LINEAR MULTI-OBJECTIVE TRANSPORTATION PROBLEM

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DEDICATED

TO

GOD, MY PARENTS

AND

SUPERVISOR

CERTIFICATE

*I hereby certify that the work which is bring presented in the thesis entitled "**Linear Multi-Objective Transportation Problem**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Mahesh Kumar Sharma.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

Multi-criteria optimization (or multiple-objective programming) also known as multi-criteria multiple-attribute optimization, is the process of simultaneously optimizing two or more objectives subject to certain constraints. A multi-objective transportation problem (MOTP) has been considered in present thesis.

The work consist three chapters. Chapter 1 is introductory. In chapter 2 MOTP given by Gupta and Gupta (1982) is reviewed and numerical example has been solved in detail. In chapter 3 the problem given in chapter 2 is extended by including one more objective which is conflicting in nature and set of non-dominated solution is obtained.

CONTENTS

CHAPTER 1

INTRODUCTION

One of many well-structured problems in operation research that has been extensively studied in literature is Transportation problem. It is the subclass of the linear programming problems for which simple and practical computational procedures have been developed that take the advantage of the special structure of the problem. Linear programming refers to the mathematical programming. In the context, it refers to a planning process that allocates material, resources labour, capital, machines, in the best possible (optimal) way so that costs are minimized or profits are maximized. In linear programming these resources are known as decision variables. Objective function is known as the criterion for selecting the best values of decision variables (e.g., minimizing costs).

1.1 COST MINIMIZING TRANSPORTATION PROBLEM

A particular class of linear programming is classical transportation problem, which is associated with day-to-day activities in our real life and mainly deals with logistics. It helps in solving transportation of resources from one place to another and problems on distribution. To meet the specific requirements, the goods are transported from a set of *m* origins (e.g., factories) to a set of *n* destinations (e.g., shops). In other words, transportation problems deal with the transportation of a single product manufactured at different plants (supply origins) to a number of different warehouses (demand destinations). The objective is to satisfy the demand at destinations from the supply constraints at the minimum transportation cost possible. To achieve this objective, we must know the quantity of available supplies and the quantities demanded. A cost minimization transportation problem is formulated as

$$\text{Minimize} \qquad Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{Subject to} \qquad \sum_{i=1}^{n} x_{ij} \le a_i \qquad a_i > 0, \quad i = 1,2,...,m$$

$$\sum_{i=1}^{n} x_{ij} \geq b_j \qquad b_j > 0, \quad j = 1,2,...,n$$

$$x_{ij} \geq 0 \qquad \forall\, i,j$$

i =1, 2,…, m is the set of origins.

j =1, 2,…, n is the set of destinations.

x_{ij} = the quantity transported from the i-th origin to the j-th destination.

c_{ij} = per unit cost in transporting goods from i-th origin to the j-th destination.

a_i = the amount available at the i-th origin

b_j = the demand of the j-th destination.

It is assumed that the total availability is equal to total demand, i.e.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Then the problem is a balanced transportation problem, otherwise it is unbalanced.
But in certain life situation, the total availability may not be equals to the total requirement, i.e.,

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

For example, if supply is greater than demand $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$,then a fictitious destination may be used to create the desired equality. If the demand exceeds supply $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$ then a fictitious source may be introduced. The aim is to minimize the objective function satisfying the above constraints. In classical transportation problem in linear programming the traditional objective is to minimize the total cost.

1.2 TIME MINIMIZING TRANSPORTATION PROBLEM

The transportation problem in which the time of transporting goods from m origins to n destinations is minimized is known as time minimizing transportation problem, satisfying certain conditions in respect of availabilities and source requirement at the destination.

Time minimization transportation problem is formulated as

Minimize $\max\ [t_{ij} / x_{ij} > 0]$

Subject to $$\sum_{i=1}^{n} x_{ij} = a_i \qquad a_i > 0, \quad i = 1,2,...,m$$

$$\sum_{i=1}^{n} x_{ij} = b_j \qquad b_j > 0, \quad j = 1,2,...,n$$

$$x_{ij} \geq 0 \qquad \forall\ i,j$$

a_i = the amount available at the i^{th} origin

b_j = the demand of the j^{th} destination.

The time of transportation remains independent of the amount of commodity sent as long as $x_{ij} > 0$. It is assumed that

(1) The carriers have sufficient capacity to carry goods from an origin to a destination in a single trip.

(2) They start simultaneously from the respective origin.

The time minimizing-transportation problems are of importance when it is required to transport perishable goods. Sometimes there may exist emergency situations such as those requiring police services, fire services, ambulance services, etc when the time of transportation is of greater importance than cost of transportation.

Some methods for minimizing the time of transportation have been established. In such situations rather than minimizing the cost, the objective is to minimize the maximum time to transport all supply to destinations satisfying certain conditions in respect of availabilities at sources and requirements at the destinations.

For any given feasible solution, X=[x_{ij}] satisfying the above constraints, the time of transportation is maximum among the cells in which there are positive allocations, corresponding to the solution X= [x_{ij}] , the time of transportation is

$$Z = \max\ [t_{ij} \,/\, x_{ij} > 0]$$

The aim is to minimize this time of transportation.

1.3 MULTI-CRITERIA OPTIMIZATION

Multi-criteria optimization (or multi-objective programming), also known as multi-attribute optimization, can be defined as:

“A vector of decision variables which satisfies constraints and optimizes a vector function whose elements represent the objective functions. Hence the term “optimizes” means finding the solution which gives the value of all the objective functions acceptable to decision maker.” Or one can say it is the process of simultaneously optimizing two or more conflicting objectives subject to certain constraints. Normally, existing multi-objective transportation models use a minimization of the total cost objective as one of their objectives. The other objectives may concern about quantity of goods delivered, energy consumption, total delivery time, etc. Consider *m* origins and *n* destinations and also the quantities available at each origin and the quantities to be transported to each destination. The total quantities required at the destinations may differ from the total quantities available at the origins .For such situations; the problem is balanced by introducing fictitious origin or destination; whichever is needed in order to get precisely the same quantities at the origins and the destinations. Specifically, a balanced transportation problem is considered as it amounts to no loss of generality.

Multi –criteria optimization problems can be found in various fields: product and process design, the oil and gas industry or wherever optimal decisions need to be taken in the presence of tradeoffs between two or more conflicting objectives.

For nontrivial multi-criteria optimization problems, one cannot identify a single solution that simultaneously minimizes each objective to its fullest. While searching for solutions, one reaches points such that, when attempting to improve an objective further, other objectives suffer as a result. A solution is called non-dominated if it cannot be eliminated from consideration because there is at least another solution which improves an objective without worsening another one.

This multi-criteria problem can be formulated as:

$$\text{Optimize} \quad f(X) = (\, f_1(X), f_2(X), \ldots, f_k(X))$$
$$\text{Subject to} \quad g_j(X) <=> b_j\,, j = 1,2,\ldots,m$$

$$X > 0$$

$$X = (x_1, x_2, \ldots\ldots, x_n)^T$$

Where, $f(X)$ is the objective function to optimize. (f_1 (X), f_2 (X), ... , f_k (X)) are k number of distinct objective functions subject to m constraints. X is a vector consisting of decision making variables $x_1, x_2, \ldots, x_n$

Multi-objective optimization has been applied in many fields of science, including engineering, economics and logistics where optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives. Minimizing weight while maximizing the strength of a component, and maximizing performance while minimizing fuel consumption and emission of pollutants of a vehicle are examples of multi-objective programming involving two or three objectives respectively.

For a non trivial multi-objective optimization problem, there does not exist a single solution that simultaneously optimizes each objective. In that case, the objective functions are said to be conflicting, and there exist a (possible infinite number of) Pareto optimal, Pareto efficient or non-inferior, if none of the objective function can be improved in value without impairment in some of the other objective values. Without additional preference information, all Pareto optimal solutions can be considered mathematically equally good (as vectors cannot be ordered completely). Researchers study multi objective optimization problems from different viewpoints and thus there exist different solution philosophies and goal when setting and solving them. The goal may be finding a representative set of Pareto optimal solutions and/or qualifying the trade-offs in satisfying the different objectives, and/or finding a single solution that satisfies the presence of human decision maker.

1.4 CONCEPT OF OPTIMAL AND EFFICIENT SOLUTIONS:

Optimal Solution

An optimal solution in the classical sense is one which attains the maximum value of all the objectives simultaneously. The solution x^* is optimal to the problem defined if and only if $x^* \in S$ and $f_l(x^*) \geq f_l(x)$ for all l and for all $x \in S$, where S is the feasible region.

In general, there is no optimal solution to a multi-objective problem. Therefore, optimality replaced by the concept of "satisfying" or the best compromise solution, which depends on the decision makers preferences with respect to object. Optimality is not an illusion only when the objectives are non-conflicting. Therefore one must be satisfied with obtaining efficient solutions in multi-objective problems.

Efficient or Non-Dominated Solutions

A set of solutions is said to be efficient if there exists no solutions that is superior to it with respect to at least one objective function but is not inferior to it with respect to any of the objective functions.

If x_1 and x_2 are two solution, then these can have any of two possibilities- one dominates the other or non-dominates the other. In a minimization problem, without the loss of generality, a solution x_1 dominates x_2 if the following two conditions are satisfied:

$$\forall\, i \in \{1,2,\ldots,N_{obj}\} : f_i(x_1) \leq f_i(x_2)$$
$$\exists\, j \in \{1,2,\ldots,N_{obj}\} : f_j(x_1) < f_j(x_2)$$

Where, $f(x_1)$ and $f(x_2)$ are the objective functions

If any the above conditions are violated, the solution x_1 does not dominate the solution x_2. If x_1 dominates the solution x_2 is called the non dominated solution with in the set $\{x_1, x_2\}$. The solutions that are non-dominated within the entire search space and denoted as pareto-optimal and constitute the pareto-optimal set or pareto-optimal front. From the entire set of efficient (non-dominated) solutions the decision maker can select the solution one believed most attractive.

1.5 MULTI-OBJECTIVE TRANSPORTATION PROBLEM

The multi-objective transportation problem model is set to solve the transportation problem simultaneously associated with several objectives. Normally, existing multi-objective transportation models use a minimization of the total cost objective as one of their objectives. The other objectives may concern about quantity of goods delivered, underused capacity, energy consumption, total delivery time, etc. Kasana and Kumar (2003) formulate the multi-objective transportation problem as follows:

Consider m origins and n destinations and also the quantities available at each origin and the quantities to be transported to each destination. The total quantities required at the destinations may differ from the total quantities available at origins. For such situations, the problem is balanced by introducing fictitious origin or destination; whichever is needed in order to get precisely the same quantities at the origin and the destinations. Specifically, a balanced transportation problem is considered as it amounts to no loss of generality.

Suppose x_{ij} = amount transported from the *i-th* origin to the *j-th* destination and for each fixed *k*: $k = 0,1,\ldots,p-1, \alpha_{ij}^{k}$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ be the units of parameter required for transporting one unit of quantity from origin to destination. What is to be determined is primary and other are classified as secondary.

The primary objective is to minimize

$$Z_0 = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}^{k} x_{ij}$$

And for $k = 1,2,\ldots,(p-1)$, also to minimize

$x_k = \max\{\alpha_{ij}^{k} : x_{ij} \geq 0, i = 1,2,...,m, j = 1,2,...,n\}$ in order of priorities to be assigned under the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad (i = 1,2,\ldots,m)$$

$$\sum_{i=1}^{n} x_{ij} = b_j \qquad (j = 1,2,\ldots,n)$$

The problem formulated above has *p* objective functions given by equations. A transportation problem with *m* supply nodes and *n* demand nodes contains *mn* variables and *m*+*n* constraint equations.

1.6 LITERATURE SURVEY

There are different types of transportation problems and the simplest of them is now standard in the literature was first presented by Hitchcock (1941). It usually aims to minimize the total transportation cost. Other objectives that can be set are a minimization of the total delivery time, a maximization of profits, etc. from the investigation; the entire existing objectives in single objective transportation model are represented by quantitative information. This may cause the negligence of some crucial points which cannot be described by quantitative data. Koopman began to spearhead research on the potentialities of linear programs for the study of the problems in economics. His historic paper "optimum utilization of the transportation potations systems" was based on his war time experience. Because of this and the work done earlier by Hitchcock, the classical case is often referred as the Hitchcock-Koopman's transportation problem. Kantorovich (1942) publishes the paper on a continuous version of the problem and later with Gavurin, an applied study of the capacitated transportation problem (Kantorovich and Gavurin 1949).

The time minimizing transportation problem has been studied by Hammer (1969),Garfinkel and Rao(1971) and Szwarc (1971).Hammer (1969) and Szwarc(1971) used labeling techniques to solve the problem. Garfunkel and Rao(1971) solved the problem by introducing a sufficiently large cost M on certain routes. Sometimes there may exists emergency situations such as those requiring police services, fire services, ambulance services, etc., when the time of transportation is of greater importance than cost of transportation. Some methods for minimizing the time of transportation have been established. Several methods for minimizing the time of transportation are also developed. Then Bhatia *et.al*. (1975) developed a technique for minimizing time in a transportation problem. The procedure involved finite number of iterations and is based on moving from one basic feasible solution to another till the last solution is arrived at. The algorithm given by them consists of determination of an initial basic feasible solution which can be found by the methods applicable in the case of the common cost minimizing transportation problem and finding an adjacent better basic feasible solution, the procedure is repeated until no better adjacent basic feasible solution can be found. This repeated procedure deals with the determination of a cell not in the basis which, when introduced ,will either reduce the time of transportation or reduce the allocation in at least one of the cells belongs to Q, where Q is the set of cell with positive allocations and corresponding time equal to the time of transportation.

The transportation problem with two-objectives known as the bi-criterion problem has been studied by many research workers. In this type of problem there are two objectives-one primary and the other secondary. The primary objective is to minimize the total cost of transportation and the secondary objective is to minimize the duration of transportation.

There are many approaches for solving multi-objective optimization problems. The various approaches for solving multi-objective optimization problems are lexicographic / prioritized and paerto optimal / efficient / non-dominated solution approach. A discussion about them can can be found in the works of Zenely (1974), Prakash (1981), Igznio (1982), Sharma and Prakash (1986), Steuer (1986), Prakash, Aggarwal and Shah (1988), Prakash and Pradeep (1991), Prakash and Gupta (2006), Balaji and Tuteja(1999), Taha(2008). The first two approaches reduce the multi-objective optimization to single objective optimization problems while the last two approaches do not alter the nature of the problems. TP with a different single objective to minimize the duration of the transportation has been studied by many researchers as Sharma and Swarup (1977). Shesan and Tikekar (1980), Prakash (1982), Sonia and Puri (2004), Sonia ,Khandelwal and Puri (2008) etc. TP with multiple-objective is discussed by Prakash (1981), Purushotam, Prakash, Dhyani (1984), Aggarwal and shah (1988). Isermann's (1979) proposed an algorithm to obtain the set of all efficient solutions for a linear multi-objective transportation problem in different phases. The algorithm starts with an initial basic feasible solutions (phase I) and while passing from one efficient solutions (phase 2), has to solve a linear sub-problem at each step to find which vector should enter to the basis. Gupta and Gupta (1982) developed a multi-criteria simplex method for a linear multi-objective transportation problem (LMOTP) which is direct generalization of the multi-criteria simplex method of Zenely (1974) to the LMOTP, by which the set of all non-dominated basic feasible solution are generated. They also proposed a technique to check the dominance or non-dominance of the solutions and show that in some cases there is no need to solve the problem completely. However the approach given by Klingman and Russell (1975) is much and more simplified to check the non-dominance character of the solution but in that case the problem has to be solved completely.

1.7 PRESENT WORK

In the present thesis a Multi-objective transportation problem (MOTP) is considered. A multi-criteria simplex method for (MOTP) given by Gupta and Gupta (1982) is reviewed. Also this problem is extended by including one more objective which is conflicting in nature and the set of non-dominated solutions is obtained.

CHAPTER 2

MULTI-CRITERIA SIMPLEX METHOD FOR A LINEAR MULTIPLE-OBJECTIVE TRANSPORTATION PROBLEM

2.1 INTRODUCTION

In this chapter a Linear Multi-Objective Transportation Problem (LMOTP) considered by Gupta and Gupta (1982) has been reviewed and only numerical example has been solved in detail. They developed a multi-criteria simplex algorithm for this problem to obtain the set of feasible solutions.

They considered the following Linear Multi-Objective Transportation Problem (LMOTP)

$$(P) \qquad \text{Minimize} \qquad Z = \left\{ \sum_{(i,j)\in J} c_{ij}^{1} x_{ij} \ , \ \sum_{(i,j)\in J} c_{ij}^{2} x_{ij} \quad ,......, \sum_{(i,j)\in J} c_{ij}^{k} x_{ij} \right\}$$

$$\text{Subject to} \qquad \sum_{j\in N} x_{ij} = a_i , \ i \in M \qquad \ldots (2.1)$$

$$\sum_{i\in M} x_{ij} = b_j , \ j \in N \qquad \ldots (2.2)$$

$$x_{ij} \geq 0 \quad \forall \quad (i,j) \in J \qquad \ldots (2.3)$$

Where $N = \{1,2,...,n\}$ $M = \{1,2,....,m\}$ and $J = \{(i,j) : i \in M, j \in N\}$

In this paper it has been assumed that $a_i > 0$, $i \in M$ and $b_j > 0$, $j \in N$ moreover it is a transportation problem.

To find the set of all non-dominated basic feasible solutions of LMOT problem in terms of following criteria.

Definition (Gupta and Gupta (1982)) - A feasible solution $\overline{X} = (\overline{x_{ij}})$ is said to be non dominated solution of (P) if there does not exist any other feasible solutions $X = (x_{ij})$ of (P), such that

$$\sum_{(i,j)\in J} c_{ij}^{l} x_{ij} \quad \leq \quad \sum_{(i,j)\in J} c_{ij}^{l} \overline{x}_{ij} \quad l=1,2,....,k ,$$

With strict inequality in at least one of the k inequalities.

2.2 ALGORITHM

To find the set of all non-dominated basic feasible solutions, they make the use of simplex-type iteration, by starting with any basic feasible solution. The algorithm ensure that the new dominating by previous one while moving from one solution to the next solution.

The description of the algorithm as follows

Let $\overline{X}$ be the solution for problem (P) with basis $\overline{B}$ and consider the following k sub-problems (P^l) $l = 1,2,...,k$, where

(P^l) Minimize $\sum_{(i,j)\in J} c_{ij}^l x_{ij}$

Subject to (2.1), (2.2) and (2.3)

Since $\overline{X}$ is a solution of problem (P), then it is clear that $\overline{X}$ is also a basic feasible solution for each of the above k sub-problems (P^l), $l = 1,2,...,k$ .Let (u_i^l, v_j^l), (l=1, 2,....,k) be the corresponding dual variable associated with the above k problems (P^l), so that

$$u_i^l + v_j^l = c_{ij}^l \quad \text{for } (i,j) \in \overline{B} \quad \forall \; l = 1,\, 2,....,k$$

To check the non-dominance character of a feasible solution the following observations hold (Gupta, (1977))

(I) $\overline{X}$ will dominate all other solutions X^* obtained by introducing the non-basic cell *(i,j)* for which $u_i^l + v_j^l - c_{ij}^l \le 0$ for $l = 1,\, 2,....,k$, with at least one inequality strictly negative, and $x_{ij}^* = \theta_{ij} > 0$ .Therefore, such a non basic cell possibly cannot enter the basis.

(II) If for a non-basic cell (*i*,*j*) $u_i^l + v_j^l - c_{ij}^l \ge 0$ for $l = 1,\, 2,....,k$, with at least one inequality, with strictly positive sign, then the solution X^* obtained by introducing the cell (*i*,*j*) will dominate $\overline{X}$ if $x_{ij}^* = \theta_{ij} > 0$

(III) If there are non-basic cells (*i*,*j*) and (*h*,*k*) such that

$$\theta_{ij}(u_i^l + v_i^l - c_{ij}^l) \ge \theta_{hk}(u_k^l + v_k^l - c_{hk}^l), \; l = 1,2,...,k$$

with strict inequality in at least one of the k inequalities, where $x^*_{pq} = \theta_{pq}$ if the cell *(p,q)* is introduced into the basis, then the solution obtained by introducing the cell (*h,k*) is dominated by the solution resulting from introducing the cell (*i,j*).

(IV) $\overline{X}$ is a non-dominated solution if for all non-basic cells (*i,j*)

$$u_i^l + v_j^l - c_{ij}^l < 0 \text{ for at least one } l$$

i.e. $\overline{X}$ is a unique optimal solution for atleast one of the k sub problems (P^l), $l=1,....,k$. Thus the above observations (ii) and (iv) help one analyzing the character of any basic feasible solution, but still, they do not cover all the possibilities. There may be the possibility when for at least one non-basic cell, $s(1 \le s \le k)$ of the k quantities $u_i^l + v_j^l - c_{ij}^l \ \forall \ l = 1, 2,....,k$ are positive, while the remaining *k-s* are negative, it being assumed that all the k quantities for the remaining non-basic cells are non-positive. In this case, to find out whether $\overline{X}$ is dominated or non-dominated, consider the linear programming problem (P_1) with one more constraint.

(P_1) Minimize $-\sum_{l=1}^{k} q_l$

Subject to (2.1), (2.2), (2.3) and also

$$\sum_{(i,j)\in J} c_{ij}^l x_{ij} + q_l = \sum_{(i,j)\in J} c_{ij}^l \overline{x_{ij}}, l = 1,2,...,k. \quad \dots (2.4)$$

$$\text{and} \quad q_l \ge 0, \quad l = 1,2,..., k. \quad \dots (2.5)$$

This can be observed that if problem (P_1) has zero optimum. Then clearly $\overline{X}$ is a non-dominated solution and otherwise it is dominated. This special structured problem has been solved by Klingman and Rusell (1975) which is much more simplified. Gupta and Gupta (1995) have shown that in some case there is no need to solve the problem (P_1) completely, but the moment we discover that some q_l becomes positive; we conclude that $\overline{X}$ is a dominated solution.

Consider the dual of problem (P_1) namely,

$$\text{Maximize} \quad \sum_{i\in M} a_i u_i + \sum_{j\in N} b_j v_j + \sum_{i=1}^{k} s_l \sum_{(i,j)\in J} c_{ij}^l x_{ij} \quad \dots (2.6)$$

Subject to
$$\left.\begin{array}{l} u_i + v_j + \sum_{i=1}^{k} s_l c_{ij}^{l} \le 0 \\ s_l \le -1 \qquad \forall\ l = 1,2,...,k \end{array}\right\} \quad \dots (2.7)$$

Let (P_2) denote the problem (P_1) without the additional constraints (2.4) and (2.5). Let $(X, q_1, q_2, q_3, ..., q_k)$ be any basic feasible solution for (P_1) with basis $\hat{B}$. Klingman and Russell (1975), the basis $\hat{B}$ can be initially partitioned in the form

$$\hat{B} = \begin{bmatrix} B & B_A \\ C_B & C_A \end{bmatrix}$$

Where B is the basis for (P_2), C_B is a $k \times (m+n\text{-}1)$ matrix containing all those components c_{ij}^{l}, $\forall\ l = 1,2,...,k$ for which $(i,j) \in B$. Also B_A is $(m+n) \times k$, zero matrix and C_A is $k \times k$ identity matrix, if $q_1, q_2, ..., q_k$ are in the basis, otherwise they contain the appropriate columns. Let $(u_i^l, v_j^l), \forall\ l = 1,2,...,k$ be determined from the equations

$$u_i^l + v_j^l = c_{ij}^l \text{ for } (i,j) \in B\ \forall\ l = 1,\ 2,....,k \quad \dots (2.8)$$

The generalized left inverse $\hat{B}^G$ of $\hat{B}$ Israel and Greville (1974) may be obtained as explained in Gupta (1977). Consider the following systems connecting the dual variables in order to find out the vector which enters the basis.

$$U = (u_i),\ V = (v_j),\ S = (s_l)$$

i.e.
$$(U^T, V^T, S^T) \begin{bmatrix} B & B_A \\ C_B & C_A \end{bmatrix} = [\,0\,, E^T\,] \quad \dots (2.9)$$

in this system if q_l is not in the basis then the l^{th} component in E^T is zero and -1 if q_l is in the basis for $l = 1,2,..,k$. Klingman and Russell (1975), derived equation (2.9) that can equivalently be expressed in the form.

$$u_i + v_j + \sum_{l=1}^{k} s_l c_{ij}^l = 0 \text{ for } (i,\ j) \in B \quad \dots (2.10)$$

$$S^T H = E_H^T \quad \dots (2.11)$$

Where H is ia a non-singular matrix, so that $H^{-1}$is the last k columns of the last k rows of $\hat{B}^G$ and E_H^T is a k component row vector such that each component of E_H^T is linear-combination of right hand side of equation (2.9), where exactly one component of E^T is multiplied by one and the others by zero.

The dual variables from (2.10), and (2.11), can be determined by first finding S_l $\forall\ l = 1,2,...,k$ from equation (2.11) and the remaining variables $u_i\text{'s and } v_j\text{'s}$ can be found out by setting

$$u_i = -\sum_{l=1}^{k} s_l u_i^l \text{ and } v_j = -\sum_{l=1}^{k} s_l v_i^l \qquad \ldots(2.12)$$

where u_i^l and v_j^l (*l=1, 2,…., k*) are determines from (2.8). The $u_i\text{'s and } v_j\text{'s}$ in (2.12) satisfy (2.10). After getting the values of dual variables, the non-basic cell (*i,j*) which enter into the basis for which

$$u_i + v_j + \sum_{l=1}^{k} s_l c_{ij}^l = -\sum_{l=1}^{k} s_l (u_i^l + v_j^l - c_{ij}^l)$$

is most positive, and this column be denoted by $\hat{P}_{hk} = [P_{hk}, c_{hk}]^T$ and can be expressed in terms of $\hat{B}$, $\hat{P}_{hk}$ as,

$$\begin{bmatrix} B & B_A \\ C_B & C_A \end{bmatrix} \begin{bmatrix} Y_{hk} \\ Y_A \end{bmatrix} = \begin{bmatrix} P_{hk} \\ c_{hk} \end{bmatrix} \qquad \ldots (2.13)$$

Let $\hat{B}_k^G$ denote the last *k* rows of $\hat{B}^G$ then

$$Y_A = \hat{B}_k^G [P_{hk}, c_{hk}]^T \qquad \ldots (2.14)$$

$$Y_{A_q} = -\sum_{l=1}^{k} a_{q_l} u_h^l - \sum_{l=1}^{k} a_{q_l} v_k^l + \sum_{l=1}^{k} a_{q_l} c_{hk}^l$$

$$= -\sum_{l=1}^{k} a_{q_l} (u_h^l + v_k^l - c_{hk}^l) \qquad \dots (2.15)$$

$$(q = 1,2,\dots,k)$$

Where $H^{-1} = (a_{ij})$ and hence $Y_A = [Y_{A_1}, Y_{A_2}, \dots\dots, Y_{A_k}]^T$ is determined completely. It is easy to see that $(\overline{X}, q_1 = 0, q_2 = 0, \dots\dots, q_k = 0)$ is the initial basic feasible solution the problem (P_1) with $q_1, q_2, \dots, q_k$ in the basis. Therefore initially, $H = I_k = H^{-1}$ where I_k is the $k \times k$ unit matrix, $E_H^T = [-1,-1,-1,\dots\dots,-1]$ and $B = \overline{B}$. Then $S_1 = -1, S_2 = -1, \dots\dots, S_k = -1$. With these values, the cell which may enter to the may be obtained as explained above and Y_A for this column can be determined from equation (2.15). It may be recalled that the only quantities we need to check the non-dominance character of $\overline{X}$, under the non-degeneracy assumption are Y_A and H^{-1}. It can be observed that if $[Y_{A_1}, Y_{A_2}, \dots\dots, Y_{A_k}]^T$ is non–positive, then $\hat{P}_{hk}$ which enters into the basis, will lead to a solution for which at least one of the *k* variables $q_1, q_2, \dots\dots, q_k$ will be at positive level. Therefore $\overline{X}$ will be a dominated solution. However if at least one of the *k*, Y_{A_q} 's is positive, then the simplex-iteration with one of the rows of H^{-1} as the pivot row, can be made and at each step, one needs to update H^{-1} and $S_1, S_2, \dots\dots, S_k$ only. Thus proceeding in the above stated manner, we may encounter any one of the following possibilities:

(a) Zero optimum for (P_1) is obtained, in which case $\overline{X}$ is a non-dominated solution. Or

(b) There exists a column $\hat{P}_{hk}$ for which $-\sum_{l=1}^{k} s_l (u_i^l + v_j^l - c_{ij}^l)$ is most positive and Y_A is non-positive, in which case $\overline{X}$ is dominated solution.

2.3 NUMERICAL PROBLEM

The above algorithm is explained in detail by considering the following linear multi-objective transportation problem (LMOTP) as considered by Gupta and Gupta (1982).

Following is the formulation for $i = 3, j = 3, l = 3$

$$\text{Minimize} \qquad Z = \left\{ \sum_{(i,j)\in J} c_{ij}^{1} x_{ij} \ , \ \sum_{(i,j)\in J} c_{ij}^{2} x_{ij} \text{and} \sum_{(i,j)\in J} c_{ij}^{3} x_{ij} \right\}$$

$$\text{Subject to} \qquad \sum_{j\in N} x_{ij} = a_i, \ i \in M$$

$$\sum_{i\in M} x_{ij} = b_j, \ j \in N$$

$$x_{ij} \geq 0 \quad \forall \quad (i,j) \in J$$

Where $N = \{1,2,3\}$ $M = \{1,2,3\}$ and $J = \{(i,j): i \in M, j \in N\}$

The following table 2.1 gives the value of cost $c_{ij}^{l}\,(i = 1,2,3;\ j = 1,2,3; \text{and } l = 1,2,3)$

Table 2.1

	c_{ij}^{1}			a_i	c_{ij}^{2}			c_{ij}^{3}		
	3	1	-1	100	2	4	3	7	7	5
	4	2	5	125	5	6	7	1	7	1
	-1	6	4	75	3	-1	3	5	7	3
b_j	60	80	160							

a_i = the quantity of material available at origin O_i $i = 1,2,3$

b_j = the quantity of material required at destination D_j $j = 1,2,3$

By using North West Corner rule first initial basic feasible solution X^1 is obtained also the dual variables $u_i^l + v_j^l - c_{ij}^l$ are obtained and are given in table 2.2.

Table 2.2

X^1	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : a_1 = 100$	3 2 7 **60** 0 0 0	1 4 7 **40** 0 0 0	-1 3 5 5 -1 4	0 0 0
$O_2 : a_2 = 125$	4 5 1 0 -1 6	2 6 7 **40** 0 0 0	5 4 1 **85** 0 0 0	1 2 0
$O_3 : a_3 = 75$	-1 3 5 4 0 9	6 -1 7 -5 6 7	4 3 8 **75** 0 0 0	0 1 7
v_j^1 v_j^2 v_j^3	3 2 7	1 4 7	4 2 1	

Where a cell (*i*,*j*) in the table 2.2 contains the information as shown below:

c_{ij}^1	c_{ij}^2	c_{ij}^2
	x_{ij}	
$u_i^1 + v_j^1 - c_{ij}^1$	$u_i^2 + v_j^2 - c_{ij}^2$	$u_i^3 + v_j^3 - c_{ij}^3$

For table 2.2 the solution is $X^1 = (60, 40, 0, 0, 40, 85, 0, 0, 75)^T$ for which $Z = (1025, 1085, 1665)$

The cell (3, 1) qualifies for entry into the basis as by taking initially,

$H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$ gives $-\sum_{l=1}^{3} S_l(u_i^l + v_j^l - c_{ij}^l)$ most positive for cell (3,1) and

also $u_i^l + v_j^l - c_{ij}^l \geq 0 \; \forall l = 1,2,3$ which gives $Y_A = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -9 \end{bmatrix} \leq 0$ thus the solution

X^1 is a *dominated solution*. Hence only one cell (3, 1) will be member of set R. $\therefore R = \{(3,1)\}$

So make a loop to find the value of θ, assign the value of θ to the cell which qualifies for entry into the basis where θ= min {60, 40, 75} =40. As a result the cell (2, 2) leaves the basis and corresponding dual variables are given in table 2.3.

Table 2.3

X^2	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : a_1 = 100$	3 2 7 **20** 0 0 0	1 4 7 **80** 0 0 0	-1 3 5 9 -1 5	0 0 0
$O_2 : a_2 = 125$	4 5 1 -4 -1 -3	2 6 7 -4 0 -9	5 4 1 **125** 0 0 0	-3 2 -9
$O_3 : a_3 = 75$	-1 3 5 **40** 0 0 0	6 -1 7 -9 6 2	4 3 8 **35** 0 0 0	-4 1 -2
v_j^1 v_j^2 v_j^3	3 2 7	1 4 7	8 2 10	

For table 2.3 the solution is $X^2 = (20, 80, 0, 0, 0, 125, 40, 0, 35)^T$ for which $Z = (865, 1085, 1305)$

Since there is no non-basic cell for which $u_i^l + v_j^l - c_{ij}^l \geq 0 \; \forall l = 1,2,3$. So, the character of this solution cell to be checked whether it is dominating and non-dominating by taking initially,

$H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$ gives $- \sum_{l=1}^{3} S_l (u_i^l + v_j^l - c_{ij}^l)$ most positive for cell (1, 3)

therefore for this cell $Y_A = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ +1 \\ -5 \end{bmatrix}$. Since, second component is most positive,

$\therefore q_2$ leaves the basis, hence $E_H^T = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}$. Performing simplex iteration, $\xi = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix}$ and

$H^{-1} = \begin{bmatrix} 1 & 9 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$ using formula $S^T = E_H^T H^{-1}$ we get $S_1 = -1, S_2 = -14$ and $S_3 = -1$ again $-\sum_{l=1}^{3} s_l(u_i^l + v_j^l - c_{ij}^l)$ is most positive for cell (3,2), so this cell enter into the basis and in similar manner $Y_A = -\begin{bmatrix} 1 & 9 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} -9 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -45 \\ -6 \\ -28 \end{bmatrix}$. Since $Y_A < 0$, hence X^2 is a *dominated solution*. So this solution will not be considered. Now $\mathrm{R} = \{(1,3), (3,2)\}$. Let us introduce the cell (1, 3) into the basis and get the new solution X^3 in table 2.4.

Table 2.4

X^3	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : a_1 = 100$	3 2 7 -9 1 -5	1 4 7 **80** 0 0 0	-1 3 5 **20** 0 0 0	0 0 0
$O_2 : a_2 = 125$	4 5 1 -4 -1 -3	2 6 7 4 -1 -4	5 4 1 **125** 0 0 0	6 1 4
$O_3 : a_3 = 75$	-1 3 5 **60** 0 0 0	6 -1 7 0 5 3	4 3 8 **15** 0 0 0	5 0 3
v_j^1 v_j^2 v_j^3	-6 3 2	1 4 7	-1 3 5	

For table 2.4 the solution is $X^3 = (0, 80, 20, 0, 0, 125, 60, 0, 15)^{\mathrm{T}}$ for which $Z = (685, 1185, 1205)$

Since for the cell (3, 2) has all $u_i^l + v_j^l - c_{ij}^l \geq 0 \; \forall l = 1,2,3$ with at least one inequality. Thus, the solution X^3 is a *dominated solution*. We can also check the character of the solution, whether dominating or non-dominating as done in previous steps. Taking initially,

$H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$ gives $-\sum_{l=1}^{3} S_l(u_i^l + v_j^l - c_{ij}^l)$ most positive for cell (3, 2) and also which gives $Y_A = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix} \leq 0$. Hence only one cell (3, 2) will be member of set R. $\therefore R = \{(3,2)\}$. So on introducing cell (3, 2) into the basis the new solution X^4 is obtained and is shown in next table 2.5.

Table 2.5

X^4	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : a_1 = 100$	3 2 7 -9 6 -2	1 4 7 **65** 0 0 0	-1 3 5 **35** 0 0 0	0 0 0
$O_2 : a_2 = 125$	4 5 1 -4 4 0	2 6 7 5 -1 -4	5 4 1 **125** 0 0 0	6 1 -4
$O_3 : a_3 = 75$	-1 3 5 **60** 0 0 0	6 -1 7 **15** 0 0 0	4 3 8 0 -5 -3	5 -5 0
v_j^1 v_j^2 v_j^3	-6 8 5	1 4 7	-1 3 5	

For table 2.5 the solution is $X^4 = (0, 65, 35, 0, 0, 125, 60, 15, 0)^T$ for which $Z = (685, 1030, 1165)$ since there is no non-basic cell having all $u_i^l + v_j^l - c_{ij}^l \geq 0 \; \forall l = 1,2,3$. So it is must to check the character of dominance or non-dominance.

Taking initially, $H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$ $-\sum_{l=1}^{3} S_l(u_i^l + v_j^l - c_{ij}^l)$ most positive for no non-basic cell. Hence there is no cell to check the dominance character of the solution. So this solution is first *non-dominated solution*. For such a case, when the character of solution is checked for any non-basic cell having at least one dual variable strictly positive will have values $S_1 = -1, S_2 = -1$ and $S_3 = -1$ therefore all the non-basic cells with at least one dual variable $u_i^l + v_j^l - c_{ij}^l \geq 0 \; \forall l = 1,2,3$ will be in set R. $\therefore R = \{(1,1),(2,1),(2,2)\}$. Let cell (1, 1) to be introduced into the cell, a new solution is obtained given in table 2.6

Table 2.6

X^5	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	$u_i^1 \; u_i^2 \; u_i^3$
$O_1 : a_1 = 100$	3 2 7 **60** 0 0 0 -	1 4 7 **5** 0 0 0	-1 3 5 **35** 0 0 0	0 0 0
$O_2 : a_2 = 125$	4 5 1 -4 4 0	2 6 7 5 -1 -4	5 4 1 **125** 0 0 0	6 1 -4
$O_3 : a_3 = 75$	-1 3 5 9 -6 -2	6 -1 7 **75** 0 0 0	4 3 8 0 -5 -3	5 -5 0
v_j^1 v_j^2 v_j^3	3 2 7	1 4 7	-1 3 5	

For table 2.6 the solution is $X^5 = (60, 5, 35, 0, 0, 125, 0, 75, 0)^T$ for which $Z = (1225, 670, 1280)$

Since there is no non-basic cell for which $u_i^l + v_j^l - c_{ij}^l \geq 0 \; \forall l$. So, the character of this solution cell to be checked whether it is dominating and non-dominating by taking initially,

$H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$ gives $-\sum_{l=1}^{3} S_l (u_i^l + v_j^l - c_{ij}^l)$ most positive for cell (2, 1),

therefore $Y_A = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix}$. Since, second component is most positive, $\therefore q_2$ leaves the basis, hence $E_H^T = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}$. Performing simplex iteration, $\xi = \begin{bmatrix} 5/2 \\ 1/2 \\ 1 \end{bmatrix}$

and $H^{-1} = \begin{bmatrix} 1 & 5/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ using formula $S^T = E_H^T H^{-1}$ we get $S_1 = -1, S_2 = -7/2$ and $S_3 = -1$

again $-\sum_{l=1}^{3} s_l(u_i^l + v_j^l - c_{ij}^l)$ is most positive for no non-basic cell. Hence this solution is also a *non-dominated solution* $\therefore R = \{(2,1),(2,2)\}$. Let (2, 1) be introduced in the basis and new solution X^6 is shown in table 2.7.

Table 2.7

X^6	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : a_1 = 100$	3 2 7 -5 2 -2 -	1 4 7 **5** 0 0 0	-1 3 5 **95** 0 0 0	0 0 0
$O_2 : a_2 = 125$	4 5 1 **65** 0 0 0	2 6 7 5 -1 -4	5 4 1 **65** 0 0 0	6 1 -4
$O_3 : a_3 = 75$	-1 3 5 4 -4 0	6 -1 7 **75** 0 0 0	4 3 8 0 -5 -3	5 -5 0
v_j^1 v_j^2 v_j^3	-2 4 5	1 4 7	-1 3 5	

For table 2.7 the sixth solution is $X^6 = (0, 5, 95, 60, 0, 65, 0, 75, 0)^T$ for which $Z = (925, 790, 1160)$

Taking initially, $H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$ - $\sum_{l=1}^{3} S_l(u_i^l + v_j^l - c_{ij}^l)$ most positive for no non-basic cell. Hence there is no cell to check the dominance character of the solution. So this solution is also a *non-dominated solution* $\therefore R = \{(2,2),(3,1)\}$. Let (2, 2) be introduced in the basis and new solution X^7 is shown in table 2.8.

Table 2.8

X^7	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : a_1 = 100$	3 2 7 -5 2 -2 -	1 4 7 -5 1 4	-1 3 5 **100** 0 0 0	-6 -1 4
$O_2 : a_2 = 125$	4 5 1 **60** 0 0 0	2 6 7 **5** 0 0 0	5 4 1 **60** 0 0 0	0 0 0
$O_3 : a_3 = 75$	-1 3 5 9 -5 -4	6 -1 7 **75** 0 0 0	4 3 8 5 -6 -7	4 -7 0
v_j^1 v_j^2 v_j^3	4 5 1	2 6 7	5 4 1	

For table 2.8 the seventh solution is $X^7 = (0, 0, 100, 60, 5, 60, 0, 75, 0)^{\mathrm{T}}$ for which $Z = (900, 795, 1180)$ taking initially, $H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1 - \sum_{l=1}^{3} S_l(u_i^l + v_j^l - c_{ij}^l)$ most positive for no non-basic cell. Hence there is no cell to check the dominance character of the solution. So this solution is also a *non-dominated solution* $\therefore R = \{((1,1), (3,1), (3,3)\}$. Let (1, 1) be introduced in the basis and new solution X^8 and corresponding dual variables is shown in table 2.9.

Table 2.9

X^8	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	$u_i^1 \quad u_i^2 \quad u_i^3$
$O_1 : a_1 = 100$	3 2 7 **60** 0 0 0 -	1 4 7 -5 1 4	-1 3 5 **40** 0 0 0	0 0 0
$O_2 : a_2 = 125$	4 5 1 5 -2 2	2 6 7 **5** 0 0 0	5 4 1 **120** 0 0 0	6 1 -4
$O_3 : a_3 = 75$	-1 3 5 14 -7 -2	6 -1 7 **75** 0 0 0	4 3 8 5 -6 -7	10 -6 -4
v_j^1 v_j^2 v_j^3	3 2 7	-4 5 11	-1 3 5	

For table 2.9 the solution is $X^8 = (60, 0, 40, 0, 5, 120, 0, 75, 0)^T$ for which $Z = (1200, 675, 1300)$

Taking initially, $H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$ gives $-\sum_{l=1}^{3} S_l(u_i^l + v_j^l - c_{ij}^l)$ most positive for cell (2, 1) and (3, 1). But cell (2, 1) was already in the basis in table 8 so introducing this cell will give the previous solution. so this cell (2, 1) will not be in R. So we check the character of the solution by cell (3, 1). Hence $Y_A = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 14 \\ -7 \\ -2 \end{bmatrix} = \begin{bmatrix} -14 \\ 7 \\ 2 \end{bmatrix}$ Since, second component is most positive, $\therefore q_2$ leaves the basis, hence $E_H^T = [-1 \quad 0 \quad -1]$. Performing simplex iteration,

$\xi = \begin{bmatrix} 2 \\ 1/7 \\ -2/7 \end{bmatrix}$ and $H^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1/7 & 0 \\ 0 & -2/7 & 1 \end{bmatrix}$,now using formula $S^T = E_H^T H^{-1}$

$[S_1 \quad S_2 \quad S_3] = [-1 \quad 0 \quad -1] \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1/7 & 0 \\ 0 & -2/7 & 1 \end{bmatrix}$ will give $S_1 = -1, S_2 = -12/7,$ and $S_3 = -1$

Using these values of S_l for $l = 1,2,3$ is most positive for cell (1, 2) which gives an older solution X^5, so leave this cell. As Y_A is not negative thus we can say the solution is *non-dominated solution.* $\therefore R = \{(3,1),(3,3)\}$. When cell (3, 3) from table 2.9 is introduced in the basis it gives a new solution but having dominated character. So (3, 1) be introduced in the basis and new solution X^9 is shown in table 2.10.

Table 2.10

X^9	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : a_1 = 100$	3 2 7 14 -7 -2	1 4 7 -5 1 4	-1 3 5 **100** 0 0 0	0 0 0
$O_2 : a_2 = 125$	4 5 1 -9 5 4	2 6 7 **65** 0 0 0	5 4 1 **60** 0 0 0	6 1 -4
$O_3 : a_3 = 75$	-1 3 5 **60** 0 0 0	6 -1 7 **15** 0 0 0	4 3 8 5 -6 -7	10 -6 -4
v_j^1 v_j^2 v_j^3	-11 9 9	-4 5 11	-1 3 5	

For table 2.10 the solution is $X^9 = (0, 0, 100, 0, 65, 60, 60, 15, 0)^T$ for which $Z = (360,1095,1420)$. This solution also needs to check its character so, taking initially $H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$. $-\sum_{l=1}^{3} s_l(u_i^l + v_j^l - c_{ij}^l)$ is not most positive for any non-basic cell in table 2.10. So this solution is clearly a *non-dominated solution*, as there is no non-basic cell that can be used to check the character of the solution. Hence all the non-basic cells having $u_i^l + v_j^l - c_{ij}^l > 0$ for at least one l will be in R but on introducing cells (1,1),(1,2),(2,1) gives the older solutions X^8 , X^4,X^7 respectively so, also that cell should not give previous solutions. Therefore the new solution is obtained by introducing cell (3, 3) in table 2.10 and the new solution X^{10} in table 2.11 is shown below

Table 2.11

X^{10}	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : a_1 = 100$	3 2 7 -4 7 2 -	1 4 7 -5 1 4	-1 3 5 **100** 0 0 0	0 0 0
$O_2 : a_2 = 125$	4 5 1 5 -2 2	2 6 7 **80** 0 0 0	5 4 1 **45** 0 0 0	6 1 -4
$O_3 : a_3 = 75$	-1 3 5 **60** 0 0 0	6 -1 7 -5 6 7	4 3 8 **15** 0 0 0	5 0 3
v_j^1 v_j^2 v_j^3	-6 3 2	-4 5 11	-1 3 5	

For table 2.11 the solution is $X^{10} = (0, 0, 100, 0, 80, 45, 60, 0, 15)^T$ for which $Z = (285, 1185, 1525)$

This solution also needs to check its character so, taking initially $H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$. $-\sum_{l=1}^{3} s_l(u_i^l + v_j^l - c_{ij}^l)$ is not most positive for any non-basic cell in table 11. Hence solution is *non-dominated solution.* Also on introducing the non-basic cells which qualify the entry for the basis gives all the older solutions, so no new solution is obtained by introducing the non-basic cells having $u_i^l + v_j^l - c_{ij}^l > 0$ for at least one *l*. For this solution the set $R = \phi$. Thus the process of finding the non-dominated solutions is terminated.

2.4 CONCLUSION

In this chapter LMOTP considered by Gupta and Gupta (1982) has been reviewed and the algorithm has been used to find the non-dominated solutions are as listed below

$X^4 = (0, 65, 35, 0, 0, 125, 60, 15, 0)^T$	$Z = (685, 1030, 1165)$
$X^5 = (60, 5, 35, 0, 0, 125, 0, 75, 0)^T$	$Z = (1225, 670, 1280)$
$X^6 = (0, 5, 95, 60, 0, 65, 0, 75, 0)^T$	$Z = (925, 790, 1160)$
$X^7 = (0, 0, 100, 60, 5, 60, 0, 75, 0)^T$	$Z = (900, 795, 1180)$
$X^8 = (60, 0, 40, 0, 5, 120, 0, 75, 0)^T$	$Z = (1200, 675, 1300)$
$X^9 = (0, 0, 100, 0, 65, 60, 60, 15, 0)^T$	$Z = (360, 1095, 1420)$
$X^{10} = (0, 0, 100, 0, 80, 45, 60, 0, 15)^T$	$Z = (285, 1185, 1525)$

CHAPTER 3

MULTI-OBJECTIVE TRANSPORTATION PROBLEM WITH ONE MORE NON-LINEAR OBJECTIVE

3.1 INTRODUCTION

The multi-objective transportation problem (MOTP) considered in chapter 2 has been extended by including one more objective which is conflicting in nature with other *k* objectives. The objective is to minimize the total cost of first *k* objectives as time of transportation is also to be minimized.

3.2 PROBLEM FORMULATION

The mathematical formulation of this problem is formulated as below:

(*P*) Minimize $$Z = \left\{ \sum_{(i,j)\in J} c_{ij}^{1} x_{ij} + \sum_{(i,j)\in J} c_{ij}^{2} x_{ij} + \ldots\ldots + \sum_{(i,j)\in J} c_{ij}^{k} x_{ij} \right\}$$

Minimize $$\max\ [t_{ij} / x_{ij} > 0]$$

Subject to $$\sum_{j\in N} x_{ij} = a_i,\ i \in M \qquad \ldots (3.1)$$

$$\sum_{i\in M} x_{ij} = b_j,\ j \in N \qquad \ldots (3.2)$$

$$x_{ij} \geq 0 \quad \forall \quad (i,j) \in J \qquad \ldots (3.3)$$

$$N = \{1,2,...,n\}, M = \{1,2,....,m\} \text{ and } J = \{(i,j): i \in M, j \in N\}$$

Also it has been assumed that $a_i > 0$, $i \in M$ and $b_j > 0$, $j \in N$ moreover it is a transportation problem,where

N is the set of all the origins and M is the set of all destinations J is the set of all the routes

i =1, 2,. . . ,m, are the origins and j = 1, 2,. . .,n, are the destinations,

t_{ij} = the time of transportation of the product from i^{th} origin to the j^{th} destination which is independent of the amount of commodity transported, so long as $x_{ij} > 0$,

c_{ij} = the variable cost per unit amount transported from i^{th} origin to the j^{th} destination,

x_{ij} = the amount transported from the i^{th} origin to the j^{th} destination,

a_i = maximum capacity at origin i,

b_j = the demand at destination j,

The objective in multi-objective transportation problem (MOTP) is to minimize the total cost which includes both the variable cost of first k objectives and the total time of transportation satisfying the above constraints.

3.3 SOLUTION PROCEDURE

By using re-optimization procedure given by Basu *et. al* (1994) the above problem is separated into two problems (P_1) and (S), where

(P_1) Minimize the total cost $$Z = \left\{ \sum_{(i,j)\in J} c_{ij}^1 x_{ij},\ \sum_{(i,j)\in J} c_{ij}^2 x_{ij}, \ldots,\ \sum_{(i,j)\in J} c_{ij}^k x_{ij} \right\}$$

Subject to (3.1), (3.2), (3.3).

and

(S_1) Minimize the total time $T = \max\,[t_{ij} \,/\, x_{ij} > 0]$

Subject to (3.1), (3.2), (3.3).

In problem (P_1) the procedure of chapter 2 has been same. Let Z_1 be the optimum minimum total cost of the k sub problems (P_1) which can be obtained by considering as single objective problem and the optimal cost can be find by summing up the cost of all objectives in all solution and T_1 be the time of the problem (S_1) with respect to Z_1, then any schedule which is competed earlier than T_1 would cost more than Z_1. (Z_1, T_1) is called the time-cost trade-off pair at the first iteration.

Using re-optimization procedure (Basu et al 1994), let after *q-th* iteration, the solution is infeasible. Therefore, we get the following complete set of time-cost trade-off pairs:

$$(Z_1, T_1), (Z_2, T_2), \ldots, (Z_q, T_q)$$

where

$$Z_1 < Z_2 < \ldots < Z_m \quad \text{and} \quad T_1 > T_2 > \ldots > T_m$$

3.4 ALGORITHM

By using the steps of algorithm 2.2 of Chapter 2 and considering that the first k cost objectives, are minimized we get an optimal minimum cost. **:** Let Z_1 be the minimum total cost of all k objectives from all solutions of problem P_1 and denoted by Z_1

Step 1: Calculate T_1, T_1 = max $[t_{ij} / x_{ij} > 0 \text{ according to } X_1]$

Then the corresponding pair (Z_1,T_1) is called the time-cost trade-off pair at the first iteration.

Step 2: Define $(C^l_{ij})_k$ = M , if $t_{ij} \geq T_k$

$= (C^l_{ij})$, if $t_{ij} < T_k$

where M is a sufficiently large positive number. Let (P_{p+1}) be the multi-objective transportation problem with variable cost $(C^l_{ij})_k$.

Step 3: Find a basic feasible solution of the problem (P_{p+1}) with respect to the variable costs. If the total variable cost ≥ M, then go to Step 4; otherwise use the algorithm 2.2 of chapter 2.

Step 4: Let Z_{p+1} be the optimal cost of problem (P_{p+1}) and X_{p+1} be the optimal solution corresponding to Z_{p+1}.

Step 5: Compute $T_{k+1} = \max\limits_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n+1}} \left[t_{ij}, x_{ij} > 0 \textit{ according to } X_{k+1}\right]$

Then the trade-off pair (Z_{p+1}, T_{p+1}) is called the cost-time trade-off pair at the $(p+1)$-th iteration. Obviously, $Z_{p+1} > Z_p$, $T_{p+1} < T_p$.

Step 5: Set $p=p+1$, go to Step 3.

Step 6: Suppose after q^{th} iteration the solution is infeasible i.e. $Z_{q+1} \geq M$. Then identify the complete set of efficient trade-off pairs

$$(Z_1,T_1), (Z_2,T_2), \ldots, (Z_q,T_q)$$

$$\text{where } Z_1 < Z_2 < \ldots < Z_q \text{ and } T_1 > T_2 > \ldots > T_q$$

3.5 NUMERICAL PROBLEM

The above algorithm is explained by considering the following 3×3 multi-objective transportation problem where the unit cost c_{ij}^{l} for all the three objectives (l = 1,2,3) as in Chapter 2 and time t_{ij} are taken in one standard scale.

Minimize $$Z = \left\{ \sum_{(i,j)\in J} c_{ij}^{1} x_{ij} + \sum_{(i,j)\in J} c_{ij}^{2} x_{ij} + \sum_{(i,j)\in J} c_{ij}^{3} x_{ij} \right\}$$

Minimize $$\max\ [t_{ij} / x_{ij} > 0]$$

Subject to (3.1), (3.2), (3.3)

Where $N = \{1,2,3\}$ $M = \{1,2,3\}$ and $J = \{(i,j): i \in M, j \in N\}$

The following objectives give the value of cost c_{ij}^{l}, t_{ij} for $(i = 1,2,3;\ j = 1,2,3; \text{and } l = 1,2,3)$

In order to solve the above problem it is presented as two parts as shown below:

Minimize the cost $$Z = \left\{ \sum_{(i,j)\in J} c_{ij}^{1} x_{ij} + \sum_{(i,j)\in J} c_{ij}^{2} x_{ij} + \sum_{(i,j)\in J} c_{ij}^{3} x_{ij} \right\}$$

Subject to (3.1), (3.2), (3.3)

and

Minimize the time $$\max\ [t_{ij} / x_{ij} > 0]$$

Subject to (3.1), (3.2), (3.3).

The following table 3.1 gives the values of cost c_{ij}^{l} for $(i = 1,2,3;\ j = 1,2,3; \text{and } l = 1,2,3)$ and the values of time t_{ij} are given in table 3.2

Table 3.1

	c_{ij}^1			a_i
	3	1	-1	100
	4	2	5	125
	-1	6	4	75
b_j	60	80	160	

c_{ij}^2		
2	4	3
5	6	7
3	-1	3

c_{ij}^3		
7	7	5
1	7	1
5	7	3

Table 3.2

	t_{ij}			a_i
	16	13	18	100
	17	20	15	125
	19	13	14	75
b_j	60	80	160	

For solution purpose the above data is written together and given in table 3.3 where each cell (*i*,*j*) represents the entries as

Table 3.3

	c_{ij}^1	c_{ij}^2	c_{ij}^2
		x_{ij}	
	$u_i^1 + v_j^1 - c_{ij}^1$	$u_i^2 + v_j^2 - c_{ij}^2$	$u_i^3 + v_j^3 - c_{ij}^3$
t_{ij}			

The upper entries of each row represent the cost c_{ij}^l and south west entry represent t_{ij}

Since two objectives are in consideration, and first objective is is the total cost of problem (P_1)

of chapter 2 and can be considered as single objective for which the optimal solution is given in table 3.4

Table 3.4

X	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : a_1 = 100$	3 2 7 -5 2 -2 -	1 4 7 **5** 0 0 0	-1 3 5 **95** 0 0 0	0 0 0
$O_2 : a_2 = 125$	4 5 1 **65** 0 0 0	2 6 7 5 -1 -4	5 4 1 **65** 0 0 0	6 1 -4
$O_3 : a_3 = 75$	-1 3 5 4 -4 0	6 -1 7 **75** 0 0 0	4 3 8 0 -5 -3	5 -5 0
v_j^1 v_j^2 v_j^3	-2 4 5	1 4 7	-1 3 5	

For this solution is $X = (0, 5, 95, 60, 0, 65, 0, 75, 0)^T$ for which $Z = (925,790,1160)$

and $Z_1 = 2875$ now from table 3.3, it can be seen that maximum time corresponding to this cost is 18. So the first trade off pair is $(Z_1, T_1) = (2875,18)$

so the cell (1,3), (2,2) and (3,1) will be blocked by the definition in step 2 of algorithm 3.4 . And the new solution using North-West Corner Rule is obtained and shown in table 3.5 with corresponding dual variables.

Table 3.5

X^1	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : a_1 = 100$	3 2 7 **60** 0 0 0 18	1 4 7 **40** 0 0 0 18	M M M -1-M 8-M 8-M	0 0 0
$O_2 : a_2 = 125$	4 5 1 5 -7 -1	M M M 7-M -M -M	5 4 1 **125** 0 0 0 18	6 -4 -7
$O_3 : a_3 = 75$	M M M 8-M -3-M 7-M	6 -1 7 **40** 0 0 0 18	4 3 8 **35** 0 0 0 18	5 -5 0
v_j^1 v_j^2 v_j^3	3 2 7	1 4 7	-1 8 8	

For this solution is $X^1 = (60, 40, 0, 0, 0, 125, 0, 40, 35)^{\mathrm{T}}$ for which $Z = (1225, 845, 1385)$ and $Z_1 = 3455$ now we check the character of the solution as explained in chapter 2. Taking initially, $H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$ gives $-\sum_{l=1}^{3} S_l(u_i^l + v_j^l - c_{ij}^l)$ most positive for no non-basic cell , this shows clearly the solution is *non-dominated* solution. only one cell (2,1) with $u_i^l + v_j^l - c_{ij}^l \geq 0$ for at least one l . Hence $R = \{(2,1)\}$. Let us introduce this cell and check the character of the new solution given in table 3.6

Table 3.6

X^2	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : \mathrm{a}_1 = 100$	3 2 7 **20** 0 0 0 18	1 4 7 **80** 0 0 0 18	M M M 4-M 1-M 7-M	0 0 0
$O_2 : \mathrm{a}_2 = 125$	4 5 1 **40** 0 0 0 18	M M M 2-M 7-M 1-M	5 4 1 **125** 0 0 0 18	1 3 -6
$O_3 : \mathrm{a}_3 = 75$	M M M 3-M 4-M 8-M	6 -1 7 -5 7 1	4 3 8 **75** 0 0 0 18	0 2 1
v_j^1 v_j^2 v_j^3	3 2 7	1 4 7	4 1 7	

For table 3.6 solution is $X^2 = (20, 80, 0, 40, 0, 125, 0, 0, 75)^{\mathrm{T}}$ for which $Z = (1025, 1125, 1425)$ and $Z_1 = 3575$ now we check the character of the solution.

Taking initially, $H^{-1} = I_3, S_1 = -1, S_2 = -1$ and $S_3 = -1$ gives $-\sum_{l=1}^{3} S_l(u_i^l + v_j^l - c_{ij}^l)$ most positive for no non-basic cell , this shows clearly the solution is *non-dominated* solution. Only one cell (3,2) with $u_i^l + v_j^l - c_{ij}^l \geq 0$ for at least one l . but introducing this cell gives the older solution, therefore no cell qualifies for entry into the basis. So $R = \phi$. Thus the process is terminated here.

Now the total cost is minimum for the solution X^1 . Therefore $Z_2 = 3455$ and now from table 3.6 it can be seen that maximum time corresponding to this cost is 17, therefore $T_2 = 17$. So the second trade off pair is $(Z_2, T_2) = (3455, 17)$

so the cells (1,3), (2,2) ,(3,1) and (2,1) will be blocked by the definition in step 2 of algorithm 3.4 for next solution in table 3.7

Table 3.7

X^3	D_1 $b_1 = 60$	D_2 $b_2 = 80$	D_3 $b_3 = 160$	u_i^1 u_i^2 u_i^3
$O_1 : \mathrm{a}_1 = 100$	3 2 7 **60** 0 0 0 17	1 4 7 **40** 0 0 0 17	M M M -1-M 8-M 8-M	0 0 0
$O_2 : \mathrm{a}_2 = 125$	M M M 9-M -2-M -M	M M M 7-M -M -M	5 4 1 **125** 0 0 0 17	6 -4 -7
$O_3 : \mathrm{a}_3 = 75$	M M M 8-M -3-M 7-M	6 -1 7 **40** 0 0 0 17	4 3 8 **35** 0 0 0 17	5 -5 0
v_j^1 v_j^2 v_j^3	3 2 7	1 4 7	-1 8 8	

For this solution is $X^3 = (60, 40, 0, 0, 0, 125, 0, 40, 35)^{\mathrm{T}}$ for which $Z = (1225, 845, 1385)$ and $Z_2 = 3455$. No non-basic cell is available in table 3.6. So this solution is non-dominated, therefore $R = \phi$. Thus process is terminated here. Now the total cost is minimum for the only solution X^3 . Therefore $Z_3 = 3455$ and now from table 3.7 it can be seen that maximum time corresponding to this cost is 16, therefore $T_3 = 16$. So the third trade off pair is $(Z_3, T_3) = (3455, 16)$ with this new maximum time $T_3 = 16$, cells (1,3), (2,2) ,(3,1) (2,1) and (1,1) will be blocked which leads to an infeasible solution when North-West Corner rule is used.

CONCLUSION

The cost- time trades off pairs obtained are as follows:

$(Z_1, T_1) = (2875,18)$
$(Z_2, T_2) = (3455,17)$
$(Z_3, T_3) = (3455,16)$

The optimal cost-time trade off pair is $(Z_3, T_3) = (3455,16)$ with minimum total cost 3455 and minimum time 16.

REFERENCES

Adlakha, V. and Kowalski, K., (2004), A simple algorithm for the source inducedtransportation problem, *Journal of operational research society*, **55(12)**, 1275-1280.

Ahuja, A. and Arora, S.R., (2001), Multi-Index fixed charge bicriterion transportation problem, *Indian Journal of Pure and Applied Mathematics*, **32(5)**, 739-746.

Aneja,Y.P. and Nair, K.P.K.,(1979), Bicriteria Transportation Problem, *Management Science*, **25(1)**, 73-78.

Arnold, P. Jones and Richard M. Soland, (1969), A branch and bound algorithm for multi-level fixed charge transportation problem, *Management Science*, **16 (1)**, 67-76.

Balinski, M.L.,(1961), Fixed cost transportation problem, *Naval Research Logistics Quarterly*, **8**, 41-54.

Barr, R.S., Glover, F., and Klingman, D.,(1981), A new optimization method for large scale fixed-charge transportation problem, *Operations Research*, **29**, 448-463.

Basu, M., Pal, B.B. and Kundu, A., (1994), An algorithm for the optimum time-cost trade-off in fixed-charge bi-criterion transportation problem, *Optimization*, **40**, 53-68.

Ben Israel, A., and Greville, Thomas, N. E. (1974). Generalized Inverses-Theory and Applications. John Wiley and Sons, New York.

Bhatia, H.L., Swarup, K. and Puri M.C., (1975), A procedure for time minimization transportation problem, *Indian Journal of Pure and Applied Mathematics*, **8**, 920-929.

Bhatia, H.L., Swarup, K. and Puri M.C., (1976),Time-cost trade-off in a transportation problem minimization transportation problem, *Opsearch*, **13(3-4)**, 129-142.

Cooper, L. and Drebes, C., (1967), An approximate solution method for the fixed charge transportation problem, *Naval Research Logistics Quarterly*, **14**, 101-113.

Garfinkal, R.S. and Rao, M.R., (1971), The bottleneck transportation problem, *Naval Research Logistics Quarterly*, **18**, 465-472.

Gottileb, J. and Paulmann, L., (1998), Genetic algorithm for the fixed charge transportation problem, *Evolutionary Computation Proceedings*, *IEEE World Congress on Computational Intelligence.*, 330-335.

Goyal, S.K., (1984), Improving VAM for unbalanced Transportation Problem, *Journal of Operations Research Society*, **35**(**12)**, 1113-1114.

Gupta, A.K. and Sharma J.K., (1983), An efficient algorithm for solving fixed charge problems, *Indian Journal of Pure and Applied Mathematics*, **14(1)**, 32-39.

Gupta and Gupta (1982) Multi-Criteria Simplex Method for a Linear Multi-Objective Transportation Problem, Indian journal of pure and applied mathematics, **14(2)**, 222-232.

Gray, P., (1971) Exact Solution of the Fixed-Charge Transportation Problem, *Operations Research*, **19**, 1529-1538.

Gupta, R.(1977).Time-cost transportation problem. Ekonomicko Mathematicky Obzor, 13(4), 431-43.

Haley, K.B., (1962), The Solid Transportation Problem, *Operations Research*, **10(4)**, 448-463.

Haley, K.B., (1963), The Multi-Index Problem, *Operations Research*, **11(3)**, 368-379.

Haley, K.B., (1965), The Existence of a Solution to the Multi-Index Problem, *Operations Research*, **16(4)**, 471-474.

Hirsch, W.M. and Dantzig, G.B.,(1954),The fixed charge problem, The RAND Corporation, RM-1383.

Hughes, A.J. and Grawiog, D.E., (1973), *Linear Programming: An Emphasis on Decision Making*, Addision-Welsey, Reading, MA, 300-312.

Isermann, H. (1979). The enumeration of all eficient solutions for a linear multiple objectivetransportation problem. Naval Res. Login. Quart., 26(1), 123-39.

Kasana, H.S. and Kumar, K.D.,(2003), *Introductory Operations Research, Theory and Applications.* Springer-Verlag, Berlin.

Klingman, D., and Russell, R.(1975). Solving constrained transportation problem. Op. Res.,23.

Moravek, J. and Vlach, M., (1967), On the Necessary Conditions for the Existence of the Solution of the Multi-Index Transportation Problem, *Operations Research*, **15(3)**, 542-545.

Murty, K.G., (1968), Solving the fixed-charge problem by ranking the extreme points, *Operations Research*, **16**, 268-279.

Prakash, S.,(1981) Transportation problem with objectives to minimize cost and duration of transportation, *Opsearch*, **18**, 235-238.

Prakash, S. and Aggarwal, A.,(1992), A bicriterion routing network problem, *The Mathematics Student*, **61**, 89-96.

Prasad, V.R., Nair, K.P.K. and Aneja,Y.P.,(1993), A Generalized Time-Cost Trade-Off Transportation Problem, *The Journal of the Operational Research Society*, **44(12)**, 1243-1248.

Puri, M.C. and Swaroop, K.,(1974), Simple enumerative method for fixed charge problem, *Indian Journal of Pure and Applied Mathematics*, 5(10), 894-901.

Ramakrishnan, C.S., (1977), A note on the time minimizing transportation problem, *Opsearch*, **14**, 207-209.

Reinfeld, N.V. and Vogel, W.R., (1958), *Mathematical Programming*. Prentice-Hall, Englewood Cliffs, New Jersey, 59-70.

Sandrock, K., (1988), A simple algorithm for solving small fixed-charge transportation problems, *Journal of the Operational Research Society,* **39**, 465-475.

Schell, E.,(1955), “Distribution of a Product by Several Properties,” Directorate of Management Analysis, “Proceedings of the Second Symposium in Linear Programming,” 2, 615, DCS/ Comptroller H.Q. U.S.A.F. Washington, D.C.

Shimshak, D.G., Kaslik, J.A. and Banclay, T.D., (1981), A modification of Vogel’s Approximation Method through the use of heuristics. Inform, 19(3), 159-163.

Smith, G. (1973), Further Conditions for the Existence of a Solution to the Multi-Index Problem, Operations Research, 21(1), 380-386.

Steinberg, D.I., (1970), The Fixed charge problem, Naval Research Logistics Quarterly, 17, 217-236.

Zeleny, M. (1974). Linear Multi-Objective Programming, Berlin-Heidelberg, New York.