FIXED POINT THEOREMS ON CONTRACTION AND COMMUTING MAPPINGS

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CERTIFICATE

Certified that the dissertation entitled, "**FIXED POINT THEOREMS ON CONTRACTION AND COMMUTING MAPPINGS**", which being submitted by **Miss. Tanupreet Kaur** (Roll No. 301003027), in the fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision of **Dr. Jatinderdeep Kaur** during the period of January 2012 to June 2012. The contents of this dissertation do not appear in any of the dissertation submitted to this university or any other university.

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ABSTRACT

The intent of this dissertation entitled, "**FIXED POINT THEOREMS ON CONTRACTION AND COMMUTING MAPPINGS**" embodies a brief account of investigations carried out by various authors on existence of fixed points of self-mappings in metric spaces under the supervision of **Dr. Jatinderdeep Kaur**, Lecturer, School of Mathematics and Computer Applications, Thapar University, Patiala.

The aim of this work is to study and obtain some result on existence and uniqueness of fixed points. Fixed point theory has wide ranging application in many areas of mathematics. For example, in finding the solution of the system of linear equations, in proving the existence of solutions of ordinary and partial differential equation, integral equations, analysis and many other disciplines.

The whole work is divided into four chapters. Chapter I is introduction which includes brief account of definitions and results which will be required for the later chapters. In chapter II, we have studied Banach Contraction Theorem which guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and application of Banach Contraction Theorem to system of linear equations. The aim of chapter III is to study the generalization of Banach Contraction Theorem obtained by Meir-Keeler in 1969. Chapter IV is concerned with common fixed point theorem for two pairs of commuting mappings satisfying Meir and Keeler type conditions.

At the end of the present dissertation, we have added some references of research papers and books cited in the dissertation.

NOTATIONS

- **R** – Set of real numbers.
- **N** – Set of natural numbers.
- $\mathbf{R}^n$ – Euclidean n-space.
- $\in$ – Belongs to.
- $\subset$ – Subset.
- $\forall$ – For all.
- $\Rightarrow$ – Implies.
- $\exists$ – There exists.
- [a, b] – Closed interval.
-] a, b [– Open interval.
-] a, b] – Semi-open interval

CONTENTS

CHAPTER-I

INTRODUCTION

1.1 INTRODUCTION

A fixed point of a function is a point that is mapped to itself by the function. Let X be any non-empty set. Given a function $f: X \to X$, a fixed point of f is a point $x \in X$ such that $f(x) = x$, that is, a point which remains invariant under the mapping f.

In graphical terms, a fixed point means the point $(x, f(x))$ is on the line $y = x$, or in other words the graph of f has a point in common with that line. The example $f(x) = x - 1$ is a case where the graph and the line are a pair of parallel lines.

Fixed point theory has wide ranging application in many areas of mathematics. For example, in finding the solution of the system of linear equations, in proving the existence of solutions of ordinary and partial differential equations, integral equations, analysis and many other disciplines.

The first fixed point theorem was given by Brouwer [2] in 1912 which states that "*Every continuous map of the closed unit ball* $\delta = (x: ||x|| \leq 1)$ *in* $\boldsymbol{R^n}$ *to itself has a fixed point*", but the credit of making concept useful and popular goes to Polish mathematician Stefan Banach who proved the famous contraction mapping theorem in 1922 which states that "*Let* (X, d) *be a complete metric space and let* $T: X \to X$ *be a contraction on* X. *Then,* T *has a unique fixed point in* X". The Banach contraction theorem [1] is one of the most important and useful results in the metric fixed point theory. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points.

This theorem was generalized by various authors including those of Rhodes, Park and Moon [12], Cho, Khan and Singh [3], Jungck [5], Kumar [6], Meir and Keeler [7], Pant [9], Pathak, Cho and Kang [10], Seesa [13] etc.

In the present dissertation, some results of above mentioned authors have been studied.

1.2. DEFINITIONS

Here, we give a brief account of definitions and results which will be required for the later chapters. However, some of the definitions and notations will be repeated occasionally in various chapters for the sake of convenience.

METRIC SPACE ([4], p.11)

Let X be a non-empty set. A function

$$d: X \times X \to \boldsymbol{R}$$

is said to be a '*metric*' or '*distance function*' on X if it satisfies the following conditions:

$(i)\ d\ (x, y) \geq 0, \ \ \forall\, x, y \in X;$

$(ii)\ d\ (x, y) = 0\ iff\ x = y, \qquad \forall\, x, y \in X;$

$(iii)\ d\ (x, y) = d\ (y, x), \qquad \forall\, x, y \in X;$ $\qquad\qquad (symmetry)$

$(iv)\ d\ (x, z\ \leq d\ (x, y) + d\ (y, z), \qquad \forall\, x, y, z \in X;$ $\qquad (triangle\ inequality)$

The ordered pair (X, d) is called a '*metric space*'.

Example: Let $X = \boldsymbol{R}$, the set of all real numbers. For $x, y \in X$, define

$$d\ (x, y) = |x - y|$$

Then (X, d) is a metric space. This is called the metric space $\boldsymbol{R}$ with the usual metric and we denote it by $\boldsymbol{R_u}$.

SEQUENCE ([4], p.5)

A function $f: \boldsymbol{N} \to X$, where X is any set, is called a sequence in X. A sequence is usually denoted by $\{x_n\}$.

CONVERGENT SEQUENCE ([4], p.47)

Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be convergent sequence if there is a point $x \in X$ such that for each $\varepsilon > 0, \exists$ a positive integer N such that

$$d(x_n, x) < \varepsilon, for\ all\ n \geq N.$$

We usually symbolize this by writing

$$x_n \to x \quad or \quad \lim_{n\to\infty} x_n = x$$

and we express it by saying that $\{x_n\}$ converges to x.

The element x is called the limit of the sequence $\{x_n\}$.

Example: Consider sequence $\{a_n\} = \frac{1}{n}$. Given $\varepsilon > 0$, however small, we choose a natural number m such that

$$m > \varepsilon \Rightarrow \frac{1}{m} < \varepsilon$$

$$\therefore \forall\, n \geq m, \quad \frac{1}{n} \leq \frac{1}{m} < \varepsilon$$

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon, \qquad \forall\, n \geq m.$$

Thus, sequence $\{a_n\}$ converges to 0.

CAUCHY SEQUENCE ([4], p.52)

A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for each $\varepsilon > 0$, $\exists$ a positive integer N such that

$$d\,(x_m, x_n) < \varepsilon, \forall\, m, n \geq N$$

REMARK: *In a metric space every Convergent sequence is a Cauchy sequence. But the converse is not true which can be shown by next example.*

Example: Let $X =]0, 1]$ be the metric space with usual metric and $\{x_n\}$, where $x_n = \frac{1}{n}$, be a sequence in X.

Then $\{x_n\}$ is Cauchy sequence since for each $\varepsilon > 0$, we have

$$d\,(x_m, x_n) = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \varepsilon, \qquad \forall\, m, n > \frac{1}{\varepsilon} \quad ,$$

$$i.e., d\,(x_m, x_n) \to 0, but\ 0 \notin X.$$

Therefore $\{x_n\}$ is not convergent sequence.

If we take $X = [0,1]$, then the sequence $\{x_n\}$ is Cauchy as well as Convergent.

COMPLETE METRIC SPACE ([4], p.55)

A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

In other words, (X, d) is a complete metric space if, whenever the sequence $\{x_n\}$ in X is such that $d\,(x_m, x_n) \to 0$ as $m, n \to \infty$, then $x \in X$ with $d(x_m, x) \to 0$ as $n \to \infty$.

Example: Let X be an arbitrary non-empty set. For $x, y \in X$, define $d: X \times X \to R$ by

$$d(x, y) = \begin{cases} 0, & if\ x = y \\ 1, & if\ x \neq y \end{cases}$$

Then (X, d) is a metric space called as the discrete metric space.

This is a complete metric space.

FIXED POINT ([4], p.127)

Let (X, d) be a metric space and $T: X \to X$ be a mapping, the point $x \in X$ is called a fixed point of T if x is mapped into itself; i.e.

$$T(x) = x$$

Examples:

- Let $X = R$ and $T: X \to X$ be a mapping defined by $T(x) = x^2$. Then the points 0 and 1 are two fixed points of T.
- A translation mapping $T: R \to R$ defined by $T(x) = x + a$ where $0 \neq a \in R$, has no fixed point.
- Let $X = R^2$ and $T: X \to X$ be a mapping defined by $T(x, y) = x$. Then mapping have infinitely many fixed points. In fact, all points of x-axis are fixed points.
- Let $X = R$ and $T: X \to X$ be a mapping defined by $T(x) = \frac{x}{3}$. Then $x = 0$ is only fixed point.

REMARK: *Mapping may not have any fixed point, it may have a unique fixed point, it may have more than one or even infinitely many fixed points.*

CONTRACTION MAPPING ([4], p.128)

Let (X, d) be a metric space. A mapping $T: X \to X$ is called a contraction of X if $\exists$ a real number α with $0 \leq \alpha < 1$ such the

$$d\,(Tx, Ty) \leq \alpha\, d(\,x, y), \qquad \forall\, x, y \in X.$$

Example: Consider the usual metric d for $\boldsymbol{R^2}$and the mapping $f: \boldsymbol{R^2} \to \boldsymbol{R^2}$; $f(x) = \frac{x}{2}$, $\forall\, x \in \boldsymbol{R^2}$, where $x = (x_1, x_2)$. We have

$$d\big(f(x), f(y)\big) = d\left(\frac{x}{2}, \frac{y}{2}\right)$$

$$= d\left(\frac{1}{2}(x_1, y_1), \frac{1}{2}(x_2, y_2)\right)$$

$$= d\left(\left(\frac{x_1}{2}, \frac{y_1}{2}\right), \left(\frac{x_2}{2}, \frac{y_2}{2}\right)\right)$$

$$= \sqrt{\frac{1}{4}(x_1 - x_2)^2 + \frac{1}{4}(y_1 - y_2)^2}$$

$$= \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \frac{1}{2} d(x, y)$$

Since,

$$d\big(f(x), f(y)\big) = \frac{1}{2} d(x, y)$$

Therefore, f is contraction mapping on $\boldsymbol{R^2}$.

STRICT CONTRACTION MAPPING [7]

Let (X, d) be a metric space. A mapping $T: X \to X$ is called a strict contraction of X if ∃ a real number α with $0 \leq \alpha < 1$ such that

$$d\,(Tx, Ty) < \alpha\, d\,(x, y), \quad \forall x, y \in X.$$

Example: Consider the usual metric d for $X =]0, \frac{1}{2}]$ and the mapping $T: X \to X$: $T(x) = \frac{x}{2}$, $\forall\, x$. Then we have

$$d\big(T(x), T(y)\big) = d\left(\frac{x}{2}, \frac{y}{2}\right)$$

$$= \left|\frac{x}{2} - \frac{y}{2}\right|$$

$$= \frac{1}{2}|x - y|$$

$$< \alpha|x - y| \qquad \textit{where } 0.5 < \alpha < 1$$

$$\Longrightarrow \quad d\big(T(x), T(y)\big) < \alpha d(x, y)$$

Hence T is a strict contraction.

WEAKLY UNIFORMLY STRICT CONTRACTION [7]

Let (X, d) be a metric space. A mapping $T: X \to X$ is called a weakly uniformly strict contraction of X if for given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq d\,(x, y) < \varepsilon + \delta \quad implies \quad d\left(f(x), f(y)\right) < \varepsilon$$

COMMUTING MAPPING [5]

Two self mappings f and g on a metric space (X, d) are called commuting if

$$(fog)\,x = (gof)\,x, for\ all\ x \in X$$

Example: Let $X = [0, 1]$ with usual metric defined by

$$d(x, y) = |x - y|, \ \ for\ all\ x, y \in X.$$

and define $f(x) = x$ and $g(x) = x^2, x \in X$,

then

$$(fog)\,x = f\,(g(x)) = f\,(x^2) = x^2$$

and

$$(gof)x = g\,(f(x)) = g\,(x) = x^2$$

Since

$$(fog)x = (gof)x, for\ all\ x \in X.$$

Thus both f and g commute for all $x \in X$.

WEAKLY COMMUTING MAPPING [13]

Two self mappings f and g on a metric space (X, d) are said to be weakly commuting if

$$d(fgx, gfx) \leq d(fx, gx), \qquad \forall\, x \in X$$

REMARK: *Commuting mappings are always weakly commuting but the converse is not true which can be shown by following example:*

EXAMPLE: Let $X = [0,1]$ with $d(x,y) = |x-y|$, for all $x, y \in X$. Define $f, g: X \rightarrow X$ by

$f(x) = \frac{x}{2-x}$ and $g(x) = \frac{x}{2}$, for all $x \in X$.

Then

$$d\,(fgx, gfx) = \left|\frac{x}{4-x} - \frac{x}{4-2x}\right|$$

$$= \frac{x^2}{(4-x)(4-2x)}$$

$$< \frac{x^2}{4-2x}\left(\because \frac{1}{4-x} < 1 \; as \; x \in [0,1]\right)$$

$$= \frac{x}{2-x} - \frac{x}{2} = d(fx, gx)$$

which shows that, f and g are weakly commuting mapping.

$$(fog)x \; \frac{x}{4-x} \neq (gof)x = \frac{x}{4-2x}$$

which implies that, f and g are not commuting mapping.

R-Weakly Commuting mapping [8]

Two self mappings f and g on a metric space (X,d) are called R-weakly commuting at a point x in X if

$$d\,(fgx, gfx) \leq Rd\,(fx, gx), \;\; for\ some\ R > 0$$

Example: Let $X = [0,1]$ with usual metric $d(x,y) = |x-y|, \forall\, x, y \in X$.

Define $f(x) = x$ and $g(x) = x^2$.

Since $d\,(fgx, gfx) = 0$, $d\,(fx, gx) = |x(x-1)|$, for all x in X.

And also $d(fgx, gfx) \leq d(fx, gx)$

Therefore, pair (f, g) is R-Weakly commuting for all positive real values of R.

R-weakly commuting of type (A_g) [10]

Two self mappings f and g on a metric space (X, d) are called R-weakly commuting of type (A_g) if ∃ some positive real number $R > 0$ such that

$$d\,(ffx, gfx) \leq \ Rd\,(fx, gx), \qquad \forall x \ \in \ X.$$

R-weakly commuting of type (A_f) [10]

Two self mappings f and g on a metric space (X, d) are called R-weakly commuting of type (A_f) if ∃ some positive real number $R > 0$ such that

$$d\,(ggx, fgx) \leq \ Rd\,(fx, gx), \qquad \forall x \in X.$$

R-weakly commuting mappings of type (P) [6]

Two self mappings f and g on a metric space (X, d) are called R-weakly commuting of type (P) if ∃ some $R > 0$ such that

$$d\,(ffx, ggx) \leq Rd\,(fx, gx), \forall x \in X$$

EXAMPLE: Let $X \ = \ [0, 1]$ with usual metric $d(x, y) = |x - y|, \forall\, x, y \in X$.

Define $f(x) \ = \ x$ and $g(x) \ = \ x^2$.

Since $\quad d\,(fgx, gfx) = \ 0, \qquad d\,(fgx, ggx) \ = \ |x^2(x-1)(x+1)|,$

$$d\,(gfx, ffx) \ = \ |x(x-1)|, \quad d\,(ffx, ggx) \ = \ |x(x-1)(x^2+x+1)|$$

$$d\,(fx, gx) \ = \ |x(x-1)|, \quad for\ all\ x\ in\ X.$$

Therefore we conclude that

(i) For $R = 3$,

Pair (f, g) is R-weakly commuting of the type (A_f),

R-weakly commuting of the type (A_g) and

R-weakly commuting of type (P).

(ii) For $R = 2$,

Pair (f, g) is R-weakly commuting of type (A_f)

and R-weakly commuting of type (A_g)

but not R-weakly commuting of type(P) $\left(\text{for } x = \frac{3}{4}\right)$.

Now, I give a brief chapter wise resume of the results contained in this dissertation.

In chapter II, we have studied Banach Contraction Theorem which guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and application of Banach Contraction Theorem to system of linear equations.

The aim of chapter III is to study the generalization of Banach Contraction Theorem obtained by Meir-Keeler in 1969.

In chapter IV, we have presented common fixed point theorem for two pairs of commuting mappings satisfying Meir and Keeler type conditions.

CHAPTER – II

BANACH'S FIXED POINT THEOREM

2.1. INTRODUCTION

In mathematics, the **Banach fixed-point theorem** [1] (also known as the **contraction mapping theorem** or **contraction mapping principle**) is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. The theorem is named after Stefan Banach (1892–1945), and was first stated by him in 1922.

FIXED POINT ([4], p.127)

Let (X, d) be a metric space and $T: X \to X$ be a mapping, the point $x \in X$ is called a fixed point of T if x is mapped into itself; i.e.

$$T(x) = x$$

Example: A mapping $T: \boldsymbol{R} \to \boldsymbol{R}$ defined by $T(x) = x^3$ have three fixed points namely $0, 1$ and -1.

CONTRACTION MAPPING ([4], p.128)

Let (X, d) be a metric space. A mapping $T: X \to X$ is called a contraction of X if $\exists$ a real number α with $0 \leq \alpha < 1$ such that

$$d\,(Tx, Ty) \leq \alpha\, d\,(x, y), \forall\, x, y \in X.$$

Example :

Consider the usual metric d for $\boldsymbol{R^2}$and the mapping $f: \boldsymbol{R^2} \to \boldsymbol{R^2}$: $f(x) = \frac{x}{4}, \forall\, x \in \boldsymbol{R^2}$, where $x = (x_1, x_2)$. We have

$$d\big(f(x), f(y)\big) = d\left(\frac{x}{4}, \frac{y}{4}\right)$$

$$= d\left(\tfrac{1}{4}(x_1, y_1), \tfrac{1}{4}(x_2, y_2)\right)$$

$$= d\left(\left(\frac{x_1}{4}, \frac{y_1}{4}\right), \left(\frac{x_2}{4}, \frac{y_2}{4}\right)\right)$$

$$= \sqrt{\frac{1}{16}(x_1 - x_2)^2 + \frac{1}{16}(y_1 - y_2)^2}$$

$$= \tfrac{1}{4}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \tfrac{1}{4} d(x, y).$$

Since,

$$d\big(f(x), f(y)\big) = \frac{1}{4} d\left(\frac{x}{4}, \frac{y}{4}\right)$$

Therefore, f is a contraction on $\boldsymbol{R^2}$.

The aim of this chapter is to study Existence and Uniqueness of Banach fixed points and also to study the application of Banach contraction theorem to system of linear equations.

2.2 MAIN THEOREM

THEOREM (Banach Contraction Theorem) ([4], p.128): - *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a contraction on* X. *Then,* T *has a unique fixed point in* X.

PROOF: Let $\alpha \in [0, 1[$ be the Lipschitz constant such that

$$d\,(Tx, Ty) \le \alpha\, d\,(x, y), \ \ \forall\, x, y \in X.$$

We prove the theorem in various steps.

Step (i) We construct a sequence $\{x_n\} \subset X$ *as follows:*

Take ant point $x_0 \in X$ and inductively construct the sequence $\{x_n\}$ of points in X as:

$$x_1 = Tx_0$$

$$x_2 = T\,x_1 = T^2\,x_0$$

$$x_3 = T\,x_2 = T^3 x_0$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$x_n = Tx_{n-1} = T^n\,x_0$$

Clearly, $\{x_n\}$ is the sequence of images of x_0 under repeated application of T.

Step (ii) $\{x_n\}$ is a Cauchy sequence in X.

Let $m < n$. Then

$$d\,(x_m, x_n) = d\,(T^m\,x_0,\ T^n x_0)$$

$$\leq \alpha\,d\,(\,T^{m-1}\,x_0, T^{n-1}x_0)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\leq \alpha^m\,d(x_0, T^{n-m}x_0)$$

$$\leq \alpha^m[d\,(x_0, Tx_0) + d\,(Tx_0, T^2x_0) + \dots. + d\,(T^{n-m-1}x_0, T^{n-m}x_0)]$$

$$(by\ triangle\ inequality)$$

$$\leq \alpha^m\,[d\,(x_0, Tx_0) + \alpha d\,(x_0, T\,x_0) + \dots. + \alpha^{n-m-1}\,d\,(x_0, Tx_0)]$$

$$= \alpha^m[1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}]\,d(\,x_0, T\,x_0)$$

$$\leq \frac{\alpha^m}{1-\alpha} d\,(x_0, T\,x_0), \quad (0 \leq \alpha < 1)$$

$$\to 0; \quad as\ m \to \infty$$

Hence $\{x_n\}$ is a Cauchy sequence.

Step (iii) Since X is complete and $\{x_n\}$ is a Cauchy sequence in X, therefore there exists a $x \in X$ such that $x_n \to x$.

Step (iv) x is a fixed point of T.

We have

$$d(x,Tx) \le d(x,x_n) + d(x_n,Tx) \qquad (\text{by triangle inequality})$$

$$\le d(x,x_n) + \alpha d(x_{n-1},x) \qquad (\because x_n = Tx_{n-1})$$
$$\to 0, \; as\; n \to \infty \qquad \text{(By } \textit{step (iii)}\text{)}$$

$$\Rightarrow \qquad d(x,Tx) = 0$$

$$x = Tx$$

Hence x is a fixed point of T.

Thus the existence of a fixed point is established.

In the last step, we verify the uniqueness of such a fixed point.

Step (v) x is a unique fixed point of T.

Let if possible, x and y be two fixed points of T in X.

Then $Tx = x$ and $Ty = y$.

Now, note that

$$d\,(x,y) = d\,(Tx,Ty)$$

Now by definition of contraction mapping, we get

$$d\,(x,y) = d\,(Tx,Ty) < d(x,y)$$

$$\Rightarrow \qquad d\,(x,y) = 0$$

$$\Rightarrow \qquad x = y$$

This completes the proof of the theorem.

REMARKS:

(i) If X is not complete in above theorem, then T may not have a fixed point.

For example, consider $X =]0,1[$ and the mapping $T: X \to X$ defined by $(x) = \frac{x}{2}$.

Then X is not complete metric space with the usual metric and T does not have any fixed point. In fact, $T\ (0) = 0 \notin X$.

(ii) If T is not contraction in above theorem, then it may not have a fixed point.

For example, consider the metric space $X = [1, \infty[$ with the usual metric and the mapping $T: X \to X$ given by $T(x) = x + \frac{1}{x}$. Then X is a complete metric space but T is not a contraction mapping. In fact,

$$|T(x) - T(y)| = |x - y| \left(1 - \frac{1}{xy}\right) < |x - y|, \qquad \forall\, x, y \in X$$

2.3 APPLICATION TO SYSTEM OF LINEAR EQUATIONS

Here we discuss an application of Banach contraction theorem to find the solution of the following system of n linear equations with n unknowns:

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array}\right\} \qquad (2.3.1)$$

This system can be written

$$\left.\begin{array}{c} x_1 = (1 - a_{11})x_1 - a_{12}x_2 - \dots - a_{1n}x_n + b_1 \\ x_2 = -a_{21}x_1 + (1 - a_{22})x_2 - a_{23}x_3 - \dots - a_{2n}x_n + b_2 \\ x_3 = -a_{31}x_1 - a_{32}x_2 + (1 - a_{33})x_3 - \dots - a_{3n}x_n + b_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n = -a_{n1}x_1 - a_{n2}x_2 - a_{n3}x_3 - \dots + (1 - a_{nn})x_n + b_n \end{array}\right\} \qquad (2.3.2)$$

By letting $\alpha_{ij} = -a_{ij} + \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 \; for \; i = j \\ 0 \; for \; i \neq j \end{cases}$$

System (2.3.2) is equivalent to the following system:

$$x_i = \sum_{j=1}^{n} \alpha_{ij} \; x_j + b_i \qquad i = 1, 2, 3, \dots, n \tag{2.3.3}$$

If $x = (x_1, x_2, \dots, x_n) \in \boldsymbol{R^n}$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

and $= (b_1, b_2, b_3, \dots, b_n) \in \boldsymbol{R^n}$, then the system (2.3.3) is equivalent to

$$x = Ax + b \tag{2.3.4}$$

In other words, the problem is to find the fixed point of the transformation $T: \boldsymbol{R^n} \to \boldsymbol{R^n}$ defined by

$$T(x) = Ax + b \tag{2.3.5}$$

If T is a contraction mapping, then we can use Banach Contraction Theorem and obtain the unique solution of $T(x) = x$ by the method of successive approximation.

The condition under which T is a contraction mapping depend on the choice of the metric on $X = \boldsymbol{R^n}$.

THEOREM1 ([11], p.128): *Let* $X = \boldsymbol{R^n}$ *be a metric space with the metric*

$$d_\infty(x, y) = \max_{1 \le i \le n} |x_i - y_i|$$

if

$$\sum_{j=1}^{n} |\alpha_{ij}| \le \alpha < 1 \qquad for \; all \; i = 1, 2, \dots, n \tag{2.3.6}$$

then the linear system (2.3.1) of n linear equations in n unknowns has a unique solution.

PROOF: Since $X = \boldsymbol{R^n}$ with respect to the metric d_∞ is complete it is sufficient to prove that the mapping T defined by (2.3.5) is a contraction

$$d_\infty\big(T(x),T(y)\big) = \max_{1\le i\le n}\left|\sum_{j=1}^{n} \alpha_{ij}\,(x_i - y_i)\right|$$

$$\le \max_{1\le i\le n}\sum_{j=1}^{n}|\alpha_{ij}||x_i - y_i|$$

$$\le \max_{1\le i\le n}\left(\max_{1\le j\le n}|x_j - y_j|\right)\sum_{j=1}^{n}|\alpha_{ij}|$$

$$= \max_{1\le i\le n}\sum_{j=1}^{n}|\alpha_{ij}|d_\infty(x,y)$$

$$\le \alpha d_\infty(x,y) \qquad \text{(By (2.3.6))}$$

Thus T is a contraction mapping. By Banach Contraction Theorem, the linear system has a unique solution.

THEOREM 2 ([11], p.128): *Let $X = \boldsymbol{R^n}$ be a metric space with the metric*

$$d_1(x,y) = \sum_{i=1}^{n}|x_i - y_i|$$

If

$$\sum_{i=1}^{n}|\alpha_{ij}| \le \alpha \quad \forall\, j = 1,2,\dots,n \tag{2.3.7}$$

Then the linear system (2.3.1) of n linear equations in n unknowns has a unique solution.

PROOF: Since $X = \boldsymbol{R^n}$ with respect to the metric d_1 is complete, it is sufficient to prove that the mapping T defined by (2.3.5) is a contraction

$$d_1\big(T(x),T(y)\big) = \sum_{i=1}^{n}\left|\sum_{j=1}^{n} \alpha_{ij}\,(x_j - y_j)\right|$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_{ij}||x_j - y_j|$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} |\alpha_{ij}||x_j - y_j|$$

$$\leq \max_{1 \leq j \leq n} \sum_{i=1}^{n} |\alpha_{ij}| d_1(x, y)$$

By using (2.3.7), we get

$$d_1\big(T(x), T(y)\big) \leq \alpha d_1(x, y).$$

Thus T is a contraction mapping. By Banach Contraction Theorem, the linear system (2.3.1) has unique solution.

THEOREM 3 ([11], p.129): *Let $X = \boldsymbol{R^n}$ be a metric space with the metric*

$$d_2(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$$

If

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_{ij}|^2 \leq \alpha^2 < 1 \tag{2.3.8}$$

Then the linear system (2.3.1) of n linear equations in n unknowns has a unique solution.

PROOF: Since $X = \boldsymbol{R^n}$ with respect to the metric d_2 is complete, it is sufficient to prove that the mapping T defined by (2.3.8) is a contraction

$$\left[d_2\big(T(x), T(y)\big)\right]^2 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \alpha_{ij} \left(x_i - y_j\right) \right|^2$$

$$\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\alpha_{ij}| \sum_{j=1}^{n} |x_j - y_j| \right)^2$$

$$\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\alpha_{ij}|^2 \sum_{j=1}^{n} |x_j - y_j|^2 \right).$$

So,

$$\left[d_2\big(T(x),T(y)\big)\right]^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\alpha_{ij}\right|^2 {d_2}^2(x,y)$$

By using (2.3.8), we get

$$\left[d_2\big(T(x),T(y)\big)\right]^2 \leq \alpha^2 {d_2}^2(x,y)$$

Therefore, T is a contraction mapping and by hypothesis of Banach Contraction Theorem, the linear system (2.3.1) has unique solution.

CHAPTER-III

GENERLISATION OF BANACH FIXED POINT THEOREM

3.1 INTRODUCTION

In 1969, Meir and Keeler [7] generalized the well-known Banach fixed point theorem [1] which states that if there exist an $\alpha < 1$ such that for all x, y ∈ X

$$d\left(f(x), f(y)\right) \leq \alpha\, d\,(x, y) \qquad \text{... (3.1.1)}$$

then f has a unique fixed point i.e. a point ξ such that $f(\xi) = \xi$.

In this section, (X, d) will be a complete metric space and f be a mapping of X into itself. For a function $f(x)$ on the points x in X, we call $f(x), f\left(f(x)\right) = f^2(x), \dots, and\ f\left(f^{n-1}(x)\right) = f^n(x)$ the iterates of x and denote them by x_n.

This well known Banach Contraction Theorem has also been extended by many other authors in different way.

In this chapter, we discussed the generalization of Banach Contraction Theorem by considering strict contraction mapping instead of contraction mapping. The strict contraction mapping is defined as follows:

STRICT CONTRACTION MAPPING [7]

Let (X, d) be a metric space. A mapping $T: X \to X$ is called a strict contraction of X if ∃ a real number α with $0 \leq \alpha < 1$ such that

$$d\,(Tx, Ty) < \alpha\, d\,(x, y), \quad \forall x, y \in X.$$

Example: Consider the usual metric d for $X =]0, \frac{1}{2}]$ and the mapping

$$T: X \to X:\ T(x) = \frac{x^2}{2}, \qquad \forall\, x \in X.$$

Then we have

$$d\big(T(x),T(y)\big) = d\left(\frac{x^2}{2},\frac{y^2}{2}\right)$$

$$= |\frac{x^2 - y^2}{2}|$$

$$= \frac{1}{2}|(x-y)(x+y)|$$

$$= \frac{1}{2}|x-y||x+y|$$

$$< \alpha|x-y|$$

$$\Rightarrow d\big(T(x),T(y)\big) < \alpha d(x,y),\ where\ 0.5 < \alpha < 1$$

Hence T is a strict contraction.

WEAKLY UNIFORMLY STRICT CONTRACTION [7]

Let (X,d) be a metric space. A mapping $T: X \to X$ is called a weakly uniformly strict contraction of X if for given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \le d\,(x,y) < \varepsilon + \delta \qquad implies \qquad d\,\big(f(x),f(y)\big) < \varepsilon \qquad \text{... (3.1.2)}$$

3.3 LEMMAS: The proof of our result is based upon the following lemmas given by Meir and Keeler [7]:

LEMMA 1 [7]: If $f: X \to X$ is a strict contraction and if, for every $x \in X$, the $f^n(x)$ form a Cauchy sequence, then f has a unique fixed point and for any $x \in X$

$$\lim_{n\to\infty} f^n\,(x) = \xi$$

holds.

PROOF: Due to the completeness of the space X each $f^n(x)$ has a limit point $\xi(x)$.

By the continuity of f we have

$$f(\xi(x)) = f(\lim_{n\to\infty} f^n(x))$$

$$= \lim_{n\to\infty} f(f^n(x))$$

$$= \lim_{n\to\infty} f^{n+1}(x)$$

$$= \xi(x)$$

Thus $\xi(x)$ is a fixed point and therefore all $\xi(x)$ are equal.

LEMMA 2 [7]: Condition (3.1.2) implies that

$$d\,(x_n, x_{n+1}) \downarrow 0$$

PROOF: We prove it by contradiction.

Let $c_n = d\,(x_n, x_{n+1})$.

Let us assume that ε be the lim inf of c_n.

Let $d(x, y) = \varepsilon$, then condition (3.1.2)

$$\Rightarrow \quad d\,(f(x), f(y)) < d\,(x, y)$$

$$\Rightarrow \quad d\,(f(x_n), f(x_{n+1})) < d\,(x_n, x_{n+1})$$

$$\Rightarrow \quad d(x_{n+1}, x_{n+2}) < d\,(x_n, x_{n+1})$$

which implies c_n is a decreasing sequence with n.

Then condition (3.1.2) implies $c_{n+1} < \varepsilon$, contradiction.

Hence c_n decreases to 0.

3.3 MAIN THEOREM

The main result of this chapter is the following theorem:

Theorem [7]: *Let (X, d) be a complete metric space and f be a mapping of X into itself, if for given $\varepsilon > 0$, $\exists\, \delta > 0$ such that*

$$\varepsilon \leq d\,(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d\left(f(x), f(y)\right) < \varepsilon \qquad \text{... (3.3.1)}$$

holds, then f has a unique fix point ξ. Moreover, for any $x \in X$

$$\lim_{n\to\infty} f^n(x) = \xi$$

PROOF: We first observe that equation (3.3.1) trivially implies that f is a strict contraction i.e. for $x \neq y$ and let $d\,(x, y) = \varepsilon$ then

$$d\,(f(x), f(y)) < d\,(x, y)$$

Thus f is a continuous and has at most one fixed point.

Now by Lemma 1, our theorem will be established if for each iterates $\{f^n(x)\} = \{x_n\}$ are Cauchy sequence and f is a strict contraction, then f has fix point ξ to which all iterates converge.

We now show that $\forall\, x$, the iterates $x_n = f^n(x)$ are Cauchy.

Select any ε, and its corresponding $\delta(\varepsilon)$.

Without loss of generality, we can choose $\delta < \varepsilon$.

Because by Lemma 2 the $c_n \downarrow 0$, we can choose M such that $c_n < \delta$ for all $n > M$.

Next we show that for $n > M$ we have, $d\left(x_n, x_{n+j}\right) < \varepsilon + \delta, \forall j$

We prove it by induction.

For $j = 1, d\,(x_n, x_{n+1}) = c_n < \delta < \varepsilon + \delta$.

Need to show if $d\,(x_n, x_{n+j}) < \varepsilon + \delta$, then $d\,(x_n, x_{n+j+1}) < \varepsilon + \delta$

There are two cases:

Case 1: If $d(x_n, x_{n+j}) < \varepsilon$, then by triangle inequality

$$d\,(x_n, x_{n+j+1}) < d\,(x_n, x_{n+j}) + d\,(x_{n+j}, x_{n+j+1})$$

$$< d\left(x_n, x_{n+j}\right) + c_{n+j}$$

$$< \varepsilon + \delta$$

Case 2: if $\varepsilon \leq d\,(x_n, x_{n+j}) < \varepsilon + \delta$

Then (3.3.1) $\Rightarrow d\,(f(x_n), f(x_{n+j})) < \varepsilon$

$$d\,(x_{n+1}, x_{n+j+1}) < \varepsilon$$

but $d\,(x_n, x_{n+j+1}) < c_n + d\,(x_{n+1}, x_{n+j+1})$

$$< \delta + \varepsilon$$

$$< 2\varepsilon \quad \text{(for any arbitrary } \varepsilon)$$

Hence sequence $\{c_n\}$ is Cauchy sequence.

Hence by Lemma 1 our main result holds.

REMARK [7]: *Meir-Keeler condition is weaker than Banach's condition, which can be shown by the following example:*

EXAMPLE: Consider $f(x) = x^2$ on $[0, 0.5]$ and $d(x, y) = .5 - x$.

It satisfies Meir-Keeler condition but not Banach's condition.

For $d\,(f(x), f(y)) = .5^2 - x^2 = (.5 + x)\,(.5 - x)$

But $(.5 + x)$ can be chosen bigger than any $\alpha < 1$.

$\Rightarrow d\big(f(x), f(y)\big) < \alpha d(x, y), \qquad where\ \alpha > 1.$

So it does not satisfy the contraction mapping.

Hence, it does not satisfy Banach's condition.

Now by condition (3.3.1)

$$\varepsilon \leq d(x,y) < \varepsilon + \delta.$$

Let $\delta(\varepsilon) = \varepsilon^2$.

$$\varepsilon \leq d(x,y) < \varepsilon + \varepsilon^2$$

$$\varepsilon \leq .5 - x < \varepsilon + \varepsilon^2$$

$$-.5 + \varepsilon \leq -x < -.5 + \varepsilon + \varepsilon^2$$

$$.5 - \varepsilon > x > .5 - \varepsilon - \varepsilon^2$$

Now $.5^2 - x^2 = (.5 - x)(.5 + x)$

$$< (\varepsilon + \varepsilon^2)(1 - \varepsilon)$$

$$= \varepsilon(1 + \varepsilon)(1 - \varepsilon)$$

$$= \varepsilon(1 - \varepsilon^2)$$

$$< \varepsilon.$$

Hence it satisfies Meir-Keeler condition (3.3.1).

CHAPTER-IV

COMMON FIXED POINTS OF TWO PAIRS OF COMMUTING MAPPINGS

4.1 INTRODUCTION

In 1969, Meir and Keeler [7] obtained a remarkable generalization of the Banach contraction principle which satisfies the following Meir and Keeler type condition:

Let (X, d) be a complete metric space and f a mapping of X into itself if for given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$\varepsilon \leq d\,(x, y) < \varepsilon + \delta \qquad \text{implies} \qquad d\,(\,f(x), f(y)\,) < \varepsilon$$

holds then f has a unique fixed point ξ. Moreover, for any $x \in X$

$$\lim_{n\to\infty} f^n(x) = \xi.$$

In 1986, R. P. Pant generalized Meir and Keeler theorem to obtain a common fixed point theorem for two pairs of commuting mappings satisfying Meir and Keeler type condition.

In this chapter, we have discussed common fixed point theorem for two pairs of commuting mappings.

COMMUTING MAPPING [5]

Two self mappings f and g on a metric space (X, d) are called commuting if

$$(fog)\,x = (gof)\,x, \ \ \forall x \in X.$$

Example: Let $X = [\frac{1}{2}, 2]$ and define $f, g: X \to X$ by $f(x) = \frac{x+1}{3}$ and $g(x) = \frac{x+2}{5}, x \in X$,

then

$$(fog)\, x = f\,(g(x)) = f\left(\frac{x+2}{5}\right) = \frac{x+7}{15}$$

and

$$(gof)\, x = g\,(f(x)) = g\left(\frac{x+1}{3}\right) = \frac{x+7}{15}$$

Since

$$(fog)x = (gof)x, \quad for\ all\ x \in X.$$

Thus both f and g are commuting mappings.

4.2 MAIN THEOREM

THEOREM [9]: *Let P and S be commuting mappings and Q and T be commuting mappings of a complete metric space (X, d) into itself satisfying the conditions:*

Given $\varepsilon > 0$, $\exists$ a $\delta\,(\varepsilon) > 0$, $\delta\,(\varepsilon)$ being a non-decreasing function of ε, such that $\forall\, x, y$ in X,

$$\varepsilon \leq max\{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\} < \varepsilon + \delta$$

$$\Rightarrow d\,(Px, Qy) < \varepsilon \qquad \text{... (4.2.1)}$$

$$Px = Qy \ \ whenever \ \ Px = Sx, Qy = Ty. \qquad \text{... (4.2.2)}$$

If the range of T contains the range of P and the range of S contains the range of Q and if one of P, Q, S, and T is continuous then P, Q, S and T have a unique common fixed point z. Further, z is the unique common fixed point of P and S and of Q and T.

Proof: From (4.2.1), we have for all x, y in X such that $Px \neq Sx, Qy \neq Ty$

$$d\,(Px, Qy) < max\{d\,(Sx, Ty), d\,(Px, Sx), d\,(Qy, Ty)\} \qquad \text{... (4.2.3)}$$

and since $\delta\,(\varepsilon)$ is non-decreasing function of $\varepsilon > 0$, there exists $\varepsilon' > 0$, such that
$\varepsilon' < \varepsilon < \varepsilon' + \delta\,(\varepsilon')$ or equivalently

$$max\ \{d\ (Sx, Ty), d\ (Px, Sx), d\ (Qy, Ty)\}\ =\ \varepsilon$$

$$\Rightarrow\ d(\ Px, Qy)\ <\ \varepsilon', \qquad \varepsilon' <\ \varepsilon \qquad \text{... (4.2.4)}$$

Let x_0 be an arbitrary point in X. Since the range of T contains the range of P and the range of S contains the range of Q, therefore we can choose a point $x_1$and the a point x in Xwe have $Px_0\ =\ Tx_1$ and $Qx_1\ =\ Sx_2$.

Proceeding in this we can choose x_{2n} choose a point x_{2n+1} and then a point x_{2n+2} such that $Px_{2n}\ =\ Tx_{2n+1}$ and $Qx_{2n+1}\ =\ Sx_{2n+1}$. Then

$$d\ (Px_{2n}, Qx_{2n+1})\ <\ max\ \{d\ (Sx_{2n}, Tx_{2n+1}), d\ (Px_{2n}, Sx_{2n}), d\ (Qx_{2n+1}, Tx_{2n+1})\}$$

$$= max\{d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n+1}, Sx_{2n}), d(Qx_{2n+1}, Px_{2n})\}$$

$$= max\{d(Sx_{2n}, Tx_{2n+1}), d(Qx_{2n+1}, Px_{2n})\}$$

$$=\ max\ \{d\ (Qx_{2n-1}, Px_{2n}), d\ (Qx_{2n+1}, Px_{2n})\}$$

$$=\ d\ (Qx_{2n-1}, Px_{2n}).$$

Similarly,

$$d\ (Qx_{2n-1}, Px_{2n})\ <\ d\ (Px_{2n-2}, Qx_{2n-1}).$$

Last two inequalities implies that both $d\ (Px_{2n}, Qx_{2n+1})$ and $d(Qx_{2n+1}, Px_{2n+2})$ are monotone decreasing sequences of positive real numbers.

Next we prove that

$$\lim_{n\to\infty} d(Px_{2n}, Qx_{2n+1}) = 0 = \lim_{n\to\infty} d(Qx_{2n+1}, Px_{2n+2}).$$

Let if possible

$$\lim_{n\to\infty} d(Px_{2n}, Qx_{2n+1}) = r, \qquad r > 0.$$

Then by definition of limit given $\delta\ >\ 0$, there exist a positive integer N such that for each integer $m \geq N$, we have

$$r \leq d(Px_{2m}, Qx_{2m+1}) < r + \delta \qquad \text{... (4.2.5)}$$

Or

$$r \leq \max\{d\ (Sx_{2m+2}, Tx_{2m+1}), d\ (Px_{2m+2}, Sx_{2m+2}), d(Qx_{2m+1}, Tx_{2m+1})\} < r + \delta$$

... (4.2.6)

By proper choice of δ we note that

$d\ (Px_{2m}, Qx_{2m+1}) < r$, which contradicts (4.2.5).

Therefore,

$$\lim_{n\to\infty} d(Px_{2n}, Qx_{2n+1}) = 0$$

Similarly,

$$\lim_{n\to\infty} d(Qx_{2n+1}, Px_{2n+2}) = 0$$

Form this it follows that

$$\{Px_0, Qx_1, Px_2, Qx_3, \dots, Px_{2n}, Qx_{2n+1}, \dots\}$$

is a Cauchy sequence in the complete metric space X and so has a limit z in X.

Therefore the sequences

$$\{Px_{2n}\} = \{Tx_{2n+1}\} \text{ and } \{Qx_{2n+1}\} = \{Sx_{2n+2}\}$$

converge to the point z.

Next we have to prove that z is common fixed point of P, Q, S and T if one of P, Q, S and T is continuous.

Therefore we first assume that the mapping S is continuous. It is given that the mappings P and S commute, then the sequence $\{SSx_{2n}\}$ and $\{PSx_{2n}\}$ converge to the point Sz.

We claim that $Sz = z$.

Assume that $z \neq Sz$, i.e.

$$d(PSx_{2n}, Qx_{2n+1}) < max\{d\ (SSx_{2n}, Tx_{2n+1}), d\ (PSx_{2n}, SSx_{2n}), d\ (Qx_{2n+1}, Tx_{2n+1})\}$$

On letting $n \to \infty$

$d\ (Sz, z) < d\ (Sz, z)$, a contradiction.

Therefore, $Sz = z$.

Similarly the inequality

$$d(Pz, Qx_{2n+1}) < max\{d\ (Sz, Tx_{2n+1}), d\ (Pz, Sz), d\ (Qx_{2n+1}, Tx_{2n+1})\}$$

On letting $n \to \infty$

$$d(Pz, z) < d(Pz, z) \text{ a contradiction}$$

Therefore, $z = Pz$.

Since $P(x) \subset T(x)$, therefore $\exists$ a point z' in X such that $Sz = z = Pz = Tz'$. So

$$QTz' = Qz = TQz' \qquad \dots (4.2.7)$$

Further it is given that Q and T commute then $z = Qz'$.

if not, then

$$d(z, Qz') = d\ (Pz, Qz')$$

$$< max\ \{d\ (Sz, Tz'), d\ (Pz, Sz), d\ (Qz', Tz')\}$$

$$= d\ (z, Qz'),$$

a contradiction. Therefore $z = Qz' = Tz'$.

From equation (4.2.7) $Qz = Tz$, which implies $z = Qz = Tz$

let if possible that $z \neq Qz \neq Tz$

$$d\ (z, Qz) = d\ (Pz, Qz)$$

$$< max\{d\ (Sz, Tz), d(Pz, Sz), d\ (Qz, Tz)\}$$

$$= d\ (z, Qz)$$

a contradiction.

Hence, z is a common fixed point of P, Q, S and T.

Similarly it can be proved that z is a common fixed point of P, Q, S and T if T is continuous instead of S.

Next we suppose that the mapping P is continuous. It is given that P and S commute, then the sequences $\{PPx_{2n}\}$ and $\{SPx_{2n}\}$ converge to Pz.

We claim that $Pz = z$.

Let if possible $Pz \neq z$.

then the inequality

$$d\ (PPx_{2n}, Qx_{2n+1}) < max\ \{d\ (SPx_{2n}, Tx_{2n+1}), d\ (PPx_{2n}, SPx_{2n}), d\ (Qx_{2n+1}, Tx_{2n+1})\}$$

implies that

$$d(Pz, z) < d(Pz, z), \text{ as } n \to \infty$$

which is a contradiction.

Hence $z = Pz$.

Similarly, the inequality

$$d\ (PPx_{2n}, Qz') < max\ \{d\ (SPx_{2n}, Tz'), d\ (PPx_{2n}, SPx_{2n}), d\ (Qz', Tz')\}$$

implies $z = Qz'$.

Further, we get

$$Tz = TQz' = QTz' = Qz.$$

Now the inequality

$$d\ (Px_{2n}, Qz) < max\{d(Sx_{2n}, Tz), d\ (Sx_{2n}, Px_{2n}), d\ (Qz, Tz)\}$$

on letting $n \to \infty$ and in view of (4.2.4) yields $z = Qz = Tz$.

Since the range of S contains the range of Q, therefore there exists a point z'' in X such that $z = Qz = Sz''$. Then

$$\begin{aligned} d\ (Pz'', z) &= d\ (Pz'', Qz) \\ &< max\ \{d\ (Sz'', Tz), d\ (Pz'', Sz''), d\ (Qz, Tz)\} \end{aligned}$$

implies $z = Pz'' = Sz''$.

Since P and S commute therefore, we get $PSz'' = Sz = Pz = z$.

Thus, we have proved that z is again a common fixed point of P, Q, S and T.

Similarly we can show that z is again a common fixed point of P, Q, S and T if Q is continuous instead of P.

Uniqueness of fixed point:

Let us assume that w is another common fixed point of P and S. Then

$$\begin{aligned} d(w, z) &= d(Pw, Qz) \\ &< max\{d(Sw, Tz), d(Pw, Sw), d(Qz, Tz)\} \\ &= d(w, z) \end{aligned}$$

a contradiction.

Hence, z is the unique common fixed point of P and S.

Similarly, z is the unique common fixed point of Q and T.

This completes the proof of the theorem.

COROLLARY [9] - *Let P and S be commuting mappings and Q and T be commuting mappings of a complete metric space (X, d) into itself satisfying the conditions:*

Given $\varepsilon > 0, \exists$ *a* $\delta(\varepsilon) > 0, \delta(\varepsilon)$ *being a nondecreasing function of* ε, *such that* $\forall\, x, y$ *in* X

$$\varepsilon \leq max\{d(Sx, Ty), d(Px, Sx), d(Qy, Ty,), \frac{1}{2}d(Px, Ty), \frac{1}{2}d(Sx, Qy)\} < \varepsilon + \delta$$

$$\Rightarrow d(Px, Qy) < \varepsilon \qquad \text{... (4.2.8)}$$

$$Px = Qy \text{ whenever } Px = Sx, Qy = Ty. \qquad \text{... (4.2.9)}$$

If the range of T contains the range of P and the range of S contains the range of Q and if one of P, Q, S and T is continuous then P, Q, S and T have a unique common fixed point z. Further, z is the unique common fixed point of P and S and of Q and T.

PROOF: For all x, y in X we have

$$\frac{1}{2} d\ (Sx, Qy) \leq \frac{1}{2} [d\ (Sx, Px) + d\ (Px, Qy)]$$

and

$$\frac{1}{2} d\ (Px, Ty) \leq \frac{1}{2} [d\ (Px, Sx) + d\ (Qy, Ty)].$$

Therefore condition (4.2.8) is equivalent to

$$\varepsilon \leq max\{d\ (Sx, Ty), d\ (Px, Sx), d\ (Qy, Ty)\} < \varepsilon + \delta$$

$$\Rightarrow d\ (Px, Qy) < \ \varepsilon.$$

The conditions of the main theorem are satisfied and therefore the result of the corollary follows.

Remark [9]: In the main theorem if we take $P = Q$, $S = T$ identity mapping and

$$\max\{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\} = d(Sx, Ty)$$

then the assumption of non-decreasing character of $\delta(\varepsilon)$ can be dropped and we obtain the Theorem of Meir and Keeler (3.3.1) discussed in chapter III as a special case of our theorem.

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