alpha(z)\exp\left(i(z^2/2)\tan\alpha\right)dz$ $if\alpha-\pi/2$ *is not a multiple of* π
Multiplication	$t^m x(t)$	$\left(u\cos\alpha - i\sin\alpha\frac{d}{du}\right)^m X_\alpha(u)$
Division	$x(t)/t$	$-i\sec\alpha\exp\left(i(u^2/2)\cot\alpha\right)\int_{-\infty}^u x(z)\exp\left(-i(z^2/2)\cot\alpha\right)dz$ $if\alpha$ *is not a multiple of* π

The convolution and product theorem for the ordinary Fourier transform do not extend in a simple way to the FrFT with arbitrary angle α . The convolution theorem for FrFT is given by Almeida [4] in 1997 and then by Zayed [106] in 1998. In the year 2010, a modified convolution theorem for FrFT is proposed by Singh and Saxena [92].

The product theorem for fractional Fourier transform is not having a closed-form expression which can be widely accepted. The product theorem for fractional Fourier transform is defined by several researchers in recent past, but, all those definitions do not generalize very appropriately the corresponding classical results for the Fourier transform. The product theorem for the ordinary Fourier transform states that the Fourier transform of product of two signals is the convolution of their individual Fourier transforms. The definition given by Almeida [4] for FrFT of product of two functions does not converge to the classical definition of product theorem under ordinary Fourier transform for fractional angle $\alpha = \pi/2$. In the definition proposed by Zayed [106], the transforms of the signals are required to be multiplied at least three times by the different chirp signals in order to evaluate FrFT of product of the signals which makes this definition difficult to realize. This is because the chirp signals can never be obtained accurately in communication systems [92].

A new definition for FrFT of product of two functions is proposed in the thesis to overcome the difficulties of the previously stated definitions. The proposed product theorem for FrFT is easy to implement as the product theorem for Continuous Fourier Transform (CFT) and requires lesser number of computations as compared to existing models. Two signals $f(u')$ and $g(u')$ in time domain are considered to define product theorem under FrFT. The notation u' is used here to denote the time coordinate.

Proposed Definition: The product of two functions $f(u')$ and $g(u')$ in time domain is equivalent to weighted convolution of their continuous fractional Fourier transforms.

Proof: Consider the FrFT of signal $f(u')$ is denoted by $F_\alpha(u)$ and the FrFT of signal $g(u')$ is denoted by $G_\alpha(u)$ i.e.

$$f(u') \xleftrightarrow{\text{FrFT with angle } \alpha} F_\alpha(u) \tag{2.22}$$

and

$$g(u') \xleftrightarrow{\text{FrFT with angle } \alpha} G_\alpha(u) \tag{2.23}$$

then fractional Fourier transform $Z_\alpha(u)$ of $z(u')$ which is equal to product of $f(u')$, $g(u')$ and a chirp given by $\exp(i\frac{u'^2}{2}\cot\alpha)$, is written as:

$$z(u') = f(u')\, g(u')\, \exp(i\frac{u'^2}{2}\cot\alpha) \xleftrightarrow{\text{FrFT with angle } \alpha} Z_\alpha(u) \tag{2.24}$$

From the definition of fractional Fourier transform given by (2.9), $Z_\alpha(u)$ becomes:

$$Z_\alpha(u) = \int_{-\infty}^{\infty} f(u')g(u')\exp(iu'^2/2)K_\alpha(u',u)du' \tag{2.25}$$

Also, from Eq. (2.10), the inverse FrFT is expressed as:

$$f(u') = \int_{-\infty}^{\infty} F_\alpha(v)K_\alpha^{*}(v,u')dv \tag{2.26}$$

The notation v in (2.26) is a separate frequency variable.

Substituting the value of $f(u')$ given by (2.26) in Eq. (2.25) so that:

$$Z_\alpha(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_\alpha(v)K_\alpha^{*}(u',v)dv.g(u')K_\alpha(u',u)\exp(iu'^2/2)du' \tag{2.27}$$

Now, substituting the value of transform kernel $K_\alpha(u',u)$ and its complex conjugate, and interchanging the order of integration thereafter, $Z_\alpha(u)$ becomes equal to:

$$Z_\alpha(u) = \sqrt{\frac{1+i\cot\alpha}{2\pi}} \times \int_{-\infty}^{\infty} F_\alpha(v)\left[\int_{-\infty}^{\infty} g(u')\sqrt{\frac{1-i\cot\alpha}{2\pi}}\exp\{i(\frac{u^2-v^2}{2})\cot\alpha - iu'(u-v)\csc\alpha\}\exp\{i(u'^2/2)\cot\alpha\}du'\right].dv \tag{2.28}$$

In Eq. (2.28), $u^2 - v^2$ can also be written as:

$$u^2 - v^2 = (u-v)^2 + 2(uv - v^2) \tag{2.29}$$

Replacing $u^2 - v^2$ by (2.29) in Eq. (2.28) and rearranging, one gets:

$$Z_\alpha(u) = \sqrt{\frac{1+i\cot\alpha}{2\pi}} \times \int_{-\infty}^{\infty} F_\alpha(v)\left[\int_{-\infty}^{\infty} g(u')\sqrt{\frac{1-i\cot\alpha}{2\pi}}\exp\{i(\frac{u'^2+(u-v)^2}{2})\cot\alpha - iu'(u-v)\csc\alpha\}du'\right]\exp\{\frac{i}{2}(2uv-2v^2)\cot\alpha\}.dv \tag{2.30}$$

$$= \sqrt{\frac{1+\mathrm{i}\cot\alpha}{2\pi}} \int_{-\infty}^{\infty} F_\alpha(v) G_\alpha(u-v) \exp\{iv(u-v)\cot\alpha\}\, dv \tag{2.31}$$

Now, considering $w(u,v) = \exp\{iv(u-v)\cot\alpha\}$ as a weighting function and defining: $\overline{G_\alpha}(u-v) = G_\alpha(u-v)w(u,v)$, the Eq. (2.31) becomes:

$$Z_\alpha(u) = \frac{\sqrt{1+i\cot\alpha}}{\sqrt{2\pi}} \left[(F_\alpha \otimes \overline{G_\alpha})(u) \right] \tag{2.32}$$

Where '$\otimes$' denotes the convolution operation for the Fourier transform and is defined as:

$$h(x) = (f \otimes g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt \tag{2.33}$$

It can be easily seen that accumulated error will be less if the product theorem is realized using the derived expression given in Eq. (2.31) as it requires only one chirp multiplication. Also, the evaluation of convolution in (2.32) using derived model is more precise, accurate and efficient.

Particular Case: Taking value of $\alpha = \pi/2$, the Eq. (2.31) becomes:

$$Z_\alpha(u) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} F_\alpha(v) G_\alpha(u-v) dv \tag{2.34}$$

or

$$Z_\alpha(u) = \frac{1}{\sqrt{2\pi}} \left[(F_\alpha \otimes G_\alpha)(u) \right] \tag{2.35}$$

From the Eq. (2.35), it can be seen that product theorem defined using (2.31) converge to the classical convolution theorem for ordinary Fourier transform at $\alpha = \pi/2$ similarly as FrFT becomes FT at an angle $\pi/2$ and is better and more relevant.

The suggested product theorem is more accurate, compact and simple. The mathematical model for the FrFT of product of two signals described can be considered as a product theorem for FrFT which proves to be a generalization of product theorem for ordinary FT.

FrFT has wide range of applications in signal and image processing [27, 41, 45, 61, 103], optics [8, 49, 62-63,], radar and sonar signal processing [7, 12, 35, 48, 57, 99] and for processing chirp-like signals [53] etc. The direct computation of the fractional Fourier transform using digital computers is required for all these practical applications [64]. In order to implement FrFT in practical applications, discrete version of continuous FrFT is defined. The discretization of continuous fractional Fourier transform for the purpose of digital computation is known as Discrete Fractional Fourier Transform (DFrFT).

DFrFT will be a generalization of the DFT that obeys the rotation rules as the continuous FrFT and provides the similar results as the FrFT [70-73]. The essential property of discrete fractional Fourier transform is that it maps the samples of function into the samples of its fractional Fourier transform to sufficient degree of accuracy. The algorithm used for computation of DFrFT needs to obey the following properties of continuous FrFT:

(i) Additivity property: $F^{\alpha_2} F^{\alpha_1} = F^{\alpha_2 + \alpha_1}$

(ii) Unitarity: $(F^{\alpha})^{-1} = (F^{\alpha})^{H} = F^{-\alpha}$, where $(\,)^{H}$ denotes the conjugate transpose of the operator

(iii) Reduction to the DFT when the order is equal to unity and

(iv) Similarity condition.

The additivity property means that application of the transform with angular parameter α_1 followed by an application of the transform with angular parameter α_2 is equivalent to the application of the transform with angular parameter $\alpha_1 + \alpha_2$. If additivity property holds, the inverse transform of DFrFT with angular parameter α is the DFrFT angular parameter $-\alpha$. The advantage to define a DFrFT which satisfies additivity property is that signals processed under FrFT can be easily transformed into time domain by a unified computer program with minus angular parameter.

The DFrFT should also satisfy the unitary property for the transform to be reversible. The unitarity and index additivity are essential properties of the continuous transform and it is desirable that the discrete transform satisfy these properties exactly. The third property is necessary for the DFrFT to be a consistent generalization of the ordinary DFT.

The similarity condition means that the transform results of DFrFT are similar to the FrFT. The DFrFT should obey similarity condition and provide the similar transform results as those of continuous case so that continuous signal processing algorithms derived in continuous fractional Fourier domains can be directly modified into the digital signal cases by replacing continuous FrFT with DFrFT [70, 77]. To sum up, for discrete signals DFrFT provides a method for computing the fractional Fourier transform [11, 82].

The width of the transition band of digital filter is inversely proportional to the filter length. To sharpen the filter transition width directly by increasing its length, impulse response coefficients are to be recomputed and filter redesign is involved. Whereas, by using fractional order a as the tuning parameter, the variability in filter characteristics can be obtained on-line without the need to redesign a filter and keeping the computational burden fixed as discussed in the section below.

2.3 Transition Bandwidth Tuning of Window-based FIR Filters using Fractional Fourier Transform

The analysis of window functions using fractional Fourier transform establishes the dependence of their main-lobe width on the adjustable rotation parameter of the transform [43, 54-55, 87-88]. As the transition-band width of window-based FIR filters is proportional to the main-lobe of the window transform, therefore, transition-width can be made directly dependent on fractional angle or order of the transform. The FrFT can be efficiently used in window-based FIR filter design to modify the filter characteristics immediately so that the desired application can be achieved [16, 54, 79, 87-88]. The filter characteristics are modified by convolving FrFT of the window function with the desired transfer function of the filter so that different frequency characteristics can be obtained simply by varying the fractional order parameter a (or the fractional angle $\alpha = a\pi/2$).

The proposed methodology for designing FrFT-based VDFs includes the following steps:

1. Take the known desired impulse response coefficients $h_d(n)$ of low-pass FIR filter of specified length and initial cut-off frequency.

2. Compute and store the FFT of the desired impulse response $h_d(n)$ to obtain the desired frequency response function $H_a(f)$ of the filter. For all values of fractional angle, the frequency response $H_a(f)$ of the filter is constant.

3. Compute the window function response coefficients $w(n)$ of the low-pass FIR filter corresponding to the initial filtering requirements.

4. Take the fractional Fourier transform of $w(n)$ for fractional order a to obtain $F^a[w(n)]$. The frequency response of window is now parameterized in transition-width with fractional order parameter a as a tuning parameter. When $a=1$, the Fourier transform of window is obtained.

5. Convolve FrFT of the window with the constant desired transfer function of the filter to obtain adjustable frequency response characteristics of the filter.

The overall process of designing window-based LP FIR filter with sharp transition width using FrFT is shown with the help of block diagram in Fig. 2.3 below.

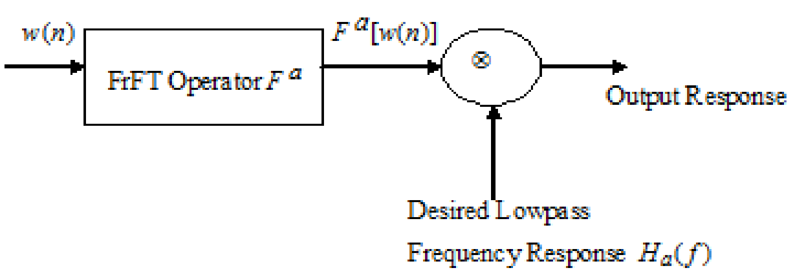

Fig. 2.3: Block Diagram of Tuning Procedure for Obtaining Sharp Transition Bandwidth.

The fractional Fourier transform operator F^a is interpreted as a system here which is applied to an input function $w(n)$. The changed frequency characteristics of a digital filter can be observed for different values of fractional order a.

2.3.1 Analysis of Window Functions using FrFT

The Triangular window, Gaussian window and Kaiser window are taken for doing the analysis with FrFT. In time domain, these windows can be written as below.

The **Triangular** window is given by**:**

$$w_T(n) = \begin{cases} 1 - \dfrac{2|n|}{N-1} & for|n| \le \dfrac{N-1}{2} \\ 0 & otherwise \end{cases} \qquad (2.36)$$

Triangular window can be seen as convolution of two half-sized Rectangular windows (for N even).

The **Gaussian window** is defined as:

$$w_G(n) = e^{-1/2\left(\frac{\rho n}{(N-1)/2}\right)^2} \qquad for|n| \le \frac{N-1}{2} \qquad (2.37)$$

where $\rho = (N-1)/2\sigma$, and σ is usual standard deviation of Gaussian value.

This window is parameterized on ρ, which acts as the reciprocal of the standard deviation, a measure of the width of its Fourier transform. Increased ρ will decrease the width of the window and reduce the severity of the discontinuity at the boundaries.

The **Kaiser window** is defined as:

$$w_K(n) = \frac{I_0\left(\gamma\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right)}{I_0(\gamma)} \qquad if\ |n| \le \frac{N-1}{2} \qquad (2.38)$$

where the Bessel function $I_0(x)$ is defined by the summation:

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2 \qquad (2.39)$$

Like all window functions, the Kaiser window is symmetric about its middle, $n = 0$, and has the value $w(0) = 1$ there. At the endpoints, $n = -(N-1)/2$ and $n = (N-1)/2$, it has the value $1/I_0(\gamma)$ because $I_0(0) = 1$. For $\gamma = 0$, the Kaiser window reduces to the rectangular one. Table 2.3 below gives the relationship of the Kaiser window to some of the commonly used windows.

Table 2.3:

Relationship of the Kaiser window to other windows

Type of window	Equivalent Kaiser window, γ
Rectangular	0
Bartlett	1.33
Hanning	3.86
Hamming	4.86
Blackman	7.04

The Kaiser window has two parameters: the window length N and a shape parameter γ. Increasing γ will reduce the side-lobe level, but the main-lobe becomes wider. Increasing N while holding γ constant causes the main lobe to reduce in width, but the amplitude of the side lobes remains unaffected. The parameter γ controls the trade-off between the main-lobe width and the peak side-lobe level. By adjusting N and γ, the side-lobe amplitude and main-lobe width can be properly designed.

There is a standard formula [6] for the Kaiser window shape parameter γ. To ensure that the Fourier transform of the Kaiser window suppresses high-frequency components to more than $-\Delta\, dB$ (attenuation), the shape parameter is set to be:

$$\gamma = \begin{cases} 0.1102(\Delta - 8.7) & if\ \Delta > 50, \\ 0.5842(\Delta - 21)^{0.4} + 0.07886(\Delta - 21) & if\ 50 \geq \Delta \geq 21, \\ 0 & if\ 21 > \Delta. \end{cases} \qquad (2.40)$$

The widely used Kaiser window is easy to compute window and has near-optimum performance (in the sense of minimizing the side-lobe energy of the window). Thus, by selecting the shape and duration of the window, the magnitude characteristics of the resulting FIR filter can be controlled.

2.3.1.1 Results

The behavior of Triangular window (length $N=201$), Gaussian window (length $N=51$ and $\rho=3.5$) and Kaiser window (length $N=11$ and $\gamma=3.5$) is analyzed by computing FrFT of these windows where, the γ and ρ are the variable shape parameters of Kaiser and Gaussian window respectively. The modified spectral characteristics of these windows are obtained for different values of fractional order a. Figures 2.4(a)-2.6(a) show the plots of Triangular, Gaussian, and Kaiser window functions in time-domain respectively. The magnitude plots of fractional Fourier transform of these windows as a function of gain (dB) versus normalized frequency (in cycles/sample), are illustrated with the help of Figs. 2.4(b) to 2.6(b) respectively. Figures 2.4(b)-2.6(b) show the variations in the main-lobe width of these windows with the variation of the fractional order parameter a. The width of the main-lobe can be defined as either the frequency point where the window response becomes 0.707 (-3dB) times of the main lobe peak gain called as -3dB main-lobe width or 0.5 (-6dB) times of the main lobe peak gain called as -6dB main-lobe width.

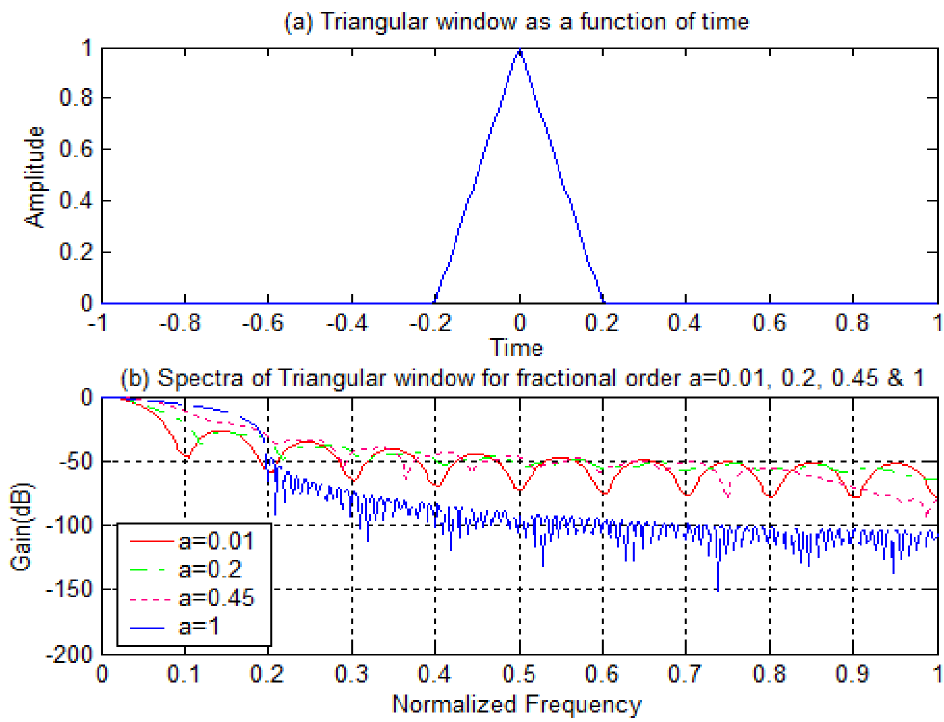

Fig. 2.4: Plot of the Triangular window (a) Triangular window in time-domain, (b) Decibel magnitude plot of FrFT of Triangular window showing variations in response for fractional order a =0.01, 0.2, 0.45 and 1.

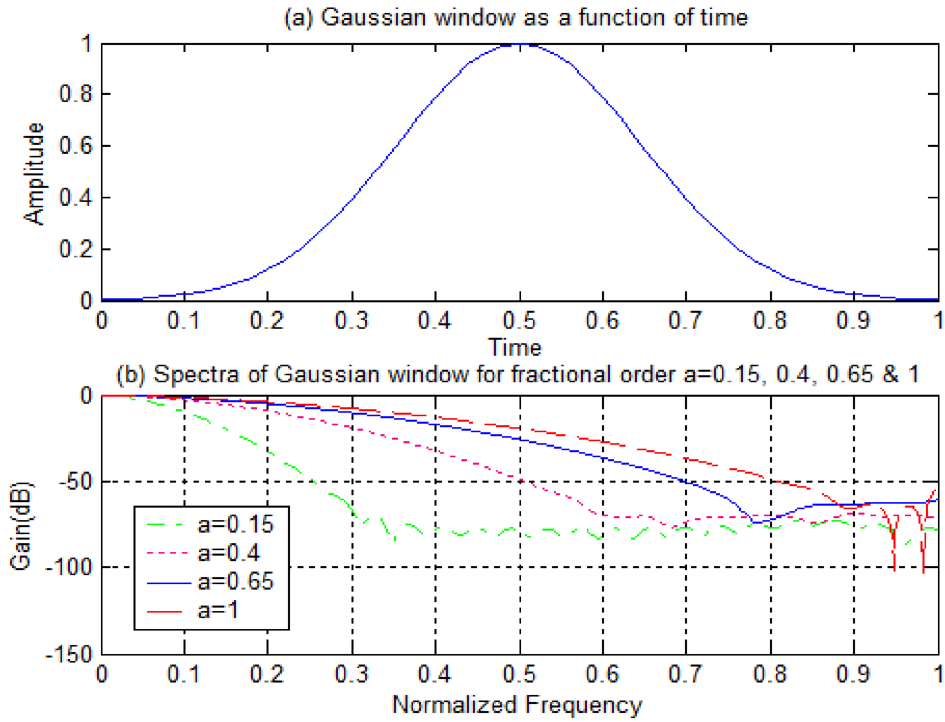

Fig. 2.5: Plot of the Gaussian window (a) Gaussian window in time-domain, (b) The magnitude plot of FrFT of Gaussian window showing variations in response for fractional order $a = 0.15$, 0.4, 0.65 and 1.

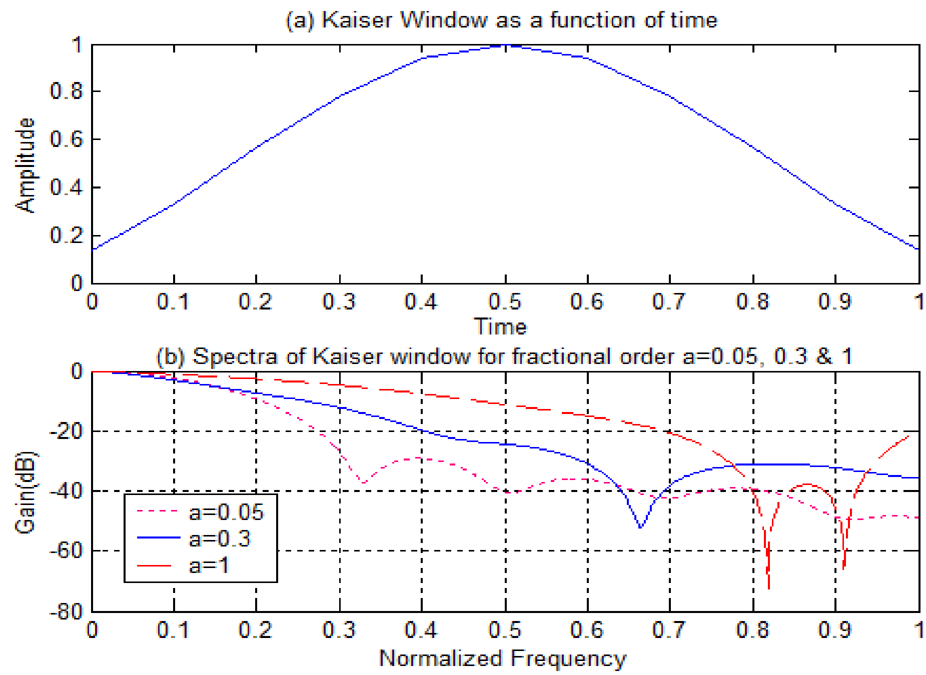

Fig. 2.6: Plot of the Kaiser window (a) Kaiser window in time-domain, (b) The magnitude plot of FrFT of Kaiser window showing variations in response for fractional order a =0.05, 0.3 and 1.

It is observed that as the fractional order a is reduced from $1\ to\ 0$, the main-lobe width shrinks. This feature is then used in tuning the transition width of low pass filter. It is well known that the transition band width is proportional to the main lobe of window transform in FIR filters designed using window functions. Therefore, the width of the transition region can be varied with adjustable parameter of the FrFT. The change in the frequency characteristics of window-based FIR filters, which is obtained by convolving the FrFT of the window function with the desired transfer function of the filter, is illustrated through examples.

2.3.2 Examples of Low-pass FIR Filter with Variable Transition-width

The magnitude response of a low-pass FIR filter with cut-off frequency = 0.5π radians for different values of tuning parameter a is shown in Figs. 2.7-2.9. Kaiser window (length N = 64, $\gamma = 3.5$), Gaussian window (length N = 128, $\rho = 3.5$) and Triangular window (length N = 64) are taken to design the desired filter.

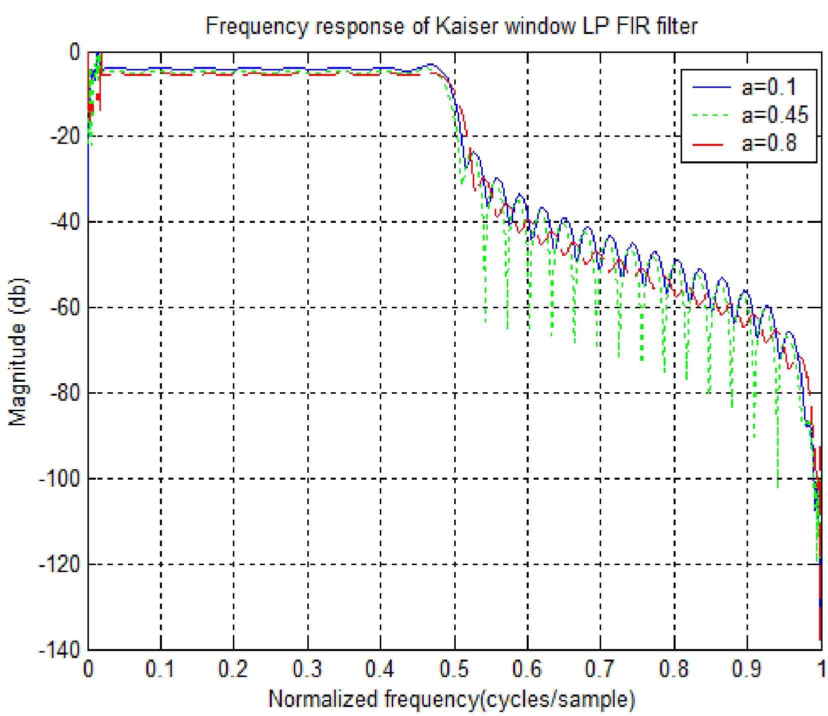

Fig. 2.7: The magnitude response of Kaiser window-based LP FIR filter for fractional order a = 0.1, 0.45 & 0.8.

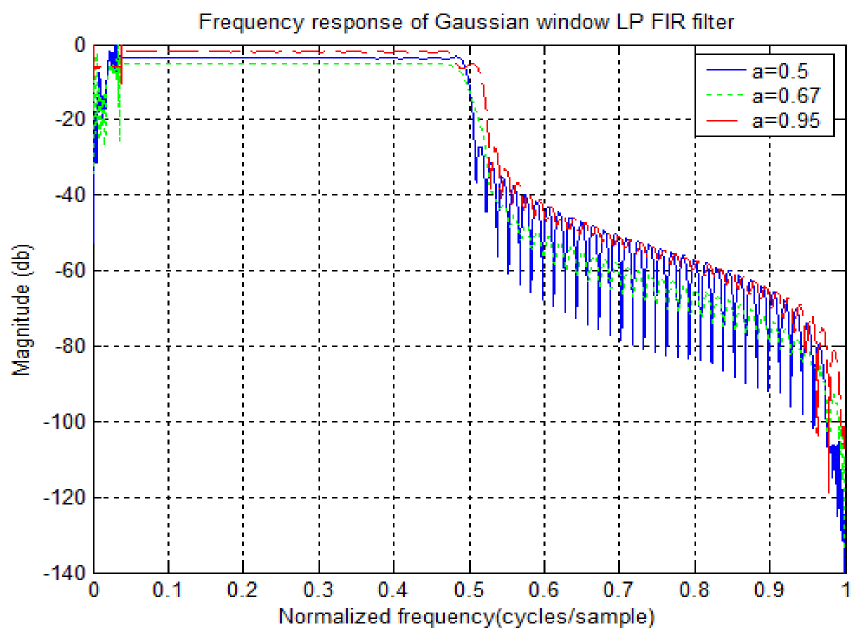

Fig. 2.8: The magnitude response of Gaussian window-based LP FIR filter for fractional order a = 0.5, 0.67 & 0.95.

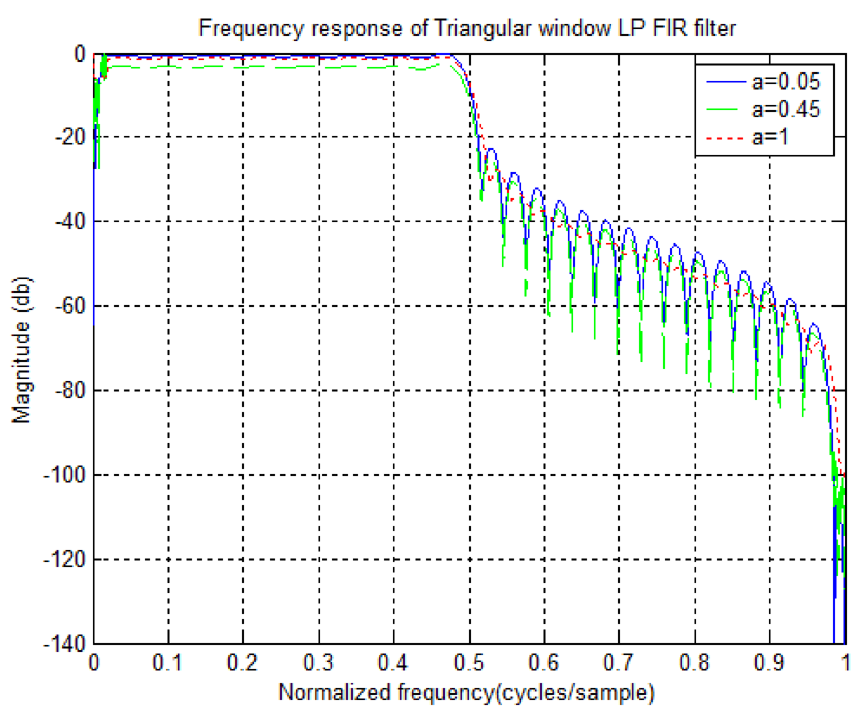

Fig. 2.9: The magnitude response of Triangular window-based LP FIR filter for fractional order a = 0.05, 0.45 & 1.

The comparison of filters' frequency response with the conventional FFT-based ($a = 1$) filter response designed with Triangular window is also presented in the Fig. 2.9. The above comparison graphs clearly show that by varying FrFT order a, transition width can be made to vary.

2.4 Comparison of FrFT based tuning versus direct method of tuning

By using the conventional methodology based on FFT, low pass variable transition-width FIR filters can be designed using two techniques: (i) by varying the filter order $M = N - 1$, where N is the length of a filter, and (ii) by applying variable windows in FIR filter design. The performance comparison of designing FIR filters with variable transition width using these two traditional techniques is compared below with the FrFT-based tuning methodology.

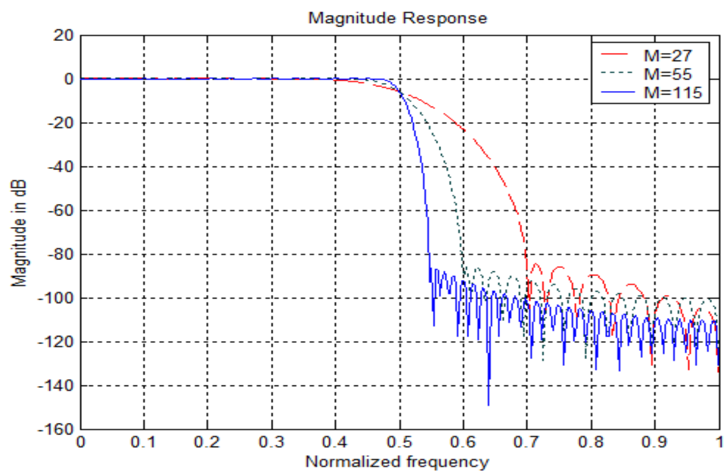

Fig. 2.10: Tuning of transition-width of Kaiser window-based LP FIR filter by varying the filter length N.

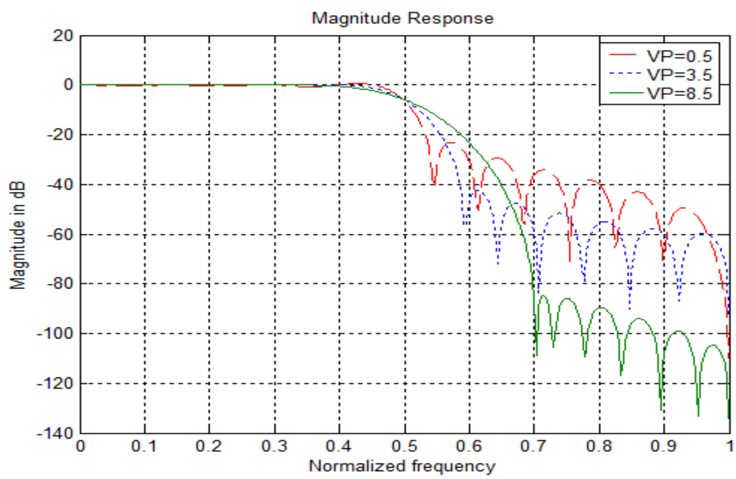

Fig. 2.11: Tuning of transition-width of Kaiser window-based LP FIR filter of order $M = 27$ by adjusting the variable shape parameter of the window denoted here by VP.

Figures 2.10 and 2.11 shows the magnitude response of LP FIR filter designed using Kaiser window by varying the length of the window, and by varying the adjustable shape parameter (denoted by VP) of the window, respectively.

To compare the performance, the following parameters are considered:

(i) Implementation complexity: With direct tuning, either window length has to be increased in order to sharpen the FIR filter transition-width or the variable parameter of adjustable windows needs to be changed. So, the window coefficients and hence the impulse response coefficients need to be recomputed to achieve modified transition-width of the designed filter. The designed variable transition bandwidth FIR filter using variable windows needs to be implemented using Farrow structures which have very high complexity [34]. Whereas, in FrFT based design, fractional order a is the only tuning parameter and neither the window coefficients nor the impulse response coefficients need to be changed. Window coefficients computed and stored once are subjected to different values of fractional order to meet the desired objective. FrFT based tuning of transition bandwidth is comparatively easier. However, it can be observed from Figs. (2.10) and (2.11) that the direct approach can introduce a lot more adjustability in the transition bandwidth than the FrFT approach.

(ii) Computational resources: The computational resources refer to the number of multipliers and adders required for implementation of a digital filter. The computational burden is directly proportional to the length N of a filter. In order to design a variable digital filter using FrFT based approach, the length of a filter remains the same so that computational resources remain constant whereas, the length (cost) needs to be increased using direct method. The computational cost is reduced by using FrFT based tuning method as sharp transitions can be obtained on-line by simply reducing the FrFT order.

(iii) Any additional requirement: No more memory is required to store large number of filter coefficients. No additional hardware is required.

2.5 Conclusions

It is concluded that transition-width can be reduced by reducing the fractional order parameter of the transform without the need of changing the length of the filter and thereby saving computational time and design time. Whereas, if sharp transition filters are designed using traditional constant filter design techniques, the computational burden is highly increased due to which real time high speed implementation will become impractical.

Few researchers worked on design of variable digital filters using FrFT and presented only simulation results but no one formulates the magnitude function of a variable digital filter in a closed-form expression where the rotation angle α of the transform is a tuning parameter.

An attempt is made to derive the same in Chapter 3.

CHAPTER 3

ANALYSIS OF RECTANGULAR WINDOWED FIR FILTER WITH VARIABLE CHARACTERISTICS USING FRACTIONAL FOURIER TRANSFORM

In this chapter, a new mathematical model is developed for computing the Rectangular windowed low-pass FIR filter transfer function where the rotation angle of fractional Fourier transform is used as a free parameter. By changing the rotation angle α in the range from 0 to $\pi/2$, the transition bandwidth and the stop-band attenuation can be tuned. The closed-form expression derived establishes a direct relationship between the magnitude response of the filter and fractional angle of the transform. It is shown that sharp transition bandwidth followed with an increase in stop band attenuation of LP FIR filter designed with Rectangular window can be achieved for small values of the rotation angle of the transform.

3.1 Introduction

As demonstrated in Chapter 2 that digital filters with variable frequency responses can be obtained by using the direct method of varying the filter length but the direct tuning is not computationally efficient. The response characteristics of FIR filter can be tuned during operation by changing only one parameter, while keeping coefficients of the fixed FIR filter unchanged by using the fractional Fourier transformation technique.

Recently, different window functions are being analyzed under FrFT by several researchers [87-88, 43, 54-55]. It is observed that the main-lobe width can be reduced by reducing the rotation angle of the transform. In 2006, Sharma et al. [87] suggested the use of FrFT in window based FIR filter design to vary the transition bandwidth and presented only simulation results. Other researchers also worked on FrFT based windowed FIR filter design in last few years [16, 54, 79]. But, the Rectangular window was not particularly examined. Besides, there is no closed-form solution for the FrFT based modified frequency response of FIR filter realized with Rectangular window.

A closed-form expression based on FrFT is developed for obtaining the adjustable frequency response of Rectangular windowed fixed length LP FIR filter. The derived tuning model establishes a direct relationship between the magnitude response of the filter and the fractional angle of the transform. The proposed tuning methodology involves first applying FrFT to the Rectangular window and then convolving it with the ideal frequency response of low pass FIR filter. The advantage of using FrFT in tuning FIR filters instead of using traditional constant filter design techniques is that the frequency response can be tuned immediately and easily with no need for redesigning a new filter from the existing one. The proposed FrFT based method does not require a filter design algorithm to compute new coefficients in order to meet the new desired application and also does not require any other additional hardware or non-computational elements such as delay elements, digital comparators, interpolators or memory block etc. Window coefficients computed and stored once are subjected to a range of fractional angle to meet the desired objective. The adjustment is provided by only one tuning parameter i.e. fractional angle α which makes FrFT based tuning comparatively easier. While tuning the filter characteristics, none of the filter specifications initially taken are disturbed. The FrFT based tuning model is derived below.

3.2 FrFT Based Tuning Model of Rectangular Windowed Low-pass FIR Filter

Consider a Rectangular window $x(t)=\begin{cases}1 & -0.5\le t\le 0.5\\ 0 & otherwise\end{cases}$ (3.1)

The FrFT $X_\alpha(u)$ of Rectangular window $x(t)$ can be computed using Eq. (2.9) given in Chapter 2:

$$X_\alpha(u)=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\exp(\frac{1}{2}iu^2\cot\alpha)\underbrace{\int_{-0.5}^{0.5}\exp\left[i\left(\frac{1}{2}t^2\cot\alpha-ut\csc\alpha\right)\right]dt}_{I_1} \tag{3.2}$$

Now, solving the integral $I_1 = \int_{-0.5}^{0.5}\exp\left[i\left(\frac{1}{2}t^2\cot\alpha-ut\csc\alpha\right)\right]dt$, the following expression results [1, 73]:

$$I_1 = \frac{\sqrt{\pi}}{\sqrt{2i\cot\alpha}} \exp\left(\frac{-iu^2}{2\sin\alpha\cos\alpha}\right)\left(erfi\left[\sqrt{i}\,\frac{(\cos\alpha - 2u)}{2\sqrt{2}\sqrt{\sin\alpha\cos\alpha}}\right] - erfi\left[\sqrt{i}\,\frac{(-\cos\alpha - 2u)}{2\sqrt{2}\sqrt{\sin\alpha\cos\alpha}}\right]\right) \quad (3.3)$$

where $erfi(z)$ is imaginary error function of z, which is defined in the whole complex z-plane.

By using $erfi(z) = -ierf(iz)$ and simplifying (3.3), one gets:

$$I_1 = \frac{\sqrt{i\pi}}{\sqrt{2\cot\alpha}} \exp\left(\frac{-iu^2}{2\sin\alpha\cos\alpha}\right)\left(-erf\left[i^{3/2}\,\frac{(\cos\alpha - 2u)}{2\sqrt{2}\sqrt{\sin\alpha\cos\alpha}}\right] + erf\left[i^{3/2}\,\frac{(-\cos\alpha - 2u)}{2\sqrt{2}\sqrt{\sin\alpha\cos\alpha}}\right]\right)$$

(3.4)

where $erf(z)$ is error function of z defined in the whole complex z-plane,

Solving for (3.2) by using (3.4), one gets after simplifying:

$$X_\alpha(u) = \frac{\sqrt{1 - i\tan\alpha}}{2}\, e^{-i\frac{u^2}{2}\tan\alpha}\left\{erf\left[\left(\frac{1-i}{4}\right)\frac{(\cos\alpha - 2u)}{\sqrt{\sin\alpha\cos\alpha}}\right] + erf\left[\left(\frac{1-i}{4}\right)\frac{(\cos\alpha + 2u)}{\sqrt{\sin\alpha\cos\alpha}}\right]\right\} \quad (3.5)$$

or

$$X_\alpha(u) = e^{-i\frac{u^2}{2}\tan\alpha}\, X_{\alpha 1}(u) \quad (3.6)$$

The first term $e^{-\frac{1}{2}iu^2\tan\alpha}$ in Eq. (3.6) is a phase factor and does not contribute to the magnitude of FrFT of Rectangular window. The other term $X_{\alpha 1}(u)$ can be written as:

$$X_{\alpha 1}(u) = \frac{\sqrt{1 - i\tan\alpha}}{2}\left\{erf\left[\left(\frac{1-i}{4}\right)\frac{(\cos\alpha - 2u)}{\sqrt{\sin\alpha\cos\alpha}}\right] + erf\left[\left(\frac{1-i}{4}\right)\frac{(\cos\alpha + 2u)}{\sqrt{\sin\alpha\cos\alpha}}\right]\right\} \quad (3.7)$$

or

$$X_{\alpha 1}(u) = \frac{\sqrt{1 - i\tan\alpha}}{2}\left\{erf[p - qu] + erf[p + qu]\right\} \quad (3.8)$$

$$p = \left(\frac{1-i}{4}\right)\frac{\cos\alpha}{\sqrt{\sin\alpha\cos\alpha}} \quad (3.9)$$

and

$$q = \left(\frac{1-i}{4}\right)\frac{2}{\sqrt{\sin\alpha\cos\alpha}} \quad (3.10)$$

The oscillatory $X_{\alpha 1}(u)$ when convolved with the ideal low-pass filter response $H_d(u)$ will result in adjustable response $H_\alpha(u)$. The parameters such as transition-width and stop-band attenuation of FIR filter can be controlled by varying the fractional angle α.

Let the response of ideal low-pass filter denoted by $H_d(u)$ is defined as:

$$H_d(u) = \begin{cases} 1 & |u| \le 0.5 \\ 0 & otherwise \end{cases} \tag{3.11}$$

and $H_\alpha(u)$ denotes the adjustable response of FIR filter with α as a tuning parameter.

$H_d(u)$ is assumed to be constant at the value of fractional angle that is used for computing the Rectangular window transform. The variable response can be obtained by convolving $H_d(u)$ with $X_{\alpha 1}(u)$ (the term contributes to the magnitude of FrFT of Rectangular window) i.e.

$$H_\alpha(u) = H_d(u) \otimes X_{\alpha 1}(u) \tag{3.12}$$

where '$\otimes$' denotes the convolution operator.

Putting (3.8) and (3.11) in (3.12) and using the definition of convolution, $H_\alpha(u)$ can be written as:

$$H_\alpha(u) = \frac{\sqrt{1 - i\tan\alpha}}{2} \int_{-0.5}^{0.5} 1\left(erf\left[p - q(u - v)\right] + erf\left[p + q(u - v)\right]\right)dv \tag{3.13}$$

Substituting $v - u = z$; $dv = dz$ in (3.13) and changing the limits of integration, the Eq. (3.13) can be rewritten as:

$$H_\alpha(u) = \frac{\sqrt{1 - i\tan\alpha}}{2} \int_{-0.5-u}^{0.5-u} 1\left(erf\left[p + qz\right] + erf\left[p - qz\right]\right)dz \tag{3.14}$$

Rearranging (3.14), $H_\alpha(u)$ becomes:

$$H_\alpha(u) = \frac{\sqrt{1 - i\tan\alpha}}{2} \int_{-0.5-u}^{0.5-u} \left(erf\left[p + qz\right]\right)dz + \frac{\sqrt{1 - i\tan\alpha}}{2} \int_{-0.5-u}^{0.5-u} \left(erf\left[p + (-qz)\right]\right)dz \tag{3.15}$$

Thereby, solving (3.15) for $H_\alpha(u)$ by applying:

$\int erf(b+az)dz = \frac{berf(b+az)}{a} + zerf(b+az) + \frac{e^{-a^2z^2-2abz-b^2}}{a\sqrt{\pi}}$ [1, 109], results in:

$$H_\alpha(u) = \frac{\sqrt{1-i\tan\alpha}}{2}\left|\left\{\begin{matrix} \frac{p}{q}erf[p+qz]+zerf[p+qz]+\frac{1}{q\sqrt{\pi}}e^{(-q^2z^2-2pqz-p^2)}+ \\ \frac{(-p)}{q}erf[p-qz]+zerf[p-qz]-\frac{1}{q\sqrt{\pi}}e^{(-q^2z^2+2pqz-p^2)} \end{matrix}\right\}\right|_{-0.5-u}^{0.5-u} \tag{3.16}$$

On rearranging (3.16), $H_\alpha(u)$ can be written as:

$$H_\alpha(u) = \frac{\sqrt{1-i\tan\alpha}}{2}\left|\left\{\begin{matrix} \left(z+\frac{p}{q}\right)erf[p+qz]+\left(z-\frac{p}{q}\right)erf[p-qz]+ \\ \frac{1}{q\sqrt{\pi}}\left(e^{(-q^2z^2-2pqz-p^2)}-e^{(-q^2z^2+2pqz-p^2)}\right) \end{matrix}\right\}\right|_{-0.5-u}^{0.5-u} \tag{3.17}$$

Equation (3.18) is obtained after simplifying (3.17) as below:

$$H_\alpha(u) = \frac{\sqrt{1-i\tan\alpha}}{2}\left|\left\{\begin{matrix} \left(z+\frac{p}{q}\right)erf[p+qz]+\left(z-\frac{p}{q}\right)erf[p-qz]+ \\ \frac{1}{q\sqrt{\pi}}\left(e^{-(p+qz)^2}-e^{-(p-qz)^2}\right) \end{matrix}\right\}\right|_{-0.5-u}^{0.5-u}$$

(3.18)

Replacing values of p and q given by (3.9) and (3.10) in (3.18) and solving for (3.18), one gets:

$$H_{\alpha}(u)=\sqrt{\frac{1-i\tan\alpha}{2}}\times\left[\begin{array}{l}\left(0.5-u+\frac{\cos\alpha}{2}\right)erf\left[\left(\frac{1-i}{4}\right)\frac{(\cos\alpha+1-2u)}{\sqrt{\sin\alpha\cos\alpha}}\right]+\left(0.5-u-\frac{\cos\alpha}{2}\right)erf\left[\left(\frac{1-i}{4}\right)\frac{(\cos\alpha-1+2u)}{\sqrt{\sin\alpha\cos\alpha}}\right]+\\ \frac{2\sqrt{\sin\alpha\cos\alpha}}{(1-i)\sqrt{\pi}}\left(e^{\frac{i(\cos\alpha+1-2u)^2}{8\sin\alpha\cos\alpha}}-e^{\frac{i(\cos\alpha-1+2u)^2}{8\sin\alpha\cos\alpha}}\right)-\left(-0.5-u+\frac{\cos\alpha}{2}\right)erf\left[\left(\frac{1-i}{4}\right)\frac{(\cos\alpha-1-2u)}{\sqrt{\sin\alpha\cos\alpha}}\right]-\\ \left(-0.5-u-\frac{\cos\alpha}{2}\right)erf\left[\left(\frac{1-i}{4}\right)\frac{(\cos\alpha+1+2u)}{\sqrt{\sin\alpha\cos\alpha}}\right]-\frac{2\sqrt{\sin\alpha\cos\alpha}}{(1-i)\sqrt{\pi}}\left(e^{\frac{i(\cos\alpha-1-2u)^2}{8\sin\alpha\cos\alpha}}-e^{\frac{i(\cos\alpha+1+2u)^2}{8\sin\alpha\cos\alpha}}\right)\end{array}\right] \tag{3.19}$$

Equation (3.19) presents the closed-form solution for the FrFT based adjustable frequency response of the Rectangular window based LP FIR filter. From Eq. (3.19), it can be seen that the magnitude response $H_{\alpha}(u)$ obtained by convolving FrFT of Rectangular window with the ideal response of low-pass FIR filter depends directly on the fractional angle α of the transform. As α takes value from 0 to $\pi/2$, temporal changes in the frequency content of the transformed signal can be observed as a function of free variable u [107]. Therefore, filter characteristics can be tuned by modifying variable parameter α. The mathematical model for computing the Rectangular windowed LP FIR transfer function is an alternative to decrease the complexity when tunable digital filters are designed. In proposed methodology, the Rectangular window function coefficients are computed according to the initial filtering requirements. The FFT of Rectangular window is then obtained by taking FrFT angle $\alpha=\pi/2$, which is further convolved with the desired frequency response so that modified frequency response of the filter can be obtained. By merely changing the FrFT angle α, different filter characteristics can be achieved as and when required. There is no need to compute filter's new impulse response coefficients and redesign a filter. The desired results are illustrated with the help of examples.

3.3 Results

A low-pass FIR filter of desired frequency response specification $H_d(u)$ as given by (3.11) is taken. Instead of convolving $H_d(u)$ with ordinary Fourier transform of Rectangular window, $H_d(u)$ is convolved with FrFT $X_\alpha(u)$ of rectangular window of specified length as given by (3.1) in order to obtain variable response $H_\alpha(u)$. The desired filter's length is restricted by choosing the length of the Rectangular window.

Figure 3.1 shows variable magnitude response of Rectangular windowed LP FIR filter for a range of fractional angle α . The magnitude response $\left|H_\alpha(u)\right|$ of FIR filter with response function given by Eq. (3.19) is plotted as function of gain (dB) versus frequency u (cycles/sec or Hz) reflecting the changes in parameters such as the transition width, the pass-band width and the stop-band attenuation of filter with change in value of fractional angle α . For fractional angle $\alpha = \pi/2$, the designed filter is analogous to ordinary Fourier transform based filter. As fractional angle α is reduced from $\pi/2$ to 0, the changed frequency characteristics of the filter are revealed for different values of α . The influence of convolving FrFT of the Rectangular window with ideal response of the FIR filter on actual frequency response of the filter is illustrated in Fig. 3.1. It can be seen that as fractional angle is reduced, sharp transition region of filter along with improved attenuation can be achieved without changing the filter length. Reducing the FrFT angle will be analogous to increasing the filter order. Also, reduction in pass-band width is observed for small values of fractional angle. Table 3.1 illustrates the parameters such as transition width, pass-band width stop-band attenuation evaluated from the Fig. 3.1. It is observed that as the value of α changes from $\pi/2$ to $0.1\pi/2$, transition-width reduces from 2.42 cycles/sec to 0.948 cycles/sec, pass-band width reduces from 2.69 cycles/sec to 0.474 cycles/sec and stop-band attenuation starts increasing from 12.9 dB to 25.98 dB.

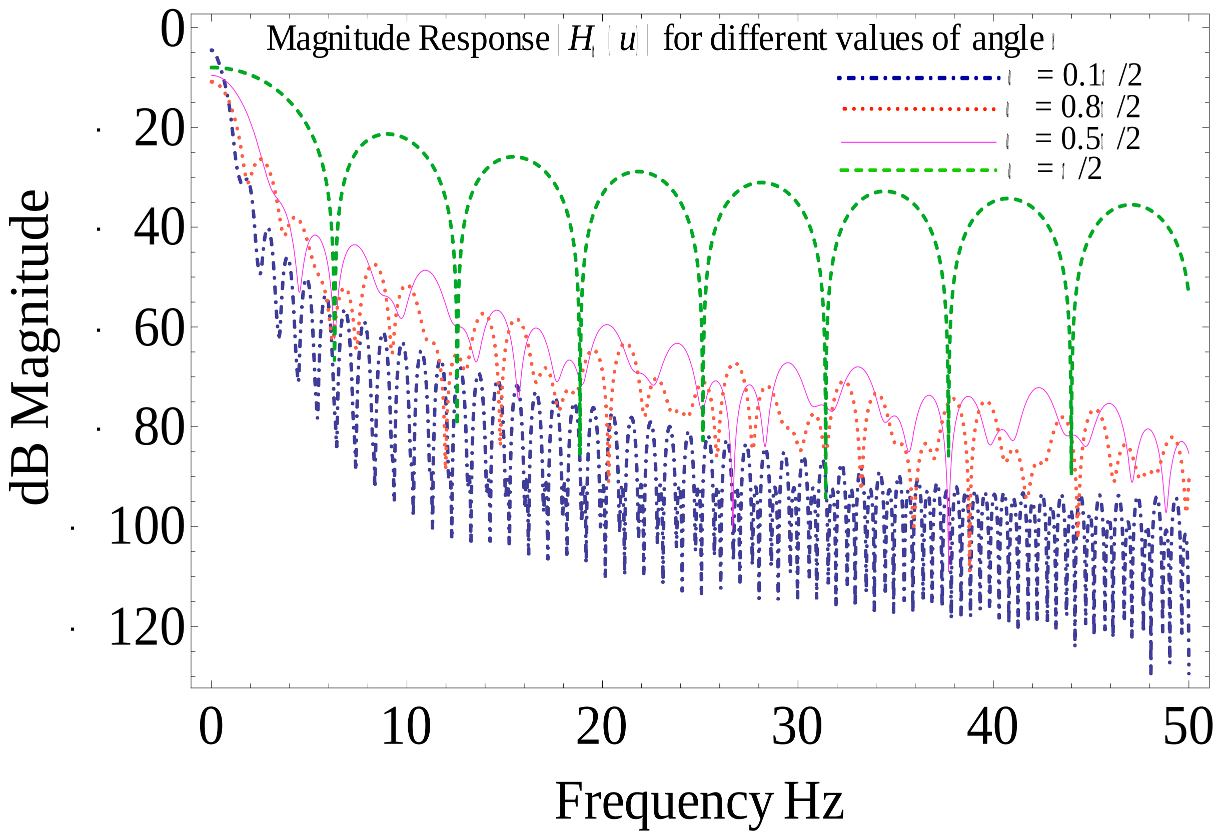

Fig. 3.1: Variable magnitude response of Rectangular window-based LP FIR filter for values of fractional angle $\alpha = 0.1\pi/2, 0.5\pi/2, 0.8\pi/2$ and $\pi/2$.

Table 3.1:

Spectral and magnitude related parameters for values of fractional angle $\alpha = 0.1\pi/2, 0.5\pi/2, 0.8\pi/2$ and $\pi/2$

Fractional angle α	Transition width (cycles/sec)	Pass-band width (cycles/sec)	Stop-band Attenuation (dB)
$0.1\pi/2$	0.948	0.474	25.98
$0.5\pi/2$	1.894	1.233	25.19
$0.8\pi/2$	0.95	0.79	15.64
$\pi/2$	2.42	2.69	12.9

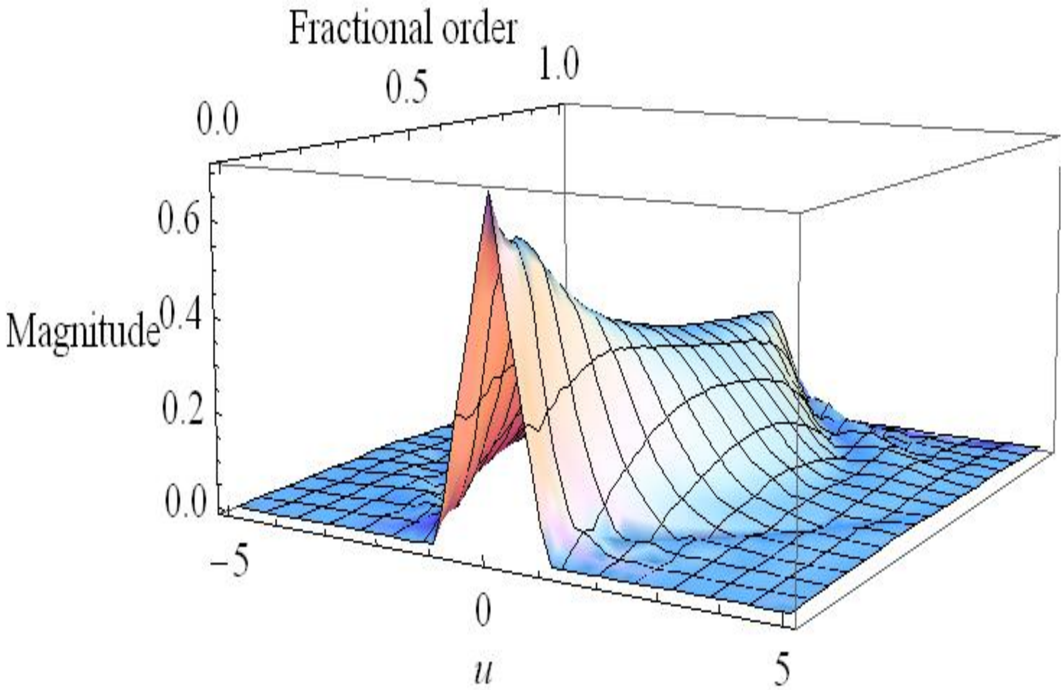

Fig. 3.2(a): 3-D perspective plot of the magnitude response of low-pass FIR VDF as a function of both fractional order a (where $a = 2\alpha/\pi$) and u.

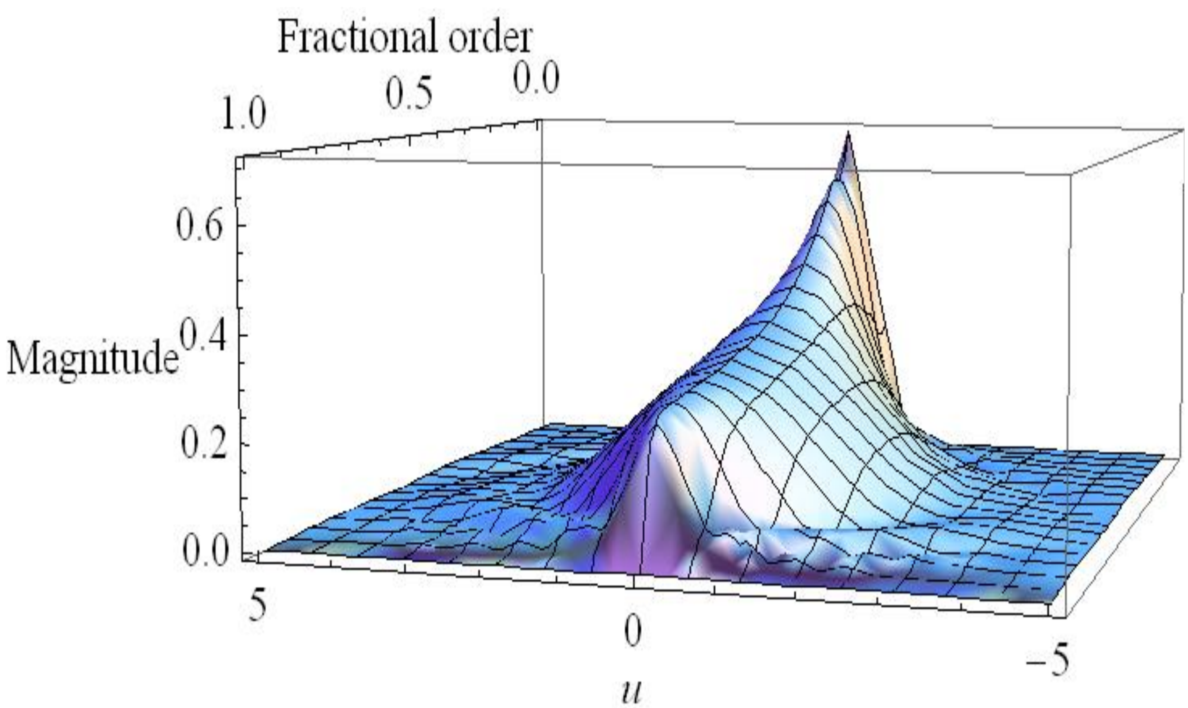

Fig. 3.2(b): 3-D perspective plot of the magnitude response of low-pass FIR VDF as a function of both fractional order a (where $a = 2\alpha / \pi$) and u .

Figures 3.2(a) and 3.2(b) show three dimensional plots for the magnitude response of the low pass FIR VDF for different values of fractional order $a \in [0, 1]$, where $a = 2\alpha / \pi$. For $a = 0$, Eq. (3.12) gives convolution of Rectangular function with itself. The result of convolution match to Triangular function as can be seen from Fig. 3.2(a). On the other hand for $a = 1$, Eq. (3.12) gives convolution of ordinary FT of rectangular window with the desired frequency response of the filter. The result of convolution is equivalent to ordinary FT based actual magnitude response of the filter realized with Rectangular window and can be seen from Fig. 3.2(b). The simulation results are shown below to present the superiority of the work over that of other works that use only the rectangular window.

3.3.1 Comparison

The performance comparison of the proposed FrFT based method over the ordinary method based on FFT (by varying window length) in tuning the above FIR filter with prescribed transfer function as given by (3.11) is shown in Fig. 3.3.

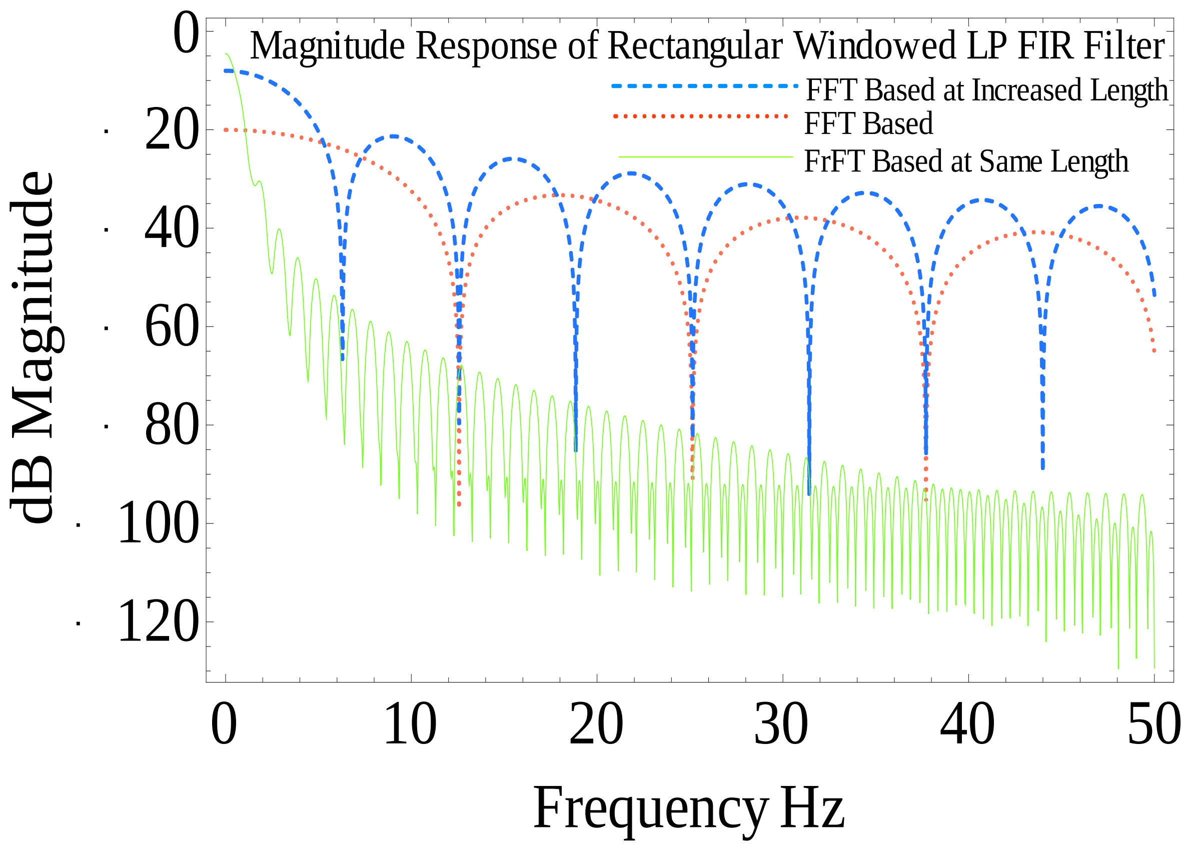

Fig. 3.3: Comparison of FrFT based approach with conventional FFT based approach in tuning FIR filter.

In order to reduce the transition width with FFT based approach, the filter length has to be increased. With FrFT, sharp transition region can be achieved by simply reducing the FrFT angle. In Fig. 3.3, filter length is increased by a factor of 2 (i.e. the duration of the Rectangular window is made twice by changing the time interval from $-0.5 \le t \le 0.5$ to $-1 \le t \le 1$) to show the reduction in the transition width using FFT with fixed rotation angle α i.e. $\pi/2$. Whereas, with FrFT based approach reduced transition width is obtained by reducing the rotation angle to $0.1\pi/2$ while keeping length of the filter fixed. It can be concluded from Fig. 3.3 that FrFT based proposed tunable technique is superior as compared to FFT as there is no need to change length of the filter in the previous one. The simulation results are obtained using Wolfram Mathematica® software.

Thus, Rectangular windowed fixed length digital filter can be tuned by using fractional Fourier transformation technique with fractional angle α as the tuning parameter. The filter impulse response coefficients need not to be re-computed in order to obtain variable responses. This method can be applied in many **practical applications** of digital signal processing in order to tune the digital filters so that the new desired specifications can be met easily. It can be used to perform a slightly modified task in accordance with the need that arises in professional digital audio applications or to fine tune the desired channel in Software Defined Radio (SDR) so that sharp transition-region filter responses with different bandwidths can be obtained. SDR implies that radio characteristics can be defined by software i.e. a single hardware solution can be adjusted to different system standards by changing software [28]. This design method can also be used in smart receiver to fine tune a bandwidth so that the level of interference from strong adjacent channels can be reduced. The need for efficient translating between various sampling frequencies is increased with the increasing number of applications of digital techniques in signal processing. In typical decimation/interpolation filtering applications, various sample rates are desired depending upon the available bandwidth of the channel and the required level of signal quality. A proper filter design is required in any sample rate conversion system in order to avoid aliasing. As converter's sampling rate changes, anti-aliasing filter's pass-band should increase or decrease accordingly. By applying this tuning method, a common fixed hardware platform for sample rate conversion can be designed which is adaptable by means of software to different modes of operation.

3.4 Conclusions

It is concluded that tuning of FIR filters using FrFT can be achieved immediately without the need for a filter design algorithm operating in the background to compute new coefficients. A low-pass FIR filter designed using Rectangular window having variable transition width characteristics followed with improved stop band attenuation is proposed by convolving FrFT of the Rectangular window with the desired frequency response of filter. A new mathematical model is presented to modify filter's characteristics on-line with fractional angle of the transform as the tuning parameter. It is shown that the proposed approach does not involve changing the filter length as reducing the FrFT angle or the fractional order parameter will be analogous to increasing the filter order. The transition width and stop band attenuation of Rectangular window based FIR filter can be tuned simply by adjusting the fractional angle to different values. As the fractional angle is reduced, the transition-width can be minimized and the stop-band attenuation can be raised to maximum.

In contrast to Chapter 3, a new method to increase the pass-band width of FIR filters by reducing the fractional order a is presented in Chapter 4. Also, adjustable phase response is achieved by using the proposed design method.

CHAPTER 4

A DESIGN TECHNIQUE FOR VARIABLE NON-RECURSIVE DIGITAL FILTER USING FrFT

This chapter presents a new design technique for obtaining window-based LP FIR (non-recursive) filters whose magnitude and phase characteristics in the pass-band can be varied simultaneously by using the fractional order a of FrFT as a tuning parameter. By changing the order parameter a in the range from 0 to 1, the pass-band width and phase response of FIR filter can be modified easily.

4.1 Introduction

There are various applications in signal processing which require digital filters with variable magnitude characteristics in the pass-band along with adjustable phase characteristics. For example in medical signal processing, digital communication, time/delay estimation, multi-standard receivers, conversion between arbitrary sampling frequencies, echo cancellation, speech coding and synthesis and many more. The modified characteristics are needed to perform a slightly modified task in accordance with the need that arises in real time processing. The pass-band width as well as phase response of a window-based FIR filter is dependent on the length of the filter at a particular frequency. This involves change in the filter length and redesign of a new filter from the existing one in order to tune the magnitude and phase response characteristics of the filter.

By analyzing the behavior of desired impulse response of FIR filter under FrFT, it is experienced that pass-band width and phase response of the filter can be made dependent on the fractional order parameter $a = 2\alpha / \pi$, where α is the rotation angle of the transform. It is put forward in the research that FrFT can be applied in tuning the frequency characteristics of the window-based FIR digital filters while keeping the same filter length. The variations in behavior of frequency response of LP FIR filter are analyzed for different values of fractional order a.

4.2 FrFT Based Tunable Methodology for FIR Filters

The analysis of the desired impulse response function of an ideal low-pass FIR filter under FrFT establishes the dependence of its frequency response on the changeable fractional order parameter a. It is observed that the pass-band edge characteristics and the phase response of filter can be varied by changing the FrFT order a from 1 to 0. Several methods are proposed in literature to control the pass-band characteristics of digital filters [95, 101]. The essence of using FrFT in variable digital filter design is that impulse response coefficients need not be recomputed and there is no need to redesign a new filter in order to bring changes in the filter characteristics. It is projected in the work that tuning of window-based FIR digital filters can be achieved easily using FrFT.

4.2.1 Proposed Novel Technique

In window-based FIR filter, the multiplication of the window function $w(n)$ with desired impulse response $h_d(n)$ is equivalent to convolution of desired (or ideal) frequency response $H_d(\omega)$ with frequency domain representation of the window function $W(\omega)$ so that the modified response $H(\omega)$ of the filter is given by:

$$H(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(v)W(\omega - v)dv \tag{4.3}$$

or $$H(\omega) = H_d(\omega) \otimes W(\omega) \tag{4.4}$$

(where the operator $\otimes$ denotes the convolution operator for ordinary Fourier transform)

The proposed method based on FrFT for tuning the magnitude and phase characteristics of FIR filter includes the following steps:

1. Compute the desired impulse response coefficients $h_d(n)$ of ideal low-pass FIR filter of specified length and initial cut-off frequency.

2. Take the *ath* order fractional Fourier transform of $h_d(n)$ to obtain $F^a[h_d(n)]$. Thus, the desired frequency response is parameterized in pass-band width with FrFT order a as a

tuning parameter. When $a=1$, the desired magnitude response corresponds to that which is obtained using ordinary Fourier transform.

3. Convolve $F^a[h_d(n)]$ with the magnitude response $W_a(f)$ of an appropriate window function to obtain adjustable magnitude as well as phase response of the filter. The magnitude response $W_a(f)$ can be obtained by taking the Fourier transform of a window function. For all values of the order parameter a, the transform of window function denoted by $W_a(f)$ is constant.

The overall procedure of designing window-based LP FIR filter with variable pass-band width using FrFT is shown with the help of block diagram in Fig. 4.1.

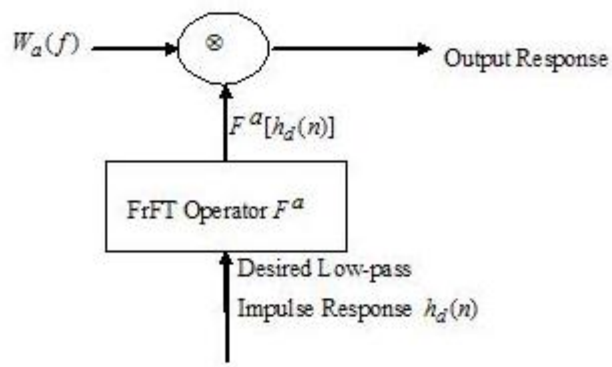

Fig. 4.1: Block Diagram of Tuning Procedure for Increasing the Pass-band Width.

In the proposed methodology, the desired impulse response coefficients $h_d(n)$ corresponding to the initial filtering requirements are computed at first. Then the desired magnitude response of filter is obtained by taking FrFT order $a=1$, and is convolved with the window response to obtain actual frequency response $H(f)$, which is stored to perform filtering. If required, the pass-band edge characteristics of filter can be varied by changing FrFT order, and thus modifying $H(f)$ to $H_a(f)$. Thus, in the proposed work filter coefficients need not to be recomputed and tuning can be achieved by exploiting the additional degree of freedom coming from the fractional order parameter a. The computational resources required for achieving variable filter characteristics using proposed method will remain same. Also, the cost of computing the fractional Fourier transform is not more expensive than computing the ordinary Fourier transform.

4.3 Examples of Low-pass FIR Filters with Variable Magnitude and Phase Characteristics

To illustrate the proposed technique, a low-pass FIR filter with variable characteristics is designed with a cut-off frequency equal to 0.5π radians and length equal to 16. The magnitude response and phase response of the filter are plotted against normalized frequency (cycles/sample) for different values of fractional order a. The performance parameter i.e. the pass-band width of the filter is evaluated and analyzed from the plot of the magnitude response. Figure 4.2 shows the variable filter magnitude response of Kaiser window-based (with variable parameter $\gamma = 3.5$) LP FIR filter (the *Kaiser window response convolved with FrFT of the desired impulse response)* for a range of fractional order.

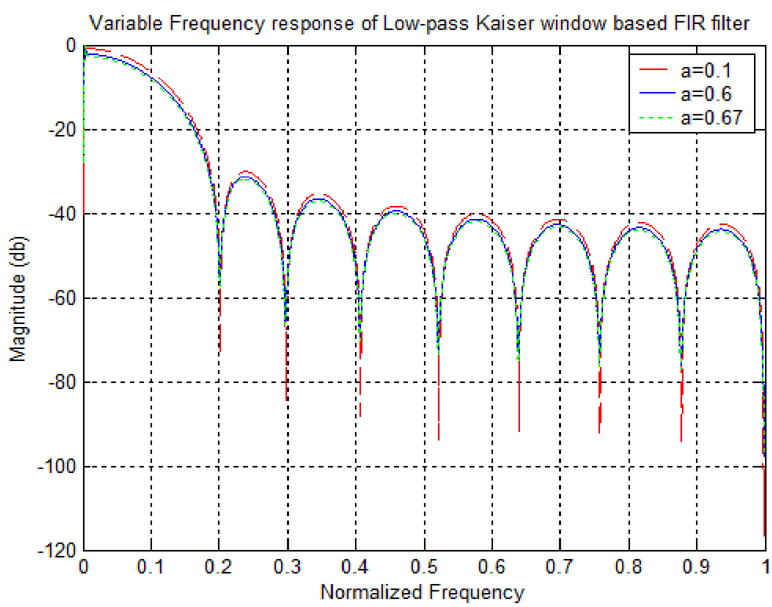

Fig. 4.2: Magnitude response of Kaiser window-based LP FIR filter with variable pass-band width for fractional order $a = 0.1, 0.6 \, and \, 0.67$ on a dB scale.

Table 4.1:

Performance features of Kaiser window-based (variable shape parameter $\gamma = 3.5$) LP FIR filter with tunable pass-band characteristics

Fractional order a	Pass-band width (cycles/sample)
0.1	0.1
0.6	0.09
0.67	0.085

The application of FrFT in tuning the phase response of FIR filter is also analyzed and illustrated with the help of Fig. 4.3. The phase response of a digital filter is dependent on the length of the filter at a particular frequency and can be tuned by changing the impulse response coefficients which is cumbersome. The results in Fig. 4.3 show that simply by varying the fractional order a of the transform, the phase response can be adjusted.

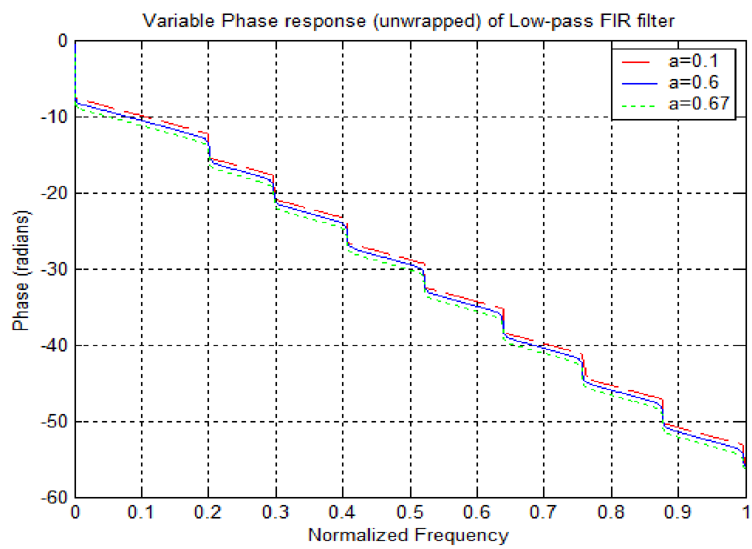

Fig. 4.3: Variable phase response (unwrap) of Kaiser window-based LP FIR filter for different values of fractional order.

One more example is illustrated in Fig. 4.4 to show output of the proposed methodology by convolving the Hamming window response with FrFT of the desired impulse response.

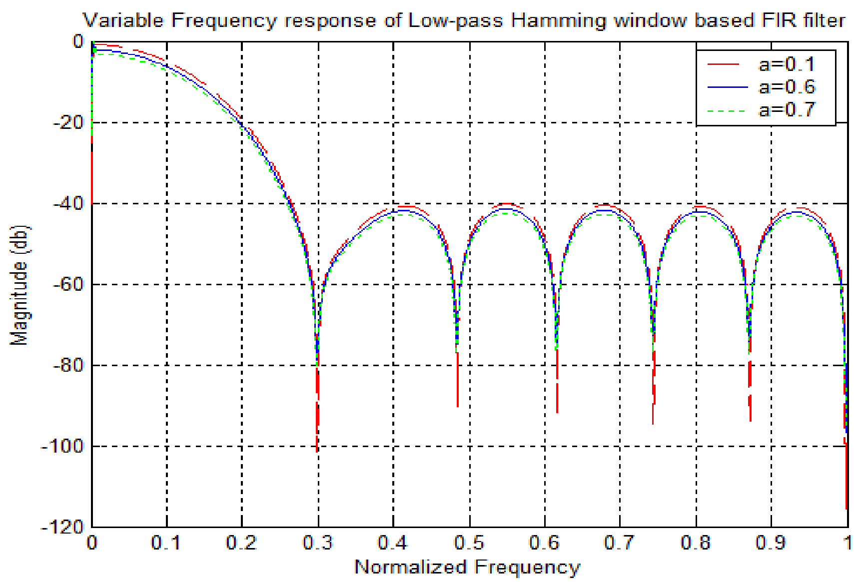

Fig. 4.4: Magnitude response of Hamming window-based LP FIR filter with variable pass-band width for fractional order $a = 0.1, 0.6 \, and \, 0.7$ on a dB scale.

Table 4.2:

Performance features of Hamming window-based LP FIR filter with tunable pass-band characteristics

Fractional order a	Pass-band width (cycles/sample)
0.1	0.13
0.6	0.11
0.7	0.098

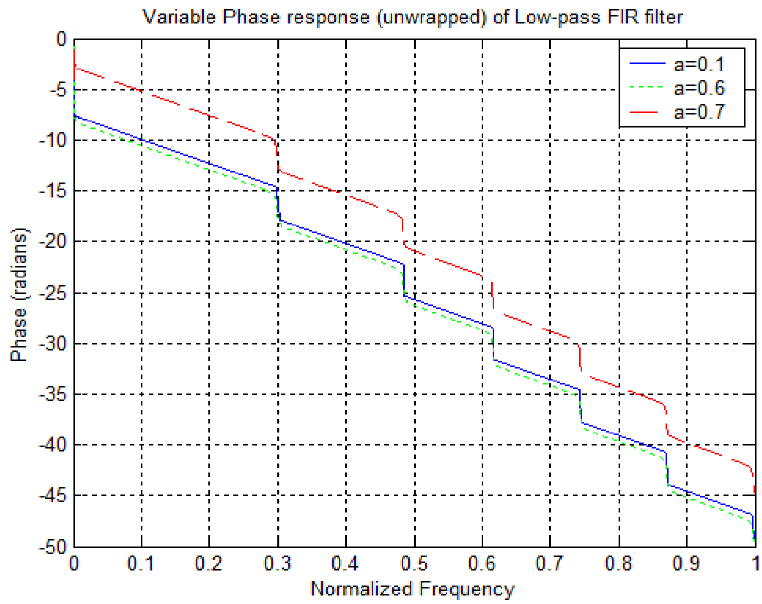

Fig. 4.5: Variable phase response (unwrap) of Hammimg window-based LP FIR filter for different values of fractional order.

The above simulated results and the performance features measured in Table 4.1 and Table 4.2, clearly show that by reducing the FrFT order a, pass-band width can be increased without the need to redesign a new filter off-line. However, the tuning range is small. Also, it is observed from Fig. 4.3 and Fig. 4.5, that the phase response can also be adjusted on-line as the FrFT order a is reduced but linearity can not be guaranteed. The designed filter with modified magnitude and phase characteristics is useful in applications where arbitrary sample rate conversion and signal delaying is required [26, 97].

4.3.1 Comparisons

In order to meet the modified specifications, digital filters with variable characteristics can be designed by changing the coefficients of the existing filter. The traditional methods of changing the filter coefficients of a digital filter are:

(i) Re-computing of coefficients: It is cumbersome to re-compute the filter coefficients and design a new filter from the old one.

(ii) Table lookup: The tables of filter coefficients can be stored as functions of some index, and then loading different coefficient sets as the index changes This way memory requirement would be more. Let n memory locations are required for the initially designed filter. If look up table is used to store the filter coefficients as a function of some index which changes m times, the memory locations required would be $m \times n$. The proposed method would require only n locations.

(iii) Polynomial approximation of coefficients: If the impulse response coefficients are approximated as polynomial of spectral parameter, there exists some error due to the approximation.

The proposed method is comparatively easier and simple.

4.4 Conclusions

It is concluded in the work that by convolving FrFT of the impulse response of LP FIR filter with magnitude response of the desired window function, the pass-band width of the filter can be increased for reduced values of fractional order a of the transform. In addition, phase response can also be varied using FrFT with no need to re-compute the filter coefficients.

The next Chapter comprises the analysis of finite duration Chirp as windowing function using FrFT so that the Chirp under FrFT can be used as an alternative window function to the existing Rectangular window in achieving better spectral characteristics.

CHAPTER 5

ANALYSIS OF CHIRP AS WINDOWING FUNCTION USING FrFT

This chapter presents a new mathematical model for obtaining the fractional Fourier transform of Chirp function. The closed-form expression for the fractional Fourier transform of finite duration Chirp establishes the dependence of FrFT of Chirp on the fractional order a of the transform and the Chirp parameter. Based on the derived expression, it can be clearly stated that Chirp function can be used as an adjustable window function with fractional order a of the transform and Chirp parameter as the tuning parameters. The analysis shows that Chirp window under FrFT can prove to be an alternative to the basic Rectangular window in improving the spectral analysis overall performance and in tuning of FIR filters' characteristics. The main-lobe width, side-lobe level and side-lobe fall-off rate of Chirp window can be controlled by changing the adjustable transform parameter a to different values. For some particular values of fractional order, Chirp can give better spectral parameters than the existing window functions. The variations in spectral parameters of Chirp window are analyzed by varying the order parameter a of the transform,. The performance of Chirp window under FrFT is also compared with some of the existing windows.

5.1 Introduction

Windows are weighting functions that are used to attenuate signals at their discontinuities. The window functions are applied to the time-domain signal and the process of multiplying the measured signal with smoothly ending window function is called windowing technique. Windowing is done to make an infinitely long signal finite in length so that the frequency content (spectrum) of a signal of interest can be measured. Window functions are mainly used in spectrum analysis and FIR filter design [74]. Window is considered to be a multiplicative operator that turns on the signal within the finite support and turns it off outside that same support.

Mathematically, the following expression can be introduced to describe the truncated signal $x_{\omega}(t)$ whose spectrum can be practically evaluated as:

$$x_{\omega}(t) = x(t)w(t) \tag{5.1}$$

where $x(t)$ is actual input signal and $w(t)$ is the window function applied.

The presence of factor $w(t)$ in Eq. (5.1) is a source of strong distortion on the spectrum of the input signal. The modified spectrum $X_{\omega}(f)$ of the truncated signal $x_{\omega}(t)$ can be written as:

$$X_{\omega}(f) = X(f) \otimes W(f) \tag{5.2}$$

where the notation $\otimes$ denotes the convolution operator for ordinary FT, $X(f)$ is the spectrum of actual input signal $x(t)$ and $W(f)$ is the Fourier transform of window $w(t)$.

The windowing of the input signal is equivalent to convolving the spectrum of the original signal with the spectrum of the window. Because of the shape of $W(f)$, the spectrum of truncated signal becomes different from the original continuous time signal. Windows can be continuous functions or discrete sequences defined over their appropriate finite supports. The effects of a discrete time window on a discrete time signal are analogous to those encountered in the continuous time case. The side lobes of the window function tend to introduce oscillations in the amplitude spectrum of the signal whose amplitude tends to decrease as the ripple ratio of the window is decreased. The main lobe width tends to even out abrupt changes in the amplitude spectrum and, as a consequence, it tends to introduce transition bands at discontinuities. To minimize this effect, the main-lobe width should be as small as possible. The main-lobe width can be reduced by increasing the length of the window both in the continuous time as well as discrete time case. This results in increased complexity.

Windowing not only distorts the spectral estimate due to leakage effects (i.e. power of the original signal that was earlier concentrated at a single frequency, now spread by window into entire range of frequency), but it also reduces spectral resolution (ability of distinguishing narrowband spectral components). Frequency domain characteristics of several window functions are being analyzed in literature to determine their suitability for a specific application and to reduce the spectral leakage that results because of limiting a time interval signal to finite duration [51, 69, 74]. The selection of an appropriate window function for a particular application depends on the performance features, such as the attenuation at the maximum height of a side lobe, generally the first side-lobe (the side-lobe level), the rate at which peak of the side-lobes decrease in magnitude (side-lobe fall-off rate) and the main-lobe width (width of main-lobe at -3 dB below main-lobe peak). The narrower the main-lobe width the better will be the frequency resolution; and the lower the side lobe level, the better will be the noise suppression. The narrow main-lobe width and reduced side-lobe level are conflicting requirements. Thus, the problem lies in deciding which window function is the best to apply on the signal being studied.

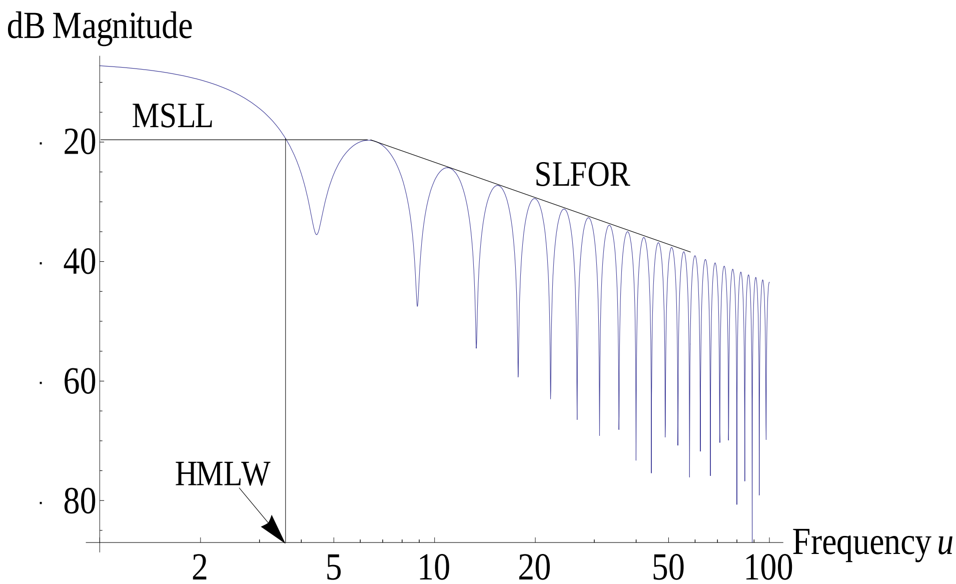

Fig. 5.1: Log-Magnitude plot of Chirp window to define spectral parameters.

There are three important parameters in spectral analysis of a function: Half Main-Lobe Width (HMLW), Maximum Side-Lobe Level (MSLL) and Side-Lobe Fall-Off Rate (SLFOR). These parameters can be defined as:

1. Half Main-Lobe Width (**HMLW**): It is the frequency at which the Main lobe drops to the peak ripple value of the side lobes.
2. Maximum Side-Lobe Level (**MSLL**): It is the largest side lobe level in decibels relative to the main lobe peak gain.
3. Side-Lobe Fall-Off Rate (**SLFOR**): It is the asymptotic decay rate of side-lobe level in decibels per decade/octave of frequency of the peaks of the side lobes.

Any improvement in these parameters can improve the spectral performance of the function. A plot of these spectral parameters for Chirp window is shown in Fig. 5.1. Numerous window functions can be preferred for the prevention of spectral leakage in the signal and to offer the specified side-lobe level [47]. But the reduction of side-lobe leakage due to the applied window function introduces leakage from the expansion of main-lobe in ordinary frequency domain. This reduces spectral resolution and also some gain is lost because of the main-lobe spreading. In order to improve spectral resolution, an increase in length of the window function is required. By applying fractional Fourier transform to the window functions, better resolution can be achieved without changing the length of the window functions [55, 36]. Thus, computational time and design time can be saved. The chirps are the signals which exhibit a change in instantaneous frequency with time. In 2009, Jain et al. [36] presented that Chirp function can give better spectral characteristics for some particular values of fractional angle than the existing window functions. The authors gave only the simulation results and no analytical relationship is derived.

The mathematical analysis of Chirp function is carried out using fractional Fourier transform in this research work. Based on which, it is found that Chirp function can be used as an adjustable window function with better spectral characteristics under FrFT. The variations in spectral parameters of Chirp window are studied by adjusting the fractional angle of the transform to different values. The performance comparison of Chirp under FrFT is also carried out with some of the existing windows.

5.2 Derivation of FrFT of Chirp Function

The FrFT of truncated Chirp $x(t)$ is derived as follows:

Let, an exponential Chirp function denoted by $x(t)$ is defined as:

$$x(t) = \begin{cases} \exp(i\beta t^2) & -T \le t \le T \\ 0 & otherwise \end{cases}, \tag{5.3}$$

where β is called the Chirp parameter or slope which indicates the rate of change of instantaneous frequency of the given Chirp.

The FrFT [5] of a signal $x(t)$ represented along time axis denoted by t, with rotation angle α is computed as :

$$F^{\alpha}[x(t)] = X_{\alpha}(u) = \int_{-\infty}^{\infty} x(t) K_{\alpha}(t,u) dt \tag{5.4}$$

The rotation angle $\alpha = a\pi/2$, where the notation a is called the fractional order parameter. Let, the transform kernel $K_\alpha(t,u)$ of FrFT is written as:

$$K_\alpha(t,u) = \begin{cases} C_\alpha \exp\left\{ i(t^2+u^2)p - iqut \right\}, & \text{if } \alpha \text{ is not a multiple of } \pi \\ \delta(t-u), & \text{if } \alpha \text{ is a multiple of } 2\pi \\ \delta(t+u), & \text{if } \alpha+\pi \text{ is a multiple of } 2\pi \end{cases} \quad (5.5a)$$

where

$$C_\alpha = \sqrt{\frac{1-i\cot\alpha}{2\pi}}, \; p = \frac{1}{2}\cot\alpha \text{ and } q = \csc\alpha \quad (5.5b)$$

Substituting $x(t)$ given by Eq. (5.3) in Eq. (5.4) results:

$$X_\alpha(u) = C_\alpha \exp(ipu^2) \int_{-T}^{T} \exp(i\beta t^2)\exp(ipt^2 - iqut)dt \quad (5.6)$$

Rearranging (5.6), one gets:

$$X_\alpha(u) = C_\alpha \exp(jpu^2) \underbrace{\int_{-T}^{T} \exp\left[i\left\{ (\beta+p)t^2 - qut \right\} \right] dt}_{I_1} \quad (5.7)$$

The integral I_1 is simplified by completing the square and rewritten as:

$$I_1 = \int_{-T}^{T} \exp\left[i\left\{ \left(\sqrt{(\beta+p)}t - \frac{qu}{2\sqrt{(\beta+p)}} \right)^2 - \frac{q^2u^2}{4(\beta+p)} \right\} \right] dt \quad (5.8)$$

$$= \exp\left(-\frac{iq^2u^2}{4(\beta+p)} \right) \int_{-T}^{T} \exp\left[i\left\{ \left(\sqrt{(\beta+p)}t - \frac{qu}{2\sqrt{(\beta+p)}} \right)^2 \right\} \right] dt \quad (5.9)$$

Let $z = \sqrt{(\beta+p)}t - \frac{qu}{2\sqrt{(\beta+p)}}$, which gives $dt = \frac{dz}{\sqrt{(\beta+p)}}$, and substituting in (5.9) gives:

$$I_1 = \frac{1}{\sqrt{(\beta+p)}} \exp\left(-\frac{iq^2u^2}{4(\beta+p)} \right) \int_{L_1}^{L_2} \exp(iz^2)dz \quad (5.10)$$

where $L_1 = -\sqrt{(\beta+p)}T - \dfrac{qu}{2\sqrt{(\beta+p)}}$ (5.11a)

and $L_2 = \sqrt{(\beta+p)}T - \dfrac{qu}{2\sqrt{(\beta+p)}}$ (5.11b)

By applying $\int_{L_1}^{L_2} \exp(iz^2)dz = \dfrac{\sqrt{\pi}}{2\sqrt{i}}\left[erfi\left(\sqrt{i}L_2\right) - erfi\left(\sqrt{i}L_1\right)\right]$ to (5.10), the following expression results [1]:

$$I_1 = \frac{1}{\sqrt{i(\beta+p)}}\exp\left(-\frac{iq^2u^2}{4(\beta+p)}\right)\frac{\sqrt{\pi}}{2}$$

$$\times\left[erfi\left\{\frac{1+i}{\sqrt{2}}\left(\sqrt{(\beta+p)}T - \frac{qu}{2\sqrt{(\beta+p)}}\right)\right\} - erfi\left\{\frac{1+i}{\sqrt{2}}\left(-\sqrt{(\beta+p)}T - \frac{qu}{2\sqrt{(\beta+p)}}\right)\right\}\right] \quad (5.12)$$

where $erfi(z)$ is imaginary error function of z, which is defined in the whole complex z-plane. By rearranging (5.7) and (5.12), one gets:

$$X_\alpha(u) = \frac{C_\alpha \exp(ipu^2)}{\sqrt{i(\beta+p)}}\exp\left(\frac{-iq^2u^2}{4(\beta+p)}\right)\frac{\sqrt{\pi}}{2}$$

$$\times\left[erfi\left\{\frac{1+i}{\sqrt{2}}\left(\sqrt{(\beta+p)}T - \frac{qu}{2\sqrt{(\beta+p)}}\right)\right\} - erfi\left\{\frac{1+i}{\sqrt{2}}\left(-\sqrt{(\beta+p)}T - \frac{qu}{2\sqrt{(\beta+p)}}\right)\right\}\right] \quad (5.13)$$

Putting values of p, q and C_α given by Eq. (5.5b) in Eq. (5.13), and simplifying results:

$$X_\alpha(u) = \sqrt{\frac{1-i\cot\alpha}{8i(\beta+0.5\cot\alpha)}}\exp\left(\frac{iu^2(2\beta-\tan\alpha)}{2(2\beta\tan\alpha+1)}\right)$$

$$\times\left[erfi\left\{\frac{1+i}{\sqrt{2}}\sqrt{\beta+0.5\cot\alpha}\left(T-\frac{u\csc\alpha}{2\beta+\cot\alpha}\right)\right\}-erfi\left\{\frac{1+i}{\sqrt{2}}\sqrt{\beta+0.5\cot\alpha}\left(-T-\frac{u\csc\alpha}{2\beta+\cot\alpha}\right)\right\}\right]$$

(5.14a)

Putting $\alpha = a\pi/2$, Eq. (5.14a) becomes:

$$X_\alpha(u)=\sqrt{\frac{1-i\cot a\pi/2}{8i(\beta+0.5\cot a\pi/2)}}\exp\left(\frac{iu^2(2\beta-\tan a\pi/2)}{2(2\beta\tan a\pi/2+1)}\right)$$

$$\times\left[erfi\left\{\frac{1+i}{\sqrt{2}}\sqrt{\beta+0.5\cot a\pi/2}\left(T-\frac{u\csc a\pi/2}{2\beta+\cot a\pi/2}\right)\right\}-erfi\left\{\frac{1+i}{\sqrt{2}}\sqrt{\beta+0.5\cot a\pi/2}\left(-T-\frac{u\csc a\pi/2}{2\beta+\cot a\pi/2}\right)\right\}\right]$$

(5.14b)

Thus, from (5.14b), it can be seen that the FrFT of Chirp function is directly dependent on FrFT order parameter a and Chirp parameter β. The adjustable spectral parameters of the Chirp function can be obtained by modifying the fractional angle.

Special Case I: If $\beta = 0$,

Substituting $\beta = 0$ in (5.3), $x(t)$ can be written as:

$$x(t)=\begin{cases}1 & -T\le t\le T\\ 0 & otherwise\end{cases} \tag{5.15}$$

The signal $x(t)$ becomes a Rectangular function. The FrFT of Rectangular window function can be obtained by substituting $\beta = 0$ in Eq. (5.14a):

$$X_\alpha(u)=\frac{\sqrt{1-i\cot\alpha}}{2\sqrt{i\cot\alpha}}\exp\left(-\frac{i}{2}u^2\tan\alpha\right)$$
$$\times\left[erfi\left(\frac{1+i}{2}\sqrt{\cot\alpha}(T-u\sec\alpha)\right)-erfi\left(\frac{1+i}{2}\sqrt{\cot\alpha}(-T-u\sec\alpha)\right)\right] \tag{5.16a}$$

By using $erfi(z) = -ierf(iz)$ and simplifying (5.16a), the FrFT of Rectangular window is given by:

$$X_\alpha(u) = \frac{\sqrt{1 - i\tan\alpha}}{2} \exp\left(-\frac{i}{2}u^2 \tan\alpha\right)$$

$$\times\left[erf\left(\frac{1-i}{2}\sqrt{\cot\alpha}\left(T - u\sec\alpha\right)\right) - erf\left(\frac{1-i}{2}\sqrt{\cot\alpha}\left(-T - u\sec\alpha\right)\right)\right] \tag{5.16b}$$

Special Case II: If $\beta = -p$ **,**

Now substituting $\beta = -p$ in (5.6), $X_\alpha(u)$ can be written as:

$$X_\alpha(u) = C_\alpha \exp(ipu^2) \int_{-T}^{T} \exp(-iqut)dt \tag{5.17}$$

$$= C_\alpha \exp(ipu^2)\, \frac{\exp(-iquT) - \exp(iquT)}{-iqu} \tag{5.18}$$

$$= C_\alpha \exp(ipu^2)\, \frac{2T\sin(quT)}{quT} \tag{5.19}$$

$$= 2T\, C_\alpha \exp(ipu^2)\, \sin c(quT) \tag{5.20}$$

From Eq. (5.20), it can be seen that FrFT of finite Chirp at fractional angle $\alpha = \cot^{-1}(-2\beta)$ gives sinc function main-lobe. The most compact support for a given Chirp signal can be obtained at this angle and is called as the optimal fractional angle for that signal.

5.3 Results and Discussion

The real part for Chirp function $x(t) = \exp(it^2)$ as a function of time is shown in Fig. 5.2. The value of Chirp parameter β is taken to be equal to 1 and the range of time interval is -0.5 $\leq$ t $\leq$ 0.5. The plots of magnitude of its FrFT i.e. $|X_\alpha(u)|$ versus frequency u (cycles/sec) are obtained using Eq. (5.14b). The plots for calculating and comparing the MSLL and SLFOR for Chirp window at different values of fractional order parameter are shown in Figs. 5.3 and 5.4 respectively. The continuum of Chirp window function for Chirp parameter β varying from 0 to 3 and at fractional order a equal to 1 is shown in Fig. 5.5. Figure 5.6 shows the continuum of Chirp window function at Chirp parameter β equal to 0 and the value of fractional order a is varied from 0 to 1.

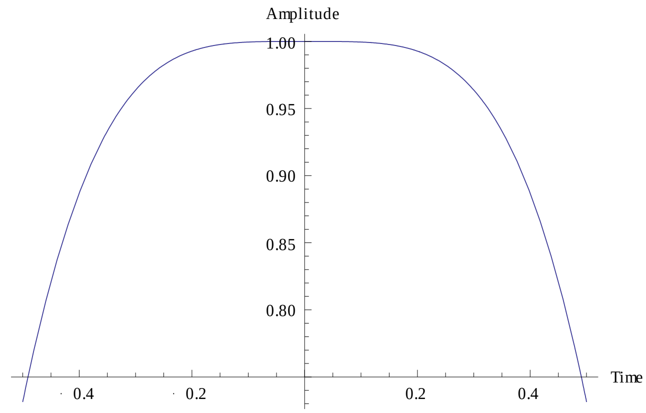

Fig. 5.2: Real part of Chirp function $x(t) = \exp(it^2)$ for -0.5 ≤ t ≤ 0.5.

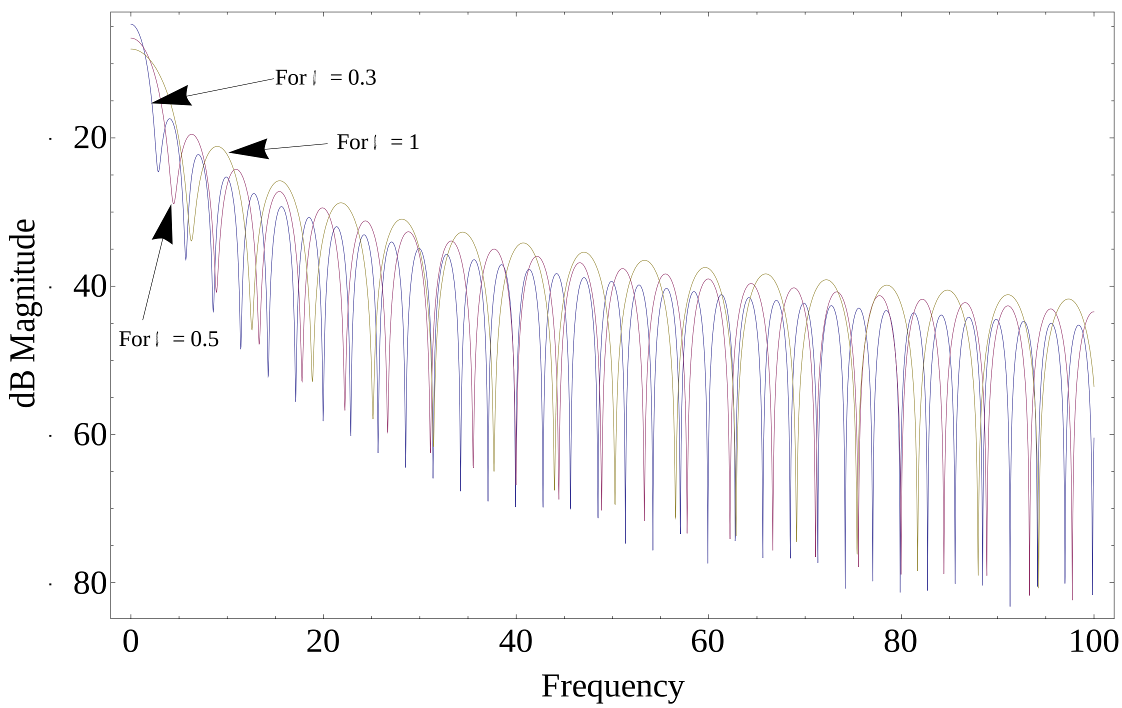

Fig. 5.3: MSLL plots for Chirp window (for $\beta = 1$) at fractional order parameter a = 0.3, 0.5 and 1.

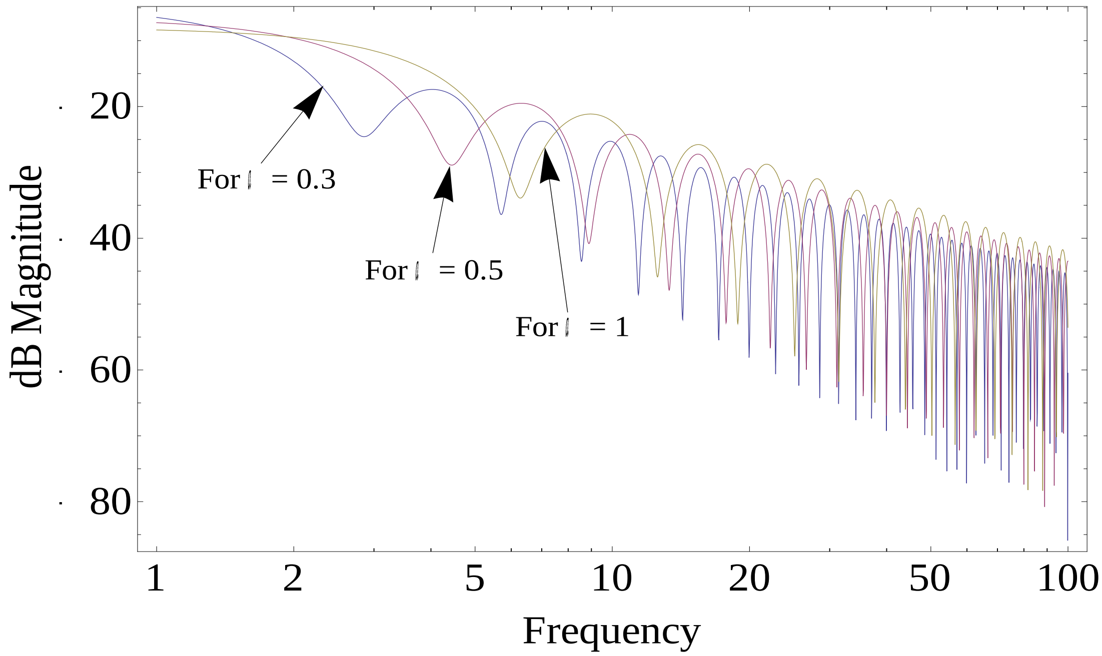

Fig. 5.4: SLFOR plot for Chirp window (for $\beta = 1$) at fractional order parameter a = 0.3, 0.5 and 1.

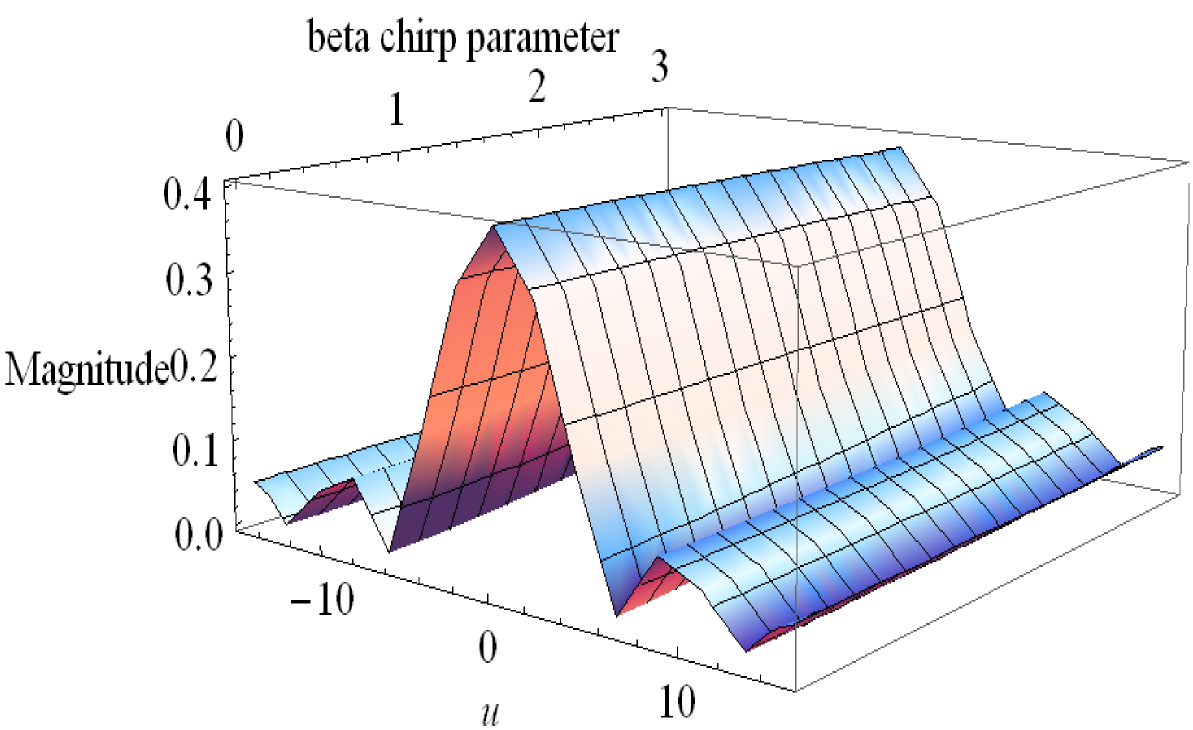

Fig. 5.5: The continuum of fractional Fourier transform of Chirp window (at fractional order a =1) for Chirp parameter β varying from 0 to 3.

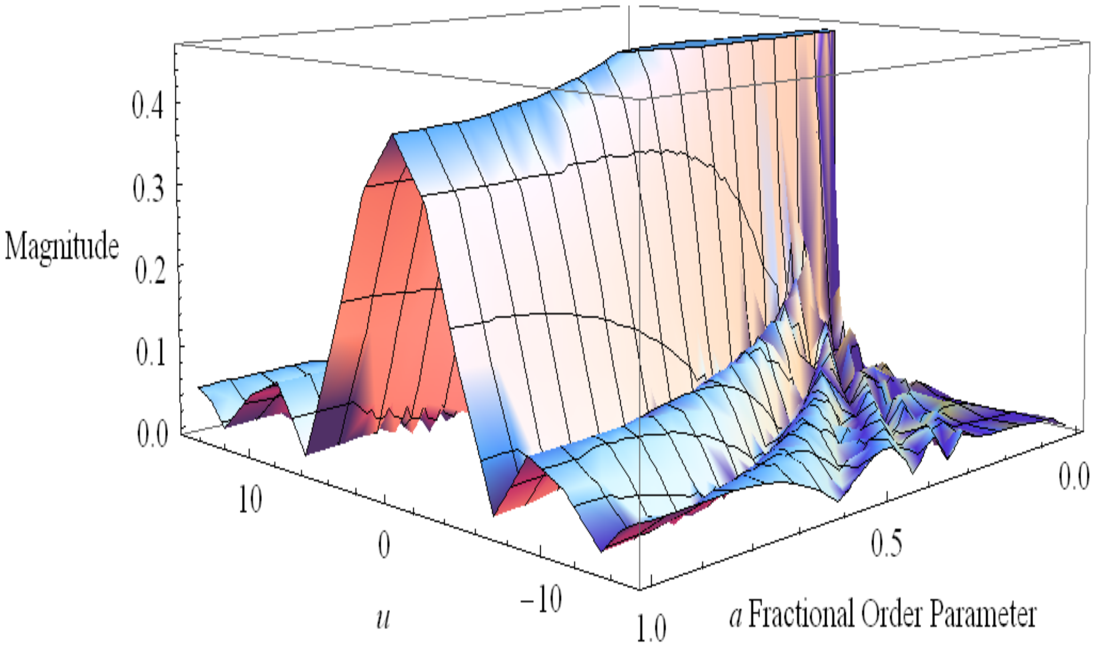

Fig. 5.6: The continuum of fractional Fourier transform of Chirp window (at Chirp parameter $\beta = 0$) for fractional order a varying from 0 to 1.

Table 5.1:

Parameters of a Chirp window (at $\beta = 1$) with variations in fractional order a.

Fractional Order a	MSLL (dB)	HMLW	SLFOR (dB/octave)
0.2	-12.12	1.58	-5.72
0.3	-12.23	2.49	-5.65
0.4	-12.55	2.94	-5.59
0.5	-12.77	3.39	-5.42
0.6	-12.80	3.85	-5.38
0.7	-12.83	4.34	-5.27
0.8	-12.91	4.41	-5.13
0.9	-13.23	4.75	-5.05
1	-13.32	5.21	-4.61

The values of MSLL and SLFOR for Chirp window function (at $\beta = 1$) are tabulated in Table 5.1 for various values of adjustable parameter a. It can be seen from Table 5.1 that the parameters of Chirp window depend upon the value of fractional order parameter a and the main-lobe width of the window shrinks regularly with decrease in adjustable fractional order a of the transform. SLFOR also shows variation between -4.61dB/octave to -5.72 dB/octave with change in order parameter a. A low side-lobe level is achieved with increase in fractional order a, e.g. MSLL for $a = 0.8$ is -12.91 db compared to -13.23 db for $a = 0.9$. The MSLL plots of Chirp window by varying Chirp parameter β and fractional order parameter a to different values are shown in Fig. 5.7.

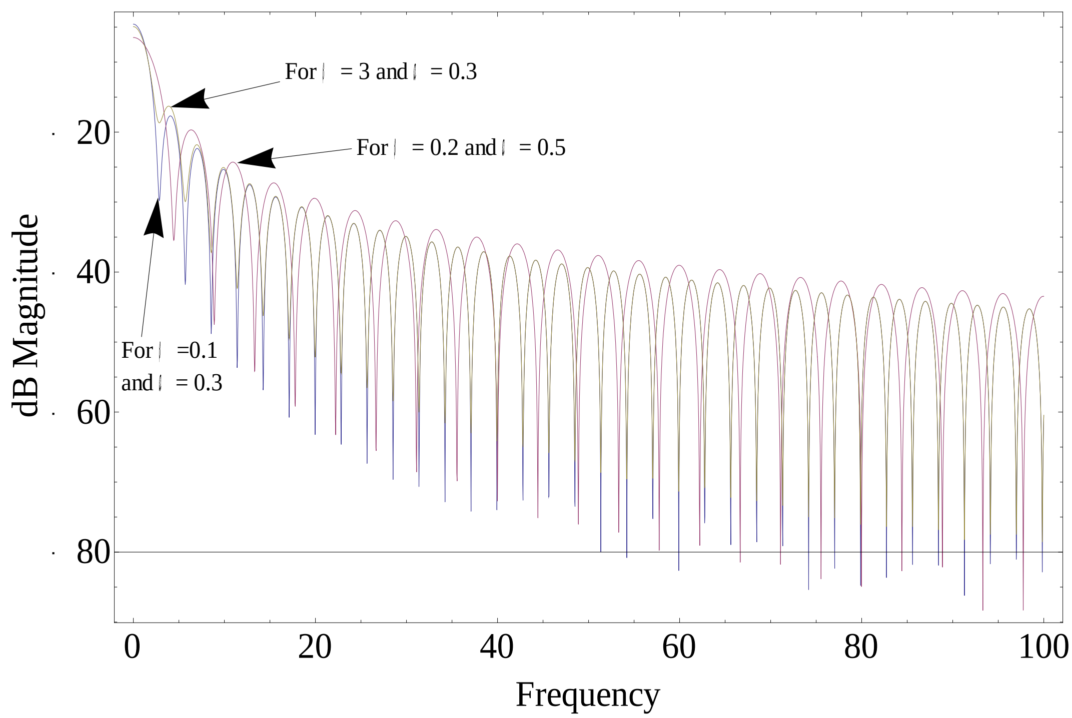

Fig. 5.7: MSLL plot of Chirp window for various values of Chirp parameter and fractional order parameter a.

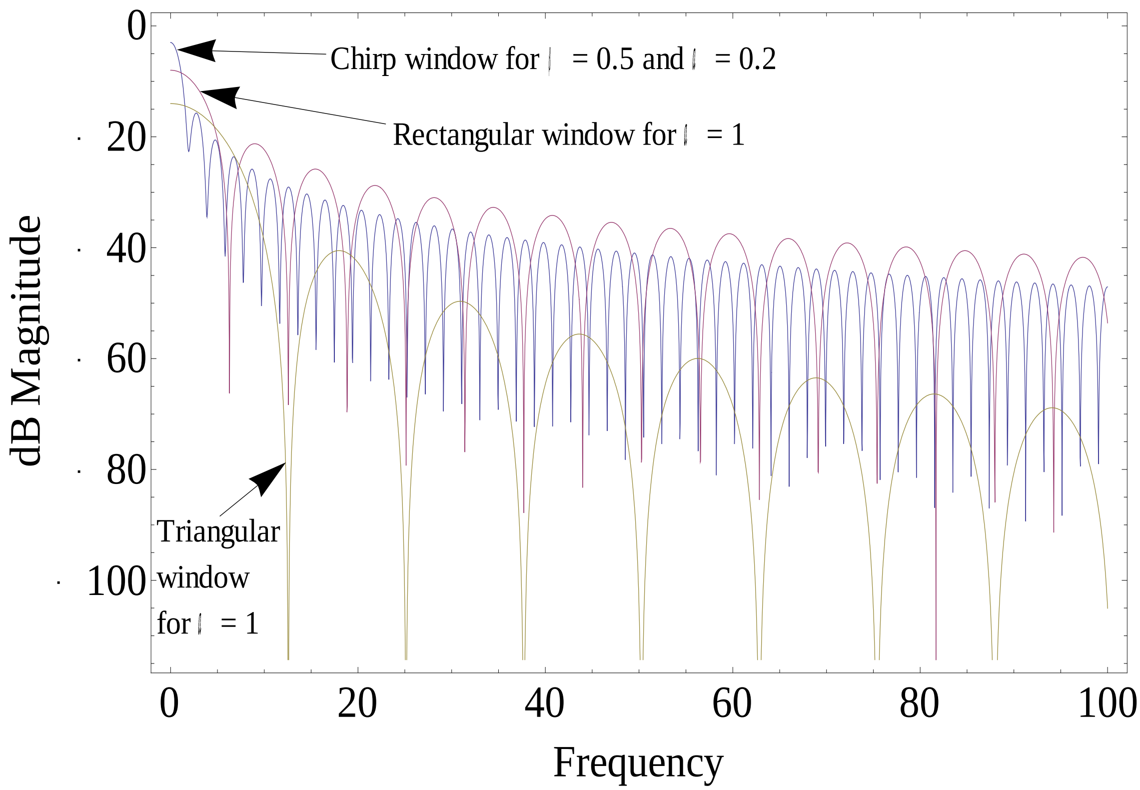

Fig. 5.8: Comparative performance of Chirp window with some existing windows.

Table 5.2:

Comparative Parameters of Chirp window under FrFT with some existing windows

Function	Fractional Order a	MSLL(dB)	HMLW	SLFOR (dB/octave)
Chirp at $\beta = 0.2$	0.5	-13.18	3.5	-4.68
Chirp at $\beta = 0.5$	0.1	-13.14	0.65	-4.88
Rectangular	1	-13	5.1	-6
Chirp at $\beta = 0.2$	0.2	-12.6	1.5	-4.8
Chirp at $\beta = 0.1$	0.3	-13.2	2.1	-4.92
Chirp at $\beta = 0.5$	0.3	-13.52	2.29	-4.2
Triangular	1	-26	10.3	-12
Chirp at $\beta = 1$	0.7	-13.378	4.753	-4.53

Figure 5.8 shows the **performance comparisons** of Rectangular and Triangular window for fractional order parameter a =1 with Chirp window for $\beta = 0.5$ and fractional order parameter a = 0.2. All the three windows are defined for time interval -0.5$\leq$ t$\leq$0.5. In Table 5.2, the spectral parameters of Chirp window under FrFT are evaluated for a range of fractional angle α and compared with the parameters of Rectangular and Triangular window. It can be seen from Table 5.2 that by adjusting the value of fractional order and Chirp parameter, spectral parameters of Chirp window can be controlled more precisely. The value of MSLL for Chirp window at $\beta = 0.2$ and $a = 0.2$ is -13.95 db which shows an improvement over Rectangular window whose MSLL is -13dB at fractional order $a = 1$. It is observed that Chirp window can achieve low side-lobe levels with minimum increase in main-lobe width for some particular values of Chirp parameter β and fractional order parameter a. Figures 5.7 and 5.8 show that Chirp window gives better spectral parameters than Rectangular window. Thus, Rectangular window can be replaced with Chirp window in order to give superior performance in a variety of applications. The analysis of Chirp window using FrFT shows that a better compromise can be easily made between increase in main-lobe width and side-lobe level reduction by choosing an optimal fractional order a (or rotation angle α). The simulation results are obtained using Wolfram Mathematica® software.

5.4 Application in Transition BW Tuning of FIR Filters using FrFT

The Chirp window-based LP FIR filter using FrFT is designed with modified transition-width characteristics by using the same methodology as is presented in Section 2.2. Consider a Chirp $x(t) = \exp(i\beta t^2)$ for the interval $-0.5 \leq t \leq 0.5$. Figures 5.9 and 5.10 show the magnitude response of Chirp window based low-pass FIR filter with cut-off frequency $= 0.5\pi$ radians for different values of fractional order parameter with chirp parameter $\beta = 0.1$ and $\beta = 0.01$ respectively.

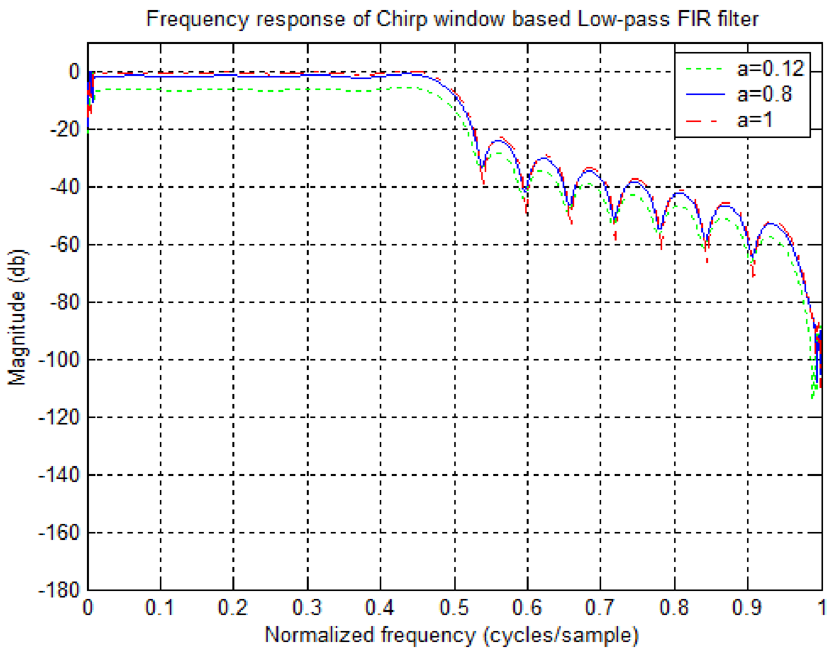

Fig. 5.9: The magnitude response of Chirp window (chirp parameter $\beta = 0.1$) based low-pass FIR filter for fractional order $a = 0.12, 0.8$ & 1.

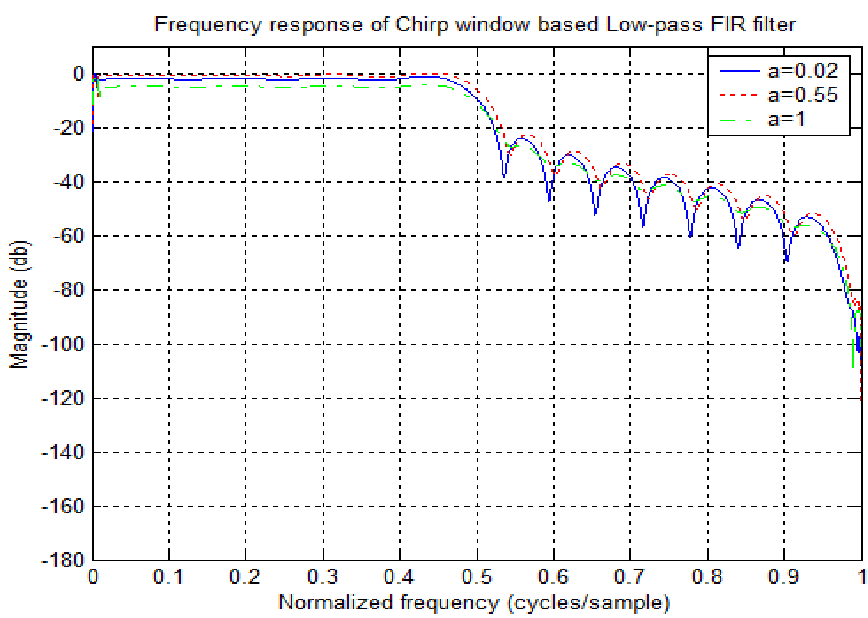

Fig. 5.10: The magnitude response of Chirp window (chirp parameter $\beta = 0.01$) based low-pass FIR filter for fractional order $a = 0.02, 0.55$ & 1.

5.5 Conclusions

It can be summarized that FrFT of Chirp function varies directly with the change in fractional order a (or fractional angle α) and Chirp parameter β. The variations in spectral parameters of Chirp are studied for different values of fractional order a. It is observed that as the fractional order is reduced, main-lobe width can be minimized and side-lobe fall-off rate can be raised to maximum. By applying FrFT to Chirp window, low side-lobe levels with minimum increase in main-lobe width can be achieved. Thus, a trade-off can be made between increase in main-lobe width and reduced side-lobe level to have an estimate of the spectrum of a signal which is best for the desired application. The results of comparison clearly show that Chirp window provides better spectral parameters for some particular values of fractional order and can replace the existing Rectangular window in a number of particular applications. Also, Chirp window under FrFT can be applied in the design of FIR filters with variable transition bandwidth.

CHAPTER 6

CONCLUSIONS AND SCOPE OF FUTURE WORK

In this thesis, the design of digital filters is considered that depend on a single parameter α . The spectral characteristics of FIR filters can be easily adjusted by using the rotation angle α (or the fractional order parameter a) inherent in the fractional Fourier transform as a control parameter. The relation between α and a is given by $\alpha = a\pi/2$. The analysis of behavior of window functions with FrFT shows that the spectral parameters of window functions depend on the fractional order parameter a and window-based FIR filters can be designed to have variable characteristics by using FrFT based methodology. The simulations as well as analytical results are presented in the work.

6.1 Contributions

(a) Design of Variable Transition-width FIR Filters using Fractional Fourier Transform

A new closed-form expression based on FrFT is developed for obtaining the adjustable frequency response function of Rectangular windowed fixed length LP FIR filters. The proposed approach does not involve changing the filter length as reducing the FrFT angle or the fractional order parameter is analogous to increasing the filter order. As the fractional angle is reduced from $\pi/2$ to 0 , the transition-width can be minimized and the stop-band attenuation can be raised to maximum. The applicability of the proposed method in real-time applications is presented. The performance comparison of the proposed FrFT based method over the ordinary method based on FFT (by varying the window length) in tuning FIR filter is also done.

The results are presented in Chapter 3 and are published in the journal *J. Signal, Image and Video Processing*.

(b) Design of Variable Cut-off/ Band-edge FIR Filters using Fractional Fourier Transform

A new design method is developed to increase the pass-band width and to achieve adjustable phase response of window-based FIR filters where the fractional order a of the transform is used as a tuning parameter. The applicability of the proposed method in practical applications is presented. The performance comparison of the proposed FrFT based method with the traditional methods for designing digital filters with variable spectral characteristics is also made.

The results are presented in Chapter 4 and are published in the journal *Journal Elektronika Ir Elektrotechnika*.

(c) Analysis of Finite Duration Chirp as Windowing Function using FrFT

A new mathematical model for obtaining the closed-form expression of fractional Fourier transform of Chirp window function is developed. The Chirp under FrFT proves to be an alternative window function to the existing Rectangular window in achieving better spectral characteristics.

The results are presented in Chapter 5 and are published in the journal *International Journal of Electronics.*

6.2 Conclusions

It is concluded on the basis of the proposed design techniques and the derived closed-form expressions that on changing the fractional order parameter a in the range from 0 to 1 (or the rotation angle α in the range from $0 \text{ to } \pi/2$), the transition-width, pass-band width and minimum stop-band attenuation parameters in the frequency response of digital can be adjusted easily. The advantage of applying FrFT is that frequency response of a digital filter can be instantly tuned without changing the length of the filter. The FrFT based tuning procedure does not require different computational resources, large number of filter coefficients, approximation of filter coefficients which leads to error and any other additional hardware or non-computational components such as delay elements, digital comparators, interpolators or memory block etc. Also, variation in the magnitude related parameter such as stop-band attenuation can be easily achieved. It is summarized that:

(i) The frequency response of digital filters can be modified easily by using FrFT in order to meet the desired application immediately. There is no need to repeat the filtering design procedure.

(ii) The derived closed-form expression for obtaining the adjustable frequency response of Rectangular windowed fixed length LP FIR filters represents advancement in the design of FrFT based VDFs.

(iii) Better spectral parameters can be obtained by analyzing the window functions with FrFT. The improvement in terms of main lobe and side ripples in the window function transform can be achieved for a range of fractional angles. Thus, the design of Chirp window-based LP FIR filter using FrFT adds to the advantage of using FrFT in adjustable FIR filter design.

6.3 Future Scope of Work

After concluding the present work, the following research areas may be explored further:

(i) Development of techniques based on FrFT to design variable recursive infinite impulse response digital filters.

(ii) Characterization of digital filter's transfer function establishing analytical relationship between the FrFT order and the filter parameters.

(iii) Analysis of Phase characteristics of the FrFT based variable transition-width FIR filter for a range of fractional order.

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