Construction of Iterated Function System using Contraction Mapping Principle

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CERTIFICATE

I hereby declare that the dissertation entitled "Construction of Iterated Function System using Contraction Mapping Principle" is an authentic record of my work carried out as requirements for the award of the degree of Master of Science in Mathematics and Computing at Thapar Institute of Engineering and Technology, Patiala under the supervision of Dr. Sumit Chandok(Assistant Professor, School of Mathematics).

No part of the matter embodied in this report has been submitted to any other university or institute for the award of any degree.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Contents

Abstract

Fractals, briefly defined as self-similar structure or mathematically they are subsets of simple geometrical spaces such as $\mathbb{R}, \mathbb{C}$ and $\mathbb{R}^2$. Fractals are viewed as significant on the grounds that they characterize pictures that are generally cannot be characterized by Euclidean geometry. Fractals are depicted utilizing calculations and manages objects that do not have whole number measurements. Some of the examples of fractals are the Cantor set, the Koch curve, the Sierpinski triangle, and the Julia set etc.

In Chapter 1, we give a brief introduction of fractals, their existence in nature or real life and some of their applications. Also we discuss the two vital properties of fractals that is self-similarity and fractional dimension with the help of Koch curve. We also give a brief introduction of Hutchinson operator, iterated function system and an attractor of iterated function system.

The study of Picard operator is similar to the study of contractive type mappings in the context of metric spaces. It is easy to see that almost all contractive type mappings on a complete metric space are Picard operators. In Chapter 2, we introduce weak θ_m-contraction and give some results on the existence of Picard operator for such class of mappings in the setting of metric spaces.

In Chapter 3, we define weak θ_m iterated function system and present some results on the existence of a unique attractor for such an iterated function system. Also we define weak θ_m multifunction iterated system and prove some results on the existence of the attractor for such iterated multifunction system.

Chapter 1

Introduction

Fractal are characterized as a rugged or divided geometric shapes which can be divided into parts that are viewed as a diminished duplicate of the entirety. Despite the fact that the investigation of fractals have existed as ahead of schedule as the seventeenth century, however the term fractal was just instituted in 1975 by Benoit Mandelbrot. It is originated from the Latin word fractus, which means broken or cracked. While a fractal is carefully a scientific develop, it is found in different non-numerical models, for example, common frameworks and fine arts.

To understand fractals, it is critical to know about its attributes, that is its shape cannot be characterized by Euclidean geometry, its structure is characterized by fine and little scales or potentially substructures which is recursive. Fractals are casually viewed as boundlessly unpredictable as they seem comparable in all degrees of amplification. There are a great deal of regular wonders that can be characterized and anticipated utilizing fractals. A portion of these shapes incorporate mists, vegetables, shading examples, lightning, and snowflakes etc.

1.1 Geometrical interpretation and dimension of fractals

Now, we give the concept of *self-similarity* geometrically with the help of a Koch curve.
To draw a Koch curve we separate a line segment into three equivalent fragments at that point, supplant the center part with different sides of a symmetrical triangle of same length as the length of center portion, along these lines we get the first iteration of Koch curve as shown in Figure 1.1. Now for second iteration we repeat the same process for each of the four segments obtained in first iteration, continuing in the way upto infinity, we get a *Koch curve.*

Now we discuss the meaning of *self-similar structure which vary under degree of magnification.* In Figure 1.1 we see that if we magnify third iteration of Koch curve then we get same structures at every stage of magnification which is similar to second iteration, hence so on.
From the geometry of Koch curve it can be seen that fractals are nothing but subsets of simple geometrical space such as $\mathbb{R}^2$, (Euclidean's space) or the space of complex numbers, $\mathbb{C}$. The word *fractal* also means that an object which has fractional dimension. We know that the dimension of a line is one and that of a plane is two but if we consider a particular fractal say *Koch curve* then we see that it is something in between a line and a plane, which means that its dimension is neither one nor two but some number between one and two.
Dimension of a Koch curve
The basic formula to calculate fractal dimension is

$$d = \frac{\log(number\ \ of\ \ pieces\ \ in\ \ which\ \ it\ \ break\ \ into)}{\log(magnification\ \ factor)}.$$

By magnification factor we mean that how many times we have to magnify a particular segment or piece in order to get initial structure. For example in Koch curve if we take the line sengment (initial structure) of unit length

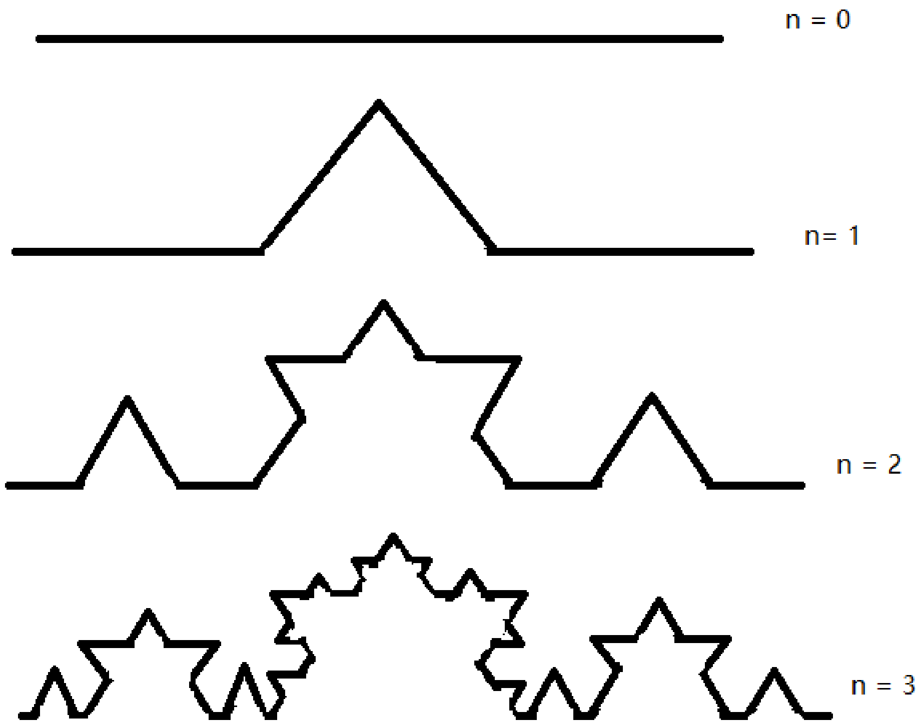

Figure 1.1: The first four iterations of Koch curve

then we know that each piece of first iteration is of length say $b = \frac{1}{3}$, be the contraction factor that is the initial curve is contracting by a factor of $\frac{1}{3}$. Also from Figure 1.1 we observe that from first iteration we get 4 equal pieces, similarly from second iteration we again get four pieces for each segment of first iteration. Thus the dimension d is

$$d = \frac{\log 4}{\log 3} = 1.262.$$

So we can say that two most important properties of fractals are self-similarity and non-integral dimension.

1.2 Banach contraction principle

The Banach contraction principle is named after Stefan Banach (1892 - 1945), and was first stated in 1922. it gives the assurance of existence and uniqueness of fixed points in contractive self-maps on a complete metric space. This principle also provides useful way to find those fixed points.

Now, we begin with the definition of contraction mapping and discuss the Banach contractionn principle.

Let $(\mathcal{M}, \rho)$ be a complete metric space and let

$$T:\mathcal{M} \to \mathcal{M}$$

be a self mapping. We call T a Lipschitz mapping with Lipschitz constant $k \geq 0$, provided that

$$\rho(T(x), T(y)) \leq k\rho(x, y), \text{ for all } x, y \in \mathcal{M}.$$

We know that the Lipschitz mappings are neccessarily continuous mappings and composition of continuous mapping is again a continuous mapping. Thus for mapping $T^n = T \circ ... \circ T$, the mapping T composed with itself n times, is a Lipschitz mapping as well. We call a Lipschitz maping T a nonexpansive mapping if $k = 1$ and a contraction mapping provided the Lipschitz constant k may be choosen so that $0 \leq k < 1$. In this case the Lipschitz constant k is called the contraction constant of T see([12]).

Now, we discuss the contraction mapping principle also known as the **Banach Fixed Point theorem**.

Theorem 1.2.1. *[12] Let $(\mathcal{M}, \rho)$ be a complete metric space and let $T:\mathcal{M} \to \mathcal{M}$ be a contraction mapping with contraction constant k. Then T has a unique fixed point $x \in \mathcal{M}$. Furthermore, if $y \in \mathcal{M}$ is arbitrary chosen, then the iterates x_n given by*

$$x_0 = y$$

$$x_n = T(x_{n-1}),\ \textit{for}\ n \geq 1$$

have the property that $\lim\limits_{n\to\infty} x_n = x$

Proof. Let $y \in \mathcal{M}$ be an arbitrary point of $\mathcal{M}$ and consider the sequence x_n given by

$$x_0 = y$$

$$x_n = T(x_{n-1}), \text{ for } n \geq 1.$$

We shall prove that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}$. For $m < n$ we use the triangle inequality and note that

$$\rho(x_m, x_n) \leq \rho(x_m, x_{m+1}) + \rho(x_{m+1}, x_{m+2}) + ... + \rho(x_{n-1}, x_n).$$

Since T is a contraction mapping , we have

$$\rho(x_p, x_{p+1}) = \rho(T(x_{p-1}), T(x_p)) \leq k\rho(x_{p-1}, x_p),$$

for any integer $p \geq 1$. Using this inequality repeatedly, we obtain

$$\rho(x_p, x_{p+1}) \leq k^p \rho(x_0, x_1).$$

Hence,

$$\rho(x_m, x_{m+1}) \leq\ (k^m + k^{m+1} + \cdots + k^{n-1})\rho(x_0, x_1),$$

that is,

$$\rho(x_m, x_n) \leq\ \tfrac{k^m}{1-k}\rho(x_0, x_1)$$

whenever $m \leq n$. From this we deduce that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}$. Since $\mathcal{M}$ is complete, $(x_n)_{n=1}^{\infty}$ sequence has a limit say $x \in M$. On the other hand, since T is continuous, it follows that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T(\lim_{n\to\infty} x_{n-1}) = T(x),$$

and thus x is a fixed point of T.

If x and z $(x \neq z)$ are two fixed points of T, we get

$$d\rho(x,z) = \rho(T(x),T(z)) \leq k\rho(x,z), \text{ where } k < 1,$$

Thus we have $x = z$, which means their exist a unique fixed point.

□

Example 1.2.2. Let $(\mathbb{R}, d)$ be a metric space where d is a usual metric $d(x,y) = |x-y|$. Let f be a mapping $f : \mathbb{R} \to \mathbb{R}$ defined as :

$$f(x) = \tfrac{x}{2}$$

then show that f has a fixed point.

We will try to prove that f is a contraction mapping

$$\begin{aligned} d(f(x), f(y) &= |f(x) - f(y)| \\ &= \left|\frac{x}{2} - \frac{y}{2}\right| \\ &= \frac{1}{2}|x-y| \\ &\leq k|x-y|, for\ each\ \ 0 \leq k \leq \frac{1}{2} \end{aligned}$$

which means that f is a contraction mapping and we know that $(\mathbb{R}, d)$ is a complete metric space so by *Banach contraction principle* f has a unique fixed point, that is 0.

Now let us consider an example in which it is not sure about having a fixed point when the contraction constant $k\ is$ 1.

Example 1.2.3. Let $(\mathcal{M}, \rho)$ be a metric space where $\mathcal{M} = [1, \infty)$ and ρ be the usual metric,

which is

$$\rho(x, y) = |x - y|, \quad \text{for all} \quad x, y \in \mathcal{M}$$

and let $T : \mathcal{M} \to \mathcal{M}$ defined as

$$T(x) = x + \tfrac{1}{x^2}$$

Then, by computation we have

$$\begin{aligned} \rho(x, y) &= |x - y| \left| 1 - \frac{(x + y)}{x^2 y^2} \right| \\ &< |x - y| = \rho(x, y) \end{aligned}$$

so we can observe that there does not exist any k where $0 \le k < 1$ such that

$$\rho(T(x), T(y)) \le k\rho(x, y), \quad \text{for all} \quad x, y \in \mathcal{M}$$

which means T does not have any fixed point in $\mathcal{M}$.

Thus from the above example it is clear that Banach contraction principle is applicable only if the contraction constant k lies in $[0, 1)$.

1.3 Preliminaries

In this section, we give some basic definitions and notations to be used in the subsequent chapters of this report.

Definition 1.3.1. Let $(\mathcal{M}, \rho)$ be a metric space, and let $K(\mathcal{M})$ be the class of all non-empty compact sets of $\mathcal{M}$. The function $\eta : K(\mathcal{M}) \times K(\mathcal{M}) \to [0, \infty)$ define by

$$\eta(A, B) = \max\{D(A, B), D(B, A)\}$$

where $D(A, B) = \sup_{a \in A} \inf_{b \in B} \rho(a, b)$ for all $A, B \in K(\mathcal{M})$ is a metric known as *Hausdorff-Pompeiu metric*. It is well known that if $(\mathcal{M}, \rho)$ is complete then $(K(\mathcal{M}), \eta)$ is also complete.

Definition 1.3.2. (see [12])**Iterated Function System**:Let $(\mathcal{M}, \rho)$ be a complete metric space, and $f_1, f_2, \ldots f_n : \mathcal{M} \to \mathcal{M}$ be contraction mappings. Then the finite set $\{f_1, f_2, \ldots f_n\}$ is called an Iterated function system, abbreviated as IFS.

Definition 1.3.3. (see [12])**Hutchinson operator**: Let $\{f_1, f_2, \dots f_n\}$ is an IFS defined on complete metric space $\mathcal{M}$. Then the function $F : H(\mathcal{M}) \to H(\mathcal{M})$ defined by :

$$F(A) = \bigcup_{i=1}^{n} f_i(A), A \in H(\mathcal{M})$$

is called an Hutchinson operator, where $H(\mathcal{M})$ is the collection of all non-empty closed and bounded subsets of $\mathcal{M}$.

Definition 1.3.4. (see [12])**Attractors**: A set A in H($\mathcal{M}$) is said to be an *attractor* of an IFS $\{f_1, f_2, \dots f_n\}$ if $A = F(A)$

$$A = \bigcup_{i=1}^{n} f_i(A)$$

Theorem 1.3.5. *(see [3]) Let $\{f_1, f_2, \dots f_n\}$ be an IFS over a complete metric space $(\mathcal{M}, \rho)$ and let F be the associated Hutchinson operator. Then there exist a unique $A \in H(\mathcal{M})$ such that $F(A) = A$. Moreover*

$$\lim_{n\to\infty} F^n(X) = A \quad \textit{for all} \quad X \in H(\mathcal{M}).$$

This theorem is just an analogy of Banach contraction principle.

Example 1.3.6. Let us take $(\mathcal{M}, \rho)$ a metric space where $\mathcal{M} = [0, 1]$ and ρ be the usual metric define two functions f_1 and f_2 as :

$$f_1, f_2 : \mathcal{M} \to \mathcal{M}$$
$$f_1(x) = \tfrac{x}{3} \text{ and } f_2(x) = \tfrac{2x}{3}$$

define Hutchinson operator over the IFS (f_1, f_2) as $F : H(\mathcal{M}) \to H(\mathcal{M})$

$$F(A) = f_1(A) \cup f_2(A)$$

both f_1 and f_2 are contraction mappings over the complete metric space $(\mathcal{M}, \rho)$ then by Theorem 1.3.5 we can conclude that F defined as $F(A) = f_1(A) \cup f_2(A)$ has a unique attractor and that attractor for F is Cantor set. This means F converges to Cantor set.

Thus in this chapter we have discussed about the fractals, Banach contraction principle and some basic definitions relating to them. We also saw the formation of Koch curve and discussed the concept of self-similarity from its geometric interpretation and by using the Banach contraction principle we have also proved that Cantor set is a fractal.

Chapter 2

Existence of Picard operator

2.1 Introduction

Many authors have generalized Banach contraction principle in many ways, one of them is introduced by *Jleli and Samet* [7] in the form of θ-contractions. Later on it was generalized by Imdad and Alfaqih [6] as weak $\theta-$ contractions. In this chapter we introduce weak θ_m- contraction and obtain some results on the existence of Picard operator for such class of mappings in the context of metric spaces.

2.2 Picard operator

To start with we have the following definition :

Definition 2.2.1. Let $(\mathcal{M}, \rho)$ be a metric space and $f : \mathcal{M} \to \mathcal{M}$ be a self mapping. A sequence $\{x_n\}$ defined by $x_n = f^n x_0$ is called a *Picard sequence* based at the point $x_0 \in \mathcal{M}$. A self-mapping f is said to be a *Picard operator* if it has a unique fixed point $y \in \mathcal{M}$ and $y = \lim_{n\to\infty} f^n x$ for all $x \in M$.

The following reults of Radenovic and Chandok [16] will be used in the later section.

Lemma 2.2.2. *[16] Let $(\mathcal{M}, \rho)$ be a metric space and let $\{x_n\}$ be a sequence in $\mathcal{M}$ such that*

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0. \tag{2.2.0}$$

If $\{x_n\}$ is not a Cauchy sequence in $\mathcal{M}$, then there exist $\varepsilon > 0$ and two sequences $\{m(t)\}$ and $\{n(t)\}$ of positive integers such that $n(t) > m(t) > t$ and the following sequences tend to ε^+ when $t \to +\infty$:

$$\rho\left(x_{m(t)}, x_{n(t)}\right), \ \rho\left(x_{m(t)}, x_{n(t)+1}\right), \ \rho\left(x_{m(t)-1}, x_{n(t)}\right), \tag{2.2.1}$$

$$\rho\left(x_{m(t)-1}, x_{n(t)+1}\right), \ \rho\left(x_{m(t)+1}, x_{n(t)+1}\right).$$

Remark 2.2.3. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). If for all $n \in \mathbb{N}$ holds $d(x_{n+1}, x_n) < d(x_n, x_{n-1})$, then $n \neq m$ implies $x_n \neq x_m$ whenever $n, m \in \mathbb{N}$.

The following results will also be used in the sequel.

Lemma 2.2.4. *[8] Let $A, B, C \in K(\mathcal{M})$. Then we have the following:*
(i) $A \subset B$ if and only if $D(A, B) = 0$;
(ii) $D(A, B) \leq D(A, C) + D(C, B)$.

Lemma 2.2.5. *[9] If $\{\mathcal{E}_i\}_{i\in\tau}$ and $\{\mathcal{F}_i\}_{i\in\tau}$ are finite collection of elements in $K(\mathcal{M})$, then $\eta(\bigcup_{i\in\tau}\mathcal{E}_i, \bigcup_{i\in\tau}\mathcal{F}_i) \leq \sup_{i\in\tau}\eta(\mathcal{E}_i, \mathcal{F}_i)$.*

2.2.1 Weak θ_m-contraction

In the this section we introduce weak θ_m- contraction, by generalizing weak θ- contraction, and prove some results on the existence of Picard operator. Now we give the definition of an auxillary function and utilize it to introduce weak θ_m-contraction.

Definition 2.2.6. (see [6, 7, 10]) Let $\theta : (0, \infty) \to (1, \infty)$ be a function and consider the following conditions:
$\Theta 1$: θ is non-decreasing.
$\Theta 2$: for each sequence $\{\alpha_n\}$ in $(0, \infty)$,

$$\lim_{n\to\infty} \theta(\alpha_n) = 1 \Leftrightarrow \lim_{n\to\infty} (\alpha_n) = 0.$$

$\Theta 3$: there exist $r \in (0, 1)$ and $l \in (0, \infty)$ such that $\lim\limits_{\alpha\to 0^+} \frac{\theta(\alpha)-1}{\alpha^r} = l$;
$\Theta 4$: θ is continuous.

The following notations to be used in the sequel.

- $\Theta_{1,2,3}$ the family of all θ that satisfy $\Theta 1 - \Theta 3$.
- $\Theta_{1,2,4}$ the family of all θ that satisfy $\Theta 1, \Theta 2$ and $\Theta 4$.
- $\Theta_{2,3}$ the family of all θ that satisfy $\Theta 2$ and $\Theta 3$.
- $\Theta_{2,4}$ the family of all θ that satisfy $\Theta 2$ and $\Theta 4$.
- Θ_2 the family of all θ that satisfy $\Theta 2$.

Example 2.2.7. [10] Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\alpha) = e^{\alpha}$, for all $\alpha \in (0, \infty)$.Then $\theta \in \Theta_{1,2,3}$.

Example 2.2.8. [6] The following function $\theta : (0, \infty) \to (1, \infty)$ are in $\Theta_{2,4}$:

1. $\theta(\alpha) = e^{\frac{\alpha}{2}+sin\alpha}$;
2. $\theta(\alpha) = \alpha^r + 1, r \in (0, \infty)$;

Example 2.2.9. Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\alpha) = 2^{\sqrt{\alpha}2^{-\frac{1}{\sqrt{\alpha}}}}$, for all $\alpha \in (0, \infty)$.Then $\theta \in \Theta_{1,2,4}$.

Now, we introduce weak θ_m-contraction mapping.

Definition 2.2.10. Let $(\mathcal{M}, \rho)$ be a metric space and $f : \mathcal{M} \to \mathcal{M}$ is a self-mapping. A mapping f is called a weak θ_m-contraction if there exist a $\theta \in \Theta_{2,4}$ and $k \in (0, 1)$, such that for all $x, y \in \mathcal{M}$, we have

$$\rho(fx, fy) > 0 \Rightarrow \quad \theta(\rho(fx, fy)) \leq [\theta(M(x, y))]^k, \tag{2.2.2}$$

where $M(x, y) = \max\{\rho(x, fx), \rho(y, fy), \rho(x, y)\}$.

Remark 2.2.11. Here, we note that weak θ_m-contraction mapping has atmost one fixed point. Assume that f has another fixed point say $y \in \mathcal{M}$, $\rho(x, y) > 0$. Using (2.2.2) we have

$$\begin{aligned}\theta(\rho(x, y)) &= \theta(\rho(fx, fy)) \\ &\leq [\theta(\max\{\rho(x, fx), \rho(y, fy), \rho(x, y)\})]^k \\ &= [\theta(\rho(x, y))]^k,\end{aligned}$$

which is a contradiction.

Lemma 2.2.12. *Let $(\mathcal{M}, \rho)$ be a metric space and $f : \mathcal{M} \to \mathcal{M}$ is a weak θ_m-contraction. Suppose that there exists a Picard sequence $\{x_n\} \subseteq \mathcal{M}$ defined by $x_{n+1} = f^n x_0 = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then $\rho(x_n, x_{n+1}) \to 0$ as $n \to \infty$, where $x_n \neq x_{n+1}$ (Here $\theta \in \Theta_{2,4}$ or $\Theta_{1,2,4}$).*

Proof. Let $x_0 \in \mathcal{M}$ be an arbitrary point. Define the Picard sequence as $\{x_n\} \subseteq \mathcal{M}$ by $x_{n+1} = f^n x_0 = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Applying (2.2.2) we have, for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}\theta(\rho(x_n, x_{n+1})) &= \theta(\rho(fx_{n-1}, fx_n)) \\ &\leq [\theta(\max\{\rho(x_{n-1}, fx_{n-1}), \rho(x_n, fx_n), \rho(x_{n-1}, x_n)\})]^k \\ &= [\theta(\max\{\rho(x_n, x_{n+1}), \rho(x_n, x_{n-1})\})]^k\end{aligned}$$

Case 1: When $\rho(x_n, x_{n+1}) > \rho(x_n, x_{n-1})$, then we have $\theta\rho(fx_{n-1}, fx_n)) = \theta(\rho(x_n, x_{n+1})) \leq [\theta(\rho(x_n, x_{n+1})]^k$, but $\alpha \geq \alpha^k, \forall \alpha \in \mathbb{R}^+, k \in (0, 1)$. Thus we get contradiction.

Case 2: When $\rho(x_n, x_{n-1}) > \rho(x_n, x_{n+1})$, we have $\theta(\rho(fx_{n-1}, fx_n)) \leq [\theta(\rho(x_n, x_{n-1})]^k$. Hence on the same lines, we have

$[\theta(\rho(fx_{n-1}, fx_{n-2}))]^k \leq [\theta(\max\{\rho(x_{n-1}, fx_{n-1}), \rho(x_{n-2}, fx_{n-2}), \rho(x_{n-1}, x_{n-2})\})]^{k^2} = [\theta(\max\{\rho(x_{n-1}, x_n), \rho(x_{n-1}, x_{n-2})\})]^{k^2} \leq [\theta(\rho(x_{n-1}, x_{n-2}))]^{k^2}$.

Proceeding on these lines, we get

$$\begin{aligned}\theta(\rho(fx_n, fx_{n-1})) &\leq [\theta(\rho(fx_{n-1}, fx_{n-2}))]^k \\ &\leq [\theta(\rho(fx_{n-2}, fx_{n-3}))]^{k^2} \\ &\leq \ldots \leq [\theta(\rho(fx_0, x_1))]^{k^n}.\end{aligned}$$

Thus, we have $\theta(\rho(x_n, x_{n+1})) \leq [\theta(\rho(x_1, x_0))]^{k^n}$. Now, taking $n \to \infty$ we have, $\lim_{n\to\infty} \theta(\rho(x_n, x_{n+1})) = 1$. Using Θ_2, we have $\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0$.

□

Lemma 2.2.13. *Let $(\mathcal{M}, \rho)$ be a metric space and $f : \mathcal{M} \to \mathcal{M}$ is a weak θ_m-contraction. Suppose that there exists a Picard sequence $\{x_n\} \subseteq \mathcal{M}$ defined by $x_{n+1} = f^n x_0 = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then Picard sequence $\{x_n\}$ is a Cauchy sequence (Here $\theta \in \Theta_{2,4}$ or $\Theta_{1,2,4}$)*

Proof. Let $x_0 \in \mathcal{M}$ be an arbitrary point. Define the Picard sequence as $\{x_n\} \subseteq \mathcal{M}$ by $x_{n+1} = f^n x_0 = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using Lemma 2.2.12, we have $\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0$. Now we have to prove that $\{x_n\}$ is a Cauchy sequence. We'll prove this by contradiction. Assume that $\{x_n\}$ is not a Cauchy sequence.

Now, since the sequence $\{x_n\}$ is not a Cauchy sequence, then by Lemma 2.2.2, we have $\rho\left(x_{m(t)}, x_{n(t)}\right)$ and $\rho\left(x_{m(t)+1}, x_{n(t)+1}\right)$ tend to $\varepsilon > 0$, as $t \to \infty$. Using (2.2.2), we have

$$\begin{aligned}\theta(\rho(x_{m(t)}, x_{n(t)})) &= \theta(\rho(fx_{m(t)-1}, fx_{n(t)-1})) \\ &\leq [\theta(\max\{\rho(x_{m(t)-1}, fx_{m(t)-1}), \rho(x_{n(t)-1}, fx_{n(t)-1}), \\ &\quad \rho(x_{m(t)-1}, x_{n(t)-1})\})]^k.\end{aligned}$$

Case 1: If $\max\{\rho(x_{m(t)-1}, fx_{m(t)-1}), \rho(x_{n(t)-1}, fx_{n(t)-1}), \rho(x_{m(t)-1}, x_{n(t)-1})\} = \rho(x_{m(t)-1}, fx_{m(t)-1})$, then we have $\theta(\rho(x_{m(t)}, x_{n(t)})) \leq [\theta(\rho(x_{m(t)-1}, fx_{m(t)-1})]^k$.

Letting $t \to \infty$, from Lemma 2.2.2 and Θ4, we have

$$\theta(\epsilon) \leq [\theta(0)]^k,$$

which is a contradiction.

Case 2: If $\max\{\rho(x_{m(t)-1}, fx_{m(t)-1}), \rho(x_{n(t)-1}, fx_{n(t)-1}), \rho(x_{m(t)-1}, x_{n(t)-1})\} = \rho(x_{n(t)-1}, fx_{n(t)-1})$, then proceeding the same way as in Case 1 we again get a contradiction.

Case 3: If $\max\{\rho(x_{m(t)-1}, fx_{m(t)-1}), \rho(x_{n(t)-1}, fx_{n(t)-1}), \rho(x_{m(t)-1}, x_{n(t)-1})\} = \rho(x_{m(t)-1}, x_{n(t)-1})$, then we have

$$\theta(\rho(x_{m(t)}, x_{n(t)})) \leq [\theta(\rho(x_{m(t)-1}, x_{n(t)-1})]^k.$$

Letting $t \to \infty$ and using Lemma 2.2.2 and Θ4, we obtain $\theta(\epsilon) \leq [\theta(\epsilon)]^k$, which is again a contradiction. Hence Picard sequence $\{x_n\}$ is a Cauchy sequence. □

Theorem 2.2.14. *Every weak θ_m-contraction on a complete metric space is a Picard operator. [Here, we consider $\theta \in \Theta_{1,2,4}$.]*

Proof. Let $x_0 \in \mathcal{M}$ be an arbitrary point. Define the Picard sequence as $\{x_n\} \subseteq \mathcal{M}$ by $x_{n+1} = f^n x_0 = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If there exist $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = fx_{n_0}$, then we are done. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using Lemma 2.2.13, we have $\{x_n\}$ is a Cauchy sequence. Now as $(\mathcal{M}, \rho)$ is a complete metric space so there exist $x \in \mathcal{M}$ such that $\{x_n\}$ converges to x. From (Θ_1) and (2.2.2), it is easy to conclude that

$$\begin{aligned} \theta(\rho(fx, fy)) &\leq [\theta(\max\{\rho(x, fx), \rho(y, fy), \rho(x, y)\})]^k \\ &\leq \theta(\max\{\rho(x, fx), \rho(y, fy), \rho(x, y)\}) \end{aligned}$$

for all $x, y \in \mathcal{M}$ with $\rho(fx, fy) > 0$. Using (Θ_1) and above inequality, we have $\rho(fx, fy) \leq \max\{\rho(x, fx), \rho(y, fy), \rho(x, y)\}$. Suppose that $x \neq fx$. Therefore, we have

$$\begin{aligned} \rho(x_{n+1}, fx) = \rho(fx_n, fx) &\leq \max\{\rho(x_n, fx_n), \rho(x, fx), \rho(x_n, x)\} \\ &= \max\{\rho(x_n, x_{n+1}), \rho(x, fx), \rho(x_n, x)\}. \end{aligned}$$

Taking $n \to \infty$, using Lemma 2.2.12 we have $\rho(x, fx) \leq \rho(x, fx)$, which is a contradiction. Hence $fx = x$, thus we get a fixed point.

Further, now we prove the uniqueness of the fixed point. Assume that f has another fixed point say $y \in \mathcal{M}$, $y \neq x$. Using (2.2.2) we have

$$\begin{aligned} \theta(\rho(fx, fy)) &\leq [\theta(\max\{\rho(x, fx), \rho(y, fy), \rho(x, y)\})]^k \\ &= [\theta(\rho(x, y))]^k, \end{aligned}$$

which is a contradiction. Hence the result. □

Theorem 2.2.15. *Every continuous weak θ_m-contraction on a complete metric space is a Picard operator. [Here, we consider $\theta \in \Theta_{2,4}$.]*

Proof. Let $x_0 \in \mathcal{M}$ be an arbitrary point. Define the Picard sequence as $\{x_n\} \subseteq \mathcal{M}$ by $x_{n+1} = f^n x_0 = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If there exist $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = fx_{n_0}$, then we are done. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Proceeding as in Theorem 2.2.14, we have Picard sequence $\{x_n\}$ is a Cauchy sequence. Now as $(\mathcal{M}, \rho)$ is a complete metric space so there exist $x \in \mathcal{M}$ such that $\{x_n\}$ converges to x. The continuity of f and uniqueness of limit implies $fx = x$, thus we get a fixed point. Hence every continuous weak θ_m-contraction on a complete metric space is a Picard operator. □

Theorem 2.2.16. *Let $(\mathcal{M}, \rho)$ be a complete metric space and let $f : \mathcal{M} \to \mathcal{M}$ be a self mapping. If there exist $n \in \mathbb{N}$ such that f^n is a continuous weak θ_m -contraction, then f is a Picard operator.*

Proof. From Theorem 2.2.15, it is obvious that f^n is a Picard operator, thus there exista a unique $x \in M$ such that $f^n x = x$ and $\lim_{m \to \infty} (f^n u)^m = x$, for all $x \in M$. Also, we observe that $f^{n+1} x = f^n x$, that is $f^n(fx) = fx$, thus fx is also a fixed point of f^n. Thus $fx = x$.
Further, if z is another fixed point of f, then it must be a fixed point of f^n. Hence $z = x$. Therefore f has a unique fixed point.

Now, let m be a positive integer greater than n. Then there exist $l \geq 1$ and $s \in \{0, 1, 2, ..., n-1\}$ such that $m = nl + s$. Here, we notice that

$$\lim_{m\to\infty} f^m u = \lim_{l\to\infty} f^{nl}(f^s u) = x.$$

Hence the result. □

Lemma 2.2.17. *Let $\mathcal{M}$ be a nonempty set and $f : \mathcal{M} \to \mathcal{M}$ a function. Then there exist a set $\mathcal{E} \subseteq \mathcal{M}$ such that $f(\mathcal{E}) = f(\mathcal{M})$ and $f : \mathcal{E} \to \mathcal{M}$ is one-to-one.*

By using above lemma, we prove common fixed point theorems for two self mappings on M as follows:

Theorem 2.2.18. *Let $(\mathcal{M}, \rho)$ be a complete metric space and f, g be two self maps on $\mathcal{M}$ satisfying*

$$\rho(fx, fy) > 0 \Rightarrow \quad \theta(\rho(fx, fy)) \leq [\theta(\max\{\rho(gx, fx), \rho(gy, fy), \rho(gx, gy)\})]^k \quad (2.2.\text{-}11)$$

for all $x, y \in \mathcal{M}$ and $\theta \in \Theta_{2,4}$. If $f(\mathcal{M}) \subseteq g(\mathcal{M})$ and $g(\mathcal{M})$ is a complete subset of $\mathcal{M}$ then f and g have a unique common fixed point in M.

Proof. By using Lemma 2.2.17, there exist $\mathcal{E} \subseteq \mathcal{M}$ such that $g(\mathcal{E}) = g(\mathcal{M})$ and $g : \mathcal{E} \to \mathcal{M}$ is one-to-one. Define $h : g(\mathcal{E}) \to g(\mathcal{M})$ by $h(gx) = fx$. Clearly, h is well defined as g is one-to-one on $\mathcal{E}$. Also, $\theta(\rho(h(gx), h(gy))) \leq [\theta(\max\{\rho(gx, fx), \rho(gy, fy), \rho(gx, gy)\})]^k$ for all $gx, gy \in g(\mathcal{E})$. Since $g(\mathcal{E}) = g(\mathcal{E})$ is complete, then by using Theorem 2.2.15, we can easily prove that f and g have a unique common fixed point in $\mathcal{M}$. □

Example 2.2.19. Let $M = \{1, 2, 3\}$. Define the metric $d : M \times M \to [0, \infty)$ by $d(x, y) = |x - y|$, for all $x, y \in M$. Define a function $f : M \to M$ as $f(1) = 2, f(2) = 2, f(3) = 1$.

Define a function $\theta : (0, \infty) \to (1, \infty)$ by $\theta(t) = e^{\sqrt{t}}$. So $\theta \in \Theta_{1,2,3,4}$.

Case 1. Consider $(u, v) = (1, 3)$. We have $\theta(d(f1, f3)) = \theta(d(2, 1)) = \theta(1) = e$. Also, $[\theta(\max\{d(1, f1), d(3, f3), d(1, 3)\})]^h = [\theta(d(1, 2), d(1, 3))]^h =$

$[\theta(2)]^h = [e^{\sqrt{2}}]^h$. Therefore $\theta(d(f1, f3)) \leq [\theta(\max\{d(1, f1), d(3, f3), d(1, 3)\})]^h$, for all $h \in [\frac{1}{\sqrt{2}}, 1)$.

Case 2. Consider $(u, v) = (2, 3)$. We have $\theta(d(f2, f3)) = \theta(d(2, 1)) = \theta(1) = e$. Also, $[\theta(\max\{d(2, f2), d(3, f3), d(2, 3)\})]^h = [\theta(d(3, 1), d(2, 3))]^h = [\theta(2)]^h = [e^{\sqrt{2}}]^h$. Therefore $\theta(d(f1, f3)) \leq [\theta(\max\{d(2, f2), d(3, f3), d(2, 3)\})]^h$, for all $h \in [\frac{1}{\sqrt{2}}, 1)$.

Thus all the conditions of Theorem 2.2.14 are satisfied and 2 is a unique fixed point of f.

Here is to note that when $(u, v) = (2, 3)$ in the above example, then

(a) f is not Banach contraction;

(b) f is not weak θ-contraction of Imdad et al. [6];

(c) f is weak θ_m-contraction.

(d) f is a Picard operator.

Chapter 3

Weak θ_m iterated function system and iterated multifunction system

3.1 Introduction

An iterated function system is the set of functions defined on a mteric space. In this chapter our main focus is on the existence of the attractor of iterated function system. As an application of results proved in the last chapter, we obtain some results on the existence and uniqueness of attractor of iterated function system composed by weak θ_m-contraction in the setting of complete metric spaces. Also, we introduce weak θ_m multifunction system and discuss the existence of its multifractal in the last section of the chapter.

3.2 Weak θ_m iterated function system

In the following sections, we consider $(\mathcal{M}, \rho)$ as a complete metric space, $N \in \mathbb{N}$ and $\theta \in \Theta_{1,2,4}$. Now we will begin with the defintion of weak θ_m

iterated function system.

Definition 3.2.1. Let $\{f_i\}_{i=1}^N$ be a finite family of self mappings on $\mathcal{M}$. If $f_i : \mathcal{M} \to \mathcal{M}$ is a weak θ_m-contraction (for each i), then the family $\{f_i\}_{i=1}^N$ is called a *weak θ_m iterated function system* (weak θ_m IFS).

The set function $G : K(\mathcal{M}) \to K(\mathcal{M})$ define by $G(B) = \bigcup_{i=1}^N f_i(B)$ (for all $B \in K(\mathcal{M})$) is said to be *associated Hutchinson operator*. A set $A \in K(\mathcal{M})$ is called an *attractor* of the weak θ_m IFS if $G(A) = A$.

Now to prove that the weak θ_m IFS has a unique attractor, we have the following results.

Lemma 3.2.2. *Let $f : \mathcal{M} \to \mathcal{M}$ is a continuous weak θ_m-contraction. Then the mapping $A \mapsto f(A)$ is also a weak θ_m-contraction from $K(\mathcal{M})$ into itself.*

Proof. Let $A, B \in K(\mathcal{M})$ be such that $\eta(f(A), f(B)) > 0$. Assume that

$$\begin{aligned}\eta(f(A), f(B)) &= D(f(A), f(B)) \\ &= \sup_{x\in A}\inf_{y\in B} \rho(fx, fy), \quad \text{for all} \quad A, B \in K(\mathcal{M}). \quad (3.2.0)\end{aligned}$$

As f is a continuous weak θ_m-contraction there exist $k \in (0,1)$ such that

$$\theta(\rho(fx, fy)) \le [\theta(\max\{\rho(x, fx), \rho(y, fy), \rho(x, y)\})]^k,$$

for all $x, y \in \mathcal{M}$.

Now using (3.2.1), compactness of A, and continuity of f, we can find $a \in A$ such that

$$D(f(A), f(B)) = \inf_{y\in B} \rho(fa, fy) > 0,$$

so that $\rho(fa, fy) > 0$, for all $y \in B$. Hence, for all $y \in B$, we have

$$\begin{aligned}\theta(\inf_{y\in B} \rho(fa, fv)) &\le \theta(\rho(fa, fy) \\ &\le [\theta(max\{\rho(a, fa), \rho(y, fy), \rho(a, y)\})]^k\end{aligned}$$

Therefore, for all $y \in B$ we get

$$\theta(\eta(f(A), F(B)) \leq [\theta(max\{\rho(a, fa), \rho(y, fy), \rho(a, y)\})]^k. \qquad (3.2.-2)$$

Case 1: If $\max\{\rho(a, fa), \rho(y, fy), \rho(a, y)\} = \rho(a, fa)$, then we have:

$$\theta(\inf_{y \in B} \rho(fa, fy)) \leq [\theta(\rho(a, fa))]^k,$$

Now let $fa' \in f(A)$ such that

$$\rho(a, fa') = \inf_{fa \in f(A)} \rho(a, fa).$$

Hence from (3.2) we have,

$$\begin{aligned} \theta(\eta(f(A), f(B))) &\leq [\theta(\rho(a, fa'))]^k \\ &= [\theta(\inf_{fa \in f(A)} \rho(a, fa))]^k \\ &\leq [\theta(\sup_{a \in A} \inf_{fa \in f(A)} \rho(a, fa))]^k \\ &\leq [\theta(\sup_{a \in A} \inf_{fa \in f(A)} \rho(a, fa))]^k \\ &= [\theta(D(A, A))]^k, \end{aligned}$$

which is a contradiction.

Case 2: If $\max\{\rho(a, fa), \rho(y, fy), \rho(a, y)\} = \rho(y, fy)$, then proceeding in the same way as in Case 1 we again get a contradiction.

Case 3: If $\max\{\rho(a, fa), \rho(y, fy), \rho(a, y)\} = \rho(a, y)$, then for all $y \in B$ we have $\theta(\eta(f(A), f(B))) \leq [\theta(\rho(a, v))]^k$.

Now let $y \in B$ be such that

$$\rho(a, y) = \inf_{y \in B} d(a, y).$$

From (3.2) we have,

$$\begin{aligned} \theta(\eta(f(A), f(B))) &\leq [\theta(\rho(a, v))]^k, \\ &= [\theta(\inf_{v \in B} \rho(a, v))]^k, \\ &\leq [\theta(\sup_{a \in A} \inf_{v \in B} \rho(a, v))]^k \\ &= [\theta(D(A, B))]^k \\ &\leq [\eta(A, B)]^k. \end{aligned}$$

Hence we get the result.

∎

Theorem 3.2.3. *If $\{f_i\}_{i=1}^N$ is a continuous weak θ_m -IFS, then it has unique attractor. Moreover, $A = \lim_{n\to\infty} G^n(B)$ for all $B \in K(\mathcal{M})$, the limit being taken with respect to the Hutchinson-Pompeiu metric.*

Proof. For each $i \in \{1, 2, \ldots N\}$, let k_i be constant such that $k_i \in (0, 1)$ and is associated with f_i. Let $B, C \in K(\mathcal{M})$ such that

$$0 < \eta(G(B), G(C)) = \eta(\bigcup_{i=1}^{N} f_i(B), \bigcup_{i=1}^{N} f_i(C)).$$

Now Lemma 2.2.5 implies that

$$\begin{aligned} \eta(G(B), G(C)) = \eta(\bigcup_{i=1}^{N} f_i(B), \bigcup_{i=1}^{N} f_i(C)) &\leq \sup_{1\leq i\leq N} \eta(f_i(B), f_i(C)) \\ &= \eta(f_{i_0}(B), f_{i_0}(C)), \end{aligned}$$

for some $i_0 \in \{1, 2, 3, \ldots, N\}$. Using $\Theta 1$ and Lemma 3.2.2, we have

$$\theta(\eta(G(B), G(C)) \leq \theta(\eta(f_{i_0}(B), f_{i_0}(C))) \leq [\theta(\eta(B, C))]^{k_{i_0}}.$$

Therefore G is also a continuous weak θ_m-contraction on the complete metric space $(K(\mathcal{M}), \eta)$. Theorem 2.2.15 ensures the existence and uniqueness of $A \in K(\mathcal{M})$ such that $G(A) = A$ and $A = \lim_{n\to\infty} G^n(B)$ for all $B \in K(\mathcal{M})$.
This completes the proof. ∎

Take the example 1.3.6 and consider the set of functions f_1, f_2 as an IFS then we have Cantor set as the unique attractor
Till now we have stated and proved all the results on a single valued function. In the coming section we will extend these concepts to multivalued function and prove some results on it.

3.3 Weak θ_m iterated multifunction system

We begin this section with the defintion of multifunction system.

Definition 3.3.1. A function $f : X \to Y$ defined as :

$$f(x) = A \text{ for all } x \in X, \text{ where } A \subset Y.$$

or we can say tht a function f is multivalued if every value of domain of f is associated to more than one values of co-domain of f.
For instance every real number greater than 0 has two real square roots so square root can be considered as a multivalued functon. For example $\sqrt{4} = 2, -2$.

Let $(\mathcal{M}, \rho)$ be a metric space and $f_1, f_2, \ldots, f_n : \mathcal{M} \to K(\mathcal{M})$ be multivalued operator. Then the system $f = (f_1, f_2, \ldots, f_n)$ is called an iterated multifunction system (abbreviated as IMS).

Definition 3.3.2. Let $\{f_i\}_{i=1}^N$ be a finite family of iterated multifunction system . If $f_i : \mathcal{M} \to K(\mathcal{M})$ is a weak θ_m-contraction (for each i), then the family $\{f_i\}_{i=1}^N$ is called a *weak θ_m iterated multifunction system* (weak θ_mIMS).

Definition 3.3.3. If $\{f_i\}_{i=1}^N$ is weak θ_m IMS such that $f_i : \mathcal{M} \to K(\mathcal{M})$ is upper semi-continuous for $i = 1, 2, \ldots, N$ then the operator

$$G_f : K(\mathcal{M}) \to K(\mathcal{M}), G_f(Y) = \bigcup_{i=1}^{N} f_i(Y)$$

is well defined and is called weak θ_m multi-fractal operator. A fixed point of G_f is called a multivalued fractal.

Now we will use the following lemma to show that a weak θ_m multi-fractal operator has a unique multivalued fractal.

Lemma 3.3.4. *Let $f : \mathcal{M} \to K(\mathcal{M})$ is a continuous weak θ_m multivalued operator. Then the mapping $A \mapsto f(A)$ is also a weak θ_m multivalued operator from $K(\mathcal{M})$ into itself.*

Proof. Let $A, B \in K(\mathcal{M})$ be such that $\eta(f(A), f(B)) > 0$. Assume that

$$\begin{aligned}\eta(f(A), f(B)) &= D(f(A), f(B)) \\ &= \sup_{x \in A} \inf_{y \in B} D(fx, fy), \quad \text{for all} \quad A, B \in K(\mathcal{M}). \quad (3.3.\text{-}1)\end{aligned}$$

As f is a continuous weak θ_m multivalued operator so there exist $k \in (0, 1)$ such that

$$\theta(D(fx, fy)) \leq [\theta(\max\{D(x, fx), D(y, fy), d(x, y)\})]^k,$$

for all $x, y \in \mathcal{M}$.

Now using (3.3.0), compactness of A, and continuity of f, we can find $a \in A$ such that $D(f(A), f(B)) = \inf_{y \in B} D(fa, fy) > 0$, so that $D(fa, fy) > 0$, for all $y \in B$. Hence, for all $y \in B$, we have

$$\theta(\inf_{y \in B} D(fa, fy)) \quad \leq \theta(D(fa, fy) \quad \leq [\theta(max\{D(a, fa), D(y, fy), \rho(a, y)\})]^k.$$

Therefore, for all $y \in B$ we get

$$\theta(\eta(f(A), f(B)) \leq [\theta(max\{D(a, fa), D(y, fy), \rho(a, y)\})]^k. \qquad (3.3.\text{-}2)$$

Case 1: If $\max\{D(a, fa), D(y, fy), \rho(a, y)\} = D(a, fa)$, then we have:

$$\theta(\inf_{y \in B} D(fa, fy)) \leq [\theta(D(a, fa))]^k,$$

Now from (3.3) we have

$$\begin{aligned}\theta(\eta(f(A), f(B))) &\leq [\theta(D(a, fa'))]^k \\ &\leq [\theta(\sup_{a \in A} \inf_{fa \in f(A)} d(a, fa))]^k \\ &= [\theta(D(A, A))]^k,\end{aligned}$$

which is a contradiction.

Case 2: If $\max\{D(a, fa), D(y, fy), \rho(a, y)\} = D(y, fy)$, then proceeding in the same way as in Case 1 we again get a contradiction.

Case 3: If $\max\{D(a, fa), D(v, fv), \rho(a, y)\} = \rho(a, y)$, then for all $y \in B$ we have $\theta(\eta(f(A), f(B))) \leq [\theta(\rho(a, y)]^k$.
Now let $y \in B$ be such that

$$\rho(a, y) = \inf_{y \in B} \rho(a, y).$$

From (3.3) we have,

$$\begin{aligned}
\theta(\eta(f(A), f(B))) &\leq [\theta(\rho(a, y))]^k, \\
&= [\theta(\inf_{b \in B} \rho(a, b))]^k, \\
&\leq [\theta(\sup_{a \in A} \inf_{y \in B} \rho(a, y))]^k \\
&= [\theta(D(A, B))]^k \\
&\leq [\eta(A, B)]^k.
\end{aligned}$$

Hence we get the result.

□

Theorem 3.3.5. *If $\{f_i\}_{i=1}^N$ is a continuous weak θ_m IMS, then it has unique multivalued fractal. Moreover, $A = \lim_{n \to \infty} G_f^n(B)$ for all $B \in K(\mathcal{M})$, the limit being taken with respect to the Hutchinson-Pompeiu metric.*

Proof. For each $i \in \{1, 2, \dots N\}$, let k_i be constant such that $k_i \in (0, 1)$ and is associated with f_i. Let $B, C \in K(\mathcal{M})$ such that

$$0 < \eta(G_f(B), G_f(C)) = \eta(\bigcup_{i=1}^N f_i(B), \bigcup_{i=1}^N f_i(C)).$$

Now Lemma 2.2.5 implies that

$$\begin{aligned}
\eta(G_f(B), G_f(C)) = \eta(\bigcup_{i=1}^N f_i(B), \bigcup_{i=1}^N f_i(C)) &\leq \sup_{1 \leq i \leq N} \eta(f_i(B), f_i(C)) \\
&= \eta(f_{i_0}(B), f_{i_0}(C)),
\end{aligned}$$

for some $i_0 \in \{1, 2, 3, \dots, N\}$. Using $\Theta 1$ and Lemma 3.3.4, we have

$$\theta(\eta(G_f(B), G_f(C)) \leq \theta(\eta(f_{i_0}(B), f_{i_0}(C))) \leq [\theta(\eta(B, C))]^{h_{i_0}}.$$

Therefore G_f is also a continuous weak θ_m-contraction on the complete metric space $(K(\mathcal{M}), \eta)$. Theorem 2.2.15 ensures the existence and uniqueness of $A \in K(\mathcal{M})$ such that $G(A) = A$ and $A = \lim_{n\to\infty} G^n(B)$ for all $B \in K(\mathcal{M})$. This proves that the multi-fractal operator G_f has a unique multivalued-fractal. $\square$

Example 3.3.6. Let M=$[0, 1] \subset \mathbb{R}$, with the metric given by the absolute value. We define, $F : K(M) \to K(M)$ by

$$F(A) = f_1(A) \cup f_2(A),$$

where

$$f_1(x) = \tfrac{1}{3}x,\ f_2(x) = \tfrac{1}{3}x + \tfrac{2}{3},\ 0 \leq x \leq 1.$$

First we verify that f_1 and f_2 are weak θ_m contraction.
Take $\theta = e^x$ and $d(x, y) = |x - y|$, thus
$d(x, f_1x) = \left|x - \frac{x}{3}\right| = \left|\frac{2x}{3}\right|$ for all $x \in M$
Thus, $\max\{d(x, f_1x), d(y, f_1y), d(x, y)\} = \max\{\frac{2x}{3}, \frac{2y}{3}, \frac{|x-y|}{3}\}$
Case 1: Let $x > y$, $\max\{d(x, f_1x), d(y, f_1y), d(x, y)\} = \frac{2x}{3}$.

We know

$$, \frac{x-y}{3} \leq \frac{2xy}{3} \quad for\ all \quad x, y \in M \tag{3.3.-12}$$

$$\Leftrightarrow e^{\frac{x-y}{3}} \leq e^{\frac{2xy}{3}} = [e^{\frac{2x}{3}}]^y = [e^{\frac{2x}{3}}]^h, where\ h = y \in (0, 1).$$

We have
$\theta(d(f_1x.f_1y)) = e^{\frac{x-y}{3}} \leq [\theta d(x, f_1x)]^h, h = y \in (0, 1)$

Now take $f_2(x) = \frac{1}{3}x + \frac{2}{3}$, then $d(x, f_2x) = \left|x - \left(\frac{1}{3}x + \frac{2}{3}\right)\right| = \left|\frac{2x}{3} - \frac{2}{3}\right|$ for all $x \in M$. We know

$$\frac{x-y}{3} - \frac{2}{3} \leq \frac{2xy}{3} - \frac{2}{3} \quad for\ all \quad x, y \in M \tag{3.3.-12}$$

$$\Leftrightarrow e^{\frac{x-y}{3}-\frac{2}{3}} \le e^{\frac{2xy}{3}-\frac{2}{3}} \le e^{\frac{2xy}{3}} = [e^{\frac{2x}{3}}]^y = [e^{\frac{2x}{3}}]^h, wehaveh = y \in (0,1)$$

Thus we have
$\theta(d(f_2x.f_2y)) = e^{\frac{x-y}{3}-\frac{2}{3}} \le [\theta d(x, f_2x)]^h, h = y \in (0,1)$
Case 2: Now take $y > x$, we have $\max\{d(x, f_1x), d(y, f_1y), d(x,y)\} = d(y, f_1y)$ in this case we arrive at same conclusion as in case 1.
$d(x,y) \neq \max\{d(x, f_1x), d(y, f_1y), d(x,y)\}$, for any value of $x, y \in [0,1]$ thus from case 1 and case 2 we can say that f_1 is a weeak θ_m-contraction for $\theta = e^x$

In the similar way, we can prove that f_2 is also a weeak θ_m-contraction for $\theta = e^x$.
Thus $f = (f_1, f_2)$ is iterated multifunction system. The unique fixed point of F must satisfy

$$A = F(A) = f_1(A) \cup f_2(A).$$

Considering the nature of the two transformations, we get a unique fractal $A \subset K(M)$ which is Cantor subset of $[0,1]$.

In this chapter we have discussed about the existence and uniqueness of attractors of an IFS and we extend this concept to the multivalued function. We have also discused about the fixed point of a multivalued function and iterated multifunction system. After giving the brief introduction about multifunction system we introduce weak θ_m iterated multifunction system and existence of multifractal in it.

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