Symmetry Analysis of Nonlinear Fractional Partial Differential Equations

Thesis

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by

KOMAL SINGLA

(Registration No.: 901311009)

to the

SCHOOL OF MATHEMATICS
THAPAR UNIVERSITY, PATIALA - 147004
PUNJAB, INDIA
JULY-2017

DEDICATED

TO

MY PARENTS

Certificate

This is to certify that the thesis entitled "**Symmetry Analysis of Nonlinear Fractional Partial Differential Equations**", submitted by **Ms. Komal Singla** in the partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy in School of Mathematics, Thapar University, Patiala, is a record of candidate's own work carried out by her under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in other University or Institute.

Attestation by Supervisor

Dr. Rajesh Kumar Gupta
Associate Professor
Centre for Mathematics and Statistics
School of Basic and Applied Sciences
Central University of Punjab, Bathinda- 151001
Punjab, INDIA

Declaration

It is certified that the thesis is entirely my own and the idea and references cited herein have been duly acknowledged.

Komal Singla

Komal Singla

(Regn. No. 901311009)

Attestation by Supervisor

Dr. Rajesh Kumar Gupta

Associate Professor

Centre for Mathematics and Statistics

School of Basic and Applied Sciences

Central University of Punjab, Bathinda- 151001

Punjab, INDIA

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Komal Singla

Komal Singla

Abstract

Fractional calculus is a branch of mathematics that deals with real number or complex number powers of the differential operator and integral operator. Although the idea of fractional calculus was born more than 300 years ago, serious efforts have been dedicated to its study recently. Fractional differential equations (FDEs) are generalization of the differential equations of integer order, studied through the theory of fractional calculus. Lie symmetry method is a powerful technique for solving integer order differential equations. In this thesis, its various extensions are proposed for the symmetry analysis of nonlinear systems of FDEs. The aim of this thesis is to extend the symmetry approach in order to apply them to a wider class of FDEs including time fractional nonlinear systems, space-time fractional nonlinear systems, higher dimensional nonlinear systems, and variable coefficient nonlinear systems.

The thesis consists of six chapters comprising various novel extensions and applications of the symmetry method. **Chapter 1** provides the history of fractional calculus, basic definitions, and properties of the Riemann-Liouville fractional operators used in this study. The main features, background and methodology of the Lie classical method by Sophus Lie are also discussed in the introductory chapter.

Chapter 2 deals with the extension of Lie symmetry method for studying

time fractional systems of partial differential equations (PDEs). The prolongation formulae given in a recent paper [86] for symmetry analysis of time fractional systems are proved incomplete and the correct formulae are suggested in this chapter. The prolongation operators are derived for time fractional systems having two independent and an arbitrary number of dependent variables. Also, the technique to investigate nonlinear self-adjointness and conservation laws is extended for time fractional systems of PDEs. The proposed methods are applied for the symmetry analysis and derivation of conserved vectors of five time fractional nonlinear systems of PDEs including Ito system, Burgers system, coupled KdV system, Hirota-Satsuma coupled KdV system, coupled Hirota equations. As a result, these systems are reduced into fractional nonlinear systems of ordinary differential equations (ODEs).

Chapter 3 is devoted to extending the Lie group method and Noether operators for computing Lie symmetries and conserved vectors of space-time fractional PDEs. The complete Lie group classification is performed and concept of nonlinear self-adjointness is extended for space-time fractional PDEs. Two space-time fractional nonlinear PDEs namely Gilson-Pickering equation and generalized KdV equation are studied for their Lie symmetries resulting in their reductions into fractional nonlinear ODEs in the Erdélyi-Kober operators. In addition, the conservation laws for both the fractional partial differential equations (FPDEs) are obtained successfully.

Chapter 4 is concerned with the investigation of space-time fractional nonlinear systems of PDEs for their Lie symmetry analysis. For this purpose, the symmetry method is proposed for space-time fractional systems of PDEs by derivation of the required prolongations. Using the extended prolongation operators, the group infinitesimals for five space-time fractional nonlinear systems are successfully calculated. The resulting group invariant solutions are used to obtain their symmetry reductions into nonlinear systems of fractional ordinary differential equations (FODEs). The discussed fractional nonlinear systems of PDEs are as follows: space-time fractional Ito system, space-time fractional coupled Burgers equations, space-time fractional coupled KdV system, space-time fractional Hirota-Satsuma coupled KdV system, space-time fractional coupled Hirota equations.

In **chapter 5**, a generalized symmetry approach is proposed for systems of FDEs having an arbitrary number of independent as well as dependent variables. The derivation of the prolongation operators is discussed for generalized fractional order systems. The symmetry analysis of higher dimensional systems can be discussed using the suggested approach. The efficiency of the presented symmetry method is proved by its application to five higher dimensional nonlinear systems namely (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov (ANNV) system, (3+1)-dimensional Burgers system, (3+1)-dimensional Navier-Stokes system, (3+1)-dimensional fractional incompressible non-hydrostatic Boussinesq system, fractional (3+1)-dimensional incompressible non-hydrostatic Boussinesq system with viscosity. Their symmetries and symmetry reductions into lower dimensional nonlinear fractional order systems are deduced in terms of extended Erdélyi-Kober fractional operators systematically.

In **chapter 6**, certain variable coefficient fractional nonlinear PDEs are investigated using the Lie group analysis. The complete group classification of fractional nonlinear variable coefficient PDEs is demonstrated for some single time fractional PDEs as well as systems of time fractional PDEs with variable coefficients. The studied variable coefficient fractional nonlinear PDEs include KdV-Burger-Kuramoto equation and generalized seventh order KdV equation. The considered time fractional nonlinear systems

of PDEs with variable coefficients are coupled Boussinesq system, coupled KdV system and Hirota-Satsuma coupled KdV system. After computing their group infinitesimals, the optimal sets of inequivalent one dimensional subalgebras are calculated. Finally, for each component of the optimal set, the similarity reductions of the considered variable coefficient fractional PDEs are obtained successfully.

List of Research Papers

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2. **Komal Singla**, R. K. Gupta, “Space time fractional nonlinear partial differential equations: symmetry analysis and conservation laws”, *Nonlinear Dynamics*, 89, 321-331 (2017) (**Springer, Impact Factor: 3.464**) (**SCI**).
3. **Komal Singla**, R. K. Gupta, “On invariant analysis of space-time fractional nonlinear systems of partial differential equations. II”, *Journal of Mathematical Physics*, 58, 051503 (2017) (**AIP Publishers, Impact Factor: 1.077**) (**SCI**).
4. **Komal Singla**, R. K. Gupta, “Comment on ‘Lie symmetries and group classification of a class of time fractional evolution systems’ [JMP 56, 123504 (2015)]”, *Journal of Mathematical Physics*, 58, 054101 (2017) (**AIP Publishers, Impact Factor: 1.077**) (**SCI**).
5. **Komal Singla**, R. K. Gupta, “Conservation laws for certain time fractional nonlinear systems of partial differential equations”, *Communications in Nonlinear Science and Numerical Simulation*, 53, 10–21 (2017) (**Elsevier, Impact Factor: 2.784**) (**SCI**).

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7. **Komal Singla**, R. K. Gupta, “On symmetry analysis of time fractional nonlinear partial differential equations with variable coefficients”, (communicated).
8. **Komal Singla**, R. K. Gupta, “Symmetry analysis of variable coefficient time fractional nonlinear systems of partial differential equations”, (communicated).

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- $\mathbb{R}$ Set of real numbers
- $\mathbb{R}^+$ Set of positive real numbers
- $\mathbb{N}$ Set of positive integers (natural numbers)
- $\mathbb{C}$ Set of complex numbers
- D Total derivative operator
- I_x^α Riemann-Liouville fractional integral of order α
- D_x^α Riemann-Liouville fractional derivative of order α
- $\left({}_aD_x^\alpha\right)$ Left-sided Riemann-Liouville fractional derivative of order α
- $\left({}_xD_b^\alpha\right)$ Right-sided Riemann-Liouville fractional derivative of order α
- $\partial_t^\alpha u$ Riemann-Liouville partial fractional derivative of u of order α
- $\left(\mathcal{P}_\delta^{\zeta,\alpha} h\right)$ Left-sided Erdélyi-Kober fractional derivative operator
- $\left(\mathcal{K}_\delta^{\zeta,\alpha} h\right)$ Right-sided Erdélyi-Kober fractional derivative operator
- $\left(\mathcal{D}_\delta^{\zeta,\beta} h\right)$ Left-sided Erdélyi-Kober fractional integral operator
- $\left(\mathcal{I}_\delta^{\zeta,\beta} h\right)$ Right-sided Erdélyi-Kober fractional integral operator

Chapter 1

Introduction

Fractional calculus is the theory of integrals and derivatives of any arbitrary order (real or complex). It is one of the developing areas of mathematical physics which is evident from the large number of publications in the last three decades [70, 106, 171, 181]. Over the years, there has been considerable interest in fractional calculus by recognising its applications in numerical analysis and different areas of physics and engineering [46, 48, 123, 155]. In recent times, it is the center for new developments and latest research in many fields of nonlinear science.

A fractional differential equation (FDE) is a differential equation which contains fractional derivatives. In the past few decades, the wide applications of FDEs in various fields of science and engineering, specifically statistical mechanics, control theory, heat transfer, problems in mechanical and dynamical systems, chaos synchronization, wave propagation, image processing, fluid flow, electrochemistry, electromagnetics, viscoelasticity, fractal phenomena, material science, and optics [106, 133, 144, 155, 157, 181] have been recognized. Some of the available methods for solving FDEs are Laplace transform method [151, 183], Mellin transform method [34], power series method [98, 109], Babenko's symbolic calculus method [157], finite element method [126], variation iteration method [160], Adomian decomposition method [192], homotopy analysis method

[14], homotopy perturbation method [135], invariant subspace method [149, 170].

This chapter is designed as an introduction to fractional calculus, Lie classical method and conservation laws. The brief outline of the proposed extensions of Lie symmetry approach is also mentioned in this chapter. The details for the developed method and its diverse applications may be found in their corresponding chapters.

1.1 History of Fractional Calculus

The origin of fractional calculus is a letter written long back in the 17th century by L'Hôpital [133]. Leibniz invented the notation $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ for nth derivative of a function $y = f(x)$. On September 30th, 1695, a question was raised by L'Hôpital in his letter to Leibniz about derivative of order $n = \frac{1}{2}$. In his reply, Leibniz responded that $\mathrm{d}^{1/2}x$ will be equal to $x\sqrt{\mathrm{d}x : x}$. He replied that this is an apparent paradox, from which one day useful consequences will be drawn [106, 133]. Since then, many scientists as well as mathematicians paid attention to fractional calculus and contributed in its development. These authors [106, 133] include L. Euler (1738) [55], J. L. Lagrange (1772) [116], P. S. Laplace (1812) [117], S. F. Lacroix (1819) [115], J. B. J. Fourier (1822) [59], N. H. Abel (1823-1826) [2], J. Liouville (1832-1873) [124], G. F. B. Riemann (1847) [161], H. R. Greer (1859) [72], H. Holmgren (1865-1867) [85], A.V. Letnikov (1868-1872) [121], H. Laurent (1884) [118], P. A. Nekrassov (1888) [140], J. Hadamard (1892) [75], O. Heaviside (1892-1912) [80].

Lagrange contributed to fractional calculus by developing the law of exponents in 1772 for integer order derivative operators [116, 133]. The first step was taken by Euler in 1738 who observed the derivative $\frac{\mathrm{d}^p(x^a)}{\mathrm{d}x^p}$ of x^a has a meaning for non-integer p [55]. The French mathematician Lacroix was the first author who published a text mentioning fractional derivative [115, 133] in 1819. He presented fractional derivatives as a generalization of integer order derivatives. For $y = x^m$, (m is a positive integer), Lacroix found the nth derivative as follows:

$$\begin{aligned}
&\frac{\mathrm{d}y}{\mathrm{d}x} = mx^{m-1}, \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = m(m-1)x^{m-2},\\
&\vdots \\
&\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = m(m-1)..(m-n+1)x^{m-n} = \frac{m!}{(m-n)!}x^{m-n}, \quad m \geq n.
\end{aligned} \tag{1.1}$$

In terms of Legendre's symbol, he developed the following:

$$\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}. \tag{1.2}$$

Using example $y(x) = x, n = \frac{1}{2}$, the following result was obtained:

$$\frac{\mathrm{d}^{1/2}y}{\mathrm{d}x^{1/2}} = \sqrt{\frac{2x}{\pi}}. \tag{1.3}$$

Euler and Fourier also mentioned the fractional derivatives in their work but without giving any examples. In 1823, Niels Henrik Abel [2] was the first author to apply the fractional operators and fractional calculus in solving an integral problem that arises in the formulation of the tautochrone problem. The first mathematician to attempt solving differential equations having fractional operators was Liouville [133]. He published three long memoirs in 1832 [124]. In 1847, Riemann wrote a paper dealing with a definition of fractional operator but was published posthumously [161]. Holmgren [85] wrote a text dealing with the applications of fractional calculus to the solutions of some ODEs.

During the 20th century, some of the contributing authors [17] are G. H. Weyl (1917) [193], H. Hardy (1917-1928) [76], J. E. Littlewood (1917-1928) [77], P. Lévy (1937) [122], A. Zygmund (1935-1945) [201], A. Marchaud (1927) [132], H. T. Davis (1924-1936) [45], E. R. Love (1938-1996) [127], M. Riesz (1936-1949) [162], A. Erdélyi (1939-1965) [53], H. Kober (1940) [108] and D. V. Widder (1941) [194]. Oldham and

Spanier [144] published the 1st text completely devoted to fractional calculus in 1974. Podlubny [157] published a book dealing with FDEs in 1999. For more facts about the development of the fractional calculus during these two centuries, the reader is referred to Refs. [107, 133, 144]. Nowadays, a wide range of books and research articles are devoted to fractional calculus and its applications [4, 12, 48, 81–83, 106, 168, 181, 182].

1.2 Riemann-Liouville Fractional Operators

The correspondence between Leibniz and L'Hôpital motivated many mathematicians and physicists to give a definition to the fractional derivatives. Over the years, many mathematicians have given various definitions to find derivatives and integrals of non-integer order. The interesting fact is that fractional integrals and derivatives are not a local property. Fractional integrals are often considered in the Riemann-Liouville sense but fractional derivatives have been introduced in several different ways, such as fractional derivatives of Liouville, Riemann, Riemann-Liouville, Grunwald-Letnikov, Caputo, Miller-Ross [106, 133, 157]. The most famous of these definitions is that belongs to Riemann and Liouville. In this thesis, the fractional derivatives and integrals are considered in Riemann-Liouville sense. The basic definitions of fractional integrals and derivatives are presented [106, 133, 157, 171] in this section.

1.2.1 Useful Mathematical Functions

One of the most important functions of fractional calculus is the Euler's Gamma function denoted by $\Gamma(z)$. It generalizes the factorial $n!$ from $n \in \mathbb{N}$ to n being any non-integer and even complex numbers. The gamma function is defined by [157]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t, \tag{1.4}$$

which converges for $\Re(z) > 0$ where $\Re(z)$ denotes the real part of z. In this thesis, only $z \in \mathbb{R}^+$ are considered. It has the following two important properties:

$$\begin{aligned} &\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{R}^+, \\ &\Gamma(z) = (z-1)!, \quad z \in \mathbb{N}. \end{aligned} \tag{1.5}$$

Note 1. The gamma function $\Gamma(z)$ is an analytic function for all complex numbers except zero and negative integers.

Another useful mathematical function in fractional calculus is the Beta function. For $\Re(p) > 0$, $\Re(q) > 0$, it is defined by [157]

$$\begin{aligned} B(p,q) &= \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t, \\ B(p,q) &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \end{aligned} \tag{1.6}$$

1.2.2 Riemann-Liouville Fractional Integrals

First, the Riemann-Liouville fractional integration is considered for a finite interval on real line. The fractional integrals can be seen as generalization of the ordinary integrals. The integral of order $n \in \mathbb{N}$ of a function $f(x)$ continuous on real line can be written as follows [106, 157]:

$$I^n f(x) = \int_0^x \int_0^z \ldots \int_0^u f(v)\mathrm{d}v\mathrm{d}u \cdots \mathrm{d}z. \tag{1.7}$$

This n-fold integral can be written as a single integral by using Cauchy's formula as follows:

$$I^n f(x) = \frac{1}{(n-1)!}\int_0^x (x-s)^{n-1} f(s)\mathrm{d}s, \quad x > 0, n \in \mathbb{N}. \tag{1.8}$$

This formula of integrals is naturally extended from positive integers to positive real numbers by using Gamma function.

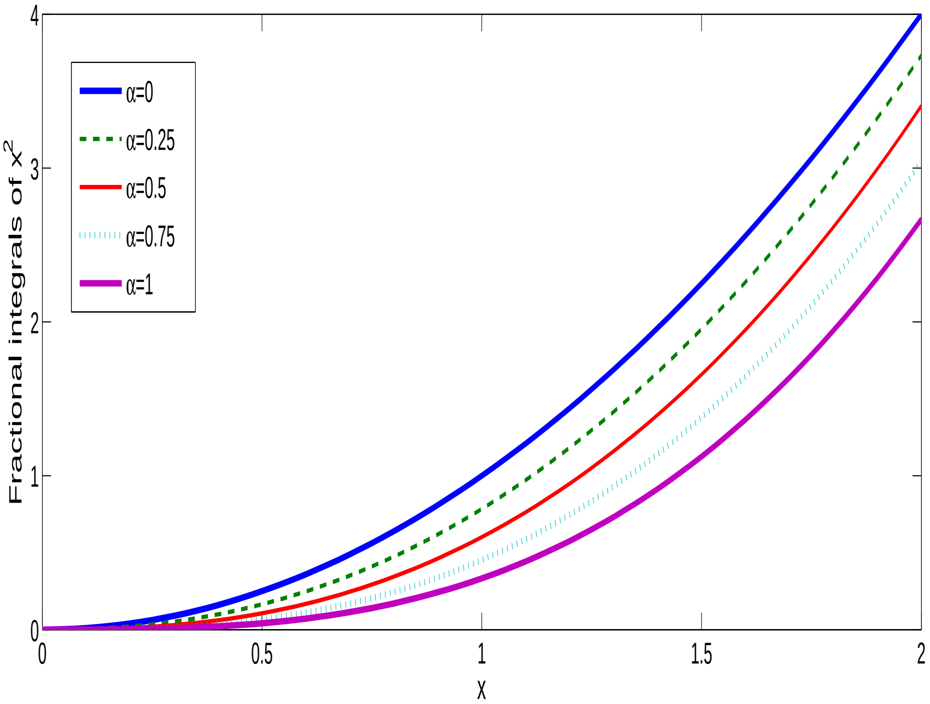

Figure 1.1: Graphical representation of the αth order fractional integrals of function $f(x) = x^2$. As $\alpha \to 1$, the curves representing αth order fractional integrals for $0 \leq \alpha < 1$ approach to curve of 1st order integral.

Therefore, the Riemann-Liouville fractional integrals for $\alpha > 0$ are defined as follows [106, 157]:

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) \mathrm{d}s, \quad x > 0, \alpha \in \mathbb{R}^+. \tag{1.9}$$

The operator I_x^α has the following features:

$$\begin{aligned} &I_x^\alpha (x^\gamma) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} x^{\gamma+\alpha}; \quad \alpha > 0, \quad \gamma > -1, \quad x > 0, \\ &I_x^\alpha C = \frac{C}{\Gamma(1+\alpha)} x^\alpha, \quad C \text{ being a constant}, \\ &I_x^0 f(x) = f(x). \end{aligned} \tag{1.10}$$

For $f(x) = x^2$, the graphical representation of αth order fractional integrals $I_x^\alpha(x^2)$ for $0 \leq \alpha \leq 1$ is presented in Figure 1.1.

1.2.2.1 Left and right-sided fractional integrals

The Riemann-Liouville fractional integrals can be defined in general for $\alpha \in \mathbb{C}$ as follows.

Definition 1.1. Let $f(x)$ be defined on a finite interval $[a,b]$ $(-\infty < a < b < \infty)$ on the real axis $\mathbb{R}$. The left-sided, right-sided Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$ denoted by ${}_aI_x^\alpha f$ and ${}_xI_b^\alpha f$ respectively, are defined as follows [106]:

$$({}_aI_x^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} \mathrm{d}t, \quad (x > a; \Re(\alpha) > 0), \tag{1.11}$$

$$({}_xI_b^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} \mathrm{d}t, \quad (x < b; \Re(\alpha) > 0). \tag{1.12}$$

1.2.2.2 Partial and mixed Riemann-Liouville fractional integrals

In this section, the definitions of multidimensional partial and mixed fractional integrals [106, 157, 171] are presented which are natural generalizations of the corresponding one-dimensional fractional integrals. These definitions involve the integrals with respect to one or several variables in the n-dimensional Euclidean space $\mathbb{R}^n (n \in \mathbb{N})$.

For $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ $(n \in \mathbb{N} \setminus \{1\})$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ $(n \in \mathbb{N} \setminus \{1\})$, the following notations are used:

$$\begin{aligned}
&\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \dots x_n^{\alpha_n} = \prod_{i=1}^n x_i^{\alpha_i}; \quad \Gamma(\boldsymbol{\alpha}) := \Gamma(\alpha_1) \dots \Gamma(\alpha_n); \quad \frac{\partial}{\partial \boldsymbol{x}} := \frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_n} = \prod_{i=1}^n \frac{\partial}{\partial x_i}; \\
&[\boldsymbol{a}, \boldsymbol{b}] = [a_1, b_1] \times \dots \times [a_n, b_n]; \quad \boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^n; \quad \boldsymbol{b} = (b_1, \dots, b_n) \in \mathbb{R}^n; \\
&\boldsymbol{x} > \boldsymbol{a} \text{ means } x_i > a_i \ \forall i = 1, ..., n.
\end{aligned} \tag{1.13}$$

Definition 1.2 (Left-sided partial Riemann-Liouville fractional integral)**.** The left-sided partial Riemann-Liouville fractional integral of order $\alpha_k \in \mathbb{C}$ $(\Re(\alpha_k) > 0)$ of functions $f(\boldsymbol{x}) = f(x_1, \dots, x_n)$ defined for $x_k > a_k$ with respect to the kth variable x_k is defined by [106]

$$({}_{a_k}I_{x_k}^\alpha f)(\boldsymbol{x}) := \frac{1}{\Gamma(\alpha_k)} \int_{a_k}^{x_k} \frac{f(x_1, \dots, x_{k-1}, t_k, x_{k+1}, \dots, x_n)}{(x_k - t_k)^{1-\alpha_k}} \mathrm{d}t_k, \quad (x_k > a_k). \tag{1.14}$$

Definition 1.3 (Right-sided partial Riemann-Liouville fractional integral)**.** The right-sided partial Riemann-Liouville fractional integral of order $\alpha_k \in \mathbb{C}$ ($\Re(\alpha_k) > 0$) of functions $f(\boldsymbol{x}) = f(x_1, \ldots, x_n)$ defined for $x_k < b_k$ with respect to the kth variable x_k is defined by [106]

$$({}_{x_k}I^{\alpha}_{b_k} f)(\boldsymbol{x}) := \frac{1}{\Gamma(\alpha_k)} \int_{x_k}^{b_k} \frac{f(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n)}{(t_k - x_k)^{1-\alpha_k}} \mathrm{d}t_k, \quad (x_k < b_k). \tag{1.15}$$

Definition 1.4 (Mixed left and right-sided Riemann-Liouville fractional integrals)**.** The mixed left and right-sided Riemann-Liouville fractional integrals of order $\boldsymbol{\alpha} \in \mathbb{C}^n$ ($\Re(\boldsymbol{\alpha}) > 0$) are defined as follows [106]:

$$({}_{\boldsymbol{a}}I^{\boldsymbol{\alpha}}_{\boldsymbol{x}} f)(\boldsymbol{x}) = ({}_{a_1}I^{\boldsymbol{\alpha}}_{x_1} \cdots {}_{a_n}I^{\boldsymbol{\alpha}}_{x_n} f)(\boldsymbol{x}) = \frac{1}{\Gamma(\boldsymbol{\alpha})} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \frac{f(\boldsymbol{t})}{(\boldsymbol{x} - \boldsymbol{t})^{1-\boldsymbol{\alpha}}} \mathrm{d}\boldsymbol{t}, \quad (\boldsymbol{x} > \boldsymbol{a}), \tag{1.16}$$

and

$$({}_{\boldsymbol{x}}I^{\boldsymbol{\alpha}}_{\boldsymbol{b}} f)(\boldsymbol{x}) = ({}_{x_1}I^{\boldsymbol{\alpha}}_{b_1} \cdots {}_{x_n}I^{\boldsymbol{\alpha}}_{b_n} f)(\boldsymbol{x}) = \frac{1}{\Gamma(\boldsymbol{\alpha})} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} \frac{f(\boldsymbol{t})}{(\boldsymbol{t} - \boldsymbol{x})^{1-\boldsymbol{\alpha}}} \mathrm{d}\boldsymbol{t}, \quad (\boldsymbol{x} < \boldsymbol{b}). \tag{1.17}$$

1.2.3 Riemann-Liouville Fractional Derivatives

The Riemann-Liouville fractional derivative can be defined by using the definition of fractional integral. The fractional derivative of a function $f(x)$ is given by [106, 157]

$$D^{\alpha}_{x} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(D_x^{-(n-\alpha)} f(x) \right), \quad (n - 1 \leq \alpha < n). \tag{1.18}$$

The integral I^{α}_{x} is also denoted as $D^{-\alpha}_{x}$. The operator D^{α}_{x} represents a derivative if α is a positive real number, and an integral if α is a negative real number. Therefore, the Riemann-Liouville fractional derivatives of a function $f(x)$ of real order $\alpha > 0$ are defined as follows [106, 157]:

$$D^{\alpha}_{x} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x (x - s)^{n-\alpha-1} f(s) \mathrm{d}s, & x > 0,\, n - 1 < \alpha < n \in \mathbb{N}, \\ \frac{d^n f}{dx^n}, & \alpha = n \in \mathbb{N}. \end{cases} \tag{1.19}$$

For $\alpha = n \in \mathbb{N}$, we have the following result:

$$D^n_x I^n_x = \mathbb{I}, \quad I^n_x D^n_x \neq \mathbb{I}, \tag{1.20}$$

where $\mathbb{I}$ is the identity operator. Therefore, D_x^n is the left inverse but not the right inverse of the integral operator I_x^n. In fact, in general the following holds:

$$D_x^\alpha I_x^\alpha = \mathbb{I}, \quad \alpha \geq 0. \tag{1.21}$$

The fractional derivative of $f(x) = x^\gamma$ is as follows:

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}; \quad \alpha > 0, \;\; \gamma > -1, \;\; t > 0. \tag{1.22}$$

Substituting $\gamma = 0$, the following result can be obtained:

$$D_x^\alpha 1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha > 0, t > 0. \tag{1.23}$$

Note that $D_x^\alpha(1) \neq 0$, in fact $D_x^\alpha(C) = \frac{Cx^{-\alpha}}{\Gamma(1-\alpha)} \neq 0$ for a constant function $f(x) = C$ for $\alpha \neq \mathbb{N}$. Of course, $D_x^\alpha(C) = 0$ for $\alpha \in \mathbb{N}$, due to the poles of the gamma function at the points $0, -1, -2, \cdots$.

The αth order fractional derivatives $D_x^\alpha(f(x))$ for $f(x) = x^2$, $0 \leq \alpha \leq 1$ are presented graphically in Figure 1.2.

1.2.3.1 Left and right-sided fractional derivatives

Similar to the left and right-sided fractional integrals (1.11), (1.12), the definitions of left and right-sided fractional derivatives for $\alpha \in \mathbb{C}$ can be easily obtained.

Definition 1.5. The left-sided and right-sided Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}$ are defined as follows [106]:

$$\begin{aligned} ({}_aD_x^\alpha f)(x) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n ({}_aI_x^{n-\alpha} f)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} \mathrm{d}t, \quad (x > a; n = [\Re(\alpha)] + 1). \end{aligned} \tag{1.24}$$

$$\begin{aligned} ({}_xD_b^\alpha f)(x) &= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n ({}_xI_b^{n-\alpha} f)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}} \mathrm{d}t, \quad (x < b; n = [\Re(\alpha)] + 1), \end{aligned} \tag{1.25}$$

where $[\Re(\alpha)]$ is the integral part of $\Re(\alpha)$.

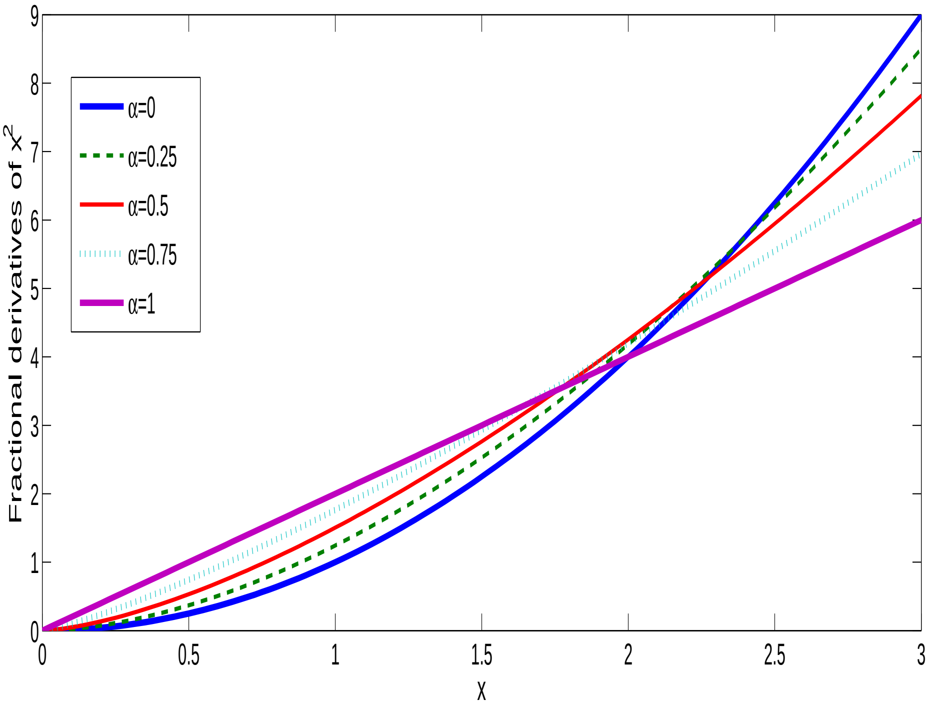

Figure 1.2: Graphical representation of the fractional derivatives of function $f(x) = x^2$. As $\alpha \to 1$, the curves representing αth order fractional derivatives for $0 \leq \alpha < 1$ approach to curve of 1st order derivative.

1.2.3.2 Partial and mixed Riemann-Liouville fractional derivatives

Definition 1.6 (Left-sided partial Riemann-Liouville fractional derivative)**.** The left-sided partial Riemann-Liouville fractional derivative of order $\alpha_k \in \mathbb{C}$ $(\Re(\alpha_k) \geq 0)$ with respect to the kth variable x_k is defined by [106]

$$\begin{aligned}({}_{a_k}D^{\alpha_k}_{x_k} f)(\boldsymbol{x}) &= \left(\frac{\partial}{\partial x_k}\right)^{L_k} ({}_{a_k}I^{L_k-\alpha_k}_{x_k} f)(\boldsymbol{x}) \\ &= \frac{1}{\Gamma(L_k-\alpha_k)}\left(\frac{\partial}{\partial x_k}\right)^{L_k}\int_{a_k}^{x_k}\frac{f(x_1,\cdots,x_{k-1},t_k,x_{k+1},\cdots,x_n)}{(x_k-t_k)^{\alpha_k-L_k+1}}\mathrm{d}t_k, \quad (x_k > a_k),\end{aligned} \tag{1.26}$$

where $L_k = [\Re(\alpha_k)] + 1$.

Definition 1.7. (Right-sided partial Riemann-Liouville fractional derivative)[106, 133] The right-sided partial Riemann-Liouville fractional derivative of order $\alpha_k \in \mathbb{C}$ $(\Re(\alpha_k) \geq 0)$ with respect to x_k is defined by [106]

$$\begin{aligned}({}_{x_k}D^{\alpha_k}_{b_k} f)(\boldsymbol{x}) &= \left(-\frac{\partial}{\partial x_k}\right)^{L_k} ({}_{x_k}I^{L_k-\alpha_k}_{b_k} f)(\boldsymbol{x}) \\ &= \frac{1}{\Gamma(L_k-\alpha_k)}\left(-\frac{\partial}{\partial x_k}\right)^{L_k}\int_{x_k}^{b_k}\frac{f(x_1,\cdots,x_{k-1},t_k,x_{k+1},\cdots,x_n)}{(t_k-x_k)^{\alpha_k-L_k+1}}\mathrm{d}t_k, \quad (x_k < b_k).\end{aligned} \tag{1.27}$$

Definition 1.8 (Mixed left and right-sided Riemann-Liouville fractional derivatives)**.** The mixed left- and right-sided Riemann-Liouville fractional derivatives of order $\alpha_k \in \mathbb{C}$ ($\Re(\alpha_k) \geq 0$), corresponding to mixed fractional integrals (1.16) and (1.17) are defined as follows [106]:

$$\begin{aligned}({}_{\boldsymbol{a}}D_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f)(\boldsymbol{x}) &= ({}_{a_1}D_{x_1}^{\boldsymbol{\alpha}} \cdots {}_{a_n}D_{x_n}^{\boldsymbol{\alpha}} f)(\boldsymbol{x}) \\ &= \frac{1}{\Gamma(\boldsymbol{L}-\boldsymbol{\alpha})}\left(\frac{\partial}{\partial x_1}\right)^{L_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{L_n} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \frac{f(\boldsymbol{t})}{(\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{\alpha}-\boldsymbol{L}+1}} \mathrm{d}\boldsymbol{t}, \quad (\boldsymbol{x} > \boldsymbol{a}),\end{aligned} \tag{1.28}$$

and

$$\begin{aligned}({}_{\boldsymbol{x}}D_{\boldsymbol{b}}^{\boldsymbol{\alpha}} f)(\boldsymbol{x}) &= \left({}_{x_1}D_{b_1}^{\boldsymbol{\alpha}} \cdots {}_{x_n}D_{b_n}^{\boldsymbol{\alpha}} f\right)(\boldsymbol{x}) \\ &= \frac{1}{\Gamma(\boldsymbol{L}-\boldsymbol{\alpha})}\left(-\frac{\partial}{\partial x_1}\right)^{L_1} \cdots \left(-\frac{\partial}{\partial x_n}\right)^{L_n} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} \frac{f(\boldsymbol{t})}{(\boldsymbol{t}-\boldsymbol{x})^{\boldsymbol{\alpha}-\boldsymbol{L}+1}} \mathrm{d}\boldsymbol{t}, \quad (\boldsymbol{x} < \boldsymbol{b}),\end{aligned} \tag{1.29}$$

where $\boldsymbol{L} = (L_1, \cdots, L_n)$ and $L_k = [\Re(\alpha_k)] + 1$, $(k = 1, \cdots, n)$

Note 2. Only left-sided derivatives ${}_aD_x^\alpha$ are considered in this thesis, with left limit $a = 0$ and $\alpha \in \mathbb{R}^+$. The operators ${}_0D_x^\alpha$ are denoted by D_x^α throughout the thesis.

For more applications, basic results, basic theorems, physical and geometric interpretation of fractional derivatives, the interested reader is referred to Refs. [44, 81, 106, 107, 158].

1.2.4 Properties of Riemann-Liouville Fractional Operators

The Riemann-Liouville fractional derivative and integral operators have some properties [106, 133, 157, 171] which are as follows. Since only left-sided fractional operators are considered in this study so properties are discussed only for left-sided operators. In all these properties, $f(x), g(x)$ are assumed to be continuous for $x \geq a$.

1.2.4.1 Linearity

The Riemann-Liouville fractional operator is a linear operator *i.e.* the following holds [106, 157]:

$$
{}_aD_x^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda \left({}_aD_x^{\alpha} f(x)\right) + \mu \left({}_aD_x^{\alpha} g(x)\right), \quad (\alpha \in \mathbb{R}), \tag{1.30}
$$

where λ, μ are constants. If $\alpha > 0$, it implies the linearity of Riemann-Liouville fractional derivative operator and for $\alpha < 0$, it shows the linearity of integral operator.

1.2.4.2 The law of exponents for fractional operators

For Riemann-Liouville fractional integrals, the following commutative rule holds:

$$
{}_aD_x^{p}[{}_aD_x^{q} f(x)] = {}_aD_x^{p+q} f(x) = {}_aD_x^{q}[{}_aD_x^{p} f(x)], \quad (p, q < 0). \tag{1.31}
$$

This property also holds for $p \geq 0, q \leq 0$ where $p+q \leq 0$. But in general for $p, q > 0, p \neq q$, *i.e.* for fractional derivatives, it does not hold. The commutative property (1.31) holds for $m-1 \leq p < m$, $n-1 \leq q < n$ $(m, n \in \mathbb{N})$ if the following condition holds:

$$
f^{(k)}(x) = 0, \quad (k = 0, 1, \cdots, r-1), \tag{1.32}
$$

where r=max(m, n).

1.2.4.3 Leibniz rule

The Leibniz rule for integer order differentiation is generalized for fractional derivatives. If the function $f(x)$ and all its derivatives are continuous in $[a, x]$, the function $g(x)$ and its $n+1$ derivatives are continuous in $[a, x]$, then the Riemann-Liouville fractional derivative of the product $f(x)g(x)$ for $\alpha > 0$ is given by [157]

$$
{}_aD_x^{\alpha}(g(x)f(x)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(x)({}_aD_x^{\alpha-k} f)(x), \tag{1.33}
$$

where $g^{(k)}(x)$ is the kth order integer derivative.

1.2.4.4 Fractional derivative of a composite function

The Riemann-Liouville fractional derivative for $\alpha > 0$ of a composite function is given by [106, 157]

$$ {}_aD_x^{\alpha} F(g(x)) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} F(g(x)) + \sum_{k=1}^{\infty} \binom{\alpha}{k} \frac{k!(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{m=1}^{k} F^{(m)}(h(t)) \sum \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{h^{(r)}(x)}{r!} \right)^{a_r}, \tag{1.34}$$

where the sum $\sum$ extends over all combinations of non-negative integer values of $a_1, a_2, \ldots, a_k$ such that

$$\sum_{r=1}^{k} r a_r = k, \quad \text{and} \quad \sum_{r=1}^{k} a_r = m. \tag{1.35}$$

1.2.4.5 Derivative of fractional integral and fractional integral of derivative

There is an important question that whether the derivative of fractional integral is same as the fractional integral of derivative of a function. It means

$$\frac{\mathrm{d}}{\mathrm{d}x} \left({}_aD_x^{-\alpha} f(x) \right) = {}_aD_x^{-\alpha} \left(\frac{\mathrm{d}f(x)}{\mathrm{d}x} \right)? \tag{1.36}$$

The answer is No. The expression (1.36) is true if $0 < \alpha < 1$.

1.2.4.6 Composition of fractional derivatives with integer order derivatives

The property of composition of fractional derivatives with nth ($n \in \mathbb{N}$) order derivatives is given by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \left({}_aD_x^{\alpha} f(x) \right) = {}_aD_x^{\alpha} \left(\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} \right) = {}_aD_x^{\alpha+n} f(x). \tag{1.37}$$

It holds if the function $f(x)$ satisfies the following:

$$f^{(j)}(a) = 0, \quad (j = 0, 1, \cdots, n-1). \tag{1.38}$$

1.3 Lie Symmetry Analysis

The Lie symmetry method is an extremely popular technique for solving nonlinear differential equations. In general, nonlinear differential equations are very difficult to solve. Therefore, to obtain exact solutions of nonlinear differential equations has always been an interesting and challenging concept in various fields of research. The symmetry method has been widely applied to deal with many nonlinear differential equations. The pioneer work on Lie symmetry analysis (also called Lie classical method) was done by a Norwegian mathematician Sophus Lie in 1872-1899. He started with the investigation of continuous groups of transformations leaving differential equations invariant for solving ODEs. He gave an algorithm to determine infinitesimal generators of Lie groups of transformations. Despite its importance in the nonlinear phenomena, the Lie theory remained unexplored for many years. Thereafter, the Lie group theory was developed by Ovsiannikov [150] in 1960. Since then, Ibragimov [89, 90, 94], Olver [147], Bluman [24, 25, 27, 28], Cantwell [36], Clarkson [40–42] and many other authors have contributed in the development and applications of Lie group theory [31, 50, 88, 101, 112, 145, 179]. Some applications of Lie group method are given by Bhutani et al. [20, 21], Broadbridge et al. [31, 49], Singh et al. [176, 177], Sharma et al. [154, 172, 173], Lakshmanan et al. [22, 184], Gupta et al. [71, 74, 105, 111, 113], Morris et al. [136, 153], Gandarias et al. [62, 63, 164], Wang et al. [185, 186], Nucci et al. [142, 143], Özemir et al. [73, 152].

A symmetry of an object is a transformation leaving that object invariant. Symmetry group of a differential equation is a group of transformations which maps any solution to another solution. Therefore, symmetry group is useful to obtain new solutions from known solutions. The Lie symmetries admitted by differential equations can be used to reduce the number of independent variables in case of PDEs and reduce its order in case of ODEs. Obviously, it is easier to solve the reduced differential equations. The Lie symmetry method has been extended for the construction of various techniques to study differential equations. Some of these generalizations are the nonclassical method by Bluman and Cole [26], higher order symmetries by Olver [147], nonlocal symmetries [146] etc. Many computer algebra packages are also available in computer softwares like Maple, Mathematica to find the Lie symmetries of nonlinear differential equations.

In the following sections, the basic definitions, methodology of Lie classical method for integer order differential equations and its latest advancements for FDEs are presented. The symmetry method is further developed in the subsequent chapters for the study of nonlinear systems of FDEs.

1.3.1 Basic Definitions

Consider a kth order system of N differential equations with $n \in \mathbb{N}$ independent variables $\boldsymbol{x} = (x_1, ..., x_n)$ and $m \in \mathbb{N}$ dependent variables $\boldsymbol{u} = (u^1, ..., u^m)$ as follows:

$$F^{\sigma}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{(1)}, ..., \boldsymbol{u}_{(k)}) = 0, \quad \sigma = 1, ..., N, \tag{1.39}$$

where $x_i, u^j \in \mathbb{R}$ for all i, j and $\boldsymbol{u}_{(1)}$ is the set of nm first order derivatives of $\boldsymbol{u}$ given as follows:

$$\boldsymbol{u}_{(1)} = \left(\frac{\partial u^1}{\partial x_1}, \frac{\partial u^1}{\partial x_2}, \cdots, \frac{\partial u^1}{\partial x_n}, \frac{\partial u^2}{\partial x_1}, \cdots, \frac{\partial u^2}{\partial x_n}, \frac{\partial u^m}{\partial x_1}, \cdots, \frac{\partial u^m}{\partial x_n}\right). \tag{1.40}$$

In general, for $k \geq 2$, $\boldsymbol{u}_{(k)}$ denote the set of all kth order derivatives of $\boldsymbol{u}$ given by

$$\boldsymbol{u}_{(k)} = \frac{\partial^k u^j}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}, \quad \text{where} \quad j = 1, 2, \cdots, m \ \text{ and } \ i_1, \cdots, i_k = 1, \cdots, n. \tag{1.41}$$

Before discussion of the methodology of its Lie symmetry analysis, the basic definitions and main features of Lie group theory are introduced [24, 89, 90, 147].

1.3.1.1 One parameter Lie group of transformations

Definition 1.9 (Group)**.** A non-empty set G with a law of composition * between its elements forms a group provided that the following axioms hold [24, 147]:

1. (Closure): For all $a_1, a_2 \in G$, $a_1 * a_2$ must be an element of G.
2. (Associative): For all elements $a_1, a_2, a_3 \in G$, $a_1 * (a_2 * a_3) = (a_1 * a_2) * a_3$.
3. (Existence of Identity): There exists a unique identity element $e \in G$ for all $a \in G$ such that $a * e = a = e * a$.
4. (Existence of Inverse): For every $a \in G$, there exists a unique inverse element $a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$.

Definition 1.10 (One parameter Lie group of transformations)**.** Consider a set G of the following transformations [94]:

$$T_\lambda : \ \ \tilde{x}_i = f^i(\boldsymbol{x}, \boldsymbol{u}; \lambda), \ \ \tilde{u}^j = g^j(\boldsymbol{x}, \boldsymbol{u}; \lambda); \quad i = 1, ..., n; \ \ j = 1, ..., m, \tag{1.42}$$

where λ is a continuous real parameter in a neighbourhood $S \subseteq \mathbb{R}$ of $\lambda = 0$, and f^i, g^j are infinitely differentiable with respect to the real variables $\boldsymbol{x}$ and an analytic function of λ in open interval S. The set G defines a one parameter Lie group of transformations in $\mathbb{R}^{n+m}$ with a law of composition $\phi : S \times S \longrightarrow S$ and an analytic function of parameters in S if the following axioms hold [24, 94, 147]:

1. **Closure**: For $T_\lambda, T_{\lambda'} \in G$ and $\lambda, \lambda' \in S' \subset S$,

$$T_\lambda T_{\lambda'} = T_{\lambda''} \in G, \quad \text{where } \lambda'' = \phi(\lambda, \lambda') \in S. \tag{1.43}$$

2. **Existence of Identity**: There exists a unique identity $T_0 \in G$ with respect to $0 \in S$ following $\phi(\lambda, 0) = \lambda = \phi(0, \lambda)$ such that

$$T_\lambda T_0 = T_\lambda = T_0 T_\lambda. \tag{1.44}$$

3. **Existence of Inverse**: For any $T_\lambda \in G$, there is a unique inverse $T_\lambda^{-1} = T_{\lambda^{-1}} \in G$ for $\lambda^{-1} \in S$ such that

$$T_\lambda T_\lambda^{-1} = T_0 = T_\lambda^{-1} T_\lambda, \tag{1.45}$$

where $\phi(\lambda, \lambda^{-1}) = 0 = \phi(\lambda^{-1}, \lambda)$.

1.3.1.2 Infinitesimal generators and prolongation formulae

Consider a one parameter Lie group of transformations as follows [24, 94, 147]:

$$T_\lambda : \ \ \tilde{x}_i = f^i(\boldsymbol{x}, \boldsymbol{u}; \lambda), \ \ \tilde{u}^j = g^j(\boldsymbol{x}, \boldsymbol{u}; \lambda), \tag{1.46}$$

where $i = 1, ..., n$; $j = 1, ..., m$, λ is the group parameter in group S with the identity $\lambda = 0$ and law of composition ϕ. Expanding (1.46) about $\lambda = 0$, the following can be obtained:

$$\begin{aligned}
\tilde{x}_i &= x_i + \lambda \left(\xi^i(\boldsymbol{x}, \boldsymbol{u}) \right) + O(\lambda^2), \\
\tilde{u}^j &= u^j + \lambda \left(\eta^j(\boldsymbol{x}, \boldsymbol{u}) \right) + O(\lambda^2), \\
\tilde{u}^j_{i_1} &= u^j_{i_1} + \lambda \left(\eta^j_{i_1}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{(1)}) \right) + O(\lambda^2), \\
&\vdots \\
\tilde{u}^j_{i_1 i_2 \cdots i_k} &= u^j_{i_1 i_2 \cdots i_k} + \lambda \left(\eta^j_{i_1 i_2 \cdots i_k}(\boldsymbol{x}, \boldsymbol{u}, \cdots, \boldsymbol{u}_{(k)}) \right) + O(\lambda^2),
\end{aligned} \tag{1.47}$$

where $i_l = 1, \cdots, n$, $l = 1, \cdots, k$ and the functions $u^j_{i_1} = \frac{\partial u^j}{\partial x_{i_1}}, u^j_{i_1 i_2} = \frac{\partial^2 u^j}{\partial x_{i_1} x_{i_2}}$, etc.

Here, $\xi^i(\boldsymbol{x}, \boldsymbol{u}), \eta^j(\boldsymbol{x}, \boldsymbol{u})$ are called the infinitesimals of the group defined by [24, 94, 147]

$$\begin{aligned}
\xi^i(\boldsymbol{x}, \boldsymbol{u}) &= \frac{\partial \tilde{x}_i}{\partial \lambda}\bigg|_{\lambda=0} = \frac{\partial f^i(\boldsymbol{x}, \boldsymbol{u}; \lambda)}{\partial \lambda}\bigg|_{\lambda=0}, \\
\eta^j(\boldsymbol{x}, \boldsymbol{u}) &= \frac{\partial \tilde{u}^j}{\partial \lambda}\bigg|_{\lambda=0} = \frac{\partial g^j(\boldsymbol{x}, \boldsymbol{u}; \lambda)}{\partial \lambda}\bigg|_{\lambda=0}.
\end{aligned} \tag{1.48}$$

Also, $\eta^j_{i_1 i_2 \cdots i_k}(\boldsymbol{x}, \boldsymbol{u}, \cdots, \boldsymbol{u}_{(k)}) = \frac{\partial \tilde{u}^j_{i_1 i_2 \cdots i_k}}{\partial \lambda}\Big|_{\lambda=0}$ are called the extended infinitesimals of order k (also called prolongation formulae or prolongations) defined as follows [24, 147]:

$$\begin{aligned}
\eta^j_i(\boldsymbol{x}, \boldsymbol{u}) &= \mathrm{D}_i(\eta^j) - u^j_l \mathrm{D}_i(\xi^l), \\
&\vdots \\
\eta^j_{i_1 i_2 \cdots i_k}(\boldsymbol{x}, \boldsymbol{u}, \cdots, \boldsymbol{u}_{(k)}) &= \mathrm{D}_{i_k}(\eta^j_{i_1 \cdots i_{k-2}}) - u^j_{i_1 \cdots i_{k-1} l} \mathrm{D}_{i_k}(\xi^l).
\end{aligned} \tag{1.49}$$

where D_i denote the total derivative operators, defined by

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + u_i^j \frac{\partial}{\partial u^j} + u_{ip}^j \frac{\partial}{\partial u_p^j} + \cdots \tag{1.50}$$

The transformations (1.47) up to first order are as follows:

$$\tilde{x}_i = x_i + \lambda \left(\frac{\partial f^i(\boldsymbol{x}, \boldsymbol{u}; \lambda)}{\partial \lambda} \bigg|_{\lambda=0} \right), \quad \tilde{u}^j = u^j + \lambda \left(\frac{\partial g^j(\boldsymbol{x}, \boldsymbol{u}; \lambda)}{\partial \lambda} \bigg|_{\lambda=0} \right). \tag{1.51}$$

The transformations (1.51) are called the infinitesimal transformations with the infinitesimal generator or symmetry operator of the following form:

$$V = \xi^i(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial x_i} + \eta^j(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial u^j}, \tag{1.52}$$

with the standard convention of summation over repeated indices.

Theorem 1.1 (First fundamental theorem of Lie)**.** *The one parameter Lie group of transformations* (1.46) *is equivalent to the solution of an initial value problem for a system of first order ODEs in the following form* [24]*:*

$$\frac{d\tilde{x}_i}{d\lambda} = \xi^i(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}), \quad \frac{d\tilde{u}^j}{d\lambda} = \eta^j(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}), \tag{1.53}$$

with the following initial conditions

$$\tilde{x}_i|_{\lambda=0} = x_i, \quad \tilde{u}^j|_{\lambda=0} = u^j. \tag{1.54}$$

The first fundamental theorem of Lie proves that the essential information for characterizing a one parameter Lie group of transformations is within the infinitesimal transformations. Therefore, the Lie group of transformations is equivalent to its infinitesimal generators (also called vector fields).

Definition 1.11. The extended infinitesimal generator $V^{(k)}$ $(k = 1, 2, \cdots)$ for a one parameter Lie group of transformations (1.46) is defined as follows [24]:

$$V^{(k)} = \xi^i(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial x_i} + \eta^j(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial u^j} + \eta_i^j(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{(1)}) \frac{\partial}{\partial u_i^j} + \cdots + \eta_{i_1 i_2 \cdots i_k}^j(\boldsymbol{x}, \boldsymbol{u}, \cdots, \boldsymbol{u}_{(k)}) \frac{\partial}{\partial u_{i_1 i_2 \cdots i_k}^j}. \tag{1.55}$$

1.3.1.3 Invariance

Definition 1.12 (Invariant function)**.** An infinitely differentiable function $F(\boldsymbol{x},\boldsymbol{u})$ is an invariant function of the Lie group of transformations (1.46) if and only if the following holds [24]:

$$F(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{u}}) = F(\boldsymbol{x},\boldsymbol{u}). \tag{1.56}$$

The function $F(\boldsymbol{x},\boldsymbol{u})$ is called invariant under the group of transformations (1.46).

Theorem 1.2. *A function $F(\boldsymbol{x},\boldsymbol{u})$ is invariant under a Lie group of transformations* (1.46) *if and only if the following holds* [24]*:*

$$VF(\boldsymbol{x},\boldsymbol{u}) = 0, \tag{1.57}$$

where V is the associated infinitesimal generator of group (1.46) *given by* (1.52)*.*

Definition 1.13 (Infinitesimal invariance criterion for differential equations)**.** The Lie group of transformations (1.46) with generator (1.52) is admitted by the system (1.39) if and only if $V^{(k)}$ leaves (1.39) invariant *i.e.* the following must hold [24]:

$$V^{(k)}F^{\sigma}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u_{(1)}},\boldsymbol{...},\boldsymbol{u_{(k)}}) = 0, \quad \text{when} \quad F^{\mu}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u_{(1)}},\boldsymbol{...},\boldsymbol{u_{(k)}}) = 0, \tag{1.58}$$

where $\sigma, \mu = 1, ..., N$. The condition (1.58) is called the infinitesimal invariance criterion for the system of differential equations (1.39).

Definition 1.14 (Invariant solution)**.** The function $\boldsymbol{u} = \boldsymbol{\Theta(x)}$ with components $u^j = \Theta^j(\boldsymbol{x})$, $j = 1, ..., m$ is an invariant solution of the system (1.39) admitting the group of transformations (1.46) with generator (1.52) if and only if the following two axioms hold [24]:

1. $\boldsymbol{u} = \boldsymbol{\Theta(x)}$ satisfies the system (1.39).
2. $u^j = \Theta^j(\boldsymbol{x})$ is an invariant surface of (1.52) for all j..

The second condition implies that $V(u^j - \Theta^j) = 0$ when $\boldsymbol{u} = \boldsymbol{\Theta}(\boldsymbol{x})$. Therefore, the following is obtained:

$$\xi^i(\boldsymbol{x},\boldsymbol{\Theta(x)})\frac{\partial\boldsymbol{\Theta}}{\partial x_i} = \eta^j(\boldsymbol{x},\boldsymbol{\Theta(x)}). \tag{1.59}$$

These are called the invariant surface conditions for the invariant solutions of system (1.39). The invariant surface conditions (1.59) can be solved by solving the following associated characteristic equations:

$$\frac{\mathrm{d}x_1}{\xi^1(\boldsymbol{x},\boldsymbol{u})}=\frac{\mathrm{d}x_2}{\xi^2(\boldsymbol{x},\boldsymbol{u})}=\cdots=\frac{\mathrm{d}x_n}{\xi^n(\boldsymbol{x},\boldsymbol{u})}=\frac{\mathrm{d}u^1}{\eta^1(\boldsymbol{x},\boldsymbol{u})}=\frac{\mathrm{d}u^2}{\eta^2(\boldsymbol{x},\boldsymbol{u})}=\cdots=\frac{\mathrm{d}u^m}{\eta^m(\boldsymbol{x},\boldsymbol{u})}. \tag{1.60}$$

The solutions of above system of equations gives $n+m-1$ new variables called group invariants. The number of independent variables is reduced into $n-1$ say $z_1(\boldsymbol{x},\boldsymbol{u}), ..., z_{n-1}(\boldsymbol{x},\boldsymbol{u})$ called similarity variables and m dependent variables say $v^1(\boldsymbol{x},\boldsymbol{u}), ..., v^m(\boldsymbol{x},\boldsymbol{u})$ with Jacobian $\frac{\partial(v^1,...,v^m)}{\partial(z_1,...,z_n)} \neq 0$. In general, they are expressed as follows:

$$v^j(\boldsymbol{x},\boldsymbol{u})=\phi^j(z_1(\boldsymbol{x},\boldsymbol{u}), ..., z_{n-1}(\boldsymbol{x},\boldsymbol{u})), \tag{1.61}$$

where ϕ^j is differentiable function of $z_1(\boldsymbol{x},\boldsymbol{u}), ..., z_{n-1}(\boldsymbol{x},\boldsymbol{u})$. Substituting (1.61) into the original system (1.39), the reduced system can be obtained in $n+m-1$ new variables.

1.3.1.4 r-parameter Lie group of transformations

An r-parameter Lie group of transformations is defined by the following transformations [24, 147]:

$$T_{\boldsymbol{\lambda}}: \ \tilde{x}_i = f^i(\boldsymbol{x},\boldsymbol{u};\boldsymbol{\lambda}), \ \ \tilde{u}^j = g^j(\boldsymbol{x},\boldsymbol{u};\boldsymbol{\lambda}); \quad i=1,...,n; \ \ j=1,...,m, \tag{1.62}$$

satisfying the properties same as the definition of one parameter Lie group of transformations with parameter λ replaced by vector $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\cdots,\lambda_r)$. The law of composition in this case can be denoted by

$$\boldsymbol{\phi}(\boldsymbol{\lambda},\boldsymbol{\lambda}')=(\phi_1(\boldsymbol{\lambda},\boldsymbol{\lambda}'),\phi_2(\boldsymbol{\lambda},\boldsymbol{\lambda}'),\cdots,\phi_r(\boldsymbol{\lambda},\boldsymbol{\lambda}')), \tag{1.63}$$

with $\boldsymbol{\lambda}'=(\lambda'_1,\cdots,\lambda'_r)$ and $\boldsymbol{\phi}(\boldsymbol{\lambda},\boldsymbol{\lambda}')$ is analytic in its domain and satisfies the axioms of group with $\boldsymbol{\lambda}=0$ implying identity $\lambda_1=\lambda_2=\cdots=\lambda_r=0$. The set of infinitesimal generators can be given by

$$V_s=\xi^i_s(\boldsymbol{x},\boldsymbol{u})\frac{\partial}{\partial x_i}+\eta^j_s(\boldsymbol{x},\boldsymbol{u})\frac{\partial}{\partial u^j}, \quad s=1,2,\cdots,r. \tag{1.64}$$

The Theorem 1.1 can also be generalized for r-parameter Lie group of transformations.

1.3.1.5 Lie algebras and classification of subalgebras

Definition 1.15 (Lie bracket)**.** Consider an r-parameter group of transformations with infinitesimal generators V_s, $s = 1, 2, \cdots, r$ then the Lie bracket (also called commutator) of any two generators V_1, V_2 is a unique infinitesimal generator satisfying the following [24]:

$$[V_1, V_2] = V_1V_2 - V_2V_1. \tag{1.65}$$

Definition 1.16 (Lie algebra)**.** A Lie algebra L is a vector space over field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) with a binary operation given by [24, 147]

$$[.,.] : L \times L \longrightarrow L \tag{1.66}$$

satisfying the following axioms

1. **Bilinearity**

$$\begin{aligned} &[a_1V_1 + a_2V_2, V_3] = a_1[V_1, V_3] + a_2[V_2, V_3], \\ &[V_1, a_2V_2 + a_3V_3] = a_2[V_1, V_2] + a_3[V_1, V_3] \quad \text{for constants} \quad a_1, a_2, a_3 \in \mathbb{F}. \end{aligned} \tag{1.67}$$

2. **Skew-Symmetry**

$$[V_1, V_2] = -[V_2, V_1]. \tag{1.68}$$

3. **Jacobi Identity**

$$[V_1, [V_2, V_3]] + [V_2, [V_3, V_1]] + [V_3, [V_1, V_2]] = 0, \tag{1.69}$$

for all $V_1, V_2, V_3 \in L$.

Remark

A Lie algebra is called abelian if and only if $[V_1, V_2] = 0$ for all $V_1, V_2 \in L$.

Definition 1.17 (Subalgebra)**.** A subalgebra [24] of a Lie algebra L is a vector subspace $H \subset L$ which is closed under Lie bracket. So, $[V_1, V_2] \in H$ for every $V_1, V_2 \in H$.

Definition 1.18 (Adjoint operator)**.** The adjoint of two vector fields $V, W \in L$ denoted by $Ad(\exp \epsilon V)W$ is defined as follows [24]:

$$Ad(\exp \epsilon V)W = W - \epsilon[V, W] + \frac{\epsilon^2}{2!}[V, [V, W]] - \cdots . \tag{1.70}$$

Classification of subalgebras

Let $V_1, V_2, \cdots, V_r$ be the vector fields of a Lie algebra L. It is obvious that any linear combination of these vector fields is again a vector field in L. The invariant solutions can be obtained for any of such linear combinations. The classification of subalgebras is a systematic process for determining a set of those linear combinations of symmetry generators of a Lie algebra which lead to inequivalent group invariant solutions. The technique for classification of one-dimensional subalgebras of a Lie algebra is as follows [24, 147]. The most general element of L is of the following form:

$$V = a_1V_1 + a_2V_2 + \cdots + a_rV_r. \tag{1.71}$$

Take its adjoint with each of the vector fields V_i for different values of a_i and simplify it as much as possible. It will result in a list of one dimensional subalgebras, called the optimal set, leading to inequivalent group invariants. Therefore, the inequivalent symmetry reductions can be determined solving which the exact solutions of the considered system of differential equations can be obtained.

1.3.2 Methodology

The stepwise procedure for Lie symmetry analysis of a system of differential equations is as follows [24]:

1. Let the one parameter Lie group of transformations (1.46) leaves the system (1.39) invariant.

2. Obtain the invariance criterion for the considered system (1.39).
3. Substitute the prolongations and equate the coefficients of various partial derivatives of dependent variables to zero resulting in an over-determined system of linear differential equations in the infinitesimals ξ^i, η^j. This system is called the set of determining equations.
4. Integrate the determining equations to obtain the infinitesimals ξ^i, η^j involving some arbitrary constants or arbitrary functions.
5. Solve the corresponding characteristic equations obtained from associated vector fields to find the dependent variables $\boldsymbol{u}$ in terms of new $n-1$ independent variables.
6. Transform the system (1.39) into a reduced system in the new variables.

1.3.3 Recent Advances of Symmetry Approach for FDEs

Recently, the Lie symmetry method has been developed for differential equations of fractional order in Riemann-Liouville sense. It has been proposed for single fractional order ODEs as well as time fractional PDEs [16, 32, 65]. New kind of prolongation formulae have been introduced for symmetry analysis of single FODEs and time fractional PDEs. The basic theory of invariance for integer order differential equations still remains applicable to FDEs. However, there are additional requirements of invariance at lower limit of integral in definition of Riemann-Liouville fractional derivative. Recently, many research articles have been discussed dealing with application of the symmetry method for single fractional order PDEs [78, 79, 99, 100, 130, 189, 196]. In continuation to work in this direction, this thesis mainly aims to develop and apply the Lie symmetry analysis for investigation of various kinds of fractional order systems of differential equations of any dimension and of any arbitrary order having integer as well as variable coefficients. The details of the executed research are provided in the corresponding chapters.

1.4 Conservation Laws

The concept of conservation laws plays an important role in the study of differential equations. The conservation laws for differential equations are mathematical formulation of the physical laws such as conservation of energy and conservation of momentum. A conservation law of a system of the form (1.39) is an n-tuple $(C^1, C^2, \cdots, C^n)$ satisfying divergence expression given as follows [147]:

$$\sum_{i=1}^{n} \mathrm{D}_i C^i(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{(1)}, \cdots, \boldsymbol{u}_{(k)}) = 0, \tag{1.72}$$

which vanishes for all solutions of the system (1.39). The law (1.72) is called a local conservation law provided that C^i are free of integral terms. The components C^i are called the fluxes of conservation law and the highest order derivative in the fluxes C^i is called the order of a conservation law. For $\boldsymbol{x} = (x, t)$, the conservation law is as follows:

$$\mathrm{D}_t(C^t) + \mathrm{D}_x(C^x) = 0. \tag{1.73}$$

The flux C^t corresponding to time variable t is called the conserved density. The conservation laws can be trivial or non-trivial. The conservation laws are called trivial provided any of the following two conditions hold:

- The n-tuple $(C^1, C^2, \cdots, C^n)$ itself vanishes for the solutions of the system (1.39).
- The expression (1.72) vanishes not only for solutions of the system (1.39) but for any arbitrary function $\boldsymbol{u}$.

There are many approaches to construct conservation laws for differential equations. In 1918, Emmy Noether [141] proved a theorem about application of continuous symmetries in proving the existence and derivation of conservation laws. Noether's theorem is a systematic procedure for determining conservation laws of PDEs having a Lagrangian function. There are some methods to find conservation laws which do not depend on the knowledge of Lagrangian. Anco and Bluman [9, 10] developed a direct method for determining conservation laws for PDEs. Kara and Mahomed [103] gave a technique to obtain conservation laws using partial Lagrangians. Ibragimov [91] proposed a formula using symmetries in which the PDE together with the adjoint equation is considered as a system. A computer package in Maple is also available to obtain conservation laws [39]. In this work, the Ibragimov's method is used for constructing conservation laws. Recently, many research articles are devoted to derivation of conservation laws for differential equations [3, 8, 33, 61, 138]. For more details on the theory, methods and applications of conservation laws, the interested reader is referred to Refs. [25, 137, 147].

Despite the importance of conservation laws in investigating integrability, internal properties and proving existence and uniqueness of solutions of differential equations [25, 95, 147], the conservation laws for fractional order PDEs are not widely discussed. Recently, the fractional generalized Noether operators are proposed [129] for time fractional PDEs not having Lagrangians to find their conservation laws using new conservation theorem [91]. The conservation laws have been discussed only for single time fractional PDEs in few papers [58, 64, 166, 167, 187, 195].

In this thesis, the conservation laws for FDEs are discussed in two chapters: chapter 2 and chapter 3.

1.5 Outline of the Thesis

The overall goal of this thesis is to extend the Lie group theory by proposing new prolongations for each extension and to demonstrate its applications to different kinds of physically significant fractional nonlinear PDEs.

In **chapter 2**, the methods to investigate Lie symmetries and conservation laws for time fractional systems of PDEs are proposed. The prolongation formulae are proposed for (1+1)-dimensional time fractional systems with an arbitrary number of dependent variables. The extended prolongation operators in Ref. [86] are proved incomplete in comparison with our suggested formulae with the help of some examples. The following time fractional nonlinear systems are investigated for determining their Lie symmetries, symmetry reductions, nonlinear self-adjointness and conservation laws.

- Time fractional Ito system
- Time fractional coupled Burgers equations
- Time fractional coupled KdV system
- Time fractional Hirota-Satsuma coupled KdV system
- Time fractional coupled Hirota equations

The extended form of classical symmetry method is introduced as well as applied for studying space-time fractional (1+1)-dimensional PDEs in **chapter 3**. The efficiency of the proposed method is proved by its application to some nonlinear space-time fractional PDEs leading to their similarity reductions into fractional order ODEs in terms of Erdélyi-Kober fractional operators. Also, by generalizing the concept of nonlinear self-adjointness and conserved vectors for space-time fractional PDEs, the conservation laws for the considered FPDEs are obtained. The following space-time fractional nonlinear PDEs are investigated:

- Space-time fractional Gilson-Pickering equation
- Space-time fractional generalized KdV equation

In **chapter 4**, the Lie symmetry approach is extended for space-time fractional systems of PDEs by developing the required prolongation formulae. The suggested formulae are applied for symmetry analysis of the following five physically significant nonlinear space-time fractional systems of PDEs:

- Space-time fractional Ito system
- Space-time fractional coupled Burgers equations
- Space-time fractional coupled KdV system
- Space-time fractional Hirota-Satsuma coupled KdV system
- Space-time fractional coupled Hirota equations

The above mentioned fractional nonlinear systems of PDEs are discussed for their Lie symmetries and reduced into fractional systems of nonlinear ODEs in Erdélyi-Kober operators.

Extending the work done in the previous chapters, **chapter 5** deals with deriving a generalized Lie symmetry approach for determining the Lie point symmetries for systems of FDEs with an arbitrary number of both independent and dependent variables. For this purpose, the required general prolongation formulae are proposed in a systematic way. In chapter 2 and chapter 4, the Lie symmetry method is proposed for time fractional systems and space-time fractional systems respectively having two independent and more than one dependent variables. The present study is an extension of the existing theory which allows to find the symmetries of higher dimensional systems of fractional ODEs as well as PDEs. The proposed symmetry approach is applied for Lie group analysis of the following five nonlinear higher dimensional fractional systems of PDEs:

- Fractional (2+1)-dimensional ANNV system
- Fractional (3+1)-dimensional Burgers system
- Fractional (3+1)-dimensional Navier-Stokes system
- Fractional (3+1)-dimensional incompressible non-hydrostatic Boussinesq system
- Fractional (3+1)-dimensional incompressible non-hydrostatic Boussinesq system with viscosity

In **chapter 6**, the applications of Lie symmetry method for FDEs is extended from FPDEs with constant coefficients to FPDEs with variable coefficients. Two single time fractional nonlinear PDEs with variable coefficients and three time fractional nonlinear systems of PDEs with variable coefficients are investigated using the symmetry approach and reduced into fractional order nonlinear ODEs. The discussed FDEs are as follows:

- Variable coefficient time fractional KdV-Burger-Kuramoto equation
- Variable coefficient time fractional generalized seventh order KdV equation
- Variable coefficient coupled Boussinesq system
- Variable coefficient coupled KdV system
- Variable coefficient Hirota-Satsuma coupled KdV system

Chapter 2

Time Fractional Nonlinear Systems of Partial Differential Equations

2.1 Introduction

Lie group analysis has been applied to FODEs [16, 104] as well as FPDEs [32, 78, 87, 100, 120, 188, 191, 197]. In the research works in literature on symmetry analysis of FDEs [163, 169, 187, 189, 190], only single fractional order differential equations have been discussed. But this chapter deals with the study of systems of time fractional PDEs with the help of Lie symmetry analysis. Huang and Shen [86] also extended the Lie symmetry approach for solving the time fractional systems of PDEs. They proposed fractional order prolongation operators for determining Lie symmetries for systems of FPDEs and presented its application to a time fractional system. In this chapter, it is shown that their prolongation formulae [86] are incomplete and are not able to derive the general form of Lie symmetries for time fractional systems. Therefore, the required modifications in the formulae are suggested in this chapter to overcome the shortcomings of their formulae.

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(ii) A part of this chapter is published in *Journal of Mathematical Physics, 58, 054101 (2017).*
(iii) A part of this chapter is published in *Communications in Nonlinear Science and Numerical Simulation 53, 10-21 (2017).*

For FDEs not having Lagrangians, the derivation of conserved vectors is limited to only single time fractional PDEs [5, 64, 129, 167, 187]. However, to the best of our knowledge, the idea of investigating conservation laws using symmetry generators has not been extended to the systems of time fractional PDEs. This chapter aims to extend the symmetry approach and method to find conservation laws for time fractional systems of PDEs. For this purpose, the fractional extended infinitesimals are proposed for the time fractional systems of PDEs with an arbitrary number of dependent variables. The concept of nonlinear self-adjointness and a generalization of fractional Noether operators are provided for formulation of conservation laws for time fractional systems with the aid of new conservation theorem [91].

The organization of the chapter is as follows. In Section 2.2, the Lie symmetry method is extended for time fractional systems of PDEs. Also, the comparison of the proposed formulae with the formulae in literature is presented using some examples. In Section 2.3, the technique to find conservation laws is proposed for time fractional systems of PDEs. Section 2.4 deals with applications of the proposed approaches for investigating Lie symmetry analysis and conserved vectors of some time fractional nonlinear systems. The conclusion of the chapter is discussed in Section 2.5.

2.2 Lie Symmetry Analysis for Time Fractional Systems of Partial Differential Equations

In this section, the Lie symmetry analysis is introduced for systems of time fractional PDEs. Consider a system of time fractional PDEs with $u(x,t)$ and $v(x,t)$ as dependent variables and $\alpha > 0$ in the following form:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= F_1(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}, ...), \\ \frac{\partial^\alpha v}{\partial t^\alpha} &= F_2(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}, ...). \end{aligned} \tag{2.1}$$

Assume the invariance of the system (2.1) under one parameter Lie group of transformations of the following form:

$$\begin{aligned} \tilde{x} &= x + \epsilon\xi(x, t, u, v) + O(\epsilon^2), \\ \tilde{t} &= t + \epsilon\tau(x, t, u, v) + O(\epsilon^2), \\ \tilde{u} &= u + \epsilon\eta(x, t, u, v) + O(\epsilon^2), \\ \tilde{v} &= v + \epsilon\phi(x, t, u, v) + O(\epsilon^2), \\ \frac{\partial^\alpha \tilde{u}}{\partial t^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta^{\alpha,t} + O(\epsilon^2), \\ \frac{\partial^\alpha \tilde{v}}{\partial t^\alpha} &= \frac{\partial^\alpha v}{\partial t^\alpha} + \epsilon\phi^{\alpha,t} + O(\epsilon^2), \\ \frac{\partial \tilde{u}}{\partial x} &= \frac{\partial u}{\partial x} + \epsilon\eta^x + O(\epsilon^2), \\ \frac{\partial \tilde{v}}{\partial x} &= \frac{\partial v}{\partial x} + \epsilon\phi^x + O(\epsilon^2), \\ &\vdots \end{aligned} \tag{2.2}$$

where $(\xi,\ \tau,\ \eta,\ \phi)$ is the set of infinitesimals, $(\eta^{\alpha,t},\ \phi^{\alpha,t})$ are extended infinitesimals of order α, $(\eta^x,\ \phi^x)$ are extended infinitesimals of order 1 and so on. The corresponding infinitesimal symmetry generator is as follows:

$$X = \xi(x, t, u, v)\frac{\partial}{\partial x} + \tau(x, t, u, v)\frac{\partial}{\partial t} + \eta(x, t, u, v)\frac{\partial}{\partial u} + \phi(x, t, u, v)\frac{\partial}{\partial v}. \tag{2.3}$$

The α^{th} order extended infinitesimal function $\eta^{\alpha,t}$ related to Riemann-Liouville fractional derivatives is as follows [15, 78, 87, 169]:

$$\eta^{\alpha,t} = D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u). \tag{2.4}$$

Since $D_t^{\alpha+1} f(t) = D_t^\alpha(D_t f(t))$, it can be further simplified to the following:

$$\eta^{\alpha,t} = D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + \tau D_t^\alpha(u_t) - D_t^\alpha(\tau u_t). \tag{2.5}$$

Using the generalized Leibniz rule (1.33), the expression (2.5) can be written as follows:

$$\eta^{\alpha,t} = D_t^\alpha(\eta) - \alpha \mathrm{D}_t(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n}\mathrm{D}_t^n(\xi)D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1}\mathrm{D}_t^{n+1}(\tau)D_t^{\alpha-n}(u). \tag{2.6}$$

Using the Osler's rule [148] and generalized chain rule [106, 120, 157], the term $D_t^\alpha(\eta)$ in (2.6) can be obtained as follows:

$$\begin{aligned} D_t^\alpha(\eta) = {} & \frac{\partial^\alpha \eta}{\partial t^\alpha} + \left(\eta_u\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha}\right) + \left(\eta_v\frac{\partial^\alpha v}{\partial t^\alpha} - v\frac{\partial^\alpha \eta_v}{\partial t^\alpha}\right) + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n}D_t^{\alpha-n}(u) \\ & + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \eta_v}{\partial t^n}D_t^{\alpha-n}(v) + \mu_1 + \mu_2, \end{aligned} \tag{2.7}$$

where

$$\mu_1 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}, \tag{2.8}$$

$$\mu_2 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-v)^r\frac{\partial^m}{\partial t^m}(v^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial v^k}. \tag{2.9}$$

Therefore, the final expression for α^{th} extended infinitesimal $\eta^{\alpha,t}$ for system of FPDEs of the form (2.1) can be calculated as follows:

$$\begin{aligned} \eta^{\alpha,t} = {} & \frac{\partial^\alpha \eta}{\partial t^\alpha} + \left(\eta_u - \alpha \mathrm{D}_t(\tau)\right)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \left(\eta_v\frac{\partial^\alpha v}{\partial t^\alpha} - v\frac{\partial^\alpha \eta_v}{\partial t^\alpha}\right) + \mu_1 + \mu_2 \\ & + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \eta_v}{\partial t^n}D_t^{\alpha-n}(v) \\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n}\mathrm{D}_t^n(\xi)D_t^{\alpha-n}(u_x). \end{aligned} \tag{2.10}$$

Analogous to the expression obtained above, the α^{th} extended infinitesimal $\phi^{\alpha,t}$ can be obtained as follows:

$$\begin{aligned} \phi^{\alpha,t} = {} & \frac{\partial^\alpha \phi}{\partial t^\alpha} + \left(\phi_v - \alpha \mathrm{D}_t(\tau)\right)\frac{\partial^\alpha v}{\partial t^\alpha} - v\frac{\partial^\alpha \phi_v}{\partial t^\alpha} + \left(\phi_u\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \phi_u}{\partial t^\alpha}\right) + \mu_3 + \mu_4 \\ & + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \phi_v}{\partial t^n} - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}(\tau)\right]D_t^{\alpha-n}(v) + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \phi_u}{\partial t^n}D_t^{\alpha-n}(u) \\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n}\mathrm{D}_t^n(\xi)D_t^{\alpha-n}(v_x), \end{aligned} \tag{2.11}$$

where μ_3 and μ_4 are as follows:

$$\mu_3 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\phi}{\partial t^{n-m}\partial u^k}, \tag{2.12}$$

and

$$\mu_4 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-v)^r\frac{\partial^m}{\partial t^m}(v^{k-r})\frac{\partial^{n-m+k}\phi}{\partial t^{n-m}\partial v^k}. \tag{2.13}$$

Following the previous steps, these prolongation rules can be easily extended in case of systems of time fractional PDEs with more than two dependent variables. The general prolongation operators for m dependent variables are proposed in the following theorem:

Theorem 2.1. *The general prolongation formulae* $\eta^{i(\alpha,0)}$ $(i=1,...,m)$ *for time fractional systems of PDEs having* m *dependent variables* $(u^1,...,u^m)$ *and independent variables* (x,t) *for Riemann-Liouville fractional derivative* $0<\alpha<1$ *are as follows:*

$$\begin{aligned}\eta^{i(\alpha,0)} = {} & \frac{\partial^\alpha \eta^i}{\partial t^\alpha} + \left(\frac{\partial \eta^i}{\partial u^i} - \alpha D_t\tau\right)\frac{\partial^\alpha u^i}{\partial t^\alpha} - u^i\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial \eta^i}{\partial u^i}\right) + \sum_{j\neq i}\left(\frac{\partial \eta^i}{\partial u^j}\frac{\partial^\alpha u^j}{\partial t^\alpha} - u^j\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial \eta^i}{\partial u^j}\right)\right)\\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x^i) + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^{n+1}\eta^i}{\partial t^n\partial u^i} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}u^i\\ & + \sum_{j\neq i}\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^{n+1}\eta^i}{\partial t^n\partial u^j}D_t^{\alpha-n}(u^j) + \sum_{p=1}^{m}\mu_p^i,\end{aligned} \tag{2.14}$$

where

$$\mu_p^i = \sum_{n=2}^{\infty}\sum_{q=2}^{n}\sum_{k=2}^{q}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{q}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u^p)^r\frac{\partial^q}{\partial t^q}(u^p)^{k-r}\frac{\partial^{n-q+k}\eta^i}{\partial t^{n-q}(\partial u^p)^k}. \tag{2.15}$$

Note 3. Note that in expression (2.14), the expressions for μ_p^i become zero if the η^i are linear in the dependent variables u^p.

2.2.1 Comments on the Prolongation Formulae in Literature

The fractional order prolongation formulae $\eta^{i(\alpha,0)}$ $(i=1,...,m)$ provided by Huang and Shen [86] for the time fractional systems of PDEs with m dependent variables $(u^1,...,u^m)$ are as follows:

$$\begin{aligned}\eta^{i(\alpha,0)} = {} & \frac{\partial^\alpha \eta^i}{\partial t^\alpha} + \left(\frac{\partial \eta^i}{\partial u^i} - \alpha \mathrm{D}_t\tau\right)\frac{\partial^\alpha u^i}{\partial t^\alpha} - u^i\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial \eta^i}{\partial u^i}\right) - \sum_{n=1}^{\infty}\binom{\alpha}{n}\mathrm{D}_t^n(\xi)D_t^{\alpha-n}(u_x^i)\\ & + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^{n+1}\eta^i}{\partial t^n\partial u^i} - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}(\tau)\right]D_t^{\alpha-n}u^i + \mu^i.\end{aligned} \tag{2.16}$$

In our point of view, the formulae (2.16) are incomplete and the correct are proposed by Theorem 2.1. The suggested prolongation formulae (2.14) for $m = 1$ *i.e.* in case of one dependent variable, coincide with the one given particularly for two independent variables in a recent work [120]. In that paper [120], the generalized prolongation formula is proposed for the symmetry analysis of single fractional PDEs with one dependent variable and arbitrary number of independent variables. On the other hand, in this chapter, the generalized prolongation formulae (2.14) are proposed for symmetry analysis of fractional systems with an arbitrary number of dependent variables and two independent variables. Our claim is that the proposed formulae (2.14) are complete in the sense of providing generalized Lie symmetries for time fractional systems with an arbitrary number of dependent variables. For justification of our claim, both kind of formulae are applied for the derivation of Lie point symmetries for some time fractional systems.

2.2.1.1 Comparison of (2.14) and (2.16) with the integer order formulae

In this subsection, the comparison of both types of prolongation formulae (2.14) and (2.16) has been done with the integer order prolongation operators. For simplification, it is performed for $m = 2$ *i.e.* in case of two dependent variables say (u, v). The symmetry operator can be written in the following form:

$$V = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \eta(x,t,u,v)\partial_u + \zeta(x,t,u,v)\partial_v. \tag{2.17}$$

It is well known that for integer order systems of PDEs, the first order prolongation operators [24, 147] are defined as follows:

$$\begin{aligned}\eta^t &= \eta_t + (\eta_u - \tau_t)\, u_t + \eta_v v_t - \tau_u u_t^2 - \tau_v v_t u_t - \xi_t u_x - \xi_u u_t u_x - \xi_v v_t u_x,\\ \zeta^t &= \zeta_t + (\zeta_v - \tau_t)\, v_t + \zeta_u u_t - \tau_u u_t v_t - \tau_v v_t^2 - \xi_t v_x - \xi_u u_t v_x - \xi_v v_t v_x.\end{aligned} \tag{2.18}$$

Substituting $\alpha = 1$ and $m = 2$, the expression (2.16) reduces into the operators as follows:

$$\begin{aligned} \eta^{(1,0)} &= \eta_t + (\eta_u - \tau_t)\, u_t - \tau_u u_t^2 - \tau_v v_t u_t - \xi_t u_x - \xi_u u_t u_x - \xi_v v_t u_x, \\ \zeta^{(1,0)} &= \zeta_t + (\zeta_v - \tau_t)\, v_t - \tau_u u_t v_t - \tau_v v_t^2 - \xi_t v_x - \xi_u u_t v_x - \xi_v v_t v_x. \end{aligned} \tag{2.19}$$

Obviously, the reduced formulae (2.19) are incomplete in comparison with (2.18). On the other hand, the reduced form of prolongation operators (2.14) for $m = 2$ and $\alpha = 1$ coincide with the integer order formulae (2.18). It proves the accuracy of the proposed prolongation formulae.

2.2.1.2 Examples

In this section, the efficiency of the proposed formulae (2.14) in comparison with formulae (2.16) is illustrated by calculating Lie symmetries for three systems of time fractional PDEs.

Example 1

The first example is the same system considered by Haung and Shen [86] and given by

$$\begin{aligned} \partial_t^\alpha u &= C^2(x) v_x, \\ \partial_t^\alpha v &= u_x, \end{aligned} \tag{2.20}$$

where $\partial_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha}$. Using the proposed prolongation operators (2.14) in the invariance conditions given by Haung and Shen [86], the determining equations can be obtained as follows:

$$\begin{aligned} &\binom{\alpha}{n}\frac{\partial^n \zeta_v}{\partial t^n} - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \\ &\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \\ &\mathrm{D}_t^n(\xi) = 0, \\ &\tau_x = \tau_u = \tau_v = 0, \\ &\eta_v - \zeta_u C^2(x) = 0, \\ &(\eta_u - \alpha\tau_t)C^2(x) - 2C(x)\dot{C}(x)\xi - C^2(x)\zeta_v + C^2(x)D_x\xi = 0, \end{aligned} \tag{2.21}$$

$$\zeta_v - \alpha\tau_t - \eta_u + \mathrm{D}_x\xi = 0,$$

$$\partial_t^\alpha \eta - u\partial_t^\alpha \eta_u - v\partial_t^\alpha \eta_v - C^2(x)\zeta_x + \mu_1^1 + \mu_2^1 = 0,$$

$$\partial_t^\alpha \zeta - u\partial_t^\alpha \zeta_u - v\partial_t^\alpha \zeta_v - \eta_x + \mu_1^2 + \mu_2^2 = 0.$$

The determining equations obtained above are generalized than those in Ref. [86]. Solving these determining equations, the set of solutions is as follows:

$$\xi = C(x)\left(\int \frac{C_1}{C(x)}\mathrm{d}x + C_2\right), \quad \tau = \frac{C_1 t}{\alpha}. \tag{2.22}$$

Here, for brevity, the values for η and ζ are solved only for the following two cases:

Case 1: If $C_1 = C_2 = 0$, $C(x)$ is arbitrary, integrating the determining equations gives the following symmetries:

$$\xi = 0, \quad \tau = 0, \quad \eta = C_3 u + (C_4 v)C^2(x) + f(x,t), \quad \zeta = C_3 v + C_4 u + g(x,t), \tag{2.23}$$

where C_3, C_4 are arbitrary constants and functions f, g satisfy the following:

$$\begin{aligned} \partial_t^\alpha f &= C^2(x) g_x, \\ \partial_t^\alpha g &= f_x. \end{aligned} \tag{2.24}$$

Case 2: If $C_1 \neq 0$, $C_2 \neq 0$ and $C(x) = 1$ then the set of solutions is as follows:

$$\xi = C_1 x + C_2, \quad \tau = \frac{C_1 t}{\alpha}, \quad \eta = C_3 u + C_4 v + f(x,t), \quad \zeta = C_3 v + C_4 u + g(x,t), \tag{2.25}$$

where C_1, C_2, C_3, C_4 are arbitrary constants and functions f, g must satisfy the following:

$$\begin{aligned} \partial_t^\alpha f &= g_x, \\ \partial_t^\alpha g &= f_x. \end{aligned} \tag{2.26}$$

Clearly, the symmetries derived in both the cases are more generalized than the symmetries obtained in Ref. [86]. Similarly, for the remaining cases in Ref. [86], the generalized form of symmetries can be calculated using the suggested formulae. Note that for the sake of simplicity, the linearity of η and ζ in both the dependent variables u, v has been assumed.

Example 2

The second considered time fractional system of PDEs is as follows:

$$\begin{aligned} &\partial_t^\alpha u + 2v(u_x^2 + v_x^2) = 0, \\ &\partial_t^\alpha v - 2u(u_x^2 + v_x^2) = 0. \end{aligned} \tag{2.27}$$

The invariance criterion with the infinitesimal operator (2.17) can be written in the following form:

$$\begin{aligned} &\left[\eta^{(\alpha,0)} + 2\zeta(u_x^2 + v_x^2) + 2v(2u_x\eta^x + 2v_x\zeta^x)\right]\Big|_{(2.27)} = 0, \\ &\left[\zeta^{(\alpha,0)} - 2\eta(u_x^2 + v_x^2) - 2u(2u_x\eta^x + 2v_x\zeta^x)\right]\Big|_{(2.27)} = 0. \end{aligned} \tag{2.28}$$

Using the prolongation operators (2.16) in (2.28) and solving the resulting determining equations, the solutions can be divided into the following cases:

Case $\alpha = \frac{1}{2}$: In this case, the Lie point symmetries are obtained as follows:

$$\xi = C_1x + C_2, \quad \tau = \tau(t), \quad \eta = \frac{u}{2}\left(2C_1 - \frac{\tau_t}{2}\right), \quad \zeta = \frac{v}{2}\left(2C_1 - \frac{\tau_t}{2}\right), \tag{2.29}$$

where C_1, C_2 being arbitrary constants and $\tau(0) = 0$.

Case $\alpha \neq \frac{1}{2}, \tau_{tt} = 0$: In this case, the obtained symmetries are of the following form:

$$\xi = C_1x + C_2, \quad \tau = C_3t, \quad \eta = \frac{u}{2}(2C_1 - \alpha C_3), \quad \zeta = \frac{v}{2}(2C_1 - \alpha C_3), \tag{2.30}$$

where C_1, C_2 and C_3 are arbitrary constants.

On the other hand, taking advantage of the suggested operators (2.14) in conditions (2.28), the Lie point symmetries are obtained as follows:

Case $\alpha = \frac{1}{2}$: In this case, the obtained Lie point symmetries are as follows:

$$\xi = C_1x + C_2, \quad \tau = \tau(t), \quad \eta = C_4v + \frac{u}{2}\left(2C_1 - \frac{\tau_t}{2}\right), \quad \zeta = -C_4u + \frac{v}{2}\left(2C_1 - \frac{\tau_t}{2}\right). \tag{2.31}$$

Case $\alpha \neq \frac{1}{2}, \tau_{tt} = 0$: In this case, the symmetries are obtained in the following form:

$$\xi = C_1x + C_2, \quad \tau = C_3t, \quad \eta = C_4v + \frac{u}{2}(2C_1 - \alpha C_3), \quad \zeta = -C_4u + \frac{v}{2}(2C_1 - \alpha C_3), \tag{2.32}$$

where C_1, C_2, C_3 and C_4 being arbitrary constants. Clearly, the symmetries (2.31), (2.32) are more generalized than (2.29), (2.30). It is also important to note that the symmetries (2.31), (2.32) are equivalent to the integer order symmetries of the system (2.27) for $\alpha = 1$, proving the validity of the introduced formulae (2.14).

Example 3

The third time fractional system is of following form:

$$\begin{aligned} &\partial_t^\alpha u + v_x + 2v(u^2+v^2) = 0, \\ &\partial_t^\alpha v - u_x - 2u(u^2+v^2) = 0. \end{aligned} \tag{2.33}$$

The invariance conditions are of the following forms:

$$\begin{aligned} &\left[\eta^{(\alpha,0)} + \zeta^x + 2\zeta(u^2+v^2) + 2v(2u\eta + 2v\zeta)\right]\Big|_{(2.33)} = 0, \\ &\left[\zeta^{(\alpha,0)} - \eta^x - 2\eta(u^2+v^2) - 2u(2u\eta + 2v\zeta)\right]\Big|_{(2.33)} = 0. \end{aligned} \tag{2.34}$$

Using the prolongation operators (2.16), the Lie point symmetries can be obtained as follows:

$$\xi = C_1 x + C_2, \quad \tau = \frac{C_1 t}{\alpha}, \quad \eta = -\frac{C_1 u}{2}, \quad \zeta = -\frac{C_1 v}{2}, \tag{2.35}$$

where C_1, C_2 being arbitrary constants. Applying the prolongation formulae (2.14) proposed in this chapter, the investigated symmetries are of the following form:

$$\xi = C_1 x + C_2, \quad \tau = \frac{C_1 t}{\alpha}, \quad \eta = -\frac{C_1 u}{2} + C_3 v, \quad \zeta = -\frac{C_1 v}{2} - C_3 u, \tag{2.36}$$

where C_1, C_2 and C_3 are arbitrary constants. The symmetries (2.36) are general than (2.35) and also equivalent with the integer order symmetries of the system (2.33) for $\alpha = 1$. Here, for simplicity, the linearity of η and ζ in both the dependent variables has been chosen.

For all these systems, the derived Lie point symmetries using the formulae (2.14) are more generalized than those using the formulae (2.16).

2.3 Conservation Laws for Time Fractional Systems of Partial Differential Equations

In this section, the technique to find conservation laws is proposed by investigating the nonlinear self-adjointness of the time fractional systems. Consider a time fractional system of the following form:

$$F_j(x,t,u_1,..,u_m,\partial_t^\alpha u_1,..,\partial_t^\alpha u_m,u_{1,x},..,u_{m,x},..)=0,\quad j=1,..,m, \tag{2.37}$$

with m dependent variables $(u_1,...,u_m)$. Assume that the system (2.37) admits the Lie group of transformations with the following vector fields:

$$V_i=\xi_i\partial_x+\tau_i\partial_t+\sum_{j=1}^{m}\eta_i^j\partial_{u_j}. \tag{2.38}$$

2.3.1 Nonlinear Self-Adjointness

The idea of nonlinear self-adjointness is very well known and well established [92, 93] for the integer order PDEs. Recently, it is presented for single time fractional PDEs [129, 167]. In this section, the concept of nonlinear self-adjointness is extended from single time fractional PDEs to the systems of time fractional PDEs. The formal Lagrangian can be written in the following form:

$$\mathcal{L}=\sum_{j=1}^{m}p_j(x,t)(F_j), \tag{2.39}$$

where $p_j(x,t)$ are the new dependent variables. The adjoint equations are defined as follows:

$$F_j^*\equiv\frac{\delta\mathcal{L}}{\delta u_j}=0,\quad j=1,...,m. \tag{2.40}$$

Here, $\frac{\delta}{\delta u_j}$ represent the Euler-Lagrange operators defined by

$$\frac{\delta}{\delta u_j}=\frac{\partial}{\partial u_j}+(D_t^\alpha)^*\frac{\partial}{\partial(D_t^\alpha u_j)}+\sum_{k=1}^{\infty}(-1)^k\mathrm{D}_{i_1}\mathrm{D}_{i_2},\ldots,\mathrm{D}_{i_k}\frac{\partial}{\partial(u_j)_{i_1,i_2,\ldots,i_k}}. \tag{2.41}$$

Here, $(D_t^\alpha)^*$ is the adjoint operator defined by

$$(D_t^\alpha)^* = (-1)^n I_c^{n-\alpha}(D_t^n) = {}_t^C D_c^\alpha, \tag{2.42}$$

with $I_c^{n-\alpha}$ is the right-hand sided fractional integral operator of order $n-\alpha$ given by

$$I_c^{n-\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_t^c \frac{f(x,s)}{(s-t)^{1+\alpha-n}} \mathrm{d}s, \quad \text{for } n = [\alpha] + 1. \tag{2.43}$$

Also, ${}_t^C D_c^\alpha$ is the right-hand sided Caputo fractional differential operator [106, 129] of order α.

The time fractional system (2.37) is called nonlinearly self-adjoint if the adjoint equations (2.40) are satisfied for all solutions of (2.37) upon the following substitutions:

$$p_j = \psi_j(x, t, u_1, .., u_m), \quad j = 1, .., m, \tag{2.44}$$

satisfying $\psi_j \neq 0$ for at least one j.

In other words, the following conditions must be satisfied:

$$\left.\frac{\delta \mathcal{L}}{\delta u_j}\right|_{(2.44)} = \sum_{i=1}^{m} \lambda_i^j (F_i), \qquad j = 1, .., m, \tag{2.45}$$

where λ_i^j are the undetermined coefficients.

2.3.2 Conservation Laws

Here, the Noether operators are generalized to calculate conservation laws for time fractional systems with the help of new conservation theorem [91]. Since there are no fractional derivatives with respect to x, the conserved vectors for the component x are same as for integer order systems, and given by [92, 93]:

$$C_i^x = \sum_{j=1}^{m} \left[W_i^j \frac{\delta L}{\delta u_{j,x}} + D_x(W_i^j) \frac{\delta L}{\delta u_{j,xx}} + D_x^2(W_i^j) \frac{\delta L}{\delta u_{j,xxx}} + \dots \right] \tag{2.46}$$

where W_i^j are the characteristic functions corresponding to vector fields V_i and dependent variables u_j, defined as follows:

$$W_i^j = \eta_i^j - \xi_i u_x - \tau_i u_t. \tag{2.47}$$

The t-component of conserved vector can be extended for time fractional systems of PDEs as follows:

$$C_i^t = \sum_{j=1}^{m} \left[\sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k}(W_i^j) D_t^k \left(\frac{\partial \mathcal{L}}{\partial (D_t^\alpha u_j)} \right) - (-1)^n J \left(W_i^j, D_t^n \left(\frac{\partial \mathcal{L}}{\partial (D_t^\alpha u_j)} \right) \right) \right], \tag{2.48}$$

where $J(f, g)$ is the integral defined by

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^p \frac{f(x,s)g(x,r)}{(r-s)^{\alpha+1-n}} \mathrm{d}r \mathrm{d}s. \tag{2.49}$$

2.4 Applications of the Proposed Methods

In this section, the efficiency of the proposed extensions of Lie symmetry method and method to find conserved vectors is illustrated by their application to some physically significant time fractional nonlinear systems. The considered time fractional nonlinear systems of PDEs are discussed for symmetry analysis for $0 < \alpha \leq 1$ and for conserved vectors with $0 < \alpha < 2, \alpha \neq 1$.

2.4.1 Time Fractional Ito System

The Ito system [19] describes the interaction process of two internal long waves and has infinitely many conserved quantities. It is a generalization of KdV for more complicated nonlinear effects on the wave propagation in KdV [19]. In [114], it is shown that this coupled system can be a member of a bi-Hamiltonian integrable hierarchy. The time fractional Ito system has been studied in different forms [110] for its approximate solutions. Here, we consider the time fractional Ito system in the following form:

$$\begin{aligned}
&\partial_t^\alpha u - 3uu_x - vu_x - u_{xxx} = 0,\\
&\partial_t^\alpha v - (uv)_x = 0.
\end{aligned} \tag{2.50}$$

2.4.1.1 Symmetry analysis

The symmetry analysis of the system (2.50) is described in the following theorem:

Theorem 2.2. *If the Ito system* (2.50) *admits the group of transformations* (2.2) *then the Lie symmetries are derived as follows:*

$$\xi = \frac{c_1 x}{3} + c_2, \quad \tau = \frac{c_1 t}{\alpha}, \quad \eta = -\frac{2}{3}c_1 u \quad \textit{and} \quad \phi = -\frac{2}{3}c_1 v, \tag{2.51}$$

where c_1, c_2 are arbitrary constants. Hence, the corresponding Lie algebra is generated by the following vector fields:

$$\begin{aligned}
V_1 &= \frac{x}{3}\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} - \frac{2}{3}u\frac{\partial}{\partial u} - \frac{2}{3}v\frac{\partial}{\partial v},\\
V_2 &= \frac{\partial}{\partial x}.
\end{aligned} \tag{2.52}$$

Proof. For the invariance of system (2.50) under group of transformations (2.2), the invariance criterion takes the following form:

$$\begin{aligned}
&[\eta^{\alpha,t} - 3(u\eta^x + \eta u_x) - (v\eta^x + \phi u_x) - \eta^{xxx}]|_{(2.50)} = 0,\\
&[\phi^{\alpha,t} - (u\phi^x + \eta v_x) - (v\eta^x + \phi u_x)]|_{(2.50)} = 0.
\end{aligned} \tag{2.53}$$

Then substituting the values of prolongations and equating the coefficients of various linearly independent variables to zero, the following set of determining equations for $0 < \alpha < 1$ can be obtained:

$$\begin{aligned}
&\xi_t = \xi_u = \xi_v = 0,\\
&\binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad n \in \mathbb{N},\\
&\binom{\alpha}{n}\partial_t^n \phi_v - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad n \in \mathbb{N},\\
&\tau_u = \tau_v = \tau_x = 0,\\
&\eta_{uu} = \eta_{vv} = \eta_{uv} = \eta_{vt} = 0,
\end{aligned} \tag{2.54}$$

$$\begin{aligned}
&\phi_u = 0,\\
&3\eta_{xxv} - (2u+v)\eta_v = 0,\\
&v\eta_v + (v+3u)(\xi_x - \alpha\tau_t) - \phi - 3\eta + 3\eta_{xxu} - \xi_{xxx} = 0,\\
&\alpha\tau_t - 3\xi_x = 0,\\
&\eta_{xu} - \xi_{xx} = 0,\\
&\eta = u(\xi_x - \alpha\tau_t) - v\eta_v,\\
&\phi - v(\phi_v - \alpha\tau_t - \eta_u + \xi_x) = 0,\\
&\partial_t^\alpha \eta - u\partial_t^\alpha \eta_u - v\partial_t^\alpha \eta_v - (3u+v)\eta_x - \eta_{xxx} = 0,\\
&\partial_t^\alpha \phi - v\partial_t^\alpha \phi_v - u\phi_x - v\eta_x = 0.
\end{aligned}$$

On solving this system of FDEs and PDEs yields the following general solution:

$$\xi = \frac{c_1 x}{3} + c_2, \quad \tau = \frac{c_1 t}{\alpha} + c_3, \quad \eta = -\frac{2}{3}c_1 u \quad \text{and} \quad \phi = -\frac{2}{3}c_1 v, \tag{2.55}$$

where c_i $(i = 1,2,3)$ are arbitrary constants. Also, for $\alpha = 1$, the obtained symmetries are same as symmetries (2.55). Because the lower limit of integral in Riemann-Liouville fractional derivative operator (1.19) is fixed, so for preservation of its structure under transformations (2.2), the following condition should hold:

$$\tau(x,t,u,v)|_{t=0} = 0.$$

Thus, it gives $c_3 = 0$ and completes the proof of the theorem. □

It is easy to check that the symmetry generators V_1 and V_2 in (2.52) form a closed Lie algebra, as we have the following:

$$[V_1, V_1] = 0 = [V_2, V_2], \qquad [V_1, V_2] = -\frac{1}{3}V_2 = -[V_2, V_1].$$

Next step is imposing the obtained Lie symmetries to find the reduced system for (2.50). For the vector field V_1, the following characteristic equations are obtained:

$$\frac{\mathrm{d}x}{\frac{x}{3}} = \frac{\mathrm{d}t}{\frac{t}{\alpha}} = \frac{\mathrm{d}u}{-\frac{2u}{3}} = \frac{\mathrm{d}v}{-\frac{2v}{3}}.$$

The obtained similarity solutions are as follows:

$$z = xt^{-\frac{\alpha}{3}}, \quad u(x,t) = t^{-\frac{2\alpha}{3}} f(z) \quad \text{and} \quad v(x,t) = t^{-\frac{2\alpha}{3}} g(z). \tag{2.56}$$

Before calculating the symmetry reductions, let us introduce the Erdélyi-Kober fractional differential operator $\left(\mathcal{P}_{\delta}^{\zeta,\alpha}\right)$ defined by [107, 169, 178]

$$\left(\mathcal{P}_{\delta}^{\zeta,\alpha} h\right)(z) := \prod_{j=0}^{m-1} \left(\zeta + j - \frac{1}{\delta} z \frac{\mathrm{d}}{\mathrm{d}z}\right) (\mathcal{K}_{\delta}^{\zeta+\alpha, m-\alpha} h)(z), \qquad z > 0,\, \delta > 0,\, \alpha > 0, \tag{2.57}$$

$$m = \begin{cases} [\alpha] + 1 & \text{if } \alpha \notin \mathbb{N}, \\ \alpha & \text{if } \alpha \in \mathbb{N}, \end{cases}$$

where

$$\left(\mathcal{K}_{\delta}^{\zeta,\alpha} h\right)(z) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_{1}^{\infty} (p-1)^{\alpha-1} p^{-(\zeta+\alpha)} h(zp^{\frac{1}{\delta}}) \mathrm{d}p & \text{if } \alpha > 0; \\ h(z) & \text{if } \alpha = 0, \end{cases} \tag{2.58}$$

is the Erdélyi-Kober fractional integral operator. Then the reduction of system (2.50) into a system of FODEs, using (2.56), is described by the following assertion:

Theorem 2.3. *The similarity transformations* $u(x,t) = t^{-\frac{2\alpha}{3}} f(z)$ *and* $v(x,t) = t^{-\frac{2\alpha}{3}} g(z)$ *with the similarity variable* $z = xt^{-\frac{\alpha}{3}}$ *reduce the time fractional Ito system* (2.50) *into the following system of nonlinear FODEs:*

$$\begin{aligned} \left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} f\right)(z) &= g(z)f'(z) + 3f(z)f'(z) + f'''(z), \\ \left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} g\right)(z) &= f(z)g'(z) + g(z)f'(z), \end{aligned} \tag{2.59}$$

where $\left(\mathcal{P}_{\delta}^{\zeta,\alpha}\right)$ *is the Erdélyi-Kober fractional differential operator defined by* (2.57) *and* (2.58).

Proof. Before calculating $\partial_t^{\alpha} u$, first let $n-1 < \alpha < n$ $(n = 1, 2, 3, ...)$ then the definition of Riemann -Liouville fractional derivatives (1.19) gives the following:

$$\partial_t^{\alpha} u = \frac{\partial^n}{\partial t^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} s^{\frac{-2\alpha}{3}} f(xs^{\frac{-\alpha}{3}}) \mathrm{d}s \right]. \tag{2.60}$$

Let $p = \frac{t}{s}$, then the above expression is transformed into the following:

$$\partial_t^{\alpha} u = \frac{\partial^n}{\partial t^n} \left[\frac{t^{n-\frac{5\alpha}{3}}}{\Gamma(n-\alpha)} \int_1^{\infty} (p-1)^{n-\alpha-1} p^{-(n-\frac{5\alpha}{3}+1)} f(zp^{\frac{\alpha}{3}}) \mathrm{d}p \right]. \tag{2.61}$$

In view of the definition of Erdélyi-Kober fractional integral operator, it can be written as follows:

$$\partial_t^\alpha u = \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{3}}\left(\mathcal{K}_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha} f\right)(z)\right]. \tag{2.62}$$

For further simplification, considering $\psi(z) \in C^1(0,\infty)$ for $z = xt^{-\frac{\alpha}{3}}$ implies the following:

$$t\frac{\partial}{\partial t}\psi(z) = tx\left(\frac{-\alpha}{3}\right)t^{-\frac{\alpha}{3}-1}\psi'(z) = -\frac{\alpha}{3}z\frac{\mathrm{d}}{\mathrm{d}z}\psi(z). \tag{2.63}$$

Therefore, (2.62) takes the following form:

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{3}}\left(\mathcal{K}_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha} f\right)(z)\right] &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{5\alpha}{3}}\left(\mathcal{K}_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha} f\right)(z)\right)\right],\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{5\alpha}{3}-1}\left(n-\frac{5\alpha}{3}-\frac{\alpha}{3}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha} f\right)(z)\right].\end{aligned} \tag{2.64}$$

Continuing in this manner leads to the following:

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{3}}\left(\mathcal{K}_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha} f\right)(z)\right] &= t^{-\frac{5\alpha}{3}}\prod_{j=0}^{n-1}\left(1-\frac{5\alpha}{3}+j-\frac{\alpha}{3}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha} f\right)(z),\\ &= t^{-\frac{5\alpha}{3}}\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} f\right)(z),\end{aligned} \tag{2.65}$$

using the definition of Erdélyi-Kober fractional differential operator. So, the required differential is as follows:

$$\partial_t^\alpha u = t^{-\frac{5\alpha}{3}}\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} f\right)(z). \tag{2.66}$$

Analogous to the result obtained above, the following result can also be concluded

$$\partial_t^\alpha v = t^{-\frac{5\alpha}{3}}\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} g\right)(z). \tag{2.67}$$

In the case $\alpha = n = 1, 2, 3, ...$, for $z = xt^{-\frac{n}{3}}$ we have the following:

$$\begin{aligned}\partial_t^\alpha u = \frac{\partial^n}{\partial t^n}(t^{-\frac{2n}{3}}f(z)) &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}(t^{-\frac{2n}{3}}f(z))\right],\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left(t^{-\frac{2n}{3}-1}\left(-\frac{2n}{3}-\frac{n}{3}z\frac{\mathrm{d}}{\mathrm{d}z}\right)f(z)\right),\\ &= ... = t^{-\frac{5n}{3}}\prod_{j=0}^{n-1}\left(1-\frac{5n}{3}+j-\frac{n}{3}z\frac{\mathrm{d}}{\mathrm{d}z}\right)f(z),\\ &= t^{-\frac{5n}{3}}\left(\mathcal{P}_{\frac{3}{n}}^{1-\frac{5n}{3},n} f\right)(z).\end{aligned} \tag{2.68}$$

Similarly, for $\alpha = n = 1, 2, 3, ...$, the following holds:

$$\partial_t^\alpha v = t^{\frac{-5n}{3}} \left(\mathcal{P}_{\frac{3}{n}}^{1-\frac{5n}{3}, n} g \right)(z). \tag{2.69}$$

Hence, the expressions (2.66) and (2.67) hold for $n - 1 < \alpha \leq n$. The result (2.59) of the theorem follows. □

2.4.1.2 Conservation laws

The formal Lagrangian for system (2.50) can be written in the following form:

$$L = p(x,t)(\partial_t^\alpha u - 3uu_x - vu_x - u_{xxx}) + q(x,t)(\partial_t^\alpha v - uv_x - vu_x), \tag{2.70}$$

where $p(x,t)$ and $q(x,t)$ are the new dependent variables.

Using (2.40), the adjoint equations can be obtained as follows:

$$\begin{aligned} \frac{\delta L}{\delta u} &= F_1^* \equiv (D_t^\alpha)^* p + 3up_x + pv_x + vp_x + vq_x + p_{xxx} = 0, \\ \frac{\delta L}{\delta v} &= F_2^* \equiv (D_t^\alpha)^* q - pu_x + uq_x = 0. \end{aligned} \tag{2.71}$$

For nonlinear self-adjointness of system (2.50), the equations (2.71) must be satisfied for all solutions of (2.50) after the following substitutions:

$$p = \psi_1(x,t,u,v), \quad q = \psi_2(x,t,u,v), \tag{2.72}$$

satisfying $\psi_i \neq 0$ for at least one i $(i = 1, 2)$. The derivatives of (2.72) are of the following forms:

$$\begin{aligned} & p_x = \psi_{1,x} + \psi_{1,u} u_x + \psi_{1,v} v_x, \\ & q_x = \psi_{2,x} + \psi_{2,u} u_x + \psi_{2,v} v_x, \\ & p_{xxx} = \psi_{1,xxx} + 6\psi_{1,uvx} u_x v_x + 3\psi_{1,uuv} v_x u_x^2 + 3\psi_{1,uu} u_x u_{xx} + 3\psi_{1,uvv} u_x v_x^2 + 3\psi_{1,uv}(u_x v_{xx} + v_x u_{xx}) \\ & \qquad + 3\psi_{1,vv} v_x v_{xx} + 3\psi_{1,xxv} v_x + 3\psi_{1,xxu} u_x + 3\psi_{xu} u_{xx} + 3\psi_{1,xv} v_{xx} + \psi_{1,u} u_{xxx} + \psi_{1,v} v_{xxx} \\ & \qquad + 3\psi_{1,xuu} u_x^2 + 3\psi_{xvv} v_x^2 + \psi_{1,uuu} u_x^3 + \psi_{1,vvv} v_x^3, \\ & \vdots \end{aligned} \tag{2.73}$$

The nonlinear self-adjointness conditions can be given as follows:

$$\frac{\delta L}{\delta u}\bigg|_{(2.72)} = \lambda_1(\partial_t^\alpha u - 3uu_x - vu_x - u_{xxx}) + \lambda_2(\partial_t^\alpha v - uv_x - vu_x), \tag{2.74}$$

and

$$\frac{\delta L}{\delta v}\bigg|_{(2.72)} = \lambda_3(\partial_t^\alpha u - 3uu_x - vu_x - u_{xxx}) + \lambda_4(\partial_t^\alpha v - uv_x - vu_x), \tag{2.75}$$

where λ_i $(i = 1, .., 4)$ are undetermined coefficients. Therefore, using (2.72) and (2.73) in both the above equations, these conditions take the following form:

$$\begin{aligned}
&(D_t^\alpha)^*\psi_1 + (3u+v)\left(\psi_{1,x} + \psi_{1,u}u_x + \psi_{1,v}v_x\right) + v_x\psi_1 + v\left(\psi_{2,x} + \psi_{2,u}u_x + \psi_{2,v}v_x\right) + \psi_{1,xxx} \\
&+ 6\psi_{1,uvx}u_xv_x + 3\psi_{1,uuv}v_xu_x^2 + 3\psi_{1,uu}u_xu_{xx} + 3\psi_{1,uvv}u_xv_x^2 + 3\psi_{1,uv}(u_xv_{xx} + v_xu_{xx}) + 3\psi_{1,vv}v_xv_{xx} \\
&+ 3\psi_{1,xxv}v_x + 3\psi_{1,xxu}u_x + 3\psi_{xu}u_{xx} + 3\psi_{1,xv}v_{xx} + \psi_{1,u}u_{xxx} + \psi_{1,v}v_{xxx} + 3\psi_{1,xuu}u_x^2 + 3\psi_{xvv}v_x^2 \\
&+ \psi_{1,uuu}u_x^3 + \psi_{1,vvv}v_x^3 = \lambda_1(u_t^\alpha - 3uu_x - vu_x - u_{xxx}) + \lambda_2(v_t^\alpha - uv_x - vu_x),
\end{aligned} \tag{2.76}$$

$$(D_t^\alpha)^*\psi_2 - \psi_1u_x + u\left(\psi_{2,x} + \psi_{2,u}u_x + \psi_{2,v}v_x\right) = \lambda_3(u_t^\alpha - 3uu_x - vu_x - u_{xxx}) + \lambda_4(v_t^\alpha - uv_x - vu_x). \tag{2.77}$$

Equating the coefficients of various derivatives and powers of u, v in (2.76), (2.77) and solving simultaneously, the solution is obtained as follows:

$$\begin{aligned}
&\lambda_i = 0 \quad (i = 1, 2, 3, 4), \\
&\psi_1(x, t, u, v) = p(x, t) = 0, \quad \psi_2(x, t, u, v) = q(x, t) = A,
\end{aligned} \tag{2.78}$$

where A is an arbitrary constant. Hence, the Ito system (2.50) is nonlinearly self-adjoint.

For generators, (2.52), the characteristic functions are of the following forms:

$$\begin{aligned}
&W_1^1 = -\frac{2}{3}u - \frac{x}{3}u_x - \frac{t}{\alpha}u_t, \quad W_1^2 = -\frac{2}{3}v - \frac{x}{3}v_x - \frac{t}{\alpha}v_t, \\
&W_2^1 = -u_x, \quad W_2^2 = -v_x.
\end{aligned} \tag{2.79}$$

Using the adjoint variables in (2.78) and choosing $A = 1$, the conserved vectors are as follows:

The x-components C_i^x corresponding to vector fields V_i $(i = 1, 2)$ in (2.52) are calculated as below:

$$\begin{aligned} C_1^x &= v\left(\frac{2}{3}u + \frac{x}{3}u_x + \frac{t}{\alpha}u_t\right) + u\left(\frac{2}{3}v + \frac{x}{3}v_x + \frac{t}{\alpha}v_t\right), \\ C_2^x &= vu_x + uv_x. \end{aligned} \tag{2.80}$$

The t-components C_i^t for vector fields V_i using (2.48) are of the following forms:

Case $0 < \alpha < 1$:

$$\begin{aligned} C_1^t &= -\frac{2}{3}I_t^{1-\alpha}(v) - \frac{x}{3}I_t^{1-\alpha}(v_x) - \frac{1}{\alpha}I_t^{1-\alpha}(tv_t), \\ C_2^t &= -I_t^{1-\alpha}(v_x). \end{aligned} \tag{2.81}$$

Case $1 < \alpha < 2$:

$$\begin{aligned} C_1^t &= -\frac{2}{3}D_t^{\alpha-1}(v) - \frac{x}{3}D_t^{\alpha-1}(v_x) - \frac{1}{\alpha}D_t^{\alpha-1}(tv_t), \\ C_2^t &= -D_t^{\alpha-1}(v_x). \end{aligned} \tag{2.82}$$

2.4.2 Time Fractional Coupled Burgers Equations

This subsection is devoted to the invariant analysis of the fractional coupled Burgers equations [160]. The Burgers equation of fractional order describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe [7]. The time fractional coupled Burgers equations are as follows:

$$\begin{aligned} \partial_t^\alpha u &= u_{xx} + 2uu_x - (uv)_x, \\ \partial_t^\alpha v &= v_{xx} + 2vv_x - (uv)_x. \end{aligned} \tag{2.83}$$

2.4.2.1 Symmetry analysis

The invariance of (2.83) under the group of transformations (2.2) results in the following invariance criterion:

$$\begin{aligned} &[\eta^{\alpha,t} - \eta^{xx} - 2(\eta u_x + u\eta^x) + (u\phi^x + v\eta^x + \eta v_x + \phi u_x)]|_{(2.83)} = 0, \\ &[\phi^{\alpha,t} - \phi^{xx} - 2(\phi v_x + v\phi^x) + (u\phi^x + v\eta^x + \eta v_x + \phi u_x)]|_{(2.83)} = 0. \end{aligned} \tag{2.84}$$

Then using all the prolongation formulae and equating the coefficients of various monomials in terms of derivatives of u, v and its powers, the set of determining equations can be obtained. Solving these equations, it can be proved that the following assertion holds:

Theorem 2.4. *The coupled Burgers equations* (2.83) *admit the Lie group of transformations* (2.2) *with the following infinitesimals:*

$$\xi = \frac{c_1 x}{2} + c_2, \quad \tau = \frac{c_1 t}{\alpha}, \quad \eta = -\frac{u c_1}{2} \quad \textit{and} \quad \phi = -\frac{v c_1}{2}, \tag{2.85}$$

where c_1*,* c_2 *are arbitrary constants along with the Lie algebra spanned by the following vector fields*

$$\begin{aligned} V_1 &= \frac{x}{2}\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u} - \frac{v}{2}\frac{\partial}{\partial v}, \\ V_2 &= \frac{\partial}{\partial x}. \end{aligned} \tag{2.86}$$

Corresponding to the vector field V_1, the resulting auxiliary equations are as follows:

$$\frac{\mathrm{d}x}{\frac{x}{2}} = \frac{\mathrm{d}t}{\frac{t}{\alpha}} = \frac{\mathrm{d}u}{\frac{-u}{2}} = \frac{\mathrm{d}v}{\frac{-v}{2}}.$$

Solving these equations, the following similarity solutions are determined:

$$z = x t^{-\frac{\alpha}{2}}, \quad u(x,t) = t^{-\frac{\alpha}{2}} f(z) \quad \text{and} \quad v(x,t) = t^{-\frac{\alpha}{2}} g(z), \tag{2.87}$$

such that the reduction to system of FODEs is given by the following theorem:

Theorem 2.5. *The similarity transformations* $u(x,t) = t^{-\frac{\alpha}{2}} f(z)$ *and* $v(x,t) = t^{-\frac{\alpha}{2}} g(z)$ *with the similarity variable* $z = x t^{-\frac{\alpha}{2}}$ *reduce the system* (2.83) *into a system of nonlinear FODEs as follows:*

$$\begin{aligned} \left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha} f\right)(z) &= f''(z) + 2f(z)f'(z) - (g(z)f'(z) + f(z)g'(z)), \\ \left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha} g\right)(z) &= g''(z) + 2g(z)g'(z) - (g(z)f'(z) + f(z)g'(z)). \end{aligned} \tag{2.88}$$

Proof. Similar to proof of Theorem 2.3. □

2.4.2.2 Conservation laws

The formal Lagrangian for system (2.83) is of the following form:

$$L = p(x,t)\left(\partial_t^\alpha u - u_{xx} - 2uu_x + (uv)_x\right) + q(x,t)\left(\partial_t^\alpha v - v_{xx} - 2vv_x + (uv)_x\right). \tag{2.89}$$

The associated adjoint equations are obtained as follows:

$$\begin{aligned} \frac{\delta L}{\delta u} &= F_1^* \equiv (D_t^\alpha)^* p + (2u+v)p_x + vq_x - p_{xx} = 0,\\ \frac{\delta L}{\delta v} &= F_2^* \equiv (D_t^\alpha)^* q + (2v+u)q_x + up_x - q_{xx} = 0. \end{aligned} \tag{2.90}$$

The self-adjointness conditions for the system (2.83) are as follows:

$$\begin{aligned} \left.\frac{\delta L}{\delta u}\right|_{(2.72)} &= \lambda_1(\partial_t^\alpha u - u_{xx} - 2uu_x + (uv)_x) + \lambda_2(\partial_t^\alpha v - v_{xx} - 2vv_x + (uv)_x),\\ \left.\frac{\delta L}{\delta v}\right|_{(2.72)} &= \lambda_3(\partial_t^\alpha u - u_{xx} - 2uu_x + (uv)_x) + \lambda_4(\partial_t^\alpha v - v_{xx} - 2vv_x + (uv)_x). \end{aligned} \tag{2.91}$$

Consequently, substituting (2.72) and its derivatives in (2.91) gives the following expressions:

$$\begin{aligned} &(D_t^\alpha)^*\psi_1 + (2u+v)\left(\psi_{1,x} + \psi_{1,u}u_x + \psi_{1,v}v_x\right) + v\left(\psi_{2,x} + \psi_{2,u}u_x + \psi_{2,v}v_x\right)\\ &- \Big(\psi_{1,xx} + 2\psi_{1,xu}u_x + 2\psi_{1,xv}v_x + \psi_{1,uu}u_x^2 + \psi_{1,u}u_{xx} + 2\psi_{1,uv}u_xv_x + \psi_{1,vv}v_x^2 + \psi_{1,v}v_{xx}\Big)\\ &= \lambda_1(\partial_t^\alpha u - u_{xx} - 2uu_x + (uv)_x) + \lambda_2(\partial_t^\alpha v - v_{xx} - 2vv_x + (uv)_x), \end{aligned} \tag{2.92}$$

$$\begin{aligned} &(D_t^\alpha)^*\psi_2 + (2v+u)\left(\psi_{2,x} + \psi_{2,u}u_x + \psi_{2,v}v_x\right) + u\left(\psi_{1,x} + \psi_{1,u}u_x + \psi_{1,v}v_x\right)\\ &- \Big(\psi_{2,xx} + 2\psi_{2,xu}u_x + 2\psi_{2,xv}v_x + \psi_{2,uu}u_x^2 + \psi_{2,u}u_{xx} + 2\psi_{2,uv}u_xv_x + \psi_{2,vv}v_x^2 + \psi_{2,v}v_{xx}\Big)\\ &= \lambda_3(\partial_t^\alpha u - u_{xx} - 2uu_x + (uv)_x) + \lambda_4(\partial_t^\alpha v - v_{xx} - 2vv_x + (uv)_x). \end{aligned} \tag{2.93}$$

Solving (2.92) and (2.93) simultaneously, leads to the following solution:

$$\begin{aligned} &\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0,\\ &p = \psi_1 = A, \quad q = \psi_2 = B, \end{aligned} \tag{2.94}$$

where A and B are arbitrary constants. Hence, proving the nonlinear self-adjointness of the system (2.83).

For the adjoint variables p, q in (2.94) with $A = B = 1$, the conserved vectors are obtained as follows:

The x-components C_i^x corresponding to vector fields V_i (2.86) are calculated in the following form

$$\begin{aligned} C_1^x &= 2(u+v)\left(\frac{u}{2}+\frac{x}{2}u_x+\frac{t}{\alpha}u_t+\frac{v}{2}+\frac{x}{2}v_x+\frac{t}{\alpha}v_t\right)+(u_x+v_x)+\frac{x}{2}(u_{xx}+v_{xx})+\frac{t}{\alpha}(u_{xt}+v_{xt}),\\ C_2^x &= 2(u+v)(u_x+v_x)+u_{xx}+v_{xx}. \end{aligned} \tag{2.95}$$

The t-components C_i^t are obtained in the following forms:

Case $0 < \alpha < 1$:

$$\begin{aligned} C_1^t &= -\frac{1}{2}(I_t^{1-\alpha}(u)+I_t^{1-\alpha}(v))-\frac{x}{2}(I_t^{1-\alpha}(u_x)+I_t^{1-\alpha}(v_x))-\frac{1}{\alpha}(I_t^{1-\alpha}(tu_t)+I_t^{1-\alpha}(tv_t)),\\ C_2^t &= -I_t^{1-\alpha}(u_x)-I_t^{1-\alpha}(v_x). \end{aligned} \tag{2.96}$$

Case $1 < \alpha < 2$:

$$\begin{aligned} C_1^t &= -\frac{1}{2}(D_t^{\alpha-1}(u)+D_t^{\alpha-1}(v))-\frac{x}{2}(D_t^{\alpha-1}(u_x)+D_t^{\alpha-1}(v_x))-\frac{1}{\alpha}(D_t^{\alpha-1}(tu_t)+D_t^{\alpha-1}(tv_t)),\\ C_2^t &= -D_t^{\alpha-1}(u_x)-D_t^{\alpha-1}(v_x), \end{aligned} \tag{2.97}$$

where the characteristic functions are as follows:

$$\begin{aligned} W_1^1 &= -\frac{u}{2}-\frac{x}{2}u_x-\frac{t}{\alpha}u_t, \quad W_1^2 = -\frac{v}{2}-\frac{x}{2}v_x-\frac{t}{\alpha}v_t,\\ W_2^1 &= -u_x, \quad W_2^2 = -v_x. \end{aligned} \tag{2.98}$$

2.4.3 Time Fractional Coupled KdV Equations

The coupled KdV equations [56, 175] have been studied for many decades because of their immense applications in various fields. The fractional KdV equation can be used to estimate the effect of the higher order of the dispersion of the regular KdV equations to increase the amplitude of the soliton [51]. The time fractional coupled KdV equations are considered in the following form:

$$\begin{aligned} &\partial_t^\alpha u + 6auu_x - 6vv_x + au_{xxx} = 0, \\ &\partial_t^\alpha v + 3auv_x + av_{xxx} = 0. \end{aligned} \tag{2.99}$$

2.4.3.1 Symmetry analysis

The application of Lie group theory gives the following invariance condition:

$$\begin{aligned} &[\eta^{\alpha,t} + 6a(u\eta^x + \eta u_x) - 6(v\phi^x + \phi v_x) + a\eta^{xxx}]|_{(2.99)} = 0, \\ &[\phi^{\alpha,t} + 3a(u\phi^x + \eta v_x) + a\phi^{xxx}]|_{(2.99)} = 0. \end{aligned} \tag{2.100}$$

Using the formulae for extended infinitesimals and equating coefficients gives the determining equations. Solving those equations, it can be proved that the following theorem holds.

Theorem 2.6. *The derived Lie symmetries of fractional system* (2.99) *are as follows:*

$$\xi = c_1 x + c_2, \quad \tau = \frac{3c_1 t}{\alpha}, \quad \eta = -2c_1 u \quad \textit{and} \quad \phi = -2c_1 v, \tag{2.101}$$

c_1, c_2 being arbitrary constants. Hence, the corresponding Lie algebra is generated by the following vector fields:

$$\begin{aligned} V_1 &= x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}, \\ V_2 &= \frac{\partial}{\partial x}. \end{aligned} \tag{2.102}$$

Solving the auxiliary equations for vector field V_1, it can be shown that the following theorem holds:

Theorem 2.7. *The similarity transformations $u(x,t) = t^{-\frac{2\alpha}{3}} f(z)$ and $v(x,t) = t^{-\frac{2\alpha}{3}} g(z)$ with the similarity variable $z = xt^{-\frac{\alpha}{3}}$ reduce the system* (2.99) *into the following system of nonlinear FODEs:*

$$\begin{aligned} &\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} f\right)(z) + 6af(z)f'(z) - 6g(z)g'(z) + af'''(z) = 0, \\ &\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} g\right)(z) + 3af(z)g'(z) + ag'''(z) = 0, \end{aligned} \tag{2.103}$$

where $\left(\mathcal{P}_\delta^{\zeta,\alpha}\right)$ is the Erdélyi-Kober fractional differential operator.

Proof. The proof is similar to Theorem 2.3. □

2.4.3.2 Conservation laws

The formal Lagrangian is of the following form:

$$L = p(x,t)\left(\partial_t^\alpha u + 6auu_x - 6vv_x + au_{xxx}\right) + q(x,t)\left(\partial_t^\alpha v + 3auv_x + av_{xxx}\right). \tag{2.104}$$

The adjoint equations are obtained as follows:

$$\begin{aligned} \frac{\delta L}{\delta u} &= F_1^* \equiv (D_t^\alpha)^* p + 3aqv_x - 6aup_x - ap_{xxx} = 0, \\ \frac{\delta L}{\delta v} &= F_2^* \equiv (D_t^\alpha)^* q + 6vp_x - 3auq_x - 3aqu_x - aq_{xxx} = 0. \end{aligned} \tag{2.105}$$

Here, the self-adjointness conditions can be written in the following form:

$$\begin{aligned} \left.\frac{\delta L}{\delta u}\right|_{(2.72)} &= \lambda_1(\partial_t^\alpha u + 6auu_x - 6vv_x + au_{xxx}) + \lambda_2(\partial_t^\alpha v + 3auv_x + av_{xxx}), \\ \left.\frac{\delta L}{\delta v}\right|_{(2.72)} &= \lambda_3(\partial_t^\alpha u + 6auu_x - 6vv_x + au_{xxx}) + \lambda_4(\partial_t^\alpha v + 3auv_x + av_{xxx}). \end{aligned} \tag{2.106}$$

Consequently, the following expressions must hold:

$$\begin{aligned} &(D_t^\alpha)^*\psi_1 + 3a\psi_2 v_x - 6au\left(\psi_{1,x} + \psi_{1,u}u_x + \psi_{1,v}v_x\right) - a\left(\psi_{1,xxx} + 6\psi_{1,uvx}u_x v_x + 3\psi_{1,uuv}v_x u_x^2\right) \\ &- a\left(3\psi_{1,uu}u_x u_{xx} + 3\psi_{1,uvv}u_x v_x^2 + 3\psi_{1,uv}(u_x v_{xx} + v_x u_{xx}) + 3\psi_{1,vv}v_x v_{xx} + 3\psi_{1,xxv}v_x + 3\psi_{1,xxu}u_x\right) \\ &- a\left(3\psi_{1,xu}u_{xx} + 3\psi_{1,xv}v_{xx} + \psi_{1,u}u_{xxx} + \psi_{1,v}v_{xxx} + 3\psi_{1,xuu}u_x^2 + 3\psi_{1,xvv}v_x^2 + \psi_{1,uuu}u_x^3 + \psi_{1,vvv}v_x^3\right) \\ &= \lambda_1(\partial_t^\alpha u + 6auu_x - 6vv_x + au_{xxx}) + \lambda_2(\partial_t^\alpha v + 3auv_x + av_{xxx}), \end{aligned} \tag{2.107}$$

$$\begin{aligned} &(D_t^\alpha)^*\psi_2 - 3a\psi_2 u_x + 6v\left(\psi_{1,x} + \psi_{1,u}u_x + \psi_{1,v}v_x\right) - 3au\left(\psi_{2,x} + \psi_{2,u}u_x + \psi_{2,v}v_x\right) \\ &- a\left(\psi_{2,xxx} + 6\psi_{2,uvx}u_x v_x + 3\psi_{2,uuv}v_x u_x^2 + 3\psi_{2,uu}u_x u_{xx} + 3\psi_{2,uvv}u_x v_x^2 + 3\psi_{2,uv}(u_x v_{xx} + v_x u_{xx})\right) \\ &- a\left(3\psi_{2,vv}v_x v_{xx} + 3\psi_{2,xxv}v_x + 3\psi_{2,xxu}u_x + 3\psi_{2,xu}u_{xx} + 3\psi_{2,xv}v_{xx} + \psi_{2,u}u_{xxx} + \psi_{2,v}v_{xxx}\right) \\ &- a\left(3\psi_{2,xuu}u_x^2 + 3\psi_{2,xvv}v_x^2 + \psi_{2,uuu}u_x^3 + \psi_{2,vvv}v_x^3\right) \\ &= \lambda_3(\partial_t^\alpha u + 6auu_x - 6vv_x + au_{xxx}) + \lambda_4(\partial_t^\alpha v + 3auv_x + av_{xxx}). \end{aligned} \tag{2.108}$$

Solving the expressions given above, we have the following solution:

$$p = A, \qquad q = B, \tag{2.109}$$

where A, B are arbitrary constants. Therefore, the coupled KdV system (2.99) is nonlinearly self-adjoint.

Using the adjoint variables given by (2.109), along with choosing $A = B = 1$, the x-components C_i^x of conserved vectors for V_i in (2.102) are obtained as follows:

$$\begin{aligned} C_1^x &= -6au\left(2u + xu_x + \frac{3t}{\alpha}u_t\right) - a\left(4(u_{xx} + v_{xx}) + x(u_{xxx} + v_{xxx}) + \frac{3t}{\alpha}(u_{xxt} + v_{xxt})\right) \\ &\quad + (-3au + 6v)\left(2v + xv_x + \frac{3t}{\alpha}v_t\right), \\ C_2^x &= -6auu_x + (6v - 3au)v_x - a(u_{xxx} + v_{xxx}). \end{aligned} \tag{2.110}$$

The t-components C_i^t can be calculated in the following forms:

Case $0 < \alpha < 1$:

$$\begin{aligned} C_1^t &= -2(I_t^{1-\alpha}(u) + I_t^{1-\alpha}(v)) - x(I_t^{1-\alpha}(u_x) + I_t^{1-\alpha}(v_x)) - \frac{3}{\alpha}(I_t^{1-\alpha}(tu_t) + I_t^{1-\alpha}(tv_t)), \\ C_2^t &= -I_t^{1-\alpha}(u_x) - I_t^{1-\alpha}(v_x). \end{aligned} \tag{2.111}$$

Case $1 < \alpha < 2$:

$$\begin{aligned} C_1^t &= -2(D_t^{\alpha-1}(u) + D_t^{\alpha-1}(v)) - x(D_t^{\alpha-1}(u_x) + D_t^{\alpha-1}(v_x)) - \frac{3}{\alpha}(D_t^{\alpha-1}(tu_t) + D_t^{\alpha-1}(tv_t)), \\ C_2^t &= -D_t^{\alpha-1}(u_x) - D_t^{\alpha-1}(v_x), \end{aligned} \tag{2.112}$$

where the characteristic functions are as follows:

$$\begin{aligned} W_1^1 &= -2u - xu_x - \frac{3t}{\alpha}u_t, \quad W_1^2 = -2v - xv_x - \frac{3t}{\alpha}v_t, \\ W_2^1 &= -u_x, \quad W_2^2 = -v_x. \end{aligned} \tag{2.113}$$

2.4.4 Time Fractional Hirota-Satsuma Coupled KdV Equations

The Hirota-Satsuma system [177] was proposed by Hirota and Satsuma [84] to describe the interaction of two long waves with different dispersion relations. Its generalized form has many applications in various branches of applied sciences and time fractional generalized Hirota-Satsuma coupled KdV system has been studied [18, 139] using various methods. In this study, the considered time fractional Hirota-Satsuma coupled KdV equations are as follows:

$$\begin{aligned}
&\partial_t^\alpha u - \frac{1}{4}u_{xxx} - 3uu_x - 3(-v^2+w)_x = 0,\\
&\partial_t^\alpha v + \frac{1}{2}v_{xxx} + 3uv_x = 0,\\
&\partial_t^\alpha w + \frac{1}{2}w_{xxx} + 3uw_x = 0.
\end{aligned} \tag{2.114}$$

2.4.4.1 Symmetry analysis

The invariance criterion obtained from the group invariance of system (2.114) is as follows:

$$\begin{aligned}
&[\eta^{\alpha,t} - \frac{1}{4}\eta^{xxx} - 3(u\eta^x + \eta u_x) - 3(\psi^x - 2\phi v_x - 2v\phi^x)]|_{(2.114)} = 0,\\
&[\phi^{\alpha,t} + \frac{1}{2}\phi^{xxx} + 3(u\phi^x + \eta v_x)]|_{(2.114)} = 0,\\
&[\psi^{\alpha,t} + \frac{1}{2}\psi^{xxx} + 3(u\psi^x + \eta w_x)]|_{(2.114)} = 0,
\end{aligned} \tag{2.115}$$

for the symmetry generator $X = \xi\partial_x + \tau\partial_t + \eta\partial_u + \phi\partial_v + \psi\partial_w$. Substituting the expressions for prolongations into (2.115) and comparing the coefficients of various powers and the derivatives of u, v, w gives the determining equations, whose general solutions prove that the following theorem holds.

Theorem 2.8. *The Lie symmetries of system* (2.114) *are obtained as follows:*

$$\xi = c_1x + c_2, \quad \tau = \frac{3c_1t}{\alpha}, \quad \eta = -2c_1u, \quad \phi = -2c_1v \quad \textit{and} \quad \psi = -4c_1w + c_3t^{\alpha-1}, \tag{2.116}$$

where c_1, c_2, c_3 are arbitrary constants. Hence, the corresponding Lie algebra is generated by the set of following vector fields:

$$\begin{aligned}
&V_1 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} - 4w\frac{\partial}{\partial w},\\
&V_2 = \frac{\partial}{\partial x},\\
&V_3 = t^{\alpha-1}\frac{\partial}{\partial w}.
\end{aligned} \tag{2.117}$$

The solutions of the characteristic equations corresponding to the vector field V_1 reduce the system (2.114) into a system of FODEs presented in the following theorem:

Theorem 2.9. *The similarity transformations* $u(x,t) = t^{-\frac{2\alpha}{3}} f(z)$, $v(x,t) = t^{-\frac{2\alpha}{3}} g(z)$ *and* $w(x,t) = t^{-\frac{4\alpha}{3}} h(z)$ *with the similarity variable* $z = xt^{-\frac{\alpha}{3}}$ *reduce the time fractional system* (2.114) *into a system of nonlinear FODEs in the following form:*

$$\begin{aligned}
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} f\right)(z) - \frac{1}{4} f'''(z) - 3(ff')(z) + 6(gg')(z) - 3h'(z) = 0,\\
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} g\right)(z) + \frac{1}{2} g'''(z) + 3(fg')(z) = 0,\\
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{7\alpha}{3},\alpha} h\right)(z) + \frac{1}{2} h'''(z) + 3(fh')(z) = 0.
\end{aligned} \tag{2.118}$$

Proof. Similar to proof of Theorem 2.3. □

2.4.4.2 Conservation laws

The formal Lagrangian can be written as follows:

$$\begin{aligned}
L = {}& p(x,t)\left(\partial_t^\alpha u - \frac{1}{4} u_{xxx} - 3uu_x - 3(-v^2 + w)_x\right) + q(x,t)\left(\partial_t^\alpha v + \frac{1}{2} v_{xxx} + 3uv_x\right)\\
&+ r(x,t)\left(\partial_t^\alpha w + \frac{1}{2} w_{xxx} + 3uw_x\right).
\end{aligned} \tag{2.119}$$

The associated adjoint equations are obtained in the following forms:

$$\begin{aligned}
\frac{\delta L}{\delta u} &= F_1^* \equiv (D_t^\alpha)^* p + 3up_x + 3qv_x + 3rw_x + \frac{1}{4} p_{xxx} = 0,\\
\frac{\delta L}{\delta v} &= F_2^* \equiv (D_t^\alpha)^* q - 3uq_x - 3qu_x - 6vp_x - \frac{1}{2} q_{xxx} = 0,\\
\frac{\delta L}{\delta w} &= F_3^* \equiv (D_t^\alpha)^* r + 3p_x - 3ru_x - 3ur_x - \frac{1}{2} r_{xxx} = 0.
\end{aligned} \tag{2.120}$$

Let us assume the following expressions:

$$p = \psi_1(x,t,u,v,w), \quad q = \psi_2(x,t,u,v,w), \quad r = \psi_3(x,t,u,v,w), \tag{2.121}$$

For nonlinear self-adjointness, substitute (2.121) in the self-adjointness conditions given by

$$\begin{aligned}
\left.\frac{\delta L}{\delta u}\right|_{(2.121)} = {}& \lambda_1\left(\partial_t^\alpha u - \frac{1}{4} u_{xxx} - 3uu_x - 3(-v^2 + w)_x\right) + \lambda_2\left(\partial_t^\alpha v + \frac{1}{2} v_{xxx} + 3uv_x\right)\\
&+ \lambda_3\left(\partial_t^\alpha w + \frac{1}{2} w_{xxx} + 3uw_x\right),
\end{aligned} \tag{2.122}$$

$$
\begin{aligned}
\left.\frac{\delta L}{\delta v}\right|_{(2.121)} &= \lambda_4\left(\partial_t^\alpha u - \frac{1}{4}u_{xxx} - 3uu_x - 3(-v^2+w)_x\right) + \lambda_5\left(\partial_t^\alpha v + \frac{1}{2}v_{xxx} + 3uv_x\right)\\
&\quad + \lambda_6\left(\partial_t^\alpha w + \frac{1}{2}w_{xxx} + 3uw_x\right),\\
\left.\frac{\delta L}{\delta w}\right|_{(2.121)} &= \lambda_7\left(\partial_t^\alpha u - \frac{1}{4}u_{xxx} - 3uu_x - 3(-v^2+w)_x\right) + \lambda_8\left(\partial_t^\alpha v + \frac{1}{2}v_{xxx} + 3uv_x\right)\\
&\quad + \lambda_9\left(\partial_t^\alpha w + \frac{1}{2}w_{xxx} + 3uw_x\right).
\end{aligned}
$$

The left-hand side of first equation in (2.122) results in the following expression:

$$
\begin{aligned}
&(D_t^\alpha)^*\psi_1 + 3u\left(\psi_{1,x} + \psi_{1,u}u_x + \psi_{1,v}v_x + \psi_{1,w}w_x\right) + 3\psi_2 v_x + 3\psi_3 w_x + \frac{1}{4}\left(\psi_{1,xxx} + 3\psi_{1,xxu}u_x\right)\\
&+ \frac{1}{4}\left(3\psi_{1,xxv}v_x + 3\psi_{1,xxw}w_x + 3(\psi_{1,xu}u_{xx} + \psi_{1,xv}v_{xx} + \psi_{1,xw}w_{xx}) + \psi_{1,u}u_{xxx} + \psi_{1,v}v_{xxx} + \psi_{1,w}w_{xxx}\right)\\
&+ \frac{1}{4}\left(6\psi_{1,uvw}u_x v_x w_x + 6(\psi_{1,xuv}u_x v_x + \psi_{1,xvw}v_x w_x + \psi_{1,xuw}u_x w_x) + 3\psi_{1,uuv}u_x^2 v_x + 3\psi_{1,uuw}u_x^2 w_x\right)\\
&+ \frac{1}{4}\left(3(\psi_{1,uvv}u_x v_x^2 + \psi_{1,uww}u_x w_x^2 + \psi_{1,vvw}v_x^2 w_x + \psi_{1,vww}v_x w_x^2) + 6(\psi_{1,xuu}u_x^2 + \psi_{1,xvv}v_x^2 + \psi_{1,xww}w_x^2)\right)\\
&+ \frac{1}{4}\left(\psi_{1,uu}u_x u_{xx} + 3\psi_{1,vv}v_x v_{xx} + 3\psi_{1,ww}w_x w_{xx} + 3\psi_{1,uv}(u_x v_{xx} + u_{xx}v_x) + 3\psi_{1,vw}(v_x w_{xx} + v_{xx}w_x)\right)\\
&+ \frac{1}{4}\left(3\psi_{1,uw}(u_x w_{xx} + u_{xx}w_x) + \psi_{1,uuu}u_x^3 + \psi_{1,vvv}v_x^3 + \psi_{1,www}w_x^3\right).
\end{aligned}
\tag{2.123}
$$

Similarly, the left-hand sides for second and third equations of (2.122) can be expanded. Solving all three resulting expressions altogether, the obtained solution is as below:

$$
p = A, \quad q = r = 0, \tag{2.124}
$$

where A is arbitrary constant. Hence, proving the nonlinear self-adjointness of the Hirota-Satsuma system (2.114).

For vector fields (2.117), the characteristic functions are written as below:

$$
\begin{aligned}
&W_1^1 = -2u - xu_x - \frac{3t}{\alpha}u_t, \quad W_1^2 = -2v - xv_x - \frac{3t}{\alpha}v_t, \quad W_1^3 = -4w - xw_x - \frac{3t}{\alpha}w_t,\\
&W_2^1 = -u_x, \quad W_2^2 = -v_x, \quad W_2^3 = -w_x,\\
&W_3^1 = 0, \quad W_3^2 = 0, \quad W_3^3 = t^{\alpha-1}.
\end{aligned}
\tag{2.125}
$$

Using $p = A = 1$, $q = r = 0$, the x-components C_i^x corresponding to V_i in (2.117) are obtained as follows:

$$\begin{aligned}
C_1^x &= 3u\left(2u + xu_x + \frac{3t}{\alpha}u_t\right) + \frac{1}{4}\left(4u_{xx} + xu_{xxx} + \frac{3t}{\alpha}u_{xxt}\right) - 6v\left(2v + xv_x + \frac{3t}{\alpha}v_t\right)\\
&\quad + 3\left(4w + xw_x + \frac{3t}{\alpha}w_t\right),\\
C_2^x &= 3uu_x + \frac{1}{4}u_{xxx} - 6vv_x + 3w_x,\\
C_3^x &= -3t^{\alpha-1}.
\end{aligned} \tag{2.126}$$

The t-components C_i^t are obtained in following forms:

Case $0 < \alpha < 1$:

$$\begin{aligned}
C_1^t &= -2I_t^{1-\alpha}(u) - xI_t^{1-\alpha}(u_x) - \frac{3}{\alpha}I_t^{1-\alpha}(tu_t),\\
C_2^t &= -I_t^{1-\alpha}(u_x), \quad C_3^t = 0.
\end{aligned} \tag{2.127}$$

Case $1 < \alpha < 2$:

$$\begin{aligned}
C_1^t &= -2D_t^{\alpha-1}(u) - xD_t^{\alpha-1}(u_x) - \frac{3}{\alpha}D_t^{\alpha-1}(tu_t),\\
C_2^t &= -D_t^{\alpha-1}(u_x), \quad C_3^t = 0.
\end{aligned} \tag{2.128}$$

2.4.5 Time Fractional Coupled Nonlinear Hirota Equations

The well known Hirota equation has many applications in physics like propagation of optical pulses in liquid crystal waveguide and single mode fibers [14, 23]. The time fractional coupled Hirota system has recently been studied for its approximate solutions with HAM [14]. The time fractional coupled nonlinear Hirota equations under consideration are as follows:

$$\begin{aligned}
\partial_t^\alpha p + p_{xxx} + 6\left(|p|^2 + |q|^2\right)p_x &= 0,\\
\partial_t^\alpha q + q_{xxx} + 6\left(|p|^2 + |q|^2\right)q_x &= 0,
\end{aligned} \tag{2.129}$$

where $p(x,t)$, $q(x,t)$ are complex valued functions. Assume that $p = u + \iota v$ and $q = w + \iota z$ then system (2.129) corresponds to the following system of equations:

$$\begin{aligned}
&\partial_t^\alpha u + u_{xxx} + 6\left(u^2 + v^2 + w^2 + z^2\right) u_x = 0,\\
&\partial_t^\alpha v + v_{xxx} + 6\left(u^2 + v^2 + w^2 + z^2\right) v_x = 0,\\
&\partial_t^\alpha w + w_{xxx} + 6\left(u^2 + v^2 + w^2 + z^2\right) w_x = 0,\\
&\partial_t^\alpha z + z_{xxx} + 6\left(u^2 + v^2 + w^2 + z^2\right) z_x = 0.
\end{aligned} \tag{2.130}$$

2.4.5.1 Symmetry analysis

The following invariance criterion for the system (2.130) can be obtained:

$$\begin{aligned}
&[\eta^{\alpha,t} + \eta^{xxx} + 6\eta^x\left(u^2 + v^2 + w^2 + z^2\right) + 12u_x\left(u\eta + v\phi + w\psi + z\delta\right)]|_{(2.130)} = 0,\\
&[\phi^{\alpha,t} + \phi^{xxx} + 6\phi^x\left(u^2 + v^2 + w^2 + z^2\right) + 12v_x\left(u\eta + v\phi + w\psi + z\delta\right)]|_{(2.130)} = 0,\\
&[\psi^{\alpha,t} + \psi^{xxx} + 6\psi^x\left(u^2 + v^2 + w^2 + z^2\right) + 12w_x\left(u\eta + v\phi + w\psi + z\delta\right)]|_{(2.130)} = 0,\\
&[\delta^{\alpha,t} + \delta^{xxx} + 6\delta^x\left(u^2 + v^2 + w^2 + z^2\right) + 12z_x\left(u\eta + v\phi + w\psi + z\delta\right)]|_{(2.130)} = 0,
\end{aligned} \tag{2.131}$$

for the symmetry generator $X = \xi\partial_x + \tau\partial_t + \eta\partial_u + \phi\partial_v + \psi\partial_w + \delta\partial_z$. Inserting the prolongation expressions in (2.131) and then equating the coefficients gives the set of determining equations. The solutions of the determining equations are described by the following theorem:

Theorem 2.10. *The Lie symmetries of time fractional system* (2.130) *are as follows:*

$$\begin{aligned}
&\xi = \frac{c_1 x}{3} + c_2, \quad \tau = \frac{c_1 t}{\alpha}, \quad \eta = -\frac{c_1 u}{3} + c_3 v + c_4 w + c_5 z, \quad \phi = -c_3 u - \frac{c_1 v}{3} + c_6 w + c_7 z,\\
&\psi = -c_4 u - c_6 v - \frac{c_1 w}{3} + c_8 z \quad \textit{and} \quad \delta = -c_5 u - c_7 v - c_8 w - \frac{c_1 z}{3},
\end{aligned} \tag{2.132}$$

where c_i $(i = 1, 2, \cdots, 8)$ are all arbitrary constants. Hence, the corresponding Lie algebra *is generated by the following vector fields:*

$$\begin{aligned}
&V_1=\frac{x}{3}\frac{\partial}{\partial x}+\frac{t}{\alpha}\frac{\partial}{\partial t}-\frac{u}{3}\frac{\partial}{\partial u}-\frac{v}{3}\frac{\partial}{\partial v}-\frac{w}{3}\frac{\partial}{\partial w}-\frac{z}{3}\frac{\partial}{\partial z},\\
&V_2=\frac{\partial}{\partial x},\quad V_3=v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v},\\
&V_4=w\frac{\partial}{\partial u}-u\frac{\partial}{\partial w},\quad V_5=z\frac{\partial}{\partial u}-u\frac{\partial}{\partial z},\\
&V_6=w\frac{\partial}{\partial v}-v\frac{\partial}{\partial w},\quad V_7=z\frac{\partial}{\partial v}-v\frac{\partial}{\partial z},\\
&V_8=z\frac{\partial}{\partial w}-w\frac{\partial}{\partial z}.
\end{aligned}\tag{2.133}$$

For vector field V_1, the solution of the characteristic equations yields the following assertion:

Theorem 2.11. *The similarity transformations $u(x,t)=t^{-\frac{\alpha}{3}}f(\theta)$, $v(x,t)=t^{-\frac{\alpha}{3}}g(\theta)$, $w(x,t)=t^{-\frac{\alpha}{3}}h(\theta)$ and $z(x,t)=t^{-\frac{\alpha}{3}}k(\theta)$ with the similarity variable $\theta=xt^{-\frac{\alpha}{3}}$ reduce the system* (2.130) *into the following system of nonlinear FODEs:*

$$\begin{aligned}
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3},\alpha}f\right)(\theta)+f'''(\theta)+6f'(\theta)(f^2+g^2+h^2+k^2)(\theta)=0,\\
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3},\alpha}g\right)(\theta)+g'''(\theta)+6g'(\theta)(f^2+g^2+h^2+k^2)(\theta)=0,\\
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3},\alpha}h\right)(\theta)+h'''(\theta)+6h'(\theta)(f^2+g^2+h^2+k^2)(\theta)=0,\\
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3},\alpha}k\right)(\theta)+k'''(\theta)+6k'(\theta)(f^2+g^2+h^2+k^2)(\theta)=0,
\end{aligned}\tag{2.134}$$

where $\left(\mathcal{P}_{\delta}^{\zeta,\alpha}\right)$ is the Erdélyi-Kober fractional differential operator.

Proof. The proof is similar to the Theorem 2.3. □

2.4.5.2 Conservation laws

The formal Lagrangian for system (2.130) is of following form:

$$\begin{aligned}
L=&A(x,t)\left(\partial_t^{\alpha}u+u_{xxx}+6(u^2+v^2+w^2+z^2)u_x\right)+B(x,t)\left(\partial_t^{\alpha}v+v_{xxx}+6(u^2+v^2+w^2+z^2)v_x\right)\\
&+C(x,t)\left(\partial_t^{\alpha}w+w_{xxx}+6(u^2+v^2+w^2+z^2)w_x\right)+P(x,t)\left(\partial_t^{\alpha}z+z_{xxx}+6(u^2+v^2+w^2+z^2)z_x\right)
\end{aligned}\tag{2.135}$$

where A,B,C,D are the adjoint variables.

Therefore, the adjoint equations are obtained as follows:

$$\begin{aligned}
\frac{\delta L}{\delta u} &= F_1^* \equiv (D_t^\alpha)^* A - A_{xxx} - 6A_x(u^2+v^2+w^2+z^2) + 12v_x(uB - vA) + 12w_x(uC - wA) \\
&\quad + 12z_x(uP - zA) = 0, \\
\frac{\delta L}{\delta v} &= F_2^* \equiv (D_t^\alpha)^* B - B_{xxx} - 6B_x(u^2+v^2+w^2+z^2) + 12u_x(vA - uB) + 12w_x(vC - wB) \\
&\quad + 12z_x(vP - zB) = 0, \\
\frac{\delta L}{\delta w} &= F_3^* \equiv (D_t^\alpha)^* C - C_{xxx} - 6C_x(u^2+v^2+w^2+z^2) + 12u_x(wA - uC) + 12v_x(wB - vC) \\
&\quad + 12z_x(wP - zC) = 0, \\
\frac{\delta L}{\delta z} &= F_4^* \equiv (D_t^\alpha)^* P - P_{xxx} - 6P_x(u^2+v^2+w^2+z^2) + 12u_x(zA - uP) + 12v_x(zB - vP) \\
&\quad + 12w_x(zC - wP) = 0.
\end{aligned} \tag{2.136}$$

For nonlinear self-adjointness of (2.130), assume the following expressions:

$$A = \psi_1(x,t,u,v,w,z), \quad B = \psi_2(x,t,u,v,w,z), \quad C = \psi_3(x,t,u,v,w,z), \quad P = \psi_4(x,t,u,v,w,z). \tag{2.137}$$

Their derivatives can be obtained in the following forms:

$$\begin{aligned}
A_x &= \psi_{1,x} + \psi_{1,u}u_x + \psi_{1,v}v_x + \psi_{1,w}w_x + \psi_{1,z}z_x, \\
B_x &= \psi_{2,x} + \psi_{2,u}u_x + \psi_{2,v}v_x + \psi_{2,w}w_x + \psi_{2,z}z_x, \\
C_x &= \psi_{3,x} + \psi_{3,u}u_x + \psi_{3,v}v_x + \psi_{3,w}w_x + \psi_{3,z}z_x, \\
P_x &= \psi_{4,x} + \psi_{4,u}u_x + \psi_{4,v}v_x + \psi_{4,w}w_x + \psi_{4,z}z_x, \\
&\vdots
\end{aligned} \tag{2.138}$$

Substitute (2.137) and (2.138) in the self-adjointness conditions of the following form:

$$\begin{aligned}
\left.\frac{\delta L}{\delta u}\right|_{(2.137)} &= \lambda_1\left(\partial_t^\alpha u + u_{xxx} + 6(u^2+v^2+w^2+z^2)u_x\right) + \lambda_2\left(\partial_t^\alpha v + v_{xxx} + 6(u^2+v^2+w^2+z^2)v_x\right) \\
&\quad + \lambda_3\left(\partial_t^\alpha w + w_{xxx} + 6(u^2+v^2+w^2+z^2)w_x\right) + \lambda_4\left(\partial_t^\alpha z + z_{xxx} + 6(u^2+v^2+w^2+z^2)z_x\right),
\end{aligned} \tag{2.139}$$

$$
\begin{aligned}
\frac{\delta L}{\delta v}\bigg|_{(2.137)} &= \lambda_5\left(\partial_t^\alpha u + u_{xxx} + 6(u^2+v^2+w^2+z^2)u_x\right) + \lambda_6\left(\partial_t^\alpha v + v_{xxx} + 6(u^2+v^2+w^2+z^2)v_x\right)\\
&\quad + \lambda_7\left(\partial_t^\alpha w + w_{xxx} + 6(u^2+v^2+w^2+z^2)w_x\right) + \lambda_8\left(\partial_t^\alpha z + z_{xxx} + 6(u^2+v^2+w^2+z^2)z_x\right),\\
\frac{\delta L}{\delta w}\bigg|_{(2.137)} &= \lambda_9\left(\partial_t^\alpha u + u_{xxx} + 6(u^2+v^2+w^2+z^2)u_x\right) + \lambda_{10}\left(\partial_t^\alpha v + v_{xxx} + 6(u^2+v^2+w^2+z^2)v_x\right)\\
&\quad + \lambda_{11}\left(\partial_t^\alpha w + w_{xxx} + 6(u^2+v^2+w^2+z^2)w_x\right) + \lambda_{12}\left(\partial_t^\alpha z + z_{xxx} + 6(u^2+v^2+w^2+z^2)z_x\right),\\
\frac{\delta L}{\delta z}\bigg|_{(2.137)} &= \lambda_{13}\left(\partial_t^\alpha u + u_{xxx} + 6(u^2+v^2+w^2+z^2)u_x\right) + \lambda_{14}\left(\partial_t^\alpha v + v_{xxx} + 6(u^2+v^2+w^2+z^2)v_x\right)\\
&\quad + \lambda_{15}\left(\partial_t^\alpha w + w_{xxx} + 6(u^2+v^2+w^2+z^2)w_x\right) + \lambda_{16}\left(\partial_t^\alpha z + z_{xxx} + 6(u^2+v^2+w^2+z^2)z_x\right).
\end{aligned}
$$

Consequently, the left hand side of 1st expression in (2.139) can be written as follows:

$$
\begin{aligned}
&(D_t^\alpha)^*\psi_1 - 6(u^2+v^2+w^2+z^2)\left(\psi_{1,x} + \psi_{1,u}u_x + \psi_{1,v}v_x + \psi_{1,w}w_x + \psi_{1,z}z_x\right) + 12v_x(u\psi_2 - v\psi_1)\\
&+ 12w_x(u\psi_3 - w\psi_1) + 12z_x(u\psi_4 - z\psi_1) - \psi_{1,xxx} - 3\psi_{1,wz}(w_x z_{xx} + w_{xx}z_x) - \psi_{1,uuu}u_x^3 - \psi_{1,vvv}v_x^3\\
&- \psi_{1,www}w_x^3 - \psi_{1,zzz}z_x^3 - 6(\psi_{1,vwz}v_x w_x z_x + \psi_{1,uvz}u_x v_x z_x + \psi_{1,uwz}u_x w_x z_x) - 3\psi_{1,uw}(u_x w_{xx} + u_{xx}w_x)\\
&- 3(\psi_{1,xww}w_x^2 + \psi_{1,xuu}u_x^2 + \psi_{1,xvv}v_x^2 + \psi_{1,xzz}z_x^2) - 3(\psi_{1,xxu}u_x + \psi_{1,xxv}v_x + \psi_{1,xxw}w_x + \psi_{1,xxz}z_x)\\
&- 3(\psi_{1,xu}u_{xx} + \psi_{1,xv}v_{xx} + \psi_{1,xw}w_{xx} + \psi_{1,xz}z_{xx}) - \psi_{1,u}u_{xxx} - \psi_{1,v}v_{xxx} - \psi_{1,www}w_{xxx} - \psi_{1,z}z_{xxx}\\
&- 3\psi_{1,vw}(v_x w_{xx} + v_{xx}w_x) - 3\psi_{1,uv}(u_x v_{xx} + u_{xx}v_x) - 3\psi_{1,uz}(u_x z_{xx} + u_{xx}z_x) - 3\psi_{1,vz}(v_x z_{xx} + v_{xx}z_x)\\
&- 3\psi_{1,vz}(v_x z_{xx} + v_{xx}z_x) - 3(\psi_{uu}u_x u_{xx} + \psi_{1,vv}v_x v_{xx} + \psi_{ww}w_x w_{xx} + \psi_{1,zz}z_x z_{xx}) - 3\psi_{1,uuw}u_x^2 w_x\\
&- 3\psi_{1,vvw}v_x^2 w_x - 3\psi_{1,wzz}w_x z_x^2 - 6\psi_{1,xuw}u_x w_x - 6\psi_{1,xvw}v_x w_x - 6\psi_{1,xwz}w_x z_x - 3\psi_{uww}u_x w_x^2\\
&- 3\psi_{1,vww}v_x w_x^2 - 3\psi_{wwz}w_x^2 z_x - 6\psi_{1,xuv}u_x v_x - 6\psi_{1,xuz}u_x z_x - 6\psi_{xvz}v_x z_x - 3\psi_{1,uvv}u_x v_x^2\\
&- 3\psi_{1,uzz}u_x z_x^2 - 3\psi_{1,uuv}u_x^2 v_x - 3\psi_{1,uuz}u_x^2 z_x - 3\psi_{1,vvz}v_x^2 z_x - 3\psi_{vzz}v_x z_x^2.
\end{aligned}
\tag{2.140}
$$

Similarly, the expanded expressions can be obtained for remaining three equations in (2.139). Solving all the resulting expressions simultaneously, gives the following solution:

$$A = \psi_1 = 0, \quad B = \psi_2 = 0, \quad C = \psi_3 = 0, \quad P = \psi_4 = 0. \tag{2.141}$$

Therefore, the system (2.130) is not nonlinearly self-adjoint. Further, the conserved vectors for the vector fields (2.133) are calculated.

Vector field V_1

For vector field V_1, the characteristic functions are of the following form:

$$\begin{aligned} W_1^1 &= \frac{u}{3} - \frac{x}{3}u_x - \frac{t}{\alpha}u_t, \quad W_1^2 = \frac{v}{3} - \frac{x}{3}v_x - \frac{t}{\alpha}v_t, \\ W_1^3 &= \frac{w}{3} - \frac{x}{3}w_x - \frac{t}{\alpha}w_t, \quad W_1^4 = \frac{z}{3} - \frac{x}{3}z_x - \frac{t}{\alpha}z_t. \end{aligned} \tag{2.142}$$

The x component of conserved vector is obtained in the following form:

$$\begin{aligned} C_1^x = W_1^1 &\left(6A(u^2+v^2+w^2+z^2) + D_x^2(A)\right) - D_x(W_1^1)D_x(A) + AD_x^2(W_1^1) \\ &+ W_1^2\left(6B(u^2+v^2+w^2+z^2) + D_x^2(B)\right) - D_x(W_1^2)D_x(B) + BD_x^2(W_1^2) \\ &+ W_1^3\left(6C(u^2+v^2+w^2+z^2) + D_x^2(C)\right) - D_x(W_1^3)D_x(C) + CD_x^2(W_1^3) \\ &+ W_1^4\left(6P(u^2+v^2+w^2+z^2) + D_x^2(P)\right) - D_x(W_1^4)D_x(P) + PD_x^2(W_1^4). \end{aligned} \tag{2.143}$$

The t components are calculated as follows:

Case $0 < \alpha < 1$:

$$\begin{aligned} C_1^t = A &\left[-\frac{1}{3}I_t^{1-\alpha}(u) - \frac{x}{3}I_t^{1-\alpha}(u_x) - \frac{1}{\alpha}I_t^{1-\alpha}(tu_t)\right] + J(W_1^1, D_t(A)) \\ &+ B\left[-\frac{1}{3}I_t^{1-\alpha}(v) - \frac{x}{3}I_t^{1-\alpha}(v_x) - \frac{1}{\alpha}I_t^{1-\alpha}(tv_t)\right] + J(W_1^2, D_t(B)) \\ &+ C\left[-\frac{1}{3}I_t^{1-\alpha}(w) - \frac{x}{3}I_t^{1-\alpha}(w_x) - \frac{1}{\alpha}I_t^{1-\alpha}(tw_t)\right] + J(W_1^3, D_t(C)) \\ &+ P\left[-\frac{1}{3}I_t^{1-\alpha}(z) - \frac{x}{3}I_t^{1-\alpha}(z_x) - \frac{1}{\alpha}I_t^{1-\alpha}(tz_t)\right] + J(W_1^4, D_t(P)). \end{aligned} \tag{2.144}$$

Case $1 < \alpha < 2$:

$$\begin{aligned} C_1^t = &-A\left(\frac{1}{3}D_t^{\alpha-1}(u) + \frac{x}{3}D_t^{\alpha-1}(u_x) + \frac{1}{\alpha}D_t^{\alpha-1}(tu_t)\right) + A_t\left(\frac{1}{3}I_t^{2-\alpha}(u) + \frac{x}{3}I_t^{2-\alpha}(u_x) + \frac{1}{\alpha}I_t^{2-\alpha}(tu_t)\right) \\ &- B\left(\frac{1}{3}D_t^{\alpha-1}(v) + \frac{x}{3}D_t^{\alpha-1}(v_x) + \frac{1}{\alpha}D_t^{\alpha-1}(tv_t)\right) + B_t\left(\frac{1}{3}I_t^{2-\alpha}(v) + \frac{x}{3}I_t^{2-\alpha}(v_x) + \frac{1}{\alpha}I_t^{2-\alpha}(tv_t)\right) \\ &- C\left(\frac{1}{3}D_t^{\alpha-1}(w) + \frac{x}{3}D_t^{\alpha-1}(w_x) + \frac{1}{\alpha}D_t^{\alpha-1}(tw_t)\right) + C_t\left(\frac{1}{3}I_t^{2-\alpha}(w) + \frac{x}{3}I_t^{2-\alpha}(w_x) + \frac{1}{\alpha}I_t^{2-\alpha}(tw_t)\right) \\ &- P\left(\frac{1}{3}D_t^{\alpha-1}(z) + \frac{x}{3}D_t^{\alpha-1}(z_x) + \frac{1}{\alpha}D_t^{\alpha-1}(tz_t)\right) + P_t\left(\frac{1}{3}I_t^{2-\alpha}(z) + \frac{x}{3}I_t^{2-\alpha}(z_x) + \frac{1}{\alpha}I_t^{2-\alpha}(tz_t)\right) \\ &- J(W_1^1, A_{tt}) - J(W_1^2, B_{tt}) - J(W_1^3, C_{tt}) - J(W_1^4, P_{tt}). \end{aligned} \tag{2.145}$$

Vector field V_2

In case of vector field V_2, we obtain the following characteristic functions:

$$W_2^1 = -u_x, \quad W_2^2 = -v_x, \quad W_2^3 = -w_x, \quad W_2^4 = -z_x. \tag{2.146}$$

The x components can be calculated as follows:

$$\begin{aligned} C_2^x &= -u_x\left(6A(u^2+v^2+w^2+z^2)+A_{xx}\right)+A_xu_{xx}-Au_{xxx}-v_x\left(6B(u^2+v^2+w^2+z^2)+B_{xx}\right)\\ &+B_xv_{xx}-Bv_{xxx}-w_x\left(6C(u^2+v^2+w^2+z^2)+C_{xx}\right)+C_xw_{xx}-Cw_{xxx}\\ &-z_x\left(6P(u^2+v^2+w^2+z^2)+P_{xx}\right)+P_xz_{xx}-Pz_{xxx}. \end{aligned} \tag{2.147}$$

The t components are obtained in following form:

Case $0<\alpha<1$:

$$C_2^t=-AI_t^{1-\alpha}(u_x)-J(u_x,A_t)-BI_t^{1-\alpha}(v_x)-J(v_x,B_t)-wI_t^{1-\alpha}(u_x)-J(w_x,C_t)-PI_t^{1-\alpha}(z_x)-J(z_x,P_t). \tag{2.148}$$

Case $1<\alpha<2$:

$$\begin{aligned} C_2^t &= -AD_t^{\alpha-1}(u_x)+A_tI_t^{2-\alpha}(u_x)+J(u_x,A_{tt})-BD_t^{\alpha-1}(v_x)+B_tI_t^{2-\alpha}(v_x)+J(v_x,B_{tt})\\ &-CD_t^{\alpha-1}(w_x)+C_tI_t^{2-\alpha}(w_x)+J(w_x,C_{tt})-PD_t^{\alpha-1}(z_x)+P_tI_t^{2-\alpha}(z_x)+J(z_x,P_{tt}). \end{aligned} \tag{2.149}$$

Vector field V_3

Here for vector field V_3, the characteristic functions are as below:

$$W_3^1=v,\quad W_3^2=-u,\quad W_3^3=0,\quad W_3^4=0. \tag{2.150}$$

The x component of conserved vector can be obtained as follows:

$$\begin{aligned} C_3^x &= v\left(6A(u^2+v^2+w^2+z^2)+A_{xx}\right)-v_xA_x+Av_{xx}-u\left(6B(u^2+v^2+w^2+z^2)+B_{xx}\right)\\ &+u_xB_x-Bu_{xx}. \end{aligned} \tag{2.151}$$

The t components are calculated as follows:

Case $0<\alpha<1$:

$$C_3^t=AI_t^{1-\alpha}(v)+J(v,A_t)-BI_t^{1-\alpha}(u)-J(u,B_t). \tag{2.152}$$

Case $1 < \alpha < 2$:

$$C_3^t = AD_t^{\alpha-1}(v) - A_tI_t^{2-\alpha}(v) - J(v, A_{tt}) - BD_t^{\alpha-1}(u) + B_tI_t^{2-\alpha}(u) + J(u, B_{tt}). \tag{2.153}$$

Vector field V_4

For V_4, the characteristic functions are of the following form:

$$W_4^1 = w, \quad W_4^2 = 0, \quad W_4^3 = -u, \quad W_4^4 = 0. \tag{2.154}$$

The x component of conserved vector is obtained as follows:

$$C_4^x = w\left(6A(u^2+v^2+w^2+z^2) + A_{xx}\right) - w_xA_x + Aw_{xx} - u\left(6C(u^2+v^2+w^2+z^2) + C_{xx}\right) + u_xC_x - Cu_{xx}. \tag{2.155}$$

The t components of conserved vectors are as follows:

Case $0 < \alpha < 1$:

$$C_4^t = AI_t^{1-\alpha}(w) + J(w, A_t) - CI_t^{1-\alpha}(u) - J(u, C_t). \tag{2.156}$$

Case $1 < \alpha < 2$:

$$C_4^t = AD_t^{\alpha-1}(w) - A_tI_t^{2-\alpha}(w) - J(w, A_{tt}) - CD_t^{\alpha-1}(u) + C_tI_t^{2-\alpha}(u) + J(u, C_{tt}). \tag{2.157}$$

Following the same procedure, the conservation laws for the remaining vector fields can be easily obtained.

2.5 Conclusion

The Lie symmetry analysis for single PDEs of fractional order is extended to systems of nonlinear time fractional PDEs in this chapter. In the present chapter, the general fractional order prolongation formulae (2.14) for time fractional systems of (1+1)-dimensional

PDEs having an arbitrary number of dependent variables have been proposed. By derivation of Lie symmetries of three systems of FPDEs, it is proved that general form of symmetries can be obtained using operators (2.14) rather than formulae (2.16). The proposed approach is successfully applied for symmetry analysis of five nonlinear systems of FPDEs. The obtained symmetries have been utilized for transforming the considered systems of FPDEs into the systems of nonlinear ODEs of fractional order. The obtained symmetries and reduced systems of FODEs are equivalent to those obtained in case of $\alpha = 1$. Hence, the results for fractional order systems of PDEs are generalized version of those for integer order systems. In this chapter, one vector field for each system has been discussed for investigating the reductions, however the reductions for remaining vector fields can also be solved. Their discussion has been avoided due to the lack of physical importance of their results.

Also, a general approach for derivation of conserved vectors for time fractional systems with the help of Lie symmetry method has been introduced. Using the derived Lie symmetries for five time fractional nonlinear systems of PDEs, their conserved vectors have been successfully obtained. The first four fractional systems are nonlinearly self-adjoint, so their local conservations laws have been investigated. But, the derived conservation laws for the time fractional coupled Hirota system are nonlocal since this system is not nonlinearly self-adjoint.

Chapter 3

Space-Time Fractional Nonlinear Partial Differential Equations

3.1 Introduction

In chapter 2, the symmetry approach has been extended from time fractional PDEs to systems of time fractional PDEs. But, the symmetry method has not been proposed for analysis of FPDEs with both space and time derivatives of fractional order. However, a few authors [128, 163] discussed the symmetry analysis of space-time fractional PDEs only considering their invariance under the Lie group of scaling transformations. In this chapter, the required prolongation formulae are derived to execute the complete group classification of the space-time fractional PDEs. Although a few works [58, 64, 166, 167, 187, 195] dealing with conservation laws of time fractional PDEs can be noticed, the investigation of conserved vectors for space-time fractional PDEs is completely unexplored. In this chapter, the Lie symmetry method is developed and a generalization of Noether operators is presented for symmetry analysis as well as conservation laws of space-time fractional PDEs.

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The efficiency of the proposed approach is proved using the following space-time fractional nonlinear PDEs:

1. Space-time fractional Gilson-Pickering (STFGP) equation

$$\partial_t^\alpha u + a\partial_x^\beta u - bu_x u_{xx} - uu_{xxx} = 0, \tag{3.1}$$

2. Space-time fractional generalized KdV (STFgKdV) equation

$$\partial_t^\alpha u + \partial_x^\beta u + 6cu_x^3 + 18cuu_x u_{xx} + 3cu^2 u_{xxx} = 0, \tag{3.2}$$

where $0 < \alpha, \beta < 2$, a, b, c are constants. The well known KdV equation was derived from the propagation of dispersive shallow water waves and used for modeling many phenomena such as shock wave formation, solitons, turbulence and mass transport [35, 52, 102, 199]. The time fractional KdV equation has been discussed widely in literature by several techniques [11, 13, 51, 54, 200]. Here, the fractional generalized KdV equation is considered [43, 165] with $m = 1, n = 3$, and both space and time derivatives of fractional order. Also, the Gilson-Pickering equation has some important applications in nonlinear physics and has been studied for its behaviour and exact travelling wave solutions [37, 57, 69]. In this chapter, the space-time fractional Gilson-Pickering equation is discussed for its symmetry analysis and conservation laws.

The chapter is organized as follows. In Section 3.2, the Lie symmetry approach is proposed for investigating the point symmetries of space-time fractional PDEs by construction of the extended infinitesimal operators. In Section 3.3, the Lie symmetries of the STFGP equation (3.1) are derived leading to its reduction into a nonlinear FODE in terms of Erdélyi-Kober operators. Also, the nonlinear self-adjointness of the equation (3.1) is proved such that the conservation laws are investigated using the proposed generalized Noether operators for space-time fractional PDEs. Section 3.4 is devoted to the

Lie symmetry analysis, nonlinear self adjointness and conservation laws for the STFgKdV equation (3.2). The last Section 3.5 contains the conclusion of the chapter.

3.2 Lie Symmetry Method for Space-Time Fractional Partial Differential Equations

Consider a space-time fractional PDE with two independent variables in the following form:

$$F(x,t,u,\partial_t^{\alpha}u,\partial_x^{\beta}u,u_{xx},u_{xxx},...)=0,\quad \alpha>0,\ \ \beta>0. \tag{3.3}$$

The admitted one parameter Lie group of transformations has the symmetry generator in the following form:

$$X=\xi(x,t,u)\partial_x+\tau(x,t,u)\partial_t+\eta(x,t,u)\partial_u. \tag{3.4}$$

The prolonged generator can be defined as follows:

$$\mathrm{pr}^{(\alpha,\beta,i)}X=X+\eta^{\alpha,t}\partial_{\partial_t^{\alpha}u}+\eta^{\beta,x}\partial_{\partial_x^{\beta}u}+\eta^{2x}\partial_{u_{2x}}+...+\eta^{ix}\partial_{u_{ix}}, \tag{3.5}$$

where i is the order of the FPDE (3.3). The operators η^{jx} are jth ($j=2,3,..$) order extended symmetry operators [24, 147] and ($\eta^{\alpha,t}$, $\eta^{\beta,x}$) are fractional extended operators defined as follows:

$$\begin{aligned}
\eta^{\alpha,t}&=D_t^{\alpha}(\eta)+\xi D_t^{\alpha}(u_x)-D_t^{\alpha}(\xi u_x)+D_t^{\alpha}(u(D_t\tau))-D_t^{\alpha+1}(\tau u)+\tau D_t^{\alpha+1}(u),\\
\eta^{\beta,x}&=D_x^{\beta}(\eta)+D_x^{\beta}(u(D_x\xi))-D_x^{\beta+1}(\xi u)+\xi D_x^{\beta+1}(u)+\tau D_x^{\beta}(u_t)-D_x^{\beta}(\tau u_t),\\
\eta^{xx}&=\mathrm{D}_x(\eta^{x})-u_{xx}\mathrm{D}_x(\xi)-u_{xt}\mathrm{D}_x(\tau),\\
\eta^{xxx}&=\mathrm{D}_x(\eta^{xx})-u_{xxx}\mathrm{D}_x(\xi)-u_{xxt}\mathrm{D}_x(\tau),\\
&\ \vdots
\end{aligned} \tag{3.6}$$

The expression for $\eta^{\beta,x}$ in (3.6) can be written as below:

$$\eta^{\beta,x}=D_x^{\beta}(\eta)+\xi D_x^{\beta}(u_x)-D_x^{\beta}(\xi u_x)+\tau D_x^{\beta}(u_t)-D_x^{\beta}(\tau u_t). \tag{3.7}$$

Using the generalized Leibniz rule (1.33), the expression (3.7) can be further simplified as follows:

$$\eta^{\beta,x}=D_x^{\beta}(\eta)-\beta\mathrm{D}_x(\xi)\frac{\partial^{\beta}u}{\partial x^{\beta}}-\sum_{n=1}^{\infty}\binom{\beta}{n}\mathrm{D}_x^{n}(\tau)D_x^{\beta-n}(u_t)-\sum_{n=1}^{\infty}\binom{\beta}{n+1}\mathrm{D}_x^{n+1}(\xi)D_x^{\beta-n}(u). \tag{3.8}$$

Since $\eta=\eta(x,t,u(x,t))$, we use the formula given as follows [120, 148]:

$$D_x^{\beta}f(x,g(x))=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{r=0}^{k}\binom{\beta}{n}\binom{n}{m}\binom{k}{r}\frac{x^{n-\beta}(-g)^{r}}{k!\Gamma(n+1-\beta)}\frac{\mathrm{d}^{m}}{\mathrm{d}x^{m}}(g^{k-r})\frac{\partial^{n-m+k}f(x,g)}{\partial x^{n-m}\partial g^{k}}. \tag{3.9}$$

The expression for $D_x^{\beta}(\eta)$ is obtained as follows:

$$D_x^{\beta}(\eta)=\frac{\partial^{\beta}\eta}{\partial x^{\beta}}+\left(\eta_u\frac{\partial^{\beta}u}{\partial x^{\beta}}-u\frac{\partial^{\beta}\eta_u}{\partial x^{\beta}}\right)+\sum_{n=1}^{\infty}\binom{\beta}{n}\frac{\partial^{n}\eta_u}{\partial x^{n}}D_x^{\beta-n}(u)+\mu_{\beta}, \tag{3.10}$$

where

$$\mu_{\beta}=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\beta}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{x^{n-\beta}}{\Gamma(n-\beta+1)}(-u)^{r}\frac{\partial^{m}}{\partial x^{m}}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial x^{n-m}\partial u^{k}}. \tag{3.11}$$

Therefore, using (3.10) in (3.8) gives that the βth order extended symmetry operator $\eta^{\beta,x}$ is introduced in the following form:

$$\begin{aligned}\eta^{\beta,x}=&\ \partial_x^{\beta}\eta+\left(\eta_u-\beta\mathrm{D}_x(\xi)\right)\partial_x^{\beta}u-u\partial_x^{\beta}\eta_u+\sum_{n=1}^{\infty}\left[\binom{\beta}{n}\partial_x^{n}\eta_u-\binom{\beta}{n+1}\mathrm{D}_x^{n+1}(\xi)\right]D_x^{\beta-n}(u)\\&-\sum_{n=1}^{\infty}\binom{\beta}{n}\mathrm{D}_x^{n}(\tau)D_x^{\beta-n}(u_t)+\mu_{\beta}.\end{aligned} \tag{3.12}$$

Equivalently, the αth order extended infinitesimal $\eta^{\alpha,t}$ can be written as follows [65, 66, 190]:

$$\begin{aligned}\eta^{\alpha,t}=&\ \partial_t^{\alpha}\eta+(\eta_u-\alpha\mathrm{D}_t(\tau))\partial_t^{\alpha}u-u\partial_t^{\alpha}\eta_u+\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\partial_t^{n}\eta_u-\binom{\alpha}{n+1}\mathrm{D}_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u)\\&-\sum_{n=1}^{\infty}\binom{\alpha}{n}\mathrm{D}_t^{n}(\xi)D_t^{\alpha-n}(u_x)+\mu_{\alpha},\end{aligned} \tag{3.13}$$

where μ_α is given by

$$\mu_\alpha = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\left[\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)} \times (-u)^r \frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}\right]. \tag{3.14}$$

The invariance criterion for the FPDE (3.3) can be written as follows:

$$\mathrm{pr}^{(\alpha,\beta,i)}X(\Delta)\Big|_{\Delta=0} = 0, \tag{3.15}$$

where $\Delta = F(x,t,u,\partial_t^\alpha u, \partial_x^\beta u, u_{xx}, u_{xxx}, ...)$.

Hence, using this extended approach, the symmetry analysis of space-time fractional PDEs can be easily investigated.

3.3 Space-Time Fractional Gilson-Pickering Equation

In this section, the STFGP equation (3.1) is considered for its symmetry analysis followed by the investigation of conserved vectors.

3.3.1 Symmetry Analysis

For generator X given by (3.4), the third order prolongation $\mathrm{pr}^{(\alpha,\beta,3)}X$ for FPDE (3.1) gives the following invariance criterion:

$$[\eta^{\alpha,t} + a\eta^{\beta,x} - b\eta^x u_{xx} - bu_x\eta^{xx} - \eta u_{xxx} - u\eta^{xxx}]\Big|_{(3.1)} = 0. \tag{3.16}$$

By substituting the extended infinitesimals and equating the coefficients of alike partial derivatives, fractional derivatives and powers of u, the set of determining equations can be obtained as follows:

$$\xi_t = \xi_u = 0,$$

$$\tau_x = \tau_u = 0,$$

$$\alpha\tau_t - \beta\xi_x = 0,$$

$$
\begin{aligned}
&\eta_{uu} = 0,\\
&\eta = u(3\xi_x - \alpha\tau_t),\\
&\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad \forall n \in \mathbb{N},\\
&\binom{\beta}{n}\partial_x^n\eta_u - \binom{\beta}{n+1}\mathrm{D}_x^{n+1}\xi = 0, \quad \forall n \in \mathbb{N},\\
&\partial_t^\alpha\eta - u\partial_t^\alpha\eta_u + a(\partial_x^\beta\eta - u\partial_x^\beta\eta_u) - u\eta_{xxx} = 0.
\end{aligned} \tag{3.17}
$$

On solving these equations, the resulting group infinitesimals are as follows:

$$\xi = \frac{c_1 x}{\beta} + c_2, \quad \tau = \frac{c_1 t}{\alpha} + c_3, \quad \eta = c_1 u\left(\frac{3-\beta}{\beta}\right), \tag{3.18}$$

where c_1, c_2 and c_3 are arbitrary constants. Since there are fractional derivatives with respect to both x and t, the invariance of the fixed lower limit 0 in the Riemann-Liouville fractional derivative operator gives the following conditions:

$$\xi(x,t,u)|_{x=0} = 0, \quad \tau(x,t,u)|_{t=0} = 0. \tag{3.19}$$

Thus the corresponding infinitesimal generator can be written as follows:

$$X = \frac{x}{\beta}\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} + u\left(\frac{3-\beta}{\beta}\right)\frac{\partial}{\partial u}. \tag{3.20}$$

The associated characteristic equations are as follows:

$$\frac{\mathrm{d}x}{\frac{x}{\beta}} = \frac{\mathrm{d}t}{\frac{t}{\alpha}} = \frac{\mathrm{d}u}{u(\frac{3-\beta}{\beta})}. \tag{3.21}$$

Solving these equations, gives the following similarity solutions:

$$z = xt^{-\frac{\alpha}{\beta}}, \quad u = t^{\frac{\alpha(3-\beta)}{\beta}}F. \tag{3.22}$$

Before finding the symmetry reductions, let us introduce the right-hand sided Erdélyi-Kober fractional differential operator [107, 128] as follows:

$$\left(\mathcal{D}_\delta^{\zeta,\beta}h\right)(z) := \left(\prod_{j=1}^{m}\left(\zeta + j + \frac{1}{\delta}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\right)(\mathcal{I}_\delta^{\zeta+\beta,m-\beta}h)(z), \tag{3.23}$$

$$m = \begin{cases} [\beta]+1 & \text{if } \beta \notin \mathbb{N}, \\ \beta & \text{if } \beta \in \mathbb{N}, \end{cases} \qquad z>0,\ \delta>0,\ \alpha>0,$$

where

$$\left(\mathcal{I}_{\delta}^{\zeta,\beta} h\right)(z) := \begin{cases} \frac{1}{\Gamma(\beta)} \int\limits_0^1 (1-s)^{\beta-1} s^{\zeta} h(zs^{\frac{1}{\delta}}) \mathrm{d}s & \text{if } \beta>0, \\ h(z) & \text{if } \beta=0, \end{cases} \tag{3.24}$$

is the right-hand sided Erdélyi-Kober fractional integral operator.

The reduced nonlinear fractional order ODE is presented in the following theorem as follows.

Theorem 3.1. *The STFGP equation* (3.1) *is reduced into a nonlinear FODE for* $\alpha, \beta > 0$ *written as follows:*

$$\left(\mathcal{P}_{\frac{\beta}{\alpha}}^{1-\alpha+\frac{\alpha}{\beta}(3-\beta),\alpha} F\right)(z) + az^{-\beta}\left(\mathcal{D}_{1}^{-\beta,\beta} F\right)(z) - bF'(z)F''(z) - (FF''')(z) = 0. \tag{3.25}$$

Proof. By the definition of Riemann-Liouville fractional derivative, the αth order fractional derivative of $u(x,t) = t^{\frac{\alpha(3-\beta)}{\beta}} F(z)$ with respect to t for $n-1 < \alpha < n$ ($n \in \mathbb{N}$) is given by

$$\partial_t^{\alpha} u = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} s^{\frac{\alpha(3-\beta)}{\beta}} F(xs^{-\frac{\alpha}{\beta}}) \mathrm{d}s\right]. \tag{3.26}$$

Taking $w = \frac{t}{s}$, it can be written in the following form:

$$\begin{aligned} \partial_t^{\alpha} u &= \frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\alpha+\frac{\alpha(3-\beta)}{\beta}}}{\Gamma(n-\alpha)} \int_1^{\infty} (w-1)^{n-\alpha-1} w^{-(n-\alpha+1+\frac{\alpha(3-\beta)}{\beta})} F(zw^{\frac{\alpha}{\beta}}) \mathrm{d}w\right], \\ &= \frac{\partial^n}{\partial t^n}\left[t^{n-\alpha+\frac{\alpha(3-\beta)}{\beta}} \left(\mathcal{K}_{\frac{\beta}{\alpha}}^{1+\frac{\alpha(3-\beta)}{\beta},n-\alpha} F\right)(z)\right], \end{aligned} \tag{3.27}$$

where the operator $\left(\mathcal{K}_{\delta}^{\zeta,\alpha}\right)$ is defined by (2.58). The relation (3.27) is also true for $\alpha = n = 1,2,3,...$ because $\left(\mathcal{K}_{\delta}^{\zeta,0} F\right)(z) = F(z)$. For $z = xt^{-\frac{\alpha}{\beta}}$ and a function $\psi(z) \in C^1(0,\infty)$, the following relation holds:

$$t\frac{\mathrm{d}}{\mathrm{d}t}\psi(z) = -\frac{\alpha}{\beta} z \frac{\mathrm{d}}{\mathrm{d}z}\psi(z). \tag{3.28}$$

It follows that (3.27) can be written as follows:

$$\partial_t^{\alpha} u = \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha-1+\frac{\alpha(3-\beta)}{\beta}} \times \left(n-\alpha+\frac{\alpha(3-\beta)}{\beta} - \frac{\alpha}{\beta} z \frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}_{\frac{\beta}{\alpha}}^{1+\frac{\alpha(3-\beta)}{\beta},n-\alpha} F\right)(z)\right]. \tag{3.29}$$

Continuing in this manner, gives the following:

$$\begin{aligned}\partial_t^\alpha u &= t^{-\alpha+\alpha\left(\frac{3-\beta}{\beta}\right)} \prod_{j=0}^{n-1} \left(1-\alpha+\alpha\left(\frac{3-\beta}{\beta}\right)+j-\frac{\alpha}{\beta}z\frac{\mathrm{d}}{\mathrm{d}z}\right) \times \left(\mathcal{K}_{\frac{\beta}{\alpha}}^{1+\frac{\alpha(3-\beta)}{\beta},n-\alpha}F\right)(z),\\ &= t^{-\alpha+\alpha\left(\frac{3-\beta}{\beta}\right)} \left(\mathcal{P}_{\frac{\beta}{\alpha}}^{1-\alpha+\frac{\alpha(3-\beta)}{\beta},\alpha}F\right)(z), \quad \forall\alpha>0,\end{aligned} \tag{3.30}$$

where $\left(\mathcal{P}_\delta^{\zeta,\alpha}\right)$ is the left hand sided Erdélyi-Kober fractional differential operator defined by (2.57).

Similarly, the βth order Riemann-Liouville fractional derivative with respect to x can be obtained as follows:

$$\partial_x^\beta u = t^{\frac{\alpha(3-\beta)}{\beta}} x^{-\beta} \left(\mathcal{D}_1^{-\beta,\beta}F\right)(z), \qquad \forall\beta>0, \tag{3.31}$$

where $\left(\mathcal{D}_\delta^{\zeta,\beta}\right)$ is the right hand sided Erdélyi-Kober fractional differential operator defined by (3.23). Hence, completing the proof. □

Next step is to find the conserved vectors with the help of the obtained Lie symmetries. Before that the nonlinear self-adjointness of equation (3.1) is investigated in next subsection.

3.3.2 Nonlinear Self-Adjointness

Here, the concept of nonlinear self-adjointness [92] is extended from time fractional PDEs [64, 129, 167] to space-time fractional PDEs by its application to equation (3.1). A formal Lagrangian [91] for the FPDE (3.1) is as follows:

$$\mathcal{L} = v(x,t)(\partial_t^\alpha u + a\partial_x^\beta u - bu_x u_{xx} - uu_{xxx}), \tag{3.32}$$

where $v(x,t)$ is a new dependent variable. The adjoint equation of the equation (3.1) is defined by

$$F^* \equiv \frac{\delta\mathcal{L}}{\delta u} = 0, \tag{3.33}$$

where $\frac{\delta}{\delta u}$ is the Euler-Lagrange operator defined as follows:

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+(D_t^\alpha)^*\frac{\partial}{\partial(D_t^\alpha u)}+(D_x^\beta)^*\frac{\partial}{\partial(D_x^\beta u)}+\sum_{k=1}^{\infty}(-1)^k \mathrm{D}_{i_1}\mathrm{D}_{i_2},\dots,\mathrm{D}_{i_k}\frac{\partial}{\partial u_{i_1 i_2,\dots,i_k}}. \tag{3.34}$$

Here $(D_t^\alpha)^*$, $(D_x^\beta)^*$ are the adjoint operators of Riemann-Liouville fractional differential operators D_t^α and D_x^β respectively, given as follows:

$$\begin{aligned}(D_t^\alpha)^*&=(-1)^n I_p^{n-\alpha}(D_t^n)={}_t^C D_p^\alpha,\\(D_x^\beta)^*&=(-1)^m I_q^{m-\beta}(D_x^m)={}_x^C D_q^\beta,\end{aligned} \tag{3.35}$$

where $I_p^{n-\alpha}$ and $I_q^{m-\beta}$ are the right-hand sided fractional integral operators of order $n-\alpha$ and $m-\beta$ respectively, defined as follows:

$$I_p^{n-\alpha}f(x,t)=\frac{1}{\Gamma(n-\alpha)}\int_t^p\frac{f(x,s)}{(s-t)^{1+\alpha-n}}\mathrm{d}s,\text{where } n=[\alpha]+1, \tag{3.36}$$

$$I_q^{m-\beta}f(x,t)=\frac{1}{\Gamma(m-\beta)}\int_x^q\frac{f(s,t)}{(s-x)^{1+\beta-m}}\mathrm{d}s,\text{where } m=[\beta]+1. \tag{3.37}$$

Also ${}_t^C D_p^\alpha$, ${}_x^C D_q^\beta$ are the right-hand sided Caputo fractional differential operators [157] of order α and β respectively. According to (3.32) and (3.33), the adjoint equation of FPDE (3.1) can be obtained as follows:

$$F^*\equiv(D_t^\alpha)^*v+a(D_x^\beta)^*v+(3-b)(u_x v_{xx}+u_{xx}v_x)+uv_{xxx}=0. \tag{3.38}$$

For nonlinear self-adjointness of equation (3.1), the equation (3.38) must be satisfied for all the solutions of equation (3.1) with the following substitution:

$$v=\phi(x,t,u),\quad \text{where } \phi(x,t,u)\neq 0. \tag{3.39}$$

The derivatives of (3.39) are given by

$$\begin{aligned}v_x&=\phi_x+\phi_u u_x,\\v_{xx}&=\phi_{xx}+2\phi_{xu}u_x+\phi_u u_{xx}+\phi_{uu}u_x^2,\\v_{xxx}&=\phi_{xxx}+3\phi_{xxu}u_x+3\phi_{xuu}u_x^2+3\phi_{xu}u_{xx}\\&\quad+3\phi_{uu}u_x u_{xx}+\phi_u u_{xxx}+\phi_{uuu}u_x^3.\end{aligned} \tag{3.40}$$

Substituting (3.39) and (3.40) in the expression (3.38) gives the following condition for nonlinear self-adjointness:

$$
\begin{aligned}
&(D_t^{\alpha})^*\phi + a(D_x^{\beta})^*\phi + (3-b)\left(\phi_x u_{xx} + \phi_{xx} u_x + 2\phi_u u_x u_{xx} + 2\phi_{xu} u_x^2 + \phi_{uu} u_x^3\right)\\
&+ u\left(\phi_{xxx} + 3\phi_{xxu} u_x + 3\phi_{xuu} u_x^2 + 3\phi_{xu} u_{xx} + 3\phi_{uu} u_x u_{xx} + \phi_u u_{xxx} + \phi_{uuu} u_x^3\right) \qquad (3.41)\\
&= \lambda(\partial_t^{\alpha} u + a\partial_x^{\beta} u - b u_x u_{xx} - u u_{xxx}),
\end{aligned}
$$

with regular undetermined coefficient λ. Comparing the coefficients and solving includes the following condition:

$$(b-3)\phi_x = 0, \tag{3.42}$$

which splits into the following two cases:

Case 1: $b = 3$

In this case, the solution of (3.41) can be obtained as follows:

$$\phi = A_1(t)x^2 + B_1(t)x + C_1(t) \quad \text{and} \quad \lambda = 0, \tag{3.43}$$

where $A_1(t)$, $B_1(t)$, $C_1(t)$ are arbitrary functions of t such that the following holds:

$$x^2\left({}_t^C D_p^{\alpha} A_1(t)\right) + x\left({}_t^C D_p^{\alpha} B_1(t)\right) + \left({}_t^C D_p^{\alpha} C_1(t)\right) + a\left[A_1(t)\left({}_x^C D_q^{\beta} x^2\right) + B_1(t)\left({}_x^C D_q^{\beta} x\right)\right] = 0. \tag{3.44}$$

The values of $\left({}_x^C D_q^{\beta} x\right)$ and $\left({}_x^C D_q^{\beta} x^2\right)$ depend on β leading to the subcases given below:

Subcase $0 < \beta < 1$:

In this subcase, using the values of right-hand sided Caputo fractional derivatives $\left({}_x^C D_q^{\beta} x\right)$ and $\left({}_x^C D_q^{\beta} x^2\right)$ in (3.44) gives the following solution:

$$A_1(t) = 0 = B_1(t), \quad C_1(t) = a_1, \tag{3.45}$$

where a_1 is an arbitrary constant. Then from (3.39), the following result is obtained:

$$v = a_1. \tag{3.46}$$

Subcase $1 < \beta < 2$:

In this subcase, using the values of $\left({}_x^C D_q^\beta x\right)$ and $\left({}_x^C D_q^\beta x^2\right)$ in (3.44) implies the following:

$$A_1(t) = 0, \quad B_1(t) = a_2, \quad C_1(t) = a_3. \tag{3.47}$$

Therefore, the following solution of $v(x,t)$ is obtained:

$$v = a_2 x + a_3, \tag{3.48}$$

where a_2 and a_3 are arbitrary constants.

Case 2: $b \neq 3$, $\phi_x = 0$

In this case, solution $v(x,t)$ is obtained as follows:

$$v = a_4, \tag{3.49}$$

where a_4 is an arbitrary constant.

These solutions of $v(x,t)$ are substituted in the formal Lagrangian (3.32) for the construction of conserved vectors in next subsection.

3.3.3 Conservation Laws

In this section, the conservation laws for the STFGP equation (3.1) are obtained using the Lie symmetry generator (3.20). The existence of fractional derivatives of both independent variables x, t indicates the requirement of the fractional generalization of the Noether operators. The fractional Noether operator for the variable t is given by [129, 167, 187]

$$C^t = \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k}(W) D_t^k \left(\frac{\partial \mathcal{L}}{\partial (D_t^\alpha u)}\right) - (-1)^n J\left(W, D_t^n \left(\frac{\partial \mathcal{L}}{\partial (D_t^\alpha u)}\right)\right), \tag{3.50}$$

where $n = [\alpha] + 1$, $W = \eta - \xi u_x - \tau u_t$ is the Lie characteristic function for generator $X = \xi\partial_x + \tau\partial_t + \eta\partial_u$ and $J(f, g)$ is the integral defined by

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^p \frac{f(x,s)g(x,r)}{(r-s)^{\alpha+1-n}} \mathrm{d}r\mathrm{d}s, \tag{3.51}$$

for any two functions $f(x,t)$ and $g(x,t)$.

Equivalently, the fractional Noether operator for the component x of the conserved vector is introduced as follows:

$$C^x = \sum_{k=0}^{m-1} (-1)^k D_x^{\beta-1-k}(W) D_x^k \left(\frac{\partial \mathcal{L}}{\partial(D_x^\beta u)}\right) - (-1)^n J_1\left(W, D_x^m \left(\frac{\partial \mathcal{L}}{\partial(D_x^\beta u)}\right)\right), \tag{3.52}$$

where $m = [\beta] + 1$ and $J_1(f, g)$ is the integral defined by

$$J_1(f,g) = \frac{1}{\Gamma(m-\beta)} \int_0^x \int_x^q \frac{f(s,t)g(r,t)}{(r-s)^{\beta+1-m}} \mathrm{d}r\mathrm{d}s. \tag{3.53}$$

Here, for the symmetry X given by (3.20), the function W is of the following form:

$$W = u\left(\frac{3-\beta}{\beta}\right) - \frac{x}{\beta}u_x - \frac{t}{\alpha}u_t. \tag{3.54}$$

Firstly, the conservation laws are obtained for the solution (3.46) of variable $v(x,t)$. Substituting the Lagrangian (3.32) in (3.52) using (3.46) with $a_1 = 1$, the x component of the conserved vector can be obtained as follows:

$$C^x = a\left[\left(\frac{3-\beta}{\beta}\right) I_x^{1-\beta}(u) - \frac{1}{\beta} I_x^{1-\beta}(xu_x) - \frac{t}{\alpha} I_x^{1-\beta}(u_t)\right]. \tag{3.55}$$

Also, the t component of the conserved vector is obtained in the following form:

case $0 < \alpha < 1$

$$C^t = \left[\left(\frac{3-\beta}{\beta}\right) I_t^{1-\alpha}(u) - \frac{x}{\beta} I_t^{1-\alpha}(u_x) - \frac{1}{\alpha} I_t^{1-\alpha}(tu_t)\right]. \tag{3.56}$$

case $1 < \alpha < 2$

$$C^t = \left[\left(\frac{3-\beta}{\beta}\right) D_t^{\alpha-1}(u) - \frac{x}{\beta} D_t^{\alpha-1}(u_x) - \frac{1}{\alpha} D_t^{\alpha-1}(tu_t)\right]. \tag{3.57}$$

Note that the conservation laws with the fractional components C^x and C^t are obtained in terms of Riemann-Liouville fractional derivative operators and do not involve the integrals (3.51) and (3.53). The fractional conserved vectors for the remaining solutions of $v(x,t)$ are calculated as follows:

In case of (3.48), the x component of the conserved vector is as follows:

$$\begin{aligned} C^x &= a(a_2x+a_3)\left[\left(\frac{3-\beta}{\beta}\right)D_x^{\beta-1}(u)-\frac{1}{\beta}D_x^{\beta-1}(xu_x)-\frac{t}{\alpha}D_x^{\beta-1}(u_t)\right] \\ &- aa_2\left[\left(\frac{3-\beta}{\beta}\right)I_x^{2-\beta}(u)-\frac{1}{\beta}I_x^{2-\beta}(xu_x)-\frac{t}{\alpha}I_x^{2-\beta}(u_t)\right]. \end{aligned} \tag{3.58}$$

Thus corresponding to the constants a_2 and a_3, the linearly independent components of the conserved vector with respect to x are given as follows:

$$\begin{aligned} C^x_{a_2} &= ax\left[\left(\frac{3-\beta}{\beta}\right)D_x^{\beta-1}(u)-\frac{1}{\beta}D_x^{\beta-1}(xu_x)-\frac{t}{\alpha}D_x^{\beta-1}(u_t)\right] \\ &- a\left[\left(\frac{3-\beta}{\beta}\right)I_x^{2-\beta}(u)-\frac{1}{\beta}I_x^{2-\beta}(xu_x)-\frac{t}{\alpha}I_x^{2-\beta}(u_t)\right]. \end{aligned} \tag{3.59}$$

$$C^x_{a_3} = a\left[\left(\frac{3-\beta}{\beta}\right)D_x^{\beta-1}(u)-\frac{1}{\beta}D_x^{\beta-1}(xu_x)-\frac{t}{\alpha}D_x^{\beta-1}(u_t)\right]. \tag{3.60}$$

For $0<\alpha<1$, the t component of conserved vector for constant a_2 is of the following form:

$$C^t_{a_2} = x\left[\left(\frac{3-\beta}{\beta}\right)I_t^{1-\alpha}(u)-\frac{x}{\beta}I_t^{1-\alpha}(u_x)-\frac{1}{\alpha}I_t^{1-\alpha}(tu_t)\right], \tag{3.61}$$

and the component $C^t_{a_3}$ coincides with vector (3.56).

For $1<\alpha<2$, the t component for a_2 is obtained as follows:

$$C^t_{a_2} = x\left[\left(\frac{3-\beta}{\beta}\right)D_t^{\alpha-1}(u)-\frac{x}{\beta}D_t^{\alpha-1}(u_x)-\frac{1}{\alpha}D_t^{\alpha-1}(tu_t)\right], \tag{3.62}$$

and the component $C^t_{a_3}$ is same as vector (3.57).

In case of (3.49), the conserved vector has the following components:

case 0 $<\beta<1$

The obtained x component of conserved vector is coincident with (3.55).

case 1 $< \beta < 2$

In this case, the x component of the conserved vector coincides with vector (3.60).

case 0 $< \alpha < 1$

The t component of conserved vector in this case is coincident with (3.56).

case 1 $< \alpha < 2$

The obtained t component coincides with vector (3.57).

3.4 Space-Time Fractional Generalized KdV Equation

In this section, the Lie symmetries and conservation laws of the STFgKdV equation (3.2) are calculated systematically.

3.4.1 Symmetry Analysis

The invariance of the FPDE (3.2) under the admitted Lie group of transformations gives the following criterion:

$$\left[\eta^{\alpha,t} + \eta^{\beta,x} + 18cu_x^2\eta^x + 18c\eta u_x u_{xx} + 18cuu_{xx}\eta^x + 18c\eta^{xx}uu_x + 6c\eta uu_{xxx} + 3cu^2\eta^{xxx}\right]\Big|_{(3.2)} = 0. \tag{3.63}$$

Inserting the extended symmetry operators and equating the coefficients, the following set of determining equations can be obtained:

$$\xi_t = 0, \quad \xi_u = 0,$$

$$\tau_x = 0, \quad \tau_u = 0,$$

$$\eta_x = 0, \quad \text{for } c \neq 0,$$

$$\alpha\tau_t - \beta\xi_x = 0, \quad 2\eta = u(3\xi_x - \alpha\tau_t),$$

$$\begin{aligned}
&2\eta_u - 3\xi_x + \alpha\tau_t + 3u\eta_{uu} + \frac{1}{2}u^2\eta_{uuu} = 0, \quad \text{for } c \neq 0,\\
&\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad \forall n \in \mathbb{N},\\
&\binom{\beta}{n}\partial_x^n\eta_u - \binom{\beta}{n+1}\mathrm{D}_x^{n+1}\xi = 0, \quad \forall n \in \mathbb{N},\\
&\partial_t^\alpha\eta - u\partial_t^\alpha\eta_u + a(\partial_x^\beta\eta - u\partial_x^\beta\eta_u) = 0.
\end{aligned} \tag{3.64}$$

Solving the determining equations, the infinitesimals are derived in the following form:

$$\xi = \frac{c_1 x}{\beta}, \quad \tau = \frac{c_1 t}{\alpha}, \quad \eta = \frac{c_1 u}{2}\left(\frac{3}{\beta} - 1\right), \tag{3.65}$$

where c_1 is an arbitrary constant. The corresponding symmetry generator is as follows:

$$X = \frac{x}{\beta}\partial_x + \frac{t}{\alpha}\partial_t + \frac{u(3-\beta)}{2\beta}\partial_u. \tag{3.66}$$

Solving the associated auxiliary equations, the symmetry invariants are obtained as follows:

$$z = xt^{\frac{-\alpha}{\beta}}, \quad u = t^{\frac{\alpha(3-\beta)}{2\beta}}F. \tag{3.67}$$

The above invariants are used for the reduction of the FPDE (3.2) into a nonlinear fractional ODE as follows:

Theorem 3.2. *The space-time fractional PDE* (3.2) *is reduced into a nonlinear ODE of fractional order in the following form:*

$$\left(\mathcal{P}_{\frac{\beta}{\alpha}}^{1-\alpha+\frac{\alpha}{2\beta}(3-\beta),\alpha}F\right)(z) + z^{-\beta}\left(\mathcal{D}_1^{-\beta,\beta}F\right)(z) + 6c(F'(z))^3 + 18cFF'(z)F''(z) + 3cF^2F'''(z) = 0, \tag{3.68}$$

where $\left(\mathcal{P}_\delta^{\zeta,\alpha}\right)$ *and* $\left(\mathcal{D}_\delta^{\zeta,\beta}\right)$ *are the left and right hand sided Erdélyi-Kober fractional differential operators respectively.*

Proof. Similar to the proof of Theorem 3.1. □

3.4.2 Nonlinear Self-Adjointness

A formal Lagrangian for the FPDE (3.2) can be written as follows:

$$\mathcal{L} = v(x,t)(\partial_t^\alpha u + \partial_x^\beta u + 6cu_x^3 + 18cuu_xu_{xx} + 3cu^2u_{xxx}). \tag{3.69}$$

In view of (3.33) and (3.69), the adjoint equation for FPDE (3.2) can be calculated as follows:

$$F^* \equiv (D_t^\alpha)^* v + (D_x^\beta)^* v - 3cu^2 v_{xxx} = 0. \tag{3.70}$$

For nonlinear self-adjointness, assume the value of v given by

$$v = \phi(x,t,u), \quad \text{where } \phi(x,t,u) \neq 0. \tag{3.71}$$

Substituting (3.71) and its derivatives in (3.70) gives the following condition for nonlinear self-adjointness:

$$\begin{aligned}&(D_t^\alpha)^*\phi + (D_x^\beta)^*\phi - 3cu^2\left(\phi_{xxx} + 3\phi_{xxu}u_x + 3\phi_{xuu}u_x^2 + 3\phi_{xu}u_{xx} + 3\phi_{uu}u_xu_{xx} + \phi_u u_{xxx} + \phi_{uuu}u_x^3\right)\\ &= \lambda(\partial_t^\alpha u + \partial_x^\beta u + 6cu_x^3 + 18cuu_xu_{xx} + 3cu^2u_{xxx}).\end{aligned} \tag{3.72}$$

Equating the coefficients and solving (3.72) leads to the following two cases:

case $0 < \beta < 1$:

In this case, the value of ϕ and hence v is attained as below:

$$v = a_5, \tag{3.73}$$

where a_5 is an arbitrary constant.

case $1 < \beta < 2$:

In this case, v can be obtained as follows:

$$v = a_6x + a_7, \tag{3.74}$$

where a_6 and a_7 are arbitrary constants.

These values of variable $v(x,t)$ are used for investigating conservation laws for the FPDE (3.2).

3.4.3 Conservation Laws

For the symmetry generator X given by (3.66), we have the following:

$$W = \frac{u}{2}\left(\frac{3-\beta}{\beta}\right) - \frac{x}{\beta}u_x - \frac{t}{\alpha}u_t. \tag{3.75}$$

For the case (3.73), the x component of the conserved vector can be calculated in the following form:

$$C^x = \left(\frac{3-\beta}{2\beta}\right) I_x^{1-\beta}(u) - \frac{1}{\beta}I_x^{1-\beta}(xu_x) - \frac{t}{\alpha}I_x^{1-\beta}(u_t). \tag{3.76}$$

The t component of the conserved vector is given in following cases:

Case $0 < \alpha < 1$**:**

$$C^t = \left(\frac{3-\beta}{2\beta}\right) I_t^{1-\alpha}(u) - \frac{x}{\beta}I_t^{1-\alpha}(u_x) - \frac{1}{\alpha}I_t^{1-\alpha}(tu_t). \tag{3.77}$$

Case $1 < \alpha < 2$**:**

$$C^t = \left(\frac{3-\beta}{2\beta}\right) D_t^{\alpha-1}(u) - \frac{x}{\beta}D_t^{\alpha-1}(u_x) - \frac{1}{\alpha}D_t^{\alpha-1}(tu_t). \tag{3.78}$$

For the case (3.74), the two independent values of v are given by

$$v = 1, \quad v = x. \tag{3.79}$$

Firstly, in case of $v = 1$, the component of conserved vector with respect to x can be obtained as follows:

$$C^x = \left(\frac{3-\beta}{2\beta}\right) D_x^{\beta-1}(u) - \frac{1}{\beta}D_x^{\beta-1}(xu_x) - \frac{t}{\alpha}D_x^{\beta-1}(u_t). \tag{3.80}$$

where for $0 < \alpha < 1$, the conserved vector component for t coincides with (3.77) and for $1 < \alpha < 2$, the t component of conserved vector coincides with (3.78).

In case of $v = x$, the x component of the conserved vector results in the following form:

$$\begin{aligned} C^x = x &\left[\left(\frac{3-\beta}{2\beta}\right) D_x^{\beta-1}(u) - \frac{1}{\beta}D_x^{\beta-1}(xu_x) - \frac{t}{\alpha}D_x^{\beta-1}(u_t)\right] \\ - &\left[\left(\frac{3-\beta}{2\beta}\right) I_x^{2-\beta}(u) - \frac{1}{\beta}I_x^{2-\beta}(xu_x) - \frac{t}{\alpha}I_x^{2-\beta}(u_t)\right]. \end{aligned} \tag{3.81}$$

In this case, for $0 < \alpha < 1$, the t component of conserved vector is obtained as follows:

$$C^t = x\left[\left(\frac{3-\beta}{2\beta}\right) I_t^{1-\alpha}(u) - \frac{x}{\beta} I_t^{1-\alpha}(u_x) - \frac{1}{\alpha} I_t^{1-\alpha}(tu_t)\right]. \tag{3.82}$$

For $1 < \alpha < 2$, the t component is calculated as below:

$$C^t = x\left[\left(\frac{3-\beta}{2\beta}\right) D_t^{\alpha-1}(u) - \frac{x}{\beta} D_t^{\alpha-1}(u_x) - \frac{1}{\alpha} D_t^{\alpha-1}(tu_t)\right]. \tag{3.83}$$

3.5 Conclusion

In this chapter, the Lie symmetry method and Noether operators have been extended to find the Lie point symmetries and conservation laws for space-time fractional PDEs. The suggested approach has been successfully applied for providing the Lie symmetries for the STFGP equation and the STFgKdV equation resulting in their reduction into the nonlinear FODEs involving left and right hand sided Erdélyi-Kober fractional operators. Also, the concept of nonlinear self-adjointness has been extended for space-time fractional FDEs and it has been shown that both the considered FPDEs are nonlinearly self-adjoint. Therefore, by using the introduced fractional Noether operators for space-time fractional PDEs, the conservation laws for both the considered fractional PDEs have been derived successfully.

Chapter 4

Space-Time Fractional Nonlinear Systems of Partial Differential Equations

4.1 Introduction

The fractional PDEs with time derivatives of fractional order are a particular case of the space-time fractional PDEs. By analyzing the space-time fractional PDEs, the generalized form of results can be obtained. Only a few researchers [128, 163] studied the invariance of single space-time fractional PDEs under Lie group of scaling transformations rather than executing their complete group classification. In chapter 3, Lie symmetry method is discussed for complete group classification of space-time fractional PDEs. Furthermore, there is no work in literature investigating Lie symmetry analysis of systems of space-time fractional PDEs. It motivated us to introduce the symmetry approach for space and time fractional systems of PDEs. For this purpose, there was a requirement of some new extended symmetry operators. These required prolongation operators are proposed in this chapter and used to study some nonlinear systems of space-time fractional PDEs. As a result, the considered systems are reduced into nonlinear systems

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of FODEs involving the left and right-hand sided Erdélyi-Kober fractional differential operators [107, 128].

The present chapter is organized as follows. In Section 4.2, the Lie symmetry method is extended for systems of space-time fractional PDEs by derivation of the required prolongation formulae. Section 4.3 is devoted to symmetry analysis of five nonlinear systems of fractional PDEs with time and space derivatives of fractional order. The conclusion is given in Section 4.4.

4.2 Lie Symmetry Method for Space-Time Fractional Systems of Partial Differential Equations

In this section, the Lie symmetry analysis for systems of space-time fractional PDEs is developed with two independent (x, t) and two dependent variables (u_1, u_2). Consider a system of space-time fractional PDEs having the following form:

$$\begin{aligned}\frac{\partial^\alpha u_1}{\partial t^\alpha} &= F_1(x, t, u_1, u_2, \frac{\partial^\beta u_1}{\partial x^\beta}, \frac{\partial^\beta u_2}{\partial x^\beta}, \frac{\partial^2 u_1}{\partial x^2}, \frac{\partial^2 u_2}{\partial x^2}, ...),\\ \frac{\partial^\alpha u_2}{\partial t^\alpha} &= F_2(x, t, u_1, u_2, \frac{\partial^\beta u_1}{\partial x^\beta}, \frac{\partial^\beta u_2}{\partial x^\beta}, \frac{\partial^2 u_1}{\partial x^2}, \frac{\partial^2 u_2}{\partial x^2}, ...).\end{aligned} \tag{4.1}$$

Assume the invariance of system (4.1) under a one parameter Lie group of transformations of the following form:

$$\begin{aligned}\tilde{x} &= x + \epsilon\xi(x, t, u_1, u_2) + O(\epsilon^2),\\ \tilde{t} &= t + \epsilon\tau(x, t, u_1, u_2) + O(\epsilon^2),\\ \tilde{u_1} &= u_1 + \epsilon\eta(x, t, u_1, u_2) + O(\epsilon^2),\\ \tilde{u_2} &= u_2 + \epsilon\phi(x, t, u_1, u_2) + O(\epsilon^2),\\ \frac{\partial^\alpha \tilde{u_1}}{\partial t^\alpha} &= \frac{\partial^\alpha u_1}{\partial t^\alpha} + \epsilon\eta^{\alpha,t} + O(\epsilon^2),\\ \frac{\partial^\alpha \tilde{u_2}}{\partial t^\alpha} &= \frac{\partial^\alpha u_2}{\partial t^\alpha} + \epsilon\phi^{\alpha,t} + O(\epsilon^2),\end{aligned}$$

$$\begin{aligned}
\frac{\partial^\beta \tilde{u_1}}{\partial x^\beta} &= \frac{\partial^\beta u_1}{\partial x^\beta} + \epsilon \eta^{\beta,x} + O(\epsilon^2),\\
\frac{\partial^\beta \tilde{u_2}}{\partial x^\beta} &= \frac{\partial^\beta u_2}{\partial x^\beta} + \epsilon \phi^{\beta,x} + O(\epsilon^2),\\
\frac{\partial^2 \tilde{u_1}}{\partial x^2} &= \frac{\partial^2 u_1}{\partial x^2} + \epsilon \eta^{xx} + O(\epsilon^2),\\
\frac{\partial^2 \tilde{u_2}}{\partial x^2} &= \frac{\partial^2 u_2}{\partial x^2} + \epsilon \phi^{xx} + O(\epsilon^2),\\
&\vdots
\end{aligned} \tag{4.2}$$

where $\eta^{\beta,x}$, $\phi^{\beta,x}$ are extended infinitesimals of order β. The associated symmetry generator can be wriiten in the following form:

$$X = \xi(x,t,u_1,u_2)\frac{\partial}{\partial x} + \tau(x,t,u_1,u_2)\frac{\partial}{\partial t} + \eta(x,t,u_1,u_2)\frac{\partial}{\partial u_1} + \phi(x,t,u_1,u_2)\frac{\partial}{\partial u_2}. \tag{4.3}$$

The prolonged generator is written as follows:

$$\mathrm{pr}^{(\alpha,\beta,m)}X = X + \eta^{\alpha,t}\partial_{\partial_t^\alpha u_1} + \phi^{\alpha,t}\partial_{\partial_t^\alpha u_2} + \eta^{\beta,x}\partial_{\partial_x^\beta u_1} + \phi^{\beta,x}\partial_{\partial_x^\beta u_2} + \eta^{xx}\partial_{u_{1,xx}} + \dots, \tag{4.4}$$

where m is the order of the system (4.1).

The integer order extended infinitesimals can be obtained in their expanded form in Refs. [24, 147]. The fractional extended infinitesimals $\eta^{\alpha,t}, \phi^{\alpha,t}$ have been derived in chapter 2. The extended infinitesimals $\eta^{\beta,x}$ and $\phi^{\beta,x}$ can be easily derived by a systematic computation. In this study, the extended infinitesimal $\eta^{\beta,x}$ of order β is introduced as follows:

$$\begin{aligned}
\eta^{\beta,x} = {} & \frac{\partial^\beta \eta}{\partial x^\beta} + (\eta_{u_1} - \beta \mathrm{D}_x(\xi))\frac{\partial^\beta u_1}{\partial x^\beta} - u_1\frac{\partial^\beta \eta_{u_1}}{\partial x^\beta} + (\eta_{u_2}\frac{\partial^\beta u_2}{\partial x^\beta} - u_2\frac{\partial^\beta \eta_{u_2}}{\partial x^\beta}) + \mu_{\eta,\beta,1} + \mu_{\eta,\beta,2}\\
& + \sum_{n=1}^{\infty}\left[\binom{\beta}{n}\frac{\partial^n \eta_{u_1}}{\partial x^n} - \binom{\beta}{n+1}\mathrm{D}_x^{n+1}(\xi)\right] D_x^{\beta-n}(u_1) + \sum_{n=1}^{\infty}\binom{\beta}{n}\frac{\partial^n \eta_{u_2}}{\partial x^n}D_x^{\beta-n}(u_2)\\
& - \sum_{n=1}^{\infty}\binom{\beta}{n}\mathrm{D}_x^n(\tau)D_x^{\beta-n}(u_{1,t}),
\end{aligned} \tag{4.5}$$

where $\mu_{\eta,\beta,i}$ for $i = 1,2$ are as follows

$$\mu_{\eta,\beta,i} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\beta}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{x^{n-\beta}}{\Gamma(n-\beta+1)}(-u_i)^r\frac{\partial^m}{\partial x^m}(u_i^{k-r})\frac{\partial^{n-m+k}\eta}{\partial x^{n-m}\partial u_i^k}. \tag{4.6}$$

Similarly, the β^{th} order extended infinitesimal function $\phi^{\beta,x}$ is introduced as follows:

$$\begin{aligned}
\phi^{\beta,x} = &\frac{\partial^\beta \phi}{\partial x^\beta} + (\phi_{u_2} - \beta \mathrm{D}_x(\xi))\frac{\partial^\beta u_2}{\partial x^\beta} - u_2 \frac{\partial^\beta \phi_{u_2}}{\partial x^\beta} + (\phi_{u_1}\frac{\partial^\beta u_1}{\partial x^\beta} - u_1 \frac{\partial^\beta \phi_{u_1}}{\partial x^\beta}) + \mu_{\phi,\beta,1} + \mu_{\phi,\beta,2} \\
&+ \sum_{n=1}^{\infty}\left[\binom{\beta}{n}\frac{\partial^n \phi_{u_2}}{\partial x^n} - \binom{\beta}{n+1}\mathrm{D}_x^{n+1}(\xi)\right] D_x^{\beta-n}(u_2) + \sum_{n=1}^{\infty}\binom{\beta}{n}\frac{\partial^n \phi_{u_1}}{\partial x^n} D_x^{\beta-n}(u_1) \\
&- \sum_{n=1}^{\infty}\binom{\beta}{n}\mathrm{D}_x^n(\tau) D_x^{\beta-n}(u_{2,t}),
\end{aligned} \tag{4.7}$$

where $\mu_{\phi,\beta,i}$ for $i = 1, 2$ are of the following form:

$$\mu_{\phi,\beta,i} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\beta}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{x^{n-\beta}}{\Gamma(n-\beta+1)}(-u_i)^r \frac{\partial^m}{\partial x^m}(u_i^{k-r})\frac{\partial^{n-m+k}\phi}{\partial x^{n-m}\partial u_i^k}. \tag{4.8}$$

The invariance criterion for system (4.1) can be written as follows:

$$\mathrm{pr}^{\alpha,\beta,m} X(\Delta)|_{\Delta=0} = 0, \tag{4.9}$$

where the system (4.1) is named as Δ. Hence, this symmetry approach can be applied for studying systems of space-time fractional PDEs.

The proposed approach can be easily generalized for space-time fractional systems with more than two dependent variables.

4.3 Application of the Proposed Symmetry Approach

In this section, the developed symmetry method for systems of space-time fractional PDEs is used for the analysis of some physically important space-time fractional nonlinear systems with fractional orders $0 < \alpha, \beta \leq 1$ as follows.

4.3.1 Space-Time Fractional Ito System

Here, the considered Ito system with space-time fractional derivatives is as follows:

$$\begin{aligned}
&\frac{\partial^\alpha u}{\partial t^\alpha} - 3u\frac{\partial^\beta u}{\partial x^\beta} - v\frac{\partial^\beta u}{\partial x^\beta} - \frac{\partial^3 u}{\partial x^3} = 0, \\
&\frac{\partial^\alpha v}{\partial t^\alpha} - u\frac{\partial^\beta v}{\partial x^\beta} - v\frac{\partial^\beta u}{\partial x^\beta} = 0.
\end{aligned} \tag{4.10}$$

The invariant analysis of system (4.10) is described by the theorem written as follows:

Theorem 4.1. *The system* (4.10) *admits the group of transformations* (4.2) *then the resulting Lie symmetries can be obtained as follows:*

$$\xi = \frac{c_1 x}{3}, \quad \tau = \frac{c_1 t}{\alpha}, \quad \eta = \frac{(\beta-3)}{3}c_1 u \quad \textit{and} \quad \phi = \frac{(\beta-3)}{3}c_1 v, \tag{4.11}$$

where c_1 is an arbitrary constant along with the corresponding Lie symmetry generator as follows:

$$V = \frac{x}{3}\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{(\beta-3)u}{3}\frac{\partial}{\partial u} + \frac{(\beta-3)v}{3}\frac{\partial}{\partial v}. \tag{4.12}$$

Proof. For the invariance of system (4.10) under group of transformations (4.2), the invariance criterion takes the following form:

$$\begin{aligned} &[\eta^{\alpha,t} - 3(u\eta^{\beta,x} + \eta u^{\beta}_{x}) - (v\eta^{\beta,x} + \phi u^{\beta}_{x}) - \eta^{xxx}]|_{(4.10)} = 0, \\ &[\phi^{\alpha,t} - (u\phi^{\beta,x} + \eta v^{\beta}_{x}) - (v\eta^{\beta,x} + \phi u^{\beta}_{x})]|_{(4.10)} = 0. \end{aligned} \tag{4.13}$$

Using the prolongation expressions developed before and equating the coefficients of various partial derivatives of u, v, the obtained set of determining equations is as follows:

$$\begin{aligned} &\xi_t = \xi_u = \xi_v = 0, \\ &\tau_x = \tau_u = \tau_v = 0, \\ &\eta_{xu} - \xi_{xx} = 0, \\ &\phi_u = \eta_v = \eta_{uu} = 0, \\ &\phi = v(\phi_v - \alpha\tau_t - \eta_u + \beta\xi_x), \\ &\eta = u(\beta\xi_x - \alpha\tau_t), \\ &3\xi_x - \alpha\tau_t = 0, \\ &\phi + 3\eta + (v + 3u)(\alpha\tau_t - \beta\xi_x) = 0, \\ &\binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad n \in \mathbb{N}, \\ &\binom{\alpha}{n}\partial_t^n \phi_v - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad n \in \mathbb{N}, \\ &\binom{\beta}{n}\partial_x^n \eta_u - \binom{\beta}{n+1}\mathrm{D}_x^{n+1}\xi = 0, \quad n \in \mathbb{N}, \\ &\binom{\beta}{n}\partial_x^n \phi_v - \binom{\beta}{n+1}\mathrm{D}_x^{n+1}\xi = 0, \quad n \in \mathbb{N}, \end{aligned} \tag{4.14}$$

$$\partial_t^\alpha \eta - u\partial_t^\alpha \eta_u - (3u+v)(\partial_x^\beta \eta - u\partial_x^\beta \eta_u) - \eta_{xxx} = 0,$$

$$\partial_t^\alpha \phi - v\partial_t^\alpha \phi_v - u(\partial_x^\beta \phi - v\partial_x^\beta \phi_v) - v(\partial_x^\beta \eta - u\partial_x^\beta \eta_u) = 0.$$

Solving these equations, the obtained infinitesimals are of the following form:

$$\xi = \frac{c_1 x}{3} + c_2, \quad \tau = \frac{c_1 t}{\alpha} + c_3, \quad \eta = \frac{(\beta-3)}{3}c_1 u \quad \text{and} \quad \phi = \frac{(\beta-3)}{3}c_1 v, \tag{4.15}$$

for c_1, c_2 and c_3 being arbitrary constants. These symmetries for $\alpha = \beta = 1$ are equivalent to those for integer order Ito system proving the validity of the proposed formulae. Also, for preserving the structure of the Riemann-Liouville fractional derivative in case of space-time fractional systems, the following conditions must hold:

$$\xi(x,t,u,v)|_{x=0} = 0, \quad \tau(x,t,u,v)|_{t=0} = 0. \tag{4.16}$$

It gives $c_2 = c_3 = 0$, completing the proof. □

For the generator V, the auxiliary equations are as follows:

$$\frac{\mathrm{d}x}{\frac{x}{3}} = \frac{\mathrm{d}t}{\frac{t}{\alpha}} = \frac{\mathrm{d}u}{\frac{(\beta-3)}{3}u} = \frac{\mathrm{d}v}{\frac{(\beta-3)}{3}v}. \tag{4.17}$$

Their solutions lead to the following result:

Theorem 4.2. *The corresponding similarity transformations* $u(x,t) = t^{\frac{\alpha}{3}(\beta-3)}f(z)$ *and* $v(x,t) = t^{\frac{\alpha}{3}(\beta-3)}g(z)$ *with the similarity variable* $z = xt^{-\frac{\alpha}{3}}$ *reduce the system* (4.10) *for* $\alpha, \beta > 0$ *into a system of nonlinear FODEs as follows:*

$$\begin{aligned} &\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(\beta-6),\alpha} f\right)(z) - z^{-\beta}(3f(z)+g(z))\left(\mathcal{D}_1^{-\beta,\beta} f\right)(z) - f'''(z) = 0, \\ &\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(\beta-6),\alpha} g\right)(z) - z^{-\beta}\left[f(z)\left(\mathcal{D}_1^{-\beta,\beta} g\right)(z) + g(z)\left(\mathcal{D}_1^{-\beta,\beta} f\right)(z)\right] = 0, \end{aligned} \tag{4.18}$$

where $\left(\mathcal{P}_\delta^{\zeta,\alpha}\right)$ *and* $\left(\mathcal{D}_\delta^{\zeta,\beta}\right)$ *are the left and right-hand sided Erdélyi-Kober fractional differential operators* [107, 128] *respectively.*

Proof. The fractional derivatives of $u(x,t)$ and $v(x,t)$ of order $\alpha > 0$ with respect to t can be obtained as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{\frac{\alpha}{3}(\beta-6)}\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(\beta-6),\alpha} f\right)(z), \tag{4.19}$$

and

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} = t^{\frac{\alpha}{3}(\beta-6)} \left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(\beta-6),\alpha} g\right)(z). \tag{4.20}$$

The fractional derivatives of $u(x,t)$ and $v(x,t)$ of order β with respect to x are calculated by the following Lemma:

Lemma 4.3. *The partial derivatives* $\frac{\partial^{\beta} u}{\partial x^{\beta}}$ *and* $\frac{\partial^{\beta} v}{\partial x^{\beta}}$ *of fractional order* $\beta > 0$ *are obtained by the following relations:*

$$\begin{aligned} \frac{\partial^{\beta} u}{\partial x^{\beta}} &= t^{\frac{\alpha}{3}(\beta-3)} x^{-\beta} \left(\mathcal{D}_{1}^{-\beta,\beta} f\right)(z), \\ \frac{\partial^{\beta} v}{\partial x^{\beta}} &= t^{\frac{\alpha}{3}(\beta-3)} x^{-\beta} \left(\mathcal{D}_{1}^{-\beta,\beta} g\right)(z). \end{aligned} \tag{4.21}$$

Proof. For $n-1 < \beta < n$, $(n = 1, 2, , ..)$, by using the definition of Riemann-Liouville fractional derivative, the following holds:

$$\frac{\partial^{\beta} u}{\partial x^{\beta}} = \frac{\partial^{n}}{\partial x^{n}} \left(\frac{1}{\Gamma(n-\beta)} \int\limits_{0}^{x} (x-\rho)^{n-\beta-1} t^{\frac{\alpha}{3}(\beta-3)} f(\rho t^{-\frac{\alpha}{3}}) \mathrm{d}\rho \right). \tag{4.22}$$

Let $s = \frac{\rho}{x}$, then the above expression is transformed into the following:

$$\frac{\partial^{\beta} u}{\partial x^{\beta}} = t^{\frac{\alpha}{3}(\beta-3)} \frac{\partial^{n}}{\partial x^{n}} \left(x^{n-\beta} \left(\mathcal{I}_{1}^{0,n-\beta} f\right)(z) \right), \tag{4.23}$$

where

$$\left(\mathcal{I}_{1}^{0,n-\beta} f\right)(z) = \frac{1}{\Gamma(n-\beta)} \int\limits_{0}^{1} (1-s)^{n-\beta-1} f(zs) \mathrm{d}s, \tag{4.24}$$

is the right-hand sided Erdélyi-Kober fractional integral operator defined by (3.24). Using $\left(\mathcal{I}_{1}^{\zeta,0} f\right)(z) = f(z)$, the relation (4.23) is also true for $\beta = n = 1, 2, 3, ...$
For $z = xt^{-\frac{\alpha}{3}}$, the following relation holds

$$x \frac{\partial}{\partial x} \psi(z) = x \psi'(z) t^{-\frac{\alpha}{3}} = z \frac{\mathrm{d}}{\mathrm{d}z} \psi(z), \tag{4.25}$$

Therefore, the following expression is deduced:

$$\begin{aligned} \frac{\partial^{n}}{\partial x^{n}} \left(x^{n-\beta} \left(\mathcal{I}_{1}^{0,n-\beta} f\right)(z) \right) &= \frac{\partial^{n-1}}{\partial x^{n-1}} \left[x^{n-\beta-1} \left(n - \beta + z \frac{d}{dz} \right) \left(\mathcal{I}_{1}^{0,n-\beta} f\right)(z) \right], \\ &= ... = x^{-\beta} \prod_{j=1}^{n} \left(-\beta + j + z \frac{d}{dz} \right) \left(\mathcal{I}_{1}^{0,n-\beta} f\right)(z). \end{aligned} \tag{4.26}$$

In view of the definition of right-hand sided Erdélyi-Kober fractional differential operator (3.23), for $n-1 < \beta \leq n$ he following must hold

$$\frac{\partial^{\beta} u}{\partial x^{\beta}} = t^{\frac{\alpha}{3}(\beta-3)} x^{-\beta} \left(\mathcal{D}_{1}^{-\beta,\beta} f\right)(z). \tag{4.27}$$

Analogous to the result (4.27) obtained, $\frac{\partial^\beta v}{\partial x^\beta}$ can be solved in the following form:

$$\frac{\partial^\beta v}{\partial x^\beta} = t^{\frac{\alpha}{3}(\beta-3)} x^{-\beta} \left(\mathcal{D}_1^{-\beta,\beta} g\right)(z). \tag{4.28}$$

This completes the proof of the Lemma. □

The statement of the theorem follows by using the expressions (4.19), (4.20) and the results proved in Lemma. □

4.3.2 Space-Time Fractional Coupled Burgers Equations

The coupled Burgers equations with space-time fractional derivatives are as follows [38]:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial^\beta u}{\partial x^\beta} - u\frac{\partial^\beta v}{\partial x^\beta} - v\frac{\partial^\beta u}{\partial x^\beta}, \\ \frac{\partial^\alpha v}{\partial t^\alpha} &= \frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial^\beta v}{\partial x^\beta} - u\frac{\partial^\beta v}{\partial x^\beta} - v\frac{\partial^\beta u}{\partial x^\beta}. \end{aligned} \tag{4.29}$$

Considering the invariance of (4.29) under the group of transformations (4.2) gives the following invariance criterion:

$$\begin{aligned} &[\eta^{\alpha,t} - \eta^{xx} - 2(\eta u_x^\beta + u\eta^{\beta,x}) + (u\phi^{\beta,x} + v\eta^{\beta,x} + \eta v_x^\beta + \phi u_x^\beta)]|_{(4.29)} = 0, \\ &[\phi^{\alpha,t} - \phi^{xx} - 2(\phi v_x^\beta + v\phi^{\beta,x}) + (u\phi^{\beta,x} + v\eta^{\beta,x} + \eta v_x^\beta + \phi u_x^\beta)]|_{(4.29)} = 0. \end{aligned} \tag{4.30}$$

Substituting the prolongations, the following result holds:

Theorem 4.4. *The infinitesimals for coupled Burgers equations* (4.29) *can be obtained in the following form:*

$$\xi = \frac{c_1 x}{2}, \quad \tau = \frac{c_1 t}{\alpha}, \quad \eta = \frac{(\beta-2)uc_1}{2} \quad \textit{and} \quad \phi = \frac{(\beta-2)vc_1}{2}, \tag{4.31}$$

where c_1 is an arbitrary constant. Hence, the associated Lie algebra is generated by the following vector field

$$V = \frac{x}{2}\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{(\beta-2)u}{2}\frac{\partial}{\partial u} + \frac{(\beta-2)v}{2}\frac{\partial}{\partial v}. \tag{4.32}$$

Proof. The following determining equations can be obtained:

$$
\begin{aligned}
&\xi_t = \xi_u = \xi_v = 0,\\
&\tau_x = \tau_u = \tau_v = 0,\\
&2\xi_x - \alpha\tau_t = 0,\\
&\eta_{vt} = \phi_{ut} = 0,\\
&\eta_{uv} = \eta_{vv} = \eta_{xv} = \eta_{uu} = 0,\\
&\phi_{uu} = \phi_{vv} = \phi_{xu} = 0,\\
&2\eta_{xu} = 2\phi_{xv} = \xi_{xx},\\
&\eta = 3(u-v)\eta_v + u(\eta_u - \phi_v - \alpha\tau_t + \beta\xi_x),\\
&\phi = 3(v-u)\phi_u + v(\phi_v - \eta_u - \alpha\tau_t + \beta\xi_x),\\
&\phi - 2\eta = u(2\alpha\tau_t - \phi_u - 2\beta\xi_x) + v(\eta_v - \alpha\tau_t + \beta\xi_x),\\
&2\phi - \eta = u(\alpha\tau_t - \phi_u - \beta\xi_x) + v(\eta_v + 2\beta\xi_x - 2\alpha\tau_t),\\
&\binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad n \in \mathbb{N},\\
&\binom{\alpha}{n}\partial_t^n \phi_v - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad n \in \mathbb{N},\\
&\partial_t^\alpha \eta - u\partial_t^\alpha \eta_u - v\partial_t^\alpha \eta_v + (v-2u)\left(\partial_x^\beta \eta - u\partial_x^\beta \eta_u - v\partial_x^\beta \eta_v\right) +\\
&u\left(\partial_x^\beta \phi - u\partial_x^\beta \phi_u - v\partial_x^\beta \phi_v\right) - \eta_{xx} = 0,\\
&\partial_t^\alpha \phi - u\partial_t^\alpha \phi_u - v\partial_t^\alpha \phi_v + (u-2v)\left(\partial_x^\beta \phi - u\partial_x^\beta \phi_u - v\partial_x^\beta \phi_v\right)\\
&\quad + v\left(\partial_x^\beta \eta - u\partial_x^\beta \eta_u - v\partial_x^\beta \eta_v\right) - \phi_{xx} = 0.
\end{aligned}
\tag{4.33}
$$

Solving these equations, the following solutions are obtained:

$$
\xi = \frac{c_1 x}{2} + c_2, \quad , \tau = \frac{c_1 t}{\alpha} + c_3, \quad \eta = u\left(\frac{\beta-2}{2}\right), \quad \phi = v\left(\frac{\beta-2}{2}\right). \tag{4.34}
$$

Using the conditions (4.16) completes the proof. ☐

The solution of auxiliary equations for the generator (4.32) leads to the following assertion:

Theorem 4.5. *The system* (4.29) *using the similarity transformations* $u(x,t) = t^{\frac{(\beta-2)\alpha}{2}} f(z)$ *and* $v(x,t) = t^{\frac{(\beta-2)\alpha}{2}} g(z)$ *with the similarity variable* $z = xt^{-\frac{\alpha}{2}}$ *can be reduced into the* *following system of nonlinear FODEs:*

$$
\begin{aligned}
&\left(\mathcal{P}_{\frac{2}{\alpha}}^{1+\frac{\alpha(\beta-4)}{2},\alpha} f\right)(z) - z^{-\beta}(2f(z)-g(z))\left(\mathcal{D}_1^{-\beta,\beta} f\right)(z) + z^{-\beta} f(z)\left(\mathcal{D}_1^{-\beta,\beta} g\right)(z) - f''(z) = 0,\\
&\left(\mathcal{P}_{\frac{2}{\alpha}}^{1+\frac{\alpha(\beta-4)}{2},\alpha} g\right)(z) - z^{-\beta}(2g(z)-f(z))\left(\mathcal{D}_1^{-\beta,\beta} g\right)(z) + z^{-\beta} g(z)\left(\mathcal{D}_1^{-\beta,\beta} f\right)(z) - g''(z) = 0,
\end{aligned}
\tag{4.35}
$$

where $\left(\mathcal{P}_{\delta}^{\zeta,\alpha}\right)$ *and* $\left(\mathcal{D}_{\delta}^{\zeta,\beta}\right)$ *are the left and right-hand sided Erdélyi-Kober fractional differential operators.*

Proof. Similar to proof of Theorem 4.2. □

4.3.3 Space-Time Fractional Coupled KdV Equations

The time fractional coupled KdV equations [68] have been studied with various techniques. We consider the coupled KdV equations of space-time fractional order in the following form:

$$
\begin{aligned}
&\frac{\partial^\alpha u}{\partial t^\alpha} + 6au\frac{\partial^\beta u}{\partial x^\beta} - 6v\frac{\partial^\beta v}{\partial x^\beta} + a\frac{\partial^3 u}{\partial x^3} = 0,\\
&\frac{\partial^\alpha v}{\partial t^\alpha} + 3au\frac{\partial^\beta v}{\partial x^\beta} + a\frac{\partial^3 v}{\partial x^3} = 0.
\end{aligned}
\tag{4.36}
$$

The invariance condition for system (4.36) can be written in the form:

$$
\begin{aligned}
&[\eta^{\alpha,t} + 6a(u\eta^{\beta,x} + \eta u_x^\beta) - 6(v\phi^{\beta,x} + \phi v_x^\beta) + a\eta^{xxx}]|_{(4.36)} = 0,\\
&[\phi^{\alpha,t} + 3a(u\phi^{\beta,x} + \eta v_x^\beta) + a\phi^{xxx}]|_{(4.36)} = 0.
\end{aligned}
\tag{4.37}
$$

The solutions of the resulting determining equations corresponding to the conditions (4.37) are presented by the following theorem:

Theorem 4.6. *The system* (4.36) *admits the Lie group of transformations* (4.2) *with symmetries as follows:*

$$\xi = c_1 x, \quad \tau = \frac{3c_1 t}{\alpha}, \quad \eta = (\beta-3)c_1 u \quad \textit{and} \quad \phi = (\beta-3)c_1 v, \tag{4.38}$$

where c_1 *being an arbitrary constant. The following vector field is obtained:*

$$V = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} + (\beta-3)u\frac{\partial}{\partial u} + (\beta-3)v\frac{\partial}{\partial v}. \tag{4.39}$$

Solving the auxiliary equations for the associated vector field (4.39), it can be shown that the following theorem holds:

Theorem 4.7. *The similarity transformations* $u(x,t) = t^{\frac{\alpha}{3}(\beta-3)}f(z)$ *and* $v(x,t) = t^{\frac{\alpha}{3}(\beta-3)}g(z)$ *with the similarity variable* $z = xt^{-\frac{\alpha}{3}}$ *reduce the system* (4.36) *into the following system of nonlinear FODEs:*

$$\begin{aligned}
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(\beta-6),\alpha}f\right)(z) + 6az^{-\beta}f(z)\left(\mathcal{D}_{1}^{-\beta,\beta}f\right)(z) - 6z^{-\beta}g(z)\left(\mathcal{D}_{1}^{-\beta,\beta}g\right)(z) + af'''(z) = 0,\\
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(\beta-6),\alpha}g\right)(z) + 3az^{-\beta}f(z)\left(\mathcal{D}_{1}^{-\beta,\beta}g\right)(z) + ag'''(z) = 0.
\end{aligned} \tag{4.40}$$

Proof. The proof is similar to the proof of Theorem 4.2. □

4.3.4 Space-Time Fractional Generalized Hirota-Satsuma Coupled KdV Equations

The space-time fractional Hirota-Satsuma coupled KdV equations [67] are considered in the following form:

$$\begin{aligned}
&\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3} - 3u\frac{\partial^\beta u}{\partial x^\beta} - 3\left(-2v\frac{\partial^\beta v}{\partial x^\beta} + \frac{\partial^\beta w}{\partial x^\beta}\right) = 0,\\
&\frac{\partial^\alpha v}{\partial t^\alpha} + \frac{1}{2}\frac{\partial^3 v}{\partial x^3} + 3u\frac{\partial^\beta v}{\partial x^\beta} = 0,\\
&\frac{\partial^\alpha w}{\partial t^\alpha} + \frac{1}{2}\frac{\partial^3 w}{\partial x^3} + 3u\frac{\partial^\beta w}{\partial x^\beta} = 0.
\end{aligned} \tag{4.41}$$

Assume that the associated symmetry generator is as follows:

$$V = \xi\partial_x + \tau\partial_t + \eta\partial_u + \phi\partial_v + \psi\partial_w. \tag{4.42}$$

The invariance criterion for system (4.41) can be obtained as follows:

$$\begin{aligned}
&[\eta^{\alpha,t} - \frac{1}{4}\eta^{xxx} - 3(u\eta^{\beta,x} + \eta u_x^\beta) - 3(\psi^{\beta,x} - 2\phi v_x^\beta - 2v\phi^{\beta,x})]|_{(4.41)} = 0,\\
&[\phi^{\alpha,t} + \frac{1}{2}\phi^{xxx} + 3(u\phi^{\beta,x} + \eta v_x^\beta)]|_{(4.41)} = 0,\\
&[\psi^{\alpha,t} + \frac{1}{2}\psi^{xxx} + 3(u\psi^{\beta,x} + \eta w_x^\beta)]|_{(4.41)} = 0.
\end{aligned} \tag{4.43}$$

Inserting the prolongations and equating the coefficients gives the set of equations whose solutions are described in the following assertion:

Theorem 4.8. *The Lie symmetries of system* (4.41) *are obtained as follows:*

$$\xi = c_1 x, \;\; \tau = \frac{3c_1 t}{\alpha}, \;\; \eta = (\beta-3)c_1 u, \;\; \phi = (\beta-3)c_1 v \;\; \textit{and} \;\; \psi = 2(\beta-3)c_1 w, \tag{4.44}$$

where c_1 is an arbitrary constant. Hence, the corresponding symmetry generator is of following form:

$$V = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} + (\beta-3)u\frac{\partial}{\partial u} + (\beta-3)v\frac{\partial}{\partial v} + 2(\beta-3)w\frac{\partial}{\partial w}. \tag{4.45}$$

Proof. The determining equations for system (4.43) are as follows:

$$\begin{aligned}
&\xi_t = \xi_u = \xi_v = 0,\\
&\tau_x = \tau_u = \tau_v = 0,\\
&3\xi_x - \alpha\tau_t = 0,\\
&\eta_v = \eta_w = \eta_{uu} = \phi_u = \psi_u = 0,\\
&\phi_{vv} = \phi_{vw} = \phi_{ww} = \phi_{wt} = \phi_{wx} = 0,\\
&\psi_{vv} = \psi_{ww} = \psi_{vw} = \psi_{vt} = \psi_{vx} = 0,\\
&\eta_{ut} = \phi_{vt} = \psi_{wt} = \left(\frac{\alpha-1}{2}\right)\tau_{tt} = 0,\\
&\eta_{ux} = \phi_{vx} = \psi_{wx} = \left(\frac{\beta-1}{2}\right)\xi_{xx} = 0,\\
&3\eta_{ux} = \phi_{vx} = \xi_{xx},\\
&\eta = u(\beta\xi_x - \alpha\tau_t),\\
&2\phi = 2v(\eta_u - \alpha\tau_t - \phi_v + \beta\xi_x) + \psi_v,\\
&2v\phi_w = \psi_w - 2\alpha\tau_t - 2\beta\xi_x,\\
&\partial_t^\alpha\eta - u\partial_t^\alpha\eta_u - 3u\left(\partial_x^\beta\eta - u\partial_x^\beta\eta_u\right) + 6v\left(\partial_x^\beta\phi - v\partial_x^\beta\phi_v - w\partial_x^\beta\phi_w\right)\\
&\;-3\left(\partial_x^\beta\psi - v\partial_x^\beta\psi_v - w\partial_x^\beta\psi_w\right) - \frac{1}{4}\eta_{xxx} = 0,\\
&\partial_t^\alpha\phi - v\partial_t^\alpha\phi_v - w\partial_t^\alpha\phi_w + 3u\left(\partial_x^\beta\phi - v\partial_x^\beta\phi_v - w\partial_x^\beta\phi_w\right) + \frac{1}{2}\phi_{xxx} = 0,\\
&\partial_t^\alpha\psi - v\partial_t^\alpha\psi_v - w\partial_t^\alpha\psi_w + 3u\left(\partial_x^\beta\psi - v\partial_x^\beta\psi_v - w\partial_x^\beta\psi_w\right) + \frac{1}{2}\psi_{xxx} = 0.
\end{aligned} \tag{4.46}$$

The solutions of these equations are as follows:

$$\xi = c_1 x + c_2, \;\; \tau = \frac{3c_1 t}{\alpha} + c_3, \;\; \eta = (\beta-3)c_1 u, \;\; \phi = (\beta-3)c_1 v \;\; \textit{and} \;\; \psi = 2(\beta-3)c_1 w + C(x,t), \tag{4.47}$$

where c_1, c_2, c_3 are arbitrary constants and $C(x,t)$ satisfies the following:

$$\partial_x^{\beta}(C(x,t)) = 0, \quad \partial_t^{\alpha}(C(x,t)) + \frac{1}{2}C_{xxx} = 0. \tag{4.48}$$

The conditions (4.16) give $c_2 = c_3 = 0$ and for simplicity, assuming $C(x,t) = 0$, the proof is completed. □

The solutions of the characteristic equations for (4.45) transform the system (4.41) into a system of FODEs presented in the following theorem:

Theorem 4.9. *The similarity transformations* $u(x,t) = t^{\frac{\alpha}{3}(\beta-3)}f(z)$, $v(x,t) = t^{\frac{\alpha}{3}(\beta-3)}g(z)$ *and* $w(x,t) = t^{\frac{2\alpha}{3}(\beta-3)}h(z)$ *with the similarity variable* $z = xt^{-\frac{\alpha}{3}}$ *reduce the system* (4.41) *into a system of nonlinear FODEs written as follows:*

$$\begin{gathered}
\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(\beta-6),\alpha} f\right)(z) - 3z^{-\beta}\left[f\left(\mathcal{D}_1^{-\beta,\beta} f\right)(z) - 2g\left(\mathcal{D}_1^{-\beta,\beta} g\right)(z)\right] - 3\left(\mathcal{D}_1^{-\beta,\beta} h\right)(z) - \frac{1}{4}f'''(z) = 0,\\
\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(\beta-6),\alpha} g\right)(z) + 3z^{-\beta}f(z)\left(\mathcal{D}_1^{-\beta,\beta} g\right)(z) + \frac{1}{2}g'''(z) = 0,\\
\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(2\beta-9),\alpha} h\right)(z) + 3z^{-\beta}f(z)\left(\mathcal{D}_1^{-\beta,\beta} h\right)(z) + \frac{1}{2}h'''(z) = 0.
\end{gathered} \tag{4.49}$$

4.3.5 Space-Time Fractional Coupled Hirota Equations

The coupled nonlinear Hirota equations [14] with space-time fractional derivatives under consideration are as follows:

$$\begin{aligned}
\frac{\partial^{\alpha} p}{\partial t^{\alpha}} + \frac{\partial^3 p}{\partial x^3} + 6\left(|p|^2 + |q|^2\right)\frac{\partial^{\beta} p}{\partial x^{\beta}} &= 0,\\
\frac{\partial^{\alpha} q}{\partial t^{\alpha}} + \frac{\partial^3 q}{\partial x^3} + 6\left(|p|^2 + |q|^2\right)\frac{\partial^{\beta} q}{\partial x^{\beta}} &= 0,
\end{aligned} \tag{4.50}$$

where $p(x,t)$, $q(x,t)$ are complex valued functions. For $p = u + \iota v$ and $q = w + \iota z$, the system (4.50) gives the following system of equations:

$$\begin{aligned}
\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \frac{\partial^3 u}{\partial x^3} + 6\left(u^2 + v^2 + w^2 + z^2\right)\frac{\partial^{\beta} u}{\partial x^{\beta}} &= 0,\\
\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + \frac{\partial^3 v}{\partial x^3} + 6\left(u^2 + v^2 + w^2 + z^2\right)\frac{\partial^{\beta} v}{\partial x^{\beta}} &= 0,\\
\frac{\partial^{\alpha} w}{\partial t^{\alpha}} + \frac{\partial^3 w}{\partial x^3} + 6\left(u^2 + v^2 + w^2 + z^2\right)\frac{\partial^{\beta} w}{\partial x^{\beta}} &= 0,\\
\frac{\partial^{\alpha} z}{\partial t^{\alpha}} + \frac{\partial^3 z}{\partial x^3} + 6\left(u^2 + v^2 + w^2 + z^2\right)\frac{\partial^{\beta} z}{\partial x^{\beta}} &= 0.
\end{aligned} \tag{4.51}$$

The invariance conditions for the system (4.51) can be obtained as follows:

$$\begin{aligned}
&[\eta^{\alpha,t}+\eta^{xxx}+6\eta^{\beta,x}\left(u^2+v^2+w^2+z^2\right)+12u_x^{\beta}\left(u\eta+v\phi+w\psi+z\delta\right)]|_{(4.51)}=0,\\
&[\phi^{\alpha,t}+\phi^{xxx}+6\phi^{\beta,x}\left(u^2+v^2+w^2+z^2\right)+12v_x^{\beta}\left(u\eta+v\phi+w\psi+z\delta\right)]|_{(4.51)}=0,\\
&[\psi^{\alpha,t}+\psi^{xxx}+6\psi^{\beta,x}\left(u^2+v^2+w^2+z^2\right)+12w_x^{\beta}\left(u\eta+v\phi+w\psi+z\delta\right)]|_{(4.51)}=0,\\
&[\delta^{\alpha,t}+\delta^{xxx}+6\delta^{\beta,x}\left(u^2+v^2+w^2+z^2\right)+12z_x^{\beta}\left(u\eta+v\phi+w\psi+z\delta\right)]|_{(4.51)}=0,
\end{aligned}\tag{4.52}$$

where the associated symmetry generator is as follows:

$$V=\xi\partial_x+\tau\partial_t+\eta\partial_u+\phi\partial_v+\psi\partial_w. \tag{4.53}$$

The solutions of the resulting set of determining equations after substituting the prolongation expressions leads to the result in the following form.

Theorem 4.10. *The explicit form of Lie symmetries of system* (4.51) *is as follows:*

$$\begin{aligned}
&\xi=\frac{c_1x}{3},\quad \tau=\frac{c_1t}{\alpha},\quad \eta=\frac{(\beta-3)}{6}c_1u+c_2v+c_3w+c_4z,\quad \phi=-c_2u+\frac{(\beta-3)}{6}c_1v+c_5w+c_6z,\\
&\psi=-c_3u-c_5v+\frac{(\beta-3)}{6}c_1w+c_7z\ \ \textit{and}\ \ \delta=-c_4u-c_6v-c_7w+\frac{(\beta-3)}{6}c_1z,
\end{aligned}\tag{4.54}$$

where c_i $(i=1,2,3,4,5,6,7)$ *are arbitrary constants. Hence, the corresponding Lie algebra is generated by the following vector fields:*

$$\begin{aligned}
&V_1=\frac{x}{3}\frac{\partial}{\partial x}+\frac{t}{\alpha}\frac{\partial}{\partial t}+\frac{(\beta-3)}{6}u\frac{\partial}{\partial u}+\frac{(\beta-3)}{6}v\frac{\partial}{\partial v}+\frac{(\beta-3)}{6}w\frac{\partial}{\partial w}+\frac{(\beta-3)}{6}z\frac{\partial}{\partial z},\\
&V_2=v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v},\\
&V_3=w\frac{\partial}{\partial u}-u\frac{\partial}{\partial w},\\
&V_4=z\frac{\partial}{\partial u}-u\frac{\partial}{\partial z},\\
&V_5=w\frac{\partial}{\partial v}-v\frac{\partial}{\partial w},\\
&V_6=z\frac{\partial}{\partial v}-v\frac{\partial}{\partial z},\\
&V_7=z\frac{\partial}{\partial w}-w\frac{\partial}{\partial z}.
\end{aligned}\tag{4.55}$$

For the sake of brevity, the symmetry reductions are discussed in this chapter only for vector field V_1. For vector field V_1, the solution of the characteristic equation leads to the following assertion:

Theorem 4.11. *The corresponding similarity transformations* $u(x,t) = t^{\frac{\alpha}{6}(\beta-3)}f(\theta)$, $v(x,t) = t^{\frac{\alpha}{6}(\beta-3)}g(\theta)$, $w(x,t) = t^{\frac{\alpha}{6}(\beta-3)}h(\theta)$ *and* $z(x,t) = t^{\frac{\alpha}{6}(\beta-3)}k(\theta)$ *with the similarity variable* $\theta = xt^{-\frac{\alpha}{3}}$ *reduce the system* (4.51) *into the following system of nonlinear FODEs:*

$$\begin{aligned}
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{6}(\beta-9),\alpha}f\right)(\theta) + f'''(\theta) + 6\theta^{-\beta}(f^2+g^2+h^2+k^2)(\theta)\left(\mathcal{D}_1^{-\beta,\beta}f\right)(\theta) = 0,\\
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{6}(\beta-9),\alpha}g\right)(\theta) + g'''(\theta) + 6\theta^{-\beta}(f^2+g^2+h^2+k^2)(\theta)\left(\mathcal{D}_1^{-\beta,\beta}g\right)(\theta) = 0,\\
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{6}(\beta-9),\alpha}h\right)(\theta) + h'''(\theta) + 6\theta^{-\beta}(f^2+g^2+h^2+k^2)(\theta)\left(\mathcal{D}_1^{-\beta,\beta}h\right)(\theta) = 0,\\
&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{6}(\beta-9),\alpha}k\right)(\theta) + k'''(\theta) + 6\theta^{-\beta}(f^2+g^2+h^2+k^2)(\theta)\left(\mathcal{D}_1^{-\beta,\beta}k\right)(\theta) = 0.
\end{aligned} \tag{4.56}$$

Proof. Following the procedure to find fractional order derivatives given in proof of Theorem 4.2, the αth order fractional derivatives of u, v, w, z can be obtained as follows:

$$\begin{aligned}
\partial_t^\alpha u &= t^{\frac{\alpha(\beta-3)}{6}-\alpha}\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{6}(\beta-9),\alpha}f\right)(\theta),\\
\partial_t^\alpha v &= t^{\frac{\alpha(\beta-3)}{6}-\alpha}\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{6}(\beta-9),\alpha}g\right)(\theta),\\
\partial_t^\alpha w &= t^{\frac{\alpha(\beta-3)}{6}-\alpha}\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{6}(\beta-9),\alpha}h\right)(\theta),\\
\partial_t^\alpha z &= t^{\frac{\alpha(\beta-3)}{6}-\alpha}\left(\mathcal{P}_{\frac{3}{\alpha}}^{1+\frac{\alpha}{6}(\beta-9),\alpha}k\right)(\theta).
\end{aligned} \tag{4.57}$$

Similarly, the βth order fractional derivatives are as follows:

$$\begin{aligned}
\partial_x^\beta u &= t^{\frac{\alpha(\beta-3)}{6}}x^{-\beta}\left(\mathcal{D}_1^{-\beta,\beta}f\right)(\theta),\\
\partial_x^\beta v &= t^{\frac{\alpha(\beta-3)}{6}}x^{-\beta}\left(\mathcal{D}_1^{-\beta,\beta}g\right)(\theta),\\
\partial_x^\beta w &= t^{\frac{\alpha(\beta-3)}{6}}x^{-\beta}\left(\mathcal{D}_1^{-\beta,\beta}h\right)(\theta),\\
\partial_x^\beta z &= t^{\frac{\alpha(\beta-3)}{6}}x^{-\beta}\left(\mathcal{D}_1^{-\beta,\beta}k\right)(\theta).
\end{aligned} \tag{4.58}$$

Using the obtained fractional derivatives, the space-time fractional Hirota equations (4.50) can be easily reduced into the system (4.56). □

4.4 Conclusion

In this study, the Lie symmetry analysis for systems of space-time fractional PDEs has been introduced. The invariance of the considered systems gives similarity solutions leading to their reduction into FODEs in terms of Erdélyi-Kober fractional differential operators. The results obtained for all the considered space-time fractional systems are generalized than their corresponding time fractional form in chapter 2 and integer order systems. Hence, the Lie symmetry technique has emerged as an efficient method in study of systems having both time and space derivatives of fractional order. There can be further extensions of this work e.g. symmetry analysis of higher dimensional systems of PDEs with more than two independent variables that is given in chapter 5.

Chapter 5

Higher Dimensional Nonlinear Systems of Fractional Differential Equations

5.1 Introduction

In most of the available works on symmetry analysis of FDEs [15, 16, 78, 87, 188, 189], the FPDEs with only one dependent variable and two independent variables have been analyzed involving fractional derivative with respect to only one independent variable. The extensions of Lie symmetry method for time fractional systems of PDEs, space-time fractional PDEs and space-time fractional systems of PDEs are presented in chapter 2, chapter 3 and chapter 4 respectively. In the previous chapters, the symmetry method is discussed for fractional PDEs having only two independent variables.

In this chapter, an extension of Lie symmetry approach is proposed and new generalized prolongation formulae are introduced for symmetry analysis of FDEs having an arbitrary number of independent as well as dependent variables. Therefore, the complete group classification of higher dimensional nonlinear systems of FDEs can be successfully done. The higher dimensional systems of FPDEs have been studied through various

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approaches [47, 96] but the idea of symmetry analysis for higher dimensional systems of FDEs is completely unexplored in literature. In the literature, a general result of existence and calculation of Lie symmetries has been presented for fractional order PDEs [119, 120] with only one dependent variable and arbitrary number of independent variables. The present chapter is an extension of the existing Lie group theory to find the symmetries of FDEs having more than one dependent variable. The proposed formulae are applied for Lie group analysis of five nonlinear fractional systems of higher dimensional PDEs.

The rest of this chapter is organized as follows. In Section 5.2, a generalized symmetry approach for systems of FDEs is proposed and the complete derivation of the required prolongation formulae for generalized systems of FDEs having p independent and q dependent variables, is presented. In Section 5.3, the proposed approach is applied to five nonlinear systems of FPDEs including fractional (2+1)-dimensional ANNV system, fractional (3+1)-dimensional Burgers equations, fractional (3+1)-dimensional Navier-Stokes system, fractional (3+1)-dimensional non-hydrostatic Boussinesq system and fractional (3+1)-dimensional incompressible non-hydrostatic Boussinesq system with viscosity. With the help of their obtained Lie symmetries, the considered systems are reduced into lower dimensional systems of fractional order in the extended Erdélyi-Kober fractional differential operators. Section 5.4 is for the concluding remarks of the chapter.

5.2 Generalized Lie Symmetry Approach for Systems of Fractional Differential Equations

Consider a system of N FDEs with p independent variables $\boldsymbol{x} = (x_1, x_2, ..., x_p)$ and q dependent variables $\boldsymbol{u} = (u^1, u^2, ..., u^q)$, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$, $q \geq 2$ in the following form:

$$F_\mu(\boldsymbol{x}, \boldsymbol{u}, {}_{\boldsymbol{a}}D_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{u}, \ldots) = 0, \quad \mu = 1, 2, ..., N, \tag{5.1}$$

such that ${}_{\boldsymbol{a}}D^{\boldsymbol{\alpha}}_{\boldsymbol{x}}\boldsymbol{u}$ denotes the set of all fractional derivatives as follows:

$${}_{a_1,\dots,a_p}D^{\alpha_1,\dots,\alpha_p}_{x_1,\dots,x_p}u^\nu, \quad \nu = 1, 2, ..., q, \tag{5.2}$$

where $\alpha_i \geq 0$ for $i = 1, 2, ..., p$.

Assume the invariance of system (5.1) under one parameter Lie group of transformations of the following form:

$$\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{u}; \epsilon), \quad \tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u}; \epsilon), \tag{5.3}$$

for ϵ being the group parameter. The infinitesimal generator for the transformations is as follows:

$$V = \sum_{i=1}^{p} \xi^{x_i}(\boldsymbol{x}, \boldsymbol{u})\partial_{x_i} + \sum_{\nu=1}^{q} \eta^\nu(\boldsymbol{x}, \boldsymbol{u})\partial_{u^\nu}, \tag{5.4}$$

where

$$\eta^\nu = \left.\frac{\mathrm{d}\tilde{u^\nu}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \text{and } \xi^{x_i} = \left.\frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}\epsilon}\right|_{\epsilon=0}. \tag{5.5}$$

The explicit expression for the prolongations $(\eta^\nu)^{\boldsymbol{\alpha},\boldsymbol{x}} = \left.\left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left({}_{\boldsymbol{a}}D^{\boldsymbol{\alpha}}_{\boldsymbol{x}}\tilde{u^\nu}(\tilde{\boldsymbol{x}})\right)\right]\right|_{\epsilon=0}$ is presented by the following theorem.

Theorem 5.1. *For infinitesimal generator* (5.4)*, the general mixed prolongations* [120] *denoted by*

$$(\eta^\nu)^{\boldsymbol{\alpha},\boldsymbol{x}} = \left.\left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left({}_{\boldsymbol{a}}D^{\boldsymbol{\alpha}}_{\boldsymbol{x}}\tilde{u^\nu}(\tilde{\boldsymbol{x}})\right)\right]\right|_{\epsilon=0} = \left.\left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left({}_{a_1,\dots,a_p}D^{\alpha_1,\dots,\alpha_p}_{x_1,\dots,x_p}\tilde{u^\nu}(\tilde{\boldsymbol{x}})\right)\right]\right|_{\epsilon=0} \tag{5.6}$$

are obtained by the following formula:

$$(\eta^\nu)^{\boldsymbol{\alpha},\boldsymbol{x}} = {}_{\boldsymbol{a}}D^{\boldsymbol{\alpha}}_{\boldsymbol{x}}(\eta^\nu) + \sum_{j=1}^{p}\xi^{x_j}\partial_{x_j}\left({}_{\boldsymbol{a}}D^{\boldsymbol{\alpha}}_{\boldsymbol{x}}(u^\nu)\right) - {}_{\boldsymbol{a}}D^{\boldsymbol{\alpha}}_{\boldsymbol{x}}\Big(\sum_{j=1}^{p}\xi^{x_j}u^\nu_{x_j}\Big), \tag{5.7}$$

where $\nu = 1, 2, ..., q$, $u^\nu_{x_j} = \frac{\partial u^\nu}{\partial x_j}$ *and the expression* ${}_{\boldsymbol{a}}D^{\boldsymbol{\alpha}}_{\boldsymbol{x}} = {}_{a_1,\dots,a_p}D^{\alpha_1,\dots,\alpha_p}_{x_1,\dots,x_p}$.

The proof of the above theorem for $q = 1$ is given in Ref. [120] since the general prolongation formula (5.7) for $q = 1$ coincides with their proposed general formula [120]. In rest of the chapter, for all the fractional derivatives, the lower extreme of Riemann-Liouville definition is taken as $a = 0$ so that ${}_0D^\alpha_x$ is denoted by D^α_x.

The expanded prolongation formulae $(\eta^\nu)^{\alpha_i,x_i} = \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(D^{\alpha_i}_{x_i}\tilde{u^\nu}(\tilde{\boldsymbol{x}})\right)\right]\Big|_{\epsilon=0}$ $\forall i = 1,2,...,p$ are derived in the following theorem:

Theorem 5.2. *The general α_ith extended infinitesimals $(\eta^\nu)^{\alpha_i,x_i} = \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(D^{\alpha_i}_{x_i}\tilde{u^\nu}(\tilde{\boldsymbol{x}})\right)\right]\Big|_{\epsilon=0}$, for $\alpha_i > 0$ ($i = 1,2,...,p$, $\nu = 1,2,...,q$), and general infinitesimal generator* (5.4)*, are proposed as follows:*

$$\begin{aligned}(\eta^\nu)^{\alpha_i,x_i} = {} & \frac{\partial^{\alpha_i}\eta^\nu}{\partial {x_i}^{\alpha_i}} + \sum_{n=1}^{\infty}\left[\binom{\alpha_i}{n}\frac{\partial^n \eta^\nu_{u^\nu}}{\partial {x_i}^n} - \binom{\alpha_i}{n+1}\mathrm{D}^{n+1}_{x_i}\xi^{x_i}\right]\partial^{\alpha_i-n}_{x_i}(u^\nu) \\ & + \sum_{s\neq\nu,s=1}^{q}\sum_{n=1}^{\infty}\binom{\alpha_i}{n}\frac{\partial^n \eta^\nu_{u^s}}{\partial {x_i}^n}\partial^{\alpha_i-n}_{x_i}(u^s) - \sum_{j\neq i}\sum_{n=1}^{\infty}\binom{\alpha_i}{n}\mathrm{D}^n_{x_i}\xi^{x_j}D^{\alpha_i-n}_{x_i}u^\nu_{x_j} \\ & + \left(\eta^\nu_{u^\nu} - \alpha_i\mathrm{D}_{x_i}\xi^{x_i}\right)\partial^{\alpha_i}_{x_i}u^\nu - u^\nu\frac{\partial^{\alpha_i}\eta^\nu_{u^\nu}}{\partial {x_i}^{\alpha_i}} + \sum_{s\neq\nu,s=1}^{q}\left(\eta^\nu_{u^s}\frac{\partial^{\alpha_i}u^s}{\partial {x_i}^{\alpha_i}} - u^s\frac{\partial^{\alpha_i}\eta^\nu_{u^s}}{\partial {x_i}^{\alpha_i}}\right) + \sum_{s=1}^{q}\mu^i_{\nu,s},\end{aligned} \tag{5.8}$$

where $\eta^\nu_{u^s} = \frac{\partial\eta^\nu}{\partial u^s}$ and $\mu^i_{\nu,s}$ are defined by

$$\mu^i_{\nu,s} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha_i}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{{x_i}^{n-\alpha_i}}{\Gamma(n-\alpha_i+1)}(-u^s)^r\frac{\partial^m}{\partial x_i^m}(u^s)^{k-r}\frac{\partial^{n-m+k}\eta^\nu}{(\partial x_i)^{n-m}(\partial u^s)^k}. \tag{5.9}$$

Proof. For the infinitesimal generator (5.4), the prolongation rules $(\eta^\nu)^{\alpha_i,x_i}$ are of the following form:

$$(\eta^\nu)^{\alpha_i,x_i} = D^{\alpha_i}_{x_i}(\eta^\nu) + \sum_{j=1}^{p}\left(\xi^{x_j}D^{\alpha_i}_{x_i}(u^\nu_{x_j}) - D^{\alpha_i}_{x_i}(\xi^{x_j}u^\nu_{x_j})\right). \tag{5.10}$$

Using the generalized Leibniz rule (1.33), the following result can be obtained:

$$\xi^{x_i}D^{\alpha_i}_{x_i}(u^\nu_{x_i}) - D^{\alpha_i}_{x_i}(\xi^{x_i}u^\nu_{x_i}) = -\alpha_i\mathrm{D}_{x_i}\xi^{x_i}\partial^{\alpha_i}_{x_i}u^\nu - \sum_{n=1}^{\infty}\binom{\alpha_i}{n+1}\mathrm{D}^{n+1}_{x_i}\xi^{x_i}D^{\alpha_i-n}_{x_i}u^\nu. \tag{5.11}$$

where D_{x_i} indicate the total derivative operators given by

$$\mathrm{D}_{x_i} = \partial_{x_i} + u^\nu_{x_i}\partial_{u^\nu} + u^\nu_{x_ix_i}\partial_{u^\nu_{x_i}} + \sum_{j\neq i}u^\nu_{x_jx_i}\partial_{u^\nu_{x_j}} + ... \tag{5.12}$$

Equivalently, for $j \neq i$, we have the following result:

$$\xi^{x_j}D^{\alpha_i}_{x_i}(u^\nu_{x_j}) - D^{\alpha_i}_{x_i}(\xi^{x_j}u^\nu_{x_j}) = -\sum_{n=1}^{\infty}\binom{\alpha_i}{n}(\mathrm{D}^n_{x_i}\xi^{x_j})D^{\alpha_i-n}_{x_i}u^\nu_{x_j}. \tag{5.13}$$

Thus (5.10) can be written as follows:

$$\begin{aligned}(\eta^\nu)^{\alpha_i,x_i} = {} & D^{\alpha_i}_{x_i}(\eta^\nu) - \alpha_i(\mathrm{D}_{x_i}\xi^{x_i})\partial^{\alpha_i}_{x_i}u^\nu - \sum_{j\neq i,j=1}^{p}\sum_{n=1}^{\infty}\binom{\alpha_i}{n}(\mathrm{D}^n_{x_i}\xi^{x_j})D^{\alpha_i-n}_{x_i}u^\nu_{x_j} \\ & - \sum_{n=1}^{\infty}\binom{\alpha_i}{n+1}(\mathrm{D}^{n+1}_{x_i}\xi^{x_i})D^{\alpha_i-n}_{x_i}u^\nu.\end{aligned} \tag{5.14}$$

The extended form of formula given by Osler [148] for $\alpha > 0$ is written as follows:

$$D_t^{\alpha} f(t, g_1(t), g_2(t), ..., g_p(t)) = \sum_{j=1}^{p} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{k=0}^{m} \sum_{r=0}^{k} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{t^{n-\alpha}}{k!\Gamma(n+1-\alpha)} (-g_j)^r \frac{d^m}{dt^m} (g_j^{k-r}) \frac{\partial^{n-m+k} f(t, g_1, ..., g_p)}{(\partial t)^{n-m} (\partial g_j)^k}, \tag{5.15}$$

Since $\eta^{\nu} = \eta^{\nu}(x_1, .., x_p, u^1, .., u^q)$, using (5.15) the term $D_{x_i}^{\alpha_i}(\eta^{\nu})$ in (5.14) can be obtained as follows:

$$D_{x_i}^{\alpha_i}(\eta^{\nu}) = \frac{\partial^{\alpha_i} \eta^{\nu}}{\partial {x_i}^{\alpha_i}} + \sum_{s=1}^{q} \left[\left(\eta^{\nu}_{u^s} \frac{\partial^{\alpha_i} u^s}{\partial {x_i}^{\alpha_i}} - u^s \frac{\partial^{\alpha_i} \eta^{\nu}_{u^s}}{\partial {x_i}^{\alpha_i}} \right) + \sum_{n=1}^{\infty} \binom{\alpha_i}{n} \frac{\partial^n \eta^{\nu}_{u^s}}{\partial {x_i}^n} D_{x_i}^{\alpha_i - n}(u^s) + \mu^i_{\nu,s} \right], \tag{5.16}$$

where $\eta^{\nu}_{u^s} = \frac{\partial \eta^{\nu}}{\partial u^s}$ and $\mu^i_{\nu,s}$ denote the following:

$$\mu^i_{\nu,s} = \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \binom{\alpha_i}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{{x_i}^{n-\alpha_i}}{\Gamma(n-\alpha_i+1)} (-u^s)^r \frac{\partial^m}{\partial x_i^m} (u^s)^{k-r} \frac{\partial^{n-m+k} \eta^{\nu}}{\partial {x_i}^{n-m} \partial (u^s)^k}. \tag{5.17}$$

Therefore, the prolongation formulae $(\eta^{\nu})^{\alpha_i, x_i}$ are expressed in the following form:

$$\begin{aligned} (\eta^{\nu})^{\alpha_i, x_i} = {} & \frac{\partial^{\alpha_i} \eta^{\nu}}{\partial {x_i}^{\alpha_i}} + \sum_{n=1}^{\infty} \left[\binom{\alpha_i}{n} \frac{\partial^n \eta^{\nu}_{u^{\nu}}}{\partial {x_i}^n} - \binom{\alpha_i}{n+1} \mathrm{D}_{x_i}^{n+1} \xi^{x_i} \right] \partial_{x_i}^{\alpha_i - n}(u^{\nu}) \\ & + \sum_{s \neq \nu, s=1}^{q} \sum_{n=1}^{\infty} \binom{\alpha_i}{n} \frac{\partial^n \eta^{\nu}_{u^s}}{\partial {x_i}^n} \partial_{x_i}^{\alpha_i - n}(u^s) - \sum_{j \neq i, j=1}^{p} \sum_{n=1}^{\infty} \binom{\alpha_i}{n} \mathrm{D}_{x_i}^{n} \xi^{x_j} D_{x_i}^{\alpha_i - n} u^{\nu}_{x_j} \\ & + \left(\eta^{\nu}_{u^{\nu}} - \alpha_i \mathrm{D}_{x_i} \xi^{x_i} \right) \partial_{x_i}^{\alpha_i} u^{\nu} - u^{\nu} \frac{\partial^{\alpha_i} \eta^{\nu}_{u^{\nu}}}{\partial {x_i}^{\alpha_i}} + \sum_{s \neq \nu, s=1}^{q} \left(\eta^{\nu}_{u^s} \frac{\partial^{\alpha_i} u^s}{\partial {x_i}^{\alpha_i}} - u^s \frac{\partial^{\alpha_i} \eta^{\nu}_{u^s}}{\partial {x_i}^{\alpha_i}} \right) + \sum_{s=1}^{q} \mu^i_{\nu,s}, \end{aligned} \tag{5.18}$$

completing the proof. ☐

Using the proposed symmetry method and new prolongation formulae, the Lie symmetry analysis for systems of FDEs can be performed in the same way as for the integer order systems of differential equations [24, 147].

The prolongation formulae (5.18) for (1+1)-dimensional systems with $p = 2, q = 2$ *i.e.* two independent and arbitrary number of dependent variables can be found in chapter 2 and chapter 4. For $p = 2, q = 1$, the formulae (5.18) are discussed in chapter 3. The prolongation formulae (5.18) can be simplified for the particular cases of systems of FODEs and (2+1)-dimensional systems of FPDEs as follows.

Corollary 5.3. *For the systems of FODEs with q dependent variables $u^{\nu}(x)$, $(\nu = 1, ..., q)$ having the symmetry generator in the following form:*

$$X = \xi\partial_x + \sum_{\nu=1}^{q} \eta^{\nu}\partial_{u^{\nu}}, \tag{5.19}$$

the proposed prolongation formulae are as follows:

$$\begin{aligned}(\eta^{\nu})^{\alpha,x} &= \frac{\partial^{\alpha}\eta^{\nu}}{\partial x^{\alpha}} + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^{n}\eta^{\nu}_{u^{\nu}}}{\partial x^{n}} - \binom{\alpha}{n+1}\mathrm{D}_x^{n+1}\xi\right]\partial_x^{\alpha-n}(u^{\nu}) + \left(\eta^{\nu}_{u^{\nu}} - \alpha\mathrm{D}_x\xi\right)\partial_x^{\alpha}u^{\nu} \\ &+ \sum_{s\neq\nu,s=1}^{q}\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^{n}\eta^{\nu}_{u^{s}}}{\partial x^{n}}\partial_x^{\alpha-n}(u^{s}) - u^{\nu}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(\eta^{\nu}_{u^{\nu}}) + \sum_{s\neq\nu,s=1}^{q}\left(\eta^{\nu}_{u^{s}}\frac{\partial^{\alpha}u^{s}}{\partial x^{\alpha}} - u^{s}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(\eta^{\nu}_{u^{s}})\right) + \sum_{s=1}^{q}\mu^{\alpha}_{\nu,s},\end{aligned} \tag{5.20}$$

where $\mu^{\alpha}_{\nu,s}$ are given by the following expression:

$$\mu^{\alpha}_{\nu,s} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u^{s})^{r}\frac{\partial^{m}}{\partial x^{m}}(u^{s})^{k-r}\frac{\partial^{n-m+k}\eta^{\nu}}{\partial x^{n-m}\partial(u^{s})^{k}}. \tag{5.21}$$

Corollary 5.4. *Consider the $(2+1)$-dimensional systems of FPDEs with $p = 3$, $q = 2$ i.e. dependent variables u, v and independent variables x_1, x_2, x_3 where $x_1 = t$ such that the associated symmetry generator is of the following form:*

$$X = \xi^{x_1}\partial_{x_1} + \xi^{x_2}\partial_{x_2} + \xi^{x_3}\partial_{x_3} + \eta\partial_u + \phi\partial_v. \tag{5.22}$$

The extended symmetry generators η^{α_i,x_i} for $i = 1, 2, 3$ are proposed as follows:

$$\begin{aligned}\eta^{\alpha_i,x_i} &= \partial^{\alpha_i}_{x_i}\eta + \sum_{n=1}^{\infty}\left[\binom{\alpha_i}{n}\partial^{n}_{x_i}\eta_u - \binom{\alpha_i}{n+1}\mathrm{D}^{n+1}_{x_i}\xi^{x_i}\right]\partial^{\alpha_i-n}_{x_i}u + \sum_{n=1}^{\infty}\binom{\alpha}{n}\partial^{n}_{x_i}\eta_v\partial^{\alpha_i-n}_{x_i}v \\ &- \sum_{j\neq i,j=1}^{3}\sum_{n=1}^{\infty}\binom{\alpha_i}{n}\mathrm{D}^{n}_{x_i}\xi^{x_j}\partial^{\alpha_i-n}_{x_i}u_{x_j} + \Big(\eta_u - \alpha_i\mathrm{D}_{x_i}\xi^{x_i}\Big)\partial^{\alpha_i}_{x_i}u + \Big(\eta_v\partial^{\alpha_i}_{x_i}v - v\partial^{\alpha_i}_{x_i}\eta_v\Big) \\ &- u\partial^{\alpha_i}_{x_i}\eta_u + \mu^{i}_{1,1} + \mu^{i}_{1,2},\end{aligned} \tag{5.23}$$

where $\mu^{i}_{1,1}$ $(i = 1, 2, 3)$ are of the following forms:

$$\mu^{i}_{1,1} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha_i}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{{x_i}^{n-\alpha_i}}{\Gamma(n-\alpha_i+1)}(-u)^{r}\frac{\partial^{m}}{\partial {x_i}^{m}}(u)^{k-r}\frac{\partial^{n-m+k}\eta}{\partial {x_i}^{n-m}\partial u^{k}}, \tag{5.24}$$

and $\mu^{i}_{1,2}$ are as follows:

$$\mu^{i}_{1,2} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha_i}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{{x_i}^{n-\alpha_i}}{\Gamma(n-\alpha_i+1)}(-v)^{r}\frac{\partial^{m}}{\partial {x_i}^{m}}(v)^{k-r}\frac{\partial^{n-m+k}\eta}{\partial {x_i}^{n-m}\partial v^{k}}. \tag{5.25}$$

Similarly, ϕ^{α_i,x_i} *are introduced as follows:*

$$\begin{aligned}\phi^{\alpha_i,x_i} = \partial_{x_i}^{\alpha_i}\phi + \sum_{n=1}^{\infty}\left[\binom{\alpha_i}{n}\partial_{x_i}^{n}\phi_v - \binom{\alpha_i}{n+1}\mathrm{D}_{x_i}^{n+1}\xi^{x_i}\right]\partial_{x_i}^{\alpha_i-n}v + \sum_{n=1}^{\infty}\binom{\alpha_i}{n}\partial_{x_i}^{n}\phi_u\partial_{x_i}^{\alpha_i-n}u \\ - \sum_{j\neq i,j=1}^{3}\sum_{n=1}^{\infty}\binom{\alpha_i}{n}\mathrm{D}_{x_i}^{n}\xi^{x_j}\partial_{x_i}^{\alpha_i-n}v_{x_j} + \Big(\phi_v - \alpha_i\mathrm{D}_{x_i}\xi^{x_i}\Big)\partial_{x_i}^{\alpha_i}v + \Big(\phi_u\partial_{x_i}^{\alpha_i}u - u\partial_{x_i}^{\alpha_i}\phi_u\Big) \\ - v\partial_{x_i}^{\alpha_i}\phi_v + \mu_{2,1}^{i} + \mu_{2,2}^{i},\end{aligned} \tag{5.26}$$

where $\mu_{2,1}^{i}$ *and* $\mu_{2,2}^{i}$ *can be defined similar to* (5.24) *and* (5.25)*.*

5.3 Application to Some Higher Dimensional Nonlinear Systems of Fractional Partial Differential Equations

In this section, the proposed symmetry approach is utilized for calculating the Lie point symmetries and reductions systematically for some well known fractional nonlinear systems of higher dimensional PDEs obtained by replacing all the first order derivatives with derivatives of fractional order $0 < \alpha, \beta, \gamma, \delta < 1$. For brevity, the details of Lie symmetries and reductions for some of the discussed systems are avoided.

5.3.1 Fractional 2D ANNV System

The ANNV equations of integer order have been studied for their solutions [174, 198] but the considered system is obtained by taking $k = 2$ in generalized ANNV system given in Ref. [198] and with fractional order derivatives. The fractional ANNV system is of the following form:

$$\begin{aligned}&\partial_t^{\alpha}u + r(u\partial_x^{\beta}v + v\partial_x^{\beta}u) + mu_{xxx} + nu_{yyy} = 0, \\ &\partial_x^{\beta}u - 2v\partial_y^{\gamma}v = 0,\end{aligned} \tag{5.27}$$

where r, m, n are arbitrary constants. For the system (5.27), let the associated symmetry generator be as follows:

$$X = \xi^x(x,y,t,u,v)\partial_x + \xi^y(x,y,t,u,v)\partial_y + \xi^t(x,y,t,u,v)\partial_t + \eta(x,y,t,u,v)\partial_u + \phi(x,y,t,u,v)\partial_v. \tag{5.28}$$

Therefore, the invariance criterion is as follows:

$$\begin{aligned} &[\eta^{\alpha,t} + r(u\phi^{\beta,x} + \eta v_x^{\beta} + v\eta^{\beta,x} + \phi u_x^{\beta}) + m\eta^{xxx} + n\eta^{yyy}]|_{(5.27)} = 0, \\ &[\eta^{\beta,x} - 2v\phi^{\gamma,y} - 2\phi v_y^{\gamma}]|_{(5.27)} = 0, \end{aligned} \tag{5.29}$$

where η^{xxx}, η^{yyy} are extended infinitesimals of order 3 and $\eta^{\alpha,t}$, $\eta^{\beta,x}$, $\phi^{\beta,x}$, $\phi^{\gamma,y}$ are extended infinitesimals of fractional order. Using all the prolongation formulae and equating the coefficients of various monomials in partial derivatives with respect to x, y, t, the set of determining equations is obtained as follows:

$$\begin{aligned} &\xi^x_u = \xi^x_v = \xi^x_t = \xi^x_y = 0, \quad \xi^y_u = \xi^y_v = \xi^y_t = \xi^y_x = 0, \\ &\xi^t_u = \xi^t_v = \xi^t_x = \xi^t_y = 0, \quad 3\xi^x_x = 3\xi^y_y = \alpha\xi^t_t, \\ &\eta_v = \phi_u = 0, \quad \eta_{xu} - \xi^x_{xx} = 0, \\ &\eta_{yu} - \xi^y_{yy} = 0, \quad \eta = u(\eta_u - \phi_v + \beta\xi^x_x - \alpha\xi^t_t), \\ &\phi = v(\beta\xi^x_x - \alpha\xi^t_t) = v(\eta_u - \phi_v - \beta\xi^x_x + \gamma\xi^y_y), \\ &\partial^\beta_x\eta - u\partial^\beta_x\eta_u - 2v(\partial^\gamma_y\phi - v\partial^\gamma_y\phi_v) = 0, \\ &\partial^\alpha_t\eta - u\partial^\alpha_t\eta_u + ru(\partial^\beta_x\phi - v\partial^\beta_x\phi_v) + rv(\partial^\beta_x\eta - u\partial^\beta_x\eta_u) + m\eta_{xxx} + n\eta_{yyy} = 0, \\ &\binom{\alpha}{n}\partial^n_t\eta_u - \binom{\alpha}{n+1}\mathrm{D}^{n+1}_t\xi^t = 0, \\ &\binom{\beta}{n}\partial^n_x\eta_u - \binom{\beta}{n+1}\mathrm{D}^{n+1}_x\xi^x = 0, \\ &\binom{\beta}{n}\partial^n_x\phi_v - \binom{\beta}{n+1}\mathrm{D}^{n+1}_x\xi^x = 0, \\ &\binom{\gamma}{n}\partial^n_y\phi_v - \binom{\gamma}{n+1}\mathrm{D}^{n+1}_y\xi^y = 0, \end{aligned} \tag{5.30}$$

where in the last four equations $n \in \mathbb{N}$. The above set of PDEs and FPDEs can be integrated to obtain the following general solution:

$$\xi^x = \frac{c_1 x}{3} + c_2, \quad \xi^y = \frac{c_1 y}{3} + c_3, \quad \xi^t = \frac{c_1 t}{\alpha} + c_4, \quad \eta = c_1 u\left(\frac{3\beta - \gamma}{3} - 2\right), \quad \phi = c_1 v\left(\frac{\beta}{3} - 1\right), \tag{5.31}$$

where c_i $(i = 1, 2, 3, 4)$ are arbitrary constants. Also, by taking $\alpha = \beta = \gamma = 1$ in the system (5.27), the same set of symmetries can be obtained. So, the generated symmetries are consistent for $0 < \alpha, \beta, \gamma \leq 1$. Further, since the system has fractional derivatives with respect to all independent variables x, y, t, it gives the following conditions for infinitesimals:

$$\xi^x|_{x=0} = 0, \quad \xi^y|_{y=0} = 0, \quad \xi^t|_{t=0} = 0. \tag{5.32}$$

Hence, the infinitesimal generator is as follows:

$$X = \left(\frac{c_1 x}{3}\right)\frac{\partial}{\partial x} + \left(\frac{c_1 y}{3}\right)\frac{\partial}{\partial y} + \left(\frac{c_1 t}{\alpha}\right)\frac{\partial}{\partial t} + c_1 u\left(\frac{3\beta - \gamma}{3} - 2\right)\frac{\partial}{\partial u} + c_1 v\left(\frac{\beta}{3} - 1\right)\frac{\partial}{\partial v}.$$

The corresponding vector field is of the following form:

$$V = \frac{x}{3}\partial_x + \frac{y}{3}\partial_y + \frac{t}{\alpha}\partial_t + \left(\frac{3\beta - \gamma}{3} - 2\right) u\partial_u + \left(\frac{\beta}{3} - 1\right) v\partial_v. \tag{5.33}$$

The associated characteristic equations are as follows:

$$\frac{\mathrm{d}x}{\frac{x}{3}} = \frac{\mathrm{d}y}{\frac{y}{3}} = \frac{\mathrm{d}t}{\frac{t}{\alpha}} = \frac{\mathrm{d}u}{\frac{(3\beta-\gamma-6)}{3}u} = \frac{\mathrm{d}v}{\frac{(\beta-3)}{3}v}. \tag{5.34}$$

Integrating these equations, results in the similarity variables and similarity solutions of the following form:

$$\rho = xt^{-\frac{\alpha}{3}}, \quad \varrho = yt^{-\frac{\alpha}{3}}, \quad F = ut^{-\frac{\alpha(3\beta-\gamma-6)}{3}}, \quad G = vt^{-\frac{\alpha(\beta-3)}{3}}. \tag{5.35}$$

Further, the reduction to the (1+1)-dimensional system of fractional order in variables $F(\rho, \varrho), G(\rho, \varrho)$ is presented in the theorem as follows:

Theorem 5.5. *The similarity transformations* (5.35) *for* $\alpha, \beta, \gamma > 0$ *transform the fractional ANNV equations* (5.27) *into the following nonlinear system of fractional* $(1+1)$-*dimensional PDEs:*

$$\begin{aligned}
&\left(\mathcal{P}^{1+\frac{\alpha}{3}(3\beta-\gamma-9),\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F\right)(\rho,\varrho)+r\rho^{-\beta}\left(G\left(\mathcal{D}^{-\beta,\beta}_{1,\infty}F\right)+F\left(\mathcal{D}^{-\beta,\beta}_{1,\infty}G\right)\right)(\rho,\varrho)+mF_{\rho\rho\rho}+nF_{\varrho\varrho\varrho}=0,\\
&\rho^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{1,\infty}F\right)(\rho,\varrho)-2\varrho^{-\gamma}G\left(\mathcal{D}^{-\gamma,\gamma}_{\infty,1}G\right)(\rho,\varrho)=0,
\end{aligned}\tag{5.36}$$

where $\left(\mathcal{P}^{\zeta,\alpha}_{\delta_1,\delta_2}\right)$ *is the extended left hand sided Erdélyi-Kober fractional differential operator* [107, 128] *defined by*

$$\left(\mathcal{P}^{\zeta,\alpha}_{\delta_1,\delta_2}h\right)(z_1,z_2):=\prod_{j=0}^{m-1}\left(\zeta+j-\frac{1}{\delta_1}z_1\frac{\partial}{\partial z_1}-\frac{1}{\delta_2}z_2\frac{\partial}{\partial z_2}\right)(\mathcal{K}^{\zeta+\alpha,m-\alpha}_{\delta_1,\delta_2}h)(z_1,z_2),\quad z_i>0,\ \delta_i>0,\ \alpha>0,\tag{5.37}$$

$$m=\begin{cases}[\alpha]+1 & \text{if } \alpha\notin\mathbb{N},\\ \alpha & \text{if } \alpha\in\mathbb{N},\end{cases}$$

where

$$\left(\mathcal{K}^{\zeta,\alpha}_{\delta_1,\delta_2}h\right)(z_1,z_2):=\begin{cases}\frac{1}{\Gamma(\alpha)}\int\limits_1^\infty(s-1)^{\alpha-1}s^{-(\zeta+\alpha)}h(z_1s^{\frac{1}{\delta_1}},z_2s^{\frac{1}{\delta_2}})\mathrm{d}s & \text{if } \alpha>0,\\ h(z_1,z_2) & \text{if } \alpha=0\end{cases}\tag{5.38}$$

is the extended left-hand sided Erdélyi-Kober fractional integral operator. Also, $\left(\mathcal{D}^{\zeta,\beta}_{\delta_1,\delta_2}\right)$ *is the extended right hand sided Erdélyi-Kober fractional differential operator* [107, 128] *defined as follows:*

$$\left(\mathcal{D}^{\zeta,\beta}_{\delta_1,\delta_2}h\right)(z_1,z_2):=\prod_{j=1}^{m}\left(\zeta+j+\frac{1}{\delta_1}z_1\frac{\partial}{\partial z_1}+\frac{1}{\delta_2}z_2\frac{\partial}{\partial z_2}\right)(\mathcal{I}^{\zeta+\beta,m-\beta}_{\delta_1,\delta_2}h)(z_1,z_2),\quad z_i>0,\ \delta_i>0,\ \beta>0,\tag{5.39}$$

$$m=\begin{cases}[\beta]+1 & \text{if } \beta\notin\mathbb{N},\\ \beta & \text{if } \beta\in\mathbb{N},\end{cases}$$

where

$$\left(\mathcal{I}^{\zeta,\beta}_{\delta_1,\delta_2}h\right)(z_1,z_2):=\begin{cases}\frac{1}{\Gamma(\beta)}\int\limits_0^1(1-s)^{\beta-1}s^{\zeta}h(z_1s^{\frac{1}{\delta_1}},z_2s^{\frac{1}{\delta_2}})\mathrm{d}s & \text{if } \beta>0,\\ h(z_1,z_2) & \text{if } \beta=0\end{cases}\tag{5.40}$$

is the extended right-hand sided Erdélyi-Kober fractional integral operator.

Proof. The proof is partitioned into the three lemmas in which the required partial derivatives of $u(x, y, t)$ and $v(x, y, t)$ of orders $\alpha, \beta, \gamma > 0$ are described:

Lemma 5.6. *The partial derivative $\partial_t^\alpha u$ of fractional order $\alpha > 0$ is obtained by the following expression:*

$$\partial_t^\alpha u = t^{\frac{\alpha}{3}(3\beta-\gamma-9)} \left(\mathcal{P}_{\frac{3}{\alpha},\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(3\beta-\gamma-9),\alpha} F\right)(\rho, \varrho), \tag{5.41}$$

with the extended left-hand sided Erdélyi-Kober fractional differential operator $\left(\mathcal{P}_{\delta_1,\delta_2}^{\zeta,\alpha}\right)$ *defined by* (5.37) *and* (5.38).

Proof. First of all, in the case $n - 1 < \alpha < n$ for $n \in \mathbb{N}$, the Riemann-Liouville fractional derivative $\partial_t^\alpha u$ is as follows:

$$\partial_t^\alpha u = \frac{\partial^n}{\partial t^n} \left(\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \tau^{\frac{\alpha}{3}(3\beta-\gamma-6)} F(x\tau^{-\frac{\alpha}{3}}, y\tau^{-\frac{\alpha}{3}}) \mathrm{d}\tau \right). \tag{5.42}$$

Taking $p = \frac{t}{\tau}$, it can be written as follows:

$$\partial_t^\alpha u = \frac{\partial^n}{\partial t^n} \left(t^{n-\alpha+\frac{\alpha}{3}(3\beta-\gamma-6)} \left(\mathcal{K}_{\frac{3}{\alpha},\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(3\beta-\gamma-6),n-\alpha} F\right)(\rho, \varrho) \right), \tag{5.43}$$

where

$$\left(\mathcal{K}_{\frac{3}{\alpha},\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(3\beta-\gamma-6),n-\alpha} F\right)(\rho, \varrho) = \frac{1}{\Gamma(n-\alpha)} \int_1^\infty (p-1)^{n-\alpha-1} p^{-(n-\alpha+1+\frac{\alpha}{3}(3\beta-\gamma-6))} F(\rho p^{\frac{\alpha}{3}}, \varrho p^{\frac{\alpha}{3}}) \mathrm{d}p. \tag{5.44}$$

The result (5.43) holds also for $\alpha = n \in \mathbb{N}$, because the definition (5.38) implies the following:

$$\left(\mathcal{K}_{\frac{3}{\alpha},\frac{3}{\alpha}}^{1+\frac{\alpha}{3}(3\beta-\gamma-6),n-\alpha} F\right)(\rho, \varrho) = F(\rho, \varrho). \tag{5.45}$$

For $\rho = \rho(x, t)$ and $\varrho = \varrho(y, t)$ given in (5.35), the chain rule for differentiable functions of the form $\phi(\rho, \varrho)$ gives the result given by

$$t \frac{\partial}{\partial t} \phi(\rho, \varrho) = -\frac{\alpha}{3} \rho \frac{\partial \phi}{\partial \rho} - \frac{\alpha}{3} \varrho \frac{\partial \phi}{\partial \varrho}. \tag{5.46}$$

In view of (5.46), the right hand side of (5.43) can be obtained as follows:

$$
\begin{aligned}
&\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha+\frac{\alpha}{3}(3\beta-\gamma-6)-1}\left(n-\alpha+\frac{\alpha(3\beta-\gamma-6)}{3}-\frac{\alpha}{3}\rho\frac{\partial}{\partial\rho}-\frac{\alpha}{3}\varrho\frac{\partial}{\partial\varrho}\right)\left(\mathcal{K}^{1+\frac{\alpha}{3}(3\beta-\gamma-6),n-\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F\right)(\rho,\varrho)\right],\\
&=\cdots\\
&=t^{-\alpha+\frac{\alpha}{3}(3\beta-\gamma-6)}\prod_{j=0}^{n}\left(1+j-\alpha+\frac{\alpha}{3}(3\beta-\gamma-6)-\frac{\alpha}{3}\rho\frac{\partial}{\partial\rho}-\frac{\alpha}{3}\varrho\frac{\partial}{\partial\varrho}\right)\left(\mathcal{K}^{1+\frac{\alpha}{3}(3\beta-\gamma-6),n-\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F\right)(\rho,\varrho),\\
&=t^{\frac{\alpha}{3}(3\beta-\gamma-9)}\left(\mathcal{P}^{1+\frac{\alpha}{3}(3\beta-\gamma-9),\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F\right)(\rho,\varrho).
\end{aligned}
\tag{5.47}
$$

Hence, completing the proof of Lemma. □

The following lemmas can be proved in the equivalent manner:

Lemma 5.7. *The partial derivatives of fractional order* $\beta > 0$ *i.e.* $\frac{\partial^\beta u}{\partial x^\beta}$ *and* $\frac{\partial^\beta v}{\partial x^\beta}$ *are derived by the following expressions:*

$$
\begin{aligned}
\partial^\beta_x u &= t^{\frac{\alpha}{3}(3\beta-\gamma-6)}x^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{1,\infty}F\right)(\rho,\varrho),\\
\partial^\beta_x v &= t^{\frac{\alpha}{3}(\beta-3)}x^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{1,\infty}G\right)(\rho,\varrho),
\end{aligned}
\tag{5.48}
$$

with the extended right-hand sided Erdélyi-Kober fractional differential operator $\left(\mathcal{D}^{\zeta,\beta}_{\delta_1,\delta_2}\right)$ *defined by* (5.39) *and* (5.40).

Lemma 5.8. *The partial derivative of order* $\gamma > 0$ *namely* $\frac{\partial^\gamma v}{\partial x^\gamma}$ *is as follows:*

$$
\partial^\gamma_y v = t^{\frac{\alpha}{3}(\beta-3)}y^{-\gamma}\left(\mathcal{D}^{-\gamma,\gamma}_{\infty,1}G\right)(\rho,\varrho),
\tag{5.49}
$$

where $\left(\mathcal{D}^{\zeta,\beta}_{\delta_1,\delta_2}\right)$ *is the the extended right-hand sided Erdélyi-Kober fractional differential operator defined by* (5.39) *and* (5.40).

Using the Lemmas given before, it can be concluded that the ANNV system (5.27) reduces into the system (5.36) of fractional (1+1)-dimensional PDEs, completing the proof of theorem. □

5.3.2 Fractional 3D Burgers System

The studied fractional Burgers system is of the following form:

$$\begin{aligned}
&\partial_t^{\alpha} u - 2u\partial_y^{\gamma} u - 2v\partial_x^{\beta} u - 2w\partial_z^{\delta} u - u_{xx} - u_{yy} - u_{zz} = 0,\\
&\partial_x^{\beta} u - \partial_y^{\gamma} v = 0,\\
&\partial_z^{\delta} u - \partial_y^{\gamma} w = 0.
\end{aligned} \tag{5.50}$$

The (3+1)-dimensional Burgers system (5.50) for $\alpha = \beta = \gamma = \delta = 1$ has already been investigated for their symmetries [131] and for conservation laws [1]. Here, the point symmetries for fractional Burgers system (5.50), in case of $0 < \alpha, \beta, \gamma, \delta < 1$, are going to be derived. Let the system (5.50) admits the group of transformations with the following symmetry generator:

$$\begin{aligned}
V = {} & \xi^x(x,y,z,t,u,v,w)\partial_x + \xi^y(x,y,z,t,u,v,w)\partial_y + \xi^z(x,y,z,t,u,v,w)\partial_z + \tau(x,y,z,t,u,v,w)\partial_t\\
& + \eta(x,y,z,t,u,v,w)\partial_u + \phi(x,y,z,t,u,v,w)\partial_v + \psi(x,y,z,t,u,v,w)\partial_w.
\end{aligned} \tag{5.51}$$

The invariance criterion takes the following form:

$$\begin{aligned}
&[\eta^{\alpha,t} - 2u\eta^{\gamma,y} - 2\eta\partial_y^{\gamma} u - 2v\eta^{\beta,x} - 2\phi\partial_x^{\beta} u - 2w\eta^{\delta,z} - 2\psi\partial_z^{\delta} u - \eta^{xx} - \eta^{yy} - \eta^{zz}]|_{(5.50)} = 0,\\
&[\eta^{\beta,x} - \phi^{\gamma,y}]|_{(5.50)} = 0,\\
&[\eta^{\delta,z} - \psi^{\gamma,y}]|_{(5.50)} = 0.
\end{aligned} \tag{5.52}$$

Inserting the prolongation formulae and equating the coefficients of linearly independent variables to zero, the over-determined set of FDEs and PDEs can be obtained. Solving the set of determining equations, the derived symmetries are as follows:

$$\begin{aligned}
&\xi^x = \frac{c_1 x}{2} + c_2, \quad \xi^y = \frac{c_1 y}{2} + c_3, \quad \xi^z = \frac{c_1 z}{2} + c_4, \quad \tau = \frac{c_1 t}{\alpha} + c_5,\\
&\eta = c_1 u\left(\frac{\gamma - 2}{2}\right), \quad \phi = c_1 v\left(\frac{\beta - 2}{2}\right), \quad \psi = c_1 w\left(\frac{\delta - 2}{2}\right),
\end{aligned} \tag{5.53}$$

where c_i $(i = 1, 2, 3, 4, 5)$ are arbitrary constants. The symmetries reported in Ref. [131] in case of $\alpha = \beta = \gamma = \delta = 1$ for the system (5.50) are different. In fact, the symmetries (5.53) are a particular case of those obtained symmetries. Continuing further, the conditions $\xi^x(0) = \xi^y(0) = \xi^z(0) = \tau(0) = 0$ imply that the symmetry generator is as follows:

$$V = \frac{x}{2}\partial_x + \frac{y}{2}\partial_y + \frac{z}{2}\partial_z + \frac{t}{\alpha}\partial_t + u\left(\frac{\beta-2}{2}\right)\partial_u + v\left(\frac{\beta-2}{2}\right)\partial_v + w\left(\frac{\beta-2}{2}\right)\partial_w, \quad (5.54)$$

since for nontrivial solutions, $c_1 \neq 0$ implies $\beta = \gamma = \delta$. The associated similarity solutions are obtained in the following form:

$$\begin{aligned} &\theta_1 = xt^{-\frac{\alpha}{2}}, \quad \theta_2 = yt^{-\frac{\alpha}{2}}, \quad \theta_3 = zt^{-\frac{\alpha}{2}}, \quad F = ut^{-\frac{\alpha(\beta-2)}{2}}, \\ &G = vt^{-\frac{\alpha(\beta-2)}{2}}, \quad H = wt^{-\frac{\alpha(\beta-2)}{2}}. \end{aligned} \quad (5.55)$$

Using these solutions, the Burgers system can be converted into a system of (2+1)-dimensional FPDEs presented in the following theorem:

Theorem 5.9. *The system* (5.50) *of FPDEs can be transformed into the following* (2+1)*-dimensional fractional nonlinear system as follows:*

$$\begin{aligned} &\left(\mathcal{P}^{1+\frac{\alpha(\beta-4)}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}} F\right)(\theta_1,\theta_2,\theta_3) - 2F\theta_2^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty} F\right)(\theta_1,\theta_2,\theta_3) - 2G\theta_1^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty} F\right)(\theta_1,\theta_2,\theta_3) \\ &- 2H\theta_3^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1} F\right)(\theta_1,\theta_2,\theta_3) - F_{\theta_1\theta_1} - F_{\theta_2\theta_2} - F_{\theta_3\theta_3} = 0, \\ &\theta_1^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty} F\right)(\theta_1,\theta_2,\theta_3) - \theta_2^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty} G\right)(\theta_1,\theta_2,\theta_3) = 0, \\ &\theta_3^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1} F\right)(\theta_1,\theta_2,\theta_3) - \theta_2^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty} H\right)(\theta_1,\theta_2,\theta_3) = 0, \end{aligned} \quad (5.56)$$

where $\left(\mathcal{P}^{\zeta,\alpha}_{\delta_1,\delta_2,\delta_3}\right)$ *is the extended left hand sided Erdélyi-Kober fractional differential operator* [107, 128] *defined as follows:*

$$\left(\mathcal{P}^{\zeta,\alpha}_{\delta_1,\delta_2,\delta_3} h\right)(z_1,z_2,z_3) := \prod_{j=0}^{m-1}\left(\zeta + j - \frac{1}{\delta_1}z_1\frac{\partial}{\partial z_1} - \frac{1}{\delta_2}z_2\frac{\partial}{\partial z_2} - \frac{1}{\delta_3}z_3\frac{\partial}{\partial z_3}\right)(\mathcal{K}^{\zeta+\alpha,m-\alpha}_{\delta_1,\delta_2,\delta_3} h)(z_1,z_2,z_3), \quad (5.57)$$

for $z_i > 0$, $\delta_i > 0$, $\alpha > 0$ *and*

$$m = \begin{cases} [\alpha] + 1 & \text{if } \alpha \notin \mathbb{N}, \\ \alpha & \text{if } \alpha \in \mathbb{N}, \end{cases}$$

where

$$\left(\mathcal{K}^{\zeta,\alpha}_{\delta_1,\delta_2,\delta_3}h\right)(z_1,z_2,z_3) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int\limits_1^\infty (s-1)^{\alpha-1}s^{-(\zeta+\alpha)}h(z_1 s^{\frac{1}{\delta_1}}, z_2 s^{\frac{1}{\delta_2}}, z_3 s^{\frac{1}{\delta_3}})\mathrm{d}s & \textit{if } \alpha > 0, \\ h(z_1,z_2,z_3) & \textit{if } \alpha = 0 \end{cases} \tag{5.58}$$

is the extended left-hand sided Erdélyi-Kober fractional integral operator. Also, $\left(\mathcal{D}^{\zeta,\beta}_{\delta_1,\delta_2,\delta_3}\right)$ *is the extended right hand sided Erdélyi-Kober fractional differential operator* [107, 128] *defined by*

$$\left(\mathcal{D}^{\zeta,\beta}_{\delta_1,\delta_2,\delta_3}h\right)(z_1,z_2,z_3) := \prod_{j=1}^{m}\left(\zeta + j + \frac{1}{\delta_1}z_1\frac{\partial}{\partial z_1} + \frac{1}{\delta_2}z_2\frac{\partial}{\partial z_2} + \frac{1}{\delta_3}z_3\frac{\partial}{\partial z_3}\right)(\mathcal{I}^{\zeta+\beta,m-\beta}_{\delta_1,\delta_2,\delta_3}h)(z_1,z_2,z_3), \tag{5.59}$$

for $z_i > 0$, $\delta_i > 0$, $\beta > 0$ *and*

$$m = \begin{cases} [\beta]+1 & \textit{if } \beta \notin \mathbb{N}, \\ \beta & \textit{if } \beta \in \mathbb{N}, \end{cases}$$

where

$$\left(\mathcal{I}^{\zeta,\beta}_{\delta_1,\delta_2,\delta_3}h\right)(z_1,z_2,z_3) := \begin{cases} \frac{1}{\Gamma(\beta)}\int\limits_0^1 (1-s)^{\beta-1}s^{\zeta}h(z_1 s^{\frac{1}{\delta_1}}, z_2 s^{\frac{1}{\delta_2}}, z_3 s^{\frac{1}{\delta_3}})\mathrm{d}s & \textit{if } \beta > 0, \\ h(z_1,z_2,z_3) & \textit{if } \beta = 0 \end{cases} \tag{5.60}$$

is the extended right-hand sided Erdélyi-Kober fractional integral operator.

Proof. The proof is similar to Theorem 5.5. □

5.3.3 Fractional 3D Navier-Stokes System

In this subsection, the primary focus is on implementing the Lie group method for determining the point symmetries for the fractional order Navier-Stokes system as follows:

$$\begin{aligned} &\partial_t^\alpha u + u\partial_x^\beta u + v\partial_y^\gamma u + w\partial_z^\delta u + \partial_x^\beta p - \mu(u_{xx}+u_{yy}+u_{zz}) = 0, \\ &\partial_t^\alpha v + u\partial_x^\beta v + v\partial_y^\gamma v + w\partial_z^\delta v + \partial_y^\gamma p - \mu(v_{xx}+v_{yy}+v_{zz}) = 0, \\ &\partial_t^\alpha w + u\partial_x^\beta w + v\partial_y^\gamma w + w\partial_z^\delta w + \partial_z^\delta p - \mu(w_{xx}+w_{yy}+w_{zz}) = 0, \\ &\partial_x^\beta u + \partial_y^\gamma v + \partial_z^\delta w = 0. \end{aligned} \tag{5.61}$$

The symmetry analysis of the Navier-Stokes system (5.61) for $\alpha = \beta = \gamma = \delta = 1$ is very well known [29]. For fractional Navier-Stokes system (5.61), assume the admitted symmetry generator is of the following form:

$$\begin{aligned} V = {} & \xi^x(x,y,z,t,u,v,w,p)\partial_x + \xi^y(x,y,z,t,u,v,w,p)\partial_y + \xi^z(x,y,z,t,u,v,w,p)\partial_z \\ & + \tau(x,y,z,t,u,v,w,p)\partial_t + \eta_1(x,y,z,t,u,v,w,p)\partial_u + \eta_2(x,y,z,t,u,v,w,p)\partial_v \\ & + \eta_3(x,y,z,t,u,v,w,p)\partial_w + \eta_4(x,y,z,t,u,v,w,p)\partial_p. \end{aligned} \tag{5.62}$$

The invariance criterion can be written as follows:

$$\begin{aligned} & [\eta_1^{\alpha,t} + u\eta_1^{\beta,x} + \eta_1\partial_x^\beta u + v\eta_1^{\gamma,y} + \eta_2\partial_y^\gamma u + w\eta_1^{\delta,z} + \eta_3\partial_z^\delta u + \eta_4^{\beta,x} - \mu(\eta_1^{xx} + \eta_1^{yy} + \eta_1^{zz})]|_{(5.61)} = 0, \\ & [\eta_2^{\alpha,t} + u\eta_2^{\beta,x} + \eta_1\partial_x^\beta v + v\eta_2^{\gamma,y} + \eta_2\partial_y^\gamma v + w\eta_2^{\delta,z} + \eta_3\partial_z^\delta v + \eta_4^{\gamma,y} - \mu(\eta_2^{xx} + \eta_2^{yy} + \eta_2^{zz})]|_{(5.61)} = 0, \\ & [\eta_3^{\alpha,t} + u\eta_3^{\beta,x} + \eta_1\partial_x^\beta w + v\eta_3^{\gamma,y} + \eta_2\partial_y^\gamma w + w\eta_3^{\delta,z} + \eta_3\partial_z^\delta w + \eta_4^{\delta,z} - \mu(\eta_3^{xx} + \eta_3^{yy} + \eta_3^{zz})]|_{(5.61)} = 0, \\ & [\eta_1^{\beta,x} + \eta_2^{\gamma,y} + \eta_3^{\delta,z}]|_{(5.61)} = 0. \end{aligned} \tag{5.63}$$

After substituting the prolongation expressions and equating the various coefficients results in the following set of determining equations:

$$\begin{aligned} & \xi^x_t = \xi^x_y = \xi^x_z = \xi^x_u = \xi^x_v = \xi^x_w = \xi^x_p = 0, \\ & \xi^y_t = \xi^y_x = \xi^y_z = \xi^y_u = \xi^y_v = \xi^y_w = \xi^y_p = 0, \\ & \xi^z_t = \xi^z_x = \xi^z_y = \xi^z_u = \xi^x_v = \xi^x_w = \xi^x_p = 0, \\ & \tau_x = \tau_y = \tau_z = \tau_u = \tau_v = \tau_w = \tau_p = 0, \\ & 2\xi^x_x - \alpha\tau_t = 0, \quad 2\xi^y_y - \alpha\tau_t = 0, \quad 2\xi^z_z - \alpha\tau_t = 0, \\ & \eta_{1,v} = \eta_{1,w} = \eta_{1,p} = 0, \quad \eta_{2,u} = \eta_{2,w} = \eta_{2,p} = 0, \\ & \eta_{3,u} = \eta_{3,v} = \eta_{3,p} = 0, \quad \eta_{4,u} = \eta_{4,v} = \eta_{4,w} = 0, \\ & \eta_{1,uu} = \eta_{2,vv} = \eta_{3,ww} = 0, \quad 2\eta_{1,xu} = 2\eta_{2,xv} = 2\eta_{3,xw} = \xi^x_{xx}, \\ & 2\eta_{1,yu} = 2\eta_{2,yv} = 2\eta_{3,yw} = \xi^y_{yy}, \end{aligned}$$

$$
\begin{aligned}
&2\eta_{1,zu} = 2\eta_{2,zv} = 2\eta_{3,zw} = \xi^z_{zz},\\
&\eta_1 = u(\beta\xi^x_x - \alpha\tau_t), \quad \eta_2 = v(\gamma\xi^y_y - \alpha\tau_t),\\
&\eta_3 = w(\delta\xi^z_z - \alpha\tau_t), \quad \eta_{4,p} = 2(\beta\xi^x_x - \alpha\tau_t),\\
&\eta_{4,p} = 2(\gamma\xi^y_y - \alpha\tau_t), \quad \eta_{4,p} = 2(\delta\xi^z_z - \alpha\tau_t),\\
&\binom{\alpha}{n}\partial^n_t\eta_{1,u} = \binom{\alpha}{n}\partial^n_t\eta_{2,v} = \binom{\alpha}{n}\partial^n_t\eta_{3,w} = \binom{\alpha}{n+1}\mathrm{D}^{n+1}_t\tau, \quad n \in \mathbb{N},\\
&\binom{\beta}{n}\partial^n_x\eta_{1,u} = \binom{\beta}{n}\partial^n_x\eta_{2,v} = \binom{\beta}{n}\partial^n_x\eta_{3,w} = \binom{\beta}{n}\partial^n_x\eta_{4,p} = \binom{\beta}{n+1}\mathrm{D}^{n+1}_x\xi^x, \quad n \in \mathbb{N},\\
&\binom{\gamma}{n}\partial^n_y\eta_{1,u} = \binom{\gamma}{n}\partial^n_y\eta_{2,v} = \binom{\gamma}{n}\partial^n_y\eta_{3,w} = \binom{\gamma}{n}\partial^n_y\eta_{4,p} = \binom{\gamma}{n+1}\mathrm{D}^{n+1}_y\xi^y, \quad n \in \mathbb{N},\\
&\binom{\delta}{n}\partial^n_z\eta_{1,u} = \binom{\delta}{n}\partial^n_z\eta_{2,v} = \binom{\delta}{n}\partial^n_z\eta_{3,w} = \binom{\delta}{n}\partial^n_z\eta_{4,p} = \binom{\delta}{n+1}\mathrm{D}^{n+1}_z\xi^z, \quad n \in \mathbb{N},\\
&\partial^\alpha_t\eta_1 - u\partial^\alpha_t\eta_{1,u} + u\left(\partial^\beta_x\eta_1 - u\partial^\beta_x\eta_{1,u}\right) + v\left(\partial^\gamma_y\eta_1 - u\partial^\gamma_y\eta_{1,u}\right) + w\left(\partial^\delta_z\eta_1 - u\partial^\delta_z\eta_{1,u}\right)\\
&+\left(\partial^\beta_x\eta_4 - p\partial^\beta_x\eta_{4,p}\right) - \mu\left(\eta_{1,xx} + \eta_{1,yy} + \eta_{1,zz}\right) = 0,\\
&\partial^\alpha_t\eta_2 - v\partial^\alpha_t\eta_{2,v} + u\left(\partial^\beta_x\eta_2 - v\partial^\beta_x\eta_{2,v}\right) + v\left(\partial^\gamma_y\eta_2 - v\partial^\gamma_y\eta_{2,v}\right) + w\left(\partial^\delta_z\eta_2 - v\partial^\delta_z\eta_{2,v}\right)\\
&+\left(\partial^\gamma_y\eta_4 - p\partial^\gamma_y\eta_{4,p}\right) - \mu\left(\eta_{2,xx} + \eta_{2,yy} + \eta_{2,zz}\right) = 0,\\
&\partial^\alpha_t\eta_3 - w\partial^\alpha_t\eta_{3,w} + u\left(\partial^\beta_x\eta_3 - w\partial^\beta_x\eta_{3,w}\right) + v\left(\partial^\gamma_y\eta_3 - w\partial^\gamma_y\eta_{3,w}\right) + w\left(\partial^\delta_z\eta_3 - w\partial^\delta_z\eta_{3,w}\right)\\
&+\left(\partial^\delta_z\eta_4 - p\partial^\delta_z\eta_{4,p}\right) - \mu\left(\eta_{3,xx} + \eta_{3,yy} + \eta_{3,zz}\right) = 0,\\
&\partial^\beta_x\eta_1 - u\partial^\beta_x\eta_{1,u} + \partial^\gamma_y\eta_2 - v\partial^\gamma_y\eta_{2,v} + \partial^\delta_z\eta_3 - w\partial^\delta_z\eta_{3,w} = 0.
\end{aligned}
\tag{5.64}
$$

On solving these equations, the infinitesimals can be derived in the following form:

$$
\begin{aligned}
&\xi^x = \frac{c_1 x}{2} + c_2, \quad \xi^y = \frac{c_1 y}{2} + c_3, \quad \xi^z = \frac{c_1 z}{2} + c_4, \quad \tau = \frac{c_1 t}{\alpha} + c_5, \quad \eta_1 = \frac{c_1 u(\beta - 2)}{2},\\
&\eta_2 = \frac{c_1 v(\gamma - 2)}{2}, \quad \eta_3 = \frac{c_1 w(\delta - 2)}{2}, \quad \eta_4 = c_1 p(\beta - 2) + \frac{x^{\beta-1}y^{\gamma-1}z^{\delta-1}A(t)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)},
\end{aligned}
\tag{5.65}
$$

where c_i $(i = 1, 2, 3, 4, 5)$ are arbitrary constants, $A(t)$ being an arbitrary function of t. Apparently, the generated symmetries (5.65) for $0 < \alpha, \beta, \gamma, \delta < 1$ are particular case of the symmetries for integer order Navier-Stokes system [29]. Using the conditions

$\xi^x(0) = \xi^y(0) = \xi^z(0) = \tau(0) = 0$, the symmetry generators of the Lie group for system (5.61) are as follows:

$$\begin{aligned} V_1 &= \frac{x}{2}\partial_x + \frac{y}{2}\partial_y + \frac{z}{2}\partial_z + \frac{t}{\alpha}\partial_t + \frac{u(\beta-2)}{2}\partial_u + \frac{v(\beta-2)}{2}\partial_v + \frac{w(\beta-2)}{2}\partial_w + p(\beta-2)\partial_p, \\ V_\infty &= \frac{x^{\beta-1}y^{\beta-1}z^{\beta-1}A(t)}{(\Gamma(\beta))^3}, \end{aligned} \tag{5.66}$$

because for nontrivial solutions $c_1 \neq 0$ gives $\beta = \gamma = \delta$. Next the reductions for vector field V_1 are discussed. The associated characteristic equations for V_1 are of the following form:

$$\frac{\mathrm{d}x}{\frac{x}{2}} = \frac{\mathrm{d}y}{\frac{y}{2}} = \frac{\mathrm{d}z}{\frac{z}{2}} = \frac{\mathrm{d}t}{\frac{t}{\alpha}} = \frac{\mathrm{d}u}{\frac{u(\beta-2)}{2}} = \frac{\mathrm{d}v}{\frac{v(\beta-2)}{2}} = \frac{\mathrm{d}w}{\frac{w(\beta-2)}{2}} = \frac{\mathrm{d}p}{p(\beta-2)}. \tag{5.67}$$

Integrating these equations, yields the following invariant solutions:

$$\begin{aligned} &\sigma_1 = xt^{-\frac{\alpha}{2}}, \quad \sigma_2 = yt^{-\frac{\alpha}{2}}, \quad \sigma_3 = zt^{-\frac{\alpha}{2}}, \quad F = ut^{-\frac{\alpha(\beta-2)}{2}}, \\ &G = vt^{-\frac{\alpha(\beta-2)}{2}}, \quad H = wt^{-\frac{\alpha(\beta-2)}{2}}, \quad J = pt^{-\alpha(\beta-2)}. \end{aligned} \tag{5.68}$$

Theorem 5.10. *The similarity transformations* (5.68) *reduce the system* (5.61) *into the following nonlinear system of* $(2+1)$*-dimensional FPDEs:*

$$\begin{aligned} &\left(\mathcal{P}^{1+\frac{\alpha(\beta-4)}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}}F\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_1^{-\beta}F\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty}F\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_2^{-\beta}G\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty}F\right)(\sigma_1,\sigma_2,\sigma_3) \\ &\quad + \sigma_3^{-\beta}H\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1}F\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_1^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty}J\right)(\sigma_1,\sigma_2,\sigma_3) - \mu\left(F_{\sigma_1\sigma_1} + F_{\sigma_2\sigma_2} + F_{\sigma_3\sigma_3}\right) = 0, \\ &\left(\mathcal{P}^{1+\frac{\alpha(\beta-4)}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}}G\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_1^{-\beta}F\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty}G\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_2^{-\beta}G\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty}G\right)(\sigma_1,\sigma_2,\sigma_3) \\ &\quad + \sigma_3^{-\beta}H\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1}G\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_2^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty}J\right)(\sigma_1,\sigma_2,\sigma_3) - \mu\left(G_{\sigma_1\sigma_1} + G_{\sigma_2\sigma_2} + G_{\sigma_3\sigma_3}\right) = 0, \\ &\left(\mathcal{P}^{1+\frac{\alpha(\beta-4)}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}}H\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_1^{-\beta}F\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty}H\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_2^{-\beta}G\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty}H\right)(\sigma_1,\sigma_2,\sigma_3) \\ &\quad + \sigma_3^{-\beta}H\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1}H\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_3^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1}J\right)(\sigma_1,\sigma_2,\sigma_3) - \mu\left(H_{\sigma_1\sigma_1} + H_{\sigma_2\sigma_2} + H_{\sigma_3\sigma_3}\right) = 0, \\ &\sigma_1^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty}F\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_2^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty}G\right)(\sigma_1,\sigma_2,\sigma_3) + \sigma_3^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1}H\right)(\sigma_1,\sigma_2,\sigma_3) = 0, \end{aligned} \tag{5.69}$$

where $\left(\mathcal{P}^{\zeta,\alpha}_{\delta_1,\delta_2,\delta_3}\right)$ *and* $\left(\mathcal{D}^{\zeta,\beta}_{\delta_1,\delta_2,\delta_3}\right)$ *are the extended left and right hand sided Erdélyi-Kober fractional differential operators defined by* (5.57) *and* (5.59) *respectively.*

Proof. The proof is similar to Theorem 5.5. □

5.3.4 Fractional 3D Incompressible Non-Hydrostatic Boussinesq Equations

The fractional (3+1)-dimensional nonlinear incompressible non-hydrostatic Boussinesq equations are as follows [156]:

$$\begin{aligned}
&\partial_t^\alpha u + u\partial_x^\beta u + v\partial_y^\gamma u + w\partial_z^\delta u + \frac{1}{a}\partial_x^\beta p = 0,\\
&\partial_t^\alpha v + u\partial_x^\beta v + v\partial_y^\gamma v + w\partial_z^\delta v + \frac{1}{a}\partial_y^\gamma p = 0,\\
&\partial_t^\alpha w + u\partial_x^\beta w + v\partial_y^\gamma w + w\partial_z^\delta w + \frac{1}{a}\partial_z^\delta p - \frac{gT}{b} = 0,\\
&\partial_t^\alpha T + u\partial_x^\beta T + v\partial_y^\gamma T + w\partial_z^\delta T + (\Gamma_d - \Gamma)w = 0,\\
&\partial_x^\beta u + \partial_y^\gamma v + \partial_z^\delta w = 0,
\end{aligned} \tag{5.70}$$

where $a, g, b, \Gamma_d, \Gamma$ are constants [156]. Let the system (5.70) admits Lie group of transformations with the following symmetry generator:

$$\begin{aligned}
V = {}&\xi^x(x,y,z,t,u,v,w,p,T)\partial_x + \xi^y(x,y,z,t,u,v,w,p,T)\partial_y + \xi^z(x,y,z,t,u,v,w,p,T)\partial_z\\
&+ \tau(x,y,z,t,u,v,w,p,T)\partial_t + \eta_1(x,y,z,t,u,v,w,p,T)\partial_u + \eta_2(x,y,z,t,u,v,w,p,T)\partial_v\\
&+ \eta_3(x,y,z,t,u,v,w,p,T)\partial_w + \eta_4(x,y,z,t,u,v,w,p,T)\partial_p + \eta_5(x,y,z,t,u,v,w,p,T)\partial_T.
\end{aligned} \tag{5.71}$$

Further, the invariance criterion of the Boussinesq system (5.70) is obtained as follows:

$$\begin{aligned}
&[\eta_1^{\alpha,t} + \eta_1 u_x^\beta + u\eta_1^{\beta,x} + \eta_2 u_y^\gamma + v\eta_1^{\gamma,y} + \eta_3 u_z^\delta + w\eta_1^{\delta,z} + \frac{1}{a}\eta_4^{\beta,x}]|_{(5.70)} = 0,\\
&[\eta_2^{\alpha,t} + \eta_1 v_x^\beta + u\eta_2^{\beta,x} + \eta_2 v_y^\gamma + v\eta_2^{\gamma,y} + \eta_3 v_z^\delta + w\eta_2^{\delta,z} + \frac{1}{a}\eta_4^{\gamma,y}]|_{(5.70)} = 0,\\
&[\eta_3^{\alpha,t} + \eta_1 w_x^\beta + u\eta_3^{\beta,x} + \eta_2 w_y^\gamma + v\eta_3^{\gamma,y} + \eta_3 w_z^\delta + w\eta_3^{\delta,z} + \frac{1}{a}\eta_4^{\delta,z} - \frac{g}{b}\eta_5]|_{(5.70)} = 0,\\
&[\eta_5^{\alpha,t} + \eta_1 T_x^\beta + u\eta_5^{\beta,x} + \eta_2 T_y^\gamma + v\eta_5^{\gamma,y} + \eta_3 T_z^\delta + w\eta_5^{\delta,z} + d_1\eta_3]|_{(5.70)} = 0,\\
&[\eta_1^{\beta,x} + \eta_2^{\gamma,y} + \eta_3^{\delta,z}]|_{(5.70)} = 0.
\end{aligned} \tag{5.72}$$

where $\Gamma_d - \Gamma = d_1$. By using the prolongation formulae and comparing the coefficients of alike differential coefficients, the determining equations can be obtained as follows:

$$
\begin{aligned}
&\xi^x_t = \xi^x_y = \xi^x_z = \xi^x_u = \xi^x_v = \xi^x_w = \xi^x_p = \xi^x_T = 0,\\
&\xi^y_t = \xi^y_x = \xi^y_z = \xi^y_u = \xi^y_v = \xi^y_w = \xi^y_p = \xi^y_T = 0,\\
&\xi^z_t = \xi^z_x = \xi^z_y = \xi^z_u = \xi^z_v = \xi^z_w = \xi^z_p = \xi^z_T = 0,\\
&\tau_x = \tau_y = \tau_z = \tau_u = \tau_v = \tau_w = \tau_p = \tau_T = 0,\\
&\eta_{1,v} = \eta_{1,w} = \eta_{1,p} = \eta_{1,T} = 0,\\
&\eta_{2,u} = \eta_{2,w} = \eta_{2,p} = \eta_{2,T} = 0,\\
&\eta_{3,u} = \eta_{3,v} = \eta_{3,p} = \eta_{3,T} = 0,\\
&\eta_{4,u} = \eta_{4,v} = \eta_{4,w} = \eta_{4,T} = 0,\\
&\eta_{5,u} = \eta_{5,v} = \eta_{5,w} = \eta_{5,p} = 0,\\
&\binom{\alpha}{n}\partial^n_t\eta_{1,u} = \binom{\alpha}{n}\partial^n_t\eta_{2,v} = \binom{\alpha}{n}\partial^n_t\eta_{3,w} = \binom{\alpha}{n}\partial^n_t\eta_{5,T} = \binom{\alpha}{n+1}\mathrm{D}^{n+1}_t\tau, \quad n \in \mathbb{N},\\
&\binom{\beta}{n}\partial^n_x\eta_{1,u} = \binom{\beta}{n}\partial^n_x\eta_{2,v} = \binom{\beta}{n}\partial^n_x\eta_{3,w} = \binom{\beta}{n}\partial^n_x\eta_{5,T} = \binom{\beta}{n+1}\mathrm{D}^{n+1}_x\xi^x, \quad n \in \mathbb{N},\\
&\binom{\gamma}{n}\partial^n_y\eta_{1,u} = \binom{\gamma}{n}\partial^n_y\eta_{2,v} = \binom{\gamma}{n}\partial^n_y\eta_{3,w} = \binom{\gamma}{n}\partial^n_y\eta_{5,T} = \binom{\gamma}{n+1}\mathrm{D}^{n+1}_y\xi^y, \quad n \in \mathbb{N},\\
&\binom{\delta}{n}\partial^n_z\eta_{1,u} = \binom{\delta}{n}\partial^n_z\eta_{2,v} = \binom{\delta}{n}\partial^n_z\eta_{3,w} = \binom{\delta}{n}\partial^n_z\eta_{5,T} = \binom{\delta}{n+1}\mathrm{D}^{n+1}_z\xi^z, \quad n \in \mathbb{N},\\
&\eta_1 = u(\beta\xi^x_x - \alpha\tau_t), \quad \eta_2 = v(\gamma\xi^y_y - \alpha\tau_t), \quad \eta_3 = w(\delta\xi^z_z - \alpha\tau_t),\\
&\eta_{4,p} = \eta_{1,u} + \beta\xi^x_x - \alpha\tau_t = \eta_{2,v} + \gamma\xi^y_y - \alpha\tau_t = \eta_{3,w} + \delta\xi^z_z - \alpha\tau_t,\\
&\partial^\alpha_t\eta_1 - u\partial^\alpha_t\eta_{1,u} + u(\partial^\beta_x\eta_1 - u\partial^\beta_x\eta_{1,u}) + v(\partial^\gamma_y\eta_1 - u\partial^\gamma_y\eta_{1,u})\\
&\quad + w(\partial^\delta_z\eta_1 - u\partial^\delta_z\eta_{1,u}) + \frac{1}{a}(\partial^\beta_x\eta_4 - p\partial^\beta_x\eta_{4,p}) = 0,\\
&\partial^\alpha_t\eta_2 - v\partial^\alpha_t\eta_{2,v} + u(\partial^\beta_x\eta_2 - v\partial^\beta_x\eta_{2,v}) + v(\partial^\gamma_y\eta_2 - v\partial^\gamma_y\eta_{2,v})\\
&\quad + w(\partial^\delta_z\eta_2 - v\partial^\delta_z\eta_{2,v}) + \frac{1}{a}(\partial^\gamma_y\eta_4 - p\partial^\gamma_y\eta_{4,p}) = 0,\\
&\partial^\alpha_t\eta_3 - w\partial^\alpha_t\eta_{3,w} + u(\partial^\beta_x\eta_3 - w\partial^\beta_x\eta_{3,w}) + v(\partial^\gamma_y\eta_3 - w\partial^\gamma_y\eta_{3,w})\\
&\quad + w(\partial^\delta_z\eta_3 - w\partial^\delta_z\eta_{3,w}) + \frac{1}{a}(\partial^\delta_z\eta_4 - p\partial^\delta_z\eta_{4,p}) + \frac{g}{b}T(\eta_{3,w} - \alpha\tau_t) - \frac{g}{b}\eta_5 = 0,
\end{aligned}
\tag{5.73}
$$

$$\begin{aligned}
&\partial_t^{\alpha}\eta_5 - T\partial_t^{\alpha}\eta_{5,T} + u(\partial_x^{\beta}\eta_5 - T\partial_x^{\beta}\eta_{5,T} + v(\partial_y^{\gamma}\eta_5 - T\partial_y^{\gamma}\eta_{5,T}),\\
&+ w(\partial_z^{\delta}\eta_5 - T\partial_z^{\delta}\eta_{5,T}) - d_1 w(\eta_{5,T} - \alpha\tau_t) + d_1\eta_3 = 0,\\
&\partial_x^{\beta}\eta_1 - u\partial_x^{\beta}\eta_{1,u} + (\partial_y^{\gamma}\eta_2 - v\partial_y^{\gamma}\eta_{2,v}) + (\partial_z^{\delta}\eta_3 - w\partial_z^{\delta}\eta_{3,w}) = 0.
\end{aligned}$$

On solving the above system systematically and using the conditions $\xi^x(0) = \xi^y(0) = \xi^z(0) = \tau(0) = 0$, the infinitesimals are obtained as follows:

$$\begin{aligned}
&\xi^x = \frac{c_1 x}{\beta}, \quad \xi^y = \frac{c_1 y}{\gamma}, \quad \xi^z = \frac{c_1 z}{\delta}, \quad \tau = c_5 t, \quad \eta_1 = u(c_1 - \alpha c_5),\\
&\eta_2 = v(c_1 - \alpha c_5), \quad \eta_3 = w(c_1 - \alpha c_5), \quad \eta_4 = 2p(c_1 - \alpha c_5) + \frac{x^{\beta-1}y^{\gamma-1}}{\Gamma(\beta)\Gamma(\gamma)}A(z,t),\\
&\eta_5 = T(c_1 - 2\alpha c_5) + \frac{b}{ga}\frac{x^{\beta-1}y^{\gamma-1}}{\Gamma(\beta)\Gamma(\gamma)}\partial_z^{\delta}(A(z,t)),
\end{aligned} \tag{5.74}$$

such that the following two conditions hold:

$$\begin{aligned}
&\partial_t^{\alpha}\partial_z^{\delta}(A(z,t)) = 0,\\
&2d_1\alpha c_5 + \frac{b}{ga}\frac{x^{\beta-1}y^{\gamma-1}}{\Gamma(\beta)\Gamma(\gamma)}\partial_z^{\delta}\partial_z^{\delta}(A(z,t)) = 0.
\end{aligned} \tag{5.75}$$

Simplifying (5.75) implies the following:

$$\partial_t^{\alpha}\partial_z^{\delta}(A(z,t)) = \partial_z^{\delta}\partial_z^{\delta}(A(z,t)) = 0, \tag{5.76}$$

and either $d_1 = 0$ or $c_5 = 0$. Solving the condition (5.76) gives $A(z,t) = \mu\frac{t^{\alpha-1}z^{2\delta-1}}{\Gamma(\alpha)\Gamma(2\delta)}$ where μ is an arbitrary constant. Therefore, the explicit form of the symmetries is derived as follows:

$$\begin{aligned}
&\xi^x = \frac{c_1 x}{\beta}, \quad \xi^y = \frac{c_1 y}{\gamma}, \quad \xi^z = \frac{c_1 z}{\delta}, \quad \tau = c_5 t, \quad \eta_1 = u(c_1 - \alpha c_5),\\
&\eta_2 = v(c_1 - \alpha c_5), \quad \eta_3 = w(c_1 - \alpha c_5), \quad \eta_4 = 2p(c_1 - \alpha c_5) + \mu\frac{x^{\beta-1}y^{\gamma-1}z^{2\delta-1}t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(2\delta)},\\
&\eta_5 = T(c_1 - 2\alpha c_5) + \mu\frac{b}{ga}\frac{x^{\beta-1}y^{\gamma-1}z^{\delta-1}t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)},
\end{aligned} \tag{5.77}$$

for c_1 and c_5 being arbitrary constants. The infinitesimal operator is obtained in the following form:

$$\begin{aligned}
V = {}&\frac{c_1 x}{\beta}\frac{\partial}{\partial x} + \frac{c_1 y}{\gamma}\frac{\partial}{\partial y} + \frac{c_1 z}{\delta}\frac{\partial}{\partial z} + c_5 t\frac{\partial}{\partial t} + u(c_1 - \alpha c_5)\frac{\partial}{\partial u} + v(c_1 - \alpha c_5)\frac{\partial}{\partial v} + w(c_1 - \alpha c_5)\frac{\partial}{\partial w}\\
&+ \left(2p(c_1 - \alpha c_5) + \mu\frac{x^{\beta-1}y^{\gamma-1}z^{2\delta-1}t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(2\delta)}\right)\frac{\partial}{\partial p} + \left(T(c_1 - 2\alpha c_5) + \mu\frac{b}{ga}\frac{x^{\beta-1}y^{\gamma-1}z^{\delta-1}t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}\right)\frac{\partial}{\partial T}.
\end{aligned} \tag{5.78}$$

Therefore, the symmetry generators spanning the associated Lie algebra are obtained as follows:

$$\begin{aligned}
V_1 &= \frac{x}{\beta}\partial_x + \frac{y}{\gamma}\partial_y + \frac{z}{\delta}\partial_z + u\partial_u + v\partial_v + w\partial_w + 2p\partial_p + T\partial_T,\\
V_2 &= t\partial_t - \alpha u\partial_u - \alpha v\partial_v - \alpha w\partial_w - 2\alpha p\partial_p - 2\alpha T\partial_T,\\
V_\mu &= \frac{x^{\beta-1}y^{\gamma-1}z^{2\delta-1}t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(2\delta)}\partial_p + \frac{b}{ga}\frac{x^{\beta-1}y^{\gamma-1}z^{\delta-1}t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}\partial_T.
\end{aligned} \tag{5.79}$$

5.3.4.1 Symmetry reduction associated with V_1

The differential invariants associated with the generator V_1 can be obtained as follows:

$$\theta_1 = x^\beta y^{-\gamma},\quad \theta_2 = x^\beta z^{-\delta},\quad \theta_3 = t,\quad u = x^\beta F_1,\quad v = x^\beta F_2,\quad w = x^\beta F_3,\quad p = x^{2\beta}F_4,\quad T = x^\beta F_5, \tag{5.80}$$

which allows the considered system to reduce (5.70) into a lower dimensional system with dependent variables $F_i(\theta_1,\theta_2,\theta_3)$, $i = 1,..,5$, described in the theorem below:

Theorem 5.11. *The similarity transformations and similarity variables* (5.80) *reduce the Boussinesq equations* (5.70) *into the following* $(2+1)$*-dimensional nonlinear system of PDEs of fractional order:*

$$\begin{aligned}
&\theta_3^{-\alpha}\left(\mathcal{D}^{-\alpha,\alpha}_{\infty,\infty,1}F_1\right)(\theta_1,\theta_2,\theta_3) + F_1\left(\mathcal{D}^{0,\beta}_{\frac{1}{\beta},\frac{1}{\beta},\infty}F_1\right)(\theta_1,\theta_2,\theta_3) + \theta_1F_2\left(\mathcal{P}^{1-\gamma,\gamma}_{\frac{1}{\gamma},\infty,\infty}F_1\right)(\theta_1,\theta_2,\theta_3)+\\
&\theta_2F_3\left(\mathcal{P}^{1-\delta,\delta}_{\infty,\frac{1}{\delta},\infty}F_1\right)(\theta_1,\theta_2,\theta_3) + \frac{1}{a}\left(\mathcal{D}^{0,\beta}_{\frac{1}{\beta},\frac{1}{\beta},\infty}F_4\right)(\theta_1,\theta_2,\theta_3) = 0,\\
&\theta_3^{-\alpha}\left(\mathcal{D}^{-\alpha,\alpha}_{\infty,\infty,1}F_2\right)(\theta_1,\theta_2,\theta_3) + F_1\left(\mathcal{D}^{0,\beta}_{\frac{1}{\beta},\frac{1}{\beta},\infty}F_2\right)(\theta_1,\theta_2,\theta_3) + \theta_1F_2\left(\mathcal{P}^{1-\gamma,\gamma}_{\frac{1}{\gamma},\infty,\infty}F_2\right)(\theta_1,\theta_2,\theta_3)+\\
&\theta_2F_3\left(\mathcal{P}^{1-\delta,\delta}_{\infty,\frac{1}{\delta},\infty}F_2\right)(\theta_1,\theta_2,\theta_3) + \frac{\theta_1}{a}\left(\mathcal{P}^{1-\gamma,\gamma}_{\frac{1}{\gamma},\infty,\infty}F_4\right)(\theta_1,\theta_2,\theta_3) = 0,\\
&\theta_3^{-\alpha}\left(\mathcal{D}^{-\alpha,\alpha}_{\infty,\infty,1}F_3\right)(\theta_1,\theta_2,\theta_3) + F_1\left(\mathcal{D}^{0,\beta}_{\frac{1}{\beta},\frac{1}{\beta},\infty}F_3\right)(\theta_1,\theta_2,\theta_3) + \theta_1F_2\left(\mathcal{P}^{1-\gamma,\gamma}_{\frac{1}{\gamma},\infty,\infty}F_3\right)(\theta_1,\theta_2,\theta_3)+\\
&\theta_2F_3\left(\mathcal{P}^{1-\delta,\delta}_{\infty,\frac{1}{\delta},\infty}F_3\right)(\theta_1,\theta_2,\theta_3) + \frac{\theta_2}{a}\left(\mathcal{P}^{1-\delta,\delta}_{\infty,\frac{1}{\delta},\infty}F_4\right)(\theta_1,\theta_2,\theta_3) - \frac{g}{b}F_5 = 0,\\
&\theta_3^{-\alpha}\left(\mathcal{D}^{-\alpha,\alpha}_{\infty,\infty,1}F_5\right)(\theta_1,\theta_2,\theta_3) + F_1\left(\mathcal{D}^{0,\beta}_{\frac{1}{\beta},\frac{1}{\beta},\infty}F_5\right)(\theta_1,\theta_2,\theta_3) + \theta_1F_2\left(\mathcal{P}^{1-\gamma,\gamma}_{\frac{1}{\gamma},\infty,\infty}F_5\right)(\theta_1,\theta_2,\theta_3)+\\
&\theta_2F_3\left(\mathcal{P}^{1-\delta,\delta}_{\infty,\frac{1}{\delta},\infty}F_5\right)(\theta_1,\theta_2,\theta_3) + d_1F_3 = 0,\\
&\left(\mathcal{D}^{0,\beta}_{\frac{1}{\beta},\frac{1}{\beta},\infty}F_1\right)(\theta_1,\theta_2,\theta_3) + \theta_1\left(\mathcal{P}^{1-\gamma,\gamma}_{\frac{1}{\gamma},\infty,\infty}F_2\right)(\theta_1,\theta_2,\theta_3) + \theta_2\left(\mathcal{P}^{1-\delta,\delta}_{\infty,\frac{1}{\delta},\infty}F_3\right)(\theta_1,\theta_2,\theta_3) = 0,
\end{aligned} \tag{5.81}$$

where $\left(\mathcal{P}^{\zeta,\alpha}_{\delta_1,\delta_2,\delta_3}\right)$ *and* $\left(\mathcal{D}^{\zeta,\beta}_{\delta_1,\delta_2,\delta_3}\right)$ *are the extended left and right hand sided Erdélyi-Kober fractional differential operators.*

Proof. The Riemann-Liouville fractional derivative of $u = x^{\beta}F_1(\theta_1,\theta_2,\theta_3)$ with respect to t for $\alpha-1<n<\alpha$ is written as follows:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^{n}}{\partial t^{n}}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{\beta}F_1(x^{\beta}y^{-\gamma},x^{\beta}z^{-\delta},s)\mathrm{d}s\right]. \tag{5.82}$$

Taking $\lambda = \frac{s}{t}$, it is converted into the following:

$$\begin{aligned}\frac{\partial^{\alpha}u}{\partial t^{\alpha}} &= x^{\beta}\frac{\partial^{n}}{\partial t^{n}}\left[\frac{t^{n-\alpha}}{\Gamma(n-\alpha)}\int_0^1 (1-\lambda)^{n-\alpha-1}F_1(\theta_1,\theta_2,\lambda\theta_3)\mathrm{d}\lambda\right],\\ &= x^{\beta}\frac{\partial^{n}}{\partial t^{n}}\left[t^{n-\alpha}\left(\mathcal{I}^{0,n-\alpha}_{\infty,\infty,1}F_1\right)(\theta_1,\theta_2,\theta_3)\right],\end{aligned} \tag{5.83}$$

using the definition of $\left(\mathcal{I}^{\zeta,\beta}_{\delta}h\right)$ in (5.60). Further, for any differentiable function $\psi(\theta_1,\theta_2,\theta_3)$ using relations (5.80), the following must hold:

$$t\frac{\partial\psi}{\partial t} = \theta_3\frac{\partial\psi}{\partial\theta_3}. \tag{5.84}$$

Then the expression (5.83) is simplified into the following:

$$\begin{aligned}\frac{\partial^{\alpha}u}{\partial t^{\alpha}} &= x^{\beta}\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha-1}\left(n-\alpha+\theta_3\frac{\partial}{\partial\theta_3}\right)\left(\mathcal{I}^{0,n-\alpha}_{\infty,\infty,1}F_1\right)(\theta_1,\theta_2,\theta_3)\right],\\ &= \cdots\\ &= x^{\beta}t^{-\alpha}\prod_{j=1}^{n}(-\alpha+j+\theta_3\frac{\partial}{\partial\theta_3})\left(\mathcal{I}^{0,n-\alpha}_{\infty,\infty,1}F_1\right)(\theta_1,\theta_2,\theta_3).\end{aligned} \tag{5.85}$$

Hence, the following result holds:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = x^{\beta}t^{-\alpha}\left(\mathcal{D}^{-\alpha,\alpha}_{\infty,\infty,1}F_1\right)(\theta_1,\theta_2,\theta_3). \tag{5.86}$$

Following the same procedure, the following derivatives of u of fractional order can be easily obtained:

$$\begin{aligned}\partial^{\beta}_{x}u &= \left(\mathcal{D}^{0,\beta}_{\frac{1}{\beta},\frac{1}{\beta},\infty}F_1\right)(\theta_1,\theta_2,\theta_3),\\ \partial^{\gamma}_{y}u &= x^{\beta}y^{-\gamma}\left(\mathcal{P}^{1-\gamma,\gamma}_{\frac{1}{\gamma},\infty,\infty}F_1\right)(\theta_1,\theta_2,\theta_3),\\ \partial^{\delta}_{z}u &= x^{\beta}z^{-\delta}\left(\mathcal{P}^{1-\delta,\delta}_{\infty,\frac{1}{\delta},\infty}F_1\right)(\theta_1,\theta_2,\theta_3).\end{aligned} \tag{5.87}$$

Proceeding in the same way, for $v = x^\beta F_2(\theta_1, \theta_2, \theta_3)$, the fractional derivatives can be obtained in the following form:

$$\begin{aligned}
\partial_t^\alpha v &= x^\beta t^{-\alpha}\left(\mathcal{D}^{-\alpha,\alpha}_{\infty,\infty,1} F_2\right)(\theta_1,\theta_2,\theta_3),\\
\partial_x^\beta v &= \left(\mathcal{D}^{0,\beta}_{\frac{1}{\beta},\frac{1}{\beta},\infty} F_2\right)(\theta_1,\theta_2,\theta_3),\\
\partial_y^\gamma v &= x^\beta y^{-\gamma}\left(\mathcal{P}^{1-\gamma,\gamma}_{\frac{1}{\gamma},\infty,\infty} F_2\right)(\theta_1,\theta_2,\theta_3),\\
\partial_z^\delta v &= x^\beta z^{-\delta}\left(\mathcal{P}^{1-\delta,\delta}_{\infty,\frac{1}{\delta},\infty} F_2\right)(\theta_1,\theta_2,\theta_3).
\end{aligned} \tag{5.88}$$

Similarly, the required fractional derivatives of w, p, T with respect to x, y, z, t can be easily calculated, completing the proof of the theorem. □

5.3.4.2 Symmetry reduction corresponding to V_2

The similarity solutions and similarity transformations corresponding to generator V_2 are obtained as follows:

$$\begin{aligned}
&\varrho_1 = x, \quad \varrho_2 = y, \quad \varrho_3 = z, \quad u = F_1 t^{-\alpha}, \quad v = F_2 t^{-\alpha},\\
&w = F_3 t^{-\alpha}, \quad p = F_4 t^{-2\alpha}, \quad T = F_5 t^{-2\alpha}.
\end{aligned} \tag{5.89}$$

The system (5.70) is reduced into a system of FPDEs in new variables $F_i(\varrho_1, \varrho_2, \varrho_3) = F_i(x, y, z)$ for $i = 1, .., 5$, presented below:

Theorem 5.12. *The similarity transformations* (5.89) *reduce the Boussinesq system* (5.70) *into a* $(2+1)$*-dimensional nonlinear system of FPDEs in the following form:*

$$\begin{aligned}
&CF_1 + F_1\partial_x^\beta F_1 + F_2\partial_y^\gamma F_1 + F_3\partial_z^\delta F_1 + \frac{1}{a}\partial_x^\beta F_4 = 0,\\
&CF_2 + F_1\partial_x^\beta F_2 + F_2\partial_y^\gamma F_2 + F_3\partial_z^\delta F_2 + \frac{1}{a}\partial_y^\gamma F_4 = 0,\\
&CF_3 + F_1\partial_x^\beta F_3 + F_2\partial_y^\gamma F_3 + F_3\partial_z^\delta F_3 + \frac{1}{a}\partial_z^\delta F_4 - \frac{g}{b}F_5 = 0,\\
&C'F_5 + F_1\partial_x^\beta F_5 + F_2\partial_y^\gamma F_5 + F_3\partial_z^\delta F_5 = 0,\\
&\partial_x^\beta F_1 + \partial_y^\gamma F_2 + \partial_z^\delta F_3 = 0.
\end{aligned} \tag{5.90}$$

where $C = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}$, $C' = \frac{\Gamma(1-2\alpha)}{\Gamma(1-3\alpha)}$ *and* $c_5 \neq 0$ *implies* $d_1 = 0$.

Furthermore, the invariant analysis of this reduced $(2+1)$-dimensional system (5.90) and the conditions $\xi^x(0)=\xi^y(0)=\xi^z(0)=0$, lead to the following infinitesimals:

$$\begin{aligned}&\xi^x=\frac{k_1x}{\beta},\quad \xi^y=\frac{k_1y}{\gamma},\quad \xi^z=\frac{k_1z}{\delta},\quad \eta_1=k_1F_1,\quad \eta_2=k_1F_2,\quad \eta_3=k_1F_3,\\&\eta_4=2k_1F_4+k_2\frac{x^{\beta-1}y^{\gamma-1}z^{\delta-1}}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)},\quad \eta_5=k_1F_5,\end{aligned}\tag{5.91}$$

where k_1 and k_2 are arbitrary constants. The symmetry generators are as follows:

$$\begin{aligned}&V_{11}=\frac{x}{\beta}\partial_x+\frac{y}{\gamma}\partial_y+\frac{z}{\delta}\partial_z+F_1\partial_{F_1}+F_2\partial_{F_2}+F_3\partial_{F_3}+2F_4\partial_{F_4}+F_5\partial_{F_5},\\&V_{12}=\frac{x^{\beta-1}y^{\gamma-1}z^{\delta-1}}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}\partial_{F_4}.\end{aligned}\tag{5.92}$$

Associated with the generator V_{11} yields the following invariant solutions:

$$\begin{aligned}&\rho_1=x^{-\beta}y^{\gamma},\quad \rho_2=x^{-\beta}z^{\delta},\quad F_1=x^{\beta}G_1,\quad F_2=x^{\beta}G_2,\\&F_3=x^{\beta}G_3,\quad F_4=x^{2\beta}G_4,\quad F_5=x^{\beta}G_5.\end{aligned}\tag{5.93}$$

Consequently, the reduced (2+1)-dimensional system (5.90) is further reduced as follows:

Theorem 5.13. *The* (2+1)*-dimensional system* (5.90) *is reduced into a* (1+1)*-dimensional nonlinear system of FPDEs in variables* $G_i(\rho_1,\rho_2)$ *as below:*

$$\begin{aligned}&CG_1+G_1\left(\mathcal{P}^{1,\beta}_{\frac{1}{\beta},\frac{1}{\beta}}G_1\right)(\rho_1,\rho_2)+\rho_1^{-1}G_2\left(\mathcal{D}^{-\gamma,\gamma}_{\frac{1}{\gamma},\infty}G_1\right)(\rho_1,\rho_2)+\rho_2^{-1}G_3\left(\mathcal{D}^{-\delta,\delta}_{\infty,\frac{1}{\delta}}G_1\right)(\rho_1,\rho_2)\\&+\frac{1}{a}\left(\mathcal{P}^{1+\beta,\beta}_{\frac{1}{\beta},\frac{1}{\beta}}G_4\right)(\rho_1,\rho_2)=0,\\&CG_2+G_1\left(\mathcal{P}^{1,\beta}_{\frac{1}{\beta},\frac{1}{\beta}}G_2\right)(\rho_1,\rho_2)+\rho_1^{-1}G_2\left(\mathcal{D}^{-\gamma,\gamma}_{\frac{1}{\gamma},\infty}G_2\right)(\rho_1,\rho_2)+\rho_2^{-1}G_3\left(\mathcal{D}^{-\delta,\delta}_{\infty,\frac{1}{\delta}}G_2\right)(\rho_1,\rho_2)\\&+\frac{\rho_1^{-1}}{a}\left(\mathcal{D}^{-\gamma,\gamma}_{\frac{1}{\gamma},\infty}G_4\right)(\rho_1,\rho_2)=0,\\&CG_3+G_1\left(\mathcal{P}^{1,\beta}_{\frac{1}{\beta},\frac{1}{\beta}}G_3\right)(\rho_1,\rho_2)+\rho_1^{-1}G_2\left(\mathcal{D}^{-\gamma,\gamma}_{\frac{1}{\gamma},\infty}G_3\right)(\rho_1,\rho_2)+\rho_2^{-1}G_3\left(\mathcal{D}^{-\delta,\delta}_{\infty,\frac{1}{\delta}}G_3\right)(\rho_1,\rho_2)\\&+\frac{\rho_2^{-1}}{a}\left(\mathcal{D}^{-\delta,\delta}_{\infty,\frac{1}{\delta}}G_4\right)(\rho_1,\rho_2)-\frac{g}{b}G_5=0,\\&C'G_5+G_1\left(\mathcal{P}^{1,\beta}_{\frac{1}{\beta},\frac{1}{\beta}}G_5\right)(\rho_1,\rho_2)+\rho_1^{-1}G_2\left(\mathcal{D}^{-\gamma,\gamma}_{\frac{1}{\gamma},\infty}G_5\right)(\rho_1,\rho_2)+\rho_2^{-1}G_3\left(\mathcal{D}^{-\delta,\delta}_{\infty,\frac{1}{\delta}}G_5\right)(\rho_1,\rho_2)=0,\\&\left(\mathcal{P}^{1,\beta}_{\frac{1}{\beta},\frac{1}{\beta}}G_1\right)(\rho_1,\rho_2)+\rho_1^{-1}\left(\mathcal{D}^{-\gamma,\gamma}_{\frac{1}{\gamma},\infty}G_2\right)(\rho_1,\rho_2)+\rho_2^{-1}\left(\mathcal{D}^{-\delta,\delta}_{\infty,\frac{1}{\delta}}G_3\right)(\rho_1,\rho_2)=0.\end{aligned}\tag{5.94}$$

5.3.5 Fractional 3D Incompressible Boussinesq Equations with Viscosity

The fractional (3+1)-dimensional nonlinear incompressible non-hydrostatic Boussinesq equations with viscosity is as follows [125]:

$$\begin{aligned}
&\partial_t^\alpha u + u\partial_x^\beta u + v\partial_y^\gamma u + w\partial_z^\delta u + \frac{1}{a}\partial_x^\beta p - \nu(u_{xx}+u_{yy}+u_{zz}) = 0,\\
&\partial_t^\alpha v + u\partial_x^\beta v + v\partial_y^\gamma v + w\partial_z^\delta v + \frac{1}{a}\partial_y^\gamma p - \nu(v_{xx}+v_{yy}+v_{zz}) = 0,\\
&\partial_t^\alpha w + u\partial_x^\beta w + v\partial_y^\gamma w + w\partial_z^\delta w + \frac{1}{a}\partial_z^\delta p - \frac{gT}{b} - \nu(w_{xx}+w_{yy}+w_{zz}) = 0,\\
&\partial_t^\alpha T + u\partial_x^\beta T + v\partial_y^\gamma T + w\partial_z^\delta T + (\Gamma_d - \Gamma)w = 0,\\
&\partial_x^\beta u + \partial_y^\gamma v + \partial_z^\delta w = 0,
\end{aligned} \tag{5.95}$$

where $a, g, b, \Gamma_d, \Gamma, \nu$ are constants [125]. The invariance of the system (5.95) with the associated symmetry generator of the form (5.71) gives the following conditions:

$$\begin{aligned}
&[\eta_1^{\alpha,t} + \eta_1 u_x^\beta + u\eta_1^{\beta,x} + \eta_2 u_y^\gamma + v\eta_1^{\gamma,y} + \eta_3 u_z^\delta + w\eta_1^{\delta,z} + \frac{1}{a}\eta_4^{\beta,x} - (\eta_1^{xx} + \eta_1^{yy} + \eta_1^{zz})]|_{(5.95)} = 0,\\
&[\eta_2^{\alpha,t} + \eta_1 v_x^\beta + u\eta_2^{\beta,x} + \eta_2 v_y^\gamma + v\eta_2^{\gamma,y} + \eta_3 v_z^\delta + w\eta_2^{\delta,z} + \frac{1}{a}\eta_4^{\gamma,y} - (\eta_2^{xx} + \eta_2^{yy} + \eta_2^{zz})]|_{(5.95)} = 0,\\
&[\eta_3^{\alpha,t} + \eta_1 w_x^\beta + u\eta_3^{\beta,x} + \eta_2 w_y^\gamma + v\eta_3^{\gamma,y} + \eta_3 w_z^\delta + w\eta_3^{\delta,z} + \frac{1}{a}\eta_4^{\delta,z} - \frac{g}{b}\eta_5 - (\eta_3^{xx} + \eta_3^{yy} + \eta_3^{zz})]|_{(5.95)} = 0,\\
&[\eta_5^{\alpha,t} + \eta_1 T_x^\beta + u\eta_5^{\beta,x} + \eta_2 T_y^\gamma + v\eta_5^{\gamma,y} + \eta_3 T_z^\delta + w\eta_5^{\delta,z} + d_1\eta_3]|_{(5.95)} = 0,\\
&[\eta_1^{\beta,x} + \eta_2^{\gamma,y} + \eta_3^{\delta,z}]|_{(5.95)} = 0.
\end{aligned} \tag{5.96}$$

Solving the corresponding determining equations, the Lie point symmetries can be calculated as follows:

$$\begin{aligned}
&\xi^x = \frac{c_1 x}{2},\quad \xi^y = \frac{c_1 y}{2},\quad \xi^z = \frac{c_1 z}{2},\quad \tau = \frac{c_1 t}{\alpha},\\
&\eta_1 = c_1 u\left(\frac{\beta}{2} - 1\right),\quad \eta_2 = c_1 v\left(\frac{\beta}{2} - 1\right),\quad \eta_3 = c_1 w\left(\frac{\beta}{2} - 1\right),\\
&\eta_4 = c_1 p(\beta - 2) + \sigma\frac{t^{\alpha-1}x^{\beta-1}y^{\beta-1}z^{2\beta-1}}{\Gamma(\alpha)\Gamma(\beta)^2\Gamma(2\beta)},\quad \eta_5 = c_1 T\left(\frac{\beta}{2} - 2\right) + \sigma\frac{b}{ga}\frac{t^{\alpha-1}x^{\beta-1}y^{\beta-1}z^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)^3},
\end{aligned} \tag{5.97}$$

where σ, c_1 are arbitrary constants. For nontrivial solutions, the constant $c_1 \neq 0$ implies $\beta = \gamma = \delta$ and $d_1 = 0$. Hence, the associated Lie algebra is generated by the following symmetry generators:

$$
\begin{aligned}
V_1 &= \frac{x}{2}\partial_x + \frac{y}{2}\partial_y + \frac{z}{2}\partial_z + \frac{t}{\alpha}\partial_t + u\left(\frac{\beta}{2}-1\right)\partial_u + v\left(\frac{\beta}{2}-1\right)\partial_v + w\left(\frac{\beta}{2}-1\right)\partial_w \\
&\quad + p(\beta-2)\partial_p + T\left(\frac{\beta}{2}-2\right)\partial_T, \\
V_\sigma &= \frac{x^{\beta-1}y^{\beta-1}z^{2\beta-1}t^{\beta-1}}{\Gamma(\beta)^3\Gamma(2\beta)}\partial_p + \frac{b}{ga}\frac{x^{\beta-1}y^{\beta-1}z^{\beta-1}t^{\beta-1}}{\Gamma(\beta)^4}\partial_T.
\end{aligned}
\tag{5.98}
$$

The similarity solutions corresponding to the generator V_1 in (5.98) can be easily calculated in the form as below:

$$
\begin{aligned}
&\varsigma_1 = xt^{-\frac{\alpha}{2}}, \quad \varsigma_2 = yt^{-\frac{\alpha}{2}}, \quad \varsigma_3 = zt^{-\frac{\alpha}{2}}, \\
&u = H^1 t^{-\frac{\alpha(\beta-2)}{2}}, \quad v = H^2 t^{-\frac{\alpha(\beta-2)}{2}}, \quad w = H^3 t^{-\frac{\alpha(\beta-2)}{2}}, \quad p = H^4 t^{-\alpha(\beta-2)}, \quad T = H^5 t^{-\frac{\alpha(\beta-4)}{2}}.
\end{aligned}
\tag{5.99}
$$

Using these invariant solutions, the reduced system is presented as follows:

Theorem 5.14. *The* (3+1)*-dimensional system* (5.95) *is reduced into a* (2+1)*-dimensional system of FPDEs with new dependent variables* $H^i(\varsigma_1,\varsigma_2,\varsigma_3), (i=1,..,5)$ *in the following form:*

$$
\begin{aligned}
&\left(\mathcal{P}^{1+\frac{\alpha(\beta-4)}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}} H^1\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_1^{-\beta} H^1\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty} H^1\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_2^{-\beta} H^2\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty} H^1\right)(\varsigma_1,\varsigma_2,\varsigma_3) \\
&\quad + \varsigma_3^{-\beta} H^3\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1} H^1\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_1^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty} H^4\right)(\varsigma_1,\varsigma_2,\varsigma_3) - \nu\left(H^1_{\varsigma_1\varsigma_1} + H^1_{\varsigma_2\varsigma_2} + H^1_{\varsigma_3\varsigma_3}\right) = 0, \\
&\left(\mathcal{P}^{1+\frac{\alpha(\beta-4)}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}} H^2\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_1^{-\beta} H^1\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty} H^2\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_2^{-\beta} H^2\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty} H^2\right)(\varsigma_1,\varsigma_2,\varsigma_3) \\
&\quad + \varsigma_3^{-\beta} H^3\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1} H^2\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_2^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty} H^4\right)(\varsigma_1,\varsigma_2,\varsigma_3) - \nu\left(H^2_{\varsigma_1\varsigma_1} + H^2_{\varsigma_2\varsigma_2} + H^2_{\varsigma_3\varsigma_3}\right) = 0, \\
&\left(\mathcal{P}^{1+\frac{\alpha(\beta-4)}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}} H^3\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_1^{-\beta} H^1\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty} H^3\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_2^{-\beta} H^2\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty} H^3\right)(\varsigma_1,\varsigma_2,\varsigma_3) - \frac{g}{b}H^5 \\
&\quad + \varsigma_3^{-\beta} H^3\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1} H^3\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_3^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1} H^4\right)(\varsigma_1,\varsigma_2,\varsigma_3) - \nu\left(H^3_{\varsigma_1\varsigma_1} + H^3_{\varsigma_2\varsigma_2} + H^3_{\varsigma_3\varsigma_3}\right) = 0, \\
&\left(\mathcal{P}^{1+\frac{\alpha(\beta-4)}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}} H^5\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_1^{-\beta} H^1\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty} H^5\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_2^{-\beta} H^2\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty} H^5\right)(\varsigma_1,\varsigma_2,\varsigma_3) \\
&\quad + \varsigma_3^{-\beta} H^3\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1} H^5\right)(\varsigma_1,\varsigma_2,\varsigma_3) = 0, \\
&\varsigma_1^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{1,\infty,\infty} H^1\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_2^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,1,\infty} H^2\right)(\varsigma_1,\varsigma_2,\varsigma_3) + \varsigma_3^{-\beta}\left(\mathcal{D}^{-\beta,\beta}_{\infty,\infty,1} H^3\right)(\varsigma_1,\varsigma_2,\varsigma_3) = 0.
\end{aligned}
\tag{5.100}
$$

5.4 Conclusion

The present chapter establishes generalized prolongation formulae for determining Lie group symmetries for systems of FDEs with Riemann-Liouville fractional derivative. The general group classification approach is proposed based on the extension of pioneer works on symmetry analysis for FDEs. The comparison of the suggested formulae with the formulae given by some recent works is provided and shown that the formulae presented in previous chapters are particular cases of our proposed formulae. Further, the Lie point symmetries of five fractional nonlinear systems of PDEs have been investigated. Using the derived symmetries, their reductions into lower dimensional fractional nonlinear systems of PDEs involving the left and right hand sided extended Erdélyi-Kober fractional differential operators have been obtained successfully.

Chapter 6

Variable Coefficient Time Fractional Nonlinear Systems of Partial Differential Equations

6.1 Introduction

The study of nonlinear differential equations with variable coefficients is very important, because they have widespread applications in physics, such as propagation of nonlinear dispersive waves in inhomogeneous media [180]. However, the FDEs with variable coefficients are not widely discussed using the symmetry approach due to the complexity of calculations involving the nonlocal fractional differential operators. The symmetry analysis of only few variable coefficient single time fractional PDEs [130] has been discussed in literature. The novelty of this chapter follows from the fact that the symmetry method has not been applied to higher order variable coefficient time fractional nonlinear PDEs and time fractional nonlinear systems of PDEs with variable coefficients so far. The main objective of the chapter is to investigate some physically significant higher order variable coefficient nonlinear time fractional PDEs and some variable coefficient time fractional nonlinear systems of PDE with the help of Lie symmetry method.

The contents of this chapter are communicated in SCI journal.

The outline of this chapter is as follows. Section 6.2 is devoted to the derivation of infinitesimal symmetries and similarity reductions of two time fractional nonlinear PDEs namely time fractional KdV-Burger-Kuramoto equation and time fractional seventh order KdV equation. Section 6.3 deals with the symmetry analysis of three time fractional nonlinear systems with variable coefficients including the variant Boussinesq system, the coupled KdV system and the Hirota-Satsuma coupled KdV system. The last section presents the conclusion of the chapter.

6.2 Symmetry Analysis of Variable Coefficient Nonlinear Partial Differential Equations

The purpose of this section is to present the application of Lie symmetry method to two important higher order time fractional nonlinear PDEs with variable coefficients. Here, the considered PDEs are investigated for their explicit Lie symmetries and reductions into FODEs in terms of Erdélyi-Kober operators.

6.2.1 Time Fractional KdV-Burgers-Kuramoto Equation

The time fractional fourth order KdV-Burgers-Kuramoto (FKBK) equation [6, 60] with variable coefficients is as follows:

$$\partial_t^\alpha u + a(t)uu_x + b(t)u_{2x} + c(t)u_{3x} + u_{4x} = 0. \tag{6.1}$$

Let the system admits the following infinitesimal generator:

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \tag{6.2}$$

The invariance criterion for FKBK equation (6.1) can be written as follows:

$$\left[\eta^{\alpha,t} + a(t)(\eta u_x + u\eta^x) + b(t)\eta^{2x} + c(t)\eta^{3x} + \tau\left(a'(t)uu_x + b'(t)u_{2x} + c'(t)u_{3x}\right) + \eta^{4x}\right]\Big|_{(6.1)} = 0. \tag{6.3}$$

By substituting all the extended infinitesimals and comparing the coefficients of various derivatives and powers of u, the following system of nonlinear PDEs and FPDEs can be obtained:

$$\begin{aligned}
&\xi_t = \xi_u = 0, \quad \tau_x = \tau_u = 0, \quad \eta_{uu} = 0,\\
&\alpha\tau_t - 4\xi_x = 0, \quad \binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0,\\
&a(t)u\left(\alpha\tau_t - \xi_x\right) + \eta a(t) + \tau a'(t)u + b(t)(2\eta_{xu} - \xi_{xx}) + c(t)(3\eta_{xxu} - \xi_{xxx}) + (4\eta_{xxu} - \xi_{xxx}) = 0,\\
&b(t)\left(\alpha\tau_t - 2\xi_x\right) + \tau b'(t) + 3c(t)(\eta_{xu} - \xi_{xx}) + (6\eta_{xxu} - 4\xi_{xxx}) = 0,\\
&c(t)u\left(\alpha\tau_t - 3\xi_x\right) + \tau c'(t)u + (4\eta_{xu} - 6\xi_{xx}) = 0,\\
&\partial_t^\alpha \eta - u\partial_t^\alpha \eta_u + a(t)u\eta_x + b(t)\eta_{xx} + c(t)u\eta_{xxx} + \eta_{4x} = 0,
\end{aligned} \tag{6.4}$$

where primes ($'$) denote the first order derivatives. On solving these equations and using the required condition $\tau|_{t=0} = 0$, the derived symmetries are as follows:

$$\xi = \frac{C_1 x}{4} + C_2, \quad \tau = \frac{C_1 t}{\alpha}, \quad \eta = C_3 u, \tag{6.5}$$

where C_1, C_2, C_3 are arbitrary constants such that the following conditions must hold:

$$\begin{aligned}
&a(t)\left(C_3 + \frac{3C_1}{4}\right) + \left(\frac{C_1 t}{\alpha}\right)a'(t) = 0,\\
&b(t)\left(\frac{C_1}{2}\right) + \left(\frac{C_1 t}{\alpha}\right)b'(t) = 0,\\
&c(t)\left(\frac{C_1}{4}\right) + \left(\frac{C_1 t}{\alpha}\right)c'(t) = 0.
\end{aligned} \tag{6.6}$$

The first order symmetry generator admitted by FPDE (6.1) can be written as follows:

$$X = \left(\frac{C_1 x}{4} + C_2\right)\partial_x + \left(\frac{C_1 t}{\alpha}\right)\partial_t + (C_3 u)\partial_u. \tag{6.7}$$

The associated vector fields are as follows:

$$X_1 = \frac{x}{4}\partial_x + \frac{t}{\alpha}\partial_t, \quad X_2 = \partial_x, \quad X_3 = u\partial_u. \tag{6.8}$$

Using the concept of optimal set [147, 150], the optimal set for generators (6.8) has the following components:

$$X_1, \quad X_3 + rX_1. \tag{6.9}$$

Table 6.1 lists the similarity variables, invariants, the variable coefficients for both these generators of the optimal set.

6.2.1.1 Symmetry reductions of FKBK equation

The next step is to calculate the symmetry reductions for both the symmetry generators of optimal set.

For generator X_1, using the similarity variables and similarity transformation described in Table 6.1, the FPDE (6.1) can be reduced as follows:

Theorem 6.1. *The similarity transformation* $u = F(z)$ *for* $z = xt^{-\frac{\alpha}{4}}$ *reduces the equation* (6.1) $\forall\alpha > 0$ *into the following nonlinear ODE of fractional order:*

$$\left(\mathcal{P}_{\frac{4}{\alpha}}^{1-\alpha,\alpha}F\right)(z) + a(FF')(z) + bF''(z) + cF'''(z) + F''''(z) = 0, \tag{6.10}$$

where $\left(\mathcal{P}_{\frac{4}{\alpha}}^{1-\alpha,\alpha}F\right)(z)$ *is the Erdélyi-Kober fractional differential operator.*

Proof. Firstly for $n-1 < \alpha < n$ $(n \in \mathbb{N})$, the Riemann-Liouville fractional derivative of $u = F(z)$ for $z = xt^{-\frac{\alpha}{4}}$ can be obtained as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}F(xs^{-\frac{\alpha}{4}})\mathrm{d}s\right]. \tag{6.11}$$

Let $w = \frac{t}{s}$ such that the above expression can be written in the following form:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\alpha}}{\Gamma(n-\alpha)}\int_1^\infty (w-1)^{n-\alpha-1}w^{-(n-\alpha+1)}F(zw^{\frac{\alpha}{4}})\mathrm{d}w\right], \\ &= \frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(\mathcal{K}_{\frac{4}{\alpha}}^{1,n-\alpha}F\right)(z)\right],\end{aligned} \tag{6.12}$$

where $\left(\mathcal{K}_{\frac{4}{\alpha}}^{1,n-\alpha}F\right)$ is the Erdélyi-Kober fractional integral operator.
The expression (6.12) also holds in case of $\alpha = n = 1, 2, ...,$ since $\left(\mathcal{K}_{\delta}^{\zeta,0}F\right)(z) = F(z)$. Also, for differentiable functions $\psi(z)$ where $z = xt^{-\frac{\alpha}{4}}$, the following holds:

$$t\frac{\partial}{\partial t}\psi(z) = -\frac{\alpha}{4}z\frac{\mathrm{d}}{\mathrm{d}z}\psi(z). \tag{6.13}$$

Thus, the expression (6.12) can be simplified as follows:

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(\mathcal{K}_{\frac{4}{\alpha}}^{1,n-\alpha}F\right)(z)\right] = \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha-1}\left(n-\alpha-\frac{\alpha}{4}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}_{\frac{4}{\alpha}}^{1,n-\alpha}F\right)(z)\right]. \tag{6.14}$$

Repeating this procedure $n-1$ times, it can be obtained in the following form:

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(\mathcal{K}_{\frac{4}{\alpha}}^{1,n-\alpha}F\right)(z)\right] = t^{-\alpha}\prod_{j=0}^{n-1}\left(1-\alpha+j-\frac{\alpha}{4}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}_{\frac{4}{\alpha}}^{1,n-\alpha}F\right)(z). \tag{6.15}$$

Substituting (6.15) in (6.12), the final expression for the time fractional derivative is as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\alpha}\left(\mathcal{P}_{\frac{4}{\alpha}}^{1-\alpha,\alpha}F\right)(z), \qquad \forall\alpha > 0. \tag{6.16}$$

For the generator X_1, solving (6.6), the variable coefficients can be obtained in the form given in Table 6.1. Therefore, it completes the proof. □

For generator X_3+rX_1, the invariant solutions can be calculated in the form presented in Table 6.1. Furthermore, following the steps of Theorem 6.1, the reduced equation is illustrated in the following theorem:

Theorem 6.2. *The FKBK equation* (6.1) *can be reduced* $\forall\alpha > 0$ *into the following nonlinear FODE as follows:*

$$\begin{aligned}&\left(\mathcal{P}_{\frac{4}{\alpha}}^{1-\alpha,\alpha}F\right)(z) + A_1(z)F(z) + A_2(z)F^2(z) + A_3(z)F'(z)\\ &+ A_4(z)(FF')(z) + A_5(z)F''(z) + A_6(z)F'''(z) + F''''(z) = 0,\end{aligned} \tag{6.17}$$

where $A_i(z)$ $(i = 1, .., 6)$ *have the following form:*

$$\begin{aligned}&A_1(z) = \left[\frac{4b}{r}\left(\frac{4}{r}-1\right)z^{-2} + \frac{4c}{r}\left(\frac{4}{r}-1\right)\left(\frac{4}{r}-2\right)z^{-3} + \frac{12}{r}\left(\frac{4}{r}-1\right)\left(\frac{4}{r}-2\right)\left(\frac{4}{r}-3\right)z^{-4}\right],\\ &A_2(z) = \frac{4a}{r}z^{\frac{4}{r}-1}, \quad A_3(z) = \left[\frac{8b}{r}z^{-1} + \frac{12c}{r}\left(\frac{4}{r}-1\right)z^{-2} + \frac{16}{r}\left(\frac{4}{r}-1\right)\left(\frac{4}{r}-1\right)z^{-3}\right],\\ &A_4(z) = az^{\frac{4}{r}}, \quad A_5(z) = \left[b + \frac{12c}{r}z^{-1} + \frac{24}{r}\left(\frac{4}{r}-1\right)z^{-2}\right], \quad A_6(z) = \left(c + \frac{16}{r}z^{-1}\right).\end{aligned} \tag{6.18}$$

Table 6.1: Symmetry reductions of the FPDE (6.1) into nonlinear FODEs

Generators	Invariants	Ansatze	Variable Coefficients	Reduced FODE
X_1	$(xt^{-\frac{\alpha}{4}}, u)$	$u = F(xt^{-\frac{\alpha}{4}})$	$a(t) = at^{-\frac{3\alpha}{4}}$ $b(t) = bt^{-\frac{\alpha}{2}}$ $c(t) = ct^{-\frac{\alpha}{4}}$	FODE (6.10)
$X_3 + rX_1$	$(xt^{-\frac{\alpha}{4}}, x^{-\frac{4}{r}}u)$	$u = x^{\frac{4}{r}}F(xt^{-\frac{\alpha}{4}})$	$a(t) = at^{-\frac{(1+\frac{3r}{4})\alpha}{r}}$ $b(t) = bt^{-\frac{\alpha}{2}}$ $c(t) = ct^{-\frac{\alpha}{4}}$	FODE (6.17)

6.2.2 Fractional Generalized Seventh Order KdV Equation

The time fractional KdV type equations arise in many processes of mathematical physics such as theory of long waves in shallow water and plasma physics [51]. The time fractional generalized seventh order KdV equation (FGSO-KdV) [159, 195] with variable coefficients is as follows:

$$\partial_t^\alpha u+a(t)u^3u_x+b(t)u_x^3+c(t)uu_xu_{xx}+d(t)u^2u_{3x}+e(t)u_{2x}u_{3x}+f(t)u_xu_{4x}+g(t)uu_{5x}+u_{7x}=0, \tag{6.19}$$

The invariance of FGSO-KdV equation (6.19) under a one parameter group of transformations with generator of the form (6.2) gives the following condition:

$$\begin{aligned}&\Big[\eta^{\alpha,t} + a(t)(3u^2\eta u_x + u^3\eta^x) + 3b(t)u_x^2\eta^x + c(t)(\eta u_x u_{xx} + u\eta^x u_{xx} + uu_x\eta^{2x}) + d(t)(2u\eta u_{3x} + u^2\eta^{3x})\\ &+ \tau(a'(t)u^3u_x + b'(t)u_x^3 + c'(t)uu_xu_{xx} + d'(t)u^2u_{3x} + e'(t)u_{2x}u_{3x} + f'(t)u_xu_{4x} + g'(t)uu_{5x})\\ &+ e(t)(\eta^{2x}u_{3x} + u_{2x}\eta^{3x}) + f(t)(\eta^x u_{4x} + u_x\eta^{4x}) + g(t)(\eta u_{5x} + u\eta^{5x}) + \eta^{7x}\Big]\Big|_{(6.19)} = 0.\end{aligned} \tag{6.20}$$

Using the extended symmetry operators and equating the coefficients, the following nonlinear undetermined system of nonlinear PDEs and FPDEs are obtained:

$$\begin{aligned}&\xi_t = \xi_u = 0, \quad \tau_x = \tau_u = 0, \quad \eta_{uu} = 0,\\ &\alpha\tau_t - 7\xi_x = 0, \quad \binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0,\\ &\eta_{xu} - 3\xi_{xx} = 0, \quad e(t)(\eta_{xu} - \xi_{xx}) = 0,\end{aligned}$$

$$
\begin{aligned}
&a(t)u^3\left(\alpha\tau_t-\xi_x\right)+3u^2\eta a(t)+\tau a'(t)+c(t)u\eta_{xx}+d(t)u^2(3\eta_{xxu}-\xi_{3x})+f(t)\eta_{3x}\\
&\;+g(t)u\left(5\eta_{4x,u}-\xi_{5x}\right)+\left(7\eta_{6x,u}-\xi_{7x}\right)=0,\\
&b(t)\left(2\eta_u+\alpha\tau_t-3\xi_x\right)+\tau b'(t)=0,\\
&c(t)u\left(\eta_u+\alpha\tau_t-3\xi_x\right)+\eta c(t)+\tau c'(t)u+e(t)\left(3\eta_{xxu}-\xi_{3x}\right)+f(t)\left(6\eta_{xxu}-4\xi_{3x}\right)=0,\\
&d(t)u^2\left(\alpha\tau_t-3\xi_x\right)+2u\eta d(t)+\tau d'(t)u^2+e(t)\eta_{xx}+g(t)u\left(10\eta_{xxu}-10\xi_{3x}\right)+\left(35\eta_{4x,u}-21\xi_{5x}\right)=0,\\
&e(t)\left(\eta_u+\alpha\tau_t-5\xi_x\right)+\tau e'(t)=0,\\
&f(t)\left(\eta_u+\alpha\tau_t-5\xi_x\right)+\tau f'(t)=0,\\
&g(t)u\left(\alpha\tau_t-5\xi_x\right)+\eta g(t)+\tau g'(t)u+\left(21\eta_{xxu}-\xi_{3x}\right)=0,\\
&f(t)\eta_x+g(t)u\left(5\eta_{xu}-10\xi_{xx}\right)+\left(35\eta_{3x,u}-10\xi_{4x}\right)=0,\\
&3b(t)\eta_x+e(t)u\left(2\eta_{xu}-\xi_{xx}\right)+f(t)\left(4\eta_{3x,u}-\xi_{4x}\right)=0,\\
&c(t)u\eta_x+e(t)\eta_{3x}+g(t)u\left(10\eta_{3x,u}-5\xi_{4x}\right)+\left(21\eta_{5x,u}-7\xi_{6x}\right)=0,\\
&e(t)\left(2\eta_{xu}-\xi_{xx}\right)+f(t)\left(\eta_{xu}-6\xi_{xx}\right)=0,\\
&\partial_t^\alpha\eta-u\partial_t^\alpha\eta_u+a(t)u^3\eta_x+d(t)u^2\eta_{3x}+g(t)u\eta_{5x}+\eta_{7x}=0,
\end{aligned}
\tag{6.21}
$$

where $\eta_{jx}=\frac{\partial^j\eta}{\partial x^j}$, $\eta_{jx,u}=\frac{\partial^{j+1}\eta}{\partial x^j\partial u}$ and $\xi_{jx}=\frac{\partial^j\xi}{\partial x^j}$. Solving the above system gives the infinitesimals in the following form:

$$
\xi=\frac{C_1x}{7}+C_2,\quad \tau=\frac{C_1t}{\alpha},\quad \eta=C_3u,
\tag{6.22}
$$

where C_1, C_2, C_3 are arbitrary constants and the variable coefficient functions must satisfy the following equations:

$$
\begin{aligned}
a(t)\left(3C_3+\frac{6C_1}{7}\right)+\left(\frac{C_1t}{\alpha}\right)a'(t)=0,\\
b(t)\left(2C_3+\frac{4C_1}{7}\right)+\left(\frac{C_1t}{\alpha}\right)b'(t)=0,\\
c(t)\left(2C_3+\frac{4C_1}{7}\right)+\left(\frac{C_1t}{\alpha}\right)c'(t)=0,\\
d(t)\left(2C_3+\frac{4C_1}{7}\right)+\left(\frac{C_1t}{\alpha}\right)d'(t)=0,
\end{aligned}
\tag{6.23}
$$

$$
\begin{aligned}
e(t)\left(C_3+\frac{2C_1}{7}\right)+\left(\frac{C_1 t}{\alpha}\right)e'(t)&=0,\\
f(t)\left(C_3+\frac{2C_1}{7}\right)+\left(\frac{C_1 t}{\alpha}\right)f'(t)&=0,\\
g(t)\left(C_3+\frac{2C_1}{7}\right)+\left(\frac{C_1 t}{\alpha}\right)g'(t)&=0.
\end{aligned}
$$

The corresponding infinitesimal generators are written as follows:

$$V_1=\frac{x}{7}\partial_x+\frac{t}{\alpha}\partial_t,\quad V_2=\partial_x,\quad V_3=u\partial_u. \tag{6.24}$$

The optimal set is spanned by the following inequivalent symmetry generators:

$$V_1,\quad V_3+rV_1. \tag{6.25}$$

In Table 6.2, the similarity variables, invariants, variable coefficients etc. for both above generators have been listed.

6.2.2.1 Symmetry reductions of FGSO-KdV equation

The symmetry reductions for both the symmetry generators have been calculated in the following theorems:

Theorem 6.3. *The similarity transformation for* V_1 *reduces the equation* (6.19) $\forall\alpha>0$ *into the following nonlinear FODE:*

$$
\begin{aligned}
&\left(\mathcal{P}_{\frac{7}{\alpha}}^{1-\alpha,\alpha}F\right)(z)+a(F^3F')(z)+bF'^3(z)+c(FF'F'')(z)+d(F^2F''')(z)+e(F''F''')(z)+f(F'F'''')(z)\\
&+g(FF''''')(z)+F'''''''(z)=0,
\end{aligned}
\tag{6.26}
$$

where $\left(\mathcal{P}_{\frac{7}{\alpha}}^{1-\alpha,\alpha}F\right)(z)$ *is the Erdélyi-Kober fractional differential operator.*

Proof. For vector field V_1, the similarity solutions are obtained as follows:

$$z=xt^{-\frac{\alpha}{7}},\quad u=F(z). \tag{6.27}$$

Following the same procedure as in Theorem 6.1, it can be concluded that the following holds:

$$
\begin{aligned}
\frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(\mathcal{K}_{\frac{7}{\alpha}}^{1,n-\alpha}F\right)(z)\right]\\
&=t^{-\alpha}\prod_{j=0}^{n-1}\left(1-\alpha+j-\frac{\alpha}{7}z\frac{\partial}{\partial z}\right)\left(\mathcal{K}_{\frac{7}{\alpha}}^{1,n-\alpha}F\right)(z),\qquad\forall\alpha>0.
\end{aligned}
\tag{6.28}
$$

Therefore, the time fractional derivative can be written in left-hand sided Erdélyi-Kober fractional differential operators as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\alpha}\left(\mathcal{P}_{\frac{7}{\alpha}}^{1-\alpha,\alpha}F\right)(z), \qquad \forall\alpha > 0. \tag{6.29}$$

Solving the conditions (6.23), the variable coefficients can be calculated of the form given in Table 6.2. Hence, it completes the proof. □

For generator $V_3 + rV_1$, the resulting invariant solutions reduce the FPDE (6.19) into a nonlinear FODE presented by the following theorem:

Theorem 6.4. *For generator V_3+rV_1, the reduced nonlinear FODE $\forall\alpha > 0$ is as follows:*

$$\begin{aligned}
&\left(\mathcal{P}_{\frac{7}{\alpha}}^{1-\alpha,\alpha}F\right)(z) + A_1 z^{-7}F(z) + A_2 z^{\frac{7}{r}-5}F^2(z) + A_3 z^{\frac{14}{r}-3}F^3(z) + \frac{7a}{r}z^{\frac{21}{r}-1}F^4(z) + A_4 z^{-6}F'(z)\\
&+ A_5 z^{\frac{7}{r}-4}(FF')(z) + A_6 z^{\frac{7}{r}-2}F'^2(z) + A_7 z^{\frac{14}{r}-1}(FF'^2)(z) + A_8 z^{\frac{14}{r}-2}(F^2F')(z) + a z^{\frac{21}{r}}(F^3F')(z)\\
&+ A_9 z^{\frac{7}{r}-3}F'^2(z) + b z^{\frac{14}{r}}F'^3(z) + A_{10} z^{-5}F''(z) + A_{11} z^{\frac{7}{r}-3}(FF'')(z) + A_{12} z^{\frac{14}{r}-1}(F^2F'')(z)\\
&+ A_{13} z^{\frac{7}{r}-2}(F'F'')(z) + \frac{21e}{r}z^{\frac{7}{r}-1}(F'')^2(z) + c z^{\frac{14}{r}}(FF'F'')(z) + A_{14} z^{-4}F'''(z) + A_{15} z^{\frac{7}{r}-2}(FF''')(z)\\
&+ d z^{\frac{14}{r}}(F^2F''')(z) + A_{16} z^{\frac{7}{r}-1}(F'F''')(z) + e z^{\frac{7}{r}}(F''F''')(z) + A_{17} z^{-3}F''''(z) + \left(\frac{7f}{r}+g\right)z^{\frac{7}{r}-1}(FF'''')(z)\\
&+ f z^{\frac{7}{r}}(F'F'''')(z) + A_{18} z^{-2}F'''''(z) + g z^{\frac{7}{r}}(FF''''')(z) + \frac{49}{r}z^{-1}F''''''(z) + F'''''''(z) = 0,
\end{aligned} \tag{6.30}$$

where A_i $(i = 1, ..., 18)$ have the following form:

$$\begin{aligned}
A_1 &= \frac{823543}{r^7} - \frac{2470629}{r^6} + \frac{2941225}{r^5} - \frac{1764735}{r^4} + \frac{557032}{r^3} - \frac{86436}{r^2} + \frac{5040}{r},\\
A_2 &= \frac{49e}{r^2}\left(\frac{7}{r}-1\right)^2\left(\frac{7}{r}-2\right) + \frac{7f}{r}\left(\frac{2901}{r^4} - \frac{2058}{r^3} + \frac{539}{r^2} - \frac{42}{r}\right) + g\left(\frac{16807}{r^5} - \frac{24010}{r^4}\right.\\
&\quad\left. + \frac{12005}{r^3} - \frac{2450}{r^2} + \frac{168}{r}\right) z^{\frac{7}{r}-5},\\
A_3 &= \frac{343b}{r^3} + c\left(\frac{343}{r^3} - \frac{49}{r^2}\right) + d\left(\frac{343}{r^3} - \frac{147}{r^2}\right),\\
A_4 &= \frac{823543}{r^6} - \frac{1764735}{r^5} + \frac{1428595}{r^4} - \frac{540225}{r^3} + \frac{93982}{r^2} - \frac{5880}{r},\\
A_5 &= \frac{49e}{r^2}\left(\frac{245}{r^2} - \frac{84}{r} + 7\right) + f\left(\frac{12005}{r^4} - \frac{6174}{r^3} + \frac{931}{r^2} - \frac{42}{r}\right) + g\left(\frac{12005}{r^4} - \frac{10290}{r^3} + \frac{2695}{r^2} - \frac{210}{r}\right),\\
A_6 &= \frac{294e}{r^2}\left(\frac{7}{r}-1\right), \quad A_7 = \frac{21b}{r} + \frac{14c}{r}, \quad A_8 = \frac{147b}{r^2} + c\left(\frac{147}{r^2} - \frac{7}{r}\right) + d\left(\frac{147}{r^2} - \frac{21}{r}\right),\\
A_9 &= f\left(\frac{1372}{r^3} - \frac{588}{r^2} + \frac{56}{r}\right), \quad A_{10} = \frac{352947}{r^5} - \frac{504210}{r^4} + \frac{252105}{r^3} - \frac{51450}{r^2} + \frac{3528}{r},
\end{aligned}$$

$$\begin{aligned}
A_{11} &= e\left(\frac{1372}{r^3}-\frac{294}{r^2}+\frac{14}{r}\right)+f\left(\frac{2058}{r^3}-\frac{294}{r^2}\right)+g\left(\frac{3430}{r^3}-\frac{1470}{r^2}+\frac{140}{r}\right),\\
A_{12} &= \frac{7c}{r}+\frac{21d}{r}, \quad A_{13}=e\left(\frac{441}{r^2}-\frac{21}{r}\right)+f\left(\frac{294}{r^2}-\frac{42}{r}\right),\\
A_{14} &= \frac{84035}{r^4}-\frac{72030}{r^3}+\frac{18865}{r^2}-\frac{1470}{r}, \quad A_{15}=e\left(\frac{49}{r^2}-\frac{7}{r}\right)+\frac{7f}{r}+g\left(\frac{490}{r^2}-\frac{70}{r}\right),\\
A_{16} &= \frac{14e}{r}+\frac{28f}{r}, \quad A_{17}=\frac{12005}{r^3}-\frac{5145}{r^2}+\frac{490}{r}, \quad A_{18}=\frac{1029}{r^2}-\frac{147}{r}.
\end{aligned} \tag{6.31}$$

Table 6.2: Symmetry reductions of the FPDE (6.19) into nonlinear FODEs

Generators	Invariants	Ansatze	Variable Coefficients	Reduced FODE
V_1	$(xt^{-\frac{\alpha}{7}}, u)$	$u = F(xt^{-\frac{\alpha}{7}})$	$a(t) = at^{-\frac{6\alpha}{7}}, b(t) = bt^{-\frac{4\alpha}{7}}$ $c(t) = ct^{-\frac{4\alpha}{7}}, d(t) = dt^{-\frac{4\alpha}{7}}$ $e(t) = et^{-\frac{2\alpha}{7}}, f(t) = ft^{-\frac{2\alpha}{7}}$ $g(t) = gt^{-\frac{2\alpha}{7}}$	FODE (6.26)
$V_3 + rV_1$	$(xt^{-\frac{\alpha}{7}}, x^{-\frac{7}{r}}u)$	$u = x^{\frac{7}{r}}F(xt^{-\frac{\alpha}{7}})$	$a(t) = at^{-(3+\frac{6r}{7})\frac{\alpha}{r}}, b(t) = bt^{-(2+\frac{4r}{7})\frac{\alpha}{r}}$ $c(t) = ct^{-(2+\frac{4r}{7})\frac{\alpha}{r}}, d(t) = dt^{-(2+\frac{4r}{7})\frac{\alpha}{r}}$ $e(t) = et^{-(1+\frac{2r}{7})\frac{\alpha}{r}}, f(t) = ft^{-(1+\frac{2r}{7})\frac{\alpha}{r}}$ $g(t) = gt^{-(1+\frac{2r}{7})\frac{\alpha}{r}}$	FODE (6.30)

6.3 Symmetry Analysis of Variable Coefficient Nonlinear Systems of Partial Differential Equations

In this section, the Lie symmetry approach proposed in chapter 2 is applied to some time fractional nonlinear systems with variable coefficients. The underlying idea is to provide a systematic procedure to identify their infinitesimal symmetries and reduce the time fractional nonlinear systems into inequivalent nonlinear ODEs of fractional order. The details of determining equations and reductions for some systems are not presented for brevity.

6.3.1 Time Fractional Variant Boussinesq System

The Boussinesq equation [30] plays an important role in many branches of physics such as vibrations in a nonlinear string. The exact solutions for integer order variant Boussinesq system with variable coefficients have already been discussed by using the Lie symmetry method. [176]. The time fractional Boussinesq system [134] with variable coefficients can be written in the following form:

$$\begin{aligned} &\partial_t^\alpha u + a(t)v_x + b(t)uu_x + c(t)u_{xx} = 0, \\ &\partial_t^\alpha v + d(t)uv_x + e(t)vu_x + f(t)v_{xx} + g(t)u_{xxx} = 0, \end{aligned} \tag{6.32}$$

where $a(t), b(t), c(t), d(t), e(t), f(t), g(t)$ are arbitrary functions of t. Using the symmetry method for time fractional systems of PDEs in chapter 2, the group invariance of system (6.32) gives the following infinitesimal invariance criterion:

$$\begin{aligned} &\Big[\eta^{\alpha,t} + a(t)\phi^x + b(t)(u\eta^x + \eta u_x) + c(t)\eta^{xx} + \tau(a'(t)v_x + b'(t)uu_x + c'(t)u_{xx})\Big]\Big|_{(6.32)} = 0, \\ &\Big[\phi^{\alpha,t} + d(t)(\eta v_x + u\phi^x) + e(t)(\phi u_x + v\eta^x) + f(t)\phi^{xx} + g(t)\eta^{xxx} \\ &+\tau(d'(t)uv_x + e'(t)vu_x + f'(t)v_{xx} + g'(t)u_{xxx})\Big]\Big|_{(6.32)} = 0, \end{aligned} \tag{6.33}$$

where the associated symmetry operator is of the following form:

$$X = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \eta(x,t,u,v)\partial_u + \phi(x,t,u,v)\partial_v. \tag{6.34}$$

Upon substitution of the prolongation operators proposed in chapter 2, and comparing the coefficients of various partial and fractional derivatives of (u, v), the obtained determining equations are as follows:

$$\begin{aligned} &\xi_t = \xi_u = \xi_v = 0, \quad \tau_x = \tau_u = \tau_v = 0, \\ &\eta_v = \eta_{uu} = 0, \quad \phi_u = \phi_{vv} = 0, \\ &\binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad \forall n \in \mathbb{N}, \\ &\binom{\alpha}{n}\partial_t^n \phi_v - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad \forall n \in \mathbb{N}, \end{aligned}$$

$$
\begin{aligned}
&a(t)(\phi_v - \eta_u + \alpha\tau_t - \xi_x) + \tau a'(t) = 0,\\
&b(t)u(\alpha\tau_t - \xi_x) + b(t)\eta + \tau b'(t)u = 0,\\
&c(t)(\alpha\tau_t - 2\xi_x) + \tau c'(t) = 0, \quad g(t)(\alpha\tau_t - 3\xi_x) + \tau g'(t) = 0,\\
&d(t)u(\alpha\tau_t - \xi_x) + d(t)\eta + \tau d'(t)u + f(t)(2\phi_{xv} - \xi_{xx}) + g(t)(3\phi_{xxv} - \xi_{xxx}) = 0,\\
&e(t)v(\eta_u - \phi_v + \alpha\tau_t - \xi_x) + e(t)\phi + \tau e'(t)v = 0,\\
&f(t)(\alpha\tau_t - 2\xi_x) + \tau f'(t) + 3g(t)(\phi_{xv} - \xi_{xx}) = 0,\\
&\partial_t^\alpha\eta - u\partial_t^\alpha\eta_u + a(t)\phi_x + b(t)u\eta_x + c(t)\eta_{xx} = 0,\\
&\partial_t^\alpha\phi - v\partial_t^\alpha\phi_v + d(t)u\phi_x + e(t)v\eta_x + f(t)\phi_{xx} + g(t)\phi_{xxx} = 0.
\end{aligned}
\tag{6.35}
$$

The general solution of the system (6.35) can be obtained in the following form:

$$
\begin{aligned}
&\xi = A_1x + A_2, \quad \tau = \frac{1}{c(t)^{\frac{1}{\alpha}}}\left(\frac{2A_1}{\alpha}\int c(t)^{\frac{1}{\alpha}}\mathrm{d}t + A_5\right),\\
&\eta = A_3(t)u, \quad \phi = A_4(t)v,
\end{aligned}
\tag{6.36}
$$

where A_1, A_2, A_5 are arbitrary constants and $A_3(t), A_4(t)$ are functions of t satisfying the following equations:

$$
\begin{aligned}
&\tau(0) = 0, \quad A_3'(t) = A_4'(t) = \left(\frac{\alpha-1}{2}\right)\tau_{tt},\\
&a(t)(A_3(t) - A_4(t) + \alpha\tau_t - A_1) + \tau a'(t) = 0,\\
&b(t)(A_3(t) + \alpha\tau_t - A_1) + \tau b'(t) = 0,\\
&d(t)(A_3(t) + \alpha\tau_t - A_1) + \tau d'(t) = 0,\\
&e(t)(A_3(t) + \alpha\tau_t - A_1) + \tau e'(t) = 0,\\
&f(t)(\alpha\tau_t - 2A_1) + \tau f'(t) = 0,\\
&g(t)(\alpha\tau_t - 3A_1) + \tau g'(t) = 0.
\end{aligned}
\tag{6.37}
$$

Clearly, the symmetries (6.36) are generalized than those obtained for integer order variant Boussinesq system [176] since the fractional system (6.32) for $\alpha = 1$ is coincident with the integer order system [176]. In general, it is quite difficult to reduce the system (6.32) further into a system of FODEs using the invariant solutions resulting from the associated vector fields. For now, in particular $c(t) = c$ (constant), is considered to overcome this problem. However, the computation of the reduced system for the general case is in progress. In this particular case, the symmetries using the condition $\tau(0) = 0$ are as follows:

$$\begin{aligned} &\xi = A_1 x + A_2, \quad \tau = \frac{2A_1 t}{\alpha}, \\ &\eta = A_3 u, \quad \phi = A_4 v, \end{aligned} \tag{6.38}$$

where A_1, A_2, A_3, A_4 are arbitrary constants such that the variable coefficients must satisfy the following:

$$\begin{aligned} &a(t)(A_4 - A_3 + A_1) + \tau a'(t) = 0, \\ &b(t)(A_3 + A_1) + \tau b'(t) = 0, \\ &d(t)(A_3 + A_1) + \tau d'(t) = 0, \\ &e(t)(A_3 + A_1) + \tau e'(t) = 0, \\ &f(t) = f \equiv \text{constant}, \\ &g(t)(-A_1) + \tau g'(t) = 0. \end{aligned} \tag{6.39}$$

The associated Lie algebra is generated by the following symmetry generators

$$V_1 = x\partial_x + \frac{2t}{\alpha}\partial_t, \quad V_2 = \partial_x, \quad V_3 = u\partial_u, \quad V_4 = v\partial_v. \tag{6.40}$$

Further, the optimal set for (6.32) has the following components:

$$\Delta_1 = V_4 + rV_3 + sV_1, \quad \Delta_2 = V_3 + pV_1, \quad \Delta_3 = V_1. \tag{6.41}$$

Firstly, for the generator $\Delta_1 = V_4 + rV_3 + sV_1$, the symmetry reduction is presented by the following theorem:

Theorem 6.5. *The time fractional variant Boussinesq system* (6.32) *can be reduced* $\forall \alpha >$

0 *into the following nonlinear system of FODEs:*

$$\begin{aligned}
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{2}{\alpha}}F\right)(z)+\frac{cr}{s}\left(\frac{r}{s}-1\right)z^{-2}F(z)+\frac{a}{s}z^{\frac{1-r}{s}-1}G(z)+\frac{br}{s}z^{\frac{r}{s}-1}F^{2}(z)+\frac{2cr}{s}z^{-1}F'(z)\\
&+az^{\frac{1-r}{s}}G'(z)+bz^{\frac{r}{s}}(FF')(z)+cF''(z)=0,\\
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{2}{\alpha}}G\right)(z)+\frac{1}{s}\left(\frac{1}{s}-1\right)\left(fz+g\left(\frac{1}{s}-2\right)\right)z^{-3}G(z)+\frac{1}{s}\left(d+er\right)z^{\frac{r}{s}-1}(FG)(z)\\
&+\frac{1}{s}\left(2fz+3g\left(\frac{1}{s}-1\right)\right)z^{-2}G'(z)+z^{\frac{r}{s}}\left(dFG'+eGF'\right)(z)+\left(fz+\frac{3g}{s}\right)z^{-1}G''(z)+cG'''(z)=0,
\end{aligned}\tag{6.42}$$

where $\left(\mathcal{P}^{\zeta,\alpha}_{\delta}\right)$ *is the Erdélyi-Kober fractional differential operator.*

Proof. For Δ_1, the similarity transformations with similarity variables F, G can be obtained as follows:

$$z=xt^{-\frac{\alpha}{2}},\quad u=x^{\frac{r}{s}}F(z),\quad v=x^{\frac{1}{s}}G(z).\tag{6.43}$$

The definition of fractional derivative in Riemann-Liouville sense and $n-1<\alpha<n$ gives the following:

$$\partial_t^{\alpha}u=x^{\frac{r}{s}}\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}F(xs^{-\frac{\alpha}{2}})\mathrm{d}s\right].\tag{6.44}$$

Taking $\frac{t}{s}=w$, it can be written in the following form:

$$\begin{aligned}
\partial_t^{\alpha}u&=x^{\frac{r}{s}}\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_1^{\infty}(w-1)^{n-\alpha-1}w^{-(n-\alpha+1)}F(zw^{\frac{\alpha}{2}})\mathrm{d}w\right],\\
&=x^{\frac{r}{s}}\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(\mathcal{K}^{1,n-\alpha}_{\frac{2}{\alpha}}F\right)(z)\right].
\end{aligned}\tag{6.45}$$

Also, for $\alpha=n=1,2,...$ the result (6.45) holds using the definition of Erdélyi-Kober operator. Therefore, the fractional derivative of $u(x,t)$ is as follows:

$$\begin{aligned}
\frac{\partial^{\alpha}u}{\partial t^{\alpha}}&=x^{\frac{r}{s}}t^{-\alpha}\prod_{j=0}^{n-1}\left(1-\alpha+j-\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}^{1,n-\alpha}_{\frac{2}{\alpha}}F\right)(z),\qquad\forall\alpha>0,\\
&=x^{\frac{r}{s}}t^{-\alpha}\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{2}{\alpha}}F\right)(z),\qquad\forall\alpha>0.
\end{aligned}\tag{6.46}$$

Similarly, the following must hold for $v(x,t)$:

$$\frac{\partial^{\alpha}v}{\partial t^{\alpha}}=x^{\frac{1}{s}}t^{-\alpha}\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{2}{\alpha}}G\right)(z),\qquad\forall\alpha>0.\tag{6.47}$$

Hence, completing the proof. ☐

Consequently, for the remaining generators $\Delta_i (i = 2, 3)$, the following assertions can be proved:

Theorem 6.6. *For generator Δ_2, the system* (6.32) *is transformed $\forall \alpha > 0$ into the following fractional system of nonlinear ODEs:*

$$\begin{aligned}
&\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\alpha,\alpha} F\right)(z) + \frac{c}{p}\left(\frac{1}{p} - 1\right) z^{-2} F(z) + \frac{b}{p} z^{\frac{1}{p}-1} F^2(z) + \frac{2c}{p} z^{-1} F'(z) + a z^{-\frac{1}{p}} G'(z) \\
&+ b z^{\frac{1}{p}} (FF')(z) + cF''(z) = 0, \\
&\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\alpha,\alpha} G\right)(z) + \frac{e}{p} z^{\frac{1}{p}-1} (FG)(z) + z^{\frac{1}{p}} \left(dFG' + eGF'\right)(z) + fG''(z) + gG'''(z) = 0.
\end{aligned} \tag{6.48}$$

Theorem 6.7. *The similarity transformations for generator Δ_3 reduce the system* (6.32) *$\forall \alpha > 0$ into a nonlinear system of FODEs as follows:*

$$\begin{aligned}
&\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\alpha,\alpha} F\right)(z) + aG'(z) + b(FF')(z) + cF'''(z) = 0, \\
&\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\alpha,\alpha} G\right)(z) + d(FG')(z) + e(GF')(z) + fG''(z) + gG'''(z) = 0.
\end{aligned} \tag{6.49}$$

For brevity, the details have been omitted and the invariant solutions, the variable coefficients etc. have been illustrated in Table 6.3.

Table 6.3: Symmetry reductions of the system (6.32) into systems of FODEs

Generators	Invariants	Ansatze	Variable Coefficients	Reduced System
Δ_1	$(xt^{-\frac{\alpha}{2}}, x^{-\frac{r}{s}}u, x^{-\frac{1}{s}}v)$	$u = x^{\frac{r}{s}}F(z)$ $v = x^{\frac{1}{s}}G(z)$	$a(t) = at^{-\frac{(1-r+s)\alpha}{2s}}$ $b(t) = bt^{-\frac{(r+s)\alpha}{2s}}$, $d(t) = dt^{-\frac{(r+s)\alpha}{2s}}$ $e(t) = et^{-(r+s)\frac{\alpha}{2s}}$, $g(t) = gt^{\frac{\alpha}{2}}$	System (6.42)
Δ_2	$(xt^{-\frac{\alpha}{2}}, x^{-\frac{1}{p}}u, v)$	$u = x^{\frac{1}{p}}F(z)$ $v = G(z)$	$a(t) = at^{\frac{(1-p)\alpha}{2p}}$ $b(t) = bt^{-\frac{(1+p)\alpha}{2p}}$, $d(t) = dt^{-\frac{(1+p)\alpha}{2p}}$ $e(t) = et^{-(1+p)\frac{\alpha}{2p}}$, $g(t) = gt^{\frac{\alpha}{2}}$	System (6.48)
Δ_3	$(xt^{-\frac{\alpha}{2}}, u, v)$	$u = F(z)$ $v = G(z)$	$a(t) = at^{-\frac{\alpha}{2}}$ $b(t) = bt^{-\frac{\alpha}{2}}$, $d(t) = dt^{-\frac{\alpha}{2}}$ $e(t) = et^{-\frac{\alpha}{2}}$, $g(t) = gt^{-\frac{\alpha}{2}}$	System (6.49)

6.3.2 Time Fractional Coupled KdV System

The integer order as well as fractional order coupled KdV systems with variable coefficients have already been discussed using different methods in literature [97, 175]. The investigated time fractional coupled KdV system with variable coefficients [175] is as follows:

$$\begin{aligned} &\partial_t^\alpha u + a(t)uu_x + b(t)vv_x + c(t)u_{xxx} = 0,\\ &\partial_t^\alpha v + d(t)uv_x + e(t)vu_x + f(t)v_{xxx} = 0. \end{aligned} \tag{6.50}$$

The infinitesimal invariance conditions are obtained in the following form:

$$\begin{aligned} &\left[\eta^{\alpha,t} + a(t)(u_x\eta + u\eta^x) + b(t)(v_x\phi + v\phi^x) + c(t)\eta^{xxx} + \tau(a'(t)uu_x + b'(t)vv_x + c'(t)u_{xxx})\right]\Big|_{(6.50)} = 0,\\ &\left[\phi^{\alpha,t} + d(t)(v_x\eta + u\phi^x) + e(t)(u_x\phi + v\eta^x) + f(t)\phi^{xxx} + (d'(t)uv_x + e'(t)vu_x + f'(t)v_{xxx})\right]\Big|_{(6.50)} = 0. \end{aligned} \tag{6.51}$$

Solving the determining equations resulting form the associated invariance criterion, the explicit form of obtained Lie symmetries is as follows:

$$\xi = \frac{A_1x}{3} + A_2, \quad \tau = \frac{1}{c(t)^{\frac{1}{\alpha}}}\left(\frac{A_1}{\alpha}\int c(t)^{\frac{1}{\alpha}}\mathrm{d}t + A_5\right), \quad \eta = A_3(t)u, \quad \phi = A_4(t)v, \tag{6.52}$$

such that the following conditions must hold for the arbitrary constants A_1, A_2, A_5 and arbitrary functions $A_3(t), A_4(t)$:

$$\begin{aligned} &\tau(0) = 0, \quad A_3'(t) = A_4'(t) = \left(\frac{\alpha-1}{2}\right)\tau_{tt},\\ &a(t)\left(\alpha\tau_t - \frac{A_1}{3} - A_3(t)\right) + \tau a'(t) = 0,\\ &b(t)\left(\alpha\tau_t - \frac{A_1}{3} - A_3(t) + 2A_4(t)\right) + \tau b'(t) = 0,\\ &d(t)\left(\alpha\tau_t - \frac{A_1}{3} + A_3(t)\right) + \tau d'(t) = 0,\\ &e(t)\left(\alpha\tau_t - \frac{A_1}{3} + A_3(t)\right) + \tau e'(t) = 0,\\ &f(t)\left(\alpha\tau_t - \frac{A_1}{3}\right) + \tau f'(t) = 0. \end{aligned} \tag{6.53}$$

To find successful reductions of (6.50) into nonlinear systems of FODEs, taking $c(t) = c \equiv$ constant, the symmetries are obtained in the following simplified form:

$$\xi = \frac{A_1 x}{3} + A_2, \quad \tau = \frac{A_1 t}{\alpha}, \quad \eta = A_3 u, \quad \phi = A_4 v, \tag{6.54}$$

such that the following equations must be satisfied:

$$\begin{aligned}
&a(t)\left(\frac{2}{3}A_1 - A_3\right) + \tau a'(t) = 0,\\
&b(t)\left(\frac{2}{3}A_1 - A_3 + 2A_4\right) + \tau b'(t) = 0,\\
&d(t)\left(\frac{2}{3}A_1 + A_3\right) + \tau d'(t) = 0,\\
&e(t)\left(\frac{2}{3}A_1 + A_3\right) + \tau e'(t) = 0,\\
&f(t) = f, \quad \text{where } f \text{ is an arbitrary constant.}
\end{aligned} \tag{6.55}$$

The corresponding vector fields are as follows:

$$V_1 = \frac{x}{3}\partial_x + \frac{t}{\alpha}\partial_t, \quad V_2 = \partial_x, \quad V_3 = u\partial_u, \quad V_4 = v\partial_v. \tag{6.56}$$

The optimal set has the following linear combinations of generators:

$$\Delta_1 = V_4 + pV_3 + qV_1, \quad \Delta_2 = V_3 + rV_1, \quad \Delta_3 = V_1. \tag{6.57}$$

For these generators $\Delta_i (i = 1, 2, 3)$, the reduced systems are illustrated in the following theorems:

Theorem 6.8. *In case of generator* Δ_1, *the time fractional coupled KdV system* (6.50) *is reduced* $\forall \alpha > 0$ *into a system of nonlinear FODEs as follows:*

$$\begin{aligned}
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}} F\right)(z) + \frac{3cp}{q}\left(\frac{3p}{q} - 1\right)\left(\frac{3p}{q} - 2\right) z^{-3}F(z) + \frac{3ap}{q} z^{\frac{3p}{q}-1}F^2(z) + \frac{3b}{q} z^{\frac{6-3p}{q}-1}G^2(z)\\
&+ \frac{9cp}{q}\left(\frac{3p}{q} - 1\right) z^{-2}F'(z) + az^{\frac{3p}{q}}(FF')(z) + bz^{\frac{6-3p}{q}}(GG')(z) + \frac{9cp}{q} z^{-1}F''(z) + cF'''(z) = 0,\\
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}} G\right)(z) + \frac{3f}{q}\left(\frac{3}{q} - 1\right)\left(\frac{3}{q} - 2\right) z^{-3}G(z) + \frac{3}{q}(ep + d) z^{\frac{3p}{q}-1}(FG)(z)\\
&+ z^{\frac{3p}{q}}\left(dFG' + eGF'\right)(z) + \frac{9f}{q}\left(\frac{3}{q} - 1\right) z^{-2}G'(z) + \frac{9f}{q} z^{-1}G''(z) + fG'''(z) = 0.
\end{aligned} \tag{6.58}$$

Proof. Using the obtained invariants for Δ_1 given in Table 6.4, the definition of Riemann-Liouville fractional derivative gives the following:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} x^{\frac{3p}{q}} F(xs^{-\frac{\alpha}{3}})\mathrm{d}s\right]. \tag{6.59}$$

Further, it can be written as follows:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= x^{\frac{3p}{q}}\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(\mathcal{K}_{\frac{3}{\alpha}}^{1,n-\alpha}F\right)(z)\right],\\ &= x^{\frac{3p}{q}}t^{-\alpha}\prod_{j=0}^{n-1}\left(1-\alpha+j-\frac{\alpha}{3}\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}_{\frac{3}{\alpha}}^{1,n-\alpha}F\right)(z), \qquad \forall\alpha>0.\end{aligned} \tag{6.60}$$

Consequently, the expression for $\partial_t^\alpha u(x,t)$ in the Erdélyi-Kober fractional operator is obtained as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x^{\frac{3p}{q}}t^{-\alpha}\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\alpha,\alpha}F\right)(z), \qquad \forall\alpha>0. \tag{6.61}$$

Equivalently, $\partial_t^\alpha v$ can be obtained in the following form:

$$\frac{\partial^\alpha v}{\partial t^\alpha} = x^{\frac{3}{q}}t^{-\alpha}\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\alpha,\alpha}G\right)(z), \qquad \forall\alpha>0. \tag{6.62}$$

The result of theorem follows. □

Theorem 6.9. *For Δ_2, the system* (6.50) *can be reduced $\forall\alpha>0$ into the following nonlinear fractional system of ODEs:*

$$\begin{aligned}&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\alpha,\alpha}F\right)(z)+\frac{3c}{r}\left(\frac{3}{r}-1\right)\left(\frac{3}{r}-2\right)z^{-3}F(z)+\frac{3a}{r}z^{\frac{3}{r}-1}F^2(z)+\frac{9c}{r}\left(\frac{3}{r}-1\right)z^{-2}F'(z)\\ &+az^{\frac{3}{r}}(FF')(z)+bz^{-\frac{3}{r}}(GG')(z)+\frac{9c}{r}z^{-1}F''(z)+cF'''(z)=0,\\ &\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\alpha,\alpha}G\right)(z)+\frac{3e}{r}z^{\frac{3}{r}-1}(FG)(z)+z^{\frac{3}{r}}\left(dFG'+eGF'\right)(z)+fG'''(z)=0.\end{aligned} \tag{6.63}$$

Theorem 6.10. *The similarity transformations for Δ_3 reduce the system* (6.50) *$\forall\alpha>0$ into a nonlinear system of FODEs as follows:*

$$\begin{aligned}&\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\alpha,\alpha}F\right)(z)+a(FF')(z)+b(GG')(z)+cF'''(z)=0,\\ &\left(\mathcal{P}_{\frac{3}{\alpha}}^{1-\alpha,\alpha}G\right)(z)+d(FG')(z)+e(GF')(z)+fG'''(z)=0.\end{aligned} \tag{6.64}$$

The invariants, variable coefficients and symmetry reductions for all generators can be summarized in the following Table 6.4.

Table 6.4: Symmetry reductions of the system (6.50) into systems of FODEs

Generators	Invariants	Ansatze	Variable Coefficients	Reduced System
Δ_1	$(xt^{-\frac{\alpha}{3}}, x^{-\frac{3p}{q}}u, x^{-\frac{3}{q}}v)$	$u = x^{\frac{3p}{q}}F(z)$ $v = x^{\frac{3}{q}}G(z)$	$a(t) = at^{-\frac{(3p+2q)\alpha}{3q}}$ $b(t) = bt^{-\frac{(6-3p+2q)\alpha}{3q}}$ $d(t) = dt^{-\frac{(3p+2q)\alpha}{3q}}$ $e(t) = et^{-\frac{(3p+2q)\alpha}{3q}}$	System (6.58)
Δ_2	$(xt^{-\frac{\alpha}{3}}, x^{-\frac{3}{r}}u, v)$	$u = x^{\frac{3}{r}}F(z)$ $v = G(z)$	$a(t) = at^{\frac{(3+2r)\alpha}{3r}}$ $b(t) = bt^{-\frac{(-3+2r)\alpha}{3r}}$ $d(t) = dt^{-\frac{(3+2r)\alpha}{3r}}$ $e(t) = et^{-(3+2r)\frac{\alpha}{3r}}$	System (6.63)
Δ_3	$(xt^{-\frac{\alpha}{3}}, u, v)$	$u = F(z)$ $v = G(z)$	$a(t) = at^{-\frac{2\alpha}{3}}$ $b(t) = bt^{-\frac{2\alpha}{3}}$ $d(t) = dt^{-\frac{2\alpha}{3}}$ $e(t) = et^{-\frac{2\alpha}{3}}$	System (6.64)

6.3.3 Time Fractional Hirota-Satsuma Coupled KdV System

The Hirota-Satsuma coupled KdV system is an important system in the nonlinear physical phenomena [177]. The integer order system with variable coefficients has also been studied using the Lie group method [177]. The time fractional Hirota-Satsuma coupled KdV system with variable coefficients is written as follows:

$$\begin{aligned} &\partial_t^\alpha u + a(t)uu_x + b(t)(vw_x + wv_x) + c(t)u_{xxx} = 0,\\ &\partial_t^\alpha v + d(t)uv_x + e(t)v_{xxx} = 0, \qquad (6.65)\\ &\partial_t^\alpha w + f(t)uw_x + g(t)w_{xxx} = 0. \end{aligned}$$

The associated symmetry generator is of the following form:

$$V = \xi(x,t,u,v,w)\partial_x + \tau(x,t,u,v,w)\partial_t + \eta(x,t,u,v,w)\partial_u + \phi(x,t,u,v,w)\partial_v + \psi(x,t,u,v,w)\partial_w. \qquad (6.66)$$

Therefore, the invariance criterion can be written as follows:

$$\begin{aligned}
&\Big[\eta^{\alpha,t} + a(t)(\eta u_x + u\eta^x) + b(t)(\phi w_x + v\psi^x + \psi v_x + w\phi^x) + \tau(a'(t)uu_x + b'(t)(vw_x + wv_x)) \\
&+\tau c'(t)u_{xxx} + c(t)\eta^{xxx}\Big]\Big|_{(6.65)} = 0, \\
&\Big[\phi^{\alpha,t} + d(t)(\eta v_x + u\phi^x) + \tau(d'(t)uv_x + e'(t)v_{xxx}) + e(t)\phi^{xxx}\Big]\Big|_{(6.65)} = 0, \\
&\Big[\psi^{\alpha,t} + f(t)(\eta w_x + u\psi^x) + \tau(f'(t)uw_x + g'(t)w_{xxx}) + g(t)\psi^{xxx}\Big]\Big|_{(6.65)} = 0,
\end{aligned} \tag{6.67}$$

With the aid of the prolongation operators proposed in chapter 2, the over-determined system of PDEs as well as FDEs in the required group infinitesimals $\xi, \tau, \eta, \phi, \psi$ can be obtained as follows:

$$\begin{aligned}
&\xi_t = \xi_u = \xi_v = \xi_w = 0, \quad \tau_x = \tau_u = \tau_v = \tau_w = 0, \\
&\eta_{uu} = \eta_{vv} = \eta_{ww} = \eta_{uv} = \eta_{vw} = \eta_{uw} = 0, \quad \eta_{xv} = \eta_{xw} = \eta_{vt} = \eta_{wt} = 0, \\
&\phi_{uu} = \phi_{vv} = \phi_{ww} = \phi_{uv} = \phi_{vw} = \phi_{uw} = 0, \quad \phi_{xu} = \phi_{xw} = \phi_{ut} = \phi_{wt} = 0, \\
&\psi_{uu} = \psi_{vv} = \psi_{ww} = \psi_{uv} = \psi_{vw} = \psi_{uw} = 0, \\
&\psi_{xu} = \psi_{xw} = \psi_{ut} = \psi_{wt} = 0, \quad \eta_{xu} = \phi_{xv} = \psi_{xw} = \xi_{xx}, \\
&\binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad n \in \mathbb{N}, \\
&\binom{\alpha}{n}\partial_t^n \phi_v - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad n \in \mathbb{N}, \\
&\binom{\alpha}{n}\partial_t^n \psi_w - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau = 0, \quad n \in \mathbb{N}, \\
&a(t)u(\alpha\tau_t - \xi_x) + a(t)\eta + \tau a'(t)u + b(t)(w\phi_u + v\psi_u) + c(t)(3\eta_{xxu} - \xi_{xxx}) = 0, \\
&b(t)v(\alpha\tau_t - \xi_x + \psi_w - \eta_u) + b(t)\phi + \tau b'(t)v + a(t)u\eta_w + b(t)w\phi_w - f(t)u\eta_w = 0, \\
&b(t)w(\alpha\tau_t - \xi_x + \phi_v - \eta_u) + b(t)\psi + \tau b'(t)w + a(t)u\eta_v + b(t)v\psi_v - d(t)u\eta_v = 0, \\
&c(t)(\alpha\tau_t - 3\xi_x) + \tau c'(t) = 0, \\
&(d(t) - f(t))u\psi_v + b(t)w\psi_u = 0,
\end{aligned}$$

$$
\begin{aligned}
&d(t)u(\alpha\tau_t-\xi_x)+d(t)\eta+\tau d'(t)u-b(t)w\phi_u+e(t)(3\phi_{xxv}-\xi_{xxx})=0,\\
&e(t)(\alpha\tau_t-3\xi_x)+\tau e'(t)=0,\quad g(t)(\alpha\tau_t-3\xi_x)+\tau g'(t)=0,\\
&f(t)u(\alpha\tau_t-\xi_x)+f(t)\eta+\tau f'(t)u-b(t)v\psi_u+g(t)(3\psi_{xxw}-\xi_{xxx})=0,\\
&(c(t)-e(t))\,\eta_v=0,\quad (c(t)-g(t))\,\eta_w=0,\quad (d(t)-a(t))\,\phi_u=0,\\
&(e(t)-c(t))\,\phi_u=0,\quad (e(t)-g(t))\,\phi_w=0,\quad (d(t)-f(t))\,\phi_w=0,\\
&(g(t)-c(t))\,\psi_u=0,\quad (f(t)-a(t))\,\psi_u=0,\quad (g(t)-e(t))\,\psi_v=0,\\
&\partial_t^\alpha\eta-u\partial_t^\alpha\eta_u-v\partial_t^\alpha\eta_v-w\partial_t^\alpha\eta_w+a(t)u\eta_x+b(t)(w\phi_x+v\psi_x)+c(t)\eta_{xxx}=0,\\
&\partial_t^\alpha\phi-u\partial_t^\alpha\phi_u-v\partial_t^\alpha\phi_v-w\partial_t^\alpha\phi_w+d(t)u\phi_x+e(t)\phi_{xxx}=0,\\
&\partial_t^\alpha\psi-u\partial_t^\alpha\psi_u-v\partial_t^\alpha\psi_v-w\partial_t^\alpha\psi_w+f(t)u\psi_x+g(t)\psi_{xxx}=0.
\end{aligned}
\tag{6.68}
$$

Solving the determining equations yields the Lie symmetries in the following form:

$$
\begin{aligned}
&\xi=A_1x+A_2,\quad \tau=\frac{1}{c(t)^{\frac{1}{\alpha}}}\left(\frac{3A_1}{\alpha}\int c(t)^{1/\alpha}\mathrm{d}t+A_6\right),\quad \eta=A_3(t)u,\\
&\phi=A_4(t)v,\quad \psi=A_5(t)w,
\end{aligned}
\tag{6.69}
$$

where A_1, A_2, A_6 are arbitrary constants and $A_3(t), A_4(t), A_5(t)$ are arbitrary functions of t satisfying the following conditions:

$$
\begin{aligned}
&\tau(0)=0,\quad A_3'(t)=A_4'(t)=A_5'(t)=\left(\frac{\alpha-1}{2}\right)\tau_{tt},\\
&a(t)(\alpha\tau_t-A_1+A_3(t))+\tau a'(t)=0,\\
&b(t)(\alpha\tau_t-A_1-A_3(t)+A_4(t)+A_5(t))+\tau b'(t)=0,\\
&d(t)(\alpha\tau_t-A_1+A_3(t))+\tau d'(t)=0,\\
&e(t)=K_1c(t),\quad f(t)(\alpha\tau_t-A_1+A_3(t))+\tau f'(t)=0,\\
&g(t)=K_2c(t),
\end{aligned}
\tag{6.70}
$$

for K_1, K_2 being arbitrary constants. In particular $c(t)=c$ (a constant) is considered to overcome the difficulty in calculating the symmetry reductions. In this case, the symmetries are simplified having the following form:

$$\begin{aligned}&\xi = A_1x + A_2, \quad \tau = \frac{3t}{\alpha}A_1, \quad \eta = A_3u,\\&\phi = A_4v, \quad \psi = A_5w,\end{aligned} \tag{6.71}$$

for A_1, A_2, A_3, A_4, A_5 are all arbitrary constants such that the following conditions must be satisfied:

$$\begin{aligned}&a(t)(2A_1 + A_3) + \left(\frac{3t}{\alpha}A_1\right)a'(t) = 0,\\&b(t)(2A_1 - A_3 + A_4 + A_5) + \left(\frac{3t}{\alpha}A_1\right)b'(t) = 0,\\&d(t)(2A_1 + A_3) + \left(\frac{3t}{\alpha}A_1\right)d'(t) = 0,\\&e(t) = e \equiv \text{constant},\\&f(t)(2A_1 + A_3) + \left(\frac{3t}{\alpha}A_1\right)f'(t) = 0,\\&g(t) = g \equiv \text{constant}.\end{aligned} \tag{6.72}$$

The associated symmetry generators are written as follows:

$$\begin{aligned}&V_1 = x\partial_x + \frac{3t}{\alpha}\partial_t, \quad V_2 = \partial_x, \quad V_3 = u\partial_u,\\&V_4 = v\partial_v, \quad V_5 = w\partial_w.\end{aligned} \tag{6.73}$$

By a systematic calculation, the optimal set has the following components:

$$\Delta_1 = V_5 + kV_4 + lV_3 + mV_1, \quad \Delta_2 = V_4 + rV_3 + sV_1, \quad \Delta_3 = V_3 + \beta V_1, \quad \Delta_4 = V_1. \tag{6.74}$$

The above mentioned generators lead to the reduced systems presented by the following theorems. To obtain the symmetry reduction for Δ_1, firstly solving the corresponding characteristic equations the invariant solutions are as follows:

$$z = xt^{-\frac{\alpha}{3}}, \quad u = x^{l/m}F(z), \quad v = x^{k/m}G(z) \quad w = x^{1/m}H(z) \tag{6.75}$$

Theorem 6.11. *The similarity transformations for Δ_1 reduce the time fractional system*

(6.65) $\forall\alpha > 0$ *into the following fractional nonlinear system of ODEs:*

$$\begin{aligned}
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}F\right)(z) + \frac{3cl}{m}\left(\frac{l}{m}-1\right)\left(\frac{l}{m}-2\right)z^{-3}F(z) + \frac{al}{m}z^{\frac{l}{m}-1}F^2(z) + bz^{\left(\frac{(1+k-l)}{m}\right)-1}(1+k)(HG)(z)\\
&+\frac{3cl}{m}\left(\frac{l}{m}-1\right)z^{-2}F'(z) + az^{\frac{l}{m}}(FF')(z) + +bz^{\frac{(1+k-l)}{m}}\left(GH'+HG'\right)(z) + \frac{3cl}{m}z^{-1}F''(z)\\
&+cF'''(z) = 0,\\
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}G\right)(z) + \frac{dk}{m}z^{\frac{l}{m}-1}(FG)(z) + \frac{3ek}{m}\left(\frac{k}{m}-1\right)\left(\frac{k}{m}-2\right)z^{-3}G(z) + dz^{\frac{l}{m}}(FG')(z)\\
&+\frac{3ek}{m}\left(\frac{k}{m}-1\right)z^{-2}G'(z) + \frac{3ek}{m}z^{-1}G''(z) + eG'''(z) = 0,\\
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}H\right)(z) + fz^{\frac{l}{m}-1}(FH)(z) + \frac{3g}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)z^{-3}H(z) + fz^{\frac{l}{m}}(FH')(z)\\
&+\frac{3g}{m}\left(\frac{1}{m}-1\right)z^{-2}H'(z) + \frac{3g}{m}z^{-1}H''(z) + gH'''(z) = 0,
\end{aligned} \tag{6.76}$$

where $\left(\mathcal{P}^{\zeta,\alpha}_{\delta}\right)$ *is the Erdélyi-Kober fractional differential operator.*

Proof. By definition of Riemann-Liouville fractional derivative, we have the following:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{\frac{l}{m}}F(xs^{-\frac{\alpha}{3}})\mathrm{d}s\right]. \tag{6.77}$$

Setting $\rho = \frac{t}{s}$, the above expression can be written as follows:

$$\begin{aligned}
\frac{\partial^\alpha u}{\partial t^\alpha} &= x^{\frac{l}{m}}\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(\mathcal{K}^{1,n-\alpha}_{\frac{3}{\alpha}}F\right)(z)\right],\\
&= x^{\frac{l}{m}}t^{-\alpha}\prod_{j=0}^{n-1}\left(1-\alpha+j-\frac{\alpha}{3}\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}^{1,n-\alpha}_{\frac{3}{\alpha}}F\right)(z), \qquad \forall\alpha > 0.
\end{aligned} \tag{6.78}$$

Therefore, $\partial^\alpha_t u$ can be written in terms of Erdélyi-Kober fractional differential operators as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x^{\frac{l}{m}}t^{-\alpha}\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}F\right)(z), \qquad \forall\alpha > 0. \tag{6.79}$$

Similarly, the following expressions can be obtained:

$$\begin{aligned}
\frac{\partial^\alpha v}{\partial t^\alpha} &= x^{\frac{k}{m}}t^{-\alpha}\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}G\right)(z), \qquad \forall\alpha > 0,\\
\frac{\partial^\alpha w}{\partial t^\alpha} &= x^{\frac{1}{m}}t^{-\alpha}\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}H\right)(z), \qquad \forall\alpha > 0.
\end{aligned} \tag{6.80}$$

Solving the expressions for the variable coefficients, we have the following:

$$a(t) = at^{-\frac{(2m+l)\alpha}{3m}}, \quad b(t) = bt^{-\frac{(1+2m-l+k)\alpha}{3m}}, \quad d(t) = dt^{-\frac{(2m+l)\alpha}{3m}}, \quad f(t) = ft^{-\frac{(2m+l)\alpha}{3m}}. \tag{6.81}$$

Hence, using these variable coefficient functions, the result of theorem follows. □

In a similar manner, the symmetry reductions corresponding to the remaining generators Δ_i, $i = 2, 3, 4$ can be obtained.

Theorem 6.12. *The reduced nonlinear system of FODEs associated with the generator* Δ_2 $\forall \alpha > 0$ *is of the following form:*

$$\begin{aligned}
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}F\right)(z) + \frac{3cr}{s}\left(\frac{r}{s}-1\right)\left(\frac{r}{s}-2\right)z^{-3}F(z) + \frac{ar}{s}z^{\frac{r}{s}-1}F^2(z) + \frac{b}{s}z^{\frac{(1-r-s)}{s}}(HG)(z)\\
&+ \frac{3cr}{s}\left(\frac{r}{s}-1\right)z^{-2}F'(z) + az^{\frac{r}{s}}(FF')(z) + bz^{\frac{(1-r)}{s}}\left(GH'+HG'\right)(z) + \frac{3cr}{s}z^{-1}F''(z) + cF'''(z) = 0,\\
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}G\right)(z) + \frac{3e}{s}\left(\frac{1}{s}-1\right)\left(\frac{1}{s}-2\right)z^{-3}G(z) + \frac{d}{s}z^{\frac{r}{s}-1}(FG)(z) + \frac{3e}{s}\left(\frac{1}{s}-1\right)z^{-2}G'(z)\\
&+ dz^{\frac{r}{s}}(FG')(z) + \frac{3e}{s}z^{-1}G''(z) + eG'''(z) = 0,\\
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}H\right)(z) + fz^{\frac{r}{s}}(FH')(z) + gH'''(z) = 0.
\end{aligned} \tag{6.82}$$

Theorem 6.13. *The time fractional system* (6.65) *for the generator* Δ_3 *is reduced into a nonlinear system of FODEs written as follows:*

$$\begin{aligned}
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}F\right)(z) + \frac{3c}{\beta}\left(\frac{1}{\beta}-1\right)\left(\frac{1}{\beta}-2\right)z^{-3}F(z) + \frac{a}{\beta}z^{\frac{1}{\beta}-1}F^2(z) + \frac{3c}{\beta}\left(\frac{1}{\beta}-1\right)z^{-2}F'(z)\\
&+ az^{\frac{1}{\beta}}(FF')(z) + bz^{-\frac{1}{\beta}}\left(GH'+HG'\right)(z) + \frac{3c}{\beta}z^{-1}F''(z) + cF'''(z) = 0,\\
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}G\right)(z) + dz^{\frac{1}{\beta}}(FG')(z) + eG'''(z) = 0,\\
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}H\right)(z) + fz^{\frac{1}{\beta}}(FH')(z) + gH'''(z) = 0.
\end{aligned} \tag{6.83}$$

Theorem 6.14. *The invariant solutions for* Δ_4 *reduce the system* (6.65) $\forall \alpha > 0$ *into the following nonlinear system of ODEs of fractional order:*

$$\begin{aligned}
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}F\right)(z) + a(FF')(z) + b(GH'+HG')(z) + cF'''(z) = 0,\\
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}G\right)(z) + d(FG')(z) + eG'''(z) = 0,\\
&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}H\right)(z) + f(FH')(z) + gH'''(z) = 0.
\end{aligned} \tag{6.84}$$

Table 6.5 gives the brief discussion of the invariant solutions and symmetry reductions for the time fractional Hirota-Satsuma coupled KdV system (6.65) as below.

Table 6.5: Symmetry reductions of the system (6.65) into systems of FODEs

Generators	Invariants	Ansatze	Variable Coefficients	Reduced System
Δ_1	$(xt^{-\frac{\alpha}{3}}, x^{-\frac{l}{m}}u, x^{-\frac{k}{m}}v, x^{-\frac{1}{m}}w)$	$u = x^{\frac{l}{m}}F(z)$ $v = x^{\frac{k}{m}}G(z)$ $w = x^{\frac{1}{m}}H(z)$	$a(t) = at^{-\frac{(2m+l)\alpha}{3m}}$ $b(t) = bt^{-\frac{(1+2m-l+k)\alpha}{3m}}$ $d(t) = dt^{-\frac{(2m+l)\alpha}{3m}}$ $f(t) = ft^{-\frac{(2m+l)\alpha}{3m}}$	System (6.76)
Δ_2	$(xt^{-\frac{\alpha}{3}}, x^{-\frac{r}{s}}u, x^{-\frac{1}{s}}v, w)$	$u = x^{\frac{r}{s}}F(z)$ $v = x^{\frac{1}{s}}G(z)$ $w = H(z)$	$a(t) = at^{-\frac{(2s+r)\alpha}{3s}}$ $b(t) = bt^{-\frac{(1+2s-r)\alpha}{3s}}$ $d(t) = dt^{-\frac{(2s+r)\alpha}{3s}}$ $f(t) = ft^{-\frac{(2s+r)\alpha}{3s}}$	System (6.82)
Δ_3	$(xt^{-\frac{\alpha}{3}}, x^{-\frac{1}{\beta}}u, v, w)$	$u = x^{\frac{1}{\beta}}F(z)$ $v = G(z)$ $w = H(z)$	$a(t) = at^{-\frac{(2\beta+1)\alpha}{3\beta}}$ $b(t) = bt^{-\frac{(2\beta-1)\alpha}{3\beta}}$ $d(t) = dt^{-\frac{(2\beta+1)\alpha}{3\beta}}$ $f(t) = ft^{-\frac{(2\beta+1)\alpha}{3\beta}}$	System (6.83)
Δ_4	$(xt^{-\frac{\alpha}{3}}, u, v, w)$	$u = F(z)$ $v = G(z)$ $w = H(z)$	$a(t) = at^{-\frac{2\alpha}{3}}$ $b(t) = bt^{-\frac{2\alpha}{3}}$ $d(t) = dt^{-\frac{2\alpha}{3}}$ $f(t) = ft^{-\frac{2\alpha}{3}}$	System (6.84)

6.4 Conclusion

In this chapter, the efficiency of Lie symmetry approach has been discussed by successful investigation of two higher order time fractional nonlinear PDEs and three time fractional nonlinear systems of PDEs with variable coefficients. The complete group classification of the fourth order KdV-Burgers-Kuramoto equation and generalized seventh order KdV equation has been discussed resulting in their reduction into nonlinear FODEs in terms of Erdélyi-Kober operators. Also, the symmetry analysis of three time fractional nonlinear systems of PDEs with variable coefficients has been discussed. In comparison with the corresponding integer order systems, the generalized results have been obtained for the time fractional systems. The inequivalent generators of the optimal sets have been used for the group invariant solutions leading to reduced fractional nonlinear systems of ODEs.

Summary

The fractional calculus was first mentioned in a discussion between the pioneers of fractional calculus, Leibniz and L'Hospital. Initially, its study has been limited to the mathematics community since fractional order derivatives are generalization of the integer order derivatives. Despite having more than three hundred years of history and wide range of applications, only a few methods to study FDEs have been proposed. This thesis mainly aims at developing the Lie symmetry method for investigating a wide variety of FDEs with Riemann-Liouville fractional derivatives. This thesis not only consists of various extensions of symmetry approach but also presents a number of examples for its demonstration. The technique to find conservation laws has also been extended from integer order systems to fractional order systems in this thesis.

The investigation of FDEs using Lie symmetry method plays an important role in the nonlinear phenomena. As mentioned earlier, the application of Lie symmetry analysis is confined to single (1+1)-dimensional time fractional PDEs in literature. The unavailability of the required extended symmetry operators for symmetry analysis of other types of fractional PDEs is the reason behind this limitation. The efficiency of various types of fractional systems of PDEs in modelling various scientific phenomena was a great motivation to carry out the work in this thesis. The first extension of Lie symmetry method has been proposed in chapter 2 for time fractional (1+1)-dimensional

systems of PDEs. The symmetry analysis has been executed for deriving similarity reductions of some physically significant time fractional nonlinear systems of PDEs. This chapter also investigates these fractional nonlinear systems for determining their nonlinear self-adjointness conditions and conservation laws. For this aim, the technique to find conserved vectors has been developed by presenting the extended Noether's formulae.

Next step is to discover the extended symmetry approach applicable to study (1+1)-dimensional fractional PDEs with both space and time derivatives of fractional order. This has been achieved in chapter 3 by introducing the prolongation operators. Its application to some space-time fractional nonlinear PDEs has been discussed leading to the reduced fractional nonlinear ODEs. Also, the method to find conserved vectors for space-time fractional PDEs is developed. The local conservation laws of the considered fractional PDEs have been determined in this chapter by proving their nonlinear self-adjointness. To perform symmetry analysis of space-time fractional systems of PDEs, another extension of Lie classical method is proposed in chapter 4. The efficiency of proposed method is proved by its successful application to five space-time fractional nonlinear systems of PDEs.

In chapters mentioned before, the symmetry analysis of FDEs has been limited to only (1+1)-dimensional fractional systems of PDEs. However, the importance of higher dimensional fractional systems in modelling many physical phenomena demands the generalization of symmetry approach for their study. Chapter 5 aims to propose the symmetry method for exploring the FDEs involving an arbitrary number of independent as well as dependent variables. The derivation of required prolongation formulae makes it possible to study higher dimensional systems of ODEs as well as PDEs. The successful demonstration of derived Lie symmetries, invariant solutions leading to symmetry reductions of higher dimensional FDEs has been presented in this chapter. The nonlinear systems of FPDEs discussed include one (2+1)-dimensional and four (3+1)-dimensional physically relevant systems. There is another limitation in literature of absence of applications of Lie symmetry method to variable coefficient higher order FDEs and variable coefficient systems of FDEs. The aim of chapter 6 is to fill this gap by using two examples of higher order nonlinear FPDEs with variable coefficients and three nonlinear systems with variable coefficients.

It is worth mentioning that the ideas and results presented in this thesis are completely original and the literature studied for the completion of thesis is fully acknowledged. It has been noted in this thesis, the results obtained from the applications of symmetry method for FDEs are quite comparable to those of the method for integer order differential equations.

The objective of the present work is to propose the extensions and applications of the Lie symmetry method for wider class of FDEs resulting in their symmetries and similarity reductions. However, the reduced fractional order nonlinear differential equations have not been further solved for their exact solutions. The reason behind is that the reduced FDEs are in terms of Erdélyi-Kober operators, so solving these complex FDEs is a difficult task. Keeping in view this limitation, the general solutions of reduced FDEs for obtaining exact solutions of FDEs will be interesting for future investigation. The Lie classical symmetries for systems of FDEs have been derived in this thesis but the investigation of nonclassical symmetries, higher order symmetries and Lie Bäcklund symmetries for systems of FDEs is not discussed in literature. This brings forth a tremendous scope of future work. In this work, the Lie symmetry method is extended only for FDEs with

Riemann-Liouville fractional derivatives so it will be interesting to find its extensions for working with FDEs with other definitions of fractional derivatives. Also, the conservation laws for only (1+1)-dimensional time fractional systems and space-time fractional PDEs have been discussed in this thesis. In future, the conserved vectors for other types of fractional systems will be derived.

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