SOME ALGORITHMS FOR DECISION-MAKING PROBLEMS BASED ON THE EXTENSIONS OF INTUITIONISTIC FUZZY SETS

A Thesis

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**Some algorithms for decision-making problems based on the extensions of intuitionistic fuzzy sets**" in partial fulfillment of the requirement for the award of Degree of Philosophy and submitted in the School of Mathematics (SoM), Thapar Institute of Engineering & Technology, Patiala is an authentic record of my own, carried out during a period from January, 2017 to August, 2021 under the supervision of Dr. Harish Garg, Associate Professor, SoM, Thapar Institute of Engineering & Technology, Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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Abstract

Decision making (DM) is a cognitive process in which actions are taken to frame rational decisions subjected to problems. These problems may be a constituted part of any discipline such as engineering, economics, psychology, management etc. However, under a decisive situation, multi-criteria decision making (MCDM) problems are a valuable format of analyzing different alternatives classified under relevant criteria information. The persistent situations of modernization and rapid advancements have made data handling and processing highly vulnerable to uncertainties. To address the influence of uncertain information, theories such as fuzzy set (FS), intuitionistic fuzzy set (IFS), interval-valued fuzzy/intuitionistic fuzzy set (IVFS/IVIFS), hesitant fuzzy set (HFS) etc, have done a remarkable job. Apparently, amid the breakneck growth in uncertainty possessing situations, there felt need of advanced genres of the existing theories so that sure decisions can be framed out of the unsure data values. For it, progressive minds involved in research developed advanced tools such as aggregation operators and information measures. Deploying these tools in the DM approaches helps to choose a suitable alternative(s) out of the available ones.

Driven by the present state-of-art, this research work focuses on building a novel environment called cubic intuitionistic fuzzy set (CIFS) and its related aggregation operators and information measures. In addition to it, this work also addresses the uncertainty quantification under probabilistic environments where non-membership hesitant information plays a dominant role. For that, DM approaches as well as algorithms have been developed under probabilistic dual hesitant fuzzy set (PDHFS) environment. Under these, various statistical tools such as correlation measure, distance measure, entropy etc, have been proposed and several aggregation operators such as generalized operators, Einstein,

Bonferroni, Maclaurin Symmteric Mean operators etc, have been formulated. For strengthening the desirability and applicability of the proposed research, all related mathematical aspects in form of theorems, properties as well as results have been investigated in detail. In order to facilitate the practical DM scenarios, the presented work has also been applied to various application domains such as CPU job scheduling, personnel recruitment/selection problems, signal processing, inventory management, analysis of consumer's buying behavior, gesture quantification in brain hemorrhage patients etc. Afterwards, each proposed aspect is checked for its superiority or alignment in comparison to the existing work.

The present thesis is organized into twelve chapters which are briefly summarized below:

A brief account of the related work of various authors in the evaluation of decision making approaches by using several approaches is presented in the first chapter, while the **Chapter 2** focusses on all the basics and preliminary concepts related to the presented research work.

In **Chapter 3**, the novel concept of CIFS is proposed along with its mathematical properties and related results. The notion of CIFS has been provided an analytic analysis and cubic intuitionistic internal and external cubic sets are introduced. Various functional capabilities of CIFS are discussed by checking the inherent properties related to P(-R) union and intersection.

In **Chapter 4**, some series of aggregation operators under the CIFSs are developed along with their suitable properties. Also, operational laws, score function, and accuracy function under the P-order and R-order are defined and some weighted averaging and geometric aggregation operators are proposed. A DM method based on these operators is proposed for ranking different set of the alternatives classified under CIFS domain. Finally, an illustrative example based on CPU job scheduling is given to demonstrate the proposed approach.

In **Chapter 5**, the structural characteristics of CIFS are enhanced by defining generalized t-norm and t-conorm aggregation operators. These are formulated in a generalized format in which they are subjected to get reduced into some existing operators. Further, to strengthen the practical applicability of the proposed operators, a DM approach and a numerical problem based on disaster management is put-forth. The superiority analysis and validation of the proposed operators is done by comparing it with some existing theories.

Chapter 6 discusses the idea of Bonferroni mean based operators for the CIFSs to aggregate the information. The major advantage of the proposed operator is an inherent ability to capture the mutual interrelationships of aggregated values. Further, their properties are discussed and some special cases in alignment to the existing theories have been shown. Finally, a DM approach has been given for ranking the different alternatives based on the proposed operators. A practical example related to inventory management is provided to verify the developed approach and to demonstrate its practicality and feasibility.

In **Chapter 7**, a novel MCDM method for multiple number of experts under CIFS environment by is proposed integrating extended TOPSIS method. In it, some series of distance measures and their various properties are investigated. Further, a group DM method based on the proposed measure is presented by taking the different priority pairs of the decision makers and it is applied to problem related to recruitment process. To demonstrate its practicality and feasibility, the results are compared with the several existing approaches outcomes.

In **Chapter 8**, a nonlinear programming model based TOPSIS approach is developed. This is intended to solve MCDM problems with incomplete weight information. The relative closeness coefficient degree of the TOPSIS method is formulated based on a distance measures. Several special cases of the proposed approach are discussed in detail. For signifying the practical rationality of the proposed models, the work is applied in the area of signal processing and comparison analysis with existing studies has been conducted.

In **Chapter 9**, an attempt has been made to capture uncertain information in form of PDHFS. Several weighted and ordered weighted averaging and geometric aggregation operators are presented under Einstein norm operations. In addition to it, two distance measures are proposed. Based on them, maximum deviation method to compute the weight vector of the different criteria is utilized. Finally, an MCGDM approach is constructed based on proposed operators and the presented algorithm is explained with the help of the numerical example based on consumer's buying behavior.

In **Chapter 10**, some weighted averaging and geometric MSM operators are proposed to address the uncertainties in the medical diagnosis problems. Some desirable properties of the operators are discussed and an optimization model based on Shanon's entropy for determining the probabilities is framed. A DM approach is developed based on the proposed operators followed by a suitable comparison analysis and a case study is conducted on gesture quantification of a patient suffering from hemorrhage strokes.

In **Chapter 11**, a method to solve the MCDM problem under PDHFS environment. For it, the informational energy and the covariance between two PDHFSs are defined and their properties are studied in detail. Additionally, correlation coefficients and the weighted correlation coefficients for PDHFSs are developed. In the formulation, PDHFSs are able to represent the information in terms of their respective degrees while the assigned probabilities give more details about the level of agreeness or disagree. A novel algorithm based on personnel selection is developed analyzing the applicability of the proposed information measure to solve MCDM problem.

In **Chapter 12**, the concluding observation discussing the contributions of the proposed research and future scope of related work is discussed.

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List of Publications

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(J1) Harish Garg, Gagandeep Kaur, Quantifying gesture information in brain hemorrhage patients using probabilistic dual hesitant fuzzy sets with unknown probability information, *Computers & Industrial Engineering, Elsevier*, 140, pp. 106211, 2020, doi: 10.1016/j.cie.2019.106211 (**SCI: Impact Factor: 5.431**)

(J2) Harish Garg, Gagandeep Kaur, A robust correlation coefficient for probabilistic dual hesitant fuzzy sets and its applications, *Neural Computing and Applications, Springer*, 32(13), pp. 8847 - 8866, 2020, doi: 10.1007/s00521-019-04362-y (**SCI: Impact Factor: 5.606**).

(J3) Harish Garg, Gagandeep Kaur, Extended TOPSIS method for multi-criteria group decision-making problems under cubic intuitionistic fuzzy environment, *Scientia Iranica*, 27 (1), pp. 396 - 410, 2020, doi: 10.24200/SCI.2018.5307.1194 (**SCI: Impact Factor: 1.435**)

(J4) Harish Garg, Gagandeep Kaur, Novel distance measures for cubic intuitionistic fuzzy sets and their applications to pattern recognitions and medical diagnosis, *Granular Computing, Springer* 5(2), pp. 169 - 184, 2020, doi: 10.1007/s41066-018-0140-3.

(J5) Harish Garg, Gagandeep Kaur, Cubic Intuitionistic fuzzy sets and its fundamental properties, *Journal of Multiple-valued Logic and Soft Computing*, 33 (6), 507 - 537, 2019 (**SCI: Impact Factor: 0.861**).

(J6) Harish Garg, Gagandeep Kaur, TOPSIS based on nonlinear-programming methodology for solving decision-making problems under cubic intuitionistic fuzzy set environment, *Computational and Applied Mathematics*, 38, 114, 2019, doi: 10.1007/s40314-019-0869-6 (**SCI: Impact Factor: 2.239**).

(J7) Gagandeep Kaur, Harish Garg, Generalized cubic intuitionistic fuzzy aggregation operators using t-norm operations and their applications to group decision-making process, *Arabian Journal for Science and Engineering*, 44(3), 2775 - 2794, 2019 (**SCI: Impact Factor: 2.334**).

(J8) Gagandeep Kaur and Harish Garg, Cubic Intuitionistic fuzzy Aggregation operators, *International Journal for Uncertainty Quantification*, 8(5), 405 427, 2018, doi: 10.1615/Int.J.UncertaintyQuantification.2018020471 (**SCI: Impact Factor: 2.083**).

(J9) Harish Garg, Gagandeep Kaur, Algorithm for Probabilistic dual hesitant fuzzy multi-criteria decision making based on aggregation operators with new distance measures, *Mathematics*, 6(12), 280, 2018; doi: 10.3390/math6120280 (**SCI: Impact Factor: 2.258**)

(J10) Gagandeep Kaur and Harish Garg, Multi-attribute decision-making based on Bonferroni mean operators under cubic intuitionistic fuzzy set environment, *Entropy.* 20(1), 65, 2018; doi:10.3390/e20010065 (**SCI: Impact Factor: 2.524**).

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Decision making (DM) is the most fundamental task associated with human mind. It is an integral part of the activities undertaken at various working domains such as academia, industry, medical field, management areas, etc. In all these working genres, sound and rational decisions are the foundation of a well-sustained and progressive system. However, each prevalent situation under which decisions are to be taken always have risk as an inescapable part. Addressing these risks becomes an important activity while framing the decisions. This brings into light the situations in which there is more than one available alternative under which risk factor may vary from different alternatives. Thus, the DM task becomes crucial with the advent of more than one available choice to be chosen from. In addition to it, each alternative is subjected under different criteria [31]. Each criterion induces its characteristics into the alternatives which are needed to be contemplated in the DM process. Reflecting all these sensitivities towards the considered criteria information, the process of ranking the alternatives fulfilling the objectives of the problem, is known as multiple criteria decision making (MCDM). In literature, the relevant criteria information is also labeled as attributes and the respective DM process is addressed as multiple attribute decision making (MADM).

A conventional MCDM problem comprises of a set of alternatives $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m\}$ characterized by various criteria $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\}$. Each criterion may have its varying priority value incorporated into a weight vector $(\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. The decision-matrix corresponding to given information about alternatives as well as criteria is given as:

$$\begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{V}_{11} & \mathcal{V}_{12} & \ldots & \mathcal{V}_{1n} \\ \mathcal{V}_{21} & \mathcal{V}_{22} & \ldots & \mathcal{V}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{V}_{m1} & \mathcal{V}_{m2} & \ldots & \mathcal{V}_{mn} \end{pmatrix} \end{array}$$

where $\mathcal{V}_{ij}$ represents the preference value of the decision-maker of the related alternative $\mathcal{V}_i$ $(i = 1, 2, \ldots, m)$ towards the criteria $\mathfrak{B}_j$ $(j = 1, 2, \ldots, n)$. Nevertheless, this structure of MCDM allows only one decision-maker to provide the preference rating values, but the concept of group decision-making allows more rational and impartial decisions. Consequently, such a domain of DM in which more than one expert provides the rating values, is called multiple criteria (or attribute) group decision making (MCGDM or MAGDM).

However, rigorous advancement and involvement of DM in day-to-day activities poses the situation in which uncertainty plays a dominant role. In order to apprehend this concept of uncertainty, the theory of fuzzy sets (FSs), developed by Zadeh [212], is used by decision-makers. In contrast to crisp sets, FSs give a flexibility to express the partial belongingness of an entity to a set, through membership grades between 0 and 1. Utilization of this theory bring revolution in capturing uncertainty. Subsequently, advanced versions of FSs such that intuitionistic FSs (IFSs) [5], interval-valued IFSs (IVIFSs) [4] are also favored by experts. In these theories, non-membership degrees (NMDs) are considered to be equally important as that of the membership degrees (MDs). Besides this, many prevalent circumstances lead to situations in which the MDs cannot be determined uniquely and expert find it difficult to capture uncertainty by single MD. Such situations are facilitated by hesitant fuzzy sets (HFSs) [146, 175] for hesitant MDs and dual HFSs (DHFSs) [236] for capturing both hesitant MDs as well as NMDs.

The traditional advanced versions of FSs, deal with the uncertainties in an efficient manner, but they have some of the inherent inadequacy in enduring the increased extent of imprecise information. In order to withstand the situations of capturing hybrid information, decision-makers use cubic FSs (CFSs) whose advanced characterization as cubic IFSs (CIFSs) provide flexibility to use the inherent features of both IVIFSs as well as IFSs.

Apart from these, the extended versions like probabilistic HFSs (PHFSs) [235] and probabilistic dual HFSs (PDHFSs) [61] contemplate the uncertain situations by taking into account the corresponding probability information. In observance of rapid advancement in the amount of uncertainty, the main aim of this work is to formulate some algorithmic approaches assisting the DM problems based on the extensions of IFSs.

1.1 Literature review

1.1.1 Review on distance and similarity measures

Distance and similarity are metrics used for satisfying complement perspectives. While distance measure is used for eliciting differences between two objects, similarity measure supports the context of outlining the closeness between them. However, both of them process the input to give output in form of a numerical value. In fuzzy set theory, both these measures hold an important place. Distance measure is used in figuring out the distinctive features and similarity measure is used for highlighting close proximities of the input values. Numerous attempts have been made by researchers in developing various distance and similarity measures. For instance, under IFS, the notion of Euclidean and Hamming distances was introduced by Szmidt and Kacprzyk [139] and Grzegorzewski [56] developed distances based on Hausdorff metrics. Wang and Xin [165] introduced a generalized interpretation of distance measure under IFS environment. However, the concept of similarity measures on IFSs was proposed by Dengfeng and Chuntian [32] and it was further modified and enhanced by Mitchell [109]. Hung and Yang [68] introduced a new version of similarity measure based on Hausdorff distance and consequently Szmidt and Kacprzyk [140] put-forth the utilization of similarity measures in medical diagnosis under IFS. Afterwards, Chen and Randyanto [24] gave new similarity measures overcoming the drawbacks of [32, 68, 109]. Moreover, some attempts on the similarity measures were related to its comprehension in advanced manner such as Chen and Chang [19] developed a novel similarity measure based on transformation techniques and Garg [46] developed distance and similarity based on intuitionistic multiplicative preference relations. Apart from them, Chen, Cheng and Lan [21] developed a similarity measure based on centroid points of transformed fuzzy numbers. Enhancing the scope of uncertainty capturing from

IFS to IVIFS, Xu and Chen [188] gave an extensive overview of the distance and similarity measures created under IFS environment and extended their scope by developing these measures for IVIFSs. Also, Zhang et al. [220] processed IVIFS information by introducing distances, similarity and inclusion measures and justified their transformations among one another by giving axiomatic definitions. Afterwards, Dugenci [34] corroborated a novel distance measure for IVIFSs with an MCDM approach processing information under incomplete weight information. Li, Pelusi and Deng [92] generate two-dimensional belief function based on an improved similarity measure.

In contrast to above stated theories, an advanced version existed as HFSs in which multiple valued opinion can be contemplated. In direction of it, Xu and Xia [190] introduced the concept of distance and similarity measures. Afterwards, Li et al. [86] and Zeng et al. [213] outlined shortcomings inherited in measures proposed in [190], and gave new measures considering the simultaneous hesitance degree between the input sets. Besides that, Farhadinia and Xu [39] introduced the concept of generalized hybrid normalized distance based on HFSs. Also, Hu et al. [64], developed a novel similarity based on interval bound footprint of an HFS. Furthermore, incorporating the hesitant NMDs along with the MDs, Wang, Xu, Wang and Ni [157] introduced Hamming distance and similarity measures for DHFSs and also formulated their generalized version. Subsequently, Su et al. [135] investigated the preference values by developing a series of Hamming, Hausdorff and Euclidean distances and their corresponding similarity equivalents. Apart from it, Zhang et al. [224] proposed the cosine similarity measure under DHFS. Holding the capacities of dealing with the probabilities, Ding et al. [33] developed Hamming and generalized normalized distances for PHFSs. Afterwards, Li, Niu, Chen and Wu [87] analyzed the inherited shortcomings in [33] and introduced Euclidean distance measure.

The informational measures such as distance and similarity act as building blocks on which various approaches such as Combinative Distance-based Assessment (CODAS) [55], Weighted Distance Based Approximation (WDBA) [120] and Technique of order preference by similarity to ideal solution (TOPSIS) [69] depend. Among these DM approaches, TOPSIS holds a peculiar place. In this, two ideal solutions (also known reference ideals) *viz.* positive ideal solution (PIS) and negative ideal solution (NIS) are determined.

Choosing PIS and NIS is a subjective process, typically based on an elementary objective of attaining maximum characteristics of benefit criteria, while on the other hand, NIS, primarily elicit minimum characteristics of cost criteria. Both these ideal solutions are considered as hypothetical reference points in the approach and all the available data values are judged for more closeness to PIS and large proximity from NIS. Keeping these features in mind, researchers have enhanced the functional abilities of their proposed measures by engaging them in TOPSIS. For example, Hung and Chen [66] developed a TOPSIS model with entropy weight consideration. However, Joshi and Kumar [75] incorporated separation measures from the ideal solutions and employ them in TOPSIS based on IFS and Chen, Cheng and Lan [20] further, listed the underlying problem of division by '0' in [75] and developed new separation measures as improvised TOPSIS version. A group DM process under IFS environment was given by Izadikhah [70]. Apart from it, magnifying the scope of uncertainty capturing ability from IFS to IVIFS, Li [85] introduced a TOPSIS based on non-linear programming model for DM. Bai [7] gave TOPSIS approach based on improved score function. Biswas and Kumar [10] introduced a TOPSIS based integrated group DM approach under IVIF rating values. However, under hesitant environment, Zeng and Xiao [215] gave method of TOPSIS for HFSs and Wang et al. [158] developed a new distance measure based TOPSIS technique for DHFSs. In addition to it, faciliting the TOPSIS technique for multiple decision-makers, Lan et al. [83] put-forth MAGDM model for HF preferences. Moreover, He and Xu [62] proposed TOPSIS model for PHFS environment. Besides the above listed theories, researchers also applied the TOPSIS technique in advanced real life domains such as credit risk evaluation [128], investment selection [214], prospective supplier selection [13], E-Commerce [78], customer satisfaction [2], OTA websites [1] etc.

1.1.2 Review on correlation coefficients

Correlation holds a peculiar place in statistical analysis that reflect the mutual dependance in two variables. It signifies, how well two variables are inter-related and captures their linear relationship. But, accomplishing the practical realms of day-to-day activities, correlation is needed to be having more features and capacity of uncertainty handling. Based on this need, the research on correlation gained momentum in capturing the situational ambiguity. For example, Gerstenkorn and Manko [54] gave correlation on IFSs. Hong and Hwang [63] modified the loopholes in the existing correlation [54], and proposed a new one in context of probability spaces. Further, Hung [67] proposed Pearson's correlation coefficient on IFSs. Similar attempt was made by Mitchell [110] in establishing a novel correlation under the IFS environment. Additionally, Zeng and Li [216] introduced the concept of correlation as an analogy to cosine intersectional angle in IFSs. Based on the statistical correspondence with the crisp correlation coefficients, Szmidt and Kacprzyk [141] analyzed that the existing correlations on IFSs [54, 63, 216] failed to derive the negative correlation among the sets. So, they made an attempt to propose a correlation ranging between -1 and 1. Apart from that, Garg [47] gave a novel correlation under IF multiplicative environment. In contrast to these theories, researchers extended the correlation-based theories to IVIFSs too, such as, Bustine and Burillo [16] introduced a correlation coefficient for IVIFSs. Furthermore, Xu [185] made a notable contribution in reviewing the correlation coefficients on IFSs and extended them to IVIFS environment. Additionally, Park et al. [115] modified the existing correlation [16], and developed an optimization model on it. Subsequently, Ye [200], provided a notion for weighted correlation coefficients based on the information obtained through entropy and applied to MAGDM problem. Besides this, Wei et al. [170] developed an approach based on deriving correlation measure through incomplete weight information.

While the above theories contemplate the MDs as well as NMDs, nevertheless, they fail to address the efficient handling of multiple hesitant values which may come from a single set. For that, Xu and Yager [193] proposed a notion of correlation coefficients under HFSs. Chen et al. [18] derived correlation and applied them in clustering analysis. These studies inherited some setbacks such that, the length of HFSs needed to be matched before computation, based on expert's attitude of optimism or pessimism. Moreover, these correlation measures did not justified the negative correlation between the input sets. So, Liao et al. [94] analyzed these shortcomings of the existing studies [18, 193] to give novel correlation measures and investigated their applicability in medical diagnosis as well as cluster analysis. Apart from them, some improved attempts were also made by Meng and Chen [107]. Also, Guan et al. [57] gave synthetic correlation coefficients under HFS environment. Taking into account the hesitancy values related to the NMDs, Farhadinia [37] introduced the concept of correlation coefficients for DHFSs. Followed by it, Wang et al. [155] also made a similar attempt and applied it to the medical diagnosis process. Further, Ye [201] proposed a notion for correlation coefficient for which length of the input sets was not needed to be matched and justified applicability in investment policy DM process. However, another advanced application based correlation coefficient was introduced by Tyagi [148] in which relationship between water in different lakes was studied. Furthermore, addressing the aspect of probabilities along with the hesitancy, Wang and Li [168] established the concept of correlation coefficients under PHFS environment. Later, Song et al. [134] devised a modified version of the previous attempt [168] by addressing negative correlation as well under the same environment.

Along with the above listed theories, several attempts have been found on TOPSIS based on correlation coefficients. For instance, Solanki et al. [132] developed a TOPSIS method based and applied it to supplier selection problem. Li et al. [89] gave a new method involving correlation based TOPSIS along with quality function deployment under IFS environment. However, Chen and Lu [26] assessed the competition among different insurance corporations by using correlation TOPSIS approach. Also, Dammak et al. [28] studied the impact of weight techniques in TOPSIS method based on correlation coefficients having IFS rating values. Additionally, Sun et al. [136] presented a TOPSIS based DM process under HFS domain. Zhang et al. [225] proposed a HFS based MADM based on linear programming and TOPSIS. Another notable attempt was made by Tian et al. [144] in applying the correlation based TOPSIS method in green performance evaluation of alternatives. Hence, from the literature, it is evident that correlation plays a dominant role in capturing attention of researchers in the associated field of study.

1.1.3 Review on aggregation operators

Aggregation operators (AOs) play a very important role in processing the available information by fusing together the input values according to the desired DM scenario. Undergoing AOs, the information containing data-values are fused together to form a single output unit. This single generated output capture the characterizations of all the input arguments. So, the output generated by AOs being a collective representative of more than one input arguments, specifically occupy an integral place in the DM process. For instance, Bustince et al. [15] proposed various generators and applied them to IFS theory. Further, Xu [186] proposed average AOs on IFSs. Also, Yager [195] gave ordered weighted average AOs and applied it to MCDM problem. Moreover, Beliakov et al. [8] also gave his remarks on averaging AOs by solving some of its desirable properties. Corresponding to these studies, Xu and Yager [191] proposed the geometric AOs based on IFSs. Proceeding towards the establishment of general layout of AOs, Xia et al. [173] addressed the issues of AOs based on Archimedean t-conorm and t-norm. Yager [196] also put-forth the generalized ordered weighted AOs on IFSs and in the similar direction, Zhao et al. [229] established the concept of generalized AOs including the ordered weighted formats. Subsequently, Einstein AOs for IFSs were given by Wang and Liu [160]. Further, Li [84] developed a DM methodology based on generalized ordered weighted AOs and Wei [169] gave some induced geometric AOs with IFS information. The above AOs are based on the independent arguments. Nevertheless, to achieve the dependency aspect among the input arguments, AOs were made based on Bonferroni [11] mean as well as Maclaurin symmetric mean [101]. Primarily, Xu and Yager [193] worked on developing Boneferroni mean operator. Further, Qin and Liu [119] gave an approach based on Maclaurin symmetric mean operator under IFSs. Contemplating the uncertainties in a wider aspect related to IVIFS, Atanassov [6] gave various operations on IVIFSs and Xu [187] proposed methods for aggregating the IVIFS information using weighted average, order weighted average as well as hybrid averaging AOs. Further, Xu and Chen [181] induced the concept of AO to implement MAGDM problems with IVIFS based judgement matrices. Xu and Chen [182] also developed some geometric averaging AOs. Wei and Wang [171] not only introduced hybrid geometric AOs but also applied them to solve MAGDM problems. Apart of them, some researchers [30, 40, 72, 103, 112, 114, 142, 163, 164, 199] developed various operators such as Einstein AOs and constructed novel score and accuracy functions for facilitating the DM techniques under IVIFS environment. Garg [48] proposed some robust geometric AOs under the same. Moving towards information fusion by capturing the dependent arguments, Xu and Chen [183] established the concept of Bonferroni mean. Subsequently, Sun and Xia [137] gave Maclaurin symmetric mean based AOs for IVIFSs. Establishing the AOs for the advanced environment such as CFSs, Khan et al. [79] gave cubic aggregation operators and Fahmi et al. [36] developed Einstein AOs and applied them to DM process.

The aforementioned studies do not handle the uncertainties available in discrete format signifying the expert's hesitation towards providing the rating values as in case of HFSs and DHFSs. For that, Xia and Xu [175] proposed information AOs under HFSs. Some quasi-arithmetic HFS based AOs were also developed by Xia et al. [174] and Yu [205] gave AOs based on Einstein average. However, Zhou and Li [233] established the concept of geometric Einstein AOs under HFSs. Moving further towards DHFSs, Yu, Zhang and Huang [210] developed AOs and Wang, Li and Zhou [151] proposed AOs with applications to MCDM problems. Contemplating the corresponding probabilistic information, Zhang et al. [221] developed various operations and evaluation techniques related to PHFSs. Based on it, Jiang and Ma [74] developed average AOs and Park et al. [117] introduced the Einstein AOs for PHFSs. From all the above listed theories, it is evident that AOs plays a central role in enhancing the computational capability of the DM techniques by fusing together the input information to give a collective output.

1.1.4 Review on dual hesitant fuzzy sets

The above sections focus on various tools and techniques which are being applied in DM scenarios for processing the information. However, the environment-centric review based on DHFSs is given in this section. DHFSs, primarily take into account, the membership and non-membership values into multiple numbers. Unlike considering the interval range as that of IVIFSs, DHFSs characterize the multiple values into discrete format. For instance, a person wants to buy a commodity and he is 20% and 25% willing to buy it, but 40% and 42% not willing to buy it. So, the corresponding DHFS is be formed as $(\{0.20, 0.25\}, \{0.40, 0.42\})$. Thus, the hesitant MDs and NMDs are encapsulated together to form a DHFS. This gives the expert a freedom to address large number of hesitant truth and falsity values particularly. The notion of DHFSs was introduced by Zhu et al. [236].

The concepts related to basic mathematical operations such as union, intersection etc, were introduced by Zhu and Xu [234], in which the difference of DHFSs from the existing FS environments was highlighted. Moreover, Farhadinia [38] presented the notions related to division and subtraction operations for the hesitant environments. In DM approaches, information is processed in several ways; one of which is to characterize the available preference values in form of information measures while the other widely used method is to process the information using AOs. In direction of this, several attempts have been made by researchers under DHFSs such as Zhang [218] gave distance and entropy measures. Similarly, Zhao and Xu [230] provided another entropy measure for DHFS information. Apart from it, Ye [202] gave the notion of cross entropy and applied it to MADM process. Yuan and Meng [211] proposed a similarity measure with prospective applications, whereas Ren and Wei [121] gave dice similarity under DHFSs. Also, Chen et al. [17] developed DM approach on the basis of information measures and an MADM method, particularly, based on correlation coefficient was presented in [27]. Nevertheless, these listed contributions focussed on development of information measures, but for computing the fused values of the input data entities, AOs were proposed for DHFS environment under different norms such as Archimedean norm [206], Einstein norm [228], Hamacher norm [76] etc. However, researchers also focussed on presenting the generalized AOs which got reduced to these norm operators after particularization. For instance, Wang, Li and Zhou [151] gave generalized AOs, Yu [204] developed generalized geometric AOs and Xing et al. [177] corroborated generalized point AOs based on DHFSs.

All the above listed operators inherit the feature of contemplating the independent characterization of the input values, however, if the expert require to fuse together the preference values by addressing the dependency factor, then AOs based on various mean operators such as Bonferroni and Maclaurin symmetric mean are utilized. Under DHFSs, Tu et al. [147] put-forth AOs based on Bonferroni mean and Zhang [227] developed the notion of AOs based on Maclaurin symmetric mean. Apart from it, Wang et al. [154] presented the generalized Bonferroni mean based AOs. However, similar attempts have been made in applying the mean based operators in real-life centric advanced scenarios like deciding the energy policy for society [71], supplier selection problem [208] etc. Furthermore, several researchers [29, 65, 97, 98, 143, 156, 179, 207, 217] worked on developing notions for various theories based on AOs under the DHFSs. Subsequently, the role of DHFSs in real life applications has been highly strengthened by some notable attempts associated to various real-life inclined domains such as evaluation of product design quality [178], portfolio selection [90], security system assessment [209], cooperative partner selection [122], evaluation of cloth creative design [91], customer service management [159], T-S fuzzy system [73], support vector [43, 125] etc. Thus, DHFSs hold an eminent position in addressing the handling the hesitant information and is very closely related to the pragmatic situations encountered in the real-life scenarios.

1.1.5 Review on probabilistic hesitant fuzzy sets

Probability theory hold a significant place in statistical analysis. Classical interpretations show its role as a building block of many application domains such as economics, behavioral sciences, trend analysis etc. Running parallel to it, the probability theory together with HFS theory gave rise to PHFSs, which show remarkable capacity in modeling the uncertainties in real life. Basically, PHFS is a set designed by Zhu and Xu [235] which encapsulates the hesitant membership values with probability. For example, two persons A and B provide their opinion regarding buying a commodity X. Suppose opinion provided by A is noted in form of HFS as $(\{0.20, 0.30\})$ and B gave opinion as $(\{0.20, 0.25\})$. Now, both the experts are providing different opinions regarding the same commodity X. This is a common problem that arises in the real life DM scenarios where different experts give different rating values. To address this case, the information is combined into PHFS by analyzing the probabilities of decision given by both the experts. The PHFS, thus formed, is given as $\left(\left\{0.20\middle|\frac{0.5+0.5}{2}, 0.30\middle|\frac{0.5}{2}, 0.25\middle|\frac{0.5}{2}\right\}\right)$. In simple form, it is $\left(\left\{0.20|0.5, 0.30|0.25, 0.25|0.25\right\}\right)$. Along with it, if the designated expert also takes into account the non-membership aspect, then the resulting information is captured in form of PDHFS. These two advanced fuzzy set environments attained the interest and attention of various researchers. For instance, Zhang et al. [221] provided operational techniques based on PHFSs. Alongside, Xu and Zhou [194] proposed weighted as well as ordered average AOs and also gave a maximizing score deviation method under the same. Furthermore, Jiang and Ma [74] gave Frank AOs whereas Park et al. [117] put-forth the Einstein AOs. Ding et al. [33] introduced distance measures along with a technique for determining the incomplete weight information. Following that, Li, Niu, Chen and Wu [87] gave a consensus based MCDM method pointing out the shortcomings and improved version of distance measures in [33]. He and Xu [62] also investigated PHFS environment for proposing a new distance measure and gave an MADM method based on reference ideal theory. Also, correlation coefficients were formulated and applied in the DM process by Wang and Li [168]. Apart from it, detailed analysis in MCDM based on dominance degree was done by Li et al. [88]. Zhou and Xu [232] established the notion for consistency improvement for the related preference relations. Besides them, some authors applied the PHFS information in venture capital project [223], in time series forecasting [58] and in measuring the group consistency encountered in DM framework [231]. A detailed overview of the benefit of utilization of probability-based expressions for DM techniques was given by Xu et al. [184]. Constituting the role of non-membership along with the membership, Hao et al. [61] initiated the notion of PDHFSs and developed AOs based on it. Later, Ren et al. [123] developed an equi-probabilistic distance measure and applied the theory in enterprize strategy management. Ren et al. [124], also, gave the strategy selection problem solving techniques based on artificial intelligence based on PDHFSs. Similar notions for strengthening the applicability of PDHFSs were given by Hao et al. [60]. All these theories elicit the remarkable capacity and role of PDHFSs in the DM process.

1.1.6 Review on cubic fuzzy set and its extensions

Cubic fuzzy set (CFS) is a prominent environment with striking features inherited from both IVFS and FS. Consequently, it holds the information capturing capabilities of the two stated environments merged together to form a single unit. For instance, a person wants to buy a commodity and is $10-20$ % sure about his decision of buying and further after evaluating some related pros and cons, he is feeling unsure or sure regarding his buying decision by 30%, then the corresponding CFS is formulated as $([0.10, 0.20], 0.30)$. It will form R-ordered CFS if 0.30 is the disagreeness towards the interval $[0.10, 0.20]$ and will be a P-ordered CFS, if 0.30 denotes the agreeness. Thus, CFS is an hybrid environment which is assimilation of two classical environments. The concept of CFS was put-forth by Jun et al. [77]. In this theory, basic axiomatic expressions of CFSs were proposed. It also established a strong mathematical investigation regarding the algebraic properties possessed by both R-ordered and P-ordered CFSs. Later, Khan et al. [79] developed AOs on cubic sets and Shakeel [126] also gave ordered weighted AOs and applied them to MAGDM problem. However, Shakeel et al. [127] also developed AOs along with distance and Fahmi et al. [36] introduced the Einstein AOs on CFSs. These proposed AOs were checked for their applicability to DM problems and justified the working of CFS as a generalized and upgraded version of the classical FS theories. Apart from it, Fahmi et al. [35] utilized the concept of CFSs employed together with CFSs in precursor selection for synthesis of titanium carbide nano-powders. This contribution strengthened the scope of CFS theory in sensitive information containing situations. The research in the field of CFSs gained momentum with the advent of cubic hesitant fuzzy sets (CHFSs) [104], in which the hesitant information related to CFS was modeled. Subsequently, Mahmood et al. [105] gave generalized AOs on CHFSs and contributions related to similarity measure was put-forth by Yong et al. [203]. Besides this, Mehmood et al. [106] developed DM method by utilizing the heronian mean operator. Nevertheless, the use of theories based on CFSs was strengthened by contributions based on application in medical field such as Fu et al. [41] evaluated the risk factors for patients of prostrate cancer. Also, the evaluation method for analyzing the benign prostatic hyperplasia symptoms was given by Fu et al. [42]. From all these related studies based CFSs, it is clear that this environment facilitates the expert in giving reasonable opinion without sticking to any one of the IVFS or FS only. An expert can model the opinion subjected to the situational diversity and can easily fit the information into the environment based on CFS.

1.2 Gaps and research motivation towards cubic intuitionistic fuzzy sets and probabilistic dual hesitant fuzzy sets

In the previous sections, we have reviewed different theories dealing with the incorporation of uncertain information in DM scenarios. Based on that, the present section discusses their incapabilities and focus on the motivation towards filling the gap after scrutinizing their underlying effects. The existing theories can efficiently handle the uncertain information minutely and it can be noticed that there is difference in the information modeling capacity among them. FSs (IFSs) being the foundational theories acknowledge information in single values associated to MDs (MDs and NMDs both). However, progressing towards the extended versions such as IVIFSs, CFSs, HFSs etc, a remarkable distinction is found in information consideration by various environments. Broadly, distinguishing between them, we categorize the information handling by environments into continuous and discrete way. Continuous environments contemplate the information by considering uninterrupted progression from one real number to another. For example, IVIFS as well as CFS have input fragments in form of intervals which start from one real number, consist of continuous intake of numbers till the one up-to which the interval lasts. Supposing the membership interval to be $[0.10, 0.20]$, all the real numbers in between 0.10 and 0.20 are considered without any break. On the other hand, the discrete environments such as HFSs and DHFSs consist of distinct values while modeling the available data. For example, taking $(\{0.10, 0.20\})$ as membership distinctively include the points 0.10 and 0.20 only without considering all the values enclosed with-in. In light of this, we derive motivation of bridging the intrinsic gaps in both continuous and discrete environments as below:

(i) The existing studies [4, 5, 14, 30, 167] do not emphasize on hybrid as well as two dimensional data consideration. In the race of proposing a theory that inclines well with the need of the DM scenarios, the existing contributions are found to be having loose grip in modeling the hybrid as well as information dispersing over more than one time phrase. Apart from that, no adequate attempt is found in demonstration of such environments which facilitate the expert to provide acceptance/rejection towards the considered membership values. Consequently, there is need of proposing a new environment that can address the existing shortcomings and can contemplate the uncertain information more minutely.

(ii) From the literature of HFSs and DHFSs [146, 151, 157, 175, 236], it is observed that in the DM framework, the subjective preference of an expert towards any particular hesitant value is left unnoticed. For instance, in situations having multiple values for membership, if any one expert provide judgement in form of HFS and he is preferring some of his hesitant values more as compared to the other ones, then the traditional theories do not consider this aspect of providing judgement. To address such cases, theories consisting the probabilistic information are preferred. For that, significant contributions of PHFSs play distinguishable but they do not accentuate the impact of NMDs into analysis. Apparently, NMDs hold a representative position in analysis rather than only focussing on the MDs as the practical situations have a considerable amount of in-born falsity which cannot be ignored completely. Thus, there is need of enhancing the conceptual notions based on contemplating the NMDs along with the probabilistic information.

(iii) Taking into account the work carried in existing environments [6, 77, 114, 235], gap has been found in the capturing uncertain information under such environments which not only consider the two-dimensional or hybrid information but also capture the associated falsity values. Also, there has been wide gap in such DM theories which contemplate the probabilistic information along with the NMDs. So, there is a need to bridge this gap for facilitating the effective data processing.

(iv) The data encountered in real life being affected by a large number of factors, often, have non-linear nature. So, in light of the previous stated points there is a need to handle non-linear formulation based techniques as well as optimization model for efficient modeling of available data sets.

1.3 Objective of the thesis

Deriving motivation from the above listed gaps, the purpose of the present thesis is to deal with the uncertain information by introducing new DM techniques under advanced extensions of fuzzy/intuitionistic fuzzy environments. The complete list of objectives is summarized as follows:

(O1) To develop some new decision-making approaches on discrete/continuous extensions of fuzzy/intuitionistic fuzzy sets.

(O2) To formulate some non-linear optimization model to solve the decision-making problems.

(O3) To test and validate the proposed technique on decision making problems in some fields.

1.4 Structure of the Thesis

The present thesis is assembled into twelve chapters including the present one that contains mainly the literature review. The rest of the chapters are described below:

In **Chapter 2** the basics and the preliminaries related to the IFS, IVIFSs, CFSs, HFSs, DHFSs, PDHFSs etc., which are to be used in the subsequent chapters are given.

Chapter 3 presents the notion of cubic intuitionistic fuzzy sets (CIFSs), its properties and related results. Also, we have introduced a new notion, called a cubic intuitionistic fuzzy set, internal (or external) cubic intuitionistic fuzzy set and have investigated their several properties. As a generalization of the IFSs and IVIFSs, CIFS is a strong and valuable tool to represent the imprecise information by embedding both the features of IFSs and IVIFSs instantaneously. Based on its inherent property, we further define the internal (external) CIFS, P(R)-union and P(R)-intersection of CIFSs. Several results are presented to show the intrinsic properties of the internal (or external) CIFS and showed that their union (or intersection) need not be an internal (or external) CIFS.

In **Chapter 4**, we developed present some series of AOs under the CIFS and their suitable properties. For it, firstly an operational law, score function, and accuracy function between the Cubic intuitionistic fuzzy numbers (CIFNs) under the P-order and R-order are defined and hence based on them, some weighted averaging and geometric aggregation operators, namely cubic intuitionistic fuzzy weighted (CIFWA), ordered weighted (CIFOWA), hybrid averaging (CIFHA) and their respective geometric (CIFWG, CIFOWG, CIFHG) are proposed. A decision-making method based on these operators is proposed for ranking different set of the alternatives classified under CIFS domain. Finally, an illustrative example is given to demonstrate the proposed approach.

In **Chapter 5**, we extend the structural characteristics of CIFS as defined in Chapter

4 by defining generalized t-norm and t-conorm aggregation operators. The aspect of generalization of these operators is captured in detail in which they are subjected to get reduced into some existing AOs. Further, to strengthen the practical applicability of the proposed operators, we also formulate a decision-making approach. The examination of the developed operators is tested by presenting an algorithm to solve the MCDM approach and explain it with a numerical example.

Chapter 6 discusses the idea of Bonferroni mean based operators for the CIFSs to aggregate the information. The major advantages of the proposed operator are that they have considered the interrelationships of aggregated values. Further, we examine the properties and develop some special cases of proposed work. Some of the existing studies have been deduced from the proposed operator which signifies that the proposed operators are more generalized than the others. Finally, a decision-making approach has been given for ranking the different alternatives based on the proposed operators. A practical example is provided to verify the developed approach and to demonstrate its practicality and feasibility.

In **Chapter 7**, we describe a novel MCGDM method under CIFS environment by integrating extended TOPSIS method. In it, we present some series of distance measures and investigated their various relationship. Further, under this environment, a group decision-making method based on the proposed measure is presented by taking the different priority pairs of the decision makers. A practical example is provided to verify the developed approach and to demonstrate its practicality and feasibility, we have compared their results with the several existing approaches' outcomes.

Chapter 8 developed a nonlinear programming model based TOPSIS approach for solving MCDM problems with incomplete weight information. The relative closeness coefficient(RCC) degree of the TOPSIS method is formulated based on the distance measures. Further, the importance of the attribute weights is taken in the form of interval numbers rather than a real single number. Several special cases of the proposed approach are discussed in detail. A comprehensive numerical example based on signal processing is given and the results are compared with various existing studies.

In **Chapter 9**, an attempt has been made to capture uncertain information in form of PDHFS. Several weighted and ordered weighted averaging and geometric aggregation operators are presented by using Einstein norm operations. Also, we have proposed two distance measures and its based maximum deviation method to compute the weight vector of the different criteria. Finally, a MCGDM approach is constructed based on proposed operators and the presented algorithm is explained with the help of the numerical example.

Chapter 10 discusses some weighted averaging and geometric MSM operators to address the uncertainties in the medical diagnosis problems and handle the gesture quantification. Some desirable properties of the operators are discussed and an optimization model based on Shanon's entropy for determining the probabilities is framed. A decision-making approach is developed based on the proposed operators followed by comparison analysis and a case study on gesture quantification of a patient suffering from hemorrhage strokes is conducted.

In **Chapter 11**, we develop a method to solve the MCDM problem under PDHFS environment. For it, firstly, we define the informational energy and the covariance between the two PDHFSs and study their properties. Secondly, we develop correlation coefficients and the weighted correlation coefficients for PDHFSs. In the formulation, PDHFSs are able to represent the information in terms of their respective degrees while the assigned probabilities give more details about the level of agreeness or disagree. Thirdly, a novel algorithm is developed based on the proposed operators to solve MCDM problems.

Chapter 12 deals with the overall concluding observations of this study and a brief discussion on the scope for future work.

Chapter 2

Preliminaries

In this chapter, we present the basic concepts and mathematical structures related to IFSs, IVIFSs, CFSs, HFSs, DHFSs, PDHFSs etc., over the universal set $\mathcal{X}$.

2.1 Fuzzy set and its extensions

Definition 2.1.1. [212] A FS $\mathcal{A}$ defined on $\mathcal{X}$ is an ordered pair given as

$$\mathcal{A} = \{(x, \zeta_{\mathcal{A}}(x)), \forall x \in \mathcal{X}\} \tag{2.1}$$

where $\zeta_{\mathcal{A}}(x) \in [0,1]$ is the degree of membership.

Definition 2.1.2. [5] An IFS in a set $\mathcal{X}$ is defined as :

$$\mathcal{A} = \{(x, \zeta_{\mathcal{A}}(x), \vartheta_{\mathcal{A}}(x)), \forall x \in \mathcal{X}\} \tag{2.2}$$

where $0 \leq \zeta_{\mathcal{A}}(x) \leq 1$, $0 \leq \vartheta_{\mathcal{A}}(x) \leq 1$ and $\zeta_{\mathcal{A}}(x) + \vartheta_{\mathcal{A}}(x) \leq 1$. We denote this pair as $\mathcal{A} = (\zeta_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ and called as an intuitionistic fuzzy number (IFN).

Definition 2.1.3. [5] For two IFNs $\mathcal{A} = (\zeta_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ and $\mathcal{B} = (\zeta_{\mathcal{B}}, \vartheta_{\mathcal{B}})$, we have

(i) $\mathcal{A} \subseteq \mathcal{B}$ if $\zeta_{\mathcal{A}}(x) \leq \zeta_{\mathcal{B}}(x)$ and $\vartheta_{\mathcal{A}}(x) \geq \vartheta_{\mathcal{B}}(x)$ $\forall x$ in $\mathcal{X}$;

(ii) $\mathcal{A} = \mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq A$.

(iii) $\mathcal{A}^c = \{x, (\vartheta_{\mathcal{A}}(x), \zeta_{\mathcal{A}}(x)), \forall x \in \mathcal{X}\}$

(iv) $\mathcal{A} \cap \mathcal{B} = \{x, (\min(\zeta_{\mathcal{A}}(x), \zeta_{\mathcal{B}}(x)), \max(\vartheta_{\mathcal{A}}(x), \vartheta_{\mathcal{B}}(x))), \forall x \in \mathcal{X}\}$

(v) $\mathcal{A} \cup \mathcal{B} = \{x, (\max(\zeta_{\mathcal{A}}(x), \zeta_{\mathcal{B}}(x)), \min(\vartheta_{\mathcal{A}}(x), \vartheta_{\mathcal{B}}(x))), \forall x \in \mathcal{X}\}$

Definition 2.1.4. [191] The score function of an IFN $\mathcal{A} = (\zeta_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ is defined as:

$$\mathcal{S}c(\mathcal{A}) = \zeta_{\mathcal{A}} - \vartheta_{\mathcal{A}} \tag{2.3}$$

and accuracy is defined as:

$$\mathcal{H}(\mathcal{A}) = \zeta_{\mathcal{A}} + \vartheta_{\mathcal{A}} \tag{2.4}$$

Definition 2.1.5. [4] An IVIFS $\mathcal{A}$ over $\mathcal{X}$ is defined as

$$\mathcal{A} = \left\{ \left(x, \left[\zeta^L_{\mathcal{A}}(x), \zeta^U_{\mathcal{A}}(x)\right], \left[\vartheta^L_{\mathcal{A}}(x), \vartheta^U_{\mathcal{A}}(x)\right]\right), \forall x \in \mathcal{X}\right\}, \tag{2.5}$$

where $0 \leq \zeta^L_{\mathcal{A}}(x) \leq \zeta^U_{\mathcal{A}}(x) \leq 1$, $0 \leq \vartheta^L_{\mathcal{A}}(x) \leq \vartheta^U_{\mathcal{A}}(x) \leq 1$ and $\zeta^U_{\mathcal{A}}(x) + \vartheta^U_{\mathcal{A}}(x) \leq 1$. This pair is often called as an interval-valued intuitionistic fuzzy number (IVIFN) and is given by $\mathcal{A} = \left(\left[\zeta^L_{\mathcal{A}}, \zeta^U_{\mathcal{A}}\right], \left[\vartheta^L_{\mathcal{A}}, \vartheta^U_{\mathcal{A}}\right]\right)$.

Definition 2.1.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two IVIFNs, then

(i) $\mathcal{A} \subseteq \mathcal{B}$ if $\zeta^L_{\mathcal{A}} \leq \zeta^L_{\mathcal{B}}$, $\zeta^U_{\mathcal{A}} \leq \zeta^U_{\mathcal{B}}$, $\vartheta^L_{\mathcal{A}} \geq \vartheta^L_{\mathcal{B}}$ and $\vartheta^U_{\mathcal{A}} \geq \vartheta^U_{\mathcal{B}}$.

(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$.

(iii) $\mathcal{A}^c = \left(\left[\vartheta^L_{\mathcal{A}}, \vartheta^U_{\mathcal{A}}\right], \left[\zeta^L_{\mathcal{A}}, \zeta^U_{\mathcal{A}}\right]\right)$.

(iv) $\mathcal{A} \cup \mathcal{B} = \left(\left[\max\{\zeta^L_{\mathcal{A}}, \zeta^L_{\mathcal{B}}\}, \max\{\zeta^U_{\mathcal{A}}, \zeta^U_{\mathcal{B}}\}\right], \left[\min\{\vartheta^L_{\mathcal{A}}, \vartheta^L_{\mathcal{B}}\}, \min\{\vartheta^U_{\mathcal{A}}, \vartheta^U_{\mathcal{B}}\}\right]\right)$.

(v) $\mathcal{A} \cap \mathcal{B} = \left(\left[\min\{\zeta^L_{\mathcal{A}}, \zeta^L_{\mathcal{B}}\}, \min\{\zeta^U_{\mathcal{A}}, \zeta^U_{\mathcal{B}}\}\right], \left[\max\{\vartheta^L_{\mathcal{A}}, \vartheta^L_{\mathcal{B}}\}, \max\{\vartheta^U_{\mathcal{A}}, \vartheta^U_{\mathcal{B}}\}\right]\right)$.

Definition 2.1.7. [182] The score function of IVIFN $\mathcal{A} = \left(\left[\zeta^L_{\mathcal{A}}, \zeta^U_{\mathcal{A}}\right], \left[\vartheta^L_{\mathcal{A}}, \vartheta^U_{\mathcal{A}}\right]\right)$ is defined as:

$$\mathcal{S}c(\mathcal{A}) = \frac{\zeta^L_{\mathcal{A}} + \zeta^U_{\mathcal{A}} - \vartheta^L_{\mathcal{A}} - \vartheta^U_{\mathcal{A}}}{2} \tag{2.6}$$

and accuracy function is

$$\mathcal{H}(\mathcal{A}) = \frac{\zeta^L_{\mathcal{A}} + \zeta^U_{\mathcal{A}} + \vartheta^L_{\mathcal{A}} + \vartheta^U_{\mathcal{A}}}{2} \tag{2.7}$$

Definition 2.1.8. [77] A CFS $\mathcal{A}$ defined on a universal set $\mathcal{X}$ is an ordered pair given by

$$\mathcal{A} = \{(x, Z_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(x)), \forall x \in \mathcal{X}\} \tag{2.8}$$

where, $Z_{\mathcal{A}}(x) = \left[\zeta^L_{\mathcal{A}}(x), \zeta^U_{\mathcal{A}}(x)\right]$ is an IVFS and $\lambda_{\mathcal{A}}(x)$ represents a FS in $\mathcal{X}$. This pair denoted $(Z_{\mathcal{A}}, \lambda_{\mathcal{A}})$ is called cubic fuzzy number (CFN).

For a family $\{\mathcal{A}_i \mid i \in \Lambda\}$ of the IFSs in $\mathcal{X}$, we define the join $(\vee)$ and the meet $(\wedge)$ operations as

$$(\vee_{i\in\Lambda}\mathcal{A}_i)(x) = \left(\max_{i\in\Lambda} \zeta_i(x), \min_{i\in\Lambda} \vartheta_i(x)\right)$$

and

$$(\wedge_{i\in\Lambda}\mathcal{A}_i)(x) = \left(\min_{i\in\Lambda} \zeta_i(x), \max_{i\in\Lambda} \vartheta_i(x)\right),$$

respectively for all $x \in \mathcal{X}$.

On the other hand, for a family $\{\mathcal{A}_i \mid i \in \Lambda\}$ of the IVIFSs in $\mathcal{X}$, we define the join $(\vee)$ and the meet $(\wedge)$ operations as

$$(\vee_{i\in\Lambda}\mathcal{A}_i)(x) = \left(\left[\max_{i\in\Lambda} \zeta^L_i(x), \max_{i\in\Lambda} \zeta^U_i(x)\right], \left[\min_{i\in\Lambda} \vartheta^L_i(x), \min_{i\in\Lambda} \vartheta^U_i(x)\right]\right)$$

and

$$(\wedge_{i\in\Lambda}\mathcal{A}_i)(x) = \left(\left[\min_{i\in\Lambda} \zeta^L_i(x), \min_{i\in\Lambda} \zeta^U_i(x)\right], \left[\max_{i\in\Lambda} \vartheta^L_i(x), \max_{i\in\Lambda} \vartheta^U_i(x)\right]\right),$$

respectively for all $x \in \mathcal{X}$.

Definition 2.1.9. [77] For CFS $\mathcal{A}_i = (Z_i, \lambda_i)$ where $i \in \Lambda$, we have

(i) P-union: $\cup_{P_{i\in\Lambda}}\mathcal{A}_i = (\cup_{i\in\Lambda}Z_i, \vee_{i\in\Lambda}\lambda_i)$.

(ii) P-intersection: $\cap_{P_{i\in\Lambda}}\mathcal{A}_i = (\cap_{i\in\Lambda}Z_i, \wedge_{i\in\Lambda}\lambda_i)$.

(iii) R-union: $\cup_{R_{i\in\Lambda}}\mathcal{A}_i = (\cup_{i\in\Lambda}Z_i, \wedge_{i\in\Lambda}\lambda_i)$.

(iv) R-intersection: $\cap_{R_{i\in\Lambda}}\mathcal{A}_i = (\cap_{i\in\Lambda}Z_i, \vee_{i\in\Lambda}\lambda_i)$.

(v) Complement: $\mathcal{A}^c = \{x, (Z^c(x), 1 - \lambda(x)), \forall x \in \mathcal{X}\}$

where $Z^c(x) = \left[1 - \zeta^U(x), 1 - \zeta^L(x)\right]$

Definition 2.1.10. [236] A DHFS $\mathcal{A}$ on universal set $\mathcal{X}$ is given as:

$$\mathcal{A} = \{(x, h(x), g(x)), \forall x \in \mathcal{X}\} \tag{2.9}$$

where h and g are membership and non-membership functions taking values in $[0,1]$. A pair of it is called DHFE and is written as $\mathcal{A} = (h, g)$ or $\bigcup_{\substack{\zeta \in h \\ \vartheta \in g}} (\{\zeta\}, \{\vartheta\})$ such that

$$0 \leq \zeta, \vartheta \leq 1; 0 \leq \zeta^+ + \vartheta^+ \leq 1 \tag{2.10}$$

in which, $\zeta \in h$; $\vartheta \in g$; $\zeta^+ \in h^+ = \bigcup_{\zeta \in h} \max\{\zeta\}$ and $\vartheta^+ \in g^+ = \bigcup_{\vartheta \in g} \max\{\vartheta\}$

Definition 2.1.11. [61] A PDHFS $\mathcal{A}$ is defined on universe of discourse $\mathcal{X}$ as:

$$\mathcal{A} = \{(x, h(x)|p_h(x), g(x)|q_g(x)), \forall x \in \mathcal{X}\} \tag{2.11}$$

The components $h(x)|p_h(x)$ and $g(x)|q_g(x)$ are the sets having possible elements where $h(x)$ and $g(x)$ represent the hesitant fuzzy membership and non-membership degrees to the set $\mathcal{X}$, respectively. Also, $p_h(x)$ and $q_g(x)$ are their corresponding probabilistic information. For sake of convenience, a pair of it is called probabilistic dual hesitant fuzzy element (PDHFE) and is written as $\mathcal{A} = (h|p_h, g|q_g)$ or $\bigcup_{\substack{\zeta \in h \\ \vartheta \in g}} (\{\zeta|p_\zeta\}, \{\vartheta|q_\vartheta\})$. Moreover,

$$0 \leq \zeta, \vartheta \leq 1; 0 \leq \zeta^+ + \vartheta^+ \leq 1 \tag{2.12}$$

and

$$p_\zeta \in [0,1], q_\vartheta \in [0,1], \sum_{\zeta \in h} p_\zeta = 1, \sum_{\vartheta \in g} q_\vartheta = 1 \tag{2.13}$$

where $\zeta \in h; \vartheta \in g; \zeta^+ \in h^+ = \bigcup_{\zeta \in h} \max\{\zeta\}; \vartheta^+ \in g^+ = \bigcup_{\vartheta \in g} \max\{\vartheta\}$.

Definition 2.1.12. [61] Let $\mathcal{A} = (h|p_h, g|q_g)$ be a PDHFE, then the complement is defined as

$$\mathcal{A}^c = \begin{cases} \bigcup_{\substack{\zeta \in h, \\ \vartheta \in g}} \left(\{\vartheta \mid q_\vartheta\}, \{\zeta \mid p_\zeta\}\right), & \text{if} \quad g \neq \phi \quad \text{and} \quad h \neq \phi \\ \bigcup_{\zeta \in h} \left(\{1 - \zeta\}, \{\phi\}\right), & \text{if} \quad g \in \phi \quad \text{and} \quad h \neq \phi \\ \bigcup_{\vartheta \in g} \left(\{\phi\}, \{1 - \vartheta\}\right), & \text{if} \quad h \in \phi \quad \text{and} \quad g \neq \phi \end{cases} \tag{2.14}$$

Definition 2.1.13. [61] For a PDHFE $\mathcal{A}$, the score function is defined as:

$$\mathcal{S}c(\mathcal{A}) = \sum_{\zeta \in h} \zeta \cdot p_\zeta - \sum_{\vartheta \in g} \vartheta \cdot q_\vartheta \tag{2.15}$$

and accuracy function is calculated as:

$$\mathcal{H}(\mathcal{A}) = \sum_{\zeta \in h} \zeta \cdot p_\zeta + \sum_{\vartheta \in g} \vartheta \cdot q_\vartheta \tag{2.16}$$

Definition 2.1.14. [186] For two IFNs or IVIFNs or PDHFSs $\mathcal{A}$ and $\mathcal{B}$, an order relation $\mathcal{A} \preceq \mathcal{B}$ referred as "$\mathcal{B}$ preferred to $\mathcal{A}$", in accordance to Definitions 2.1.4, 2.1.7 and 2.1.13, is defined if anyone of the following conditions hold:

(i) $\mathcal{S}c(\mathcal{A}) \leq \mathcal{S}c(\mathcal{B})$.

(ii) If $\mathcal{S}c(\mathcal{A}) = \mathcal{S}c(\mathcal{B})$, then $\mathcal{H}(\mathcal{A}) \leq \mathcal{H}(\mathcal{B})$.

2.2 Information measures

Information measures play a significant role in different DM algorithms. There a number of different information measures from which some prominent ones such as distance, similarity and correlation measures have been discussed. For it, let $\Phi(\mathcal{X})$ be a collection of the sets in consideration over the universal set $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$

2.2.1 Distance measures on IVIFSs

Definition 2.2.1. [157] Let $\mathcal{A}$ and $\mathcal{B}$ be two sets, then the distance measure d satisfies the following conditions:

(P1) $0 \leq d(\mathcal{A}, \mathcal{B}) \leq 1$;

(P2) $d(\mathcal{A}, \mathcal{B}) = d(\mathcal{B}, \mathcal{A})$;

(P3) $d(\mathcal{A}, \mathcal{B}) = 0$ if $\mathcal{A} = \mathcal{B}$;

(P4) If $\mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$, then $d(\mathcal{A}, \mathcal{B}) \leq d(\mathcal{A}, \mathcal{C})$ and $d(\mathcal{B}, \mathcal{C}) \leq d(\mathcal{A}, \mathcal{C})$.

Under IVIFS environment, distance measures for two IVIFS $\mathcal{A} = \{(x_j, [\zeta_{\mathcal{A}}^L, \zeta_{\mathcal{A}}^U](x_j), [\vartheta_{\mathcal{A}}^L(x_j), \vartheta_{\mathcal{A}}^U(x_j)])\}$ and $\mathcal{B} = \{(x_j\, [\zeta_{\mathcal{B}}^L(x_j), \zeta_{\mathcal{B}}^U(x_j)], [\vartheta_{\mathcal{B}}^L(x_j), \vartheta_{\mathcal{B}}^U(x_j)])\}$ subjected to the weighted criteria $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, where $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$ are defined as:

(i) Park et al. [116] gave weighted Hamming distance as:

$$d_h(\mathcal{A}, \mathcal{B}) = \frac{1}{4} \sum_{j=1}^{n} \omega_j \begin{pmatrix} \left|\zeta_{\mathcal{A}}^L(x_j) - \zeta_{\mathcal{B}}^L(x_j)\right| + \left|\zeta_{\mathcal{A}}^U(x_j) - \zeta_{\mathcal{B}}^U(x_j)\right| \\ + \left|\vartheta_{\mathcal{A}}^L(x_j) - \vartheta_{\mathcal{B}}^L(x_j)\right| + \left|\vartheta_{\mathcal{A}}^U(x_j) - \vartheta_{\mathcal{B}}^U(x_j)\right| \end{pmatrix} \tag{2.17}$$

(ii) Park et al. [116] also formulated weighted Euclidean distance measure as:

$$d_e(\mathcal{A}, \mathcal{B}) = \frac{1}{4} \sum_{j=1}^{n} \omega_j \begin{pmatrix} \left|\zeta_{\mathcal{A}}^L(x_j) - \zeta_{\mathcal{B}}^L(x_j)\right|^2 + \left|\zeta_{\mathcal{A}}^U(x_j) - \zeta_{\mathcal{B}}^U(x_j)\right|^2 \\ + \left|\vartheta_{\mathcal{A}}^L(x_j) - \vartheta_{\mathcal{B}}^L(x_j)\right|^2 + \left|\vartheta_{\mathcal{A}}^U(x_j) - \vartheta_{\mathcal{B}}^U(x_j)\right|^2 \end{pmatrix}^{\frac{1}{2}} \tag{2.18}$$

(iii) For $\eta > 0$, Xu and Chen [188] gave weighted generalized distance measure as:

$$d_g(\mathcal{A}, \mathcal{B}) = \frac{1}{4} \sum_{j=1}^{n} \omega_j \begin{pmatrix} \left|\zeta_{\mathcal{A}}^L(x_j) - \zeta_{\mathcal{B}}^L(x_j)\right|^\eta + \left|\zeta_{\mathcal{A}}^U(x_j) - \zeta_{\mathcal{B}}^U(x_j)\right|^\eta \\ + \left|\vartheta_{\mathcal{A}}^L(x_j) - \vartheta_{\mathcal{B}}^L(x_j)\right|^\eta + \left|\vartheta_{\mathcal{A}}^U(x_j) - \vartheta_{\mathcal{B}}^U(x_j)\right|^\eta \end{pmatrix}^{\frac{1}{\eta}} \tag{2.19}$$

2.2.2 Similarity measure for IVIFSs

Definition 2.2.2. [219] Let $\mathcal{A}$ and $\mathcal{B}$ be two sets, then the similarity measure 'S' satisfies the following conditions:

(P1) $0 \leq S(\mathcal{A}, \mathcal{B}) \leq 1$;

(P2) $S(\mathcal{A}, \mathcal{B}) = S(\mathcal{B}, \mathcal{A})$;

(P3) $S(\mathcal{A}, \mathcal{B}) = 1$ if $\mathcal{A} = \mathcal{B}$;

(P4) If $\mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$, then $S(\mathcal{A}, \mathcal{B}) \geq S(\mathcal{A}, \mathcal{C})$ and $S(\mathcal{B}, \mathcal{C}) \geq S(\mathcal{A}, \mathcal{C})$.

Under IVIFS environment, distance measures for two IVIFS $\mathcal{A}$ and $\mathcal{B}$ subjected to the weighted criteria $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, where $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$ are defined as:

(i) Zhang and Jiang [219] gave Euclidean similarity measure as:

$$S_e(\mathcal{A}, \mathcal{B}) = 1 - \frac{1}{4} \begin{pmatrix} \left|\zeta_{\mathcal{A}}^L(x_j) - \zeta_{\mathcal{B}}^L(x_j)\right|^2 + \left|\zeta_{\mathcal{A}}^U(x_j) - \zeta_{\mathcal{B}}^U(x_j)\right|^2 \\ + \left|\vartheta_{\mathcal{A}}^L(x_j) - \vartheta_{\mathcal{B}}^L(x_j)\right|^2 + \left|\vartheta_{\mathcal{A}}^U(x_j) - \vartheta_{\mathcal{B}}^U(x_j)\right|^2 \end{pmatrix}^{\frac{1}{2}} \tag{2.20}$$

(ii) On the basis of set-theoretic approach, Zhang and Jiang [219] also gave the following similarity measure:

$$S_s(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{j=1}^{n}\left(\min\left\{\zeta_{\mathcal{A}}^{L}(x_j),\zeta_{\mathcal{B}}^{L}(x_j)\right\}+\min\left\{\zeta_{\mathcal{A}}^{U}(x_j),\zeta_{\mathcal{B}}^{U}(x_j)\right\}\right)}{\sum\limits_{j=1}^{n}\left(\min\left\{\vartheta_{\mathcal{A}}^{L}(x_j),\vartheta_{\mathcal{B}}^{L}(x_j)\right\}+\min\left\{\vartheta_{\mathcal{A}}^{U}(x_j),\vartheta_{\mathcal{B}}^{U}(x_j)\right\}\right)} \tag{2.21}$$

(iii) Meng and Chen [108] gave similarity measure as:

$$S(\mathcal{A},\mathcal{B}) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\min\left\{\zeta_{\mathcal{A}}^{L}(x_j),\zeta_{\mathcal{B}}^{L}(x_j)\right\}+\min\left\{\zeta_{\mathcal{A}}^{U}(x_j),\zeta_{\mathcal{B}}^{U}(x_j)\right\}}{\min\left\{\vartheta_{\mathcal{A}}^{L}(x_j),\vartheta_{\mathcal{B}}^{L}(x_j)\right\}+\min\left\{\vartheta_{\mathcal{A}}^{U}(x_j),\vartheta_{\mathcal{B}}^{U}(x_j)\right\}}\right) \tag{2.22}$$

2.2.3 Correlation coefficient for IVIFSs

Definition 2.2.3. [16] For two sets $\mathcal{A}$ and $\mathcal{B}$, the correlation coefficient satisfies the following properties:

(P1) $0 \leq \mathcal{K}(\mathcal{A},\mathcal{B}) \leq 1$

(P2) $\mathcal{K}(\mathcal{A},\mathcal{B}) = \mathcal{K}(\mathcal{B},\mathcal{A})$

(P3) $\mathcal{K}(\mathcal{A},\mathcal{B}) = 1$, if $\mathcal{A} = \mathcal{B}$

Under IVIFS environment, correlation for two IVIFS $\mathcal{A}$ and $\mathcal{B}$ is defined as:

(i) Bustine and Burillo [16] gave correlation measure as:

$$\mathcal{K}_1(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{j=1}^{n}\left(\zeta_{\mathcal{A}}^{L}(x_j)\zeta_{\mathcal{B}}^{L}(x_j)+\zeta_{\mathcal{A}}^{U}(x_j)\zeta_{\mathcal{B}}^{U}(x_j)+\vartheta_{\mathcal{A}}^{L}(x_j)\vartheta_{\mathcal{B}}^{L}(x_j)+\vartheta_{\mathcal{A}}^{U}(x_j)\vartheta_{\mathcal{B}}^{U}(x_j)\right)}{\left(\begin{array}{c}\sqrt{\sum\limits_{j=1}^{n}\left(\left(\zeta_{\mathcal{A}}^{L}(x_j)\right)^2+\left(\zeta_{\mathcal{A}}^{U}(x_j)\right)^2+\left(\vartheta_{\mathcal{A}}^{L}(x_j)\right)^2+\left(\vartheta_{\mathcal{A}}^{U}(x_j)\right)^2\right)}\\ \cdot\sqrt{\sum\limits_{j=1}^{n}\left(\left(\zeta_{\mathcal{B}}^{L}(x_j)\right)^2+\left(\zeta_{\mathcal{B}}^{U}(x_j)\right)^2+\left(\vartheta_{\mathcal{B}}^{L}(x_j)\right)^2+\left(\vartheta_{\mathcal{B}}^{U}(x_j)\right)^2\right)}\end{array}\right)} \tag{2.23}$$

(ii) Park et al. [115] included the hesitation interval defined as $\left[\pi_{\Lambda}^{L},\pi_{\Lambda}^{U}\right] = \left[1-\zeta_{\Lambda}^{U}-\vartheta_{\Lambda}^{U},\right.$ $\left.1-\zeta_{\Lambda}^{L}-\vartheta_{\Lambda}^{L}\right]$ where $_{\Lambda}$ denotes the set in consideration.

$$\mathcal{K}_2(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{j=1}^{n}\left(\begin{array}{c}\zeta_{\mathcal{A}}^{L}(x_j)\zeta_{\mathcal{B}}^{L}(x_j)+\zeta_{\mathcal{A}}^{U}(x_j)\zeta_{\mathcal{B}}^{U}(x_j)+\vartheta_{\mathcal{A}}^{L}(x_j)\vartheta_{\mathcal{B}}^{L}(x_j)\\+\vartheta_{\mathcal{A}}^{U}(x_j)\vartheta_{\mathcal{B}}^{U}(x_j)+\pi_{\mathcal{A}}^{L}(x_j)\pi_{\mathcal{B}}^{L}(x_j)+\pi_{\mathcal{A}}^{U}(x_j)\pi_{\mathcal{B}}^{U}(x_j)\end{array}\right)}{\left(\begin{array}{c}\sqrt{\sum\limits_{j=1}^{n}\left(\left(\zeta_{\mathcal{A}}^{L}(x_j)\right)^2+\left(\zeta_{\mathcal{A}}^{U}(x_j)\right)^2+\left(\vartheta_{\mathcal{A}}^{L}(x_j)\right)^2+\left(\vartheta_{\mathcal{A}}^{U}(x_j)\right)^2\right)+\left(\pi_{\mathcal{A}}^{L}(x_j)\right)^2+\left(\pi_{\mathcal{A}}^{U}(x_j)\right)^2}\\ \cdot\sqrt{\sum\limits_{j=1}^{n}\left(\left(\zeta_{\mathcal{B}}^{L}(x_j)\right)^2+\left(\zeta_{\mathcal{B}}^{U}(x_j)\right)^2+\left(\vartheta_{\mathcal{B}}^{L}(x_j)\right)^2+\left(\vartheta_{\mathcal{B}}^{U}(x_j)\right)^2\right)+\left(\pi_{\mathcal{B}}^{L}(x_j)\right)^2+\left(\pi_{\mathcal{B}}^{U}(x_j)\right)^2}\end{array}\right)} \tag{2.24}$$

2.2.4 Distance measures for DHFSs

Under DHFS environment, distance measures are used to extract information from the hesitant values. For two DHFSs, $\mathcal{A} = (x_j, h_{\mathcal{A}}(x_j), g_{\mathcal{A}}(x_j))$ and $\mathcal{B} = (x_j, h_{\mathcal{B}}(x_j), g_{\mathcal{B}}(x_j))$, M and N denotes total number of hesitant values in $\mathcal{A}$ and $\mathcal{B}$ respectively. Also, for $s = 1, 2, \ldots, M$ and $t = 1, 2, \ldots, N$, $\sigma(s)$ and $\sigma(t)$ denoting the s^{th} and t^{th} largest membership and non-membership values respectively, the distance measures are listed below:

(i) For $\eta > 0$, Wang, Xu, Wang and Ni [157] gave generalized Hamming distance as:

$$d_{gh}(\mathcal{A}, \mathcal{B}) = \sum_{j=1}^{n} \left(\frac{1}{n} \left(\frac{1}{M+N} \left(\begin{array}{c} \sum_{s=1}^{M} \left| \zeta_{\mathcal{A}}^{\sigma(s)}(x_j) - \zeta_{\mathcal{B}}^{\sigma(s)}(x_j) \right|^{\eta} \\ + \sum_{t=1}^{N} \left| \vartheta_{\mathcal{A}}^{\sigma(t)}(x_j) - \vartheta_{\mathcal{B}}^{\sigma(t)}(x_j) \right|^{\eta} \end{array} \right) \right) \right)^{\frac{1}{\eta}} \tag{2.25}$$

(ii) Su et al. [135] gave Hamming-Hausdorff distance as:

$$d_{hh}(\mathcal{A}, \mathcal{B}) = \left(\frac{1}{n} \sum_{j=1}^{n} \max_{j} \left\{ \begin{array}{c} \max_{s} \left| \zeta_{\mathcal{A}}^{\sigma(s)}(x_j) - \zeta_{\mathcal{B}}^{\sigma(s)}(x_j) \right|, \\ \max_{t} \left| \vartheta_{\mathcal{A}}^{\sigma(t)}(x_j) - \vartheta_{\mathcal{B}}^{\sigma(t)}(x_j) \right| \end{array} \right\} \right) \tag{2.26}$$

(iii) Su et al. [135],also, gave Euclidean-Hausdorff distance as:

$$d_{eh}(\mathcal{A}, \mathcal{B}) = \left(\frac{1}{n} \sum_{j=1}^{n} \max_{j} \left\{ \begin{array}{c} \max_{s} \left| \zeta_{\mathcal{A}}^{\sigma(s)}(x_j) - \zeta_{\mathcal{B}}^{\sigma(s)}(x_j) \right|^{2}, \\ \max_{t} \left| \vartheta_{\mathcal{A}}^{\sigma(t)}(x_j) - \vartheta_{\mathcal{B}}^{\sigma(t)}(x_j) \right|^{2} \end{array} \right\} \right)^{\frac{1}{2}} \tag{2.27}$$

2.2.5 Similarity measures on DHFSs

In contrast to the distance measures, researchers focussed on formulating different similarity measures which are listed below:

(i) For $\eta > 0$, Wang, Xu, Wang and Ni [157] gave generalized similarity measure based on Hamming distance as

$$S_{gh}(\mathcal{A}, \mathcal{B}) = 1 - \sum_{j=1}^{n} \left(\frac{1}{n} \left(\frac{1}{M+N} \left(\begin{array}{c} \sum_{s=1}^{M} \left| \zeta_{\mathcal{A}}^{\sigma(s)}(x_j) - \zeta_{\mathcal{B}}^{\sigma(s)}(x_j) \right|^{\eta} \\ + \sum_{t=1}^{N} \left| \vartheta_{\mathcal{A}}^{\sigma(t)}(x_j) - \vartheta_{\mathcal{B}}^{\sigma(t)}(x_j) \right|^{\eta} \end{array} \right) \right) \right)^{\frac{1}{\eta}} \tag{2.28}$$

(ii) Su et al. [135] gave Euclidean-Hausdorff similarity as:

$$S_{eh}(\mathcal{A},\mathcal{B}) = 1 - \left(\frac{1}{n}\sum_{j=1}^{n} \max_j \left\{ \begin{array}{c} \max_s \left| \zeta_{\mathcal{A}}^{\sigma(s)}(x_j) - \zeta_{\mathcal{B}}^{\sigma(s)}(x_j) \right|^2, \\ \max_t \left| \vartheta_{\mathcal{A}}^{\sigma(t)}(x_j) - \vartheta_{\mathcal{B}}^{\sigma(t)}(x_j) \right|^2 \end{array} \right\} \right)^{\frac{1}{2}} \tag{2.29}$$

2.2.6 Correlation coefficient for DHFSs

In addition to the stated information measures, on DHFSs correlation coefficients are given below:

(i) Wang and Liu [163] introduced correlation coefficient as:

$$K_d(\mathcal{A},\mathcal{B}) = \frac{\sum_{j=1}^{n}\left(\frac{1}{M}\sum_{s=1}^{M} \zeta_{\mathcal{A}}^{\sigma(s)}(x_j)\zeta_{\mathcal{B}}^{\sigma(s)}(x_j) + \frac{1}{N}\sum_{t=1}^{N} \vartheta_{\mathcal{A}}^{\sigma(s)}(x_j)\vartheta_{\mathcal{B}}^{\sigma(s)}(x_j) \right)}{\sqrt{\sum_{j=1}^{n}\left(\frac{1}{M}\left(\zeta_{\mathcal{A}}^{\sigma(s)}(x_j)\right)^2 + \frac{1}{N}\left(\vartheta_{\mathcal{A}}^{\sigma(t)}(x_j)\right)^2 \right)}\sqrt{\sum_{j=1}^{n}\left(\frac{1}{M}\left(\zeta_{\mathcal{B}}^{\sigma(s)}(x_j)\right)^2 + \frac{1}{N}\left(\vartheta_{\mathcal{B}}^{\sigma(t)}(x_j)\right)^2 \right)}} \tag{2.30}$$

(ii) Ye [201] gave weighted correlation coefficient as:

$$K_{wd}(\mathcal{A},\mathcal{B}) = \frac{\sum_{j=1}^{n}\omega_j\left(\frac{1}{M}\sum_{s=1}^{M} \zeta_{\mathcal{A}}^{\sigma(s)}(x_j)\zeta_{\mathcal{B}}^{\sigma(s)}(x_j) + \frac{1}{N}\sum_{t=1}^{N} \vartheta_{\mathcal{A}}^{\sigma(s)}(x_j)\vartheta_{\mathcal{B}}^{\sigma(s)}(x_j) \right)}{\sqrt{\sum_{j=1}^{n}\omega_j\left(\frac{1}{M}\left(\zeta_{\mathcal{A}}^{\sigma(s)}(x_j)\right)^2 + \frac{1}{N}\left(\vartheta_{\mathcal{A}}^{\sigma(t)}(x_j)\right)^2 \right)}\sqrt{\sum_{j=1}^{n}\omega_j\left(\frac{1}{M}\left(\zeta_{\mathcal{B}}^{\sigma(s)}(x_j)\right)^2 + \frac{1}{N}\left(\vartheta_{\mathcal{B}}^{\sigma(t)}(x_j)\right)^2 \right)}} \tag{2.31}$$

2.3 Archimedean t-norm and t-conorm

A t-norm (fuzzy intersection) '$\mathcal{T}$' is a binary operation on $[0,1]$ i.e.,

$$\mathcal{T} : [0,1] \times [0,1] \to [0,1] \tag{2.32}$$

defined by

$$(\mathcal{A}_1 \cap \mathcal{A}_2)(x) = \mathcal{T}(\mathcal{A}_1(x), \mathcal{A}_2(x)) \quad \forall x \in \mathcal{X} \tag{2.33}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are arbitrary fuzzy sets. The mapping preserves the following axioms for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in [0,1]$

Axiom 1: $\mathcal{T}(\mathfrak{a}, 1) = \mathfrak{a}$ (Boundary Condition)

Axiom 2: If $\mathfrak{b} \leq \mathfrak{c}$ then $\mathcal{T}(\mathfrak{a}, \mathfrak{b}) \leq \mathcal{T}(\mathfrak{a}, \mathfrak{c})$ (Monotonicity)

Axiom 3: $\mathcal{T}(\mathfrak{a}, \mathcal{T}(\mathfrak{b}, \mathfrak{c})) = \mathcal{T}(\mathcal{T}(\mathfrak{a}, \mathfrak{b}), \mathfrak{c})$ (Associativity)

Axiom 4: $\mathcal{T}(\mathfrak{a}, \mathfrak{b}) = \mathcal{T}(\mathfrak{b}, \mathfrak{a})$ (Commutativity)

Subsequently, T-conorm (fuzzy union) '$\mathcal{S}$' is also a binary operation on $[0, 1]$ given by

$$\mathcal{S} : [0, 1] \times [0, 1] \rightarrow [0, 1] \tag{2.34}$$

defined by

$$(\mathcal{A}_1 \cup \mathcal{A}_2)(x) = \mathcal{S}(\mathcal{A}_1(x), \mathcal{A}_2(x)) \quad \forall x \in \mathcal{X} \tag{2.35}$$

Alike t-norm, it also satisfies the conditions of boundary, monotonicity, associativity and commutativity. Both the norms are inter-related to each other and their relation is given as

$$\mathcal{S}(\mathfrak{a}, \mathfrak{b}) = 1 - \mathcal{T}(1 - \mathfrak{a}, 1 - \mathfrak{b}) \tag{2.36}$$

A class of fuzzy intersection (t-norm) is obtained if t-norm also satisfies the additional axioms [80, 111] i.e.,

Axiom 5: $\mathcal{T}$ is continuous function (Continuity)

Axiom 6: $\mathcal{T}(\mathfrak{a}, \mathfrak{a}) < \mathfrak{a}$ (Subidempotency)

Axiom7: If $\mathfrak{a}_1 < \mathfrak{a}_2$ and $\mathfrak{b}_1 < \mathfrak{b}_2$ then $\mathcal{T}(\mathfrak{a}_1, \mathfrak{b}_1) < \mathcal{T}(\mathfrak{a}_2, \mathfrak{b}_2)$ (Strict monotonicity)

Similarly, for t-conorm, the axiom of subidempotency is replaced by $\mathcal{S}(\mathfrak{a}, \mathfrak{a}) > \mathfrak{a}$ and is called superidempotency. A continuous t-norm that satisfy the subidempotency i.e., $\mathcal{T}(\mathfrak{a}, \mathfrak{a}) < \mathfrak{a}$ is called an Archimedean t-norm (AT) [80, 111]. If it also satisfies the strict monotonicity then it is called strict Archimedean t-norm. On the other hand, a continuous t-conorm that satisfies the superidempotency i.e., $\mathcal{S}(\mathfrak{a}, \mathfrak{a}) > \mathfrak{a}$ is called an Archimedean t-conorm (AC) [80, 111]. If it also satisfies the strict monotonicity then it is called strict Archimedean t-conorm.

Further, strict AT and AC can be expressed in the form of continuous function $y : [0, 1) \rightarrow [0, 1]$ and $u = (0, 1] \rightarrow [0, 1]$ respectively for $\mathfrak{a}, \mathfrak{b} \in [0, 1]$ as $\mathcal{T}(\mathfrak{a}, \mathfrak{b}) = y^{-1}(y(\mathfrak{a} + y(\mathfrak{b})))$

and $\mathcal{S}(\mathfrak{a},\mathfrak{b}) = u^{-1}\left(u(\mathfrak{a}+u(\mathfrak{b}))\right)$ where y(or u) is decreasing(or increasing) function with $y(1)=0$, $u(0)=0$ and $y(\mathfrak{a})=u(1-\mathfrak{a})$. However, some standard union and intersection form for $\mathfrak{a},\mathfrak{b}\in[0,1]$ are defined as [80, 111]:

(i) Standard intersection and union [80, 111]:

$$\mathcal{T}(\mathfrak{a},\mathfrak{b}) = \min(\mathfrak{a},\mathfrak{b}) \quad ; \quad \mathcal{S}(\mathfrak{a},\mathfrak{b}) = \max(\mathfrak{a},\mathfrak{b}) \tag{2.37}$$

(ii) Algebraic product and algebraic sum

$$\mathcal{T}(\mathfrak{a},\mathfrak{b}) = \mathfrak{a}\mathfrak{b} \quad ; \quad \mathcal{S}(\mathfrak{a},\mathfrak{b}) = \mathfrak{a}+\mathfrak{b}-\mathfrak{a}\mathfrak{b} \tag{2.38}$$

(iii) Bounded Difference and Sum

$$\mathcal{T}(\mathfrak{a},\mathfrak{b}) = \max(0,\mathfrak{a}+\mathfrak{b}-1) \quad ; \quad \mathcal{S}(\mathfrak{a},\mathfrak{b}) = \min(1,\mathfrak{a}+\mathfrak{b}) \tag{2.39}$$

(iv) Drastic intersection and union

$$\mathcal{T}(\mathfrak{a},\mathfrak{b}) = \begin{cases} \mathfrak{a} & ;\text{when} \quad \mathfrak{b}=1 \\ \mathfrak{b} & ;\text{when} \quad \mathfrak{a}=1 \\ 0 & ;\text{otherwise} \end{cases} \quad ; \quad \mathcal{S}(\mathfrak{a},\mathfrak{b}) = \begin{cases} \mathfrak{a} & ;\text{when} \quad \mathfrak{b}=0 \\ \mathfrak{b} & ;\text{when} \quad \mathfrak{a}=0 \\ 1 & ;\text{otherwise} \end{cases} \tag{2.40}$$

(v) Yager class of t-norm and t-conorm ($\xi > 0$)

$$\mathcal{T}(\mathfrak{a},\mathfrak{b}) = 1 - \min\left\{1, \left[(1-\mathfrak{a})^{\xi} + (1-\mathfrak{b})^{\xi}\right]^{1/\xi}\right\}$$

$$\mathcal{S}(\mathfrak{a},\mathfrak{b}) = \min\left\{1, (\mathfrak{a}^{\xi}+\mathfrak{b}^{\xi})^{1/\xi}\right\}$$

Apart from it, some other ATs and ACs with their generator function are given in Table 2.1.

2.4 Aggregation Operators

The operations based on AT and AC are used to fuse together several fuzzy sets into a single unit using a desirable pattern called aggregation operators (AOs).

Table 2.1: Some AT and AC with relative additive generators

Name	T-norm $\mathcal{T}(\mathfrak{a},\mathfrak{b})$	Additive generator $y(t)$	S-norm $\mathcal{S}(\mathfrak{a},\mathfrak{b})$	Additive generator $u(t)$
Algebraic	$\mathfrak{a}\mathfrak{b}$	$-\log(t)$	$\mathfrak{a}+\mathfrak{b}-\mathfrak{a}+\mathfrak{b}$	$-\log(1-t)$
Einstein	$\frac{\mathfrak{a}\mathfrak{b}}{1+(1-\mathfrak{a})(1-\mathfrak{b})}$	$\log\left(\frac{2-t}{t}\right)$	$\frac{\mathfrak{a}+\mathfrak{b}}{1+\mathfrak{a}\mathfrak{b}}$	$\log\left(\frac{1+t}{1-t}\right)$
Hamacher ($\gamma>0$)	$\frac{\mathfrak{a}\mathfrak{b}}{\gamma+(1-\gamma)(\mathfrak{a}+\mathfrak{b}-\mathfrak{a}\mathfrak{b})}$	$\log\left(\frac{\gamma+(1-\gamma)t}{t}\right)$	$\frac{\mathfrak{a}+\mathfrak{b}-\mathfrak{a}\mathfrak{b}-(1-\gamma)\mathfrak{a}\mathfrak{b}}{1-(1-\gamma)\mathfrak{a}\mathfrak{b}}$	$\log\left(\frac{\gamma+(1-\gamma)(1-t)}{1-t}\right)$

Definition 2.4.1. An AO on 'n' fuzzy sets ($n \geq 2$) is defined as

$$\mathcal{L} : [0,1]^n \to [0,1] \tag{2.41}$$

Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be 'n' fuzzy sets defined on $\mathcal{X}$, then function '$\mathcal{L}$' produce an aggregated fuzzy set $\mathcal{A}$ by operating the membership degrees of these sets for each $x \in \mathcal{X}$, i.e.,

$$\mathcal{A}(x) = \mathcal{L}\left(\mathcal{A}_1(x), \mathcal{A}_2(x), \ldots, \mathcal{A}_n(x)\right) \tag{2.42}$$

An AO $\mathcal{L}$ may satisfy the following axiomatic conditions:

Axiom 1: $\mathcal{L}(0,0,\ldots,0) = 0$ and $\mathcal{L}(1,1,\ldots,1) = 1$ (Boundary condition)

Axiom 2: If $\mathfrak{a}_i \leq \mathfrak{b}_i$ for all i, then $\mathcal{L}\left(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n\right) \leq \mathcal{L}\left(\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_n\right)$ (Monotonicity)

Axiom 3: $\mathcal{L}$ is continuous. (Continuity)

Based on the AT and AC operation as defined in Table 2.1, researchers have presented several kinds of AOs under the IVIFSs, CFSs and PDHFSs, which are defined as follows.

Definition 2.4.2. [6] Let $\mathcal{A}_i = \left(\left[\zeta_i^L, \zeta_i^U\right], \left[\vartheta_i^L, \vartheta_i^U\right]\right)$, $(i = 1, 2, \ldots, n)$ be the collection of n IVIFNs, $\mathcal{A} = \left(\left[\zeta^L, \zeta^U\right], \left[\vartheta^L, \vartheta^U\right]\right)$ and $\xi > 0$ be a real number then the operational laws on these IVIFNs are defined as below:

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \left(\left[\zeta_1^L + \zeta_2^L - \zeta_1^L\zeta_2^L, \zeta_1^U + \zeta_2^U - \zeta_1^U\zeta_2^U\right], \left[\prod\limits_{i=1}^{2} \vartheta_i^L, \prod\limits_{i=1}^{2} \vartheta_i^U\right]\right)$

(ii) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \left(\left[\prod\limits_{i=1}^{2} \zeta_i^L, \prod\limits_{i=1}^{2} \zeta_i^U\right], \left[\vartheta_1^L + \vartheta_2^L - \vartheta_1^L\vartheta_2^L, \vartheta_1^U + \vartheta_2^U - \vartheta_1^U\vartheta_2^U\right]\right)$

(iii) $\xi\mathcal{A} = \left(\left[1 - (1-\zeta^L)^\xi, 1 - (1-\zeta^U)^\xi\right], \left[(\vartheta^L)^\xi, (\vartheta^U)^\xi\right]\right)$

(iv) $\mathcal{A}^{\xi} = \left(\left[(\zeta^L)^{\xi}, (\zeta^U)^{\xi}\right], \left[1-(1-\vartheta^L)^{\xi}, 1-(1-\vartheta^U)^{\xi}\right]\right)$

Based on these operations, some AO on IVIFSs are listed below:

(i) Xu and Chen [182] defined the following weighted arithmetic AO:

$$\text{IVIFWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \left(\begin{array}{c} \left[1-\prod_{i=1}^{n}(1-\zeta_i^L)^{\omega_i}, 1-\prod_{i=1}^{n}(1-\zeta_i^U)^{\omega_i}\right], \\ \left[\prod_{i=1}^{n}(\vartheta_i^L)^{\omega_i}, \prod_{i=1}^{n}(\vartheta_i^U)^{\omega_i}\right] \end{array}\right) \tag{2.43}$$

(ii) Xu and Chen [181] defined the following ordered weighted arithmetic AO:

$$\text{IVIFOWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \left(\begin{array}{c} \left[1-\prod_{i=1}^{n}(1-\zeta_{\psi(i)}^L)^{\omega_i}, 1-\prod_{i=1}^{n}(1-\zeta_{\psi(i)}^U)^{\omega_i}\right], \\ \left[\prod_{i=1}^{n}(\vartheta_{\psi(i)}^L)^{\omega_i}, \prod_{i=1}^{n}(\vartheta_{\psi(i)}^U)^{\omega_i}\right] \end{array}\right) \tag{2.44}$$

where $(\psi(1), \psi(2), \ldots, \psi(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\mathcal{A}_{\psi(i-1)} \geq \mathcal{A}_{\psi(i)}$ for $(i = 2, 3, \ldots, n)$

2.4.1 Cubic fuzzy set

Let $\mathcal{A}_i = \left(\left[\zeta_i^L, \zeta_i^U\right], \lambda_i\right)$, $(i = 1, 2, \ldots, n)$ be the collections of n CFNs, $\mathcal{A} = \left(\left[\zeta^L, \zeta^U\right], \lambda\right)$ and $\xi > 0$ be a real number then the operational laws on these CFNs are defined as below:

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \left(\left[\zeta_1^L + \zeta_2^L - \zeta_1^L\zeta_2^L, \zeta_1^U + \zeta_2^U - \zeta_1^U\zeta_2^U\right], \lambda_1\lambda_2\right)$

(ii) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \left(\left[\zeta_1^L\zeta_2^L, \zeta_1^U\zeta_2^U\right], \lambda_1 + \lambda_2 - \lambda_1\lambda_2\right)$

(iii) $\xi\mathcal{A} = \left(\left[1-(1-\zeta^L)^{\xi}, 1-(1-\zeta^U)^{\xi}\right], (\lambda)^{\xi}\right)$

(iv) $\mathcal{A}^{\xi} = \left(\left[(\zeta^L)^{\xi}, (\zeta^U)^{\xi}\right], 1-(1-\lambda)^{\xi}\right)$

Based on the above stated basic operations, the proposed AOs are given below:

(i) Khan et al. [79] gave cubic weighted average (CWA) operator as

$$\text{CWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \left(\left[1-\prod_{i=1}^{n}(1-\zeta_i^L)^{\omega_i}, 1-\prod_{i=1}^{n}(1-\zeta_i^U)^{\omega_i}\right], \prod_{i=1}^{n}(\lambda_i)^{\omega_i}\right) \tag{2.45}$$

(ii) Khan et al. [79] gave cubic ordered weighted average (COWA) operator as

$$\text{COWA}\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\left(\left[1-\prod_{i=1}^{n}(1-\zeta^{L}_{\psi(i)})^{\omega_i},1-\prod_{i=1}^{n}(1-\zeta^{U}_{\psi(i)})^{\omega_i}\right],\prod_{i=1}^{n}(\lambda_{\psi(i)})^{\omega_i}\right) \quad (2.46)$$

where $(\psi(1),\psi(2),\ldots,\psi(n))$ is a permutation of $(1,2,\ldots,n)$ such that $\mathcal{A}_{\psi(i-1)} \geq \mathcal{A}_{\psi(i)}$ for $(i=2,3,\ldots,n)$

2.4.2 Probabilistic dual hesitant fuzzy sets

Definition 2.4.3. [61] Let $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ be three PDHFEs such that $\mathcal{A}=(h|p_h,g|q_g)$, $\mathcal{A}_1=(h_1|p_{h_1},g_1|q_{g_1})$ and $\mathcal{A}_2=(h_2|p_{h_2},g_2|q_{g_2})$. Then, for $\xi>0$, operational laws are given as follows:

(i) $\mathcal{A}_1\oplus\mathcal{A}_2=\bigcup\limits_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}}\left(\{\zeta_1+\zeta_2-\zeta_1\zeta_2|\,p_{\zeta_1}p_{\zeta_2}\},\{\vartheta_1\vartheta_2|q_{\vartheta_1}q_{\vartheta_2}\}\right)$

(ii) $\mathcal{A}_1\otimes\mathcal{A}_2=\bigcup\limits_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}}\left(\{\zeta_1\zeta_2|p_{\zeta_1}p_{\zeta_2}\},\{\vartheta_1+\vartheta_2-\vartheta_1\vartheta_2|\,q_{\vartheta_1}q_{\vartheta_2}\}\right)$

(iii) $\xi\mathcal{A}=\bigcup\limits_{\substack{\zeta\in h,\\ \vartheta\in g}}\left(\left\{1-(1-\zeta)^{\xi}|p_{\zeta}\right\},\left\{\vartheta^{\xi}|q_{\vartheta}\right\}\right)$

(iv) $\mathcal{A}^{\xi}=\bigcup\limits_{\substack{\zeta\in h,\\ \vartheta\in g}}\left(\left\{\zeta^{\xi}|p_{\zeta}\right\},\left\{1-(1-\vartheta)^{\xi}|q_{\vartheta}\right\}\right)$

The above stated operations induced over PDHFSs gave AO as given below:

(i) Hao et al. [61] introduced AO as:

$$\text{PDHFWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)=\bigcup_{\substack{\zeta_i\in h_i,\\ \vartheta_i\in g_i}}\left(\begin{array}{c}\left\{1-\prod\limits_{i=1}^{n}(1-\zeta_i)^{\frac{1}{n}}\,\middle|\,\prod\limits_{i=1}^{n}p_{\zeta_i}\right\},\\ \left\{\prod\limits_{i=1}^{n}(\vartheta_i)^{\frac{1}{n}}\,\middle|\,\prod\limits_{i=1}^{n}q_{\vartheta_i}\right\}\end{array}\right) \quad (2.47)$$

In addition to the above stated AOs, significant research has been done on the AO based on mean-based operators. Two prominent mean operators are based on Bonferroni mean (BM) and Maclaurin Symmetric mean (MSM). The relevant concepts are given below:

Definition 2.4.4. For $p, q \geq 0$ and numbers a_i $(i = 1, 2, \ldots, n)$

$$\mathrm{BM}^{\mathfrak{p},\mathfrak{q}}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} {a_i}^{\mathfrak{p}} {a_j}^{\mathfrak{q}} \right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \tag{2.48}$$

is known as Bonferroni mean (BM).

It satisfies the following properties:

(i) $\mathrm{BM}^{\mathfrak{p},\mathfrak{q}}(0, 0, \ldots, 0) = 0$;

(ii) $\mathrm{BM}^{\mathfrak{p},\mathfrak{q}}(a, a, \ldots, a) = a$;

(iii) $\mathrm{BM}^{\mathfrak{p},\mathfrak{q}}(a_1, a_2, \ldots, a_n) \leq \mathrm{BM}^{\mathfrak{p},\mathfrak{q}}(b_1, b_2, \ldots, b_n)$, if $a_i \leq b_i \forall i$;

(iv) $\min_i\{a_i\} \leq \mathrm{BM}^{\mathfrak{p},\mathfrak{q}}(a_1, a_2, \ldots, a_n) \leq \max_i\{a_i\}$.

Definition 2.4.5. [101] Let a_i $(i = 1, 2, \ldots, n)$ be a collection of non-negative real numbers, and $k = 1, 2, \ldots, n$. If

$$\mathrm{MSM}^{(k)}(a_1, a_2, \ldots, a_n) = \left(\frac{\sum\limits_{\substack{1 \leq i_1 < \cdots \\ < i_k \leq n}} \prod\limits_{j=1}^{k} a_{i_j}}{\binom{n}{k}} \right)^{\left(\frac{1}{k}\right)} \tag{2.49}$$

then $\mathrm{MSM}^{(k)}$ is called the Maclaurin symmetric mean (MSM), where $(i_1, i_2, \ldots, i_k)$ traversal all the k-tuple combination of $(1, 2, \ldots, n)$, $\binom{n}{k}$ is the binomial coefficient.

Obviously, the MSM has the following properties:

(i) $\mathrm{MSM}^{(k)}(0, 0, \ldots, 0) = 0$;

(ii) $\mathrm{MSM}^{(k)}(a, a, \ldots, a) = a$;

(iii) $\mathrm{MSM}^{(k)}(a_1, a_2, \ldots, a_n) \leq \mathrm{MSM}^{(k)}(b_1, b_2, \ldots, b_n)$, if $a_i \leq b_i \forall i$;

(iv) $\min_i\{a_i\} \leq \mathrm{MSM}^{(k)}(a_1, a_2, \ldots, a_n) \leq \max_i\{a_i\}$.

2.5 Description of DM model

Consider $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m\}$ be the set of 'm' alternatives which are to be evaluated under 'n' criteria denoted by $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\}$ and experts give their preference values as CIFSs $\mathcal{A}_{ij} = (([\zeta_{ij}^L, \zeta_{ij}^U], [\vartheta_{ij}^L, \vartheta_{ij}^U]), (\zeta_{ij}, \vartheta_{ij}))$ or PDHFSs $\mathcal{A}_{ij} = \bigcup_{\substack{\zeta_{ij} \in h_{ij} \\ \vartheta_{ij} \in g_{ij}}} \left(\left\{ \zeta_{ij} \middle| p_{\zeta_{ij}} \right\}, \left\{ \vartheta_{ij} \middle| q_{\vartheta_{ij}} \right\} \right)$; $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ where ζ_{ij} represents the membership degree and ϑ_{ij} denotes the non-membership degree. There may be multiple decision makers denoted as 'd' for which the complete decision matrix is given as $\mathcal{M}^{(d)} = \left(\mathcal{A}_{ij}^{(d)} \right)_{m \times n}$. However, these preference values by multiple number of experts are fused together into a single decision-matrix $(\mathcal{A}_{ij})_{m \times n}$. Subsequently, it is evident that in the situations having a single decision-maker i.e., for value of $d = 1$, there will be no need to fusion of preference matrices into a single one. Based on this, the general procedure of approach is given as below:

Step 1: Arrange the information related to alternatives in term of the considered fuzzy environment as follows:

$$\mathcal{M}_{m \times n}^{(d)} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{bmatrix} \mathcal{A}_{11}^{(d)} & \mathcal{A}_{12}^{(d)} & \ldots & \mathcal{A}_{1n}^{(d)} \\ \mathcal{A}_{21}^{(d)} & \mathcal{A}_{22}^{(d)} & \ldots & \mathcal{A}_{2n}^{(d)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{m1}^{(d)} & \mathcal{A}_{m2}^{(d)} & \ldots & \mathcal{A}_{mn}^{(d)} \end{bmatrix} \end{array}$$

Step 2: If $d > 1$, then fuse the decision-matrices into a single one by undergoing the suitable fusion method.

$$\mathcal{M}_{m \times n} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \ldots & \mathcal{A}_{1n} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \ldots & \mathcal{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{m1} & \mathcal{A}_{m2} & \ldots & \mathcal{A}_{mn} \end{bmatrix} \end{array}$$

Step 3: Normalize the rating values from cost to benefit type or vice-versa according to the nature of problem. Based on normalization from cost to benefit criteria the normalization equation is given below:

$$r_{ij} = \begin{cases} \mathcal{A}_{ij} & ; \quad \text{if } j \in \text{benefit type criterion} \\ \mathcal{A}_{ij}^{c} & ; \quad \text{if } j \in \text{cost type criterion} \end{cases}$$

and hence obtain a normalized matrix $\mathcal{R} = (r_{ij})_{m\times n}$

Step 4: Aggregate the preference values related to alternatives into collective values using the suitable tool/technique.

Step 5: Utilize the appropriate defuzzification method to compute the crisp value related to each alternative.

Step 6: Rank the alternatives on the basis of defuzzified values and select the best alternative(s).

Chapter 3

Cubic Intuitionistic Fuzzy Sets and its Fundamental Properties[1]

In this chapter, we propose the notion of cubic intuitionistic fuzzy sets (CIFSs), its properties and related results. Traditionally, the information related to an element is collected either by an intuitionistic fuzzy set (IFS) or interval-valued IFS (IVIFS) which may lose some useful information and hence may affect the decision results. As a generalization of the IFSs and IVIFSs, CIFS is a strong and valuable tool to represent the imprecise information by embedding both the features of IFSs and IVIFSs instantaneously. Based on its inherent property, we further define the internal (external) CIFS, P(R)-union and P(R)-intersection of CIFSs. Several desirable properties of these operations are defined along with their proof.

3.1 Introduction

The primitive theories based on IFS, IVIFS etc facilitate the uncertainties to a great extent, but still, they cannot withstand the situations where the decision-maker has to consider the falsity corresponding to the truth value ranging over an interval. For signifying the situational need and applicability of the proposed CIFS, consider an example. Suppose that an experimentalist has to perform some experiment E and before performing it i.e. at time T_1, he is unsure that the outcome measure may range between $[0.40, 0.50]$, and

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after completion of the experiment at time T_2, he obtain a crisp value that may agree (P-order) or disagree (R-order) to the pre-assumed interval. Say at this point he obtained such values that disagree 50 percent to the interval $[0.40, 0.50]$, so he may formulate an R-order CFS by merging these two time constraints as $\big(([0.40, 0.50]), 0.50\big)$. Thus, this environment increases the level of precision by enhancing the scope of the membership interval by considering a fuzzy set membership value corresponding to it. But, it has been seen from the above considerations the degree of acceptance region during the analysis is considered, but the degree of rejection is completely ignored. However, in our day-to-day life decision, it is quite obvious that the degree of rejection, plays a dominant role during the performance evaluation. Revisited the above-considered experiment, if the same experimentalist is also unsure of the non-membership portion that, at time T_1, the output values may not lie between $[0.10, 0.20]$, and at time T_2 he obtain a crisp value that may agree (P-order) or disagree (R-order) to the pre-assumed interval. Say at this point, he obtained such values that agree 20 percent to the interval $[0.10, 0.20]$. This complete situation may be formulated as an R-order CIFS by merging these two time constraints into a single one as $\big(([0.40, 0.50], [0.10, 0.20]), (0.50, 0.20)\big)$.

Thus, by keeping in mind the features of the existing fuzzy set and IFS, the objective of the present work is to present a new notion named as a cubic intuitionistic fuzzy set (CIFS). Also, we have introduced internal (or external) cubic intuitionistic fuzzy set and have investigated their several properties. Under these, the P(R)- union, the intersection between the family of the CIFS have been introduced. Several results are presented to show the intrinsic properties of the internal (or external) CIFS and showed that their union (or intersection) need not be an internal (or external) CIFS. Finally, we propose some conditions on to the two internal (or external) CIFSs, which makes the P (R) -union or intersection of the two sets to be an internal or external CIFS.

3.2 Cubic Intuitionistic fuzzy set

In this section, the concept of the CIFS is proposed with its related basic definitions.

Definition 3.2.1. A CIFS $\mathcal{A}$ defined over the universal set $\mathcal{X}$ is an ordered pair which is defined as follows

$$\mathcal{A} = \{(x, A(x), \lambda(x)) \mid x \in \mathcal{X}\} \tag{3.1}$$

where $A = \{x, \big([\zeta_A^L(x), \zeta_A^U(x)], [\vartheta_A^L(x), \vartheta_A^U(x)]\big), \mid x \in \mathcal{X}\}$ represents the IVIFS defined on $\mathcal{X}$ while $\lambda(x) = \{x, (\zeta_A(x), \vartheta_A(x)) \mid x \in \mathcal{X}\}$ represents an IFS such that $0 \leq \zeta_A^L(x) \leq \zeta_A^U(x) \leq 1$, $0 \leq \vartheta_A^L(x) \leq \vartheta_A^U(x) \leq 1$ and $0 \leq \zeta_A^U(x) + \vartheta_A^U(x) \leq 1$. Also, $0 \leq \zeta_A(x), \vartheta_A(x) \leq 1$ and $\zeta_A(x) + \vartheta_A(x) \leq 1$. For the sake of simplicity, we denote these pairs as $\mathcal{A} = \big(A, \lambda\big)$, where $A = ([\zeta_A^L, \zeta_A^U], [\vartheta_A^L, \vartheta_A^U])$ and $\lambda = (\zeta_A, \vartheta_A)$.

Analytic aspect of Definition 3.2.1

For an element, $x \in \mathcal{X}$, a CIFS is modeled by combining the IVIFS with that of the IFS. Analytically, these both (IVIFS and IFS) combined to form CIFS, can capture information in two ways:

(i) The element x is evaluated at different time zones. For instance, the rating values regarding the element x are recorded in IVIFS at time zone t_1 whereas recorded in form of IFS at another time zone t_2. In context to Definition 3.2.1, this concept can be interpreted as that, at a time zone t_1, the interval-valued ratings for membership (or non-membership) are recorded as $[\zeta_A^L(x), \zeta_A^U(x)]$ (or $[\vartheta_A^L(x), \vartheta_A^U(x)]$) whereas for the time zone t_2 the same element x is recorded for IFS value in form of $(\zeta_A(x), \vartheta_A(x))$.

(ii) Another way to record hybrid information in form of CIFS is that it gave us a way to record two-level information and these two levels can be of two different evaluators also giving their opine ratings at different or same time zones. An analytic overlook, with respect to the Definition 3.2.1, can be explained like that, the said IVIFS can be the rating values given by the decision-maker 1, whereas the IFS can be that of the decision-maker 2, regarding the same element x. Thus, with this notion, we are having a facility to store two opined values in a single unit called CIFS.

(iii) Moreover, it also provides us the scope of the two-level judgment. The IFS within a CIFS can be the degree of assurance of IVIFS values in a CIFS. In context to Definition 3.2.1, it can be explained as IFS values (ζ_A, ϑ_A) recorded can also be agreeness and dis-agreeness towards the pre-recorded IVIFS values.

Remark 3.2.1. The following special cases are to be considered from Eq. (3.1), which are summarized as follows.

(i) A CIFS $\mathcal{A}$ with $A(x) = ([0,0],[1,1])$ and $\lambda(x) = (1,0)$ for all $x \in \mathcal{X}$ is denoted by $\ddot{0}$.

(ii) A CIFS $\mathcal{A}$ with $A(x) = ([1,1],[0,0])$ and $\lambda(x) = (0,1)$ for all $x \in \mathcal{X}$ is denoted by $\ddot{1}$.

(iii) A CIFS $\mathcal{A}$ with $A(x) = ([0,0],[1,1])$ and $\lambda(x) = (0,1)$ for all $x \in \mathcal{X}$ is denoted by $\hat{0}$.

(iv) A CIFS $\mathcal{A}$ with $A(x) = ([1,1],[0,0])$ and $\lambda(x) = (1,0)$ for all $\in \mathcal{X}$ is denoted by $\hat{1}$

Definition 3.2.2. (Internal Cubic Intuitionistic fuzzy set (ICIFS):) A CIFS $\mathcal{A}$ defined in Eq. (3.1) is said to be ICIFS if $\zeta_A^L(x) \leq \zeta_A(x) \leq \zeta_A^U(x)$ and $\vartheta_A^L(x) \leq \vartheta_A(x) \leq \vartheta_A^U(x)$ for all $x \in \mathcal{X}$.

Definition 3.2.3. (External Cubic Intuitionistic fuzzy set (ECIFS):) A CIFS $\mathcal{A}$ defined in Eq. (3.1) is said to be ECIFS if $\zeta_A(x) \notin \left(\zeta_A^L(x), \zeta_A^U(x)\right)$ and $\vartheta_A(x) \notin \left(\vartheta_A^L(x), \vartheta_A^U(x)\right)$ for all $x \in \mathcal{X}$

Example 3.2.1. Let $\mathcal{A}$ be CIFS defined on $\mathcal{X}$. If we take an IVIFS as $A = ([0.3, 0.5], [0.2, 0.4])$ and an IFS as $\lambda = (0.4, 0.3)$ then $\mathcal{A}$ is called an ICIFS because $0.4 \in [0.3, 0.5]$ and $0.3 \in [0.2, 0.4]$. On the other hand, if we take an IFS as $\lambda = (0.2, 0.7)$ for the same IVIFS then $\mathcal{A}$ is called an ECIFS because $0.2 \notin (0.3, 0.5)$ and $0.7 \notin (0.2, 0.4)$.

Theorem 3.2.1. Let $\mathcal{A} = (A, \lambda)$ be a CIFS in $\mathcal{X}$ which is not an ICIFS. Then there exist $x \in \mathcal{X}$ such that $\zeta_A(x) \notin (\zeta_A^L, \zeta_A^U)$ and $\vartheta_A(x) \notin (\vartheta_A^L, \vartheta_A^U)$.

Proof. Straightforward. □

Theorem 3.2.2. Let $\mathcal{A} = (A, \lambda)$ be both an ICIFS and ECIFS, for all $x \in \mathcal{X}$ then $\zeta_A(x) \in U_M(x) \cup L_M(x)$ and $\vartheta_A(x) \in U_N(x) \cup L_N(x)$ where $U_M = \{\zeta_A^U(x) \mid x \in \mathcal{X}\}$, $L_M = \{\zeta_A^L(x) \mid x \in \mathcal{X}\}$, $U_N = \{\vartheta_A^U(x) \mid x \in \mathcal{X}\}$ and $L_N = \{\vartheta_A^L(x) \mid x \in \mathcal{X}\}$.

Proof. Since $\mathcal{A}$ is both ICIFS and ECIFS, we have $\zeta_A^L(x) \leq \zeta_A(x) \leq \zeta_A^U(x)$, $\zeta_A \notin (\zeta_A^L(x), \zeta_A^U(x))$, $\vartheta_A^L(x) \leq \vartheta_A \leq \vartheta_A^U(x)$ and $\vartheta_A \notin (\vartheta_A^L(x), \vartheta_A^U(x))$. Therefore, $\zeta_A(x) = \zeta_A^L(x)$ or $\zeta_A^U(x)$. Similarly, $\vartheta_A(x) = \vartheta_A^L(x)$ or $\vartheta_A^U(x)$. Thus, $\zeta_A(x) \in U_M(x) \cup L_M(x)$ and $\vartheta_A(x) \in U_N(x) \cup L_N(x)$. □

Definition 3.2.4. Let $\mathcal{A} = (A, \lambda)$ and $\mathcal{B} = (B, \mu)$ be two CIFSs in $\mathcal{X}$. Then we define:

(a) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow [\zeta_A^L, \zeta_A^U] = [\zeta_B^L, \zeta_B^U]$, $[\vartheta_A^L, \vartheta_A^U] = [\vartheta_B^L, \vartheta_B^U]$, $\zeta_A = \zeta_B$ and $\vartheta_A = \vartheta_B$.

(b) (P-order) $\mathcal{A} \subseteq_P \mathcal{B}$ if $[\zeta_A^L, \zeta_A^U] \subseteq [\zeta_B^L, \zeta_B^U]$,$[\vartheta_A^L, \vartheta_A^U] \supseteq [\vartheta_B^L, \vartheta_B^U]$, $\zeta_A \leq \zeta_B$ and $\vartheta_A \geq \vartheta_B$

(c) (R-order) $\mathcal{A} \subseteq_R \mathcal{B}$ if $[\zeta_A^L, \zeta_A^U] \subseteq [\zeta_B^L, \zeta_B^U]$, $[\vartheta_A^L, \vartheta_A^U] \supseteq [\vartheta_B^L, \vartheta_B^U]$, $\zeta_A \geq \zeta_B$ and $\vartheta_A \leq \vartheta_B$

Definition 3.2.5. For a family of CIFS $\{\mathcal{A}_i, i \in \Lambda\}$, we have defined the P-union, P-intersection, R-union and R-intersection between them as follows:

(a) (P-union):

$$\begin{aligned}
\bigcup_{\substack{P\\ i\in\Lambda}} \mathcal{A}_i &= \left\{ \left(x, \left(\bigcup_{i\in\Lambda} A_i \right)(x), \left(\bigvee_{i\in\Lambda} \lambda_i \right)(x) \right) \mid x \in \mathcal{X} \right\} \\
&= \left\{ x, \left(\sup_{i\in\Lambda} \left[\zeta_{A_i}^L, \zeta_{A_i}^U\right](x), \inf_{i\in\Lambda} \left[\vartheta_{A_i}^L, \vartheta_{A_i}^U\right](x) \right), \left(\sup_{i\in\Lambda} \zeta_{A_i}(x), \inf_{i\in\Lambda} \vartheta_{A_i}(x) \right) \right\}
\end{aligned}$$

(b) (P-intersection):

$$\begin{aligned}
\bigcap_{\substack{P\\ i\in\Lambda}} \mathcal{A}_i &= \left\{ \left(x, \left(\bigcap_{i\in\Lambda} A_i \right)(x), \left(\bigwedge_{i\in\Lambda} \lambda_i \right)(x) \right) \mid x \in \mathcal{X} \right\} \\
&= \left\{ x, \left(\inf_{i\in\Lambda} \left[\zeta_{A_i}^L, \zeta_{A_i}^U\right](x), \sup_{i\in\Lambda} \left[\vartheta_{A_i}^L, \vartheta_{A_i}^U\right](x) \right), \left(\inf_{i\in\Lambda} \zeta_{A_i}(x), \sup_{i\in\Lambda} \vartheta_{A_i}(x) \right) \right\}
\end{aligned}$$

(c) (R-union):

$$\begin{aligned}
\bigcup_{\substack{R\\ i\in\Lambda}} \mathcal{A}_i &= \left\{ \left(x, \left(\bigcup_{i\in\Lambda} A_i \right)(x), \left(\bigwedge_{i\in\Lambda} \lambda_i \right)(x) \right) \mid x \in \mathcal{X} \right\} \\
&= \left\{ x, \left(\sup_{i\in\Lambda} \left[\zeta_{A_i}^L, \zeta_{A_i}^U\right](x), \inf_{i\in\Lambda} \left[\vartheta_{A_i}^L, \vartheta_{A_i}^U\right](x) \right), \left(\inf_{i\in\Lambda} \zeta_{A_i}(x), \sup_{i\in\Lambda} \vartheta_{A_i}(x) \right) \right\}
\end{aligned}$$

(d) (R-intersection):

$$\begin{aligned}
\bigcap_{\substack{R\\ i\in\Lambda}} \mathcal{A}_i &= \left\{ \left(x, \left(\bigcap_{i\in\Lambda} A_i \right)(x), \left(\bigvee_{i\in\Lambda} \lambda_i \right)(x) \right) \mid x \in \mathcal{X} \right\} \\
&= \left\{ x, \left(\inf_{i\in\Lambda} \left[\zeta_{A_i}^L, \zeta_{A_i}^U\right](x), \sup_{i\in\Lambda} \left[\vartheta_{A_i}^L, \vartheta_{A_i}^U\right](x) \right), \left(\sup_{i\in\Lambda} \zeta_{A_i}(x), \inf_{i\in\Lambda} \vartheta_{A_i}(x) \right) \right\}
\end{aligned}$$

Definition 3.2.6. The complement of CIFS $\mathcal{A} = (A, \lambda)$ is defined to be a cubic intuitionistic set $\mathcal{A}^c = \big(A^c, \lambda^c\big)$, where $A^c = ([\vartheta_A^L, \vartheta_A^U], [\zeta_A^L, \zeta_A^U])$ be the complement of the IVIFS $A = ([\zeta_A^L, \zeta_A^U], [\vartheta_A^L, \vartheta_A^U])$ and $\lambda^c = (\vartheta_A, \zeta_A)$ be the complement of IFS $\lambda = (\zeta_A, \vartheta_A)$.

Remark 3.2.2. $(\mathcal{A}^c)^c = \mathcal{A}, \hat{0}^c = \hat{1}, \hat{1}^c = \hat{0}, \ddot{0}^c = \ddot{1}, \ddot{1}^c = \ddot{0}$. For any $\mathcal{A}_i = \{\big(x, A_i(x), \lambda_i(x)\big) \mid x \in \mathcal{X}\}, i \in \Lambda$, we have $\big(\cup_P \mathcal{A}_i\big)^c = \cap_P(\mathcal{A}_i)^c$ and $\big(\cap_P \mathcal{A}_i\big)^c = \cup_P(\mathcal{A}_i)^c, i \in \Lambda$. Also, we have $\big(\cup_R \mathcal{A}_i\big)^c = \cap_R(\mathcal{A}_i)^c$ and $\big(\cap_R \mathcal{A}_i\big)^c = \cup_R(\mathcal{A}_i)^c, i \in \Lambda$.

Theorem 3.2.3. For CIFSs $\mathcal{A} = \big(A, \lambda\big), \mathcal{B} = \big(B, \mu\big), \mathcal{C} = \big(C, \gamma\big)$ and $\mathcal{D} = \big(D, \rho\big)$, where A, B, C and D are IVIFSs and λ, μ, γ and ρ are IFSs in $\mathcal{X}$, we have

(i) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{B} \subseteq_P \mathcal{C}$ then $\mathcal{A} \subseteq_P \mathcal{C}$.

(ii) if $\mathcal{A} \subseteq_P \mathcal{B}$ then $\mathcal{B}^c \subseteq_P \mathcal{A}^c$.

(iii) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{A} \subseteq_P \mathcal{C}$ then $\mathcal{A} \subseteq_P \mathcal{B} \cap_P \mathcal{C}$.

(iv) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{C} \subseteq_P \mathcal{B}$ then $\mathcal{A} \cup_P \mathcal{C} \subseteq_P \mathcal{B}$.

(v) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{C} \subseteq_P \mathcal{D}$ then . $\mathcal{A} \cup_P \mathcal{C} \subseteq_P \mathcal{B} \cup_P \mathcal{D}$ and . $\mathcal{A} \cap_P \mathcal{C} \subseteq_P \mathcal{B} \cap_P \mathcal{D}$.

(vi) if $\mathcal{A} \subseteq_R \mathcal{B}$ and if $\mathcal{B} \subseteq_R \mathcal{C}$ then $\mathcal{A} \subseteq_R \mathcal{C}$.

(vii) if $\mathcal{A} \subseteq_R \mathcal{B}$ then $\mathcal{B}^c \subseteq_R \mathcal{A}^c$.

(viii) if $\mathcal{A} \subseteq_R \mathcal{B}$ and $\mathcal{A} \subseteq_R \mathcal{C}$ then $\mathcal{A} \subseteq_R \mathcal{B} \cap_R \mathcal{C}$.

(ix) if $\mathcal{A} \subseteq_R \mathcal{B}$ and $\mathcal{C} \subseteq_R \mathcal{B}$ then $\mathcal{A} \cup_R \mathcal{C} \subseteq_R \mathcal{B}$.

(x) if $\mathcal{A} \subseteq_R \mathcal{B}$ and $\mathcal{C} \subseteq_R \mathcal{D}$ then $\mathcal{A} \cup_R \mathcal{C} \subseteq_R \mathcal{B} \cup_R \mathcal{D}$ and $\mathcal{A} \cap_R \mathcal{C} \subseteq_R \mathcal{B} \cap_R \mathcal{D}$.

Proof. Straightforward □

Theorem 3.2.4. Let $\mathcal{A}$ be a CIFS in $\mathcal{X}$. If $\mathcal{A}$ is an ICIFS, then $\mathcal{A}^c$ is also an ICIFS.

Proof. Since $\mathcal{A} = \big(A, \lambda\big)$ be a ICIFS in $\mathcal{X}$ where $A = ([\zeta_A^L, \zeta_A^U], [\vartheta_A^L, \vartheta_A^U])$ be the IVIFS and $\lambda = (\zeta_A, \vartheta_A)$ be the IFS. Since $A^c = ([\vartheta_A^L, \vartheta_A^U], [\zeta_A^L, \zeta_A^U])$ be compliment of A which is again an IVIFS and $\lambda^c = (\vartheta_A, \zeta_A)$ be the complement of λ which is also IFS and hence $\mathcal{A}^c = \big(A^c, \lambda^c\big)$ is also an ICIFS. □

Theorem 3.2.5. Let $\mathcal{A}$ be a CIFS in $\mathcal{X}$. If $\mathcal{A}$ is an ECIFS, then $\mathcal{A}^c$ is also an ECIFS.

Proof. Follows from the Theorem 3.2.4 and hence we omit here. □

3.3 Properties of ICIFS & ECIFS

Property 3.3.1. Let $\mathcal{A}_i = \big(A_i, \lambda_i\big), i \in \Lambda$ be the family of ICIFS in $\mathcal{X}$. Then the P-union and the P-intersection of $\mathcal{A}_i$ are also ICIFS in $\mathcal{X}$.

Proof. Since $\mathcal{A}_i = \big(A_i, \lambda_i\big)$ is an ICIFS in $\mathcal{X}$, where $A_i = ([\zeta^L_{A_i}, \zeta^U_{A_i}],\ [\vartheta^L_{A_i},\ \vartheta^U_{A_i}])$ and $\lambda_i = \big(\zeta_{A_i}, \vartheta_{A_i}\big)$ which implies that for each i, $\zeta_{A_i} \in [\zeta^L_{A_i}, \zeta^U_{A_i}]$ and $\vartheta_{A_i} \in [\vartheta^L_{A_i}, \vartheta^U_{A_i}]$ and hence $\big(\bigcup_i \zeta_{A_i}\big)^L \leq \bigvee_i \lambda_i \leq \big(\bigcup_i \zeta_{A_i}\big)^U$ and $\big(\bigcap_i \vartheta_{A_i}\big)^L \leq \bigwedge_i \lambda_i \leq \big(\bigcap_i \vartheta_{A_i}\big)^U$. Hence, $\bigcup_P \mathcal{A}_i$ and $\bigcap_P \mathcal{A}_i$ are ICIFS in $\mathcal{X}$. □

Remark 3.3.1. But the R-union and R-intersection of two ICIFSs need not to be an ICIFS, which has been explained with the following example.

Example 3.3.1. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ICIFSs in which $A = ([0.3, 0.5], [0.2, 0.4])$, $\lambda = (0.4, 0.3)$ and $B = ([0.7, 0.8], [0.1, 0.2])$ and $\mu = (0.8, 0.1)$. Then, clearly $\mathcal{A} \cup_R \mathcal{B} = \big(([0.7, 0.8], [0.1, 0.2]),\ (0.4, 0.3)\big)$ and $\mathcal{A} \cap_R \mathcal{B} = \big(([0.3, 0.5], [0.2, 0.4]), (0.8, 0.1)\big)$ are not ICIFSs.

Remark 3.3.2. Also, P-union and P-intersection of ECIFSs need not to be an ECIFS. This has been explained with the following example.

Example 3.3.2. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ECIFSs such that $A = ([0.3, 0.5], [0.2, 0.4])$, $\lambda = (0.8, 0.1)$ and $B = ([0.7, 0.8],\ [0.1, 0.2])$, $\mu = (0.4, 0.3)$. Then, $\mathcal{A} \cup_P \mathcal{B} = \big(([0.7, 0.8],\ [0.1, 0.2]),\ (0.8, 0.1)\big)$ and $\mathcal{A} \cap_P \mathcal{B} = \big(([0.3, 0.5],\ [0.2, 0.4]),\ (0.4, 0.3)\big)$ which are clearly not an ECIFSs.

Remark 3.3.3. The R-union and R-intersection of ECIFSs need not to be an ECIFS which has been explained as below.

Example 3.3.3. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ECIFSs in which A=([0.2,0.3], [0.4,0.5]), $\lambda = \big(0.50, 0.25\big)$ and $B = ([0.4, 0.6], [0.2, 0.4])$, $\mu = \big(0.7, 0.2\big)$. Then we have

$\mathcal{A} \cup_R \mathcal{B} = \big(([0.4, 0.6], [0.2, 0.4]), \ (0.50, 0.25)\big)$ which is not an ECIFS. On the other hand, if we take $A = ([0.2, 0.3], [0.4, 0.5]), \ \lambda = \big(0.10, 0.25\big)$ and $B = ([0.4, 0.6], [0.2, 0.4]), \ \mu = (0.25, 0.20)$ then $\mathcal{A} \cap_R \mathcal{B} = \big(([0.2, 0.3], [0.4, 0.5]), (0.25, 0.20)\big)$ which is not an ECIFS.

From Example 3.3.1, we have seen that R-union of two ICIFSs need not be an ICIFS. Here, we provide a condition for the R-union of two ICIFSs to be an ICIFS.

Property 3.3.2. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ICIFSs in $\mathcal{X}$ such that $\max\{\zeta_A^L(x), \zeta_B^L(x)\} \leq (\zeta_A \wedge \zeta_B)(x)$ and $\min\{\vartheta_A^U(x), \vartheta_B^U(x)\} \geq (\vartheta_A \vee \vartheta_B)(x)$ for all $x \in \mathcal{X}$. Then the R-union of $\mathcal{A}$ and $\mathcal{B}$ is an ICIFS in $\mathcal{X}$.

Proof. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ICIFSs in $\mathcal{X}$ which satisfy the given conditions. Then $\zeta_A^L(x) \leq \zeta_A(x) \leq \zeta_A^U(x)$, $\vartheta_A^L(x) \leq \vartheta_A(x) \leq \vartheta_A^U(x)$, $\zeta_B^L(x) \leq \zeta_B(x) \leq \zeta_B^U(x)$ and $\vartheta_B^L(x) \leq \vartheta_B(x) \leq \vartheta_B^U(x)$ which implies that $(\zeta_A \wedge \zeta_B)(x) \leq (\zeta_A^U \cup \zeta_B^U)(x)$ and $(\vartheta_A \vee \vartheta_B)(x) \geq (\vartheta_A^L \cap \vartheta_B^L)(x)$. Thus, it follows from the given condition that

$$\begin{aligned}
(\zeta_A^L \cup \zeta_B^L)(x) &= \max\{\zeta_A^L(x), \zeta_B^L(x)\} \\
&\leq (\zeta_A \wedge \zeta_B)(x) \\
&\leq (\zeta_A^U \cup \zeta_B^U)(x) \\
\text{and} \quad (\vartheta_A^U \cap \vartheta_B^U)(x) &= \min\{\vartheta_A^U(x), \vartheta_B^U(x)\} \\
&\geq (\vartheta_A \vee \vartheta_B)(x) \\
&\geq (\vartheta_A^L \cap \vartheta_B^L)(x)
\end{aligned}$$

and hence $\mathcal{A} \cup_R \mathcal{B} = (A \cup B, \lambda \wedge \mu)$ is an ICIFS in $\mathcal{X}$. □

Again from Example 3.3.1, we have seen that R-intersection of two ICIFSs need not be an ICIFS. Now, we provide a condition on two ICIFSs so that their R-intersection becomes an ICIFS.

Property 3.3.3. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ICIFSs defined in $\mathcal{X}$ which satisfies the conditions $\min\{\zeta_A^U(x), \zeta_B^U(x)\} \geq (\zeta_A \vee \zeta_B)(x)$ and $\max\{\vartheta_A^L(x), \vartheta_B^L(x)\} \leq (\vartheta_A \wedge \vartheta_B)(x)$ for all $x \in \mathcal{X}$. Then the R-intersection of $\mathcal{A}$ and $\mathcal{B}$ is an ICIFS in $\mathcal{X}$.

Proof. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ICIFSs in $\mathcal{X}$ which satisfy the given conditions. Then for all $x \in \mathcal{X}$, $\zeta_A^L(x) \leq \zeta_A(x) \leq \zeta_A^U(x)$ and $\vartheta_A^L(x) \leq \vartheta_A(x) \leq \vartheta_A^U(x)$

and $\zeta_B^L(x) \leq \zeta_B(x) \leq \zeta_B^U(x)$ and $\vartheta_B^L(x) \leq \vartheta_B(x) \leq \vartheta_B^U(x)$ and therefore $(\zeta_A^L \cap \zeta_B^L)(x) \leq (\zeta_A \vee \zeta_B)(x)$ and $(\vartheta_A^U \cup \vartheta_B^U)(x) \geq (\vartheta_A \wedge \vartheta_B)(x)$. Using the given condition, we have

$$\begin{aligned} (\zeta_A^L \cap \zeta_B^L)(x) &\leq (\zeta_A \vee \zeta_B)(x) \\ &\leq \min\{\zeta_A^U(x), \zeta_B^U(x)\} = (\zeta_A^U \cap \zeta_B^U)(x) \\ \text{and} \qquad (\vartheta_A^U \cup \vartheta_B^U)(x) &\geq (\vartheta_A \wedge \vartheta_B)(x) \\ &\geq \max\{\vartheta_A^L(x), \vartheta_B^L(x)\} = (\vartheta_A^L \cup \vartheta_B^L)(x) \end{aligned}$$

and hence $\mathcal{A} \cap_R \mathcal{B} = \big(A \cap B, \lambda \vee \mu\big)$ is an ICIFS in $\mathcal{X}$. □

Given two CIFSs $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ in $\mathcal{X}$, if we exchange μ with λ and then we denote these newly formulated CIFS by $\mathcal{A}^* = \big(A, \mu\big)$ and $\mathcal{B}^* = \big(B, \lambda\big)$ respectively throughout the chapter.

Property 3.3.4. For any two ECIFSs $\mathcal{A}$ and $\mathcal{B}$ defined over $\mathcal{X}$, two CIFSs $\mathcal{A}^*$ and $\mathcal{B}^*$ may neither be ICIFS and nor be ECIFS.

Proof. It is enough to give counter examples to support the results. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be ECIFS in which $A = ([0.6, 0.7], [0.2, 0.3])$, $\lambda = \big(0.8, 0.1\big)$ and $B = ([0.4, 0.6], [0.3, 0.4])$, $\mu = \big(0.2, 0.5\big)$ for all x. Then clearly $\mathcal{A}^* = \big(([0.6, 0.7],\ [0.2, 0.3]), \big(0.2, 0.5\big)\big)$ and $\mathcal{B}^* = \big(([0.4, 0.6],\ [0.3, 0.4]), \big(0.8, 0.1\big)\big)$ are not ICIFSs in $\mathcal{X}$ because $0.2 \notin [0.6, 0.7]$ and $0.8 \notin [0.4, 0.6]$.

On the other hand, if we consider $\mathcal{A} = \big(([0.2, 0.35],\ [0.4, 0.5]), (0.45, 0.25)\big)$ and $\mathcal{B} = \big(([0.3, 0.5],\ [0.2, 0.4]), (0.3, 0.4)\big)$ be two ECIFSs in $\mathcal{X}$, then clearly $\mathcal{A}^* = \big(([0.2, 0.35], [0.4, 0.5]), (0.3, 0.4)\big)$ and $\mathcal{B}^* = \big(([0.3, 0.5], [0.2, 0.4]), (0.45, 0.25)\big)$ are not ECIFSs in $\mathcal{X}$. □

The following example shows that the P-union of two ECIFS in $\mathcal{X}$ need not to be an ICIFS in $\mathcal{X}$.

Example 3.3.4. Let $\mathcal{X} = \{a, b, c\}$ be a set. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ECIFSs in $\mathcal{X}$ defined as

$$\mathcal{A} = \left\{ \begin{array}{c} \big(a, ([0.3, 0.5], [0.4, 0.5]), (0.20, 0.60)\big), \big(b, ([0.2, 0.4], [0.3, 0.5]), (0.10, 0.25)\big), \\ \big(c, ([0.3, 0.4], [0.5, 0.6]), (0.20, 0.40)\big) \end{array} \right\}$$

$$\text{and} \quad \mathcal{B} = \left\{ \begin{array}{c} \big(a, ([0.7, 0.8], [0.1, 0.2]), (0.3, 0.5)\big), \big(b, ([0.2, 0.4], [0.3, 0.4]), (0.60, 0.25)\big), \\ \big(c, ([0.6, 0.7], [0.25, 0.30]), (0.40, 0.25)\big) \end{array} \right\}$$

then by definition of P-union, we get

$$\mathcal{A} \cup_{\mathcal{P}} \mathcal{B} = \left\{ \begin{array}{c} \big(a, ([0.7, 0.8], [0.1, 0.2]), (0.3, 0.5)\big), \big(b, ([0.2, 0.4], [0.3, 0.4]), (0.60, 0.25)\big), \\ \big(c, ([0.6, 0.7], [0.25, 0.30]), (0.40, 0.25)\big) \end{array} \right\}$$

It is clearly seen from it that $\zeta_A(b) = 0.6 \notin (0.2, 0.4)$ and $\vartheta_A(b) = 0.25 \notin (0.3, 0.4)$. Hence, P-union of two ECIFS need not to be an ICIFS in $\mathcal{X}$.

In the next, we have provided a condition for the P-union of two ECIFSs to be an ICIFS.

Property 3.3.5. For any two ECIFSs $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ in $\mathcal{X}$, if $\mathcal{A}^* = \big(A, \mu\big)$ and $\mathcal{B}^* = \big(B, \lambda\big)$ are ICIFSs in $\mathcal{X}$, then the P-union $\mathcal{A} \cup_P \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is an ICIFS in $\mathcal{X}$.

Proof. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ECIFSs in $\mathcal{X}$ such that, $\mathcal{A}^* = \big(A, \mu\big)$ and $\mathcal{B}^* = \big(B, \lambda\big)$ are ICIFSs in $\mathcal{X}$. Then by the definitions of ECIFS, we have $\zeta_A(x) \notin (\zeta_A^L(x), \zeta_A^U(x))$, $\zeta_B(x) \notin (\zeta_B^L(x), \zeta_B^U(x))$ and $\vartheta_A(x) \notin (\vartheta_A^L(x), \vartheta_A^U(x))$, $\vartheta_B(x) \notin (\vartheta_B^L(x), \vartheta_B^U(x))$ for all $x \in \mathcal{X}$. Now, by the definition of ICIFS for $\mathcal{A}^*$ and $\mathcal{B}^*$, we have $\zeta_B^L(x) \leq \zeta_A(x) \leq \zeta_B^U(x)$, $\zeta_A^L(x) \leq \zeta_B(x) \leq \zeta_A^U(x)$ and $\vartheta_B^U(x) \geq \vartheta_A(x) \geq \vartheta_B^U(x)$, $\vartheta_A^U(x) \geq \vartheta_B(x) \geq \vartheta_A^L(x)$ for all $x \in \mathcal{X}$. Thus, for a given $x \in \mathcal{X}$, we can consider the following cases:

(i) $\zeta_A(x) \leq \zeta_A^L(x) \leq \zeta_B(x) \leq \zeta_A^U(x)$, $\zeta_B(x) \leq \zeta_B^L(x) \leq \zeta_A(x) \leq \zeta_B^U(x)$ and $\vartheta_A(x) \geq \vartheta_A^U(x) \geq \vartheta_B(x) \geq \vartheta_A^U(x)$, $\vartheta_B(x) \geq \vartheta_B^U(x) \geq \vartheta_A(x) \geq \vartheta_B^L(x)$.

(ii) $\zeta_A^L(x) \leq \zeta_B(x) \leq \zeta_A^U(x) \leq \zeta_A(x)$, $\zeta_B^L(x) \leq \zeta_A(x) \leq \zeta_B^U(x) \leq \zeta_B(x)$ and $\vartheta_A^U(x) \geq \vartheta_B(x) \geq \vartheta_A^L(x) \geq \vartheta_A(x)$, $\vartheta_B^U(x) \geq \vartheta_A(x) \geq \zeta_B^L(x) \geq \vartheta_B(x)$.

(iii) $\zeta_A(x) \leq \zeta_A^L(x) \leq \zeta_B(x) \leq \zeta_A^U(x)$,$\zeta_B^L(x) \leq \zeta_A(x) \leq \zeta_B^U(x) \leq \zeta_B(x)$ and $\vartheta_A(x) \geq \vartheta_A^U(x) \geq \vartheta_B(x) \geq \vartheta_A^L(x)$,$\vartheta_B^U(x) \geq \vartheta_A(x) \geq \vartheta_B^L(x) \geq \vartheta_B(x)$.

(iv) $\zeta_A^L(x) \leq \zeta_B(x) \leq \zeta_A^U(x) \leq \zeta_A(x)$, $\zeta_B(x) \leq \zeta_B^L(x) \leq \zeta_A(x) \leq \zeta_B^U(x)$ and $\vartheta_A^U(x) \geq \vartheta_B(x) \geq \vartheta_A^L(x) \geq \vartheta_A(x)$, $\vartheta_B(x) \geq \vartheta_B^U(x) \geq \vartheta_A(x) \geq \vartheta_B^L(x)$.

We consider the first case only. For remaining cases, it is similar to the first case. For the first case, we have $\zeta_B(x) = \zeta_A^L(x) = \zeta_B^L(x) = \zeta_A(x)$ and $\vartheta_B(x) = \vartheta_A^U(x) = \vartheta_B^U(x) = \vartheta_A(x)$. Since $\mathcal{A}^* = (A, \mu)$ and $\mathcal{B}^* = (B, \lambda)$ are ICIFSs in $\mathcal{X}$, we have $\zeta_B(x) \leq \zeta_A^U(x)$, $\vartheta_B(x) \geq \vartheta_A^L(x)$ and $\zeta_A(x) \leq \zeta_B^U(x)$, $\vartheta_A(x) \geq \vartheta_B^L(x)$. It follows that

$$\begin{aligned}(\zeta_A^L \cup \zeta_B^L)(x) &= \max\{\zeta_A^L(x), \zeta_B^L(x)\} = (\zeta_A \vee \zeta_B)(x) \\ &\leq \max\{\zeta_A^U(x), \zeta_B^U(x)\} = (\zeta_A^U \cup \zeta_B^U)(x)\end{aligned}$$

and

$$\begin{aligned}(\vartheta_A^U \cap \vartheta_B^U)(x) &= \min\{\vartheta_A^U(x), \vartheta_B^U(x)\} = (\vartheta_A \wedge \vartheta_B)(x) \\ &\geq \min\{\vartheta_A^L(x), \vartheta_B^L(x)\} = (\vartheta_A^L \cap \vartheta_B^L)(x)\end{aligned}$$

Hence, $\mathcal{A} \cup_P \mathcal{B}$ is an ICIFS in $\mathcal{X}$. □

We provide a condition for the P-intersection of two ECIFSs to be an ICIFS.

Property 3.3.6. Let $\mathcal{A}$ and $\mathcal{B}$ be ECIFSs in $\mathcal{X}$ such that $\mathcal{A}^*$ and $\mathcal{B}^*$ are ICIFSs. Then the P-intersection of $\mathcal{A}$ and $\mathcal{B}$ is an ICIFS in $\mathcal{X}$.

Proof. It is similar to the proof of Property 3.3.5. □

Property 3.3.7. Let $\mathcal{A} = (A, \lambda)$ and $\mathcal{B} = (B, \mu)$ be two ECIFSs in $\mathcal{X}$ such that $\mathcal{A}^* = (A, \mu)$ and $\mathcal{B}^* = (B, \lambda)$ are ECIFSs in $\mathcal{X}$. Then the P-union of $\mathcal{A}$ and $\mathcal{B}$ is an ECIFS in $\mathcal{X}$.

Proof. Since $\mathcal{A}, \mathcal{B}$ and $\mathcal{A}^*, \mathcal{B}^*$ are ECIFSs which implies that for any $x \in \mathcal{X}$, we have $\zeta_A(x) \notin (\zeta_A^L(x), \zeta_A^U(x))$, $\zeta_B(x) \notin (\zeta_B^L(x), \zeta_B^U(x))$ and $\vartheta_A(x) \notin (\vartheta_A^L(x), \vartheta_A^U(x))$, $\vartheta_B(x) \notin (\vartheta_B^L(x), \vartheta_B^U(x))$, $\zeta_B(x) \notin (\zeta_A^L(x), \zeta_A^U(x))$, $\zeta_A(x) \notin (\zeta_B^L(x), \zeta_B^U(x))$, $\vartheta_B(x) \notin (\vartheta_A^L(x), \vartheta_A^U(x))$, and $\vartheta_A(x) \notin (\vartheta_B^L(x), \vartheta_B^U(x))$. Hence,

$$\begin{aligned}&(\zeta_A \vee \zeta_B)(x) \notin \big(\max\{\zeta_A^L(x), \zeta_B^L(x)\}, \max\{\zeta_A^U(x), \zeta_B^U(x)\}\big) \\ \text{and} \quad &(\vartheta_A \wedge \vartheta_B)(x) \notin \big(\min\{\vartheta_A^L(x), \vartheta_B^L(x)\}, \min\{\vartheta_A^U(x), \vartheta_B^U(x)\}\big)\end{aligned}$$

which means that $(\zeta_A \vee \zeta_B)(x) \notin ((\zeta_A^L \cup \zeta_B^L)(x), (\zeta_A^U \cup \zeta_B^U)(x))$ and $(\vartheta_A \wedge \vartheta_B)(x) \notin ((\vartheta_A^U \cap \vartheta_B^U)(x), (\vartheta_A^L \cap \vartheta_B^L)(x))$. Therefore, $\mathcal{A} \cup_P \mathcal{B}$ is an ECIFS in $\mathcal{X}$. □

As it has been concluded from Example 3.3.2 that P-intersection of two ECIFSs may not be an ECIFS. Now, here we provide a condition for the P-intersection of two ECIFSs to be an ECIFS.

Property 3.3.8. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ECIFSs in $\mathcal{X}$ such that

$$\begin{aligned}\min\big\{\max\{\zeta_A^U(x), \zeta_B^L(x)\}, \max\{\zeta_A^L(x), \zeta_B^U(x)\}\big\} &\geq (\zeta_A \wedge \zeta_B)(x)\\ &> \max\big\{\min\{\zeta_A^U(x), \zeta_B^L(x)\}, \min\{\zeta_A^L(x), \zeta_B^U(x)\}\big\}\end{aligned}$$

and

$$\begin{aligned}\max\big\{\min\{\vartheta_A^L(x), \vartheta_B^U(x)\}, \min\{\vartheta_A^U(x), \vartheta_B^L(x)\}\big\} &\leq (\vartheta_A \vee \vartheta_B)(x)\\ &< \min\big\{\max\{\vartheta_A^L(x), \vartheta_B^U(x)\}, \max\{\vartheta_A^U(x), \vartheta_B^L(x)\}\big\}\end{aligned}$$

for all $x \in \mathcal{X}$. Then P-intersection, $\mathcal{A} \cap_P \mathcal{B}$ is an ECIFS in $\mathcal{X}$.

Proof. For each $x \in \mathcal{X}$, take

$$\alpha_x = \min\big\{\max\{\zeta_A^U(x), \zeta_B^L(x)\}, \max\{\zeta_A^L(x), \zeta_B^U(x)\}\big\} \quad (3.2)$$

$$\beta_x = \max\big\{\min\{\zeta_A^U(x), \zeta_B^L(x)\}, \min\{\zeta_A^L(x), \zeta_B^U(x)\}\big\} \quad (3.3)$$

$$\alpha_y = \max\big\{\min\{\vartheta_A^L(x), \vartheta_B^U(x)\}, \min\{\vartheta_A^U(x), \vartheta_B^L(x)\}\big\} \quad (3.4)$$

$$\beta_y = \min\big\{\max\{\vartheta_A^L(x), \vartheta_B^U(x)\}, \max\{\vartheta_A^U(x), \vartheta_B^L(x)\}\big\} \quad (3.5)$$

Then α_x is one of the $\zeta_A^U(x)$, $\zeta_B^L(x)$, $\zeta_A^L(x)$ and $\zeta_B^U(x)$ and α_y is one of the $\vartheta_A^L(x)$, $\vartheta_B^U(x)$, $\vartheta_A^U(x)$ and $\vartheta_B^L(x)$. Without loss of generality, we consider $\alpha_x = \zeta_A^U(x)$ or $\zeta_A^L(x)$ and $\alpha_y = \vartheta_A^U(x)$ or $\vartheta_A^L(x)$.

Case I: If $\alpha_x = \zeta_A^L(x)$ then from Eq. (3.2), we get $\zeta_B^L(x) \leq \zeta_B^U(x) \leq \zeta_A^L(x) \leq \zeta_A^U(x)$. Therefore, from Eq. (3.3), we get $\beta_x = \zeta_B^U(x)$. It follows that

$$\begin{aligned}\zeta_B^L(x) &= (\zeta_A^L \cap \zeta_B^L)(x)\\ &\leq (\zeta_A^U \cap \zeta_B^U)(x) = \zeta_B^U(x) = \beta_x < (\zeta_A \wedge \zeta_B)(x)\end{aligned}$$

and hence $(\zeta_A \wedge \zeta_B)(x) \notin ((\zeta_A^L \cap \zeta_B^L)(x), (\zeta_A^U \cap \zeta_B^U)(x))$.

Similarly, if we take $\alpha_y = \vartheta_A^U(x)$ then from Eq. (3.4), we get $\vartheta_B^U(x) \geq \vartheta_B^L(x) \geq \vartheta_A^U(x) \geq \vartheta_A^L(x)$ and hence by Eq. (3.5), we have $\beta_y = \vartheta_B^L(x)$. Therefore, it follows that,

$$\begin{aligned} \vartheta_B^U(x) &= (\vartheta_A^U \cup \vartheta_B^U)(x) \\ &\geq (\vartheta_A^L \cup \vartheta_B^L)(x) = \vartheta_B^L(x) = \beta_y > (\vartheta_A \vee \vartheta_B)(x) \end{aligned}$$

and hence $(\vartheta_A \vee \vartheta_B)(x) \notin ((\vartheta_A^L \cup \vartheta_B^L)(x), (\vartheta_A^U \cup \vartheta_B^U)(x))$.

Case II: If $\alpha_x = \zeta_A^U(x)$ then by Eq. (3.2), we have

$$\zeta_B^L(x) \leq \zeta_A^U(x) \leq \zeta_B^U(x).$$

Thus, from Eq. (3.3), we get $\beta_x = \max\{\zeta_A^L(x), \zeta_B^L(x)\}$.

Subcase I: Assume that $\beta_x = \zeta_A^L(x)$ then we have

$$\zeta_B^L(x) \leq \zeta_A^L(x) < (\zeta_A \wedge \zeta_B)(x) \leq \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.6}$$

From the above inequality (3.6), we have

$$\zeta_B^L(x) \leq \zeta_A^L(x) < (\zeta_A \wedge \zeta_B)(x) < \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.7}$$

$$\text{or} \qquad \zeta_B^L(x) \leq \zeta_A^L(x) < (\zeta_A \wedge \zeta_B)(x) = \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.8}$$

But inequality (3.7) give us that $(\zeta_A \wedge \zeta_B)(x) \in \big((\zeta_A^L \cap \zeta_B^L), (\zeta_A^U \cap \zeta_B^U)\big)$ which is contradiction to that fact that $\mathcal{A}$ and $\mathcal{B}$ are ECIFS. Thus, we have only the case for which inequality (3.8) have $(\zeta_A \wedge \zeta_B)(x) \notin \big((\zeta_A^L \cap \zeta_B^L), (\zeta_A^U \cap \zeta_B^U)\big)$, since $(\zeta_A \wedge \zeta_B)(x) = \zeta_A^U(x) = (\zeta_A^U \cap \zeta_B^U)(x)$.

Subcase II: If $\beta_x = \zeta_B^L(x)$, then from Eq. (3.3), we have

$$\zeta_A^L(x) \leq \zeta_B^L(x) < (\zeta_A \wedge \zeta_B)(x) \leq \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.9}$$

From this inequality (3.9), we have either

$$\zeta_A^L(x) \leq \zeta_B^L(x) < (\zeta_A \wedge \zeta_B)(x) < \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.10}$$

$$\text{or} \qquad \zeta_A^L(x) \leq \zeta_B^L(x) < (\zeta_A \wedge \zeta_B)(x) = \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.11}$$

For the case given in inequality (3.10), it contradicts to the fact that $\mathcal{A}$ and $\mathcal{B}$ are ECIFS. On the other hand, from the inequality (3.11) we get

$$(\zeta_A \wedge \zeta_B)(x) \notin \big((\zeta_A^L \cap \zeta_B^L)(x)), (\zeta_A^U \cap \zeta_B^U)(x)\big)$$

since $(\zeta_A \wedge \zeta_B)(x) = \zeta_A^U(x) = (\zeta_A^U \cap \zeta_B^U)(x)$.

Case III: If $\alpha_y = \vartheta_A^L(x)$ then by Eq. (3.4), we have

$$\vartheta_B^U(x) \geq \vartheta_A^L(x) \geq \vartheta_B^L(x).$$

Thus, from Eq. (3.5), we get $\beta_y = \min\{\vartheta_A^U(x), \vartheta_B^U(x)\}$.

Subcase I: Assume that $\beta_y = \vartheta_A^U(x)$, then we have

$$\vartheta_B^U(x) \geq \vartheta_A^U(x) > (\vartheta_A \vee \vartheta_B)(x) \geq \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.12}$$

From this inequality, we have

$$\vartheta_B^U(x) \geq \vartheta_A^U(x) > (\vartheta_A \vee \vartheta_B)(x) > \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.13}$$

$$\text{or} \qquad \vartheta_B^U(x) \geq \vartheta_A^U(x) > (\vartheta_A \vee \vartheta_B)(x) = \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.14}$$

For the inequality (3.13), we observe that

$$(\vartheta_A \vee \vartheta_B)(x) \in \left((\vartheta_A^L \cup \vartheta_B^L)(x), (\vartheta_A^U \cup \vartheta_B^U)(x)\right)$$

which is contradiction to the the property of $\mathcal{A}$ and $\mathcal{B}$ of being ECIFSs. On the other hand, inequality (3.14) tells us that $(\vartheta_A \vee \vartheta_B)(x) \notin \left((\vartheta_A^L \cup \vartheta_B^L)(x), (\vartheta_A^U \cup \vartheta_B^U)(x)\right)$, since $(\vartheta_A \vee \vartheta_B)(x) = \vartheta_A^L(x) = (\vartheta_A^L \cup \vartheta_B^L)(x)$.

Subcase II: If $\beta_y = \vartheta_B^U(x)$, then from Eq. (3.5), we have

$$\vartheta_A^U(x) \geq \vartheta_B^U(x) > (\vartheta_A \vee \vartheta_B)(x) \geq \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.15}$$

From this inequality (3.15), we have either

$$\vartheta_A^U(x) \geq \vartheta_B^U(x) > (\vartheta_A \vee \vartheta_B)(x) > \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.16}$$

$$\text{or} \qquad \vartheta_A^U(x) \geq \vartheta_B^U(x) > (\vartheta_A \vee \vartheta_B)(x) = \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.17}$$

For the case given in inequality (3.16), it contradicts to the fact that $\mathcal{A}$ and $\mathcal{B}$ are ECIFSs. On the other hand, from the inequality (3.17) we get

$$(\vartheta_A \vee \vartheta_B)(x) \notin \left((\vartheta_A^L \cup \vartheta_B^L)(x), (\vartheta_A^U \cup \vartheta_B^U)(x)\right),$$

since $(\vartheta_A \vee \vartheta_B)(x) = \vartheta_A^L(x) = (\vartheta_A^L \cup \vartheta_B^L)(x)$. Hence, combining all the cases, we get P-intersection of $\mathcal{A}$ and $\mathcal{B}$ is an ECIFS in $\mathcal{X}$. □

In the following we have shown that for any two ECIFSs $\mathcal{A}$ and $\mathcal{B}$ which satisfies the conditions (3.18) and (3.19),

$$\begin{aligned} \min\{\max\{\zeta_A^U(x),\zeta_B^L(x)\},\max\{\zeta_A^L(x),\zeta_B^U(x)\}\} &> (\zeta_A\wedge\zeta_B)(x)\\ &= \max\{\min\{\zeta_A^U(x),\zeta_B^L(x)\},\min\{\zeta_A^L(x),\zeta_B^U(x)\}\} \quad (3.18)\\ \text{and} \quad \max\{\min\{\vartheta_A^L(x),\vartheta_B^U(x)\},\min\{\vartheta_A^U(x),\vartheta_B^L(x)\}\} &< (\vartheta_A\vee\vartheta_B)(x)\\ &= \min\{\max\{\vartheta_A^L(x),\vartheta_B^U(x)\},\max\{\vartheta_A^U(x),\vartheta_B^L(x)\}\} \quad (3.19)\end{aligned}$$

for all $x\in\mathcal{X}$, then P-intersection of $\mathcal{A}$ and $\mathcal{B}$, i.e., $\mathcal{A}\cap_P\mathcal{B}$ may not be an ECIFS in $\mathcal{X}$.

Example 3.3.5. Let $\mathcal{A}=\big(([0.2,0.6],[0.1,0.2]),\ (0.7,0.3)\big)$ and $\mathcal{B}=\big(([0.3,0.7],\ [0.2,0.3]),\ (0.3,0.2)\big)$ be two ECIFSs which satisfy the conditions as given in Eqs. (3.18) and (3.19). But $\mathcal{A}\cap_P\mathcal{B}=\big(([0.2,0.6],[0.2,0.3]),\ (0.3,0.3)\big)$ is not an ECIFS because $0.3\in[0.2,0.6]$ and $0.3\in[0.2,0.3]$.

Now, we provide a condition for the P-intersection of two CIFSs to be both ECIFS and ICIFS in $\mathcal{X}$.

Property 3.3.9. Let $\mathcal{A}=\big(A,\lambda\big)$ and $\mathcal{B}=\big(B,\mu\big)$ be two CIFSs in $\mathcal{X}$ such that

$$\begin{aligned}\min\{\max\{\zeta_A^U(x),\zeta_B^L(x)\},\max\{\zeta_A^L(x),\zeta_B^U(x)\}\} &= (\zeta_A\wedge\zeta_B)(x)\\ &= \max\{\min\{\zeta_A^U(x),\zeta_B^L(x)\},\min\{\zeta_A^L(x),\zeta_B^U(x)\}\} \quad (3.20)\end{aligned}$$

and

$$\begin{aligned}\max\{\min\{\vartheta_A^L(x),\vartheta_B^U(x)\},\min\{\vartheta_A^U(x),\vartheta_B^L(x)\}\} &= (\vartheta_A\vee\vartheta_B)(x)\\ &= \min\{\max\{\vartheta_A^L(x),\vartheta_B^U(x)\},\max\{\vartheta_A^U(x),\vartheta_B^L(x)\}\} \quad (3.21)\end{aligned}$$

for all $x\in\mathcal{X}$. Then the P-intersection of $\mathcal{A}$ and $\mathcal{B}$ is both an ECIFS and ICIFS in $\mathcal{X}$.

Proof. For each $x\in\mathcal{X}$, take

$$\alpha_x=\min\big\{\max\{\zeta_A^U(x),\zeta_B^L(x)\},\max\{\zeta_A^L(x),\zeta_B^U(x)\}\big\} \quad (3.22)$$

$$\beta_x=\max\big\{\min\{\zeta_A^U(x),\zeta_B^L(x)\},\min\{\zeta_A^L(x),\zeta_B^U(x)\}\big\} \quad (3.23)$$

$$\alpha_y=\max\big\{\min\{\vartheta_A^L(x),\vartheta_B^U(x)\},\min\{\vartheta_A^U(x),\vartheta_B^L(x)\}\big\} \quad (3.24)$$

$$\beta_y=\min\big\{\max\{\vartheta_A^L(x),\vartheta_B^U(x)\},\max\{\vartheta_A^U(x),\vartheta_B^L(x)\}\big\} \quad (3.25)$$

Then α_x is one of the $\zeta_A^U(x)$, $\zeta_B^L(x)$, $\zeta_A^L(x)$, $\zeta_B^U(x)$ and $\alpha_y = \vartheta_A^L(x)$, $\vartheta_B^U(x)$, $\vartheta_A^U(x)$, $\vartheta_B^L(x)$. For the sake of simplicity, we consider $\alpha_x = \zeta_A^L(x)$ and $\zeta_A^U(x)$ while $\alpha_y = \vartheta_A^L(x)$ and $\vartheta_A^U(x)$ only. For the remaining ones, it follows in the similar manner.

Case I: If $\alpha_x = \zeta_A^L(x)$ and $\alpha_y = \vartheta_A^U(x)$, then from Eq. (3.22) and Eq. (3.24), we get

$$\zeta_B^L(x) \leq \zeta_B^U(x) \leq \zeta_A^L(x) \leq \zeta_A^U(x) \tag{3.26}$$

$$\text{and} \qquad \vartheta_B^U(x) \geq \vartheta_B^L(x) \geq \vartheta_A^U(x) \geq \vartheta_A^L(x) \tag{3.27}$$

and hence from Eq. (3.23), we get $\beta_x = \zeta_B^U(x)$ and from Eq. (3.25), we get $\beta_y = \vartheta_B^L(x)$. Therefore, from Eq. (3.20), we have $\zeta_A^L(x) = \alpha_x = (\zeta_A \wedge \zeta_B)(x) = \beta_x = \zeta_B^U(x)$ and from Eq. (3.21), we have $\vartheta_A^U(x) = \alpha_y = (\vartheta_A \vee \vartheta_B)(x) = \beta_y = \vartheta_B^L(x)$. Thus, inequality (3.26) and (3.27), respectively, becomes

$$\zeta_B^L(x) \leq \zeta_B^U(x) = (\zeta_A \wedge \zeta_B)(x) = \zeta_A^L(x) \leq \zeta_A^U(x)$$

$$\text{and} \qquad \vartheta_B^U(x) \geq \vartheta_B^L(x) = (\vartheta_A \vee \vartheta_B)(x) = \vartheta_A^U(x) \geq \vartheta_A^L(x)$$

This implies that $(\zeta_A \wedge \zeta_B)(x) = \zeta_B^U(x) = (\zeta_A^U \cap \zeta_B^U)(x)$ and $(\vartheta_A \vee \vartheta_B)(x) = \vartheta_B^L(x) = (\vartheta_A^L \cup \vartheta_B^L)(x)$. Hence, $(\zeta_A \wedge \zeta_B)(x) \notin \big((\zeta_A^L \cap \zeta_B^L)(x), (\zeta_A^U \cap \zeta_B^U)(x)\big)$ and $(\zeta_A^L \cap \zeta_B^L)(x) \leq (\zeta_A \wedge \zeta_B)(x) \leq (\zeta_A^U \cap \zeta_B^U)(x)$. On the other hand, $(\vartheta_A \vee \vartheta_B)(x) \notin ((\vartheta_A^L \cup \vartheta_B^L)(x), (\vartheta_A^U \cup \vartheta_B^U)(x))$ and $(\vartheta_A^U \cup \vartheta_B^U)(x) \geq (\vartheta_A \vee \vartheta_B)(x) \geq (\vartheta_A^L \cup \vartheta_B^L)(x)$.

Case II: If $\alpha_x = \zeta_A^U(x)$ and $\alpha_y = \vartheta_A^L(x)$, then from Eqs. (3.22) and (3.24), we get

$$\zeta_B^L(x) \leq \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.28}$$

$$\text{and} \qquad \vartheta_B^U(x) \geq \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.29}$$

and hence from Eq. (3.20) and Eq. (3.21), we get $(\zeta_A \wedge \zeta_B)(x) = \zeta_A^U(x) = (\zeta_A^U \cap \zeta_B^U)(x)$ and $(\vartheta_A \vee \vartheta_B)(x) = \vartheta_A^L(x) = (\vartheta_A^L \cup \vartheta_B^L)(x)$. Hence, $(\zeta_A \wedge \zeta_B)(x) \notin \big((\zeta_A^L \cap \zeta_B^L)(x), (\zeta_A^U \cap \zeta_B^U)(x)\big)$ and $(\zeta_A^L \cap \zeta_B^L)(x) \leq (\zeta_A \wedge \zeta_B)(x) \leq (\zeta_A^U \cap \zeta_B^U)(x)$, while $(\vartheta_A \vee \vartheta_B)(x) \notin \big((\vartheta_A^L \cup \vartheta_B^L)(x), (\vartheta_A^U \cup \vartheta_B^U)(x)\big)$ and $(\vartheta_A^U \cup \vartheta_B^U)(x) \geq (\vartheta_A \vee \vartheta_B)(x) \geq (\vartheta_A^L \cup \vartheta_B^L)(x)$. Thus, by combining all these cases, we get P-intersection of $\mathcal{A}$ and $\mathcal{B}$ is both ECIFS as well as ICIFS in $\mathcal{X}$. □

As it has been concluded from Example 3.3.2 that P-union of two ECIFSs may not be an ECIFS. Now, here we provide a condition for the P-union of two ECIFSs to be an ECIFS.

Property 3.3.10. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be ECIFSs in $\mathcal{X}$ such that

$$\begin{aligned}\min\{\max\{\zeta_A^U(x), \zeta_B^L(x)\}, \max\{\zeta_A^L(x), \zeta_B^U(x)\}\} &> (\zeta_A \vee \zeta_B)(x) \\ &\geq \max\{\min\{\zeta_A^U(x), \zeta_B^L(x)\}, \min\{\zeta_A^L(x), \zeta_B^U(x)\}\} \end{aligned} \tag{3.30}$$

and

$$\begin{aligned}\max\{\min\{\vartheta_A^L(x), \vartheta_B^U(x)\}, \min\{\vartheta_A^U(x), \vartheta_B^L(x)\}\} &< (\vartheta_A \wedge \vartheta_B)(x) \\ &\leq \min\{\max\{\vartheta_A^L(x), \vartheta_B^U(x)\}, \max\{\vartheta_A^U(x), \vartheta_B^L(x)\}\} \end{aligned} \tag{3.31}$$

for all $x \in \mathcal{X}$. Then P-union of $\mathcal{A}$ and $\mathcal{B}$, written by $\mathcal{A} \cup_P \mathcal{B}$, is an ECIFS in $\mathcal{X}$.

Proof. For each $x \in \mathcal{X}$, take

$$\alpha_x = \min\{\max\{\zeta_A^U(x), \zeta_B^L(x)\}, \max\{\zeta_A^L(x), \zeta_B^U(x)\}\} \tag{3.32}$$

$$\beta_x = \max\{\min\{\zeta_A^U(x), \zeta_B^L(x)\}, \min\{\zeta_A^L(x), \zeta_B^U(x)\}\} \tag{3.33}$$

$$\alpha_y = \max\{\min\{\vartheta_A^L(x), \vartheta_B^U(x)\}, \min\{\vartheta_A^U(x), \vartheta_B^L(x)\}\} \tag{3.34}$$

$$\beta_y = \min\{\max\{\vartheta_A^L(x), \vartheta_B^U(x)\}, \max\{\vartheta_A^U(x), \vartheta_B^L(x)\}\} \tag{3.35}$$

Then α_x is one of the $\zeta_A^U(x)$, $\zeta_B^L(x)$, $\zeta_A^L(x)$ and $\zeta_B^U(x)$ and α_y is one of the $\vartheta_A^L(x)$, $\vartheta_B^U(x)$, $\vartheta_A^U(x)$ and $\vartheta_B^L(x)$. Here, we have proved the result by considering the case $\alpha_x = \zeta_A^L(x)$ or $\zeta_A^U(x)$ and $\alpha_y = \vartheta_A^U(x)$ or $\vartheta_A^L(x)$ only. For remaining cases, it is similar to this one.

Case I: If $\alpha_x = \zeta_A^L(x)$ and $\alpha_y = \vartheta_A^U(x)$, then from Eq. (3.32), we get $\zeta_B^L(x) \leq \zeta_B^U(x) \leq \zeta_A^L(x) \leq \zeta_A^U(x)$ and from Eq. (3.34), we get $\vartheta_B^U(x) \geq \vartheta_B^L(x) \geq \vartheta_A^U(x) \geq \vartheta_A^L(x)$. Thus, Eq. (3.33) gives $\beta_x = \zeta_B^U(x)$ and Eq. (3.35) gives $\beta_y = \vartheta_B^L(x)$. Therefore by using the given conditions,

$$(\zeta_A^L \cup \zeta_B^L)(x) = \zeta_A^L(x) = \alpha_x > (\zeta_A \vee \zeta_B)(x)$$

and

$$(\vartheta_A^U \cap \vartheta_B^U)(x) = \vartheta_A^U(x) = \alpha_y < (\vartheta_A \wedge \vartheta_B)(x)$$

and hence $(\zeta_A \vee \zeta_B)(x) \notin ((\zeta_A^L \cup \zeta_B^L)(x), (\zeta_A^U \cup \zeta_B^U)(x))$ and $(\vartheta_A \wedge \vartheta_B)(x) \notin ((\vartheta_A^L \cap \vartheta_B^L)(x), (\vartheta_A^U \cap \vartheta_B^U)(x))$.

Case II: If $\alpha_x = \zeta_A^U(x)$ and $\alpha_y = \vartheta_A^L(x)$, then from Eq. (3.32), we get $\zeta_B^L(x) \leq \zeta_A^U(x) \leq \zeta_B^U(x)$ and from Eq. (3.34), we get $\vartheta_B^U(x) \geq \vartheta_A^L(x) \geq \vartheta_B^L(x)$. Thus, Eq. (3.33) gives $\beta_x = \max\{\zeta_A^L(x), \zeta_B^L(x)\}$ and Eq. (3.35) gives $\beta_y = \min\{\vartheta_A^U(x), \vartheta_B^U(x)\}$.

Subcase I: Assume that $\beta_x = \zeta_A^L(x)$ and $\beta_y = \vartheta_A^U(x)$. Then, from Eqs. (3.30) and (3.31), we have

$$\begin{aligned} &\zeta_B^L(x) \leq \zeta_A^L(x) \leq (\zeta_A \vee \zeta_B)(x) < \zeta_A^U(x) \leq \zeta_B^U(x) \\ \text{and} \quad &\vartheta_B^U(x) \geq \vartheta_A^U(x) \geq (\vartheta_A \wedge \vartheta_B)(x) > \vartheta_A^L(x) \geq \vartheta_B^L(x) \end{aligned}$$

From these inequalities, we have

$$\zeta_B^L(x) \leq \zeta_A^L(x) < (\zeta_A \vee \zeta_B)(x) < \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.36}$$

$$\text{or} \quad \zeta_B^L(x) \leq \zeta_A^L(x) = (\zeta_A \vee \zeta_B)(x) < \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.37}$$

and

$$\vartheta_B^U(x) \geq \vartheta_A^U(x) > (\vartheta_A \wedge \vartheta_B)(x) > \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.38}$$

$$\text{or} \quad \vartheta_B^U(x) \geq \vartheta_A^U(x) = (\vartheta_A \wedge \vartheta_B)(x) > \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.39}$$

Here, Eqs. (3.36) and (3.38) contradicts to the fact that $\mathcal{A}$ and $\mathcal{B}$ are ECIFS in $\mathcal{X}$. On the other hand, Eqs. (3.37) and (3.39) gives that $(\zeta_A \vee \zeta_B)(x) \notin ((\zeta_A^L \cup \zeta_B^L)(x), (\zeta_A^U \cup \zeta_B^U)(x))$ and $(\vartheta_A \wedge \vartheta_B)(x) \notin ((\vartheta_A^L \cap \vartheta_B^L)(x), (\vartheta_A^U \cap \vartheta_B^U)(x))$, since $(\zeta_A^L \cup \zeta_B^L)(x) = \zeta_A^L(x) = (\zeta_A \vee \zeta_B)(x)$ and $(\vartheta_A^U \cap \vartheta_B^U)(x) = \vartheta_A^U(x) = (\vartheta_A \wedge \vartheta_B)(x)$.

Subcase II: Assume that $\beta_x = \zeta_B^L(x)$ and $\beta_y = \vartheta_B^U(x)$. Then, from Eqs. (3.30) and (3.31), we have

$$\begin{aligned} &\zeta_A^L(x) \leq \zeta_B^L(x) \leq (\zeta_A \vee \zeta_B)(x) < \zeta_A^U(x) \leq \zeta_B^U(x) \\ \text{and} \quad &\vartheta_A^U(x) \geq \vartheta_B^U(x) \geq (\vartheta_A \wedge \vartheta_B)(x) > \vartheta_A^L(x) \geq \vartheta_B^L(x) \end{aligned}$$

which implies that

$$\zeta_A^L(x) \leq \zeta_B^L(x) < (\zeta_A \vee \zeta_B)(x) < \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.40}$$

$$\text{or} \quad \zeta_A^L(x) \leq \zeta_B^L(x) = (\zeta_A \vee \zeta_B)(x) < \zeta_A^U(x) \leq \zeta_B^U(x) \tag{3.41}$$

and

$$\vartheta_A^U(x) \geq \vartheta_B^U(x) > (\vartheta_A \wedge \vartheta_B)(x) > \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.42}$$

$$\text{or} \quad \vartheta_A^U(x) \geq \vartheta_B^U(x) = (\vartheta_A \wedge \vartheta_B)(x) > \vartheta_A^L(x) \geq \vartheta_B^L(x) \tag{3.43}$$

The inequalities (3.40) and (3.42) contradicts the fact that $\mathcal{A}$ and $\mathcal{B}$ are ECIFSs in $\mathcal{X}$. On the other hand, inequalities (3.41) and (3.43) gives that $(\zeta_A \vee \zeta_B)(x) \notin \big((\zeta_A^L \cup \zeta_B^L)(x), (\zeta_A^U \cup \zeta_B^U)(x)\big)$ and $(\vartheta_A \wedge \vartheta_B)(x) \notin \Big((\vartheta_A^L \cap \vartheta_B^L)(x),\ (\vartheta_A^U \cap \vartheta_B^U)(x)\Big)$ since $(\zeta_A^L \cup \zeta_B^L)(x) = \zeta_B^L(x) = (\zeta_A \vee \zeta_B)(x)$ and $(\vartheta_A^U \cap \vartheta_B^U)(x) = \vartheta_B^U(x) = (\vartheta_A \wedge \vartheta_B)(x)$.

Hence, by combining all the cases, we get the P-union of $\mathcal{A}$ and $\mathcal{B}$ is an ECIFS in $\mathcal{X}$. □

As it has been seen from Example 3.3.3 that the R-union of two ECIFSs is not an ECIFS. Now, here we have provided the condition for the R-union of two ECIFSs to be an ECIFS.

Property 3.3.11. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be ECIFSs in $\mathcal{X}$. If for each $x \in \mathcal{X}$,

$$\begin{aligned} \min\Big(\max\big(\zeta_A^U(x), \zeta_B^L(x)\big), \max\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big) &> (\zeta_A \wedge \zeta_B)(x) \\ &\geq \max\Big(\min\big(\zeta_A^U(x), \zeta_B^L(x)\big), \min\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big) \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} \max\Big(\min\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \min\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big) &< (\vartheta_A \vee \vartheta_B)(x) \\ &\leq \min\Big(\max\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \max\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big), \end{aligned} \tag{3.45}$$

then the R-union of $\mathcal{A}$ and $\mathcal{B}$ i.e., $\mathcal{A} \cup_R \mathcal{B}$, is an ECIFS in $\mathcal{X}$.

Proof. For each $x \in \mathcal{X}$, take

$$\alpha_x = \min\Big(\max\big(\zeta_A^U(x), \zeta_B^L(x)\big), \max\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big), \tag{3.46}$$

$$\beta_x = \max\Big(\min\big(\zeta_A^U(x), \zeta_B^L(x)\big), \min\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big), \tag{3.47}$$

$$\alpha_y = \max\Big(\min\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \min\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big), \tag{3.48}$$

$$\beta_y = \min\Big(\max\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \max\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big) \tag{3.49}$$

Then α_x is one of the $\zeta_A^U(x)$, $\zeta_B^L(x)$, $\zeta_A^L(x)$, $\zeta_B^U(x)$ and α_y is one of the $\vartheta_A^L(x)$, $\vartheta_B^U(x)$, $\vartheta_A^U(x)$, $\vartheta_B^L(x)$. Without loss of generality, we have considered the cases of $\alpha_x = \zeta_B^L(x)$ or $\zeta_B^U(x)$ and $\alpha_y = \vartheta_B^U(x)$ or $\vartheta_B^L(x)$. while it is similar for the remaining ones.

Case I: If $\alpha_x = \zeta_B^L(x)$ and $\alpha_y = \vartheta_B^U(x)$, then from Eqs. (3.46) and (3.48), we get $\zeta_A^L(x) \leq \zeta_A^U(x) \leq \zeta_B^L(x) \leq \zeta_B^U(x)$ and $\vartheta_A^U(x) \geq \vartheta_A^L(x) \geq \vartheta_B^U(x) \geq \vartheta_B^L(x)$ and thus $\beta_x = \zeta_A^U(x)$ and $\beta_y = \vartheta_A^L(x)$. Now by using the inequalities (3.44) and (3.45), we get

$$(\zeta_A^L \cup \zeta_B^L)(x) = \zeta_B^L(x) = \alpha_x > (\zeta_A \wedge \zeta_B)(x) \tag{3.50}$$

$$\text{and} \quad (\vartheta_A^U \cap \vartheta_B^U)(x) = \vartheta_B^U(x) = \alpha_y < (\vartheta_A \vee \vartheta_B)(x) \tag{3.51}$$

and hence $(\zeta_A \wedge \zeta_B)(x) \notin \big((\zeta_A^L \cup \zeta_B^L)(x), (\zeta_A^U \cup \zeta_B^U)(x)\big)$ and $(\vartheta_A \vee \vartheta_B)(x) \notin \big((\vartheta_A^L \cap \vartheta_B^L)(x), (\vartheta_A^U \cap \vartheta_B^U)(x)\big)$.

Case II: If $\alpha_x = \zeta_B^U(x)$ and $\alpha_y = \vartheta_B^L(x)$ then from Eq. (3.46) and Eq. (3.48), we get $\zeta_A^L(x) \leq \zeta_B^U(x) \leq \zeta_A^U(x)$ and $\vartheta_A^U(x) \geq \vartheta_B^L(x) \geq \vartheta_A^L(x)$. Thus, by Eq. (3.47), we get $\beta_x = \max(\zeta_A^L(x), \zeta_B^L(x))$ and from Eq. (3.49), we get $\beta_y = \min(\vartheta_A^U(x), \vartheta_B^U(x))$.

Subcase I: Assume that $\beta_x = \zeta_B^L(x)$ and $\beta_y = \vartheta_B^U(x)$, then from the given inequalities (3.44) and (3.45), we have

$$\zeta_A^L(x) \leq \zeta_B^L(x) \leq (\zeta_A \wedge \zeta_B)(x) < \zeta_B^U(x) \leq \zeta_A^U(x)$$
$$\text{and} \quad \vartheta_A^U(x) \geq \vartheta_B^U(x) \geq (\vartheta_A \vee \vartheta_B)(x) > \vartheta_B^L(x) \geq \vartheta_A^L(x)$$

which implies that

$$\zeta_A^L(x) \leq \zeta_B^L(x) < (\zeta_A \wedge \zeta_B)(x) < \zeta_B^U(x) \leq \zeta_A^U(x) \tag{3.52}$$

$$\text{or} \quad \zeta_A^L(x) \leq \zeta_B^L(x) = (\zeta_A \wedge \zeta_B)(x) < \zeta_B^U(x) \leq \zeta_A^U(x) \tag{3.53}$$

and

$$\vartheta_A^U(x) \geq \vartheta_B^U(x) > (\vartheta_A \vee \vartheta_B)(x) > \vartheta_B^L(x) \geq \vartheta_A^L(x) \tag{3.54}$$

$$\text{or} \quad \vartheta_A^U(x) \geq \vartheta_B^U(x) = (\vartheta_A \vee \vartheta_B)(x) > \vartheta_B^L(x) \geq \vartheta_A^L(x) \tag{3.55}$$

From it, the inequalities (3.52) and (3.54) contradicts the statement that $\mathcal{A}$ and $\mathcal{B}$ are ECIFSs, while from inequalities (3.53) and (3.55), we obtain $(\zeta_A \wedge \zeta_B)(x) \notin ((\zeta_A^L \cup \zeta_B^L)(x), (\zeta_A^U \cup \zeta_B^U)(x))$ since $(\zeta_A^L \cup \zeta_B^L)(x) = \zeta_B^L(x) = (\zeta_A \wedge \zeta_B)(x)$ and $(\vartheta_A \vee \vartheta_B)(x) \notin ((\vartheta_A^L \cap \vartheta_B^L)(x), (\vartheta_A^U \cap \vartheta_B^U)(x))$ because $(\vartheta_A^U \cap \vartheta_B^U)(x) = \vartheta_B^U(x) = (\vartheta_A \vee \vartheta_B)(x)$.

Subcase II: Assume that $\beta_x = \zeta_A^L(x)$ and $\beta_y = \vartheta_A^U(x)$, then from the given inequalities (3.44) and (3.45), we have

$$\begin{aligned} \zeta_B^L(x) \leq \zeta_A^L(x) \leq (\zeta_A \wedge \zeta_B)(x) < \zeta_B^U(x) \leq \zeta_A^U(x) \\ \text{and} \quad \vartheta_B^U(x) \geq \vartheta_A^U(x) \geq (\vartheta_A \vee \vartheta_B)(x) > \vartheta_B^L(x) \geq \vartheta_A^L(x) \end{aligned}$$

which implies that

$$\zeta_B^L(x) \leq \zeta_A^L(x) < (\zeta_A \wedge \zeta_B)(x) < \zeta_B^U(x) \leq \zeta_A^U(x) \tag{3.56}$$

$$\text{or} \quad \zeta_B^L(x) \leq \zeta_A^L(x) = (\zeta_A \wedge \zeta_B)(x) < \zeta_B^U(x) \leq \zeta_A^U(x) \tag{3.57}$$

and

$$\vartheta_B^U(x) \geq \vartheta_A^U(x) > (\vartheta_A \vee \vartheta_B)(x) > \vartheta_B^L(x) \geq \vartheta_A^L(x) \tag{3.58}$$

$$\text{or} \quad \vartheta_B^U(x) \geq \vartheta_A^U(x) = (\vartheta_A \vee \vartheta_B)(x) > \vartheta_B^L(x) \geq \vartheta_A^L(x) \tag{3.59}$$

The inequalities (3.56) and (3.58) contradicts the given statement that both $\mathcal{A}$ and $\mathcal{B}$ are ECIFSs. On the other hand, from the inequalities (3.57) and (3.59), we have $(\zeta_A \wedge \zeta_B)(x) \notin \big((\zeta_A^L \cup \zeta_B^L)(x), (\zeta_A^U \cup \zeta_B^U)(x)\big)$ and $(\vartheta_A \vee \vartheta_B)(x) \notin \big((\vartheta_A^L \cap \vartheta_B^L)(x), (\vartheta_A^U \cap \vartheta_B^U)(x)\big)$ since $(\zeta_A^L \cup \zeta_B^L)(x) = \zeta_A^L(x) = (\zeta_A \wedge \zeta_B)(x)$ and $(\vartheta_A^U \cap \vartheta_B^U)(x) = \vartheta_A^U(x) = (\vartheta_A \vee \vartheta_B)(x)$.

Hence, by combining all the cases, we get R-union of $\mathcal{A}$ and $\mathcal{B}$ is an ECIFS in $\mathcal{X}$. $\square$

The following example show that for two ECIFSs $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ which satisfies the conditions

$$\begin{aligned} \min\Big(\max\big(\zeta_A^U(x), \zeta_B^L(x)\big), \max\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big) &= (\zeta_A \wedge \zeta_B)(x) \\ > \max\Big(\min\big(\zeta_A^U(x), \zeta_B^L(x)\big), \min\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big) \end{aligned} \tag{3.60}$$

and

$$\begin{aligned} \max\Big(\min\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \min\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big) &= (\vartheta_A \vee \vartheta_B)(x) \\ < \min\Big(\max\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \max\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big) \end{aligned} \tag{3.61}$$

for all $x \in \mathcal{X}$, the R-union of $\mathcal{A}$ and $\mathcal{B}$ may not be an ECIFS in $\mathcal{X}$.

Example 3.3.6. Let $\mathcal{A} = \big(([0.1, 0.8], [0.15, 0.20]), (0.9, 0.1)\big)$ and $\mathcal{B} = \big(([0.2, 0.7], [0.1, 0.2]), (0.7, 0.1)\big)$ be two ECIFSs satisfying the given conditions (3.60) and (3.61) then, $\mathcal{A} \cup_R \mathcal{B} = \big(([0.2, 0.8], [0.1, 0.2]), (0.7, 0.1)\big)$ is not ECIFS because $0.7 \in [0.2, 0.8]$ and $0.1 \in [0.1, 0.2]$.

Also, we have seen from Example 3.3.3 that R-intersection of two ECIFSs need not be an ECIFS. Now, here we have provided a condition for the R-intersection of two ECIFSs to be an ECIFS.

Property 3.3.12. Let $\mathcal{A} = (A, \lambda)$ and $\mathcal{B} = (B, \mu)$ be two ECIFSs such that

$$\min\Big(\max\big(\zeta_A^U(x), \zeta_B^L(x)\big), \max\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big) \geq (\zeta_A \vee \zeta_B)(x) > \max\Big(\min\big(\zeta_A^U(x), \zeta_B^L(x)\big), \min\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big) \tag{3.62}$$

and

$$\max\Big(\min\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \min\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big) \leq (\vartheta_A \wedge \vartheta_B)(x) < \min\Big(\max\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \max\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big), \tag{3.63}$$

then the R-intersection of $\mathcal{A}$ and $\mathcal{B}$ is an ECIFS in $\mathcal{X}$.

Proof. Proof is similar as that of Property 3.3.11, so we omit here. □

The following example shows that for two ECIFSs $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ which satisfies the condition

$$\min\Big(\max\big(\zeta_A^U(x), \zeta_B^L(x)\big), \max\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big) > (\zeta_A \vee \zeta_B)(x) = \max\Big(\min\big(\zeta_A^U(x), \zeta_B^L(x)\big), \min\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big) \tag{3.64}$$

and

$$\max\Big(\min\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \min\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big) < (\vartheta_A \wedge \vartheta_B)(x) = \min\Big(\max\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \max\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big) \tag{3.65}$$

for all $x \in \mathcal{X}$, the R-intersection of $\mathcal{A}$ and $\mathcal{B}$ may not be an ECIFS in $\mathcal{X}$.

Example 3.3.7. Let $\mathcal{A} = \big(([0.2, 0.4], [0.1, 0.2]), (0.1, 0.25)\big)$ and $\mathcal{B} = \big(([0.3, 0.6], [0.25, 0.4]), (0.3, 0.6)\big)$ be two ECIFSs satisfying the given conditions (3.64) and (3.65), then we get $\mathcal{A} \cap_R \mathcal{B} = \big(([0.2, 0.4], [0.25, 0.40]), (0.3, 0.25)\big)$ is not an ECIFS because $0.3 \in [0.2, 0.4]$.

Property 3.3.13. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two CIFSs in $\mathcal{X}$ such that

$$\min\Big(\max\big(\zeta_A^U(x), \zeta_B^L(x)\big), \max\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big) = (\zeta_A \vee \zeta_B)(x)$$
$$= \max\Big(\min\big(\zeta_A^U(x), \zeta_B^L(x)\big), \min\big(\zeta_A^L(x), \zeta_B^U(x)\big)\Big) \tag{3.66}$$

and

$$\max\Big(\min\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \min\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big) = (\vartheta_A \wedge \vartheta_B)(x)$$
$$= \min\Big(\max\big(\vartheta_A^L(x), \vartheta_B^U(x)\big), \max\big(\vartheta_A^U(x), \vartheta_B^L(x)\big)\Big) \tag{3.67}$$

for all $x \in \mathcal{X}$, then the R-intersection of $\mathcal{A}$ and $\mathcal{B}$ is both an ECIFS and ICIFS in $\mathcal{X}$.

Proof. Proof is similar to that of Property 3.3.9, so we omit here. □

Remark 3.3.4. The R-union of two ICIFSs need not be an ECIFS which has been explained as below.

Example 3.3.8. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ICIFSs in which A=([0.1, 0.4], [0.5, 0.6]), $\lambda = (0.3, 0.5)$ and $B = ([0.1, 0.3], [0.20, 0.55])$ and $\mu = (0.2, 0.3)$. Then $\mathcal{A} \cup_R \mathcal{B} = \big(([0.1, 0.4], [0.20, 0.55]),\ (0.2, 0.5)\big)$ which is not an ECIFS.

Now, we provide condition for R-union of two ICIFSs to be an ECIFS.

Property 3.3.14. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be ICIFSs in $\mathcal{X}$. If

$$(\zeta_A \wedge \zeta_B)(x) \leq \max\big(\zeta_A^L(x), \zeta_B^L(x)\big)$$

and

$$(\vartheta_A \vee \vartheta_B)(x) \geq \min\big(\vartheta_A^U(x), \vartheta_B^U(x)\big)$$

for all $x \in \mathcal{X}$, then the R-union of $\mathcal{A}$ and $\mathcal{B}$ is an ECIFS in $\mathcal{X}$.

Proof. Straightforward. □

Remark 3.3.5. The R-intersection of two ICIFSs need not be an ECIFS which has been explained as below.

Example 3.3.9. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ICIFSs in which $A = ([0.2,\ 0.4], [0.5,\ 0.6])$, $\lambda = (0.30, 0.55)$ and $B = ([0.1, 0.3], [0.20, 0.55])$ and $\mu = (0.20, 0.55)$. Then $\mathcal{A} \cap_R \mathcal{B} = \Big(([0.1, 0.3], [0.5, 0.6]),\ (0.30, 0.55)\Big)$ which is not an ECIFS.

Now, we provide a condition for the R- intersection of two ICIFSs to be an ECIFS.

Property 3.3.15. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be ICIFSs in $\mathcal{X}$. If

$$(\zeta_A \vee \zeta_B)(x) \geq \min\big(\zeta_A^U(x), \zeta_B^U(x)\big)$$

and

$$(\vartheta_A \wedge \vartheta_B)(x) \leq \max\big(\vartheta_A^L(x), \vartheta_B^L(x)\big)$$

for all $x \in \mathcal{X}$, then the R-intersection of $\mathcal{A}$ and $\mathcal{B}$ is an ECIFS in $\mathcal{X}$.

Remark 3.3.6. The R-union of two ECIFSs need not be an ICIFS which has been explained as below.

Example 3.3.10. Let $\mathcal{A} = \big(A, \lambda\big)$ and $\mathcal{B} = \big(B, \mu\big)$ be two ECIFSs in which $A = ([0.2, 0.3], [0.4, 0.5])$, $\lambda = (0.8, 0.1)$ and $B = ([0.4, 0.6], [0.2, 0.4])$ and $\mu = (0.7, 0.1)$. Then $\mathcal{A} \cup_R \mathcal{B} = \big(([0.4, 0.6], [0.2, 0.4]),\ (0.7, 0.1)\big)$ which is not an ICIFS.

3.4 Conclusion

The key contribution of this chapter is summarized as follows:

1) A concept of cubic intuitionistic fuzzy set, an extension of the cubic fuzzy set, is presented with its several properties. The existing environment, IFSs, and IVIFSs, to describe the information are the special cases of CIFS environment.

2) In this presented CIFS, the preferences of an object corresponding to each element have been expressed by means of IVIFS and IFS simultaneously. The major advantages of this set are in terms of representing the data where an element is evaluated under the consideration of a disagreement degree (in terms of IFNs) corresponding to the agreed interval region (in terms of IVIFNs). Thus, this set considers the significance of IVIFS to get progressively proper outcomes through IFS.

3) Based on the features of CIFS, the concept of internal or external CIFS and their corresponding operations such as P-union, R-union, P-intersection, and R-intersection have been proposed. The desirable relations between these operations are investigated in detail. From the computed results, it has been concluded that P - union and P-intersection of an external CIFS need not be a CIFS.

4) Furthermore, we provide additional conditions on P-union and P-intersection (respectively for R-union and R-intersection) of external CIFS to be internal CIFS and conditions for an external CIFS. Finally, we give the conditions which define the P-intersection as well as R-intersection of two CIFSs to be internal as well as external CIFSs. The applicability of the results has been demonstrated with some numerical examples.

Chapter 4

Cubic Intuitionistic Fuzzy Aggregation Operators[1]

The objective of this chapter is to present some series of aggregation operators (AOs) under the cubic intuitionistic fuzzy set (CIFS) and their suitable properties. For it, firstly an operational law, score function, and accuracy function between the cubic intuitionistic fuzzy numbers (CIFNs) under the P-order and R-order are defined and hence based on them, some weighted averaging and geometric aggregation operators, namely cubic intuitionistic fuzzy weighted (CIFWA), ordered weighted (CIFOWA), hybrid averaging (CIFHA) and their respective geometric (CIFWG, CIFOWG, CIFHG) AOs are proposed. A decision-making method based on these operators is proposed for ranking different set of the alternatives classified under CIFS domain. Finally, an illustrative example is given to demonstrate the proposed approach.

4.1 Introduction

In the field of the aggregation process, various researchers have developed different AOs under the different environments. A comprehensive literature review on the various AOs is given in Section 1.1.3 of Chapter 1 which shows that AOs have a dominant role in existing theories. These AOs cover all the necessary aspects of fusing the information captured within the considered environment into a single unit. However, the theory proposed by Jun

[1]The content of this chapter is published as "Cubic Intuitionistic fuzzy Aggregation operators", *International Journal for Uncertainty Quantification*, 8(5), 405 427, 2018 (**SCI: Impact Factor: 2.083**).

et al. [77] initiated the concept of CFSs but it predominately fails to capture the relevant non-membership information. For that, the concept of CIFS introduced in Chapter 3 addresses the non-membership degrees and the associated analytical meaning makes CIFSs fit for the capturing information in practical situations. But, to further enhance the functionality of CIFSs, the need of information aggregation is inescapable. Gradually, the aggregated information serve the purpose of achieving objective of real-life problems when various comparison facilitating units such as score and accuracy functions are used altogether to address the obtained fused values. Due to this fact, this chapter not only deals with introducing the relevant AOs but also put-forth the score and accuracy functions for de-fuzzification of the processed CIFS information.

Thus, by keeping the features of CIFSs as compared to existing fuzzy set theory we present a notion of aggregating different cubic intuitionistic fuzzy numbers (CIFNs) by using algebraic P-order and R-order operations. For it, firstly P-order and R-order operation laws, score and accuracy functions have been defined and further based on them, some cubic intuitionistic fuzzy AOs, namely cubic intuitionistic fuzzy weighted averaging (CIFWA) operator, cubic intuitionistic fuzzy ordered weighted averaging (CIFOWA) operator, cubic intuitionistic fuzzy hybrid averaging (CIFHA) operators under the R-order and P-order operations are proposed. Further, these AOs are extended from averaging to the geometric AOs. Various desirable properties of these AOs have been investigated in detail. Finally, an illustrative example from the field of the decision-making has been given to show the developed method.

4.2 Score, accuracy and operational laws of CIFSs

In this section, we introduce score as well as accuracy functions for P-order and R-order CIFSs along with their operational laws.

Definition 4.2.1. Score function corresponding to CIFN $\mathcal{A} = \left(\left([\zeta^L, \zeta^U], [\vartheta^L, \vartheta^U]\right), (\zeta, \vartheta)\right)$ for R-order is defined as

$$\mathcal{S}c(\mathcal{A}) = \frac{\zeta^L + \zeta^U - \vartheta^L - \vartheta^U}{2} + \vartheta - \zeta \tag{4.1}$$

while for P-order as

$$\mathcal{S}c(\mathcal{A}) = \frac{\zeta^L + \zeta^U - \vartheta^L - \vartheta^U}{2} + \zeta - \vartheta \tag{4.2}$$

It is evident that $-2 \leq \mathcal{S}c(\mathcal{A}) \leq 2$.

Example 4.2.1. Consider two R-order CIFNs $\mathcal{A}_1 = (([0.4, 0.5], [0.3, 0.4]), (0.3, 0.2))$ and $\mathcal{A}_2 = (([0.2, 0.3], [0.4, 0.5]),\ (0.2, 0.6))$ then by using Eq. (4.1), we get $\mathcal{S}c(\mathcal{A}_1) = 0$ and $\mathcal{S}c(\mathcal{A}_2) = 0.2$. As $\mathcal{S}c(\mathcal{A}_2) > \mathcal{S}c(\mathcal{A}_1)$ so, we have $\mathcal{A}_2 \succ \mathcal{A}_1$.

However, in some situations, the above function is unable to rank the CIFNs. For instance, if $\mathcal{A}_1 = (([0.2, 0.3], [0.4, 0.5]), (0.2, 0.4))$ and $\mathcal{A}_2 = (([0.1, 0.2], [0.3, 0.4]), (0.3, 0.5))$, then it is impossible to know which one is bigger because $\mathcal{S}c(\mathcal{A}_1) = \mathcal{S}c(\mathcal{A}_2)$. For this, an accuracy function $\mathcal{H}(\mathcal{A})$ is defined as

$$\mathcal{H}(\mathcal{A}) = \frac{\zeta^L + \zeta^U + \vartheta^L + \vartheta^U}{2} + \zeta + \vartheta \tag{4.3}$$

It is clearly seen that $0 \leq \mathcal{H}(\mathcal{A}) \leq 2$.

Based on these functions, a comparison method for two CIFNs $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:

Definition 4.2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two CIFNs corresponding to CIFSs A and B then, an order relation between them is defined as: if $\mathcal{S}c(\mathcal{A}) < \mathcal{S}c(\mathcal{B})$ then $\mathcal{A}$ is smaller than $\mathcal{B}$ denoted by $\mathcal{A} \prec \mathcal{B}$ and if $\mathcal{S}c(\mathcal{A}) = \mathcal{S}c(\mathcal{B})$ then, if $\mathcal{H}(\mathcal{A}) < \mathcal{H}(\mathcal{B})$ then $\mathcal{A} \prec \mathcal{B}$ and if $\mathcal{H}(\mathcal{A}) = \mathcal{H}(\mathcal{B})$ then $\mathcal{A}$ and $\mathcal{B}$ represent the same information denoted by $\mathcal{A} \sim \mathcal{B}$.

To study the properties of score function and accuracy function, we propose the following propositions.

Theorem 4.2.1. (Monotonicity of score function). Let $\mathcal{A} = \left(\left([\zeta^L, \zeta^U], [\vartheta^L, \vartheta^U]\right), (\zeta, \vartheta)\right)$ be P-order CIFN, the score function $\mathcal{S}c(\mathcal{A}) = \frac{\zeta^L + \zeta^U - \vartheta^L - \vartheta^U}{2} + \zeta - \vartheta$ is a monotonic increasing function with ζ^L, ζ^U, and ζ, and a monotone decreasing function with ϑ^L, ϑ^U, and ϑ.

Theorem 4.2.2. (Monotonicity of score function). Let $\mathcal{A} = \left(\left([\zeta^L, \zeta^U], [\vartheta^L, \vartheta^U]\right), (\zeta, \vartheta)\right)$ be R-order CIFN, the score function $\mathcal{S}c(\mathcal{A}) = \frac{\zeta^L + \zeta^U - \vartheta^L - \vartheta^U}{2} - \zeta + \vartheta$ is a monotonic increasing function with ζ^L, ζ^U, and ϑ, and a monotone decreasing function with ϑ^L, ϑ^U, and ζ.

Proof. Omitted. □

Theorem 4.2.3. (Symmetry of score function) Let $\mathcal{A}_i = \left(\left([\zeta_i^L, \zeta_i^U], [\vartheta_i^L, \vartheta_i^U]\right), (\zeta_i, \vartheta_i)\right)$ $(i = 1, 2)$ be two CIFNs, $\mathcal{A}_i^c = \left(\left([\vartheta_i^L, \vartheta_i^U], [\zeta_i^L, \zeta_i^U]\right), (\vartheta_i, \zeta_i)\right)$ $(i = 1, 2)$ be their associated inverse function, respectively, then we have the following conclusion $\mathcal{S}c(\mathcal{A}_1) \leq \mathcal{S}c(\mathcal{A}_2)$ $\Leftrightarrow \mathcal{S}c(\mathcal{A}_1^c) \geq \mathcal{S}c(\mathcal{A}_2^c)$.

Proof. By taking P-order CIFN, and from Definition 4.2.1, we obtain

$$\mathcal{S}c(\mathcal{A}_1) = \frac{\zeta_1^L + \zeta_1^U - \vartheta_1^L - \vartheta_1^U}{2} + \zeta_1 - \vartheta_1 \text{ and } \mathcal{S}c(\mathcal{A}_2) = \frac{\zeta_2^L + \zeta_2^U - \vartheta_2^L - \vartheta_2^U}{2} + \zeta_2 - \vartheta_2$$

Since $\mathcal{S}c(\mathcal{A}_1) \leq \mathcal{S}c(\mathcal{A}_2)$, then

$$\begin{aligned} \Leftrightarrow \quad & \frac{\zeta_1^L + \zeta_1^U - \vartheta_1^L - \vartheta_1^U}{2} + \zeta_1 - \vartheta_1 \leq \frac{\zeta_2^L + \zeta_2^U - \vartheta_2^L - \vartheta_2^U}{2} + \zeta_2 - \vartheta_2 \\ \Leftrightarrow \quad & \frac{\vartheta_1^L + \vartheta_1^U - \zeta_1^L - \zeta_1^U}{2} + \vartheta_1 - \zeta_1 \geq \frac{\vartheta_2^L + \vartheta_2^U - \zeta_2^L - \zeta_2^U}{2} + \vartheta_2 - \zeta_2 \\ \Leftrightarrow \quad & \mathcal{S}c(\mathcal{A}_1^c) \geq \mathcal{S}c(\mathcal{A}_2^c) \end{aligned}$$

Similarly, we can obtain for R-order CIFN. □

Theorem 4.2.4. (Monotonicity of accuracy function) Let $\mathcal{A} = \left(\left([\zeta^L, \zeta^U], [\vartheta^L, \vartheta^U]\right), (\zeta, \vartheta)\right)$ be CIFN, the score function $\mathcal{H}(\mathcal{A}) = \frac{\zeta^L + \zeta^U + \vartheta^L + \vartheta^U}{2} + \zeta + \vartheta$ is a monotonic increasing function with ζ^L, ζ^U, ζ, ϑ^L, ϑ^U, and ϑ.

Theorem 4.2.5. (Symmetry of accuracy function) Let $\mathcal{A} = \left(\left([\zeta^L, \zeta^U], [\vartheta^L, \vartheta^U]\right), (\zeta, \vartheta)\right)$ be CIFN and $\mathcal{A}^c = \left(\left([\vartheta^L, \vartheta^U]\ ,\ [\zeta^L, \zeta^U]\right),\ (\vartheta, \zeta)\right)$ be their associated complement, then we have $\mathcal{H}(\mathcal{A}) = \mathcal{H}(\mathcal{A}^c)$.

Proof. Omitted. □

Definition 4.2.3. (Operational laws of CIFNs) Let $\mathcal{A}_i = \left(\left([\zeta_i^L, \zeta_i^U], [\vartheta_i^L, \vartheta_i^U]\right), (\zeta_i, \vartheta_i)\right)$, $(i = 1, 2, \ldots, n)$ be the collections of n CIFNs, $\mathcal{A} = \left(\left([\zeta^L, \zeta^U], [\vartheta^L, \vartheta^U]\right), (\zeta, \vartheta)\right)$ and $\xi > 0$ be a real number then the operational laws on these CIFNs, based on the R-order, are defined as below:

(i) $\mathcal{A}_1 \oplus_R \mathcal{A}_2 = \left(\left(\left[\zeta_1^L + \zeta_2^L - \zeta_1^L\zeta_2^L, \zeta_1^U + \zeta_2^U - \zeta_1^U\zeta_2^U\right], \left[\vartheta_1^L\vartheta_2^L, \vartheta_1^U\vartheta_2^U\right]\right), (\zeta_1\zeta_2, \vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2)\right)$

(ii) $\mathcal{A}_1 \otimes_R \mathcal{A}_2 = \left(\left(\left[\zeta_1^L\zeta_2^L, \zeta_1^U\zeta_2^U\right], \left[\vartheta_1^L + \vartheta_2^L - \vartheta_1^L\vartheta_2^L, \vartheta_1^U + \vartheta_2^U - \vartheta_1^U\vartheta_2^U\right]\right), (\zeta_1 + \zeta_2 - \zeta_1\zeta_2, \vartheta_1\vartheta_2)\right)$

(iii) $\xi\mathcal{A} = \left(\left(\left[1 - (1-\zeta^L)^\xi, 1 - (1-\zeta^U)^\xi\right], \left[(\vartheta^L)^\xi, (\vartheta^U)^\xi\right]\right), \left((\zeta)^\xi, 1 - (1-\vartheta)^\xi\right)\right)\ ;\ \xi > 0$

(iv) $\mathcal{A}^\xi = \left(\left(\left[(\zeta^L)^\xi, (\zeta^U)^\xi\right], \left[1 - (1-\vartheta^L)^\xi, 1 - (1-\vartheta^U)^\xi\right]\right), \left(1 - (1-\zeta)^\xi, (\vartheta)^\xi\right)\right);\ \xi > 0$

while based on P-order, are defined as below

(i) $\mathcal{A}_1 \oplus_P \mathcal{A}_2 = \left(\left(\left[\zeta_1^L + \zeta_2^L - \zeta_1^L\zeta_2^L, \zeta_1^U + \zeta_2^U - \zeta_1^U\zeta_2^U\right], \left[\vartheta_1^L\vartheta_2^L, \vartheta_1^U\vartheta_2^U\right]\right), (\zeta_1 + \zeta_2 - \zeta_1\zeta_2, \vartheta_1\vartheta_2)\right)$.

(ii) $\mathcal{A}_1 \otimes_P \mathcal{A}_2 = \left(\left(\left[\zeta_1^L\zeta_2^L, \zeta_1^U\zeta_2^U\right], \left[\vartheta_1^L + \vartheta_2^L - \vartheta_1^L\vartheta_2^L, \vartheta_1^U + \vartheta_2^U - \vartheta_1^U\vartheta_2^U\right]\right), (\zeta_1\zeta_2, \vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2)\right)$.

(iii) $\xi\mathcal{A} = \left(\left(\left[1 - (1-\zeta^L)^\xi, 1 - (1-\zeta^U)^\xi\right], \left[(\vartheta^L)^\xi, (\vartheta^U)^\xi\right]\right), \left(1 - (1-\zeta)^\xi, (\vartheta)^\xi\right)\right)\ ;\ \xi > 0$

(iv) $\mathcal{A}^\xi = \left(\left(\left[(\zeta^L)^\xi, (\zeta^U)^\xi\right], \left[1 - (1-\vartheta^L)^\xi, 1 - (1-\vartheta^U)^\xi\right]\right), \left((\zeta)^\xi, 1 - (1-\vartheta)^\xi\right)\right);\ \xi > 0$

Theorem 4.2.6. For two CIFNs $\mathcal{A}_1$ and $\mathcal{A}_2$, $\xi > 0$ be a real number then $\mathcal{A}_1 \oplus_R \mathcal{A}_2$, $\mathcal{A}_1 \otimes_R \mathcal{A}_2$, $\xi\mathcal{A}_1$ and $\mathcal{A}_1^\xi$ are also CIFNs.

Proof. Since $\mathcal{A}_1 = \left(\left([\zeta_1^L, \zeta_1^U], [\vartheta_1^L, \vartheta_1^U]\right), (\zeta_1, \vartheta_1)\right)$ and $\mathcal{A}_2 = \left(\left([\zeta_2^L, \zeta_2^U], [\vartheta_2^L, \vartheta_2^U]\right), (\zeta_2, \vartheta_2)\right)$ are two CIFNs such that $0 \le \zeta_1^L, \zeta_2^L, \zeta_1^U, \zeta_2^U, \vartheta_1^L, \vartheta_2^L, \vartheta_1^U, \vartheta_2^U \le 1$ and $\zeta_1^U + \vartheta_1^U \le 1$, $\zeta_2^U + \vartheta_2^U \le 1$ which implies that $0 \le \left(1 - \zeta_1^L\right)\left(1 - \zeta_2^L\right) \le 1$ and hence $0 \le \zeta_1^L + \zeta_2^L - \zeta_1^L\zeta_2^L \le 1$. Similarly, we can prove that $0 \le \zeta_1^U + \zeta_2^U - \zeta_1^U\zeta_2^U \le 1$, $0 \le \vartheta_1^L\vartheta_2^L \le 1$ and $0 \le \vartheta_1^U\vartheta_2^U \le 1$. Also, $0 \le \zeta_1, \zeta_2, \vartheta_1, \vartheta_2 \le 1$ and $\zeta_1 + \vartheta_1 \le 1$, $\zeta_2 + \vartheta_2 \le 1$ which implies that $\zeta_1\zeta_2 \le 1$ and $0 \le \vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2 \le 1$. Finally, we have $\zeta_1^U + \zeta_2^U - \zeta_1^U\zeta_2^U + \vartheta_1^U\vartheta_2^U = 1 - \left(1 - \zeta_1^U\right)\left(1 - \zeta_2^U\right) + \vartheta_1^U\vartheta_2^U \le 1 - \vartheta_1^U\vartheta_2^U + \vartheta_1^U\vartheta_2^U \le 1$ and $\zeta_1\zeta_2 + \vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2 = \zeta_1\zeta_2 + 1 - (1-\vartheta_1)(1-\vartheta_2) \le \zeta_1\zeta_2 + 1 - \zeta_1\zeta_2 \le 1$. Therefore, $\mathcal{A}_1 \oplus_R \mathcal{A}_2$ is CIFN.

Further, for any positive number ξ and CIFN $\mathcal{A}$, we have $0 \le \zeta_1^\xi \le 1$, $0 \le 1 - (1-\vartheta_1)^\xi \le 1$, $0 \le (\vartheta_1^L)^\xi, (\vartheta_1^U)^\xi \le 1$ and $0 \le 1 - \left(1 - \zeta_1^L\right)^\xi, 1 - \left(1 - \zeta_1^U\right)^\xi \le 1$. Thus, $\xi\mathcal{A}_1$ is also CIFN. Similarly, we can prove that $\mathcal{A}_1 \otimes_R \mathcal{A}_2$ and $\mathcal{A}_1^\xi$ are also CIFNs. □

Theorem 4.2.7. For two CIFNs $\mathcal{A}_1$ and $\mathcal{A}_2$, $\xi > 0$ be a real number then $\mathcal{A}_1 \oplus_P \mathcal{A}_2$, $\mathcal{A}_1 \otimes_P \mathcal{A}_2$, $\xi\mathcal{A}_1$ and $\mathcal{A}_1^\xi$ by using P-order operations, are also CIFNs.

Proof. Same as that of the above, so we omit here. □

Theorem 4.2.8. For CIFNs $\mathcal{A}_1$ and $\mathcal{A}_2$, and $\xi, \xi_1, \xi_2 > 0$ be three real numbers:

(i) $\xi(\mathcal{A}_1 \oplus_R \mathcal{A}_2) = \xi\mathcal{A}_1 \oplus_R \xi\mathcal{A}_2$

(ii) $(\mathcal{A}_1^\xi \otimes_R \mathcal{A}_2^\xi) = (\mathcal{A}_1 \otimes_R \mathcal{A}_2)^\xi$

(iii) $(\xi_1\xi_2)\mathcal{A}_1 = \xi_1(\xi_2\mathcal{A}_1)$

(iv) $\mathcal{A}_1^{\xi_1\xi_2} = (\mathcal{A}_1^{\xi_1})^{\xi_2}$

Proof. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are two CIFNs as defined in Definition 3.2.1 of Chapter 3

(i) For real number $\xi > 0$, we have

$$\begin{aligned}
&\xi(\mathcal{A}_1 \oplus_R \mathcal{A}_2)\\
= &\left(\left(\begin{bmatrix} 1-\left(\left(1-\zeta_1^L\right)\left(1-\zeta_2^L\right)\right)^\xi, \\ 1-\left(\left(1-\zeta_1^U\right)\left(1-\zeta_2^U\right)\right)^\xi \end{bmatrix}, \begin{bmatrix} (\vartheta_1^L\vartheta_2^L)^\xi, \\ (\vartheta_1^U\vartheta_2^U)^\xi \end{bmatrix}\right), \begin{pmatrix} (\zeta_1\zeta_2)^\xi, \\ 1-((1-\vartheta_1)(1-\vartheta_2))^\xi \end{pmatrix}\right)\\
= &\left(\left(\begin{bmatrix} 1-\left(1-\zeta_1^L\right)^\xi, \\ 1-\left(1-\zeta_1^U\right)^\xi \end{bmatrix}, \left[(\vartheta_1^L)^\xi, (\vartheta_1^U)^\xi\right]\right), \left((\zeta_1)^\xi, 1-(1-\vartheta_1)^\xi\right)\right)\\
\oplus_R &\left(\left(\begin{bmatrix} 1-\left(1-\zeta_2^L\right)^\xi, \\ 1-\left(1-\zeta_2^U\right)^\xi \end{bmatrix}, \left[(\vartheta_2^L)^\xi, (\vartheta_2^U)^\xi\right]\right), \left((\zeta_2)^\xi, 1-(1-\vartheta_2)^\xi\right)\right)\\
= &\ \xi\mathcal{A}_1 \oplus_R \xi\mathcal{A}_2
\end{aligned}$$

(ii) For two CIFNs $\mathcal{A}_1$ and $\mathcal{A}_2$, we have

$$\begin{aligned}
&\mathcal{A}_1^\xi \otimes_R \mathcal{A}_2^\xi\\
= &\left(\left(\left[\left(\zeta_1^L\right)^\xi, \left(\zeta_1^U\right)^\xi\right], \begin{bmatrix} 1-\left(1-\vartheta_1^L\right)^\xi, \\ 1-\left(1-\vartheta_1^U\right)^\xi \end{bmatrix}\right), \left(1-(1-\zeta_1)^\xi, (\vartheta_1)^\xi\right)\right)\\
\otimes_R &\left(\left(\left[\left(\zeta_2^L\right)^\xi, \left(\zeta_2^U\right)^\xi\right], \begin{bmatrix} 1-\left(1-\vartheta_2^L\right)^\xi, \\ 1-\left(1-\vartheta_2^U\right)^\xi \end{bmatrix}\right), \left(1-(1-\zeta_2)^\xi, (\vartheta_2)^\xi\right)\right)\\
= &\left(\left(\left[(\zeta_1^L\zeta_2^L)^\xi, (\zeta_1^U\zeta_2^U)^\xi\right], \begin{bmatrix} 1-\left(\left(1-\vartheta_1^L\right)\left(1-\vartheta_2^L\right)\right)^\xi, \\ 1-\left(\left(1-\vartheta_1^U\right)\left(1-\vartheta_2^U\right)\right)^\xi \end{bmatrix}\right),\right.\\
&\left.\left(1-((1-\zeta_1)(1-\zeta_2))^\xi, (\vartheta_1\vartheta_2)^\xi\right)\right)\\
= &\ (\mathcal{A}_1 \otimes_R \mathcal{A}_2)^\xi
\end{aligned}$$

(iii) For two real numbers $\xi_1, \xi_2 > 0$, we have

$$\xi_2\mathcal{A}_1 = \left(\begin{array}{c} \left([1-\left(1-\zeta_1^L\right)^{\xi_2}, 1-\left(1-\zeta_1^U\right)^{\xi_2}], [(\vartheta_1^L)^{\xi_2}, (\vartheta_1^U)^{\xi_2}]\right), \\ \left((\zeta_1)^{\xi_2}, 1-(1-\vartheta_1)^{\xi_2}\right) \end{array} \right)$$

$$\begin{aligned} \xi_1(\xi_2\mathcal{A}_1) &= \left(\begin{array}{c} \left(\begin{bmatrix} 1-\left(1-\left(1-\left(1-\zeta_1^L\right)^{\xi_2}\right)\right)^{\xi_1}, \\ 1-\left(1-\left(1-\left(1-\zeta_1^U\right)^{\xi_2}\right)\right)^{\xi_1} \end{bmatrix}, \left[(\vartheta_1^L)^{\xi_2\xi_1}, (\vartheta_1^U)^{\xi_2\xi_1}\right] \right), \\ \left((\zeta_1)^{\xi_2\xi_1}, 1-\left((1-\vartheta_1)^{\xi_2}\right)^{\xi_1}\right) \end{array} \right) \\ &= \left(\begin{array}{c} \left(\begin{bmatrix} 1-\left(1-\zeta_1^L\right)^{\xi_1\xi_2}, \\ 1-\left(1-\zeta_1^U\right)^{\xi_1\xi_2} \end{bmatrix}, \left[(\vartheta_1^L)^{\xi_1\xi_2}, (\vartheta_1^U)^{\xi_1\xi_2}\right] \right), \\ \left((\zeta_1)^{\xi_1\xi_2}, 1-(1-\vartheta_1)^{\xi_1\xi_2}\right) \end{array} \right) \\ &= (\xi_1\xi_2)\mathcal{A}_1 \end{aligned}$$

(iv) For two real numbers $\xi_1, \xi_2 > 0$, we have

$$\mathcal{A}_1^{\xi_2} = \left(\begin{array}{c} \left([\left(\zeta_1^L\right)^{\xi_2}, \left(\zeta_1^U\right)^{\xi_2}], \begin{bmatrix} 1-(1-\vartheta_1^L)^{\xi_2}, \\ 1-\left(1-\vartheta_1^U\right)^{\xi_2} \end{bmatrix} \right), \\ \left(1-(1-\zeta_1)^{\xi_2}, (\vartheta_1)^{\xi_2}\right) \end{array} \right)$$

$$\begin{aligned} (\mathcal{A}_1{}^{\xi_2})^{\xi_1} &= \left(\begin{array}{c} \left(\left[\left(\zeta_1^L\right)^{\xi_2\xi_1}, \left(\zeta_1^U\right)^{\xi_2\xi_1}\right], \begin{bmatrix} 1-\left(1-\left(1-(1-\vartheta_1^L)^{\xi_2}\right)\right)^{\xi_1}, \\ 1-\left(1-\left(1-\left(1-\vartheta_1^U\right)^{\xi_2}\right)\right)^{\xi_1} \end{bmatrix} \right), \\ \left(1-\left(1-\left(1-(1-\zeta_1)^{\xi_2}\right)\right)^{\xi_1}, (\vartheta_1)^{\xi_2\xi_1}\right) \end{array} \right) \\ &= \left(\begin{array}{c} \left(\left[\left(\zeta_1^L\right)^{\xi_1\xi_2}, \left(\zeta_1^U\right)^{\xi_1\xi_2}\right], \begin{bmatrix} 1-(1-\vartheta_1^L)^{\xi_1\xi_2}, \\ 1-\left(1-\vartheta_1^U\right)^{\xi_1\xi_2} \end{bmatrix} \right), \\ \left(1-(1-\zeta_1)^{\xi_1\xi_2}, (\vartheta_1)^{\xi_1\xi_2}\right) \end{array} \right) \\ &= (\mathcal{A}_1)^{\xi_1\xi_2} \end{aligned}$$

□

Theorem 4.2.9. For two CIFNs $\mathcal{A}_1$ and $\mathcal{A}_2$, the following results hold:

(i) $\mathcal{A}_1 \oplus_R \mathcal{A}_2 = \mathcal{A}_2 \oplus_R \mathcal{A}_1$

(ii) $\mathcal{A}_1 \otimes_R \mathcal{A}_2 = \mathcal{A}_2 \otimes_R \mathcal{A}_1$

(iii) $(\mathcal{A}_1 \oplus_R \mathcal{A}_2)^c = (\mathcal{A}_1)^c \otimes_R (\mathcal{A}_2)^c$

(iv) $(\mathcal{A}_1 \otimes_R \mathcal{A}_2)^c = (\mathcal{A}_1)^c \oplus_R (\mathcal{A}_2)^c$

(v) $(\mathcal{A}_1 \cup_R \mathcal{A}_2)^c = (\mathcal{A}_1)^c \cap_R (\mathcal{A}_2)^c$

(vi) $(\mathcal{A}_1 \cap_R \mathcal{A}_2)^c = (\mathcal{A}_1)^c \cup_R (\mathcal{A}_2)^c$

Proof. We prove the parts (i), (iii) and (v) while others can be proved similarly.

(i) Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are two CIFNs then,

$$\begin{aligned} & \mathcal{A}_1 \oplus_R \mathcal{A}_2 \\ = & \left(\left([\zeta_1^L + \zeta_2^L - \zeta_1^L\zeta_2^L, \zeta_1^U + \zeta_2^U - \zeta_1^U\zeta_2^U], [\vartheta_1^L\vartheta_2^L, \vartheta_1^U\vartheta_2^U]\right), (\zeta_1\zeta_2, \vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2)\right) \\ = & \left(\left([\zeta_2^L + \zeta_1^L - \zeta_2^L\zeta_1^L, \zeta_2^U + \zeta_1^U - \zeta_2^U\zeta_1^U], [\vartheta_2^L\vartheta_1^L, \vartheta_2^U\vartheta_1^U]\right), (\zeta_2\zeta_1, \vartheta_2 + \vartheta_1 - \vartheta_2\vartheta_1)\right) \\ = & \mathcal{A}_2 \oplus_R \mathcal{A}_1 \end{aligned}$$

(iii) For CIFSs $\mathcal{A}_1$ and $\mathcal{A}_2$, we have

$$\begin{aligned} & (\mathcal{A}_1)^c \otimes_R (\mathcal{A}_2)^c \\ = & \left(\left([\vartheta_1^L, \vartheta_1^U], [\zeta_1^L, \zeta_1^U]\right), (\vartheta_1, \zeta_1)\right) \otimes_R \left(\left([\vartheta_2^L, \vartheta_2^U], [\zeta_2^L, \zeta_2^U]\right), (\vartheta_2, \zeta_2)\right) \\ = & \begin{pmatrix} \left([\vartheta_1^L\vartheta_2^L, \vartheta_1^U\vartheta_2^U], [1 - \left(1 - \zeta_1^L\right)\left(1 - \zeta_2^L\right), 1 - \left(1 - \zeta_1^U\right)\left(1 - \zeta_2^U\right)]\right), \\ (1 - (1 - \vartheta_1)(1 - \vartheta_2), \zeta_1\zeta_2) \end{pmatrix} \\ = & (\mathcal{A}_1 \oplus_R \mathcal{A}_2)^c \end{aligned}$$

(v) It can be obtained from the Definition 3.2.5 of Chapter 3.

□

Theorem 4.2.10. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two CIFNs, and $\xi, \xi_1, \xi_2 > 0$ be three real numbers, then the following results hold:

(i) $\xi(\mathcal{A}_1 \oplus_P \mathcal{A}_2) = \xi\mathcal{A}_1 \oplus_P \xi\mathcal{A}_2$

(ii) $(\mathcal{A}_1^\xi \otimes_P \mathcal{A}_2^\xi) = (\mathcal{A}_1 \otimes_P \mathcal{A}_2)^\xi$

(iii) $(\xi_1\xi_2)\mathcal{A}_1 = \xi_1(\xi_2\mathcal{A}_1)$

(iv) $\mathcal{A}_1^{\xi_1\xi_2} = (\mathcal{A}_1^{\xi_1})^{\xi_2}$

Proof. As similar to that of above, so we omit here. □

Theorem 4.2.11. For any two CIFNs $\mathcal{A}_1$ and $\mathcal{A}_2$, the following results hold:

(i) $\mathcal{A}_1 \oplus_P \mathcal{A}_2 = \mathcal{A}_2 \oplus_P \mathcal{A}_1$

(ii) $\mathcal{A}_1 \otimes_P \mathcal{A}_2 = \mathcal{A}_2 \otimes_P \mathcal{A}_1$

(iii) $(\mathcal{A}_1 \oplus_P \mathcal{A}_2)^c = (\mathcal{A}_1)^c \otimes_P (\mathcal{A}_2)^c$

(iv) $(\mathcal{A}_1 \otimes_P \mathcal{A}_2)^c = (\mathcal{A}_1)^c \oplus_P (\mathcal{A}_2)^c$

(v) $(\mathcal{A}_1 \cup_P \mathcal{A}_2)^c = (\mathcal{A}_1)^c \cap_P (\mathcal{A}_2)^c$

(vi) $(\mathcal{A}_1 \cap_P \mathcal{A}_2)^c = (\mathcal{A}_1)^c \cup_P (\mathcal{A}_2)^c$

Proof. As similar to that of above, so we omit here. □

4.3 Aggregation Operators

In this section, some strategic decision-making methods for solving the decision-making problem under the CIFSs environment are presented by using R-order and P-order operations. In order to achieve it, in the following, we have proposed some weighted averaging AOs under the cubic intuitionistic fuzzy environment which include: CIFWA operator, CIFOWA operator and CIFHA operator. Similarly, we can obtain these operators corresponding to the P-order operations.

4.3.1 Averaging operators based on R-order operations

Definition 4.3.1. Let $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ be the collection of CIFNs, and CIFWA: $\Phi^n \to \Phi$, if

$$\text{CIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1\mathcal{A}_1 \oplus_R \omega_2\mathcal{A}_2 \oplus_R \ldots \oplus_R \omega_n\mathcal{A}_n \tag{4.4}$$

where Φ is the set of CIFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ such that, $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$, then CIFWA is called cubic intuitionisitic fuzzy weighted averaging operator.

Theorem 4.3.1. Let $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ be the collection of CIFNs, then the aggregated value by using CIFWA operator becomes

$$\text{CIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\begin{array}{c} \left(\left[1 - \prod_{i=1}^{n} \left(1 - \zeta_i^L\right)^{\omega_i}, 1 - \prod_{i=1}^{n} \left(1 - \zeta_i^U\right)^{\omega_i} \right], \left[\prod_{i=1}^{n} \left(\vartheta_i^L\right)^{\omega_i}, \prod_{i=1}^{n} \left(\vartheta_i^U\right)^{\omega_i} \right] \right), \\ \left(\prod_{i=1}^{n} (\zeta_i)^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - \vartheta_i)^{\omega_i} \right) \end{array} \right) \tag{4.5}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ such that, $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$.

Proof. Based on the R-order operational laws of CIFNs, we can derive the result.

Next, in order to show that the aggregated value by CIFWA is again CIFN, for this, assume that $\text{CIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\left([\zeta_C^L, \zeta_C^U], [\vartheta_C^L, \vartheta_C^U] \right), (\zeta_C, \vartheta_C) \right)$ where $\zeta_C^U = 1 - \prod_{i=1}^{n} \left(1 - \zeta_i^U\right)^{\omega_i}$ and $\vartheta_C^U = \prod_{i=1}^{n} \left(\vartheta_i^U\right)^{\omega_i}$ then we have to prove that $0 \leq \zeta_C^U \leq 1$, $0 \leq \vartheta_C^U \leq 1$ and $\zeta_C^U + \vartheta_C^U \leq 1$. Since for each i, $\mathcal{A}_i = \left(\left([\zeta_i^L, \zeta_i^U], [\vartheta_i^L, \vartheta_i^U] \right), (\zeta_i, \vartheta_i) \right)$ is CIFN which implies that $0 \leq \zeta_i^L, \zeta_i^U, \vartheta_i^L, \vartheta_i^U \leq 1$, $0 \leq \zeta_i, \vartheta_i \leq 1$ and $\zeta_i^U + \vartheta_i^U \leq 1$ and $\zeta_i + \vartheta_i \leq 1$. Therefore, $0 \leq 1 - \zeta_i^U \leq 1$ and for weight vector $\omega_i > 0$, we have $0 \leq \left(1 - \zeta_i^U\right)^{\omega_i} \leq 1$ which implies that $0 \leq 1 - \prod_{i=1}^{n} \left(1 - \zeta_i^U\right)^{\omega_i} \leq 1$. Similarly, $0 \leq \prod_{i=1}^{n} \left(\vartheta_i^U\right)^{\omega_i} \leq 1$. Finally,

$$\begin{aligned} \zeta_C^U + \vartheta_C^U &= 1 - \prod_{i=1}^{n} \left(1 - \zeta_i^U\right)^{\omega_i} + \prod_{i=1}^{n} \left(\vartheta_i^U\right)^{\omega_i} \\ &\leq 1 - \prod_{i=1}^{n} \left(\vartheta_i^U\right)^{\omega_i} + \prod_{i=1}^{n} \left(\vartheta_i^U\right)^{\omega_i} \\ &\leq 1 \end{aligned}$$

Hence, the aggregated value by using CIFWA operator is again CIFN, which completes the proof. □

Example 4.3.1. Consider four CIFNs $\mathcal{A}_1 = (([0.2, 0.4], [0.3, 0.5]), (0.5, 0.2))$, $\mathcal{A}_2 = (([0.1, 0.2], [0.3, 0.4]), (0.3, 0.2))$, $\mathcal{A}_3 = (([0.2, 0.6], [0.1, 0.2]), (0.1, 0.3))$ and $\mathcal{A}_4 = (([0.3,0.4], [0.2,0.5]), (0.5,0.1))$ and $\omega = (0.2, 0.1, 0.3, 0.4)^T$ be their associated weight vector. Then,

$$\begin{aligned}
\prod_{i=1}^{4}\left(1-\zeta_i^L\right)^{\omega_i} &= (1-0.2)^{0.2}\times(1-0.1)^{0.1}\times(1-0.2)^{0.3}\times(1-0.3)^{0.4}=0.7674\\
\prod_{i=1}^{4}\left(1-\zeta_i^U\right)^{\omega_i} &= (1-0.4)^{0.2}\times(1-0.2)^{0.1}\times(1-0.6)^{0.3}\times(1-0.4)^{0.4}=0.5468\\
\prod_{i=1}^{4}\left(\vartheta_i^L\right)^{\omega_i} &= (0.3)^{0.2}\times(0.3)^{0.1}\times(0.1)^{0.3}\times(0.2)^{0.4}=0.1835\\
\prod_{i=1}^{4}\left(\vartheta_i^U\right)^{\omega_i} &= (0.5)^{0.2}\times(0.4)^{0.1}\times(0.2)^{0.3}\times(0.5)^{0.4}=0.3714\\
\prod_{i=1}^{4}(\zeta_i)^{\omega_i} &= (0.5)^{0.2}\times(0.3)^{0.1}\times(0.1)^{0.3}\times(0.5)^{0.4}=0.2932\\
\prod_{i=1}^{4}(1-\vartheta_i)^{\omega_i} &= (1-0.2)^{0.2}\times(1-0.2)^{0.1}\times(1-0.3)^{0.3}\times(1-0.1)^{0.4}=0.8057
\end{aligned}$$

Therefore, by using Eq. (4.5), we obtain

$$\text{CIFWA}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4)=\left(\left([0.2326,0.4532],[0.1835,0.3714]\right),(0.2932,0.1943)\right).$$

Further, it has been observed that the proposed CIFWA operators satisfies the properties of the boundedness, idempotent and monotonicity, for a collection of CIFNs $\mathcal{A}_i$, $i=1,2,\ldots,n$, which can be demonstrated as follows:

Property 4.3.1. (Idempotency) If $\mathcal{A}_i=\mathcal{A}=\left(\left([\zeta^L,\zeta^U],[\vartheta^L,\vartheta^U]\right),(\zeta,\vartheta)\right)$ for all i, then, we have

$$\text{CIFWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)=\mathcal{A}$$

Proof. Since $\mathcal{A}_i=\mathcal{A}=\left(\left([\zeta^L,\zeta^U],[\vartheta^L,\vartheta^U]\right),(\zeta,\vartheta)\right)$ for all i and $\sum_{i=1}^{n}\omega_i=1$. Then, by using Theorem 4.3.1, we have

$$\begin{aligned}
&\text{CIFWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\
&=\left(\begin{array}{c}\left(\left[1-\prod_{i=1}^{n}(1-\zeta^L)^{\omega_i},1-\prod_{i=1}^{n}(1-\zeta^U)^{\omega_i}\right],\left[\prod_{i=1}^{n}(\vartheta^L)^{\omega_i},\prod_{i=1}^{n}(\vartheta^U)^{\omega_i}\right]\right),\\ \left(\prod_{i=1}^{n}(\zeta)^{\omega_i},1-\prod_{i=1}^{n}(1-\vartheta)^{\omega_i}\right)\end{array}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\begin{array}{c} \left(\left[1-(1-\zeta^L)^{\sum_{i=1}^{n}\omega_i}, 1-(1-\zeta^U)^{\sum_{i=1}^{n}\omega_i} \right], \left[(\vartheta^L)^{\sum_{i=1}^{n}\omega_i}, (\vartheta^U)^{\sum_{i=1}^{n}\omega_i} \right] \right), \\ \left(\zeta^{\sum_{i=1}^{n}\omega_i}, 1-(1-\vartheta)^{\sum_{i=1}^{n}\omega_i} \right) \end{array} \right) \\
&= \left(\left([\zeta^L, \zeta^U], [\vartheta^L, \vartheta^U] \right), (\zeta, \vartheta) \right) \\
&= \mathcal{A}
\end{aligned}
$$

□

Property 4.3.2. (Boundedness) Let $\mathcal{A}^- = \left(\left([\zeta^L_{\min}, \zeta^U_{\min}], [\vartheta^L_{\max}, \vartheta^U_{\max}] \right), (\zeta_{\max}, \vartheta_{\min}) \right)$ and $\mathcal{A}^+ = \left(\left([\zeta^L_{\max}, \zeta^U_{\max}], [\vartheta^L_{\min}, \vartheta^U_{\min}] \right), (\zeta_{\min}, \vartheta_{\max}) \right)$ be two CIFNs. Then, we have

$$\mathcal{A}^- \leq \mathrm{CIFWA}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+.$$

Proof. Since each $\mathcal{A}_i$ is a CIFN, and it is obvious that $\min\{\zeta_i^U\} \leq \zeta_i^U \leq \max\{\zeta_i^U\}$ which implies that $\prod_{i=1}^{n}(1-\min\{\zeta_i^U\})^{\omega_i} \geq \prod_{i=1}^{n}\left(1-\zeta_i^U\right)^{\omega_i} \geq \prod_{i=1}^{n}(1-\max\{\zeta_i^U\})^{\omega_i}$, i.e., $1-\min\{\zeta_i^U\} \geq \prod_{i=1}^{n}\left(1-\zeta_i^U\right)^{\omega_i} \geq 1-\max\{\zeta_i^U\}$ and thus, we have

$$\min\{\zeta_i^U\} \leq 1 - \prod_{i=1}^{n}\left(1-\zeta_i^U\right)^{\omega_i} \leq \max\{\zeta_i^U\} \tag{4.6}$$

In the similar manner, we can prove that $\min\{\zeta_i^L\} \leq 1 - \prod_{i=1}^{n}\left(1-\zeta_i^L\right)^{\omega_i} \leq \max\{\zeta_i^L\}$, $\min\{\vartheta_i^L\} \leq \prod_{i=1}^{n}\left(\vartheta_i^L\right)^{\omega_i} \leq \max\{\vartheta_i^L\}$, $\min\{\vartheta_i^U\} \leq \prod_{i=1}^{n}\left(\vartheta_i^U\right)^{\omega_i} \leq \max\{\vartheta_i^U\}$, $\min\{\zeta_i\} \leq \prod_{i=1}^{n}(\zeta_i)^{\omega_i} \leq \max\{\zeta_i\}$ and $\min\{\vartheta_i\} \leq 1 - \prod_{i=1}^{n}(1-\vartheta_i)^{\omega_i} \leq \max\{\vartheta_i\}$.

Let $\mathrm{CIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \equiv \mathcal{A} = \left(\left([\zeta^L_{\mathcal{A}}, \zeta^U_{\mathcal{A}}], [\vartheta^L_{\mathcal{A}}, \vartheta^U_{\mathcal{A}}] \right), (\zeta_{\mathcal{A}}, \vartheta_{\mathcal{A}}) \right)$ and take $\zeta^L_{\min} = \min\{\zeta_i^L\}$, $\zeta^U_{\min} = \min\{\zeta_i^U\}$, $\vartheta^L_{\min} = \min\{\vartheta_i^L\}$, $\vartheta^U_{\min} = \min\{\vartheta_i^U\}$, $\zeta^L_{\max} = \max\{\zeta_i^L\}$, $\zeta^U_{\max} = \max\{\zeta_i^U\}$, $\vartheta^L_{\max} = \max\{\vartheta_i^L\}$, $\vartheta^U_{\max} = \max\{\vartheta_i^U\}$, $\zeta_{\min} = \min\{\zeta_i\}$, $\zeta_{\max} = \max\{\zeta_i\}$, $\vartheta_{\min} = \min\{\vartheta_i\}$ and $\vartheta_{\max} = \max\{\vartheta_i\}$. Therefore, by the definition of score function given in Definition 4.2.1, we get

$$
\begin{aligned}
\mathcal{S}c(\mathcal{A}) &= \frac{\zeta^L_{\mathcal{A}} + \zeta^U_{\mathcal{A}} - \vartheta^L_{\mathcal{A}} - \vartheta^U_{\mathcal{A}}}{2} - \zeta_{\mathcal{A}} + \vartheta_{\mathcal{A}} \\
&\leq \frac{\zeta^L_{\max} + \zeta^U_{\max} - \vartheta^L_{\min} - \vartheta^U_{\min}}{2} - \zeta_{\min} + \vartheta_{\max} \\
&\leq \mathcal{S}c(\mathcal{A}^+)
\end{aligned}
$$

Further,

$$\begin{aligned}\mathcal{S}c(\mathcal{A}) &= \frac{\zeta^L_{\mathcal{A}} + \zeta^U_{\mathcal{A}} - \vartheta^L_{\mathcal{A}} - \vartheta^U_{\mathcal{A}}}{2} - \zeta_{\mathcal{A}} + \vartheta_{\mathcal{A}} \\ &\geq \frac{\zeta^L_{\min} + \zeta^U_{\min} - \vartheta^L_{\max} - \vartheta^U_{\max}}{2} - \zeta_{\max} + \vartheta_{\min} \\ &\geq \mathcal{S}c(\mathcal{A}^-)\end{aligned}$$

Thus, $\mathcal{S}c(\mathcal{A}^-) \leq \mathcal{S}c(\mathcal{A}) \leq \mathcal{S}c(\mathcal{A}^+)$ and therefore by ranking order, we get

$$\mathcal{A}^- \leq \text{CIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$$

□

Property 4.3.3. (Monotonicity) Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be two CIFNs such that $\mathcal{A}_i \leq \mathcal{B}_i$ for all $i = 1, 2, \ldots, n$, then

$$\text{CIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{CIFWA}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n).$$

Proof. Follows from the above and hence we omit here. □

Definition 4.3.2. Let $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ be the collection of CIFNs, and CIFOWA: $\Phi^n \to \Phi$, if

$$\text{CIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1 \mathcal{A}_{\psi(1)} \oplus_R \omega_2 \mathcal{A}_{\psi(2)} \ldots \oplus_R \omega_n \mathcal{A}_{\psi(n)} \tag{4.7}$$

where Φ is the set of CIFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ such that $\omega_i > 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$. $(\psi(1), \psi(2), \ldots, \psi(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\mathcal{A}_{\psi(i-1)} \geq \mathcal{A}_{\psi(i)}$ for $i = 2, 3, \ldots, n$, then CIFOWA is called cubic intuitionsitic fuzzy ordered weighted averaging operator.

Theorem 4.3.2. Let $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ be the collection of CIFNs, then

$$\begin{aligned}&\text{CIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ &= \left(\begin{array}{c} \left(\left[1 - \prod\limits_{i=1}^{n} (1 - \zeta^L_{\psi(i)})^{\omega_i}, 1 - \prod\limits_{i=1}^{n} (1 - \zeta^U_{\psi(i)})^{\omega_i} \right], \left[\prod\limits_{i=1}^{n} (\vartheta^L_{\psi(i)})^{\omega_i}, \prod\limits_{i=1}^{n} (\vartheta^U_{\psi(i)})^{\omega_i} \right] \right), \\ \left(\prod\limits_{i=1}^{n} (\zeta_{\psi(i)})^{\omega_i}, 1 - \prod\limits_{i=1}^{n} (1 - \vartheta_{\psi(i)})^{\omega_i} \right) \end{array} \right)\end{aligned} \tag{4.8}$$

Proof. It can be proved in a similar manner as that of Theorem 4.3.1. □

Example 4.3.2. Consider four CIFNs $\mathcal{A}_1 = (([0.2, 0.4], [0.3, 0.5]), (0.5, 0.2))$, $\mathcal{A}_2 = (([0.1, 0.2], [0.3, 0.4]), (0.3, 0.2))$, $\mathcal{A}_3 = (([0.2, 0.6], [0.1, 0.2]), (0.1, 0.3))$ and $\mathcal{A}_4 = (([0.3, 0.4], [0.2, 0.5]), (0.5, 0.1))$ and $\omega = (0.2, 0.1, 0.3, 0.4)^T$ be the associated weight vector. Then, score function of each CIFN is calculated as $\mathcal{S}c(\mathcal{A}_1) = -0.40$, $\mathcal{S}c(\mathcal{A}_2) = -0.30$, $\mathcal{S}c(\mathcal{A}_3) = 0.45$, $\mathcal{S}c(\mathcal{A}_4) = -0.40$. Since $\mathcal{S}c(\mathcal{A}_1) = \mathcal{S}c(\mathcal{A}_4)$ so we compute the accuracy function and get $\mathcal{H}(\mathcal{A}_1) = 1.4$ and $\mathcal{H}(\mathcal{A}_4) = 1.3$. Thus, by comparison law we get $\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$. So, the ordered CIFNs are

$$\begin{aligned} \mathcal{A}_{\psi(1)} &= (([0.2, 0.6], [0.1, 0.2]), (0.1, 0.3)), \\ \mathcal{A}_{\psi(2)} &= (([0.1, 0.2], [0.3, 0.4]), (0.3, 0.2)), \\ \mathcal{A}_{\psi(3)} &= (([0.2, 0.4], [0.3, 0.5]), (0.5, 0.2)), \\ \text{and } \mathcal{A}_{\psi(4)} &= (([0.3, 0.4], [0.2, 0.5]), (0.5, 0.1)). \end{aligned}$$

Based on these information, we have

$$\begin{aligned} \prod_{i=1}^{4} \left(1 - \zeta_i^L\right)^{\omega_i} &= (1 - 0.2)^{0.2} \times (1 - 0.1)^{0.1} \times (1 - 0.2)^{0.3} \times (1 - 0.3)^{0.4} = 0.7674 \\ \prod_{i=1}^{4} \left(1 - \zeta_i^U\right)^{\omega_i} &= (1 - 0.6)^{0.2} \times (1 - 0.2)^{0.1} \times (1 - 0.4)^{0.3} \times (1 - 0.4)^{0.4} = 0.5694 \\ \prod_{i=1}^{4} \left(\vartheta_i^L\right)^{\omega_i} &= (0.1)^{0.2} \times (0.3)^{0.3} \times (0.3)^{0.1} \times (0.2)^{0.4} = 0.2048 \\ \prod_{i=1}^{4} \left(\vartheta_i^U\right)^{\omega_i} &= (0.2)^{0.2} \times (0.4)^{0.1} \times (0.5)^{0.3} \times (0.5)^{0.4} = 0.4071 \\ \prod_{i=1}^{4} (\zeta_i)^{\omega_i} &= (0.1)^{0.2} \times (0.3)^{0.1} \times (0.5)^{0.3} \times (0.5)^{0.4} = 0.3443 \\ \prod_{i=1}^{4} (1 - \vartheta_i)^{\omega_i} &= (1 - 0.3)^{0.2} \times (1 - 0.2)^{0.1} \times (1 - 0.2)^{0.3} \times (1 - 0.1)^{0.4} = 0.8165 \end{aligned}$$

Thus, using Eq. (4.8), we obtain

$$\text{CIFOWA} = (([0.2326, 0.4306], [0.2048, 0.4071]), (0.3443, 0.1835)).$$

Property 4.3.4. For a collection of CIFNs $\mathcal{A}_i$, $(i = 1, 2, \ldots, n)$ and for an associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, such that each $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$, we have

(P1) (Idempotency) If $\mathcal{A}_i = \mathcal{A}$ for all i, then, we have

$$\text{CIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$$

(P2) (Boundedness) Let $\mathcal{A}^L$ and $\mathcal{A}^U$ be the bounds of the collections of CIFNs $\mathcal{A}_i (i = 1, 2, \ldots, n)$ then

$$\mathcal{A}^L \leq \text{CIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^U$$

(P3) (Monotonicity) Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be two CIFNs such that $\mathcal{A}_i \leq \mathcal{B}_i$ for all i, then we have

$$\text{CIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{CIFOWA}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n).$$

Proof. It can be proved in a similar manner as that of Properties 4.3.1, 4.3.2 and 4.3.3. □

Definition 4.3.3. Let $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ be the collection of CIFNs, and CIFHA: $\Phi^n \to \Phi$, if

$$\text{CIFHA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \delta_1 \dot{\mathcal{A}}_{\psi(1)} \oplus_R \delta_2 \dot{\mathcal{A}}_{\psi(2)} \oplus_R \ldots \oplus_R \delta_n \dot{\mathcal{A}}_{\psi(n)} \tag{4.9}$$

where Φ is the set of CIFNs and $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^T$ is the weight vector associated with CIFHA such that $\delta_i > 0$ and $\sum_{i=1}^{n} \delta_i = 1$. $(\psi(1), \psi(2), \ldots, \psi(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\dot{\mathcal{A}}_{\psi(i-1)} \geq \dot{\mathcal{A}}_{\psi(i)}$ for $i = 2, 3, \ldots, n$ and $\dot{\mathcal{A}}_{\psi(i)}$ is the i^{th} weighted CIFN given by $\dot{\mathcal{A}}_i = n\omega_i \mathcal{A}_i$ where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector associated with $\mathcal{A}_i$ such that $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$ then, CIFHA is called cubic intuitionsitic fuzzy hybrid averaging operator.

Theorem 4.3.3. Let $\mathcal{A}_i$, $(i = 1, 2, \ldots, n)$ be the collection of CIFNs, then the aggregated value by using CIFHA operator is given by

$$\begin{aligned} &\text{CIFHA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ = &\left(\begin{array}{c} \left(\left[1 - \prod_{i=1}^{n} (1 - \dot{\zeta}^L_{\psi(i)})^{\delta_i}, 1 - \prod_{i=1}^{n} (1 - \dot{\zeta}^U_{\psi(i)})^{\delta_i} \right], \left[\prod_{i=1}^{n} (\dot{\vartheta}^L_{\psi(i)})^{\delta_i}, \prod_{i=1}^{n} (\dot{\vartheta}^U_{\psi(i)})^{\delta_i} \right] \right), \\ \left(\prod_{i=1}^{n} (\dot{\zeta}_{\psi(i)})^{\delta_i}, 1 - \prod_{i=1}^{n} (1 - \dot{\vartheta}_{\psi(i)})^{\delta_i} \right) \end{array} \right) \end{aligned} \tag{4.10}$$

Proof. It can be proved in a similar manner as that of Theorem 4.3.1. □

Example 4.3.3. Consider four CIFNs $\mathcal{A}_1 = (([0.2, 0.4], [0.3, 0.5]), (0.5, 0.2))$, $\mathcal{A}_2 = (([0.1, 0.2], [0.3, 0.4]), (0.3, 0.2))$, $\mathcal{A}_3 = (([0.2, 0.6], [0.1, 0.2]), (0.1, 0.3))$, $\mathcal{A}_4 = (([0.3, 0.4], [0.2, 0.5]), (0.5, 0.1))$ and $\omega = (0.2, 0.1, 0.3, 0.4)^T$ be the weight vector of these CIFNs, then $\dot{\mathcal{A}}_i = 4\omega_i \mathcal{A}_i$ for $i = 1, 2, 3, 4$ are computed as:

$$\begin{aligned}\dot{\mathcal{A}}_1 &= \left(\left(\begin{bmatrix} 1-(1-0.2)^{0.8}, \\ 1-(1-0.4)^{0.8} \end{bmatrix}, \left[(0.3)^{0.8}, (0.5)^{0.8}\right] \right), \left((0.5)^{0.8}, 1-(1-0.2)^{0.8}\right) \right) \\ &= (([0.1635, 0.3355], [0.3817, 0.5743]), (0.5743, 0.1635))\end{aligned}$$

Similarly, we get

$$\begin{aligned}\dot{\mathcal{A}}_2 &= (([0.0413, 0.0854], [0.6178, 0.6931]), (0.6178, 0.0854)), \\ \dot{\mathcal{A}}_3 &= (([0.2349, 0.6670], [0.0631, 0.1450]), (0.0631, 0.3482)), \\ \dot{\mathcal{A}}_4 &= (([0.4349, 0.5584], [0.0761, 0.3299]), (0.3299, 0.1551)).\end{aligned}$$

The score values corresponding to these numbers are $\mathcal{S}c(\dot{\mathcal{A}}_1) = -0.6394$, $\mathcal{S}c(\dot{\mathcal{A}}_2) = -1.1246$, $\mathcal{S}c(\dot{\mathcal{A}}_3) = 0.6320$ and $\mathcal{S}c(\dot{\mathcal{A}}_4) = 0.1189$. Thus, based on it the ranking order of the given CIFNs is $\mathcal{S}c(\dot{\mathcal{A}}_3) > \mathcal{S}c(\dot{\mathcal{A}}_4) > \mathcal{S}c(\dot{\mathcal{A}}_1) > \mathcal{S}c(\dot{\mathcal{A}}_2)$ and therefore, $\dot{\mathcal{A}}_{\psi(1)} = (([0.2349, 0.6670], [0.0631, 0.1450]), (0.0631, 0.3482))$, $\dot{\mathcal{A}}_{\psi(2)} = (([0.4349, 0.5584], [0.0761, 0.3299]), (0.3299, 0.1551))$, $\dot{\mathcal{A}}_{\psi(3)} = (([0.1635, 0.3355], [0.3817, 0.5743]), (0.5743, 0.1635))$ and $\dot{\mathcal{A}}_{\psi(4)} = (([0.0413, 0.0854], [0.6178, 0.6931]), (0.6178, 0.0854))$. Let $\delta = (0.3, 0.1, 0.1, 0.5)^T$ be the position weighted vector and based on it we have,

$$\prod_{i=1}^{4}(1-\dot{\zeta}^L_{\psi(i)})^{\delta_i} = (1-0.2349)^{0.3} \times (1-0.4349)^{0.1} \times (1-0.1635)^{0.1} \times (1-0.0413)^{0.5} = 0.8383$$

$$\prod_{i=1}^{4}(1-\dot{\zeta}^U_{\psi(i)})^{\delta_i} = (1-0.6670)^{0.3} \times (1-0.5584)^{0.1} \times (1-0.3355)^{0.1} \times (1-0.0854)^{0.5} = 0.6083$$

$$\prod_{i=1}^{4}(\dot{\vartheta}^L_{\psi(i)})^{\delta_i} = (0.0631)^{0.3} \times (0.0761)^{0.1} \times (0.3817)^{0.1} \times (0.6178)^{0.5} = 0.2409$$

$$\prod_{i=1}^{4}(\dot{\vartheta}^U_{\psi(i)})^{\delta_i} = (0.1450)^{0.3} \times (0.3299)^{0.1} \times (0.5743)^{0.1} \times (0.6931)^{0.5} = 0.3949$$

$$\prod_{i=1}^{4}(\dot{\zeta}_{\psi(i)})^{\delta_i} = (0.0631)^{0.3} \times (0.3299)^{0.1} \times (0.5743)^{0.1} \times (0.6178)^{0.5} = 0.2905$$

$$\prod_{i=1}^{4}(1-\dot{\vartheta}_{\psi(i)})^{\delta_i} = (1-0.3482)^{0.3} \times (1-0.1551)^{0.1} \times (1-0.1635)^{0.1} \times (1-0.0854)^{0.5} = 0.8124$$

So, using Eq. (4.10) we get the aggregated value of the CIFNs by using CIFHA operator as

$$\text{CIFHA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) = (([0.1617, 0.3917], [0.2409, 0.3949]), (0.2905, 0.1876))$$

Also, it is clearly seen that the CIFHA operator is also satisfies the boundedness, idempotent and monotonicity properties.

4.3.2 Averaging operators based on P-order operations

Now, in the following, we have proposed averaging AOs by using P-order operations as follows:

Definition 4.3.4. Let $\mathcal{A}_i = \left(([\zeta_i^L, \zeta_i^U], [\vartheta_i^L, \vartheta_i^U]), (\zeta_i, \vartheta_i)\right)$, $(i = 1, 2, \ldots, n)$ be a collection of CIFNs. An CIFWA$_P$ operator of dimension n is a mapping CIFWA$_P$: $\Phi^n \to \Phi$ such that

$$\text{CIFWA}_P(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1\mathcal{A}_1 \oplus_P \omega_2\mathcal{A}_2 \oplus_P \ldots \oplus_P \omega_n\mathcal{A}_n$$
$$= \left(\begin{array}{c} \left(\left[1 - \prod_{i=1}^{n} (1-\zeta_i^L)^{\omega_i}, 1 - \prod_{i=1}^{n} (1-\zeta_i^U)^{\omega_i} \right], \left[\prod_{i=1}^{n} (\vartheta_i^L)^{\omega_i}, \prod_{i=1}^{n} (\vartheta_i^U)^{\omega_i} \right] \right), \\ \left(1 - \prod_{i=1}^{n}(1-\zeta_i)^{\omega_i}, \prod_{i=1}^{n}(\vartheta_i)^{\omega_i} \right) \end{array} \right) \quad (4.11)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$, such that $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$.

Definition 4.3.5. Let $\mathcal{A}_i = \left(([\zeta_i^L, \zeta_i^U], [\vartheta_i^L, \vartheta_i^U]), (\zeta_i, \vartheta_i)\right)$, $(i = 1, 2, \ldots, n)$ be a collection of CIFNs. A CIFOWA$_P$ operator of dimension n is a mapping CIFOWA$_P$: $\Phi^n \to \Phi$ such that

$$\text{CIFOWA}_P(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1\mathcal{A}_{\psi(1)} \oplus_P \omega_2\mathcal{A}_{\psi(2)} \oplus_P \ldots \oplus_P \omega_n\mathcal{A}_{\psi(n)}$$
$$= \left(\begin{array}{c} \left(\left[1 - \prod_{i=1}^{n}(1-\zeta_{\psi(i)}^L)^{\omega_i}, 1 - \prod_{i=1}^{n}(1-\zeta_{\psi(i)}^U)^{\omega_i} \right], \left[\prod_{i=1}^{n}(\vartheta_{\psi(i)}^L)^{\omega_i}, \prod_{i=1}^{n}(\vartheta_{\psi(i)}^U)^{\omega_i} \right] \right), \\ \left(1 - \prod_{i=1}^{n}(1-\zeta_{\psi(i)})^{\omega_i}, \prod_{i=1}^{n}(\vartheta_{\psi(i)})^{\omega_i} \right) \end{array} \right) \quad (4.12)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$, such that $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$.

Definition 4.3.6. Let $\mathcal{A}_i = \left(\left([\zeta_i^L, \zeta_i^U], [\vartheta_i^L, \vartheta_i^U]\right), (\zeta_i, \vartheta_i)\right)$, $(i = 1, 2, \ldots, n)$ be a collection of CIFNs. A CIFHA$_P$ operator of dimension n is a mapping CIFHA$_P : \Phi^n \to \Phi$ such that

$$\text{CIFHA}_P(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \xi_1 \dot{\mathcal{A}}_{\psi(1)} \oplus_P \xi_2 \dot{\mathcal{A}}_{\psi(2)} \oplus_P \ldots \oplus_P \xi_n \dot{\mathcal{A}}_{\psi(n)}$$
$$= \left(\begin{array}{c} \left(\left[1 - \prod_{i=1}^{n} (1 - \dot{\zeta}_{\psi(i)}^L)^{\xi_i}, 1 - \prod_{i=1}^{n} (1 - \dot{\zeta}_{\psi(i)}^U)^{\xi_i} \right], \left[\prod_{i=1}^{n} (\dot{\vartheta}_{\psi(i)}^L)^{\xi_i}, \prod_{i=1}^{n} (\dot{\vartheta}_{\psi(i)}^U)^{\xi_i} \right] \right), \\ \left(1 - \prod_{i=1}^{n} (1 - \dot{\zeta}_{\psi(i)})^{\xi_i}, \prod_{i=1}^{n} (\dot{\vartheta}_{\psi(i)})^{\xi_i} \right) \end{array} \right)$$

4.3.3 Geometric Operators based on R-order operations

Now, in the following, we have proposed some geometric AOs under the cubic intuitionistic fuzzy environment which include: cubic intuitionistic fuzzy weighted geometric (CIFWG) operator, cubic intuitionistic fuzzy ordered weighted geometric (CIFOWG) operator and cubic intuitionistic fuzzy hybrid geometric (CIFHG) operator.

Definition 4.3.7. A CIFWG operator, defined on a collections of CIFNs $\mathcal{A}_i$, $(i = 1, 2, \ldots, n)$, is a mapping CIFWG : $\Phi^n \to \Phi$ as

$$\text{CIFWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \;=\; \mathcal{A}_1^{\omega_1} \otimes_R \mathcal{A}_2^{\omega_2} \otimes_R \ldots \otimes_R \mathcal{A}_n^{\omega_n} \tag{4.13}$$

where Φ is the set of CIFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ with each $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$, then CIFWG is called cubic intuitionistic fuzzy weighted averaging operator.

Theorem 4.3.4. The aggregated value by using CIFWG operator for a collection of 'n' CIFNs $\mathcal{A}_i$, $(i = 1, 2, \ldots, n)$ is still CIFN and is given by

$$\text{CIFWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$
$$= \left(\begin{array}{c} \left(\left[\prod_{i=1}^{n} (\zeta_i^L)^{\omega_i}, \prod_{i=1}^{n} (\zeta_i^U)^{\omega_i} \right], \left[1 - \prod_{i=1}^{n} (1 - \vartheta_i^L)^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - \vartheta_i^U)^{\omega_i} \right] \right) \\ \left(1 - \prod_{i=1}^{n} (1 - \zeta_i)^{\omega_i}, \prod_{i=1}^{n} (\vartheta_i)^{\omega_i} \right) \end{array} \right) \tag{4.14}$$

Proof. It can be proved in a similar manner as that of Theorem 4.3.1. □

Definition 4.3.8. A CIFOWG operator, defined on a collections of CIFNs $\mathcal{A}_i$, $(i = 1, 2, \ldots, n)$, is a mapping CIFOWG : $\Phi^n \to \Phi$ as

$$\text{CIFOWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}_{\psi(1)}^{\omega_1} \otimes_R \mathcal{A}_{\psi(2)}^{\omega_2} \cdots \otimes_R \mathcal{A}_{\psi(n)}^{\omega_n} \quad (4.15)$$

where Φ is the set of CIFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ with $\omega_i > 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$. $(\psi(1), \psi(2), \ldots, \psi(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\mathcal{A}_{\psi(i-1)} \geq \mathcal{A}_{\psi(i)}$ for $i = 2, 3, \ldots, n$, then CIFOWG is called cubic intuitionsitic fuzzy ordered weighted geometric operator.

Theorem 4.3.5. The aggregated value of a collection of CIFNs $\mathcal{A}_i$, $(i = 1, 2, \ldots, n)$ by using CIFOWG operator is still CIFN and is given by

$$\text{CIFOWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}_{\psi(1)}^{\omega_1} \otimes_R \mathcal{A}_{\psi(2)}^{\omega_2} \cdots \otimes_R \mathcal{A}_{\psi(n)}^{\omega_n}$$

$$= \left(\begin{array}{c} \left(\left[\prod\limits_{i=1}^{n} (\zeta_{\psi(i)}^{L})^{\omega_i}, \prod\limits_{i=1}^{n} (\zeta_{\psi(i)}^{U})^{\omega_i} \right], \left[1 - \prod\limits_{i=1}^{n} (1 - \vartheta_{\psi(i)}^{L})^{\omega_i}, 1 - \prod\limits_{i=1}^{n} (1 - \vartheta_{\psi(i)}^{U})^{\omega_i} \right] \right), \\ \left(1 - \prod\limits_{i=1}^{n} (1 - \zeta_{\psi(i)})^{\omega_i}, \prod\limits_{i=1}^{n} (\vartheta_{\psi(i)})^{\omega_i} \right) \end{array} \right)$$

Proof. It can be proved in a similar manner as that of Theorem 4.3.4. □

Definition 4.3.9. A CIFHG operator, defined on a collections of CIFNs $\mathcal{A}_i$, $(i = 1, 2, \ldots, n)$, is a mapping CIFHG : $\Phi^n \to \Phi$ as

$$\text{CIFHG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = (\dot{\mathcal{A}}_{\psi(1)})^{\delta_1} \otimes_R (\dot{\mathcal{A}}_{\psi(2)})^{\delta_2} \otimes_R \ldots \otimes_R (\dot{\mathcal{A}}_{\psi(n)})^{\delta_n} \quad (4.16)$$

where Φ is the set of CIFNs and $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^T$ is the weight vector associated with CIFHG such that $\delta_i > 0$ and $\sum\limits_{i=1}^{n} \delta_i = 1$. $(\psi(1), \psi(2), \ldots, \psi(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\dot{\mathcal{A}}_{\psi(i-1)} \geq \dot{\mathcal{A}}_{\psi(i)}$ for $i = 2, 3, \ldots, n$ and $\dot{\mathcal{A}}_{\psi(i)}$ is the i^{th} weighted CIFN given by $\dot{\mathcal{A}}_i = (\mathcal{A}_i)^{n\omega_i}$, $i = 1, 2, \ldots, n$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector associated with $\mathcal{A}_i$ such that $\omega_i > 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$ then, CIFHG is called cubic intuitionsitic fuzzy hybrid geometric operator.

Theorem 4.3.6. Let $\mathcal{A}_i$, $(i = 1, 2, \ldots, n)$ be the collection of CIFNs, then the aggregated value by using CIFHG operator is still CIFN and is given by

$$\begin{aligned}&\text{CIFHG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\\ =& \left(\begin{array}{c}\left(\left[\prod_{i=1}^{n}(\dot{\zeta}^L_{\psi(i)})^{\delta_i}, \prod_{i=1}^{n}(\dot{\zeta}^U_{\psi(i)})^{\delta_i}\right], \left[1-\prod_{i=1}^{n}(1-\dot{\vartheta}^L_{\psi(i)})^{\delta_i}, 1-\prod_{i=1}^{n}(1-\dot{\vartheta}^U_{\psi(i)})^{\delta_i}\right]\right), \\ \left(1-\prod_{i=1}^{n}(1-\dot{\zeta}_{\psi(i)})^{\delta_i}, \prod_{i=1}^{n}(\dot{\vartheta}_{\psi(i)})^{\delta_i}\right)\end{array}\right)\end{aligned} \tag{4.17}$$

Proof. It can be proved in a similar manner as that of Theorem 4.3.1. □

Further, it can be easily obtained from the above weighted geometric operators that they also satisfy the boundedness, idempotency as well as monotonicity properties for a collection of CIFNs.

4.3.4 Geometric operators based on P-order operations

Now, in the following, we have proposed geometric AOs by using P-order operations as follows:

Definition 4.3.10. Let $\mathcal{A}_i = \left(\left([\zeta^L_i, \zeta^U_i], [\vartheta^L_i, \vartheta^U_i]\right), (\zeta_i, \vartheta_i)\right)$, $(i = 1, 2, \ldots, n)$ be a collection of CIFNs. An CIFWG$_P$ operator of dimension n is a mapping CIFWG$_P : \Phi^n \to \Phi$ such that

$$\begin{aligned}&\text{CIFWG}_P(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = (\mathcal{A}_1)^{\omega_1} \otimes_P (\mathcal{A}_2)^{\omega_2} \otimes_P \ldots \otimes_P (\mathcal{A}_n)^{\omega_n}\\ =& \left(\begin{array}{c}\left(\left[\prod_{i=1}^{n}(\zeta^L_i)^{\omega_i}, \prod_{i=1}^{n}(\zeta^U_i)^{\omega_i}\right], \left[1-\prod_{i=1}^{n}(1-\vartheta^L_i)^{\omega_i}, 1-\prod_{i=1}^{n}(1-\vartheta^U_i)^{\omega_i}\right]\right), \\ \left(\prod_{i=1}^{n}(\zeta_i)^{\omega_i}, 1-\prod_{i=1}^{n}(1-\vartheta_i)^{\omega_i}\right)\end{array}\right)\end{aligned} \tag{4.18}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$, such that $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$.

Definition 4.3.11. Let $\mathcal{A}_i = \left(\left([\zeta^L_i, \zeta^U_i], [\vartheta^L_i, \vartheta^U_i]\right), (\zeta_i, \vartheta_i)\right)$, $(i = 1, 2, \ldots, n)$ be a collection of CIFNs. A CIFOWG$_P$ operator of dimension n is a mapping CIFOWG$_P : \Phi^n \to \Phi$ such that

$$\text{CIFOWG}_P(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = (\mathcal{A}_{\psi(1)})^{\omega_1} \otimes_P (\mathcal{A}_{\psi(2)})^{\omega_2} \otimes_P \ldots \otimes_P (\mathcal{A}_{\psi(n)})^{\omega_n}$$
$$= \left(\begin{array}{c} \left(\left[\prod_{i=1}^{n} (\zeta^L_{\psi(i)})^{\omega_i}, \prod_{i=1}^{n} (\zeta^U_{\psi(i)})^{\omega_i} \right], \left[1 - \prod_{i=1}^{n} (1 - \vartheta^L_{\psi(i)})^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - \vartheta^U_{\psi(i)})^{\omega_i} \right] \right), \\ \left(\prod_{i=1}^{n} (\zeta_{\psi(i)})^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - \vartheta_{\psi(i)})^{\omega_i} \right) \end{array} \right)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$, $(i = 1, 2, \ldots, n)$, such that $\omega_i > 0$, $\sum\limits_{i=1}^{n} \omega_i = 1$.

Definition 4.3.12. Let $\mathcal{A}_i = \left(\left([\zeta^L_i, \zeta^U_i], [\vartheta^L_i, \vartheta^U_i] \right), (\zeta_i, \vartheta_i) \right)$, $(i = 1, 2, \ldots, n)$ be a collection of CIFNs. A CIFHG_P operator of dimension n is a mapping $\text{CIFHG}_P : \Phi^n \to \Phi$ such that

$$\text{CIFHG}_P(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = (\dot{\mathcal{A}}_{\psi(1)})^{\xi_1} \otimes_P (\dot{\mathcal{A}}_{\psi(2)})^{\xi_2} \otimes_P \ldots \otimes_P (\dot{\mathcal{A}}_{\psi(n)})^{\xi_n}$$
$$= \left(\begin{array}{c} \left(\left[\prod_{i=1}^{n} (\dot{\zeta}^L_{\psi(i)})^{\xi_i}, \prod_{i=1}^{n} (\dot{\zeta}^U_{\psi(i)})^{\xi_i} \right], \left[1 - \prod_{i=1}^{n} (1 - \dot{\vartheta}^L_{\psi(i)})^{\xi_i}, 1 - \prod_{i=1}^{n} (1 - \dot{\vartheta}^U_{\psi(i)})^{\xi_i} \right] \right), \\ \left(\prod_{i=1}^{n} (\dot{\zeta}_{\psi(i)})^{\xi_i}, 1 - \prod_{i=1}^{n} (1 - \dot{\vartheta}_{\psi(i)})^{\xi_i} \right) \end{array} \right) \quad (4.19)$$

4.4 Decision making approach using proposed operators

In this section, a decision-making method by using above defined AOs for CIFSs has been presented followed by an illustrative example for demonstrating the approach.

4.4.1 Decision-making approach

The general description of MCDM problem is same as Section 2.5 of Chapter 2. The preferences are given in the form of the CIFNs $\mathcal{A}_{ij} = \left(\left([\zeta^L_{ij}, \zeta^U_{ij}], [\vartheta^L_{ij}, \vartheta^U_{ij}] \right), (\zeta_{ij}, \vartheta_{ij}) \right)$ such that $[\zeta^L_{ij}, \zeta^U_{ij}], [\vartheta^L_{ij}, \vartheta^U_{ij}] \subseteq [0, 1]$, $\zeta_{ij}, \vartheta_{ij} \in [0, 1]$ and $\zeta^+_{ij} + \vartheta^+_{ij} \leq 1$, $\zeta_{ij} + \vartheta_{ij} \leq 1$ for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. Thus, the rating values corresponding to each alternative are represented in the form of CIFNs over the criteria set $\mathfrak{B}$ as follows:

$$\mathcal{V}_i = \left\{ \left(x, \left([\zeta^L_{ij}(x), \zeta^U_{ij}(x)], [\vartheta^L_{ij}(x), \vartheta^U_{ij}(x)] \right), (\zeta_{ij}(x), \vartheta_{ij}(x)) \right) \mid x \in \mathfrak{B} \right\}$$

for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. Let $\omega_j (j = 1, 2, \ldots, n)$ be the weight of the criteria $\mathfrak{B}_j$ such that $\omega_j > 0$ and $\sum\limits_{j=1}^{n} \omega_j = 1$. In the following, we develop an approach based on the proposed operators with cubic intuitionistic fuzzy information, to find the best-fit among the available alternatives, which involves the following steps:

Step 1: *(Construction of CIFN decision-making matrix)* Collect all the information corresponding to each alternative in terms of CIFNs and hence an overall decision matrix $\mathcal{M}$ is expressed as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \dots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \dots & \mathcal{A}_{1n} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \dots & \mathcal{A}_{2n} \\ \dots & \dots & \ddots & \vdots \\ \mathcal{A}_{m1} & \mathcal{A}_{m2} & \dots & \mathcal{A}_{mn} \end{pmatrix} \end{array}$$

Step 2: *(Normalize the decision matrix)* If all the attributes are of same type, then there is no need of normalization. But, if there are different types of criterion say profit and cost, then we convert cost type criterion into profit type by the following normalized formula:

$$r_{ij} = \begin{cases} \left(\left([\zeta_{ij}^L, \zeta_{ij}^U], [\vartheta_{ij}^L, \vartheta_{ij}^U]\right), (\zeta_{ij}, \vartheta_{ij})\right) & ; \quad \text{for benefit type criterion} \\ \left(\left([\vartheta_{ij}^L, \vartheta_{ij}^U], [\zeta_{ij}^L, \zeta_{ij}^U]\right), (\vartheta_{ij}, \zeta_{ij})\right) & ; \quad \text{for cost type criterion} \end{cases} \tag{4.20}$$

Hence, we obtain the normalized CIFN decision matrix, $R = (r_{ij})_{m \times n}$.

Step 3: *(Compute the aggregated value of the alternatives)* Aggregate all the preference value of the matrix R corresponding to each alternative $\mathcal{V}_i$, $(i = 1, 2, \dots, m)$, making use of CIFWA, CIFOWA, CIFHA, CIFWG, CIFOWG or CIFHG operator as given in Eqs. (4.5), (4.8), (4.10), (4.14), (4.16) or (4.17) respectively and hence the aggregated value denoted as r_i.

Step 4: *(Computation of score value)* Determine the score of the aggregated values r_i, $(i = 1, 2, \dots, m)$ by using the Definition 4.2.1.

Step 5: *(Ranking of alternatives)* Rank the alternatives based on the score values and according to the comparison law defined in Definition 4.2.2.

4.4.2 Illustrative Example

Job-Scheduling is the process of allocating resources to different jobs by an Operating System (OS). The job-scheduler schedules the order of jobs in the job queue and the system allocates CPU to the prioritized jobs among the queue. Job-scheduling makes sure that all jobs are completed fairly on time and no job has to starve for CPU allocation as well. Most OSs like UNIX, Windows, IOS, Android etc. include standard job scheduling abilities. Not only this, a number of programs including Database Management System (DBMS), Enterprise Resource Planning(ERP), Business Process Management(BPM) feature specific job scheduling capabilities as well. Suppose a programmer has to schedule jobs $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ and $\mathcal{V}_4$ in accordance of the criterion $\mathfrak{B}_1, \mathfrak{B}_2$ and $\mathfrak{B}_3$, where $\mathfrak{B}_1$: Automated restart in case of failure; $\mathfrak{B}_2$: Number of parallel jobs permitted for a user and $\mathfrak{B}_3$: Execution time assigned to a user. These jobs are prioritized in accordance to the weight vector $\omega = (0.20, 0.38, 0.42)^T$ and $\delta = (0.48, 0.29, 0.23)^T$ be the positional weight vector. The main aim of the programmer is to schedule the jobs in such a way that it saves the execution time and no job has to wait for CPU allocation for an elongated period of time.

Step 1: The evaluation of these alternatives are taken using the CIFNs by the decision makers under the above three general characteristics and hence construct the decision matrix as given in Table 4.1.

Step 2: As $\mathfrak{B}_1$ and $\mathfrak{B}_3$ are of the cost types criteria so it can be normalized by using Eq. (4.20) and hence get the normalized decision matrix R given in Table 4.2.

Step 3: Aggregate these normalized data by using CIFWA operator (for the sake of simplicity) and hence get the aggregated CIFNs denoted by $r_i (i = 1, 2, 3, 4)$ as

$$\begin{aligned}
r_1 &= \left(\left([0.1809, 0.2811], [0.4183, 0.5930]\right), (0.4795, 0.2447)\right), \\
r_2 &= \left(\left([0.5314, 0.6696], [0.0641, 0.2644]\right), (0.4764, 0.3307)\right), \\
r_3 &= \left(\left([0.2828, 0.4341], [0.3530, 0.4919]\right), (0.3938, 0.3220)\right) \\
\text{and } r_4 &= \left(\left([0.2658, 0.6658], [0.1714, 0.3132]\right), (0.3000, 0.4512)\right)
\end{aligned}$$

Step 4: The score values of these aggregated numbers are obtained by using the Eq. (4.1) as $\mathcal{S}c(r_1) = -0.5094$, $\mathcal{S}c(r_2) = 0.2905$, $\mathcal{S}c(r_3) = -0.1358$ and $\mathcal{S}c(r_4) = 0.3747$.

Step 5: Thus, from these score values, the ranking order of the considered schedule jobs is $\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$. Therefore, we conclude that $\mathcal{V}_4$ is to be scheduled first.

However, if we aggregate the given data by using CIFOWA operator, then the following steps have been executed for finding the best alternative:

Step 1: The information related to each alternative is summarized in the form of CIFNs in Table 4.1.

Step 2: The normalized data of the considered problem is given in Table 4.2.

Step 3: By utilizing the CIFOWA operator as given in Eq. (4.8) for aggregating the i^{th} row and hence get the overall performance value of each alternative, denoted by r_i, as

$$\begin{aligned} r_1 &= (([0.1809, 0.2811], [0.4183, 0.5930]), (0.4795, 0.2447)), \\ r_2 &= (([0.4289, 0.5585], [0.1099, 0.3222]), (0.5125, 0.3645)), \\ r_3 &= (([0.2447, 0.3741], [0.4163, 0.5573]), (0.4147, 0.2859)) \\ \text{and } r_4 &= (([0.2064, 0.5257], [0.2339, 0.4473]), (0.3000, 0.3902)) \end{aligned}$$

Step 4: The score values of these alternatives are $\mathcal{S}c(r_1) = -0.5094$, $\mathcal{S}c(r_2) = 0.1297$, $\mathcal{S}c(r_3) = -0.3062$ and $\mathcal{S}c(r_4) = 0.1156$.

Step 5: Therefore, the ranking order of the alternative is $\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$ and thus $\mathcal{V}_2$ is the firstly assigned job.

On the other hand, if we apply the other proposed AOs namely CIFHA, CIFWG, CIFOWG, CIFHG, CIFWA$_P$, CIFOWA$_P$, CIFHA$_P$, CIFWG$_P$, CIFOWG$_P$ and CIFHG$_P$ for aggregating the different preferences of the decision makers then, the score values corresponding to each alternative are summarized in Table 4.3. According to score function of the aggregated values, the ordering of the alternatives is shown in Table 4.3 in which $\succ$ means "preferred to". On the other hand, in order to compare the proposed approach results with the existing approaches under the IVIFS environment, we have taken the fuzzy judgements as zero in CIFS so that it get reduced to IVIFS. Based on these observations, we have conducted an experiment by using the existing approaches [25, 44, 45, 114, 131, 180–182, 198, 199] and their corresponding results are summarized in Table 4.4. From these results, it has been seen that the most preferable alternative is $\mathcal{V}_2$ by all the operators while the different AOs have different ranking strategies which are slightly different. Thus, based on the decision makers preference in terms of their AOs used, the results may lead to the different decisions.

Compared with these existing approaches with general intuitionistic sets (IVIFSs or IFSs), the proposed decision-making method under CIFS environment contains much more evaluation information on the alternatives by considering both the IVIFSs and IFSs simultaneously, while the existing approaches contain either IFS or IVIFS information. Therefore, the approaches under the IVIFSs or IFSs may lose some useful information, either IVIFNs or IFNs, of alternatives which may affect the decision results. Furthermore, it is noted from the study that the computational procedure of the proposed approach is different from the existing approaches under the different environment, but the proposed result in this chapter is more rational to reality in the decision process due to the consideration of the consistent priority degree between the pairs of the arguments. Also, the corresponding studies under the IVIFS or IFS environment can be considered as a special case of the proposed operators. Finally, the existing decision-making methods under IVIFSs or IFSs cannot deal with the decision-making problem with CIFS.

4.5 Conclusion

The fundamental contributions of this chapter are summarized below:

(i) The extended theory of the IFS to the cubic IFS is put-forth with enhanced features where preference corresponding to an element is expressed by means of the IVIFS and IFS which shows the importance of IVIFS to get the more appropriate results through IFS. Also, to compare the different CIFSs, a score and accuracy function is defined with their desired relations.

(ii) Based on the intrinsic features of P(R)-order CIFSs, some basic operational laws between the pairs of the CIFNs are defined and hence stated the series of the aggregation operators, namely cubic intuitionistic fuzzy weighted averaging (geometric), ordered weighted averaging (geometric) and hybrid averaging (geometric). The Various desirable properties of these AOs have been investigated in detail.

(iii) On the basis of the standardized decision-matrix and the proposed AOs, a decision-making method has been presented for solving the practical problem is demonstrated in an efficient way under the CIFS environment. The proposed operators will make an optimistic as well as pessimistic choices to the decision-makers according to their preferences levels and hence the proposed approach can be applied as an alternative way to solve the problem in real-life situations.

(iv) To demonstrate the efficiency of the proposed operators, an example from the field of job-scheduling has been taken. From these studies, it has been concluded that they can easily handle the real-life decision-making problems with their targets and hence beneficial for the system analysts.

Table 4.1: Preferences of the decision makers to each job in the form of CIFN

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$(([0.50, 0.80], [0.10, 0.20]), (0.40, 0.20))$	$(([0.20, 0.30], [0.40, 0.50]), (0.50, 0.20))$	$(([0.40, 0.60], [0.20, 0.30]), (0.20, 0.70))$
$\mathcal{V}_2$	$(([0.20, 0.30], [0.40, 0.50]), (0.40, 0.60))$	$(([0.80, 0.90], [0.01, 0.10]), (0.40, 0.20))$	$(([0.20, 0.60], [0.10, 0.20]), (0.40, 0.50))$
$\mathcal{V}_3$	$(([0.50, 0.60], [0.20, 0.30]), (0.20, 0.40))$	$(([0.40, 0.60], [0.20, 0.30]), (0.30, 0.40))$	$(([0.50, 0.70], [0.20, 0.30]), (0.30, 0.50))$
$\mathcal{V}_4$	$(([0.30, 0.70], [0.10, 0.30]), (0.10, 0.30))$	$(([0.40, 0.90], [0.05, 0.10]), (0.30, 0.40))$	$(([0.40, 0.60], [0.20, 0.30]), (0.60, 0.30))$

Table 4.2: Normalized decision-making data values in the form of CIFN

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$(([0.10, 0.20], [0.50, 0.80]), (0.20, 0.40))$	$(([0.20, 0.30], [0.40, 0.50]), (0.50, 0.20))$	$(([0.20, 0.30], [0.40, 0.60]), (0.70, 0.20))$
$\mathcal{V}_2$	$(([0.40, 0.50], [0.20, 0.30]), (0.60, 0.40))$	$(([0.80, 0.90], [0.01, 0.10]), (0.40, 0.20))$	$(([0.10, 0.20], [0.20, 0.60]), (0.50, 0.40))$
$\mathcal{V}_3$	$(([0.20, 0.30], [0.50, 0.60]), (0.40, 0.20))$	$(([0.40, 0.60], [0.20, 0.30]), (0.30, 0.40))$	$(([0.20, 0.30], [0.50, 0.70]), (0.50, 0.30))$
$\mathcal{V}_4$	$(([0.10, 0.30], [0.30, 0.70]), (0.30, 0.10))$	$(([0.40, 0.90], [0.05, 0.10]), (0.30, 0.40))$	$(([0.20, 0.30], [0.40, 0.60]), (0.30, 0.60))$

Table 4.3: Influence of the different operators on score values

Proposed Operators	Score value of the alternatives				Order of the alternatives
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
CIFWA	-0.5094	0.2905	-0.1358	0.3747	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
CIFOWA	-0.5094	0.1297	-0.3062	0.1156	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
CIFHA	-0.4656	0.4553	0.0536	0.5517	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
CIFWG	-0.6229	-0.0839	-0.2587	0.0258	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
CIFOWG	-0.6229	-0.1473	-0.3944	-0.2073	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
CIFHG	-0.4569	0.0799	-0.1332	0.1320	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
CIFWA$_P$	0.0524	0.6164	0.0381	0.1641	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
CIFOWA$_P$	0.0524	0.4531	-0.0229	0.0648	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
CIFHA$_P$	-0.0121	0.7887	0.2248	0.3657	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
CIFWG$_P$	-0.0611	0.2420	-0.0847	-0.1848	$\mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
CIFOWG$_P$	-0.0611	0.1761	-0.1111	-0.2582	$\mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
CIFHG$_P$	-0.1336	0.3661	-0.0497	0.0041	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$

Table 4.4: Comparative studies with existing approaches

Existing approaches	Overall value of the alternatives				Order of the alternatives
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
Xu and Chen [181]	-0.2746	0.4363	-0.0640	0.2235	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Xu and Chen [182]	-0.2959	0.0963	-0.1565	-0.0336	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Ye [199]	-0.0324	0.3652	0.1394	0.1739	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Nayagam et al. [114]	-0.1535	0.5418	0.0927	0.3506	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Sivaraman et al. [131]	0.1704	0.5477	0.2801	0.4023	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Garg [44]	0.2850	0.7301	0.4260	0.5476	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Chen et al. [25]	0.1289	0.2344	0.1362	0.1512	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Xu [180]	0.3581	0.0458	0.2334	0.1859	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Garg [45]	0.7442	0.9962	0.8961	0.9349	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Yager and Abbasov [198]	-0.2717	0.4864	-0.0488	0.2537	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$

Chapter 5

Generalized cubic intuitionistic fuzzy aggregation operators using t-norm operations[1]

In the present chapter, we extend the structural characteristics of CIFS as defined in Chapter 4 by defining generalized t-norm and t-conorm aggregation operators. The aspect of generalization of these operators is captured in detail in which they are subjected to get reduced into some existing AOs. Further, to strengthen the practical applicability of the proposed operators, we also formulate a decision-making approach. Finally, an illustrative example is provided to discuss the reliability of the proposed approach and an extensive comparison analysis is conducted.

5.1 Introduction

Cubic intuitionistic fuzzy set (CIFS) is an efficient tool in handling possible disagreeness of the agreed interval values and vice-versa. This environment increases the level of precision by enhancing the scope of the membership (and non-membership) interval by considering a fuzzy set value corresponding to it. However, in the real world, it is regularly hard to express the estimation of membership degree by an exact value in a fuzzy set. In such cases, CIFSs make it easier to depict vagueness and uncertainty in the real world using an

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interval value and an exact value, instead of unique interval/exact values. Consequently, the hybrid form of an interval value might be extremely valuable to depict the uncertainties because of expert's reluctant judgment in complicated decision-making problems. For this reason, the idea of the cubic intuitionistic fuzzy set (CIFS), which is described by two parts simultaneously, where one represents the membership degrees by an IVIFS and the other represents the membership degrees by an IFS, holds an eminent position. Henceforth, a CIFS is the hybrid set joined by both an IVIFS and an IFS. Clearly, the advantage of the CIFS is that it can contain substantially more data to express the IVIFS and IFS at the same time. As CIFSs have the ability to represent two-dimensional information in a single set and hence, it is a useful tool for handling the imprecise and ambiguous information during the decision-making process under the uncertain environment.

Keeping the advantages of the CIFS to express the uncertainty and fuzzy decision process more precisely and objectively, in this chapter, we study an MCDM problem under CIF setting and explore the structural characteristics of the set. For it, we first introduced some generalized operational laws based on the t-norm operations and hence based on these operations, some AOs named as generalized CIF weighted average (GCIFWA), generalized CIF ordered weighted average (GCIFOWA) and generalized CIF hybrid average (GCIFHA) are proposed. Various properties of these AOs are studied. Furthermore, efforts have been put forth to solve MCDM problems by considering these multi-dimensional data sets.

5.2 Generalized operational laws and AOs

In this section, generalized operational laws and AOs are defined for aggregating CIFNs. However, based on Section 2.3 of Chapter 2, we summarize the related definitions as below:

Definition 5.2.1. [80] A t-norm may be defined as $\mathcal{T} : [0,1] \times [0,1] \to [0,1]$, if it satisfies the conditions *viz.* boundary, monotonicity, commutativity and associativity, while a function $\mathcal{S}$ defined by $\mathcal{S}(\mathfrak{a}, \mathfrak{b}) = 1 - \mathcal{T}(1-\mathfrak{a}, 1-\mathfrak{b})$ is called t-conorm.

Definition 5.2.2. [80] A function $\mathcal{T}$(or $\mathcal{S}$) is known as Archimedian t-norm (or t-conorm) if it is a continuous function and $\mathcal{T}(\mathfrak{a}, \mathfrak{a}) < \mathfrak{a}$ (or $\mathcal{S}(\mathfrak{a}, \mathfrak{a}) < \mathfrak{a}$), for $\mathfrak{a} \in (0, 1)$. It is said to be a strict Archimedian t-norm (t-conorm) if it increases strictly in $(0,1)\times(0,1)$. Further, strict Archimedian t-norms ($\mathcal{T}$) and t-conorms ($\mathcal{S}$) can be expressed in form of continuous functions $u, y : (0,1] \to [0,\infty)$ respectively as $\mathcal{T}(\mathfrak{a},\mathfrak{b}) = y^{-1}\big(y(\mathfrak{a}) + y(\mathfrak{b})\big)$ and $\mathcal{S}(\mathfrak{a},\mathfrak{b}) = u^{-1}\big(u(\mathfrak{a}) + u(\mathfrak{b})\big)$ where y is a decreasing function with $y(1) = 0$; u is an increasing function with $u(0) = 0$ and $y(\mathfrak{a}) = u(1-\mathfrak{a})$.

Definition 5.2.3. Let $\mathcal{A}_1 = \left(\left(\left[\zeta_1^L,\zeta_1^U\right],\left[\vartheta_1^L,\vartheta_1^U\right]\right),(\zeta_1,\vartheta_1)\right)$, $\mathcal{A}_2 = \left(\left(\left[\zeta_2^L,\zeta_2^U\right],\left[\vartheta_2^L,\vartheta_2^U\right]\right),\right.$ $\left.(\zeta_2,\vartheta_2)\right)$ and $\mathcal{A} = \left(\left(\left[\zeta^L,\zeta^U\right],\left[\vartheta^L,\vartheta^U\right]\right),(\zeta,\vartheta)\right)$ be three CIFNs, then for any real number $\xi > 0$ we have

(i) $$\mathcal{A}_1 \oplus \mathcal{A}_2 = \left(\begin{pmatrix} \left[u^{-1}\left(u\left(\zeta_1^L\right)+u\left(\zeta_2^L\right)\right), u^{-1}\left(u\left(\zeta_1^U\right)+u\left(\zeta_2^U\right)\right)\right], \\ \left[y^{-1}\left(y\left(\vartheta_1^L\right)+y\left(\vartheta_2^L\right)\right), y^{-1}\left(y\left(\vartheta_1^U\right)+y\left(\vartheta_2^U\right)\right)\right] \end{pmatrix}, \begin{pmatrix} y^{-1}\left(y\left(\zeta_1\right)+y\left(\zeta_2\right)\right), \\ u^{-1}\left(u\left(\vartheta_1\right)+u\left(\vartheta_2\right)\right) \end{pmatrix}\right)$$

(ii) $$\mathcal{A}_1 \otimes \mathcal{A}_2 = \left(\begin{pmatrix} \left[y^{-1}\left(y\left(\zeta_1^L\right)+y\left(\zeta_2^L\right)\right), y^{-1}\left(y\left(\zeta_1^U\right)+y\left(\zeta_2^U\right)\right)\right], \\ \left[u^{-1}\left(u\left(\vartheta_1^L\right)+u\left(\vartheta_2^L\right)\right), u^{-1}\left(u\left(\vartheta_1^U\right)+u\left(\vartheta_2^U\right)\right)\right] \end{pmatrix}, \begin{pmatrix} u^{-1}\left(u\left(\zeta_1\right)+u\left(\zeta_2\right)\right), \\ y^{-1}\left(y\left(\vartheta_1\right)+y\left(\vartheta_2\right)\right) \end{pmatrix}\right)$$

(iii) $$\xi\mathcal{A} = \left(\begin{pmatrix} \left[u^{-1}\left(\xi u\left(\zeta^L\right)\right), u^{-1}\left(\xi u\left(\zeta^U\right)\right)\right], \\ \left[y^{-1}\left(\xi y\left(\vartheta^L\right)\right), y^{-1}\left(\xi y\left(\vartheta^U\right)\right)\right] \end{pmatrix}, \begin{pmatrix} y^{-1}\left(\xi y\left(\zeta\right)\right), \\ u^{-1}\left(\xi u\left(\vartheta\right)\right) \end{pmatrix}\right)$$

(iv) $$\mathcal{A}^{\xi} = \left(\begin{pmatrix} \left[y^{-1}\left(\xi y\left(\zeta^L\right)\right), y^{-1}\left(\xi y\left(\zeta^U\right)\right)\right], \\ \left[u^{-1}\left(\xi u\left(\vartheta^L\right)\right), u^{-1}\left(\xi u\left(\vartheta^U\right)\right)\right] \end{pmatrix}, \begin{pmatrix} u^{-1}\left(\xi u\left(\zeta\right)\right), \\ y^{-1}\left(\xi y\left(\vartheta\right)\right) \end{pmatrix}\right)$$

Theorem 5.2.1. For CIFNs, $\mathcal{A}_1 = \left(\left([\zeta_1^L,\zeta_1^U],[\vartheta_1^L,\vartheta_1^U]\right),(\zeta_1,\vartheta_1)\right)$, $\mathcal{A}_2 = \left(\left([\zeta_2^L,\zeta_2^U],[\vartheta_2^L,\vartheta_2^U]\right),\right.$ $\left.(\zeta_2,\vartheta_2)\right)$, $\mathcal{A} = \left(\left([\zeta^L,\zeta^U],[\vartheta^L,\vartheta^U]\right),(\zeta,\vartheta)\right)$ and $\xi > 0$ be real number; $\mathcal{A}_1 \oplus \mathcal{A}_2$, $\mathcal{A}_1 \otimes \mathcal{A}_2$, $\xi\mathcal{A}$ and $\mathcal{A}^{\xi}$ are also CIFNs.

Proof. Let $\mathcal{A}_3 = \mathcal{A}_1 \oplus \mathcal{A}_2 = \left(\left([\zeta_3^L,\zeta_3^U],[\vartheta_3^L,\vartheta_3^U]\right),(\zeta_3,\vartheta_3)\right)$. By using Definition 5.2.3, we obtain $\zeta_3^L = u^{-1}\left(u\left(\zeta_1^L\right)+u\left(\zeta_2^L\right)\right)$, $\zeta_3^U = u^{-1}\left(u\left(\zeta_1^U\right)+u\left(\zeta_2^U\right)\right)$, $\vartheta_3^L = y^{-1}\left(y\left(\vartheta_1^L\right)+y\left(\vartheta_2^L\right)\right)$, $\vartheta_3^U = y^{-1}\left(y\left(\vartheta_1^U\right)+ y\left(\vartheta_2^U\right)\right)$, $\zeta_3 = y^{-1}\left(y\left(\zeta_1\right)+y\left(\zeta_2\right)\right)$, $\vartheta_3 = u^{-1}\left(u\left(\vartheta_1\right)+u\left(\vartheta_2\right)\right)$. By the basic nature of u and v as mentioned in Definition 5.2.2, it is obvious that $0 \leq \zeta_3^L,\zeta_3^U,\vartheta_3^L,\vartheta_3^U,\zeta_3,\vartheta_3 \leq 1$. Furthur, as $\zeta_i^U + \vartheta_i^U \leq 1$, for $i = 1,2$, and from the fact that u is an increasing function and $u(x,z) = \mathcal{T}(1-x,1-z)$, it follows that:

$$\begin{aligned} \zeta_3^U + \vartheta_3^U &= u^{-1}\left(u\left(\zeta_1^U\right)+u\left(\zeta_2^U\right)\right) + y^{-1}\left(y\left(\vartheta_1^U\right)+y\left(\vartheta_2^U\right)\right) \\ &\leq u^{-1}\left(u\left(1-\vartheta_1^U\right)+u\left(1-\vartheta_2^U\right)\right) + y^{-1}\left(y\left(\vartheta_1^U\right)+y\left(\vartheta_2^U\right)\right) \end{aligned}$$

$$\begin{aligned} &\le 1 - y^{-1}\left(y\left(\vartheta_1^U\right) + y\left(\vartheta_2^U\right)\right) + y^{-1}\left(y\left(\vartheta_1^U\right) + y\left(\vartheta_2^U\right)\right) \\ &= 1 \end{aligned}$$

Thus, $\zeta_3^U + \vartheta_3^U \le 1$. Similarly, $0 \le \zeta_3 + \vartheta_3 \le 1$. Hence, $\mathcal{A}_3$ is a CIFN. In the similar manner, $\mathcal{A}_1 \otimes \mathcal{A}_2$, $\xi\mathcal{A}$ and $\mathcal{A}^\xi$ are also CIFNs. □

Theorem 5.2.2. For CIFNs $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}$, and for three real numbers $\xi_1, \xi_2, \xi > 0$ we have

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$.

(ii) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A}_2 \otimes \mathcal{A}_1$.

(iii) $\xi(\mathcal{A}_1 \oplus \mathcal{A}_2) = \xi\mathcal{A}_1 \oplus \xi\mathcal{A}_2$.

(iv) $(\mathcal{A}_1 \otimes \mathcal{A}_2)^\xi = \mathcal{A}_1^\xi \otimes \mathcal{A}_2^\xi$.

(v) $\xi_1\mathcal{A} \oplus \xi_2\mathcal{A} = (\xi_1 + \xi_2)\mathcal{A}$.

(vi) $\mathcal{A}^{\xi_1} \otimes \mathcal{A}^{\xi_2} = \mathcal{A}^{\xi_1+\xi_2}$.

Proof. We shall prove the parts (iii) and (v) and others can be proceeded likewise.

(iii) For $\xi > 0$, we have

$$\begin{aligned} &\xi(\mathcal{A}_1 \oplus \mathcal{A}_2) \\ = \;& \xi\left(\left(\begin{matrix} \left[u^{-1}\left(u\left(\zeta_1^L\right) + u\left(\zeta_2^L\right)\right), u^{-1}\left(u\left(\zeta_1^U\right) + u\left(\zeta_2^U\right)\right)\right], \\ \left[y^{-1}\left(y\left(\vartheta_1^L\right) + y\left(\vartheta_2^L\right)\right), y^{-1}\left(y\left(\vartheta_1^U\right) + y\left(\vartheta_2^U\right)\right)\right] \end{matrix}\right), \left(\begin{matrix} y^{-1}\left(y\left(\zeta_1\right) + y\left(\zeta_2\right)\right), \\ u^{-1}\left(u\left(\vartheta_1\right) + u\left(\vartheta_2\right)\right) \end{matrix}\right)\right) \\ = \;& \left(\left(\begin{matrix} \left[\begin{matrix} u^{-1}\left(\xi u\left(u^{-1}\left(u\left(\zeta_1^L\right) + u\left(\zeta_2^L\right)\right)\right)\right), \\ u^{-1}\left(\xi u\left(u^{-1}\left(u\left(\zeta_1^U\right) + u\left(\zeta_2^U\right)\right)\right)\right) \end{matrix}\right], \\ \left[\begin{matrix} y^{-1}\left(\xi y\left(y^{-1}\left(y\left(\vartheta_1^L\right) + y\left(\vartheta_2^L\right)\right)\right)\right), \\ y^{-1}\left(\xi y\left(y^{-1}\left(y\left(\vartheta_1^U\right) + y\left(\vartheta_2^U\right)\right)\right)\right) \end{matrix}\right] \end{matrix}\right), \left(\begin{matrix} y^{-1}\left(\xi y\left(y^{-1}\left(y\left(\zeta_1\right) + y\left(\zeta_2\right)\right)\right)\right), \\ u^{-1}\left(\xi u\left(u^{-1}\left(u\left(\vartheta_1\right) + u\left(\vartheta_2\right)\right)\right)\right) \end{matrix}\right)\right) \\ = \;& \left(\left(\begin{matrix} \left[\begin{matrix} u^{-1}\left(\xi\left(u\left(\zeta_1^L\right) + u\left(\zeta_2^L\right)\right)\right), \\ u^{-1}\left(\xi\left(u\left(\zeta_1^U\right) + u\left(\zeta_2^U\right)\right)\right) \end{matrix}\right], \\ \left[\begin{matrix} y^{-1}\left(\xi\left(y\left(\vartheta_1^L\right) + y\left(\vartheta_2^L\right)\right)\right), \\ y^{-1}\left(\xi\left(y\left(\vartheta_1^U\right) + y\left(\vartheta_2^U\right)\right)\right) \end{matrix}\right] \end{matrix}\right), \left(\begin{matrix} y^{-1}\left(\xi\left(y\left(\zeta_1\right) + y\left(\zeta_2\right)\right)\right), \\ u^{-1}\left(\xi\left(u\left(\vartheta_1\right) + u\left(\vartheta_2\right)\right)\right) \end{matrix}\right)\right) \end{aligned}$$

$$= \left(\begin{pmatrix} \begin{bmatrix} u^{-1}\left(u\left(u^{-1}\left(\xi u\left(\zeta_1^L\right)+\xi u\left(\zeta_2^L\right)\right)\right)\right), \\ u^{-1}\left(u\left(u^{-1}\left(\xi u\left(\zeta_1^U\right)+\xi u\left(\zeta_2^U\right)\right)\right)\right) \end{bmatrix}, \\ \begin{bmatrix} y^{-1}\left(y\left(y^{-1}\left(\xi y\left(\vartheta_1^L\right)+\xi y\left(\vartheta_2^L\right)\right)\right)\right), \\ y^{-1}\left(y\left(y^{-1}\left(\xi y\left(\vartheta_1^U\right)+\xi y\left(\vartheta_2^U\right)\right)\right)\right) \end{bmatrix} \end{pmatrix}, \begin{pmatrix} y^{-1}\left(y\left(y^{-1}\left(\xi y\left(\zeta_1\right)+\xi y\left(\zeta_2\right)\right)\right)\right), \\ u^{-1}\left(u\left(u^{-1}\left(\xi u\left(\vartheta_1\right)+\xi u\left(\vartheta_2\right)\right)\right)\right) \end{pmatrix} \right)$$

$$= \begin{pmatrix} \begin{pmatrix} \left[u^{-1}\left(\xi u\left(\zeta_1^L\right)\right), u^{-1}\left(\xi u\left(\zeta_1^U\right)\right)\right], \\ \left[y^{-1}\left(\xi y\left(\vartheta_1^L\right)\right), y^{-1}\left(\xi y\left(\vartheta_1^U\right)\right)\right] \end{pmatrix}, \\ \left(y^{-1}\left(\xi y\left(\zeta_1\right)\right), u^{-1}\left(\xi u\left(\vartheta_1\right)\right)\right) \end{pmatrix} \oplus \begin{pmatrix} \begin{pmatrix} \left[u^{-1}\left(\xi u\left(\zeta_2^L\right)\right), u^{-1}\left(\xi u\left(\zeta_2^U\right)\right)\right], \\ \left[y^{-1}\left(\xi y\left(\vartheta_2^L\right)\right), y^{-1}\left(\xi y\left(\vartheta_2^U\right)\right)\right] \end{pmatrix}, \\ \left(y^{-1}\left(\xi y\left(\zeta_2\right)\right), u^{-1}\left(\xi u\left(\vartheta_2\right)\right)\right) \end{pmatrix}$$

$$= \xi\mathcal{A}_1 \oplus \xi\mathcal{A}_2$$

(v) For $\xi_1, \xi_2 > 0$, we have

$$\xi_1\mathcal{A} \oplus \xi_2\mathcal{A}$$

$$= \begin{pmatrix} \begin{pmatrix} \left[u^{-1}\left(\xi_1 u\left(\zeta^L\right)\right), u^{-1}\left(\xi_1 u\left(\zeta^U\right)\right)\right], \\ \left[y^{-1}\left(\xi_1 y\left(\vartheta^L\right)\right), y^{-1}\left(\xi_1 y\left(\vartheta^U\right)\right)\right] \end{pmatrix}, \\ \left(y^{-1}\left(\xi_1 y\left(\zeta\right)\right), u^{-1}\left(\xi_1 u\left(\vartheta\right)\right)\right) \end{pmatrix} \oplus \begin{pmatrix} \begin{pmatrix} \left[u^{-1}\left(\xi_2 u\left(\zeta^L\right)\right), u^{-1}\left(\xi_2 u\left(\zeta^U\right)\right)\right], \\ \left[y^{-1}\left(\xi_2 y\left(\vartheta^L\right)\right), y^{-1}\left(\xi_2 y\left(\vartheta^U\right)\right)\right] \end{pmatrix}, \\ \left(y^{-1}\left(\xi_2 y\left(\zeta\right)\right), u^{-1}\left(\xi_2 u\left(\vartheta\right)\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \begin{bmatrix} u^{-1}\left(u\left(u^{-1}\left(\xi_1 u\left(\zeta^L\right)\right)\right)+u\left(u^{-1}\left(\xi_2 u\left(\zeta^L\right)\right)\right)\right), \\ u^{-1}\left(u\left(u^{-1}\left(\xi_1 u\left(\zeta^U\right)\right)\right)+u\left(u^{-1}\left(\xi_2 u\left(\zeta^U\right)\right)\right)\right) \end{bmatrix}, \\ \begin{bmatrix} y^{-1}\left(y\left(y^{-1}\left(\xi_1 y\left(\vartheta^L\right)\right)\right)+y\left(y^{-1}\left(\xi_2 y\left(\vartheta^L\right)\right)\right)\right), \\ y^{-1}\left(y\left(y^{-1}\left(\xi_1 y\left(\vartheta^U\right)\right)\right)+y\left(y^{-1}\left(\xi_2 y\left(\vartheta^U\right)\right)\right)\right) \end{bmatrix} \end{pmatrix}, \\ \begin{pmatrix} y^{-1}\left(y\left(y^{-1}\left(\xi_1 y\left(\zeta\right)\right)\right)+y\left(y^{-1}\left(\xi_2 y\left(\zeta\right)\right)\right)\right), \\ u^{-1}\left(u\left(u^{-1}\left(\xi_1 u\left(\vartheta\right)\right)\right)+u\left(u^{-1}\left(\xi_2 u\left(\vartheta\right)\right)\right)\right) \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \begin{bmatrix} u^{-1}\left(\left(\xi_1 u\left(\zeta^L\right)+\xi_2 u\left(\zeta^L\right)\right)\right), \\ u^{-1}\left(\left(\xi_1 u\left(\zeta^U\right)+\xi_2 u\left(\zeta^U\right)\right)\right) \end{bmatrix}, \begin{bmatrix} y^{-1}\left(\left(\xi_1 y\left(\vartheta^L\right)+\xi_2 y\left(\vartheta^L\right)\right)\right), \\ y^{-1}\left(\left(\xi_1 y\left(\vartheta^U\right)+\xi_2 y\left(\vartheta^U\right)\right)\right) \end{bmatrix} \end{pmatrix}, \\ \left(y^{-1}\left(\left(\xi_1 y\left(\zeta\right)+\xi_2 y\left(\zeta\right)\right)\right), u^{-1}\left(\left(\xi_1 u\left(\vartheta\right)+\xi_2 u\left(\vartheta\right)\right)\right)\right) \end{pmatrix}$$

$$= (\xi_1+\xi_2)\mathcal{A}$$

□

Theorem 5.2.3. For any three CIFNs $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}$, and scalar $\xi > 0$, we have

(i) $(\mathcal{A}^c)^\xi = (\xi\mathcal{A})^c$;

(ii) $\xi(\mathcal{A}^c) = (\mathcal{A}^\xi)^c$;

(iii) $\mathcal{A}_1^c \oplus \mathcal{A}_2^c = (\mathcal{A}_1 \otimes \mathcal{A}_2)^c$;

(iv) $\mathcal{A}_1^c \otimes \mathcal{A}_2^c = (\mathcal{A}_1 \oplus \mathcal{A}_2)^c$.

Proof. Using Definition 5.2.3, we have

$$
\begin{aligned}
(\mathcal{A}^c)^\xi &= \left(\begin{pmatrix} \left[y^{-1}\left(\xi y\left(\vartheta^L\right)\right), y^{-1}\left(\xi y\left(\vartheta^U\right)\right)\right], \\ \left[u^{-1}\left(\xi u\left(\zeta^L\right)\right), u^{-1}\left(\xi u\left(\zeta^U\right)\right)\right] \end{pmatrix}, \begin{pmatrix} u^{-1}\left(\xi u\left(\vartheta\right)\right), \\ y^{-1}\left(\xi y\left(\zeta\right)\right) \end{pmatrix} \right) = (\xi\mathcal{A})^c \\
\xi(\mathcal{A}^c) &= \left(\begin{pmatrix} \left[u^{-1}\left(\xi u\left(\vartheta^L\right)\right), u^{-1}\left(\xi u\left(\vartheta^U\right)\right)\right], \\ \left[y^{-1}\left(\xi y\left(\zeta^L\right)\right), y^{-1}\left(\xi y\left(\zeta^U\right)\right)\right] \end{pmatrix}, \begin{pmatrix} y^{-1}\left(\xi y\left(\vartheta\right)\right), \\ u^{-1}\left(\xi u\left(\zeta\right)\right) \end{pmatrix} \right) = (\mathcal{A}^\xi)^c \\
\mathcal{A}_1^c \oplus \mathcal{A}_2^c &= \begin{pmatrix} \begin{pmatrix} \left[u^{-1}\left(u\left(\vartheta_1^L\right) + u\left(\vartheta_2^L\right)\right), u^{-1}\left(u\left(\vartheta_1^U\right) + u\left(\vartheta_2^U\right)\right)\right], \\ \left[y^{-1}\left(y\left(\zeta_1^L\right) + y\left(\zeta_2^L\right)\right), y^{-1}\left(y\left(\zeta_1^U\right) + y\left(\zeta_2^U\right)\right)\right] \end{pmatrix}, \\ \left(y^{-1}\left(y\left(\vartheta_1\right) + y\left(\vartheta_2\right)\right), u^{-1}\left(u\left(\zeta_1\right) + u\left(\zeta_2\right)\right)\right) \end{pmatrix} = (\mathcal{A}_1 \otimes \mathcal{A}_2)^c \\
\mathcal{A}_1^c \otimes \mathcal{A}_2^c &= \begin{pmatrix} \begin{pmatrix} \left[y^{-1}\left(y\left(\vartheta_1^L\right) + y\left(\vartheta_2^L\right)\right), y^{-1}\left(y\left(\vartheta_1^U\right) + y\left(\vartheta_2^U\right)\right)\right], \\ \left[u^{-1}\left(u\left(\zeta_1^L\right) + u\left(\zeta_2^L\right)\right), u^{-1}\left(u\left(\zeta_1^U\right) + u\left(\zeta_2^U\right)\right)\right] \end{pmatrix}, \\ \left(u^{-1}\left(u\left(\vartheta_1\right) + u\left(\vartheta_2\right)\right), y^{-1}\left(y\left(\zeta_1\right) + y\left(\zeta_2\right)\right)\right) \end{pmatrix} = (\mathcal{A}_1 \oplus \mathcal{A}_2)^c
\end{aligned}
$$

□

Remark 5.2.1. The following special cases have been observed for some special values of ξ and $\mathcal{A}$

(i) If $\mathcal{A} = (([1,1],[0,0]),(0,1))$, then

$$
\xi\mathcal{A} = \begin{pmatrix} \begin{pmatrix} [u^{-1}(\xi u(1)), u^{-1}(\xi u(1))], \\ [y^{-1}(\xi y(0)), y^{-1}(\xi y(0))] \end{pmatrix}, \\ \left(y^{-1}(\xi y(0)), u^{-1}(\xi u(1))\right) \end{pmatrix} = (([1,1],[0,0]),(0,1))
$$

(ii) If $\mathcal{A} = (([0,0],[1,1]),(1,0))$, then

$$
\xi\mathcal{A} = \begin{pmatrix} \begin{pmatrix} [u^{-1}(\xi u(0)), u^{-1}(\xi u(0))], \\ [y^{-1}(\xi y(1)), y^{-1}(\xi y(1))] \end{pmatrix}, \\ \left(y^{-1}(\xi y(1)), u^{-1}(\xi u(0))\right) \end{pmatrix} = (([0,0],[1,1]),(1,0))
$$

(iii) If $\mathcal{A} = (([0,0],[0,0]),(0,0))$, then

$$\xi\mathcal{A} = \left(\begin{array}{l} \left(\begin{array}{l} [u^{-1}(\xi u(0)), u^{-1}(\xi u(0))], \\ [y^{-1}(\xi y(0)), y^{-1}(\xi y(0))] \end{array} \right), \\ \left(y^{-1}(\xi y(0)), u^{-1}(\xi u(0))\right) \end{array} \right) = (([0,0],[0,0]),(0,0))$$

(iv) If $\xi \to 0$, then

$$\xi\mathcal{A} = \left(\begin{array}{l} \left(\begin{array}{l} [u^{-1}(\xi u(\zeta^L)), u^{-1}(\xi u(\zeta^U))], \\ [y^{-1}(\xi y(\vartheta^L)), y^{-1}(\xi y(\vartheta^U))] \end{array} \right), \\ \left(y^{-1}(\xi y(\zeta)), u^{-1}(\xi u(\vartheta))\right) \end{array} \right) \to (([0,0],[1,1]),(1,0)).$$

(v) If $\xi \to \infty$, then

$$\xi\mathcal{A} = \left(\begin{array}{l} \left(\begin{array}{l} [u^{-1}(\xi u(\zeta^L)), u^{-1}(\xi u(\zeta^U))], \\ [y^{-1}(\xi y(\vartheta^L)), y^{-1}(\xi y(\vartheta^U))] \end{array} \right), \\ \left(y^{-1}(\xi y(\zeta)), u^{-1}(\xi u(\vartheta))\right) \end{array} \right) \to (([1,1],[0,0]),(0,1))$$

(vi) If $\xi = 1$, then

$$\xi\mathcal{A} = \left(\begin{array}{l} \left(\begin{array}{l} [u^{-1}(\xi u(\zeta^L)), u^{-1}(\xi u(\zeta^U))], \\ [y^{-1}(\xi y(\vartheta^L)), y^{-1}(\xi y(\vartheta^U))] \end{array} \right), \\ \left(y^{-1}(\xi y(\zeta)), u^{-1}(\xi u(\vartheta))\right) \end{array} \right) = \left(\left([\zeta^L,\zeta^U],[\vartheta^L,\vartheta^U]\right),(\zeta,\vartheta)\right)$$

5.3 Averaging Aggregation Operators

In this section, based on the above operational laws, some new averaging operators named as GCIFWA, GCIFOWA and GCIFHA have been proposed under CIF environment as follows.

5.3.1 GCIFWA operator

Definition 5.3.1. A GCIFWA operator is a mapping GCIFWA: $\Phi^n \to \Phi$ defined as

$$\text{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1\mathcal{A}_1 \oplus \omega_2\mathcal{A}_2 \oplus \ldots \oplus \omega_n\mathcal{A}_n \tag{5.1}$$

where Φ is the collections of CIFNs $\mathcal{A}_i (i = 1, 2, \ldots, n)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ such that $\omega_i > 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$.

Theorem 5.3.1. For CIFNs $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$, the value obtained by GCIFWA is again a CIFN which is given by

$$\text{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\begin{array}{c} \left(\begin{array}{c} \left[u^{-1}\left(\sum\limits_{i=1}^{n} \omega_i u\left(\zeta_i^L\right)\right), u^{-1}\left(\sum\limits_{i=1}^{n} \omega_i u\left(\zeta_i^U\right)\right) \right], \\ \left[y^{-1}\left(\sum\limits_{i=1}^{n} \omega_i y\left(\vartheta_i^L\right)\right), y^{-1}\left(\sum\limits_{i=1}^{n} \omega_i y\left(\vartheta_i^U\right)\right) \right] \end{array} \right), \\ \left(y^{-1}\left(\sum\limits_{i=1}^{n} \omega_i y\left(\zeta_i\right)\right), u^{-1}\left(\sum\limits_{i=1}^{n} \omega_i u\left(\vartheta_i\right)\right) \right) \end{array} \right) \tag{5.2}$$

Proof. Based on the operational laws for CIFNs as defined in Definition 5.2.3, we can directly obtain the result. □

Property 5.3.1. If the IFS argument in a CIFS i.e., $(\zeta_i, \vartheta_i) = (0, 0)$ $\forall i$, then the GCIFWA operator reduces to weighted averaging operator in IVIFS environment.

Proof. Since, $(\zeta_i, \vartheta_i) = (0, 0)$ and thus, Eq. (5.2) becomes

$$\begin{aligned} \text{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \left(\begin{array}{c} \left(\begin{array}{c} \left[u^{-1}\left(\sum\limits_{i=1}^{n} \omega_i u\left(\zeta_i^L\right)\right), u^{-1}\left(\sum\limits_{i=1}^{n} \omega_i u\left(\zeta_i^U\right)\right) \right], \\ \left[y^{-1}\left(\sum\limits_{i=1}^{n} \omega_i y\left(\vartheta_i^L\right)\right), y^{-1}\left(\sum\limits_{i=1}^{n} \omega_i y\left(\vartheta_i^U\right)\right) \right] \end{array} \right), \\ \left(y^{-1}\left(\sum\limits_{i=1}^{n} \omega_i y(0)\right), u^{-1}\left(\sum\limits_{i=1}^{n} \omega_i u(0)\right) \right) \end{array} \right) \\ &= \left(\begin{array}{c} \left[u^{-1}\left(\sum\limits_{i=1}^{n} \omega_i u\left(\zeta_i^L\right)\right), u^{-1}\left(\sum\limits_{i=1}^{n} \omega_i u\left(\zeta_i^U\right)\right) \right], \\ \left[y^{-1}\left(\sum\limits_{i=1}^{n} \omega_i y\left(\vartheta_i^L\right)\right), y^{-1}\left(\sum\limits_{i=1}^{n} \omega_i y\left(\vartheta_i^U\right)\right) \right] \end{array} \right) \end{aligned}$$

which is weighted averaging operator in IVIFS environment. □

Property 5.3.2. If the IFS argument i.e., $(\zeta_i, \vartheta_i) = (0, 0)$ and $\zeta_i^L = \zeta_i^U$ and $\vartheta_i^L = \vartheta_i^U$ $\forall i$, then the GCIFWA operator reduces to weighted operator in IFS environment.

Proof. Follows from Proposition 5.3.1. □

From Theorem 5.3.1, it has been observed that properties of boundedness, idempotency, monotonicity etc holds for the GCIFWA operator. These properties are demonstrated as below:

Property 5.3.3. If, for all i, $\mathcal{A}_i = \mathcal{A}$ where $\mathcal{A} = \left(\left(\left[\zeta^L, \zeta^U\right], \left[\vartheta^L, \vartheta^U\right]\right), (\zeta, \vartheta)\right)$, then

$$\mathrm{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$$

This property is called an Idempotency.

Proof. Since $\mathcal{A}_i = \mathcal{A}$ for all i, and $\sum_{i=1}^{n} \omega_i = 1$, then we have

$$\begin{aligned}
\mathrm{GCIFWA}(\mathcal{A}, \mathcal{A}, \ldots, \mathcal{A}) &= \left(\begin{array}{c} \left(\begin{array}{c} \left[u^{-1}\left(\sum_{i=1}^{n} \omega_i u\left(\zeta^L\right)\right), u^{-1}\left(\sum_{i=1}^{n} \omega_i u\left(\zeta^U\right)\right)\right], \\ \left[y^{-1}\left(\sum_{i=1}^{n} \omega_i y\left(\vartheta^L\right)\right), y^{-1}\left(\sum_{i=1}^{n} \omega_i y\left(\vartheta^U\right)\right)\right] \end{array} \right), \\ \left(y^{-1}\left(\sum_{i=1}^{n} \omega_i y(\zeta)\right), u^{-1}\left(\sum_{i=1}^{n} \omega_i u(\vartheta)\right) \right) \end{array} \right) \\
&= \left(\begin{array}{c} \left(\begin{array}{c} \left[u^{-1}\left(u(\zeta^L)\right), u^{-1}\left(u(\zeta^U)\right)\right], \\ \left[y^{-1}\left(y\left(\vartheta^L\right)\right), y^{-1}\left(y\left(\vartheta^U\right)\right)\right] \end{array} \right), \\ \left(y^{-1}\left(y\left(\zeta\right)\right), u^{-1}\left(u\left(\vartheta\right)\right) \right) \end{array} \right) \\
&= \left(\left(\left[\zeta^L, \zeta^U\right], \left[\vartheta^L, \vartheta^U\right]\right), (\zeta, \vartheta)\right) \\
&= \mathcal{A}
\end{aligned}$$

□

Property 5.3.4. Let $\mathcal{A}_i = \left(\left(\left[\zeta^L_{\mathcal{A}_i}, \zeta^U_{\mathcal{A}_i}\right], \left[\vartheta^L_{\mathcal{A}_i}, \vartheta^U_{\mathcal{A}_i}\right]\right), (\zeta_{\mathcal{A}_i}, \vartheta_{\mathcal{A}_i})\right)$ and $\mathcal{B}_i = \left(\left(\left[\zeta^L_{\mathcal{B}_i}, \zeta^U_{\mathcal{B}_i}\right], \left[\vartheta^L_{\mathcal{B}_i}, \vartheta^U_{\mathcal{B}_i}\right]\right), (\zeta_{\mathcal{B}_i}, \vartheta_{\mathcal{B}_i})\right)$ be two CIFNs where $(i = 1, 2, \ldots, n)$, such that $\mathcal{A}_i \leq \mathcal{B}_i$, then

$$\mathrm{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathrm{GCIFWA}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n)$$

This property is called Monotonicity.

Proof. For the sake of convenience, let us denote

$$u^{-1}\left(\sum_{i=1}^{n} \omega_i u\left(\zeta^L_{\mathcal{A}_i}\right)\right) = a; \quad u^{-1}\left(\sum_{i=1}^{n} \omega_i u\left(\zeta^U_{\mathcal{A}_i}\right)\right) = b; \quad y^{-1}\left(\sum_{i=1}^{n} \omega_i y\left(\vartheta^L_{\mathcal{A}_i}\right)\right) = c;$$

$$y^{-1}\left(\sum_{i=1}^{n}\omega_i y\left(\vartheta^U_{\mathcal{A}_i}\right)\right)=d;\quad y^{-1}\left(\sum_{i=1}^{n}\omega_i y\left(\zeta_{\mathcal{A}_i}\right)\right)=e;\quad u^{-1}\left(\sum_{i=1}^{n}\omega_i u\left(\vartheta_{\mathcal{A}_i}\right)\right)=f$$
$$u^{-1}\left(\sum_{i=1}^{n}\omega_i u\left(\zeta^L_{\mathcal{B}_i}\right)\right)=a';\quad u^{-1}\left(\sum_{i=1}^{n}\omega_i u\left(\zeta^U_{\mathcal{B}_i}\right)\right)=b';\quad y^{-1}\left(\sum_{i=1}^{n}\omega_i y\left(\vartheta^L_{\mathcal{B}_i}\right)\right)=c';$$
$$y^{-1}\left(\sum_{i=1}^{n}\omega_i y\left(\vartheta^U_{\mathcal{B}_i}\right)\right)=d';\quad y^{-1}\left(\sum_{i=1}^{n}\omega_i y\left(\zeta_{\mathcal{B}_i}\right)\right)=e';\quad u^{-1}\left(\sum_{i=1}^{n}\omega_i u\left(\vartheta_{\mathcal{B}_i}\right)\right)=f'$$

Also, $\mathcal{A}_i \subseteq \mathcal{B}_i$ for all i, then we have $\zeta^L_{\mathcal{A}_i} \le \zeta^L_{\mathcal{B}_i}$, $\zeta^U_{\mathcal{A}_i} \le \zeta^U_{\mathcal{B}_i}$, $\vartheta^L_{\mathcal{A}_i} \ge \vartheta^L_{\mathcal{B}_i}$, $\vartheta^U_{\mathcal{A}_i} \ge \vartheta^U_{\mathcal{B}_i}$, $\zeta_{\mathcal{A}_i} \ge \zeta_{\mathcal{B}_i}$, $\vartheta_{\mathcal{A}_i} \le \vartheta_{\mathcal{B}_i}$. Further, y and u are decreasing and increasing functions respectively, then we have, $a \le a'$; $b \le b'$; $c \ge c'$; $d \ge d'$; $e \ge e'$; $f \le f'$. Thus, by using the score function of CIFNs, we obtain

$$\begin{aligned}\mathcal{S}c\big(\mathrm{GCIFWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\big) &= \frac{a+b-c-d}{2}+(f-e)\\ &\le \frac{a'+b'-c'-d'}{2}+(f'-e')\\ &= \mathcal{S}c\big(\mathrm{GCIFWA}(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_n)\big)\end{aligned}$$

Hence, $\mathrm{GCIFWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \le \mathrm{GCIFWA}(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_n)$ □

Property 5.3.5. For a collection of CIFNs $\mathcal{A}_i$. If $\mathcal{A}^- = \Big(([\min_i \zeta^L_i, \min_i \zeta^U_i], [\max_i \vartheta^L_i, \max_i \vartheta^U_i]), \langle \max_i \zeta_i, \min_i \vartheta_i\rangle\Big)$ and $\mathcal{A}^+ = \Big(([\max_i \zeta^L_i, \max_i \zeta^U_i], [\min_i \vartheta^L_i, \min_i \vartheta^U_i]), \langle \min_i \zeta_i, \max_i \vartheta_i\Big)$, then

$$\mathcal{A}^- \le \mathrm{GCIFWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \le \mathcal{A}^+.$$

This property is called as Boundedness.

Proof. Since for all i, we have $\min_i \zeta^L_i \le \zeta^L_i \le \max_i \zeta^L_i$; $\min_i \zeta^U_i \le \zeta^U_i \le \max_i \zeta^U_i$; $\min_i \vartheta^L_i \le \vartheta^L_i \le \max_i \vartheta^L_i$; $\min_i \vartheta^U_i \le \vartheta^U_i \le \max_i \vartheta^U_i$; $\min_i \zeta_i \le \zeta_i \le \max_i \zeta_i$; $\min_i \vartheta_i \le \vartheta_i \le \max_i \vartheta_i$ and the generators y and u are decreasing and increasing functions respectively, then

$$u^{-1}\left(\sum_i \omega_i u\left(\min_i\left(\zeta^L_i\right)\right)\right) \le u^{-1}\left(\sum_i \omega_i u\left(\zeta^L_i\right)\right) \le u^{-1}\left(\sum_i \omega_i u\left(\max_i\left(\zeta^L_i\right)\right)\right);$$
$$u^{-1}\left(\sum_i \omega_i u\left(\min_i\left(\zeta^U_i\right)\right)\right) \le u^{-1}\left(\sum_i \omega_i u\left(\zeta^U_i\right)\right) \le u^{-1}\left(\sum_i \omega_i u\left(\max_i\left(\zeta^U_i\right)\right)\right);$$
$$y^{-1}\left(\sum_i \omega_i y\left(\max_i\left(\vartheta^L_i\right)\right)\right) \le y^{-1}\left(\sum_i \omega_i y\left(\vartheta^L_i\right)\right) \le y^{-1}\left(\sum_i \omega_i y\left(\min_i\left(\vartheta^L_i\right)\right)\right);$$
$$y^{-1}\left(\sum_i \omega_i y\left(\max_i\left(\vartheta^U_i\right)\right)\right) \le y^{-1}\left(\sum_i \omega_i y\left(\vartheta^U_i\right)\right) \le y^{-1}\left(\sum_i \omega_i y\left(\min_i\left(\vartheta^U_i\right)\right)\right);$$

$$y^{-1}\left(\sum_i \omega_i y\left(\max_i(\zeta_i)\right)\right) \le y^{-1}\left(\sum_i \omega_i y(\zeta_i)\right) \le y^{-1}\left(\sum_i \omega_i y\left(\min_i(\zeta_i)\right)\right);$$

$$\text{and}\quad u^{-1}\left(\sum_i \omega_i u\left(\min_i(\vartheta_i)\right)\right) \le u^{-1}\left(\sum_i \omega_i u(\vartheta_i)\right) \le u^{-1}\left(\sum_i \omega_i u\left(\max_i(\vartheta_i)\right)\right)$$

that is $\min_i(\zeta_i^L) \le u^{-1}\left(\sum_i \omega_i u(\zeta_i^L)\right) \le \max_i(\zeta_i^L)$; $\min_i(\zeta_i^U) \le u^{-1}\left(\sum_i \omega_i u(\zeta_i^U)\right) \le \max_i(\zeta_i^U)$; $\max_i(\vartheta_i^L) \le y^{-1}\left(\sum_i \omega_i y(\vartheta_i^L)\right) \le \min_i(\vartheta_i^L)$; $\max_i(\vartheta_i^U) \le y^{-1}\left(\sum_i \omega_i y(\vartheta_i^U)\right) \le \min_i(\vartheta_i^U)$; $\max_i(\zeta_i) \le y^{-1}\left(\sum_i \omega_i y(\zeta_i)\right) \le \min_i(\zeta_i)$; $\min_i(\vartheta_i) \le u^{-1}\left(\sum_i \omega_i u(\vartheta_i)\right) \le \max_i(\vartheta_i)$.

Therefore, $\mathcal{A}^- \le \text{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \le \mathcal{A}^+$.

□

Property 5.3.6. For CIFNs $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, and $\mathcal{B} = \left(\left(\left[\zeta_{\mathcal{B}}^L, \zeta_{\mathcal{B}}^U\right], \left[\vartheta_{\mathcal{B}}^L, \vartheta_{\mathcal{B}}^U\right]\right), (\zeta_{\mathcal{B}}, \vartheta_{\mathcal{B}})\right)$, we have

$$\text{GCIFWA}(\mathcal{A}_1 \oplus \mathcal{B}, \mathcal{A}_2 \oplus \mathcal{B}, \ldots, \mathcal{A}_n \oplus \mathcal{B}) = \text{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \oplus \mathcal{B}.$$

This property is called Shift Invariance.

Proof. Let $\mathcal{A}_i = \left(\left(\left[\zeta_i^L, \zeta_i^U\right], \left[\vartheta_i^L, \vartheta_i^U\right]\right), (\zeta_i, \vartheta_i)\right)$, where $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} &\mathcal{A}_i \oplus \mathcal{B} \\ =& \left(\begin{pmatrix} \left[u^{-1}\left(u(\zeta_i^L) + u(\zeta_{\mathcal{B}}^L)\right), u^{-1}\left(u(\zeta_i^U) + u(\zeta_{\mathcal{B}}^U)\right)\right], \\ \left[y^{-1}\left(y(\vartheta_i^L) + y(\vartheta_{\mathcal{B}}^L)\right), y^{-1}\left(y(\vartheta_i^U) + y(\vartheta_{\mathcal{B}}^U)\right)\right] \end{pmatrix}, \begin{pmatrix} y^{-1}\left(y(\zeta_i) + y(\zeta_{\mathcal{B}})\right), \\ u^{-1}\left(u(\vartheta_i) + u(\vartheta_{\mathcal{B}})\right) \end{pmatrix}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} &\text{GCIFWA}(\mathcal{A}_1 \oplus \mathcal{B}, \mathcal{A}_2 \oplus \mathcal{B}, \ldots, \mathcal{A}_n \oplus \mathcal{B}) \\ =& \begin{pmatrix} \begin{pmatrix} \left[u^{-1}\left(\sum_i \omega_i u\left(u^{-1}\left(u(\zeta_i^L) + u(\zeta_{\mathcal{B}}^L)\right)\right)\right), u^{-1}\left(\sum_i \omega_i u\left(u^{-1}\left(u(\zeta_i^U) + u(\zeta_{\mathcal{B}}^U)\right)\right)\right)\right], \\ \left[y^{-1}\left(\sum_i \omega_i y\left(y^{-1}\left(y(\vartheta_i^L) + y(\vartheta_{\mathcal{B}}^L)\right)\right)\right), y^{-1}\left(\sum_i \omega_i y\left(y^{-1}\left(y(\vartheta_i^U) + y(\vartheta_{\mathcal{B}}^U)\right)\right)\right)\right] \end{pmatrix}, \\ \left(y^{-1}\left(\sum_i \omega_i y\left(y^{-1}\left(y(\zeta_i) + y(\zeta_{\mathcal{B}})\right)\right)\right), u^{-1}\left(\sum_i \omega_i u\left(u^{-1}\left(u(\vartheta_i) + u(\vartheta_{\mathcal{B}})\right)\right)\right)\right) \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \left(\begin{array}{l} \left(\begin{array}{l} \left[u^{-1}\left(\sum_i \omega_i \left(u\left(\zeta_i^L\right)+u\left(\zeta_{\mathcal{B}}^L\right)\right)\right), u^{-1}\left(\sum_i \omega_i \left(u\left(\zeta_i^U\right)+u\left(\zeta_{\mathcal{B}}^U\right)\right)\right)\right], \\ \left[y^{-1}\left(\sum_i \omega_i \left(y\left(\vartheta_i^L\right)+y\left(\vartheta_{\mathcal{B}}^L\right)\right)\right), y^{-1}\left(\sum_i \omega_i \left(y\left(\vartheta_i^U\right)+y\left(\vartheta_{\mathcal{B}}^U\right)\right)\right)\right] \end{array} \right), \\ \left(y^{-1}\left(\sum_i \omega_i \left(y\left(\zeta_i\right)+y\left(\zeta_{\mathcal{B}}\right)\right)\right), u^{-1}\left(\sum_i \omega_i \left(u\left(\vartheta_i\right)+u\left(\vartheta_{\mathcal{B}}\right)\right)\right)\right) \end{array} \right) \\
&= \left(\begin{array}{l} \left(\begin{array}{l} \left[u^{-1}\left(\sum_i \omega_i u\left(\zeta_i^L\right)+u\left(\zeta_{\mathcal{B}}^L\right)\right), u^{-1}\left(\sum_i \omega_i u\left(\zeta_i^U\right)+u\left(\zeta_{\mathcal{B}}^U\right)\right)\right], \\ \left[y^{-1}\left(\sum_i \omega_i y\left(\vartheta_i^L\right)+y\left(\vartheta_{\mathcal{B}}^L\right)\right), y^{-1}\left(\sum_i \omega_i y\left(\vartheta_i^U\right)+y\left(\vartheta_{\mathcal{B}}^U\right)\right)\right] \end{array} \right), \\ \left(y^{-1}\left(\sum_i \omega_i y\left(\zeta_i\right)+y\left(\zeta_{\mathcal{B}}\right)\right), u^{-1}\left(\sum_i \omega_i u\left(\vartheta_i\right)+u\left(\vartheta_{\mathcal{B}}\right)\right)\right) \end{array} \right) \\
&= \left(\begin{array}{l} \left(\begin{array}{l} \left[u^{-1}\left(\sum_i \omega_i u\left(\zeta_i^L\right)\right), u^{-1}\left(\sum_i \omega_i u\left(\zeta_i^U\right)\right)\right], \\ \left[y^{-1}\left(\sum_i \omega_i y\left(\vartheta_i^L\right)\right), y^{-1}\left(\sum_i \omega_i y\left(\vartheta_i^U\right)\right)\right] \end{array} \right), \\ \left(y^{-1}\left(\sum_i \omega_i y\left(\zeta_i\right)\right), u^{-1}\left(\sum_i \omega_i u\left(\vartheta_i\right)\right)\right) \end{array} \right) \oplus \left(\begin{array}{l} \left(\begin{array}{l} \left[u^{-1}\left(u\left(\zeta_{\mathcal{B}}^L\right)\right), u^{-1}\left(u\left(\zeta_{\mathcal{B}}^U\right)\right)\right], \\ \left[y^{-1}\left(y\left(\vartheta_{\mathcal{B}}^L\right)\right), y^{-1}\left(y\left(\vartheta_{\mathcal{B}}^U\right)\right)\right] \end{array} \right), \\ \left(y^{-1}\left(y\left(\zeta_{\mathcal{B}}\right)\right), u^{-1}\left(u\left(\vartheta_{\mathcal{B}}\right)\right)\right) \end{array} \right) \\
&= \text{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \oplus \mathcal{B}
\end{aligned}
$$

□

Property 5.3.7. For any real number $\xi > 0$, the homogeneity property holds i.e.,

$$
\text{GCIFWA}(\xi\mathcal{A}_1, \xi\mathcal{A}_2, \ldots, \xi\mathcal{A}_n) = \xi\text{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)
$$

This property is called as Homogeneity.

Proof. Let $\mathcal{A}_i = \left(\left([\zeta_i^L, \zeta_i^U], [\vartheta_i^L, \vartheta_i^U]\right), (\zeta_i, \vartheta_i)\right)$, where $i = 1, 2, \ldots, n$ and $\xi > 0$ be the real number then we have,

$$
\xi\mathcal{A} = \left(\left(\begin{bmatrix} u^{-1}\left(\xi u\left(\zeta^L\right)\right), \\ u^{-1}\left(\xi u\left(\zeta^U\right)\right) \end{bmatrix}, \begin{bmatrix} y^{-1}\left(\xi y\left(\vartheta^L\right)\right), \\ y^{-1}\left(\xi y\left(\vartheta^U\right)\right) \end{bmatrix} \right), \begin{pmatrix} y^{-1}\left(\xi y\left(\zeta\right)\right), \\ u^{-1}\left(\xi u\left(\vartheta\right)\right) \end{pmatrix} \right)
$$

Thus,

$$\begin{aligned}
&\mathrm{GCIFWA}(\xi\mathcal{A}_1, \xi\mathcal{A}_2, \ldots, \xi\mathcal{A}_n) \\
= &\left(\begin{array}{l}\left(\left[\begin{array}{c} u^{-1}\left(\sum_i \omega_i u\left(u^{-1}\left(\xi u\left(\zeta_i^L\right)\right)\right)\right), \\ u^{-1}\left(\sum_i \omega_i u\left(u^{-1}\left(\xi u\left(\zeta_i^U\right)\right)\right)\right)\end{array}\right], \left[\begin{array}{c} y^{-1}\left(\sum_i \omega_i y\left(y^{-1}\left(\xi y\left(\vartheta_i^L\right)\right)\right)\right), \\ y^{-1}\left(\sum_i \omega_i y\left(y^{-1}\left(\xi y\left(\vartheta_i^U\right)\right)\right)\right)\end{array}\right]\right), \\ \left(y^{-1}\left(\sum_i \omega_i y\left(y^{-1}\left(\xi y\left(\zeta_i\right)\right)\right)\right), u^{-1}\left(\sum_i \omega_i u\left(u^{-1}\left(\xi u\left(\vartheta_i\right)\right)\right)\right)\right)\end{array}\right) \\
= &\left(\begin{array}{l}\left(\begin{array}{l}\left[u^{-1}\left(\sum_i \omega_i\left(\xi u\left(\zeta_i^L\right)\right)\right), u^{-1}\left(\sum_i \omega_i\left(\xi u\left(\zeta_i^U\right)\right)\right)\right], \\ \left[y^{-1}\left(\sum_i \omega_i\left(\xi y\left(\vartheta_i^L\right)\right)\right), y^{-1}\left(\sum_i \omega_i\left(\xi y\left(\vartheta_i^U\right)\right)\right)\right]\end{array}\right), \\ \left(y^{-1}\left(\sum_i \omega_i\left(\xi y\left(\zeta_i\right)\right)\right), u^{-1}\left(\sum_i \omega_i\left(\xi u\left(\vartheta_i\right)\right)\right)\right)\end{array}\right) \\
= &\left(\begin{array}{l}\left(\begin{array}{l}\left[u^{-1}\left(\xi u\left(u^{-1}\sum_i \omega_i \xi u\left(\zeta_i^L\right)\right)\right), u^{-1}\left(\xi u\left(u^{-1}\sum_i \omega_i \xi u\left(\zeta_i^U\right)\right)\right)\right], \\ \left[y^{-1}\left(\xi y\left(y^{-1}\sum_i \omega_i \xi y\left(\vartheta_i^L\right)\right)\right), y^{-1}\left(\xi y\left(y^{-1}\sum_i \omega_i \xi y\left(\vartheta_i^U\right)\right)\right)\right]\end{array}\right), \\ \left(y^{-1}\left(\xi y\left(y^{-1}\sum_i \omega_i \xi y\left(\zeta_i\right)\right)\right), u^{-1}\left(\xi u\left(u^{-1}\sum_i \omega_i \xi u\left(\vartheta_i\right)\right)\right)\right)\end{array}\right) \\
= &\ \xi\mathrm{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)
\end{aligned}$$

□

The GCIFWA operator, as discussed above, can be particularized by assigning different values to generator y as follows:

(i) If $y(t) = -\log(t)$, then Eq. (5.2) reduces to :

$$\begin{aligned}
&\mathrm{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\
= &\left(\left(\begin{array}{c}\left[1-\prod_{i=1}^n\left(1-\zeta_i^L\right)^{\omega_i}, 1-\prod_{i=1}^n\left(1-\zeta_i^U\right)^{\omega_i}\right], \\ \left[\prod_{i=1}^n\left(\vartheta_i^L\right)^{\omega_i}, \prod_{i=1}^n\left(\vartheta_i^U\right)^{\omega_i}\right]\end{array}\right), \left(\begin{array}{c}\prod_{i=1}^n\left(\zeta_i\right)^{\omega_i}, \\ 1-\prod_{i=1}^n\left(1-\vartheta_i\right)^{\omega_i}\end{array}\right)\right)
\end{aligned}$$

and is known as CIF Archimedean weighted averaging operator.

(ii) If $y(t) = \log\left(\frac{2-t}{t}\right)$, then Eq. (5.2) reduces to:

$$
\begin{aligned}
&\text{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\
&= \left(\begin{array}{l}
\left(\left[\frac{\prod_{i=1}^{n}\left(1+\zeta_i^L\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\zeta_i^L\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\zeta_i^L\right)^{\omega_i} + \prod_{i=1}^{n}\left(1-\zeta_i^L\right)^{\omega_i}}, \frac{\prod_{i=1}^{n}\left(1+\zeta_i^U\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\zeta_i^U\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\zeta_i^U\right)^{\omega_i} + \prod_{i=1}^{n}\left(1-\zeta_i^U\right)^{\omega_i}} \right], \right. \\
\left. \left[\frac{2\prod_{i=1}^{n}\left(\vartheta_i^L\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\vartheta_i^L\right)^{\omega_i} + \prod_{i=1}^{n}\left(\vartheta_i^L\right)^{\omega_i}}, \frac{2\prod_{i=1}^{n}\left(\vartheta_i^U\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\vartheta_i^U\right)^{\omega_i} + \prod_{i=1}^{n}\left(\vartheta_i^U\right)^{\omega_i}} \right] \right), \\
\left(\frac{2\prod_{i=1}^{n}\left(\zeta_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\zeta_i\right)^{\omega_i} + \prod_{i=1}^{n}\left(\zeta_i\right)^{\omega_i}}, \frac{\prod_{i=1}^{n}\left(1+\vartheta_i\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\vartheta_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\vartheta_i\right)^{\omega_i} + \prod_{i=1}^{n}\left(1-\vartheta_i\right)^{\omega_i}} \right)
\end{array} \right)
\end{aligned}
$$

and is called CIF Einstein averaging operator.

(iii) If $y(t) = \log\left(\frac{\gamma+(1-\gamma)t}{t}\right), \gamma \in (0, \infty)$ then the Eq. (5.2) reduces to:

$$
\begin{aligned}
&\text{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\
&= \left(\begin{array}{l}
\left(\left[\begin{array}{c} \frac{\prod_{i=1}^{n}\left(1+(\gamma-1)\zeta_i^L\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\zeta_i^L\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1)\zeta_i^L\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}\left(1-\zeta_i^L\right)^{\omega_i}}, \\ \frac{\prod_{i=1}^{n}\left(1+(\gamma-1)\zeta_i^U\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\zeta_i^U\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1)\zeta_i^U\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}\left(1-\zeta_i^U\right)^{\omega_i}} \end{array} \right], \left[\begin{array}{c} \frac{\gamma\prod_{i=1}^{n}\left(\vartheta_i^L\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1)(1-\vartheta_i^L)\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}\left(\vartheta_i^L\right)^{\omega_i}}, \\ \frac{\gamma\prod_{i=1}^{n}\left(\vartheta_i^U\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1)(1-\vartheta_i^U)\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}\left(\vartheta_i^U\right)^{\omega_i}} \end{array} \right] \right) \\
\left(\frac{\gamma\prod_{i=1}^{n}\left(\zeta_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1)(1-\zeta_i)\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}\left(\zeta_i\right)^{\omega_i}}, \frac{\prod_{i=1}^{n}\left(1+(\gamma-1)\vartheta_i\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\vartheta_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1)\vartheta_i\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}\left(1-\vartheta_i\right)^{\omega_i}} \right)
\end{array} \right)
\end{aligned}
$$

and is called as CIF Hamacher weighted averaging operator.

5.3.2 GCIFOWA operator

Definition 5.3.2. A GCIFOWA is a mapping defined as GCIFOWA: $\Phi^n \to \Phi$ on a collection of CIFNs $\mathcal{A}_i, (i = 1, 2, \ldots, n)$ as follows

$$\text{GCIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1\mathcal{A}_{\psi(1)} \oplus \omega_2\mathcal{A}_{\psi(2)} \oplus \ldots \oplus \omega_n\mathcal{A}_{\psi(n)} \tag{5.3}$$

where, ψ is a permutation of $(1, 2, \ldots, n)$, such that, $\mathcal{A}_{\psi(i-1)} \geq \mathcal{A}_{\psi(i)}$ for $i = 2, 3, \ldots, n$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is its weight vector, such that, $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$. Moreover, the i^{th} largest CIFN among $\mathcal{A}_i$'s is $\mathcal{A}_{\psi(i)}$.

Theorem 5.3.2. For CIFNs $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$, the value obtained by utilizing GCIFOWA operator is again a CIFN and is given by

$$\text{GCIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$
$$= \left(\begin{array}{c} \left(\begin{array}{c} \left[u^{-1}\left(\sum_{i=1}^{n} \omega_i u\left(\zeta^L_{\psi(i)}\right)\right), u^{-1}\left(\sum_{i=1}^{n} \omega_i u\left(\zeta^U_{\psi(i)}\right)\right) \right], \\ \left[y^{-1}\left(\sum_{i=1}^{n} \omega_i y\left(\vartheta^L_{\psi(i)}\right)\right), y^{-1}\left(\sum_{i=1}^{n} \omega_i y\left(\vartheta^U_{\psi(i)}\right)\right) \right] \end{array} \right), \\ \left(y^{-1}\left(\sum_{i=1}^{n} \omega_i y\left(\zeta_{\psi(i)}\right)\right), u^{-1}\left(\sum_{i=1}^{n} \omega_i u\left(\vartheta_{\psi(i)}\right)\right) \right) \end{array} \right) \tag{5.4}$$

Proof. Similar to Theorem 5.3.1, so we omit here. □

Furthermore, it has been noticed that the GCIFOWA operator beholds the properties such that idempotency, monotonicity, boundedness, shift invariance as well as homogeneity in same manner as that of GCIFWA operator.

Property 5.3.8. For a collection of CIFNs $\mathcal{A}_i$, we have the following:

(i) If $\omega = (1, 0, \ldots, 0)^T$ then $\text{GCIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \max\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$;

(ii) If $\omega = (0, 0, \ldots, 1)^T$ then $\text{GCIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \min\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$;

(iii) If $\omega_i = 1$ and $\omega_j = 0 (j \neq i)$, then $\text{GCIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}_{\psi(i)}$ where $\mathcal{A}_{\psi(i)}$ is the i^{th} largest of $\mathcal{A}_i$.

5.3.3 GCIFHA operator

Definition 5.3.3. For CIFNs $\mathcal{A}_i$, $(i = 1, 2, \ldots, n)$; the operator GCIFHA: $\Phi^n \to \Phi$, is given as

$$\text{GCIFHA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1 \dot{\mathcal{A}}_{\psi(1)} \oplus \omega_2 \dot{\mathcal{A}}_{\psi(2)} \oplus \ldots \oplus \omega_n \dot{\mathcal{A}}_{\psi(n)} \tag{5.5}$$

where, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector, such that, $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$ and i^{th} largest of $\dot{\mathcal{A}}_i$'s $(\dot{\mathcal{A}}_i = n\delta_i \mathcal{A}_i)$ is $\dot{\mathcal{A}}_{\psi(i)}$, where n is the number of CIFNs and $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^T$ is the vector corresponding to $\mathcal{A}_i$ with $\delta_i > 0$ and $\sum_{i=1}^{n} \delta_i = 1$.

Theorem 5.3.3. For a collection of CIFNs $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$, the value obtained by utilizing GCIFHA operator is again a CIFN. It is given by

$$\begin{aligned} &\text{GCIFHA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ = &\left(\begin{array}{l} \left(\begin{array}{l} \left[u^{-1}\left(\sum_{i=1}^{n} \omega_i u\left(\dot{\zeta}^L_{\psi(i)}\right)\right), u^{-1}\left(\sum_{i=1}^{n} \omega_i u\left(\dot{\zeta}^U_{\psi(i)}\right)\right) \right], \\ \left[y^{-1}\left(\sum_{i=1}^{n} \omega_i y\left(\dot{\vartheta}^L_{\psi(i)}\right)\right), y^{-1}\left(\sum_{i=1}^{n} \omega_i y\left(\dot{\vartheta}^U_{\psi(i)}\right)\right) \right] \end{array} \right), \\ \left(y^{-1}\left(\sum_{i=1}^{n} \omega_i y\left(\dot{\zeta}_{\psi(i)}\right)\right), u^{-1}\left(\sum_{i=1}^{n} \omega_i u\left(\dot{\vartheta}_{\psi(i)}\right)\right) \right) \end{array} \right) \end{aligned} \tag{5.6}$$

Proof. Similar to Theorem 5.3.1. □

Similar to the previously proposed GCIFWA and GCIFOWA, this GCIFHA operator follows the bounded, monotonic, idempotent, shift invariance and homogeneity properties.

Theorem 5.3.4. For $\omega = \big(1/n, 1/n \ldots, 1/n\big)^T$, the operator GCIFHA get reduced to GCIFWA.

Proof. As $\dot{\mathcal{A}}_{\psi(i)} = n\delta_i \mathcal{A}_i$ and $\omega = (1/n, 1/n, \ldots, 1/n)^T$, thus $\omega_i \dot{\mathcal{A}}_{\psi(i)} = \delta_i \mathcal{A}_i$ and hence

$$\begin{aligned} \text{GCIFHA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \omega_1 \dot{\mathcal{A}}_{\psi(1)} \oplus \omega_2 \dot{\mathcal{A}}_{\psi(2)} \oplus \ldots \oplus \omega_n \dot{\mathcal{A}}_{\psi(n)} \\ &= \delta_1 \mathcal{A}_1 \oplus \delta_2 \mathcal{A}_2 \oplus \ldots \oplus \delta_n \mathcal{A}_n \\ &= \text{GCIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \end{aligned}$$

□

Theorem 5.3.5. For $\delta = (1/n, 1/n, \ldots, 1/n)^T$, GCIFHA becomes GCIFOWA operator.

Proof. Follows from Theorem 5.3.4. □

5.4 Decision making approach based on the proposed operators

Decision-making approach provides us with the facility of aligning our study to the real world. It is a useful approach to find the best alternative under a set of some feasible criterion. The general description of MCDM problem is same as Section 2.5 of Chapter 2. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of the criteria such that $\omega_j > 0$ and $\sum\limits_{j=1}^{n} \omega_j = 1$. Then, we have presented an efficient approach for solving the decision making problems under CIFN environment which involves the following steps.

Step 1: Arrange the rating values of each alternative in the form of CIFNs $\mathcal{A}_{ij} = \Big(\Big([\zeta_{ij}^L, \zeta_{ij}^U], [\vartheta_{ij}^L, \vartheta_{ij}^U]\Big), (\zeta_{ij}, \vartheta_{ij})\Big) : i = 1, 2, \ldots, m\,;\, j = 1, 2, \ldots, n$ as a decision matrix $\mathcal{M}$ given by

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \ldots & \mathcal{A}_{1n} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \ldots & \mathcal{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{m1} & \mathcal{A}_{m2} & \ldots & \mathcal{A}_{mn} \end{pmatrix} \end{array} \tag{5.7}$$

Step 2: Convert the cost type criteria into the benefit type by:

$$r_{ij} = \begin{cases} \Big(\Big(\Big[\zeta_{ij}^L, \zeta_{ij}^U\Big], \Big[\vartheta_{ij}^L, \vartheta_{ij}^U\Big]\Big), (\zeta_{ij}, \vartheta_{ij})\Big) & ;\text{if benefit type criteria} \\ \Big(\Big(\Big[\vartheta_{ij}^L, \vartheta_{ij}^U\Big], \Big[\zeta_{ij}^L, \zeta_{ij}^U\Big]\Big), (\vartheta_{ij}, \zeta_{ij})\Big) & ;\text{if cost type criteria} \end{cases} \tag{5.8}$$

Step 3: Utilize either GCIFWA, GCIFOWA or GCIFHA operator to get the aggregated value $r_i = \big(\big([\zeta_i^L, \zeta_i^U], [\vartheta_i^L, \vartheta_i^U]\big), (\zeta_i, \vartheta_i)\big)$ of the alternative $\mathcal{V}_i$, $(i = 1, 2, \ldots, m)$ such as

(a) by GCIFWA operator

$$\begin{aligned} r_i &= \text{GCIFWA}(r_{i1}, r_{i2}, \ldots, r_{in}) \\ &= \left(\begin{array}{c} \left(\begin{array}{c} \left[u^{-1}\left(\sum\limits_{j=1}^{n} \omega_j u\left(\zeta_{ij}^L\right)\right), u^{-1}\left(\sum\limits_{j=1}^{n} \omega_j u\left(\zeta_{ij}^U\right)\right) \right], \\ \left[y^{-1}\left(\sum\limits_{j=1}^{n} \omega_j y\left(\vartheta_{ij}^L\right)\right), y^{-1}\left(\sum\limits_{j=1}^{n} \omega_j y\left(\vartheta_{ij}^U\right)\right) \right] \end{array} \right), \\ \left(y^{-1}\left(\sum\limits_{j=1}^{n} \omega_j y\left(\zeta_{ij}\right)\right), u^{-1}\left(\sum\limits_{j=1}^{n} \omega_j u\left(\vartheta_{ij}\right)\right) \right) \end{array} \right) \end{aligned} \tag{5.9}$$

(b) by GCIFOWA operator

$$
\begin{aligned}
r_i &= \text{GCIFOWA}(r_{\psi(i1)}, r_{\psi(i2)}, \ldots, r_{\psi(in)}) \\
&= \left(\begin{array}{l} \left(\begin{array}{l} \left[u^{-1}\left(\sum_{j=1}^{n}\omega_j u\left(\zeta^L_{\psi(ij)}\right)\right), u^{-1}\left(\sum_{j=1}^{n}\omega_j u\left(\zeta^U_{\psi(ij)}\right)\right)\right], \\ \left[y^{-1}\left(\sum_{j=1}^{n}\omega_j y\left(\vartheta^L_{\psi(ij)}\right)\right), y^{-1}\left(\sum_{j=1}^{n}\omega_j y\left(\vartheta^U_{\psi(ij)}\right)\right)\right] \end{array} \right), \\ \left(y^{-1}\left(\sum_{j=1}^{n}\omega_j y\left(\zeta_{\psi(ij)}\right)\right), u^{-1}\left(\sum_{j=1}^{n}\omega_j u\left(\vartheta_{\psi(ij)}\right)\right)\right) \end{array} \right)
\end{aligned} \tag{5.10}
$$

(c) by GCIFHA operator

$$
\begin{aligned}
r_i &= \text{GCIFHA}(\dot{r}_{\psi(i1)}, \dot{r}_{\psi(i2)}, \ldots, \dot{r}_{\psi(in)}) \\
&= \left(\begin{array}{l} \left(\begin{array}{l} \left[u^{-1}\left(\sum_{j=1}^{n}\omega_j u\left(\dot{\zeta}^L_{\psi(ij)}\right)\right), u^{-1}\left(\sum_{j=1}^{n}\omega_j u\left(\dot{\zeta}^U_{\psi(ij)}\right)\right)\right], \\ \left[y^{-1}\left(\sum_{j=1}^{n}\omega_j y\left(\dot{\vartheta}^L_{\psi(ij)}\right)\right), y^{-1}\left(\sum_{j=1}^{n}\omega_j y\left(\dot{\vartheta}^U_{\psi(ij)}\right)\right)\right] \end{array} \right), \\ \left(y^{-1}\left(\sum_{j=1}^{n}\omega_j y\left(\dot{\zeta}_{\psi(ij)}\right)\right), u^{-1}\left(\sum_{j=1}^{n}\omega_j u\left(\dot{\vartheta}_{\psi(ij)}\right)\right)\right) \end{array} \right)
\end{aligned} \tag{5.11}
$$

Step 4: Compute the score values of the overall aggregated values $r_i (i = 1, 2, \ldots, m)$ by using equation

$$\mathcal{S}c(r_i) = \frac{\zeta_i^L + \zeta_i^U - \vartheta_i^L - \vartheta_i^U}{2} + (\vartheta_i - \zeta_i).$$

If for ant two indices i_1, i_2 we have $\mathcal{S}c(r_{i_1}) = \mathcal{S}c(r_{i_2})$, then compute accuracy values as

$$\mathcal{H}(r_i) = \frac{\zeta_i^L + \zeta_i^U + \vartheta_i^L + \vartheta_i^U}{2} + (\vartheta_i + \zeta_i)$$

Step 5: Rank all the alternatives based on the descending order of the score values and select the most desirable one(s).

5.5 Illustrative Example

In this section, a numerical example associated with the case of disaster-management is taken to test the efficiency of the proposed method.

5.5.1 Case study

Natural hazards and calamities are beyond man's control. One can either take the preventive measures before its occurrence or can ensure less damage by formulating an efficient disaster management strategy so as to avoid loss of human life and other necessary resources. Consider the Indian subcontinent where floods hit several states and lead to huge loss of life and property. Focusing on the disaster management, related particularly to four Indian States, taken as alternatives $\mathcal{V}_i, (i = 1, 2, 3, 4)$, *viz.*, ("Bihar"; "Assam"; "Gujrat" and "Arunachal Pradesh") which were largely devastated during first half of the year 2017, suppose the Government of India is trying to make an optimal decision on allocating funds to these four states. When the whole situation was analyzed for major areas of fund allocation, it was found that money should be allocated in such a way that three major factors represented by $\mathfrak{B}_j, (j = 1, 2, 3)$ namely: "Food scarcity", "Number of people rescued" and "Lack of infrastructure reconstruction facilities" needed to be coped up. Suppose, different factors are prioritized state-wise in accordance to the weight vector $\omega = (0.30, 0.38, 0.32)^T$ and $\delta = (0.48, 0.29, 0.23)^T$. Thus, the aim of the problem is to determine the order in which states should be allocated the relief fund.

In order to do so, concerned authority person of the Government has constituted a committee which has taken the whole responsibility for finding the possible state(s). For it, they have hired an expert who have investigated these states and rate their information with the given three factors in terms of the CIFN. Based on the information, the steps of the proposed method are executed as follows:

Step 1: The rating values towards each alternative are summarized in Table 5.1 under CIF environment.

Step 2: The values of the cost criteria $\mathfrak{B}_1$ and $\mathfrak{B}_3$ are changed to the benefit type by using Eq. (5.8) and result is summarized in Table 5.2.

Step 3: Without loss of generality, we have taken $y(t) = -\log(t)$ and aggregate the different preferences into collective values $r_i, (i = 1, 2, 3, 4)$ as

(i) by using GCIFWA operator, as given in Eq. (5.9), we get

$$\begin{aligned} r_1 &= (([0.2122, 0.2758], [0.2718, 0.3537]), (0.1892, 0.3000)); \\ r_2 &= (([0.2230, 0.2794], [0.2471, 0.3188]), (0.3586, 0.1701)); \\ r_3 &= (([0.7816, 0.8372], [0.1000, 0.1397]), (0.1072, 0.7959)); \\ r_4 &= (([0.2088, 0.2970], [0.2128, 0.2821]), (0.4178, 0.1693)) \end{aligned}$$

(ii) by using GCIFOWA operator, as given in Eq. (5.10), we get

$$\begin{aligned} r_1 &= (([0.2019, 0.2665], [0.2620, 0.3426]), (0.2028, 0.3000)); \\ r_2 &= (([0.2175, 0.2743], [0.2471, 0.3182]), (0.3586, 0.1697)); \\ r_3 &= (([0.7812, 0.8377], [0.1000, 0.1378]), (0.1056, 0.7860)); \\ r_4 &= (([0.2088, 0.2970], [0.2128, 0.2821]), (0.4178, 0.1693)) \end{aligned}$$

(iii) by using GCIFHA operator, as given in Eq. (5.11), we get

$$\begin{aligned} r_1 &= (([0.2435, 0.3037], [0.2925, 0.3800]), [0.1499, 0.3081)); \\ r_2 &= (([0.2620, 0.3166], [0.2539, 0.3297]), [0.3642, 0.1776)); \\ r_3 &= (([0.7996, 0.8527], [0.0928, 0.1325]), [0.1025, 0.8185)); \\ r_4 &= (([0.2470, 0.3334], [0.2204, 0.2919]), [0.3923, 0.1759)) \end{aligned}$$

Step 4: The score values of the aggregate numbers obtained during Step 3 are obtained as

(i) $\mathcal{S}c(r_1) = 0.0421$, $\mathcal{S}c(r_2) = -0.2203$, $\mathcal{S}c(r_3) = 1.3783$ and $\mathcal{S}c(r_4) = -0.2431$.

(ii) $\mathcal{S}c(r_1) = 0.0291$, $\mathcal{S}c(r_2) = -0.2256$, $\mathcal{S}c(r_3) = 1.3709$ and $\mathcal{S}c(r_4) = -0.2431$.

(iii) $\mathcal{S}c(r_1) = 0.0956$, $\mathcal{S}c(r_2) = -0.1891$, $\mathcal{S}c(r_3) = 1.4295$ and $\mathcal{S}c(r_4) = -0.1823$.

Step 5: Since $\mathcal{S}c(r_3) > \mathcal{S}c(r_1) > \mathcal{S}c(r_2) > \mathcal{S}c(r_4)$ by GCIFWA and GCIFOWA operator while $\mathcal{S}c(r_3) > \mathcal{S}c(r_1) > \mathcal{S}c(r_4) > \mathcal{S}c(r_2)$ by other and hence the ranking order of the alternative corresponding to its are $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$ and $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$, where "$\succ$" refers to "preferred to". Thus, the state "Gujrat" needs maximum fund allocation whereas the state "Arunachal Pradesh's" and "Assam" needs can be fulfilled with less amount of allocated money respectively.

On the other hand, instead of taking a specific generator function $y(t) = -\log(t)$, other generators also have been taken in the Step 3 of the proposed approach. Then, by applying the steps, the results corresponding to it are summarized in Table 5.3 along with the ranking order of the alternatives. From this table, it has been observed that the best alternative remains same.

5.5.2 Comparative studies under IVIFS environment

To compare the results of the proposed method with some of the existing theories under the IVIF environment, we set the fuzzy judgments of the CIFNs as zero and hence the obtained information reduces to IVIFS. Then, on this transformed data, we apply the several existing existing methods [22, 23, 25, 44, 95, 114, 131, 161, 162, 164, 171, 180–182, 199] to find the most suitable alternative(s) and results are represented in Table 5.4 along with the ranking order. From these results, it has been seen that the best alternative remains $\mathcal{V}_3$. But the worst alternative is changed for all. This is due to the fact that during these existing studies they have taken the preferences of the alternative initially only. In other words, these existing theories cannot deal with the problems where the evaluator or decision maker have to consider the falsity degree corresponding to his earlier assigned truth degree ranging over an interval. Thus, if an experiment has been conducted by completely ignoring the second judgments of the preferences then the worst alternative is $\mathcal{V}_1$ and the alternative $\mathcal{V}_4$ is superior to $\mathcal{V}_2$. In contrast to this, by taking all the features of the decision maker and represented in terms of CIFSs then the best alternative remains $\mathcal{V}_3$ but $\mathcal{V}_4$ is the worst alternatives and preferences of the remaining alternatives is in order of $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_4$. Hence, we conclude that under the only IVIFSs environment, the best alternative coincides with the proposed one while the other alternatives are completely altered and hence it will lead to the different decisions. Thus, this CIFS environment increases the level of precision by enhancing the scope of the membership and non-membership intervals by considering an IFS membership values corresponding to it. Therefore, based on the decision makers preferences in terms of aggregation operators and the aggregation environments used, the different ranking strategies may have used according to their own desired goals.

5.5.3 Validity Test

The following three test criteria are useful to validate the MCDM method [166].

Test criterion 1: "If we replace the rating values of non-optimal alternative with worse alternative then the best alternative should not change, provided the relative weighted criteria remains unchanged."

Test criterion 2: "Method should possess transitive nature."

Test criterion 3: "When a given problem is decomposed into smaller ones and the same MCDM method has been applied, then the combined ranking of the alternatives should be identical to the ranking of un-decomposed one."

Validity check with criterion 1

The ranking order of the given alternative is found as $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$ corresponding to GCIFWA operator. To test the analogous nature of proposed method under test criterion 1, the values of non-optimal alternative say, $\mathcal{V}_2$ is replaced with its arbitrary worst rating as $\mathcal{V}_2' = \{(\mathfrak{B}_1, ([0.29, 0.36], [0.19, 0.20]), (0.50, 0.10))\}$; $\{(\mathfrak{B}_2$, ([0.09 , 0.11], [0.15 , 0.30]), (0.40, 0.05))}; $\{\mathfrak{B}_3, ([0.35, 0.45], [0.30, 0.35]), (0.45, 0.20)\}$. Then, by following the steps of the proposed method corresponding to $y(t) = -\log(t)$, we get the final score values of the alternatives as $\mathcal{S}c(r_1) = 0.0421$, $\mathcal{S}c(r_2') = 0.3803$, $\mathcal{S}c(r_3) = 1.3783$ and $\mathcal{S}c(r_4) = -0.2431$. Thus, the ranking order of the alternatives is $\mathcal{V}_3 \succ \mathcal{V}_2' \succ \mathcal{V}_1 \succ \mathcal{V}_4$ and coincides with original one.

Validity check with criteria 2 and 3

Under this test, we decompose original MCDM problem into the three smaller MCDM problem which consists of $\{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$, $\{\mathcal{V}_1, \mathcal{V}_3, \mathcal{V}_2\}$ and $\{\mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_1\}$ alternatives. Now, the proposed method is applied on each parts and final ranking orders are obtained as $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$, $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2$ and $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$, respectively. By combining we get $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$ which is similar to original problem ranking order. Thus, proposed method follow transitive property and hence satisfy test criteria 2 and 3.

5.6 Conclusion

This key contributions of this work are listed below:

1) This chapter concluded the inherited characteristics of CIFS in which preference corresponding to an element is expressed by means of IVIFS and IFS which shows degree of the disagreement (in the form of IFS values) corresponding to the agreed interval region (in form of IVIFS). However, this remarkable feature is studied further for the generalized operational laws.

2) In light of the proposed generalized operational laws based on t-norm and t-conorm, the presented study has given insight to AOs formulated into a generalized format and their relevant desirable properties have been thoroughly investigated.

3) Additionally, a multi-criteria decision-making model is supported by constructing a novel approach on the basis of proposed AOs.

4) Adding value to above stated points, an illustrative example demonstrating the practical relevance of the proposed approach is given. Apart from it, strong foundation of the obtained results is justified by carrying on comparison studies on the existing environments as well as the practical stability of the proposed approach has been shown by conducting a suitable validity test on the data entities. From these studies, it has been analyzed that proposed operators can handle the problems in a more profitable ways than the existing studies based only on IVIFS or IFS theories.

Table 5.1: Information related to the alternative in terms of CIF environment

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$(([0.15, 0.20], [0.10, 0.20]), (0.30, 0.40))$	$(([0.30, 0.35], [0.40, 0.50]), (0.10, 0.30))$	$(([0.30, 0.40], [0.20, 0.25]), (0.30, 0.20))$
$\mathcal{V}_2$	$(([0.30, 0.35], [0.15, 0.22]), (0.20, 0.40))$	$(([0.10, 0.15], [0.18, 0.25]), (0.30, 0.10))$	$(([0.30, 0.39], [0.40, 0.45]), (0.22, 0.40))$
$\mathcal{V}_3$	$(([0.10, 0.15], [0.70, 0.75]), (0.80, 0.10))$	$(([0.80, 0.85], [0.10, 0.15]), (0.12, 0.85))$	$(([0.10, 0.12], [0.82, 0.88]), (0.70, 0.10))$
$\mathcal{V}_4$	$(([0.20, 0.28], [0.40, 0.48]), (0.20, 0.30))$	$(([0.10, 0.20], [0.30, 0.35]), (0.40, 0.20))$	$(([0.15, 0.22], [0.12, 0.20]), (0.10, 0.60))$

Table 5.2: Normalized decision matrix for the alternatives

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$(([0.10, 0.20], [0.15, 0.20]), (0.40, 0.30))$	$(([0.30, 0.35], [0.40, 0.50]), (0.10, 0.30))$	$(([0.20, 0.25], [0.30, 0.40]), (0.20, 0.30))$
$\mathcal{V}_2$	$(([0.15, 0.22], [0.30, 0.35]), (0.40, 0.20))$	$(([0.10, 0.15], [0.18, 0.25]), (0.30, 0.10))$	$(([0.40, 0.45], [0.30, 0.39]), (0.40, 0.22))$
$\mathcal{V}_3$	$(([0.70, 0.75], [0.10, 0.15]), (0.10, 0.80))$	$(([0.80, 0.85], [0.10, 0.15]), (0.12, 0.85))$	$(([0.82, 0.88], [0.10, 0.12]), (0.10, 0.70))$
$\mathcal{V}_4$	$(([0.40, 0.48], [0.20, 0.28]), (0.30, 0.20))$	$(([0.10, 0.20], [0.30, 0.35]), (0.40, 0.20))$	$(([0.12, 0.20], [0.15, 0.22]), (0.60, 0.10))$

Table 5.3: Score values and ranking order of the alternatives for different generators

Generators	Operators	Score				Ranking
		$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
$y(t) = -\log(t)$	GCIFWA	0.0421	-0.2203	1.3783	-0.2431	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
	GCIFOWA	0.0291	-0.2256	1.3709	-0.2431	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$y(t) = \log\left(\frac{2-t}{t}\right)$	GCIFWA	0.0320	-0.2304	1.3765	-0.2567	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
	GCIFOWA	0.0229	-0.2355	1.3756	-0.2149	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	GCIFHA	-1.1334	-1.2827	-0.3027	-1.1822	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$

Table 5.4: Comparative studies with existing approaches

Existing approaches	Overall value of the alternatives				Ordering
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
Xu and Chen [181]	-0.0687	-0.0318	0.6896	0.0054	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Xu and Chen [182]	-0.1174	-0.0845	0.6810	-0.0467	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Ye [199]	-0.1998	-0.2928	0.7228	-0.3222	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
Nayagam et al. [114]	0.0089	0.0403	0.7871	0.0695	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Sivaraman et al. [131]	0.2052	0.2151	0.7606	0.2208	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Garg [44]	0.3499	0.3664	0.8654	0.3758	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Chen et al. [25]	0.1932	0.2133	0.3878	0.2184	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Xu [180]	0.4356	0.4270	0	0.4309	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1$
Wang and Liu [161]	-0.0752	-0.0400	0.6889	-0.0028	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Wang and Liu [162]	-0.1111	-0.0793	0.6822	-0.0417	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Liu [95] ($\gamma = 3$)	-0.0785	-0.0449	0.6886	-0.0078	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Wei and Wang [171]	-0.1174	-0.0845	0.6810	-0.0467	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Wang et al. [164]	-0.0950	-0.0506	0.6828	-0.0145	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Chen, Cheng and Tsai [22]	-0.1304	-0.0689	0.6761	-0.0093	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$
Chen, Cheng and Tsai [23]	-0.0715	-0.0129	0.7022	0.0009	$\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$

Chapter 6

Bonferroni Mean Operators under Cubic Intuitionistic Fuzzy Set Environment[1]

In this chapter, Bonferroni mean and weighted Bonferroni mean averaging operators between the cubic intuitionistic fuzzy numbers are developed for aggregating the different preferences of the decision-maker. Some of the desirable properties of the proposed operators are investigated in detail. In addition to it, a decision-making method based on the proposed operators under the cubic intuitionistic fuzzy environment is devised and is illustrated with a numerical example based on inventory management. Finally, a comparison analysis between the proposed and the existing approaches have been performed.

6.1 Introduction

In our real-life situation, there always exist situations in which a relationship between the different criteria are related to each other. Some situations permit the characteristics such as prioritization, support, and impact among the criteria information and they cannot be treated as independent entities in the aggregation process. For handling it and to incorporate into the DM analysis, Yager [197] proposed the concept of the Bonferroni Mean (BM) [11] whose main characteristic is its capability to capture the interrelationship between

[1]The content of this chapter is published as "Multi-attribute decision-making based on Bonferroni mean operators under cubic intuitionistic fuzzy set environment", *Entropy.* 20(1), 65, 2018; doi:10.3390/e20010065 (**SCI: Impact Factor: 2.524**)

the input arguments. Beliakov et al. [9] introduced the generalized BM to overcome the drawback of former BM. Xu and Yager [192] developed an intuitionistic fuzzy Bonferroni mean to aggregate the intuitionistic fuzzy information. Xu and Chen [189] extended these mean operators to the IVIFSs environment. Xia et al. [176] proposed the generalized intuitionistic fuzzy BMs. Liu et al. [96] presented the partitioned BM operators under IFSs environment. Shi and He [129] threw light on optimizing BMs with their applications to various decision-making processes. Garg and Arora [49] presented BM aggregation operator under intuitionistic fuzzy soft set environment.

Inspired by above existing literature, the impact of BM operators in successfully aggregating available information cannot be denied. Consequently, we present the concept of BM operator associated with CIFS. Obviously, the advantage of the CIFS is that it can contain much more information to express the IVIFN and IFN simultaneously and utilizing the concept of BM along with it, the interrelationships between criteria pairs can be successfully addressed. In the present contribution, we propose some new AOs called the cubic intuitionistic fuzzy Bonferroni mean (CIFBM), as well as weighted cubic intuitionistic fuzzy Bonferroni mean (WCIFBM) operator to aggregate the preferences of decision-makers. Various desirable properties of these operators are investigated in details. Further, we examine the properties and develop some special cases of proposed work. Some of the existing studies have been deduced from the proposed operator which signifies that the proposed operators are more generalized than the others. Finally, a DM approach has been given for ranking the different alternatives based on the proposed operators.

6.2 Cubic Intuitionistic Fuzzy Bonferroni Mean Operator

Definition 6.2.1. A cubic intuitionistic fuzzy Bonferroni mean (CIFBM) operator is a mapping CIFBM: $\Phi^n \to \Phi$ defined on the collection of CIFNs $\mathcal{A}_i$, and is given by

$$\text{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} (\mathcal{A}_i^{\mathfrak{p}} \otimes \mathcal{A}_j^{\mathfrak{q}}) \right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \tag{6.1}$$

where $\mathfrak{p}, \mathfrak{q} > 0$ are the real numbers.

Theorem 6.2.1. The aggregated value by using CIFBM operator for CIFNs $\mathcal{A}_i = \left(\left([\zeta_i^L, \zeta_i^U],\right.\right.$ $\left.\left.[\vartheta_i^L, \vartheta_i^U]\right), (\zeta_i, \vartheta_i)\right)$ is still CIFN and is given by

$$\text{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\left([\zeta^L, \zeta^U], [\vartheta^L, \vartheta^U]\right), (\zeta, \vartheta)\right) \tag{6.2}$$

where

$$\zeta^L = \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (\zeta_i^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}; \zeta^U = \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

$$\vartheta^L = 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1 - \vartheta_i^L)^{\mathfrak{p}}(1 - \vartheta_j^L)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

$$\vartheta^U = 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1 - \vartheta_i^U)^{\mathfrak{p}}(1 - \vartheta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

$$\zeta = 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1 - \zeta_i)^{\mathfrak{p}}(1 - \zeta_j)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

$$\vartheta = \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (\vartheta_i)^{\mathfrak{p}}(\vartheta_j)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

Proof is given in Appendix.

From CIFBM operator, it is observed that they satisfies certain properties for a collection of CIFN $\mathcal{A}_i$, which are stated as follows:

Property 6.2.1. (Idempotency) If $\mathcal{A}_i = \mathcal{A}$ for all i, then CIFBM satsifies

$$\text{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}, \mathcal{A}, \ldots \mathcal{A}) = \mathcal{A}$$

Proof. Assume $\mathcal{A}_i = \mathcal{A}$ for all i, then from the Definition of $\text{CIFBM}^{\mathfrak{p},\mathfrak{q}}$, we have

$$\begin{aligned} \text{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} (\mathcal{A}^{\mathfrak{p}} \otimes \mathcal{A}^{\mathfrak{q}})\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \\ &= \left(\frac{1}{n(n-1)} n(n-1)(\mathcal{A})^{\mathfrak{p}+\mathfrak{q}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \\ &= \mathcal{A} \end{aligned}$$

□

Property 6.2.2. (Monotonicity) Let $\mathcal{A}_i = \left(\left([\zeta^L_{\mathcal{A}_i}, \zeta^U_{\mathcal{A}_i}], [\vartheta^L_{\mathcal{A}_i}, \vartheta^U_{\mathcal{A}_i}]\right), (\zeta_{\mathcal{A}_i}, \vartheta_{\mathcal{A}_i})\right)$ and $\mathcal{B}_i = \left(\left([\zeta^L_{\mathcal{B}_i}, \zeta^U_{\mathcal{B}_i}], [\vartheta^L_{\mathcal{B}_i}, \vartheta^U_{\mathcal{B}_i}]\right),\ (\zeta_{\mathcal{B}_i}, \vartheta_{\mathcal{B}_i})\right)$ be any two CIFNs such that $\zeta^L_{\mathcal{A}_i} \leq \zeta^L_{\mathcal{B}_i}$, $\zeta^U_{\mathcal{A}_i} \leq \zeta^U_{\mathcal{A}_i}$, $\vartheta^L_{\mathcal{A}_i} \geq \vartheta^U_{\mathcal{B}_i}$, $\vartheta^U_{\mathcal{A}_i} \geq \vartheta^U_{\mathcal{B}_i}$ and $\zeta_{\mathcal{A}_i} \geq \zeta_{\mathcal{B}_i}$, $\vartheta_{\mathcal{A}_i} \leq \vartheta_{\mathcal{B}_i}$, then

$$\text{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n)$$

Proof. Let $\text{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\left([\zeta^L_{\mathcal{A}}, \zeta^U_{\mathcal{A}}], [\vartheta^L_{\mathcal{A}}, \vartheta^U_{\mathcal{A}}]\right), (\zeta_{\mathcal{A}}, \vartheta_{\mathcal{A}})\right)$ and $\text{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n) = \left(\left([\zeta^L_{\mathcal{B}}, \zeta^U_{\mathcal{B}}],\ [\vartheta^L_{\mathcal{B}}, \vartheta^U_{\mathcal{B}}]\right), (\zeta_{\mathcal{B}}, \vartheta_{\mathcal{B}})\right)$. Now, for any two CIFNs $\mathcal{A}_i$ and $\mathcal{A}_j$, and by using relation that $\zeta^U_{\mathcal{A}_i} \leq \zeta^U_{\mathcal{A}_i}$, we have

$$\begin{aligned}
& \left(\zeta^U_{\mathcal{A}_i}\right)^{\mathfrak{p}} \left(\zeta^U_{\mathcal{A}_j}\right)^{\mathfrak{q}} \leq \left(\zeta^U_{\mathcal{B}_i}\right)^{\mathfrak{p}} \left(\zeta^U_{\mathcal{B}_j}\right)^{\mathfrak{q}} \\
\Leftrightarrow\ & \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(\zeta^U_{\mathcal{B}_i}\right)^{\mathfrak{p}} \left(\zeta^U_{\mathcal{B}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(\zeta^U_{\mathcal{A}_i}\right)^{\mathfrak{p}} \left(\zeta^U_{\mathcal{A}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \\
\Leftrightarrow\ & 1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(\zeta^U_{\mathcal{A}_i}\right)^{\mathfrak{p}} \left(\zeta^U_{\mathcal{A}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \leq 1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(\zeta^U_{\mathcal{B}_i}\right)^{\mathfrak{p}} \left(\zeta^U_{\mathcal{B}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \\
\Leftrightarrow\ & \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(\zeta^U_{\mathcal{A}_i}\right)^{\mathfrak{p}} \left(\zeta^U_{\mathcal{A}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(\zeta^U_{\mathcal{B}_i}\right)^{\mathfrak{p}} \left(\zeta^U_{\mathcal{B}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \\
& \textit{i.e.,}\quad \zeta^U_{\mathcal{A}} \leq \zeta^U_{\mathcal{B}}
\end{aligned}$$

Similarly,

$$\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(\zeta^L_{\mathcal{A}_i}\right)^{\mathfrak{p}} \left(\zeta^L_{\mathcal{A}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(\zeta^L_{\mathcal{B}_i}\right)^{\mathfrak{p}} \left(\zeta^L_{\mathcal{B}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

and $$\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(\vartheta_{\mathcal{A}_i}\right)^{\mathfrak{p}} \left(\vartheta_{\mathcal{A}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(\vartheta_{\mathcal{B}_i}\right)^{\mathfrak{p}} \left(\vartheta_{\mathcal{B}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

On the other hand, for $\vartheta^U_{\mathcal{A}_i} \leq \vartheta^U_{\mathcal{B}_i}$ and hence for any two CIFNs $\mathcal{A}_i$ and $\mathcal{A}_j$, we have

$$\begin{aligned}
& \left(1 - \vartheta^U_{\mathcal{A}_i}\right)^{\mathfrak{p}} \left(1 - \vartheta^U_{\mathcal{A}_j}\right)^{\mathfrak{q}} \leq \left(1 - \vartheta^U_{\mathcal{B}_i}\right)^{\mathfrak{p}} \left(1 - \vartheta^U_{\mathcal{B}_j}\right)^{\mathfrak{q}} \\
\Leftrightarrow\ & \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(1 - \vartheta^U_{\mathcal{B}_i}\right)^{\mathfrak{p}} \left(1 - \vartheta^U_{\mathcal{B}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \left(1 - \vartheta^U_{\mathcal{A}_i}\right)^{\mathfrak{p}} \left(1 - \vartheta^U_{\mathcal{A}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}
\end{aligned}$$

$$\Leftrightarrow \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\vartheta_{\mathcal{A}_i}^{U}\right)^{\mathfrak{p}}\left(1-\vartheta_{\mathcal{A}_j}^{U}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

$$\leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\vartheta_{\mathcal{B}_i}^{U}\right)^{\mathfrak{p}}\left(1-\vartheta_{\mathcal{B}_j}^{U}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

$$\Leftrightarrow 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\vartheta_{\mathcal{B}_i}^{U}\right)^{\mathfrak{p}}\left(1-\vartheta_{\mathcal{B}_j}^{U}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

$$\leq 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\vartheta_{\mathcal{A}_i}^{U}\right)^{\mathfrak{p}}\left(1-\vartheta_{\mathcal{A}_j}^{U}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

Similarly,

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\vartheta_{\mathcal{B}_i}^{L}\right)^{\mathfrak{p}}\left(1-\vartheta_{\mathcal{B}_j}^{L}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

$$\leq 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\vartheta_{\mathcal{A}_i}^{L}\right)^{\mathfrak{p}}\left(1-\vartheta_{\mathcal{A}_j}^{L}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

and

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\zeta_{\mathcal{B}_i}\right)^{\mathfrak{p}}\left(1-\zeta_{\mathcal{B}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

$$\leq 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\zeta_{\mathcal{A}_i}\right)^{\mathfrak{p}}\left(1-\zeta_{\mathcal{A}_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

Thus,

$$\begin{aligned}
\mathcal{S}c(\mathrm{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)) &= \frac{\zeta_{\mathcal{A}}^{L}+\zeta_{\mathcal{A}}^{U}-\vartheta_{\mathcal{A}}^{L}-\vartheta_{\mathcal{A}}^{U}}{2}-\zeta_{\mathcal{A}}+\vartheta_{\mathcal{A}}\\
&\leq \frac{\zeta_{\mathcal{B}}^{L}+\zeta_{\mathcal{B}}^{U}-\vartheta_{\mathcal{B}}^{L}-\vartheta_{\mathcal{B}}^{U}}{2}-\zeta_{\mathcal{B}}+\vartheta_{\mathcal{B}}\\
&= \mathcal{S}c(\mathrm{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_n))
\end{aligned}$$

Hence, by comparison law, we get the result. □

Property 6.2.3. (Commutativity) If $(\dot{\mathcal{A}}_1, \dot{\mathcal{A}}_2, \ldots, \dot{\mathcal{A}}_n)$ be any permutation of CIFNs $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$, then $\mathrm{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathrm{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\dot{\mathcal{A}}_1, \dot{\mathcal{A}}_2, \ldots, \dot{\mathcal{A}}_n)$.

Proof. For permutation $(\dot{\mathcal{A}}_1, \dot{\mathcal{A}}_2, \ldots, \dot{\mathcal{A}}_n)$ of $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$,

$$\begin{aligned}\mathrm{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\mathcal{A}_i^{\mathfrak{p}} \otimes \mathcal{A}_j^{\mathfrak{q}}\right)\right)\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\\ &= \left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\dot{\mathcal{A}}_i^{\mathfrak{p}} \otimes \dot{\mathcal{A}}_j^{\mathfrak{q}}\right)\right)\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\\ &= \mathrm{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\dot{\mathcal{A}}_1, \dot{\mathcal{A}}_2, \ldots, \dot{\mathcal{A}}_n)\end{aligned}$$

□

Property 6.2.4. (Boundedness) Let $\mathcal{A}^- = \left(\left([\zeta^L_{\min}, \zeta^U_{\min}], [\vartheta^L_{\max}, \vartheta^U_{\max}]\right), (\zeta_{\max}, \vartheta_{\min})\right)$, $\mathcal{A}^+ = \left(\left([\zeta^L_{\max}, \zeta^U_{\max}], [\vartheta^L_{\min}, \vartheta^U_{\min}]\right), (\zeta_{\min}, \vartheta_{\max})\right)$ are the lower and upper bounds for the collection of CIFNs $\mathcal{A}_i$, then

$$\mathcal{A}^- \leq \mathrm{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+.$$

Proof. Since, $\zeta^L_{\min} \leq \zeta^L_i \leq \zeta^L_{\max}$ and $\zeta^U_{\min} \leq \zeta^U_i \leq \zeta^U_{\max}$ which implies that

$$\begin{aligned}&\left(\zeta^L_{\min}\right)^{\mathfrak{p}+\mathfrak{q}} \leq (\zeta^U_i)^{\mathfrak{p}}(\zeta^L_j)^{\mathfrak{q}} \leq \left(\zeta^L_{\max}\right)^{\mathfrak{p}+\mathfrak{q}}\\ \Leftrightarrow\ & \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\zeta^L_{\max}\right)^{\mathfrak{p}+\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\zeta^L_i\right)^{\mathfrak{p}}\left(\zeta^L_j\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\zeta^L_{\min}\right)^{\mathfrak{p}+\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\\ \Leftrightarrow\ & 1-\left(\zeta^L_{\max}\right)^{\mathfrak{p}+\mathfrak{q}} \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\zeta^L_i\right)^{\mathfrak{p}}\left(\zeta^L_j\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \leq 1-\left(\zeta^L_{\min}\right)^{\mathfrak{p}+\mathfrak{q}}\\ \Leftrightarrow\ & \left(\zeta^L_{\min}\right)^{\mathfrak{p}+\mathfrak{q}} \leq 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\zeta^L_i\right)^{\mathfrak{p}}\left(\zeta^L_j\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \leq \left(\zeta^L_{\max}\right)^{\mathfrak{p}+\mathfrak{q}}\\ \Leftrightarrow\ & \zeta^L_{\min} \leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\zeta^L_i\right)^{\mathfrak{p}}\left(\zeta^L_j\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq \zeta^L_{\max}\end{aligned}$$

Similarly, we get

$$\zeta^U_{\min} \leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\zeta^U_i\right)^{\mathfrak{p}}\left(\zeta^U_j\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq \zeta^U_{\max}$$

$$\text{and} \qquad \vartheta_{\min} \leq \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (\vartheta_i)^{\mathfrak{p}} (\vartheta_j)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq \vartheta_{\max}$$

On the other hand, for $\vartheta^L_{\min} \leq \vartheta^L_i \leq \vartheta^L_{\max}$ and $\vartheta^U_{\min} \leq \vartheta^U_i \leq \vartheta^U_{\max}$, we have

$$\begin{aligned}
& \left(1 - \vartheta^U_{\max}\right)^{\mathfrak{p}+\mathfrak{q}} \leq \left(1 - \vartheta^U_i\right)^{\mathfrak{p}} \left(1 - \vartheta^U_j\right)^{\mathfrak{q}} \leq \left(1 - \vartheta^U_{\min}\right)^{\mathfrak{p}+\mathfrak{q}} \\
\Leftrightarrow \quad & 1 - \left(1 - \vartheta^U_{\min}\right)^{\mathfrak{p}+\mathfrak{q}} \leq 1 - \left(1 - \vartheta^U_i\right)^{\mathfrak{p}} \left(1 - \vartheta^U_j\right)^{\mathfrak{q}} \leq 1 - \left(1 - \vartheta^U_{\max}\right)^{\mathfrak{p}+\mathfrak{q}} \\
\Leftrightarrow \quad & 1 - \left(1 - \vartheta^U_{\min}\right)^{\mathfrak{p}+\mathfrak{q}} \leq \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \vartheta^U_i\right)^{\mathfrak{p}} \left(1 - \vartheta^U_j\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \leq 1 - \left(1 - \vartheta^U_{\max}\right)^{\mathfrak{p}+\mathfrak{q}} \\
\Leftrightarrow \quad & \left(1 - \vartheta^U_{\max}\right)^{\mathfrak{p}+\mathfrak{q}} \leq 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \vartheta^U_i\right)^{\mathfrak{p}} \left(1 - \vartheta^U_j\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \leq \left(1 - \vartheta^U_{\min}\right)^{\mathfrak{p}+\mathfrak{q}} \\
\Leftrightarrow \quad & 1 - \vartheta^U_{\max} \leq \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \vartheta^U_i\right)^{\mathfrak{p}} \left(1 - \vartheta^U_j\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq 1 - \vartheta^U_{\min} \\
\Leftrightarrow \quad & \vartheta^U_{\min} \leq 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \vartheta^U_i\right)^{\mathfrak{p}} \left(1 - \vartheta^U_j\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq \vartheta^U_{\max}
\end{aligned}$$

Similarly, we have

$$\vartheta^L_{\min} \leq 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \vartheta^L_i\right)^{\mathfrak{p}} \left(1 - \vartheta^L_j\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq \vartheta^L_{\max}$$

$$\text{and} \qquad \zeta_{\min} \leq 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \zeta_i)^{\mathfrak{p}} (1 - \zeta_j)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq \zeta_{\max}$$

Thus, by comparing the two CIFNs, we get $\mathcal{A}^- \leq \text{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$. □

In the following, we will discuss some special cases of CIFBM operator by taking different values of $\mathfrak{p}$ and $\mathfrak{q}$.

Case 1: As $\mathfrak{q} \to 0$, then Eq. (6.1) reduces to generalized cubic intuitionistic fuzzy mean which is defined as follows:

$$\begin{aligned}
&\mathrm{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\
=&\left(\begin{array}{l}\left(\begin{array}{l}\left[\left(1-\prod_{i=1}^{n}\left(1-(\zeta_i^L)^{\mathfrak{p}}\right)^{\frac{1}{n}}\right)^{\frac{1}{\mathfrak{p}}},\left(1-\prod_{i=1}^{n}\left(1-(\zeta_i^U)^{\mathfrak{p}}\right)^{\frac{1}{n}}\right)^{\frac{1}{\mathfrak{p}}}\right],\\
\left[1-\left(1-\prod_{i=1}^{n}\left(1-(1-\vartheta_i^L)^{\mathfrak{p}}\right)^{\frac{1}{n}}\right)^{\frac{1}{\mathfrak{p}}},1-\left(1-\prod_{i=1}^{n}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}\right)^{\frac{1}{n}}\right)^{\frac{1}{\mathfrak{p}}}\right]\end{array}\right),\\
\left(1-\left(1-\prod_{i=1}^{n}\left(1-(1-\zeta_i)^{\mathfrak{p}}\right)^{\frac{1}{n}}\right)^{\frac{1}{\mathfrak{p}}},\left(1-\prod_{i=1}^{n}\left(1-(\vartheta_i)^{\mathfrak{p}}\right)^{\frac{1}{n}}\right)^{\frac{1}{\mathfrak{p}}}\right)\end{array}\right)\\
=&\left(\frac{1}{n}\left(\bigoplus_{i=1}^{n}\mathcal{A}_i^{\mathfrak{p}}\right)\right)^{\frac{1}{\mathfrak{p}}}\\
=&\mathrm{CIFBM}^{p,0}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)
\end{aligned}$$

Case 2: If $\mathfrak{p}=2$ and as $\mathfrak{q}\to 0$, then Eq. (6.1) reduces to cubic intuitionistic fuzzy square mean which is given as follows:

$$\begin{aligned}
&\mathrm{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\
=&\left(\begin{array}{l}\left(\begin{array}{l}\left[\left(1-\left(\prod_{i=1}^{n}(1-(\zeta_i^L)^2)\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{i=1}^{n}\left(1-(\zeta_i^U)^2\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right],\\
\left[1-\left(1-\left(\prod_{i=1}^{n}(1-(1-\vartheta_i^L)^2)\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},1-\left(1-\left(\prod_{i=1}^{n}(1-(1-\vartheta_i^U)^2)\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right]\end{array}\right),\\
\left(1-\left(1-\left(\prod_{i=1}^{n}(1-(1-\zeta_i)^2)\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{i=1}^{n}(1-(\vartheta_i)^2)\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right)\end{array}\right)\\
=&\left(\frac{1}{n}\bigoplus_{i=1}^{n}\mathcal{A}_i^2\right)^{\frac{1}{2}}
\end{aligned}$$

Case 3: For $\mathfrak{p}=1$ and $\mathfrak{q}\to 0$, Eq. (6.1) becomes cubic intuitionistic fuzzy average operator as:

$$\mathrm{CIFBM}^{1,0}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)=\frac{1}{n}\bigoplus_{i=1}^{n}\mathcal{A}_i$$

Case 4: For $\mathfrak{p}=\mathfrak{q}=1$, Eq. (6.1) reduces to cubic intuitionistic interrelated square mean which is defined as

$$
\begin{aligned}
&\mathrm{CIFBM}^{1,1}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\
&= \left(\begin{array}{l}
\left(\begin{array}{l}
\Big[\Big(\frac{1}{n(n-1)}\Big(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}(1-\zeta_i^L\zeta_j^L)\Big)^{\frac{1}{n(n-1)}}\Big)^{\frac{1}{2}},\Big(\frac{1}{n(n-1)}\Big(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}(1-\zeta_i^U\zeta_j^U)\Big)^{\frac{1}{n(n-1)}}\Big)^{\frac{1}{2}}\Big],\\
\Big[1-\Big(1-\Big(\prod_{\substack{i,j=1\\ i\neq j}}(1-(1-\vartheta_i^L)(1-\vartheta_i^L))\Big)^{\frac{1}{n(n-1)}}\Big)^{\frac{1}{2}},\\
1-\Big(1-\Big(\prod_{\substack{i,j=1\\ i\neq j}}(1-(1-\vartheta_i^U)(1-\vartheta_i^U))\Big)^{\frac{1}{n(n-1)}}\Big)^{\frac{1}{2}}\Big]
\end{array}\right),\\
\left(1-\Big(1-\Big(\prod_{\substack{i,j=1\\ i\neq j}}(1-(1-\zeta_i)(1-\zeta_i))\Big)^{\frac{1}{n(n-1)}}\Big)^{\frac{1}{2}},\Big(\frac{1}{n(n-1)}\Big(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}(1-\vartheta_i\vartheta_j)\Big)^{\frac{1}{n(n-1)}}\Big)^{\frac{1}{2}}\right)
\end{array}\right)
\end{aligned}
$$

6.2.1 Weighted BM Operator of CIFNs

Definition 6.2.2. For CIFNs $\mathcal{A}_i$ $(i=1,2,\ldots,n)$ and weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ such that each $\omega_i>0$ and $\sum\limits_{i=1}^{n}\omega_i=1$, a weighted CIFBM defined over family of CIFNs Φ as WCIFBM : $\Phi^n\to\Phi$ and is given by

$$
\mathrm{WCIFBM}_{\omega}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)=\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left((\omega_i\mathcal{A}_i)^{\mathfrak{p}}\otimes(\omega_j\mathcal{A}_j)^{\mathfrak{q}}\right)\right)\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \tag{6.3}
$$

for a positive real number $\mathfrak{p}$ ad $\mathfrak{q}$.

Theorem 6.2.2. The aggregated value by using WCIFBM operator for collection of CIFNs $\mathcal{A}_i=\left(\left([\zeta_i^L,\zeta_i^U],\ \ [\vartheta_i^L,\vartheta_i^U]\right),(\zeta_i,\vartheta_i)\right)$, $(i=1,2,\ldots,n)$ is also CIFN and can be expressed as

$$
\mathrm{WCIFBM}_{\omega}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)=\left(\left([\zeta^L,\zeta^U],[\vartheta^L,\vartheta^U]\right),(\zeta,\vartheta)\right) \tag{6.4}
$$

where

$$
\begin{aligned}
\zeta^L &= \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-(1-\zeta_i^L)^{\omega_i}\right)^{\mathfrak{p}}\left(1-(1-\zeta_j^L)^{\omega_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\\
\zeta^U &= \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-(1-\zeta_i^U)^{\omega_i}\right)^{\mathfrak{p}}\left(1-(1-\zeta_j^U)^{\omega_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}
\end{aligned}
$$

$$\begin{aligned}
\vartheta^L &= 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\vartheta_i^L\right)^{\omega_i}\right)^{\mathfrak{p}}\left(1-\left(\vartheta_j^L\right)^{\omega_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\\
\vartheta^U &= 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\vartheta_i^U\right)^{\omega_i}\right)^{\mathfrak{p}}\left(1-\left(\vartheta_j^U\right)^{\omega_j}\right)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\\
\zeta &= 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(\zeta_i)^{\omega_i})^{\mathfrak{p}}(1-(\zeta_j)^{\omega_j})^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\\
\vartheta &= \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(1-\vartheta_i)^{\omega_i})^{\mathfrak{p}}(1-(1-\vartheta_j)^{\omega_j})^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}
\end{aligned}$$

and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ be the associated weight vector such that each $\omega_i>0$ and $\sum\limits_{i=1}^{n}\omega_i=1$.

Proof. Proof is similar to that of Theorem 6.2.1, so we omit here. □

6.3 Proposed Decision-Making Approach Based of Cubic Intuitionistic Fuzzy Bonferroni Mean Operator

In this section, we shall utilize the proposed Bonferroni mean aggregation operator to solve the multi-attribute decision making under the CIFSs environment. For it, the general description of MCDM problem is same as Section 2.5 of Chapter 2 where $\mathcal{A}_{ij}=\left(\left([\zeta_{ij}^L,\zeta_{ij}^U],[\vartheta_{ij}^L,\vartheta_{ij}^U]\right),(\zeta_{ij},\vartheta_{ij})\right)$ represents the priority values of alternative $\mathcal{V}_i$ given by decision maker such that $[\zeta_{ij}^L,\zeta_{ij}^U],[\vartheta_{ij}^L,\vartheta_{ij}^U]\subseteq[0,1]$, $\zeta_{ij},\vartheta_{ij}\in[0,1]$ and $\zeta_{ij}^U+\vartheta_{ij}^U\leq 1$, $\zeta_{ij}+\vartheta_{ij}\leq 1$ for $i=1,2,\ldots,m; j=1,2,\ldots,n$. Let $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ be the weight vector of the criteria such that $\omega_j>0$ and $\sum\limits_{j=1}^{n}\omega_j=1$. Then, the proposed method has been summarized into the various steps which are described as follows to find the best alternative(s).

Step 1: Collect the information rating of alternatives corresponding to criteria and summarize in the form of CIFN $\mathcal{A}_{ij}=\left(\left([\zeta_{ij}^L,\zeta_{ij}^U],[\vartheta_{ij}^L,\vartheta_{ij}^U]\right),(\zeta_{ij},\vartheta_{ij})\right)$: $i=1,2,\ldots,m$;

$j = 1, 2, \ldots, n$. These rating values are expressed as a decision matrix $\mathcal{M} = (\mathcal{A}_{ij})$.

Step 2: Normalize these collective information decision matrix by transforming the rating values of cost type into benefit type, if any, by using the normalization formula:

$$r_{ij} = \begin{cases} \left(\left([\zeta_{ij}^L, \zeta_{ij}^U], [\vartheta_{ij}^L, \vartheta_{ij}^U]\right), (\zeta_{ij}, \vartheta_{ij})\right) & ; \quad \text{for benefit type criterion} \\ \left(\left([\vartheta_{ij}^L, \vartheta_{ij}^U], [\zeta_{ij}^L, \zeta_{ij}^U]\right), (\vartheta_{ij}, \zeta_{ij})\right) & ; \quad \text{for cost type criterion} \end{cases} \tag{6.5}$$

and hence summarize it into the decision matrix $R = (r_{ij})_{m\times n}$.

Step 3: Aggregate the different preference values $r_{ij}, (j = 1, 2, \ldots, n)$ of the alternatives $\mathcal{V}_i$ into the collective one $r_i, (i = 1, 2, \ldots, m)$ by using WCIFBM aggregation operator for a real positive number $\mathfrak{p}, \mathfrak{q}$ as

$$\begin{aligned} r_i &= \left(\left([\zeta_{ij}^L, \zeta_{ij}^U], [\vartheta_{ij}^L, \vartheta_{ij}^U]\right), ([\zeta_{ij}, \vartheta_{ij}])\right) \\ &= \text{WCIFBM}^{\mathfrak{p},\mathfrak{q}}(r_{i1}, r_{i2}, \ldots, r_{in}) \end{aligned}$$

Step 4: Compute the score value of the aggregated CIFN r_i by using Eq. (4.1) of Chapter 4 as

$$\mathcal{S}c(r_i) = \frac{\zeta_i^L + \zeta_i^U - \vartheta_i^L - \vartheta_i^U}{2} + (\vartheta_i - \zeta_i) \tag{6.6}$$

Step 5: Rank the alternative $\mathcal{V}_i, (i = 1, 2, \ldots, m)$ with the order of their score value $\mathcal{S}c(r_i)$.

6.4 Illustrative Example

For demonstrating the real-life application of the proposed approach, a numerical example has been illustrated below:

6.4.1 Case Study

Inventory management is an issue of great concern these days. From an industrial viewpoint, a company cannot excel in desired levels of manufacturing until its inventory is not managed properly. Therefore, proper inventory management is the first step of the ladder of good production levels. Any shortage of raw material in inventory may disrupt the whole manufacturing cycle which in-turn can incur a huge loss to the company.

Suppose, a Food company wants to keep track of various inventory items. The company produces mainly four kinds of food $(\mathcal{V}_i)$'s namely "Beverages", "Edible oils", "Pickles" and "Bakery items". For manufacturing these food items, the stock re-ordering decisions for ingredients in inventory are to be taken on account of three factors $\mathfrak{B}_j$'s as "Cost Price", "Storage facilities" and "Staleness level". The weight vector of these factors is taken as $\omega = (0.20, 0.38, 0.42)^T$. The given alternatives are evaluated under these three factors and rate their values in terms of CIFNs. In each CIFN, the IVIFNs shows the existing stock level in the inventory and the IFNs represent the estimate of agreeness as well as disagreeness towards the present stock level for a coming week. Since the company does not compromise with the quality of production, therefore maximum priority is given to reduce staleness levels. Then, the aim is to identify the food-items whose ingredients' stock is needed to be re-ordered frequently. For it, the following steps of the proposed approach have been executed as follows.

Step 1: The preferences information related to each alternative are summarized in CIFNs and the collection rating are given in the decision matrix as shown in Table 6.1.

Step 2: By using Eq. (6.5), we obtain normalized CIFNs and summarized in Table 6.2.

Step 3: For the sake of simplicity, we choose $\mathfrak{p} = \mathfrak{q} = 1$ and then by using Eq. (6.4) to compute the overall value of each alternative as

$$\begin{aligned} r_1 &= (([0.0601, 0.0988], [0.7589, 0.8305]), (0.6681, 0.0892)) \\ r_2 &= (([0.0648, 0.1069], [0.6265, 0.7146]), (0.7503, 0.1361)) \\ r_3 &= (([0.0758, 0.1215], [0.7480, 0.8254]), (0.7436, 0.1132)) \\ \text{and } r_4 &= (([0.0844, 0.1463], [0.6609, 0.7358]), (0.6908, 0.1264)) \end{aligned}$$

Step 4: By using Eq. (4.1) of Chapter 4, the score value of each alternative is obtained as $\mathcal{S}c(r_1) = -1.2942$, $\mathcal{S}c(r_2) = -1.1989$, $\mathcal{S}c(r_3) = -1.3185$ and $\mathcal{S}c(r_4) = -1.1474$.

Step 5: The ranking order of the alternatives based on the score values is found to be $\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$. Thus, Bakery items' stock needs maximum re-ordering.

The proposed aggregation operators are symmetric with respect to the parameters $\mathfrak{p}$ and $\mathfrak{q}$. However, in order to analyze the effect of these parameters on to the final ranking of the alternatives, an investigation has been done by varying it simultaneously and their score values along with ranking order are summarized in Table 6.3. From this table, we can find that by assigning different pairs of the parameters $\mathfrak{p}$ and $\mathfrak{q}$, the score values of the aggregated numbers are different; however, the ranking orders of the alternatives remain same. This feature of the proposed operators is more crucial in real decision-making problems. For instance, it has been seen that with the increase of the parameters, the score values of the alternative increases, which gives us optimism view to the decision makers'. Therefore, if the decision makers are optimistic then the higher values can be assigned to these parameters during the aggregation process. On the other hand, if the decision makers are pessimistic then lower values can be assigned to these parameters and the score values of the overall values are decreasing. However, the best alternative is the same, which influenced that the results are objective and cannot be changed by decision makes' preference of pessimism and optimism. Thus, the ranking results are reliable.

On the other hand, the variations of the complete score values of each alternative by varying one of the parameter $\mathfrak{p}$ are summarized in Figure 6.1. It can be analyzed from Figure 6.1, that the maximum score possessing alternative remains $\mathcal{V}_4$ for all cases. However, in Figure 6.1(a), by fixing the parameter $\mathfrak{p} = 1$, and varying $\mathfrak{q}$ from 0 to 10, it is observed that when $\mathfrak{q} < 2.0307$, alternative $\mathcal{V}_3$ shows least scores whereas for $\mathfrak{q} > 2.0307$, $\mathcal{V}_1$ possesses least score values. However, at $\mathfrak{q} = 2.0307$, Sc($\mathcal{V}_1$)=Sc($\mathcal{V}_3$)=-1.2811 and thus, from the accuracy function, we get $\mathcal{H}(\mathcal{V}_1) = 1.6286$ and $\mathcal{H}(\mathcal{V}_3) = 1.6410$. which implies that the ranking order of the alternatives at $\mathfrak{p} = 1$ and $\mathfrak{q} = \mathfrak{q}' = 2.0307$ is given as $\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$. Therefore, the worst alternative changes from $\mathcal{V}_3$ to $\mathcal{V}_1$. Similarly, in Figure 6.1(b), we observed that when $\mathfrak{q} < 2.742$, the worst alternative is $\mathcal{V}_3$ white it is $\mathcal{V}_1$ when $\mathfrak{q} > 2.742$ corresponding to $\mathfrak{p} = 2$. Further, $\mathfrak{q} = \mathfrak{q}' = 2.742$, the ranking order of the alternatives is $\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$. The complete rating values for all the alternatives are summarized in Table 6.4.

6.4.2 Validity Test

To demonstrate our approach's viability in the dynamic working environment, following test criteria, corroborated by Wang and Triantaphyllou [166], are accomplished

Test criterion 1: "If we replace the rating values of non-optimal alternative with worse alternative then the best alternative should not change, provided the relative weighted criteria remains unchanged."

Test criterion 2: "Method should possess transitive nature."

Test criterion 3: "When a given problem is decomposed into smaller ones and the same MADM method has been applied, then the combined ranking is same."

Validity Check with Criterion 1: The non-optimal alternative $\mathcal{V}_1$ is replaced with a worst alternative $\mathcal{V}_1'$ where rating value of $\mathcal{V}_1'$ under the three considered criteria are expressed as $\{\big(([0.11, 0.15],[0.60, 0.70]),\ (0.30,\ 0.10)\big);\ \big(([0.22, 0.25],[0.45, 0.55]),\ (0.40,\ 0.10)\big)$ and $\big(([0.20, 0.28],[0.45, 0.70]),\ (0.40,\ 0.15)\big)\}$. Based on these observation, the proposed approach has been applied and hence the final score values of the alternatives are obtained as $\mathcal{S}c(r_1') = -1.4431$, $\mathcal{S}c(r_2) = -1.1989$, $\mathcal{S}c(r_3) = -1.3185$ and $\mathcal{S}c(r_4) = -1.1474$. Therefore, the ranking order is $\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1'$ in which the best alternative remains same as that of the proposed approach. Thus, our approach is fetching out consistent results with respect to the test criterion 1.

Validity Check with Criteria 2 and 3: For checking validity corresponding to criteria 2 and 3 , the fragmented MADM subproblems are taken as $\{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$, $\{\mathcal{V}_1, \mathcal{V}_3, \mathcal{V}_2\}$ and $\{\mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_1\}$. Then, following the stated procedure of the approach their ranking is obtained as: $\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3$, $\mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$ and $\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1$, respectively. The overall ranking by clubbing all of them is $\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$ which is same as that of the results of the proposed original MADM problem, hence it beholds the transitive property. Therefore, the proposed method is valid under the test criterion 2 and test criterion 3.

6.4.3 Graphical Analysis of Obtained Score Values

Figure 6.2 gives an outlook to influence of variable $\mathfrak{p}$ and $\mathfrak{q}$ values on the scores obtained by utilizing WCIFBM operator. It clearly shows that the score function possess different values for different values of parameters $\mathfrak{p}$ and $\mathfrak{q}$ ranging between 1 to 10. For an alternative

$\mathcal{V}_1$, the score value lies between -1.45 to -1.2 while from its corresponding rear-view plot, it is seen that the score value undergoes surface change thrice. The first surface starts at $\mathfrak{p} = \mathfrak{q} = 1$ and ends at (the leftmost coordinate measure) $\mathfrak{p} = 10, \mathfrak{q} = 4.5$ having score value -1.226. The second surface starts at $\mathfrak{p} = 10$ and $\mathfrak{q} = 5.5$ having $\mathcal{S}c = -1.353$ whereas it ends at $\mathfrak{p} = 10, \mathfrak{q} = 7.5$ possessing $\mathcal{S}c = -1.35$. The third surface begins at $\mathfrak{p} = 10, \mathfrak{q} = 8.5$ bearing $\mathcal{S}c = -1.412$ and ends at $\mathfrak{p} = 10, \mathfrak{q} = 10$ with $\mathcal{S}c = -1.41$. The alternatives surface readings (Sr) with values of $\mathfrak{p}$ and $\mathfrak{q}$ are given in Table 6.5 alongwith their corresponding score values. In this table Sr(d)(B) denotes the beginning of dth surface and Sr(d)(E) denotes ending of dth surface where "d" is an integer. It can be seen that $\mathcal{V}_4$ has score values ranging over two surfaces whereas $\mathcal{V}_2$ has score values spread over 4 surfaces with only one value on the 4th surface i.e., $(10, 10)$. On the other hand, both alternatives $\mathcal{V}_1$ and $\mathcal{V}_3$ covers three surfaces with $\mathcal{V}_3$ having only one point i.e., $(10, 10)$ lying on 3rd surface.

6.4.4 Comparative Studies

In order to justify the superiority of our proposed mean operator with respect to the existing approaches namely, Bonferroni mean operator [129, 189], averaging operator [22, 162], geometric operator [23, 150], ranking method [34, 44, 131], an analysis has been conducted under the IVIFSs environment by taking intuitionistic fuzzy judgements of CIFSs as zero and the weight vector is $\omega = (0.20, 0.38, 0.42)^T$. The optimal score values and the ranking order of the alternatives are summarized in Table 6.6. From this table, we observed that the best alternative coincides with the proposed approach results which validate the stability of the approach with respect to state-of-art. Compared with these existing approaches with general intuitionistic sets (IVIFSs or IFSs), the proposed decision-making method under CIFS environment contains much more evaluation information on the alternatives by considering both the IVIFSs and IFSs simultaneously, while the existing approaches contain either IFS or IVIFS information. Therefore, the approaches under the IVIFSs or IFSs may lose some useful information, either IVIFNs or IFNs, of alternatives which may affect the decision results. Furthermore, it is noted from the study that the computational procedure of the proposed approach is different from the existing approaches under the different environment, but the proposed result in this chapter is more rational to reality in the decision process due to the consideration of the consistent priority degree between the pairs of the arguments. In the end, it is concluded that proposed operators consider the decision makers' parameters $\mathfrak{p}$ and $\mathfrak{q}$, which provide, the more choices to the decision makers to avail their desirable alternatives depending upon the different score values of the alternatives for the different parametric values of $\mathfrak{p}$ and $\mathfrak{q}$. Also, the corresponding studies under the IVIFS or IFS environment can be considered as a special case of the proposed operators.

6.5 Conclusion

The major concluded aspects of this chapter are summarized below:

1) Driven by the functionalities of CIFSs, in dealing with the uncertainty and fuzziness, we have presented two BM operators i.e., the CIFBM operator and the WCIFBM operator, to aggregate the different preferences of experts over the different criteria. Also, various desirable characteristics of these operators are studied.

2) The remarkable characteristic of CIFBM and WCIFBM to capture the relationships between the paired arguments is investigated for practical applicability by constructing a DM approach. It has been analyzed that different values of parameters $\mathfrak{p}$ and $\mathfrak{q}$, makes the proposed operators more flexible and offers the various choices to the decision-maker for assessing the decisions.

3) An illustrative case study has been done on inventory management to strengthen the real-life application of the proposed operators. A comparative study with some existing operators has been carried out which shows that the proposed operators along with their corresponding techniques provide a more stable, practical, and optimistic nature to the decision-maker during the aggregation process.

Thus, we conclude that the proposed operators can be applied as an alternative way to solve the problem in real-life situations.

6.6 Appendix: Proofs of the Theorem

Proof of the Theorem 6.2.1:

Proof. For any two positive real numbers $\mathfrak{p}, \mathfrak{q}$ and CIFNs $\mathcal{A}_i$, $\mathcal{A}_j$, we have from the basic operational laws between CIFNs given in Definition 4.2.8 of Chapter 4,

$$\mathcal{A}_i^{\mathfrak{p}} = \left(\left(\left[(\zeta_i^L)^{\mathfrak{p}}, (\zeta_i^U)^{\mathfrak{p}}\right], \left[1-(1-\vartheta_i^L)^{\mathfrak{p}}, 1-(1-\vartheta_i^U)^{\mathfrak{p}}\right]\right), (1-(1-\zeta_i)^{\mathfrak{p}}, (\vartheta_i)^{\mathfrak{p}})\right) \quad (6.7)$$

$$\text{and } \mathcal{A}_j^{\mathfrak{q}} = \left(\left(\left[(\zeta_j^L)^{\mathfrak{q}}, (\zeta_j^U)^{\mathfrak{q}}\right], \left[1-(1-\vartheta_j^L)^{\mathfrak{q}}, 1-(1-\vartheta_j^U)^{\mathfrak{q}}\right]\right), (1-(1-\zeta_j)^{\mathfrak{q}}, (\vartheta_j)^{\mathfrak{q}})\right) \quad (6.8)$$

Therefore,

$$\mathcal{A}_i^{\mathfrak{p}} \otimes \mathcal{A}_j^{\mathfrak{q}} = \left(\begin{pmatrix} \left[(\zeta_i^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}, (\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right], \\ \begin{bmatrix} 1-(1-\vartheta_i^L)^{\mathfrak{p}}(1-\vartheta_j^L)^{\mathfrak{q}}, \\ 1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}} \end{bmatrix} \end{pmatrix}, \begin{pmatrix} 1-(1-\zeta_i)^{\mathfrak{p}}(1-\zeta_j)^{\mathfrak{q}}, \\ (\vartheta_i)^{\mathfrak{p}}(\vartheta_j)^{\mathfrak{q}} \end{pmatrix}\right) \quad (6.9)$$

Firstly, we prove

$$\bigoplus_{\substack{i,j=1 \\ i\neq j}}^{n} (\mathcal{A}_i^{\mathfrak{p}} \otimes \mathcal{A}_j^{\mathfrak{q}}) = \left(\begin{array}{l} \left(\begin{array}{l} \left[1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}(1-(\zeta_i^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}), 1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}})\right], \\ \left[\prod_{\substack{i,j=1 \\ i\neq j}}^{n}(1-(1-\vartheta_i^L)^{\mathfrak{p}}(1-\vartheta_j^L)^{\mathfrak{q}}), \prod_{\substack{i,j=1 \\ i\neq j}}^{n}(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}})\right] \end{array}\right), \\ \left(\prod_{\substack{i,j=1 \\ i\neq j}}^{n}(1-(1-\zeta_i)^{\mathfrak{p}}(1-\zeta_j)^{\mathfrak{q}}), 1-\prod_{\substack{i,j=1 \\ i\neq j}}^{n}(1-(\vartheta_i)^{\mathfrak{p}}(\vartheta_j)^{\mathfrak{q}})\right) \end{array}\right)$$

by induction on n.

For $n = 2$ we get,

$$\bigoplus_{\substack{i,j=1 \\ i\neq j}}^{2} (\mathcal{A}_i^{\mathfrak{p}} \otimes \mathcal{A}_j^{\mathfrak{q}}) = \left(\begin{array}{l} \left(\begin{array}{l} \left[1-\prod_{\substack{i,j=1 \\ i\neq j}}^{2}(1-(\zeta_i^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}), 1-\prod_{\substack{i,j=1 \\ i\neq j}}^{2}(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}})\right], \\ \left[\prod_{\substack{i,j=1 \\ i\neq j}}^{2}\left(1-(1-\vartheta_i^L)^{\mathfrak{p}}(1-\vartheta_j^L)^{\mathfrak{q}}\right), \prod_{\substack{i,j=1 \\ i\neq j}}^{2}(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}})\right] \end{array}\right), \\ \left(\prod_{\substack{i,j=1 \\ i\neq j}}^{2}(1-(1-\zeta_i)^{\mathfrak{p}}(1-\zeta_j)^{\mathfrak{q}}), 1-\prod_{\substack{i,j=1 \\ i\neq j}}^{2}(1-(\vartheta_i)^{\mathfrak{p}}(\vartheta_j)^{\mathfrak{q}})\right) \end{array}\right)$$

Thus, it holds for $n=2$. Assuming result is true for $n=k$ i.e.,

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}\left(\mathcal{A}_i^{\mathfrak{p}}\otimes\mathcal{A}_j^{\mathfrak{q}}\right)=\left(\begin{array}{c}\left(\begin{array}{c}\left[1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(1-(\zeta_i^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}\right),1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)\right],\\ \left[\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(1-(1-\vartheta_i^L)^{\mathfrak{p}}(1-\vartheta_j^L)^{\mathfrak{q}}\right),\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}}\right)\right]\end{array}\right),\\ \left(\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(1-(1-\zeta_i)^{\mathfrak{p}}(1-\zeta_j)^{\mathfrak{q}}\right),1-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(1-(\vartheta_i)^{\mathfrak{p}}(\vartheta_j)^{\mathfrak{q}}\right)\right)\end{array}\right) \tag{6.10}$$

Now, for $n=k+1$, we have

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\mathcal{A}_i^{\mathfrak{p}}\otimes\mathcal{A}_j^{\mathfrak{q}}\right)=\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}\left(\mathcal{A}_i^{\mathfrak{p}}\otimes\mathcal{A}_j^{\mathfrak{q}}\right)\right)\oplus\left(\bigoplus_{i=1}^{k}\left(\mathcal{A}_i^{\mathfrak{p}}\otimes\mathcal{A}_{k+1}^{\mathfrak{q}}\right)\right)\oplus\left(\bigoplus_{j=1}^{k}\left(\mathcal{A}_{k+1}^{\mathfrak{p}}\otimes\mathcal{A}_j^{\mathfrak{q}}\right)\right) \tag{6.11}$$

Now, we shall prove

$$\bigoplus_{i=1}^{k}\left(\mathcal{A}_i^{\mathfrak{p}}\otimes\mathcal{A}_{k+1}^{\mathfrak{q}}\right)=\left(\begin{array}{c}\left(\begin{array}{c}\left[1-\prod\limits_{i=1}^{k}(1-(\zeta_i^L)^{\mathfrak{p}}(\zeta_{k+1}^L)^{\mathfrak{q}}),1-\prod\limits_{i=1}^{k}(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_{k+1}^U)^{\mathfrak{q}})\right],\\ \left[\prod\limits_{i=1}^{k}\left(1-(1-\vartheta_i^L)^{\mathfrak{p}}(1-\vartheta_{k+1}^L)^{\mathfrak{q}}\right),\prod\limits_{i=1}^{k}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_{k+1}^U)^{\mathfrak{q}}\right)\right]\end{array}\right),\\ \left(\prod\limits_{i=1}^{k}\left(1-(1-\zeta_i)^{\mathfrak{p}}(1-\zeta_{k+1})^{\mathfrak{q}}\right),1-\prod\limits_{i=1}^{k}(1-(\vartheta_i)^{\mathfrak{p}}(\vartheta_{k+1})^{\mathfrak{q}})\right)\end{array}\right) \tag{6.12}$$

Again, for $k=2$, using Eq. (6.9), we have

$$\mathcal{A}_i^{\mathfrak{p}}\otimes\mathcal{A}_{2+1}^{\mathfrak{q}} = \left(\begin{array}{c}\left(\begin{array}{c}\left[(\zeta_i^L)^{\mathfrak{p}}(\zeta_{2+1}^L)^{\mathfrak{q}},(\zeta_i^U)^{\mathfrak{p}}(\zeta_{2+1}^U)^{\mathfrak{q}}\right],\\ \left[1-(1-\vartheta_i^L)^{\mathfrak{p}}(1-\vartheta_{2+1}^L)^{\mathfrak{q}},1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_{2+1}^U)^{\mathfrak{q}}\right]\end{array}\right),\\ (1-(1-\zeta_i)^{\mathfrak{p}}(1-\zeta_{2+1})^{\mathfrak{q}},(\vartheta_i)^{\mathfrak{p}}(\vartheta_{2+1})^{\mathfrak{q}})\end{array}\right) \tag{6.13}$$

and thus,

$$\begin{aligned}&\bigoplus_{i=1}^{2}\left(\mathcal{A}_i^{\mathfrak{p}}\otimes\mathcal{A}_{2+1}^{\mathfrak{q}}\right)\\ =\;&\left(\mathcal{A}_1^{\mathfrak{p}}\otimes\mathcal{A}_{2+1}^{\mathfrak{q}}\right)\oplus\left(\mathcal{A}_2^{\mathfrak{p}}\otimes\mathcal{A}_{2+1}^{\mathfrak{q}}\right)\\ =\;&\left(\begin{array}{c}\left(\begin{array}{c}\left[1-\prod\limits_{i=1}^{2}\left(1-(\zeta_i^L)^{\mathfrak{p}}(\zeta_3^L)^{\mathfrak{q}}\right),1-\prod\limits_{i=1}^{2}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_3^U)^{\mathfrak{q}}\right)\right],\\ \left[\prod\limits_{i=1}^{2}\left(1-(1-\vartheta_i^L)^{\mathfrak{p}}(1-\vartheta_3^L)^{\mathfrak{q}}\right),\prod\limits_{i=1}^{2}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_3^U)^{\mathfrak{q}}\right)\right]\end{array}\right),\\ \left(\prod\limits_{i=1}^{2}\left(1-(1-\zeta_i)^{\mathfrak{p}}(1-\zeta_3)^{\mathfrak{q}}\right),1-\prod\limits_{i=1}^{2}\left(1-(\vartheta_i)^{\mathfrak{p}}(\vartheta_3)^{\mathfrak{q}}\right)\right)\end{array}\right)\end{aligned}$$

If Eq. (6.12) holds for $k = k_0$ i.e.,

$$\begin{aligned}
&\bigoplus_{i=1}^{k_0} \left(\mathcal{A}_i^{\mathfrak{p}} \otimes \mathcal{A}_{k_0+1}^{\mathfrak{q}}\right) \\
&= \left(\begin{array}{l} \left(\begin{array}{l} \left[1 - \prod_{i=1}^{k_0}(1 - (\zeta_i^L)^{\mathfrak{p}}(\zeta_{k_0+1}^L)^{\mathfrak{q}}), 1 - \prod_{i=1}^{k_0}(1 - (\zeta_i^U)^{\mathfrak{p}}(\zeta_{k_0+1}^U)^{\mathfrak{q}}) \right], \\ \left[\prod_{i=1}^{k_0}\left(1 - (1 - \vartheta_i^L)^{\mathfrak{p}}(1 - \vartheta_{k_0+1}^L)^{\mathfrak{q}}\right), \prod_{i=1}^{k_0}\left(1 - (1 - \vartheta_i^U)^{\mathfrak{p}}(1 - \vartheta_{k_0+1}^U)^{\mathfrak{q}}\right) \right] \end{array} \right), \\ \left(\prod_{i=1}^{k_0}\left(1 - (1 - \zeta_i)^{\mathfrak{p}}(1 - \zeta_{k_0+1})^{\mathfrak{q}}\right), 1 - \prod_{i=1}^{k_0}(1 - (\vartheta_i)^{\mathfrak{p}}(\vartheta_{k_0+1})^{\mathfrak{q}}) \right) \end{array} \right)
\end{aligned}$$

then, for $k = k_0 + 1$, we have:

$$\begin{aligned}
&\bigoplus_{i=1}^{k_0+1} \left(\mathcal{A}_i^{\mathfrak{p}} \otimes \mathcal{A}_{k_0+2}^{\mathfrak{q}}\right) \\
&= \bigoplus_{i=1}^{k_0} \left(\mathcal{A}_i^{\mathfrak{p}} \otimes \mathcal{A}_{k_0+2}^{\mathfrak{q}}\right) \oplus \left(\mathcal{A}_{k_0+1}^{\mathfrak{p}} \otimes \mathcal{A}_{k_0+2}^{\mathfrak{q}}\right) \\
&= \left(\begin{array}{l} \left(\begin{array}{l} \left[1 - \prod_{i=1}^{k_0+1}\left(1 - (\zeta_i^L)^{\mathfrak{p}}(\zeta_{k_0+2}^L)^{\mathfrak{q}}\right), \left(1 - (\zeta_i^U)^{\mathfrak{p}}(\zeta_{k_0+2}^U)^{\mathfrak{q}}\right) \right], \\ \left[\prod_{i=1}^{k_0+1}\left(1 - (1 - \vartheta_i^L)^{\mathfrak{p}}(1 - \vartheta_{k_0+2}^L)^{\mathfrak{q}}\right), \prod_{i=1}^{k_0+1}\left(1 - (1 - \vartheta_i^U)^{\mathfrak{p}}(1 - \vartheta_{k_0+2}^U)^{\mathfrak{q}}\right) \right] \end{array} \right), \\ \left(\prod_{i=1}^{k_0+1}\left(1 - (1 - \zeta_i)^{\mathfrak{p}}(1 - \zeta_{k_0+2})^{\mathfrak{q}}\right), 1 - \prod_{i=1}^{k_0+1}\left(1 - (\vartheta_i)^{\mathfrak{p}}(\vartheta_{k_0+2})^{\mathfrak{q}}\right) \right) \end{array} \right)
\end{aligned}$$

and hence Eq. (6.12) holds for $k = k_0 + 1$. Thus, it holds true for every k. Similarly,

$$\bigoplus_{j=1}^{k} \left(\mathcal{A}_{k+1}^{\mathfrak{p}} \otimes \mathcal{A}_j^{\mathfrak{q}}\right) = \left(\begin{array}{l} \left(\begin{array}{l} \left[1 - \prod_{j=1}^{k}(1 - (\zeta_{k+1}^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}), 1 - \prod_{j=1}^{k}(1 - (\zeta_{k+1}^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}) \right], \\ \left[\prod_{j=1}^{k}\left(1 - (1 - \vartheta_{k+1}^L)^{\mathfrak{p}}(1 - \vartheta_j^L)^{\mathfrak{q}}\right), \prod_{j=1}^{k}\left(1 - (1 - \vartheta_{k+1}^U)^{\mathfrak{p}}(1 - \vartheta_j^U)^{\mathfrak{q}}\right) \right] \end{array} \right), \\ \left(\prod_{j=1}^{k}\left(1 - (1 - \zeta_{k+1})^{\mathfrak{p}}(1 - \zeta_j)^{\mathfrak{q}}\right), 1 - \prod_{j=1}^{k}(1 - (\vartheta_{k+1})^{\mathfrak{p}}(\vartheta_j)^{\mathfrak{q}}) \right) \end{array} \right) \quad (6.14)$$

Therefore, by using Eqs. (6.10), (6.12) and (6.14), Eq. (6.11) becomes

$$\bigoplus_{\substack{i,j=1\\i\neq j}}^{k}\left(\mathcal{A}_i^{\mathfrak{p}}\otimes\mathcal{A}_j^{\mathfrak{q}}\right)$$

$$=\left(\begin{array}{l}\left(\begin{array}{l}\left[1-\prod\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(1-(\zeta_i^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}\right),1-\prod\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)\right],\\ \left[\prod\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(1-(1-\vartheta_i^L)^{\mathfrak{p}}(1-\vartheta_j^L)^{\mathfrak{q}}\right),\prod\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}}\right)\right]\end{array}\right),\\ \left(\prod\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(1-(1-\zeta_i)^{\mathfrak{p}}(1-\zeta_j)^{\mathfrak{q}}\right),1-\prod\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(1-(\vartheta_i)^{\mathfrak{p}}(\vartheta_j)^{\mathfrak{q}}\right)\right)\end{array}\right)$$

$$\oplus\left(\begin{array}{l}\left(\begin{array}{l}\left[1-\prod\limits_{i=1}^{k}\left(1-(\zeta_i^L)^{\mathfrak{p}}(\zeta_{k+1}^L)^{\mathfrak{q}}\right),1-\prod\limits_{i=1}^{k}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_{k+1}^U)^{\mathfrak{q}}\right)\right],\\ \left[\prod\limits_{i=1}^{k}\left(1-\left(1-\vartheta_i^L\right)^{\mathfrak{p}}\left(1-\vartheta_{k+1}^L\right)^{\mathfrak{q}}\right),\prod\limits_{i=1}^{k}\left(1-\left(1-\vartheta_i^U\right)^{\mathfrak{p}}\left(1-\vartheta_{k+1}^U\right)^{\mathfrak{q}}\right)\right]\end{array}\right),\\ \left(\prod\limits_{i=1}^{k}\left(1-\left(1-\zeta_i\right)^{\mathfrak{p}}\left(1-\zeta_{k+1}\right)^{\mathfrak{q}}\right),1-\prod\limits_{i=1}^{k}\left(1-(\vartheta_i)^{\mathfrak{p}}(\vartheta_{k+1})^{\mathfrak{q}}\right)\right)\end{array}\right)$$

$$\oplus\left(\begin{array}{l}\left(\begin{array}{l}\left[1-\prod\limits_{j=1}^{k}\left(1-(\zeta_{k+1}^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}\right),1-\prod\limits_{j=1}^{k}\left(1-(\zeta_{k+1}^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)\right],\\ \left[\prod\limits_{i=1}^{k}\left(1-\left(1-\vartheta_{k+1}^L\right)^{\mathfrak{p}}\left(1-\vartheta_j^L\right)^{\mathfrak{q}}\right),\prod\limits_{i=1}^{k}\left(1-\left(1-\vartheta_{k+1}^U\right)^{\mathfrak{p}}\left(1-\vartheta_j^U\right)^{\mathfrak{q}}\right)\right]\end{array}\right),\\ \left(\prod\limits_{i=1}^{k}\left(1-\left(1-\zeta_{k+1}\right)^{\mathfrak{p}}\left(1-\zeta_j\right)^{\mathfrak{q}}\right),1-\left(1-\vartheta_{k+1}\right)^{\mathfrak{p}}\left(1-\vartheta_j\right)^{\mathfrak{q}}\right)\end{array}\right)$$

$$=\left(\begin{array}{l}\left(\begin{array}{l}\left[1-\prod\limits_{\substack{i,j=1\\i\neq j}}^{k+1}\left(1-(\zeta_i^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}\right),1-\prod\limits_{\substack{i,j=1\\i\neq j}}^{k+1}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)\right],\\ \left[\prod\limits_{\substack{i,j=1\\i\neq j}}^{k+1}\left(1-\left(1-\vartheta_i^L\right)^{\mathfrak{p}}\left(1-\vartheta_j^L\right)^{\mathfrak{q}}\right),\prod\limits_{\substack{i,j=1\\i\neq j}}^{k+1}\left(1-\left(1-\vartheta_i^U\right)^{\mathfrak{p}}\left(1-\vartheta_j^U\right)^{\mathfrak{q}}\right)\right]\end{array}\right),\\ \left(\prod\limits_{\substack{i,j=1\\i\neq j}}^{k+1}\left(1-\left(1-\zeta_i\right)^{\mathfrak{p}}\left(1-\zeta_j\right)^{\mathfrak{q}}\right),1-\prod\limits_{\substack{i,j=1\\i\neq j}}^{k+1}\left(1-(\vartheta_i)^{\mathfrak{p}}(\vartheta_j)^{\mathfrak{q}}\right)\right)\end{array}\right)$$

which is true for $n = k + 1$ and hence by principle of mathematical induction, Eq. (6.12) holds for all positive integers n.

Now,

$$\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\i\neq j}}^{n}(\mathcal{A}_i^{\mathfrak{p}}\otimes\mathcal{A}_j^{\mathfrak{q}})\right)$$

$$=\left(\begin{array}{l}\left(\begin{array}{l}\left[1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(\zeta_i^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}\right)\right)^{\frac{1}{n(n-1)}},1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)\right)^{\frac{1}{n(n-1)}}\right],\\ \left[\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta_i^L)^{\mathfrak{p}}(1-\vartheta_j^L)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}},\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right]\end{array}\right),\\ \left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\zeta_i)^{\mathfrak{p}}(1-\zeta_j)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}},1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(\vartheta_i)^{\mathfrak{p}}(\vartheta_j)^{\mathfrak{q}}\right)\right)^{\frac{1}{n(n-1)}}\right)\end{array}\right)$$

So by definition of CIFBM, we get

$$\text{CIFBM}^{\mathfrak{p},\mathfrak{q}}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)=\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\i\neq j}}^{n}\left(\mathcal{A}_i^{\mathfrak{p}}\otimes\mathcal{A}_j^{\mathfrak{q}}\right)\right)\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}$$

$$=\left(\begin{array}{l}\left(\begin{array}{l}\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(\zeta_i^L)^{\mathfrak{p}}(\zeta_j^L)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\right],\\ \left[\begin{array}{l}1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta_i^L)^{\mathfrak{p}}(1-\vartheta_j^L)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}},\\ 1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\end{array}\right]\end{array}\right),\\ \left(1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\zeta_i)^{\mathfrak{p}}(1-\zeta_j)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(\vartheta_i)^{\mathfrak{p}}(\vartheta_j)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\right)\end{array}\right)$$

Hence, the result.

Finally, in order to show the aggregated value by using CIFBM is also CIFN, it is necessary to satisfy the CIFN property. For it, since $\mathcal{A}_i=\left(\left([\zeta_i^L,\zeta_i^U],[\vartheta_i^L,\vartheta_i^U]\right),(\zeta_i,\vartheta_i)\right)$ is CIFN which implies that $[\zeta_i^L,\zeta_i^U],[\vartheta_i^L,\vartheta_i^U]\subseteq[0,1]$ and $\zeta_i^U+\vartheta_i^U\leq 1$, $\zeta_i,\vartheta_i\in[0,1]$ and $\zeta_i+\vartheta_i\leq 1$. Thus, for any positive number $\mathfrak{p}$ and $\mathfrak{q}$, we have $0\leq 1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\leq 1$ which turns $0\leq\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\leq 1$ and hence $0\leq\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\leq$

1. On the other hand, $0 \leq \vartheta_i^U, \vartheta_j^U \leq 1$, thus $0 \leq (1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}} \leq 1$, which further gives $0 \leq 1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} \leq 1$. Lastly, from $\zeta_i^U+\vartheta_i^U \leq 1$, we have $(\zeta_i^U)^{\mathfrak{p}} \leq (1-\vartheta_i^U)^{\mathfrak{p}}$ and $(\zeta_j^U)^{\mathfrak{q}} \leq (1-\vartheta_j^U)^{\mathfrak{q}}$ and thus follows that $\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}} \geq \prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}$ which in turns leads us to

$$\begin{aligned}
&\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} + 1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\\
= \;& 1-\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta_i^U)^{\mathfrak{p}}(1-\vartheta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}} - \left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(\zeta_i^U)^{\mathfrak{p}}(\zeta_j^U)^{\mathfrak{q}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\right]\\
\leq \;& 1
\end{aligned}$$

Similarly, we can prove for remaining components of CIFBM and hence the aggregated value by CIFBM operator is again CIFN. This completes the proof. □

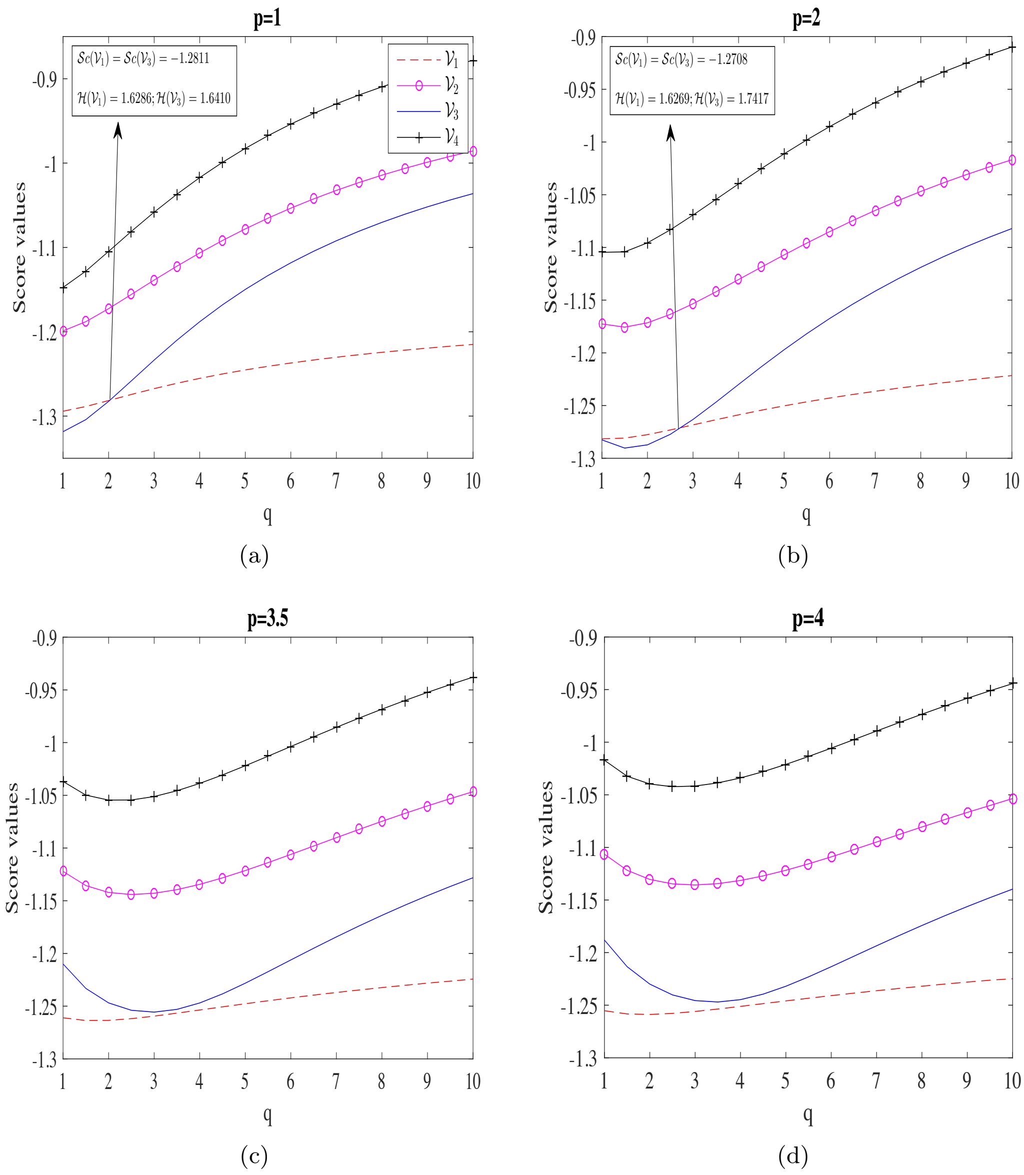

Figure 6.1: Effect of the parameter $\mathfrak{q}$ on to the score value by fixing the parameter $\mathfrak{p}$.

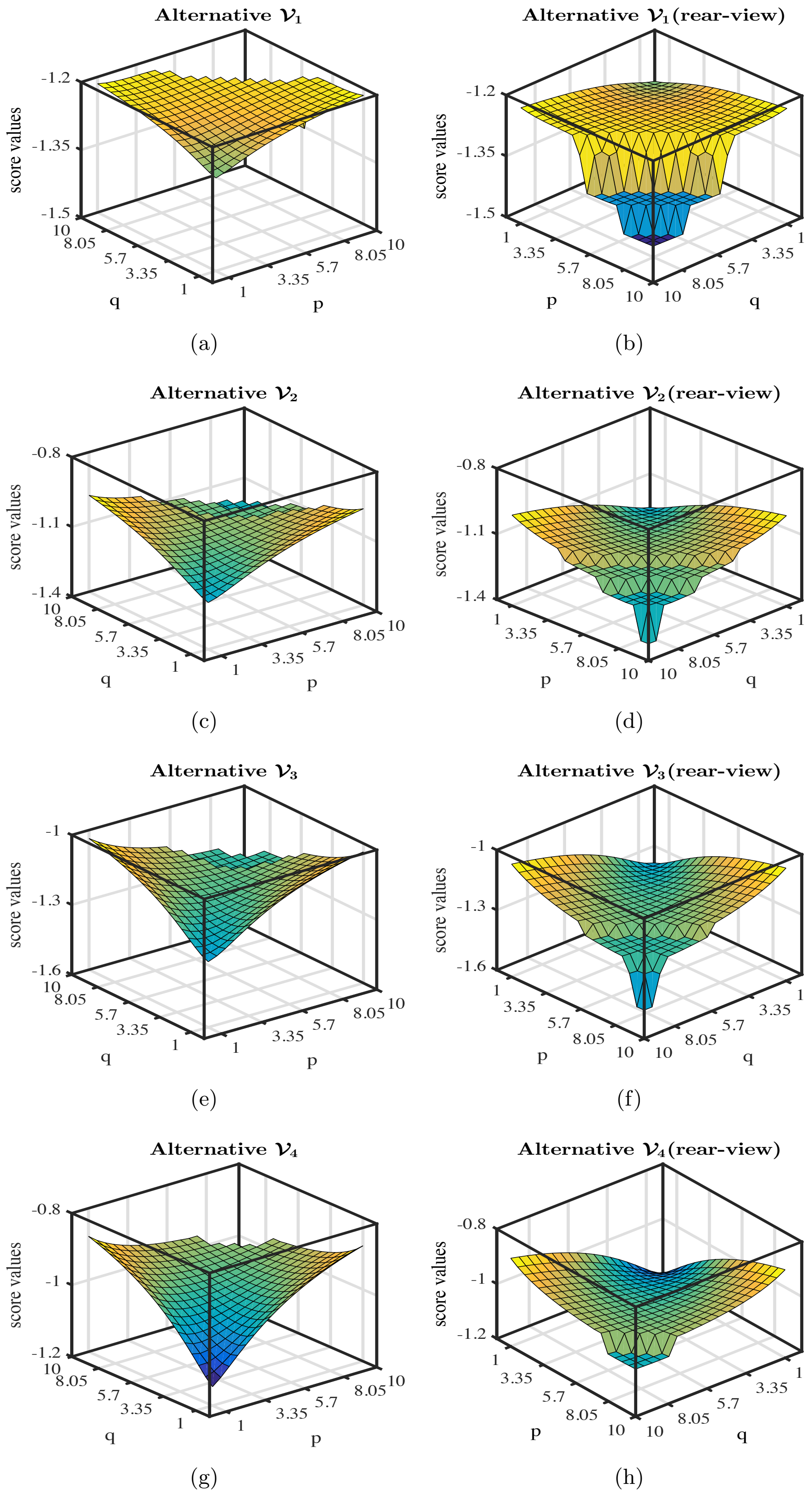

Figure 6.2: Score values of alternative for different values of $\mathfrak{p}$ and $\mathfrak{q}$.

Table 6.1: Rating values of the alternatives in terms of CIFNs.

Alternatives	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$(([0.50, 0.60], [0.10, 0.20]), (0.40, 0.20))$	$(([0.20, 0.30], [0.40, 0.50]), (0.30, 0.20))$	$(([0.40, 0.60], [0.20, 0.30]), (0.20, 0.35))$
$\mathcal{V}_2$	$(([0.20, 0.30], [0.40, 0.50]), (0.40, 0.60))$	$(([0.15, 0.25], [0.30, 0.35]), (0.20, 0.30))$	$(([0.20, 0.40], [0.10, 0.20]), (0.40, 0.50))$
$\mathcal{V}_3$	$(([0.50, 0.60], [0.20, 0.30]), (0.20, 0.40))$	$(([0.40, 0.60], [0.25, 0.35]), (0.30, 0.40))$	$(([0.50, 0.70], [0.10, 0.15]), (0.30, 0.50))$
$\mathcal{V}_4$	$(([0.30, 0.50], [0.10, 0.30]), (0.10, 0.30))$	$(([0.40, 0.55], [0.15, 0.20]), (0.30, 0.40))$	$(([0.40, 0.50], [0.20, 0.30]), (0.45, 0.35))$

Table 6.2: Normalized decision ratings.

Alternatives	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$(([0.10, 0.20], [0.50, 0.60]), (0.20, 0.40))$	$(([0.20, 0.30], [0.40, 0.50]), (0.30, 0.20))$	$(([0.20, 0.30], [0.40, 0.60]), (0.35, 0.20))$
$\mathcal{V}_2$	$(([0.40, 0.50], [0.20, 0.30]), (0.60, 0.40))$	$(([0.15, 0.25], [0.30, 0.35]), (0.20, 0.30))$	$(([0.10, 0.20], [0.20, 0.40]), (0.50, 0.40))$
$\mathcal{V}_3$	$(([0.20, 0.30], [0.50, 0.60]), (0.40, 0.20))$	$(([0.40, 0.60], [0.25, 0.35]), (0.30, 0.40))$	$(([0.10, 0.15], [0.50, 0.70]), (0.50, 0.30))$
$\mathcal{V}_4$	$(([0.10, 0.30], [0.30, 0.50]), (0.30, 0.10))$	$(([0.40, 0.55], [0.15, 0.20]), (0.30, 0.40))$	$(([0.20, 0.30], [0.40, 0.50]), (0.35, 0.45))$

Table 6.3: Effects on the ranking with the variation of the parameters $\mathfrak{p}$ and $\mathfrak{q}$.

$\mathfrak{p}$	$\mathfrak{q}$	$Sc(\mathcal{V}_1)$	$Sc(\mathcal{V}_2)$	$Sc(\mathcal{V}_3)$	$Sc(\mathcal{V}_4)$	Ranking Order
$\mathfrak{p}=1$	$\mathfrak{q}=1$	−1.2942	−1.1989	−1.3185	−1.1474	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathfrak{q}=2$	−1.2815	−1.1726	−1.2825	−1.1047	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathfrak{q}=3$	−1.2674	−1.1386	−1.2335	−1.0582	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
	$\mathfrak{q}=4$	−1.2552	−1.1065	−1.1881	−1.0172	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
$\mathfrak{p}=2$	$\mathfrak{q}=1$	−1.2815	−1.1726	−1.2825	−1.1047	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathfrak{q}=2$	−1.2776	−1.1715	−1.2872	−1.0956	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathfrak{q}=3$	−1.2683	−1.1534	−1.2632	−1.0693	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
	$\mathfrak{q}=4$	−1.2588	−1.1304	−1.2300	−1.0397	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
$\mathfrak{p}=3$	$\mathfrak{q}=1$	−1.2674	−1.1386	−1.2335	−1.0582	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
	$\mathfrak{q}=2$	−1.2683	−1.1534	−1.2632	−1.0693	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
	$\mathfrak{q}=3$	−1.2628	−1.1487	−1.2627	−1.0594	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
	$\mathfrak{q}=4$	−1.2559	−1.1357	−1.2456	−1.0417	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
$\mathfrak{p}=4$	$\mathfrak{q}=1$	−1.2552	−1.1065	−1.1881	−1.0172	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
	$\mathfrak{q}=2$	−1.2588	−1.1304	−1.2300	−1.0397	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
	$\mathfrak{q}=3$	−1.2559	−1.1357	−1.2456	−1.0417	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
	$\mathfrak{q}=4$	−1.2511	−1.1314	−1.2447	−1.0340	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$

Table 6.4: Ranking order of the alternatives with accuracy value at $\mathfrak{q}'$.

Figure	Value of $\mathfrak{q}'$	Accuracy Value at $\mathfrak{q} = \mathfrak{q}'$	Ranking Order		
			When $\mathfrak{q} < \mathfrak{q}'$	When $\mathfrak{q} = \mathfrak{q}'$	When $\mathfrak{q} > \mathfrak{q}'$
6.1a	2.0307	$H(\mathcal{V}_1) = 1.6286$, $H(\mathcal{V}_3) = 1.6410$	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
6.1b	2.742	$H(\mathcal{V}_1) = 1.6269$, $H(\mathcal{V}_3) = 1.7417$	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
6.1c	-	-		$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$	
6.1d	-	-		$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$	

Table 6.5: Surface readings (Sr's) for each alternative's rear view.

	$\mathcal{V}_1$		$\mathcal{V}_2$		$\mathcal{V}_3$		$\mathcal{V}_4$	
	(p, q)	Sc($\mathcal{V}_1$)	(p, q)	Sc($\mathcal{V}_2$)	(p, q)	Sc($\mathcal{V}_3$)	(p, q)	Sc($\mathcal{V}_4$)
Sr(1)(B)	$(1, 1)$	-1.294	$(1, 1)$	-1.199	$(1, 1)$	-1.302	$(1, 1)$	-1.147
Sr(1)(E)	$(10, 4.5)$	-1.226	$(10, 4)$	-1.053	$(10, 5.5)$	-1.166	$(10, 7.5)$	-0.9665
Sr(2)(B)	$(10, 5.5)$	-1.353	$(10, 5)$	-1.102	$(10, 6.5)$	-1.227	$(10, 8)$	-1.029
Sr(2)(E)	$(10, 7.5)$	-1.35	$(10, 6.5)$	-1.115	$(10, 8.5)$	-1.239	$(10, 10)$	-1.027
Sr(3)(B)	$(10, 8.5)$	-1.412	$(10, 7.5)$	-1.173	$(10, 10)$	-1.458	–	–
Sr(3)(E)	$(10, 10)$	-1.41	$(10, 9)$	-1.174	$(10, 10)$	-1.458	–	–
Sr(4)(B)	–	–	$(10, 10)$	-1.323	–	–	–	–
Sr(4)(E)	–	–	$(10, 10)$	-1.323	–	–	–	–

Table 6.6: Comparison analysis with some of the existing approaches.

Comparison with	Score Values				Ranking
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
Xu and Chen [189]	-0.7152	-0.5847	-0.6880	-0.5830	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Shi and He [129]	-0.2680	-0.0685	-0.2244	-0.0301	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Wang and Liu [162]	-0.2593	-0.0635	-0.1552	0.0194	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Chen, Cheng and Tsai [22]	-0.2633	-0.0630	-0.1425	0.0096	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Chen, Cheng and Tsai [23]	-0.2608	-0.0613	-0.2154	-0.0315	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Sivaraman et al. [131]	-0.2120	-0.0792	-0.1910	0.0214	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Wan et al. [150]	-0.2610	-0.0705	-0.1835	-0.0100	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Dugenci [34]	0.7940	0.7316	0.7693	0.7106	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Garg [44]	0.1082	0.1101	0.1230	0.1649	$\mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1$

Chapter 7

Extended TOPSIS method for group decision-making problems under cubic intuitionistic fuzzy environment[1]

In this chapter, we present a novel multi-criteria group decision making (MCGDM) method under cubic intuitionistic fuzzy (CIF) environment by integrating extended technique for order preference with respect to the similarity to the ideal solution (TOPSIS) method. We presented some series of distance measures between the pairs of CIFSs and investigated their various relationship. Further, under this environment, a group decision-making method based on the proposed measure is presented by taking the different priority pairs of the decision makers. A practical example is provided to verify the developed approach and to demonstrate its practicality and feasibility, we have compared their results with the several existing approaches' outcomes.

7.1 Introduction

In decision-making (DM) approaches information measures constituted with various methodologies act as a boon for the person who has to reach some conclusion, by keeping all the favorable as well as unfavorable conditions in their mind. On such methodology, technique

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for order preference with respect to the similarity to the ideal solution (TOPSIS), developed by Hwang and Yoon [69], is a well-known multicriteria decision-making (MCDM) method. The aim of this method is to choose the best alternative whose distance from its positive ideal solution is shortest. After its existence, numerous attempts are made by the researchers to apply the TOPSIS method under the fuzzy and IFS environment. For instance, Boran et al. [12] applied the TOPSIS method to solve the problem of Human Resource Personnel selection. Dugenci [34] presented a distance measure for interval-valued IF (IVIF) set and their applications to MCDM with incomplete weight information. Garg [44] presented a generalized improved score function for IVIFSs and their based TOPSIS method for solving the decision-making problems. Mohammadi et al. [113] presented a gray relational analysis and TOPSIS approach to solving the DM problems. Biswas and Kumar [10] presented an integrated TOPSIS approach for solving the DM problems with IVIFS environment. Vommi [149] presented a TOPSIS method using statistical distances to solve DM problems. Li [85] presented a nonlinear programming methodology based TOPSIS method for solving MADM problems under IVIFS environment. Garg and Kumar [53] presented new similarity measures for IFSs based on the connection number of the set pair analysis. Askarifar et al. [3] presented an approach to studying the framework of Iran's seashores using TOPSIS and best-worst MCDM methods. In [59, 152], authors developed a group decision-making method under IVIF environment by integrating extended TOPSIS and linear programming methods. Kumar and Garg [81, 82] presented TOPSIS approach for solving the DM problems by using connection number of the set pair analysis theory. Since all these facilitate the uncertainties to a great extent, but still they cannot withstand the situations where the decision-maker has to consider the falsity corresponding to the truth value ranging over an interval. For this reason, CIFS is described by two parts simultaneously; where one represents the membership degrees by an IVIF number (IVIFN) and the other represents the membership degrees by IF number (IFN).

Keeping the advantages of the CIFS, in this chapter, we study the MCGDM problem under CIF setting and propose a methodology that utilizes extended TOPSIS method where each of the element is characterized by CIF numbers (CIFNs). CIFNs combine the advantages of both IVIFNs and IFNs. Furthermore, we propose some new weighted and generalized weighted distance measures in order to signify the level of resemblance between two CIF values based upon the decision values and both the CIF positive ideal alternative (CIF-PIA) and CIF negative ideal alternative (CIF-NIA). Several desirable properties of the proposed distance and weighted distance measures are investigated. Multiple decision makers have been included in the decision-making process, highlighting the impetus of different perspectives which makes the proposed approach more realistic for an MCGDM process. The presented approach has been illustrated with a numerical example to verify its feasibility and effectiveness. Finally, the computed results obtained by the presented approach are compared with several existing approaches results to show the superiority of the approach.

7.2 Distance measures for CIFS

In this section, we propose some new distance measures for the non-zero CIFN over the finite universal set $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$. For it, we consider $\Phi(\mathcal{X})$ to be family of CIFSs over the set $\mathcal{X}$.

Definition 7.2.1. A real-valued function $d : \Phi(\mathcal{X}) \times \Phi(\mathcal{X}) \to [0, 1]$ is called the distance measure if it satisfies the following properties for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Phi(\mathcal{X})$:

(P1) $0 \leq d(\mathcal{A}, \mathcal{B}) \leq 1$;

(P2) $d(\mathcal{A}, \mathcal{B}) = 0$ if and only if $\mathcal{A} = \mathcal{B}$;

(P3) $d(\mathcal{A}, \mathcal{B}) = d(\mathcal{B}, \mathcal{A})$;

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $d(\mathcal{A}, \mathcal{B}) \leq d(\mathcal{A}, \mathcal{C})$ and $d(\mathcal{B}, \mathcal{C}) \leq d(\mathcal{A}, \mathcal{C})$.

where $\Phi(\cdot)$ represents the set of all CIFSs.

Definition 7.2.2. Let $\mathcal{A} = \big(([\zeta_A^L(x),\, \zeta_A^U(x)],\, [\vartheta_A^L(x),\, \vartheta_A^U(x)]),\, (\zeta_A(x),\, \vartheta_A(x))\big)$ and $\mathcal{B} = \big(([\zeta_B^L(x),\, \zeta_B^U(x)],\, [\vartheta_B^L(x),\, \vartheta_B^U(x)]),\, (\zeta_B(x),\, \vartheta_B(x))\big)$ be two CIFNs, then, for $\eta \geq 1$, the following distance measures are defined as

(i) Distance measures :

$$d''_\eta(\mathcal{A},\mathcal{B}) = \left(\frac{1}{6}\sum_{i=1}^{n}\left\{\begin{array}{l}|\zeta_A^L(x_i)-\zeta_B^L(x_i)|^\eta+|\zeta_A^U(x_i)-\zeta_B^U(x_i)|^\eta+|\vartheta_A^L(x_i)-\vartheta_B^L(x_i)|^\eta\\ +|\vartheta_A^U(x_i)-\vartheta_B^U(x_i)|^\eta+|\zeta_A(x_i)-\zeta_B(x_i)|^\eta+|\vartheta_A(x_i)-\vartheta_B(x_i)|^\eta\end{array}\right\}\right)^{1/\eta} \quad (7.1)$$

(ii) Normalized distance measures:

$$d'_\eta(\mathcal{A},\mathcal{B}) = \left(\frac{1}{6n}\sum_{i=1}^{n}\left\{\begin{array}{l}|\zeta_A^L(x_i)-\zeta_B^L(x_i)|^\eta+|\zeta_A^U(x_i)-\zeta_B^U(x_i)|^\eta+|\vartheta_A^L(x_i)-\vartheta_B^L(x_i)|^\eta\\ +|\vartheta_A^U(x_i)-\vartheta_B^U(x_i)|^\eta+|\zeta_A(x_i)-\zeta_B(x_i)|^\eta+|\vartheta_A(x_i)-\vartheta_B(x_i)|^\eta\end{array}\right\}\right)^{1/\eta} \quad (7.2)$$

Next, we validate that the above defined measures are valid distance measures.

Theorem 7.2.1. The measure d'_η between two CIFSs $\mathcal{A}$ and $\mathcal{B}$ satisfies the properties (P1)–(P4) as defined in Definition 7.2.1.

Proof. In order to prove that measure defined in Eq. (7.1) is a valid distance measure, we shall prove that it satisfies the properties (P1) - (P4) as defined in Definition 7.2.1 for a collection of CIFNs $\mathcal{A} = \big(([\zeta_A^L(x),\ \zeta_A^U(x)],\ [\vartheta_A^L(x),\ \vartheta_A^U(x)]),\ (\zeta_A(x),\ \vartheta_A(x))\big)$ and $\mathcal{B} = \big(([\zeta_B^L(x),\ \zeta_B^U(x)],\ [\vartheta_B^L(x),\ \vartheta_B^U(x)]),\ (\zeta_B(x),\ \vartheta_B(x))\big)$. For any real number $\eta \geq 1$ and a collection of CIFSs $\mathcal{A}$ and $\mathcal{B}$, we have

(P1) By definition of d'_η, we have $d'_\eta(\mathcal{A},\mathcal{B}) \geq 0$, so for an arbitrary CIFSs $\mathcal{A}$ and $\mathcal{B}$, it is enough to show that $d'_\eta(\mathcal{A},\mathcal{B}) \leq 1$. Since $\mathcal{A}$ and $\mathcal{B}$ are two CIFSs, so we have, $0 \leq \zeta_A^L(x_i),\ \zeta_A^U(x_i),\ \vartheta_A^L(x_i),\ \vartheta_A^U(x_i) \leq 1$, $0 \leq \zeta_A(x_i),\ \vartheta_A(x_i) \leq 1$, $0 \leq \zeta_B^L(x_i),\ \zeta_B^U(x_i),\ \vartheta_B^L(x_i),\ \vartheta_B^U(x_i) \leq 1$ and $0 \leq \zeta_B(x_i),\ \vartheta_B(x_i) \leq 1$. This implies that $0 \leq |\zeta_A^L(x_i)-\zeta_B^L(x_i)|^\eta \leq 1$, $0 \leq |\zeta_A^U(x_i)-\zeta_B^U(x_i)|^\eta \leq 1$, $0 \leq |\vartheta_A^L(x_i)-\vartheta_B^L(x_i)|^\eta \leq 1$ and $0 \leq |\vartheta_A^U(x_i)-\vartheta_B^U(x_i)|^\eta \leq 1$. Similarly, $0 \leq |\zeta_A(x_i)-\zeta_B(x_i)|^\eta \leq 1$ and $0 \leq |\vartheta_A(x_i)-\vartheta_B(x_i)|^\eta \leq 1$. Thus, it follows that $0 \leq d'_\eta(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) For any two CIFSs $\mathcal{A}$ and $\mathcal{B}$,

$$\begin{aligned} & d'_\eta(\mathcal{A},\mathcal{B}) = 0 \\ \Leftrightarrow\ & \frac{1}{6n}\sum_{i=1}^{n}\left\{\begin{array}{l}|\zeta_A^L(x_i)-\zeta_B^L(x_i)|^\eta+|\zeta_A^U(x_i)-\zeta_B^U(x_i)|^\eta+|\vartheta_A^L(x_i)-\vartheta_B^L(x_i)|^\eta\\ +|\vartheta_A^U(x_i)-\vartheta_B^U(x_i)|^\eta+|\zeta_A(x_i)-\zeta_B(x_i)|^\eta+|\vartheta_A(x_i)-\vartheta_B(x_i)|^\eta\end{array}\right\} = 0\end{aligned}$$

$$\Leftrightarrow \quad |\zeta_A^L(x_i) - \zeta_B^L(x_i)|^\eta = 0, |\zeta_A^U(x_i) - \zeta_B^U(x_i)|^\eta = 0,$$
$$|\vartheta_A^L(x_i) - \vartheta_B^L(x_i)|^\eta = 0, |\vartheta_A^U(x_i) - \vartheta_B^U(x_i)|^\eta = 0,$$
$$|\zeta_A(x_i) - \zeta_B(x_i)|^\eta = 0 \text{ and } |\vartheta_A(x_i) - \vartheta_B(x_i)|^\eta = 0; \text{ for all } i$$
$$\Leftrightarrow \quad \zeta_A^L(x_i) = \zeta_B^L(x_i), \zeta_A^U(x_i) = \zeta_B^U(x_i), \vartheta_A^L(x_i) = \vartheta_B^L(x_i), \vartheta_A^U(x_i) = \vartheta_B^U(x_i),$$
$$\zeta_A(x_i) = \zeta_B(x_i), \text{ and } \vartheta_A(x_i) = \vartheta_B(x_i) \qquad ; \text{for all } i$$
$$\Leftrightarrow \quad \mathcal{A} = \mathcal{B}$$

(P3) For any two real numbers a and b, we have $|a - b| = |b - a|$. Thus, we have $d'_\eta(\mathcal{A}, \mathcal{B}) = d'_\eta(\mathcal{B}, \mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ are R-order CIFNs then for all i, we have $[\zeta_A^L(x_i), \zeta_A^U(x_i)] \subseteq [\zeta_B^L(x_i), \zeta_B^U(x_i)] \subseteq [\zeta_C^L(x_i), \zeta_C^U(x_i)]$, $[\vartheta_A^L(x_i), \vartheta_A^U(x_i)] \supseteq [\vartheta_B^L(x_i), \vartheta_B^U(x_i)] \supseteq [\vartheta_C^L(x_i), \vartheta_C^U(x_i)]$ $\zeta_A(x_i) \geq \zeta_B(x_i) \geq \zeta_C(x_i)$ and $\vartheta_A(x_i) \leq \vartheta_B(x_i) \leq \vartheta_C(x_i)$. Therefore,

$$|\zeta_A^L(x_i) - \zeta_B^L(x_i)|^\eta \leq |\zeta_A^L(x_i) - \zeta_C^L(x_i)|^\eta, \quad |\zeta_A^U(x_i) - \zeta_B^U(x_i)|^\eta \leq |\zeta_A^U(x_i) - \zeta_C^U(x_i)|^\eta,$$
$$|\vartheta_A^L(x_i) - \vartheta_B^L(x_i)|^\eta \geq |\vartheta_A^L(x_i) - \vartheta_C^L(x_i)|^\eta, \quad |\vartheta_A^U(x_i) - \vartheta_B^U(x_i)|^\eta \geq |\vartheta_A^U(x_i) - \vartheta_C^U(x_i)|^\eta,$$
$$|\zeta_A(x_i) - \zeta_B(x_i)|^\eta \geq |\zeta_A(x_i) - \zeta_C(x_i)|^\eta, \text{and} \quad |\vartheta_A(x_i) - \vartheta_B(x_i)|^\eta \leq |\vartheta_A(x_i) - \vartheta_C(x_i)|^\eta.$$

Thus,

$$\begin{aligned} d'_\eta(\mathcal{A}, \mathcal{C}) &= \left[\frac{1}{6n} \sum_{i=1}^n \left\{ |\zeta_A^L(x_i) - \zeta_C^L(x_i)|^\eta + |\zeta_A^U(x_i) - \zeta_C^U(x_i)|^\eta + |\vartheta_A^L(x_i) - \vartheta_C^L(x_i)|^\eta \right.\right. \\ &\quad \left.\left. + |\vartheta_A^U(x_i) - \vartheta_C^U(x_i)|^\eta + |\zeta_A(x_i) - \zeta_C(x_i)|^\eta + |\vartheta_A(x_i) - \vartheta_C(x_i)|^\eta \right\}\right]^{1/\eta} \\ &\geq \left[\frac{1}{6n} \sum_{i=1}^n \left\{ |\zeta_A^L(x_i) - \zeta_B^L(x_i)|^\eta + |\zeta_A^U(x_i) - \zeta_B^U(x_i)|^\eta + |\vartheta_A^L(x_i) - \vartheta_B^L(x_i)|^\eta \right.\right. \\ &\quad \left.\left. + |\vartheta_A^U(x_i) - \vartheta_B^U(x_i)|^\eta + |\zeta_A(x_i) - \zeta_B(x_i)|^\eta + |\vartheta_A(x_i) - \vartheta_B(x_i)|^\eta \right\}\right]^{1/\eta}. \end{aligned}$$

Hence, $d'_\eta(\mathcal{A}, \mathcal{C}) \geq d'_\eta(\mathcal{A}, \mathcal{B})$. Similarly, $d'_\eta(\mathcal{A}, \mathcal{C}) \geq d'_\eta(\mathcal{B}, \mathcal{C})$. Similarly, we can prove it for P-order CIFNs.

Hence, $d'_\eta(\eta \geq 1)$ is a valid distance measure. □

Theorem 7.2.2. The measures d''_η satisfies the inequality $d''_\eta \leq n^{1/\eta}$.

Proof. For any real number $\eta \geq 1$, and for two CIFSs $\mathcal{A}$ and $\mathcal{B}$, we have $|\zeta_A^L(x_i) - \zeta_B^L(x_i)|^\eta \leq 1$, $|\zeta_A^U(x_i) - \zeta_B^U(x_i)|^\eta \leq 1$ and so on. Therefore, we get

$$\begin{aligned} d''_\eta(\mathcal{A},\mathcal{B}) &= \left(\frac{1}{6}\sum_{i=1}^{n}\left\{\begin{array}{l} |\zeta_A^L(x_i) - \zeta_B^L(x_i)|^\eta + |\zeta_A^U(x_i) - \zeta_B^U(x_i)|^\eta + |\vartheta_A^L(x_i) - \vartheta_B^L(x_i)|^\eta \\ + |\vartheta_A^U(x_i) - \vartheta_B^U(x_i)|^\eta + |\zeta_A(x_i) - \zeta_B(x_i)|^\eta + |\vartheta_A(x_i) - \vartheta_B(x_i)|^\eta \end{array}\right\}\right)^{1/\eta} \\ &\leq \left(\frac{1}{6}\sum_{i=1}^{n}(1+1+1+1+1+1)\right)^{1/\eta} \\ &\leq n^{1/\eta} \end{aligned}$$

Hence, the result. □

Theorem 7.2.3. The measures d''_η and d'_η satisfy the inequality $d'_\eta \leq \sqrt[\eta]{d'_1}$ and $d''_\eta \leq \sqrt[\eta]{d''_1}$.

Proof. For any real number $\eta \geq 1$, and for two CIFSs $\mathcal{A}$ and $\mathcal{B}$, we have $|\zeta_A^L(x_i) - \zeta_B^L(x_i)|^\eta \leq |\zeta_A^L(x_i) - \zeta_B^L(x_i)|$, $|\zeta_A^U(x_i) - \zeta_B^U(x_i)|^\eta \leq |\zeta_A^U(x_i) - \zeta_B^U(x_i)|$ and so on. Therefore, we get

$$\begin{aligned} d'_\eta(\mathcal{A},\mathcal{B}) &= \left(\frac{1}{6n}\sum_{i=1}^{n}\left\{\begin{array}{l} |\zeta_A^L(x_i) - \zeta_B^L(x_i)|^\eta + |\zeta_A^U(x_i) - \zeta_B^U(x_i)|^\eta + |\vartheta_A^L(x_i) - \vartheta_B^L(x_i)|^\eta \\ + |\vartheta_A^U(x_i) - \vartheta_B^U(x_i)|^\eta + |\zeta_A(x_i) - \zeta_B(x_i)|^\eta + |\vartheta_A(x_i) - \vartheta_B(x_i)|^\eta \end{array}\right\}\right)^{1/\eta} \\ &\leq \left(\frac{1}{6n}\sum_{i=1}^{n}\left\{\begin{array}{l} |\zeta_A^L(x_i) - \zeta_B^L(x_i)| + |\zeta_A^U(x_i) - \zeta_B^U(x_i)| + |\vartheta_A^L(x_i) - \vartheta_B^L(x_i)| \\ + |\vartheta_A^U(x_i) - \vartheta_B^U(x_i)| + |\zeta_A(x_i) - \zeta_B(x_i)| + |\vartheta_A(x_i) - \vartheta_B(x_i)| \end{array}\right\}\right)^{1/\eta} \\ &\leq (d'_1(\mathcal{A},\mathcal{B}))^{1/\eta} \end{aligned}$$

Similarly, we can prove $d''_\eta \leq \sqrt[\eta]{d''_1}$. □

Theorem 7.2.4. The measures d''_η and d'_η satisfy the equation $d''_\eta = n^{1/\eta} d'_\eta$.

Proof. Easily follows from the definitions of d'_η and d''_η. □

Remark 7.2.1. From the proposed measure, it has been observed that

(i) When $\eta = 1$, Eq. (7.2) reduces to the normalized hamming distance measure, and

(ii) When $\eta = 2$, Eq. (7.2) reduces to the normalized Euclidean distance measure.

As in practical situations, many times we have to deal with such situations in which various CIFSs may have weights assigned to them. So, taking into account weights $\omega_i (i = 1, 2, \ldots, n)$, where each $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$, we define generalized weighted distances between two CIFSs $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$d_\eta(\mathcal{A}, \mathcal{B}) = \left(\frac{1}{6} \sum_{i=1}^{n} \omega_i \left\{ \begin{array}{l} |\zeta_A^L(x_i) - \zeta_B^L(x_i)|^\eta + |\zeta_A^U(x_i) - \zeta_B^U(x_i)|^\eta + |\vartheta_A^L(x_i) - \vartheta_B^L(x_i)|^\eta \\ + |\vartheta_A^U(x_i) - \vartheta_B^U(x_i)|^\eta + |\zeta_A(x_i) - \zeta_B(x_i)|^\eta + |\vartheta_A(x_i) - \vartheta_B(x_i)|^\eta \end{array} \right\} \right)^{1/\eta} \tag{7.3}$$

Theorem 7.2.5. The weighted distance measure d_η, $(1 \leq \eta < \infty)$, defined in Eq. (7.3), satisfies the following properties:

(P1) $0 \leq d_\eta(\mathcal{A}, \mathcal{B}) \leq 1$;

(P2) $d_\eta(\mathcal{A}, \mathcal{B}) = 0 \Leftrightarrow A = B$;

(P3) $d_\eta(\mathcal{A}, \mathcal{B}) = d_\eta(\mathcal{B}, \mathcal{A})$;

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $d_\eta(\mathcal{A}, \mathcal{B}) \leq d_\eta(\mathcal{A}, \mathcal{C})$ and $d_\eta(\mathcal{B}, \mathcal{C}) \leq d_\eta(\mathcal{A}, \mathcal{C})$.

Proof. The proof is similar to the Theorem 7.2.1, so we omit here. □

Theorem 7.2.6. The measures d'_η, d''_η and d_η satisfy the following inequalities:

(i) $d'_\eta \leq d''_\eta \leq \sqrt[\eta]{d''_1}$;

(ii) $d_\eta \leq d''_\eta \leq \sqrt[\eta]{d''_1}$.

Proof. Since $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$ and hence we follows the results from their definitions. □

Remark 7.2.2. From this weighted measure, it has been observed that

(i) If $\eta = 1$, then Eq. (7.3) reduces to weighted Hamming distance,

(ii) If $\eta = 2$, then Eq. (7.3) is called as weighted Euclidean distance.

(iii) Especially, when $\omega_i = 1/n$, for $i = 1, 2, \ldots, n$, then Eq. (7.3) reduces to Eq. (7.2).

7.3 An extended TOPSIS approach based on the proposed distance

In this section, we present a TOPSIS approach under the CIFNs environment for solving MCGDM problems based on the proposed distance measure.

7.3.1 Description of the problem

The general description of MCDM problem is same as Section 2.5 of Chapter 2, such that their rating values are summarized in the form of CIFNs $\mathcal{A}_{ij} = (\mathcal{V}_{ij}, \lambda_{ij})$ where $\mathcal{V}_{ij} = \left([\zeta_{ij}^L, \zeta_{ij}^U], [\vartheta_{ij}^L, \vartheta_{ij}^U]\right)$ represent the IVIFNs and $\lambda_{ij} = (\zeta_{ij}, \vartheta_{ij})$ represent the IFNs. Here, the components $[\zeta_{ij}^L, \zeta_{ij}^U]$ and ϑ_{ij} represent the degree up to which the given alternative $\mathcal{V}_i$ satisfies the criterion $\mathfrak{B}_j$ whereas the components $[\vartheta_{ij}^L, \vartheta_{ij}^U]$ and ζ_{ij} indicate the dissatisfaction degree of alternative $\mathcal{V}_i$ regarding the criterion $\mathfrak{B}_j$. Thus, the overall representation of these rating values can be framed into the CIFN environment and hence the collective decision matrix is summarized as $\mathcal{M} = (\mathcal{A}_{ij})_{m\times n}$.

7.3.2 Compute the CIF-PIA and CIF-NIA

As all the rating values of the alternatives are CIFNs, so the CIF-positive ideal alternative (CIF-PIA), and CIF-negative ideal alternative (CIF-NIA) on the alternative $\mathcal{V}_i(i = 1, 2, \ldots, m)$ may be chosen as 1, and 0 respectively. Thus, rating values of CIF-PIA and CIF-NIA are expressed as $\mathcal{V}^+ = (([1,1],[0,0]), (0,1))_{1\times n}$ and $\mathcal{V}^- = (([0,0],[1,1]), (1,0))_{1\times n}$. From these, it has been seen that $\mathcal{V}^+$ and $\mathcal{V}^-$ are complement to each other.

However, if we take the fixed a *priori* CIF-PIA and CIF-NIA reference points, then the overall performance value and hence the ranking order of the alternatives could not change if the alternatives are changed. Instead of it, if decision-maker wants to define these references points as

$$\mathcal{V}_j^+ = \left(\left([g_j^{L+}, g_j^{U+}], [h_j^{L+}, h_j^{U+}]\right), \left(r_j^+, s_j^+\right)\right), \tag{7.4}$$

$$\text{and} \qquad \mathcal{V}_j^- = \left(\langle [g_j^{L-}, g_j^{U-}], [h_j^{L-}, h_j^{U-}]\rangle, \langle r_j^-, s_j^-\rangle\right) \tag{7.5}$$

where $g_j^{L+} = \max_j\{\zeta_{ij}^L\}$, $g_j^{U+} = \max_j\{\zeta_{ij}^U)\}$, $h_j^{L+} = \min_j\{\vartheta_{ij}^L)\}$, $h_j^{U+} = \min_j\{\vartheta_{ij}^U)\}$, $r_j^+ =$

$\min_j\{\zeta_{ij})\}$, $s_j^+ = \max_j\{\vartheta_{ij})\}$, $g_j^{L-} = \min_j\{\zeta_{ij}^L)\}$, $g_j^{U-} = \min_j\{\zeta_{ij}^U)\}$, $h_j^{L-} = \max_j\{\vartheta_{ij}^L)\}$, $h_j^{U-} = \max_j\{\vartheta_{ij}^U)\}$, $r_j^- = \max_j\{\zeta_{ij})\}$ and $s_j^- = \min_j\{\vartheta_{ij})\}$ for all i.

7.3.3 Compute distance measures between the alternatives

By considering the importance of the criteria in terms of weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ along with the $\mathcal{V}^+$ and $\mathcal{V}^-$, we compute the weighted distances between the alternative $\mathcal{V}_i$ and $\mathcal{V}^+$ as well as $\mathcal{V}^-$ as:

$$d_\eta(\mathcal{V}_i, \mathcal{V}^+) = \left(\frac{1}{6}\sum_{j=1}^{n}\omega_j\left\{\begin{array}{l}|g_j^{L+} - \zeta_{ij}^L|^\eta + |g_j^{U+} - \zeta_{ij}^U|^\eta + |\vartheta_{ij}^L - h_j^{L+}|^\eta \\ + |\vartheta_{ij}^U - h_j^{U+}|^\eta + |\zeta_{ij} - r_j^+|^\eta + |s_j^+ - \vartheta_{ij}|^\eta\end{array}\right\}\right)^{\frac{1}{\eta}} \tag{7.6}$$

and

$$d_\eta(\mathcal{V}_i, \mathcal{V}^-) = \left(\frac{1}{6}\sum_{j=1}^{n}\omega_j\left\{\begin{array}{l}|\zeta_{ij}^L - g_j^{L-}|^\eta + |\zeta_{ij}^U - g_j^{U-}|^\eta + |h_j^{L+} - \vartheta_{ij}^L|^\eta \\ + |h_j^{U+} - \vartheta_{ij}^U|^\eta + |r_j^- - \zeta_{ij}|^\eta + |\vartheta_{ij} - s_j^-|^\eta\end{array}\right\}\right)^{\frac{1}{\eta}} \tag{7.7}$$

where $\eta \geq 1$ be a real number.

Based on these weighted distances, the relative closeness coefficient of alternative $\mathcal{V}_i (i = 1, 2, \ldots, n)$ with respect to CIF-PIA $\mathcal{V}^+$ is given as follows:

$$\mathfrak{C}_i = \frac{d_\eta(\mathcal{V}_i, \mathcal{V}^-)}{d_\eta(\mathcal{V}_i, \mathcal{V}^-) + d_\eta(\mathcal{V}_i, \mathcal{V}^+)}; \quad d_\eta(\mathcal{V}_i, \mathcal{V}^+) \neq 0. \tag{7.8}$$

Further, it has been seen that, $0 \leq d_\eta(\mathcal{V}_i, \mathcal{V}^-) \leq d_\eta(\mathcal{V}_i, \mathcal{V}^-) + d_\eta(\mathcal{V}_i, \mathcal{V}^+)$ and hence $0 \leq \mathfrak{C}_i \leq 1$.

7.3.4 Proposed group decision making TOPSIS approach

Based on the above analysis, we have presented an approach for solving the group decision making problems under the CIFN environment. For it, consider that there are '$\mathcal{D}$' decision makers $\{\mathcal{M}^{(1)}, \mathcal{M}^{(2)}, \ldots, \mathcal{M}^{(\mathcal{D})}\}$ which are evaluating the given set of 'm' alternatives $\mathcal{V}_i (i = 1, 2, \ldots, m)$ under the set of 'n' criteria $\mathfrak{B}_j (j = 1, 2, \ldots, n)$. These decision makers give their preferences in terms of CIFNs $(\mathcal{A}_{ij})^{(d)} = \Big(\Big([(\zeta_{ij}^L)^{(d)}, (\zeta_{ij}^U)^{(d)}], [(\vartheta_{ij}^L)^{(d)}, (\vartheta_{ij}^U)^{(d)}]\Big), \Big((\zeta_{ij})^{(d)}, (\vartheta_{ij})^{(d)}\Big)\Big)$ where $d = 1, 2, \ldots, \mathcal{D}$. Further, assume that $\omega^{(d)} = (\omega_1^{(d)}, \omega_2^{(d)}, \ldots, \omega_n^{(d)})^T$ such that each $\omega_j^{(d)} > 0$ and $\sum\limits_{j=1}^{n} \omega_j^{(d)} = 1$ be the weight vector of the criteria. Also, in order to overcome the diverse judgements by different experts, their opinion is prioritized in accordance to the weight vector $\tau = \big(\tau_1, \tau_2, \ldots, \tau_{\mathcal{D}}\big)$ such that $\tau_d > 0$ and $\sum\limits_{d=1}^{\mathcal{D}} \tau_d = 1$. Then the following steps of the proposed approach has been summarized as follows:

Step 1: Arrange the rating values of the alternative given by each decision maker in the matrix form as

$$\mathcal{M}^{(d)} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \dots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{V}_{11}^{(d)} & \mathcal{V}_{12}^{(d)} & \dots & \mathcal{V}_{1n}^{(d)} \\ \mathcal{V}_{21}^{(d)} & \mathcal{V}_{22}^{(d)} & \dots & \mathcal{V}_{2n}^{(d)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{V}_{m1}^{(d)} & \mathcal{V}_{m2}^{(d)} & \dots & \mathcal{V}_{mn}^{(d)} \end{pmatrix} \end{array}$$

Step 2: For each decision maker $\mathcal{M}^{(d)}, (d = 1, 2, \dots, \mathcal{D})$, compute CIF-PIA and CIF-NIA corresponding to alternative $\mathcal{V}_i; i = 1, 2, \dots, m$ by using Eqs. (7.4) and (7.5) respectively and are defined as

$$(\mathcal{V}^+)^{(d)} = \left(\left(\left[(g_j^{L+})^{(d)}, (g_j^{U+})^{(d)} \right], \left[(h_j^{L+})^{(d)}, (h_j^{U+})^{(d)} \right] \right), \left(\left[(r_j^+)^{(d)}, (s_j^+)^{(d)} \right] \right) \right) \quad (7.9)$$

and

$$(\mathcal{V}^-)^{(d)} = \left(\left(\left[(g_j^{L-})^{(d)}, (g_j^{U-})^{(d)} \right], \left[(h_j^{L-})^{(d)}, (h_j^{U-})^{(d)} \right] \right), \left(\left[(r_j^-)^{(d)}, (s_j^-)^{(d)} \right] \right) \right) \quad (7.10)$$

where $(g_j^{L+})^{(d)} = \max_j \{(\zeta_{ij}^L)^{(d)}\}$, $(g_j^{U+})^{(d)} = \max_j \{(\zeta_{ij}^U)^{(d)}\}$, $(h_j^{L+})^{(d)} = \min_j \{(\vartheta_{ij}^L)^{(d)}\}$, $(h_j^{U+})^{(d)} = \min_j \{(\vartheta_{ij}^U)^{(d)}\}$, $(r_j^+)^{(d)} = \min_j \{(\zeta_{ij})^{(d)}\}$, $(s_j^+)^{(d)} = \max_j \{(\vartheta_{ij})^{(d)}\}$, $(g_j^{L-})^{(d)} = \min_j \{(\zeta_{ij}^L)^{(d)}\}$, $(g_j^{U-})^{(d)} = \min_j \{(\zeta_{ij}^U)^{(d)}\}$, $(h_j^{L-})^{(d)} = \max_j \{(\vartheta_{ij}^L)^{(d)}\}$, $(h_j^{U-})^{(d)} = \max_j \{(\vartheta_{ij}^U)^{(d)}\}$, $(r_j^-)^{(d)} = \max_j \{(\zeta_{ij})^{(d)}\}$ and $(s_j^-)^{(d)} = \min_j \{(\vartheta_{ij})^{(d)}\}$.

Step 3: For each decision maker, compute the separation measures between the alternatives $\mathcal{V}_i$ from its CIF-PIA and CIF-NIA and are denoted by $d_\eta\big((\mathcal{V}_i)^{(d)}, (\mathcal{V}^+)^{(d)}\big)$ and $d_\eta\left((\mathcal{V}_i)^{(d)}, (\mathcal{V}^-)^{(d)}\right)$ respectively.

Step 4: For each decision maker, the relative closeness coefficient is determined as

$$\mathfrak{C}_i^{(d)} = \frac{d_\eta\left((\mathcal{V}_i)^{(d)}, (\mathcal{V}^-)^{(d)}\right)}{d_\eta\left((\mathcal{V}_i)^{(d)}, (\mathcal{V}^+)^{(d)}\right) + d_\eta\left((\mathcal{V}_i)^{(d)}, (\mathcal{V}^-)^{(d)}\right)} \quad ; \quad k = 1, 2, \dots, K. \quad (7.11)$$

where $d_\eta\big((\mathcal{V}_i)^{(d)}, (\mathcal{V}^+)^{(d)}\big) \neq 0$.

Step 5: Since each decision maker may have obtained the different ranking towards the alternatives and hence the overall finding of the best alternative remain unclear.

In order to overcome these variable rankings, different values of the experts are aggregated by assigning a priority value, $\tau = (\tau_1, \tau_2, \ldots, \tau_{\mathcal{D}})^T$ such that $\tau_d > 0$ and $\sum_{d=1}^{\mathcal{D}} \tau_d = 1$, to each expert. The separation measures of each expert are aggregated by using these weight vector, and get the overall measurement values of the alternative as follows

$$D_i^+ = \sum_{d=1}^{\mathcal{D}} \tau_d \; d_\eta \left((\mathcal{V}_i)^{(d)}, (\mathcal{V}^+)^{(d)} \right) \quad \text{and} \quad D_i^- = \sum_{d=1}^{\mathcal{D}} \tau_d \; d_\eta \left((\mathcal{V}_i)^{(d)}, (\mathcal{V}^-)^{(d)} \right) \tag{7.12}$$

Step 6: Based on these values, D_i^+ and D_i^-, the closeness coefficient for an alternative $\mathcal{V}_i (i = 1, 2, \ldots, m)$ is determined as

$$\mathfrak{C}_i = \frac{D_i^-}{D_i^+ + D_i^-}; \quad D_i^+ \neq 0 \tag{7.13}$$

Step 7: Rank the alternative(s) based on the descending values of $\mathfrak{C}_i$'s.

7.4 Illustrative example

In order to demonstrate the above mentioned approach, an illustrative example has been taken as below:

7.4.1 Case study

A multinational company has started its recruitment process for selecting the best candidate for the new project. For this, a company has published notification in the newspaper and based on it, different candidates have applied for it. Out of that, four candidates $\mathcal{V}_i; (i = 1, 2, 3, 4)$ are to be selected for the interview. To evaluate the candidates, company manager has invited four decision-makers $\mathcal{M}^{(1)}$, $\mathcal{M}^{(2)}$, $\mathcal{M}^{(3)}$ and $\mathcal{M}^{(4)}$ and give them responsibilities to find the best candidate for the company. The panel has decided to evaluate the candidates $\mathcal{V}_i; (i = 1, 2, 3, 4)$ on the basis of four criteria namely, $\mathfrak{B}_1$: 'Educational qualification'; $\mathfrak{B}_2$: 'Technical knowledge'; $\mathfrak{B}_3$: 'Communication skills'; $\mathfrak{B}_4$: 'Work experience'. For it, they firstly conducted group discussions (GDs) with all the candidates and the results for each candidate are formulated by a panel in the form of IVIFNs. Among the pool of applicants appearing for GD, four candidates were shortlisted for personal interview and the results for this stage of the recruitment process are recorded in the form of IFNs. Then the following steps of the proposed approach are executed in order to find the best candidate(s) for the required post.

Step 1: The rating values of each decision maker towards the evaluation of the given alternatives are summarized in Table 7.1. In this table, rating values under both the recruitment stages are clubbed, that is the previously obtained IVIFNs (from GD sessions) and IFNs (from the personal interview round), in the form of CIFNs.

Step 2: By using Eq. (7.9) and Eq. (7.10), the ideal alternatives namely CIF-PIA and CIF-NIA are determined for each decision maker. The corresponding values are summarized in Table 7.2.

Step 3: Without loss of generality, we choose $\eta = 2$, compute the distance measure values by using Eq. (7.3) for each decision maker and their results are summarized in Table 7.3.

Step 4: Utilize Eq. (7.11) to compute the closeness coefficients with respect to each decision maker. The results and the corresponding ranking order of the alternatives are summarized in Table 7.4 and observed that $\mathcal{V}_3$ is the best candidate for the decision maker $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ while $\mathcal{V}_1$ for the other decision makers.

Step 5: To overcome the ambiguity about the best alternatives w.r.t. the decision makers, aggregate the ideal distance measurement values, as given in Table 7.3, of every decision-maker by using Eq. (7.12) corresponding to the five different priority pairs $(\tau_1, \tau_2, \tau_3, \tau_4)$ of decision makers. The results are summarized in the fourth and fifth column of the Table 7.5.

Step 6: For each priority pair, the values of $\mathfrak{C}_i$'s are computed by using Eq. (7.13) and their results are summarized in sixth column of the Table 7.5.

Step 7: Based on the values of $\mathfrak{C}_i$'s, the ranking order of the alternatives is summarized in the last column of the Table 7.5. From this table, we can see that corresponding to the different pairs, the best alternative is either $\mathcal{V}_1$ or $\mathcal{V}_3$.

7.4.2 Validity test

The following test criteria are presented by Wang and Triantaphyllou [166] to validate the approach.

Test criterion 1: "If we replace the rating values of non-optimal alternative with worse alternative then the best alternative should not change, provided the relative weighted criteria remains unchanged."

Test criterion 2: "Method should possess transitive nature."

Test criterion 3: "When a given problem is decomposed into smaller ones and the same MCDM method has been applied, then the combined ranking of the alternatives should be identical to the ranking of un-decomposed one."

Below, we have validated these test criteria on our proposed method.

Validity test by test criterion 1

Without loss of generality, we have considered the case 5 of the above-discussed analysis (similarly for the other cases) where the priority level of the decision makers has been taken as 0.42, 0.36, 0.12, 0.10 respectively. The original ranking order for the case is $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4$. Now, in order to validate it with respect to criterion 1, the following decision makers, given in Table 7.6, are obtained from the original matrices after replacing the non-optimal alternative ($\mathcal{V}_1$) with an arbitrary worst alternative ($\mathcal{V}_1'$). Then, by applying the proposed approach to this data closeness coefficients $\mathfrak{C}_i$'s of each candidate $\mathcal{V}_i (i = 1, 2, 3, 4)$ are obtained as 0.3601, 0.5246, 0.5465 and 0.4111. Thus, the ranking order of the candidate is $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1'$, which shows that the best alternative remains the same i.e., $\mathcal{V}_3$.

Validity test by test criteria 2 and 3

Under this test, if we decomposed the given problem into a sub-problems, namely $\{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_1\}$, $\{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$ and $\{\mathcal{V}_3, \mathcal{V}_1, \mathcal{V}_4\}$ and the same procedure steps of the approach has been applied, then we get the ranking orders of these sub-problems as $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2$, $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$ and $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4$, respectively. Therefore, by combining these, we get the overall ranking order of the alternative is $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$ which is same as that of the original ranking order, hence it beholds the transitive property. Thus, the proposed approach is valid under the test criteria 2 and 3.

7.4.3 Comparative Studies:

In order to compare the performance of the proposed approach with respect to the existing approaches [10, 34, 35, 59, 100, 152] under the CFSs, IVIFSs, IFSs, interval-valued FSs environment, an analysis has been conducted. To apply these existing approaches on to the considered data, we first convert the rating values of CIFNs into these numbers by taking the rating corresponding to IFNs be zero. Further, without loss of generality, we take the case by taking the weight vector of the decision makers as $\tau = (0.42, 0.36, 0.12, 0.10)^T$ and hence the existing approaches are applied to the considered data. The results computed by these different approaches are summarized in Table 7.7 and conclude that the ranking order of the given alternatives is $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1$ and hence the best alternative is $\mathcal{V}_3$ which coincides with the proposed approach results given in Table 7.5, which validates the stability of our approach.

Compared with these existing approaches with general intuitionistic sets (IVIFSs or IFSs), the proposed decision-making method under cubic intuitionistic fuzzy set environment contains much more evaluation information on the alternatives by considering both the IVIFSs and IFSs simultaneously, while the existing approaches contain either IFS or IVIFS information. Therefore, the approaches under the IVIFSs or IFSs may lose some useful information, either IVIFNs or IFNs, of alternatives which may affect the decision results. Furthermore, it is noted from the study that the computational procedure of the proposed approach is different from the existing approaches under the different environment, but the proposed result in this chapter is more rational to reality in the decision process due to the consideration of the consistent priority degree between the pairs of the arguments as well as between different experts. Also, the corresponding studies under the IVIFS or IFS environment can be considered as a special case of the proposed operators. Finally, the existing decision-making methods under IVIFSs or IFSs cannot deal with the decision-making problem with CIFS. Therefore, the proposed approach is more generalized and suitable to capture the real-life fuzziness more accurately than the existing ones.

In addition to these, we give some characteristics comparison of our proposed method and the aforementioned methods, which are listed in Table 7.8.

7.5 Conclusion

The concluded contribution of this chapter is given below:

1) CIFS is one of the successful extensions of the IFS in which a degree of the disagreement (in the form of IFS values) corresponding to the agreed interval region (in form of IVIFS) has been used to represent the data. By taking the advantages of it, in this chapter, an extended TOPSIS approach to solve the group decision-making problems under the CIFS environment has been presented.

2) For it, some generalized distance measures between the pairs of the CIFNs have been introduced. The prominent characteristics holding properties of these distance measures have also been studied in detail.

3) Based on these measures, an extended TOPSIS group decision-making approach for solving MCGDM problem under CIFS environment has been formulated.

4) The proposed approach have been illustrated with a numerical example and their results have been compared with some of the existing approaches. In addition to these, the characteristics comparison of the proposed approach with the existing approaches have been summarized.

From the computed study, it is obtained that the several approaches under CFSs, IVIFSs and/or IFSs are the special case of the proposed approach. Thus, the proposed approach is more generalized and suitable to capture the real-life fuzziness more accurately than the existing ones.

Table 7.1: Rating values of each decision maker in terms of CIFNs

Decision maker	Candidates & weights	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{M}^{(1)}$	$\mathcal{V}_1$	$(([0.15,0.30],[0.35,0.40]),(0.20,0.65))$	$(([0.13,0.25],[0.40,0.45]),(0.30,0.60))$	$(([0.30,0.45],[0.25,0.30]),(0.55,0.33))$	$(([0.10,0.30],[0.25,0.35]),(0.11,0.20))$
	$\mathcal{V}_2$	$(([0.10,0.15],[0.35,0.40]),(0.40,0.17))$	$(([0.15,0.22],[0.27,0.30]),(0.15,0.29))$	$(([0.40,0.45],[0.21,0.33]),(0.16,0.35))$	$(([0.50,0.60],[0.15,0.20]),(0.35,0.19))$
	$\mathcal{V}_3$	$(([0.14,0.25],[0.35,0.65]),(0.10,0.40))$	$(([0.35,0.45],[0.15,0.20]),(0.30,0.50))$	$(([0.45,0.55],[0.15,0.25]),(0.20,0.80))$	$(([0.30,0.50],[0.10,0.30]),(0.20,0.35))$
	$\mathcal{V}_4$	$(([0.30,0.35],[0.25,0.45]),(0.20,0.30))$	$(([0.20,0.55],[0.40,0.45]),(0.20,0.45))$	$(([0.15,0.25],[0.20,0.35]),(0.60,0.20))$	$(([0.10,0.29],[0.40,0.50]),(0.30,0.40))$
	Weights	0.17	0.30	0.13	0.40
$\mathcal{M}^{(2)}$	$\mathcal{V}_1$	$(([0.10,0.30],[0.35,0.45]),(0.60,0.10))$	$(([0.15,0.20],[0.25,0.29]),(0.18,0.66))$	$(([0.44,0.50],[0.20,0.30]),(0.18,0.35))$	$(([0.10,0.30],[0.25,0.35]),(0.11,0.20))$
	$\mathcal{V}_2$	$(([0.20,0.30],[0.40,0.50]),(0.10,0.40))$	$(([0.30,0.40],[0.10,0.60]),(0.20,0.40))$	$(([0.40,0.50],[0.20,0.30]),(0.60,0.30))$	$(([0.10,0.50],[0.20,0.30]),(0.40,0.30))$
	$\mathcal{V}_3$	$(([0.10,0.20],[0.30,0.60]),(0.40,0.20))$	$(([0.25,0.30],[0.45,0.50]),(0.60,0.30))$	$(([0.30,0.45],[0.20,0.25]),(0.10,0.80))$	$(([0.40,0.50],[0.10,0.30]),(0.30,0.70))$
	$\mathcal{V}_4$	$(([0.15,0.45],[0.25,0.30]),(0.40,0.60))$	$(([0.20,0.25],[0.30,0.35]),(0.15,0.20))$	$(([0.45,0.60],[0.20,0.25]),(0.29,0.63))$	$(([0.16,0.20],[0.25,0.30]),(0.15,0.30))$
	Weights	0.20	0.25	0.15	0.40
$\mathcal{M}^{(3)}$	$\mathcal{V}_1$	$(([0.20,0.30],[0.25,0.40]),(0.15,0.20))$	$(([0.30,0.35],[0.40,0.45]),(0.40,0.30))$	$(([0.32,0.40],[0.35,0.45]),(0.30,0.50))$	$(([0.15,0.18],[0.19,0.30]),(0.30,0.60))$
	$\mathcal{V}_2$	$(([0.30,0.50],[0.20,0.40]),(0.10,0.30))$	$(([0.40,0.50],[0.10,0.30]),(0.20,0.10))$	$(([0.40,0.45],[0.30,0.35]),(0.60,0.20))$	$(([0.10,0.30],[0.20,0.50]),(0.40,0.30))$
	$\mathcal{V}_3$	$(([0.25,0.32],[0.40,0.45]),(0.20,0.30))$	$(([0.30,0.35],[0.38,0.49]),(0.20,0.62))$	$(([0.37,0.42],[0.20,0.29]),(0.30,0.10))$	$(([0.20,0.35],[0.30,0.60]),(0.20,0.42))$
	$\mathcal{V}_4$	$(([0.40,0.44],[0.50,0.52]),(0.30,0.20))$	$(([0.40,0.45],[0.35,0.40]),(0.30,0.10))$	$(([0.10,0.18],[0.15,0.30]),(0.40,0.50))$	$(([0.30,0.40],[0.50,0.55]),(0.30,0.70))$
	Weights	0.18	0.12	0.25	0.45
$\mathcal{M}^{(4)}$	$\mathcal{V}_1$	$(([0.30,0.40],[0.20,0.30]),(0.40,0.60))$	$(([0.18,0.30],[0.19,0.34]),(0.40,0.32))$	$(([0.30,0.38],[0.40,0.45]),(0.30,0.40))$	$(([0.30,0.60],[0.20,0.40]),(0.40,0.20))$
	$\mathcal{V}_2$	$(([0.10,0.30],[0.20,0.50]),(0.20,0.10))$	$(([0.25,0.29],[0.32,0.45]),(0.60,0.10))$	$(([0.40,0.45],[0.47,0.50]),(0.30,0.20))$	$(([0.10,0.15],[0.20,0.25]),(0.30,0.50))$
	$\mathcal{V}_3$	$(([0.20,0.31],[0.35,0.42]),(0.30,0.10))$	$(([0.30,0.40],[0.52,0.59]),(0.30,0.40))$	$(([0.18,0.36],[0.20,0.25]),(0.20,0.40))$	$(([0.30,0.35],[0.40,0.45]),(0.20,0.70))$
	$\mathcal{V}_4$	$(([0.10,0.15],[0.30,0.40]),(0.20,0.10))$	$(([0.20,0.30],[0.40,0.50]),(0.20,0.50))$	$(([0.23,0.32],[0.40,0.45]),(0.30,0.60))$	$(([0.16,0.32],[0.17,0.34]),(0.30,0.40))$
	Weights	0.35	0.40	0.12	0.13

Table 7.2: Positive and Negative ideals for each decision maker

Decision maker	PIA NIA	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{M}^{(1)}$	$\mathcal{V}_1^+$	$(([0.30,0.35],[0.25,0.40]),(0.10,0.65))$	$(([0.35,0.55],[0.15,0.20]),(0.15,0.60))$	$(([0.45,0.55],[0.15,0.25]),(0.16,0.80))$	$(([0.50,0.60],[0.10,0.20]),(0.11,0.40))$
	$\mathcal{V}_1^-$	$(([0.10,0.15],[0.35,0.65]),(0.40,0.17))$	$(([0.13,0.22],[0.40,0.45]),(0.30,0.29))$	$(([0.15,0.25],[0.25,0.35]),(0.60,0.20))$	$(([0.10,0.29],[0.40,0.50]),(0.35,0.19))$
$\mathcal{M}^{(2)}$	$\mathcal{V}_2^+$	$(([0.20,0.45],[0.25,0.30]),(0.10,0.60))$	$(([0.30,0.40],[0.10,0.29]),(0.15,0.66))$	$(([0.45,0.60],[0.20,0.25]),(0.10,0.80))$	$(([0.40,0.50],[0.10,0.30]),(0.11,0.70))$
	$\mathcal{V}_2^-$	$(([0.10,0.20],[0.40,0.60]),(0.60,0.10))$	$(([0.15,0.20],[0.45,0.60]),(0.60,0.20))$	$(([0.30,0.45],[0.20,0.30]),(0.60,0.30))$	$(([0.10,0.20],[0.25,0.35]),(0.40,0.20))$
$\mathcal{M}^{(3)}$	$\mathcal{V}_3^+$	$(([0.40,0.50],[0.20,0.40]),(0.10,0.30))$	$(([0.40,0.50],[0.10,0.30]),(0.20,0.62))$	$(([0.40,0.45],[0.15,0.29]),(0.30,0.50))$	$(([0.30,0.40],[0.19,0.30]),(0.20,0.70))$
	$\mathcal{V}_3^-$	$(([0.20,0.30],[0.50,0.52]),(0.30,0.20))$	$(([0.30,0.35],[0.40,0.49]),(0.40,0.10))$	$(([0.10,0.18],[0.35,0.45]),(0.60,0.10))$	$(([0.10,0.18],[0.50,0.60]),(0.40,0.30))$
$\mathcal{M}^{(4)}$	$\mathcal{V}_4^+$	$(([0.30,0.40],[0.20,0.30]),(0.20,0.60))$	$(([0.30,0.40],[0.19,0.34]),(0.20,0.50))$	$(([0.40,0.45],[0.20,0.25]),(0.20,0.60))$	$(([0.30,0.60],[0.17,0.25]),(0.20,0.70))$
	$\mathcal{V}_4^-$	$(([0.10,0.15],[0.35,0.50]),(0.40,0.10))$	$(([0.18,0.29],[0.52,0.59]),(0.60,0.10))$	$(([0.18,0.32],[0.47,0.50]),(0.30,0.20))$	$(([0.10,0.15],[0.40,0.45]),(0.40,0.20))$

Table 7.3: Separation measures from ideal solutions corresponding to each decision maker

Alternatives	$\mathcal{M}^{(1)}$		$\mathcal{M}^{(2)}$		$\mathcal{M}^{(3)}$		$\mathcal{M}^{(4)}$	
	$D_i^{(1+)}$	$D_i^{(1-)}$	$D_i^{(2+)}$	$D_i^{(2-)}$	$D_i^{(3+)}$	$D_i^{(3-)}$	$D_i^{(4+)}$	$D_i^{(4-)}$
$\mathcal{V}_1$	0.2167	0.1539	0.2344	0.1852	0.1361	0.1998	0.1359	0.2214
$\mathcal{V}_2$	0.1921	0.1977	0.2166	0.1889	0.1899	0.1625	0.2378	0.1203
$\mathcal{V}_3$	0.1082	0.2258	0.1966	0.2173	0.1658	0.1689	0.1909	0.1609
$\mathcal{V}_4$	0.2451	0.1262	0.1970	0.1983	0.1780	0.1819	0.1868	0.1803

Table 7.4: Closeness coefficients and ranking order with respect to decision maker

	$\mathcal{M}^{(1)}$		$\mathcal{M}^{(2)}$		$\mathcal{M}^{(3)}$		$\mathcal{M}^{(4)}$	
Alternatives	$\mathfrak{C}_i^{(1)}$	Ranking	$\mathfrak{C}_i^{(2)}$	Ranking	$\mathfrak{C}_i^{(3)}$	Ranking	$\mathfrak{C}_i^{(4)}$	Ranking
$\mathcal{V}_1$	0.4152	3	0.4414	4	0.5948	1	0.6196	1
$\mathcal{V}_2$	0.5071	2	0.4659	3	0.4612	4	0.3359	4
$\mathcal{V}_3$	0.6761	1	0.5250	1	0.5047	3	0.4696	3
$\mathcal{V}_4$	0.3398	4	0.5016	2	0.5054	2	0.4912	2

Table 7.5: Aggregated closeness coefficient and ranking for each candidate

	Weights	τ	Candidates	distance measures D_i^+	distance measures D_i^-	$\mathfrak{C}_i$	Ranking	Selected Candidate
Case 1	$\mathcal{M}^{(1)}$	0.20	$\mathcal{V}_1$	0.1817	0.1884	0.5090	2	$\mathcal{V}_3$
	$\mathcal{M}^{(2)}$	0.30	$\mathcal{V}_2$	0.2031	0.1732	0.4603	4	
	$\mathcal{M}^{(3)}$	0.40	$\mathcal{V}_3$	0.1660	0.1948	0.5399	1	
	$\mathcal{M}^{(4)}$	0.10	$\mathcal{V}_4$	0.1980	0.1755	0.4699	3	
Case 2	$\mathcal{M}^{(1)}$	0.20	$\mathcal{V}_1$	0.1718	0.1963	0.5333	1	$\mathcal{V}_1$
	$\mathcal{M}^{(2)}$	0.20	$\mathcal{V}_2$	0.2148	0.1579	0.4237	4	
	$\mathcal{M}^{(3)}$	0.20	$\mathcal{V}_3$	0.1705	0.1900	0.5271	2	
	$\mathcal{M}^{(4)}$	0.40	$\mathcal{V}_4$	0.1988	0.1734	0.4660	3	
Case 3	$\mathcal{M}^{(1)}$	0.13	$\mathcal{V}_1$	0.1612	0.2007	0.5545	1	$\mathcal{V}_1$
	$\mathcal{M}^{(2)}$	0.15	$\mathcal{V}_2$	0.2143	0.1533	0.4170	4	
	$\mathcal{M}^{(3)}$	0.30	$\mathcal{V}_3$	0.1735	0.1836	0.5142	2	
	$\mathcal{M}^{(4)}$	0.42	$\mathcal{V}_4$	0.1933	0.1765	0.4773	3	
Case 4	$\mathcal{M}^{(1)}$	0.35	$\mathcal{V}_1$	0.1957	0.1827	0.4828	2	$\mathcal{V}_3$
	$\mathcal{M}^{(2)}$	0.32	$\mathcal{V}_2$	0.2074	0.1761	0.4592	3	
	$\mathcal{M}^{(3)}$	0.16	$\mathcal{V}_3$	0.1598	0.2043	0.5612	1	
	$\mathcal{M}^{(4)}$	0.17	$\mathcal{V}_4$	0.2091	0.1674	0.4446	4	
Case 5	$\mathcal{M}^{(1)}$	0.42	$\mathcal{V}_1$	0.2053	0.1774	0.4635	3	$\mathcal{V}_3$
	$\mathcal{M}^{(2)}$	0.36	$\mathcal{V}_2$	0.2052	0.1826	0.4708	2	
	$\mathcal{M}^{(3)}$	0.12	$\mathcal{V}_3$	0.1552	0.2102	0.5753	1	
	$\mathcal{M}^{(4)}$	0.10	$\mathcal{V}_4$	0.2139	0.1643	0.4343	4	

Table 7.6: Rating values of the worse alternative A_1' for each decision maker

Decision Maker	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{M}^{(1)}$	$(([0.15, 0.20], [0.30, 0.45]), (0.25, 0.50))$	$(([0.13, 0.20], [0.40, 0.48]), (0.35, 0.50))$	$(([0.30, 0.35], [0.25, 0.35]), (0.60, 0.30))$	$(([0.10, 0.20], [0.20, 0.35]), (0.15, 0.10))$
$\mathcal{M}^{(2)}$	$(([0.10, 0.15], [0.30, 0.45]), (0.62, 0.05))$	$(([0.15, 0.18], [0.25, 0.35]), (0.20, 0.50))$	$(([0.44, 0.48], [0.20, 0.35]), (0.20, 0.30))$	$(([0.10, 0.25], [0.20, 0.39]), (0.20, 0.10))$
$\mathcal{M}^{(3)}$	$(([0.20, 0.25], [0.25, 0.42]), (0.20, 0.15))$	$(([0.32, 0.35], [0.40, 0.50]), (0.44, 0.20))$	$(([0.32, 0.38], [0.35, 0.50]), (0.40, 0.20))$	$(([0.15, 0.17], [0.19, 0.32]), (0.35, 0.50))$
$\mathcal{M}^{(4)}$	$(([0.30, 0.35], [0.20, 0.35]), (0.50, 0.40))$	$(([0.18, 0.25], [0.19, 0.39]), (0.50, 0.30))$	$(([0.30, 0.35], [0.40, 0.50]), (0.40, 0.20))$	$(([0.30, 0.50], [0.20, 0.45]), (0.45, 0.15))$

Table 7.7: Comparison analysis with some of the existing approaches

Existing approaches	Aggregated closeness coefficients				Ranking
	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	
Fahmi et al. [35]	0.4350	0.5103	0.5618	0.4693	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1$
Lu and Ye [100]	0.5293	0.4911	0.4829	0.5171	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1$
Biswas and Kumar [10]	0.5471	0.5729	0.5867	0.5553	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1$
Gupta et al. [59]	0.5648	0.5021	0.4453	0.5356	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1$
Dugenci [34]	0.3510	0.5396	0.5803	0.4187	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1$
Wang and Chen [152]	0.5300	0.4917	0.4865	0.5161	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1$

Table 7.8: The characteristic comparisons of different methods

Methods	Whether flexible to express a wider range of information	Whether consider more than one decision-maker	Whether describe hybrid information at same level	Whether have the characteristic of generalization
Lu and Ye [100]	✓	×	×	×
Wang and Chen [152]	✓	×	×	×
Gupta et al. [59]	✓	✓	×	×
Dugenci [34]	✓	✓	×	×
Biswas and Kumar [10]	✓	×	×	×
Fahmi et al. [35]	✓	✓	×	×
The proposed method	✓	✓	✓	✓

Chapter 8

Nonlinear-programming methodology for solving decision - making problems[1]

In this chapter, we discuss a novel nonlinear-programming (NLP) models based on the TOPSIS method to solve the decision making problems under the cubic intuitionistic fuzzy sets (CIFSs) environment. For it, we modeled the NLP models by considering the interval weights as well as the concept of the relative-closeness coefficient and weighted distance measures. Some of the salient features of these models are also examined. Furthermore, a novel MCDM method is presented and to illustrate with a problem related to signal processing in sound navigation and ranging (SONAR).

8.1 Introduction

Cubic intuitionistic fuzzy set (CIFS) is an efficient tool in handling possible disagreeness of the agreed interval values and vice-versa. This environment increases the level of precision by enhancing the scope of the membership (and non-membership) interval by considering a fuzzy set value corresponding to it. The prominent characteristic of CIFSs is to take the elements of IFSs and IVIFSs simultaneously. From the CIFS structure, it is noticed that the CIFS contains more information than the existing sets and thus IFSs, IVIFSs

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and CFSs are the special cases of the CIFSs. In order to explain its practical utility, suppose a person wants to invest money in an international market. Before investing, he wants to analyze the estimation of the interest on his investment and found that it would be 60-70% tending towards the profit side while 20-25% towards the loss side. However, after the completion of the certain months, they found that his return to be 35% agreeing to his earlier profit estimate while towards the loss estimates it is 20% disagreeing. Such information is represented as $(([0.60, 0.70], [0.20, 0.25]), (0.35, 0.20))$ and termed as P-order CIFS. Similarly, if instead of 35% and 20% agreeing and disagreeing, the person found 20% disagree towards the profit while 40% agree toward the loss estimates then it is represented as $(([0.60, 0.70], [0.20, 0.25]), (0.20, 0.40))$ and termed as R-order CIFS. Therefore, CIFS is a useful way to address the information more precisely rather than the IFSs or IVIFSs only during the DM problem.

Therefore, motivated by the structure of the CIFS sets and the importance of the TOPSIS method, we develop an approach of TOPSIS based on NLP models to access the DM problems. Non-linear systems frames the dynamic uncertain situations and provide decision-maker with the liberty to closely analyze the available information. In this chapter, non-linear model for applying TOPSIS is considered in which CIFS features by taking into account the interval weights information. So, in this chapter, we design a framework to formulate two auxiliary NLP models and hence a TOPSIS approach to solve DM problems. The primary objectives of this work are to formulate two NLP models by using the weighted distance measures. Further, some special cases of the proposed models have been deduced from it. Further, an algorithm is presented in which information is processed by using relative closeness coefficient (RCC) and possibility degree to solve DM problems. In addition to it, the performance of the developed approach is demonstrated and the way in which it outperforms the different methods is discussed.

8.2 Proposed nonlinear-programming model

This section presents a TOPSIS approach based on NLP models under the R-ordered CIFS environment to solve DM problems. Beforehand, we list the following definition:

Definition 8.2.1. [130] For two interval numbers $a = [a^L, a^U]$ and $b = [b^L, b^U]$, the likelihood of $a \succeq b$ is defined as

$$\mathfrak{z}(a \succeq b) = \max\left\{1 - \max\left\{\frac{b^U - a^L}{\mathcal{L}(a) + \mathcal{L}(b)}, 0\right\}, 0\right\} \tag{8.1}$$

where $\mathcal{L}(a) = a^U - a^L$ and $\mathcal{L}(b) = b^U - b^L$.

8.2.1 Description of the MCDM problem

The general description of MCDM problem is same as Section 2.5 of Chapter 2. During evaluation phase of each alternative, an expert give his rating in terms of CIFNs $\mathcal{A}_{ij} = (A_{ij}, \lambda_{ij})$ where $A_{ij} = ([\zeta_{ij}^L, \zeta_{ij}^U], [\vartheta_{ij}^L, \vartheta_{ij}^U])$ and $\lambda_{ij} = (\zeta_{ij}, \vartheta_{ij})$ represents the rating value of i^{th} alternative under the criteria $\mathfrak{B}_j$. Here, the components $[\zeta_{ij}^L, \zeta_{ij}^U]$ and ϑ_{ij} represent the degree up to which alternative $\mathcal{V}_i$ agrees to the criterion $\mathfrak{B}_j$ whereas the components $[\vartheta_{ij}^L, \vartheta_{ij}^U]$ and ζ_{ij} indicate the disagreeness of alternative $\mathcal{V}_i$ regarding the criterion $\mathfrak{B}_j$. Thus, the collective preferences corresponding to each alternative are framed into the decision matrix and summarized as $\mathcal{M} = (\mathcal{A}_{ij})_{m\times n}$.

Further, it is noted that the different criteria plays an important role during the evaluations of the alternatives thus it is necessary to consider their different priority levels, instead of giving equal priorities, in real-life problems. To do this, assume that an expert may think that an importance of the membership degrees during the evaluation of the alternatives under the criterion $\mathfrak{B}_j$ ranges from ω_j^L to ω_j^U while for non-membership its varies from ρ_j^L to ρ_j^U such that $0 \leq \omega_j^L \leq \omega_j^U \leq 1$ and $0 \leq \rho_j^L \leq \rho_j^U \leq 1$ and $\omega_j^U + \rho_j^U \leq 1$. Thus, the weight vector possessed by the criterion $\mathfrak{B}_j$ corresponding to A_{ij} ratings is represented as $([\omega_j^L, \omega_j^U], [\rho_j^L, \rho_j^U])$. Similarly, we can set the weight vector for the values λ_{ij} associated with CIFNs as $([\varepsilon_j^L, \varepsilon_j^U], [\kappa_j^L, \kappa_j^U])$ where $0 \leq \varepsilon_j^L, \varepsilon_j^U, \kappa_j^L, \kappa_j^U \leq 1$, and $\varepsilon_j^U + \kappa_j^U \leq 1$. Therefore, the overall weight of all the criteria $\mathfrak{B}_j$ can be concisely expressed as $W = \big(([\omega_j^L, \omega_j^U], [\rho_j^L, \rho_j^U]), ([\varepsilon_j^L, \varepsilon_j^U], [\kappa_j^L, \kappa_j^U])\big)_{1\times n}$.

8.2.2 Measurement of ideal solutions

The concept of the ideal measures also called as reference points play a significant role in the DM problems to select the optimal alternative(s). Since the decision matrix $\mathcal{M} =$

$(\mathcal{A}_{ij})$ captures the rating values in the form of CIFNs, and hence the two ideals namely, positive and negative ideal solutions, denoted by PIS and NIS respectively may be taken as $\mathcal{V}^+ = \big(([1,1],[0,0]),(0,1)\big)$ and $\mathcal{V}^- = (([0,0],[1,1]),(1,0))$ under the CIFS environment. From these ideal values, it is clearly seen that they are a complement to each other.

Furthermore, instead of fixing the MD and NMDs of $\mathcal{V}^+$ and $\mathcal{V}^-$ to the ideal states i.e., 1 or 0, an expert may be varying it by defining these ideals as $\mathcal{V}_j^+ = \big(([f_j^+, f_j^+],[t_j^+, t_j^+]),$ $(r_j^+,\ s_j^+)\big)_{1\times n}$ and $\mathcal{V}_j^- = \big(([f_j^-, f_j^-],[t_j^-, t_j^-]),(r_j^-,\ s_j^-)\big)_{1\times n}$ where $f_j^+ = \max_j\{\zeta_{ij}^U \mid i = 1,2,\ldots,m\}$, $t_j^+ = \min_j\{\vartheta_{ij}^L \mid i = 1,2,\ldots,m\}$, $r_j^+ = \min_j\{\zeta_{ij} \mid i = 1,2,\ldots,m\}$, $s_j^+ = \max_j\{\vartheta_{ij} \mid i = 1,2,\ldots,m\}$, $f_j^- = \min_j\{\zeta_{ij}^L \mid i = 1,2,\ldots,m\}$, $t_j^- = \max_j\{\vartheta_{ij}^U \mid i = 1,2,\ldots,m\}$, $r_j^- = \max_j\{\zeta_{ij} \mid i = 1,2,\ldots,m\}$ and $s_j^- = \min_j\{\vartheta_{ij} \mid i = 1,2,\ldots,m\}$. From this, it is concluded that $\mathcal{V}_j^- \subseteq \big(([\zeta_{ij}^L,\zeta_{ij}^U],[\vartheta_{ij}^L,\vartheta_{ij}^U]),(\zeta_{ij},\vartheta_{ij})\big) \subseteq \mathcal{V}_j^+$.

8.2.3 Separation measurement values

The weighted Euclidean measure [52] for CIFNs are taken to compute measurement values between the alternatives $\mathcal{V}_i$ from its reference values $\mathcal{V}^+$ and $\mathcal{V}^-$. For it, assume that for any $\zeta_{ij}^I \in [\zeta_{ij}^L,\zeta_{ij}^U]$, $\vartheta_{ij}^I \in [\vartheta_{ij}^L,\vartheta_{ij}^U]$, $\omega_j \in [\omega_j^L,\omega_j^U]$, $\rho_j \in [\rho_j^L,\rho_j^U]$, $\varepsilon_j \in [\varepsilon_j^L,\varepsilon_j^U]$ and $\kappa_j \in [\kappa_j^L,\kappa_j^U]$, the separation measure of $\mathcal{V}_i$ from $\mathcal{V}^+ = \big(([1,1],[0,0]),(0,1)\big)$ and $\mathcal{V}^- = (([0,0],[1,1]),(1,0))$ are defined as

$$d(\mathcal{V}_i,\mathcal{V}^+) = \left(\sum_{j=1}^{n}\left\{\omega_j(1-\zeta_{ij}^I)^2+\rho_j(\vartheta_{ij}^I)^2+\varepsilon_j(\zeta_{ij})^2+\kappa_j(1-\vartheta_{ij})^2\right\}\right)^{\frac{1}{2}} \tag{8.2}$$

and
$$d(\mathcal{V}_i,\mathcal{V}^-) = \left(\sum_{j=1}^{n}\left\{\omega_j(\zeta_{ij}^I)^2+\rho_j(1-\vartheta_{ij}^I)^2+\varepsilon_j(1-\zeta_{ij})^2+\kappa_j(\vartheta_{ij})^2\right\}\right)^{\frac{1}{2}} \tag{8.3}$$

However, if an expert utilize $\mathcal{V}_j^+ = \big(([f_j^+, f_j^+],\ [t_j^+, t_j^+]),\ (r_j^+,\ s_j^+)\big)_{1\times n}$ and $\mathcal{V}_j^- = \big(([f_j^-, f_j^-],$ $[t_j^-, t_j^-]),\ (r_j^-,\ s_j^-)\big)_{1\times n}$, then the measurement values are given as

$$d(\mathcal{V}_i,\mathcal{V}^+) = \left(\sum_{j=1}^{n}\left\{\begin{array}{l}\omega_j(f_j^+-\zeta_{ij}^I)^2+\rho_j(t_j^+-\vartheta_{ij}^I)^2\\ +\,\varepsilon_j(r_j^+-\zeta_{ij})^2+\kappa_j(s_j^+-\vartheta_{ij})^2\end{array}\right\}\right)^{\frac{1}{2}} \tag{8.4}$$

and
$$d(\mathcal{V}_i,\mathcal{V}^-) = \left(\sum_{j=1}^{n}\left\{\begin{array}{l}\omega_j(f_j^--\zeta_{ij}^I)^2+\rho_j(t_j^--\vartheta_{ij}^I)^2\\ +\,\varepsilon_j(r_j^--\zeta_{ij})^2+\kappa_j(s_j^--\vartheta_{ij})^2\end{array}\right\}\right)^{\frac{1}{2}} \tag{8.5}$$

8.2.4 Relative-closeness coefficient and its nature

To measure the relative strength of the alternatives $\mathcal{V}_i, (i = 1, 2, \ldots, m)$ with respect to $\mathcal{V}^+$ and $\mathcal{V}^-$, we define the closeness-coefficients of the alternatives by Eq. (8.6) as

$$\mathfrak{R}_i \begin{pmatrix} (\zeta^I_{ij})_{m\times n}, (\vartheta^I_{ij})_{m\times n}, (\omega_j)_{1\times n}, \\ (\rho_j)_{1\times n}, (\varepsilon_j)_{1\times n}, (\kappa_j)_{1\times n} \end{pmatrix} = \frac{d(\mathcal{V}_i, \mathcal{V}^-)}{d(\mathcal{V}_i, \mathcal{V}^-) + d(\mathcal{V}_i, \mathcal{V}^+)} \tag{8.6}$$

provided $d(\mathcal{V}_i, \mathcal{V}^+) \neq 0$. Here, ζ^I_{ij}, ϑ^I_{ij} are $m \times n$ matrices, (ω_j), (ρ_j), (ε_j), (κ_j) are the $n-$ dimensional column weights corresponding to MD and NMD. Further, it can be noticed that, $0 \leq d(\mathcal{V}_i, \mathcal{V}^-) \leq d(\mathcal{V}_i, \mathcal{V}^-) + d(\mathcal{V}_i, \mathcal{V}^+)$ and hence $0 \leq \mathfrak{R}_i((\zeta^I_{ij}), (\vartheta^I_{ij}), (\omega_j), (\rho_j), (\varepsilon_j), (\kappa_j)) \leq 1$. From Eq. (8.6), it can be clearly seen that $\mathfrak{R}_i$ is a function of $2n(m + 2)$ unknown variables bearing continuous nature, which rises sharply even with small values of m and n. To explain it, let us assume that $m = 2$ and $n = 3$, i.e., there are only two alternatives which are evaluated under three criteria, then under such conditions and by formulating Eq. (8.6), we have 24 unknown variables. On the other hand, if we considered a problem with $m = 4$ and $n = 5$, then count of such unknown variables increases to 60. Therefore, it is a tedious task to invest effort leading to large computational time and excess cost. So, there is a need for construction of a time-efficient algorithm to resolve this problem so that the number of unknowns gets reduced.

In order to do so, firstly, we will check the monotonicity, boundedness as well as continuity of the function $\mathfrak{R}_i$ with respect to the unknown variables $\zeta^I_{ij} \in [\zeta^L_{ij}, \zeta^U_{ij}]$ and $\vartheta^I_{ij} \in [\vartheta^L_{ij}, \vartheta^U_{ij}]$. For it, by using measures given in Eqs. (8.2) and (8.3), we explicit the expression of $\mathfrak{R}_i$ given in Eq. (8.6) as

$$\mathfrak{R}_i = \frac{\left(\sum\limits_{j=1}^{n} \left\{\omega_j(\zeta^I_{ij})^2 + \rho_j(1-\vartheta^I_{ij})^2 + \varepsilon_j(1-\zeta_{ij})^2 + \kappa_j(\vartheta_{ij})^2\right\}\right)^{\frac{1}{2}}}{\left(\sum\limits_{j=1}^{n} \left\{\begin{matrix} \omega_j(\zeta^I_{ij})^2 + \rho_j(1-\vartheta^I_{ij})^2 \\ + \varepsilon_j(1-\zeta_{ij})^2 + \kappa_j(\vartheta_{ij})^2 \end{matrix}\right\}\right)^{\frac{1}{2}} + \left(\sum\limits_{j=1}^{n} \left\{\begin{matrix} \omega_j(1-\zeta^I_{ij})^2 + \rho_j(\vartheta^I_{ij})^2 \\ + \varepsilon_j(\zeta_{ij})^2 + \kappa_j(1-\vartheta_{ij})^2 \end{matrix}\right\}\right)^{\frac{1}{2}}} \tag{8.7}$$

To address the monotonic behavior of $\mathfrak{R}_i$, we partially differentiate it with respect to ζ^I_{ij} and get

$$\frac{\partial \mathfrak{R}_i}{\partial \zeta^I_{ij}} = \frac{d(\mathcal{V}_i, \mathcal{V}^+)\omega_j(\zeta^I_{ij})(1 - d(\mathcal{V}_i, \mathcal{V}^-)) + d^2(\mathcal{V}_i, \mathcal{V}^-)\omega_j(1-\zeta^I_{ij})}{\left(d^-(\mathcal{V}_i, \mathcal{V}^-) + d^+(\mathcal{V}_i, \mathcal{V}^+)\right)^2 \cdot d(\mathcal{V}_i, \mathcal{V}^+) \cdot d(\mathcal{V}_i, \mathcal{V}^-)} \tag{8.8}$$

Since, $0 \leq d(\mathcal{V}_i, \mathcal{V}^-), d(\mathcal{V}_i, \mathcal{V}^+) \leq 1$, $0 \leq \zeta_{ij}^I \leq 1$ which implies that $\frac{\partial \mathfrak{R}_i}{\partial \zeta_{ij}^I} \geq 0$. However, for $\omega_j \neq 0$, $\frac{\partial \mathfrak{R}_i}{\partial \zeta_{ij}^I} > 0$. Therefore, $\mathfrak{R}_i$ is a monotonic and increasing function of ζ_{ij}^I.

Proceeding in a similar manner for $\mathfrak{R}_i$ with respect to ϑ_{ij}^I, the function is computed to be:

$$\frac{\partial \mathfrak{R}_i}{\partial \vartheta_{ij}^I} = -\left\{ \frac{d(\mathcal{V}_i, \mathcal{V}^+)\rho_j(1-\vartheta_{ij}^I)(1-d(\mathcal{V}_i, \mathcal{V}^-)) + d^2(\mathcal{V}_i, \mathcal{V}^-)\rho_j(\vartheta_{ij}^I)}{\left(d^-(\mathcal{V}_i, \mathcal{V}^-) + d^+(\mathcal{V}_i, \mathcal{V}^+)\right)^2 \cdot d(\mathcal{V}_i, \mathcal{V}^+) \cdot d(\mathcal{V}_i, \mathcal{V}^-)} \right\} \tag{8.9}$$

From $\rho_j, \varepsilon_j, \kappa_j \in [0,1]$, we follow that $\frac{\partial \mathfrak{R}_i}{\partial \vartheta_{ij}^I} \leq 0$, while if $\rho_j \neq 0$, $\varepsilon_j \neq 0$ and $\kappa_j \neq 0$, then $\frac{\partial \mathfrak{R}_i}{\partial \vartheta_{ij}^I} < 0$. Therefore, $\mathfrak{R}_i$ shows monotonic non-increasing nature for ϑ_{ij}^I.

Since $\mathfrak{R}_i$ are the continuous functions because of closed and bounded nature of elements $\zeta_{ij}^I, \vartheta_{ij}^I, \zeta_{ij}, \vartheta_{ij}$, $\omega_j, \rho_j, \varepsilon_j$ and κ_j of subintervals of $[0,1]$. Thus, the $\mathfrak{R}_i$ values must also lie in range $[0,1]$ denoted by $[\mathfrak{R}_i^L, \mathfrak{R}_i^U]$ respectively. Therefore, by the basic definition as well as the continuous nature of the function $\mathfrak{R}_i$, we have $0 \leq \mathfrak{R}_i^L \leq \mathfrak{R}_i^U \leq 1$ for $\zeta_{ij}^I \in [\zeta_{ij}^L, \zeta_{ij}^U]$ and so on. Further, it can be seen that $\mathfrak{R}_i^L + (1 - \mathfrak{R}_i^U) = 1 - (\mathfrak{R}_i^U - \mathfrak{R}_i^L) \leq 1$ and thus the set $[\mathfrak{R}_i^L, \mathfrak{R}_i^U]$ can be equivalently expressed as an IFS $\mathfrak{R}_i = (\mathfrak{R}_i^L, 1 - \mathfrak{R}_i^U)$ which shows that closeness degree of alternatives $\mathcal{V}_i$ to $\mathcal{V}^+$ is $\mathfrak{R}_i^L$ whereas their non-closeness is $1 - \mathfrak{R}_i^U$. Hence, the set $\mathfrak{R}_i = [\mathfrak{R}_i^L, \mathfrak{R}_i^U]$ can be used to determine the rankings.

8.2.5 Auxiliary NLP models

Since $\mathfrak{R}_i$ is the bounded and continuous function of ζ_{ij}^I, ϑ_{ij}^I and $\mathfrak{R}_i^L$, $\mathfrak{R}_i^U$ are its lower and upper bounds. So, by the continuous property of $\mathfrak{R}_i$, we get that $\mathfrak{R}_i^L$ is attained at lower bounds of ζ_{ij}^I and at upper bounds of ϑ_{ij}^I. On the other hand, $\mathfrak{R}_i^U$ is attained at upper bounds of ζ_{ij}^I and lower bounds of ϑ_{ij}^I. Hence, the two NLP models can be constructed from Eq. (8.7) as

$$\mathfrak{R}_i^L = \min \frac{\left(\sum\limits_{j=1}^n \left\{\omega_j(\zeta_{ij}^L)^2 + \rho_j(1-\vartheta_{ij}^U)^2 + \varepsilon_j(1-\zeta_{ij})^2 + \kappa_j(\vartheta_{ij})^2\right\}\right)^{\frac{1}{2}}}{\left(\sum\limits_{j=1}^n \left\{\begin{array}{l} \omega_j(\zeta_{ij}^L)^2 + \rho_j(1-\vartheta_{ij}^U)^2 \\ + \varepsilon_j(1-\zeta_{ij})^2 + \kappa_j(\vartheta_{ij})^2 \end{array}\right\}\right)^{\frac{1}{2}} + \left(\sum\limits_{j=1}^n \left\{\begin{array}{l} \omega_j(1-\zeta_{ij}^L)^2 + \rho_j(\vartheta_{ij}^U)^2 \\ + \varepsilon_j(\zeta_{ij})^2 + \kappa_j(1-\vartheta_{ij})^2 \end{array}\right\}\right)^{\frac{1}{2}}} \tag{8.10}$$

and

$$\mathfrak{R}_i^U = \max \frac{\left(\sum_{j=1}^{n}\left\{\omega_j(\zeta_{ij}^U)^2 + \rho_j(1-\vartheta_{ij}^L)^2 + \varepsilon_j(1-\zeta_{ij})^2 + \kappa_j(\vartheta_{ij})^2\right\}\right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n}\left\{\begin{array}{l}\omega_j(\zeta_{ij}^U)^2 + \rho_j(1-\vartheta_{ij}^L)^2\\ +\varepsilon_j(1-\zeta_{ij})^2 + \kappa_j(\vartheta_{ij})^2\end{array}\right\}\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n}\left\{\begin{array}{l}\omega_j(1-\zeta_{ij}^U)^2 + \rho_j(\vartheta_{ij}^L)^2\\ +\varepsilon_j(\zeta_{ij})^2 + \kappa_j(1-\vartheta_{ij})^2\end{array}\right\}\right)^{\frac{1}{2}}} \tag{8.11}$$

$$\text{s.t.} \begin{cases} \omega_j^L \leq \omega_j \leq \omega_j^U \quad ; \quad \rho_j^L \leq \rho_j \leq \rho_j^U; \\ \varepsilon_j^L \leq \varepsilon_j \leq \varepsilon_j^U \quad ; \quad \kappa_j^L \leq \kappa_j \leq \kappa_j^U \quad ; \quad \forall j = 1, 2, \ldots, n \end{cases}$$

The models (8.10) and (8.11) are the nonlinear fractional programming models and it contains $4n$ unknown variables. This count is, obviously, less than that of unknown variables in the model (8.6). Further, the RCC of alternatives $\mathcal{V}_i$ can be extracted in $\mathfrak{R}_i = [\mathfrak{R}_i^L, \mathfrak{R}_i^U]$ which leads the inclusion-comparison probability of sets $\mathfrak{R}_i$ and $\mathfrak{R}_k$, denoted by $\mathfrak{z}(\mathfrak{R}_i \supseteq \mathfrak{R}_k)$ and hence the likeliness of alternatives $\mathfrak{z}(\mathcal{V}_i \succeq \mathcal{V}_k) = \mathfrak{z}(\mathfrak{R}_i \supseteq \mathfrak{R}_k)$, used to rank the alternatives, is defined by using Eq. (8.1) as

$$\mathfrak{z}(\mathcal{V}_i \succeq \mathcal{V}_k) = \mathfrak{z}(\mathfrak{R}_i \supseteq \mathfrak{R}_k) = \max\left\{1 - \max\left\{\frac{\mathfrak{R}_k^U - \mathfrak{R}_i^L}{\mathcal{L}(\mathfrak{R}_i) + \mathcal{L}(\mathfrak{R}_k)}, 0\right\}, 0\right\} \tag{8.12}$$

where $\mathcal{L}(\mathfrak{R}_i) = \mathfrak{R}_i^U - \mathfrak{R}_i^L$ and $\mathcal{L}(\mathfrak{R}_k) = \mathfrak{R}_k^U - \mathfrak{R}_k^L$. Therefore, the likelihood matrix is represented by $\mathcal{Z} = (\mathfrak{z}^{(ik)})_{m\times m}$ where $\mathfrak{z}^{(ik)} = \mathfrak{z}(\mathcal{V}_i \succeq \mathcal{V}_k); (i, k = 1, 2, \ldots, m)$. Also, the noticeable fact is that $0 \leq \mathfrak{z}^{(ik)} \leq 1$ and $\mathfrak{z}^{(ik)} + \mathfrak{z}^{(ki)} = 1$. The optimal value of each alternative is determined by using χ_i defined as

$$\chi_i = \frac{1}{m(m-1)}\left(\sum_{k=1}^{m}\mathfrak{z}^{(ik)} + \frac{m}{2} - 1\right) \tag{8.13}$$

Then, the ranking of the alternatives is found in accordance with the decreasing values of χ_i's and the best alternative(s) is selected.

On the basis of the above analysis, the various steps associated with a TOPSIS method relying on the NLP method are summarized in an Algorithm 8.1.

8.3 Additional features of the proposed NLP models

The proposed designed models (8.10) and (8.11) also has some additional features which are enlisted below:

Algorithm 8.1 Proposed framework of TOPSIS method to select the best alternatives

Input: Given alternatives $\mathcal{V}_i$ $(i = 1, 2, \ldots, m)$ and criteria $\mathfrak{B}_j$ $(j = 1, 2, \ldots, n)$

Output: The optimal alternative.

1: Summarize the evaluated information for each alternative $\mathcal{V}_i (i = 1, 2, \ldots, m)$ provided by an expert under the CIFS environment as $\mathcal{M} = \big(([\zeta_{ij}^L,\, \zeta_{ij}^U],\, [\vartheta_{ij}^L,\, \vartheta_{ij}^U]),\, (\zeta_{ij}, \vartheta_{ij})\big)$.

2: Arrange the importance of each criterion $\mathfrak{B}_j$ $(j = 1, 2, \ldots, n)$ in terms of their subjective weights as $W = \big(([\omega_j^L, \omega_j^U], [\rho_j^L, \rho_j^U]), (\varepsilon_j, \kappa_j)\big)$ in the form of CIFNs.

3: For each alternative $\mathcal{V}_i (i = 1, 2, \ldots, m)$, construct the optimization models (8.10) and (8.11) and hence obtain their RCCs in terms of interval sets $\mathfrak{R}_i = [\mathfrak{R}_i^L, \mathfrak{R}_i^U]$.

4: Use Eq. (8.12) to form likelihood matrix $\mathcal{Z}$ as

$$\mathcal{Z} = \begin{pmatrix} \mathfrak{z}^{(11))} & \mathfrak{z}^{(12)} & \ldots & \mathfrak{z}^{(1m)} \\ \mathfrak{z}^{(21)} & \mathfrak{z}^{(22)} & \ldots & \mathfrak{z}^{(2m)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{z}^{(m1)} & \mathfrak{z}^{(m2)} & \ldots & \mathfrak{z}^{(mm)} \end{pmatrix}$$

where $\mathfrak{z}^{(ik)} = \mathfrak{z}(\mathcal{V}_i \succeq \mathcal{V}_k); (i, k = 1, 2, \ldots, m)$ and hence compute the values of χ_i by Eq. (8.13).

5: Based on the descending values of χ_i $(i = 1, 2, \ldots, m)$, select the most optimal alternative(s).

1) It is noticed that the discussion given in section 8.2.4 is based on fixed $\mathcal{V}^+$ and $\mathcal{V}^-$ which results that RCCs could not show any variation even if the alternatives are changed. To address it completely, we have updated the ideal values to $\mathcal{V}_j^+ = \big(([f_j^+, f_j^+],\, [t_j^+, t_j^+]),\, (r_j^+,\, s_j^+)\big)_{1\times n}$ and $\mathcal{V}_j^- = \big(([f_j^-,\, f_j^-],\, [t_j^-,\, t_j^-]),\, (r_j^-,\, s_j^-)\big)_{1\times n}$ in the measures given in Eq. (8.4) and Eq. (8.5). Based on it, RCC of the alternatives $\mathcal{V}_i$ is defined as

$$\mathfrak{R}_i = \frac{\left(\sum\limits_{j=1}^{n} \left\{\omega_j (f_j^{\,-} - \zeta_{ij}^I)^2 + \rho_j (t_j^- - \vartheta_{ij}^I)^2 + \varepsilon_j (r_j^- - \zeta_{ij})^2 + \kappa_j (s_j^- - \vartheta_{ij})^2\right\}\right)^{\frac{1}{2}}}{\left(\sum\limits_{j=1}^{n} \left\{\begin{matrix} \omega_j (f_j^{\,-} - \zeta_{ij}^I)^2 + \rho_j (t_j^- - \vartheta_{ij}^I)^2 \\ + \varepsilon_j (r_j^- - \zeta_{ij})^2 + \kappa_j (s_j^- - \vartheta_{ij})^2 \end{matrix}\right\}\right)^{\frac{1}{2}} + \left(\sum\limits_{j=1}^{n} \left\{\begin{matrix} \omega_j (f_j^+ - \zeta_{ij}^I)^2 + \rho_j (t_j^+ - \vartheta_{ij}^I)^2 \\ + \varepsilon_j (r_j^+ - \zeta_{ij})^2 + \kappa_j (s_j^+ - \vartheta_{ij})^2 \end{matrix}\right\}\right)^{\frac{1}{2}}} \tag{8.14}$$

Thus, in view of models (8.10) and (8.11), $\mathfrak{R}_i^L$ and $\mathfrak{R}_i^U$ can similarly be obtained as

$$\begin{aligned} &\mathfrak{R}_i^L = \min\left\{\mathfrak{R}_i\left((\zeta_{ij}^L)_{m\times n}, (\vartheta_{ij}^U)_{m\times n}, (\omega_j)_{1\times n}, (\rho_j)_{1\times n}, (\varepsilon_j)_{1\times n}, (\kappa_j)_{1\times n}\right)\right\} \\ \text{and} \quad &\mathfrak{R}_i^U = \max\left\{\mathfrak{R}_i\left((\zeta_{ij}^U)_{m\times n}, (\vartheta_{ij}^L)_{m\times n}, (\omega_j)_{1\times n}, (\rho_j)_{1\times n}, (\varepsilon_j)_{1\times n}, (\kappa_j)_{1\times n}\right)\right\} \\ &\text{s.t.} \begin{cases} \omega_j^L \leq \omega_j \leq \omega_j^U \quad ; \quad \rho_j^L \leq \rho_j \leq \rho_j^U; \\ \varepsilon_j^L \leq \varepsilon_j \leq \varepsilon_j^U \quad ; \quad \kappa_j^L \leq \kappa_j \leq \kappa_j^U \quad ; \quad \forall j = 1, 2, \ldots, n \end{cases} \end{aligned}$$

respectively.

2) If we measure the separation between the alternatives and the ideals by using weighted Hamming distance [52], then we get

$$d(\mathcal{V}_i, \mathcal{V}^+) = \sum_{j=1}^{n} \left\{\omega_j(1-\zeta_{ij}^I) + \rho_j(\vartheta_{ij}^I) + \varepsilon_j(\zeta_{ij}) + \kappa_j(1-\vartheta_{ij})\right\} \tag{8.15}$$

and
$$d(\mathcal{V}_i, \mathcal{V}^-) = \sum_{j=1}^{n} \left\{\omega_j(\zeta_{ij}^I) + \rho_j(1-\vartheta_{ij}^I) + \varepsilon_j(1-\zeta_{ij}) + \kappa_j(\vartheta_{ij})\right\} \tag{8.16}$$

Hence, the RCC of $\mathcal{V}_i$ is given as

$$\begin{aligned}\mathfrak{R}_i &= \frac{\sum_{j=1}^{n} \left\{\omega_j(\zeta_{ij}^I) + \rho_j(1-\vartheta_{ij}^I) + \varepsilon_j(1-\zeta_{ij}) + \kappa_j(\vartheta_{ij})\right\}}{\sum_{j=1}^{n}\left\{\begin{array}{l}\omega_j(\zeta_{ij}^I) + \rho_j(1-\vartheta_{ij}^I) \\ +\varepsilon_j(1-\zeta_{ij}) + \kappa_j(\vartheta_{ij})\end{array}\right\} + \sum_{j=1}^{n}\left\{\begin{array}{l}\omega_j(1-\zeta_{ij}^I) + \rho_j(\vartheta_{ij}^I) \\ +\varepsilon_j(\zeta_{ij}) + \kappa_j(1-\vartheta_{ij})\end{array}\right\}} \\ &= \frac{\sum_{j=1}^{n} \{\omega_j(\zeta_{ij}^I) + \rho_j(1-\vartheta_{ij}^I) + \varepsilon_j(1-\zeta_{ij}) + \kappa_j(\vartheta_{ij})\}}{\sum_{j=1}^{n}(\omega_j + \rho_j + \varepsilon_j + \kappa_j)}\end{aligned} \tag{8.17}$$

Further, by the monotonicity as well as bounded property of $\mathfrak{R}_i$, we can obtain the bounds $\mathfrak{R}_i^L$ and $\mathfrak{R}_i^U$ of each alternative by solving the optimization models as below

$$\mathfrak{R}_i^L = \min\left\{\frac{\sum_{j=1}^{n} \{\omega_j(\zeta_{ij}^L) + \rho_j(1-\vartheta_{ij}^U) + \varepsilon_j(1-\zeta_{ij}) + \kappa_j(\vartheta_{ij})\}}{\sum_{j=1}^{n}(\omega_j + \rho_j + \varepsilon_j + \kappa_j)}\right\} \tag{8.18}$$

$$\mathfrak{R}_i^U = \max\left\{\frac{\sum_{j=1}^{n} \{\omega_j(\zeta_{ij}^U) + \rho_j(1-\vartheta_{ij}^L) + \varepsilon_j(1-\zeta_{ij}) + \kappa_j(\vartheta_{ij})\}}{\sum_{j=1}^{n}(\omega_j + \rho_j + \varepsilon_j + \kappa_j)}\right\} \tag{8.19}$$

$$\text{s.t.} \begin{cases}\omega_j^L \leq \omega_j \leq \omega_j^U & ; \quad \rho_j^L \leq \rho_j \leq \rho_j^U; \\ \varepsilon_j^L \leq \varepsilon_j \leq \varepsilon_j^U & ; \quad \kappa_j^L \leq \kappa_j \leq \kappa_j^U\end{cases}$$

This model can be simplified by letting $z = \frac{1}{\sum_{j=1}^{n}(\omega_j+\rho_j+\varepsilon_j+\kappa_j)}$, and $a_j = z\omega_j$, $b_j = z\rho_j$,

$c_j = z\varepsilon_j$, $d_j = z\kappa_j$. Therefore, the above models can be rewritten as

$$\mathfrak{R}_i^L = \min \sum_{j=1}^{n} \left\{ a_j(\zeta_{ij}^L) + b_j(1 - \vartheta_{ij}^U) + c_j(1 - \zeta_{ij}) + d_j(\vartheta_{ij}) \right\}$$

$$\mathfrak{R}_i^U = \max \sum_{j=1}^{n} \left\{ a_j(\zeta_{ij}^U) + b_j(1 - \vartheta_{ij}^L) + c_j(1 - \zeta_{ij}) + d_j(\vartheta_{ij}) \right\}$$

$$\text{s.t.} \begin{cases} z\omega_j^L \leq a_j \leq z\omega_j^U & ; z\rho_j^L \leq b_j \leq z\rho_j^U \\ z\varepsilon_j^L \leq c_j \leq z\varepsilon_j^U & ; z\kappa_j^L \leq d_j \leq z\kappa_j^U \\ \sum_{j=1}^{n} (a_j + b_j + c_j + d_j) = 1 & ; z \geq 0 \end{cases}$$

After solving these models, we can get the interval set $[\mathfrak{R}_i^L, \mathfrak{R}_i^U]$ of each alternative $\mathcal{V}_i (i = 1, 2, \ldots, m)$.

3) The proposed model can equivalently be well addressed for all those MCDM problems where the information about the criteria weights are taken as *a priori* and a constants real number say $\omega_j > 0$. Under it, the bounds of RCC can be obtained as

$$\mathfrak{R}_i^L = \frac{1}{4} \sum_{j=1}^{n} \omega_j \Big\{ \zeta_{ij}^L + 1 - \vartheta_{ij}^U + 1 - \zeta_{ij} + \vartheta_{ij} \Big\} \tag{8.20}$$

$$\text{and} \quad \mathfrak{R}_i^U = \frac{1}{4} \sum_{j=1}^{n} \omega_j \Big\{ \zeta_{ij}^U + 1 - \vartheta_{ij}^L + 1 - \zeta_{ij} + \vartheta_{ij} \Big\} \tag{8.21}$$

respectively.

4) If ideal values $\mathcal{V}_j^+ = (([f_j^+, f_j^+], [t_j^+, t_j^+]), ([r_j^+, s_j^+]))$, and $\mathcal{V}_j^- = (([f_j^-, f_j^-], [t_j^-, t_j^-]), ([r_j^-, s_j^-]))$ and a constant known real weight ω_j are utilized then, the bounds of RCCs are obtained as

$$\mathfrak{R}_i^L = \frac{\sum_{j=1}^{n} \Big(\omega_j(\zeta_{ij}^L - f_j^-) + \rho_j(t_j^- - \vartheta_{ij}^U) + \varepsilon_j(r_j^- - \zeta_{ij}) + \kappa_j(\vartheta_{ij} - s_j^-) \Big)}{\sum_{j=1}^{n} \Big(\omega_j(f_j^+ - f_j^-) + \rho_j(t_j^- - t_j^+) + \varepsilon_j(r_j^- - r_j^+) + \kappa_j(s_j^+ - s_j^-) \Big)} \tag{8.22}$$

$$\text{and} \quad \mathfrak{R}_i^U = \frac{\sum_{j=1}^{n} \Big(\omega_j(\zeta_{ij}^U - f_j^-) + \rho_j(t_j^- - \vartheta_{ij}^L) + \varepsilon_j(r_j^- - \zeta_{ij}) + \kappa_j(\vartheta_{ij} - s_j^-) \Big)}{\sum_{j=1}^{n} \Big(\omega_j(f_j^+ - f_j^-) + \rho_j(t_j^- - t_j^+) + \varepsilon_j(r_j^- - r_j^+) + \kappa_j(s_j^+ - s_j^-) \Big)} \tag{8.23}$$

respectively.

8.4 Numerical Example

In this section, we analyze the results of the developed MCDM approach with a numerical example on signal processing which can be read as

8.4.1 Case study

SONAR is an acronym for **So**und **Na**vigation and **R**anging which is based on the principle that sound travels faster than light inside water sources. Thus, SONAR is a technique that uses sound propagation (usually underwater, as in submarine navigation) to navigate and communicate with or to detect objects on or under the surface of the water. SONAR consists of a sound beam generator, a transmitter, and a receiver. Ultrasonic waves (mostly called pings) are generated and propagated inside the sea. On getting stricken to any of the underwater object (such as aquatic animals, rocks, sea-surface, other submarines etc), they got reflected back and the ultrasonic pulse signals are received in form of signals (mostly called the echo-signals). They are converted into the electrical signals and are analyzed by the scientists on their monitor screens. SONARs are basically of two types: Active and Passive.

(i) **Active SONARs:** These SONARs emits the pulses of sound waves that travel through water and return back to the ship.

(ii) **Passive SONARs:** These SONARs involves listening to sounds generated by the target. For example, dolphins make use of sound waves to locate their prey. This is somewhat a natural SONAR phenomenon that these dolphins use.

The main criteria to judge the objects beneath the water surface are:

(i) **Time Compression:** It is the time interval between the transmission and return of the signal. The scientists locate the distance of the object from them by simply calculating the distance using the time taken by the signal to bounce back. More the time, more distant the object is from the ship.

(ii) **Signal Strength:** The strength of the signal is another parameter to detect the position of the object. Since, there are no amplifiers installed in the way of the sound wave path, so the strength of the signal provides a lot of information about the object's location. Stronger the received signal as compared to that of the transmitted one, nearer the object is.

(iii) **Expected size of the object:** The aperture of the echo pulse gives a rough idea about the size and material of the object.

Suppose a Naval Submarine is using a SONAR for detecting the nearby rocks material and other objects like enemy submarines (if any). They are already having a sea-map of the location of the possible obstacles on the way of the submarine. But as the underwater surface bears a dynamic nature, it keeps on changing frequently, so they can't completely rely on the sea-map provided to them for submarine navigation. The four possible objects $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$, in the vicinity, are to be checked for their distances from the submarine. According to the available data, the information is formulated in form CIFNs, under the set of three criteria; $\mathfrak{B}_1$: 'Time Compression', $\mathfrak{B}_2$: 'Signal strength' and $\mathfrak{B}_3$: 'Expected size of the object'. Then the objective of the problem is to locate the nearby object so that it can't hinder the smooth journey of the ship. For it, the steps of the Algorithm 8.1 are executed as below:

Step 1: The evaluation information given by the expert are summarized in Table 8.1 under the CIFS environment.

Step 2: The collective information about the each criterion $\mathfrak{B}_j$,$(j = 1, 2, 3)$ is taken as $W = \big\{\big(([0.10, 0.40],\ [0.20, 0.55]),\ ([0.10, 0.30], [0.20, 0.60])\big),\ \big(([0.20, 0.50],\ [0.15, 0.45]), ([0.20, 0.50], [0.30, 50])\big),\ \big(([0.20, 0.50],\ [0.15, 0.38]),\ ([0.10,\ 0.40],\ [0.20, 0.30])\big)\big\}$.

Step 3: Formulate the models (8.10) and (8.11) for an alternative $\mathcal{V}_1$ as

$$\mathfrak{R}_1^L = \min \frac{\sqrt{\begin{pmatrix} 0.01\omega_1 + 0.04\rho_1 + 0.64\varepsilon_1 + 0.16\kappa_1 \\ +0.04\omega_2 + 0.25\rho_2 + 0.25\varepsilon_2 + 0.04\kappa_2 \\ +0.04\omega_3 + 0.16\rho_3 + 0.09\varepsilon_3 + 0.04\kappa_3 \end{pmatrix}}}{\sqrt{\begin{pmatrix} 0.01\omega_1 + 0.04\rho_1 + 0.64\varepsilon_1 + 0.16\kappa_1 \\ +0.04\omega_2 + 0.25\rho_2 + 0.25\varepsilon_2 + 0.04\kappa_2 \\ +0.04\omega_3 + 0.16\rho_3 + 0.09\varepsilon_3 + 0.04\kappa_3 \end{pmatrix}} + \sqrt{\begin{pmatrix} 0.81\omega_1 + 0.64\rho_1 + 0.04\varepsilon_1 + 0.36\kappa_1 \\ +0.64\omega_2 + 0.25\rho_2 + 0.25\varepsilon_2 + 0.64\kappa_2 \\ +0.64\omega_3 + 0.36\rho_3 + 0.49\varepsilon_3 + 0.64\kappa_3 \end{pmatrix}}}$$

$$\mathfrak{R}_1^U = \max \frac{\sqrt{\begin{pmatrix} 0.04\omega_1 + 0.25\rho_1 + 0.64\varepsilon_1 + 0.16\kappa_1 \\ +0.09\omega_2 + 0.36\rho_2 + 0.25\varepsilon_2 + 0.04\kappa_2 \\ +0.09\omega_3 + 0.36\rho_3 + 0.09\varepsilon_3 + 0.04\kappa_3 \end{pmatrix}}}{\sqrt{\begin{pmatrix} 0.04\omega_1 + 0.25\rho_1 + 0.64\varepsilon_1 + 0.16\kappa_1 \\ +0.09\omega_2 + 0.36\rho_2 + 0.25\varepsilon_2 + 0.04\kappa_2 \\ +0.09\omega_3 + 0.36\rho_3 + 0.09\varepsilon_3 + 0.04\kappa_3 \end{pmatrix}} + \sqrt{\begin{pmatrix} 0.64\omega_1 + 0.25\rho_1 + 0.04\varepsilon_1 + 0.36\kappa_1 \\ +0.49\omega_2 + 0.16\rho_2 + 0.25\varepsilon_2 + 0.64\kappa_2 \\ +0.49\omega_3 + 0.16\rho_3 + 0.49\varepsilon_3 + 0.64\kappa_3 \end{pmatrix}}}$$

$$\text{s.t.} \begin{cases} 0.10 \leq \omega_1 \leq 0.40 \quad ; \quad 0.10 \leq \varepsilon_1 \leq 0.30 \quad ; \quad 0.20 \leq \rho_1 \leq 0.55 \quad ; \\ 0.20 \leq \kappa_1 \leq 0.60 \quad ; \quad 0.20 \leq \omega_2 \leq 0.50 \quad ; \quad 0.20 \leq \varepsilon_2 \leq 0.50 \quad ; \\ 0.15 \leq \rho_2 \leq 0.45 \quad ; \quad 0.30 \leq \kappa_2 \leq 0.50 \quad ; \quad 0.20 \leq \omega_3 \leq 0.50 \quad ; \\ 0.10 \leq \varepsilon_3 \leq 0.40 \quad ; \quad 0.15 \leq \rho_3 \leq 0.38 \quad ; \quad 0.20 \leq \kappa_3 \leq 0.30 \end{cases}$$

After solving these models, we get $\mathfrak{R}_1^L = 0.2792$ and $\mathfrak{R}_1^U = 0.4675$ and hence the RCC interval for alternative $\mathcal{V}_1$ is $\mathfrak{R}_1 = [0.2792, 0.4675]$. Similarly, making use of models (8.10) and (8.11) for other alternatives, the RCC intervals are obtained as $\mathfrak{R}_2 = [0.4181, 0.6201]$; $\mathfrak{R}_3 = [0.3491, 0.5345]$ and $\mathfrak{R}_4 = [0.3671, 0.6316]$.

Step 4: By using Eq. (8.12), the likelihood matrix of the alternatives is formulated as

$$\mathcal{Z} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \mathcal{V}_3 \\ \mathcal{V}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{V}_1 & \mathcal{V}_2 & \mathcal{V}_3 & \mathcal{V}_4 \end{array} \\ \begin{pmatrix} 0.5 & 0.1266 & 0.3168 & 0.2217 \\ 0.8734 & 0.5 & 0.6995 & 0.5423 \\ 0.6832 & 0.3005 & 0.5 & 0.3721 \\ 0.7783 & 0.4577 & 0.6279 & 0.5 \end{pmatrix} \end{array}$$

Based on this matrix and Eq. (8.13), we can get $\chi_1 = 0.1804$; $\chi_2 = 0.3013$; $\chi_3 = 0.2380$ and $\chi_4 = 0.2803$.

Step 5: According to $\chi_2 > \chi_4 > \chi_3 > \chi_1$, the ranking order is $\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$. So, $\mathcal{V}_2$ is the nearby object.

By taking the weight vector of each criterion as a known real number instead as an interval. Then for implementing the steps of Algorithm 8.1, we revisited the above example and the steps followed under it are summarized as below:

Step 1: The rating values possessed by alternatives are tabulated in Table 8.1.

Step 2: Assume that $\omega = (0.3, 0.5, 0.2)^T$ be the weight vector of the criteria.

Step 3: Based on ω, Eqs. (8.20) and (8.21), the RCCs interval of the alternatives are computed as $\mathfrak{R}_1 = [0.3425, 0.4125]$, $\mathfrak{R}_2 = [0.5250, 0.5888]$, $\mathfrak{R}_3 = [0.4450, 0.5125]$ and $\mathfrak{R}_4 = [0.4850, 0.6137]$.

Step 4: The likelihood probability values by Eq. (8.12) are calculated as

$$\mathcal{Z} = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 1 & 0.5 & 1 & 0.5392 \\ 1 & 0 & 0.5 & 0.1402 \\ 1 & 0.4608 & 0.8598 & 0.5 \end{pmatrix}$$

and hence by Eq. (8.13), we can get $\chi_1 = 0.1250$, $\chi_2 = 0.3366$, $\chi_3 = 0.2200$, and $\chi_4 = 0.3184$.

Step 5: Since $\chi_2 > \chi_4 > \chi_3 > \chi_1$, the ordering of the alternatives is $\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$. So, the nearby object is $\mathcal{V}_2$.

8.4.2 Comparative analysis with interval weight approaches

The IVIFS is one of the special cases of CIFSs, so to demonstrate the effectiveness of the algorithm we compare the performance with Li [85] approach using the interval weight information. For it, first we convert the CIFS information into IVIFS information by setting the intuitionistic fuzzy judgments corresponding to each criterion as zero. Thus, the resultant information is given in Table 8.2. Based on it, the steps of approach given by Li [85] are implemented as below to collect the best alternative(s).

Step 1: The rating values under the IVIFS is represented in Table 8.2.

Step 2: The weight information about the criteria are represented in interval sets as $W = \big\{([0.10, 0.40], [0.20, 0.55]), ([0.20, 0.50], [0.15, 0.45]), ([0.20, 0.50], [0.15, 0.38])\big\}$.

Step 3: Formulate the two models to find the bounds of RCCs by using Euclidean distance for IVIFSs. For example, the models for $\mathcal{V}_1$ are given as

$$\mathfrak{R}_1^L = \min \frac{\sqrt{0.01\omega_1 + 0.04\rho_1 + 0.04\omega_2 + 0.25\rho_2 + 0.04\omega_3 + 0.16\rho_3}}{\sqrt{\begin{array}{l}0.01\omega_1 + 0.04\rho_1 + 0.04\omega_2 \\ + 0.25\rho_2 + 0.04\omega_3 + 0.16\rho_3\end{array}} + \sqrt{\begin{array}{l}0.81\omega_1 + 0.64\rho_1 + 0.64\omega_2 \\ + 0.25\rho_2 + 0.64\omega_3 + 0.36\rho_3\end{array}}}$$

$$\Re_1^U = \max \frac{\sqrt{0.04\omega_1 + 0.25\rho_1 + 0.09\omega_2 + 0.36\rho_2 + 0.09\omega_3 + 0.36\rho_3}}{\sqrt{\begin{array}{c}0.04\omega_1 + 0.25\rho_1 + 0.09\omega_2 \\ + 0.36\rho_2 + 0.09\omega_3 + 0.36\rho_3\end{array}} + \sqrt{\begin{array}{c}0.64\omega_1 + 0.25\rho_1 + 0.49\omega_2 \\ + 0.16\rho_2 + 0.49\omega_3 + 0.16\rho_3\end{array}}}$$

$$\text{s.t.} \begin{cases} 0.10 \leq \omega_1 \leq 0.40 \quad ; \quad 0.20 \leq \rho_1 \leq 0.55 \\ 0.20 \leq \omega_2 \leq 0.50 \quad ; \quad 0.15 \leq \rho_2 \leq 0.45 \\ 0.20 \leq \omega_3 \leq 0.50 \quad ; \quad 0.15 \leq \rho_3 \leq 0.38 \end{cases}$$

After solving these models, the optimal values are obtained as $\Re_1^L = 0.2313$ and $\Re_1^U = 0.4865$. Hence, the RCC interval is $\Re_1 = [0.2313, 0.4865]$. In a similar way, for other alternatives, these intervals are computed as $\Re_2 = [0.4499, 0.7331]$, $\Re_3 = [0.3221, 0.5696]$ and $\Re_4 = [0.3293, 0.6936]$.

Step 4: The likelihood matrix is obtained by Eq. (8.12) for the alternatives as

$$\mathcal{Z} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \mathcal{V}_3 \\ \mathcal{V}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{V}_1 & \mathcal{V}_2 & \mathcal{V}_3 & \mathcal{V}_4 \end{array} \\ \begin{pmatrix} 0.5 & 0.0680 & 0.3270 & 0.2538 \\ 0.9320 & 0.5 & 0.7744 & 0.6236 \\ 0.6730 & 0.2256 & 0.5 & 0.3928 \\ 0.7462 & 0.3764 & 0.6072 & 0.5 \end{pmatrix} \end{array}$$

and by Eq. (8.13), we get $\chi_1 = 0.1791$; $\chi_2 = 0.3192$; $\chi_3 = 0.2326$ and $\chi_4 = 0.2692$.

Step 5: The ranking order obtain as $\mathcal{V}_2 \succeq \mathcal{V}_4 \succeq \mathcal{V}_3 \succeq \mathcal{V}_1$ which implies that $\mathcal{V}_2$ is the nearby object which is same as the proposed approach.

8.4.3 Comparative analysis with real weight approaches

In order to elucidate the multiple advantages of the proposed algorithm, we have taken the weight vector as $\omega = (0.3, 0.5, 0.2)^T$ and the existing approaches [25, 34, 44, 114, 115, 131, 170, 180, 186, 191, 199, 220] under IVIFS environment. The results computed by these approaches for the data set (see Table 8.2) are represented in Table 8.3. From this table, it is seen that the optimal alternative coincides with proposed one which verifies the effectiveness and the rationality of the novel method. Further, it is observed that the principles along with the notions of these existing approaches are exceptionally different from the proposed ones. The existing methods summarized the alternatives in form of IVIFSs using the basic set measures, distance measures and so on, while the proposed approach considers not only the ideal values $\mathcal{V}^+$ and $\mathcal{V}^-$ but also judges the relative significance of the weighted distance measures. Also, the former approaches do not capture DM problems with the interval weights information. Hence, the proposed algorithm can efficiently solve the DM problems. Lastly, it is also seen that approaches under IFSs or IVIFSs are considered as a special cases of the proposed work.

8.5 Conclusion

The key contributions of this chapter are summarized below:

1) In the present chapter, a framework has been proposed to accommodate more uncertainties in order to express the fuzzy information in the MCDM problem under the CIFS environment. The existing environment, IFSs and IVIFSs, to describe the information are the special cases of CIFS environment. The major advantages of this set are in terms of representing the data where an element is evaluated under the consideration of a disagreement degree (in terms of IFNs) corresponding to the agreed interval region (in terms of IVIFNs). Thus, this set considered the significance of IVIFS to get progressively proper outcomes through IFS.

2) Based on these features of CIFS, the present chapter addressed the solve the MCDM problem by using the concept of the TOPSIS method. For it, unlike considering the unify weight to each criterion, an interval weight is considered and hence by taking the concept of relative closeness-coefficient in TOPSIS, the two NLP models are constructed based on the distance measures from its ideal values. Additionally, the relevant properties of the proposed models are also examined.

3) Further, an algorithm to solve the decision making problems is presented and a practical example is considered to verify its feasibility with the several existing approaches.

Table 8.1: Information about the alternatives using CIFNs

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$(([0.10,0.20],[0.50,0.80]),(0.20,0.40))$	$(([0.20,0.30],[0.40,0.50]),(0.50,0.20))$	$(([0.20,0.30],[0.40,0.60]),(0.70,0.20))$
$\mathcal{V}_2$	$(([0.40,0.50],[0.20,0.30]),(0.60,0.40))$	$(([0.80,0.90],[0.01,0.10]),(0.40,0.20))$	$(([0.10,0.20],[0.20,0.60]),(0.50,0.40))$
$\mathcal{V}_3$	$(([0.20,0.30],[0.50,0.60]),(0.40,0.20))$	$(([0.40,0.60],[0.20,0.30]),(0.30,0.40))$	$(([0.20,0.30],[0.50,0.70]),(0.50,0.30))$
$\mathcal{V}_4$	$(([0.10,0.30],[0.30,0.70]),(0.30,0.10))$	$(([0.40,0.90],[0.05,0.10]),(0.30,0.40))$	$(([0.20,0.30],[0.40,0.60]),(0.30,0.60))$

Table 8.2: Reduced decision matrix in terms of IVIFN

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$([0.10,0.20],[0.50,0.80])$	$([0.20,0.30],[0.40,0.50])$	$([0.20,0.30],[0.40,0.60])$
$\mathcal{V}_2$	$([0.40,0.50],[0.20,0.30])$	$([0.80,0.90],[0.01,0.10])$	$([0.10,0.20],[0.20,0.60])$
$\mathcal{V}_3$	$([0.20,0.30],[0.50,0.60])$	$([0.40,0.60],[0.20,0.30])$	$([0.20,0.30],[0.50,0.70])$
$\mathcal{V}_4$	$([0.10,0.30],[0.30,0.70])$	$([0.40,0.90],[0.05,0.10])$	$([0.20,0.30],[0.40,0.60])$

Table 8.3: Comparison of ranking with different methods

Method Ref. No.	Overall value of alternatives				Ranking order
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
Park et al. [115]	0.6602	0.5065	0.5044	0.3254	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Wei et al. [170]	0.7564	0.5906	0.5431	0.3636	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Dugenci [34]	1.4053	1.3038	1.2196	1.0618	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Zhang et al. [220]	0.6788	0.5958	0.5710	0.4699	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Xu [180]	0.1860	0.0110	-0.0320	-0.1758	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Xu [186]	0.5675	0.3157	0.0121	-0.2911	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Xu and Yager [191]	0.2934	0.0452	-0.0804	-0.3203	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Ye [199]	0.5005	0.2108	0.1549	-0.0450	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Nayagam et al. [114]	0.6565	0.4284	0.1637	-0.1734	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Sivaraman et al. [131]	0.6448	0.4525	0.3132	0.1625	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Garg [44]	0.8103	0.5948	0.4684	0.2735	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Chen et al. [25]	0.2786	0.1613	0.1497	0.1269	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$

Chapter 9

Algorithm for probabilistic dual hesitant fuzzy multi-criteria decision-making based on aggregation operators[1]

In this chapter, attempt has been made to capture uncertain information in form of probabilistic dual hesitant fuzzy set (PDHFS). Several weighted and ordered weighted averaging and geometric AOs are presented by using Einstein norm operations. Also, we have proposed two distance measures and its based maximum deviation method to compute the weight vector of the different criteria. Finally, a MCGDM approach is constructed based on proposed operators and an algorithm is explained with the help of the numerical example. The reliability of the presented DM method is explored with the help of testing criteria.

9.1 Introduction

After the appearance of the IFSs, several researchers are working on its extension to deal with the uncertain and imprecise information in a more wider manner. Among them, Hesitant fuzzy sets (HFSs) [146] play a remarkable role to model available information by providing the decision-maker with the ability of representing more than one membership

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values. However, with growth of the conceptual notions, dual hesitant fuzzy sets (DHFSs) [236] occupy a peculiar position in considering the non-membership hesitant information too. In the field of AOs, Xia and Xu [175] established different operators to aggregated their values. Garg and Arora [50] presented some AOs under the dual hesitant fuzzy soft set environment and applied them to solve the MCDM problems. Although, these approaches are able to capture the uncertainties in an efficient way, yet these works are unable to model the situations in which the refusal of an expert in providing the decision plays a dominant role. For example, suppose a panel of 6 experts is approached to select the best candidate during the recruitment process and 2 of them refused to provide any decision. While evaluating the informational data using the existing approaches, the number of decision makers is considered to be 4 instead of 6 i.e., the refusal providing experts are completely ignored and the decision is framed using the preferences given by the 4 decision-providing experts only. This cause a significant loss of information and may lead to inadequate results. In order to address such refusal-oriented cases, Zhu and Xu [235] corroborated probabilistic hesitant fuzzy sets (PHFSs). Wu et al. [172] gave the notion of AOs on interval-valued PHFSs (IVPHFSs) whereas Zhang, Xu and Wu [222] worked on preference relations based on IVPHFSs and accessed the findings by applying to real life decision scenarios. Hao et al. [61] corroborated the concept of PDHFSs. Apart from them, several researchers [117, 133, 168, 184, 194, 231, 232] have shown a keen interest in applying probabilistic hesitant fuzzy set environments to different decision making approaches. Based on these existing studies, PDHFSs are found to be offering advanced alternative to process the DM methodologies. For instance, suppose a person has to buy a commodity $\mathcal{X}$, and he is confused that either he is 10% sure or 20% sure to buy it, and is uncertain about 30% or 40% in not buying it. Thus, under DHFS environment, this information is captured as $(\{0.10, 0.20\},\ \{0.30, 0.40\})$. Here, in DHFS, each hesitant value is assumed to have probability 0.5. So, mentioning the same probability value repeatedly is omitted in DHFSs. But, if the buyer is more confident about 10% agreeness than that of 20% i.e., suppose he is certain that his agreeness towards buying the commodity is 70% towards 10% and 30% towards 20% and similarly, for the non-membership case, he is 60% favoring to the 40% rejection level and 40% favoring the 30% rejection level. Thus, PDHFS is formulated as $\left(\{0.10\big|0.70, 0.20\big|0.30\},\ \ \{0.30\big|0.4, 0.40\big|0.6\}\right)$. So, to address such cases, in which even the hesitation has some preference over the another hesitant value, PDHFS acts as an efficient tool to model them. Apart from it, in the multi-expert DM problems, there may often arise conflicts in the preferences given by different experts. These issues can easily be resolved using PDHFSs. For example, let $\mathcal{A}$ and $\mathcal{B}$ be two experts giving their opinion about buying a commodity $\mathcal{X}$. Suppose opinion provided by $\mathcal{A}$ is noted in form of DHFS as $(\{0.20, 0.30\}, \{0.10, 0.15\})$ and similarly $\mathcal{B}$ gave opinion as $(\{0.20, 0.25\}, \{0.10\})$. Now, both the experts are providing different opinions regarding the same commodity $\mathcal{X}$. This is a common problem that arises in the real life DM scenarios. To address this case, the information is combined into PDHFS by analyzing the probabilities of decision given by both the experts. The PDHFS, thus formed, is given as $\left(\left\{0.20\big|\frac{0.5+0.5}{2}, 0.30\big|\frac{0.5}{2}, 0.25\big|\frac{0.5}{2}\right\},\ \ \left\{0.10\big|\frac{0.5+1}{2}, 0.15\big|\frac{0.5}{2}\right\}\right)$. In simple form, it is $\left(\left\{0.20\big|0.5, 0.30\big|0.25, 0.25\big|0.25\right\},\ \ \left\{0.10\big|0.75, 0.15\big|0.25\right\}\right)$. Thus, this chapter is motivated by the need of capturing the more favorable values among the hesitant values.

In this chapter, we consider PDHFS environment to extract data. We have proposed two distance measures; in which the size of two PDHFSs should be the same whereas in the second one the size may vary. Moreover, to aggregate the information probabilistic dual hesitant weighted Einstein average (PDHFWEA) and geometric (PDHFWEG) AOs are proposed. In addition to it, a non-linear model is solved to capture the weighted information. A real-life based case-study is conducted and its comparative analysis with the prevailing environments is carried out.

9.2 Proposed Distance Measures for PDHFEs

In this section, we propose some measures to calculate the distance between two PDHFEs defined over a universal set $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$.

Let $\mathcal{A} = \left\{\left(x_i, h_{\mathcal{A}_s}(x_i)\big|p_{\mathcal{A}_s}(x_i), g_{\mathcal{A}_t}(x_i)\big|q_{\mathcal{A}_t}(x_i)\right) \mid x_i \in \mathcal{X}\right\}$ and $\mathcal{B} = \{(x_i, h_{\mathcal{B}_{s'}}(x_i)\big|p_{\mathcal{B}_{s'}}(x_i), g_{\mathcal{B}_{t'}}(x_i)\big|q_{\mathcal{B}_{t'}}(x_i)) \mid x_i \in \mathcal{X}\}$ where $s = 1, 2, \ldots, M_{\mathcal{A}}$; $t = 1, 2, \ldots, N_{\mathcal{A}}$; $s' = 1, 2, \ldots, M_{\mathcal{B}}$ and $t' = 1, 2, \ldots, N_{\mathcal{B}}$, be two PDHFSs. Also, let $M_{\max} = \max\{M_{\mathcal{A}}, M_{\mathcal{B}}\}, N_{\max} = \max\{N_{\mathcal{A}}, N_{\mathcal{B}}\}$, be two real numbers, then for a real-number $\eta > 0$, we define distance between $\mathcal{A}$ and $\mathcal{B}$ as:

$$d_1(\mathcal{A},\mathcal{B}) = \left(\sum_{i=1}^{n}\frac{1}{n}\left(\frac{1}{M_{\max}+N_{\max}}\left(\begin{array}{c}\sum\limits_{s=1}^{M_{\max}}\left|\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i)-\zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i)\right|^{\eta}\\ +\sum\limits_{t=1}^{N_{\max}}\left|\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i)-\vartheta_{\mathcal{B}_t}(x_i)q_{\mathcal{B}_t}(x_i)\right|^{\eta}\end{array}\right)\right)\right)^{\frac{1}{\eta}} \tag{9.1}$$

where $\zeta_{\mathcal{A}_s} \in h_{\mathcal{A}_s}$, $\zeta_{\mathcal{B}_s} \in h_{\mathcal{B}_{s'}}$, $\vartheta_{\mathcal{A}_s} \in g_{\mathcal{A}_s}$, $\vartheta_{\mathcal{B}_s} \in g_{\mathcal{B}_{s'}}$. It is noticeable that, there may arise the cases in which $M_{\mathcal{A}} \neq M_{\mathcal{B}}$ as well as $N_{\mathcal{A}} \neq N_{\mathcal{B}}$. Under such situations, for operating distance d_1, the lengths of these elements should be equal to each other. To achieve this, under the hesitant environments, the experts repeat the least or the greatest values among all the hesitant values, in the smaller set, till the length of both $\mathcal{A}$ and $\mathcal{B}$ becomes equal. In other words, if $M_{\mathcal{A}} > M_{\mathcal{B}}$, then repeat the smallest value in set $h_{\mathcal{B}}$ till $M_{\mathcal{B}}$ becomes equal to $M_{\mathcal{A}}$ and if $M_{\mathcal{A}} < M_{\mathcal{B}}$, then repeat the smallest value in set $h_{\mathcal{A}}$ till $M_{\mathcal{A}}$ becomes equal to $M_{\mathcal{B}}$. Alike the smallest values, the largest values may also be repeated. This choice of the smallest or largest value's repetition entirely depends on decision-makers optimistic or pessimistic approach. If the expert opts for the optimistic approach then he will expect the highest membership values and thus will repeat the largest values. However, if the expert chooses to follow the pessimistic approach, then he will expect the least favoring values and will go with repeating the smallest values till the same length is achieved. But sometimes, length of $\mathcal{A}$ and $\mathcal{B}$ cannot be matched by increasing the numbers of elements, then in such cases, the distance d_1 can be unappropriate for the data evaluations. To handle such cases, we propose another distance measure d_2 in which there is no need to repeat the values for matching the length of the elements under consideration. This distance d_2 is calculated as:

$$d_2(\mathcal{A},\mathcal{B}) = \left(\sum_{i=1}^{n}\frac{1}{n}\left(\begin{array}{c}\dfrac{\left|\frac{1}{M_{\mathcal{A}}}\sum\limits_{s=1}^{M_{\mathcal{A}}}\left(\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i)\right)-\frac{1}{M_{\mathcal{B}}}\sum\limits_{s'=1}^{M_{\mathcal{B}}}\left(\zeta_{\mathcal{B}_{s'}}(x_i)p_{\mathcal{B}_{s'}}(x_i)\right)\right|^{\eta}}{2}\\ +\dfrac{\left|\frac{1}{N_{\mathcal{A}}}\sum\limits_{t=1}^{N_{\mathcal{A}}}\left(\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i)\right)-\frac{1}{N_{\mathcal{B}}}\sum\limits_{t'=1}^{N_{\mathcal{B}}}\left(\vartheta_{\mathcal{B}_{t'}}(x_i)q_{\mathcal{B}_{t'}}(x_i)\right)\right|^{\eta}}{2}\end{array}\right)\right)^{\frac{1}{\eta}} \tag{9.2}$$

The distance measures proposed above satisfy the axiomatic statement given below:

Theorem 9.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two PDHFSs, then the distance measure d_1 satisfies the following conditions:

(P1) $0 \leq d_1(\mathcal{A}, \mathcal{B}) \leq 1$;

(P2) $d_1(\mathcal{A}, \mathcal{B}) = d_1(\mathcal{B}, \mathcal{A})$;

(P3) $d_1(\mathcal{A}, \mathcal{B}) = 0$ if $\mathcal{A} = \mathcal{B}$;

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $d_1(\mathcal{A}, \mathcal{B}) \leq d_1(\mathcal{A}, \mathcal{C})$ and $d_1(\mathcal{B}, \mathcal{C}) \leq d_1(\mathcal{A}, \mathcal{C})$.

Proof. Let $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ be the universal set and $\mathcal{A}$, $\mathcal{B}$ be two PDHFSs defined over $\mathcal{X}$. Then for each $x_i, (i = 1, 2, \ldots, n)$, we have

(P1) Since, $0 \leq \zeta_{\mathcal{A}_s}(x_i) \leq 1$ and $0 \leq p_{\mathcal{A}_s}(x_i) \leq 1$, for all $s = 1, 2, \ldots, M_{\max}$, this implies that $0 \leq \zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i) \leq 1$ and $0 \leq \zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i) \leq 1$. Thus, for any $\eta > 0$, we have $0 \leq |\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i) - \zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i)|^{\eta} \leq 1$. Further, $\sum_{s=1}^{M_{\max}} 0 \leq \sum_{s=1}^{M_{\max}} |\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i) - \zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i)|^{\eta} \leq \sum_{s=1}^{M_{\max}} 1$ which leads to

$$0 \leq \sum_{s=1}^{M_{\max}} |\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i) - \zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i)|^{\eta} \leq M_{\max}.$$

Similarly, for $t = 1, 2, \ldots, N_{\max}$, we have $0 \leq \sum_{t=1}^{N_{\max}} |\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i) - \vartheta_{\mathcal{B}_t}(x_i)q_{\mathcal{B}_t}(x_i)|^{\eta} \leq N_{\max}$ which yields

$$0 \leq \left(\begin{array}{c} \sum_{s=1}^{M_{\max}} |\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i) - \zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i)|^{\eta} \\ + \sum_{t=1}^{N_{\max}} |\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i) - \vartheta_{\mathcal{B}_t}(x_i)q_{\mathcal{B}_t}(x_i)|^{\eta} \end{array} \right) \leq M_{\max} + N_{\max}.$$

Thus,

$$0 \leq \left(\sum_{i=1}^{n} \frac{1}{n} \left(\frac{1}{M_{\max} + N_{\max}} \left(\begin{array}{c} \sum_{s=1}^{M_{\max}} \left|\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i) - \zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i)\right|^{\eta} \\ + \sum_{t=1}^{N_{\max}} \left|\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i) - \vartheta_{\mathcal{B}_t}(x_i)q_{\mathcal{B}_t}(x_i)\right|^{\eta} \end{array} \right) \right) \right)^{\frac{1}{\eta}} \leq 1,$$

which clearly implies that $0 \leq d_1(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) Since

$$d_1(\mathcal{A},\mathcal{B}) = \left(\sum_{i=1}^{n}\frac{1}{n}\left(\frac{1}{M_{\max}+N_{\max}}\left(\begin{array}{c}\sum_{s=1}^{M_{\max}}|\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i)-\zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i)|^{\eta}\\+\sum_{t=1}^{N_{\max}}|\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i)-\vartheta_{\mathcal{B}_t}(x_i)q_{\mathcal{B}_t}(x_i)|^{\eta}\end{array}\right)\right)\right)^{\frac{1}{\eta}}$$

$$= \left(\sum_{i=1}^{n}\frac{1}{n}\left(\frac{1}{M_{\max}+N_{\max}}\left(\begin{array}{c}\sum_{s=1}^{M_{\max}}|\zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i)-\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i)|^{\eta}\\+\sum_{t=1}^{N_{\max}}|\vartheta_{\mathcal{B}_t}(x_i)q_{\mathcal{B}_t}(x_i)-\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i)|^{\eta}\end{array}\right)\right)\right)^{\frac{1}{\eta}}$$

$$= d_1(\mathcal{B},\mathcal{A})$$

Hence, the distance measure d_1 possess a symmetric nature.

(P3) For $\mathcal{A}=\mathcal{B}$, we have $\zeta_{\mathcal{A}_s}(x_i)=\zeta_{\mathcal{B}_s}(x_i)$ and $p_{\mathcal{A}_s}(x_i)=p_{\mathcal{B}_s}(x_i)$. Also, $\vartheta_{\mathcal{A}_t}(x_i)=\vartheta_{\mathcal{B}_t}(x_i)$ and $q_{\mathcal{A}_t}(x_i)=q_{\mathcal{B}_t}(x_i)$. Thus, we have $|\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i)-\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i)|^{\eta}=0$ and $|\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i)-\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i)|^{\eta}=0$. Hence, it implies that

$$\left(\sum_{i=1}^{n}\frac{1}{n}\left(\frac{1}{M_{\max}+N_{\max}}\left(\begin{array}{c}\sum_{s=1}^{M_{\max}}|\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i)-\zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i)|^{\eta}\\+\sum_{t=1}^{N_{\max}}|\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i)-\vartheta_{\mathcal{B}_t}(x_i)q_{\mathcal{B}_t}(x_i)|^{\eta}\end{array}\right)\right)\right)^{\frac{1}{\eta}}=0$$

$$\Rightarrow \quad d_1(\mathcal{A},\mathcal{B})=0.$$

(P4) Since, $\mathcal{A}\subseteq\mathcal{B}\subseteq\mathcal{C}$, therefore $\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i)\le\zeta_{\mathcal{B}_s}(x_i)p_{\mathcal{B}_s}(x_i)\le\zeta_{\mathcal{C}_s}(x_i)p_{\mathcal{C}_s}(x_i)$ and $\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i)\ge\vartheta_{\mathcal{B}_t}(x_i)\ q_{\mathcal{B}_t}(x_i)\ge\vartheta_{\mathcal{C}_t}(x_i)q_{\mathcal{C}_t}(x_i)$.

Further, $|\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i)-\zeta_{\mathcal{B}_s}(x_i)q_{\mathcal{B}_s}(x_i)|^{\eta}\le|\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s}(x_i)-\zeta_{\mathcal{C}_s}(x_i)q_{\mathcal{C}_s}(x_i)|^{\eta}$ and $|\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i)-\vartheta_{\mathcal{B}_t}(x_i)q_{\mathcal{B}_t}(x_i)|^{\eta}\ge|\vartheta_{\mathcal{A}_t}(x_i)q_{\mathcal{A}_t}(x_i)\ -\vartheta_{\mathcal{C}_t}(x_i)q_{\mathcal{C}_t}(x_i)|^{\eta}$. Therefore, $d_1(\mathcal{A},\mathcal{B})\le d_1(\mathcal{A},\mathcal{C})$ and $d_1(\mathcal{B},\mathcal{C})\le d_1(\mathcal{A},\mathcal{C})$.

□

Theorem 9.2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two PDHFSs, then the distance measure d_2 satisfies the following conditions:

(P1) $0\le d_2(\mathcal{A},\mathcal{B})\le 1$

(P2) $d_2(\mathcal{A},\mathcal{B}) = d_2(\mathcal{B},\mathcal{A})$

(P3) $d_2(\mathcal{A},\mathcal{B}) = 0$ if $\mathcal{A} = \mathcal{B}$

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $d_2(\mathcal{A},\mathcal{B}) \leq d_2(\mathcal{A},\mathcal{C})$ and $d_2(\mathcal{B},\mathcal{C}) \leq d_2(\mathcal{A},\mathcal{C})$.

Proof. The proof is similar to Theorem 9.2.1, so we omit it here. □

9.3 Einstein Aggregation Operational laws for PDHFSs

In this section, we propose some operational laws and the investigate some of their properties associated with PDHFEs.

Definition 9.3.1. Let $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ be three PDHFEs such that $\mathcal{A} = (h|p_h, g|q_g)$, $\mathcal{A}_1 = (h_1|p_{h_1}, g_1|q_{g_1})$ and $\mathcal{A}_2 = (h_2|p_{h_2}, g_2|q_{g_2})$. Then, for $\xi > 0$, we define the Einstein operational laws for them as follows:

(i) $$\mathcal{A}_1 \oplus \mathcal{A}_2 = \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{\frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2} \,\middle|\, p_{\zeta_1}p_{\zeta_2}\right\}, \left\{\frac{\vartheta_1\vartheta_2}{1+(1-\vartheta_1)(1-\vartheta_2)} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2}\right\}\right);$$

(ii) $$\mathcal{A}_1 \otimes \mathcal{A}_2 = \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{\frac{\zeta_1\zeta_2}{1+(1-\zeta_1)(1-\zeta_2)} \,\middle|\, p_{\zeta_1}p_{\zeta_2}\right\}, \left\{\frac{\vartheta_1+\vartheta_2}{1+\vartheta_1\vartheta_2} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2}\right\}\right);$$

(iii) $$\xi\mathcal{A} = \bigcup_{\substack{\zeta\in h,\\ \vartheta\in g}} \left(\left\{\frac{(1+\zeta)^\xi-(1-\zeta)^\xi}{(1+\zeta)^\xi+(1-\zeta)^\xi} \,\middle|\, p_\zeta\right\}, \left\{\frac{2(\vartheta)^\xi}{(2-\vartheta)^\xi+(\vartheta)^\xi} \,\middle|\, q_\vartheta\right\}\right);$$

(iv) $$\mathcal{A}^\xi = \bigcup_{\substack{\zeta\in h,\\ \vartheta\in g}} \left(\left\{\frac{2(\zeta)^\xi}{(2-\zeta)^\xi+(\zeta)^\xi} \,\middle|\, p_\zeta\right\}, \left\{\frac{(1+\vartheta)^\xi-(1-\vartheta)^\xi}{(1+\vartheta)^\xi+(1-\vartheta)^\xi} \,\middle|\, q_\vartheta\right\}\right)$$

Theorem 9.3.1. For real value $\xi > 0$, the operational laws for PDHFEs given in Definition 9.3.1 that is $\mathcal{A}_1 \oplus \mathcal{A}_2$, $\mathcal{A}_1 \otimes \mathcal{A}_2$, $\xi\mathcal{A}$, and $\mathcal{A}^\xi$ are also PDHFEs.

Proof. For two PDHFEs $\mathcal{A}_1$ and $\mathcal{A}_2$, we have

$$\mathcal{A}_1 \oplus \mathcal{A}_2 = \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{\frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2} \,\middle|\, p_{\zeta_1}p_{\zeta_2}\right\}, \left\{\frac{\vartheta_1\vartheta_2}{1+(1-\vartheta_1)(1-\vartheta_2)} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2}\right\}\right)$$

As $0 \leq \zeta_1, \zeta_2, \vartheta_1, \vartheta_2 \leq 1$, thus it is evident that $0 \leq \zeta_1+\zeta_2 \leq 2$ and $1 \leq 1+\zeta_1\zeta_2 \leq 2$, thus it follows that $0 \leq \frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2} \leq 1$. On the other hand, $0 \leq \vartheta_1\vartheta_2 \leq 1$ and $1 \leq 1+(1-$

$\vartheta_1)(1-\vartheta_2) \leq 2$. Thus, $0 \leq \frac{\vartheta_1\vartheta_2}{1+(1-\vartheta_1)(1-\vartheta_2)} \leq 1$ Also, since $0 \leq p_{\zeta_1}, p_{\zeta_2}, q_{\vartheta_1}, q_{\vartheta_2} \leq 1$, thus $0 \leq p_{\zeta_1}p_{\zeta_2} \leq 1$ and $0 \leq q_{\vartheta_1}q_{\vartheta_2} \leq 1$. Similarly, $\mathcal{A}_1 \otimes \mathcal{A}_2$, $\xi\mathcal{A}$ and $\mathcal{A}^{\xi}$ are also PDHFEs. $\square$

Theorem 9.3.2. Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be three PDHFEs and $\xi, \xi_1, \xi_2 > 0$ be three real numbers, then following results hold:

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$

(ii) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A}_2 \otimes \mathcal{A}_1$

(iii) $(\mathcal{A}_1 \oplus \mathcal{A}_2) \oplus \mathcal{A}_3 = \mathcal{A}_1 \oplus (\mathcal{A}_2 \oplus \mathcal{A}_3)$

(iv) $(\mathcal{A}_1 \otimes \mathcal{A}_2) \otimes \mathcal{A}_3 = \mathcal{A}_1 \otimes (\mathcal{A}_2 \otimes \mathcal{A}_3)$

(v) $\xi(\mathcal{A}_1 \oplus \mathcal{A}_2) = \xi\mathcal{A}_1 \oplus \xi\mathcal{A}_2$

(vi) $\mathcal{A}_1^{\xi} \otimes \mathcal{A}_1^{\xi} = (\mathcal{A}_1 \otimes \mathcal{A}_2)^{\xi}$.

Proof. Let $\mathcal{A}_1 = (h_1|p_{h_1}, g_1|q_{g_1})$, $\mathcal{A}_2 = (h_2|p_{h_2}, g_2|q_{g_2})$, $\mathcal{A}_3 = (h_3|p_{h_3}, g_3|q_{g_3})$ be three PDHFEs. Then, we have

(i) For two PDHFEs $\mathcal{A}_1$ and $\mathcal{A}_2$, from Definition 9.3.1, we have

$$\begin{aligned}
\mathcal{A}_1 \oplus \mathcal{A}_2 &= \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{ \frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2} \,\middle|\, p_{\zeta_1}p_{\zeta_2} \right\}, \left\{ \frac{\vartheta_1\vartheta_2}{1+(1-\vartheta_1)(1-\vartheta_2)} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2} \right\} \right) \\
&= \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{ \frac{\zeta_2+\zeta_1}{1+\zeta_2\zeta_1} \,\middle|\, p_{\zeta_2}p_{\zeta_1} \right\}, \left\{ \frac{\vartheta_2\vartheta_1}{1+(1-\vartheta_2)(1-\vartheta_1)} \,\middle|\, q_{\vartheta_2}q_{\vartheta_1} \right\} \right) \\
&= \mathcal{A}_2 \oplus \mathcal{A}_1
\end{aligned}$$

(ii) Proof is obvious so we omit it here.

(iii) For three PDHFEs $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$, consider L.H.S. i.e.,

$$\begin{aligned}
&(\mathcal{A}_1 \oplus \mathcal{A}_2) \oplus \mathcal{A}_3 \\
&= \left(\bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{ \frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2} \,\middle|\, p_{\zeta_1}p_{\zeta_2} \right\}, \left\{ \frac{\vartheta_1\vartheta_2}{1+(1-\vartheta_1)(1-\vartheta_2)} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2} \right\} \right) \right) \oplus \mathcal{A}_3
\end{aligned}$$

$$= \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2\\ \zeta_3\in h_3,\vartheta_3\in g_3}} \left(\begin{array}{c} \left\{ \dfrac{\zeta_1+\zeta_2+\zeta_3+\zeta_1\zeta_2\zeta_3}{1+\zeta_1\zeta_2+\zeta_2\zeta_3+\zeta_3\zeta_1} \,\middle|\, p_{\zeta_1}p_{\zeta_2}p_{\zeta_3} \right\}, \\ \left\{ \dfrac{\vartheta_1\vartheta_2\vartheta_3}{4-2\vartheta_1-2\vartheta_2-2\vartheta_3+\vartheta_1\vartheta_2+\vartheta_2\vartheta_3+\vartheta_1\vartheta_3} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2}q_{\vartheta_3} \right\} \end{array} \right) \quad (9.3)$$

Also, on considering the R.H.S., we have

$$\mathcal{A}_1 \oplus (\mathcal{A}_2 \oplus \mathcal{A}_3)$$

$$= \mathcal{A}_1 \oplus \left(\bigcup_{\substack{\zeta_2\in h_2,\vartheta_2\in g_2\\ \zeta_3\in h_3,\vartheta_3\in g_3}} \left(\left\{ \frac{\zeta_2+\zeta_3}{1+\zeta_2\zeta_3} \,\middle|\, p_{\zeta_2}p_{\zeta_3} \right\}, \left\{ \frac{\vartheta_2\vartheta_3}{1+(1-\vartheta_2)(1-\vartheta_3)} \,\middle|\, q_{\vartheta_2}q_{\vartheta_3} \right\} \right) \right)$$

$$= \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2\\ \zeta_3\in h_3,\vartheta_3\in g_3}} \left(\begin{array}{c} \left\{ \dfrac{\zeta_1+\zeta_2+\zeta_3+\zeta_1\zeta_2\zeta_3}{1+\zeta_1\zeta_2+\zeta_2\zeta_3+\zeta_3\zeta_1} \,\middle|\, p_{\zeta_1}p_{\zeta_2}p_{\zeta_3} \right\}, \\ \left\{ \dfrac{\vartheta_1\vartheta_2\vartheta_3}{4-2\vartheta_1-2\vartheta_2-2\vartheta_3+\vartheta_1\vartheta_2+\vartheta_2\vartheta_3+\vartheta_1\vartheta_3} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2}q_{\vartheta_3} \right\} \end{array} \right) \quad (9.4)$$

From Eqs. (9.3) and (9.4), the required result is obtained.

(iv) Proof is obvious so we omit it here.

(v) For $\xi > 0$, consider

$$\xi(\mathcal{A}_1 \oplus \mathcal{A}_2) = \xi \left(\bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{ \frac{(1+\zeta_1)(1+\zeta_2)-(1-\zeta_1)(1-\zeta_2)}{(1+\zeta_1)(1+\zeta_2)+(1-\zeta_1)(1-\zeta_2)} \,\middle|\, p_{\zeta_1}p_{\zeta_2} \right\}, \left\{ \frac{2\vartheta_1\vartheta_2}{(2-\vartheta_1)(2-\vartheta_2)+\vartheta_1\vartheta_2} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2} \right\} \right) \right)$$

For sake of convenience, put $(1+\zeta_1)(1+\zeta_2) = a$; $(1-\zeta_1)(1-\zeta_2) = b$; $\vartheta_1\vartheta_2 = c$ and $(2-\vartheta_1)(2-\vartheta_2) = d$. This implies

$$\begin{aligned} \xi(\mathcal{A}_1 \oplus \mathcal{A}_2) &= \xi \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{ \frac{a-b}{a+b} \,\middle|\, p_{\zeta_1}p_{\zeta_2}, \right\} \left\{ \frac{2c}{d+c} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2} \right\} \right) \\ &= \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{ \frac{\left(1+\frac{a-b}{a+b}\right)^\xi - \left(1-\frac{a-b}{a+b}\right)^\xi}{\left(1+\frac{a-b}{a+b}\right)^\xi + \left(1-\frac{a-b}{a+b}\right)^\xi} \,\middle|\, p_{\zeta_1}p_{\zeta_2} \right\}, \left\{ \frac{2\left(\frac{2c}{d+c}\right)^\xi}{\left(2-\frac{2c}{d+c}\right)^\xi + \left(\frac{2c}{d+c}\right)^\xi} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2} \right\} \right) \\ &= \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{ \frac{\left(\frac{2a}{a+b}\right)^\xi - \left(\frac{2b}{a+b}\right)^\xi}{\left(\frac{2a}{a+b}\right)^\xi + \left(\frac{2b}{a+b}\right)^\xi} \,\middle|\, p_{\zeta_1}p_{\zeta_2} \right\}, \left\{ \frac{2\left(\frac{2c}{d+c}\right)^\xi}{\left(\frac{2d}{d+c}\right)^\xi + \left(\frac{2a}{d+c}\right)^\xi} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2} \right\} \right) \\ &= \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{ \frac{(a^\xi-b^\xi)}{(a^\xi+b^\xi)} \,\middle|\, p_{\zeta_1}p_{\zeta_2} \right\}, \left\{ \frac{2c^\xi}{d^\xi+c^\xi} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2} \right\} \right) \end{aligned}$$

Re-substituting a, b, c and d we have

$$\begin{aligned} &= \bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}} \left(\left\{ \frac{(1+\zeta_1)^\xi(1+\zeta_2)^\xi-(1-\zeta_1)^\xi(1-\zeta_2)^\xi}{(1+\zeta_1)^\xi(1+\zeta_2)^\xi+(1-\zeta_1)^\xi(1-\zeta_2)^\xi} \,\middle|\, p_{\zeta_1}p_{\zeta_2} \right\}, \left\{ \frac{2(\vartheta_1\vartheta_2)^\xi}{(2-\vartheta_1)^\xi(2-\vartheta_2)^\xi+\vartheta_1\vartheta_2} \,\middle|\, q_{\vartheta_1}q_{\vartheta_2} \right\} \right) \\ &= \xi\mathcal{A}_1 \oplus \xi\mathcal{A}_2 \end{aligned}$$

(vi) For $\xi > 0$,

$$(\mathcal{A}_1 \otimes \mathcal{A}_2)^\xi = \left(\bigcup_{\substack{\zeta_1 \in h_1, \vartheta_1 \in g_1 \\ \zeta_2 \in h_2, \vartheta_2 \in g_2}} \left(\left\{ \frac{2\zeta_1\zeta_2}{1+(1-\zeta_1)(1-\zeta_2)} \,\middle|\, p_{\zeta_1} p_{\zeta_2} \right\}, \left\{ \frac{(1+\vartheta_1)(1+\vartheta_2)-(1-\vartheta_1)(1-\vartheta_2)}{(1+\vartheta_1)(1+\vartheta_2)+(1-\vartheta_1)(1-\vartheta_2)} \,\middle|\, q_{\vartheta_1} q_{\vartheta_2} \right\} \right) \right)^\xi$$

For sake of convenience, put

$$\zeta_1\zeta_2 = a; (2-\zeta_1)(2-\zeta_2) = b; (1+\vartheta_1)(1+\vartheta_2) = c \text{ and } (1-\vartheta_1)(1-\vartheta_2) = d$$

So we obtain

$$\begin{aligned}
(\mathcal{A}_1 \otimes \mathcal{A}_2)^\xi &= \left(\bigcup_{\substack{\zeta_1 \in h_1, \vartheta_1 \in g_1 \\ \zeta_2 \in h_2, \vartheta_2 \in g_2}} \left(\left\{ \frac{2a}{b+a} \,\middle|\, p_{\zeta_1} p_{\zeta_2} \right\}, \left\{ \frac{c-d}{c+d} \,\middle|\, q_{\vartheta_1} q_{\vartheta_2} \right\} \right) \right)^\xi \\
&= \bigcup_{\substack{\zeta_1 \in h_1, \vartheta_1 \in g_1 \\ \zeta_2 \in h_2, \vartheta_2 \in g_2}} \left(\left\{ \frac{2\left(\frac{2a}{b+a}\right)^\xi}{\left(2-\frac{2a}{b+a}\right)^\xi + \left(\frac{2a}{b+a}\right)^\xi} \,\middle|\, p_{\zeta_1} p_{\zeta_2} \right\}, \left\{ \frac{\left(1+\frac{c-d}{c+d}\right)^\xi - \left(1-\frac{c-d}{c+d}\right)^\xi}{\left(1+\frac{c-d}{c+d}\right)^\xi + \left(1-\frac{c-d}{c+d}\right)^\xi} \,\middle|\, q_{\vartheta_1} q_{\vartheta_2} \right\} \right) \\
&= \bigcup_{\substack{\zeta_1 \in h_1, \vartheta_1 \in g_1 \\ \zeta_2 \in h_2, \vartheta_2 \in g_2}} \left(\left\{ \frac{2\left(\frac{2a}{b+a}\right)^\xi}{\left(\frac{2b}{b+a}\right)^\xi + \left(\frac{2a}{b+a}\right)^\xi} \,\middle|\, p_{\zeta_1 \zeta_2} \right\}, \left\{ \frac{\left(\frac{2c}{c+d}\right)^\xi - \left(\frac{2d}{c+d}\right)^\xi}{\left(\frac{2c}{c+d}\right)^\xi + \left(\frac{2d}{c+d}\right)^\xi} \,\middle|\, q_{\vartheta_1} q_{\vartheta_2} \right\} \right) \\
&= \bigcup_{\substack{\zeta_1 \in h_1, \vartheta_1 \in g_1 \\ \zeta_2 \in h_2, \vartheta_2 \in g_2}} \left(\left\{ \frac{2a^\xi}{b^\xi + a^\xi} \,\middle|\, p_{\zeta_1} p_{\zeta_2} \right\}, \left\{ \frac{c^\xi - d^\xi}{c^\xi + d^\xi} \,\middle|\, q_{\vartheta_1} q_{\vartheta_2} \right\} \right)
\end{aligned}$$

Re-substituting values of a, b, c and d we get

$$\begin{aligned}
&= \bigcup_{\substack{\zeta_1 \in h_1, \vartheta_1 \in g_1 \\ \zeta_2 \in h_2, \vartheta_2 \in g_2}} \left(\left\{ \frac{2(\zeta_1\zeta_2)^\xi}{(2-\zeta_1)^\xi (2-\zeta_2)^\xi + (\zeta_1\zeta_2)^\xi} \,\middle|\, p_{\zeta_1} p_{\zeta_2} \right\}, \left\{ \frac{(1+\vartheta_1)^\xi (1+\vartheta_2)^\xi - (1-\vartheta_1)^\xi (1-\vartheta_2)^\xi}{(1+\vartheta_1)^\xi (1+\vartheta_2)^\xi + (1-\vartheta_1)^\xi (1-\vartheta_2)^\xi} \,\middle|\, q_{\vartheta_1} q_{\vartheta_2} \right\} \right) \\
&= \mathcal{A}_1^\xi \otimes \mathcal{A}_2^\xi
\end{aligned}$$

□

Theorem 9.3.3. Let $\mathcal{A} = (h|p_h, g|q_g)$ $\mathcal{A}_1 = (h_1|p_{h_1}, g_1|q_{g_1})$, and $\mathcal{A}_2 = (h_2|p_{h_2}, g_2|q_{g_2})$ be three PDHFEs, and $\xi > 0$ be a real number, then

(i) $(\mathcal{A}^c)^\xi = \xi\mathcal{A}^c$

(ii) $\xi(\mathcal{A}^c) = (\mathcal{A}^\xi)^c$

(iii) $\mathcal{A}_1^c \oplus \mathcal{A}_2^c = (\mathcal{A}_1 \otimes \mathcal{A}_2)^c$

(iv) $\mathcal{A}_1^c \otimes \mathcal{A}_2^c = (\mathcal{A}_1 \oplus \mathcal{A}_2)^c$.

Proof. (i) Let $\mathcal{A} = (h|p_h, g|q_g)$ be a PDHFE, then using Definition 2.1.12, the proof for the three possible cases is given as:

(Case 1:) If $h \neq \phi$; $g \neq \phi$ then for a PDHFE $\mathcal{A} = (h|p_h, g|q_g)$, from Eq. (2.14) we have

$$\begin{aligned}
(\mathcal{A}^c)^\xi &= \left(\bigcup_{\substack{\zeta\in h\\ \vartheta\in g}} \left(\left\{\vartheta \,\middle|\, q_\vartheta\right\}, \left\{\zeta \,\middle|\, p_\zeta\right\}\right)\right)^\xi \\
&= \bigcup_{\substack{\zeta\in h\\ \vartheta\in g}} \left(\left\{\frac{2(\vartheta)^\xi}{(2-\vartheta)^\xi + (\vartheta)^\xi} \,\middle|\, q_\vartheta,\right\}, \left\{\frac{(1+\zeta)^\xi - (1-\zeta)^\xi}{(1+\zeta)^\xi + (1-\zeta)^\xi} \,\middle|\, p_\zeta\right\}\right) \\
&= \left(\bigcup_{\substack{\zeta\in h\\ \vartheta\in g}} \left(\left\{\frac{(1+\zeta)^\xi - (1-\zeta)^\xi}{(1+\zeta)^\xi + (1-\zeta)^\xi} \,\middle|\, p_\zeta\right\}, \left\{\frac{2(\vartheta)^\xi}{(2-\vartheta)^\xi + (\vartheta)^\xi} \,\middle|\, q_\vartheta\right\}\right)\right)^c \\
&= \left(\xi\left(\bigcup_{\substack{\zeta\in p\\ \vartheta\in q}} \left\{\zeta \,\middle|\, p_\zeta\right\}, \left\{\vartheta \,\middle|\, q_\vartheta\right\}\right)\right)^c = (\xi\mathcal{A})^c
\end{aligned}$$

(Case 2:) If $g = \phi, h \neq \phi$, then

$$\begin{aligned}
(\mathcal{A}^c)^\xi &= \left(\bigcup_{\zeta\in h} \left(\left\{1-\zeta \,\middle|\, p_\zeta\right\}, \{\phi\}\right)\right)^\xi \\
&= \bigcup_{\zeta\in h} \left(\left\{\frac{2(1-\zeta)^\xi}{(2-(1-\zeta))^\xi + (1-\zeta)^\xi} \,\middle|\, p_\zeta\right\}, \{\phi\}\right) \\
&= (\xi\mathcal{A})^c
\end{aligned}$$

(Case 3:) If $h = \phi, g = \phi$, then

$$\begin{aligned}
(\mathcal{A}^c)^\xi &= \left(\bigcup_{\vartheta\in g} \left(\{\phi\}, \left\{1-\vartheta \,\middle|\, q_\vartheta\right\}\right)\right)^\xi \\
&= \bigcup_{\vartheta\in g} \left(\{\phi\}, \left\{\frac{(1+(1-\vartheta))^\xi - (1-(1-\vartheta))^\xi}{(1+(1-\vartheta))^\xi + (1-(1-\vartheta))^\xi} \,\middle|\, q_\vartheta\right\}\right) \\
&= \left(\bigcup_{\vartheta\in g} \left(\left\{\frac{(2-\vartheta)^\xi - (\vartheta)^\xi}{(2-\vartheta)^\xi + (\vartheta)^\xi} \,\middle|\, q_\vartheta\right\}, \{\phi\}\right)\right)^c \\
&= \left(\xi\bigcup_{\vartheta\in g} \left\{(1-\vartheta) \,\middle|\, q_\vartheta\right\}, \{\phi\}\right)^c = (\xi\mathcal{A})^c
\end{aligned}$$

(ii) Similar to above, so it is omitted.

(iii) For two PDHFEs $\mathcal{A}_1$, $\mathcal{A}_2$ and a real number $\xi > 0$, using Definitions 2.1.12 and 2.4.3 we have,

(Case 1:) If $h_1 \neq \phi, g_1 \neq \phi, h_2 \neq \phi$ and $g_2 \neq \phi$

$$\begin{aligned}
&\mathcal{A}_1^c \oplus \mathcal{A}_2^c \\
&= \bigcup_{\substack{\zeta_1 \in h_1 \\ \vartheta_1 \in g_1}} \left(\left\{ \vartheta_1 \,\middle|\, q_{\vartheta_1} \right\}, \left\{ \zeta_1 \,\middle|\, p_{\zeta_1} \right\} \right) \oplus \bigcup_{\substack{\zeta_2 \in h_2 \\ \vartheta_2 \in g_2}} \left(\left\{ \vartheta_2 \,\middle|\, q_{\vartheta_2} \right\}, \left\{ \zeta_2 \,\middle|\, p_{\zeta_2} \right\} \right) \\
&= \bigcup_{\substack{\zeta_1 \in h_1, \vartheta_1 \in g_1 \\ \zeta_2 \in h_2, \vartheta_2 \in g_2}} \left(\left\{ \frac{\vartheta_1 + \vartheta_2}{1 + \vartheta_1 \vartheta_2} \,\middle|\, q_{\vartheta_1} q_{\vartheta_2} \right\}, \left\{ \frac{\zeta_1 \zeta_2}{1 + (1 - \zeta_1)(1 - \zeta_2)} \,\middle|\, p_{\zeta_1} p_{\zeta_2} \right\} \right) \\
&= \bigcup_{\substack{\zeta_1 \in h_1, \vartheta_1 \in g_1 \\ \zeta_2 \in h_2, \vartheta_2 \in g_2}} \left(\left\{ \frac{\zeta_1 \zeta_2}{1 + (1 - \zeta_1)(1 - \zeta_2)} \,\middle|\, p_{\zeta_1} p_{\zeta_2} \right\}, \left\{ \frac{\vartheta_1 + \vartheta_2}{1 + \vartheta_1 \vartheta_2} \,\middle|\, q_{\vartheta_1} q_{\vartheta_2} \right\} \right)^c \\
&= (\mathcal{A}_1 \otimes \mathcal{A}_2)^c
\end{aligned}$$

(Case 2:) If $h_1 \neq \phi, g_1 = \phi, h_2 \neq \phi$ and $g_2 = \phi$, then

$$\begin{aligned}
\mathcal{A}_1^c \oplus \mathcal{A}_2^c &= \bigcup_{\substack{\zeta_1 \in h_1, \\ \vartheta_1 \in g_1}} \left(\left\{ 1 - \zeta_1 \,\middle|\, p_{\zeta_1} \right\}, \{\phi\} \right) \oplus \bigcup_{\substack{\zeta_2 \in h_2, \\ \vartheta_2 \in g_2}} \left(\left\{ 1 - \zeta_2 \,\middle|\, p_{\zeta_2} \right\}, \{\phi\} \right) \\
&= \bigcup_{\substack{\zeta_1 \in h_1, \vartheta_1 \in g_1 \\ \zeta_2 \in h_2, \vartheta_2 \in g_2}} \left(\left\{ \frac{(1 - \zeta_1) + (1 - \zeta_2)}{1 + (1 - \zeta_1)(1 - \zeta_2)} \,\middle|\, p_{\zeta_1} p_{\zeta_2} \right\}, \{\phi\} \right) \\
&= (\mathcal{A}_1 \otimes \mathcal{A}_2)^c
\end{aligned}$$

(Case 3:) If $h_1 = \phi, g_1 \neq \phi, h_2 = \phi, g_2 \neq \phi$

$$\begin{aligned}
\mathcal{A}_1^c \oplus \mathcal{A}_2^c &= \bigcup_{\substack{\zeta_1 \in h_1 \\ \vartheta_1 \in g_1}} \left(\{\phi\}, \left\{ 1 - \vartheta_1 \,\middle|\, q_{\vartheta_1} \right\} \right) \oplus \bigcup_{\substack{\zeta_2 \in h_2 \\ \vartheta_2 \in g_2}} \left(\{\phi\}, \left\{ 1 - \vartheta_2 \,\middle|\, q_{\vartheta_2} \right\} \right) \\
&= \bigcup_{\substack{\zeta_1 \in h_1, \vartheta_1 \in g_1 \\ \zeta_2 \in h_2, \vartheta_2 \in g_2}} \left(\{\phi\}, \left\{ \frac{(1 - \vartheta_1)(1 - \vartheta_2)}{1 + \vartheta_1 \vartheta_2} \,\middle|\, q_{\vartheta_1} q_{\vartheta_2} \right\} \right) \\
&= (\mathcal{A}_1 \otimes \mathcal{A}_2)^c
\end{aligned}$$

(iv) Similar, so we omit it here.

□

9.4 Probabilistic Dual Hesitant Weighted Einstein AOs

In this section, we have defined some weighted aggregation operators by using aforementioned laws for a collection of PDHFEs. For it, let Φ be the family of PDHFEs.

Definition 9.4.1. Let Φ be the family of PDHFEs $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ with the corresponding weights $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, such that $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$. If PDHFWEA: $\Phi^n \to \Phi$, is a mapping defined by

$$\text{PDHFWEA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1\mathcal{A}_1 \oplus \omega_2\mathcal{A}_2 \oplus \ldots \oplus \omega_n\mathcal{A}_n \tag{9.5}$$

then, PDHFWEA is called probabilistic dual hesitant fuzzy weighted Einstein average operator.

Theorem 9.4.1. For a family of PDHFEs $\mathcal{A}_i = \left(h_i \,\middle|\, p_{h_i}, g_i \,\middle|\, q_{g_i}\right), (i = 1, 2, \ldots, n)$, the aggregated value obtained by using PDHFWEA operator is still a PDHFE and is given as

$$\text{PDHFWEA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigcup_{\substack{\zeta_i \in h_i \\ \vartheta_i \in g_i}} \left(\begin{array}{c} \left\{ \dfrac{\prod_{i=1}^{n}(1+\zeta_i)^{\omega_i} - \prod_{i=1}^{n}(1-\zeta_i)^{\omega_i}}{\prod_{i=1}^{n}(1+\zeta_i)^{\omega_i} + \prod_{i=1}^{n}(1-\zeta_i)^{\omega_i}} \,\middle|\, \prod_{i=1}^{n} p_{\zeta_i} \right\}, \\ \left\{ \dfrac{2\prod_{i=1}^{n}(\vartheta_i)^{\omega_i}}{\prod_{i=1}^{n}(2-\vartheta_i)^{\omega_i} + \prod_{i=1}^{n}(\vartheta_i)^{\omega_i}} \,\middle|\, \prod_{i=1}^{n} q_{\vartheta_i} \right\} \end{array} \right) \tag{9.6}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is a weight vector such that $\sum_{i=1}^{n} \omega_i = 1$ where $0 < \omega_i < 1$.

Proof. We will prove the Eq. (9.6) by following the steps mathematical induction on n, and the proof is executed as below:

Step 1: For $n = 2$, we have $\mathcal{A}_1 = \left(h_1 \,\middle|\, p_{h_1}, g_1 \,\middle|\, q_{g_1}\right)$ and $\mathcal{A}_2 = \left(h_2 \,\middle|\, p_{h_2}, g_2 \,\middle|\, q_{g_2}\right)$. Using operational laws on PDHFEs as stated in Definition 9.3.1 we get

$$\omega_1\mathcal{A}_1 = \bigcup_{\zeta_1 \in h_1, \vartheta_1 \in g_1} \left(\begin{array}{c} \left\{ \dfrac{(1+\zeta_1)^{\omega_1} - (1-\zeta_1)^{\omega_1}}{(1+\zeta_1)^{\omega_1} + (1-\zeta_1)^{\omega_1}} \,\middle|\, p_{\zeta_1} \right\}, \\ \left\{ \dfrac{2(\vartheta_1)^{\omega_1}}{(2-\vartheta_1)^{\omega_1} + (\vartheta_1)^{\omega_1}} \,\middle|\, q_{\vartheta_1} \right\} \end{array} \right)$$

and

$$\omega_2\mathcal{A}_2 = \bigcup_{\zeta_2 \in h_2, \vartheta_2 \in g_2} \left(\begin{array}{c} \left\{ \dfrac{(1+\zeta_2)^{\omega_2} - (1-\zeta_2)^{\omega_2}}{(1+\zeta_2)^{\omega_2} + (1-\zeta_2)^{\omega_2}} \,\middle|\, p_{\zeta_2} \right\}, \\ \left\{ \dfrac{2(\vartheta_2)^{\omega_2}}{(2-\vartheta_2)^{\omega_2} + (\vartheta_2)^{\omega_2}} \,\middle|\, q_{\vartheta_2} \right\} \end{array} \right)$$

Hence, by addition of PDHFEs, we get

$$\begin{aligned}&\text{PDHFWEA}(\mathcal{A}_1,\mathcal{A}_2)=\omega_1\mathcal{A}_1\oplus\omega_2\mathcal{A}_2\\&=\bigcup_{\substack{\zeta_1\in h_1,\vartheta_1\in g_1\\ \zeta_2\in h_2,\vartheta_2\in g_2}}\left(\begin{array}{c}\left\{\dfrac{\prod_{i=1}^{2}(1+\zeta_i)^{\omega_i}-\prod_{i=1}^{2}(1-\zeta_i)^{\omega_i}}{\prod_{i=1}^{2}(1+\zeta_i)^{\omega_i}+\prod_{i=1}^{2}(1-\zeta_i)^{\omega_i}}\,\middle|\,\prod_{i=1}^{2}p_{\zeta_i}\right\},\\ \left\{\dfrac{2\prod_{i=1}^{2}(\vartheta_i)^{\omega_i}}{\prod_{i=1}^{2}(2-\vartheta_i)^{\omega_i}+\prod_{i=1}^{2}(\vartheta_i)^{\omega_i}}\,\middle|\,\prod_{i=1}^{2}q_{\vartheta_i}\right\}\end{array}\right)\end{aligned}$$

Thus, the result holds for $n=2$.

Step 2: If Eq. (9.6) holds for $n=k$, then for $n=k+1$, we have

$$\begin{aligned}&\text{PDHFWEA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_{k+1})=\left(\bigoplus_{i=1}^{k}\omega_i\mathcal{A}_i\right)\oplus(\omega_{k+1}\mathcal{A}_{k+1})\\ =&\bigcup_{\zeta_i\in h_i,\vartheta_i\in g_i}\left(\left\{\frac{\prod_{i=1}^{k}(1+\zeta_i)^{\omega_i}-\prod_{i=1}^{k}(1-\zeta_i)^{\omega_i}}{\prod_{i=1}^{k}(1+\zeta_i)^{\omega_i}+\prod_{i=1}^{k}(1-\zeta_i)^{\omega_i}}\,\middle|\,\prod_{i=1}^{k}p_{\zeta_i}\right\},\left\{\frac{2\prod_{i=1}^{k}(\vartheta_i)^{\omega_i}}{\prod_{i=1}^{k}(2-\vartheta_i)^{\omega_i}+\prod_{i=1}^{k}(\vartheta_i)^{\omega_i}}\,\middle|\,\prod_{i=1}^{k}q_{\vartheta_i}\right\}\right)\\ \oplus&\bigcup_{\substack{\zeta_{k+1}\in h_{k+1},\\ \vartheta_{k+1}\in g_{k+1}}}\left(\left\{\frac{(1+\zeta_{k+1})^{\omega_{k+1}}-(1-\zeta_{k+1})^{\omega_{k+1}}}{(1+\zeta_{k+1})^{\omega_{k+1}}+(1-\zeta_{k+1})^{\omega_{k+1}}}\,\middle|\,p_{\zeta_{k+1}},\right\},\left\{\frac{2(\vartheta_{k+1})^{\omega_{k+1}}}{(2-\vartheta_{k+1})^{\omega_{k+1}}+(\vartheta_{k+1})^{\omega_{k+1}}}\,\middle|\,q_{\vartheta_{k+1}}\right\}\right)\\ =&\bigcup_{\zeta_i\in h_i,\vartheta_i\in g_i}\left(\left\{\frac{\prod_{i=1}^{k+1}(1+\zeta_i)^{\omega_i}-\prod_{i=1}^{k+1}(1-\zeta_i)^{\omega_i}}{\prod_{i=1}^{k+1}(1+\zeta_i)^{\omega_i}+\prod_{i=1}^{k+1}(1-\zeta_i)^{\omega_i}}\,\middle|\,\prod_{i=1}^{k+1}p_{\zeta_i}\right\},\left\{\frac{2\prod_{i=1}^{k+1}(\vartheta_i)^{\omega_i}}{\prod_{i=1}^{k+1}(2-\vartheta_i)^{\omega_i}+\prod_{i=1}^{k+1}(\vartheta_i)^{\omega_i}}\,\middle|\,\prod_{i=1}^{k+1}q_{\vartheta_i}\right\}\right)\end{aligned}$$

Thus,

$$\begin{aligned}&\text{PDHFWEA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\&=\bigcup_{\substack{\zeta_i\in h_i\\ \vartheta_i\in g_i}}\left(\begin{array}{c}\left\{\dfrac{\prod_{i=1}^{n}(1+\zeta_i)^{\omega_i}-\prod_{i=1}^{n}(1-\zeta_i)^{\omega_i}}{\prod_{i=1}^{n}(1+\zeta_i)^{\omega_i}+\prod_{i=1}^{n}(1-\zeta_i)^{\omega_i}}\,\middle|\,\prod_{i=1}^{n}p_{\zeta_i}\right\},\\ \left\{\dfrac{2\prod_{i=1}^{n}(\vartheta_i)^{\omega_i}}{\prod_{i=1}^{n}(2-\vartheta_i)^{\omega_i}+\prod_{i=1}^{n}(\vartheta_i)^{\omega_i}}\,\middle|\,\prod_{i=1}^{n}q_{\vartheta_i}\right\}\end{array}\right)\end{aligned}$$

which completes the proof. □

Further, it is observed that the proposed PDHFWEA operator satisfies the properties of boundedness and monotonicity, for a family of PDHFEs $\mathcal{A}_i,(i=1,2,\ldots,n)$ which can be demonstrated as follows:

Property 9.4.1. (Boundedness) For $\mathcal{A}_i = \left(h_i \,\middle|\, p_{h_i}, g_i \,\middle|\, q_{g_i}\right)$ where $i = 1, 2, \ldots, n$, let $\mathcal{A}^- = \left(\min(h_i) \,\middle|\, \min(p_{h_i})\ ,\ \max(g_i) \,\middle|\, \max(q_{g_i})\right) = \left(\left\{\zeta_{\min} \,\middle|\, p_{\min}\right\}, \left\{\vartheta_{\max} \,\middle|\, q_{\max}\right\}\right)$ and $\mathcal{A}^+ = \left(\max(h_i) \,\middle|\, \max(p_{h_i}), \min(g_i) \,\middle|\, \min(q_{g_i})\right) = \left(\left\{\zeta_{\max} \,\middle|\, p_{\max}\right\}, \left\{\vartheta_{\min} \,\middle|\, q_{\min}\right\}\right)$ be PDHFEs, then $\mathcal{A}^- \leq \text{PDHFWEA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$.

Proof. Since each $\mathcal{A}_i$ is a PDHFE, it is obvious that $\min(h_i) \leq h_i \leq \max(h_i), \min(g_i) \leq g_i \leq \max(g_i), p_{\min} \leq p_i \leq p_{\max}$ and $q_{min} \leq q_i \leq q_{\max}$. Let $f(x) = \frac{1-x}{1+x}, x \in [0,1]$, $f'(x) = \frac{-2}{(1+x)^2} < 0$ i.e., $f(x)$ is a decreasing function. Since, $\zeta_{\min} \leq \zeta_i \leq \zeta_{\max}$, for all i, then $f(\zeta_{\max}) \leq f(\zeta_i) \leq f(\zeta_{\min})$ i.e., $\frac{1-\zeta_{\max}}{1+\zeta_{\max}} \leq \frac{1-\zeta_i}{1+\zeta_i} \leq \frac{1-\zeta_{\max}}{1+\zeta_{\max}}$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$ such that each $\omega_i \in (0,1)$ and $\sum\limits_{i=1}^{n} \omega_i = 1$, then we have $\left(\frac{1-\zeta_{\max}}{1+\zeta_{\max}}\right)^{\omega_i} \leq \left(\frac{1-\zeta_i}{1+\zeta_i}\right)^{\omega_i} \leq \left(\frac{1-\zeta_{\min}}{1+\zeta_{\min}}\right)^{\omega_i}$. Thus, we get

$$1 + \left(\frac{1-\zeta_{\max}}{1+\zeta_{\max}}\right) \leq 1 + \prod_{i=1}^{n} \left(\frac{1-\zeta_i}{1+\zeta_i}\right)^{\omega_i} \leq 1 + \left(\frac{1-\zeta_{\min}}{1+\zeta_{\min}}\right)$$

$$\Rightarrow \frac{2}{1+\zeta_{\max}} \leq \frac{\prod\limits_{i=1}^{n}(1+\zeta_i)^{\omega_i} + \prod\limits_{i=1}^{n}(1-\zeta_i)^{\omega_i}}{\prod\limits_{i=1}^{n}(1+\zeta_i)^{\omega_i}} \leq \frac{2}{1+\zeta_{\min}}$$

$$\Rightarrow \zeta_{\min} \leq \frac{1 - \prod\limits_{i=1}^{n}\left(\frac{1-\zeta_i}{1+\zeta_i}\right)^{\omega_i}}{1 + \prod\limits_{i=1}^{n}\left(\frac{1-\zeta_i}{1+\zeta_i}\right)^{\omega_i}} \leq \zeta_{\max}$$

$$\Rightarrow \zeta_{\min} \leq \frac{\prod\limits_{i=1}^{n}(1+\zeta_i)^{\omega_i} - \prod\limits_{i=1}^{n}(1-\zeta_i)^{\omega_i}}{\prod\limits_{i=1}^{n}(1+\zeta_i)^{\omega_i} + \prod\limits_{i=1}^{n}(1-\zeta_i)^{\omega_i}} \leq \zeta_{\max}$$

Now, for non-membership, let $c(y) = \frac{2-y}{y}$, $y \in (0,1]$, then $c'(y) < 0$ i.e., $c(y)$ is the decreasing function. Since, $\vartheta_{\min} \leq \vartheta_i \leq \vartheta_{\max}$, then for all i, we have $c(\vartheta_{\max}) \leq c(\vartheta_i) \leq c(\vartheta_{\min})$, that is $\frac{2-\vartheta_{\max}}{\vartheta_{\max}} \leq \frac{2-\vartheta_i}{\vartheta_i} \leq \frac{2-\vartheta_{\min}}{\vartheta_{\min}}$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$ such that $\omega_i \in (0,1)$ and $\sum\limits_{i=1}^{n} \omega_i = 1$, then $\left(\frac{2-\vartheta_{\max}}{\vartheta_{\max}}\right)^{\omega_i} \leq \left(\frac{2-\vartheta_i}{\vartheta_i}\right)^{\omega_i} \leq \left(\frac{2-\vartheta_{\min}}{\vartheta_{\min}}\right)^{\omega_i}$. Thus,

$$\prod_{i=1}^{n}\left(\frac{2-\vartheta_{\max}}{\vartheta_{\max}}\right)^{\omega_i} \leq \prod_{i=1}^{n}\left(\frac{2-\vartheta_i}{\vartheta_i}\right)^{\omega_i} \leq \prod_{i=1}^{n}\left(\frac{2-\vartheta_{\min}}{\vartheta_{\min}}\right)^{\omega_i}$$

$$\Rightarrow \frac{2}{\vartheta_{\min}} \leq \frac{1}{1+\prod_{i=1}^{n}\left(\frac{2-\vartheta_i}{\vartheta_i}\right)^{\omega_i}} \leq \frac{2}{\vartheta_{\max}}$$

$$\Rightarrow \vartheta_{\min} \leq \frac{2\prod_{i=1}^{n}(\vartheta_i)^{\omega_i}}{\prod_{i=1}^{n}(\vartheta_i)^{\omega_i}+\prod_{i=1}^{n}(2-\vartheta_i)^{\omega_i}} \leq \vartheta_{\max}$$

Now, for probabilities, since $p_{\min} \leq p_i \leq p_{\max}$ and $q_{\min} \leq q_i \leq q_{\max}$ this implies that $\prod_{i=1}^{n} p_{\min} \leq \prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_{\max}$ and $\prod_{i=1}^{n} q_{\min} \leq \prod_{i=1}^{n} q_i \leq \prod_{i=1}^{n} q_{\max}$. According to the score function, as defined in Definition 2.1.13 of Chapter 2, we obtain $\mathcal{S}c(\mathcal{A}^-) \leq \mathcal{S}c(\mathcal{A}) \leq \mathcal{S}c(\mathcal{A}^+)$. Hence, from all the above notions, $\mathcal{A}^- \leq \text{PDHFWEA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$. □

Property 9.4.2. (Monotonicity) Let $\mathcal{A}_i = \left(h_i \;\middle|\; p_{h_i}, g_i \;\middle|\; q_{g_i}\right)$ and $\mathcal{A}_i^* = \left(h_i^* \;\middle|\; p_{h_i^*}, g_i^* \;\middle|\; q_{g_i^*}\right)$, for all $i = 1, 2, \ldots, n$ be two families of PDHFEs where for each element in $\mathcal{A}_i$ and $\mathcal{A}_i^*$, there are $\zeta_{\mathcal{A}_i} \leq \zeta_{\mathcal{A}_i^*}$ and $\vartheta_{\mathcal{A}_i} \geq \vartheta_{\mathcal{A}_i^*}$ while the probabilities remain the same i.e., $p_{h_i} = p_{h_i^*}, q_{g_i} = q_{g_i^*}$ then $\text{PDHFWEA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{PDHFWEA}(\mathcal{A}_1^*, \mathcal{A}_2^*, \ldots, \mathcal{A}_n^*)$.

Proof. Similar to that of Property 9.4.1, so we omit it here. □

However, the PDHFWEA operator does not satisfy the idempotency. To illustrate this, we give the following example:

Example 9.4.1. Let $\mathcal{A}_1 = \mathcal{A}_2 = \left(\left\{0.3|0.25, 0.4|0.75\right\}, \left\{0.2|0.4, 0.3|0.6\right\}\right)$ be two PDHFEs and $\omega = (0.2, 0.8)^T$ be the weight vector, then for $(i = 1, 2)$ the aggregated value using PDHFWEA operator is obtained as

$$\text{PDHFWEA}(\mathcal{A}_1, \mathcal{A}_2) = \bigcup_{\substack{\zeta_i \in h_i \\ \vartheta_i \in g_i}} \left(\begin{array}{l} \left\{ \frac{\prod_{i=1}^{2}(1+\zeta_i)^{\omega_i} - \prod_{i=1}^{2}(1-\zeta_i)^{\omega_i}}{\prod_{i=1}^{n}(1+\zeta_i)^{\omega_i} + \prod_{i=1}^{2}(1-\zeta_i)^{\omega_i}} \;\middle|\; \prod_{i=1}^{2} p_{\zeta_i} \right\}, \\ \left\{ \frac{2\prod_{i=1}^{2}(\vartheta_i)^{\omega_i}}{\prod_{i=1}^{2}(2-\vartheta_i)^{\omega_i} + \prod_{i=1}^{2}(\vartheta_i)^{\omega_i}} \;\middle|\; \prod_{i=1}^{2} q_{\vartheta_i} \right\} \end{array} \right)$$

$$= \left(\begin{array}{l} \left\{0.3|0.625, 0.3807|0.1875, 0.3206|0.1875, 0.4|0.5625\right\}, \\ \left\{0.2|0.16, 0.2772|0.24, 0.2173|0.24, 0.30|0.36\right\} \end{array} \right)$$

which clearly shows that PDHFWEA$(\mathcal{A}_1, \mathcal{A}_1) \neq \mathcal{A}_1$. Thus, it does not satisfy idempotency.

Definition 9.4.2. Let $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ be the collection of PDHFEs, and PDHFOWEA: $\Phi^n \to \Phi$, if

$$\text{PDHFOWEA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1 \mathcal{A}_{\psi(1)} \oplus \omega_2 \mathcal{A}_{\psi(2)} \oplus \ldots \oplus \omega_n \mathcal{A}_{\psi(n)} \tag{9.7}$$

where Φ is the set of PDHFEs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ such that $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$. $(\psi(1), \psi(2), \ldots, \psi(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\mathcal{A}_{\psi(i-1)} \geq \mathcal{A}_{\psi(i)}$ for $(i = 2, 3, \ldots, n)$, then PDHFOWEA is called probabilistic dual hesitant fuzzy ordered weighted Einstein AO.

Theorem 9.4.2. For a family of PDHFEs $\mathcal{A}_i = \left(h_i \,\middle|\, p_{h_i}, g_i \,\middle|\, q_{g_i}\right), (i = 1, 2, \ldots, n)$, the combined value obtained by using PDHFOWEA operator is given as

$$\text{PDHFOWEA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigcup_{\substack{\zeta_{\psi(i)} \in h_{\psi(i)}, \\ \vartheta_{\psi(i)} \in g_{\psi(i)}}} \left(\left\{ \frac{\prod_{i=1}^{n}(1+\zeta_{\psi(i)})^{\omega_{\psi(i)}} - \prod_{i=1}^{n}(1-\zeta_{\psi(i)})^{\omega_{\psi(i)}}}{\prod_{i=1}^{n}(1+\zeta_{\psi(i)})^{\omega_{\psi(i)}} + \prod_{i=1}^{n}(1-\zeta_{\psi(i)})^{\omega_{\psi(i)}}} \,\middle|\, \prod_{i=1}^{n} p_{\zeta_{\psi(i)}} \right\}, \right.$$
$$\left. \left\{ \frac{2\prod_{i=1}^{n}(\vartheta_{\psi(i)})^{\omega_{\psi(i)}}}{\prod_{i=1}^{n}(2-\vartheta_{\psi(i)})^{\omega_{\psi(i)}} + \prod_{i=1}^{n}(\vartheta_{\psi(i)})^{\omega_{\psi(i)}}} \,\middle|\, \prod_{i=1}^{n} q_{\vartheta_{\psi(i)}} \right\} \right) \tag{9.8}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is a weight vector such that $\sum_{i=1}^{n} \omega_i = 1$ where $0 < \omega_i < 1$.

Proof. Similar to Theorem 9.4.1. □

Property 9.4.3. For all PDHFEs, $\mathcal{A}_i = \left(h_i \,\middle|\, p_{h_i}, g_i \,\middle|\, q_{g_i}\right)$ where $i = 1, 2, \ldots, n$ and for an associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, such that each $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$, we have

(P1) (Boundedness) For $\mathcal{A}_i = \left(h_i \,\middle|\, p_{h_i}, g_i \,\middle|\, q_{g_i}\right)$ where $i = 1, 2, \ldots, n$, let
$\mathcal{A}^- = \left(\min(h_i) \,\middle|\, \min(p_{h_i}),\ \max(g_i) \,\middle|\, \max(q_{g_i})\right) = \left(\left\{\zeta_{\min} \,\middle|\, p_{\min}\right\}, \left\{\vartheta_{\max} \,\middle|\, q_{\max}\right\}\right)$
and $\mathcal{A}^+ = \left(\max(h_i) \,\middle|\, \max(p_{h_i}), \min(g_i) \,\middle|\, \min(q_{g_i})\right) = \left(\left\{\zeta_{\max} \,\middle|\, p_{\max}\right\}, \left\{\vartheta_{\min} \,\middle|\, q_{\min}\right\}\right)$
be PDHFEs, then $\mathcal{A}^- \leq \text{PDHFOWEA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$.

(P2) (Monotonicity) Let $\mathcal{A}_i = \left(h_i \,\middle|\, p_{h_i}, g_i \,\middle|\, q_{g_i}\right)$ and $\mathcal{A}_i^* = \left(h_i^* \,\middle|\, p_{h_i^*}, g_i^* \,\middle|\, q_{g_i^*}\right)$, for all $i = 1, 2, \ldots, n$ be two families of PDHFEs where for each element in $\mathcal{A}_i$ and $\mathcal{A}_i^*$, there are $\zeta_{\mathcal{A}_i} \leq \zeta_{\mathcal{A}_i^*}$ and $\vartheta_{\mathcal{A}_i} \geq \vartheta_{\mathcal{A}_i^*}$ while the probabilities remain the same i.e., $p_{h_i} = p_{h_i^*}, q_{g_i} = q_{g_i^*}$ then PDHFOWEA$(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq$ PDHFOWEA$(\mathcal{A}_1^*, \mathcal{A}_2^*, \ldots, \mathcal{A}_n^*)$.

Proof. Similar to Properties 9.4.1 and 9.4.2. □

Definition 9.4.3. Let Φ be a family of all PDHFEs $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ with the corresponding weights $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, such that $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$. If PDHFWEG: $\Phi^n \to \Phi$, is a mapping defined by

$$\text{PDHFWEG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}_1^{\omega_1} \otimes \mathcal{A}_2^{\omega_2} \otimes \ldots \otimes \mathcal{A}_n^{\omega_n} \tag{9.9}$$

then, PDHFWEG is called probabilistic dual hesitant fuzzy weighted Einstein geometric operator.

Theorem 9.4.3. For a collection of PDHFEs $\mathcal{A}_i = \left(h_i \,\middle|\, p_{h_i}, g_i \,\middle|\, q_{g_i}\right), (i = 1, 2, \ldots, n)$, the combined value obtained by using PDHFWEG operator is still a PDHFE and is given as

$$\text{PDHFWEG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigcup_{\zeta_i \in h_i, \vartheta_i \in g_i} \left(\begin{array}{c} \left\{ \dfrac{2\prod_{i=1}^{n} (\zeta_i)^{\omega_i}}{\prod_{i=1}^{n} (2 - \zeta_i)^{\omega_i} + \prod_{i=1}^{n} (\zeta_i)^{\omega_i}} \,\middle|\, \prod_{i=1}^{n} p_{\zeta_i} \right\}, \\ \left\{ \dfrac{\prod_{i=1}^{n} (1 + \vartheta_i)^{\omega_i} - \prod_{i=1}^{n} (1 - \vartheta_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + \vartheta_i)^{\omega_i} + \prod_{i=1}^{n} (1 - \vartheta_i)^{\omega_i}} \,\middle|\, \prod_{i=1}^{n} q_{\vartheta_i} \right\} \end{array} \right) \tag{9.10}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is a weight vector such that $\sum_{i=1}^{n} \omega_i = 1$ where $0 < \omega_i < 1$.

Proof. Same as Theorem 9.4.1. □

Also, it has been seen that the PDHFWEG operator satisfies the properties of boundedness and monotonicity.

Definition 9.4.4. Let $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ be the family of PDHFEs, and PDHFOWEG: $\Phi^n \to \Phi$, if

$$\text{PDHFOWEG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}_{\psi(1)}^{\omega_1} \oplus \mathcal{A}_{\psi(2)}^{\omega_2} \cdots \oplus \mathcal{A}_{\psi(n)}^{\omega_n} \tag{9.11}$$

where Φ is the set of PDHFEs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_i$ such that $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$. $(\psi(1), \psi(2), \ldots, \psi(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\mathcal{A}_{\psi(i-1)} \geq \mathcal{A}_{\psi(i)}$ for $(i = 2, 3, \ldots, n)$, then PDHFOWEG is called probabilistic dual hesitant fuzzy ordered weighted Einstein geometric operator.

Theorem 9.4.4. For a family of PDHFEs $\mathcal{A}_i = \left(h_i \,\middle|\, p_{h_i}, g_i \,\middle|\, q_{g_i}\right), (i = 1, 2, \ldots, n)$, the combined value obtained by using PDHFOWEG operator is given as

$$\begin{aligned} &\text{PDHFOWEG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ &= \bigcup_{\substack{\zeta_{\psi(i)} \in h_{\psi(i)}, \\ \vartheta_{\psi(i)} \in g_{\psi(i)}}} \left(\begin{array}{c} \left\{ \dfrac{2 \prod_{i=1}^{n} (\zeta_{\psi(i)})^{\omega_{\psi(i)}}}{\prod_{i=1}^{n} (2 - \zeta_{\psi(i)})^{\omega_{\psi(i)}} + \prod_{i=1}^{n} (\zeta_{\psi(i)})^{\omega_{\psi(i)}}} \,\middle|\, \prod_{i=1}^{n} p_{\zeta_{\psi(i)}} \right\}, \\ \left\{ \dfrac{\prod_{i=1}^{n} (1 + \vartheta_{\psi(i)})^{\omega_{\psi(i)}} - \prod_{i=1}^{n} (1 - \vartheta_{\psi(i)})^{\omega_{\psi(i)}}}{\prod_{i=1}^{n} (1 + \vartheta_{\psi(i)})^{\omega_{\psi(i)}} + \prod_{i=1}^{n} (1 - \vartheta_{\psi(i)})^{\omega_{\psi(i)}}} \,\middle|\, \prod_{i=1}^{n} q_{\vartheta_{\psi(i)}} \right\} \end{array} \right) \end{aligned} \tag{9.12}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is a weight vector such that $\sum_{i=1}^{n} \omega_i = 1$ where $0 < \omega_i < 1$.

Proof. Similar to Theorem 9.4.1. □

Also, it has been seen that the PDHFOWEG operator satisfies the properties of boundedness and monotonicity.

9.5 Maximum Deviation Method for Weights Determination

The choice of weights directly affects the performance of weighted aggregation operators. For this purpose, in this subsection, the effective maximizing deviation method is adapted to calculate the weights in MCDM when the weights are unknown or partially known.

Given the set of alternatives $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m\}$ and the set of criteria $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\}$ which is being evaluated by a decision maker under the PDHFS environment over the universal set $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$. Assume that the rating values corresponding to each alternative is expressed in terms of PDHFSs as

$$\mathcal{V}_i = \left\{ (\mathfrak{B}_1, \mathcal{A}_{i1}), (\mathfrak{B}_2, \mathcal{A}_{i2}), \ldots, (\mathfrak{B}_n, \mathcal{A}_{in}) \right\}, \tag{9.13}$$

where $\mathcal{A}_{ij} = \big(h_{ij}(x_i)\big|p_{ij}(x_i), g_{ij}(x_i)\big|q_{ij}(x_i)\big)$, where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Assume that the importance of each criterion are given in the form of weights as $(\omega_1, \omega_2, \ldots, \omega_n)^T$ respectively such that $0 < \omega_j \leq 1$ and $\sum\limits_{j=1}^{n} \omega_j = 1$. Now, by using the proposed distances d_1 in Eq. (9.1) or d_2 in Eq. (9.2) ; the deviation measure between the alternative $\mathcal{V}_i$ and all other alternatives with respect to the criteria $\mathfrak{B}_j$ is given as:

$$D_{ij}(\omega) = \sum_{b=1}^{m} \omega_j D(\mathcal{A}_{ij}, \mathcal{A}_{bj}) \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n \tag{9.14}$$

In accordance to the notion of maximizing deviation method, if the distance between the alternatives is smaller for a criteria, then it should have smaller weight. This one shows that the alternatives are homologous to the criterion. Contrarily, it should have larger weights. Let,

$$D_j(\omega) = \sum_{i=1}^{m} D_{ij}(\omega) = \sum_{i=1}^{m}\sum_{b=1}^{m} \omega_j D(\mathcal{A}_{ij}, \mathcal{A}_{bj}), \quad j = 1, 2, \ldots, n \tag{9.15}$$

Here $D_j(\omega)$ represents the distance of all the alternatives to the other alternatives under the criteria $\mathfrak{B}_j \in \mathfrak{B}$. Moreover, '$D$' represents either distance d_1 or d_2 as given in Eqs. (9.1) and (9.2) respectively. Based on the concept of maximum deviation, we have to choose a weight vector 'ω' to maximize all the deviations measures for the criteria. For this, we construct a non-linear programming model as given below:

$$\begin{cases} \max \;\; D(\omega) = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} D_{ij}(\omega) = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{b=1}^{m} D(\mathcal{A}_{ij}, \mathcal{A}_{bj})\omega_j \\ \text{s.t.} \;\; \omega_j > 0; \quad \sum\limits_{j=1}^{n} \omega_j = 1; \quad j = 1, 2, \ldots, n \end{cases} \tag{9.16}$$

where 'D' can be either d_1 or d_2.

If $D = d_1$, then for $\eta > 0$, we have

$$D(\omega) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{b=1}^{m} \left(\frac{1}{n} \left(\frac{\omega_j}{M_{\max} + N_{\max}} \left(\begin{array}{c} \sum\limits_{s=1}^{M_{\max}} |(\zeta_{\mathcal{A}_s} p_{\mathcal{A}_s})(x_{ij}) - (\zeta_{\mathcal{B}_s} p_{\mathcal{B}_s})(x_{bj})|^{\eta} \\ + \sum\limits_{t=1}^{N_{\max}} |(\vartheta_{\mathcal{A}_t} q_{\mathcal{A}_t})(x_{ij}) - (\vartheta_{\mathcal{B}_t} q_{\mathcal{B}_t})(x_{bj})|^{\eta} \end{array} \right) \right) \right)^{\frac{1}{\eta}};$$

and if $D = d_2$, then

$$D(\omega) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{b=1}^{m}\left(\frac{\omega_j}{n}\left(\frac{\left|\frac{1}{M_{\mathcal{A}}}\sum_{s=1}^{M_{\mathcal{A}}}(\zeta_{\mathcal{A}_s}(x_i)p_{\mathcal{A}_s})(x_{ij}) - \frac{1}{M_{\mathcal{B}}}\sum_{s'=1}^{M_{\mathcal{B}}}\left(\zeta_{\mathcal{B}_{s'}}p_{\mathcal{B}_{s'}}\right)(x_{bj})\right|^{\eta}}{2} + \frac{\left|\frac{1}{N_{\mathcal{A}}}\sum_{t=1}^{N_{\mathcal{A}}}(\vartheta_{\mathcal{A}_t}q_{\mathcal{A}_t})(x_{ij}) - \frac{1}{N_{\mathcal{B}}}\sum_{t'=1}^{N_{\mathcal{B}}}\left(\vartheta_{\mathcal{B}_{t'}}q_{\mathcal{B}_{t'}}\right)(x_{bj})\right|^{\eta}}{2}\right)\right)^{\frac{1}{\eta}}$$

If the information about criteria weights is completely unknown, then another programming method can be established as:

$$\begin{cases} \max \;\; D(\omega) = \sum_{j=1}^{n}\sum_{i=1}^{m} D_{ij}(\omega) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{b=1}^{m} D(\mathcal{A}_{ij}, \mathcal{A}_{bj})\omega_j \\ \text{s.t.} \quad \omega_j \geq 0; \quad \sum_{j=1}^{n}\omega_j^2 = 1; \quad j = 1, 2, \ldots, n \end{cases} \tag{9.17}$$

To solve this, a Lagrange's function is constructed as

$$L(\omega, \varepsilon) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{b=1}^{m} D(\mathcal{A}_{ij}, \mathcal{A}_{bj})\omega_j + \frac{\varepsilon}{2}\left(\sum_{j=1}^{n}\omega_j^2 - 1\right) \tag{9.18}$$

where ε is the Lagrange's parameter. Computing the partial derivatives of Lagrange's function w.r.t ω_j as well as ε and letting them equal to zero.

$$\begin{cases} \frac{\partial L}{\partial \omega_j} = \sum_{i=1}^{m}\sum_{b=1}^{m} D(\mathcal{A}_{ij}, \mathcal{A}_{bj}) + \varepsilon\omega_j = 0; \quad j = 1, 2, \ldots, n \\ \frac{\partial L}{\partial \varepsilon} = \sum_{j=1}^{n}\omega_j^2 - 1 = 0 \end{cases} \tag{9.19}$$

Solving, Eq. (9.19) we can obtain,

$$\omega_j = \frac{\sum_{i=1}^{m}\sum_{b=1}^{m} D(\mathcal{A}_{ij}, \mathcal{A}_{bj})}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{b=1}^{m} D(\mathcal{A}_{ij}, \mathcal{A}_{bj})\right)^2}}; \quad j = 1, 2, \ldots, n \tag{9.20}$$

Normalizing Eq. (9.20) we get

$$\omega_j = \frac{\sum_{i=1}^{m}\sum_{b=1}^{m} D(\mathcal{A}_{ij}, \mathcal{A}_{bj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{b=1}^{m} D(\mathcal{A}_{ij}, \mathcal{A}_{bj})} \tag{9.21}$$

In DM process, the data values for evaluation are available as DHFSs or PDHFSs which are integrated to form the PDHFSs. In order to gather the information, the probability values are assigned to each possible membership or non-membership value. An algorithm followed for this information fusion is outlined in Algorithm 9.1.

Algorithm 9.1 Aggregating probabilities for more than one Probabilistic fuzzy sets.

Input: $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \ldots, \mathcal{A}^{(d)}$ where $\mathcal{A}^{(d)} = \left(h^{(d)} \middle| p^{(d)}\right)$ where $d = 1, 2, \ldots, \mathcal{D}$ such that $\mathcal{D}$ is the total number of elements to be fused together.

Output: $\mathcal{A}^{(out)} = \left(h^{(out)} \middle| p^{(out)}\right)$

1: Let $u = \frac{1}{\mathcal{D}}$, be the normalized unit.

2: List all the probabilistic membership values in a set and represent it as $M = \{m_l \big| s_l\}$, where $m_l \big| s_l = h^{(d)} \big| p^{(d)}$, $\forall d = 1, 2, \ldots, \mathcal{D}$, and $l = 1, 2, \ldots, \#L$, such that $\#L$ is the total number of probabilistic membership values of all the considered elements.

3: Set $s = 1$

4: Set $m_e = m_i$

5: $f^{(l)}_{(mem)} = \begin{cases} 1, & \text{if} \quad m_e = m_l \\ 0, & \text{if} \quad m_e \neq m_l \end{cases}$

6: Set $l = l + 1$ and repeat Step 5, until $l = \#L$

7: Set $h^{(out)} = \bigcup_i m_e$

8: $p^{(out)} = \left(\sum_l \left(f^{(l)}_{(mem)} \cdot s_l \right) \cdot u \right)$

9: Set $s = i + 1$ and goto Step 4, until $s = \#L$

To demonstrate the working of aforementioned algorithm, an example is given below.

Example 9.5.1. Let $\mathcal{A}^{(1)} = \left(\{0.1|0.1, 0.2|0.5, 0.3|0.4\}, \{0.5|1\}\right)$; $\mathcal{A}^{(2)} = \left(\{0.2|0.4, 0.3|0.6\}\right.$, $\left.\{0.5|0.2, 0.6|0.8\}\right)$ and $\mathcal{A}^{(3)} = \left(\{0.1|0.4, 0.2|0.4, 0.4|0.2\}, \{0.1|1\}\right)$ be three PDHFEs. Since, $\left(h^{(1)}, p^{(1)}\right) = \left(\{0.1|0.1, 0.2|0.5, 0.3|0.4\}\right)$, $\left(h^{(2)}, p^{(2)}\right) = \left(\{0.2|0.4, 0.3|0.6\}\right)$ and $\left(h^{(3)}, p^{(3)}\right) = \left(\{0.1|0.4,\ 0.2|0.4, 0.4|0.2\}\right)$, so we get $M = \{0.1|0.1, 0.2|0.5, 0.3|0.4, 0.2|0.4, 0.3|0.6, 0.1|0.4, 0.2|0.4, 0.4|0.2\}$ where $\#L = 8$ and thus $l = 1, 2, \ldots, 8$. Clearly, here $\mathcal{D} = 3$. Now, by following Algorithm 9.1 for both membership and non-membership degrees, we obtained the final PDHFE as:

$$\mathcal{A}^{(out)} = \left(\left\{ \begin{array}{c} 0.1\big|0.1667, 0.2\big|0.4333, \\ 0.3\big|0.3333, 0.4\big|0.066 \end{array} \right\}, \left\{ \begin{array}{c} 0.5\big|0.4, 0.6\big|0.2666, \\ 0.1\big|0.3333 \end{array} \right\} \right)$$

9.6 Decision Making Approach Using the Proposed Operators

In this section, a DM approach based on proposed AOs is given followed by a numerical example.

9.6.1 Approach Based on the Proposed Operators

The general description of MCDM problem is same as Section 2.5 of Chapter 2. The ratings for each alternative in PDHFEs are given as:

$$\mathcal{V}_i = \Big\{ (\mathfrak{B}_1, \mathcal{A}_{i1}), (\mathfrak{B}_2, \mathcal{A}_{i2}), \ldots, (\mathfrak{B}_n, \mathcal{A}_{in}) \Big\}, \tag{9.22}$$

where $\mathcal{A}_{ij} = \big(h_{ij}\big|p_{ij}, g_{ij}\big|q_{ij}\big)$, where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. In order to get the best alternative(s) for a problem, DM approach is summarized in the following steps by utilizing proposed AOs as:

Step 1: Construct decision matrices $\mathcal{M}^{(d)}$ for 'd' number of decision makers in form of PDHFEs as:

$$\mathcal{M}^{(d)} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathfrak{B}_1 \quad & \quad \mathfrak{B}_2 \quad & \ldots & \quad\mathfrak{B}_n \end{array} \\ \left(\begin{array}{cccc} \Big(h_{11}^{(d)}|p_{11}^{(d)}, g_{11}^{(d)}|q_{11}^{(d)}\Big) & \Big(h_{12}^{(d)}|p_{12}^{(d)}, g_{12}^{(d)}|q_{12}^{(d)}\Big) & \ldots & \Big(h_{1n}^{(d)}|p_{1n}^{(d)}, g_{1n}^{(d)}|q_{1n}^{(d)}\Big) \\ \Big(h_{21}^{(d)}|p_{21}^{(d)}, g_{21}^{(d)}|q_{21}^{(d)}\Big) & \Big(h_{22}^{(d)}|p_{22}^{(d)}, g_{22}^{(d)}|q_{22}^{(d)}\Big) & \ldots & \Big(h_{2n}^{(d)}|p_{2n}^{(d)}, g_{2n}^{(d)}|q_{2n}^{(d)}\Big) \\ \vdots & \vdots & \ddots & \vdots \\ \Big(h_{m1}^{(d)}|p_{m1}^{(d)}, g_{m1}^{(d)}|q_{m1}^{(d)}\Big) & \Big(h_{m2}^{(d)}|p_{m2}^{(d)}, g_{m2}^{(d)}|q_{m2}^{(d)}\Big) & \ldots & \Big(h_{mn}^{(d)}|p_{mn}^{(d)}, g_{mn}^{(d)}|q_{mn}^{(d)}\Big) \end{array} \right) \end{array}$$

where $\Big(h_{ij}^{(d)}|p_{ij}^{(d)}, g_{ij}^{(d)}|q_{ij}^{(d)}\Big) = \bigcup\limits_{\substack{\zeta_{ij} \in h_{ij} \\ \vartheta_{ij} \in g_{ij}}} \Big(\big\{\zeta_{ij}^{(d)}|p_{\zeta_{ij}}^{(d)}\big\}, \big\{\vartheta_{ij}^{(d)}|q_{\vartheta_{ij}}^{(d)}\big\}\Big)$, such that $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 2: If $d = 1$, then $\left(h_{ij}^{(d)}|p_{ij}^{(d)}, g_{ij}^{(d)}|q_{ij}^{(d)}\right)$ is equal to $\left(h_{ij}|p_{ij}, g_{ij}|q_{ij}\right)$, where $\left(h_{ij}|p_{ij}, g_{ij}|q_{ij}\right) = \bigcup_{\substack{\zeta_{ij}\in h_{ij}\\ \vartheta_{ij}\in g_{ij}}} \left(\{\zeta_{ij}|p_{\zeta_{ij}}\}, \{\vartheta_{ij}|q_{\vartheta_{ij}}\}\right)$; such that $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ and goto Step 3. If $d \geq 2$, then a matrix is formed by combining the probabilities in accordance to the Algorithm 9.1. The comprehensive matrix so obtained is given as:

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \cdots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \left(h_{11}|p_{11}, g_{11}|q_{11}\right) & \left(h_{12}|p_{12}, g_{12}|q_{12}\right) & \cdots & \left(h_{1n}|p_{1n}, g_{1n}|q_{1n}\right) \\ \left(h_{21}|p_{21}, g_{21}|q_{21}\right) & \left(h_{22}|p_{22}, g_{22}|q_{22}\right) & \cdots & \left(h_{2n}|p_{2n}, g_{2n}|q_{2n}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(h_{m1}|p_{m1}, g_{m1}|q_{m1}\right) & \left(h_{m2}|p_{m2}, g_{m2}|q_{m2}\right) & \cdots & \left(h_{mn}|p_{mn}, g_{mn}|q_{mn}\right) \end{pmatrix} \end{array}$$

where $\left(h_{ij}|p_{ij}, g_{ij}|q_{ij}\right) = \bigcup_{\substack{\zeta_{ij}\in h_{ij}\\ \vartheta_{ij}\in g_{ij}}} \left(\{\zeta_{ij}|p_{\zeta_{ij}}\}, \{\vartheta_{ij}|q_{\vartheta_{ij}}\}\right)$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 3: Choose the appropriate distance measure among d_1 or d_2 as given in Eqs. (9.1) and (9.2), on the basis of need the expert. If the repeated values of the largest or smallest dual-hesitant probabilistic values can be repeated according to the optimistic or pessimistic behavior of the expert then choose measure d_1 otherwise choose measure d_2 and determine the weights of different criteria using Eq. (9.21).

Step 4: Compute the overall aggregated assessment 'r_i' of alternatives using PDHFWEA or PDHFOWEA or PDHFWEG or PDHFOWEG operators as given below in Eqs. (9.23)–(9.26) respectively.

$$\begin{aligned} r_i &= \text{PDHFWEA}(\mathcal{A}_{r1}, \mathcal{A}_{r2}, \ldots, \mathcal{A}_{ij}) \\ &= \bigcup_{\substack{\zeta_{ij}\in h_{ij}\\ \vartheta_{ij}\in g_{ij}}} \left(\begin{array}{c} \left\{ \dfrac{\prod_{j=1}^{n}(1+\zeta_{ij})^{\omega_j} - \prod_{j=1}^{n}(1-\zeta_{ij})^{\omega_j}}{\prod_{j=1}^{n}(1+\zeta_{ij})^{\omega_j} + \prod_{j=1}^{n}(1-\zeta_{ij})^{\omega_j}} \middle| \prod_{j=1}^{n} p_{\zeta_{ij}} \right\}, \\ \left\{ \dfrac{2\prod_{j=1}^{n}(\vartheta_{ij})^{\omega_j}}{\prod_{j=1}^{n}(2-\vartheta_{ij})^{\omega_j} + \prod_{j=1}^{n}(\vartheta_{ij})^{\omega_j}} \middle| \prod_{j=1}^{n} q_{\vartheta_{ij}} \right\} \end{array} \right) \end{aligned} \tag{9.23}$$

or

$$
\begin{aligned}
r_i &= \text{PDHFOWEA}(\mathcal{A}_{r1}, \mathcal{A}_{r2}, \ldots, \mathcal{A}_{ij}) \\
&= \bigcup_{\substack{\zeta_{\psi(ij)} \in h_{\psi(ij)} \\ \vartheta_{\psi(ij)} \in g_{\psi(ij)}}} \left(\begin{array}{l} \left\{ \dfrac{\prod_{j=1}^{n}(1+\zeta_{\psi(ij)})^{\omega_{\psi(j)}} - \prod_{j=1}^{n}(1-\zeta_{\psi(ij)})^{\omega_{\psi(j)}}}{\prod_{j=1}^{n}(1+\zeta_{\psi(ij)})^{\omega_{\psi(j)}} + \prod_{j=1}^{n}(1-\zeta_{\psi(ij)})^{\omega_{\psi(j)}}} \,\middle|\, \prod_{j=1}^{n} p_{\zeta_{\psi(ij)}} \right\}, \\ \left\{ \dfrac{2\prod_{j=1}^{n}(\vartheta_{\psi(ij)})^{\omega_{\psi(j)}}}{\prod_{j=1}^{n}(2-\vartheta_{\psi(ij)})^{\omega_{\psi(j)}} + \prod_{j=1}^{n}(\vartheta_{\psi(ij)})^{\omega_{\psi(j)}}} \,\middle|\, \prod_{j=1}^{n} q_{\vartheta_{\psi(ij)}} \right\} \end{array} \right)
\end{aligned} \tag{9.24}
$$

or

$$
\begin{aligned}
r_i &= \text{PDHFWEG}(\mathcal{A}_{r1}, \mathcal{A}_{r2}, \ldots, \mathcal{A}_{ij}) \\
&= \bigcup_{\substack{\zeta_{ij} \in h_{ij} \\ \vartheta_{ij} \in g_{ij}}} \left(\begin{array}{l} \left\{ \dfrac{2\prod_{j=1}^{n}(\zeta_{ij})^{\omega_j}}{\prod_{j=1}^{n}(2-\zeta_{ij})^{\omega_j} + \prod_{j=1}^{n}(\zeta_{ij})^{\omega_j}} \,\middle|\, \prod_{j=1}^{n} p_{\zeta_{ij}} \right\}, \\ \left\{ \dfrac{\prod_{j=1}^{n}(1+\vartheta_{ij})^{\omega_j} - \prod_{j=1}^{n}(1-\vartheta_{ij})^{\omega_j}}{\prod_{j=1}^{n}(1+\vartheta_{ij})^{\omega_j} + \prod_{j=1}^{n}(1-\vartheta_{ij})^{\omega_j}} \,\middle|\, \prod_{j=1}^{n} q_{\vartheta_{ij}} \right\} \end{array} \right)
\end{aligned} \tag{9.25}
$$

or

$$
\begin{aligned}
r_i &= \text{PDHFOWEG}(\mathcal{A}_{r1}, \mathcal{A}_{r2}, \ldots, \mathcal{A}_{ij}) \\
&= \bigcup_{\substack{\zeta_{\psi(ij)} \in h_{\psi(ij)} \\ \vartheta_{\psi(ij)} \in g_{\psi(ij)}}} \left(\begin{array}{l} \left\{ \dfrac{2\prod_{j=1}^{n}(\zeta_{\psi(ij)})^{\omega_{\psi(j)}}}{\prod_{j=1}^{n}(2-\zeta_{\psi(ij)})^{\omega_{\psi(j)}} + \prod_{j=1}^{n}(\zeta_{\psi(ij)})^{\omega_{\psi(j)}}} \,\middle|\, \prod_{j=1}^{n} p_{\zeta_{\psi(ij)}} \right\}, \\ \left\{ \dfrac{\prod_{j=1}^{n}(1+\vartheta_{\psi(ij)})^{\omega_{\psi(j)}} - \prod_{j=1}^{n}(1-\vartheta_{\psi(ij)})^{\omega_{\psi(j)}}}{\prod_{j=1}^{n}(1+\vartheta_{\psi(ij)})^{\omega_{\psi(j)}} + \prod_{j=1}^{n}(1-\vartheta_{\psi(ij)})^{\omega_{\psi(j)}}} \,\middle|\, \prod_{j=1}^{n} q_{\vartheta_{\psi(ij)}} \right\}, \end{array} \right)
\end{aligned} \tag{9.26}
$$

Step 5: Utilize Definition 2.1.13 of Chapter 2 to rank the overall aggregated values and select the most desirable alternative(s).

9.6.2 Illustrative Example

An illustrative example (based on consumer's buying behavior) for eliciting the numerical applicability of our proposed approach is given below:

In a company's production oriented DM processes, consumers or buyers play a vital role. In order to increase sales and to be in good books of every customer, every production company pays a great attention to customer's buying behavior. This consumer behavior is the main driving force behind the change of trends, need of updation in the products etc., to which the production company must remain in contact to have a great mutual relationship with the customers and to maintain a strong position in the competitive market environment.

Suppose a multi-national company wants to launch the new products on the basis of different consumers in different countries. For that, they have delegated works to the company heads of three different countries viz. India, Canada, and Australia. The company heads of these countries have to analyze the customer's buying behavior and for that, they have information available in the form of PDHFEs. Each expert ($d = 1, 2, 3$) from the three different countries accessed the available information oriented to four company products $\mathcal{V}_i$'s where ($i = 1, 2, 3, 4$) classified under four criteria determining the customer's buying behavior namely $\mathfrak{B}_1$: 'Suitability to cultural environment'; $\mathfrak{B}_2$: 'Global trend accordance'; $\mathfrak{B}_3$: 'Suitability to weather conditions' ; $\mathfrak{B}_4$: 'Good quality after-sale services'. The aim of the company is to access the main criteria which affect the customer's buying behavior so as to figure out which product among $\mathcal{V}_i$'s ($i = 1, 2, 3, 4$) has to be launched first. Following steps are adopted to find the most suitable product for the first launch.

Step 1: The preference information corresponding to three decision-makers ($d = 1, 2, 3$) is given in Tables 9.1 - 9.3.

Step 2: Since number of decision makers i.e., $d \geq 2$, therefore, using Algorithm 9.1, the comprehensive matrix obtained after integrating all the preferences given by the panel of experts is given in Table 9.4.

Step 3: The experts chose to have an optimistic behavior towards the analysis and thus utilizing distance d_1 in Eq. (9.21), the weights are determined as $\omega = (0.4385, 0.1986, 0.1815, 0.1814)^T$.

Step 4: The aggregated values for each alternative $\mathcal{V}_i, (i = 1, 2, 3, 4)$ by using PDHFWEA operator as given in Eq. (9.23) are :

$$r_1 = \left(\left\{ \begin{array}{c} 0.5213|0.0056, 0.5439|0.0006, \\ 0.5546|0.0154, 0.5760|0.0017, \\ \ldots\ldots\ldots\ldots, 0.6347|0.0037 \end{array} \right\}, \left\{ \begin{array}{c} 0.2617|0.0444, 0.2531|0.0222, \\ 0.1909|0.0222, 0.1844|0.0111, \\ \ldots\ldots\ldots\ldots, 0.3120|0.0074 \end{array} \right\} \right)$$

$$r_2 = \left(\left\{ \begin{array}{c} 0.6080|0.0469, 0.6201|0.0614, \\ 0.4838|0.0253, 0.4985|0.0331, \\ \ldots\ldots\ldots\ldots, 0.4240|0.0157 \end{array} \right\}, \left\{ \begin{array}{c} 0.2531|0.0123, 0.2359|0.0198, \\ 0.2266|0.0049, 0.5372|0.0062, \\ \ldots\ldots\ldots\ldots, 0.6427|0.0025 \end{array} \right\} \right)$$

$$r_3 = \left(\left\{ \begin{array}{c} 0.3384|0.0173, 0.3352|0.0173, \\ 0.3515|0.0173, 0.3963|0.0074, \\ \ldots\ldots\ldots\ldots, 0.7379|0.0123 \end{array} \right\}, \left\{ \begin{array}{c} 0.4391|0.0444, 0.4251|0.0346, \\ 0.4256|0.0691, 0.2226|0.0222, \\ \ldots\ldots\ldots\ldots, 0.1540|0.0026 \end{array} \right\} \right)$$

$$r_4 = \left(\left\{ \begin{array}{c} 0.4225|0.0474, 0.4036|0.0474, \\ 0.3947|0.0474, 0.4667|0.0435, \\ \ldots\ldots\ldots\ldots, 0.3110|0.0078 \end{array} \right\}, \left\{ \begin{array}{c} 0.3016|0.0017, 0.3413|0.0039, \\ 0.3698|0.0111, 0.2533|0.0006, \\ \ldots\ldots\ldots\ldots, 0.5259|0.0197 \end{array} \right\} \right)$$

Step 5: The score values are obtained as $\mathcal{S}c(r_1) = 0.1810, \mathcal{S}c(r_2) = 0.1799, \mathcal{S}c(r_3) = 0.1739$ and $\mathcal{S}c(r_4) = -0.0002$

Step 6: Since, the ranking order is $\mathcal{S}c(r_1) > \mathcal{S}c(r_2) > \mathcal{S}c(r_3) > \mathcal{S}c(r_4)$, thus the ranking is obtained as $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$.

Thus, it is clear that according to the experts product $\mathcal{V}_1$ should be launched first.

However, on the other hand, if we utilize the PDHFWEG operator instead of PDHFWEA operator to aggregate the different preferences, then the following steps of the proposed approach are executed to reach the optimal alternative(s) as.

Step 1: Similar as above Step 1.

Step 2: Similar as above Step 2.

Step 3: Similar as above Step 3.

Step 4: The aggregated values for each alternative $\mathcal{V}_i, (i = 1, 2, 3, 4)$ by using PDHFWEG operator as given in Eq. (9.25) are :

$$r_1 = \left(\left\{ \begin{array}{l} 0.3959|0.0056, 0.4092|0.0006, \\ 0.4642|0.0154, 0.4792|0.0017, \\ \ldots\ldots\ldots\ldots, 0.5908|0.0037 \end{array} \right\}, \left\{ \begin{array}{l} 0.2917|0.0444, 0.2827|0.0222, \\ 0.2008|0.0222, 0.1913|0.0111, \\ \ldots\ldots\ldots\ldots, 0.3541|0.0074 \end{array} \right\} \right)$$

$$r_2 = \left(\left\{ \begin{array}{l} 0.5090|0.0469, 0.5415|0.0614, \\ 0.4391|0.0253, 0.4685|0.0331, \\ \ldots\ldots\ldots\ldots, 0.2959|0.0157 \end{array} \right\}, \left\{ \begin{array}{l} 0.3950|0.0123, 0.3312|0.0198, \\ 0.3078|0.0049, 0.6376|0.0062, \\ \ldots\ldots\ldots\ldots, 0.6516|0.0025 \end{array} \right\} \right)$$

$$r_3 = \left(\left\{ \begin{array}{l} 0.1667|0.0173, 0.1615|0.0173, \\ 0.1828|0.0173, 0.2950|0.0074, \\ \ldots\ldots\ldots\ldots, 0.6164|0.0123 \end{array} \right\}, \left\{ \begin{array}{l} 0.4890|0.0444, 0.4646|0.0346, \\ 0.5203|0.0691, 0.3256|0.0222, \\ \ldots\ldots\ldots\ldots, 0.2742|0.0026 \end{array} \right\} \right)$$

$$r_4 = \left(\left\{ \begin{array}{l} 0.4150|0.0474, 0.3981|0.0474, \\ 0.3886|0.0474, 0.4580|0.0435, \\ \ldots\ldots\ldots\ldots, 0.2774|0.0078 \end{array} \right\}, \left\{ \begin{array}{l} 0.3395|0.0017, 0.3744|0.0039, \\ 0.4157|0.0111, 0.2974|0.0006, \\ \ldots\ldots\ldots\ldots, 0.5656|0.0197 \end{array} \right\} \right)$$

Step 5: The score values are obtained as $\mathcal{S}c(r_1) = 0.0937, \mathcal{S}c(r_2) = -0.0073, \mathcal{S}c(r_3) = -0.0202$ and $\mathcal{S}c(r_4) = -0.0545$

Step 6: Since, the ranking order is $\mathcal{S}c(r_1) > \mathcal{S}c(r_3) > \mathcal{S}c(r_2) > \mathcal{S}c(r_4)$, thus the ranking is obtained as $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$. The most desirable alternative is $\mathcal{V}_1$.

If we analyze the impact of the all the proposed operators along with the distance d_1 and d_2 onto the final ranking order of the alternative, we perform an experiment where the steps of the proposed algorithms are executed. The final score values of each alternative $\mathcal{V}_i$ $(i = 1, 2, 3, 4)$, are obtained and are summarized in Table 9.5. It is seen that utilizing different distance measures i.e., d_1 and d_2 do not affect the best alternative $\mathcal{V}_1$ in most of the cases. Moreover, the score values obtained by the proposed operators namely: PDHFWEA, PDHFWEG, and PDHFOWEG represent the same alternative $\mathcal{V}_1$ as the best alternative which is to be launched first while the operator PDHOWEA represents the alternative $\mathcal{V}_3$ as the best one. However, it can be seen that corresponding average PDHFWEA, PDHFOWEA score values are greater than that of PDHFWEG, PDHFOWEG aggregation operators showing that the average AOs offer the decision maker more optimistic score-values as compared to the geometric ones. Also, it can be seen that both the distances, despite providing, a huge variation in numerical evaluation and data processing flexibility lead to the same result as $\mathcal{V}_1$ as the best choice in most of the cases among the alternatives to be launched first.

9.6.3 Comparative Studies

In order to analyze the alignment of the proposed approach's results with the existing theories and to validate our proposed results, the score values corresponding to different operators are given in Table 9.6. The operators in the considered existing theories are: probabilistic dual hesitant fuzzy weighted average (PDHFWA) by Hao et al. [61], hesitant probabilistic fuzzy Einstein weighted average and Einstein weighted geometric (HPFEWEA, HPFEWEG) by Park et al. [117] and hesitant probabilistic fuzzy weighted average (HPFWA), hesitant probabilistic fuzzy weighted geometric (HPFWG), hesitant probabilistic fuzzy ordered weighted average (HPFOWA) , hesitant probabilistic fuzzy ordered weighted geometric (HPFOWG) aggregation operators by Xu and Zhou [194]. Noticeably, the approach outlined by Hao et al. [61] by utilizing PDHFWA operator figures out $\mathcal{V}_2$ as the best alternative and the least preferred alternative $\mathcal{V}_4$ remains same as that of our proposed approach. However, if we consider only the probabilistic hesitant fuzzy information and ignores the non-membership probabilistic hesitant values, then the best alternative starts fluctuating among $\mathcal{V}_1$ and $\mathcal{V}_3$ by varying the different AOs and the least preferred alternative remains same as $\mathcal{V}_4$, which coincides the outcomes of our proposed approach. This variation is due to the negligence of the non-membership values and their corresponding probabilities. Thus, the proposed approach is advantageous among the traditional approaches because it remains firm on the same output ranking for different operators. Moreover, the best alternative chosen by the proposed approach remains the same as that with that of the existing approaches.

Further, a deep insight into the comparison of our method with the existing ones is given by comparing the characteristics of all the approaches with the proposed one. In Table 9.7, it can be seen that the approaches put-forth by Hao et al. [61] and Xu and

Zhou [194] considers multiple experts in analysis process whereas Park et al. [117] does not consider the multi-expert problems. All the existing approaches are the probabilistic approaches so they consider probabilities corresponding to their considered membership or non-membership values. Moreover, it is analyzed that the method proposed by [61] considers the non-membership probabilistic information but the rest two only considers the hesitant values and their probabilities. In all the three existing approaches, the weights are not derived by using any non-linear technique such as maximum deviation method for determination of weights but the weights corresponding to two different distance measures are considered in the proposed methodology.

In addition to above comparison studies, we elicit some characteristic comparison of our approach with existing DM methods proposed in [61, 117, 194] which are tabulated in Table 9.7. In this table , the symbol '✓' describes that the corresponding DM approach considers more than one decision maker, handles probabilities, accounts for non-membership entities and has weights derived by the non-linear approach, whereas the symbol '×' means that the associated method fails. The symbols tabulated in Table 9.7 depicts that the MCDM mentioned in [61] as well as [194] consider multiple multiple decision-makers whereas the approach utilized by [117] consists of preference evaluations through single expert. It is seen that all the three considered approaches considers the probabilities along with their respective fuzzy environments whereas only [61] considers only the non-membership values along with the membership ones while the other two considers only the membership value ratings. On the other hand, none of the existing approach among the specified ones, adopt a non-linear weight determination technique. Thus, it is analyzed that our proposed approach consists of all the four said characteristics and thus it deals with the real life situations, more efficiently as compared to the existing approaches [61, 117, 194].

9.7 Conclusion

The primary contribution of this chapter is summarized as follows:

1) In this chapter, we have utilized the concept of PDHFS to handle the uncertainty in the data so as to capture the information with some more degree of freedom. For

it, we introduce the two new distance measures between the pairs of the PDFHEs and explore their properties. Further, some basic operational laws for this proposed structure are discussed and explore the various relationships among them using Einstein norm operations.

2) To obtain the optimal selection in the group decision making (GDM) under the probabilistic dual hesitant fuzzy environment, we have proposed a maximum deviation method (MDM) algorithm and developed several weighted AOs. In this case, the MDM method has been used to determine the optimal weight of each criterion.

3) Four new aggregation operators, namely, the PDHFWEA, PDHFOWEA, PDHFWEG, and PDHFOWEG operators have been developed to aggregate the PDHFE information. The major advantages of the proposed operators are that it considers the probability information to each dual hesitant membership degrees which give more information and help for the decision maker to take a decision more clearly. In addition to it, on a comprehensive scrutiny of DHFSs and PDHFSs, we have devised an algorithm to formulate PDHFSs from the given probabilistic fuzzy information. Based on the decision maker preferences in order to optimize their desired goals, the person can choose the required proposed distance measures and/or aggregation operators.

4) Finally, the presented group DM approach is explained with the help of numerical example and an extensive comparative analysis has been conducted with the existing DM theories [61, 117, 194] to show the advantages of the proposed approach.

Thus, we can conclude that the proposed notion about the PDHFSs is widely used in the different scenarios such as when a person provides the information about the fact that 'how much he/she sure about the uncertain information evaluated by him/her?'; in the situations, when the evaluators have no knowledge of the importance of their decision as well the considered criteria.

Table 9.1: Preference values provided by decision-maker 1.

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\left(\{0.2\vert 0.4, 0.3\vert 0.6\}, \{0.4\vert 1\}\right)$	$\left(\{0.45\vert 0.42, 0.60\vert 0.58\}, \{0.2\vert 0.4, 0.3\vert 0.6\}\right)$	$\left(\{0.9\vert 1\}, \{0.1\vert 1\}\right)$	$\left(\{0.6\vert 1\}, \{0.3\vert 1\}\right)$
$\mathcal{V}_2$	$\left(\{0.8\vert 0.9, 0.6\vert 0.1\}, \{0.1\vert 1\}\right)$	$\left(\{0.30\vert 1\}, \{0.6\vert 1\}\right)$	$\left(\{0.6\vert 1\}, \{0.2\vert 0.5, 0.1\vert 0.5\}\right)$	$\left(\{0.2\vert 1\}, \{0.8\vert 1\}\right)$
$\mathcal{V}_3$	$\left(\{0.05\vert 0.7, 0.2\vert 0.3\}, \{0.5\vert 1\}\right)$	$\left(\{0.50\vert 1\}, \{0.5\vert 1\}\right)$	$\left(\{0.8\vert 0.6, 0.6\vert 0.4\}, \{0.15\vert 1\}\right)$	$\left(\{0.12\vert 1\}, \{0.7\vert 0.9, 0.6\vert 0.1\}\right)$
$\mathcal{V}_4$	$\left(\{0.4\vert 1\}, \{0.3\vert 0.5, 0.2\vert 0.5\}\right)$	$\left(\{0.50\vert 1\}, \{0.2\vert 0.3, 0.4\vert 0.7\}\right)$	$\left(\{0.3\vert 1\}, \{0.65\vert 1\}\right)$	$\left(\{0.5\vert 1\}, \{0.2\vert 0.3, 0.4\vert 0.7\}\right)$

Table 9.2: Preference values provided by decision-maker 2.

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\left(\{0.3\vert 0.5, 0.5\vert 0.5\}, \{0.4\vert 1\}\right)$	$\left(\{0.20\vert 1\}, \{0.7\vert 0.1\}\right)$	$\left(\{0.2\vert 1\}, \{0.4\vert 0.8, 0.6\vert 0.2\}\right)$	$\left(\{0.6\vert 0.7, 0.7\vert 0.3\}, \{0.25\vert 1\}\right)$
$\mathcal{V}_2$	$\left(\{0.2\vert 1\}, \{0.7\vert 1\}\right)$	$\left(\{0.30\vert 0.5, 0.2\vert 0.5\}, \{0.20\vert 0.5, 0.15\vert 0.5\}\right)$	$\left(\{0.2\vert 1\}, \{0.6\vert 1\}\right)$	$\left(\{0.2\vert 0.3, 0.3\vert 0.7\}, \{0.6\vert 1\}\right)$
$\mathcal{V}_3$	$\left(\{0.4\vert 0.4, 0.5\vert 0.6\}, \{0.5\vert 1\}\right)$	$\left(\{0.45\vert 1\}, \{0.5\vert 1\}\right)$	$\left(\{0.8\vert 0.4, 0.6\vert 0.6\}, \{0.2\vert 0.7, 0.1\vert 0.3\}\right)$	$\left(\{0.1\vert 1\}, \{0.6\vert 0.6, 0.8\vert 0.4\}\right)$
$\mathcal{V}_4$	$\left(\{0.4\vert 0.2, 0.5\vert 0.8\}, \{0.3\vert 1\}\right)$	$\left(\{0.2\vert 0.4, 0.5\vert 0.6\}, \{0.4\vert 0.2, 0.3\vert 0.8\}\right)$	$\left(\{0.4\vert 0.1, 0.5\vert 0.9\}, \{0.3\vert 1\}\right)$	$\left(\{0.4\vert 1\}, \{0.6\vert 1\}\right)$

Table 9.3: Preference values provided by decision-maker 3.

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\left(\{0.75\vert 1\}, \{0.2\vert 1\}\right)$	$\left(\{0.50\vert 1\}, \{0.2\vert 0.5, 0.5\vert 0.5\}\right)$	$\left(\{0.3\vert 1\}, \{0.6\vert 1\}\right)$	$\left(\{0.6\vert 1\}, \{0.3\vert 1\}\right)$
$\mathcal{V}_2$	$\left(\{0.6\vert 0.6, 0.8\vert 0.4\}, \{0.1\vert 1\}\right)$	$\left(\{0.20\vert 1\}, \{0.7\vert 1\}\right)$	$\left(\{0.9\vert 1\}, \{0.1\vert 1\}\right)$	$\left(\{0.3\vert 1\}, \{0.5\vert 0.4, 0.6\vert 0.6\}\right)$
$\mathcal{V}_3$	$\left(\{0.9\vert 1\}, \{0.1\vert 1\}\right)$	$\left(\{0.6\vert 1\}, \{0.25\vert 0.5, 0.1\vert 0.5\}\right)$	$\left(\{0.8\vert 1\}, \{0.2\vert 1\}\right)$	$\left(\{0.2\vert 1\}, \{0.8\vert 1\}\right)$
$\mathcal{V}_4$	$\left(\{0.3\vert 0.7, 0.5\vert 0.3\}, \{0.4\vert 0.6, 0.5\vert 0.4\}\right)$	$\left(\{0.1\vert 1\}, \{0.8\vert 1\}\right)$	$\left(\{0.3\vert 1\}, \{0.3\vert 1\}\right)$	$\left(\{0.35\vert 1\}, \{0.6\vert 1\}\right)$

Table 9.4: Comprehensive matrix

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\left(\{0.2\mid 0.1333, 0.3\mid 0.3667, 0.5\mid 0.1667, 0.75\mid 0.3333\}, \{0.4\mid 0.6667, 0.2\mid 0.3333\}\right)$	$\left(\{0.45\mid 0.14, 0.6\mid 0.1934, 0.2\mid 0.3333, 0.5\mid 0.3333\}, \{0.2\mid 0.3, 0.3\mid 0.2, 0.7\mid 0.3333, 0.5\mid 0.1667\}\right)$	$\left(\{0.9\mid 0.3333, 0.2\mid 0.3333, 0.3\mid 0.3334\}, \{0.1\mid 0.3333, 0.4\mid 0.2667, 0.6\mid 0.4\}\right)$	$\left(\{0.6\mid 0.9, 0.7\mid 0.1\}, \{0.3\mid 0.6667, 0.25\mid 0.3333\}\right)$
$\mathcal{V}_2$	$\left(\{0.8\mid 0.4333, 0.6\mid 0.2334, 0.2\mid 0.3333\}, \{0.1\mid 0.6667, 0.7\mid 0.3333\}\right)$	$\left(\{0.30\mid 0.75, 0.2\mid 0.5\}, \{0.6\mid 0.3333, 0.2\mid 0.1667, 0.15\mid 0.1667, 0.7\mid 0.3333\}\right)$	$\left(\{0.6\mid 0.3333, 0.2\mid 0.3334, 0.9\mid 0.3333\}, \{0.2\mid 0.1667, 0.1\mid 0.6667, 0.6\mid 0.1666\}\right)$	$\left(\{0.2\mid 0.4333, 0.3\mid 0.5667\}, \{0.8\mid 0.3333, 0.6\mid 0.3333, 0.5\mid 0.1333\}\right)$
$\mathcal{V}_3$	$\left(\{0.05\mid 0.2334, 0.2\mid 0.1, 0.4\mid 0.1333, 0.5\mid 0.2, 0.9\mid 0.3333\}, \{0.5\mid 0.6667, 0.1\mid 0.3333\}\right)$	$\left(\{0.5\mid 0.3333, 0.45\mid 0.3333, 0.6\mid 0.3334\}, \{0.5\mid 0.6667, 0.2\mid 0.1667, 0.1\mid 0.1666\}\right)$	$\left(\{0.8\mid 0.6667, 0.6\mid 0.3333\}, \{0.15\mid 0.3333, 0.2\mid 0.5666, 0.1\mid 0.1\}\right)$	$\left(\{0.12\mid 0.3333, 0.1\mid 0.3333, 0.2\mid 0.3334\}, \{0.7\mid 0.3, 0.6\mid 0.2333, 0.8\mid 0.4667\}\right)$
$\mathcal{V}_4$	$\left(\{0.4\mid 0.4, 0.5\mid 0.3667, 0.3\mid 0.2333\}, \{0.3\mid 0.5, 0.2\mid 0.1667, 0.4\mid 0.2, 0.5\mid 0.1333\}\right)$	$\left(\{0.5\mid 0.5333, 0.2\mid 0.1333, 0.1\mid 0.3334\}, \{0.2\mid 0.1, 0.4\mid 0.3, 0.3\mid 0.2667, 0.8\mid 0.3333\}\right)$	$\left(\{0.30\mid 0.6667, 0.4\mid 0.0333, 0.5\mid 0.3\}, \{0.65\mid 0.3333, 0.3\mid 0.6667\}\right)$	$\left(\{0.5\mid 0.3333, 0.4\mid 0.3333, 0.35\mid 0.3334\}, \{0.2\mid 0.1, 0.4\mid 0.2334, 0.6\mid 0.6666\}\right)$

Table 9.5: Score values of proposed approach

	Operator	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	Ranking
Distance d_1	PDHFWEA	0.1810	0.1799	0.1739	−0.0002	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
	PDHFOWEA	0.2293	0.2239	0.2940	0.0013	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
	PDHFWEG	0.0937	−0.0073	−0.0202	−0.0545	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
	PDHFOWEG	0.1458	0.0283	0.0856	−0.0515	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
Distance d_2	PDHFWEA	0.1968	0.0754	0.1213	−0.0459	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
	PDHFOWEA	0.1684	0.0832	0.0971	−0.0472	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
	PDHFWEG	0.1006	−0.1189	−0.1072	−0.1056	$\mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_2$
	PDHFOWEG	0.0691	−0.1118	−0.1268	−0.1091	$\mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3$

Table 9.6: Comparison of overall rating values and ranking order of alternatives.

Existing Approaches	Operators	Score Values				
		$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	Ranking
Hao et al. [61]	PDHFWA	0.1985	0.2135	0.2061	0.0098	$\mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4$
Park et al. [117]	HPFEWA	0.5131	0.4915	0.5243	0.3917	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
	HPFEWG	0.4569	0.4094	0.4056	0.3723	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
Xu and Zhou [194]	HPFWA	0.5253	0.5091	0.5445	0.3953	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
	HPFWG	0.4457	0.3937	0.3837	0.3685	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
	HPFOWA	0.5585	0.5215	0.6078	0.3957	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
	HPFOWG	0.4826	0.3998	0.4385	0.3699	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$

Table 9.7: Characteristic comparison of the proposed approach with different methods.

Methods	Whether Consider More Than One Decision Maker	Whether Considers Probabilities	Whether Considers Non-Membership	Weights Derived By Non-Linear Approach
Hao et al. [61]	✓	✓	✓	×
Park et al. [117]	×	✓	×	×
Xu and Zhou [194]	✓	✓	×	×
Our proposed approach	✓	✓	✓	✓

Chapter 10

Quantifying gesture information in brain hemorrhage patients using probabilistic dual hesitant fuzzy sets[1]

This chapter put emphasis on proposing some weighted averaging and geometric Maclaurin symmetric mean (MSM) aggregation operators to address the uncertainties in the medical diagnosis problems and handle the gesture quantification. PDHFS is a good ways to express the information along with the recognizance of the existence of the probabilities of the ingredients in the own set. An optimization model based on Shanon's entropy is framed for determining the probabilities. Later, a decision-making approach is developed based on the proposed operators followed by comparison analysis and a case study on gesture quantification of a patient suffering from hemorrhage strokes is conducted.

10.1 Introduction

This chapter opens up the realms of practical relevance of PDHFSs in medical problems. In these days, due to various conditions and environment, human beings suffer from numerable health problems which sometimes be very dangerous. One such health disorder

[1]The content of this chapter is published as "Quantifying gesture information in brain hemorrhage patients using probabilistic dual hesitant fuzzy sets with unknown probability information", *Computers & Industrial Engineering, Elsevier*, 140, pp. 106211, 2020, doi: 10.1016/j.cie.2019.106211 (**SCI: Impact Factor: 5.431**)

is hemorrhage stroke, which is entirely different from the normal stroke. In terms of the medical field, a stroke occurs due to lack of blood circulation towards the human brain cells, but hemorrhage stroke might occur by rupture of a blood vessel in the brain which results to increase the flow of blood to the brain. As a consequence, the infected area of the brain might result in the malfunctioning of other body parts too and hence a patient may suffer from the various medical problems such as paralysis attack, lack of vision problems, speech-related problems, etc. Under such situations, the patient can't speak and hence they cannot respond to the doctor's queries efficiently. Further, to understand the patient's symptoms frequently, a doctor tries to understand it with the help of the gesture made by him. Under it, Liang et al. [93] focused on identifying nine kinds of gestures while Wang [153] focussed on the preclinical and clinical research on various inflammation factors arising because of intracerebral hemorrhage. Also, Lok et al. [99] focussed on the mechanisms of secondary brain injury resulting from hemorrhage strokes. Some other researchers [102, 138, 145] focused their vision towards the research in hemorrhage stroke. Although the problem of brain hemorrhage is very serious, the relevant management rules are not yet perfect. Therefore, it is necessary to set up a standardized DM approach to quantify the gestures so that uniformity can be made throughout to analyze the symptoms of every patient.

PDHFS is a good ways to express the uncertain and imprecise information which is not only the extension of the several other existing sets, but also recognize the existence of the probabilities of the ingredients in the own set. Therefore, under this environment, the features of the considered problems are extracted in a more accurate manner. Thus, this study intends to give a novel DM approach to quantifying the gesture information for the brain hemorrhage problems using PDHFSs with robust AOs to order the numbers.To address it, the prime motive of the work is abstracted below.

i) In the theories related to the medical diagnosis, there is always occurs a challenge to collect the information form a patient who cannot communicate their disease symptoms properly with the doctors. As a result, there always occur an ambiguity in noting down the symptoms of the speech deprived patients or in those patients, who under the severe effect of the medication cannot pass their actual disease-oriented instincts to the experts. Under such cases, a doctor tries to quantify the patient's gestures into his medical practice knowledge and want to capture the patient's opinion along with his medical experience. To provide more liberty to the expert to understand the patient through gestures and hence to serve the knowledge using PDHFS features. In PDHFS, each observation is represented as $(x|p, y|q)$ which consists of two parts, namely membership parts $(x,\, y)$ of the elements and the corresponding probabilities $(p,\, q)$.Thus, this erudition can be managed as a probability distribution function with x, y as random variables and p, q are the probabilities.

ii) During the DM process, an expert can determine all the risk status, but he or she cannot determine which risk status will happen. This will lead to the question regarding the probability of the risk status. However it is difficult to determine the probability only by subjective estimation of the decision-maker under probabilistic dual hesitant fuzzy (PDHF) environment and hence it will be another primary issue, which is also another focus of this chapter. Therefore, to determine the probability of risk status under PDHF environment, we constructed an optimization model to determine it.

iii) In real-life situations, there occur several parameters which affect each other during aggregating the information. In other words, the different parameters associated with the DM problems are not independent of each other. To address it properly, there is a need to consider the interrelationships between the multiple-pairs of the different arguments. To incorporate such a feature into the analysis, we established the concept of Maclaurin symmetric mean (MSM) operator to aggregate the PDHF elements (PDHFEs). The presented MSM operator involves an additional parameter which suggests to the decision-makers regarding the selection of the appropriate parameters during the process to reduce the risk preferences.

In modern DM problems, the problems are often accompanied by the interrelationships between the attributes. At the same time, some experts may have a strong personal bias towards the evaluation process because of different levels of knowledge. To address it completely and encouraged from the features of PDHFS, this chapter center on to investigate the MSM operators for PDHFSs by using the uncertain and fuzzy information. For it, we developed a non-linear optimization model based on the entropy and score functions of PDHFEs to determine the probabilities associated with the PDHFEs. Then, based on the PDHFEs, we define some weighted averaging and geometric MSM aggregation operators and studies their some characteristics. Afterward, we develop an efficient method to solve the MCDM problem based on the proposed operators and efficiency of it is discussed with a numerical example with the aid of a case study on gesture quantification of the patient suffering from hemorrhage strokes. The validity of the proposed method is examined with several predominating studies.

10.2 PDHF Maclaurin symmetric mean operators

In this section, we fascinating averaging and geometric MSM aggregation operators for PDHFSs, denoted by PDHFMSMA and PDHFMSMG, respectively. For it, let Φ be the collection of PDHFEs, $\mathcal{A}_i = (h_i|p_{h_i}, g_i|q_{g_i})$ such that neither $h_i = 0$ or 1 nor $g_i = 0$ or 1.

10.2.1 PDHFMSM averaging operator

Definition 10.2.1. For "n" PDHFEs $\mathcal{A}_i$, a map PDHFMSMA: $\Phi^n \to \Phi$, stated as

$$\text{PDHFMSMA}^{(k)}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \left(\frac{1}{\binom{n}{k}} \bigoplus_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(\bigotimes_{v=1}^{k} \mathcal{A}_{i_v}\right)\right)^{\frac{1}{k}} \tag{10.1}$$

where $(i_1, i_2, \ldots, i_k)$ includes all the k- tuples of $(1, 2, \ldots, k)$ and $\binom{n}{k}$ is the binomial coefficient. Then, PDHFMSMA is called PDHF Maclaurin symmetric mean (PDHFMSM) averaging operator.

Theorem 10.2.1. For a family of "n" PDHFEs $\mathcal{A}_i = (h_i|p_{h_i}, g_i|q_{g_i})$, the aggregated value using Definition 10.2.1 is still a PDHFE and given as

$$\text{PDHFMSMA}^{(k)}\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)$$
$$= \bigcup_{\substack{\zeta_{i_v}\in h_{i_v},\\ \vartheta_{i_v}\in g_{i_v}}} \left(\begin{array}{l} \left\{ \left(1-\prod\limits_{\substack{1\le i_1<\ldots\\<i_k\le n}}\left(1-\prod\limits_{v=1}^{k}\zeta_{i_v}\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \middle| \prod\limits_{\substack{1\le i_1<\ldots\\<i_k\le n}}\left(\prod\limits_{v=1}^{k}p_{\zeta_{i_v}}\right)\right\}, \\ \left\{ 1-\left(1-\prod\limits_{\substack{1\le i_1<\ldots\\<i_k\le n}}\left(1-\prod\limits_{v=1}^{k}(1-\vartheta_{i_v})\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \middle| \prod\limits_{\substack{1\le i_1<\ldots\\<i_k\le n}}\left(\prod\limits_{v=1}^{k}q_{\vartheta_{i_v}}\right)\right\} \end{array} \right) \tag{10.2}$$

Proof is given in Appendix 10.5.

Based on Theorem 10.2.1, we discuss some of its properties.

Property 10.2.1. (Monotonicity) Let $\mathcal{A}_i = (h_i|p_{h_i}, g_i|q_{g_i})$ and $\mathcal{A}_i^* = \left(h_i^*|p_{h_i^*}, g_i^*|q_{g_i^*}\right)$ be two families of PDHFEs, for $i = 1, 2, \ldots, n$, such that $\zeta_i \le \zeta_i^*$ and $\vartheta_i \ge \vartheta_i^*$ for all $\zeta_i \in h_i$, $\zeta_i^* \in h_i^*$, $\vartheta_i \in g_i$ and $\vartheta_i^* \in g_i^*$ while $p_{h_i} = p_{h_i^*}$, $q_{g_i} = q_{g_i^*}$, then

$$\text{PDHFMSMA}^{(k)}\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right) \le \text{PDHFMSMA}^{(k)}\left(\mathcal{A}_1^*,\mathcal{A}_2^*,\ldots,\mathcal{A}_n^*\right).$$

Proof is given in Appendix 10.5.

Property 10.2.2. (Boundedness) For "n" PDHFEs $\mathcal{A}_i = (h_i|p_{h_i}, g_i|q_{g_i})$, we have

$$\mathcal{A}^- \le \text{PDHFMSMA}^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \le \mathcal{A}^+$$

where $\mathcal{A}^- = \bigcup\limits_{\substack{\zeta_{\min}\in\min(h_i)\\ \vartheta_{\max}\in\max(g_i)}} \left(\left\{\zeta_{\min}|p_{\zeta_{\min}}\right\}, \left\{\vartheta_{\max}|q_{\vartheta_{\max}}\right\}\right)$ and
$\mathcal{A}^+ = \bigcup\limits_{\substack{\zeta_{\max}\in\max(h_i)\\ \vartheta_{\min}\in\min(g_i)}} \left(\left\{\zeta_{\max}|p_{\zeta_{\max}}\right\}, \left\{\vartheta_{\min}|q_{\vartheta_{\min}}\right\}\right)$ be PDHFEs.

Proof is given in Appendix 10.5.

Further, to study the monotonicity of the PDHFMSMA operator regarding the parameter k, we introduce two lemmas, which will be used in further notions.

Lemma 10.2.1. [118] Let $a_i > 0$ be n real numbers, then

$$\text{MSM}^{(1)}\left(a_1,a_2,\ldots,a_n\right) \ge \text{MSM}^{(2)}\left(a_1,a_2,\ldots,a_n\right) \ge \ldots \ge \text{MSM}^{(n)}\left(a_1,a_2,\ldots,a_n\right) \tag{10.3}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

Lemma 10.2.2. [118] Let $a_i \geq 0$ and $b_i > 0$ with $\sum_{i=1}^{n} b_i = 1$, then

$$\prod_{i=1}^{n} {a_i}^{b_i} \leq \sum_{i=1}^{n} a_i b_i \tag{10.4}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

Theorem 10.2.2. For "n" PDHFEs $\mathcal{A}_i$ having membership (non-membership) probabilities equal within themselves, and for $k = 1, 2, \ldots, n$, the PDHFMSMA is monotonically decreasing regarding the parameter k.

Proof is given in Appendix 10.5.

Next, we discuss some important cases of PDHFMSMA operator regarding the different values of k.

Case 1: If $k = 1$, then based on Definition 10.2.1, it reduces to PDHF averaging operator [61] as shown below:

$$\begin{aligned}
&\text{PDHFMSMA}^{(1)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\
&= \bigcup_{\substack{\zeta_{i_v} \in h_{i_v}, \\ \vartheta_{i_v} \in g_{i_v}}} \left(\begin{array}{l} \left\{ \left(1 - \prod\limits_{1 \leq i_1 \leq n} \left(1 - \prod\limits_{v=1}^{1} \zeta_{i_v}\right)^{\frac{1}{\binom{n}{1}}}\right)^{\frac{1}{1}} \middle| \prod\limits_{1 \leq i_1 \leq n} \left(\prod\limits_{v=1}^{1} p_{\zeta_{i_v}}\right) \right\}, \\ \left\{ 1 - \left(1 - \prod\limits_{1 \leq i_1 \leq n} \left(1 - \prod\limits_{v=1}^{1} (1 - \vartheta_{i_v})\right)^{\frac{1}{\binom{n}{1}}}\right)^{\frac{1}{1}} \middle| \prod\limits_{1 \leq i_1 \leq n} \left(\prod\limits_{v=1}^{1} q_{\vartheta_{i_v}}\right) \right\} \end{array} \right) \\
&= \bigcup_{\substack{\zeta_{i_1} \in h_{i_1}, \\ \vartheta_{i_1} \in g_{i_1}}} \left(\left\{ 1 - \prod_{i_1=1}^{n} (1 - \zeta_{i_1})^{\frac{1}{n}} \middle| \prod_{i_1=1}^{n} p_{\zeta_{i_1}} \right\}, \left\{ \prod_{i_1=1}^{n} (\vartheta_{i_1})^{\frac{1}{n}} \middle| \prod_{i_1=1}^{n} q_{\vartheta_{i_1}} \right\} \right)
\end{aligned}$$

Case 2: If $k = 2$, then based on Definition 10.2.1, it reduces to PDHF Bonferroni mean operator as shown below:

$$\begin{aligned}
&\text{PDHFMSMA}^{(2)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\
&= \bigcup_{\substack{\zeta_{i_v} \in h_{i_v}, \\ \vartheta_{i_v} \in g_{i_v}}} \left(\begin{array}{l} \left\{ \left(1 - \prod\limits_{1 \leq i_1 < i_2 \leq n} \left(1 - \prod\limits_{v=1}^{2} \zeta_{i_v}\right)^{\frac{1}{\binom{n}{2}}}\right)^{\frac{1}{2}} \middle| \prod\limits_{1 \leq i_1 < i_2 \leq n} \left(\prod\limits_{v=1}^{2} p_{\zeta_{i_v}}\right) \right\}, \\ \left\{ 1 - \left(1 - \prod\limits_{1 \leq i_1 < i_2 \leq n} \left(1 - \prod\limits_{v=1}^{2} (1 - \vartheta_{i_v})\right)^{\frac{1}{\binom{n}{2}}}\right)^{\frac{1}{2}} \middle| \prod\limits_{1 \leq i_1 < i_2 \leq n} \left(\prod\limits_{v=1}^{2} q_{\vartheta_{i_v}}\right) \right\} \end{array} \right)
\end{aligned}$$

$$= \bigcup_{\substack{\zeta_{i_v} \in h_{i_v}, \\ \vartheta_{i_v} \in g_{i_v}}} \left(\begin{array}{l} \left\{ \left(1 - \prod_{1 \le i_1 < i_2 \le n} (1 - \zeta_{i_1}\zeta_{i_2})^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}} \middle| \prod_{1 \le i_1 < i_2 \le n} \left(p_{\zeta_{i_1}} p_{\zeta_{i_2}} \right) \right\}, \\ \left\{ 1 - \left(1 - \prod_{1 \le i_1 < i_2 \le n} (1 - (1 - \vartheta_{i_1})(1 - \vartheta_{i_2}))^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}} \middle| \prod_{1 \le i_1 < i_2 \le n} \left(q_{\vartheta_{i_1}} q_{\vartheta_{i_2}} \right) \right\} \end{array} \right)$$

$$= \bigcup_{\substack{\zeta_{i_v} \in h_{i_v}, \\ \vartheta_{i_v} \in g_{i_v}}} \left(\begin{array}{l} \left\{ \left(1 - \prod_{\substack{i_1, i_2 = 1 \\ i_1 \neq i_2}}^{n} (1 - \zeta_{i_1}\zeta_{i_2})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \middle| \prod_{\substack{i_1, i_2 = 1 \\ i_1 \neq i_2}}^{n} \left(p_{\zeta_{i_1}} p_{\zeta_{i_2}} \right) \right\}, \\ \left\{ 1 - \left(1 - \prod_{\substack{i_1, i_2 = 1 \\ i_1 \neq i_2}}^{n} (1 - (1 - \vartheta_{i_1})(1 - \vartheta_{i_2}))^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \middle| \prod_{\substack{i_1, i_2 = 1 \\ i_1 \neq i_2}}^{n} \left(q_{\vartheta_{i_1}} q_{\vartheta_{i_2}} \right) \right\} \end{array} \right)$$

Case 3: If $k = n$, then based on Definition 10.2.1, it reduces to PDHF geometric mean operator as shown below:

$$\text{PDHFMSMA}^{(n)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

$$= \bigcup_{\substack{\zeta_{i_v} \in h_{i_v}, \\ \vartheta_{i_v} \in g_{i_v}}} \left(\begin{array}{l} \left\{ \left(1 - \prod_{\substack{1 \le i_1 < i_2 \\ \ldots < i_k \le n}} \left(1 - \prod_{v=1}^{n} \zeta_{i_v} \right)^{\frac{1}{\binom{n}{n}}} \right)^{\frac{1}{n}} \middle| \prod_{\substack{1 \le i_1 < i_2 \\ \ldots < i_k \le n}} \left(\prod_{v=1}^{n} p_{\zeta_{i_v}} \right) \right\}, \\ \left\{ 1 - \left(1 - \prod_{\substack{1 \le i_1 < i_2 \\ \ldots < i_k \le n}} \left(1 - \prod_{v=1}^{n} (1 - \vartheta_{i_v}) \right)^{\frac{1}{\binom{n}{n}}} \right)^{\frac{1}{n}} \middle| \prod_{\substack{1 \le i_1 < i_2 \\ \ldots < i_k \le n}} \left(\prod_{v=1}^{n} q_{\vartheta_{i_v}} \right) \right\} \end{array} \right)$$

$$= \bigcup_{\substack{\zeta_{i_v} \in h_{i_v}, \\ \vartheta_{i_v} \in g_{i_v}}} \left(\begin{array}{l} \left\{ \left(1 - \prod_{\substack{1 \le i_1 < i_2 \\ \ldots < i_k \le n}} \left(1 - \prod_{v=1}^{n} \zeta_{i_v} \right) \right)^{\frac{1}{n}} \middle| \prod_{\substack{1 \le i_1 < i_2 \\ \ldots < i_k \le n}} \left(\prod_{v=1}^{n} p_{\zeta_{i_v}} \right) \right\}, \\ \left\{ 1 - \left(1 - \prod_{\substack{1 \le i_1 < i_2 \\ \ldots < i_k \le n}} \left(1 - \prod_{v=1}^{n} (1 - \vartheta_{i_v}) \right) \right)^{\frac{1}{n}} \middle| \prod_{\substack{1 \le i_1 < i_2 \\ \ldots < i_k \le n}} \left(\prod_{v=1}^{n} q_{\vartheta_{i_v}} \right) \right\} \end{array} \right)$$

Definition 10.2.2. For a collection of PDHFEs $\mathcal{A}_i$, a mapping WPDHFMSMA : $\Phi^n \rightarrow \Phi$ defined as

$$\text{WPDHFMSMA}_{\omega}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\frac{\bigoplus_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(\bigotimes_{v=1}^{k} \omega_{i_v} \mathcal{A}_{i_v} \right)}{\binom{n}{k}} \right)^{\frac{1}{k}} \tag{10.5}$$

where ω_i represents the weight vector satisfying the property $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$, is known as weighted PDHFMSM averaging operator.

Theorem 10.2.3. For a family of PDHFEs $\mathcal{A}_i = (h_i|p_{h_i}, g_i|q_{g_i})$, the aggregated value using WPDHFMSMA operator is still a PDHFE and is given as

$$\text{WPDHFMSMA}_{\omega}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigcup_{\substack{\zeta_{i_v}\in h_{i_v},\\ \vartheta_{i_v}\in g_{i_v}}} \left(\begin{array}{l} \left\{ \left(1 - \left(\prod_{\substack{1\le i_1<\ldots\\<i_k\le n}} \left(1 - \prod_{v=1}^{k} \left(1-(1-\zeta_{i_v})^{\omega_{i_v}}\right) \right) \right)^{\frac{1}{\binom{n}{k}}} \right)^{\frac{1}{k}} \middle| \prod_{\substack{1\le i_1<\ldots\\<i_k\le n}} \left(\prod_{v=1}^{k} p_{\zeta_{i_v}} \right) \right\}, \\ \left\{ 1 - \left(1 - \left(\prod_{\substack{1\le i_1<\ldots\\<i_k\le n}} \left(1 - \prod_{v=1}^{k} \left(1-(\vartheta_{i_v})^{\omega_{i_v}}\right) \right) \right)^{\frac{1}{\binom{n}{k}}} \right)^{\frac{1}{k}} \middle| \prod_{\substack{1\le i_1<\ldots\\<i_k\le n}} \left(\prod_{v=1}^{k} q_{\zeta_{i_v}} \right) \right\} \end{array} \right) \tag{10.6}$$

Proof. Similar proof as Theorem 10.2.1. □

10.2.2 PDHFMSM geometric operator

Definition 10.2.3. For collection of "n" PDHFEs $\mathcal{A}_i$, a map PDHFMSMG: $\Phi^n \to \Phi$, stated as

$$\text{PDHFMSMG}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\frac{1}{k} \bigotimes_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(\bigoplus_{v=1}^{k} \mathcal{A}_{i_v} \right) \right)^{\binom{n}{k}} \tag{10.7}$$

is called as PDHF Maclaurin symmetric mean geometric operator.

Theorem 10.2.4. For a family of PDHFEs $\mathcal{A}_i = (h_i|p_{h_i}, g_i|q_{g_i})$, the aggregated value using PDHFMSMG operator is still a PDHFE and is given as

$$\text{PDHFMSMG}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigcup_{\substack{\zeta_{i_v}\in h_{i_v},\\ \vartheta_{i_v}\in g_{i_v}}} \left(\begin{array}{l} \left\{ 1 - \left(1 - \prod_{\substack{1\le i_1<\ldots\\<i_k\le n}} \left(1 - \prod_{v=1}^{k} (1-\zeta_{i_v}) \right)^{\frac{1}{\binom{n}{k}}} \right)^{\frac{1}{k}} \middle| \prod_{\substack{1\le i_1<\ldots\\<i_k\le n}} \left(\prod_{v=1}^{k} p_{\zeta_{i_v}} \right) \right\}, \\ \left\{ \left(1 - \prod_{\substack{1\le i_1<\ldots\\<i_k\le n}} \left(1 - \prod_{v=1}^{k} \vartheta_{i_v} \right)^{\frac{1}{\binom{n}{k}}} \right)^{\frac{1}{k}} \middle| \prod_{\substack{1\le i_1<\ldots\\<i_k\le n}} \left(\prod_{v=1}^{k} q_{\vartheta_{i_v}} \right) \right\} \end{array} \right) \tag{10.8}$$

Proof. Similar to proof of Theorem 10.2.1. □

The PDHFMSMG operator also meets the boundary and monotonicity qualities, which are asserted (without proof) as below:

Property 10.2.3. (Monotonicity) Let $\mathcal{A}_i = (h_i|p_{h_i}, g_i|q_{g_i})$ and $\mathcal{A}_i^* = \left(h_i^*|p_{h_i^*}, g_i^*|q_{g_i^*}\right)$ be two families of PDHFEs. Then $\zeta_i \leq \zeta_i^*$ and $\vartheta_i \geq \vartheta_i^*$ for all $\zeta_i \in h_i$, $\zeta_i^* \in h_i^*$, $\vartheta_i \in g_i$, $\vartheta_i^* \in g_i^*$, while $p_{h_i} = p_{h_i^*}$, $q_{g_i} = q_{g_i^*}$ for $i = 1, 2, \ldots, n$, then

$$\text{PDHFMSMG}^{(k)}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) \leq \text{PDHFMSMG}^{(k)}\left(\mathcal{A}_1^*, \mathcal{A}_2^*, \ldots, \mathcal{A}_n^*\right).$$

Property 10.2.4. (Boundedness) For PDHFEs $\mathcal{A}_i = (h_i|p_{h_i}, g_i|q_{g_i})$ we have

$$\mathcal{A}^- \leq \text{PDHFMSMG}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$$

where $\mathcal{A}^- = \bigcup\limits_{\substack{\zeta_{\min}\in\min(h_i)\\ \vartheta_{\max}\in\max(g_i)}} \begin{pmatrix} \{\zeta_{\min}|p_{\zeta_{\min}}\}, \\ \{\vartheta_{\max}|q_{\vartheta_{\max}}\} \end{pmatrix}$ and $\mathcal{A}^+ = \bigcup\limits_{\substack{\zeta_{\max}\in\max(h_i)\\ \vartheta_{\min}\in\min(g_i)}} \begin{pmatrix} \{\zeta_{\max}|p_{\zeta_{\max}}\}, \\ \{\vartheta_{\min}|q_{\vartheta_{\min}}\} \end{pmatrix}$ be PDHFEs.

Definition 10.2.4. For collection of PDHFEs $\mathcal{A}_i = (h_i|p_{h_i}, g_i|q_{g_i})$, a mapping PDHFMSMG: $\Phi^n \to \Phi$, defined by

$$\text{WPDHFMSMG}_{\omega}^{(k)}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \left(\frac{1}{k}\bigotimes_{\substack{1\leq i_1<\\ \ldots<i_k\leq n}}\left(\bigoplus_{v=1}^{k}\left(\mathcal{A}_{i_v}\right)^{\omega_{i_v}}\right)\right)^{\binom{n}{k}}$$

$$= \bigcup_{\substack{\zeta_{i_v}\in h_{i_v},\\ \vartheta_{i_v}\in g_{i_v}}} \left(\begin{array}{c} \left\{ 1-\left(1-\left(\prod\limits_{\substack{1\leq i_1<\ldots\\ <i_k\leq n}}\left(1-\prod\limits_{v=1}^{k}\left(1-(\zeta_{i_v})^{\omega_{i_v}}\right)\right)\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \Bigg| \prod\limits_{\substack{1\leq i_1<\ldots\\ <i_k\leq n}}\left(\prod\limits_{v=1}^{k} p_{\zeta_{i_v}}\right) \right\}, \\ \left\{ \left(1-\left(\prod\limits_{\substack{1\leq i_1<\ldots\\ <i_k\leq n}}\left(1-\prod\limits_{v=1}^{k}\left(1-(1-\vartheta_{i_v})^{\omega_{i_v}}\right)\right)\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \Bigg| \prod\limits_{\substack{1\leq i_1<\ldots\\ <i_k\leq n}}\left(\prod\limits_{v=1}^{k} q_{\vartheta_{i_v}}\right) \right\} \end{array} \right) \quad (10.9)$$

is called as Weighted PDHFMSM geometric operator.

10.2.3 Method for probability determination in PDHFSs

In the practical problems, the probabilities associated with the membership and non-membership values of a PDHFE are difficult to determine. This section focusses on developing an optimization model to serve the purpose of probability determination. Because of the subjective judgments of the decision-makers, the occurrence and non-occurrence probabilities are hard to drive along with the truth and falsity values simultaneously. To plan the optimization model for determining the probabilities of elements in a PDHFE $\mathcal{A} = (h_{\mathcal{A}} \mid p_{\mathcal{A}}, g_{\mathcal{A}} \mid q_{\mathcal{A}}) = \bigcup\limits_{\zeta_s\in h_{\mathcal{A}},\vartheta_t\in g_{\mathcal{A}}} (\{\zeta_s \mid p_{\zeta_s}\}, \{\vartheta_t \mid q_{\vartheta_t}\})$ where

$s = 1, 2, \ldots, M_{\mathcal{A}}; t = 1, 2, \ldots, N_{\mathcal{A}}$, we extended the maximum entropy principle to accommodate PDHF information by adding the score function into it. The larger the expected value of PDHFE, the better is PDHFE. Also, more the value of score function, more optimistic behavior is adopted by the expert. Therefore, considering these features, a mathematical programming model is constructed as

$$\begin{cases} \max f(p,q) = \sum\limits_{i=1}^{M_{\max}} \Big(-p_i \ln(p_i) - q_i \ln(q_i) + \zeta_i p_i - \vartheta_i q_i \Big) \\ s.t. \quad 0 \le p_i \le 1 \quad ; \quad 0 \le q_i \le 1, \qquad \text{where} \quad i = 1, 2, \ldots, M_{\max} \\ \quad \sum\limits_{i=1}^{M_{\max}} p_i = 1 \quad ; \quad \sum\limits_{i=1}^{M_{\max}} q_i = 1; \end{cases} \tag{10.10}$$

where $M_{\max} = \max\{M_{\mathcal{A}}, N_{\mathcal{A}}\}$, and $M_{\mathcal{A}}$, $N_{\mathcal{A}}$ are the total hesitant membership and non-membership values in $\mathcal{A}$, respectively.

Theorem 10.2.5. The optimal solution of the mathematical model (10.10) exists.

Proof. The function $f(p,q)$ in model (10.10) is bounded over the domain $\Gamma = \{p_i, q_i \mid 0 \le p_i \le 1; 0 \le q_i \le 1, i = 1, 2, \ldots, M_{\max}, \sum\limits_{i=1}^{M_{\max}} p_i = 1; \sum\limits_{i=1}^{M_{\max}} q_i = 1\}$. Further, the function f is continuous on Γ and hence there exists minimum and maximum values of f over Γ. Also, it is clearly seen that the feasible region Γ is non-empty convex set. The model (10.10) can be transformed into the following form:

$$\begin{cases} \min f(p,q) = \sum\limits_{i=1}^{M_{\max}} (p_i \ln(p_i) + q_i \ln(q_i) - \zeta_i p_i + \vartheta_i q_i) \\ s.t. \quad p_i \ge 0; 1 - p_i \ge 0, q_i \ge 0, 1 - q_i \ge 0, \\ \quad \sum\limits_{i=1}^{M_{\max}} p_i = 1; \sum\limits_{i=1}^{M_{\max}} q_i = 1, \\ i = 1, 2, \ldots, M_{\max} \end{cases} \tag{10.11}$$

which implies that it is a convex programming model and the feasible region Γ is convex set. Moreover, the function $f(p,q)$ is strictly convex and hence every local solution of model (10.11) is also a global optimal solution. Therefore, the model (10.11) has the optimal solution and exist in nature, which complete the proof. □

To draw the functioning of this model, we supply a numerical case below.

Example 10.2.1. Let $\mathcal{A} = (h|p_h, g|q_g) = (\{0.2|p_1, 0.3|p_2\}, \{0.2|q_1, 0.4|q_2\})$ be a PDHFE whose probabilities are to be determined. The optimization model Eq. (10.10) is constructed for this PDHFE and get

$$\begin{aligned} \max f(p_1, p_2, q_1, q_2) &= \left(\begin{array}{l} -p_1 \ln(p_1) - p_2 \ln(p_2) - q_1 \ln(q_1) - q_2 \ln(q_2) \\ +0.2p_1 + 0.3p_2 - 0.2q_1 - 0.4q_2 \end{array} \right) \\ s.t. \qquad & p_1 + p_2 = 1 \quad ; \quad q_1 + q_2 = 1 \quad ; \\ & 0 \leq p_1, p_2 \leq 1 \quad ; \quad 0 \leq q_1, q_2 \leq 1 \end{aligned}$$

After solving this model by mathematical software namely Maple, we obtain $p_1 = 0.4750$, $p_2 = 0.5250$, $q_1 = 0.4493$ and $q_2 = 0.5507$. Thus, the PDHFE becomes $\mathcal{A} = (\{0.2|0.4750, 0.3|0.5250\}, \{0.2|0.4493, 0.4|0.5507\})$.

In the DM process, an authority may contribute their erudition either in terms of DHFSs or PDHFSs. So to combine their conditions into the PDHFSs, we charge the probabilities to every component and then aggregate their values as per the steps specified in Algorithm 9.1 of Chapter 9.

10.3 Decision making approach using the proposed operators

In this section, a DM approach by using the above-defined operators for PDHFEs has been presented followed by an illustrative example and comparison study.

10.3.1 Proposed approach

The general description of MCDM problem is same as Section 2.5 of Chapter 2, where weights $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j > 0$ and $\sum\limits_{j=1}^{n} \omega_j = 1$. These given alternatives are evaluated by the "d" experts and give their preferences in terms of PDHFEs which are summarized as

$$\mathcal{V}_i^{(d)} = \left\{ \left(\mathfrak{B}_1, \mathcal{A}_{i1}^{(d)}\right), \left(\mathfrak{B}_2, \mathcal{A}_{i2}^{(d)}\right), \ldots, \left(\mathfrak{B}_n, \mathcal{A}_{in}^{(d)}\right) \right\}, \tag{10.12}$$

where $\mathcal{A}_{ij}^{(d)} = \left(h_{ij}^{(d)} \middle| p_{ij}^{(d)}, g_{ij}^{(d)} \middle| q_{ij}^{(d)}\right)$ represent the PDHFE, for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. To get the most suitable alternative(s) for a problem, the DM procedure is compiled by using proposed AOs in the following steps:

Step 1: Arrange the information of each expert using PDHFEs in matrices $\mathcal{M}^{(d)}$ as

$$\mathcal{M}^{(d)} = \begin{matrix} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{matrix} \begin{matrix} \mathfrak{B}_1 & \mathfrak{B}_2 & \cdots & \mathfrak{B}_n \\ \left(\begin{matrix} \left(h_{11}^{(d)} \middle| p_{11}^{(d)}, g_{11}^{(d)} \middle| q_{11}^{(d)}\right) \\ \left(h_{21}^{(d)} \middle| p_{21}^{(d)}, g_{21}^{(d)} \middle| q_{21}^{(d)}\right) \\ \vdots \\ \left(h_{m1}^{(d)} \middle| p_{m1}^{(d)}, g_{m1}^{(d)} \middle| q_{m1}^{(d)}\right) \end{matrix}\right. & \begin{matrix} \left(h_{12}^{(d)} \middle| p_{12}^{(d)}, g_{12}^{(d)} \middle| q_{12}^{(d)}\right) \\ \left(h_{22}^{(d)} \middle| p_{22}^{(d)}, g_{22}^{(d)} \middle| q_{22}^{(d)}\right) \\ \vdots \\ \left(h_{m2}^{(d)} \middle| p_{m2}^{(d)}, g_{m2}^{(d)} \middle| q_{m2}^{(d)}\right) \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \ddots \\ \cdots \end{matrix} & \left.\begin{matrix} \left(h_{1n}^{(d)} \middle| p_{1n}^{(d)}, g_{1n}^{(d)} \middle| q_{1n}^{(d)}\right) \\ \left(h_{2n}^{(d)} \middle| p_{2n}^{(d)}, g_{2n}^{(d)} \middle| q_{2n}^{(d)}\right) \\ \vdots \\ \left(h_{mn}^{(d)} \middle| p_{mn}^{(d)}, g_{mn}^{(d)} \middle| q_{mn}^{(d)}\right) \end{matrix}\right) \end{matrix}$$

where $\left(h_{ij}^{(d)} \middle| p_{ij}^{(d)}, g_{ij}^{(d)} \middle| q_{ij}^{(d)}\right) = \bigcup_{\substack{\zeta_{ij,s} \in h_{ij} \\ \vartheta_{ij,t} \in g_{ij}}} \left(\left\{\zeta_{ij,s}^{(d)} \middle| p_{\zeta_{ij,s}}^{(d)}\right\}, \left\{\vartheta_{ij,t}^{(d)} \middle| q_{\vartheta_{ij,t}}^{(d)}\right\}\right)$ for $s = 1, 2, \ldots, M_{ij}^{(d)}$, $t = 1, 2, \ldots, N_{ij}^{(d)}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 2: Construct the nonlinear optimization model, to determine the probabilities of PDHFEs, as

$$\begin{aligned} \max f(p_{ij}^{(d)}, q_{ij}^{(d)}) &= \sum_{s=1}^{M_{\max}} \left(-p_{\zeta_{ij,s}}^{(d)} \ln(p_{\zeta_{ij,s}}^{(d)}) - q_{\vartheta_{ij,s}}^{(d)} \ln(q_{\vartheta_{ij,s}^{(d)}}^{(d)})\right) + \zeta_{ij,s}^{(d)} p_{\zeta_{ij,s}}^{(d)} - \vartheta_{ij,s}^{(d)} q_{\vartheta_{ij,s}}^{(d)} \\ s.t. \quad & 0 \leq p_{\zeta_{ij,s}}^{(d)} \leq 1, \quad 0 \leq q_{\vartheta_{ij,s}}^{(d)} \leq 1 \\ & \sum_{s=1}^{M_{\max}} p_{\zeta_{ij,s}}^{(d)} = 1, \quad \sum_{s=1}^{M_{\max}} q_{\vartheta_{ij,s}}^{(d)} = 1 \end{aligned}$$

where $M_{\max} = \max\left\{M_{ij}^{(d)}, N_{ij}^{(d)}\right\}$. After solving this model, we get $p_{ij}^{(d)}$ and $q_{ij}^{(d)}$ for each PDHFE.

Step 3: If $d = 1$, then $\left(h_{ij}^{(d)} \middle| p_{ij}^{(d)}, g_{ij}^{(d)} \middle| q_{ij}^{(d)}\right) = \left(h_{ij} \middle| p_{ij}, g_{ij} \middle| q_{ij}\right)$. If $d \geq 2$, then aggregate their values $\mathcal{M}^{(d)}$ using Algorithm 9.1 of Chapter 9 to $\mathcal{M} = (\mathcal{A}_{ij})$ as

$$\mathcal{M} = \begin{matrix} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{matrix} \begin{matrix} \mathfrak{B}_1 & \mathfrak{B}_2 & \cdots & \mathfrak{B}_n \\ \left(\begin{matrix} (h_{11} | p_{11}, g_{11} | q_{11}) \\ (h_{21} | p_{21}, g_{21} | q_{21}) \\ \vdots \\ (h_{m1} | p_{m1}, g_{m1} | q_{m1}) \end{matrix}\right. & \begin{matrix} (h_{12} | p_{12}, g_{12} | q_{12}) \\ (h_{22} | p_{22}, g_{22} | q_{22}) \\ \vdots \\ (h_{m2} | p_{m2}, g_{m2} | q_{m2}) \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \ddots \\ \cdots \end{matrix} & \left.\begin{matrix} (h_{1n} | p_{1n}, g_{1n} | q_{1n}) \\ (h_{2n} | p_{2n}, g_{2n} | q_{2n}) \\ \vdots \\ (h_{mn} | p_{mn}, g_{mn} | q_{mn}) \end{matrix}\right) \end{matrix}$$

where $\mathcal{A}_{ij} = \left(h_{ij}|p_{ij}, g_{ij}|q_{ij}\right) = \bigcup\limits_{\zeta_{ij,s}\in h_{ij},\vartheta_{ij,t}\in g_{ij}} \left(\{\zeta_{ij,s}|p_{\zeta_{ij,s}}\}, \{\vartheta_{ij,t}|q_{\vartheta_{ij,t}}\}\right)$ for $s = 1, 2, \ldots, M_{ij}$, $t = 1, 2, \ldots, N_{ij}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 4: Compute the overall aggregated values '$r_i = (h_i|p_i, g_i|q_i)$' of alternatives using either WPDHFMSMA or WPDHFMSMG operators as given in Eqs. (10.13) and (10.14).

$$r_i = \text{WPDHFMSMA}_{\omega}^{(k)}\left(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{ij}\right) \tag{10.13}$$

$$= \bigcup_{\substack{\zeta_{ij,s_v}\in h_{ij_v},\\ \vartheta_{ij,t_v}\in g_{ij_v}}} \left(\begin{array}{l} \left\{ \left(1 - \left(\prod\limits_{\substack{1\le ij_1<\ldots\\<ij_k\le n}} \left(1 - \prod\limits_{v=1}^{k}\left(1-(1-\zeta_{ij,s_v})^{\omega_{ij_v}}\right)\right)\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \Bigg| \prod\limits_{\substack{1\le ij_1<\ldots\\<ij_k\le n}} \left(\prod\limits_{v=1}^{k} p_{\zeta_{ij,s_v}}\right) \right\}, \\ \left\{ 1 - \left(1 - \left(\prod\limits_{\substack{1\le ij_1<\ldots\\<ij_k\le n}} \left(1 - \prod\limits_{v=1}^{k}\left(1-(\vartheta_{ij,t_v})^{\omega_{ij_v}}\right)\right)\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \Bigg| \prod\limits_{\substack{1\le ij_1<\ldots\\<ij_k\le n}} \left(\prod\limits_{v=1}^{k} q_{\zeta_{ij,t_v}}\right) \right\} \end{array} \right)$$

or

$$r_i = \text{WPDHFMSMG}_{\omega}^{(k)}\left(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{ij}\right) \tag{10.14}$$

$$= \bigcup_{\substack{\zeta_{ij,s_v}\in h_{ij_v},\\ \vartheta_{ij,t_v}\in g_{ij_v}}} \left(\begin{array}{l} \left\{ 1 - \left(1 - \left(\prod\limits_{\substack{1\le ij_1<\ldots\\<ij_k\le n}} \left(1 - \prod\limits_{v=1}^{k}\left(1-(\zeta_{ij,s_v})^{\omega_{ij_v}}\right)\right)\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \Bigg| \prod\limits_{\substack{1\le ij_1<\ldots\\<ij_k\le n}} \left(\prod\limits_{v=1}^{k} p_{\zeta_{ij,s_v}}\right) \right\}, \\ \left\{ \left(1 - \left(\prod\limits_{\substack{1\le ij_1<\ldots\\<ij_k\le n}} \left(1 - \prod\limits_{v=1}^{k}\left(1-(1-\vartheta_{ij,t_v})^{\omega_{ij_v}}\right)\right)\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \Bigg| \prod\limits_{\substack{1\le ij_1<\ldots\\<ij_k\le n}} \left(\prod\limits_{v=1}^{k} q_{\vartheta_{ij,t_v}}\right) \right\} \end{array} \right)$$

Step 5: Compute the score value of $r_i = (h_i|p_i, g_i|q_i)$ as given in Definition 2.1.13 of Chapter 2 and hence rank them.

10.3.2 Illustrative Example

To demonstrate the above-stated method, we take an example based on medical diagnosis in tackling with the brain hemorrhage patients under uncertain medical situations.

Advancements in the medical study have shown progressive improvement in the last few decades. Many new therapies, advanced treatment techniques using the ultra-modern equipment are made available to the patients. Every treatment begins with the doctor's interrogating session with the patient, in which the doctor tries to capture the first-hand information about the symptoms of the patent by directly asking some disease-related questions from him. Under normal circumstances, the patient can answer the doctor's questions efficiently. But, if the patient suffers from such a disease that he is speech-deprived, then it becomes difficult for the doctor to judge the symptoms. The present case study focusses on quantifying the patient's gestures related to the doctor's queries. One such disease in which the patient might not answer the doctor's queries by speaking is hemorrhage strokes. Hemorrhagic stroke occurs when blood from an artery bleeds into the brain. This leads to an emergent pressure onto the brain cells and weakens them. The blood vessel bursts into the brain and the surrounded brain area got severely damaged. The two main types of hemorrhagic strokes are Intracerebral hemorrhage and Subarachnoid Hemorrhage.

(i) **Intracerebral hemorrhage:** This hemorrhage occurs when the blood vessel inside the brain itself got burst and become a reason for the stroke to occur. This kind of stroke can mostly occur in a person because of high blood pressure, heavy alcohol use, drugs or amphetamines. This can also be because of a weak blood vessel inside the brain since birth. The infected blood vessel causes severe stroke and dangerous consequences may be encountered.

(ii) **Subarachnoid hemorrhage:** This kind of hemorrhage occurs not inside the brain itself. It is the result of a clot in the brain nerves by excess blood disposition from a damaged blood vessel in any other part of the body. This hemorrhage takes time to occur as the blood slowly gathers inside the brain and mixes with the cerebrospinal fluid, resulting in the formation of a cushioned layers inside the brain and spinal cord. In a few days, following the vessel rupture, the clots begin to form inside the brain and cut the reach of oxygen to that particular area. This gradually leads to brain hemorrhage. In early symptoms, the patient feels severe headache and may be followed by other symptoms such as vomiting, difficulty in walking, less working of the sensory organs, etc.

The analysis is carried on four speech deprived patient's suffering from a brain hemorrhage. Since the patients cannot answer the doctor's questions by speaking, so there are chances that the doctor might not fully understand the symptoms. There is also possibility that the patient's gesture interpretation by the doctor is wrong. Analyzing a patient's case-study the doctor might guess the symptoms all by himself. It is also possible that the patient might express opposite symptoms what the doctor is expecting from his experience. In such cases, we make use of PDHFSs to capture the information provided by the patient. To keep the uniformity in capturing the patient's responses, we introduce a linguistic scale to determine the preference values as given below:

Rating values	Meaning
0	A clear denial (ACD)
0.2	Denial with a head node and a hand gesture (DHH)
0.4	Slight denial (SD)
0.6	Slight approval (SA)
0.8	Approval with a head node and a hand gesture (AHH)
1	A clear approval (ACA)

As shown in the symptom capturing scale, it is clear that the respective preference values stand for the corresponding listed gesture. However, if the doctor is confused about judging on these aspects clearly, he can record the response in the form of any real number lying between the given points. In case the doctor is hesitant about recording response in the form of anyone real number between 0 and 1, he can consider the multiple values as per his convenience. Four patients $\mathcal{V}_i$'s, $(i = 1, 2, 3, 4)$ are to be examined by the doctor with respect to three symptoms namely, $\mathfrak{B}_1$: 'Headache' ; $\mathfrak{B}_2$: 'Vomiting' and $\mathfrak{B}_3$: 'Irregular breathing' so that proper medication can be provided to them classified under criteria weight $\omega = (0.18, 0.35, 0.47)^T$. The target of the problem is to figure out the patient which is more prone to the classified criteria so that their medication can be started accordingly. Then, for it, the following steps of the proposed method are executed as below.

Step 1: The rating values given by the two doctors towards the evaluation of four patients under each symptom. The results of them are given in Tables 10.1 and 10.2 under PDHFS environment.

Step 2: To determine the unknown probabilities, we construct the optimization model (10.10) for each pair of patients with symptom and every doctor. For example, the model for the probability determination corresponding to $\mathcal{V}_1$ under $\mathfrak{B}_1$ in case of first doctor is constructed as

$$\max f\left(p_1,p_2,q_1,q_2\right) = \begin{pmatrix} -p_1\ln(p_1)-p_2\ln(p_2)-q_1\ln(q_1)-q_2\ln(q_2) \\ +0.3p_1+0.6p_2-0.2q_1-0.3q_2 \end{pmatrix}$$

$$s.t. \quad 0\leq p_1,p_2\leq 1 \quad ; \quad 0\leq q_1,q_2\leq 1,$$

$$p_1+p_2=1 \quad ; \quad q_1+q_2=1$$

After solving this model by Maple software, we obtain $p_1 = 0.4256$, $p_2 = 0.5744$, $q_1 = 0.5250$ and $q_2 = 0.4750$. Thus, the PDHFE corresponding to $\mathcal{V}_1$ and $\mathfrak{B}_1$ by first doctor becomes ($\{0.3|0.4256, 0.6|0.5744\}$, $\{0.2|0.5250, 0.3|0.4750\}$). Similarly, for all other pairs of each doctor, we compute the respective probabilities and results are summarized in Tables 10.3 and 10.4.

Step 3: As the number of doctors is more than one, and thus by Algorithm 9.1 of Chapter 9, the collective PDHFEs are gathered and compiled in Table 10.5.

Step 4: Without loss of generality, we take $k = 2$ and WPDHFMSMA operator as given in Eq. (10.13), to get the collective information r_i, $(i = 1, 2, 3, 4)$ as

$$r_1 = \left(\left\{ \begin{array}{l} 0.1226|0.0001, 0.1269|0.0001, \\ 0.1275|0.0001, 0.1316|0.0003, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.2470|0.0016, 0.2529|0.0032 \end{array} \right\}, \left\{ \begin{array}{l} 0.5975|0.0110, 0.6041|0.0011, \\ 0.6049|0.0034, 0.6050|0.0007, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.6997|0.0032, 0.7071|0.0032 \end{array} \right\} \right)$$

$$r_2 = \left(\left\{ \begin{array}{l} 0.1012|0.0025, 0.1059|0.0011, \\ 0.1068|0.0007, 0.1113|0.0007, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.2598|0.0036, 0.2695|0.0036 \end{array} \right\}, \left\{ \begin{array}{l} 0.5629|0.0156, 0.5712|0.156, \\ 0.5727|0.0084, 0.5810|0.0084, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.7344|0.0032, 0.7394|0.0032 \end{array} \right\} \right)$$

$$r_3 = \left(\left\{ \begin{array}{l} 0.0334|0.0127, 0.0368|0.0074, \\ 0.0369|0.0126, 0.0380|0.0126, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.1465|0.0012, 0.1486|0.0012 \end{array} \right\}, \left\{ \begin{array}{l} 0.6324|0.0012, 0.6420|0.0034, \\ 0.6506|0.0034, 0.6507|0.0030, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.8118|0.016, 0.8156|0.0036 \end{array} \right\} \right)$$

$$r_4 = \left(\left\{ \begin{array}{l} 0.1407|0.0007, 0.1425|0.0015, \\ 0.1451|0.0015, 0.1464|0.0008, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.2460|0.0192, 0.2523|0.0112 \end{array} \right\}, \left\{ \begin{array}{l} 0.4867|0.0043, 0.4989|0.0044, \\ 0.5009|0.0039, 0.5045|0.0039, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.6731|0.0040, 0.6812|0.0036 \end{array} \right\} \right)$$

On the other hand, if we take WPDHFMSMG operator, then these values are computed as

$$r_1 = \left(\left\{ \begin{array}{l} 0.0638|0.0110, 0.0663|0.0107, \\ 0.0666|0.0034, 0.0684|0.011, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.1187|0.0032, 0.1226|0.0032 \end{array} \right\}, \left\{ \begin{array}{l} 0.7071|0.0001, 0.7135|0.0001, \\ 0.7163|0.0001, 0.7189|0.0001, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.8443|0.0016, 0.8492|0.0032 \end{array} \right\} \right)$$

$$r_2 = \left(\left\{ \begin{array}{l} 0.0541|0.0156, 0.0556|0.0084, \\ 0.0567|0.0156, 0.0571|0.0084, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.1097|0.0032, 0.1124|0.0032 \end{array} \right\}, \left\{ \begin{array}{l} 0.6730|0.0025, 0.6787|0.0011, \\ 0.6822|0.0011, 0.6824|0.0012, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.8518|0.0036, 0.8547|0.0036 \end{array} \right\} \right)$$

$$r_3 = \left(\left\{ \begin{array}{l} 0.0830|0.0012, 0.0862|0.0034, \\ 0.0863|0.0011, 0.0900|0.0023, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.1975|0.016, 0.2000|0.0336 \end{array} \right\}, \left\{ \begin{array}{l} 0.4897|0.0127, 0.4989|0.0074, \\ 0.5009|0.0126, 0.5099|0.0074, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.7290|0.0012, 0.7366|0.0012 \end{array} \right\} \right)$$

$$r_4 = \left(\left\{ \begin{array}{l} 0.0334|0.0043, 0.0368|0.0044, \\ 0.0369|0.0039, 0.0380|0.0039, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.1038|0.0016, 0.1065|0.0036 \end{array} \right\}, \left\{ \begin{array}{l} 0.7431|0.0007, 0.7450|0.0015, \\ 0.7485|0.0008, 0.7489|0.0015, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ 0.8463|0.0192, 0.8500|0.0112 \end{array} \right\} \right)$$

Step 5: The score estimation of each $\mathcal{V}_i (i = 1, 2, 3, 4)$ derived from the overall evaluated values $r_i, i = 1, 2, 3, 4$ are: $\mathcal{S}c(r_1) = -0.4454$, $\mathcal{S}c(r_2) = -0.4273$, $\mathcal{S}c(r_3) = -0.5861$, $\mathcal{S}c(r_4) = -0.3835$ while the score estimations of each $\mathcal{V}_i (i = 1, 2, 3, 4)$ based on the aggregated values obtained from WPDHFMSMG operator are: $\mathcal{S}c(r_1) = 0.7110$, $\mathcal{S}c(r_2) = 0.7007$, $\mathcal{S}c(r_3) = 0.5011$, $\mathcal{S}c(r_4) = 0.7379$. Thus, based on these score values, we obtain that the ranking order is $\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$ by WPDHFMSMA operator while $\mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3$ by WPDHFMSMG operator.

From the above-ranking order, we conclude that the most infest patient is $\mathcal{V}_4$ than others by using both the AOs and thus the treatment method should include a more an emergent focus of patient $\mathcal{V}_4$. The relation between the patients $\mathcal{V}_1$ and $\mathcal{V}_2$ is selected according to the risk aversive and the risk preferable states will give variable results in accordance with the decision maker's attitude. As the most infected person remains the same by the operators but the influence of the proposed AOs is different. For instance, if the decision-maker wants to pay more attention to the degrees of membership than non-membership than they can prefer to choose either PDHFMSMA or WPDHFMSMA operators. If a person pays more attention to the degrees of rejection then they can choose PDHFMSMG and WPDHFMSMG operators accordingly to the weight assigned to the different criteria or parameters or not. Further, to see the influence of the parameters k on to the four proposed AOs, we perform an experiment which is described in the next section.

10.3.3 Sensitivity analysis with respect to parameter k

To observe the impact of parameters k on the DM process, an investigation has been made by considering distinct values of k. Based on it, the change in corresponding score values regarding changes in the parameter value is shown in Table 10.6. From this table, it is observed that the optimal alternative concluded by changing k values in both WPDHFMSMA and WPDHFMSMG operators remains unchanged. This shows the robustness of the proposed operators regarding the change in the value of k. However, a notable change in the corresponding score values can be seen. This shows that with an increase in the parameter's value, the scores got by WPDHFMSMA operator gradually decreases. In contrast to it, the values got using WPDHFMSMG operator by increasing the value of k increases. This reflects the decision maker's risk-bearing attitude. More the value of parameter k in WPDHFMSMA, more risk aversive the expert is and vice versa. The choice of parameter k helps us to capture the inter-relationship among the PDHFEs. For instance, if the doctor has taken $k = 1$, then he is analyzing the aggregated value only w.r.t., individual criteria i.e., he is not considering the effect of criterion $\mathfrak{B}_1$ on either of $\mathfrak{B}_2$ or $\mathfrak{B}_3$. However, if the doctor is taking the value of $k = 2$, then he is considering the effect of criterion $\mathfrak{B}_1$ on both $\mathfrak{B}_2$ and $\mathfrak{B}_3$, and effect of $\mathfrak{B}_2$ w.r.t., $\mathfrak{B}_3$ too. Thus, more the value of k be taken (up to n), more inter-relationships among the criteria the decision-maker is capturing.

10.3.4 Comparative studies

To study the proposed approach's nature in contrast to the existing methodologies [51, 61, 226], a comparison analysis is summarized in Table 10.7. In this analysis, the operators considered from the existing theories are PDHFWA ("probabilistic dual hesitant fuzzy weighted average") by Hao et al. [61], PDHFWEA("probabilistic dual hesitant fuzzy weighted Einstein averaging") and PDHFWEG ("probabilistic dual hesitant fuzzy weighted Einstein geometric") operators by Garg and Kaur [51] and weighted dual hesitant fuzzy MSM average and geometric (WDHFMSMA and WDHFMSMG) by Zhang [226]. These existing studies address the DM problems using different algorithmic approaches based on PDHFSs and DHFSs. The best alternative on comparing with all these approaches remains the same as that of the proposed approach's outlined results. The helpful nature of the proposed approach is inferred from its firm behavior regarding the best alternative regarding all the variable parameters.

10.3.5 Managerial implications

Under the DM scenarios, the existing approaches [51, 61, 226] can also address the same medical problem but with different extent of data coverage as compared to our approach. So, to statistically strengthen the enhanced working efficiency of our approach compared to the existing theories, the complexities and inputs details regarding the prevailing studies are outlined below:

1) The methodologies proposed by Hao et al. [61], Garg and Kaur [51] are based on PDHFSs. The former focussed on the information fusion using AOs based on the algebraic norm and the later highlighted the information fusion based on Einstein norm. These theories require input from the DM in the form of PDHFEs, but in the

working atmosphere, it is not always possible to attain information from the respondent in the desired form. For instance, in the considered case-study the symptom-details from a patient can be easily gained by the doctor using the linguistic scale, but the doctor can find it difficult to assign the probabilities in the mean-time to reform the input into the desired format of PDHFS. Thus, in our approach, we propose a non-linear method to determine the probabilities so that the data can be gained easily and then can be easily processed into the PDHFS.

2) On the other hand, the approach proposed by Zhang [226] is based on DHFSs and aggregates the input values using the MSM operator. Although their proposed MSM operator possesses the generalized advantages as that of PDHFMSMA operator, the considered environment plays a dominant role in different working capabilities of both the approaches. PDHFS has a striking feature to get reduced to DHFS by taking all the probabilities associated the membership and non-membership equal within themselves. This increase the superiority of our approach over the existing one, because the proposed methodology can model the information present in DHFS also. But the vice-versa information modeling is not possible. Moreover, focus on the aspect of input requirement by both the approaches, it is analyzed that the information that can be fed to the theory proposed by Zhang [226] must be in the form of DHFS. Thus, the prospective situation of variable priority of different hesitant values is not taken into account. But, the proposed method gives the decision maker a facility to capture the uncertain information more efficiently.

Further, a deep insight into the comparison is taken by comparing the characteristics of the proposed approach with that of the existing ones. As shown in Table 10.8, the symbol '✓' describes that the corresponding DM approach considers "over one decision-maker, handles probabilities, accounts for non-membership entities, has weights derived by the non-linear approach" and is a generalization over the existing approaches, whereas the symbol '×' means that "the associated method fails". It can be seen that the approach outlined by Hao et al. [61] does not consider the weights derived from a non-linear optimization model. However, the approach proposed by Garg and Kaur [51] is efficient in all the considered aspects to expect that it cannot be particularized to get any of the existing approaches which can be done by fixing parameter value in our approach and it does not account for probability determination using the non-linear optimization model. The theory outlined by Zhang [226] does not consider multi-expert problems and is not analyzing the associated probabilities.

10.4 Conclusion

The key contribution of the work can be summarized below.

1) PDHFSs is an exceptional DHFS that encapsulates the membership and the non-membership hesitant values with the associated probabilities and can relate closely towards the practical DM scenarios by efficiently handling the vagueness and uncertainty. Also, several existing sets such as HFSs, DHFSs, PHFSs are special cases of the PDHFSs. This makes it a more generalized and strong idea for controlling the ambiguities with both stochastic and fuzzy characteristics.

2) By taking the advantages of PDHFSs and to address the growing need of gesture quantification, we have focussed on proposing a quantifying scale based on the signs or actions shown by those patients who are speech deprived and cannot communicate their symptoms to the doctors directly.

3) A concept of MSM operators is formed to aggregate the different preferences of the experts in terms of PDHFSs, to capture the inter-relationship among all the input arguments. For it, we have defined average and geometric MSM (PDHFMSMA & PDHFMSMG) operators along with the weighted ones. The relevant properties of such operators are also studied in detail. Also, in the work, we formulate a non-linear optimization model based on the entropy and score function to compute the probabilities associated with DHFS.

4) A DM algorithm based on the proposed operators is explained, which is more generalized and flexible. The advantages of the suggested model are that it not only includes

the power between two or more PDHFEs but concurrently it circumvent the discrepancy in the results made because of the loss of the information. The applicability of the model is demonstrated through a case study related to the human gesture quantification in a medical case based on hemorrhage attacks. The comparative analysis with some of the existing studies has been directed to determine the availability and benefits of the proposed method. It is noted from the computed results that the presented theory can model the uncertainties with more enhancements as compared to the current environments.

10.5 Appendix: Proof of the Theorems

Proof of Theorem 10.2.1:

Proof. The first part of the theorem can easily be obtained from Definition 2.4.3. So, the Eq. (10.2), in accordance to the same Definition 2.4.3, is given below:

$$\bigotimes_{v=1}^{k} \mathcal{A}_{i_v} = \bigcup_{\substack{\zeta_{i_v} \in h_{i_v}, \\ \vartheta_{i_v} \in g_{i_v}}} \left(\left\{ \prod_{v=1}^{k} \zeta_{i_v} \,\middle|\, \prod_{v=1}^{k} p_{\zeta_{i_v}} \right\}, \left\{ 1 - \prod_{v=1}^{k} (1 - \vartheta_{i_v}) \,\middle|\, \prod_{v=1}^{k} q_{\vartheta_{i_v}} \right\} \right)$$

$$\bigoplus_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(\bigotimes_{v=1}^{k} \mathcal{A}_{i_v} \right) = \bigcup_{\substack{\zeta_{i_v} \in h_{i_v}, \\ \vartheta_{i_v} \in g_{i_v}}} \left(\begin{array}{l} \left\{ 1 - \prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \prod\limits_{v=1}^{k} \zeta_{i_v} \right) \,\middle|\, \prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(\prod\limits_{v=1}^{k} p_{\zeta_{i_v}} \right) \right\}, \\ \left\{ \prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \prod\limits_{v=1}^{k} (1 - \vartheta_{i_v}) \right) \,\middle|\, \prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(\prod\limits_{v=1}^{k} q_{\vartheta_{i_v}} \right) \right\} \end{array} \right)$$

$$\left(\frac{\bigoplus\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(\bigotimes\limits_{v=1}^{k} \mathcal{A}_{i_v} \right)}{\binom{n}{k}} \right)^{\frac{1}{k}} = \bigcup_{\substack{\zeta_{i_v} \in h_{i_v}, \\ \vartheta_{i_v} \in g_{i_v}}} \left(\begin{array}{l} \left\{ \left(1 - \prod\limits_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \left(1 - \prod\limits_{v=1}^{k} \zeta_{i_v} \right)^{\frac{1}{\binom{n}{k}}} \right)^{\frac{1}{k}} \,\middle|\, \prod\limits_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \left(\prod\limits_{v=1}^{k} p_{\zeta_{i_v}} \right) \right\}, \\ \left\{ 1 - \left(1 - \prod\limits_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \left(1 - \prod\limits_{v=1}^{k} (1 - \vartheta_{i_v}) \right)^{\frac{1}{\binom{n}{k}}} \right)^{\frac{1}{k}} \,\middle|\, \prod\limits_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \left(\prod\limits_{v=1}^{k} q_{\vartheta_{i_v}} \right) \right\} \end{array} \right)$$

which is same as Eq. (10.2). This completes the proof.

□

Proof of the Property 10.2.1:

Proof. For $\zeta_i \in h_i$ and $\zeta_i^* \in h_i^*$, we have $\zeta_i \leq \zeta_i^*$, therefore , $\prod_{v=1}^{k} \zeta_{i_v} \leq \prod_{v=1}^{k} \zeta_{i_v}^*$ and it follows that

$$\left(1 - \prod_{\substack{1 \leq i_1 < \ldots \\ < i_k \leq n}} \left(1 - \prod_{v=1}^{k} \zeta_{i_v}\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \leq \left(1 - \prod_{\substack{1 \leq i_1 < \ldots \\ < i_k \leq n}} \left(1 - \prod_{v=1}^{k} \zeta_{i_v}^*\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \tag{10.15}$$

Also for $\vartheta_i \in g_i$ and $\vartheta_i^* \in g_i^*$, we have $\vartheta_i \geq \vartheta_i^*$. It follows that $\prod_{v=1}^{k} (1 - \vartheta_{i_v}) \leq \prod_{v=1}^{k} \left(1 - \vartheta_{i_v}^*\right)$ which implies that

$$\prod_{\substack{1 \leq i_1 < \ldots \\ < i_k \leq n}} \left(1 - \prod_{v=1}^{k} (1 - \vartheta_{i_v})\right)^{\frac{1}{\binom{n}{k}}} \geq \prod_{\substack{1 \leq i_1 < \ldots \\ < i_k \leq n}} \left(1 - \prod_{v=1}^{k} \left(1 - \vartheta_{i_v}^*\right)\right)^{\frac{1}{\binom{n}{k}}}$$

Thus, we obtain

$$1 - \left(1 - \prod_{\substack{1 \leq i_1 < \ldots \\ < i_k \leq n}} \left(1 - \prod_{v=1}^{k} (1 - \vartheta_{i_v})\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \geq 1 - \left(1 - \prod_{\substack{1 \leq i_1 < \ldots \\ < i_k \leq n}} \left(1 - \prod_{v=1}^{k} \left(1 - \vartheta_{i_v}^*\right)\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \tag{10.16}$$

Let, $\mathcal{A} = \text{PDHFMSMA}^{(k)} (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = (h|p_h, g|q_g)$ and $\mathcal{A}^* = \text{PDHFMSMA}^{(k)}(\mathcal{A}_1^*, \mathcal{A}_2^*, \ldots, \mathcal{A}_n^*) = (h^*|p_{h^*}, g^*|q_{g^*})$. Also, let $\mathcal{S}c(\mathcal{A})$ and $\mathcal{S}c(\mathcal{A}^*)$ be the score values of $\mathcal{A}$ and $\mathcal{A}^*$ respectively, then by using Definition 2.1.13 for Eqs. (10.15) and (10.16), we get $\mathcal{S}c(\mathcal{A}) \leq \mathcal{S}c(\mathcal{A}^*)$. Therefore, following cases are to be discussed:

Case 1: If $\mathcal{S}c(\mathcal{A}) < \mathcal{S}c(\mathcal{A}^*)$, then by Definition 2.1.13 of Chapter 2,

$$\text{PDHFMSMA}^{(k)} (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) < \text{PDHFMSMA}^{(k)}(\mathcal{A}_1^*, \mathcal{A}_2^*, \ldots, \mathcal{A}_n^*).$$

Case 2: If $\mathcal{S}c(\mathcal{A}) = \mathcal{S}c(\mathcal{A}^*)$, then we have

$$\sum_{\zeta \in h} \zeta \cdot p_\zeta - \sum_{\vartheta \in g} \vartheta \cdot q_\vartheta = \sum_{\zeta^* \in h^*} \zeta^* \cdot p_{\zeta^*} - \sum_{\vartheta^* \in g^*} \vartheta^* \cdot q_{\vartheta^*} \tag{10.17}$$

So, we have, $0 \leq \zeta \cdot p_\zeta \leq \zeta^* \cdot p_{\zeta^*}$ and $\vartheta \cdot q_\vartheta \geq \vartheta^* \cdot q_{\vartheta^*} \geq 0$, then from Eq. (10.17), we can deduce $\sum_{\zeta \in h} \zeta \cdot p_\zeta = \sum_{\zeta^* \in h^*} \zeta^* \cdot p_{\zeta^*}$ and $\sum_{\vartheta \in g} \vartheta \cdot q_\vartheta = \sum_{\vartheta^* \in g^*} \vartheta^* \cdot q_{\vartheta^*}$. Therefore, using Definition 2.1.13, it follows that $\mathcal{H}(\mathcal{A}) = \sum_{\zeta \in h} \zeta \cdot p_\zeta + \sum_{\vartheta \in g} \vartheta \cdot q_\vartheta = \sum_{\zeta^* \in h^*} \zeta^* \cdot p_{\zeta^*} + \sum_{\vartheta^* \in g^*} \vartheta^* \cdot q_{\vartheta^*} = \mathcal{H}(\mathcal{A}^*)$, Thus, by combining both the cases, we get the required result. □

Proof of the Property 10.2.2:

Proof. Since, $\zeta_{\min} \leq \zeta_i \leq \zeta_{\max}$ and $\vartheta_{min} \leq \vartheta_i \leq \vartheta_{\max}$, then for all $i = 1, 2, \ldots, n$, we can obtain $\prod_{v=1}^{k} \zeta_{\min} \leq \prod_{v=1}^{k} \zeta_{i_v} \leq \prod_{v=1}^{k} \zeta_{\max}$ and $\prod_{v=1}^{k} (1-\vartheta_{\min}) \leq \prod_{v=1}^{k} (1-\vartheta_{i_v}) \leq \prod_{v=1}^{k} (1-\vartheta_{\max})$. We have,

$$\left(1-\prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(1-\prod_{v=1}^{k}\zeta_{\min}\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \leq \left(1-\prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(1-\prod_{v=1}^{k}\zeta_{i_v}\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \leq \left(1-\prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(1-\prod_{v=1}^{k}\zeta_{\max}\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}}$$

which implies that $\zeta_{\min} \leq \left(1-\prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(1-\prod_{v=1}^{k}\zeta_{i_v}\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \leq \zeta_{\max}$. Similarly, we have

$$\vartheta_{\min} \leq 1-\left(1-\prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(1-\prod_{v=1}^{k}(1-\vartheta_{i_v})\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \leq \vartheta_{\max}.$$

Now, for probabilities,

$$\prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(\prod_{v=1}^{k}p_{\zeta_{\min}}\right) \leq \prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(\prod_{v=1}^{k}p_{i_v}\right) \leq \prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(\prod_{v=1}^{k}p_{\zeta_{\max}}\right)$$

and

$$\prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(\prod_{v=1}^{k}q_{\vartheta_{\min}}\right) \leq \prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(\prod_{v=1}^{k}q_{i_v}\right) \leq \prod_{\substack{1\leq i_1<\ldots\\<i_k\leq n}}\left(\prod_{v=1}^{k}q_{\vartheta_{\max}}\right).$$

Let $\mathcal{A} = \mathrm{PDHFMSMA}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = (h|p_h, g|q_g) = \bigcup_{\substack{\zeta\in h\\ \vartheta\in g}} \left(\{\zeta \mid p_\zeta\}, \{\vartheta \mid q_\vartheta\}\right)$, then from Definition 2.1.13, we have $\mathcal{S}c(\mathcal{A}^-) = \zeta_{\min}\cdot p_{\zeta_{\min}} - \vartheta_{\max}\cdot q_{\vartheta_{\max}} \leq \mathcal{S}c(\mathcal{A}) = \sum_{\zeta\in h}\zeta\cdot p_\zeta - \sum_{\vartheta\in g}\vartheta\cdot q_\vartheta \leq \zeta_{\max}\cdot p_{\zeta_{\max}} - \vartheta_{\min}\cdot q_{\vartheta_{\min}} = \mathcal{S}c(\mathcal{A}^+)$. Thus, the following cases are to be discussed :

Case 1: If $\mathcal{S}c(\mathcal{A}^-) < \mathcal{S}c(\mathcal{A}) < \mathcal{S}c(\mathcal{A}^+)$, then $\mathcal{A}^- < \mathrm{PDHFMSMA}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) < \mathcal{A}^+$.

Case 2: If $\mathcal{S}c(\mathcal{A}^-) = \mathcal{S}c(\mathcal{A})$, then we have $\zeta_{\min} \cdot p_{\zeta_{\min}} - \vartheta_{\max} \cdot q_{\vartheta_{\max}} = \sum\limits_{\zeta \in h} \zeta \cdot p_\zeta - \sum\limits_{\vartheta \in g} \vartheta \cdot q_\vartheta$. Consequently, we have $\mathcal{H}(\mathcal{A}^-) = \mathcal{H}(\mathcal{A})$ i.e., $\zeta_{\min} \cdot p_{\zeta_{\min}} + \vartheta_{\max} \cdot q_{\vartheta_{\max}} = \sum\limits_{\zeta \in h} \zeta \cdot p_\zeta + \sum\limits_{\vartheta \in g} \vartheta \cdot q_\vartheta$. Thus, we obtain $\zeta_{\min} \cdot p_{\zeta_{\min}} = \sum\limits_{\zeta \in h} \zeta \cdot p_\zeta$ and $\vartheta_{\max} \cdot q_{\vartheta_{\max}} = \sum\limits_{\vartheta \in g} \vartheta \cdot q_\vartheta$. It follows that $\mathcal{A}^- = \text{PDHFMSMA}^{(k)}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right)$.

Case 3: If $\mathcal{S}c(\mathcal{A}) = \mathcal{S}c(\mathcal{A}^+)$, then in similar manner we have $\mathcal{A}^+ = \text{PDHFMSMA}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$. This completes the proof.

□

Proof of Theorem 10.2.2:

Proof. For sake of simplicity, let

$$f_\zeta(k) = \left(1 - \prod_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \left(1 - \prod_{v=1}^{k} \zeta_{i_v}\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \tag{10.18}$$

$$\text{and} \quad f'_\vartheta(k) = 1 - \left(1 - \prod_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \left(1 - \prod_{v=1}^{k} (1 - \vartheta_{i_v})\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \tag{10.19}$$

Now, we will prove $f_\zeta(k)$ is monotonically decreasing with respect to k. Using Lemma 10.2.2, we have

$$\begin{aligned} f_\zeta(k) &= \left(1 - \prod_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \left(1 - \prod_{v=1}^{k} \zeta_{i_v}\right)^{\frac{1}{\binom{n}{k}}}\right)^{\frac{1}{k}} \\ &\ge \left(1 - \sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \frac{1}{\binom{n}{k}} \left(1 - \prod_{v=1}^{k} \zeta_{i_v}\right)\right)^{\frac{1}{k}} = \left(\sum_{\substack{1 \le i_1 < \ldots \\ < i_k \le n}} \frac{1}{\binom{n}{k}} \left(\prod_{v=1}^{k} \zeta_{i_v}\right)\right)^{\frac{1}{k}} \end{aligned} \tag{10.20}$$

We prove the result by contradiction method. Assume that $f_\zeta(k)$ is monotonically increasing w.r.t. k, then we have

$$f_\zeta(n) > f_\zeta(n-1) > \ldots > f_\zeta(1) \tag{10.21}$$

From Eq. (10.20), we have

$$f_\zeta(1) \ge \left(\sum_{1 \le i_1 \le n} \frac{1}{\binom{n}{1}} \left(\prod_{v=1}^{1} \zeta_{i_v}\right)\right)^{\frac{1}{1}} = \sum_{1 \le i_1 \le n} \frac{1}{n} \zeta_{i_1} \tag{10.22}$$

Also, from Eq. (10.18), we have

$$f_{\zeta}(n) \;=\; \left(1-\prod_{\substack{1\leq i_1<\ldots\\<i_n\leq n}}\left(1-\prod_{v=1}^{n}\zeta_{i_v}\right)^{\frac{1}{\binom{n}{n}}}\right)^{\frac{1}{n}} = \left(\prod_{i_v=1}^{n}\zeta_{i_v}\right)^{\frac{1}{n}} \tag{10.23}$$

Using Eqs. (10.21),(10.22) and (10.23), we get $\left(\prod_{i_v=1}^{n}\zeta_{i_v}\right)^{\frac{1}{n}} \geq \sum_{1\leq i_1\leq n}\frac{1}{n}\zeta_{i_1}$. Clearly, it contradicts the inequality given in Lemma 10.2.2. Hence, the function $f_{\zeta}(k)$ is monotonically decreasing w.r.t. k. Similarly, we can prove $f'_{\vartheta}(k)$ is monotonically increasing w.r.t. k. By using Definition 2.1.13, for total number of hesitant membership and non-membership (M and N) values, we have

$$\mathcal{S}c(k) = \frac{1}{M}\sum_{\zeta\in h} f_{\zeta}(k) - \frac{1}{N}\sum_{\vartheta\in g} f'_{\vartheta}(k)$$

For any integral value of $k \in [1, n]$ we can obtain

$$\mathcal{S}c(k+1) - \mathcal{S}c(k) = \frac{1}{M}\sum_{\zeta\in h}(f_{\zeta}(k+1) - f_{\zeta}(k)) - \frac{1}{N}\sum_{\vartheta\in g}(f'_{\vartheta}(k+1) - f'_{\vartheta}(k))$$

Since $f_{\zeta}(k)$ is monotonically decreasing and $f'_{\vartheta}(k)$ is monotonically increasing w.r.t., k. Thus, for increasing values of k, we have $\mathcal{S}c(k+1) - \mathcal{S}c(k) < 0$. Thus, the PDHFMSM operator is monotonically decreasing with respect to the parameter k. □

Table 10.1: Preference values given by Doctor 1

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$\begin{pmatrix} \{0.3 \mid p_{1,11}, 0.6 \mid p_{2,11}\} \\ \{0.2 \mid q_{1,11}, 0.3 \mid q_{2,11}\} \end{pmatrix}$	$\begin{pmatrix} \{0.4 \mid p_{1,12}, 0.5 \mid p_{2,12}\} \\ \{0.4 \mid q_{1,12}\} \end{pmatrix}$	$\begin{pmatrix} \{0.3 \mid p_{1,13}, 0.4 \mid p_{2,13}\} \\ \{0.1 \mid q_{1,13}, 0.3 \mid q_{2,13}\} \end{pmatrix}$
$\mathcal{V}_2$	$\begin{pmatrix} \{0.5 \mid p_{1,21}, 0.8 \mid p_{2,21}\} \\ \{0.1 \mid q_{1,21}\} \end{pmatrix}$	$\begin{pmatrix} \{0.5 \mid p_{1,22}\} \\ \{0.3 \mid q_{1,22}\} \end{pmatrix}$	$\begin{pmatrix} \{0.2 \mid p_{1,23}, 0.6 \mid p_{2,23}\} \\ \{0.15 \mid q_{1,23}, 0.3 \mid q_{2,23}\} \end{pmatrix}$
$\mathcal{V}_3$	$\begin{pmatrix} \{0.1 \mid p_{1,31}\} \\ \{0.2 \mid q_{1,31}, 0.3 \mid q_{2,31}\} \end{pmatrix}$	$\begin{pmatrix} \{0.1 \mid p_{1,32}, 0.4 \mid p_{2,32}\} \\ \{0.2 \mid q_{1,32}, 0.3 \mid q_{2,32}\} \end{pmatrix}$	$\begin{pmatrix} \{0.1 \mid p_{1,33}\} \\ \{0.2 \mid q_{1,33}\} \end{pmatrix}$
$\mathcal{V}_4$	$\begin{pmatrix} \{0.45 \mid p_{1,41}, 0.6 \mid p_{2,41}\} \\ \{0.3 \mid q_{1,41}\} \end{pmatrix}$	$\begin{pmatrix} \{0.7 \mid p_{1,42}\} \\ \{0.1 \mid q_{1,42}\} \end{pmatrix}$	$\begin{pmatrix} \{0.2 \mid p_{1,43}\} \\ \{0.3 \mid q_{1,43}, 0.4 \mid q_{2,43}\} \end{pmatrix}$

Table 10.2: Preference values given by Doctor 2

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$\begin{pmatrix} \{0.55 \mid p_{1,11}, 0.6 \mid p_{2,11}\} \\ \{0.2 \mid q_{1,11}\} \end{pmatrix}$	$\begin{pmatrix} \{0.6 \mid p_{1,12}\} \\ \{0.3 \mid q_{1,12}\} \end{pmatrix}$	$\begin{pmatrix} \{0.5 \mid p_{1,13}\} \\ \{0.15 \mid q_{1,13}, 0.3 \mid q_{2,13}\} \end{pmatrix}$
$\mathcal{V}_2$	$\begin{pmatrix} \{0.4 \mid p_{1,21}\} \\ \{0.2 \mid q_{1,21}\} \end{pmatrix}$	$\begin{pmatrix} \{0.3 \mid p_{1,22}\} \\ \{0.4 \mid q_{1,22}\} \end{pmatrix}$	$\begin{pmatrix} \{0.4 \mid p_{1,23}, 0.5 \mid p_{2,23}, 0.6 \mid p_{3,23}\} \\ \{0.1 \mid q_{1,23}\} \end{pmatrix}$
$\mathcal{V}_3$	$\begin{pmatrix} \{0.2 \mid p_{1,31}\} \\ \{0.4 \mid q_{1,31}\} \end{pmatrix}$	$\begin{pmatrix} \{0.1 \mid p_{1,32}, 0.2 \mid p_{2,32}\} \\ \{0.3 \mid q_{1,32}\} \end{pmatrix}$	$\begin{pmatrix} \{0.45 \mid p_{1,33}, 0.52 \mid p_{2,33}\} \\ \{0.3 \mid q_{1,33}\} \end{pmatrix}$
$\mathcal{V}_4$	$\begin{pmatrix} \{0.5 \mid p_{1,41}\} \\ \{0.1 \mid q_{1,41}, 0.3 \mid q_{2,41}\} \end{pmatrix}$	$\begin{pmatrix} \{0.55 \mid p_{1,42}, 0.7 \mid p_{2,42}\} \\ \{0.2 \mid q_{1,42}\} \end{pmatrix}$	$\begin{pmatrix} \{0.5 \mid p_{1,43}\} \\ \{0.1 \mid q_{1,43}\} \end{pmatrix}$

Table 10.3: Preference values by Doctor 1 after probability determination

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$\begin{pmatrix} \{0.3 \mid 0.4256, 0.6 \mid 0.5744\} \\ \{0.2 \mid 0.5250, 0.3 \mid 0.4750\} \end{pmatrix}$	$\begin{pmatrix} \{0.4 \mid 0.4750, 0.5 \mid 0.5250\} \\ \{0.4 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.3 \mid 0.2894, 0.4 \mid 3199, 0.6 \mid 0.3907\} \\ \{0.1 \mid 0.5498, 0.3 \mid 0.4502\} \end{pmatrix}$
$\mathcal{V}_2$	$\begin{pmatrix} \{0.5 \mid 0.4256, 0.8 \mid 0.5744\} \\ \{0.1 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.5 \mid 1\} \\ \{0.3 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.2 \mid 0.4013, 0.6 \mid 0.5987\} \\ \{0.15 \mid 0.5374, 0.3 \mid 0.4626\} \end{pmatrix}$
$\mathcal{V}_3$	$\begin{pmatrix} \{0.1 \mid 1\} \\ \{0.2 \mid 0.5250, 0.3 \mid 0.4750\} \end{pmatrix}$	$\begin{pmatrix} \{0.1 \mid 0.4256, 0.4 \mid 0.5744\} \\ \{0.2 \mid 0.5250, 0.3 \mid 0.4750\} \end{pmatrix}$	$\begin{pmatrix} \{0.1 \mid 1\} \\ \{0.2 \mid 1\} \end{pmatrix}$
$\mathcal{V}_4$	$\begin{pmatrix} \{0.45 \mid 0.4626, 0.6 \mid 0.5374\} \\ \{0.3 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.7 \mid 1\} \\ \{0.1 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.2 \mid 1\} \\ \{0.3 \mid 0.5250, 0.4 \mid 0.4750\} \end{pmatrix}$

Table 10.4: Preference values by Doctor 2 after probability determination

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$\begin{pmatrix} \{0.55 \mid 0.4875, 0.6 \mid 0.5125\} \\ \{0.2 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.6 \mid 1\} \\ \{0.3 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.5 \mid 1\} \\ \{0.15 \mid 0.5374, 0.3 \mid 0.4626\} \end{pmatrix}$
$\mathcal{V}_2$	$\begin{pmatrix} \{0.4 \mid 1\} \\ \{0.2 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.3 \mid 1\} \\ \{0.4 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.4 \mid 0.3006, 0.5 \mid 0.3322, 0.6 \mid 0.3672\} \\ \{0.1 \mid 1\} \end{pmatrix}$
$\mathcal{V}_3$	$\begin{pmatrix} \{0.2 \mid 1\} \\ \{0.4 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.1 \mid 0.4750, 0.2 \mid 0.5250\} \\ \{0.3 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.45 \mid 0.4826, 0.52 \mid 0.5174\} \\ \{0.3 \mid 1\} \end{pmatrix}$
$\mathcal{V}_4$	$\begin{pmatrix} \{0.5 \mid 1\} \\ \{0.1 \mid 0.5250, 0.3 \mid 0.4750\} \end{pmatrix}$	$\begin{pmatrix} \{0.55 \mid 0.4626, 0.7 \mid 0.5374\} \\ \{0.2 \mid 1\} \end{pmatrix}$	$\begin{pmatrix} \{0.5 \mid 1\} \\ \{0.1 \mid 1\} \end{pmatrix}$

Table 10.5: Comprehensive matrix

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$\begin{pmatrix} \{0.3 \mid 0.2128, 0.55 \mid 0.2437, 0.6 \mid 0.5435\} \\ \{0.2 \mid 0.7625, 0.3 \mid 0.2375\} \end{pmatrix}$	$\begin{pmatrix} \{0.4 \mid 0.2375, 0.5 \mid 0.2625, 0.6 \mid 0.5\} \\ \{0.3 \mid 0.5, 0.4 \mid 0.5\} \end{pmatrix}$	$\begin{pmatrix} \{0.3 \mid 0.1447, 0.4 \mid 0.16, 0.5 \mid 0.5, 0.6 \mid 0.1953\} \\ \{0.1 \mid 0.2749, 0.15 \mid 0.2687, 0.3 \mid 0.4564\} \end{pmatrix}$
$\mathcal{V}_2$	$\begin{pmatrix} \{0.4 \mid 0.5, 0.5 \mid 0.2128, 0.8 \mid 0.2872\} \\ \{0.1 \mid 0.5, 0.2 \mid 0.5\} \end{pmatrix}$	$\begin{pmatrix} \{0.3 \mid 0.5, 0.5 \mid 0.5\} \\ \{0.3 \mid 0.5, 0.4 \mid 0.5\} \end{pmatrix}$	$\begin{pmatrix} \begin{Bmatrix} 0.2 \mid 0.2007, 0.4 \mid 0.1503, \\ 0.5 \mid 0.1661, 0.6 \mid 0.4829 \end{Bmatrix} \\ \{0.1 \mid 0.5, 0.15 \mid 0.2687, 0.3 \mid 0.2313\} \end{pmatrix}$
$\mathcal{V}_3$	$\begin{pmatrix} \{0.10 \mid 0.5, 0.2 \mid 0.5\} \\ \{0.2 \mid 0.2625, 0.3 \mid 0.2375, 0.4 \mid 0.5\} \end{pmatrix}$	$\begin{pmatrix} \{0.1 \mid 0.4503, 0.2 \mid 0.2625, 0.4 \mid 0.2872\} \\ \{0.2 \mid 0.2625, 0.3 \mid 0.7375\} \end{pmatrix}$	$\begin{pmatrix} \{0.1 \mid 0.5, 0.45 \mid 0.2413, 0.52 \mid 0.2587\} \\ \{0.2 \mid 0.5, 0.3 \mid 0.5\} \end{pmatrix}$
$\mathcal{V}_4$	$\begin{pmatrix} \{0.45 \mid 0.2313, 0.5 \mid 0.5, 0.6 \mid 0.2687\} \\ \{0.1 \mid 0.2625, 0.2 \mid 0.2375, 0.3 \mid 0.5\} \end{pmatrix}$	$\begin{pmatrix} \{0.55 \mid 0.2313, 0.7 \mid 0.7687\} \\ \{0.1 \mid 0.5, 0.2 \mid 0.5\} \end{pmatrix}$	$\begin{pmatrix} \{0.2 \mid 0.5, 0.5 \mid 0.5\} \\ \{0.1 \mid 0.5, 0.3 \mid 0.2625, 0.4 \mid 0.2375\} \end{pmatrix}$

Table 10.6: Score values and ranking of alternatives for different values of k

	With WPDHFMSMA operator				
	$\mathcal{S}c(\mathcal{V}_1)$	$\mathcal{S}c(\mathcal{V}_2)$	$\mathcal{S}c(\mathcal{V}_3)$	$\mathcal{S}c(\mathcal{V}_4)$	Ranking
$k=1$	-0.4094	-0.3856	-0.5485	-0.3423	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
$k=2$	-0.4454	-0.4273	-0.5861	-0.3835	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
$k=3$	-0.4668	-0.4505	-0.6171	-0.4175	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	With WPDHFMSMG operator				
	$\mathcal{S}c(\mathcal{V}_1)$	$\mathcal{S}c(\mathcal{V}_2)$	$\mathcal{S}c(\mathcal{V}_3)$	$\mathcal{S}c(\mathcal{V}_4)$	Ranking
$k=1$	0.6919	0.6735	0.4785	0.6931	$\mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3$
$k=2$	0.7110	0.7007	0.5011	0.7379	$\mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3$
$k=3$	0.7298	0.7315	0.5206	0.7594	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$

Table 10.7: Score values and ranking using existing methodologies

Existing Approaches	$\mathcal{S}c(\mathcal{V}_1)$	$\mathcal{S}c(\mathcal{V}_2)$	$\mathcal{S}c(\mathcal{V}_3)$	$\mathcal{S}c(\mathcal{V}_4)$	Ranking
Hao et al. [61] method based on PDHFWA operator	-0.4094	-0.3856	-0.5485	-0.3423	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
Garg and Kaur [51] method based on PDHFWEA operator	0.2565	0.2670	-0.0227	0.3244	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
Garg and Kaur [51] method based on PDHFWEG operator	0.2313	0.2195	-0.0609	0.2661	$\mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3$
Zhang [226] method based on WD-HFMSM operator	-0.4432	-0.4122	-0.5509	-0.3772	$\mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3$
Zhang [226] method based on WD-HFGMSM operator	0.7090	0.7118	0.5442	0.7435	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$

Abbreviations. PDHFWA: Probabilistic dual hesitant fuzzy weighted average; PDHFWEA: Probabilistic dual hesitant fuzzy weighted Einstein average; PDHFWEG: Probabilistic dual hesitant fuzzy weighted Einstein geometric; WDHFMSM: weighted dual hesitant fuzzy Maclaurin symmetric mean; WDHFGMSM: weighted dual hesitant fuzzy geometric Maclaurin symmetric mean.

Table 10.8: Characteristic comparison of the proposed approach with different methods

Methods	Whether consider more than one decision maker	Whether considers probabilities	Whether considers non-membership	Probabilities derived by non-linear approach	Generalization of existing approaches
Hao et al. [61]	✓	✓	✓	×	×
Garg and Kaur [51]	✓	✓	✓	×	×
Zhang [226]	×	×	✓	×	✓
Our proposed approach	✓	✓	✓	✓	✓

Chapter 11

A robust correlation coefficient for probabilistic dual hesitant fuzzy sets and its applications[1]

In this chapter, we develop a method to solve the MCDM problem under PDHFS environment. For it, firstly, we define the informational energy and the covariance between the two PDHFSs and study their properties. Secondly, we develop correlation coefficients and the weighted correlation coefficients for PDHFSs. In the formulation, PDHFSs are able to represent the information in terms of their respective degrees while the assigned probabilities give more details about the level of agreeness or disagree. Thirdly, a novel algorithm is developed based on the proposed operators to solve MCDM problems. A practical example is provided to verify the developed approach and to demonstrate its practicality and feasibility.

11.1 Introduction

Utilization of correlation coefficients (CCs) is a supreme choice of decision-makers to locate the finest alternative which assume an overwhelming job to quantify the level of dependency between the two sets. In statistical analysis, the CCs measure the linear relationship, whereas in FS theory, it determines the degree of dependency. Although HFSs

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and DHFSs are successfully applied in the various DM theories based on CCs, they do not address the issue regarding the occurrence of the probabilities of the elements in the HFSs or DHFSs. Here, we take the following examples as an illustration: Consider a person which give their preferences towards the comfortless of an object in terms of HFE as {0.3, 0.5, 0.6}. During its ratings, he suggested that the comfort level corresponding to 0.5 is the most desirable as compared to others, while the comfort level associated with 0.3 is more desirable than 0.6. Thus, under such circumstance, the HFE {0.3, 0.5, 0.6} is not suitable to describe the information. Similarly, consider a rating of a person towards the evaluation of the quality of a product in terms of DHFE ({0.3, 0.4, 0.5}, {0.2, 0.3, 0.4}). During the ratings, the decision maker believes that their comfortable towards the object rating 0.3, 0.4 is double than 0.5 in membership degrees while triple towards the 0.4 in the non-membership degrees with respect to the others. Thus, again such DHFE ({0.3, 0.4, 0.5}, {0.2, 0.3, 0.4}) is not suitable to describe the information. To resolve such problems, the concepts of PDHFS provides an accurate description which can be easily used to describe the information in the above-stated examples.

Therefore, motivated by the structure of the PDHFS, this chapter focuses on the CC between the pairs of PDHFEs. For it, we define the informational energies and the covariance between the pairs of PDHFEs. The north-west corner method has been utilized to compute the joint probabilities of the sets. The major advantages of present such a method is that there is no need to match the length of the considered PDHFSs by repeating the values, as in the case of DHFEs. Then, based on these, we define four CCs, based on the inherent characteristics, and obtain some properties. Further, a novel MCDM method based on the proposed CCs is presented and illustrate it with some practical problems. The performance of the proposed measure is compared with several existing theories.

11.2 Correlation coefficient on PDHFSs

For a universal set $\mathcal{X} = \{x_i \mid i = 1, , 2 \ldots, n\}$, we define the concept of the informational energy, covariance and CC for PDHFSs. For notational convenience, in this chapter, we are denoting $\mathcal{A}_\Lambda(x_i) = \mathcal{A}_{i,\Lambda}$, $\mathcal{B}_\Lambda(x_i) = \mathcal{B}_{i,\Lambda}$ (where the indexing Λ may vary among s, t, s', t'),

$M_{\mathcal{A}_i}(\text{or}N_{\mathcal{A}_i}) = M_i(\text{or } N_i)$ and $M_{\mathcal{B}_i}(\text{or}N_{\mathcal{B}_i}) = M_i'(\text{or } N_i{}')$

Definition 11.2.1. Let two PDHFEs $\mathcal{A} = \begin{pmatrix} x_i, h_{\mathcal{A}}(x_i)|p_{\mathcal{A}}(x_i), \\ g_{\mathcal{A}}(x_i)|q_{\mathcal{A}}(x_i) \end{pmatrix} = \bigcup_{\substack{\zeta_{\mathcal{A}_{i,s}} \in h_{\mathcal{A}}, \\ \vartheta_{\mathcal{A}_{i,t}} \in g_{\mathcal{A}}}} (x_i, \{\zeta_{\mathcal{A}_{i,s}} | p_{\mathcal{A}_{i,s}}\},$ $\{\vartheta_{\mathcal{A}_{i,t}} | q_{\mathcal{A}_{i,t}}\})$ and $\mathcal{B} = \begin{pmatrix} x_i, h_{\mathcal{B}}(x_i)|p_{\mathcal{B}}(x_i), \\ g_{\mathcal{B}}(x_i)|q_{\mathcal{B}}(x_i) \end{pmatrix} = \bigcup_{\substack{\zeta_{\mathcal{B}_{i,s'}} \in h_{\mathcal{B}}, \\ \vartheta_{\mathcal{B}_{i,t'}} \in g_{\mathcal{B}}}} \left(x_i, \left\{\zeta_{\mathcal{B}_{i,s'}} | p_{\mathcal{B}_{i,s'}}\right\}, \left\{\vartheta_{\mathcal{B}_{i,t'}} | q_{\mathcal{B}_{i,t'}}\right\}\right)$ where $s = 1, 2, \ldots, M_i; t = 1, 2, \ldots, N_i; s' = 1, 2, \ldots, M_i'; t = 1, 2, \ldots, N_i'$, the informational energies of them are defined as

$$\mathcal{I}(\mathcal{A}) = \sum_{i=1}^{n} \left(\sum_{s=1}^{M_i} \left(\zeta_{\mathcal{A}_{i,s}}\right)^2 p_{\mathcal{A}_{i,s}} + \sum_{t=1}^{N_i} \left(\vartheta_{\mathcal{A}_{i,t}}\right)^2 q_{\mathcal{A}_{i,t}} \right) \tag{11.1}$$

and

$$\mathcal{I}(\mathcal{B}) = \sum_{i=1}^{n} \left(\sum_{s'=1}^{M_i'} \left(\zeta_{\mathcal{B}_{i,s'}}\right)^2 p_{\mathcal{B}_{i,s'}} + \sum_{t'=1}^{N_i'} \left(\vartheta_{\mathcal{B}_{i,t'}}\right)^2 q_{\mathcal{B}_{i,t'}} \right) \tag{11.2}$$

Further, the covariance between $\mathcal{A}$ and $\mathcal{B}$ is given as:

$$\mathcal{C}ov(\mathcal{A}, \mathcal{B}) = \sum_{i=1}^{n} \left(\sum_{s=1}^{M_i} \sum_{s'=1}^{M_i'} \zeta_{\mathcal{A}_{i,s}} \zeta_{\mathcal{B}_{i,s'}} p_{i,ss'} + \sum_{t=1}^{N_i} \sum_{t'=1}^{N_i'} \vartheta_{\mathcal{A}_{i,t}} \vartheta_{\mathcal{B}_{i,t'}} q_{i,tt'} \right) \tag{11.3}$$

where $p_{i,ss'}$ and $q_{i,tt'}$ are the joint probabilities of $\mathcal{A}$ and $\mathcal{B}$ calculated as below:

Joint probability distribution between $\mathcal{A}$ and $\mathcal{B}$												
Membership Values						Non-membership Values						
	$\zeta_{\mathcal{B}_{i,1'}}$	$\zeta_{\mathcal{B}_{i,2'}}$	$\cdots$	$\zeta_{\mathcal{B}_{i,M_i'}}$			$\vartheta_{\mathcal{B}_{i,1'}}$	$\vartheta_{\mathcal{B}_{i,2'}}$	$\cdots$	$\vartheta_{\mathcal{B}_{i,N_i'}}$	
$\zeta_{\mathcal{A}_{i,1}}$	$p_{i,11'}$	$p_{i,12'}$	$\cdots$	$p_{i,1M_i'}$	$p_{\mathcal{A}_{i,1}}$	$\vartheta_{\mathcal{A}_{i,1}}$	$q_{i,11'}$	$q_{i,12'}$	$\cdots$	$q_{i,1N_i'}$	$q_{\mathcal{A}_{i,1}}$
$\zeta_{\mathcal{A}_{i,2}}$	$p_{i,21'}$	$p_{i,22'}$	$\cdots$	$p_{i,2M_i'}$	$p_{\mathcal{A}_{i,2}}$	$\vartheta_{\mathcal{A}_{i,2}}$	$q_{i,21'}$	$q_{i,22'}$	$\cdots$	$q_{i,2N_i'}$	$q_{\mathcal{A}_{i,2}}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\zeta_{\mathcal{A}_{i,M_i}}$	$p_{i,M_i1'}$	$p_{i,M_i2'}$	$\cdots$	$p_{i,M_iM_i'}$	$p_{\mathcal{A}_{i,M_i}}$	$\vartheta_{\mathcal{A}_{i,N_i}}$	$q_{i,N_i1'}$	$q_{i,N_i2'}$	$\cdots$	$q_{i,N_iN_i'}$	$q_{\mathcal{A}_{i,N_i}}$
	$p_{\mathcal{B}_{i,1'}}$	$p_{\mathcal{B}_{i,2'}}$	$\cdots$	$p_{\mathcal{B}_{i,M_i'}}$	1		$q_{\mathcal{B}_{i,1'}}$	$q_{\mathcal{B}_{i,2'}}$	$\cdots$	$q_{\mathcal{B}_{i,N_i'}}$	1

From Eq. (11.3), it is seen that $\mathcal{C}ov(\mathcal{A}, \mathcal{B}) = \mathcal{C}ov(\mathcal{B}, \mathcal{A})$ and $\mathcal{C}ov(\mathcal{A}, \mathcal{A}) = \mathcal{I}(\mathcal{A})$. However, it is also being noted that when $\mathcal{A} = \mathcal{B}$ then the joint probability distribution of $\mathcal{C}ov(\mathcal{A}, \mathcal{B})$ is equal to $\mathcal{I}(\mathcal{A})$ i.e., $p_{i,ss'} = p_{\mathcal{A}_{i,s}}$ which is computed by using the North-west corner rule as demonstrated below:

Joint probabilities computed using North-west corner rule											
Membership Values						Non-membership Values					
	$\zeta_{\mathcal{A}_{i,1}}$	$\zeta_{\mathcal{A}_{i,2}}$	$\cdots$	$\zeta_{\mathcal{A}_{i,M_i}}$			$\vartheta_{\mathcal{A}_{i,1}}$	$\vartheta_{\mathcal{A}_{i,2}}$	$\cdots$	$\vartheta_{\mathcal{A}_{i,N_i}}$	
$\zeta_{\mathcal{A}_{i,1}}$	$p_{\mathcal{A}_{i,1}}$	0	$\cdots$	0	$p_{\mathcal{A}_{i,1}}$	$\vartheta_{\mathcal{A}_{i,1}}$	$q_{\mathcal{A}_{i,1}}$	0	$\cdots$	0	$q_{\mathcal{A}_{i,1}}$
$\zeta_{\mathcal{A}_{i,2}}$	0	$p_{\mathcal{A}_{i,2}}$	$\cdots$	0	$p_{\mathcal{A}_{i,2}}$	$\vartheta_{\mathcal{A}_{i,2}}$	0	$q_{\mathcal{A}_{i,2}}$	$\cdots$	0	$q_{\mathcal{A}_{i,2}}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\zeta_{\mathcal{A}_{i,M_i}}$	0	0	$\cdots$	$p_{\mathcal{A}_{i,M_i}}$	$p_{\mathcal{A}_{i,M_i}}$	$\vartheta_{\mathcal{A}_{i,N_i}}$	0	0	$\cdots$	$q_{\mathcal{A}_{i,N_i}}$	$q_{\mathcal{A}_{i,N_i}}$
	$p_{\mathcal{A}_{i,1}}$	$p_{\mathcal{A}_{i,2}}$	$\cdots$	$p_{\mathcal{A}_{i,M_i}}$	1		$q_{\mathcal{A}_{i,1}}$	$q_{\mathcal{A}_{i,2}}$	$\cdots$	$q_{\mathcal{A}_{i,N_i}}$	1

By utilizing the above concept, we define the CC for PDHFEs $\mathcal{A}$ and $\mathcal{B}$ as follows:

Definition 11.2.2. For two PDHFSs $\mathcal{A} = \begin{pmatrix} x_i, h_{\mathcal{A}}(x_i)|p_{\mathcal{A}}(x_i), \\ g_{\mathcal{A}}(x_i)|q_{\mathcal{A}}(x_i) \end{pmatrix} = \bigcup_{\substack{\zeta_{\mathcal{A}_{i,s}} \in h_{\mathcal{A}}, \\ \vartheta_{\mathcal{A}_{i,t}} \in g_{\mathcal{A}}}} \left(x_i, \left\{\zeta_{\mathcal{A}_{i,s}} \middle| p_{\mathcal{A}_{i,s}}\right\}, \left\{\vartheta_{\mathcal{A}_{i,t}} \middle| q_{\mathcal{A}_{i,t}}\right\}\right)$ and $\mathcal{B} = \begin{pmatrix} x_i, h_{\mathcal{B}}(x_i)|p_{\mathcal{B}}(x_i), \\ g_{\mathcal{B}}(x_i)|q_{\mathcal{B}}(x_i) \end{pmatrix} = \bigcup_{\substack{\zeta_{\mathcal{B}_{i,s'}} \in h_{\mathcal{B}}, \\ \vartheta_{\mathcal{B}_{i,t'}} \in g_{\mathcal{B}}}} \left(x_i, \left\{\zeta_{\mathcal{B}_{i,s'}} \middle| p_{\mathcal{B}_{i,s'}}\right\}, \left\{\vartheta_{\mathcal{B}_{i,t'}} \middle| q_{\mathcal{B}_{i,t'}}\right\}\right)$ defined on $\mathcal{X}$, the CC denoted by $\mathcal{K}_1(\mathcal{A}, \mathcal{B})$, is defined as

$$
\begin{aligned}
\mathcal{K}_1(\mathcal{A}, \mathcal{B}) &= \frac{\mathcal{C}ov(\mathcal{A}, \mathcal{B})}{\sqrt{\mathcal{I}(\mathcal{A}) \times \mathcal{I}(\mathcal{B})}} \\
&= \frac{\sum_{i=1}^{n} \left(\sum_{s=1}^{M_i} \sum_{s'=1}^{M_i'} \zeta_{\mathcal{A}_{i,s}} \zeta_{\mathcal{B}_{i,s'}} p_{i,ss'} + \sum_{t=1}^{N_i} \sum_{t'=1}^{N_i'} \vartheta_{\mathcal{A}_{i,t}} \vartheta_{\mathcal{B}_{i,t'}} q_{i,tt'} \right)}{\sqrt{\sum_{i=1}^{n} \left(\sum_{s=1}^{M_i} \left(\zeta_{\mathcal{A}_{i,s}}\right)^2 p_{\mathcal{A}_{i,s}} + \sum_{t=1}^{N_i} \left(\vartheta_{\mathcal{A}_{i,t}}\right)^2 q_{\mathcal{A}_{i,t}} \right)} \sqrt{\sum_{i=1}^{n} \left(\sum_{s'=1}^{M_i'} \left(\zeta_{\mathcal{B}_{i,s'}}\right)^2 p_{\mathcal{B}_{i,s'}} + \sum_{t'=1}^{N_i'} \left(\vartheta_{\mathcal{B}_{i,t'}}\right)^2 q_{\mathcal{B}_{i,t'}} \right)}}
\end{aligned} \tag{11.4}
$$

Theorem 11.2.1. The CC $\mathcal{K}_1$ satisfies the following properties for PDHFEs $\mathcal{A}$ and $\mathcal{B}$:

(P1) $0 \leq \mathcal{K}_1(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{K}_1(\mathcal{A}, \mathcal{B}) = \mathcal{K}_1(\mathcal{B}, \mathcal{A})$.

(P3) If $\mathcal{A} = \mathcal{B}$, then $\mathcal{K}_1(\mathcal{A}, \mathcal{B}) = 1$.

Proof. For two PDHFSs $\mathcal{A} = \begin{pmatrix} x_i, h_{\mathcal{A}}(x_i)|p_{\mathcal{A}}(x_i), \\ g_{\mathcal{A}}(x_i)|q_{\mathcal{A}}(x_i) \end{pmatrix} = \bigcup_{\substack{\zeta_{\mathcal{A}_{i,s}} \in h_{\mathcal{A}}, \\ \vartheta_{\mathcal{A}_{i,t}} \in g_{\mathcal{A}}}} \left(x_i, \left\{\zeta_{\mathcal{A}_{i,s}} \middle| p_{\mathcal{A}_{i,s}}\right\}, \left\{\vartheta_{\mathcal{A}_{i,t}} \middle| q_{\mathcal{A}_{i,t}}\right\}\right)$ and $\mathcal{B} = \begin{pmatrix} x_i, h_{\mathcal{B}}(x_i)|p_{\mathcal{B}}(x_i), \\ g_{\mathcal{B}}(x_i)|q_{\mathcal{B}}(x_i) \end{pmatrix} = \bigcup_{\substack{\zeta_{\mathcal{B}_{i,s'}} \in h_{\mathcal{B}}, \\ \vartheta_{\mathcal{B}_{i,t'}} \in g_{\mathcal{B}}}} \left(x_i, \left\{\zeta_{\mathcal{B}_{i,s'}} \middle| p_{\mathcal{B}_{i,s'}}\right\}, \left\{\vartheta_{\mathcal{B}_{i,t'}} \middle| q_{\mathcal{B}_{i,t'}}\right\}\right)$ defined on $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$

(P1) The inequality $\mathcal{K}_1(\mathcal{A},\mathcal{B}) \geq 0$ holds straightforward, therefore, $\mathcal{K}_1(\mathcal{A},\mathcal{B}) \geq 0$. Now, from Eq. (11.3), we have

$$
\begin{aligned}
&\mathcal{C}ov(\mathcal{A},\mathcal{B})\\
=\;& \sum_{i=1}^{n}\left(\sum_{s=1}^{M_i}\sum_{s'=1}^{M'_i}\zeta_{\mathcal{A}_{i,s}}\zeta_{\mathcal{B}_{i,s'}}p_{i,ss'}+\sum_{t=1}^{N_i}\sum_{t'=1}^{N'_i}\vartheta_{\mathcal{A}_{i,t}}\vartheta_{\mathcal{B}_{i,t'}}q_{i,tt'}\right)\\
=\;& \sum_{i=1}^{n}\left(\begin{array}{l}\left\{\sum_{s=1}^{M_i}\left(\left(\zeta_{\mathcal{A}_{i,s}}\right)\left(\left(\zeta_{\mathcal{B}_{i,1}}\right)p_{i,s1}\right)+\sum_{s=1}^{M_i}\left(\left(\left(\zeta_{\mathcal{A}_{i,s}}\right)\left(\left(\zeta_{\mathcal{B}_{i,2}}\right)p_{i,s2}\right)+\ldots+\sum_{s=1}^{M_i}\left(\left(\zeta_{\mathcal{A}_{i,s}}\right)\left(\zeta_{\mathcal{B}_{i,M'_i}}\right)p_{i,sM'_i}\right)\right\}\\+\left\{\sum_{t=1}^{N_i}\left(\left(\vartheta_{\mathcal{A}_{i,t}}\right)\left(\vartheta_{\mathcal{B}_{i,1}}\right)q_{i,t1}\right)+\sum_{t=1}^{N_i}\left(\left(\vartheta_{\mathcal{A}_{i,t}}\right)\left(\vartheta_{\mathcal{B}_{i,2}}\right)q_{i,t2}\right)+\ldots+\sum_{t=1}^{N_i}\left(\left(\vartheta_{\mathcal{A}_{i,t}}\right)\left(\vartheta_{\mathcal{B}_{i,N'_i}}\right)q_{i,tN'_i}\right)\right\}\end{array}\right)\\
=\;& \sum_{i=1}^{n}\left(\begin{array}{l}\left(\left(\zeta_{\mathcal{A}_{i,1}}\right)\left(\zeta_{\mathcal{B}_{i,1}}\right)p_{i,11}\right)+\left(\left(\zeta_{\mathcal{A}_{i,2}}\right)\left(\zeta_{\mathcal{B}_{i,1}}\right)p_{i,21}\right)+\ldots+\left(\left(\zeta_{\mathcal{A}_{i,M_i}}\right)\left(\zeta_{\mathcal{B}_{i,1}}\right)p_{i,M_i1}\right)+\ldots\\+\left(\left(\zeta_{\mathcal{A}_{i,1}}\right)\left(\zeta_{\mathcal{B}_{i,M'_i}}\right)p_{i,1M'_i}\right)+\left(\left(\zeta_{\mathcal{A}_{i,2}}\right)\left(\zeta_{\mathcal{B}_{i,M'_i}}\right)p_{i,2M'_i}\right)+\ldots+\left(\left(\zeta_{\mathcal{A}_{i,M_i}}\right)\left(\zeta_{\mathcal{B}_{i,M'_i}}\right)p_{i,M_iM'_i}\right)\end{array}\right)\\
+\;& \sum_{i=1}^{n}\left(\begin{array}{l}\left(\left(\vartheta_{\mathcal{A}_{i,1}}\right)\left(\vartheta_{\mathcal{B}_{i,1}}\right)q_{i,11}\right)+\left(\left(\vartheta_{\mathcal{A}_{i,2}}\right)\left(\vartheta_{\mathcal{B}_{i,1}}\right)q_{i,21}\right)+\ldots+\left(\left(\vartheta_{\mathcal{A}_{i,N_i}}\right)\left(\vartheta_{\mathcal{B}_{i,1}}\right)q_{i,N_i1}\right)+\ldots\\+\left(\left(\vartheta_{\mathcal{A}_{i,1}}\right)\left(\vartheta_{\mathcal{B}_{i,N'_i}}\right)q_{i,1N'_i}\right)+\left(\left(\vartheta_{\mathcal{A}_{i,2}}\right)\left(\vartheta_{\mathcal{B}_{i,N'_i}}\right)q_{i,2N'_i}\right)+\ldots+\left(\left(\vartheta_{\mathcal{A}_{i,N_i}}\right)\left(\vartheta_{\mathcal{B}_{i,N'_i}}\right)q_{i,N_iN'_i}\right)\end{array}\right)\\
=\;& \left(\begin{array}{l}\zeta_{\mathcal{A}_{1,1}}\zeta_{\mathcal{B}_{1,1}}p_{1,11}+\\\ldots+\zeta_{\mathcal{A}_{n,1}}\zeta_{\mathcal{B}_{n,1}}p_{n,11}\end{array}\right)+\ldots+\left(\begin{array}{l}\zeta_{\mathcal{A}_{1,M_1}}\zeta_{\mathcal{B}_{1,1}}p_{1,M_11}+\\\ldots+\zeta_{\mathcal{A}_{n,M_n}}\zeta_{\mathcal{B}_{n,1}}p_{n,M_n1}\end{array}\right)+\left(\begin{array}{l}\zeta_{\mathcal{A}_{1,1}}\zeta_{\mathcal{B}_{1,M'_1}}p_{1,1M'_1}+\\\ldots+\zeta_{\mathcal{A}_{n,1}}\zeta_{\mathcal{B}_{n,M'_n}}p_{n,1M'_n}\end{array}\right)\\
+\;& \ldots+\left(\begin{array}{l}\zeta_{\mathcal{A}_{1,M_1}}\zeta_{\mathcal{B}_{1,M'_1}}p_{1,M_1M'_1}+\\\ldots+\zeta_{\mathcal{A}_{n,M_n}}\zeta_{\mathcal{B}_{n,M'_n}}p_{n,M_nM'_1}\end{array}\right)+\left(\begin{array}{l}\vartheta_{\mathcal{A}_{1,1}}\vartheta_{\mathcal{B}_{1,1}}q_{1,11}+\\\ldots+\vartheta_{\mathcal{A}_{n,1}}\vartheta_{\mathcal{B}_{n,1}}q_{n,11}\end{array}\right)+\ldots\\
+\;& \left(\begin{array}{l}\vartheta_{\mathcal{A}_{1,N_1}}\vartheta_{\mathcal{B}_{1,1}}q_{1,N_11}+\\\ldots+\vartheta_{\mathcal{A}_{n,N_n}}\vartheta_{\mathcal{B}_{n,1}}q_{n,N_n1}\end{array}\right)+\left(\begin{array}{l}\vartheta_{\mathcal{A}_{1,1}}\vartheta_{\mathcal{B}_{1,N'_1}}q_{1,1N'_1}+\\\ldots+\vartheta_{\mathcal{A}_{n,1}}\vartheta_{\mathcal{B}_{n,N'_n}}q_{n,1N'_n}\end{array}\right)+\ldots+\left(\begin{array}{l}\vartheta_{\mathcal{A}_{1,N_1}}\vartheta_{\mathcal{B}_{1,N'_1}}q_{1,N_1N'_1}+\\\ldots+\vartheta_{\mathcal{A}_{n,N_n}}\vartheta_{\mathcal{B}_{n,N'_n}}q_{n,N_nN'_1}\end{array}\right)\\
=\;& \left(\begin{array}{l}\zeta_{\mathcal{A}_{1,1}}\sqrt{p_{1,11}}\zeta_{\mathcal{B}_{1,1}}\sqrt{p_{1,11}}+\\\ldots+\zeta_{\mathcal{A}_{n,1}}\sqrt{p_{n,11}}\zeta_{\mathcal{B}_{n,1}}\sqrt{p_{n,11}}\end{array}\right)+\ldots+\left(\begin{array}{l}\zeta_{\mathcal{A}_{1,M_1}}\sqrt{p_{1,M_11}}\zeta_{\mathcal{B}_{1,1}}\sqrt{p_{1,M_11}}+\\\ldots+\zeta_{\mathcal{A}_{n,M_n}}\sqrt{p_{n,M_n1}}\zeta_{\mathcal{B}_{n,1}}\sqrt{p_{n,M_n1}}\end{array}\right)\\
+\;& \left(\begin{array}{l}\zeta_{\mathcal{A}_{1,1}}\sqrt{p_{1,1M'_1}}\zeta_{\mathcal{B}_{1,M'_1}}\sqrt{p_{1,1M'_1}}+\\\ldots+\zeta_{\mathcal{A}_{n,1}}\sqrt{p_{n,1M'_n}}\zeta_{\mathcal{B}_{n,M'_n}}\sqrt{p_{n,1M'_n}}\end{array}\right)+\ldots+\left(\begin{array}{l}\zeta_{\mathcal{A}_{1,M_1}}\sqrt{p_{1,M_1M'_1}}\zeta_{\mathcal{B}_{1,M'_1}}\sqrt{p_{1,M_1M'_1}}+\\\ldots+\zeta_{\mathcal{A}_{n,M_n}}\sqrt{p_{n,M_nM'_1}}\zeta_{\mathcal{B}_{n,M'_n}}\sqrt{p_{n,M_nM'_1}}\end{array}\right)\\
+\;& \left(\begin{array}{l}\vartheta_{\mathcal{A}_{1,1}}\sqrt{q_{1,11}}\vartheta_{\mathcal{B}_{1,1}}\sqrt{q_{1,11}}+\\\ldots+\vartheta_{\mathcal{A}_{n,1}}\sqrt{q_{n,11}}\vartheta_{\mathcal{B}_{n,1}}\sqrt{q_{n,11}}\end{array}\right)+\ldots+\left(\begin{array}{l}\vartheta_{\mathcal{A}_{1,N_1}}\sqrt{q_{1,N_11}}\vartheta_{\mathcal{B}_{1,1}}\sqrt{q_{1,N_11}}+\\\ldots+\vartheta_{\mathcal{A}_{n,N_n}}\sqrt{q_{n,N_n1}}\vartheta_{\mathcal{B}_{n,1}}\sqrt{q_{n,N_n1}}\end{array}\right)\\
+\;& \left(\begin{array}{l}\vartheta_{\mathcal{A}_{1,1}}\sqrt{q_{1,1N'_1}}\vartheta_{\mathcal{B}_{1,N'_1}}\sqrt{q_{1,1N'_1}}+\\\ldots+\vartheta_{\mathcal{A}_{n,1}}\sqrt{q_{n,1N'_n}}\vartheta_{\mathcal{B}_{n,N'_n}}\sqrt{q_{n,1N'_n}}\end{array}\right)+\ldots+\left(\begin{array}{l}\vartheta_{\mathcal{A}_{1,N_1}}\sqrt{q_{1,N_1N'_1}}\vartheta_{\mathcal{B}_{1,N'_1}}\sqrt{q_{1,N_1N'_1}}+\\\ldots+\vartheta_{\mathcal{A}_{n,N_n}}\sqrt{q_{n,N_nN'_1}}\vartheta_{\mathcal{B}_{n,N'_n}}\sqrt{q_{n,N_nN'_1}}\end{array}\right)
\end{aligned}
$$

Using Cauchy-Schwarz inequality, $(x_1y_1+x_2y_2+\ldots+x_ny_n)^2 \leq (x_1^2+x_2^2+\ldots+x_n^2)\cdot(y_1^2+y_2^2+\ldots+y_n^2)$, where $(x_1+x_2+\ldots+x_n)$ and $(y_1+y_2+\ldots+y_n) \in \mathbb{R}^n$, we get

$$
\begin{aligned}
&(\mathcal{C}ov(\mathcal{A},\mathcal{B})^2 \\
\leq\ & \left(\begin{array}{l}
\left(\begin{array}{l}\left(\zeta_{\mathcal{A}_{1,1}}\right)^2 p_{1,11}+\\ \ldots+\left(\zeta_{\mathcal{A}_{n,1}}\right)^2 p_{n,11}\end{array}\right)+\ldots+\left(\begin{array}{l}\left(\zeta_{\mathcal{A}_{1,M_1}}\right)^2 p_{1,M_11}+\\ \ldots+\left(\zeta_{\mathcal{A}_{n,M_n}}\right)^2 p_{n,M_n1}\end{array}\right)+\left(\begin{array}{l}\left(\zeta_{\mathcal{A}_{1,1}}\right)^2 p_{1,1M_1'}+\\ \ldots+\left(\zeta_{\mathcal{A}_{n,1}}\right)^2 p_{n,1M_n'}\end{array}\right)+\ldots\\
+\left(\begin{array}{l}\left(\zeta_{\mathcal{A}_{1,M_1}}\right)^2 p_{1,M_1M_1'}+\\ \ldots+\left(\zeta_{\mathcal{A}_{n,M_n}}\right)^2 p_{n,M_nM_n'}\end{array}\right)+\left(\begin{array}{l}\left(\vartheta_{\mathcal{A}_{1,1}}\right)^2 q_{1,11}+\\ \ldots+\left(\vartheta_{\mathcal{A}_{n,1}}\right)^2 q_{n,11}\end{array}\right)+\ldots+\left(\begin{array}{l}\left(\vartheta_{\mathcal{A}_{1,N_1}}\right)^2 q_{1,N_11}+\\ \ldots+\left(\vartheta_{\mathcal{A}_{n,N_n}}\right)^2 q_{n,N_n1}\end{array}\right)\\
+\left(\begin{array}{l}\left(\vartheta_{\mathcal{A}_{1,1}}\right)^2 q_{1,1N_1'}+\\ \ldots+\left(\vartheta_{\mathcal{A}_{n,1}}\right)^2 q_{n,1N_n'}\end{array}\right)+\ldots+\left(\begin{array}{l}\left(\vartheta_{\mathcal{A}_{1,N_1}}\right)^2 q_{1,N_1N_1'}+\\ \ldots+\left(\vartheta_{\mathcal{A}_{n,N_n}}\right)^2 q_{n,N_nN_n'}\end{array}\right)
\end{array}\right)\\
\times\ & \left(\begin{array}{l}
\left(\begin{array}{l}\left(\zeta_{\mathcal{B}_{1,1}}\right)^2 p_{1,11}+\\ \ldots+\left(\zeta_{\mathcal{B}_{n,1}}\right)^2 p_{n,11}\end{array}\right)+\ldots+\left(\begin{array}{l}\left(\zeta_{\mathcal{B}_{1,M_1}}\right)^2 p_{1,M_11}+\\ \ldots+\left(\zeta_{\mathcal{B}_{n,M_n}}\right)^2 p_{n,M_n1}\end{array}\right)+\left(\begin{array}{l}\left(\zeta_{\mathcal{B}_{1,1}}\right)^2 p_{1,1M_1'}+\\ \ldots+\left(\zeta_{\mathcal{B}_{n,1}}\right)^2 p_{n,1M_n'}\end{array}\right)+\ldots\\
+\left(\begin{array}{l}\left(\zeta_{\mathcal{B}_{1,M_1}}\right)^2 p_{1,M_1M_1'}+\\ \ldots+\left(\zeta_{\mathcal{B}_{n,M_n}}\right)^2 p_{n,M_nM_n'}\end{array}\right)+\left(\begin{array}{l}\left(\vartheta_{\mathcal{B}_{1,1}}\right)^2 q_{1,11}+\\ \ldots+\left(\vartheta_{\mathcal{B}_{n,1}}\right)^2 q_{n,11}\end{array}\right)+\ldots+\left(\begin{array}{l}\left(\vartheta_{\mathcal{B}_{1,N_1}}\right)^2 q_{1,N_11}+\\ \ldots+\left(\vartheta_{\mathcal{B}_{n,N_n}}\right)^2 q_{n,N_n1}\end{array}\right)\\
+\left(\begin{array}{l}\left(\vartheta_{\mathcal{B}_{1,1}}\right)^2 q_{1,1N_1'}+\\ \ldots+\left(\vartheta_{\mathcal{B}_{n,1}}\right)^2 q_{n,1N_n'}\end{array}\right)+\ldots+\left(\begin{array}{l}\left(\vartheta_{\mathcal{B}_{1,N_1}}\right)^2 q_{1,N_1N_1'}+\\ \ldots+\left(\vartheta_{\mathcal{B}_{n,N_n}}\right)^2 q_{n,N_nN_n'}\end{array}\right)
\end{array}\right)\\
=\ & \left(\begin{array}{l}
\sum_{i=1}^{n}\left(\begin{array}{l}\sum_{s'=1}^{M_i'}\left(\zeta_{\mathcal{A}_{i,1}}\right)^2 p_{i,1s'}+\sum_{s'=1}^{M_i'}\left(\zeta_{\mathcal{A}_{i,2}}\right)^2 p_{i,2s'}+\ldots+\sum_{s'=1}^{M_i'}\left(\zeta_{\mathcal{A}_{i,M_i}}\right)^2 p_{i,M_is'}\\ +\sum_{t'=1}^{N_i'}\left(\vartheta_{\mathcal{A}_{i,1}}\right)^2 q_{i,1t'}+\sum_{t'=1}^{N_i'}\left(\vartheta_{\mathcal{A}_{i,2}}\right)^2 q_{i,2t'}+\ldots+\sum_{t'=1}^{N_i'}\left(\vartheta_{\mathcal{A}_{i,N_i}}\right)^2 q_{i,N_it'}\end{array}\right)\\
\times\sum_{i=1}^{n}\left(\begin{array}{l}\sum_{s'=1}^{M_i'}\left(\zeta_{\mathcal{B}_{i,1}}\right)^2 p_{i,1s'}+\sum_{s'=1}^{M_i'}\left(\zeta_{\mathcal{B}_{i,2}}\right)^2 p_{i,2s'}+\ldots+\sum_{s'=1}^{M_i'}\left(\zeta_{\mathcal{B}_{i,M_i}}\right)^2 p_{i,M_is'}\\ +\sum_{t'=1}^{N_i'}\left(\vartheta_{\mathcal{B}_{i,1}}\right)^2 q_{i,1t'}+\sum_{t'=1}^{N_i'}\left(\vartheta_{\mathcal{B}_{i,2}}\right)^2 q_{i,2t'}+\ldots+\sum_{t'=1}^{N_i'}\left(\vartheta_{\mathcal{B}_{i,N_i}}\right)^2 q_{i,N_it'}\end{array}\right)
\end{array}\right)\\
=\ & \left(\begin{array}{l}
\sum_{i=1}^{n}\left(\sum_{s=1}^{M_i}\left(\zeta_{\mathcal{A}_{i,s}}\right)^2(p_{i,ss})+\sum_{t=1}^{N_i}\left(\vartheta_{\mathcal{A}_{i,t}}\right)^2(q_{i,tt})\right)\\
\times\sum_{i=1}^{n}\left(\sum_{s'=1}^{M_i'}(\zeta_{\mathcal{B}_{i,s'}})^2(p_{i,s's'})+\sum_{t'=1}^{N_i'}(\vartheta_{\mathcal{B}_{i,t'}})^2(q_{i,t't'})\right)
\end{array}\right)\\
=\ & \mathcal{I}(\mathcal{A})\times\mathcal{I}(\mathcal{B})
\end{aligned}
$$

Therefore, $(\mathcal{C}ov(\mathcal{A},\mathcal{B}))^2 \leq \mathcal{I}(\mathcal{A})\times\mathcal{I}(\mathcal{B})$. Thus, from Eq. (11.4), it follows that $\mathcal{K}_1(\mathcal{A},\mathcal{B})\leq 1$. Hence, $0\leq\mathcal{K}_1(\mathcal{A},\mathcal{B})\leq 1$.

(P2) Proof is obvious, so we omit it here.

(P3) Since $\mathcal{A} = \mathcal{B}$, i.e., for all $s = 1, 2, \ldots, M_i; t = 1, 2, \ldots, N_i; s' = 1, 2, \ldots, M_i'; t' = 1, 2, \ldots, N_i'$ we have $\zeta_{\mathcal{A}_{i,s}} = \zeta_{\mathcal{B}_{i,s'}}$, $p_{\mathcal{A}_{i,s}} = p_{\mathcal{B}_{i,s'}}$, $\vartheta_{\mathcal{A}_{i,t}} = \vartheta_{\mathcal{B}_{i,t'}}$, $q_{i,t} = q_{\mathcal{B}_{i,t'}}$, therefore,

$$\mathcal{I}(\mathcal{A}) = \sum_{i=1}^{n} \left(\sum_{s=1}^{M_i} \left(\zeta_{\mathcal{A}_{i,s}}\right)^2 p_{\mathcal{A}_{i,s}} + \sum_{t=1}^{N_i} \left(\vartheta_{\mathcal{A}_{i,t}}\right)^2 q_{\mathcal{A}_{i,t}} \right) = \mathcal{I}(\mathcal{B})$$

Since, $\mathcal{C}ov(\mathcal{A}, \mathcal{A}) = \mathcal{I}(\mathcal{A})$, thus we obtain $\mathcal{K}_1(\mathcal{A}, \mathcal{B}) = 1$.

□

In order to incorporate the pessimistic feature of the decision maker towards the process, we define a new CC by taking the maximum among the energies of the set. This is defined as below.

Definition 11.2.3. For two PDHFSs $\mathcal{A}$ and $\mathcal{B}$, the CC $\mathcal{K}_2$ is defined as:

$$\begin{aligned} \mathcal{K}_2(\mathcal{A}, \mathcal{B}) &= \frac{\mathcal{C}ov(\mathcal{A}, \mathcal{B})}{\max\{\mathcal{I}(\mathcal{A}), \mathcal{I}(\mathcal{B})\}} \\ &= \frac{\sum_{i=1}^{n} \left(\sum_{s=1}^{M_i} \sum_{s'=1}^{M_i'} \zeta_{\mathcal{A}_{i,s}} \zeta_{\mathcal{B}_{i,s'}} p_{i,ss'} + \sum_{t=1}^{N_i} \sum_{t'=1}^{N_i'} \vartheta_{\mathcal{A}_{i,t}} \vartheta_{\mathcal{B}_{i,t'}} q_{i,tt'} \right)}{\max\left\{ \sum_{i=1}^{n} \left(\sum_{s=1}^{M_i} \left(\zeta_{\mathcal{A}_{i,s}}\right)^2 p_{\mathcal{A}_{i,s}} + \sum_{t=1}^{N_i} \left(\vartheta_{\mathcal{A}_{i,t}}\right)^2 q_{\mathcal{B}_{i,t}} \right), \sum_{i=1}^{n} \left(\sum_{s'=1}^{M_i'} \left(\zeta_{\mathcal{B}_{i,s'}}\right)^2 p_{\mathcal{B}_{i,s'}} + \sum_{t'=1}^{N_i'} \left(\vartheta_{\mathcal{B}_{i,t'}}\right)^2 q_{\mathcal{B}_{i,t'}} \right) \right\}} \end{aligned} \quad (11.5)$$

Theorem 11.2.2. The CC $\mathcal{K}_2$ have the following properties:

(P1) $0 \leq \mathcal{K}_2(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{K}_2(\mathcal{A}, \mathcal{B}) = \mathcal{K}_2(\mathcal{B}, \mathcal{A})$.

(P3) $\mathcal{K}_2(\mathcal{A}, \mathcal{B}) = 1$, if $\mathcal{A} = \mathcal{B}$.

Proof. By Cauchy-Schwarz inequality:

$$\begin{aligned} \sum_{j=1}^{n} x_j y_j &\leq \sqrt{\left(\sum_{j=1}^{n} x_j^2\right) \cdot \left(\sum_{j=1}^{n} y_j^2\right)} \\ &\leq \sqrt{\left(\max\left\{\sum_{j=1}^{n} x_j^2, \sum_{j=1}^{n} y_j^2\right\}\right)^2} = \max\left\{\sum_{j=1}^{n} x_j^2, \sum_{j=1}^{n} y_j^2\right\} \end{aligned}$$

with equality if and only if the two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are linearly dependent.

Thus, by Eq. (11.5), we get $0 \leq \mathcal{K}_2(\mathcal{A},\mathcal{B}) \leq 1$.

Also,

$$\begin{aligned}
\mathcal{K}_2(\mathcal{A},\mathcal{B}) &= \frac{\sum_{i=1}^{n}\left(\sum_{s=1}^{M_i}\sum_{s'=1}^{M_i'}\zeta_{\mathcal{A}_{i,s}}\zeta_{\mathcal{B}_{i,s'}}p_{i,ss'}+\sum_{t=1}^{N_i}\sum_{t'=1}^{N_i'}\vartheta_{\mathcal{A}_{i,t}}\vartheta_{\mathcal{B}_{i,t'}}q_{i,tt'}\right)}{\max\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{M_i}\left(\zeta_{\mathcal{A}_{i,s}}\right)^2 p_{\mathcal{A}_{i,s}}+\sum_{t=1}^{N_i}\left(\vartheta_{\mathcal{A}_{i,t}}\right)^2 q_{\mathcal{A}_{i,t}}\right),\sum_{i=1}^{n}\left(\sum_{s'=1}^{M_i'}\left(\zeta_{\mathcal{B}_{i,s'}}\right)^2 p_{\mathcal{B}_{i,s'}}+\sum_{t'=1}^{N_i'}\left(\vartheta_{\mathcal{B}_{i,t'}}\right)^2 q_{\mathcal{B}_{i,t'}}\right)\right\}}\\
&= \frac{\sum_{i=1}^{n}\left(\sum_{s=1}^{M_i}\sum_{s'=1}^{M_i'}\zeta_{\mathcal{B}_{i,s'}}\zeta_{\mathcal{A}_{i,s}}p_{i,ss'}+\sum_{t=1}^{N_i}\sum_{t'=1}^{N_i'}\vartheta_{\mathcal{B}_{i,t'}}\vartheta_{\mathcal{A}_{i,t}}q_{i,tt'}\right)}{\max\left\{\sum_{i=1}^{n}\left(\sum_{s'=1}^{M_i'}\left(\zeta_{\mathcal{B}_{i,s'}}\right)^2 p_{\mathcal{B}_{i,s'}}+\sum_{t'=1}^{N_i'}\left(\vartheta_{\mathcal{B}_{i,t'}}\right)^2 q_{\mathcal{B}_{i,t'}}\right),\sum_{i=1}^{n}\left(\sum_{s=1}^{M_i}\left(\zeta_{\mathcal{A}_{i,s}}\right)^2 p_{\mathcal{A}_{i,s}}+\sum_{t=1}^{N_i}\left(\vartheta_{\mathcal{A}_{i,t}}\right)^2 q_{\mathcal{A}_{i,t}}\right)\right\}}\\
&= \mathcal{K}_2(\mathcal{B},\mathcal{A})
\end{aligned}$$

□

To illustrate the working of it, we give a numerical example as follows.

Example 11.2.1. Let $\mathcal{A}=\Big\{\big(x_1,(\{0.6|0.2,0.2|0.8\},\{0.3|0.6,0.4|0.4\})\big),\big(x_2,(\{0.2|0.3,0.4|0.7\},\{0.5|1\})\big)\Big\}$ and $\mathcal{B}=\Big\{\big(x_1,(\{0.3|0.5,0.4|0.5\},\ \{0.4|1\})\big),\big(x_2,(\{0.4|1\},\ \{0.3|0.6,0.5|0.4\})\big)\Big\}$ be two PDHFSs defined over $\mathcal{X}=\{x_1,x_2\}$. Then, by Eq. (11.1), we get

$$\begin{aligned}
\mathcal{I}(\mathcal{A}) &= \sum_{i=1}^{2}\left(\sum_{s=1}^{M_i}(\zeta_{i,s})^2 p_{i,s}+\sum_{t=1}^{M_i'}(\vartheta_{i,t})^2 q_{i,t}\right)\\
&= (0.6)^2\times 0.2+(0.2)^2\times 0.8+(0.3)^2\times 0.6+(0.4)^2\times 0.4\\
&+ (0.2)^2\times 0.3+(0.4)^2\times 0.7+(0.5)^2\times 1\\
&= 0.5960
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\mathcal{I}(\mathcal{B}) &= (0.3)^2\times 0.5+(0.4)^2\times 0.5+(0.4)^2\times 1+(0.4)^2\times 1\\
&+ (0.3)^2\times 0.6+(0.5)^2\times 0.4\\
&= 0.5990
\end{aligned}$$

Furthermore, by using North-west corner rule, we compute joint probabilities as

Joint probability calculation for membership values

x_1	0.3	0.4	$p_{\mathcal{A}_{1,M_i}}(i=1,2)$	x_2	0.4	$p_{\mathcal{A}_{2,M_i}}(i=1,2)$
0.6	0.2	0	0.2	0.3	0.6	0.6
0.2	0.3	0.5	0.8	0.4	0.4	0.4
$p_{\mathcal{B}_{1,M_i'}}(i=1,2)$	0.5	0.5		$p_{\mathcal{B}_{2,M_i'}}(i=1)$	1	

Based on it,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{s=1}^{M_i}\sum_{s'=1}^{M_i'}\zeta_{\mathcal{A}_{i,s}}\zeta_{\mathcal{B}_{i,s'}}p_{i,ss'} &= (0.6\times 0.3\times 0.2)+(0.2\times 0.3\times 0.3)+(0.2\times 0.4\times 0.5)\\ &+ (0.3\times 0.4\times 0.6)+(0.4\times 0.4\times 0.4)\\ &= 0.23\end{aligned}$$

Similarly , for non-membership values of $\mathcal{A}$ and $\mathcal{B}$, we have

Joint probability calculation for non-membership values

x_1	0.4	$q_{\mathcal{A}_{1,N_i}}(i=1,2)$	x_2	0.3	0.5	$q_{\mathcal{A}_{2,N_i}}(i=1)$
0.3	0.6	0.6	0.5	0.6	0.4	1
0.4	0.4	0.4				
$q_{\mathcal{B}_{1,N_i'}}(i=1)$	1		$q_{\mathcal{B}_{2,N_i'}}(i=1,2)$	0.6	0.4	

Thus,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{t=1}^{N_i}\sum_{t'=1}^{N_i'}\vartheta_{\mathcal{A}_{i,t}}\vartheta_{\mathcal{B}_{i,t'}}q_{i,tt'} &= (0.3\times 0.4\times 0.6)+(0.4\times 0.4\times 0.4)\\ &+ (0.5\times 0.3\times 0.6)+(0.5\times 0.5\times 0.4)\\ &= 0.3260\end{aligned}$$

Hence, by Eq. (11.3), we obtain $\mathcal{C}ov(\mathcal{A},\mathcal{B}) = 0.2300+0.3260 = 0.5560$. Therefore, Eq. (11.4) and Eq. (11.5) becomes $\mathcal{K}_1(\mathcal{A},\mathcal{B}) = \dfrac{0.5560}{\sqrt{0.5990}\sqrt{0.5960}} = 0.9305$ and $\mathcal{K}_2(\mathcal{A},\mathcal{B}) = \dfrac{0.5560}{\max\{0.5990, 0.5960\}} = 0.9282$.

In all above stated correlation formulae, equal priority is given to all the elements of universal set. This may not be relevant to the real life scenario, as we often, come across such entities which are given more weightage as compared to the other ones. To tackle such cases, we assign the weight $\omega_i > 0$ with $\sum_{i=1}^{n}\omega_i = 1$ to each of the element of $\mathcal{X}$ and define weighted CCs between two PDHFSs as follows:

$$\begin{aligned}\mathcal{K}_3(\mathcal{A},\mathcal{B}) &= \frac{\mathcal{C}ov_\omega(\mathcal{A},\mathcal{B})}{\sqrt{\mathcal{I}_\omega(\mathcal{A})}\sqrt{\mathcal{I}_\omega(\mathcal{B})}}\\ &= \frac{\sum_{i=1}^{n}\omega_i\left(\sum_{s=1}^{M_i}\sum_{s'=1}^{M_i'}\left(\zeta_{\mathcal{A}_{i,s}}\zeta_{\mathcal{B}_{i,s'}}p_{i,ss'}\right)+\sum_{t=1}^{N_i}\sum_{t'=1}^{N_i'}\left(\vartheta_{\mathcal{A}_{i,t}}\vartheta_{\mathcal{B}_{i,t'}}q_{i,tt'}\right)\right)}{\sqrt{\sum_{i=1}^{n}\omega_i\left(\sum_{s=1}^{M_i}\left(\left(\zeta_{\mathcal{A}_{i,s}}\right)^2p_{\mathcal{A}_{i,s}}\right)+\sum_{t=1}^{N_i}\left(\left(\vartheta_{\mathcal{A}_{i,t}}\right)^2q_{\mathcal{A}_{i,t}}\right)\right)}\sqrt{\sum_{i=1}^{n}\omega_i\left(\sum_{s'=1}^{M_i'}\left(\left(\zeta_{\mathcal{B}_{i,s'}}\right)^2p_{\mathcal{B}_{i,s'}}\right)+\sum_{t'=1}^{N_i'}\left(\left(\vartheta_{\mathcal{B}_{i,t'}}\right)^2q_{\mathcal{B}_{i,t'}}\right)\right)}}\end{aligned} \quad (11.6)$$

and

$$\begin{aligned}\mathcal{K}_4(\mathcal{A},\mathcal{B}) &= \frac{\mathcal{C}ov_\omega(\mathcal{A},\mathcal{B})}{\max\{\mathcal{I}_\omega(\mathcal{A}),\mathcal{I}_\omega(\mathcal{B})\}} \\ &= \frac{\sum_{i=1}^{n}\omega_i\left(\sum_{s=1}^{M_i}\sum_{s'=1}^{M_i'}\zeta_{\mathcal{A}_{i,s}}\zeta_{\mathcal{B}_{i,s'}}p_{i,ss'}+\sum_{t=1}^{N_i}\sum_{t'=1}^{N_i'}\vartheta_{\mathcal{A}_{i,t}}\vartheta_{\mathcal{B}_{i,t'}}q_{i,tt'}\right)}{\max\left\{\sum_{i=1}^{n}\omega_i\left(\sum_{s=1}^{M_i}\left(\left(\zeta_{\mathcal{A}_{i,s}}\right)^2p_{\mathcal{A}_{i,s}}\right)+\sum_{t=1}^{N_i}\left(\left(\vartheta_{\mathcal{A}_{i,t}}\right)^2q_{\mathcal{A}_{i,t}}\right)\right),\sum_{i=1}^{n}\omega_i\left(\sum_{s'=1}^{M_i'}\left(\left(\zeta_{\mathcal{B}_{i,s'}}\right)^2p_{\mathcal{B}_{i,s'}}\right)+\sum_{t'=1}^{N_i'}\left(\left(\vartheta_{\mathcal{B}_{i,t'}}\right)^2q_{\mathcal{B}_{i,t'}}\right)\right)\right\}}\end{aligned} \tag{11.7}$$

Also, if $\omega=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, then Eqs. (11.6) and (11.7) reduces to Eqs. (11.4) and (11.5) respectively.

Theorem 11.2.3. The coefficient defined in Eq. (11.6) have the following properties:

(P1) $0 \leq \mathcal{K}_3(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $\mathcal{K}_3(\mathcal{A},\mathcal{B}) = \mathcal{K}_3(\mathcal{B},\mathcal{A})$.

(P3) $\mathcal{K}_3(\mathcal{A},\mathcal{B}) = 1$, if $\mathcal{A}=\mathcal{B}$.

Proof. The properties (P2) and (P3) are straightforward, so we omit their proofs. Also, the inequality $\mathcal{K}_3(\mathcal{A},\mathcal{B}) \geq 0$ is evident since $\mathcal{C}ov_w(\mathcal{A},\mathcal{B}) \geq 0$, we shall show that $\mathcal{K}_3(\mathcal{A},\mathcal{B}) \leq 1$. For it, by definition of $\mathcal{C}ov_w(\mathcal{A},\mathcal{B})$ and by Cauchy-Schwarz inequality, we can easily deduce that

$$\begin{aligned}\mathcal{C}ov_w(\mathcal{A},\mathcal{B}) &= \sum_{i=1}^{n}\omega_i\left(\sum_{s=1}^{M_i}\sum_{s'=1}^{M_i'}\left(\zeta_{\mathcal{A}_{i,s}}\zeta_{\mathcal{B}_{i,s'}}p_{i,ss'}\right)+\sum_{t=1}^{N_i}\sum_{t'=1}^{N_i'}\left(\vartheta_{\mathcal{A}_{i,t}}\vartheta_{\mathcal{B}_{i,t'}}q_{i,tt'}\right)\right) \\ &\leq \sqrt{\sum_{i=1}^{n}\omega_i\left(\sum_{s=1}^{M_i}\left(\left(\zeta_{\mathcal{A}_{i,s}}\right)^2p_{\mathcal{A}_{i,s}}\right)+\sum_{t=1}^{N_i}\left(\left(\vartheta_{\mathcal{A}_{i,t}}\right)^2q_{\mathcal{A}_{i,t}}\right)\right)} \\ &\quad\times\sqrt{\sum_{i=1}^{n}\omega_i\left(\sum_{s'=1}^{M_i'}\left(\left(\zeta_{\mathcal{B}_{i,s'}}\right)^2p_{\mathcal{B}_{i,s'}}\right)+\sum_{t'=1}^{N_i'}\left(\left(\vartheta_{\mathcal{B}_{i,t'}}\right)^2q_{\mathcal{B}_{i,t'}}\right)\right)} \\ &\leq \sqrt{\mathcal{I}_w(\mathcal{A})\times\mathcal{I}_w(\mathcal{B})}\end{aligned}$$

Therefore, $\mathcal{K}_3(\mathcal{A},\mathcal{B}) \leq 1$. Hence, (P1) holds. □

Theorem 11.2.4. The CC $\mathcal{K}_4$ also satisfies the same properties as of Theorem 11.2.3.

Proof. Similar to above. □

From Definitions 11.2.2 and 11.2.3, it is observed that the CCs formulated in Eq. (11.4) uses the geometric mean of the informational energies of PDHFSs, whereas Eq. (11.5) considers the maximum energy possessing PDHFS. Thus, for the decision maker who is adopting an optimistic behavior, the CCs are given in Eq. (11.4) works well for without weighted criterion information and Eq. (11.6) works appropriately under the weighted criteria information. However, if the expert possesses pessimistic behavior, then Eqs. (11.5) and (11.7) works efficiently for the non-weighted and weighted criterion information respectively. Furthermore, in the DM process, an expert may provide their information either in terms of DHFSs or PDHFSs. So in order to integrate their values into the PDHFSs, we assign the probabilities to each element and then aggregate their values according to the procedure describe in Algorithm 9.1 of Chapter 9.

11.3 Decision making approach based on PDHF information

The general description of DM problem is same as Section 2.5 of Chapter 2 by a set of "d" decision makers. Each decision maker evaluate $\mathcal{V}_i$ under $\mathfrak{B}_j$ and provide their preferences in terms of PDHFEs $\mathcal{A}_{ij}^{(d)} = \left(h_{ij}^{(d)}|p_{ij}^{(d)}, g_{ij}^{(d)}|q_{ij}^{(d)}\right)$ where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Then, the rating of each alternative $\mathcal{V}_i$ under $\mathfrak{B}_j$ is expressed as

$$\mathcal{V}_i = \left\{ (\mathfrak{B}_1, \mathcal{A}_{i1}), (\mathfrak{B}_2, \mathcal{A}_{i2}), \ldots, (\mathfrak{B}_n, \mathcal{A}_{in}) \right\}, \tag{11.8}$$

Let $\omega_j > 0$ be the normalized weight vector of criteria $\mathfrak{B}_j$. Then, the following steps are executed to compute the best alternative based on the proposed measure.

Step 1: Arrange the information of each decision maker towards $\mathcal{V}_i$ in terms of decision matrices $\mathcal{M}^{(d)}$ as:

$$\mathcal{M}^{(d)} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathfrak{B}_1 \quad & \quad \mathfrak{B}_2 \quad & \ldots & \quad \mathfrak{B}_n \quad \end{array} \\ \begin{pmatrix} \left(h_{11}^{(d)}|p_{11}^{(d)}, g_{11}^{(d)}|q_{11}^{(d)}\right) & \left(h_{12}^{(d)}|p_{12}^{(d)}, g_{12}^{(d)}|q_{12}^{(d)}\right) & \ldots & \left(h_{1t}^{(d)}|p_{1t}^{(d)}, g_{1t}^{(d)}|q_{1t}^{(d)}\right) \\ \left(h_{21}^{(d)}|p_{21}^{(d)}, g_{21}^{(d)}|q_{21}^{(d)}\right) & \left(h_{22}^{(d)}|p_{22}^{(d)}, g_{22}^{(d)}|q_{22}^{(d)}\right) & \ldots & \left(h_{2t}^{(d)}|p_{2t}^{(d)}, g_{2t}^{(d)}|q_{2t}^{(d)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(h_{m1}^{(d)}|p_{m1}^{(d)}, g_{m1}^{(d)}|q_{m1}^{(d)}\right) & \left(h_{m2}^{(d)}|p_{m2}^{(d)}, g_{m2}^{(d)}|q_{m2}^{(d)}\right) & \ldots & \left(h_{mt}^{(d)}|p_{mt}^{(d)}, g_{mt}^{(d)}|q_{mt}^{(d)}\right) \end{pmatrix} \end{array}$$

Step 2: If $d = 1$, then $\left(h_{ij}^{(d)}|p_{ij}^{(d)}, g_{ij}^{(d)}|q_{ij}^{(d)}\right) = \left(h_{ij}|p_{ij}, g_{ij}|q_{ij}\right)$. On the other hand, if there are more than one decision maker i.e., when $d \geq 2$, then applying the Algorithm 9.1 of Chapter 9 to obtain the aggregated decision matrix $\mathcal{M} = (\mathcal{A}_{ij})$ from $\mathcal{M}^{(d)}$ as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \cdots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \left(h_{11}|p_{11}, g_{11}|q_{11}\right) & \left(h_{12}|p_{12}, g_{12}|q_{12}\right) & \cdots & \left(h_{1n}|p_{1n}, g_{1n}|q_{1n}\right) \\ \left(h_{21}|p_{21}, g_{21}|q_{21}\right) & \left(h_{22}|p_{22}, g_{22}|q_{22}\right) & \cdots & \left(h_{2n}|p_{2n}, g_{2n}|q_{2n}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(h_{m1}|p_{m1}, g_{m1}|q_{m1}\right) & \left(h_{m2}|p_{m2}, g_{m2}|q_{m2}\right) & \cdots & \left(h_{mn}|p_{mn}, g_{mn}|q_{mn}\right) \end{pmatrix} \end{array}$$

where $\mathcal{A}_{ij} = \left(h_{ij}|p_{ij}, g_{ij}|q_{ij}\right) = \bigcup_{\zeta_{\mathcal{A}_{ij,s}} \in h_{ij}, \vartheta_{\mathcal{A}_{ij,t}} \in g_{ij}} \left(\{\zeta_{\mathcal{A}_{ij,s}}|p_{\mathcal{A}_{ij,s}}\}, \{\vartheta_{\mathcal{A}_{ij,t}}|q_{\mathcal{A}_{ij,t}}\}\right)$, where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $s = 1, 2, \ldots, M_{ij}$ and $t = 1, 2, \ldots, N_{ij}$.

Step 3: Construct the ideal alternative $\mathcal{V}^*$ under the criteria $\mathfrak{B}_j$ as:

$$\mathcal{V}^* = \bigcup_{\zeta_{\mathcal{A}_{ij,s}} \in h_{ij}, \vartheta_{\mathcal{A}_{ij,t}} \in g_{ij}} \left(\max\left\{\left(\zeta_{\mathcal{A}_{ij,s}} \cdot p_{\mathcal{A}_{ij,s}}\right)\right\}, \min\left\{\left(\vartheta_{\mathcal{A}_{ij,t}} \cdot q_{\mathcal{A}_{ij,t}}\right)\right\} \right) \quad (11.9)$$

where $s = 1, 2, \ldots, M_{ij}$ and $t = 1, 2, \ldots, N_{ij}$.

Step 4: Compute the measurement values between $\mathcal{V}_i$ and $\mathcal{V}^*$ by utilizing either $\mathcal{K}_1$ or $\mathcal{K}_2$ or $\mathcal{K}_3$ or $\mathcal{K}_4$ as given in Eqs.(11.4), (11.5), (11.6) and (11.7). respectively .

Step 5: Ordering the alternatives with the maximum value of "$\arg\max \mathcal{K}$".

A pictorial representation of the proposed approach is given in the flowchart illustrated in Figure 11.1.

11.4 Case Study

For justifying the practical applicability of the approach proposed above, a case study based on personnel selection is considered in which a DM panel has to select a prospective candidate which suits best for the job. Personnel selection is a very prominent area of DM problems. In the practical DM processes, there arise many cases in which the best candidate has to be selected among a pool of contenders. Due to the complexity in rating criteria, there arise a lot of uncertain data which is needed to be addressed carefully to reach the desired accurate results.

Recruiting a prospective candidate for the survey projects is a prominent task carried out by Multi-national companies. Such kind of projects is basically survey oriented which can be broadly classified into two types: external survey projects and internal survey projects. In the external survey projects, the company analyzes the position of the external environment in accordance to which the company has to adapt itself to survive in the business market whereas the internal survey projects, thoroughly, focus on the internal environment of the company. In this, internal analysis of the company is conducted in figuring out several issues faced by the company such as employee turnover, job satisfaction level, company's revenue returns, etc. Preferably, for the unbiased internal survey, often a company hire an individual from outside the company so that an honest evaluation of the company's internal working can be made.

Suppose a Software Company desired to hire a Project Manager to pay his services in fulfillment of an Internal Survey project. In order to select the prospective candidate for the job, three experts were decided to give their assessment values. From a pool of applicants, 4 prospective candidates were shortlisted for the personal interviews. The panel has decided to evaluate the candidates $\mathcal{V}_i; (i = 1, 2, 3, 4)$ based on four criteria namely, $\mathfrak{B}_1$: 'Educational qualification'; $\mathfrak{B}_2$: 'Technical knowledge'; $\mathfrak{B}_3$: 'Communication skills'; $\mathfrak{B}_4$: 'Work experience'. All these criteria are accessed under the weighted criteria $\omega = (0.30, 0.40, 0.20, 0.10)^T$. The aim of the company is to recruit the best candidate for the post of Project Manager so that the project can be assigned to him and an internal survey can be conducted smoothly in the company. For it, the assessment ratings of applicants were provided by a panel of three experts in the form of PDHFEs which are given in Tables 11.1, 11.2, and 11.3. Since a number of decision makers are more than one, so by using Algorithm 9.1 of Chapter 9, the group PDHFEs are obtained and summarized in Table 11.4.

The set of information given in $\mathcal{V}^*$ is considered as reference standards. For it, by utilizing Eq. (11.9), we compute the rating values of this set and summarized in Table 11.5, which is used to compute the correlation indices for the alternatives.

By taking these preferences, the indices values corresponding to $\mathcal{K}_1$ and $\mathcal{K}_2$ are computed from set $\mathcal{V}^*$ to $\mathcal{V}_i$, $(i = 1, 2, 3, 4)$ and get:

$$\mathcal{K}_1(\mathcal{V}_1, \mathcal{V}^*) = 0.8920, \quad \mathcal{K}_1(\mathcal{V}_2, \mathcal{V}^*) = 0.9245,$$
$$\mathcal{K}_1(\mathcal{V}_3, \mathcal{V}^*) = 0.9196, \quad \mathcal{K}_1(\mathcal{V}_4, \mathcal{V}^*) = 0.9057$$

and

$$\mathcal{K}_2(\mathcal{V}_1, \mathcal{V}^*) = 0.9119, \quad \mathcal{K}_2(\mathcal{V}_2, \mathcal{V}^*) = 0.9170,$$
$$\mathcal{K}_2(\mathcal{V}_3, \mathcal{V}^*) = 0.9329, \quad \mathcal{K}_2(\mathcal{V}_4, \mathcal{V}^*) = 0.8939$$

Thus, the ordering is $\mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_1$ when $\mathcal{K}_1$ index has been used while $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4$ when $\mathcal{K}_2$ index has been utilized, where "$\succ$" refer "preferred to". As the ranking order is different by both the coefficients, so based on the inherent properties of these proposed coefficients, the decision maker may choose their goals according to their desire.

On the other hand, if $\omega = (0.30, 0.40, 0.20, 0.10)^T$ is taken, then by the expressions of $\mathcal{K}_3$ and $\mathcal{K}_4$, we get

$$\mathcal{K}_3(\mathcal{V}_1, \mathcal{V}^*) = 0.7485, \quad \mathcal{K}_3(\mathcal{V}_2, \mathcal{V}^*) = 0.8572,$$
$$\mathcal{K}_3(\mathcal{V}_3, \mathcal{V}^*) = 0.7533, \quad \mathcal{K}_3(\mathcal{V}_4, \mathcal{V}^*) = 0.7734$$

and

$$\mathcal{K}_4(\mathcal{V}_1, \mathcal{V}^*) = 0.7700, \quad \mathcal{K}_4(\mathcal{V}_2, \mathcal{V}^*) = 0.8530,$$
$$\mathcal{K}_4(\mathcal{V}_3, \mathcal{V}^*) = 0.8061, \quad \mathcal{K}_4(\mathcal{V}_4, \mathcal{V}^*) = 0.7843$$

Therefore, from the computed results we obtain the ranking $\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$ by utilizing correlation coefficient $\mathcal{K}_3$ and $\mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_1$ by using the correlation coefficient $\mathcal{K}_4$. Hence, by the maximum recognition principle, we conclude that best alternative is $\mathcal{V}_2$ while the relation between the alternatives $\mathcal{V}_3$ and $\mathcal{V}_4$ is selected according the risk aversive and the risk preferable states will give variable results in accordance to the decision-maker's attitude.

11.4.1 Comparative studies

To justify the superiority of our approach, this section consists of the comparative analysis with the existing approaches. It is noticeable that the probabilistic dual hesitant fuzzy sets can be compared to the existing studies [61, 155, 168, 201] under the different environments. To analyze our approach by analyzing it parallel to these approaches, a comparative analysis is listed below:

(i) The PDHFEs can be reduced to DHFEs by making probabilities of the membership and the non-membership portions equal within themselves. That is, for a PDHFE $\mathcal{A} = \bigcup (\{\zeta_i|p_i\}, \{\vartheta_k|q_k\})$ where $i = 1, 2, \ldots, M_i$ and $k = 1, 2, \ldots, N_i$, if $p_1 = p_2 = \ldots = p_{M_i} = p$ and $q_1 = q_2 = \ldots = q_{N_i} = q$, then it reduces to a DHFE. Based on this reduction, in accordance to the approach proposed by Wang et al. [155], the CC '$\mathcal{K}_3$' for the four alternatives is obtained as:

$$\mathcal{K}_3(\mathcal{V}_1, \mathcal{V}^*) = 0.9352, \quad \mathcal{K}_3(\mathcal{V}_2, \mathcal{V}^*) = 0.9033,$$

$$\mathcal{K}_3(\mathcal{V}_3, \mathcal{V}^*) = 0.9442, \quad \mathcal{K}_3(\mathcal{V}_4, \mathcal{V}^*) = 0.8892$$

Thus, alternatives are ranked as $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$. It is noticeable that alternatives' ranking varies with a huge difference from that of our proposed approach's outcome. This is due to the fact that in the existing theory probabilities corresponding to agreenness and dis-agreeness are not considered. The proposed approach is advantageous over the existing one [155] because, in the numerical evaluation of the existing theory, the length of two DHFEs is matched by repeating a particular entry in both membership and non-membership parts. This is done by first ordering the DHFEs into either descending order or ascending order and then according to the expert's optimistic or pessimistic viewpoint, the smaller value or the larger value is repeated until the length of two DHFEs under consideration becomes equal. This leads to redundancy of same data entries and increases the computational effort as well as different results from the proposed one. But, this repetition of the smaller or larger value to make the length equal is not required in our approach. It reduces the calculation overheads as each element is having its associated probability which can't be repeated over and over again and hence, it makes our approach inclined more towards the real-life scenarios.

(ii) Secondly, by converting the PDHFEs to DHFEs and by comparing the outcomes to that of approach given by Ye [201], it is noticed that the CC '$\mathcal{K}_3$' is obtained as:

$$\mathcal{K}_3(\mathcal{V}_1, \mathcal{V}^*) = 0.8831, \quad \mathcal{K}_3(\mathcal{V}_2, \mathcal{V}^*) = 0.8720,$$

$$\mathcal{K}_3(\mathcal{V}_3, \mathcal{V}^*) = 0.9121, \quad \mathcal{K}_3(\mathcal{V}_4, \mathcal{V}^*) = 0.8757$$

Thus, the alternatives are ranked as $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$. It can be seen that the best alternative does not remain the same as that of our proposed approach. This difference arises due to the difference in the ideal determination technique. In the existing approach [201], the ideal alternative is taken as $(\{1\}, \{0\})$, but in our approach, the ideal is determined in accordance with the informational energy. The alternatives attribute showing the largest informational energy are taken as the membership ideal value whereas the smallest information energy possessing alternative attribute is taken as non-membership ideal value as given in Eq. (11.9). By choosing an ideal alternative in such a way, there is no need to repeat the values to match the length as well as the ideal alternative is chosen in such a way that it synchronizes with the associated real-life DM problems.

(iii) The PDHFEs can be reduced to PHFEs by not considering the non-membership part. So, by converting the PDHFEs into PHFEs and by evaluating the alternatives in accordance to the approach proposed by Wang and Li [168], it is observed that in the existing approach, the CCs are calculated corresponding to each criterion separately. As per their outlined approach the obtained values for weighted CCs $(\mathcal{K}_\omega)$ are summarized in Table 11.6.

As per the ranking index (r_i) where $i = 1, 2, 3, 4$ proposed by Wang and Li [168] in which all the CC values corresponding to each alternative are added separately, the r_i's are obtained as:

$$r_1 = 3.5771, \quad r_2 = 3.7192, \quad r_3 = 3.3879, \quad r_4 = 3.4945$$

Hence, the alternatives are ranked as $\mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_3$. The best alternative coincides with our proposed approach. This is certainly because of considering the probabilistic membership values in PHFE. But the successive ranking order differs and this variation clearly signifies that by ignoring the degrees of disagreeness, results can show great deviations as compared to the case when the disagreeness is paid equal attention. Clearly, by considering the non-membership probabilistic hesitant values, the information can be knitted more closely to the practical situations giving the best alternative same as that of the existing theory. Thus, it is better to give equal priority to the non-membership values during the DM processes.

(iv) The approach followed by Hao et al. [61] is based on aggregating information available in form of PDHFEs. According to it, the alternatives are ranked as $\mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4$. So, the proposed approach's best alternative coincides with the existing theory. However, it is seen that the rest of the ranking obtained by this existing approach differs from that of the evaluation using the CC '$\mathcal{K}_3$'. This significant difference is because of the variation in processing the available PDHFEs in both the theories. In the existing one, an aggregation operator is used to compute the score values which leads to the ranking of the alternatives, but the proposed approach works on the CC.

11.4.2 Further discussion

In the below, we study the characteristic measures of the proposed approach with the existing approaches [61, 155, 168, 201] and results are tabulated in Table 11.7. In this, the symbol '$\checkmark$' represents that the associated properties satisfy while the symbol $\times$ represents the associated property fails. For example, it is clearly seen from the Table 11.7 that the MCDM method mentioned in Wang and Li [168] considers more than one experts, takes into account probabilistic values, have no data redundancy and also takes the non-trivial ideal alternative. But this approach does not consider the falsity values which is overcome in the proposed approach. On the other hand, the approaches by Wang et al. [155] and Ye [201] have considered the non-membership but they fail to capture all other characteristic features possessed by our approach such as one of them are not multi-expert DM approaches and do not capture probabilistic information with repeated data values in which their ordering, as well as the non-membership values, plays a significant role. Whereas, the approach outlined by Hao et al. [61] models multi-expert information and also contains non-membership. But, our approach is superior in the aspects of data-redundancy and ordering of data values before the evaluation. This discussion leads to the conclusion that the proposed method will efficiently handled and the solve the problems with respect to the existing methods [61, 155, 168, 201].

11.4.3 Advantages of the proposed approach

From the computed results, we highlights the following advantages of the method under the PDHFS environment.

(i) Since, PDHFEs can model the probabilities of each membership and non-membership values separately, so an expert can give a more flexible rating in which he is free to provide probabilistic preferences to the hesitant values of agreeness as well as disagreeness.

(ii) In the numerical calculation of the CCs using the proposed approach, there is no need of repeating the hesitant values of one set to match with the number of hesitant values of the second set in consideration. So, the computational overheads get reduced and the hesitant inputs become more inclined towards the real-life scenarios, i.e. the calculations are done with the elements and their respective probabilities by keeping the same as they have been acquired by the expert, but not by repeating the values again and again to get the desired results.

(iii) In the proposed approach the ideal alternative is chosen logically by considering the alternative having maximum informational energy for the membership hesitant values and the alternative with minimum informational energy values for the non-membership ones, classified under different criteria. So, the ideal alternative is viable in accordance to the practical situational gravity rather than fixing it to the extreme ideal alternative values such as $(\{1(1)\}, \{0(1)\})$ in case of PDHFEs and $(\{1\}, \{0\})$ in case of DHFEs.

(iv) Since, a PDHFE can be reduced to PHFE by not considering the non-membership values along with their associated probabilities and it can also be reduced to DHFE by considering the membership and non-membership hesitant values to be equi-probabilistic, thus, the proposed approach is a generalized version of the existing approaches based on these environments [155, 168, 201].

(v) Often in the multi-expert decision making approaches, there is a need of two weighted criteria one is the subjective weighted criteria and the other one is the objective weighted criteria. The subjective weighted criteria are used to aggregate the variable decisions taken by different experts to reach one final conclusion. But in the proposed approach, although there are more than one decision makers, still there is no need for any additional weighted criteria to figure out the collective decision taken by all the experts. Thus, it makes our approach more flexible, time-saving as well as having less computational overheads.

11.5 Conclusion

The key summary and contribution of this chapter is listed below:

1) PDHFS is a special DHFS where the membership and non-membership degrees of the element of the set has associated with the probabilities and can more easily describe the vagueness and uncertainty in the real world. Also, the several existing sets such as DHFS, HFSs, PHFS are considered as a special case of the PDHFSs. Thus, the PDHFS is a more generalized and successful concept for handling the uncertainties with both stochastic and fuzzy features.

2) In this chapter, the concept of CCs has been presented for measuring the relationships between two or more values. The advantages of the proposed measures are that it not only measures the strength between two or more PDHFEs but simultaneously it avoid the inconsistency of the decision makers results due to the loss of the information. Further, in the study, a method of north-west corner rule is utilized to compute the joint probabilities of the set.

3) a DM approach is developed for MCDM problem with probabilistic dual hesitant fuzzy information. Finally, to justify the practical resilience, the proposed method has been exemplified by a case study based on personnel selection. The comparative analysis with some of the existing studies [61, 155, 168, 201] has been conducted to show the availability and advantages of the proposed method.

From this chapter, it is concluded that PDHFS not only capture the decision maker preferences but also the corresponding probabilities under uncertain environment. Thus, due to these probabilities, this model can keep more detailed information and valuable results to the decision makers as compared to the other existing theories.

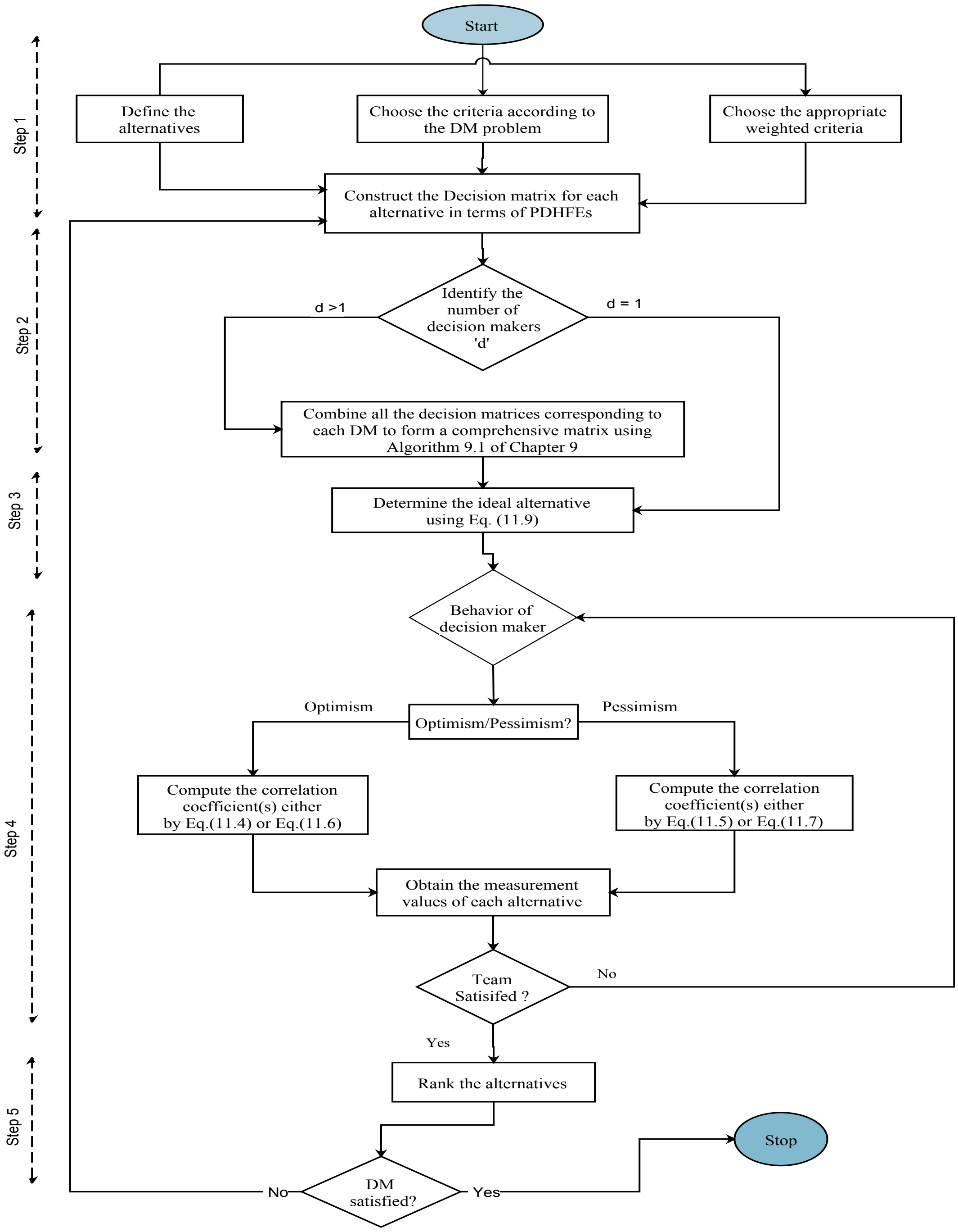

Figure 11.1: Flowchart of the proposed approach

Table 11.1: Probabilistic dual hesitant decision matrix for 1st decision maker

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\left(\{0.7\mid 0.7, 0.5\mid 0.3\}, \{0.3\mid 0.5, 0.2\mid 0.5\}\right)$	$\left(\{0.3\mid \frac{1}{3}, 0.2\mid \frac{1}{3}, 0.1\mid \frac{1}{3}\}, \{0.45\mid 0.5, 0.40\mid 0.5\}\right)$	$\left(\{0.4\mid \frac{1}{3}, 0.3\mid \frac{1}{3}, 0.1\mid \frac{1}{3}\}, \{0.2\mid 1\}\right)$	$\left(\{0.1\mid 1\}, \{0.4\mid 0.5, 0.3\mid 0.5\}\right)$
$\mathcal{V}_2$	$\left(\{0.4\mid 0.5, 0.35\mid 0.5\}, \{0.2\mid 0.5, 0.1\mid 0.5\}\right)$	$\left(\{0.5\mid 0.5, 0.4\mid 0.5\}, \{0.3\mid 0.5, 0.2\mid 0.5\}\right)$	$\left(\{0.4\mid 0.5, 0.2\mid 0.5\}, \{0.1\mid 1\}\right)$	$\left(\{0.55\mid \frac{1}{3}, 0.5\mid \frac{1}{3}, 0.40\mid \frac{1}{3}\}, \{0.2\mid 0.5, 0.1\mid 0.5\}\right)$
$\mathcal{V}_3$	$\left(\{0.5\mid 0.5, 0.4\mid 0.5\}, \{0.2\mid 0.5, 0.1\mid 0.5\}\right)$	$\left(\{0.4\mid 0.5, 0.1\mid 0.5\}, \{0.40\mid 0.5, 0.25\mid 0.5\}\right)$	$\left(\{0.40\mid 0.5, 0.30\mid 0.5\}, \{0.10\mid 1\}\right)$	$\left(\{0.2\mid 0.5, 0.1\mid 0.5\}, \{0.4\mid 0.5, 0.3\mid 0.5\}\right)$
$\mathcal{V}_4$	$\left(\{0.6\mid \frac{1}{3}, 0.5\mid \frac{1}{3}, 0.4\mid \frac{1}{3}\}, \{0.3\mid 0.5, 0.1\mid 0.5\}\right)$	$\left(\{0.30\mid 0.25, 0.1\mid 0.75\}, \{0.50\mid 0.5, 0.40\mid 0.5\}\right)$	$\left(\{0.50\mid 0.5, 0.40\mid 0.5\}, \{0.25\mid 0.5, 0.20\mid 0.5\}\right)$	$\left(\{0.25\mid 0.5, 0.15\mid 0.5\}, \{0.30\mid 1\}\right)$

Table 11.2: Probabilistic dual hesitant decision matrix provided for 2nd decision maker

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\left(\{0.40\mid 1\}, \{0.15\mid 1\}\right)$	$\left(\{0.40\mid 0.5, 0.20\mid 0.5\}, \{0.10\mid 1\}\right)$	$\left(\{0.20\mid 0.5, 0.10\mid 0.5\}, \{0.30\mid 1\}\right)$	$\left(\{0.30\mid 0.5, 0.10\mid 0.5\}, \{0.50\mid 1\}\right)$
$\mathcal{V}_2$	$\left(\{0.30\mid 0.5, 0.10\mid 0.5\}, \{0.60\mid 1\}\right)$	$\left(\{0.60\mid 0.5, 0.20\mid 0.5\}, \{0.15\mid 0.5, 0.10\mid 0.5\}\right)$	$\left(\{0.25\mid 0.5, 0.15\mid 0.5\}, \{0.10\mid 1\}\right)$	$\left(\{0.75\mid \frac{1}{3}, 0.65\mid \frac{1}{3}, 0.60\mid \frac{1}{3}\}, \{0.25\mid 0.5, 0.10\mid 0.5\}\right)$
$\mathcal{V}_3$	$\left(\{0.40\mid 0.5, 0.30\mid 0.5\}, \{0.10\mid 1\}\right)$	$\left(\{0.5\mid 1\}, \{0.3\mid 0.5, 0.2\mid 0.5\}\right)$	$\left(\{0.20\mid 0.5, 0.15\mid 0.5\}, \{0.40\mid 0.5, 0.30\mid 0.5\}\right)$	$\left(\{0.35\mid 0.50, 0.30\mid 0.50\}, \{0.20\mid 1\}\right)$
$\mathcal{V}_4$	$\left(\{0.30\mid 0.5, 0.10\mid 0.5\}, \{0.40\mid 1\}\right)$	$\left(\{0.30\mid 0.5, 0.10\mid 0.5\}, \{0.40\mid 1\}\right)$	$\left(\{0.20\mid \frac{1}{3}, 0.15\mid \frac{1}{3}, 0.10\mid \frac{1}{3}\}, \{0.05\mid 1\}\right)$	$\left(\{0.30\mid 1\}, \{0.20\mid 0.5, 0.10\mid 0.5\}\right)$

Table 11.3: Probabilistic dual hesitant decision matrix for 3rd decision maker

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\left(\{0.40\|1\}, \{0.2\|0.5, 0.1\|0.5\}\right)$	$\left(\{0.20\|0.5, 0.10\|0.5\}, \{0.30\|1\}\right)$	$\left(\{0.25\|0.5, 0.20\|0.5\}, \{0.10\|1\}\right)$	$\left(\{0.5\|\frac{1}{3}, 0.4\|\frac{1}{3}, 0.3\|\frac{1}{3}\}, \{0.2\|0.5, 0.1\|0.5\}\right)$
$\mathcal{V}_2$	$\left(\{0.30\|0.10, 0.20\|0.90\}, \{0.10\|1\}\right)$	$\left(\{0.5\|0.5, 0.4\|0.5\}, \{0.3\|1\}\right)$	$\left(\{0.3\|1\}, \{0.5\|0.5, 0.4\|0.5\}\right)$	$\left(\{0.4\|\frac{1}{3}, 0.3\|\frac{1}{3}, 0.1\|\frac{1}{3}\}, \{0.15\|1\}\right)$
$\mathcal{V}_3$	$\left(\{0.4\|1\}, \{0.20\|0.5, 0.10\|0.5\}\right)$	$\left(\{0.3\|\frac{1}{3}, 0.2\|\frac{1}{3}, 0.1\|\frac{1}{3}\}, \{0.40\|1\}\right)$	$\left(\{0.20\|0.5, 0.10\|0.5\}, \{0.40\|1\}\right)$	$\left(\{0.3\|0.5, 0.2\|0.5\}, \{0.15\|1\}\right)$
$\mathcal{V}_4$	$\left(\{0.45\|0.5, 0.30\|0.5\}, \{0.25\|0.5, 0.20\|0.5\}\right)$	$\left(\{0.30\|0.5, 0.20\|0.5\}, \{0.10\|1\}\right)$	$\left(\{0.70\|\frac{1}{3}, 0.60\|\frac{1}{3}, 0.50\|\frac{1}{3}\}, \{0.30\|1\}\right)$	$\left(\{0.35\|0.5, 0.20\|0.5\}, \{0.10\|1\}\right)$

Table 11.4: Group probabilistic dual hesitant decision matrix

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$(\{0.70\mid 0.233, 0.50\mid 0.1, 0.40\mid 0.6667\}, \{0.30\mid 0.1667, 0.20\mid 0.3333, 0.15\mid 0.3333, 0.10\mid 0.1667\})$	$(\{0.40\mid 0.1667, 0.30\mid 0.1111, 0.20\mid 0.4444, 0.10\mid 0.2778\}, \{0.45\mid 0.1667, 0.40\mid 0.1667, 0.30\mid 0.3333, 0.10\mid 0.3333\})$	$(\{0.40\mid 0.1111, 0.30\mid 0.1111, 0.25\mid 0.1667, 0.20\mid 0.3333, 0.10\mid 0.2778\}, \{0.30\mid 0.3333, 0.20\mid 0.3333, 0.10\mid 0.3334\})$	$(\{0.50\mid 0.1111, 0.40\mid 0.1111, 0.30\mid 0.2778, 0.10\mid 0.5\}, \{0.50\mid 0.3333, 0.40\mid 0.1667, 0.30\mid 0.1667, 0.20\mid 0.1667, 0.10\mid 0.1666\})$
$\mathcal{V}_2$	$(\{0.40\mid 0.1667, 0.35\mid 0.1667, 0.30\mid 0.20, 0.20\mid 0.30, 0.10\mid 0.1666\}, \{0.60\mid 0.3333, 0.20\mid 0.1667, 0.10\mid 0.5000\})$	$(\{0.60\mid 0.1667, 0.50\mid 0.3333, 0.40\mid 0.3333, 0.20\mid 0.1667,\}, \{0.30\mid 0.5, 0.20\mid 0.1667, 0.15\mid 0.1667, 0.10\mid 0.1666\})$	$(\{0.40\mid 0.1667, 0.30\mid 0.3333, 0.25\mid 0.1666, 0.20\mid 0.1667, 0.15\mid 0.1667\}, \{0.50\mid 0.1667, 0.40\mid 0.1667, 0.10\mid 0.6666\})$	$(\{0.75\mid 0.1111, 0.65\mid 0.1111, 0.60\mid 0.1111, 0.55\mid 0.1111, 0.50\mid 0.1111, 0.40\mid 0.2223, 0.30\mid 0.1111, 0.10\mid 0.1111\}, \{0.25\mid 0.1667, 0.20\mid 0.1667, 0.15\mid 0.3333, 0.10\mid 0.3333\})$
$\mathcal{V}_3$	$(\{0.50\mid 0.1667, 0.40\mid 0.6667, 0.30\mid 0.1666\}, \{0.20\mid 0.3333, 0.10\mid 0.6667\})$	$(\{0.50\mid 0.3333, 0.40\mid 0.1667, 0.30\mid 0.1111, 0.20\mid 0.1111, 0.10\mid 0.2778\}, \{0.40\mid 0.5000, 0.30\mid 0.1667, 0.25\mid 0.1667, 0.20\mid 0.1666\})$	$(\{0.40\mid 0.1667, 0.30\mid 0.1667, 0.20\mid 0.3333, 0.15\mid 0.1667, 0.10\mid 0.1666\}, \{0.40\mid 0.5, 0.30\mid 0.1667, 0.10\mid 0.3333\})$	$(\{0.35\mid 0.1667, 0.30\mid 0.3333, 0.20\mid 0.3333, 0.10\mid 0.1667\}, \{0.40\mid 0.1667, 0.30\mid 0.1667, 0.20\mid 0.3333, 0.15\mid 0.3333\})$
$\mathcal{V}_4$	$(\{0.60\mid 0.1111, 0.50\mid 0.1111, 0.45\mid 0.1667, 0.40\mid 0.1111, 0.30\mid 0.3333, 0.10\mid 0.1667\}, \{0.40\mid 0.3333, 0.30\mid 0.1667, 0.25\mid 0.1667, 0.20\mid 0.1667, 0.10\mid 0.1666\})$	$(\{0.30\mid 0.4167, 0.20\mid 0.1667, 0.10\mid 0.4166\}, \{0.50\mid 0.1667, 0.40\mid 0.5000, 0.10\mid 0.3333\})$	$(\{0.70\mid 0.1111, 0.60\mid 0.1111, 0.50\mid 0.2778, 0.40\mid 0.1666, 0.20\mid 0.1111, 0.15\mid 0.1111, 0.10\mid 0.1111\}, \{0.30\mid 0.3333, 0.25\mid 0.1667, 0.20\mid 0.1667, 0.05\mid 0.3333\})$	$(\{0.35\mid 0.1667, 0.30\mid 0.3333, 0.25\mid 0.1667, 0.20\mid 0.1667, 0.15\mid 0.1666\}, \{0.30\mid 0.3333, 0.20\mid 0.1667, 0.10\mid 0.5\})$

Table 11.5: Rating values of the ideal set $\mathcal{V}^*$

$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\left(\left\{\begin{matrix}0.70\|0.2333,\\0.50\|0.1,\\0.40\|0.6667\end{matrix}\right\},\left\{\begin{matrix}0.20\|0.3333,\\0.10\|0.6667\end{matrix}\right\}\right)$	$\left(\left\{\begin{matrix}0.60\|0.1667,\\0.50\|0.3333,\\0.40\|0.3333,\\0.20\|0.1667\end{matrix}\right\},\left\{\begin{matrix}0.30\|0.50,\\0.20\|0.1666,\\0.15\|0.1666,\\0.10\|0.1666\end{matrix}\right\}\right)$	$\left(\left\{\begin{matrix}0.70\|0.1111, 0.60\|0.1111,\\0.50\|0.2766, 0.40\|0.1666,\\0.20\|0.1111, 0.15\|0.1111,\\0.10\|0.1111\end{matrix}\right\},\left\{\begin{matrix}0.30\|0.3333,\\0.20\|0.3333,\\0.10\|0.3334\end{matrix}\right\}\right)$	$\left(\left\{\begin{matrix}0.75\|0.1111, 0.65\|0.1111,\\0.60\|0.1111, 0.55\|0.1111,\\0.50\|0.1111, 0.40\|0.2223,\\0.30\|0.1111, 0.10\|0.1111\end{matrix}\right\},\left\{\begin{matrix}0.25\|0.1667,\\0.20\|0.1667,\\0.15\|0.3333,\\0.10\|0.3333\end{matrix}\right\}\right)$

Table 11.6: Correlation coefficients in accordance to Wang and Li [168]

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{K}_\omega(\mathcal{V}_1, \mathcal{V}^*)$	1	0.8348	0.9322	0.8100
$\mathcal{K}_\omega(\mathcal{V}_2, \mathcal{V}^*)$	0.7553	1	0.9639	1
$\mathcal{K}_\omega(\mathcal{V}_3, \mathcal{V}^*)$	0.6935	0.8684	0.9072	0.9189
$\mathcal{K}_\omega(\mathcal{V}_4, \mathcal{V}^*)$	0.7461	0.8052	1	0.9433

Table 11.7: Characteristic comparison of the proposed approach with different methods

Methods	Whether consider more than one decision maker	Whether considers the probabilities	No data redundancy	No ordering of data values	Non - trivial ideal alternative	Considers non- -membership degrees
Wang et al. [155]	×	×	×	×	×	✓
Ye [201]	×	×	×	×	×	✓
Wang and Li [168]	✓	✓	✓	✓	✓	×
Hao et al. [61]	✓	✓	×	×	×	✓
Our proposed approach	✓	✓	✓	✓	✓	✓

Chapter 12

Summary and Future Work

This chapter recapitulate and summarizes the research contribution made during the period of thesis. It outlines a clear vision on association of the chapters constituted in our presented research work. In addition to it, the scope of possible realms of future work is provided.

12.1 Summary of the work

In this thesis, we have discussed the concept of cubic intuitionistic fuzzy sets (CIFSs) and probabilistic dual hesitant fuzzy sets (PDHFSs). Both these environments imbibe a remarkable property to capture the uncertain information in which non-membership entities hold a significant place. In Chapters 3 to 8, we have extensively covered the concept of CIFS with its aggregation operators as well as information measures, however, Chapters 9 to 11 accommodate the work on PDHFSs. The conclusion and the relevant contributions of the work are summarized below:

1) The research work equipped in the thesis is an attempt to contemplate the available information in an efficient manner. As per the literature, several shortcomings are found in the existing studies in capturing the uncertain information in sensitive DM scenarios. We present the notion of CIFSs in Chapter 3 to contemplate the two-dimensional information. To carry out smooth DM process based on CIFSs, in Chapters 4 to 8, work is done in regard of introducing various DM algorithms and techniques which inherit the features of both IFS as well as IVIFSs. Subsequently, we also gauged the

advanced functionality of PDHFSs in preventing the information loss in delicate DM situations, so we worked on framing advanced techniques based on it in Chapters 9 to 11. With the complete descriptive study of these two environments, we can accomplish the task of addressing the uncertainties in a powerful manner with the characteristics of generalization of the existing fuzzy set environments.

2) In literature, authors have worked on proposing different aggregation operators subjected under various norm operations. Moreover, the existing theories being deficient of capturing the minute details of the situational factors motivated us to work in the direction of proposing various aggregation operators and their generalized formats, listed in Chapters 3 and 4, subjected under CIFS preferences. In addition to it, the advanced operators in capturing the inter-relationship among different pairs subjected under variable criteria information occupied a huge significance in literature. So, in Chapter 6, we worked on proposing Bonferroni mean operators based on CIFS environment. A brief list of the proposed operators under CIFSs is given below:

   a) Cubic intuitionistic fuzzy weighted averaging (CIFWA) and geometric (CIFWG) operators

   b) Cubic intuitionistic fuzzy ordered weighted averaging (CIFOWA) and geometric (CIFOWG) operators

   c) Cubic intuitionistic fuzzy hybrid averaging (CIFHA) and geometric (CIFHG) operators

   d) Generalized cubic intuitionistic fuzzy weighted averaging (GCIFWA) operator

   e) Generalized cubic intuitionistic fuzzy ordered weighted averaging (GCIFOWA) operator

   f) Generalized cubic intuitionistic fuzzy hybrid averaging (GCIFHA) operator

   g) Weighted cubic intuitionistic fuzzy Bonferroni mean (WCIFBM) operator

3) Envisioned with the characteristics possessed by information measures, in Chapter 7 to 8 work has been undertaken on proposing various information measures and their

application in advanced DM methodologies such as TOPSIS. In light of it, work has been done on both linear and non-linear versions of TOPSIS approach.

4) Foreseeing the need of constituting the delicate DM situations under the hesitant environments, we worked on proposing both Einstein aggregation operators as well as distance measures under PDHFSs, in Chapter 9, suppressing the shortcomings of data redundancy in the existing hesitant environment related computational techniques. In addition to it, we worked on proposing Maclaurin symmetric mean average operators in Chapter 10 and introduced correlation coefficients in Chapter 11. A brief list of the proposed operators under PDHFSs is given below:

    a) Probabilistic dual hesitant fuzzy weighted Einstein average (PDHFWEA) and geometric (PDHFWEG) operators

    b) Probabilistic dual hesitant fuzzy ordered weighted Einstein average (PDHFOWEA) and geometric (PDHFOWEG) operators

    c) Weighted probabilistic dual hesitant fuzzy Maclaurin symmetric mean averaging (WPDHFMSMA) and geometric (WPDHFMSMG) operators

5) For the relevance of the proposed methodologies and justifying their application capabilities we have worked on quantifying and capturing uncertainties in different fields such as job scheduling (Chapter 4), recruitment/selection process (Chapters 5, 7 and 11), inventory management (Chapter 6), SONAR signal processing (Chapter 8), consumer's buying behavior (Chapter 9) and gesture quantification in brain hemorrhage patients (Chapter 10).

12.2 Future scope of the work

In light of the work done for the presented research, the outline of the possible future directions related to the field are given below:

1) The presented methodologies based on aggregation operators can be further extended to averaging and geometric versions of power, prioritized and Hamy mean operators.

2) The basic aggregation operators based on PDHFSs can be extended for generalized versions based on t-norm and t-conorm containing operators.

3) The presented methodologies can be further enhanced by introducing different information measures such as divergence and similarity measures.

4) There is scope of attaching the linguistic preference information containing corresponding environments and work can be proceeded in similar manner.

5) There are never-ending potential applied genres such as deep learning, artificial intelligence, pattern recognition, economical problems, medical problems, supplier selection etc, under which the applications of the proposed work can be investigated.

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