Inverse Retrival of Parameters in Inverse Heat Conduction Problems

Thesis submitted in partial fulfillment of the requirements

for the award of the degree of

Masters of Science

in

Mathematics and Computing

Submitted by

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July 2018

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CERTIFICATE

*I hereby certify that the dissertation entitled as "***Inverse Retrival of Parameters in Inverse Heat-Conduction Problems***" in the partial fulfillment of the requirements for the award of degree of Master of Science at School of Mathematics, Thapar Institute of Engineering and Technology, Patiala is an authentic record of my own work carried out under the supervision of Dr. Kavita during the period from July 2017 to July 2018.*

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University/Institute for the award of any degree.

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This to certify that the above statement made by the candidate is right and true to the best of my knowledge.

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ACKNOWLEDGEMENTS

The thesis entitled "**Inverse Retrival of Parameters in Inverse Heat-Conduction Problems** " contains my research work carried out at the School of Mathematics at Thapar Institute of Engineering and Technology in Patiala. First of all, I would like to thank the almighty for granting perseverance. Every project is successful only because of the efforts helping people who have always given their valuable advice to me. I would like to thanks to my supervisor Dr. Kavita, Assistant Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, for her guidance, motivation, immense knowledge and engagement through the learning process of this master thesis.

Besides my supervisor, I am thankful to Dr. Satish Kumar Sharma, Head, SOM, Thapar Institute of Engineering and Technology, Patiala, for providing all the necessary facilities in the department. His support and sincere attitude towards department have always been encouraging and helped me pave my way for the dissertation. I would also like to thank Dr. A. K. Lal for his support and motivation.

I owe my gratitude to my family, who were always with me throughout entire process, both by helping me putting pieces together & keeping me harmonious. Without their support and endless blessings, it is impossible for me to complete my education seamlessly. I will be always thankful for their love.

I am also heartfelt to friends for the stimulating discussions, for the times we work together before deadlines, and for all the moments we have fun together. I greatly value their friendship and deeply appreciate their belief in me. Thank you.

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ABSTRACT

In this thesis the direct problem is related to heat conduction problem which is used to determine the temperature distribution from the intial boundary conditions and intial temperature which leads to the well-posed problems. Generally, it is impossible to specify the intial boundary conditions and intial temperature in many situations so inverse problems arrived. Here, in this thesis we used two techniques for solving the Inverse Heat Conduction Problems.

First technique is the Singular Value Decomposition where decomposition of the matrices is done and also repesented the Truncated Singular Value Decomposition but this method has lots of limitations and also does not provide the stable solution of the problem. This problem is overcome by using Tikhonov regularization method.

Second method is the regularization of the solution to the inverse heat conduction problems in discrete fourier fixed domain and given only boundary condition which is also known to be as modified Tikhonov regularization method which leads to more stable and accurate solution. We have also represented the modified regularization method to solve the inverse heat conduction problem with the only boundary conditions value in the bounded domain where the boundary value is given which is x=0. The soluton soughted in the interval $x \in (0,1]$. This method of modified regularization method is introduced in order to recover the stability of the solution. The order optimal error is estimated between the exact solution and approximate solution.

In this thesis also represented the integral solution of the inverse heat conduction problems which involves the calculation of the surface heat flux and temperature from transient, measured temperatures at an interior point of the thermally conducting bodies. The inverse heat conduction problems is a mathematically improper ill -posed problems which means a small error in the interior of data induce a large error in the surface heat flux solutions and heat surface temperature.

The steps involved to solve the inverse heat conduction problems are to derive the temperature & heat fluxes of the body by changing the temperatures inside the solid. The literature reviews presented in this thesis discussed about the one-dimensional inverse heat conduction problems. The various methods prooves are very much effective and useful when a measurement of temperature and heat flux(directly) is very much tough, in several working conditions.

KEY WORDS

Inverse problems, Well-posed problems, Ill-posed problems, Inverse heat conduction problems, Singular value decomposition, Truncated singualr value decomposition, Tikhonov regularization method, Integral solution of inverse heat conduction problems.

Contents

Chapter 1

Introduction

1.1 What are Inverse Problems?

Inverse problem is the process which involves the calculation of the observed set of data from the factor which produces it in the casual manner.
Basically, its aim is to obtain the causes whereas on the other hand in the forward problem it begins with causes and then aim is to obtain the result.

Example:- Calculation of the image in the computer tomography, reconstruction of the source in acoustics & calculation of the density of Earth of its gravity field from its measurement. The terms inverse problems and ill-posed problems have been steadily and are very much popular now a days in the field of modern science .

Applications of Inverse Problems:-
1. Radio-astronomical imaging, image analysis.
2. Sesmic exploration, seismic tomography.
3. X-ray tomography, ultrasound tomography, laser tomography.
4. The inverse problem of potential theory.
5. The use of magneto-cardiography and electrocardiography.
6. Deconvolution, Reconstruction of truncated signals, Fredholm integral equations.

The reason behind that solutions of the inverse problems explains the necessary properties of

the media which under observation, such properties & location of in-homogeneities in inaccessible areas etc.
There are many refrences related to inverse problems. Some of the refrences are the textbook by Engl which provides the topic but mathematically it is quite formal. In this it provides a discussion of the of noise which differentiate between the inverse problem and the forward problem.

Example of Inverse Problems:-
1. Fredholm Integral :-
Let us take the Fredholm Integral Equation of Ist kind:

$$d(z) = \int_p^q f(z,w)n(w)dw, \tag{1.1.1}$$

where, f is a operator on the Spaces (Banach) such as $\mathbb{L}^\infty$ space, $\mathbb{L}^P$ space etc. If mapping is one-one then it is not necessarily that its inverse will not be continious. If d is an ill-posed component, then the minute error in given data gets amplified in solution n.
2. Deconvolution:-
It is also the example of the inverse problem of image deblurring known as the deconvolution problem (in plane).
The Forward problem known as Convolution problem while the Inverse problem here is known to be as deconvolution problems.
Let the Integral Equation of Fredholm of first type:-

$$d(z) = \int K(z,w)f(w)dw; \quad z \in \mathbb{R}^2, \tag{1.1.2}$$

here, K=Kernel known as Convulation Kernel, $z = (z_1, z_2)$ and $w = (w_1, w_2)$. The aim is to reconstruct the original image$f(w)$ and corresponding blurred image $d(z)$.

Applications:-
1. Used in optics.
2. Used in procession of the signal & image.
3. Used in radar, procession of natural language.
4. It is also has wide use in computer vision and astronomy etc.
In order to study the problems of this types we have to show their are large number of problems of many branches of classical mathematics which can be differentiate as inverse & ill-posed.

There are two types of theory :-
(a) Forward Theory
(b) Inverse Theory

(a) Forward Theory:

The process of predicting data based on mathematical structure with the given parameters.

parameters$\rightarrow$ model $\rightarrow$ predicted data

The process of the predection of the mathematical data in which we study how the sesmic waves can travel and also to get the time taken to travel (t) for the given set of parameters model which we are going to choose in particular case in the forward problems. Let the thickness of each of the layer is assumed to be known for some reason. Only then the M velocities are considered to be as model parameters. Thus, in forward problem the motive is to find out the output of the given system of parameters and to find out which input has led to this output. It tends to cause-effect sequences.

(b) Inverse Theory:

The process of estimation of the values of set of parameters of assumed model based on observation.
model$\rightarrow$ data $\rightarrow$ predicted (or estimated model parameters)

Goals are as follows:
1. How sensitive is the solution to small changes.
2. Estimates uncertainties in the model parameters.
3. Bounds on range of model parameters.
4. Obtain set of model parameters.

There are two basic branches of inverse theory:
(i) Reconstruction Problem:
Given the system of parameters and the output of the system, find out which input has been

leads to output.
(ii) Identifcation Problem:
Given the system of input and output, determine the system of parameters which are in the relation with input and with output as well.

There are two types of Inverse problems:
(a) Well-posed Problems
(b) Ill-posed Problems

(a) Well-posed Problems:

The mathematical terms is said to be well-posed problem according to the definition given by Jacques Hadamard if it follows some of the properties given below:

1. The solution should be unique always.
2. With the continious changes in the initial conditions behaviour of solution changes.
3. A solution exists.

Well-posed problem examples :-

It includes the heat equation with intially specified conditions and Dirichlet problem for Laplace's equation. These problems are also known to be as the natural problems in which it has physical processes which are modelled by these problems. The Problems which are not well-posed according to the defination are known to be as ill-posed problems. Generally maximum time the inverse problems are the ill-posed problems.

Example:-
1. When previous distribution of temperature is deduced by using the data at final level.
2. The inverse heat equation & solution very much sensitive with the change in final data being the not well-posed.

In order to achieve the solution numerically the continuum models should be discretized necessarily. And with the refrence to the initial conditions the solution might be continious & can also solved with errors in the data.

Their is better chance of solution on a computer by using the stable algo if the problem is well-posed. While if it is not well-posed, then their is requirement to re-formulate it for numerically treatment. The most usual method is of the regularization which used for stablity.

(b) Ill posed Problems:

The problems which are not well-posed according to defination of Hadamard are known to be ill-posed problems. Inverse problems are generally the ill-posed problems. There could be a mean number of things. It could be that your problem is completely nonsensical, by using the undefined terms that make no sense, like triangular fraction or you are asking about mathematical objects that form an empty set which don't exist, like all numbers that are both odd and even or you ask to derive a statement from a set of contradictory axioms. You could also be looking at an ill-formed expression, like asking what's the truth value of a proposition with unbound logical variable or missing some required part, like x - missing the actual proposition or you can be applying a function to a value outside of its domain, as in the real number system.
For example:-
a) The deduction of the temperature (previous) distribution from the results obtained finally.
b) Inverse heat equation.
If any one of the above conditions is violated then it is called to be ill-posed or it is not the well posed problem.

As, analytical methods can't be applied for finding solution of most of the differential equations, we go with numerical techniques. Numerical techniques may converge to a exact solution but there are several parameters like consistency, convergence, stability etc. to choose that particular technique for given differential equation. Also when we apply numerical methods, we also want less computations. Orthogonal functions have played a very significant role in deciding numerical methods with less computations to solve the differential equations & getting the exact solution of differential equations has always been a major task.

Chapter 2

Inverse Heat Conduction problems: Literature Review

Taler and Ziama[1] presents the space-marching method. Temprature sensor located away from surface as here which is uncovered then temprature calculated gets delayed and it is damped in comparision with change in time which is occured as the covered surfaces. This thesis represents the space marching procedure for multi-dimensional problems. In order to get higher accuracy in comparision to the problems of control volumes. It is not possible to examine accuretely intial solution as the calculated time derivatives are less accurate of the measured temprature. The importance of this method is that, it does not needs no knowledge related intial distribution of temprature. The output of inverse problem represented here is dependable on the intial temprature distribution but it is in the direct region.

Colaco[2] presents the basic concepts of the inverse and optimization problems. Here, loss and profits of each one of the two are discussed and a hybrid techiniques are also introduced. Deterministic Minimization technique are infinite. The inverse problems in heat conduction are presented and their application of technique of inverse are also discussed here.

Zhou et al[3] presents the Conjugate-gradient method. It solved in a way that heat flux on the front-surface is unknown function which is to be recovered and the temperature of the front-surface calculated by inverse heat conduction problem. To get better heat flux and temperature at the heated surface (front) of object with thermo physical properties at the back surface which is opposite side of the heated surface. Thus shows that the heat flux and temperature distribution on the front surface can br reconstructed with high accuracy by applying the inverse algorithm. The numerical results presents in this study demeonstrate excellent way that the proposed approach is a numerical algorithm, with temperature-dependent thermo physical properties. Investigations are carried out for the reduction in the number of sensors of the heat flux which requires on the back surface. The effect of the uncertainities in thermal prperties on inverse solutions are also verified. Conjugate gradient method algorithm is used to get better temprature at front surface and heat flux of 3-d object with temprature dependent thermal physical properties.

Alhama[4] presented a variant method applied on sequential function specification method. To determine different types of incident Heat fluxes. In order to do this, one network is i.e, it produces a piece-wise time dependent function was needed in conjuction. The solution take the form of the piece-wise linear function for each case. These method accuracy and effectiveness in both the exact and error affects the input tempratures. The network model required for numerical techniques is simple because very few devices (electrical) required for their implementation. Their are several tests to test the possibilities of those method like constant, triangular etc. are stimulated and solved. Their are also number of terms, effect of the parameters random error functions on results are also presented.

Beck[5] presents a method which could be convinient for heat a composite body with specific heat and thermal conductivity with time-variable components. It shows tha the good results can be obtained in some situations, discrete methods are generally not able to handle the more complicated problems without some accuracy. Discrete approximation by difference infinitily was able to applied to non-linear problems, but this method is successful in very limited cases as it had ill-possed mass. The objective of this thesis was to derive the method using the non-linear estimation so that it could demonstrate the applicability of non-linear estimation. So, that in order to determine the time dependent quantities at the boundary. This thesis has objective is to present a method which is suitable for use with digital computer for heating composite body. This method was the most successful and consistent approch in use. It is based in part upon the concept of the technique i.e, is used for solving inverse problems called as non-linear estimation.

Monde[6] presents the method which first approximates the temprature data. The resultant terms have an objective temprature and flux was explicitely gets in the form of Power series of time. This procedure used Laplace transform technique and it was for the one dimensional problems only. The analytical way presented here for heat conduction problems. Their is one important difference between the former methods is that to employ equation expressed by a half polynomial series of time to approximate given values. In this thesis the calculations of the result became easy and quick as it can be obtained explicitely so that no iterative way is needed. Overall, this thesis presents a theoretical method .

Weber[7] presented for same problem representing geometries, slab, spherical, cylinderical, as well as temperature and constant dependent physical properties. Here, he developed a complete formulation of the problem clearly and in addition to a better knowledge and concerning with the behaviour of this new stable and problem, higher order methods are also available.

The detailed analysis of the ill-posed problems was solved here. In this thesis, their is complete clear representation of reformulation procedure. Their is representation of the results of the problem analysis and the solution of the ill-posed inverse heat conduction problem.
The space marching method utilizes the exact matching of temperature (evaluated) with the temperature (at location) while the temperature are obtained experimentally. This procedure is naturally incorporated in the future time analysis. This procedure is very much sensitive to the error measurement. In this thesis weber represented the result analysis and the output of the ill-posed inverse heat conduction problems.

Taler[8],[9] presented a semi-numericaltechnique for the solution of the inverse heat conduction problems in composite bodies and in homogeneous. The solution does not requires both the whole temperature time at the temperature sensor locations and the intial temperature distribution in the body.
In this thesis it also presented accurate and simple method without the sequential step in the forward time and in the addition to it is not necessary to solve all the nodal temperatures at each of the time steps. The time distribution and the surface heat flux can be calculated at any interval of time at anytime without knowing of intial temperature distribution.
The aim was to represent a technique for the accurate and simple evaluation of the time-

changing heat transfer coeffcient which is given in an exact temperature of the body which is selected at the point of beneath to the surface of the body. The thesis describes the unified procedures mathematically of the transient method for the measurement of surface heat transfer rates.

The temperature interior calculations are changedd to a local heat transfer coefficient instateneously and by solving the inverse heat conduction problems for the gauge. The effect of the measurement of the interior temperatures was removed or changed by the cubic spine smoothing of the digital filtering of the interior temprature raw prior to use it in the analysis of the heat conduction.

Chapter 3

Techniques For Solving Inverse Problems

The heat conduction problem involves in determining the temperature distribution and heat flux in the body from transient measured. The direct calculation of flux or temperature change on surface is nearly not possible to measure so predictions will help dependencies of the solution of inverse heat conduction problem. Procedure to solve the inverse conduction problem is important in order to determine the unknown heat flux and surface temperature which are generally measured as the function of time and space, under various condtions.

In evaluation of the new materials, it is a times important to determine the surface temperature from a temperature history which is measured at some location inside the body. Also in the distinction of the transient heat conduction or diffussion conventional problems this is known to be as the inverse problems.

3.1 Singular Value Decomposition

Consider the transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$Ax = b. \tag{3.1.1}$$

We have given $m \times n$ matrix function A(t) , such that $m >= n$. The form of A(t) is given below:-

$$A(t) = U(t)S(t)V^T(t), \tag{3.1.2}$$

where, A is an $m \times n$ matrix, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ where U = $m \times m$ orthogonal matrix, V is an $n \times n$ matrix and S = $m \times n$ diagonal matrix with diagonal elements in increasing order such that

$\sigma_i \geq \sigma_{i+1} \geq 0$ which are all functions of t and are called the singular values. The dependence on t will not be shows not clearly, it should be clear from only context. We assumed that the function is analytic where more derivation is independent on this. The columns of U are left u_i and V are called to be right v_i singular vectors. We can perform the singular value decomposition by eigen value decomposition of AA^T and A^TA. As both AA^T and A^TA are hermitian and positive semi-definite which ensures real non-negative eigenvalues, thus $\sigma \geq 0$. We can write equation (3.1.2) which singular value decomposition of A which is written as follows:-

$$Ax = US(V^T x) \tag{3.1.3}$$

$$\Rightarrow Ax = \sum_{i=1}^{min(m,n)} (v_i^T x)\sigma_i u_i \tag{3.1.4}$$

$$\Rightarrow Ax = b \tag{3.1.5}$$

$$\Rightarrow x = A^{-1}b \tag{3.1.6}$$

$$\Rightarrow x = VS^{-1}U^T b \tag{3.1.7}$$

$$\Rightarrow Ax = \sum_{i=1}^{min(m,n)} (\frac{1}{\sigma_i} u_i^T b)v_i. \tag{3.1.8}$$

The equation (3.1.8) represents the inverse of the matrix A using the singular value decomposition expansion.

Truncated Singular Value Decomposition Method :-

We consider the SVD of operator A:

$$A = \sum_{l=1}^{s} \lambda_l u_l v_l^T, \tag{3.1.9}$$

This TSVD, method is depend on the survey that for greater singular values of A, the ingredients of the again construction along the corresponding singular vector is well-calculated by data, but the other ingredients are not properly determined. An integer $k \leq n$ is selected for which the singular values are imagined to be significant and the solution vector $\tilde{x}$ is selected such hat:

$$v_i^T \tilde{x} = \sum_{i}^{k} \frac{u_l^T b}{\lambda_l}, \tag{3.1.10}$$

for $l = 1, 2......., k$

Let represented by V_k then $n \times (n-k)$ matrix column is $\{v_l\}$ for $l = k+1, ..., n$ so that V_k is the matrix whose columns span the effective null space of A. The components along the remaining singular-vector directions $\{v_l\}$, selected so total solution vector $\tilde{x}$ satisfies some criterion of optimality. The reconstruction which has zero projection in the effective null space is,

$$x' = \sum_{l=1}^{k} \frac{(u_l^T b)}{\lambda_l} v_l, \tag{3.1.11}$$

The reconstruction $\tilde{x}$ = sum of x' and a vector of columns of V_k. Which given by,

$$\tilde{x} = x' + \sum_{l=k+1}^{n} d_l v_l = x' + V_k d, \tag{3.1.12}$$

for a n- k element column vector d. The solution of reconstruction is,

$$\Omega(\tilde{x}) = ||L(\tilde{x} - x^{\infty}||^2 = ||L(\tilde{x} + V_k d - x^{\infty})||^2 = ||L(x' - x^{\infty}) + (LV_k)d||^2. \tag{3.1.13}$$

The vector d minimizes the semi-norm ,

$$d = -(LV_k) * L(x' - x^{\infty}), \tag{3.1.14}$$

We define $A^* = (A^T A)^{-1} A^T$. Then explict expression for the truncated method is,

$$\tilde{x} = x' - V_k(LV_k) * L(x' - x^{\infty}), \tag{3.1.15}$$

where, x' is given by equation(3.1.15) above. When L is taken to be the identity matrix then called modified truncated singular value decomposition. Points of Truncated Singular Value Decomposition:-

1. It requires a scalar (k) specifying.
2. It removes the high frequency modes.
3. This method extends to compact operators, Hilbert spaces.
4. It needs singular-value decmposition.

The only ways to obtain the stable solution are Iterative regularization, Tikhonov regullarization, Generalised Tikhonov regularization.

3.2 The Tikhonov Regularization Method

Let us form the weighted sum of C(x) and $\Omega(x)$ using a weightining factor α^2 and find the image $\tilde{x}$ & the aim of the problem is to minimize the given sum, i.e,

$$\tilde{x}_\alpha = arg \quad \min\{\alpha^2||L(x-x)||^2 + ||b-Ax||^2\}, \tag{3.2.1}$$

When their is great amount of regularization, we can effectively and easily ignore the data or any type of noise completely and our aim is only to attempt to minimize the output semi-norm which is feasible by selecting it in defaul toutput manner. The weighting factor α^2 parameterizes the complete family of solutions by this term. The α denotes the regularization parameter. The impact of the term C(x) is very much non-effective in comparision to $\Omega(x)$ only if the regularization parameter is very large and we come to conclusion that $lim_{\alpha\to\infty}\tilde{x}_\alpha = x^\infty$. While if α is small, the weighting of semi-norm is small in solution and the value of the misfit at the solution forms more urgent. If α is going to zero, the problem reduces to the least-squares case considered earlier with its extreme sensitivity toerror on the data. Then we set,

$$\frac{\partial}{\partial x_k}\left\{\alpha^2(x-x^\infty)L^T L(x-x^\infty) + (b-Ax)^T(b-Ax)\right\} = 0, \tag{3.2.2}$$

for k=1 to n, which bring to the simultaneously equations,

$$2\alpha^2 L^T L(x-x^\infty) - 2A^T(b-Ax) = 0, \tag{3.2.3}$$

or

$$(\alpha^2 L^T L + A^T A)x = \alpha^2 L^T L x^\infty + A^T b. \tag{3.2.4}$$

By setting, α=0 brings equations to the normal equations related with least square problems. When values of α are non-zero then their is the addition of the term $\alpha^2 L^T L$ on the left-handside of the matrix which changes the eigenvalues(eigenvectors) of $A^T T$ alone. As long as the term $\alpha^2 L^T T + A^T A$ is non-singular, there always exists the unique solution.

Solution of Tikhonov regularization in the explict form:-

Suppose L to be identity matrix.Tikhonov regularization is analyzed in the explicit form to see how the regularization parameter α effects the solution of the inverse problems. Now, apply the Decomposition method on A in the form

$$A = \sum_{l=1}^{s} \lambda_l u_l v_l^T, \tag{3.2.5}$$

the L.H.S above equation can be written as

$$(\lambda^2 I + A^T A)\tilde{x} = \alpha^2 \sum_{l=1}^{m} \tilde{x}_l v_l + \sum_{l=1}^{s} \lambda_l^2 \tilde{x}_l v_l = \sum_{l=1}^{s} (\alpha^2 + \lambda_l^2)\tilde{x}_l v_l + \alpha^2 \sum_{l=s+1}^{m} x_l^{\infty} v_l, \tag{3.2.6}$$

where, $\tilde{x}_l$=$v_l^T \tilde{x}$

and R.H.S of (3.2.5) can be written as

$$\alpha^2 x^{\tilde{\infty}} + A^T b = \alpha^2 \sum_{l=1}^{m} x_l^{\infty} v_l + \sum_{l=1}^{s} \lambda_l b_l v_l = \sum_{l=1}^{s} \left[\alpha^2 x_l^{\infty} v_l + \lambda_l^2 \frac{b_l}{\lambda_l} \right] v_l + \alpha^2 \sum_{l=s+1}^{m} x_l^{\infty} v_l, \tag{3.2.7}$$

where, $x_l^{\infty} = v_l^T x^{\infty}$, $b_l = u_l^T b$ and we here use that $I = \sum_{l=1}^{n} v_l v_l^T$ since the $v_{l}{}_{l=1}^{n}$ form the orthogonal basis of $\mathbb{R}^n$ equating (3.2.6) & (3.2.7) we see that,

$$\tilde{x}_l = \begin{cases} \frac{\alpha^2}{\alpha^2+\lambda^2} x_l^{\infty} + \frac{\lambda^2}{\alpha^2+\lambda^2} \frac{b_l}{\lambda_l} & \text{if } l = 1, 2, ...s \\ x_l^{\infty} & \text{if } l = s+1....., n \end{cases}$$

Choosing the regularization parameter:-

The big problem in Tikhonov regularization method is the choice of α which is regularization parameter. The regularization parameter (α) sets the balance between minimizing residual norm and the minimizing roughness.

For the choice of regularization parameter (α) the most convienient tool is the $L-curve$. When the filtering level is very much small, their are many parts which are contained in solution that is divisible by singular value i.e, is small in nature & it is corresponding the regularization(indequate). However the solution does not changes in large amount in the horizontal part, it is so because as the regularization parameter changes whereas on the other side, the semi-norm $||L(x - x^{\infty})||$ in the solution is too senstive term of the regularization (α) parameter in the solution of problem it is so beacuse, solution is goes under a big change corresponding to the change with the choice of the regularization parameter (α) in an attempt to fit data in a better way.

Chapter 4

Matlab coding of methods used for solving inverse problems

Let us consider an inverse problem and solve this problem by matlab coding by Singular value decomposition and Tikhonov regularization method.
Problem:

$$f_{true} = \begin{cases} 0.75, 0.1 < x < 0.25, \\ 0.25, 0.3 < x < 0.32, \\ cos^4(2/pix), 0.5 < x < 1, \\ 0, otherwise \end{cases}$$

The code of method for the true solution of above inverse problem is given below:-

4.1 Matlab coding of Singular value decomposition method

```
clc;
clear all;
n=4;
h=1/n;
ita=1/10;
g=0.05; %gamma
C=1/(g*(sqrt(2*pi)));
```

```
k=zeros(n);
for i=1:n
    for j=1:n
        k(i,j)=h*C*(exp(-(((i-j)*h)^2)/(2*g^2)));
    end
end
f=[0.75;0.25;cos(2*pi*0.8)^4;0];
d=(k*f)+ita;
kt=k'; %transpose
K=k*kt;
M=kt*k;
[U,S,V]=svd(k);
V'
R=U*S*V'
R

Output:

ans =

   -0.3717   -0.6015   -0.6015   -0.3717
    0.6015    0.3717   -0.3717   -0.6015
   -0.6015    0.3717    0.3717   -0.6015
   -0.3717    0.6015   -0.6015    0.3717


R =

    1.9947    0.0000    0.0000         0
    0.0000    1.9947    0.0000    0.0000
   -0.0000    0.0000    1.9947    0.0000
    0.0000    0.0000    0.0000    1.9947
```

4.2 Matlab coding of Tikhonov regularization method

```
clear variables
close all
clc
h=0.25;
gamma=0.05;
pi=22/7;
c=1/(gamma*sqrt(2*pi));
for i=1:4
  for j=1:4
K(i,j)=h*c*(exp(-(((i-j)*h)^2)/(2*(gamma)^2)));
  end
end
K
itta=0.01;
ftrue=[0.75;0.25;cos(2*pi*0.8)^4;0];
d=(K*ftrue)+itta;
d
alpha=10^(-2);
I=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1]
P=transpose(K);
P
falpha=(((P*K)+ (alpha*I))^-1)*P*d
Output:

K=
        1.9943 0.0000 0.0000 0.0000
         0.0000 1.9943 0.0000 0.0000
         0.0000 0.0000 1.9943 0.0000
         0.0000 0.0000 0.0000 1.9943
```

```
d =
         1.5057
         0.5086
         1.6373
         0.0100


 I =
        1 0 0 0
        0 1 0 0
        0 0 1 0
        0 0 0 1

P =
        1.9943 0.0000 0.0000 0.0000
        0.0000 1.9943 0.0000 0.0000
        0.0000 0.0000 1.9943 0.0000
        0.0000 0.0000 0.0000 1.9943

falphा =
                0.7531
                0.2544
                0.8189
                0.0050
```

Chapter 5

Method for solving Inverse Heat Conduction Problems

5.1 Singular Value Decomposition

Let coefficient matrix B i.e., $B = (\phi(x_i - x_j, t_i - t_j)$ is ill-conditioned and it shows that output is sensitive in case of numerical error which is given follwing $\tilde{b}$

$$\tilde{b} = \begin{pmatrix} \tilde{Y} \\ h(x_i, t_i) \\ g_1(x_i, t_i) \\ g_2(x_i, t_i) \end{pmatrix}$$

& number of collocation points. We notice that their is increase in number of condition of matrix B w. r. t the total number of collocation points.

Singular value decomposition is very much helpful in case of direct problems but it is not possible to give the accuracy and the stability to the solution of the system provided. Their is huge amount of regularization techniques which have been developed in order to solve the ill-conditioned problem. Hence, we conclude i.e, it is very much beneficial to use singular value decomposition.

Let N = a + b + c + d. The singular value decomposition of the $N \times N$ matrix B whose decomposition is of the form,

$$B = W \sum V^T = \sum_{i}^{N} w_i \sigma_i v_i^T, \tag{5.1.1}$$

with $W = (w_1, w_2, ...w_N)$ and $V = (v_1, v_2, ..., v_N)$ which satisfies that $W^T W = V^T V = I_N$. and here, T shows matrix transposition. We know that $\sum = diag(\sigma_1, \sigma_2, ...\sigma_N)$ has non-negative diagonal elements which satisfies the inequality,

$$\sigma_1 \geq \sigma_2 \geq ...... \geq \sigma_N \geq 0. \tag{5.1.2}$$

The values σ_i denotes the singular values of A and w_i & v_i vectors known to be left singular vector & right singular vectors of matrix A, respectively. Their is huge decrease in singular values in (5.1.2) when their is the less reconstruction reliablity for a given error level. Whereas, in order to achieve a better remaking when the singular values decreases continously, then huge large signal-to-error ratio in input is needed.

The singular values decompose continously towards to zero of matrix A & the ratio between smallest & highest non-zero singular values is very much large in nature. On the basis of this the singular value decomposition, we can easily know that the for the system (5.1.1) solution is provided by,

$$\tilde{\lambda} = \sum_{i=1}^{N} \frac{w_i^T \tilde{b}}{\sigma_i} v_i. \tag{5.1.3}$$

For the small singular values, such type always brings to reconstruction of the vector $\tilde{\lambda}$ which is bad by nature i.e, bad reconstruction. So, we prefer only to let small singular values which is zero and with the regard of the components along such directions as being free variables which are not calculated by the data.

However, from above arguments we come to the conclusion that the singular value decomposition is not that much great to deal with the inverse problems. Hence, we always prefer to use here Tikhonov regularization method.

5.2 The Tikhonov Regularization Method

Instead of having the direct output for an ill-posed problem $A\tilde{\lambda} = \tilde{b}$

Let, we consider a minimum of a functional,

$$J[\tilde{\lambda}] = ||A\tilde{\lambda} - \tilde{b}||^2 + \alpha^2||\tilde{\lambda} - \tilde{\lambda_0}||^2, \tag{5.2.1}$$

where, α^2 is called the regularization parameter, above equation with $\tilde{\lambda_0}$ which is a known vector and $||.||$= Euclidean norm.

$$A^T(A\tilde{\lambda} - \tilde{b}) + \alpha^2(\tilde{\lambda} - \tilde{\lambda_0}) = 0, \tag{5.2.2}$$

the above equation is obtained by using the functional necessary condition of minimum in (5.2.1) . Hence,

$$\tilde{\lambda} = (A^T A + \alpha^2 I)^{-1}(A^T\tilde{b} + \alpha^2\tilde{\lambda_0}), \tag{5.2.3}$$

taking into account A=$W\sum V^T$ i.e, equation(5.1.1) we get the following form of the functional J & transformated form is:

$$\begin{aligned} J[\tilde{\lambda}] &= ||W\sum V^T\tilde{\lambda} - WW^T\tilde{b}||^2 + \alpha^2||VV^T(\tilde{\lambda} - \tilde{\lambda_0}||^2 \\ &= ||W(\sum y - c)||^2 + \alpha^2||V(y - y_0)||^2 = J[y], \end{aligned} \tag{5.2.4}$$

where, $y = V^T\tilde{\lambda}$, $y_0 = V^T\tilde{\lambda}$ and $c = W^T\tilde{b}$ and by using the properties W^TW=$V^TV = I_N$. Minimization of $J[y]$ functional gives the equation:

$$\sum^T(\sum y - c) + \alpha^2||(y - y_0)||^2 = 0or(\sum^T\sum y + \alpha^2 y) = \sum^T c + \alpha^2 y_0, \tag{5.2.5}$$

Hence,

$$y_i = \frac{\sigma}{\sigma^2 + \alpha^2}c_i + \frac{\alpha^2}{\sigma_i^2 + \alpha^2}y_{0i}, \tag{5.2.6}$$

where, i=1,2,......N
or

$$\tilde{\lambda} = \sum_{i=1}^{N}\left(\frac{\sigma_i}{\sigma_i^2 + \alpha^2}w_i^T\tilde{b}v_i + \frac{\alpha^2}{\sigma^2 + \alpha^2}\tilde{\lambda_0}\right). \tag{5.2.7}$$

If $\tilde{\lambda_0} = 0$ the Tikhonov regularized solution of ($A\tilde{\lambda} = \tilde{b}$) depend on singular value decomposition of the $N \times N$ matrix A can be represents as:

$$\tilde{\lambda_\alpha} = \sum_{i=1}^{N} \frac{\sigma_i}{\sigma_i^2 + \alpha^2} w_i^T \tilde{b} v_i. \tag{5.2.8}$$

The determination of a suitable value of the regularization parameter α^2 is important and is still under intensive research. Recently the L-curve criterion is frequently used to select a better regularization parameter. Define a curve L by

$$L = \left\{ \left(log||\tilde{\lambda_\alpha}||^2 \right), log\left(||A\tilde{(\lambda)}_\alpha - \tilde{b}||^2 \right) \right\}, \tag{5.2.9}$$

a suitable regularzation parameter α^2 is the one near the corner of the L-curve.

Now, we consider an inverse problem with only boundary values in the bounded domain in $0 < x \leq 1$. This problem is ill-posed. Now we the modified Tikhonov method in order to stablise the solution. This is done with the suitable choice of regularization parameter, this is done to achieve an optimal error in between exact solution and the approximate solution.

PROBLEM:-

Let we suppose the Inverse Heat Conduction with the boundary values(only)

$$\begin{gathered} v_t = v_{xx}; \quad 0 < x \leq 1, \quad 0 < t < 2\pi, \\ v(0,t) = f(t); \quad 0 \leq t \leq 2\pi, \\ v_x(0,t) = g(t); \quad 0 \leq t \leq 2\pi, \end{gathered} \tag{5.2.10}$$

where f and g are known.

This problem is highly ill-posed. Our aim is to recover the distribution of temprature $v(x,.)$ between $0 < x \leq 1$ for f and g which is the boundary data. As this problem is highly ill-posed according to Hadamard defination i.e, the solution if exists is independent of the continously given data.

Here, we find the solution of the problem without using the intial value (5.2.10) and obtain the optimal error between exact solution and the approximate solution.

Solution:

For Integral Heat Conduction Problem in (5.2.10) their is need to determine the distribution of temperature $v(x,0)$ for $0 < x \leq 1$ from f and g which is the Cauchy data. Now, we measure the cauchy data f^{δ}, $g^{\delta} \int L^2[0,2\pi]$, i.e, $||f - f^{\delta}|| \leq \delta$, $||g - g^{\delta}|| \leq \delta$

Now, equation (5.2.10) can also be written as in two independent inverse heat conduction problems as follows

$$\begin{aligned} u_t = u_{xx}; \quad 0 < x \leq 1, \quad 0 < t < 2\pi, \\ u(0,t) = f(t); \quad 0 \leq t \leq 2\pi, \\ u_x(0,t) = 0; \quad 0 \leq t \leq 2\pi, \end{aligned} \tag{5.2.11}$$

and

$$\begin{aligned} w_t = w_{xx}; \quad 0 < x \leq 1, \quad 0 < t < 2\pi, \\ w_x(0,t) = g(t); \quad 0 \leq t \leq 2\pi, \\ v_x(0,t) = g(t); \quad 0 \leq t \leq 2\pi, \end{aligned} \tag{5.2.12}$$

$u(x,t)$ be the solution of the equation problem (5.2.11) and $w(x,t)$ be solution of the problem (5.2.12) repectively. Then $v = u + w$ be solution of (5.2.10). This means (5.2.11) and (5.2.12) are splitted form of (5.2.10) and indicates that we need solution of (5.2.11) and (5.2.12) to find the solution of (5.2.10).

Now, applythe technique of separation of variables, so the proper solution of problems (5.2.11) and (5.2.12) are obtained from,

$$u(x,t) = \sum_{m=-\infty}^{\infty} (f(t), e^{imt}) e^{imt} cosh(\sqrt{im}x), \tag{5.2.13}$$

and

$$w(x,t) = \sum_{m=-\infty}^{\infty} \frac{1}{\sqrt{im}} (g(t), e^{imt}) e^{imt} sinh(\sqrt{im}x), \tag{5.2.14}$$

then the exact solution of problem (5.2.10) is given by,

$$v(x,t) = \sum_{m=-\infty}^{\infty} \left[(f(t), e^{imt}) e^{imt} cosh(\sqrt{im}x) + \frac{(g(t), e^{imt})}{\sqrt{im}} e^{imt} sinh(\sqrt{in}x) \right]. \tag{5.2.15}$$

Now, apply the regularization method for error estimation and to obtain stable solution.First we find out the regularization and error estimation for (5.2.11) and then for (5.2.12).
For problem (5.2.11) let us define an operator K : u(x,.) $\rightarrow$ f(.) now rewriting equation (5.2.11) in operator form as:

$$Ku(x,t) = f(t); \quad 0 < x \leq 1, \tag{5.2.16}$$

Now, combine with (5.2.13), we get

$$Ku(x,t) = \sum_{m=-\infty}^{\infty} u(x,t), e^{imt} (cosh(\sqrt{im}x))^{-1} e^{imt}, \tag{5.2.17}$$

where K is an operator with the eigen values

$$k_m = (cosh(\sqrt{im}x))^{-1}. \tag{5.2.18}$$

For distributed data $f^{\delta}(t)$, we use the Tikhonov regularization functional j_{α} has the minimum $v_{\alpha}^{\delta}(x,.)$ that has the unique solution of

$$K^* K v_{\alpha}^{\delta} + \alpha^2 v_{\alpha}^{\delta} = K^* f^{\delta},$$

where, $\alpha \geq 0$ and here K^* represents the adjoint of K we obtain the eigen-values of the operator K^* by using the properties of the inner product

$$\bar{k}_m = \overline{(cosh(\sqrt{im}x))^{-1}}, \tag{5.2.19}$$

where, $h(.)$ symbol denotes the complex conjugate for the function $h(.)$ and combining (5.2.17), (5.2.18), (5.2.19) with (5.2.20) we obtain,

$$\sum_{m=-\infty}^{\infty}(\bar{k}_m k_m+\alpha^2)(v^{\delta}_{alpha}(x,t),e^{imt})e^{imt}=\sum_{m=-\infty}^{+\infty}\bar{k}_m(f^{\delta}(t),e^{imt})e^{imt}, \tag{5.2.20}$$

this gives

$$\alpha^{\delta}(x,t)=\sum_{m=-\infty}^{+\infty}(v^{\delta}_{\alpha}(x,t).e^{imt})e^{imt}=\sum_{m=-\infty}^{+\infty}\frac{\bar{k}_m}{|k_m|^2+\alpha^2}(f^{\delta}(t),e^{imt})e^{imt}$$
$$=\sum_{m=-\infty}^{+\infty}\frac{cosh(sqrtimt)}{1+\alpha^2|cosh(\sqrt{im})|^2}(f^{\delta}(t),e^{imt})e^{imt}. \tag{5.2.21}$$

In order to obtain thenoise estimation between regularization solution and the exact solution we replace this filter with $\frac{1}{1+\alpha^2|cosh(\sqrt{in})|}^2$ and we get,

$$(v^{\delta}_{\alpha}(x,t):=\sum_{m=-\infty}^{+\infty}\frac{cosh(sqrtimt)}{1+\alpha^2|cosh(\sqrt{im})|^2}(f^{\delta}(t),e^{imt})e^{imt}. \tag{5.2.22}$$

Regularization and error estimate for problem (5.2.12):-

For problem (5.2.12), the method Tikhonov includes minimizing the quadratic functional:-

$$||Tw^{\delta}-g^{\delta}||^2+\alpha^2||w^{\delta}||^2, \tag{5.2.23}$$

where, $T: w(x,.)\rightarrow g(.)$ is the forward operator. The Tikhonov regularization functional has the unique $x^{\delta}_{\alpha}(x,.)$ which is the unique solution of the equation which is normal,

$$T^*Tw^{\delta}_{\alpha}+\alpha^2 w^{\delta}_{\alpha}=T^*g^{\delta}, \tag{5.2.24}$$

where, $\alpha>0$. Now, we can obtain the Tikhonov regularization solution of the problem (5.2.10)

$$w^{\delta}_{\alpha}(x,t)=\sum_{n=-\infty}^{+\infty}\frac{t_n^{-1}}{1+\alpha^2|\frac{sinh(\sqrt{in}x)}{\sqrt{in}}|^2}, \tag{5.2.25}$$

and we get the modified regularization of this problem is :

$$w^{\delta,*}_{\alpha}(x,t)=\sum_{n=-\infty}^{+\infty}\frac{t_n^{-1}}{1+\alpha^2|\frac{sinh(\sqrt{in}x)}{\sqrt{in}}|^2}, \tag{5.2.26}$$

the above solution is obtained by replacing the original filter with $\frac{1}{1+\alpha^2|\frac{sinh(\sqrt{in}x)}{\sqrt{in}x}|^2}$.

Result of the problem:

Here, the Inverse Heat Conduction Problem with only Boundary value are solved in a boundary domain. Where the stability condition is given already. We used this method of Modified Tikhonov regularization method in order to obtain a regularized solution of the problem. This technique is effective and suitable with the order optimal error estimation is obtainedwith a suitable choice of regularization parameter.

Chapter 6

The Integral Solution of Inverse Heat Conduction Problems

A new method for solution of the Inverse Heat Conduction Problem is on the basis of hyperbolic approximations. In this method we combines hyperbolic function model with the numerical solution of Second Volterra Integral equation .

Problem: Let linear heat conduction problem is taken. Here we taken the semi-infinite slab along with constant thermal properties & by using dimensionless quantities this is problem is normalized . Mathematically, the problem is explained as follows:

Let, the temprature $v(y,t)$ (unknown) which satisfies the following equations:

$$v_t = v_{yy}; \quad 0 < y < \infty, \quad t > 0 \tag{6.0.1}$$

$$v(1,t) = G(t); \quad t > 0, \tag{6.0.2}$$

& $G(t)$ is the data approximate function.

$$v(y,0) = 0; \qquad 0 \leq y < \infty, \tag{6.0.3}$$

$$-v_y(0,t) = p(t); \qquad t > 0, \tag{6.0.4}$$

where, $v(y,t)$ is unknown and is bounded as $y \to \infty$, $t > 0$ where y denotes the measured distance from the heated surface and t is the time. Now, we replace the equation (6.0.4) by the following data ,

$$v(0,t) = g(t), \qquad t > 0. \tag{6.0.5}$$

Here, the objective of the problems are:-

(a) To determine the surface heat flux $p(t)$

(b) Surface temperature $g(t)$ given and $G(t)$ denotes the interior temprature measurements when y=1.

Now, we use the Volterra Integral Equations of first kind to convert the inverse problems to the equivalent equations,

$$G(t) = \int_0^t q_1(1, t-\tau)p(\tau)d\tau, \tag{6.0.6}$$

$$G(t) = \int_0^t q_2(1, t-\tau)g(\tau)d\tau, \tag{6.0.7}$$

and these unknown heat flux $p(t)$ and the unknown temperature $g(t)$ satisfies (6.0.6) and (6.0.7) where,

$$q_1(1, t-\tau) = \frac{\partial}{\partial t}\left[\frac{2}{\sqrt{\pi}}(t-\tau)^{1/2}exp[-1/4(t-\tau)] - erfc(1/2\sqrt{t-\tau}\right], \tag{6.0.8}$$

and

$$q_2(1, t-\tau) = \frac{1}{2\sqrt{\pi}}(t-\tau)^{-3/2}exp[-1/4(t-\tau)], \tag{6.0.9}$$

In hyperbolic approximation, if $P(y,t)$ and $T(y,t)$ represents the heat flux for unit area and the temperature in a solid respectively in the dimensionless quantities, the heat flow is given by

$$P_y + T_t = 0, \tag{6.0.10}$$

(6.0.10) obtained by using the first law thermodyanamics for one-dimensional heat flow. And

$$P + T_y = 0, \tag{6.0.11}$$

These (6.0.11) is given by the Fourier equation of thermal conductivity for heat flow implicity assuming that the thermal propogation speed is infinite. However, their are several authors have hypothesized that a finite propogation speed must be accounted. The addittion of these two equations from the governing parabolic differential equations for transient heat conduction (6.0.1). For example, Vernotte have proposed the modified Fourier law,

$$\frac{1}{d^2}P_t + P + T_y = 0, \tag{6.0.12}$$

where, d is a non-negative constant and $1/d^2$ can be interpreted as a relaxation time parameter, as $d \to \infty$ equation (6.0.11) becames the Fourier law (6.0.10). Combining equations (6.0.10) and (6.0.11), we get hyperbolic damped wave equation,

$$\frac{1}{d^2}T_{tt} + T_t = Tyy, \tag{6.0.13}$$

with $\frac{dy}{dt} = \pm d$, as its characterstic lines .

In most of the Conduction problems, the effect of finite speed of propogation i.e, d is neglegible in equation(6.0.12). At very low temperatures, the effect became important. In order to stablize Inverse Heat Conduction Problem, the approximate hyperbolic problem is

$$\frac{1}{d^2}u_{tt} + u_t = u_{yy}; \quad 0 < y < \infty, \quad t > 0, \tag{6.0.14}$$

,

$$u(1,t) = G(t); \quad t > 0, \tag{6.0.15}$$

where, $G(t)$=corresponding approximation data.

$$u(y,0) = 0; \quad 0 \le y < \infty, \tag{6.0.16}$$

$$u_t(y,0) = 0; \;\; 0 \le y < \infty, \tag{6.0.17}$$

$$-u_y(0,t) = \phi(t); \;\; t > 0, \tag{6.0.18}$$

where, $u(y,t)$ is unknown and is bounded as y $\to \infty$, $t > 0$. Now, take the Laplace of the equation (6.0.14) with respect to t (time) using (6.0.16) & (6.0.17) and also using (6.0.18) & (6.0.19) we get,

$$L[u(y,t)] = L[\phi(t)](r^2/d^2 + r)^{-1/2} exp[-(r^2/d^2 + ry)^{1/2}], \tag{6.0.19}$$

by setting,

$$\alpha = [(r + d^2)^2 - d^4/4]^1/2, \tag{6.0.20}$$

we get,

$$L[u(y,t)] = L[\phi(t)] exp[-y\alpha/d]/(\alpha/d), \tag{6.0.21}$$

Using the Inverse Laplace transform in equation (6.0.21), we get,

$$u(y,t) = d\phi(t)^{*}exp[-d^2t/2]I_0\left[\frac{d^2}{2}(t^2-(y/d)^2)^{1/2}\right], \tag{6.0.22}$$

where I_0 = Modified Bessel functions and is of first kind Volterra integral equation of $\phi(t)$ and * denotes the convolution operation.

$$u(y,t) = \begin{cases} d\int_0^{t-y/d}\phi(\tau)exp[-(t-d)d^2]I_0[\frac{d^2}{2}((t-\tau)^2-(y/d))^{1/2}], & \text{if t > y/d} \\ 0, & \text{otherwise} \end{cases} \tag{6.0.23}$$

Now using equation (6.0.12), gives the heat flux for direct integration we get,

$$P(y,t) = d^2\int_0^t(\frac{-\partial u}{\partial y}(y,\tau))exp[-(t-\tau)d^2]d\tau. \tag{6.0.24}$$

If we use (6.0.25) at point y=0 to boundary conditions which are prescribed in (6.0.18) with $P(t) = P(0,t)$ we obtain,

$$\begin{aligned} u(y,t) = \frac{1}{d}P(t-y/d)exp[-yd/2] + \frac{d}{2}\int_0^{t-y/d} & P(\tau)exp[-(t-\tau)d^2/2] \\ \times I_0[\frac{d^2}{2}((t-\tau)^2-(y-d)^2)^{1/2} & + \frac{t-\tau}{((t-\tau)^2-(y/d)^2)^{1/2}} \\ & \times I_1\left[\frac{d^2}{2}((t-\tau)^2-(y-d)^2)^{1/2}\right]d\tau, \quad \text{if t>y/d} \end{aligned} \tag{6.0.25}$$

0,otherwise.

Where, I_1 represents the modified Bessel equations of the first kind of order one. In the same way, if we use the boundary conditions prescribed in equation (6.0.18) and is of the form

$$u(0,t) = \phi(t), \tag{6.0.26}$$

Then, the resultant integral equation for the hyperbolic surface temperature which is unknown and is given by

$$\begin{aligned} u(y,t) = \phi(t-y/d)exp[-dy/2] + \frac{dy}{2}\int_0^{t-y/d} & \phi(\tau)exp[-d^2(t-\tau)/2] \\ \times \frac{1}{[(t-\tau)^2-(y/d)^2]^{1/2}}I_1[\frac{d^2}{2}((t-\tau)^2 & -(y/d)^2)^{1/2}]d\tau, \quad \text{if t>0} \end{aligned} \tag{6.0.27}$$

0,otherwise.

The hyperbolic surface heat flux $P(t)$ and the hyperbolic surface tempertaure $\phi(t)$ satisfies linear Volterra Integral equations of second type as shown in equations (6.0.25) and (6.0.27) and leads to well posed problem.

To use the Fourier Integral analysis we extend $G(t).g(t).\tilde{G}(t)$, $p(t)$, $\phi(t)$ and $P(t)$ to real t-axis and by assuming them for $t < 0$ and also that all the functions including all L^2 function in$(-\infty,\infty)$ and also use the corresponding L^2 norm which is used to measure errors. If,

$$f(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)exp(-i\omega t)dt, \tag{6.0.28}$$

where, (6.0.28) denotes the fourier transform of $h(t)$ and the Parseval identity corresponding to it is given as,

$$||f||^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |f(\omega)|^2 d\omega = ||f||^2, \tag{6.0.29}$$

and we also assumed that the data error satisfies,

$$||G(t) - \bar{G}(t)|| \leq \varepsilon, \tag{6.0.30}$$

where ε is called as the positive upper bound. The inverse problem for the surface temperature including the inverse heat conduction hyperbolic (6.0.14) along with boundary condition (6.0.26) and the parabolic inverse heat conduction problem (6.0.1) with (6.0.5). Let us take the Fourier transform of (6.0.14) with (6.0.18) as its conditions which is replaced by equation (6.0.26) and we get,

$$\tilde{\phi_d}(w) = \tilde{G}(w)exp\left[\sqrt{\frac{|w|}{2}} I(w,d)\right], \tag{6.0.31}$$

where,

$$I(w,d) = \left\{(\frac{w^2}{d^4}+1) - \frac{|w|}{2}\right\}^{1/2} + i\sigma\left\{(\frac{w^2}{d^4}+1) + \frac{|w|}{2}\right\}^{1/2}, \tag{6.0.32}$$

and $i = \sqrt{-1}$, $\sigma = sign(w)$.

The Fourier transform of the hyperbolic surface temperature corresponding along with noisy data F(t) denoted as $\tilde{\phi}_d(w)$ which is given in equation below:

$$\tilde{\phi}_d(w) = \tilde{G}(w)exp\left[\sqrt{\frac{|w|}{2}} I(w,d)\right], \tag{6.0.33}$$

and for the fixed value of d,

$$|exp\left[\sqrt{\frac{|w|}{2}}I(w,d)\right]| \leq exp(d/2), \tag{6.0.34}$$

subtracting equation (6.0.31)-(6.0.33) and taking the norms and using (6.0.29), (6.0.30) and (6.0.34) condition we obtain,

$$||\phi - \phi_d|| \leq \varepsilon exp(d/2), \tag{6.0.35}$$

which shows that the hyperbolic inverse heat conduction problem is well-posed problem.
And if noise is absent in this then by using the exact data F(t), then their will be possiblity to compare the hyperbolic temperature $\phi(t)$ and the parabolic surface temperature $g(t)$,

$$||g - \phi|| = \frac{\beta}{d}, \tag{6.0.36}$$

where, β=the constant independent of d.

$$||g - \phi_d|| \leq \frac{\beta}{d} + \varepsilon e^{d/2}, \tag{6.0.37}$$

The above equation is obtained as $d \to \infty$, the solution of the hyperbolic problem approaches to the solution of the parabolic inverse heat conduction problem. Now combining the equation (6.0.35) and (6.0.36) and also using the triangular inequality we get the above equation (6.0.37)

This shows that their is one and only one optional choice of parameter d, $\bar{d}$ which minimizes the function and there is unique optimility as shown in last equation.

$$E(d) = \frac{\beta}{d} + \varepsilon e^{d/2}, \tag{6.0.38}$$

In order to emphasize its dependency on d, the surface temperature (hyperbolic) $\phi_d(t)$ represented by $\phi_d^d(t)$. In general, α is not known. The rate of change of $E(d)$ for $d < \tilde{d}$ is smaller than $d > \tilde{d}$. Thus, from equation (6.0.33) and (6.0.34) we get,

$$||\phi_d^d|| \leq ||\bar{G}||e^{d/2}, \tag{6.0.39}$$

This shows that monotone increasing function of d, $||\phi_d^d||$ is over-sensitive to the change in d for the function $E(d)$ when $d > \tilde{d}$. The choice of parameter creterion became important for the solution of the problem, in the existence of errors in data.

Numerical Way For Discretized Problem:-

Let, $\gamma = 1/d$ and $\tau = t - \gamma$ then the equations (6.0.22) and (6.0.25) is written as follows:

$$G(\tau+\gamma) = \begin{cases} \phi(\tau)exp(-1/2\gamma) + \int_0^\tau Q(\tau - r + \gamma, \gamma)\phi(r)dr, & \tau > 0 \\ 0, & otherwise \end{cases}, \tag{6.0.40}$$

and

$$G(\tau+\gamma) = \begin{cases} \gamma P(\tau)exp(-1/2\gamma) + \int_0^\tau \bar{Q}(\tau - r + \gamma, \gamma)P(r)dr, & \tau > 0 \\ 0, & otherwise \end{cases}, \tag{6.0.41}$$

where,

$$Q(t,\gamma) = \begin{cases} \frac{1}{2}exp(-t/2\gamma^2)\frac{I_1\left(\frac{1}{2\gamma^2}\sqrt{t^2-\gamma^2}\right)}{\sqrt{t^2-\gamma^2}}, & t \geq \gamma \\ 0, & otherwise \end{cases}, \tag{6.0.42}$$

and

$$\bar{Q}(t,\gamma) = \begin{cases} \frac{1}{2\gamma}exp(-t/2\gamma)\left[I_0(\frac{1}{2\gamma^2}\sqrt{t^2-\delta^2}) + \frac{tI_1(\frac{1}{2\gamma^2}\sqrt{t^2-\gamma^2})}{\sqrt{t^2-\gamma^2}}\right], & t > \gamma \\ 0, & otherwise \end{cases}. \tag{6.0.43}$$

There are many ways to descretize the integral equations (6.0.40) and (6.0.41). Then we use the trapezoid rule such that $\tau_q = q\Delta t$ and k=0,1,....M and $M = \frac{1}{\Delta t}$ and we will also assumed $\tilde{F}$ data function (discrete function) in $[0, 1+\gamma]$ and N+1 in the points $t_i = i\Delta t$ & i is from 0 to N and N=$[1+\gamma/\Delta t]$

$$G(\gamma) = \phi_0 exp(-1/2\gamma) + \frac{\Delta t}{2}Q(\gamma,\gamma)\phi_0, \tag{6.0.44}$$

$$G(\tau_1+\gamma) = \phi_1 exp(-1/2\gamma) + \frac{\Delta t}{2}Q(\tau_1+\gamma,\gamma)\phi_0 + \frac{\Delta t}{2}Q(\gamma,\gamma)\phi_1, \tag{6.0.45}$$

$$G(\tau_q+\gamma)=\phi_q exp(-1/2\gamma)+\frac{\Delta t}{2}Q(\tau_q+\gamma,\gamma)\phi_0+\Delta t\sum_{i=1}^{q-1}Q(\tau_{q-i}+\gamma,\gamma)\phi_i+\frac{\Delta t}{2}Q(\gamma,\gamma)\phi_q; \quad \text{for k=2 to M,} \tag{6.0.46}$$

and

$$G(\gamma)=\gamma P_0 exp(-1/2\gamma)+\frac{\Delta t}{2}\bar{Q}(\gamma,\gamma)P_0, \tag{6.0.47}$$

$$G(\tau_1+\gamma)=\gamma P_1 exp(-1/2\gamma)+\frac{\Delta t}{2}\bar{Q}(\tau_1+\gamma,\gamma)P_0+\frac{\Delta t}{2}\bar{Q}(\gamma,\gamma)P_1, \tag{6.0.48}$$

$$G(\tau_q+\gamma)=\gamma P_q exp(-1/2\gamma)+\frac{\Delta t}{2}\bar{Q}(\tau_q+\gamma,\gamma)P_0+\Delta t\sum_{i=1}^{q-1}\bar{Q}(\tau_{q-i}+\gamma,\gamma)P_i+\frac{\Delta t}{2}\bar{Q}(\gamma,\gamma)P_q, \tag{6.0.49}$$

where we define it as,

$$Q(\gamma,\gamma)= \ lim_{t\to\gamma}Q(t,\gamma)=\frac{1}{8\gamma^3}exp(-1/2\gamma), \tag{6.0.50}$$

and

$$\bar{Q}(\gamma,\gamma)= \ lim_{t\to\gamma}\bar{Q}(t,\gamma)=\frac{1}{\gamma}exp(-1/2\gamma)(1+\frac{1}{4\gamma}). \tag{6.0.51}$$

Here, we observed that while solving the (6.0.46) and (6.0.49), data function is increased by factor $exp(1/2\gamma)$ which is error in the discretized problem & as $\gamma\to 0$.

Conclusion

In order to solve the inverse heat conduction problems a complete understanding of the procedures is very necessary for the determination of the unknown heat flux and heat transfer problem from known values in the body.

This thesis presents a method of solving the inverse heat conduction problem. The problem is ill-posed and its solution is unstable. In order to improve the solution stability, the Tikhonov regularization is proposed. The method of regularization is applied to identify the heat flux at the front surface of thick plate and at the plate back surface on temperature(measured), which is insulated.

This thesis also presents the modified Tikhonov regularization method to obtain regularised output which is completely based on a priori assumption for the exact solution of the problem which is only possible by the proper choice of the regularization parameter.

In order to make the test more realisitic, noisy measured data are obtained by adding normal error to exact measured data.

This thesis also consists of integral solution of the inverse heat conduction problems which shows that their is one and only one optional choice of parameter d, $\bar{d}$ which minimizes the function and there is unique optimility.

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