EFFICIENT ALGORITHMS FOR SOLVING SOME FUZZY NETWORK FLOW PROBLEMS

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CERTIFICATE

This is to certify that the thesis entitled, **"Efficient Algorithms for Solving Some Fuzzy Network Flow Problems"**, submitted by Ms. Manjot Kaur in the fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of candidate's own work carried out by her under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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DECLARATION

It is certified that the thesis is entirely my own and the ideas and references cited herein have been duly acknowledged.

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DEDICATED

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THE ALMIGHTY

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MY PARENTS

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Patiala (Manjot Kaur)

September 20, 2012

Abstract

In this thesis, shortcomings and limitations of existing methods for solving fuzzy maximum flow problems (maximum flow problems in which each arc capacity is represented by a fuzzy number) as well as the shortcomings and the limitations of the existing methods for solving fully fuzzy capacitated minimum cost flow problems (capacitated minimum cost flow problems in which all the parameters as well as the decision variables are represented by fuzzy numbers) are pointed out. To overcome the limitations of the existing methods as well as to resolve the shortcomings of the existing methods, new methods are proposed for solving fuzzy maximum flow problems, fully fuzzy single objective capacitated minimum cost flow problems, fully fuzzy multi-objective capacitated minimum cost flow problems, fully fuzzy multi-objective capacitated solid minimum cost flow problems and intuitionistic fully fuzzy single and multi-objective capacitated solid minimum cost flow problems.

The chapter wise summary of the thesis is as follows:

Chapter 1

In this chapter, a brief review of the work done in the area of fuzzy maximal flow problems as well as fuzzy single and multi-objective capacitated minimal cost flow problems is presented.

Chapter 2

In this chapter, shortcomings and limitations of the existing method [73]for solving fuzzy maximum flow problems are pointed out. Also, to overcome these shortcomings and limitations a new method, based on fuzzy linear programming formulation, is proposed for solving fuzzy maximum flow problems. To show the application of the proposed method in real life problems a real life fuzzy maximum flow problem is solved by using the proposed method.

Chapter 3

To the best of my knowledge, only the existing method [38] is proposed in literature to find the fuzzy optimal solution of fully fuzzy capacitated minimum cost flow problems. In this chapter, shortcomings and limitations of this method are pointed out. Also, to resolve the shortcomings as well as to overcome the limitations of the existing method [38], a new method is proposed for solving fully fuzzy capacitated minimum cost flow problems. The advantages of the proposed method over the existing method [38] are discussed. To illustrate the proposed method a fully fuzzy capacitated minimum cost flow problem is solved and to show the application of the proposed method in real life problems an existing real life fully fuzzy capacitated minimum cost flow problem is solved by using the proposed method.

Chapter 4

Gupta et al. [49] claimed that there is no method in the literature for solving fully fuzzy multi-objective transportation problems and proposed a method for the same. In this chapter, the limitations of this existing method [49] and the limitations of the method, proposed in Chapter 3, are pointed out and to overcome these limitations, a new method is proposed for solving fully fuzzy multi-objective capacitated minimum cost flow problems. Also, the advantages of the proposed method over the existing method [49] and over the method, proposed in Chapter 3, are discussed.

Chapter 5

In single and multi-objective capacitated minimum cost flow problems it is assumed that there is only one conveyance which can be used for transporting the product. However, in real life problems more than one conveyances are used for transporting the product. To the best of my knowledge till now there is no method in the literature for solving such fully fuzzy single and multi-objective capacitated minimum cost flow problems in which more than one conveyances are used for transporting the product. Since, in the literature [51] such transportation problems in which more than one conveyances are used for transporting the product are named as solid transportation problems so, in this chapter, such type of fully fuzzy single and multi-objective capacitated minimum cost flow problems in which more than one conveyances are used for transporting the product are named as fully fuzzy single and multi-objective capacitated solid minimum cost flow problems and a new method is proposed for solving these problems. The proposed method is illustrated with the help of a numerical example.

Chapter 6

In real life, a person may assume that an object belongs to a set but it is possible that he (she) is not sure about it. In other words, there may be hesitation or confusion that whether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or confusion. A possible solution is to use intuitionistic fuzzy set [9]. In the methods, proposed in previous chapters, an existing ranking approach for comparing fuzzy numbers is used for converting fully fuzzy linear programming problem into crisp linear programming problem. In this chapter, it is pointed out that it is not genuine to use any of the existing ranking approaches for comparing intuitionistic fuzzy numbers for converting intuitionistic fully fuzzy linear programming problem into crisp linear programming problem and a new ranking approach is proposed for comparing intuitionistic fuzzy numbers. Also, with the help of proposed ranking approach, a new method for solving intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problems as well as a new method for solving intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problems is proposed. The proposed methods are illustrated with the help of numerical examples.

Chapter 7

Finally, in this chapter, based on the presented study future work have been suggested.

List of Research Papers

1. **M. Kaur**, A. Kumar, Fuzzy optimal solution for unbalanced fully fuzzy minimal cost flow problems, International Journal of Fuzzy Systems, 14 (2012) 1-10.
2. **M. Kaur**, A. Kumar, Optimal compromise solution of multi-objective minimal cost flow problems in fuzzy environment, Applied Mathematical Modelling, (2012) DOI: 10.1016/j.apm.2012.04.040
3. **M. Kaur**, A. Kumar, Method for solving unbalanced fully fuzzy multi-objective solid minimal cost flow problems, Applied Intelligence, (2012) DOI: 10.1007/s10 489-012-0368-6
4. A. Kumar, **M. Kaur**, An improved algorithm for solving fuzzy maximal flow problems, International Journal of Applied Science and Engineering, 10 (2012) 19-27.
5. A. Kumar, **M. Kaur**, A new method for solving fuzzy shortest path problems, Journal of Applied Mathematics and Informatics, 30 (2012) 571-591.
6. A. Kumar, **M. Kaur**, Solution of fuzzy maximal flow problems using fuzzy linear programming, World Academy of Science Engineering and Technology, 78 (2011) 28-32.

7. A. Kumar, **M. Kaur**, An algorithm for solving fuzzy maximal flow problems using generalized triangular fuzzy numbers, International Journal of Hybrid Intelligent Systems, 8 (2011) 15-24.
8. A. Kumar, **M. Kaur**, A new algorithm for solving network flow problems with fuzzy arc lengths, Turkish Journal of Fuzzy Systems, 2 (2011) 1-13.
9. A. Kumar, **M. Kaur**, A new algorithm for solving shortest path problem on a network with imprecise edge weight, Applications and Applied Mathematics: An International Journal, 6 (2011) 602- 619.
10. A. Kumar, **M. Kaur**, A fuzzy linear programming approach to solve fuzzy maximal flow problems, International Journal of Physical and Mathematical Sciences, 1 (2010) 49-54.
11. A. Kumar, **M. Kaur**, An algorithm for solving fuzzy maximal flow problems using generalized trapezoidal fuzzy numbers, International Journal of Applied Science and Engineering, 8 (2010) 109-118.
12. A. Kumar, N. Bhatia, **M. Kaur**, A new approach for solving fuzzy maximal flow problems, Lecture Notes in Computer Science, Springer-Verlag Berlin Heidelberg, 5908 (2009) 278-286.
13. A. Kumar, **M. Kaur**, A. Kaur, A new ranking approach for intuitionistic fuzzy numbers and its application, Journal of Applied Research and Technology (Accepted).
14. **M. Kaur**, A. Kumar, Maximal flow problem for Jet Airlines under uncertain atmospheric conditions, Iranian Journal of Fuzzy Systems (Submitted after

revision).

15. A. Kumar, **M. Kaur**, A new method for solving intuitionistic fully fuzzy multi-objective solid minimal cost flow problems, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems (Communicated).

Papers Presented in International Conferences

1. A. Kumar, **M. Kaur**, A new algorithm for solving fuzzy shortest path problems, International Congress on Productivity, Quality, Reliability, Optimization and Modelling, Feb 7-8, 2011.

2. A. Kumar, N. Bhatia, **M. Kaur**, A new approach for solving fuzzy maximal flow problems, Twelfth International Conference on Rough Sets, Fuzzy Sets, Data Mining and Granular Computing, Dec 15-18, 2009.

3. A. Kumar, **M. Kaur**, Fuzzy maximal flow problems using trapezoidal fuzzy numbers, Eighth Trinnial Conference of Association of Asian Pacific Operational Research Societies, Dec 6-9, 2009.

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Chapter 1

INTRODUCTION

Network flow models provide a rich and powerful framework from which many engineering and management problems can be formulated and solved.

Minimum cost flow problem is a general form of the network flow problem whose aim is to find the least cost of the shipment of a commodity through a capacitated network in order to satisfy demands at certain nodes from available supplies at other nodes. The minimum cost flow problem provides a unified approach to many applications because of its general structure which includes, as special cases, the shortest path, maximum flow, work assignment, project scheduling and transportation problems.

The minimum cost flow problem is also very practical, it has been used to solve several real-world applicational problems such as multi-stage production inventory planning, mold allocation, nurse scheduling, project assignment, faculty course assignment, and automobile routing [3].

In classical form the minimum cost flow problem minimizes the cost of transporting some product that is available at some sources and required at some destinations. However, in most real world problems due to the complexity of the social and economic environment there may also be need to consider the explicit objective functions other than cost. These objectives are frequently in conflict, measured in different scales and difficult to combine in one overall utility function. Multi-objective minimum cost flow problems have been significantly studied in the literature [52].

In actual practice, the costs and the capacities of the network may not be known precisely due to insufficient information. To deal quantitatively with imprecise information, the concepts and techniques of probability could be employed. There are articles discussing the network flow problems where arc capacities are random variable. However, probability distributions require either a priori predictable regularity or a posteriori frequency determination to construct. Moreover, the premise that imprecision can be equated with randomness is still questionable [84].

As an alternative, uncertain values can be represented by membership functions of the fuzzy set theory [126]. The main advantages of methodologies based on fuzzy theory are that they do not require prior predictable regularities or posterior frequency distributions and they can deal with imprecise input information containing feelings and emotions quantified based on the decision-makers subjective judgment.

Due to the same reason in the literature different methods are proposed for solving network flow problems with fuzzy parameters.

1.1 Literature review

Although, maximum flow problems and minimum cost flow problems are important network flow problems but the solutions of these problems in fuzzy environment is almost neglected. Very few researchers have tried to develop the methods for solving these problems in fuzzy environment. In this section, a brief review of these methods is presented.

Chanas and Kolodziejczyk [21] presented an algorithm for a graph with crisp structure and fuzzy capacities, i.e., the arcs have a membership function associated in their flow. This problem was studied by Chanas and Kolodziejczyk [22] again, by assuming the flow as a real number and the capacities have upper and lower bounds with a satisfaction function. Chanas and Kolodziejczyk [23] studied the integer flow and proposed an algorithm. Chanas et al. [20] studied the maximum flow problem when the underlying associated structure is not well defined and must be modeled as a fuzzy graph. Diamond [29] developed interval-valued version of the maximum flow minimum cut theorem and provided robustness estimates for flows in networks in an imprecise or uncertain environment. Kumar et al. [73] proposed a method for solving such fuzzy maximum flow problems in which each arc capacity is represented by triangular (or trapezoidal) fuzzy numbers.

Shih and Lee [110] proposed a fuzzy version of minimum cost flow problems by using multi-level linear programming formulation. But, they did not use the nice structure of network constraints. Moreover, their scheme was inefficient because the multi-level programming which was made applying their scheme, is non-convex and so it is NP-hard. Liu and Kao [84] solved minimum cost flow problems with fuzzy costs using Yager's ranking index. Ji et al. [60] considered a generalized fuzzy version of maximum flow problems in which arc capacities are represented by fuzzy numbers.

Hashemi et al. [53] considered the minimum cost flow problem with interval costs and extended some combinatorial algorithms for this problem. Ghatee and Hashemi [38] transformed the fuzzy minimum cost flow problem into three crisp problems. Ghatee et al. [42] extended the duality theorems for the fully fuzzy minimum cost flow problem. Ghatee and Hashemi [39] studied three models; minimum cost flow problem with fuzzy costs, minimum cost flow problem with fuzzy supply-demands and combination of two cases. Ghatee et al. [43] converted minimum cost flow problem with fuzzy cost into into a three-objective minimum cost flow problem and provided two efficient algorithms to find the preemptive priority based solution(s) using a lexicographical ordering. Ghiyasvand [45] proposed a method to solve the minimum cost flow problem with interval and fuzzy data.

After reviewing the literature, it can be concluded that there are some shortcomings and limitations in the existing methods for solving fuzzy maximum flow problems as well as in the existing methods for solving fully fuzzy minimum cost flow problems. In this thesis, these shortcomings and limitations are pointed out. To overcome the limitations of the existing methods as well as to resolve the shortcomings of the existing methods, new methods are proposed for solving fuzzy maximum flow problems, fully fuzzy single objective capacitated minimum cost flow problems, fully fuzzy multi-objective capacitated minimum cost flow problems, fully fuzzy multi-objective capacitated solid minimum cost flow problems and intuitionistic fully fuzzy single and multi-objective capacitated solid minimum cost flow problems.

1.2 Organization of the thesis

The chapter wise summary of the thesis is as follows:

Chapter 2

In this chapter, shortcomings and limitations of the existing method [73] for solving fuzzy maximum flow problems are pointed out. Also, to overcome these shortcomings and limitations a new method, based on fuzzy linear programming formulation, is proposed for solving fuzzy maximum flow problems. To show the application of the proposed method in real life problems a real life fuzzy maximum flow problem is solved by using the proposed method.

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In real life, a person may assume that an object belongs to a set but it is possible that he (she) is not sure about it. In other words, there may be hesitation or confusion that whether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or confusion. A possible solution is to use intuitionistic fuzzy set [9]. In the methods, proposed in previous chapters, an existing ranking approach for comparing fuzzy numbers is used for converting fully fuzzy linear programming problem into crisp linear programming problem. In this chapter, it is pointed out that it is not genuine to use any of the existing ranking approaches for comparing intuitionistic fuzzy numbers for converting intuitionistic fully fuzzy linear programming problem into crisp linear programming problem and a new ranking approach is proposed for comparing intuitionistic fuzzy numbers. Also, with the help of proposed ranking approach, a new method for solving intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problems as well as a new method for solving intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problems is proposed. The proposed methods are illustrated with the help of numerical examples.

Chapter 7

Finally, in this chapter, based on the presented study future work have been suggested.

Chapter 2

A New Method For Solving Fuzzy Maximum Flow Problems

In this chapter, shortcomings and limitations of the existing method [73] for solving fuzzy maximum flow problems are pointed out. Also, to overcome these shortcomings and limitations a new method, based on fuzzy linear programming formulation, is proposed for solving fuzzy maximum flow problems. To show the application of the proposed method in real life problems a real life fuzzy maximum flow problem is solved by using the proposed method.

2.1 Preliminaries

In the literature [32, 130] it is pointed out that the computational efforts required to solve a fuzzy linear programming problem can be reduced, if the decision maker can express his subjective impression using *LR* flat fuzzy numbers. All kinds of crisp numbers, triangular and trapezoidal fuzzy numbers are *LR* flat fuzzy numbers.

In this section, some basic definitions and arithmetic operations of *LR* flat fuzzy numbers are presented.

The contents of this chapter are submitted after revision in *Iranian Journal of Fuzzy Systems.*

2.1.1 Basic definitions

In this section, some basic definitions are presented [32].

Definition 2.1 Let X be a classical set of objects. Then, the set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}} : X \to [0, 1]$, is called a fuzzy set in X. The evaluation function $\mu_{\tilde{A}}(x)$ is called the membership function.

Definition 2.2 Let $\tilde{A}$ be a fuzzy set in X and $\lambda \in [0, 1]$ be a real number. Then, the classical set $A^{\lambda} = \{x \in X : \mu_{\tilde{A}}(x) \geq \lambda\}$ is called a λ-cut of $\tilde{A}$.

Definition 2.3 A fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is called a normalized fuzzy set if and only if $\underset{x \in X}{\text{Supremum}}\{\mu_{\tilde{A}}(x)\} = 1$.

Definition 2.4 A fuzzy set $\tilde{A}$ is called a convex fuzzy set if and only if

$\mu_{\tilde{A}}(\alpha x_1 + (1 - \alpha)x_2) \geq \text{Minimum}\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\},\ \forall\ x_1, x_2 \in X,\ \alpha \in [0, 1]$.

Definition 2.5 A convex normalized fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is called a fuzzy number if and only if $\mu_{\tilde{A}}(x)$ is piecewise continuous in X.

Definition 2.6 A function $L : [0, \infty) \to [0, 1]$ (or $R : [0, \infty) \to [0, 1]$) is said to be reference function of fuzzy number if and only if

(i) $L(0) = 1$ (or $R(0) = 1$)

(ii) L (or R) is non-increasing on $[0, \infty)$

Definition 2.7 A fuzzy number $\widetilde{A}$ defined on universal set of real numbers $\mathbb{R}$, denoted as $(\underline{a}, \overline{a}, a^L, a^R)_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{\widetilde{A}}(x)$ is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\frac{\underline{a}-x}{a^L}), & x \leq \underline{a}, a^L > 0 \\ R(\frac{x-\overline{a}}{a^R}), & x \geq \overline{a}, a^R > 0 \\ 1, & \underline{a} \leq x \leq \overline{a} \end{cases}$$

Definition 2.8 Let $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ be an LR flat fuzzy number and α be a real number in the interval $[0,1]$. Then, the crisp set $A_\alpha = \{x \in X : \mu_{\widetilde{A}}(x) \geq \alpha\} = [\underline{a} - a^L L^{-1}(\alpha), \overline{a} + a^R R^{-1}(\alpha)]$, is said to be an α-cut of $\widetilde{A}$.

Definition 2.9 An LR flat fuzzy number $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is said to be a zero LR flat fuzzy number if and only if $\underline{a} = 0, \overline{a} = 0, a^L = 0$ and $a^R = 0$.

Definition 2.10 Two LR flat fuzzy numbers $\tilde{A}_1 = (\underline{a}_1, \overline{a}_1, a_1^L, a_1^R)_{LR}$ and $\tilde{A}_2 = (\underline{a}_2, \overline{a}_2, a_2^L, a_2^R)_{LR}$ are said to be equal i.e., $\tilde{A}_1 = \tilde{A}_2$ if and only if $\underline{a}_1 = \underline{a}_2, \overline{a_1} = \overline{a_2}, a_1^L = a_2^L$ and $a_1^R = a_2^R$.

Definition 2.11 An LR flat fuzzy number $\tilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is said to be a non-negative LR flat fuzzy number if and only if $\underline{a} - a^L \geq 0$.

Remark 2.1 If $\underline{a} = \overline{a} = a$ (say) then an LR flat fuzzy number $(\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is said to be an LR fuzzy number and is denoted as $(a, a^L, a^R)_{LR}$.

Remark 2.2 If $\underline{a} = \overline{a} = a$ (say) and $L(x) = R(x) =$ maximum $\{0, 1-x\}$ then an LR flat fuzzy number $(\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is said to be a triangular fuzzy number and is denoted as (a, b, c).

where, $a = \underline{a} - a^L$ (or, $\overline{a} - a^L$), $b = \underline{a}$ (or $\overline{a}$), $c = \underline{a} + a^R$ (or $\overline{a} + a^R$)

Remark 2.3 If $\underline{a} \neq \overline{a}$ and $L(x) = R(x) =$ maximum $\{0, 1-x\}$ then an LR flat fuzzy number $(\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is said to be a trapezoidal fuzzy number and is denoted as (a, b, c, d).

where, $a = \underline{a} - a^L, b = \underline{a}, c = \overline{a}, d = \overline{a} + a^R$

2.1.2 Arithmetic operations

In this section, some arithmetic operations between two LR flat fuzzy numbers are presented [32].

Let $\tilde{A}_1 = (\underline{a}_1, \overline{a}_1, a_1^L, a_1^R)_{LR}$ and $\tilde{A}_2 = (\underline{a}_2, \overline{a}_2, a_2^L, a_2^R)_{LR}$ be two LR flat fuzzy numbers. Then,

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (\underline{a}_1 + \underline{a}_2, \overline{a}_1 + \overline{a}_2, a_1^L + a_2^L, a_1^R + a_2^R)_{LR}$

(ii) $\lambda \tilde{A}_1 = \begin{cases} (\lambda \underline{a}_1, \lambda \overline{a}_1, \lambda a_1^L, \lambda a_1^R)_{LR} & \lambda \geq 0 \\ (\lambda \overline{a}_1, \lambda \underline{a}_1, -\lambda a_1^R, -\lambda a_1^L)_{RL} & \lambda \leq 0 \end{cases}$

Let $\tilde{A}_1 = (\underline{a}_1, \overline{a}_1, a_1^L, a_1^R)_{LR}$ and $\tilde{A}_2 = (\underline{a}_2, \overline{a}_2, a_2^L, a_2^R)_{LR}$ be two non-negative LR flat fuzzy numbers. Then,

(iii) $\tilde{A}_1 \otimes \tilde{A}_2 = (\underline{a}_1 \underline{a}_2, \overline{a}_1 \overline{a}_2, \underline{a}_1 a_2^L + \underline{a}_2 a_1^L - a_1^L a_2^L, \overline{a}_1 a_2^R + \overline{a}_2 a_1^R + a_1^R a_2^R)_{LR}$

2.1.3 Liou and Wang ranking approach

In this section, the existing ranking approach [82] used in the existing algorithm [73] for comparing fuzzy numbers is presented.

Let $\tilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ and $\tilde{B} = (\underline{b}, \overline{b}, b^L, b^R)_{LR}$ be two LR flat fuzzy numbers. Then,

(i) $\widetilde{A} \succ \widetilde{B}$ if $\Re(\widetilde{A}) > \Re(\widetilde{B})$

(ii) $\widetilde{A} \approx \widetilde{B}$ if $\Re(\widetilde{A}) = \Re(\widetilde{B})$

(iii) $\widetilde{A} \succeq \widetilde{B}$ if $\Re(\widetilde{A}) \geq \Re(\widetilde{B})$

where,

$\Re(\tilde{A}) = \lambda \int_0^1 (\underline{a} - a^L L^{-1}(\rho)) d\rho + (1-\lambda) \int_0^1 (\overline{a} + a^R R^{-1}(\rho)) d\rho$

$\Re(\tilde{B}) = \lambda \int_0^1 (\underline{b} - b^L L^{-1}(\rho)) d\rho + (1-\lambda) \int_0^1 (\overline{b} + b^R R^{-1}(\rho)) d\rho$

$\lambda \in [0, 1]$

Remark 2.4 If $\lambda = \frac{1}{2}$, $L(x) = R(x) = \text{maximum}\{0, 1-x\}$. Then, $\Re(\widetilde{A}) = \frac{1}{4}(2\underline{a} + 2\overline{a} - a^L + a^R)$

2.2 Existing fuzzy Ford and Fulkerson algorithm

Kumar et al. [73] proposed a Fuzzy Ford and Fulkerson algorithm by mod

-ifying the classical Ford and Fulkerson algorithm [114] for solving such maximum flow problems in which each arc capacity is represented by a triangular (or trapezoidal) fuzzy number

The steps of the existing algorithm [73] are as follows:

Step 1 For all arcs (i, j), set the residual fuzzy capacity equal to the initial fuzzy capacity i.e.,$(\tilde{f}c_{ij}, \tilde{f}c_{ji})$=$(\tilde{f}\bar{c}_{ij}, \tilde{f}\bar{c}_{ji})$. Let $\tilde{f}a_1 = (\infty, \infty, \infty)$ (or $(\infty, \infty, \infty, \infty)$) and label source 1 with $[(\infty, \infty, \infty), -]$ (or $[(\infty, \infty, \infty, , \infty), -]$). Set $i = 1$, and go to Step 2.

Step 2 Determine S_i, the set of unlabeled nodes j that can be reached directly from node i by arcs with positive residuals (i.e., $\tilde{f}c_{ij}$ is a non-negative triangular (or trapezoidal) fuzzy number for all $j \in S_i$). If $S_i \neq \phi$, go to Step 3. Otherwise, go to Step 4.

Step 3 Determine $k \in S_i$ such that

$$\max_{j \in S_i}\{\Re(\tilde{f}c_{ij})\} = \Re(\tilde{f}c_{ik})$$

Set $\tilde{f}a_k = \tilde{f}c_{ik}$ and label node k with $[\tilde{f}a_k, i]$. If $k = n$, the sink node has been labeled, and a breakthrough path is found, go to Step 5. Otherwise, set $i = k$, and go to Step 2.

Step 4 Backtracking: If $i = 1$, no breakthrough is possible; go to Step 6. Otherwise, let r be the node that has been labeled immediately before current node i and remove i from the set of nodes adjacent to r. Set $i = r$ and go to Step 2.

Step 5 Determination of Residuals: Let $N_p = \{1, k_1, k_2, \ldots, n\}$ define the nodes of the p^{th} breakthrough path from source node 1 to sink node n. Then, the maximum flow along the path is computed as

$$\tilde{f}_p = \min\{\tilde{f}a_1, \tilde{f}a_{k_1}, \tilde{f}a_{k_2}, \ldots, \tilde{f}a_n\}$$

The residual capacity of each arc along the breakthrough path is decreased by $\tilde{f}_p$ in the direction of the flow and increased by $\tilde{f}_p$ in the reverse direction i.e., for nodes i and j on the path, the residual flow is changed from the current$(\tilde{f}c_{ij}, \tilde{f}c_{ji})$ to

(a) $(\tilde{f}c_{ij} \ominus \tilde{f}_p, \tilde{f}c_{ji} \oplus \tilde{f}_p)$ if the flow is from i to j

(b) $(\tilde{f}c_{ij} \oplus \tilde{f}_p, \tilde{f}c_{ji} \ominus \tilde{f}_p)$ if the flow is from j to i

Reinstate any nodes that were removed in Step 4. Set $i = 1$, and return to Step 2 to attempt a new breakthrough path.

Step 6 Solution:

(a) Let m breakthrough paths be determined. Then, the fuzzy maximum flow in the network is

$$\widetilde{F} = \tilde{f}_1 \oplus \tilde{f}_2 \oplus \ldots \oplus \tilde{f}_m$$

where, m is the number of iteration to get no breakthrough.

(b) Using the initial and final fuzzy residuals of arc (i, j),$(\tilde{f}\bar{c}_{ij}, \tilde{f}\bar{c}_{ji})$ and $(\tilde{f}c_{ij}, \tilde{f}c_{ji})$ respectively, the fuzzy optimal flow in arc (i, j) is computed as follows:

Let $(\tilde{\alpha}, \tilde{\beta}) = (\tilde{f}\bar{c}_{ij} \ominus \tilde{f}c_{ij}, \tilde{f}\bar{c}_{ji} \ominus \tilde{f}c_{ji})$. If $\Re(\tilde{\alpha}) > 0$, the fuzzy optimal flow from i to j is $\tilde{\alpha}$. Otherwise, if $\Re(\tilde{\beta}) > 0$, the fuzzy optimal flow from j to i is $\tilde{\beta}$. (It is impossible to have both $\Re(\tilde{\alpha})$ and $\Re(\tilde{\beta})$ positive.)

2.3 Shortcomings and limitations of existing fuzzy Ford and Fulkerson algorithm

In this section, the shortcomings and the limitations of the existing fuzzy Ford and Fulkerson algorithm [73] are pointed out.

2.3.1 Shortcomings of existing fuzzy Ford and Fulkerson algorithm

In this section shortcomings of the existing algorithm [73] are pointed out.

(1) On solving the fuzzy maximum flow problem, chosen in Example 2.1, by using the existing algorithm [73] the obtained fuzzy maximum flow between source node 1 and destination node 5 is $(-15, 45, 105, 165)$.

It is obvious that there exist negative part in the trapezoidal fuzzy number $(-15, 45, 105, 165)$, representing the fuzzy maximum flow between source node 1 to destination node 5, which represents that the fuzzy maximum flow may be negative. But, in real life problems the negative quantity of the flow has no physical meaning. So, it is not genuine to use the existing algorithm [73] for solving fuzzy maximum flow problems.

Example 2.1 Determine the fuzzy maximum flow between node 1 (say source node) and node 5 (say destination node) for the network shown in Figure 2.1. The fuzzy capacities $(\tilde{u}_{ij})$ of the arcs (i, j) are represented by the following trapezoidal fuzzy numbers:

$\tilde{u}_{12} = (10, 20, 30, 40)$, $\tilde{u}_{13} = (15, 30, 45, 60)$, $\tilde{u}_{14} = (5, 10, 15, 20)$, $\tilde{u}_{23} = (30, 40, 50, 70)$, $\tilde{u}_{25} = (25, 30, 35, 40)$, $\tilde{u}_{34} = (5, 10, 15, 20)$, $\tilde{u}_{35} = (10, 20, 30, 40)$, $\tilde{u}_{45} = (0, 20, 30, 50)$ and remaining values of $\tilde{u}_{ij}$ are $(0, 0, 0, 0)$.

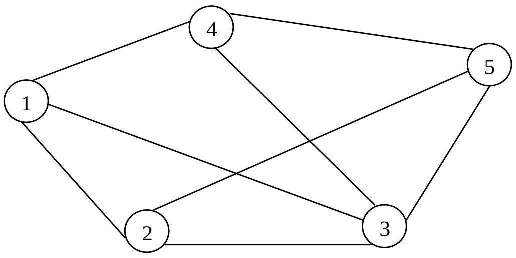

Figure 2.1 A network for the maximum flow problem

(2) Since, Garcia and Lamata [36] have proposed some modification in Liou and Wang [82] ranking approach. So, it is not genuine to apply Liou and Wang [82] ranking approach for comparing fuzzy numbers. However, in the existing algorithm [73], Liou and Wang [82] ranking approach is used for comparing fuzzy numbers so it is not genuine to use existing algorithm [73] for solving fuzzy maximum flow problems.

2.3.2 Limitations of the existing fuzzy Ford and Fulkerson algorithm

In the literature, it is pointed out that only a RL flat fuzzy number $\tilde{A}_2$ can be subtracted from an LR flat fuzzy number $\tilde{A}_1$ i.e., if $\tilde{A}_1$ and $\tilde{A}_2$ both are LR flat fuzzy numbers such that $L(\cdot) \neq R(\cdot)$ then $\tilde{A}_1 \ominus \tilde{A}_2$ does not exist. So, if all the fuzzy capacities of fuzzy maximum flow problem are represented by such LR flat fuzzy numbers for which $L(\cdot) \neq R(\cdot)$ then due to the existence of $\ominus$ in the expressions $(\tilde{f}c_{ij} \ominus \tilde{f}_p, \tilde{f}c_{ji} \oplus \tilde{f}_p)$, $(\tilde{f}c_{ij} \oplus \tilde{f}_p, \tilde{f}c_{ji} \ominus \tilde{f}_p)$ and $(\tilde{f}\bar{c}_{ij} \ominus \tilde{f}c_{ij})$, $(\tilde{f}\bar{c}_{ji} \ominus \tilde{f}c_{ji})$ used in Step 5 and Step 6 respectively, the existing fuzzy Ford and Fulkerson algorithm [73] can not be used e.g, the fuzzy maximum flow problem, chosen in Example 2.2, in which each arc capacity is represented by such LR flat fuzzy number for which $L(x) \neq R(x)$ can not be solved by using the existing fuzzy Ford and Fulkerson algorithm [73].

Example 2.2 Determine the fuzzy maximum flow between node 1 (say source node) and node 5 (say destination node) for the network as shown in Figure 2.1. The fuzzy capacities $(\tilde{u}_{ij})$ of the arcs (i, j) are represented by the following LR flat fuzzy numbers:

$\tilde{u}_{12} = (20, 30, 10, 10)_{LR}$, $\tilde{u}_{13} = (30, 45, 15, 15)_{LR}$, $\tilde{u}_{14} = (10, 15, 5, 5)_{LR}$, $\tilde{u}_{23} = (40, 50, 10, 20)_{LR}$, $\tilde{u}_{25} = (30, 35, 5, 5)_{LR}$, $\tilde{u}_{34} = (10, 15, 5, 5)_{LR}$, $\tilde{u}_{35} = (20, 30, 10, 10)_{LR}$,

$\tilde{u}_{45} = (20, 30, 20, 20)_{LR}$ and remaining values of $\tilde{u}_{ij}$ are $(0, 0, 0, 0)_{LR}$.

where, $L(x) = \text{maximum}\{0, 1 - x\}$, $R(x) = \text{maximum}\{0, 1 - x^4\}$

2.4 Garcia and Lamata ranking approach

In this section, Garcia and Lamata [36] ranking approach for comparing fuzzy numbers is presented.

Let $\tilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ and $\tilde{B} = (\underline{b}, \overline{b}, b^L, b^R)_{LR}$ be two LR flat fuzzy numbers. Then,

(i) $\widetilde{A} \succ \widetilde{B}$ if $\Re(\widetilde{A}) > \Re(\widetilde{B})$

(ii) $\widetilde{A} \approx \widetilde{B}$ if $\Re(\widetilde{A}) = \Re(\widetilde{B})$

(iii) $\widetilde{A} \succeq \widetilde{B}$ if $\Re(\widetilde{A}) \geq \Re(\widetilde{B})$

where, $\Re(\tilde{A}) = \gamma\Big[\int_0^1 (\underline{a}\lambda + \overline{a}(1-\lambda))d\rho\Big] + (1-\gamma)\Big[\lambda \int_0^1 (\underline{a} - a^L L^{-1}(\rho))d\rho + (1-\lambda)\int_0^1 (\overline{a} + a^R R^{-1}(\rho))d\rho\Big]$

$\Re(\tilde{B}) = \gamma\Big[\int_0^1 (\underline{b}\lambda + \overline{b}(1-\lambda))d\rho\Big] + (1-\gamma)\Big[\lambda \int_0^1 (\underline{b} - b^L L^{-1}(\rho))d\rho + (1-\lambda)\int_0^1 (\overline{b} + b^R R^{-1}(\rho))d\rho\Big]$

$\gamma,\ \lambda \in [0, 1]$

Remark 2.5 If $\gamma = \frac{1}{3}$, $\lambda = \frac{1}{2}$, $L(x) = \text{maximum}\{0, 1-x\}$ and $R(x) = \text{maximum}\{0, 1-x^4\}$. Then, $\Re(\widetilde{A}) = \frac{1}{2}(\underline{a} + \overline{a} - a^L) + \frac{4}{15}(a^R)$

If $\gamma = \frac{1}{3}$, $\lambda = \frac{1}{2}$ and $L(x) = R(x) = \text{maximum}\{0, 1 - x^4\}$. Then, $\Re(\widetilde{A}) = \frac{1}{2}(\underline{a} + \overline{a}) + \frac{4}{15}(a^R - a^L)$

If $\gamma = \frac{1}{3}$, $\lambda = \frac{1}{2}$ and $L(x) = R(x) = \text{maximum}\{0, 1 - x\}$. Then, $\Re(\widetilde{A}) = \frac{1}{2}(\underline{a} + \overline{a}) + \frac{1}{6}(a^R - a^L)$

Remark 2.6 For the ranking index '$\Re$', proposed by Garcia and Lamata [36] the property $\Re(k_1\tilde{A} \oplus k_2\tilde{B}) = k_1\Re(\tilde{A}) + k_2\Re(\tilde{B})\ \forall k_1, k_2 \in \mathbb{R}^+$ is satisfied.

2.5 Linear programming formulation of maximum flow problems

In this section, the linear programming formulation of maximum flow problems in crisp and fuzzy environment is presented.

2.5.1 Linear programming formulation of crisp maximum flow problems

Any crisp maximum flow problem can be formulated into the following crisp linear programming problem [112]:

$$\begin{aligned}
&\text{Minimize} \sum_{(i,j)\in A} u_{ij}x_{ij}\\
&\text{subject to}\\
&\quad v_t - v_s = 1\\
&\quad v_i - v_j + x_{ij} \geq 0 \qquad\qquad (P_{2.1})\\
&\quad x_{ij} \geq 0 \ \ \forall\ (i,j) \in A\\
&\quad v_i \text{ is unrestricted in sign } \forall\ i \in V
\end{aligned}$$

V : Set of nodes

A: The set of arcs (i,j)

v_i: The variable corresponding to node i (s and t represents the source and destination node respectively.)

x_{ij}: The decision variable denoting the flow through arc (i,j)

u_{ij}: The capacity for arc (i,j)

2.5.2 Proposed fuzzy linear programming formulation of fuzzy maximum flow problems

If the capacity of arc (i,j) is represented by fuzzy number $\tilde{u}_{ij}$ then the crisp linear programming problem $(P_{2.1})$ is converted into the fuzzy linear programming problem $(P_{2.2})$.

$$\begin{aligned}
&\text{Minimize} \sum_{(i,j)\in A} \tilde{u}_{ij}x_{ij}\\
&\text{subject to}\\
&\quad v_t - v_s = 1\\
&\quad v_i - v_j + x_{ij} \geq 0 \qquad\qquad (P_{2.2})\\
&\quad x_{ij} \geq 0 \ \ \forall\ (i,j) \in A\\
&\quad v_i \text{ is unrestricted in sign } \forall\ i \in V
\end{aligned}$$

$\tilde{u}_{ij}$: The fuzzy capacity for arc (i,j)

2.6 Proposed method based on fuzzy linear programming formulation

In this section, to overcome the shortcomings and limitations of the existing fuzzy Ford and Fulkerson algorithm [73], pointed out in Section 2.3, a new method, based on fuzzy linear programming formulation, is proposed for solving fuzzy maximum flow problems.

The steps of the proposed method are as follows:

Step 1 Assuming $\tilde{u}_{ij} = (\underline{u}_{ij}, \overline{u}_{ij}, u_{ij}^L, u_{ij}^R)_{LR}$ the fuzzy linear programming problem $(P_{2.2})$ can be written as:

$$\begin{aligned}
&\text{Minimize} \sum_{(i,j)\in A} (\underline{u}_{ij}, \overline{u}_{ij}, u_{ij}^L, u_{ij}^R)_{LR}x_{ij}\\
&\text{subject to}\\
&\quad v_t - v_s = 1\\
&\quad v_i - v_j + x_{ij} \geq 0 \qquad\qquad (P_{2.3})\\
&\quad x_{ij} \geq 0 \ \ \forall\ (i,j) \in A\\
&\quad v_i \text{ is unrestricted in sign } \forall\ i \in V
\end{aligned}$$

Step 2 Suppose the fuzzy linear programming problem $(P_{2.3})$ has k basic feasible solutions and $\{x^w_{ij}\}$ be the w^{th} basic feasible solution then the goal is to find the basic feasible solution with the smallest objective value i.e., $\underset{1\leq w\leq k}{\text{minimum}}\left\{\sum\limits_{(i,j)\in E}(\underline{u}_{ij},\overline{u}_{ij},u^L_{ij},u^R_{ij})_{LR}x^w_{ij}\right\}$. Garcia and Lamata [36] pointed out the shortcomings of several existing methods for comparing fuzzy numbers and proposed the concept that if $\underset{1\leq w\leq k}{\text{minimum}}\left\{\sum\limits_{(i,j)\in A}\Re(\underline{u}_{ij},\overline{u}_{ij},u^L_{ij},u^R_{ij})_{LR}x^w_{ij}\right\}$ is $\sum\limits_{(i,j)\in E}\Re(\underline{u}_{ij},\overline{u}_{ij},u^L_{ij},u^R_{ij})_{LR}x^\eta_{ij}$ then $\underset{1\leq w\leq k}{\text{minimum}}\left\{\sum\limits_{(i,j)\in A}(\underline{u}_{ij},\overline{u}_{ij},u^L_{ij},u^R_{ij})_{LR}x^w_{ij}\right\}$ will be $\sum\limits_{(i,j)\in E}(\underline{u}_{ij},\overline{u}_{ij},u^L_{ij},u^R_{ij})_{LR}x^\eta_{ij}$.

In other words, by using the existing method [36], the fuzzy optimal solution of the fuzzy linear programming problem $(P_{2.3})$ can be obtained by solving the following crisp linear programming problem:

$$
\begin{aligned}
&\text{Minimize} \sum_{(i,j)\in A}\Re(\underline{u}_{ij},\overline{u}_{ij},u^L_{ij},u^R_{ij})_{LR}x_{ij}\\
&\text{subject to}\\
&\quad v_t - v_s = 1\\
&\quad v_i - v_j + x_{ij} \geq 0 \qquad\qquad (P_{2.4})\\
&\quad x_{ij} \geq 0 \ \ \forall\ (i,j)\in A\\
&\quad v_i \text{ is unrestricted in sign } \forall\ i\in V
\end{aligned}
$$

Step 3 Using Section 2.4, the crisp linear programming problem $(P_{2.4})$ can be written as:

Minimize $[\gamma(\int_0^1(\underline{u}_{ij}\lambda+\overline{u}_{ij}(1-\lambda))d\rho)+(1-\gamma)(\lambda\int_0^1(\underline{u}_{ij}-u^L_{ij}L^{-1}(\rho))d\rho+(1-\lambda)\int_0^1(\overline{u}_{ij}+u^R_{ij}R^{-1}(\rho))d\rho)x_{ij}]$

subject to

$$
\begin{aligned}
&v_t - v_s = 1\\
&v_i - v_j + x_{ij} \geq 0 \qquad\qquad (P_{2.5})
\end{aligned}
$$

$x_{ij} \geq 0 \ \ \forall \ (i,j) \in A$

v_i is unrestricted in sign $\forall \ i \in V$

Step 4 Solve the crisp linear programming problem ($P_{2.5}$), to find the optimal value of x_{ij}. Put the optimal values of x_{ij} in objective function $\sum\limits_{(i,j)\in A} (\underline{u}_{ij}, \overline{u}_{ij}, u^L_{ij}, u^R_{ij})_{LR} x_{ij}$, to find the fuzzy maximum flow.

2.7 Advantages of the proposed method over the existing algorithm

The main advantage of the method proposed in this chapter over the existing algorithm [73] is that on solving the fuzzy maximum flow problems by using the proposed method all the shortcomings, occurring in the results, due to applying the existing algorithm [73] are resolved. Also, all the fuzzy maximum flow problems which can be solved by the existing algorithm [73] can also be solved by the method proposed in this chapter. However, as discussed in Section 2.3.2, there exist several fuzzy maximum flow problems which can not be solved by using the existing algorithm [73] but can be solved by the method proposed in this chapter. To show the advantage of the proposed method the fuzzy maximum flow problem, chosen in Example 2.2, which can not be solved by using the existing algorithm [73] is solved by using the proposed method.

2.7.1 Fuzzy optimal solution of the chosen problem

The fuzzy optimal solution of the fuzzy maximum flow problem, chosen in Example 2.2, by using the proposed method can be obtained as follows:

Step 1 The chosen fuzzy maximum flow problem can be formulated into the following fuzzy linear programming problem:

Minimize $((20,30,10,10)_{LR}x_{12} \oplus (30,45,15,15)_{LR}x_{13} \oplus (10,15,5,5)_{LR}x_{14} \oplus (40,50,10,20)_{LR}x_{23} \oplus (30,35,5,5)_{LR}x_{25} \oplus (10,15,5,5)_{LR}x_{34} \oplus (20,30,10,10)_{LR}x_{35} \oplus (20,30,20,20)_{LR}x_{45})$

subject to

$v_5 - v_1 = 1,$ $\quad v_1 - v_2 + x_{12} \geq 0,$ $\quad v_1 - v_3 + x_{13} \geq 0$

$v_1 - v_4 + x_{14} \geq 0,$ $\quad v_2 - v_3 + x_{23} \geq 0,$ $\quad v_2 - v_5 + x_{25} \geq 0$

$v_3 - v_4 + x_{34} \geq 0,$ $\quad v_3 - v_5 + x_{35} \geq 0,$ $\quad v_4 - v_5 + x_{45} \geq 0$

$v_2 - v_1 + x_{21} \geq 0,$ $\quad v_3 - v_1 + x_{31} \geq 0,$ $\quad v_4 - v_1 + x_{41} \geq 0$

$v_3 - v_2 + x_{32} \geq 0,$ $\quad v_5 - v_2 + x_{52} \geq 0,$ $\quad v_4 - v_3 + x_{43} \geq 0$

$v_5 - v_3 + x_{53} \geq 0,$ $\quad v_5 - v_4 + x_{54} \geq 0$

$x_{12}, x_{13}, x_{14}, x_{23}, x_{25}, x_{34}, x_{35}, x_{45}, x_{21}, x_{31}, x_{41}, x_{32}, x_{52}, x_{43}, x_{53}, x_{54} \geq 0$

v_1, v_2, v_3, v_4, v_5 are unrestricted in sign.

Step 2 Using Step 2 of the proposed method, the optimal solution of the fuzzy linear programming problem, obtained in Step 1, can be obtained by solving the following crisp linear programming problem:

Minimize $(\Re(20,30,10,10)_{LR}x_{12} + \Re(30,45,15,15)_{LR}x_{13} + \Re(10,15,5,5)_{LR}x_{14} + \Re(40,50,10,20)_{LR}x_{23} + \Re(30,35,5,5)_{LR}x_{25} + \Re(10,15,5,5)_{LR}x_{34} + \Re(20,30,10,10)_{LR}x_{35} + \Re(20,30,20,20)_{LR}x_{45})$

subject to

$v_5 - v_1 = 1,$ $\quad v_1 - v_2 + x_{12} \geq 0,$ $\quad v_1 - v_3 + x_{13} \geq 0$

$v_1 - v_4 + x_{14} \geq 0,$ $\quad v_2 - v_3 + x_{23} \geq 0,$ $\quad v_2 - v_5 + x_{25} \geq 0$

$v_3 - v_4 + x_{34} \geq 0,$ $\quad v_3 - v_5 + x_{35} \geq 0,$ $\quad v_4 - v_5 + x_{45} \geq 0$

$v_2 - v_1 + x_{21} \geq 0,$ $\quad v_3 - v_1 + x_{31} \geq 0,$ $\quad v_4 - v_1 + x_{41} \geq 0$

$v_3 - v_2 + x_{32} \geq 0,$ $\quad v_5 - v_2 + x_{52} \geq 0,$ $\quad v_4 - v_3 + x_{43} \geq 0$

$v_5 - v_3 + x_{53} \geq 0, \qquad v_5 - v_4 + x_{54} \geq 0$

$x_{12}, x_{13}, x_{14}, x_{23}, x_{25}, x_{34}, x_{35}, x_{45}, x_{21}, x_{31}, x_{41}, x_{32}, x_{52}, x_{43}, x_{53}, x_{54} \geq 0$

v_1, v_2, v_3, v_4, v_5 are unrestricted in sign.

Step 3 Using Remark 2.5, the crisp linear programming problem, obtained in Step 2, can be written as:

Minimize $(22.67x_{12} + 34x_{13} + 11.33x_{14} + 45.33x_{23} + 31.33x_{25} + 11.33x_{34} + 22.67x_{35} + 20.33x_{45})$

subject to

$v_5 - v_1 = 1, \qquad v_1 - v_2 + x_{12} \geq 0, \qquad v_1 - v_3 + x_{13} \geq 0$

$v_1 - v_4 + x_{14} \geq 0, \qquad v_2 - v_3 + x_{23} \geq 0, \qquad v_2 - v_5 + x_{25} \geq 0$

$v_3 - v_4 + x_{34} \geq 0, \qquad v_3 - v_5 + x_{35} \geq 0, \qquad v_4 - v_5 + x_{45} \geq 0$

$v_2 - v_1 + x_{21} \geq 0, \qquad v_3 - v_1 + x_{31} \geq 0, \qquad v_4 - v_1 + x_{41} \geq 0$

$v_3 - v_2 + x_{32} \geq 0, \qquad v_5 - v_2 + x_{52} \geq 0, \qquad v_4 - v_3 + x_{43} \geq 0$

$v_5 - v_3 + x_{53} \geq 0, \qquad v_5 - v_4 + x_{54} \geq 0$

$x_{12}, x_{13}, x_{14}, x_{23}, x_{25}, x_{34}, x_{35}, x_{45}, x_{21}, x_{31}, x_{41}, x_{32}, x_{52}, x_{43}, x_{53}, x_{54} \geq 0$

v_1, v_2, v_3, v_4, v_5 are unrestricted in sign.

Step 4 On solving the crisp linear programming problem, obtained in Step 3, the obtained optimal solution is $x_{12} = 1, x_{35} = 1, x_{45} = 1, x_{32} = 1, v_1 = -1, v_3 = -1, v_4 = -1$ and remaining are zero. Putting the values of x_{ij} in objective function $((20, 30, 10, 10)_{LR}x_{12} \oplus (30, 45, 15, 15)_{LR}x_{13} \oplus (10, 15, 5, 5)_{LR}x_{14} \oplus (40, 50, 10, 20)_{LR}$ $x_{23} \oplus (30, 35, 5, 5)_{LR}x_{25} \oplus (10, 15, 5, 5)_{LR}x_{34} \oplus (20, 30, 10, 10)_{LR}x_{35} \oplus (20, 30, 20, 20)_{LR}$ $x_{45})$ the fuzzy maximum flow between source node 1 and destination node 5 is $(60, 90, 40, 40)_{LR}$.

2.8 Comparative study

To show the advantages of proposed method over the existing algorithm [73]the fuzzy maximum flow for the problems, chosen in Example 2.1 and Example 2.2, obtained by using the existing algorithm [73] and the method, proposed in this chapter, are shown in Table 2.1.

Table 2.1 Results obtained by using the existing method and method proposed in this chapter

Example	Existing algorithm [73]	Method proposed in this chapter
2.1	$(-15, 45, 105, 165)$	$(30, 60, 90, 120)$
2.2	Not applicable	$(60, 90, 40, 40)_{LR}$

The results, presented in Table 2.1, can be explained as follows:

(1) The existing algorithm [73] can be used for solving such maximum flow problem in which each arc capacity is represented by triangular (or trapezoidal) fuzzy number. However, as discussed in Section 2.3.2, the existing algorithm [73] can not be used for solving such maximum flow problems in which each arc capacity is represented by *LR* flat fuzzy number (or *LR* fuzzy number). Since, in the problem, chosen in Example 2.1, each arc capacity is represented by a trapezoidal fuzzy number so, it can be solved by using the existing algorithm [73]. However, in the problem, chosen in Example 2.2, each arc capacity is represented by an *LR* flat fuzzy number. So, it can not be solved by using the existing algorithm [73].

(2) The method proposed in this chapter can be used for solving such fuzzy maximum flow problems in which each arc capacity is represented by an *LR* flat

fuzzy number. Since, triangular fuzzy numbers and trapezoidal fuzzy numbers are particular type of LR flat fuzzy numbers so such fuzzy maximum flow problems in which each arc capacity is represented by triangular (or trapezoidal) fuzzy number can also be solved by the proposed method. Due to the same reason both the problems, chosen in Example 2.1 and Example 2.2, can be solved by using the proposed method.

2.9 Case study

To show the application of the proposed method, a problem of Jet airways airline (major commercial air carrier offering passenger's service between most large cities in the India), to find the maximum number of flights from Amritsar to Mumbai for the coming month on the basis of the number of flights used in the previous months, is solved by using proposed method.

In airline system there always exist uncertainty about the number of flights due to atmospheric conditions (e.g., poor visibility due to thick cover of fog), technical reasons etc. Due to which the number of flights in the future required may not be represented by a real numbers. On the basis of the perception of the experts the appropriate numbers of monthly flights between pairs of cities, represented by LR flat fuzzy number, are shown in Table 2.2 and the different stoppage between Amritsar and Mumbai are shown in Figure 2.2.

Table 2.2. Fuzzy data for Jet airways airline

Jet airways airlines between different cities	Arc capacity for Jet airways airlines
Amritsar-Delhi (A, D)	$(140, 171, 22, 9)_{LR}$
Delhi-Ahmedabad (D, Ah)	$(160, 170, 10, 10)_{LR}$
Delhi-Lucknow (D, L)	$(155, 175, 8, 5)_{LR}$
Ahmedabad-Mumbai (Ah, M)	$(83, 88, 5, 2)_{LR}$
Lucknow-Mumbai (L, M)	$(23, 25, 3, 5)_{LR}$

where, $L(x) = \text{maximum}\{0, 1 - x\}$, $R(x) = \text{maximum}\{0, 1 - x\}$

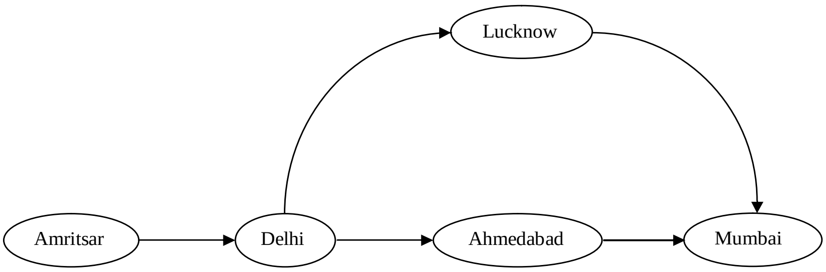

Figure 2.2 Stoppage between Amritsar and Mumbai

On solving the chosen fuzzy maximum flow problem by using the proposed method the obtained fuzzy maximum flow is $(106, 113, 8, 7)_{LR}$. Thus, Jet airways should arrange $(106, 113, 8, 7)_{LR}$ flights monthly connecting Amritsar and Mumbai.

2.9.1 Physical interpretation of the results

Using the proposed method, the maximum number of monthly flights connecting Amritsar to Mumbai is $(106, 113, 8, 7)_{LR}$, which can be physically interpreted as follows:

(1) The least number of maximum monthly flights connecting Amritsar to Mumbai is 98.

(2) The most possible number of maximum monthly flights connecting Amritsar to Mumbai lies between 106 and 113.

(3) The greatest number of maximum monthly flights connecting Amritsar to Mumbai ia 120 i.e., the maximum number of maximum monthly flights connecting Amritsar to Mumbai will always be greater than 98 and less than 120 and maximum chances are that the number of maximum monthly flights will lie between 106 and 113.

The variation in maximum number of monthly flights connecting Amritsar to Mumbai with respect to chances is shown in Figure 2.3.

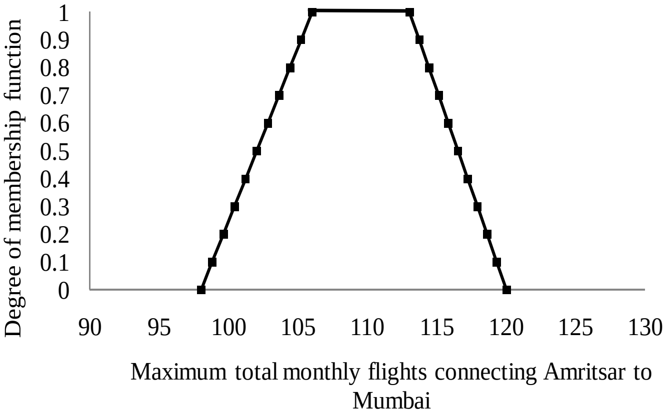

Figure 2.3 Membership function of *LR* flat fuzzy number representing the maximum total monthly flights connecting Amritsar to Mumbai

2.10 Conclusions

On the basis of the present study, it can be concluded that it is better to use proposed method as compared to the existing fuzzy Ford and Fulkerson algorithm [73] for solving fuzzy maximum flow problems.

Chapter 3

A New Method For Solving Fully Fuzzy Capacitated Minimum Cost Flow Problems

To the best of my knowledge, only the existing method [38] is proposed in literature to find the fuzzy optimal solution of fully fuzzy capacitated minimum cost flow problems. In this chapter, shortcomings and limitations of this method are pointed out. Also, to resolve the shortcomings as well as to overcome the limitations of the existing method [38], a new method is proposed for solving fully fuzzy capacitated minimum cost flow problems. The advantages of the proposed method over the existing method [38] are discussed. To illustrate the proposed method a fully fuzzy capacitated minimum cost flow problem is solved and to show the application of the proposed method in real life problems an existing real life fully fuzzy capacitated minimum cost flow problem is solved by using the proposed method.

3.1 Existing linear programming formulations of balanced crisp and fully fuzzy capacitated minimum cost flow problems

In this section, existing linear programming formulations of balanced crisp

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and fully fuzzy capacitated minimum cost flow problems are presented.

3.1.1 Classification of nodes

The nodes used in capacitated minimum cost flow problems can be categorized as follows:

Purely source node: A node S is said to be a purely source node if there exist at least one node S' such that the product may be supplied from S to S' but there does not exist any node S'' such that product may be supplied from S'' to S. The set of all such nodes is represented by N_{PS}.

Purely destination node: A node D is said to be a purely destination node if there does not exist any node D' such that the product may be supplied from D to D' but there exist at least one node D'' such that product may be supplied from D'' to D. The set of all such nodes is represented by N_{PD}.

Intermediate node: The following nodes in the network are said to be intermediate nodes:

(i) A node S at which some quantity of the product is available to transship at other nodes and also there exist some nodes such that some quantity of the product is supplied from those nodes to node S. All such intermediate nodes are said to be source nodes and the set of all such nodes is represented by N_S.

(ii) A node D at which some quantity of the product is required and also there exist some nodes such that the product is supplied from node D to those nodes. All such nodes D are said to be destination nodes and the set of all such intermediate nodes is represented by N_D.

(iii) A node T at which neither any quantity of the product is available to transship at other nodes nor any quantity of the product is required but there exist some nodes such that some quantity of the product is supplied from that nodes to node T and the same quantity of the product is supplied from T to some other nodes. All such nodes T are said to be transition nodes and the set of all such intermediate nodes is represented by N_T.

3.1.2 Existing linear programming formulation of balanced crisp capacitated minimum cost flow problems

Any balanced crisp capacitated minimum cost flow problem (total supply = total demand) can be formulated into the crisp linear programming problem $(P_{3.1})$ [3]:

$$\begin{aligned}
&\text{Minimize} \sum_{(i,j)\in A} (c_{ij}x_{ij})\\
&\text{subject to}\\
&\qquad \sum_{j:(i,j)\in A} x_{ij} = a_i && i \in N_{PS}\\
&\qquad \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = e_i && i \in N_S\\
&\qquad \sum_{i:(i,j)\in A} x_{ij} = b_j && j \in N_{PD} && (P_{3.1})\\
&\qquad \sum_{i:(i,j)\in A} x_{ij} - \sum_{i:(j,i)\in A} x_{ji} = d_j && j \in N_D\\
&\qquad \sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ji} && i \in N_T\\
&\qquad l_{ij} \le x_{ij} \le u_{ij},\ x_{ij} \ge 0 && \forall\ (i,j) \in A
\end{aligned}$$

where,

A: The set of arcs (i,j)

x_{ij}: Decision variable denoting the flow through arc (i,j)

c_{ij}: Cost per unit flow through arc (i,j)

a_i: Supply of the product at i^{th} purely source node

e_i: Supply of the product at i^{th} source node

b_j: Demand of the product at j^{th} purely destination node

d_j: Demand of the product at j^{th} destination node

l_{ij}: Minimum amount that can flow through arc (i,j)

u_{ij}: Maximum amount that can flow through arc (i,j)

Remark 3.1 A minimum cost flow problem is said to be un-capacitated minimum cost flow problem if $l_{ij} = 0$ and $u_{ij} = \infty \ \forall \ (i,j) \in A$. Otherwise, it is said to be capacitated minimum cost flow problem.

3.1.3 Existing linear programming formulation of balanced fully fuzzy capacitated minimum cost flow problems

Replacing the parameters x_{ij}, c_{ij}, a_i, e_i, b_j, d_j, l_{ij} and u_{ij} by $\tilde{x}_{ij}$, $\tilde{c}_{ij}$, $\tilde{a}_i$, $\tilde{e}_i$, $\tilde{b}_j$, $\tilde{d}_j$, $\tilde{l}_{ij}$ $(= \tilde{0})$ and $\tilde{u}_{ij}$ respectively, the crisp balanced capacitated minimum cost flow problem $(P_{3.1})$ is converted into the fuzzy linear programming problem $(P_{3.2})$ [38].

$$
\begin{aligned}
&\text{Minimize} \sum_{(i,j)\in A} (\tilde{c}_{ij} \otimes \tilde{x}_{ij})\\
&\text{subject to}\\
&\qquad \sum_{j:(i,j)\in A} \tilde{x}_{ij} = \tilde{a}_i && i \in N_{PS}\\
&\qquad \sum_{j:(i,j)\in A} \tilde{x}_{ij} \ominus_H \sum_{j:(j,i)\in A} \tilde{x}_{ji} = \tilde{e}_i && i \in N_S\\
&\qquad \sum_{i:(i,j)\in A} \tilde{x}_{ij} = \tilde{b}_j && j \in N_{PD} \qquad\qquad (P_{3.2})\\
&\qquad \sum_{i:(i,j)\in A} \tilde{x}_{ij} \ominus_H \sum_{i:(j,i)\in A} \tilde{x}_{ji} = \tilde{d}_j && j \in N_D\\
&\qquad \sum_{j:(i,j)\in A} \tilde{x}_{ij} = \sum_{j:(j,i)\in A} \tilde{x}_{ji} && i \in N_T\\
&\qquad \tilde{0} \preceq \tilde{x}_{ij} \preceq \tilde{u}_{ij} && \forall \ (i,j) \in A\\
&\qquad \tilde{x}_{ij} \text{ is a non-negative } LR \text{ fuzzy number } \forall \ (i,j) \in A
\end{aligned}
$$

where,

A: The set of arcs (i,j)

$\tilde{x}_{ij}$: Decision variable denoting the fuzzy flow through arc (i,j)

$\tilde{c}_{ij}$: Fuzzy cost per unit flow through arc (i,j)

$\tilde{a}_i$: Fuzzy supply of the product at i^{th} purely source node

$\tilde{e}_i$: Fuzzy supply of the product at i^{th} source node

$\tilde{b}_j$: Fuzzy demand of the product at j^{th} purely destination node

$\tilde{d}_j$: Fuzzy demand of the product at j^{th} destination node

$\tilde{l}_{ij}$: Minimum fuzzy amount that can flow through arc (i,j)

$\tilde{u}_{ij}$: Maximum fuzzy amount that can flow through arc (i,j)

3.2 Ghatee and Hashemi method

Ghatee and Hashemi [38] proposed a method to find the fuzzy optimal solution of such balanced fully fuzzy capacitated minimum cost flow problems for which $\tilde{l}_{ij} = \tilde{0}\ \forall\ (i,j) \in A$. Ghatee and Hashemi [39, 41] and Ghatee et al. [42] applied the existing method [38] for solving real life problems.

The steps of the existing method [38] are as follows:

Step 1 Assuming $\tilde{c}_{ij} = (c_{ij}, c_{ij}^L, c_{ij}^R)_{LR}$, $\tilde{x}_{ij} = (x_{ij}, x_{ij}^L, x_{ij}^R)_{LR}$, $\tilde{a}_i = (a_i, a_i^L, a_i^R)_{LR}$, $\tilde{e}_i = (e_i, e_i^L, e_i^R)_{LR}$, $\tilde{b}_j = (b_j, b_j^L, b_j^R)_{LR}$, $\tilde{d}_j = (d_j, d_j^L, d_j^R)_{LR}$, $\tilde{l}_{ij} = (0,0,0)_{LR}$ and $\tilde{u}_{ij} = (u_{ij}, u_{ij}^L, u_{ij}^R)_{LR}$ the fuzzy linear programming problem $(P_{3.2})$ can be written as:

Minimize $\sum\limits_{(i,j)\in A} ((c_{ij}, c_{ij}^L, c_{ij}^R)_{LR} \otimes (x_{ij}, x_{ij}^L, x_{ij}^R)_{LR})$

subject to

$$\begin{aligned}
&\sum_{j:(i,j)\in A} (x_{ij}, x_{ij}^L, x_{ij}^R)_{LR} = (a_i, a_i^L, a_i^R)_{LR} && i \in N_{PS} \\
&\sum_{j:(i,j)\in A} (x_{ij}, x_{ij}^L, x_{ij}^R)_{LR} \ominus_H \sum_{j:(j,i)\in A} (x_{ji}, x_{ji}^L, x_{ji}^R)_{LR} = (e_i, e_i^L, e_i^R)_{LR} && i \in N_S \\
&\sum_{i:(i,j)\in A} (x_{ij}, x_{ij}^L, x_{ij}^R)_{LR} = (b_j, b_j^L, b_j^R)_{LR} && j \in N_{PD} \quad (P_{3.3}) \\
&\sum_{i:(i,j)\in A} (x_{ij}, x_{ij}^L, x_{ij}^R)_{LR} \ominus_H \sum_{i:(j,i)\in A} (x_{ji}, x_{ji}^L, x_{ji}^R)_{LR} = (d_j, d_j^L, d_j^R)_{LR} && j \in N_D
\end{aligned}$$

$$\sum_{j:(i,j)\in A}(x_{ij},x_{ij}^L,x_{ij}^R)_{LR}=\sum_{j:(j,i)\in A}(x_{ji},x_{ji}^L,x_{ji}^R)_{LR} \qquad i\in N_T$$

$$(0,0,0)_{LR}\preceq(x_{ij},x_{ij}^L,x_{ij}^R)_{LR}\preceq(u_{ij},u_{ij}^L,u_{ij}^R)_{LR} \qquad \forall\ (i,j)\in A$$

$(x_{ij},x_{ij}^L,x_{ij}^R)_{LR}$ is a non-negative LR fuzzy number $\forall\ (i,j)\in A$

Step 2 Using the arithmetic operations [38] $(c_{ij},c_{ij}^L,c_{ij}^R)_{LR}\otimes(x_{ij},x_{ij}^L,x_{ij}^R)_{LR}=(c_{ij}x_{ij},c_{ij}x_{ij}^L+c_{ij}^Lx_{ij}-\kappa_1c_{ij}^Lx_{ij}^L,c_{ij}x_{ij}^R+c_{ij}^Rx_{ij}+\kappa_2c_{ij}^Rx_{ij}^R)_{LR}$ where, $\kappa_1=\frac{\int_0^1[L^{-1}(t)]^3dt}{\int_0^1[L^{-1}(t)]^2dt}$, $\kappa_2=\frac{\int_0^1[R^{-1}(t)]^3dt}{\int_0^1[R^{-1}(t)]^2dt}$ and $\sum_{j:(i,j)\in A}(x_{ij},x_{ij}^L,x_{ij}^R)_{LR}=(\sum_{j:(i,j)\in A}x_{ij},\sum_{j:(i,j)\in A}x_{ij}^L,\sum_{j:(i,j)\in A}x_{ij}^R)_{LR}$ the fuzzy linear programming problem $(P_{3.3})$, can be written as:

Minimize $\sum_{(i,j)\in A}(c_{ij}x_{ij},c_{ij}x_{ij}^L+c_{ij}^Lx_{ij}-\kappa_1c_{ij}^Lx_{ij}^L,c_{ij}x_{ij}^R+c_{ij}^Rx_{ij}+\kappa_2c_{ij}^Rx_{ij}^R)_{LR}$

subject to

$$(\sum_{j:(i,j)\in A}x_{ij},\sum_{j:(i,j)\in A}x_{ij}^L,\sum_{j:(i,j)\in A}x_{ij}^R)_{LR}=(a_i,a_i^L,a_i^R)_{LR} \qquad i\in N_{PS}$$

$$(\sum_{j:(i,j)\in A}x_{ij},\sum_{j:(i,j)\in A}x_{ij}^L,\sum_{j:(i,j)\in A}x_{ij}^R)_{LR}\ominus_H(\sum_{j:(j,i)\in A}x_{ji},\sum_{j:(j,i)\in A}x_{ji}^L,\sum_{j:(j,i)\in A}x_{ji}^R)_{LR}=(e_i,e_i^L,e_i^R)_{LR} \qquad i\in N_S$$

$$(\sum_{i:(i,j)\in A}x_{ij},\sum_{i:(i,j)\in A}x_{ij}^L,\sum_{i:(i,j)\in A}x_{ij}^R)_{LR}=(b_j,b_j^L,b_j^R)_{LR} \qquad j\in N_{PD} \quad (P_{3.4})$$

$$(\sum_{i:(i,j)\in A}x_{ij},\sum_{i:(i,j)\in A}x_{ij}^L,\sum_{i:(i,j)\in A}x_{ij}^R)_{LR}\ominus_H(\sum_{i:(j,i)\in A}x_{ji},\sum_{i:(j,i)\in A}x_{ji}^L,\sum_{i:(j,i)\in A}x_{ji}^R)_{LR}=(d_j,d_j^L,d_j^R)_{LR} \qquad j\in N_D$$

$$(\sum_{j:(i,j)\in A}x_{ij},\sum_{j:(i,j)\in A}x_{ij}^L,\sum_{j:(i,j)\in A}x_{ij}^R)_{LR}=(\sum_{j:(j,i)\in A}x_{ji},\sum_{j:(j,i)\in A}x_{ji}^L,\sum_{j:(j,i)\in A}x_{ji}^R)_{LR} \quad i\in N_T$$

$$(0,0,0)_{LR}\preceq(x_{ij},x_{ij}^L,x_{ij}^R)_{LR}\preceq(u_{ij},u_{ij}^L,u_{ij}^R)_{LR} \qquad \forall\ (i,j)\in A$$

$(x_{ij},x_{ij}^L,x_{ij}^R)_{LR}$ is a non-negative LR fuzzy number $\forall\ (i,j)\in A$

Step 3 Using Definition 2.10, Definition 2.11, Remark 2.1, Remark 3.2 and Remark 3.3, the fuzzy linear programming problem $(P_{3.4})$ can be converted into the fuzzy linear programming problem $(P_{3.5})$:

Minimize $\sum_{(i,j)\in A}(c_{ij}x_{ij},c_{ij}x_{ij}^L+c_{ij}^Lx_{ij}-\kappa_1c_{ij}^Lx_{ij}^L,c_{ij}x_{ij}^R+c_{ij}^Rx_{ij}+\kappa_2c_{ij}^Rx_{ij}^R)_{LR}$

subject to

$$
\begin{array}{lll}
\sum\limits_{j:(i,j)\in A} x_{ij} = a_i & i \in N_{PS} & \\
\sum\limits_{j:(i,j)\in A} x^L_{ij} = a^L_i & i \in N_{PS} & \\
\sum\limits_{j:(i,j)\in A} x^R_{ij} = a^R_i & i \in N_{PS} & \\
\sum\limits_{j:(i,j)\in A} x_{ij} - \sum\limits_{j:(j,i)\in A} x_{ji} = e_i & i \in N_S & \\
\sum\limits_{j:(i,j)\in A} x^L_{ij} - \sum\limits_{j:(j,i)\in A} x^L_{ji} = e^L_i & i \in N_S & \\
\sum\limits_{j:(i,j)\in A} x^R_{ij} - \sum\limits_{j:(j,i)\in A} x^R_{ji} = e^R_i & i \in N_S & \\
\sum\limits_{i:(i,j)\in A} x_{ij} = b_j & j \in N_{PD} & \\
\sum\limits_{i:(i,j)\in A} x^L_{ij} = b^L_j & j \in N_{PD} & (P_{3.5}) \\
\sum\limits_{i:(i,j)\in A} x^R_{ij} = b^R_j & j \in N_{PD} & \\
\sum\limits_{i:(i,j)\in A} x_{ij} - \sum\limits_{i:(j,i)\in A} x_{ji} = d_j & j \in N_D & \\
\sum\limits_{i:(i,j)\in A} x^L_{ij} - \sum\limits_{i:(j,i)\in A} x^L_{ji} = d^L_j & j \in N_D & \\
\sum\limits_{i:(i,j)\in A} x^R_{ij} - \sum\limits_{i:(j,i)\in A} x^R_{ji} = d^R_j & j \in N_D & \\
\sum\limits_{j:(i,j)\in A} x_{ij} = \sum\limits_{j:(j,i)\in A} x_{ji} & i \in N_T & \\
\sum\limits_{j:(i,j)\in A} x^L_{ij} = \sum\limits_{j:(j,i)\in A} x^L_{ji} & i \in N_T & \\
\sum\limits_{j:(i,j)\in A} x^R_{ij} = \sum\limits_{j:(j,i)\in A} x^R_{ji} & i \in N_T & \\
\multicolumn{2}{l}{0 \le x_{ij} \le u_{ij}, 0 \le x^L_{ij} \le u^L_{ij}, 0 \le x^R_{ij} \le u^R_{ij} \;\; \forall\ (i,j) \in A} & \\
\multicolumn{2}{l}{x_{ij} - x^L_{ij}, x^L_{ij}, x^R_{ij} \ge 0 \;\; \forall\ (i,j) \in A} &
\end{array}
$$

Step 4 The fuzzy optimal solution of fuzzy linear programming $(P_{3.5})$ can be obtained by solving the following crisp linear programming problem:

$$
\text{Minimize} \sum_{(i,j)\in A} \left(k(c_{ij}x_{ij}) + l(c_{ij}x^L_{ij} + c^L_{ij}x_{ij} - \kappa_1 c^L_{ij}x^L_{ij}) + r(c_{ij}x^R_{ij} + c^R_{ij}x_{ij} + \kappa_2 c^R_{ij}x^R_{ij})\right)
$$

subject to $(P_{3.6})$

constraints of problem $(P_{3.5})$

where, $k = q_1\vartheta^{n_1}, l = q_2\vartheta^{n_2}$ and $r = q_3\vartheta^{n_3}$, $q_1, q_2, q_3 \in \mathbb{Q}^+$, $n_1 \ne n_2 \ne n_3$ are non-negative integers and ϑ is a non-algebraic positive real number.

Step 5 Solve the crisp linear programming problem $(P_{3.6})$ to find the optimal solution $\{x_{ij}, x_{ij}^L, x_{ij}^R\}$.

Step 6 Put the obtained optimal values of x_{ij}, x_{ij}^L and x_{ij}^R in $\tilde{x}_{ij} = (x_{ij}, x_{ij}^L, x_{ij}^R)_{LR}$ to find the fuzzy optimal solution $\{\tilde{x}_{ij}\}$.

Step 7 Put the fuzzy optimal values of $\tilde{x}_{ij}$, obtained from Step 6, in $\sum\limits_{(i,j)\in A} (\tilde{c}_{ij} \otimes \tilde{x}_{ij})$, to find the minimum total fuzzy transportation cost.

Remark 3.2 Ghatee and Hashemi [38] have replaced the fuzzy restriction $\tilde{0} \preceq \tilde{x}_{ij} \preceq \tilde{u}_{ij}$ i.e., $(0,0,0)_{LR} \preceq (x_{ij}, x_{ij}^L, x_{ij}^R)_{LR} \preceq (u_{ij}, u_{ij}^L, u_{ij}^R)_{LR}$ by the following crisp restrictions:

$$0 \leq x_{ij} \leq u_{ij}, 0 \leq x_{ij}^L \leq u_{ij}^L, 0 \leq x_{ij}^R \leq u_{ij}^R$$

Remark 3.3 [38] Let $\tilde{A} = (a, a^L, a^R)_{LR}$ and $\tilde{B} = (b, b^L, b^R)_{LR}$ be two LR fuzzy numbers such that $a^L \geq b^L$ and $a^R \geq b^R$. Then, $\tilde{A} \ominus_H \tilde{B} = (a-b, a^L - b^L, a^R - a^R)_{LR}$.

Remark 3.4 [41] Let $\tilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ and $\tilde{B} = (\underline{b}, \overline{b}, b^L, b^R)_{LR}$ be two LR flat fuzzy numbers such that $a^L \geq b^L$ and $a^R \geq b^R$. Then, $\tilde{A} \ominus_H \tilde{B} = (\underline{a} - \underline{b}, \overline{a} - \overline{b}, a^L - b^L, a^R - b^R)_{LR}$.

3.3 Shortcomings of existing linear programming formulation

In this section, the shortcomings of the existing linear programming formulation $(P_{3.2})$ of balanced fully fuzzy capacitated minimum cost flow problems is pointed out.

In the formulation $(P_{3.2})$ the following type of equation is used:

$$\tilde{A} \ominus_H \tilde{B} = \tilde{C} \tag{3.1}$$

where, $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are LR flat fuzzy numbers i.e., in the existing formulation

$(P_{3.2})$, it is assumed that if $\tilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ and $\tilde{B} = (\underline{b}, \overline{b}, b^L, b^R)_{LR}$ are two LR flat fuzzy numbers such that $a^L \geq b^L$ and $a^R \geq b^R$ then Hukuhara's difference $\tilde{C} = \tilde{A} \ominus_H \tilde{B} = (\underline{a} - \underline{b}, \overline{a} - \overline{b}, a^L - b^L, a^R - b^R)_{LR}$ will also be an LR flat fuzzy number. However, the Hukuhara's difference of two LR flat fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is not necessarily an LR flat fuzzy number e.g., for the LR flat fuzzy numbers $\tilde{A} = (2, 5, 3, 4)_{LR}$ and $\tilde{B} = (1, 6, 1, 2)_{LR}$, $\tilde{C} = \tilde{A} \ominus_H \tilde{B} = (1, -1, 2, 2)_{LR}$ is not an LR flat fuzzy number.

Since, in the constraints of fuzzy linear programming formulation $(P_{3.2})$ the Hukuhara's difference $\tilde{A} \ominus_H \tilde{B}$ is used. So, it is not genuine to use the existing formulation $(P_{3.2})$ for solving fully fuzzy capacitated minimum cost flow problems.

3.4 Shortcomings of Ghatee and Hashemi method

It is not genuine to apply the existing method [38], presented in Section 3.2, for solving fully fuzzy capacitated minimum cost flow problems due to the following reasons:

(1) To solve the fuzzy linear programming problem $(P_{3.2})$, Ghatee and Hashemi [38] pointed out that the average, left and right spreads of a feasible flow have to be less than the maximal value of the corresponding quantities. Hence, in Step 3 of Ghatee and Hashemi [38] method the fuzzy restriction $\tilde{0} \preceq \tilde{x}_{ij} \preceq \tilde{u}_{ij}$ i.e., $(0, 0, 0)_{LR} \preceq (x_{ij}, x_{ij}^L, x_{ij}^R)_{LR} \preceq (u_{ij}, u_{ij}^L, u_{ij}^R)_{LR}$ are replaced by the crisp restrictions $0 \leq x_{ij} \leq u_{ij}, 0 \leq x_{ij}^L \leq u_{ij}^L, 0 \leq x_{ij}^R \leq u_{ij}^R$ i.e., in Step 3 of the Ghatee and Hashemi method, it is assumed that if $\tilde{A} = (a, a^L, a^R)_{LR}$ and $\tilde{B} = (b, b^L, b^R)_{LR}$ are two LR fuzzy numbers then $\tilde{A} \preceq \tilde{B}$ if and only if $a \leq b, a^L \leq b^L, a^R \leq b^R$ i.e.,

$$\text{minimum}\{\tilde{A}, \tilde{B}\} = \begin{cases} \tilde{A} & \text{if } a \leq b, a^L \leq b^L, a^R \leq b^R \\ \tilde{B} & \text{if } b \leq a, b^L \leq a^L, b^R \leq a^R \end{cases}$$

While, in the Step 4 of the Ghatee and Hashemi method where, there is need to find such feasible solution out of all the possible feasible solutions corresponding to which the value of the objective function is minimum i.e., to find the minimum of fuzzy numbers, representing the values of objective function corresponding to all the feasible solutions, Ghatee and Hashemi [38] have assumed that if $\tilde{A} = (a, a^L, a^R)_{LR}$ and $\tilde{B} = (b, b^L, b^R)_{LR}$ are two LR fuzzy numbers then $\tilde{A} \preceq \tilde{B}$ if and only if $ka + la^L + ra^R \leq kb + lb^L + rb^R$ i.e.,

$$\text{minimum}\{\tilde{A}, \tilde{B}\} = \begin{cases} \tilde{A} & \text{if } ka + la^L + ra^R \leq kb + lb^L + rb^R \\ \tilde{B} & \text{if } kb + lb^L + rb^R \leq ka + la^L + ra^R \end{cases}$$

Hence, in the same method two different approaches are used for finding the minimum of LR fuzzy numbers which is not genuine. Although, it seems that this shortcoming of the Ghatee and Hashemi method can be resolved by using the same method for finding the minimum of fuzzy numbers in both steps. But, it is not possible to do so due to the following reasons:

If $\tilde{A} = (a, a^L, a^R)_{LR}$ and $\tilde{B} = (b, b^L, b^R)_{LR}$ are values of the objective function corresponding to two feasible solutions of the fuzzy linear programming problem $(P_{3.5})$ such that neither $a \leq b, a^L \leq b^L, a^R \leq b^R$ nor $b \leq a, b^L \leq a^L, b^R \leq a^R$ then the minimum$\{\tilde{A}, \tilde{B}\}$ i.e., fuzzy optimal value of the fuzzy linear programming problem $(P_{3.5})$ can not be obtained by using

$$\text{minimum}\{\tilde{A}, \tilde{B}\} = \begin{cases} \tilde{A} & \text{if } a \leq b, a^L \leq b^L, a^R \leq b^R \\ \tilde{B} & \text{if } b \leq a, b^L \leq a^L, b^R \leq a^R \end{cases}$$

e.g., $\tilde{x}_1 = (1, 0, 0)_{LR}, \tilde{x}_2 = (0, 0, 0)_{LR}, \tilde{x}_3 = (1, 0, 0)_{LR}$ and $\tilde{x}_1 = (0, 0, 0)_{LR}, \tilde{x}_2 = (1, 0, 0)_{LR}, \tilde{x}_3 = (0, 0, 0)_{LR}$ both are the feasible solutions of the fuzzy linear programming problem $(P_{3.7})$ and the values of the objective function corresponding to these feasible solutions are $(10,4,1)_{LR}$ and $(9,2,2)_{LR}$ respectively. Since, in these fuzzy numbers neither the condition $a \leq b, a^L \leq b^L, a^R \leq b^R$ nor the condition $b \leq a, b^L \leq a^L, b^R \leq a^R$ is satisfying so, it is not possible to find the minimum of fuzzy numbers $(10,4,1)_{LR}$ and $(9,2,2)_{LR}$.

Minimize $((5,1,1)_{LR} \otimes \tilde{x}_1 \oplus (10,4,1)_{LR} \otimes \tilde{x}_2 \oplus (4,1,1)_{LR} \otimes \tilde{x}_3)$

subject to

$\tilde{x}_1 \oplus \tilde{x}_2 = (1,0,0)_{LR}$

$\tilde{x}_1 = \tilde{x}_3$ $(P_{3.7})$

$\tilde{x}_2 \oplus \tilde{x}_3 = (1,0,0)_{LR}$

$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are non-negative LR fuzzy numbers.

(2) Since, the method minimum$\{\tilde{A}, \tilde{B}\} = \begin{cases} \tilde{A} & \text{if } ka + la^L + ra^R \leq kb + lb^L + rb^R \\ \tilde{B} & \text{if } kb + lb^L + rb^R \leq ka + la^L + ra^R \end{cases}$ can be used only for comparing LR fuzzy numbers so if the parameters are replaced by LR flat fuzzy numbers then it is not possible to apply this method.

3.5 Limitations of Ghatee and Hashemi method

Since, in Step 3 of the existing method [38], discussed in Section 3.2, the fuzzy restriction $\tilde{l}_{ij} \preceq \tilde{x}_{ij} \preceq \tilde{u}_{ij}$ i.e., $(\underline{l}_{ij}, \overline{l}_{ij}, l^L_{ij}, l^R_{ij})_{LR} \preceq (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} \preceq (\underline{u}_{ij}, \overline{u}_{ij}, u^L_{ij}, u^R_{ij})_{LR}$ is replaced by the crisp restrictions $\underline{l}_{ij} \leq \underline{x}_{ij} \leq \underline{u}_{ij}$, $\overline{l}_{ij} \leq \overline{x}_{ij} \leq \overline{u}_{ij}$, $l^L_{ij} \leq x^L_{ij} \leq u^L_{ij}$ and $l^R_{ij} \leq x^R_{ij} \leq u^R_{ij}$, so the existing method [38] can be used only for solving such fully fuzzy capacitated minimum cost flow problems for which the restrictions $\underline{l}_{ij} \leq \underline{x}_{ij} \leq \underline{u}_{ij}$, $\overline{l}_{ij} \leq \overline{x}_{ij} \leq \overline{u}_{ij}$, $l^L_{ij} \leq x^L_{ij} \leq u^L_{ij}$ and $l^R_{ij} \leq x^R_{ij} \leq u^R_{ij}$ are satisfied. If $\tilde{l}_{ij} = (0,0,0,0)_{LR}$ $\forall$ $(i,j) \in A$ then these restrictions will always be satisfied i.e., such fully fuzzy capacitated minimum cost flow problems for which

$\tilde{l}_{ij} = (0,0,0,0)_{LR}\ \forall\ (i,j) \in A$ can always be solved by the existing method [38]. However, if the existing method [38] will be used for solving such fully fuzzy capacitated minimum cost flow problems for which these restrictions are not satisfied then no feasible solution will be obtained e.g., for the values of $\tilde{l}_{ij}$ and $\tilde{u}_{ij}$ chosen in Example 3.1 and Example 3.2, these conditions are not satisfying so no feasible solution of these problems can be obtained by using the existing method [38].

Example 3.1 Find the fuzzy optimal solution of the balanced fully fuzzy capacitated minimum cost flow problem depicted in Figure 3.1. The data is listed in Table 3.1 and Table 3.2.

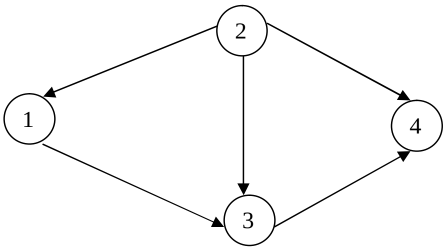

Figure 3.1 Network representing fully fuzzy capacitated minimum cost flow problem

Table 3.1 Fuzzy cost ($\tilde{c}_{ij}$), minimum fuzzy amount ($\tilde{l}_{ij}$) and maximum fuzzy amount ($\tilde{u}_{ij}$)

Arc (i,j)	$\tilde{c}_{ij}$	$\tilde{l}_{ij}$	$\tilde{u}_{ij}$
$(2,1)$	$(12,15,2,5)_{LR}$	$(0,0,0,0)_{LR}$	$(70,80,10,10)_{LR}$
$(2,3)$	$(9,11,2,19)_{LR}$	$(10,15,2,3)_{LR}$	$(251,260,1,2)_{LR}$
$(2,4)$	$(15,19,7,4)_{LR}$	$(15,17,3,10)_{LR}$	$(302,350,2,5)_{LR}$
$(1,3)$	$(10,12,3,7)_{LR}$	$(0,0,0,0)_{LR}$	$(90,100,10,20)_{LR}$
$(3,4)$	$(10,15,5,5)_{LR}$	$(0,0,0,0)_{LR}$	$(60,70,10,10)_{LR}$

Table 3.2 Fuzzy supply ($\tilde{a}_i/\tilde{e}_i$) and fuzzy demand ($\tilde{b}_j/\tilde{d}_j$)

Nodes	$(\tilde{a}_i/\tilde{e}_i)$	$(\tilde{b}_j/\tilde{d}_j)$
1	–	–
2	$(150,250,100,50)_{LR}$	–
3	–	$(100,150,80,50)_{LR}$
4	–	$(50,100,20,0)_{LR}$

where, $L(x) = \text{maximum}\{0, 1 - x\}$, $R(x) = \text{maximum}\{0, 1 - x^4\}$

Example 3.2 Find the fuzzy optimal solution of the unbalanced fully fuzzy capacitated minimum cost flow problem with the same network flow structure as in Example 3.1. The data is listed in Table 3.3 and Table 3.4.

Table 3.3 Fuzzy cost ($\tilde{c}_{ij}$), minimum fuzzy amount ($\tilde{l}_{ij}$) and maximum fuzzy amount ($\tilde{u}_{ij}$)

Arc (i,j)	$\tilde{c}_{ij}$	$\tilde{l}_{ij}$	$\tilde{u}_{ij}$
$(2,1)$	$(12,15,2,5)_{LR}$	$(0,0,0,0)_{LR}$	$(70,80,10,10)_{LR}$
$(2,3)$	$(9,11,2,19)_{LR}$	$(10,15,2,3)_{LR}$	$(251,260,1,2)_{LR}$
$(2,4)$	$(15,19,7,4)_{LR}$	$(15,17,3,10)_{LR}$	$(302,350,2,5)_{LR}$
$(1,3)$	$(10,12,3,7)_{LR}$	$(5,10,10,3)_{LR}$	$(90,100,10,20)_{LR}$
$(3,4)$	$(10,15,5,5)_{LR}$	$(0,0,0,0)_{LR}$	$(60,70,10,10)_{LR}$

Table 3.4 Fuzzy supply ($\tilde{a}_i/\tilde{e}_i$) and fuzzy demand ($\tilde{b}_j/\tilde{d}_j$)

Nodes	$(\tilde{a}_i/\tilde{e}_i)$	$(\tilde{b}_j/\tilde{d}_j)$
1	–	–
2	$(200,250,100,50)_{LR}$	–
3	–	$(100,150,80,50)_{LR}$
4	–	$(50,100,20,0)_{LR}$

where, $L(x) = \text{maximum}\{0, 1 - x\}$, $R(x) = \text{maximum}\{0, 1 - x^4\}$

3.6 Modified representation of linear programming formulation of balanced crisp and fully fuzzy capacitated minimum cost flow problems

In this section, to overcome the shortcoming of existing linear programming formulation of balanced fully fuzzy capacitated minimum cost flow problems, pointed out in Section 3.3, the existing linear programming formulation of balanced crisp and fully fuzzy capacitated minimum cost flow problems, presented in Section 3.1, is modified in such a manner so that the physical meaning of existing and modified linear programming formulation of balanced crisp capacitated minimum cost flow problems are same while the modified representation of balanced fully fuzzy capacitated minimum cost flow problem represents the balanced fully fuzzy capacitated minimum cost flow problems in a more realistic manner as compared to the existing linear programming formulation of balanced fully fuzzy capacitated minimum cost flow problems.

3.6.1 Modified representation of linear programming formulation of balanced crisp capacitated minimum cost flow problems

The crisp linear programming problem ($P_{3.1}$) can be converted into crisp linear programming problem ($P_{3.8}$):

$$\text{Minimize} \sum_{(i,j)\in A} (c_{ij}x_{ij})$$

subject to

$$\begin{array}{ll}
\sum\limits_{j:(i,j)\in A} x_{ij} = a_i & i \in N_{PS} \\
\sum\limits_{j:(i,j)\in A} x_{ij} = \sum\limits_{j:(j,i)\in A} x_{ji} + e_i & i \in N_S \\
\sum\limits_{i:(i,j)\in A} x_{ij} = b_j & j \in N_{PD} \\
\sum\limits_{i:(i,j)\in A} x_{ij} = \sum\limits_{i:(j,i)\in A} x_{ji} + d_j & j \in N_D \\
\sum\limits_{j:(i,j)\in A} x_{ij} = \sum\limits_{j:(j,i)\in A} x_{ji} & i \in N_T \\
l_{ij} \leq x_{ij} \leq u_{ij},\ x_{ij} \geq 0 & \forall\ (i,j) \in A
\end{array} \qquad (P_{3.8})$$

3.6.2 Modified representation of linear programming formulation of balanced fully fuzzy capacitated minimum cost flow problems

The linear programming formulation of balanced fully fuzzy capacitated minimum cost flow problems, presented in Section 3.1.3, is obtained by using the linear programming formulation of balanced crisp capacitated minimum cost flow problems, presented in Section 3.1.2. On the same direction fuzzy linear programming problem $(P_{3.9})$ can be obtained by using the modified crisp linear programming problem $(P_{3.8})$:

$$
\begin{aligned}
&\text{Minimize} \sum_{(i,j)\in A} (\tilde{c}_{ij} \otimes \tilde{x}_{ij})\\
&\text{subject to}\\
&\qquad \sum_{j:(i,j)\in A} \tilde{x}_{ij} = \tilde{a}_i && i \in N_{PS}\\
&\qquad \sum_{j:(i,j)\in A} \tilde{x}_{ij} = \sum_{j:(j,i)\in A} \tilde{x}_{ji} \oplus \tilde{e}_i && i \in N_S\\
&\qquad \sum_{i:(i,j)\in A} \tilde{x}_{ij} = \tilde{b}_j && j \in N_{PD} \qquad (P_{3.9})\\
&\qquad \sum_{i:(i,j)\in A} \tilde{x}_{ij} = \sum_{i:(j,i)\in A} \tilde{x}_{ji} \oplus \tilde{d}_j && j \in N_D\\
&\qquad \sum_{j:(i,j)\in A} \tilde{x}_{ij} = \sum_{j:(j,i)\in A} \tilde{x}_{ji} && i \in N_T\\
&\qquad \tilde{l}_{ij} \preceq \tilde{x}_{ij} \preceq \tilde{u}_{ij} && \forall\ (i,j) \in A\\
&\qquad \tilde{x}_{ij} \text{ is a non-negative } LR \text{ flat fuzzy number } \forall\ (i,j) \in A
\end{aligned}
$$

3.7 Proposed method

In this section, to resolve the shortcomings of existing method [38], discussed in Section 3.4, as well as to overcome the limitations of the existing method [38], discussed in Section 3.5, a new method is proposed for finding the fuzzy optimal solution of balanced and unbalanced fully fuzzy capacitated minimum cost flow problems by representing all the parameters by LR flat fuzzy numbers. If the supply of product at i^{th} purely source node and i^{th} source node is $\tilde{a}_i$ and $\tilde{e}_i$ respectively and the demand of the product at j^{th} purely destination node and j^{th} destination node is $\tilde{b}_j$ and $\tilde{d}_j$ respectively then the exact fuzzy optimal solution of fully fuzzy capacitated minimum cost flow problems can be obtained by using the following steps:

Step 1 Find the total fuzzy supply $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i$ and the total fuzzy demand $\sum_{j\in N_{PD}} \tilde{b}_j \oplus \sum_{j\in N_D} \tilde{d}_j$. Let $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i = (\underline{m}, \overline{m}, m^L, m^R)_{LR}$ and $\sum_{j\in N_{PD}} \tilde{b}_j \oplus \sum_{j\in N_D} \tilde{d}_j = (\underline{n}, \overline{n}, n^L, n^R)_{LR}$. Examine that the problem is balanced or not, i.e., $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i = \sum_{j\in N_{PD}} \tilde{b}_j \oplus \sum_{j\in N_D} \tilde{d}_j$ or $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i \neq \sum_{j\in N_{PD}} \tilde{b}_j \oplus \sum_{j\in N_D} \tilde{d}_j$.

Case (i) If the problem is balanced, i.e., $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i = \sum_{j\in N_{PD}} \tilde{b}_j \oplus \sum_{j\in N_D} \tilde{d}_j$ then Go to Step 2.

Case (ii) If $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i \neq \sum_{j\in N_{PD}} \tilde{b}_j \oplus \sum_{j\in N_D} \tilde{d}_j$ then convert the unbalanced problem into balanced problem as follows [65]:

Case (a) If $\underline{m}-m^L \leq \underline{n}-n^L$, $m^L \leq n^L$, $\overline{m}-\underline{m} \leq \overline{n}-\underline{n}$, and $m^R \leq n^R$ then introduce a dummy purely source node with fuzzy supply $(\underline{n}-\underline{m}, \overline{n}-\overline{m}, n^L-m^L, n^R-m^R)_{LR}$. Assume the fuzzy cost for transporting one unit quantity of the product from the introduced dummy purely source node to all purely destination nodes and all intermediate nodes as zero LR flat fuzzy number. Go to Step 2.

Case (b) If $\underline{m}-m^L \geq \underline{n}-n^L$, $m^L \geq n^L$, $\overline{m}-\underline{m} \geq \overline{n}-\underline{n}$, and $m^R \geq n^R$ then introduce a dummy purely destination node with fuzzy demand $(\underline{m}-\underline{n}, \overline{m}-\overline{n}, m^L-n^L, m^R-n^R)_{LR}$. Assume the fuzzy cost for transporting one unit quantity of the product from all purely source nodes and intermediate nodes to the introduced dummy purely destination node as zero LR flat fuzzy number. Go to Step 2.

Case (c) If neither Case (a) nor Case (b) is satisfied then introduce a dummy source with fuzzy supply (maximum $\{0, (\underline{n}-n^L)-(\underline{m}-m^L)\}$ + maximum $\{0, (n^L-m^L)\}$, maximum $\{0, (\underline{n}-n^L)-(\underline{m}-m^L)\}$ + maximum $\{0, (n^L-m^L)\}$ + maximum $\{0, (\overline{n}-\underline{n})-(\overline{m}-\underline{m})\}$, maximum $\{0, (n^L-m^L)\}$, maximum $\{0, (n^R-m^R)\})_{LR}$ and dummy purely destination with fuzzy demand (maximum $\{0, (\underline{m}-m^L)-(\underline{n}-n^L)\}$ + maximum $\{0, (m^L-n^L)\}$, maximum $\{0, (\underline{m}-m^L)-(\underline{n}-n^L)\}$ + maximum

$\{0,(m^L-n^L)\}$ + maximum $\{0,(\overline{m}-\underline{m})-(\overline{n}-\underline{n})\}$, maximum $\{0,(m^L-n^L)\}$, maximum $\{0,(m^R-n^R)\}\})_{LR}$. Assume the fuzzy cost for transporting one unit quantity of the product from the introduced dummy purely source node to all intermediate nodes, existing purely destination nodes and introduced dummy purely destination node as zero LR flat fuzzy number. Similarly, assume the fuzzy cost for transporting one unit quantity of the product from all intermediate nodes, existing purely source nodes and introduced dummy purely source node to the introduced dummy purely destination node as zero LR flat fuzzy number. Go to Step 2

Step 2 Formulate the balanced fully fuzzy capacitated minimum cost flow problem, obtained in Step 1, into the fuzzy linear programming problem $(P_{3.9})$.

Step 3 Assuming $\tilde{c}_{ij}=(\underline{c}_{ij},\overline{c}_{ij},c^L_{ij},c^R_{ij})_{LR}$, $\tilde{x}_{ij}=(\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij})_{LR}$, $\tilde{a}_i=(\underline{a}_i,\overline{a}_i,a^L_i,a^R_i)_{LR}$, $\tilde{e}_i=(\underline{e}_i,\overline{e}_i,e^L_i,e^R_i)_{LR}$, $\tilde{b}_j=(\underline{b}_j,\overline{b}_j,b^L_j,b^R_j)_{LR}$, $\tilde{d}_j=(\underline{d}_j,\overline{d}_j,d^L_j,d^R_j)_{LR}$, $\tilde{l}_{ij}=(\underline{l}_{ij},\overline{l}_{ij},l^L_{ij},l^R_{ij})_{LR}$ and $\tilde{u}_{ij}=(\underline{u}_{ij},\overline{u}_{ij},u^L_{ij},u^R_{ij})_{LR}$, the fuzzy linear programming problem $(P_{3.9})$ can be written as:

Minimize $\sum\limits_{(i,j)\in A}((\underline{c}_{ij},\overline{c}_{ij},c^L_{ij},c^R_{ij})_{LR}\otimes(\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij})_{LR})$

subject to

$$\begin{array}{ll}
\sum\limits_{j:(i,j)\in A}(\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij})_{LR}=(\underline{a}_i,\overline{a}_i,a^L_i,a^R_i)_{LR} & i\in N_{PS}\\
\sum\limits_{j:(i,j)\in A}(\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij})_{LR}=\sum\limits_{j:(j,i)\in A}(\underline{x}_{ji},\overline{x}_{ji},x^L_{ji},x^R_{ji})_{LR}\oplus(\underline{e}_i,\overline{e}_i,e^L_i,e^R_i)_{LR} & i\in N_S\\
\sum\limits_{i:(i,j)\in A}(\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij})_{LR}=(\underline{b}_j,\overline{b}_j,b^L_j,b^R_j)_{LR} & j\in N_{PD}\\
\sum\limits_{i:(i,j)\in A}(\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij})_{LR}=\sum\limits_{i:(j,i)\in A}(\underline{x}_{ji},\overline{x}_{ji},x^L_{ji},x^R_{ji})_{LR}\oplus(\underline{d}_j,\overline{d}_j,d^L_j,d^R_j)_{LR} & j\in N_D\ \ (P_{3.10})\\
\sum\limits_{j:(i,j)\in A}(\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij})_{LR}=\sum\limits_{j:(j,i)\in A}(\underline{x}_{ji},\overline{x}_{ji},x^L_{ji},x^R_{ji})_{LR} & i\in N_T\\
(\underline{l}_{ij},\overline{l}_{ij},l^L_{ij},l^R_{ij})_{LR}\preceq(\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij})_{LR}\preceq(\underline{u}_{ij},\overline{u}_{ij},u^L_{ij},u^R_{ij})_{LR} & \forall\ (i,j)\in A
\end{array}$$

$(\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij})_{LR}$ is a non-negative LR flat fuzzy number $\forall\ (i,j)\in A$

Step 4 Using the arithmetic operations of LR flat fuzzy numbers, defined in Section

2.1.2, the fuzzy linear programming problem ($P_{3.10}$) can be written as:

Minimize $\sum\limits_{(i,j)\in A} (\underline{c}_{ij}\underline{x}_{ij}, \overline{c}_{ij}\overline{x}_{ij}, \underline{c}_{ij}x^L_{ij} + c^L_{ij}\underline{x}_{ij} - c^L_{ij}x^L_{ij}, \overline{c}_{ij}x^R_{ij} + c^R_{ij}\overline{x}_{ij} + c^R_{ij}x^R_{ij})_{LR}$

subject to

$$(\sum_{j:(i,j)\in A} \underline{x}_{ij}, \sum_{j:(i,j)\in A} \overline{x}_{ij}, \sum_{j:(i,j)\in A} x^L_{ij}, \sum_{j:(i,j)\in A} x^R_{ij})_{LR} = (\underline{a}_i, \overline{a}_i, a^L_i, a^R_i)_{LR} \qquad i \in N_{PS}$$

$$(\sum_{j:(i,j)\in A} \underline{x}_{ij}, \sum_{j:(i,j)\in A} \overline{x}_{ij}, \sum_{j:(i,j)\in A} x^L_{ij}, \sum_{j:(i,j)\in A} x^R_{ij})_{LR} = (\sum_{j:(j,i)\in A} \underline{x}_{ji} + \underline{e}_i, \sum_{j:(j,i)\in A} \overline{x}_{ji} + \overline{e}_i,$$
$$\sum_{j:(j,i)\in A} x^L_{ji} + e^L_i, \sum_{j:(j,i)\in A} x^R_{ji} + e^R_i)_{LR}) \qquad i \in N_S$$

$$(\sum_{i:(i,j)\in A} \underline{x}_{ij}, \sum_{i:(i,j)\in A} \overline{x}_{ij}, \sum_{i:(i,j)\in A} x^L_{ij}, \sum_{i:(i,j)\in A} x^R_{ij})_{LR} = (\underline{b}_j, \overline{b}_j, b^L_j, b^R_j)_{LR} \qquad j \in N_{PD}$$

$$(\sum_{i:(i,j)\in A} \underline{x}_{ij}, \sum_{i:(i,j)\in A} \overline{x}_{ij}, \sum_{i:(i,j)\in A} x^L_{ij}, \sum_{i:(i,j)\in A} x^R_{ij})_{LR} = (\sum_{i:(j,i)\in A} \underline{x}_{ji} + \underline{d}_j, \sum_{i:(j,i)\in A} \overline{x}_{ji} + \overline{d}_j,$$
$$\sum_{i:(j,i)\in A} x^L_{ji} + d^L_j, \sum_{i:(j,i)\in A} x^R_{ji} + d^R_j)_{LR} \qquad j \in N_D \qquad (P_{3.11})$$

$$(\sum_{j:(i,j)\in A} \underline{x}_{ij}, \sum_{j:(i,j)\in A} \overline{x}_{ij}, \sum_{j:(i,j)\in A} x^L_{ij}, \sum_{j:(i,j)\in A} x^R_{ij})_{LR} = (\sum_{j:(j,i)\in A} \underline{x}_{ji}, \sum_{j:(j,i)\in A} \overline{x}_{ji}, \sum_{j:(j,i)\in A} x^L_{ji},$$
$$\sum_{j:(j,i)\in A} x^R_{ji})_{LR} \qquad i \in N_T$$

$$(\underline{l}_{ij}, \overline{l}_{ij}, l^L_{ij}, l^R_{ij})_{LR} \preceq (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} \preceq (\underline{u}_{ij}, \overline{u}_{ij}, u^L_{ij}, u^R_{ij})_{LR} \qquad \forall\ (i,j) \in A$$

$(\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR}$ is a non-negative LR flat fuzzy number $\forall\ (i,j) \in A$

Step 5 Using Definition 2.10 and Definition 2.11, the fuzzy linear programming problem ($P_{3.11}$) can be converted into the fuzzy linear programming problem ($P_{3.12}$):

Minimize $\sum\limits_{(i,j)\in A} (\underline{c}_{ij}\underline{x}_{ij}, \overline{c}_{ij}\overline{x}_{ij}, \underline{c}_{ij}x^L_{ij} + c^L_{ij}\underline{x}_{ij} - c^L_{ij}x^L_{ij}, \overline{c}_{ij}x^R_{ij} + c^R_{ij}\overline{x}_{ij} + c^R_{ij}x^R_{ij})_{LR}$

subject to

$$\sum_{j:(i,j)\in A} \underline{x}_{ij} = \underline{a}_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \overline{x}_{ij} = \overline{a}_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} x^L_{ij} = a^L_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} x^R_{ij} = a^R_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \underline{x}_{ij} = \sum_{j:(j,i)\in A} \underline{x}_{ji} + \underline{e}_i \qquad i \in N_S$$

$$\sum_{j:(i,j)\in A} \overline{x}_{ij} = \sum_{j:(j,i)\in A} \overline{x}_{ji} + \overline{e}_i \qquad i \in N_S$$

$$\sum_{j:(i,j)\in A} x^L_{ij} = \sum_{j:(j,i)\in A} x^L_{ji} + e^L_i \qquad i \in N_S$$

$$\begin{array}{lll}
\sum\limits_{j:(i,j)\in A} x_{ij}^R = \sum\limits_{j:(j,i)\in A} x_{ji}^R + e_i^R & i \in N_S & \\
\sum\limits_{i:(i,j)\in A} \underline{x}_{ij} = \underline{b}_j & j \in N_{PD} & \\
\sum\limits_{i:(i,j)\in A} \overline{x}_{ij} = \overline{b}_j & j \in N_{PD} & \\
\sum\limits_{i:(i,j)\in A} x_{ij}^L = b_j^L & j \in N_{PD} & (P_{3.12}) \\
\sum\limits_{i:(i,j)\in A} x_{ij}^R = b_j^R & j \in N_{PD} & \\
\sum\limits_{i:(i,j)\in A} \underline{x}_{ij} = \sum\limits_{i:(j,i)\in A} \underline{x}_{ji} + \underline{d}_j & j \in N_D & \\
\sum\limits_{i:(i,j)\in A} \overline{x}_{ij} = \sum\limits_{i:(j,i)\in A} \overline{x}_{ji} + \overline{d}_j & j \in N_D & \\
\sum\limits_{i:(i,j)\in A} x_{ij}^L = \sum\limits_{i:(j,i)\in A} x_{ji}^L + d_j^L & j \in N_D & \\
\sum\limits_{i:(i,j)\in A} x_{ij}^R = \sum\limits_{i:(j,i)\in A} x_{ji}^R + d_j^R & j \in N_D & \\
\sum\limits_{j:(i,j)\in A} \underline{x}_{ij} = \sum\limits_{j:(j,i)\in A} \underline{x}_{ji} & i \in N_T & \\
\sum\limits_{j:(i,j)\in A} \overline{x}_{ij} = \sum\limits_{j:(j,i)\in A} \overline{x}_{ji} & i \in N_T & \\
\sum\limits_{j:(i,j)\in A} x_{ij}^L = \sum\limits_{j:(j,i)\in A} x_{ji}^L & i \in N_T & \\
\sum\limits_{j:(i,j)\in A} x_{ij}^R = \sum\limits_{j:(j,i)\in A} x_{ji}^R & i \in N_T & \\
\multicolumn{2}{l}{\underline{x}_{ij} - x_{ij}^L \geq 0, \overline{x}_{ij} - \underline{x}_{ij} \geq 0, x_{ij}^L, x_{ij}^R \geq 0 \ \ \forall\ (i,j) \in A} &
\end{array}$$

$$\left.(\underline{l}_{ij}, \overline{l}_{ij}, l_{ij}^L, l_{ij}^R)_{LR} \preceq (\underline{x}_{ij}, \overline{x}_{ij}, x_{ij}^L, x_{ij}^R)_{LR} \preceq (\underline{u}_{ij}, \overline{u}_{ij}, u_{ij}^L, u_{ij}^R)_{LR}\ \forall\ (i,j) \in A \ \right\}\ (C_{3.1})$$

Step 6 Suppose the fuzzy linear programming problem $(P_{3.12})$ has k basic feasible solutions and $\{((\underline{x}_{ij})^w, (\overline{x}_{ij})^w, (x_{ij}^L)^w, (x_{ij}^R)^w)_{LR}\}$ be the w^{th} basic feasible solution then the goal is to find such basic feasible solution corresponding to which the value of the objective function is minimum i.e, $\underset{1\leq w\leq k}{\text{minimum}} \sum\limits_{(i,j)\in A} (\underline{c}_{ij}(\underline{x}_{ij})^w, \overline{c}_{ij}(\overline{x}_{ij})^w, \underline{c}_{ij}(x_{ij}^L)^w + c_{ij}^L(\underline{x}_{ij})^w - c_{ij}^L(x_{ij}^L)^w, \overline{c}_{ij}(x_{ij}^R)^w + c_{ij}^R(\overline{x}_{ij})^w + c_{ij}^R(x_{ij}^R)^w)_{LR}$. Garcia and Lamata [36] pointed out the shortcomings of several existing methods for comparing fuzzy numbers and proposed the concept that if $\underset{1\leq w\leq k}{\text{minimum}} \sum\limits_{(i,j)\in A} \Re(\underline{c}_{ij}(\underline{x}_{ij})^w, \overline{c}_{ij}(\overline{x}_{ij})^w, \underline{c}_{ij}(x_{ij}^L)^w + c_{ij}^L(\underline{x}_{ij})^w - c_{ij}^L(x_{ij}^L)^w, \overline{c}_{ij}(x_{ij}^R)^w + c_{ij}^R(\overline{x}_{ij})^w + c_{ij}^R(x_{ij}^R)^w)_{LR}$ is $\sum\limits_{(i,j)\in A} \Re(\underline{c}_{ij}(\underline{x}_{ij})^\phi, \overline{c}_{ij}(\overline{x}_{ij})^\phi, \underline{c}_{ij}(x_{ij}^L)^\phi + c_{ij}^L(\underline{x}_{ij})^\phi - c_{ij}^L(x_{ij}^L)^\phi, \overline{c}_{ij}(x_{ij}^R)^\phi + c_{ij}^R(\overline{x}_{ij})^\phi + c_{ij}^R(x_{ij}^R)^\phi)_{LR}$ then $\underset{1\leq w\leq k}{\text{minimum}} \sum\limits_{(i,j)\in A} (\underline{c}_{ij}$

$(\underline{x}_{ij})^w, \overline{c}_{ij}(\overline{x}_{ij})^w, \underline{c}_{ij}(x^L_{ij})^w + c^L_{ij}(\underline{x}_{ij})^w - c^L_{ij}(x^L_{ij})^w, \overline{c}_{ij}(x^R_{ij})^w + c^R_{ij}(\overline{x}_{ij})^w + c^R_{ij}(x^R_{ij})^w)_{LR}$ is $\sum\limits_{(i,j)\in A} (\underline{c}_{ij}(\underline{x}_{ij})^\phi, \overline{c}_{ij}(\overline{x}_{ij})^\phi, \underline{c}_{ij}(x^L_{ij})^\phi + c^L_{ij}(\underline{x}_{ij})^\phi - c^L_{ij}(x^L_{ij})^\phi, \overline{c}_{ij}(x^R_{ij})^\phi + c^R_{ij}(\overline{x}_{ij})^\phi + c^R_{ij}(x^R_{ij})^\phi)_{LR}$.

In other words, by using the existing method [36], the fuzzy optimal solution of the fuzzy linear programming problem $(P_{3.12})$ can be obtained by solving the following crisp linear programming problem:

Minimize $\sum\limits_{(i,j)\in A} \Re(\underline{c}_{ij}\underline{x}_{ij}, \overline{c}_{ij}\overline{x}_{ij}, \underline{c}_{ij}x^L_{ij} + c^L_{ij}\underline{x}_{ij} - c^L_{ij}x^L_{ij}, \overline{c}_{ij}x^R_{ij} + c^R_{ij}\overline{x}_{ij} + c^R_{ij}x^R_{ij})_{LR}$

subject to $(P_{3.13})$

$\Re(\underline{l}_{ij}, \overline{l}_{ij}, l^L_{ij}, l^R_{ij})_{LR} \leq \Re(\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} \leq \Re(\underline{u}_{ij}, \overline{u}_{ij}, u^L_{ij}, u^R_{ij})_{LR}\ \forall\ (i,j) \in A$

as well as all the constraints of problem $(P_{3.12})$ except $(C_{3.1})$

Step 7 Using Section 2.4, the crisp linear programming problem $(P_{3.13})$ can be written as:

Minimize $[\gamma(\int_0^1((\underline{c}_{ij}\underline{x}_{ij})\lambda + (\overline{c}_{ij}\overline{x}_{ij})(1-\lambda))d\rho) + (1-\gamma)(\lambda\int_0^1((\underline{c}_{ij}\underline{x}_{ij}) - (\underline{c}_{ij}x^L_{ij} + c^L_{ij}\underline{x}_{ij} - c^L_{ij}x^L_{ij})L^{-1}(\rho))d\rho + (1-\lambda)\int_0^1(\overline{c}_{ij}\overline{x}_{ij} + (\overline{c}_{ij}x^R_{ij} + c^R_{ij}\overline{x}_{ij} + c^R_{ij}x^R_{ij})R^{-1}(\rho))d\rho)]$

subject to $(P_{3.14})$

$\Big(\gamma(\int_0^1(\underline{l}_{ij}\lambda + \overline{l}_{ij}(1-\lambda))d\rho) + (1-\gamma)(\lambda\int_0^1(\underline{l}_{ij} - l^L_{ij}L^{-1}(\rho))d\rho + (1-\lambda)\int_0^1(\overline{l}_{ij} + l^R_{ij}R^{-1}(\rho))d\rho)\Big)$ $\leq \Big(\gamma(\int_0^1(\underline{x}_{ij}\lambda + \overline{x}_{ij}(1-\lambda))d\rho) + (1-\gamma)(\lambda\int_0^1(\underline{x}_{ij} - x^L_{ij}L^{-1}(\rho))d\rho + (1-\lambda)\int_0^1(\overline{x}_{ij} + x^R_{ij}R^{-1}(\rho))d\rho)\Big) \leq \Big(\gamma(\int_0^1(\underline{u}_{ij}\lambda + \overline{u}_{ij}(1-\lambda))d\rho) + (1-\gamma)(\lambda\int_0^1(\underline{u}_{ij} - u^L_{ij}L^{-1}(\rho))d\rho + (1-\lambda)\int_0^1(\overline{u}_{ij} + u^R_{ij}R^{-1}(\rho))d\rho)\Big)$

as well as all the constraints of problem $(P_{3.12})$ except $(C_{3.1})$

Step 8 Solve the crisp linear programming problem $(P_{3.14})$, to find the optimal solution $\{\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij}\}$.

Step 9 Put the optimal values of $\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}$ and x^R_{ij}, obtained from Step 8, in $\tilde{x}_{ij} = (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR}$ to find the fuzzy optimal solution $\{\tilde{x}_{ij}\}$.

Step 10 Put the fuzzy optimal values of $\tilde{x}_{ij}$, obtained from Step 9, in $\sum\limits_{(i,j)\in A} (\tilde{c}_{ij} \otimes$

$\tilde{x}_{ij}$) to find the minimum total fuzzy transportation cost.

3.8 Advantages of the proposed method over existing method

In this section, advantages of the proposed method over existing method [38] are discussed.

(1) Since, in the proposed method instead of fuzzy linear programming formulation ($P_{3.2}$) the modified fuzzy linear programming formulation ($P_{3.9}$) of balanced fully fuzzy capacitated minimum cost flow problems is used. So, all the shortcomings, pointed out in Section 3.3, occurring due to Hukuhara's difference are resolved.

(2) All the problems which can be solved by the existing method [38] can also be solved be the proposed method. However, there exist several problems which can be solved by the proposed method but can not be solved by using the existing method [38]. To illustrate the proposed method and to show its advantage the unbalanced fully fuzzy capacitated minimum cost flow problem, chosen in Example 3.2 which can not be solved by using the existing method [38], is solved by the proposed method.

3.8.1 Optimal solution of the chosen fully fuzzy capacitated minimum cost flow problem

The fully fuzzy capacitated minimum cost flow problem, chosen in Example 3.2, can be solved by using the following steps of the proposed method:

Step 1 Total fuzzy supply $= (200, 250, 100, 50)_{LR}$ and total fuzzy demand $= (150, 250, 100, 50)_{LR}$. Since total fuzzy supply $\neq$ total fuzzy demand, so it is an unbalanced fully fuzzy capacitated minimum cost flow problem.

Now, as described in the proposed method (using Case (c) of Step 1), the unbalanced fully fuzzy capacitated minimum cost flow problem can be converted into a balanced fully fuzzy capacitated minimum cost flow problem, by introducing a purely dummy source node (5) with fuzzy supply $(0, 50, 0, 0)_{LR}$ and a purely dummy destination node (6) with fuzzy demand $(50, 50, 0, 0)_{LR}$. So that total fuzzy supply = total fuzzy demand i.e., $(200, 250, 100, 50)_{LR} \oplus (0, 50, 0, 0)_{LR} = (150, 250, 100, 50)_{LR} \oplus (50, 50, 0, 0)_{LR}$.

Assume the fuzzy transportation cost for the one unit quantity of the product from the introduced purely dummy source node (5) to all intermediate nodes (1 and 3), existing purely destination node (4) and introduced purely dummy destination node (6) as zero LR flat fuzzy number. Similarly, assume the fuzzy transportation cost for the one unit quantity of the product from all intermediate nodes (1 and 3), existing purely source node (2) and introduced purely dummy source node (5) to the introduced purely dummy destination node (6) as zero LR flat fuzzy number i.e., $\tilde{c}_{51} = \tilde{c}_{53} = \tilde{c}_{54} = \tilde{c}_{56} = \tilde{c}_{36} = \tilde{c}_{26} = \tilde{c}_{16} = (0, 0, 0, 0)_{LR}$.

Step 2 The balanced fully fuzzy capacitated minimum cost flow problem, obtained from Step 1, can be formulated into the following fuzzy linear programming problem:

Minimize $((12, 15, 2, 5)_{LR} \otimes \tilde{x}_{21} \oplus (9, 11, 2, 19)_{LR} \otimes \tilde{x}_{23} \oplus (15, 19, 7, 4)_{LR} \otimes \tilde{x}_{24} \oplus (10, 12, 3, 7)_{LR} \otimes \tilde{x}_{13} \oplus (10, 15, 5, 5)_{LR} \otimes \tilde{x}_{34} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{16} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{26} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{36} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{51} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{53} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{54} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{56})$

subject to

$\tilde{x}_{13} \oplus \tilde{x}_{16} = \tilde{x}_{21} \oplus \tilde{x}_{51}$

$\tilde{x}_{21} \oplus \tilde{x}_{23} \oplus \tilde{x}_{24} \oplus \tilde{x}_{26} = (200, 250, 100, 50)_{LR}$

$\tilde{x}_{13} \oplus \tilde{x}_{23} \oplus \tilde{x}_{53} = \tilde{x}_{34} \oplus \tilde{x}_{36} \oplus (100, 150, 80, 50)_{LR}$

$\tilde{x}_{24} \oplus \tilde{x}_{34} \oplus \tilde{x}_{54} = (50, 100, 20, 0)_{LR}$

$\tilde{x}_{51} \oplus \tilde{x}_{53} \oplus \tilde{x}_{54} \oplus \tilde{x}_{56} = (0, 50, 0, 0)_{LR}$

$\tilde{x}_{16} \oplus \tilde{x}_{26} \oplus \tilde{x}_{36} \oplus \tilde{x}_{56} = (50, 50, 0, 0)_{LR}$

$\tilde{x}_{21} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{21} \preceq (70, 80, 10, 10)_{LR}$, $\tilde{x}_{23} \succeq (10, 15, 2, 3)_{LR}$

$\tilde{x}_{23} \preceq (251, 260, 1, 2)_{LR}$, $\tilde{x}_{24} \succeq (15, 17, 3, 10)_{LR}$, $\tilde{x}_{24} \preceq (302, 350, 2, 5)_{LR}$

$\tilde{x}_{13} \succeq (5, 10, 10, 3)_{LR}$, $\tilde{x}_{13} \preceq (90, 100, 10, 20)_{LR}$, $\tilde{x}_{34} \succeq (0, 0, 0, 0)_{LR}$

$\tilde{x}_{34} \preceq (60, 70, 10, 10)_{LR}$, $\tilde{x}_{16} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{26} \succeq (0, 0, 0, 0)_{LR}$

$\tilde{x}_{36} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{51} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{54} \succeq (0, 0, 0, 0)_{LR}$

$\tilde{x}_{53} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{56} \succeq (0, 0, 0, 0)_{LR}$

$\tilde{x}_{ij}$ is a non-negative LR flat fuzzy number $\forall\ (i, j) \in A$

Step 3 Using Step 3 to Step 7, of the method, proposed in Section 3.7, the fuzzy linear programming problem, obtained in Step 2, can be converted into the following crisp linear programming problem:

Minimize $\frac{1}{30}(150\underline{x}_{21} + 105\underline{x}_{23} + 120\underline{x}_{24} + 105\underline{x}_{13} + 75\underline{x}_{34} + 265\overline{x}_{21} + 317\overline{x}_{23} + 317\overline{x}_{24} + 236\overline{x}_{13} + 265\overline{x}_{34} - 150x^{L}_{21} - 105x^{L}_{23} - 120x^{L}_{24} - 105x^{L}_{13} - 75x^{L}_{34} + 160x^{R}_{21} + 240x^{R}_{23} + 184x^{R}_{24} + 152x^{R}_{13} + 160x^{R}_{34})$

subject to

$\underline{x}_{13} + \underline{x}_{16} - \underline{x}_{21} - \underline{x}_{51} = 0$, $\underline{x}_{21} + \underline{x}_{23} + \underline{x}_{24} + \underline{x}_{26} = 200$

$\underline{x}_{13} + \underline{x}_{23} + \underline{x}_{53} - \underline{x}_{34} - \underline{x}_{36} = 100$, $\underline{x}_{24} + \underline{x}_{34} + \underline{x}_{54} = 50$

$\underline{x}_{51} + \underline{x}_{53} + \underline{x}_{54} + \underline{x}_{56} = 0$, $\underline{x}_{16} + \underline{x}_{26} + \underline{x}_{36} + \underline{x}_{56} = 50$

$\overline{x}_{13} + \overline{x}_{16} - \overline{x}_{21} - \overline{x}_{51} = 0$, $\overline{x}_{21} + \overline{x}_{23} + \overline{x}_{24} + \overline{x}_{26} = 250$

$$\overline{x}_{13}+\overline{x}_{23}+\overline{x}_{53}-\overline{x}_{34}-\overline{x}_{36}=150, \qquad \overline{x}_{24}+\overline{x}_{34}+\overline{x}_{54}=100$$

$$\overline{x}_{51}+\overline{x}_{53}+\overline{x}_{54}+\overline{x}_{56}=50, \qquad \overline{x}_{16}+\overline{x}_{26}+\overline{x}_{36}+\overline{x}_{56}=50$$

$$x_{13}^{L}+x_{16}^{L}-x_{21}^{L}-x_{51}^{L}=0, \qquad x_{21}^{L}+x_{23}^{L}+x_{24}^{L}+x_{26}^{L}=100$$

$$x_{13}^{L}+x_{23}^{L}+x_{53}^{L}-x_{34}^{L}-x_{36}^{L}=80, \qquad x_{24}^{L}+x_{34}^{L}+x_{54}^{L}=20$$

$$x_{51}^{L}+x_{53}^{L}+x_{54}^{L}+x_{56}^{L}=0, \qquad x_{16}^{L}+x_{26}^{L}+x_{36}^{L}+x_{56}^{L}=0$$

$$x_{13}^{R}+x_{16}^{R}-x_{21}^{R}-x_{51}^{R}=0, \qquad x_{21}^{R}+x_{23}^{R}+x_{24}^{R}+x_{26}^{R}=50$$

$$x_{13}^{R}+x_{23}^{R}+x_{53}^{R}-x_{34}^{R}-x_{36}^{R}=50, \qquad x_{24}^{R}+x_{34}^{R}+x_{54}^{R}=0$$

$$x_{51}^{R}+x_{53}^{R}+x_{54}^{R}+x_{56}^{R}=0, \qquad x_{16}^{R}+x_{26}^{R}+x_{36}^{R}+x_{56}^{R}=0$$

$$15\underline{x}_{21}+15\overline{x}_{21}-15x_{21}^{L}+8x_{21}^{R}\le 2180, \quad 15\underline{x}_{23}+15\overline{x}_{23}-15x_{23}^{L}+8x_{23}^{R}\le 7666$$

$$15\underline{x}_{24}+15\overline{x}_{24}-15x_{24}^{L}+8x_{24}^{R}\le 9790, \quad 15\underline{x}_{13}+15\overline{x}_{13}-15x_{13}^{L}+8x_{13}^{R}\le 2860$$

$$15\underline{x}_{34}+15\overline{x}_{34}-15x_{34}^{L}+8x_{34}^{R}\le 1880, \quad 15\underline{x}_{16}+15\overline{x}_{16}-15x_{16}^{L}+8x_{16}^{R}\ge 0$$

$$15\underline{x}_{36}+15\overline{x}_{36}-15x_{36}^{L}+8x_{36}^{R}\ge 0, \qquad 15\underline{x}_{53}+15\overline{x}_{53}-15x_{53}^{L}+8x_{53}^{R}\ge 0$$

$$15\underline{x}_{56}+15\overline{x}_{56}-15x_{56}^{L}+8x_{56}^{R}\ge 0, \qquad 15\underline{x}_{26}+15\overline{x}_{26}-15x_{26}^{L}+8x_{26}^{R}\ge 0$$

$$15\underline{x}_{51}+15\overline{x}_{51}-15x_{51}^{L}+8x_{51}^{R}\ge 0, \qquad 15\underline{x}_{54}+15\overline{x}_{54}-15x_{54}^{L}+8x_{54}^{R}\ge 0$$

$$15\underline{x}_{21}+15\overline{x}_{21}-15x_{21}^{L}+8x_{21}^{R}\ge 0, \qquad 15\underline{x}_{23}+15\overline{x}_{23}-15x_{23}^{L}+8x_{23}^{R}\ge 369$$

$$15\underline{x}_{24}+15\overline{x}_{24}-15x_{24}^{L}+8x_{24}^{R}\ge 515, \qquad 15\underline{x}_{13}+15\overline{x}_{13}-15x_{13}^{L}+8x_{13}^{R}\ge 99$$

$$15\underline{x}_{34}+15\overline{x}_{34}-15x_{34}^{L}+8x_{34}^{R}\ge 0$$

$$\overline{x}_{21}-\underline{x}_{21}\ge 0, \quad \overline{x}_{23}-\underline{x}_{23}\ge 0, \quad \overline{x}_{24}-\underline{x}_{24}\ge 0, \quad \overline{x}_{26}-\underline{x}_{26}\ge 0, \quad \overline{x}_{13}-\underline{x}_{13}\ge 0$$

$$\overline{x}_{34}-\underline{x}_{34}\ge 0, \quad \overline{x}_{36}-\underline{x}_{36}\ge 0, \quad \overline{x}_{16}-\underline{x}_{16}\ge 0, \quad \overline{x}_{51}-\underline{x}_{51}\ge 0, \quad \overline{x}_{53}-\underline{x}_{53}\ge 0$$

$$\overline{x}_{54}-\underline{x}_{54}\ge 0, \quad \overline{x}_{56}-\underline{x}_{56}\ge 0, \quad \underline{x}_{21}-x_{21}^{L}\ge 0, \quad \underline{x}_{23}-x_{23}^{L}\ge 0, \quad \underline{x}_{24}-x_{24}^{L}\ge 0$$

$$\underline{x}_{26}-x_{26}^{L}\ge 0, \quad \underline{x}_{13}-x_{13}^{L}\ge 0, \quad \underline{x}_{34}-x_{34}^{L}\ge 0, \quad \underline{x}_{36}-x_{36}^{L}\ge 0, \quad \underline{x}_{16}-x_{16}^{L}\ge 0$$

$$\underline{x}_{51}-x_{51}^{L}\ge 0, \quad \underline{x}_{53}-x_{53}^{L}\ge 0, \quad \underline{x}_{54}-x_{54}^{L}\ge 0, \quad \underline{x}_{56}-x_{56}^{L}\ge 0$$

$$x_{21}^{L},\ x_{23}^{L},\ x_{24}^{L},\ x_{26}^{L},\ x_{13}^{L},\ x_{34}^{L},\ x_{36}^{L},\ x_{16}^{L},\ x_{51}^{L},\ x_{53}^{L},\ x_{54}^{L},\ x_{56}^{L},\ x_{21}^{R},\ x_{23}^{R},\ x_{24}^{R},\ x_{26}^{R},\ x_{13}^{R},\ x_{34}^{R}$$

$$x_{36}^{R},\ x_{16}^{R},\ x_{51}^{R},\ x_{53}^{R},\ x_{54}^{R},\ x_{56}^{R}\ge 0$$

Step 4 The optimal solution of the crisp linear programming problem, obtained in Step 3, is $\underline{x}_{23} = 100, \underline{x}_{24} = 50, \underline{x}_{26} = 50, \overline{x}_{23} = 150, \overline{x}_{24} = 50, \overline{x}_{26} = 50, \overline{x}_{54} = 50, x^L_{23} = 80, x^L_{24} = 20, x^R_{21} = 12.37, x^R_{23} = 37.62, x^R_{13} = 12.37$ and the remaining values of $\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij}$ are zero respectively.

Step 5 Putting the optimal values of $\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}$ and x^R_{ij}, obtained from Step 4, in $\tilde{x}_{ij} = (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR}$, the fuzzy optimal solution is $\tilde{x}_{21} = (0, 0, 0, 12.37)_{LR}$, $\tilde{x}_{23} = (100, 150, 80, 37.62)_{LR}$, $\tilde{x}_{24} = (50, 50, 20, 0)_{LR}$, $\tilde{x}_{13} = (0, 0, 0, 12.37)_{LR}$, $\tilde{x}_{26} = (50, 50, 0, 0)_{LR}$, $\tilde{x}_{54} = (0, 50, 0, 0)_{LR}$ and remaining values of $\tilde{x}_{ij}$ are $(0, 0, 0, 0)_{LR}$.

Step 6 Putting the fuzzy optimal values of $\tilde{x}_{21}, \tilde{x}_{23}, \tilde{x}_{24}, \tilde{x}_{26}, \tilde{x}_{13}, \tilde{x}_{34}, \tilde{x}_{36}, \tilde{x}_{16}, \tilde{x}_{51}, \tilde{x}_{53}, \tilde{x}_{54}, \tilde{x}_{56}$ in $((12, 15, 2, 5)_{LR} \otimes \tilde{x}_{21} \oplus (9, 11, 2, 19)_{LR} \otimes \tilde{x}_{23} \oplus (15, 19, 7, 4)_{LR} \otimes \tilde{x}_{24} \oplus (10, 12, 3, 7)_{LR} \otimes \tilde{x}_{13} \oplus (10, 15, 5, 5)_{LR} \otimes \tilde{x}_{34} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{16} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{26} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{36} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{51} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{53} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{54} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{56})$, the minimum total fuzzy transportation cost is $(1650, 2600, 1270, 4661.37)_{LR}$.

3.8.2 Physical interpretation of the results

In this section, the minimum total fuzzy transportation cost, obtained by using the proposed method, is physically interpreted. Similarly, the obtained fuzzy optimal solution can also be physically interpreted.

Using the proposed method the minimum total fuzzy transportation cost is $(1650, 2600, 1270, 4661.37)_{LR}$, which can be physically interpreted as follows:

(1) The least amount of minimum total transportation cost is 380.

(2) The most possible amount of minimum total transportation cost lies between 1650 and 2600.

(3) The greatest amount of minimum total transportation cost is 7261.37 i.e., the

minimum total transportation cost will always be greater than 380 and less than 7261.37 and maximum chances are that the minimum total transportation cost will lie between 1650 and 2600.

The variation in minimum total transportation cost with respect to chances is shown in Figure 3.2.

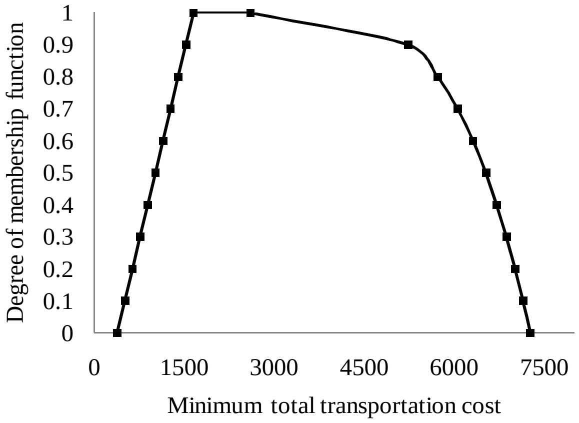

Figure 3.2 Membership function of *LR* flat fuzzy number representing the minimum total fuzzy transportation cost

3.9 Comparative study

To show the advantages of proposed method over existing method [38], the results of an existing balanced fully fuzzy capacitated minimum cost flow problem and the results of fully fuzzy capacitated minimum cost flow problems, chosen in Examples 3.1 and Example 3.2, obtained by using the existing method [38] and the proposed method are shown in Table 3.5.

Table 3.5 Results obtained by using the existing method and method proposed in this chapter

Example	Existing method [38]	Method proposed in this chapter
3.5 [39, pp. 2498]	$(1924000, 1903300, 7299800)_{LR}$	$(1924000, 1903300, 7299800)_{LR}$
3.1	No feasible solution	$(1650, 3550, 1270, 4750)_{LR}$
3.2	No feasible solution	$(1650, 2600, 1270, 4661.375)_{LR}$

The results, presented in Table 3.5, can be explained as follows:

(1) The existing method [38] can be used only for solving such fully fuzzy capacitated minimum cost flow problems for which the restrictions $\underline{l}_{ij} \leq \underline{x}_{ij} \leq \underline{u}_{ij}$, $\overline{l}_{ij} \leq \overline{x}_{ij} \leq \overline{u}_{ij}$, $l^L_{ij} \leq x^L_{ij} \leq u^L_{ij}$ and $l^R_{ij} \leq x^R_{ij} \leq u^R_{ij}$ are satisfied. However, no feasible solution is obtained by using the existing method [38] on solving such fully fuzzy capacitated minimum cost flow problems for which these restrictions are not satisfied. Since, for the values of $\tilde{l}_{ij}$ and $\tilde{u}_{ij}$ chosen in the existing problem [39, Example 3.5, pp. 2498] the restrictions $l_{ij} \leq x_{ij} \leq u_{ij}$, $l^L_{ij} \leq x^L_{ij} \leq u^L_{ij}$ and $l^R_{ij} \leq x^R_{ij} \leq u^R_{ij}$ are satisfying, so as discussed in Section 3.5, it can be solved by using the existing method [38]. However, for the values of $\tilde{l}_{ij}$ and $\tilde{u}_{ij}$ chosen in Example 3.1 and Example 3.2, the restrictions $\underline{l}_{ij} \leq \underline{x}_{ij} \leq \underline{u}_{ij}$, $\overline{l}_{ij} \leq \overline{x}_{ij} \leq \overline{u}_{ij}$, $l^L_{ij} \leq x^L_{ij} \leq u^L_{ij}$ and $l^R_{ij} \leq x^R_{ij} \leq u^R_{ij}$ are not satisfying so as discussed in Section 3.5 by using the existing method [38] no feasible solution is obtained for these problems.

(2) Since, the fully fuzzy capacitated minimum cost flow problems for which the restrictions $\underline{l}_{ij} \leq \underline{x}_{ij} \leq \underline{u}_{ij}$, $\overline{l}_{ij} \leq \overline{x}_{ij} \leq \overline{u}_{ij}$, $l^L_{ij} \leq x^L_{ij} \leq u^L_{ij}$ and $l^R_{ij} \leq x^R_{ij} \leq u^R_{ij}$ are satisfied can be solved by the proposed method. Also, by using the proposed method a feasible solution can be obtained for such fully fuzzy capacitated minimum cost flow problems for which these restrictions are not satisfied. So, the existing problem [39, Example 3.5, pp. 2498] can be solved by the proposed method as well as for the problems, chosen in Example 3.1 and Example 3.2, feasible solutions can be obtained by the proposed method.

Remark 3.5 Since, in the existing methods [38, 39] and proposed method different type of multiplication and ranking approach are used in the objective function so the different fuzzy optimal values are obtained by using the existing and proposed methods. But, to compare the results of existing method [39] and proposed method same type of multiplication and ranking approach is used for solving fully fuzzy capacitated minimum cost flow problems.

3.10 Case study

Ghatee and Hashemi [41, Definition 5.2, pp. 804] have claimed that if $\tilde{a}$ and $\tilde{b}$ and are two non-negative fuzzy numbers such that $\tilde{a} \neq \tilde{b}$. Then, $\tilde{a} \neq \tilde{b}$ can be converted in $\tilde{a} = \tilde{b}$ by the following manner:

Find $\tilde{e} = \tilde{a} \ominus \tilde{b}$ and check that $\tilde{e}$ is negative or positive.

Case (i) If $\tilde{e}$ is positive then $\tilde{b} \oplus \tilde{e} = \tilde{a}$.

Case (ii) If $\tilde{e}$ is negative then $\tilde{a} \oplus \tilde{e'} = \tilde{b}$, where, $\tilde{e'} = \ominus_H \tilde{e}$.

However, it is not always possible to convert $\tilde{a} \neq \tilde{b}$ into $\tilde{a} = \tilde{b}$ by using the described method due to the following reasons:

If $\tilde{a}$ and $\tilde{b}$ are two non-negative fuzzy numbers such that $\tilde{a} \neq \tilde{b}$ then $\tilde{e} = \tilde{a} \ominus \tilde{b}$ may be neither negative nor positive. i.e., neither $\tilde{b} \oplus \tilde{e} = \tilde{a}$ nor $\tilde{a} \oplus \tilde{e'} = \tilde{b}$. e.g., in the existing real life fully fuzzy capacitated minimum cost flow problem [41], described in Section 3.10.1, total fuzzy supply $\tilde{a} = (1580, 49, 100)_{LR}$ is not equal to the total fuzzy demand $\tilde{b} = (1498.9, 64, 59)_{LR}$ so it is an unbalanced fully fuzzy capacitated minimum cost flow problem. However, $\tilde{e} = \tilde{a} \ominus \tilde{b} = (81.1, 108, 164)_{LR}$ is neither negative nor positive fuzzy number and neither $\tilde{b} \oplus \tilde{e} = \tilde{a}$ nor $\tilde{a} \oplus \tilde{e'} = \tilde{b}$.

Since, Ghatee and Hashemi [41] have used the existing method [38] to obtain the fuzzy optimal solution of this unbalanced real life problem without converting it into the balanced fully fuzzy capacitated minimum cost flow problem. So, the results of this real life problem, obtained by Ghatee and Hashemi [41], are not genuine.

In this section, the method, proposed in this chapter, is used to find the fuzzy optimal solution of this real life problem.

3.10.1 Description of problem

A simplified network of petroleum industry distribution system Iran, shown in Figure 3.3, which is operated to transport crude oil from production units and import terminals to refineries, export terminals and storage tanks, and from there to destinations with minimum cost is depicted in Figure 3.4.

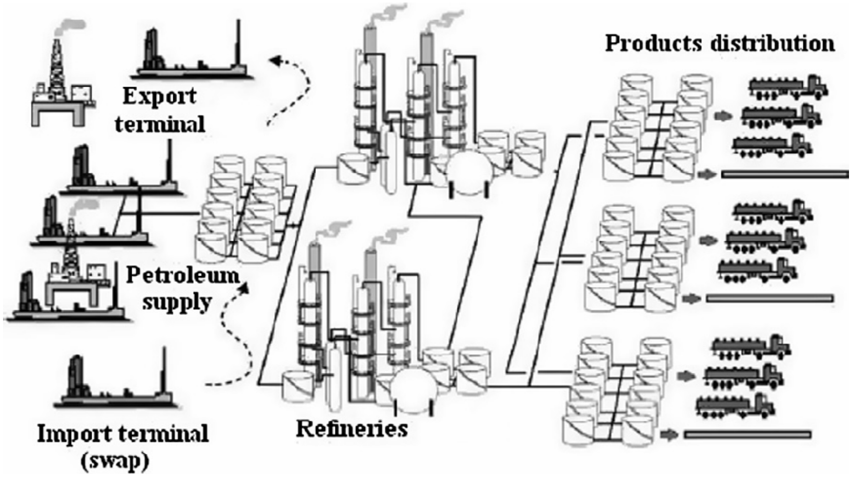

Figure 3.3 General petroleum supply chain, including the suppliers and demanders of petroleum

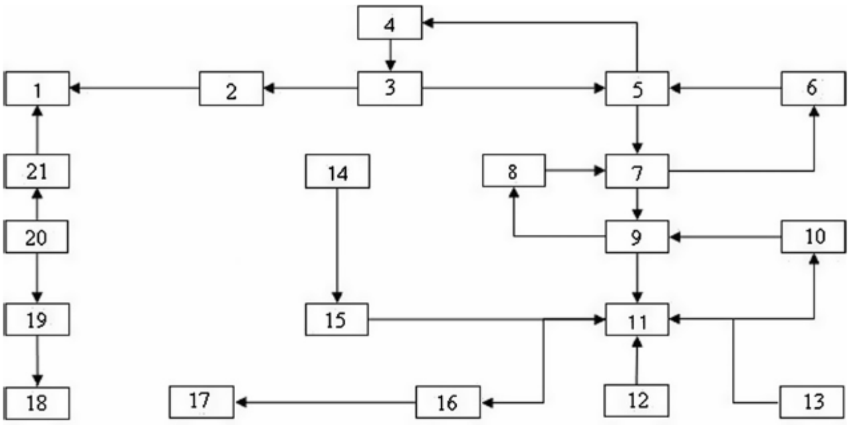

Figure 3.4 A simple diagram of a pilot in Iranian petroleum industry

The fuzzy supply of the crude oil at different sources and destinations are shown in Table 3.6 and the fuzzy cost and the fuzzy capacity for transporting one unit quantity of the crude oil from different sources to different destinations are shown in Table 3.7. with $L(x) = R(x) = \text{maximum } \{0, 1 - x\}$

Table 3.6 The fuzzy supply and fuzzy demand

Node	Fuzzy supply	Fuzzy demand
1	-	$(588, 10, 5)_{LR}$
2	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$
3	$(200, 10, 10)_{LR}$	-
4	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$
5	$(80, 4, 7)_{LR}$	-
6	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$
7	$(220, 4, 12)_{LR}$	-
8	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$
9	$(150, 5, 9)_{LR}$	-
10	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$
11	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$
12	$(220, 5, 16)_{LR}$	-
13	$(70, 2, 5)_{LR}$	-
14	$(50, 3, 5)_{LR}$	-
15	$(70, 1, 4)_{LR}$	-
16	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$
17	-	$(400, 17, 20)_{LR}$
18	-	$(158.9, 3, 15)_{LR}$
19	$(0, 0, 0)_{LR}$	(0,0,0)
20	$(520, 15, 32)_{LR}$	-
21	-	$(352, 34, 19)_{LR}$

Table 3.7 The fuzzy cost and maximum fuzzy quantity

Node	Node	Fuzzy cost	Maximum fuzzy quantity
2	1	$(4550, 220.68, 1419.6)_{LR}$	$(10, 0.78, 1.75)_{LR}$
3	2	$(4550, 22.75, 1656.2)_{LR}$	$(15, 1.365, 1.515)_{LR}$
3	5	$(5000, 160, 940)_{LR}$	$(200, 0.94, 2.22)_{LR}$
5	4	$(10000, 595, 2360)_{LR}$	$(30, 1.77, 5.34)_{LR}$
4	3	$(10000, 830, 440)_{LR}$	$(40, 0.44, 7.08)_{LR}$
5	7	$(10000, 525, 80)_{LR}$	$(50, 0.1, 5.35)_{LR}$
7	6	$(10000, 610, 960)_{LR}$	$(60, 1.44, 7.08)_{LR}$
6	5	$(10000, 150, 2840)_{LR}$	$(70, 4.97, 7.07)_{LR}$
7	9	$(10000, 440, 920)_{LR}$	$(80, 1.84, 8.88)_{LR}$
9	8	$(2500, 76.25, 940)_{LR}$	$(90, 8.46, 13.95)_{LR}$
8	7	$(10000, 625, 2280)_{LR}$	$(100, 5.7, 18.2)_{LR}$
9	11	$(10000, 590, 1920)_{LR}$	$(110, 5.28, 18.26)_{LR}$
11	10	$(10000, 430, 1040)_{LR}$	$(120, 3.12, 13.44)_{LR}$
10	9	$(10000, 515, 960)_{LR}$	$(130, 3.12, 16.51)_{LR}$
13	11	$(10000, 770, 240)_{LR}$	$(140, 0.84, 22.4)_{LR}$
14	15	$(10000, 895, 400)_{LR}$	$(150, 1.5, 28.35)_{LR}$
15	11	$(2000, 131, 368)_{LR}$	$(160, 7.36, 28.32)_{LR}$
11	16	$(5000, 127.5, 1460)_{LR}$	$(390, 12.41, 21.08)_{LR}$
16	17	$(5000, 227.5, 1520)_{LR}$	$(390, 13.68, 30.06)_{LR}$
12	11	$(5000, 167.5, 1520)_{LR}$	$(190, 14.44, 27.17)_{LR}$
20	19	$(1600, 89.6, 96)_{LR}$	$(200, 3, 25.4)_{LR}$
20	21	$(2500, 148.75, 700)_{LR}$	$(410, 14.7, 39.69)_{LR}$
21	1	$(1600, 103.2, 89.6)_{LR}$	$(588, 10.08, 31.46)_{LR}$
19	18	$(30000, 585, 7800)_{LR}$	$(230, 14.95, 23.92)_{LR}$

3.10.2 Results

The fuzzy quantity of crude oil that should be transported from one node to another node, obtained by using the proposed method, is shown in Table 3.8.

Table 3.8 The fuzzy optimal flow between each couple of node in the simplified pilot in Iran

Node $\longrightarrow$ Node	Fuzzy flow	Node $\longrightarrow$ Node	Fuzzy flow
2 $\longrightarrow$ 1	$(0,0,0)_{LR}$	11 $\longrightarrow$ 10	$(2.44,2.44,0)_{LR}$
3 $\longrightarrow$ 2	$(0,0,0)_{LR}$	11 $\longrightarrow$ 16	$(388.55,5.55,20)_{LR}$
3 $\longrightarrow$ 5	$(200,10,0)_{LR}$	12 $\longrightarrow$ 11	$(191.81,5,11)_{LR}$
3 $\longrightarrow$ 23	$(0,0,10)_{LR}$	12 $\longrightarrow$ 23	$(28.18,0,5)_{LR}$
4 $\longrightarrow$ 3	$(0,0,0)_{LR}$	13 $\longrightarrow$ 11	$(70,2,5)_{LR}$
5 $\longrightarrow$ 4	$(0,0,0)_{LR}$	14 $\longrightarrow$ 15	$(0,0,0)_{LR}$
5 $\longrightarrow$ 7	$(0,0,0)_{LR}$	14 $\longrightarrow$ 18	$(50,3,5)_{LR}$
5 $\longrightarrow$ 21	$(280,14,7)_{LR}$	15 $\longrightarrow$ 11	$(70,1,4)_{LR}$
6 $\longrightarrow$ 5	$(0,0,0)_{LR}$	16 $\longrightarrow$ 17	$(388.55,5.55,20)_{LR}$
7 $\longrightarrow$ 6	$(0,0,0)_{LR}$	19 $\longrightarrow$ 18	$(108.9,0,10)_{LR}$
7 $\longrightarrow$ 21	$(245.33,11.44,12)_{LR}$	20 $\longrightarrow$ 19	$(108.9,0,10)_{LR}$
7 $\longrightarrow$ 9	$(0,0,0)_{LR}$	20 $\longrightarrow$ 21	$(411.1,15,5)_{LR}$
8 $\longrightarrow$ 7	$(25.34,7.44,0)_{LR}$	20 $\longrightarrow$23	$(0,0,17)_{LR}$
9 $\longrightarrow$ 8	$(25.34,7.44,0)_{LR}$	21 $\longrightarrow$ 1	$(584.44,6.44,5)_{LR}$
9 $\longrightarrow$ 11	$(59.18,0,0)_{LR}$	22 $\longrightarrow$ 1	$(3.55,3.55,0)_{LR}$
9 $\longrightarrow$ 23	$(67.91,0,9)_{LR}$	22 $\longrightarrow$ 17	$(11.44,11.44,0)_{LR}$
10 $\longrightarrow$ 9	$(2.44,2.44,0)_{LR}$		

3.10.3 Discussion

Since, it is obvious from Figure 3.3 that node 5, node 7 and node 14 are not connected to node 21, node 21 and node 18 respectively. So, in the fuzzy optimal solution the fuzzy quantity of the crude oil that should be transported from node 5, node 7 and node 14 to node 21, node 21 and node 18 respectively should be zero LR fuzzy number. However, it is obvious from the results, shown in Table 3.8, that these quantities are not zero LR fuzzy number. So, the obtained fuzzy optimal solution of the chosen real life problem is a pseudo fuzzy optimal solution.

Since, the fuzzy cost for transporting one unit quantity of the crude oil from node 5, node 7 and node 14 to node 21, node 21 and node 18 respectively are not given in the existing data [41]. So, assuming this cost as M the obtained minimum total fuzzy transportation cost is $(12429794+575.33M, 781344.46+28.44M, 3775337.62+24M)_{LR}$.

Remark 3.6 Since the chosen real life problem is an unbalanced problem so to find the fuzzy optimal solution of the chosen problem a purely dummy source node (22) and a purely dummy destination node (23) is introduced. In the results, presented in Section 3.10.2, $i \longrightarrow 23$, where, $i = 3, 9, 12, 20$ represents the fuzzy quantity of the crude oil that should be transported from i^{th} node to purely destination node (23). Similarly, $22 \longrightarrow j$, where, $j = 1, 17$ represents the fuzzy quantity of the crude oil that should be transported from purely dummy source node 22 to j^{th} node.

3.11 Conclusions

On the basis of present study, it can be concluded that it is better to use the method, proposed in this chapter, as compared to the existing method [38].

Chapter 4

A New Method For Solving Fully Fuzzy Multi - Objective Capacitated Minimum Cost Flow Problems

Gupta et al. [49] claimed that there is no method in the literature for solving fully fuzzy multi-objective transportation problems and proposed a method for the same. In this chapter, the limitations of this existing method [49] and the limitations of the method, proposed in Chapter 3, are pointed out and to overcome these limitations, a new method is proposed for solving fully fuzzy multi-objective capacitated minimum cost flow problems. Also, the advantages of the proposed method over the existing method [49] and over the method, proposed in Chapter 3, are discussed.

4.1 Existing method

Gupta et al. [49] proposed a method to find the fuzzy optimal solution of fully fuzzy multi-objective transportation problems.

The steps of the proposed method are as follows:

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Step 1 Find the total fuzzy supply $\sum_{i=1}^{p}\tilde{a}_i$ and the total fuzzy demand $\sum_{j=1}^{q}\tilde{b}_j$. Let $\sum_{i=1}^{p}\tilde{a}_i=(a,a',a'',a''')$ and $\sum_{j=1}^{q}\tilde{b}_j=(b,b',b'',b''')$. Examine that the problem is balanced or not, i.e., $\sum_{i=1}^{p}\tilde{a}_i=\sum_{j=1}^{q}\tilde{b}_j$ or $\sum_{i=1}^{p}\tilde{a}_i\neq\sum_{j=1}^{q}\tilde{b}_j$.

Case (i) If the problem is balanced, i.e., $\sum_{i=1}^{p}\tilde{a}_i=\sum_{j=1}^{q}\tilde{b}_j$, then Go to Step 2.

Case (ii) If $\sum_{i=1}^{p}\tilde{a}_i\neq\sum_{j=1}^{q}\tilde{b}_j$ then convert the unbalanced problem $\sum_{i=1}^{p}\tilde{a}_i\neq\sum_{j=1}^{q}\tilde{b}_j$ into balanced problem $\sum_{i=1}^{m}\tilde{a}_i=\sum_{j=1}^{n}\tilde{b}_j, m=p$ or $p+1$ and $n=q$ or $q+1$ by using the existing method [75].

Step 2 Formulate the balanced fully fuzzy multi-objective transportation problem, obtained from Step 1, into the multi-objective fuzzy linear programming problem $(P_{4.1})$.

Minimize $\sum_{i=1}^{m}\sum_{j=1}^{n}(\tilde{c}_{ij}^{k}\otimes\tilde{x}_{ij}) \qquad ;k=1,2,\ldots,P$

subject to

$$\begin{aligned}&\sum_{j=1}^{n}\tilde{x}_{ij}=\tilde{a}_i, && i=1,2,\ldots,m;\ m=p \text{ or } p+1\\&\sum_{i=1}^{m}\tilde{x}_{ij}=\tilde{b}_j, && j=1,2,\ldots,n;\ n=q \text{ or } q+1\end{aligned} \qquad (P_{4.1})$$

$\tilde{x}_{ij}$ is a non-negative trapezoidal fuzzy number.

Step 3 Assuming $\tilde{c}_{ij}^{k},\tilde{a}_i,\tilde{b}_j$ and $\tilde{x}_{ij}$ as trapezoidal fuzzy numbers $(c_{ij}^{k},c'^{k}_{ij},c''^{k}_{ij},c'''^{k}_{ij})$, (a_i,a'_i,a''_i,a'''_i), (b_j,b'_j,b''_j,b'''_j) and $(x_{ij},y_{ij},z_{ij},w_{ij})$ respectively, the fuzzy multi-objective linear programming problem $(P_{4.1})$ can be written as:

Minimize $\sum_{i=1}^{m}\sum_{j=1}^{n}((c_{ij}^{k},c'^{k}_{ij},c''^{k}_{ij},c'''^{k}_{ij})\otimes(x_{ij},y_{ij},z_{ij},w_{ij})) \qquad ;k=1,2,\ldots,P$

subject to

$$\begin{aligned}&\sum_{j=1}^{n}(x_{ij},y_{ij},z_{ij},w_{ij})=(a_i,a'_i,a''_i,a'''_i), && i=1,2,\ldots,m\\&\sum_{i=1}^{m}(x_{ij},y_{ij},z_{ij},w_{ij})=(b_j,b'_j,b''_j,b'''_j), && j=1,2,\ldots,n\end{aligned} \qquad (P_{4.2})$$

$(x_{ij},y_{ij},z_{ij},w_{ij})$ is a non-negative trapezoidal fuzzy number.

Step 4 Assuming $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}(\tilde{c}_{ij}^{k} \otimes \tilde{x}_{ij}) = (x_0^k, y_0^k, z_0^k, w_0^k)$ and using the arithmetic operations discussed in Section 2.1.2 and Remark 2.3, the fuzzy multi-objective linear programming problem $(P_{4.2})$ can be written as:

Minimize $(x_0^k, y_0^k, z_0^k, w_0^k) \qquad ; k = 1, 2, \ldots, P$

subject to

$$\begin{aligned}
&(\sum_{j=1}^{n} x_{ij}, \sum_{j=1}^{n} y_{ij}, \sum_{j=1}^{n} z_{ij}, \sum_{j=1}^{n} w_{ij}) = (a_i, a'_i, a''_i, a'''_i),\ i = 1, 2, \ldots, m \\
&(\sum_{i=1}^{m} x_{ij}, \sum_{i=1}^{m} y_{ij}, \sum_{i=1}^{m} z_{ij}, \sum_{i=1}^{m} w_{ij}) \ = (b_j, b'_j, b''_j, b'''_j),\ j = 1, 2, \ldots, n
\end{aligned} \qquad (P_{4.3})$$

$(x_{ij}, y_{ij}, z_{ij}, w_{ij})$ is a non-negative trapezoidal fuzzy number.

Step 5 Using Definition 2.10 and Definition 2.11 and Remark 2.3, the fuzzy linear programming problem $(P_{4.3})$ can be converted into the fuzzy linear programming problem $(P_{4.4})$:

Minimize $(x_0^k, y_0^k, z_0^k, w_0^k) \qquad ; k = 1, 2, \ldots, P$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m && \sum_{j=1}^{n} y_{ij} = a'_i, \quad i = 1, 2, \ldots, m \\
&\sum_{j=1}^{n} z_{ij} = a''_i, \ \ i = 1, 2, \ldots, m && \sum_{j=1}^{n} w_{ij} = a'''_i, \ \ i = 1, 2, \ldots, m \\
&\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n && \sum_{i=1}^{m} y_{ij} = b'_j, \quad j = 1, 2, \ldots, n \qquad (P_{4.4}) \\
&\sum_{i=1}^{m} z_{ij} = b''_j, \ \ j = 1, 2, \ldots, n && \sum_{i=1}^{m} w_{ij} = b'''_j, \ \ j = 1, 2, \ldots, n \\
&x_{ij}, y_{ij} - x_{ij}, z_{ij} - y_{ij}, w_{ij} - z_{ij} \geq 0 \ \ \forall \ \ i, j
\end{aligned}$$

Step 6 The fuzzy optimal compromise solution of fuzzy linear programming problem $(P_{4.4})$ can be obtained by solving the following crisp multi-objective linear programming problem:

Minimize $\frac{1}{4}(x_0^k + y_0^k + z_0^k + w_0^k) \qquad ; k = 1, 2, \ldots, P$

subject to $\qquad (P_{4.5})$

constraints of problem $(P_{4.4})$

Step 7 Solve the crisp linear programming problem $(P_{4.5})$ to find the optimal compromise solution $\{x_{ij}, y_{ij}, z_{ij}, w_{ij}\}$.

Step 8 Put the obtained values of $x_{ij}, y_{ij}, z_{ij}, w_{ij}$ in $\tilde{x}_{ij} = (x_{ij}, y_{ij}, z_{ij}, w_{ij})$ to find the fuzzy optimal compromise solution $\{\tilde{x}_{ij}\}$.

Step 9 Find the fuzzy optimal value of each objective function by putting the values of $\tilde{x}_{ij}$ in $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}(\tilde{c}_{ij}^{k} \otimes \tilde{x}_{ij}) \quad ; k = 1, 2, \ldots, P.$

4.2 Limitations of the existing method and the method proposed in the previous chapter

In this section, the limitations of the existing method [49] and the method proposed in the Chapter 3, are pointed out.

(1) The existing method [49] can be used for solving fully fuzzy single and multi-objective un-capacitated transportation problems. However, the existing method [49] can neither be used for solving fully fuzzy single objective capacitated transportation problems nor for solving fully fuzzy multi-objective capacitated transportation problems e.g., neither the fully fuzzy single objective capacitated transportation problem nor the fully fuzzy multi-objective capacitated transportation problem chosen in Example 4.1 and Example 4.2 respectively can be solved by using the existing method [49].

Example 4.1 A company has two sources S_1, S_2 and three destinations D_1, D_2, D_3. The fuzzy supply of the product at sources S_1, S_2 is $(90, 90, 20, 10)_{LR}$ and $(60, 70, 20, 10)_{LR}$ respectively and the fuzzy demand of the product at destinations D_1, D_2, D_3 is $(40, 50, 10, 20)_{LR}$, $(30, 40, 10, 10)_{LR}$ and $(50, 50, 10, 30)_{LR}$ respectively. The fuzzy transportation cost $(\tilde{c}_{ij})$ required for transporting one unit quantity of product from i^{th} source to j^{th} destination, minimum fuzzy quantity of the product $(\tilde{l}_{ij})$ and maximum fuzzy quantity of the product $(\tilde{u}_{ij})$ are listed in Table 4.1, Table 4.2 and Table 4.3 respectively. The company wants to determine the fuzzy quantities of the product to be transported from i^{th} source to j^{th} destination in order to minimize the total fuzzy transportation cost.

Table 4.1 Fuzzy transportation cost $(\tilde{c}_{ij})$

Destination→ Source↓	D_1	D_2	D_3
S_1	$(7,8,1,3)_{LR}$	$(8,9,1,3)_{LR}$	$(9,10,1,3)_{LR}$
S_2	$(5,8,1,2)_{LR}$	$(4,5,1,3)_{LR}$	$(3,4,1,4)_{LR}$

Table 4.2 Minimum fuzzy quantity of the product $(\tilde{l}_{ij})$

Destination→ Source↓	D_1	D_2	D_3
S_1	$(2,3,1,1)_{LR}$	$(3,4,1,2)_{LR}$	$(0,0,0,0)_{LR}$
S_2	$(1,2,1,1)_{LR}$	$(0,1,0,1)_{LR}$	$(6,8,1,1)_{LR}$

Table 4.3 Maximum fuzzy quantity of the product $(\tilde{u}_{ij})$

Destination→ Source↓	D_1	D_2	D_3
S_1	$(40,50,20,20)_{LR}$	$(25,30,10,20)_{LR}$	$(55,60,50,30)_{LR}$
S_2	$(40,50,10,20)_{LR}$	$(45,50,40,10)_{LR}$	$(60,70,20,30)_{LR}$

where, $L(x) = \text{maximum}\{0, 1-x\},\ R(x) = \text{maximum}\ \{0, 1-x^4\}$

Example 4.2 Consider a fully fuzzy bi-objective capacitated transportation problem with the same data as in Example 4.1. The first objective is the same as in Example 4.1 and the second objective is to minimize the total fuzzy transportation time. The fuzzy transportation time $(\tilde{t}_{ij})$ required for transporting one unit quantity of product from i^{th} source to j^{th} destination is listed in Table 4.4.

Table 4.4 Fuzzy transportation time $(\tilde{t}_{ij})$

Destination→ Source↓	D_1	D_2	D_3
S_1	$(8,9,1,3)_{LR}$	$(1,2,1,1)_{LR}$	$(3,5,1,1)_{LR}$
S_2	$(8,9,1,3)_{LR}$	$(3,5,1,1)_{LR}$	$(5,6,1,3)_{LR}$

where, $L(x) = \text{maximum}\{0, 1 - x\},\ R(x) = \text{maximum}\ \{0, 1 - x^4\}$

(2) Since, fully fuzzy minimum cost flow problem is the generalization of fully fuzzy transportation problem so the existing method [49] can neither be used for solving fully fuzzy single and multi-objective un-capacitated minimum cost flow problems nor for solving fully fuzzy single and multi-objective capacitated minimum cost flow problems e.g., the fully fuzzy single objective un-capacitated minimum cost flow problem, fully fuzzy single objective capacitated minimum cost flow problem, fully fuzzy multi-objective un-capacitated minimum cost flow problem and fully fuzzy multi-objective capacitated minimum cost flow problem chosen in Example 4.3, Example 3.2, Example 4.4 and Example 4.5 respectively can not be solved by using the existing method [49].

Example 4.3 Find the fuzzy optimal solution of the fully fuzzy un-capacitated minimum cost flow problem depicted in Figure 4.1. The data is listed in Table 4.5.

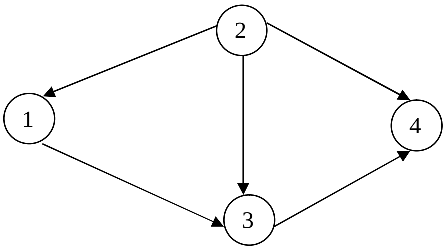

Figure 4.1 Network representing fully fuzzy capacitated minimum cost flow problem

Table 4.5 Fuzzy cost ($\tilde{c}_{ij}$), fuzzy supply ($\tilde{a}_i/\tilde{e}_i$) and fuzzy demand ($\tilde{b}_j/\tilde{d}_j$)

Node	$\tilde{a}_i/\tilde{e}_i$	$\tilde{b}_j/\tilde{d}_j$	Arc (i, j)	$\tilde{c}_{ij}$
1	–	–	$(2, 1)$	$(12, 15, 2, 5)_{LR}$
2	$(200, 250, 100, 50)_{LR}$	–	$(2, 3)$	$(9, 11, 2, 19)_{LR}$
3	–	$(100, 150, 80, 50)_{LR}$	$(2, 4)$	$(15, 19, 7, 4)_{LR}$
4	–	$(50, 100, 20, 0)_{LR}$	$(1, 3)$	$(10, 12, 3, 7)_{LR}$
–	–	–	$(3, 4)$	$(10, 15, 5, 5)_{LR}$

where, $L(x) = \text{maximum}\{0, 1 - x\},\ R(x) = \text{maximum}\{0, 1 - x^4\}$

Example 4.4 Find the fuzzy optimal compromise solution of the fully fuzzy multi-objective un-capacitated minimum cost flow problem with the same network flow structure as in Example 4.3. The first objective is to minimize the total fuzzy transportation cost and second objective is to minimize the total fuzzy passing time. The data is listed in Table 4.6.

Table 4.6 Fuzzy cost ($\tilde{c}^1_{ij}$), fuzzy time ($\tilde{c}^2_{ij}$), fuzzy supply ($\tilde{a}_i/\tilde{e}_i$) and fuzzy demand ($\tilde{b}_j/\tilde{d}_j$)

Node	$\tilde{a}_i/\tilde{e}_i$	$\tilde{b}_j/\tilde{d}_j$	Arc (i,j)	$\tilde{c}^1_{ij}$	$\tilde{c}^2_{ij}$
1	–	–	$(2,1)$	$(10,12,3,7)_{LR}$	$(10,12,3,7)_{LR}$
2	$(200,250,100,50)_{LR}$	–	$(2,3)$	$(12,15,2,5)_{LR}$	$(15,19,2,5)_{LR}$
3	–	$(100,150,80,50)_{LR}$	$(2,4)$	$(9,11,2,19)_{LR}$	$(12,14,7,4)_{LR}$
4	–	$(50,100,20,0)_{LR}$	$(1,3)$	$(15,19,7,4)_{LR}$	$(9,13,5,5)_{LR}$
–	–	–	$(3,4)$	$(10,15,5,5)_{LR}$	$(9,11,2,19)_{LR}$

where, $L(x) = \text{maximum}\{0, 1-x\}$, $R(x) = \text{maximum}\{0, 1-x^4\}$

Example 4.5 Find the fuzzy optimal compromise solution of the fully fuzzy multi-objective capacitated minimum cost flow problem with the same network flow structure as in Example 4.3. The first objective is to minimize the total fuzzy transportation cost and second objective is to minimize the total fuzzy passing time. The data is summarized in Table 4.7 and Table 4.8.

Table 4.7 Fuzzy cost ($\tilde{c}^1_{ij}$), fuzzy time ($\tilde{c}^2_{ij}$), minimum fuzzy amount ($\tilde{u}_{ij}$) and maximum fuzzy amount ($\tilde{l}_{ij}$)

Arc (i,j)	$\tilde{c}^1_{ij}$	$\tilde{c}^2_{ij}$	$\tilde{l}_{ij}$	$\tilde{u}_{ij}$
$(2,1)$	$(10,12,3,7)_{LR}$	$(10,12,3,7)_{LR}$	$(0,0,0,0)_{LR}$	$(70,90,20,10)_{LR}$
$(2,3)$	$(12,15,2,5)_{LR}$	$(15,19,2,5)_{LR}$	$(5,20,3,2)_{LR}$	$(255,270,3,2)_{LR}$
$(2,4)$	$(9,11,2,19)_{LR}$	$(12,14,7,4)_{LR}$	$(12,15,2,5)_{LR}$	$(301,351,2,5)_{LR}$
$(1,3)$	$(15,19,7,4)_{LR}$	$(9,13,5,5)_{LR}$	$(0,0,0,0)_{LR}$	$(80,110,10,20)_{LR}$
$(3,4)$	$(10,15,5,5)_{LR}$	$(9,11,2,19)_{LR}$	$(0,0,0,0)_{LR}$	$(65,70,15,10)_{LR}$

Table 4.8 Fuzzy supply ($\tilde{a}_i/\tilde{e}_i$) and fuzzy demand ($\tilde{b}_j/\tilde{d}_j$)

Node	$\tilde{a}_i/\tilde{e}_i$	$\tilde{b}_j/\tilde{d}_j$
1	–	–
2	$(200,250,100,50)_{LR}$	–
3	–	$(100,150,80,50)_{LR}$
4	–	$(50,100,20,0)_{LR}$

where, $L(x) = \text{maximum}\{0, 1-x\}$, $R(x) = \text{maximum}\{0, 1-x^4\}$

(3) The method, proposed in Chapter 3, can be used for solving fully fuzzy single objective un-capacitated and capacitated transportation problems as well as fully fuzzy single objective un-capacitated and capacitated minimum cost flow problems but can neither be used for solving fully fuzzy multi-objective un-capacitated and capacitated transportation problems nor for solving fully fuzzy multi-objective un-capacitated and capacitated minimum cost flow problems e.g., none of the problems chosen in Example 4.2, Example 4.4 and Example 4.5 can be solved by the method proposed in Chapter 3.

4.3 Fuzzy programming technique to solve multi-objective linear programming problems

In this section, the fuzzy programming technique [129] which is used in this chapter for solving multi-objective problems is presented. The optimal compromise solution of the multi-objective problems can be obtained by using the following steps:

Step 1 Formulate the chosen multi-objective problem into the following multi-objective linear programming problem:

$$\begin{aligned}&\text{Minimize } \sum C^{\eta}X \qquad ;\eta = 1, 2, ..., P\\&\text{subject to}\\&\qquad AX \leq, =, \geq b\\&\qquad X \geq 0\end{aligned}$$

Step 2 Solve the multi-objective problem, obtained in Step 1, as a single-objective problem P times, by taking one of the objectives at a time, to find P optimal solutions $X^1, X^2, ..., X^P$.

Step 3 Find the value of each objective function corresponding to the each optimal solution obtained in Step 2. Let the value of i^{th} objective function Z_i corresponding to j^{th} optimal solution X^j be denoted by $Z_i(X^j)$.

Step 4 Find $U_i = \underset{1\le j\le P}{\text{maximum}}\{Z_i(X^j)\} \;\; \forall\, i = 1, 2, ..., P$

$$L_i = \underset{1\le j\le P}{\text{minimum}}\{Z_i(X^j)\} \;\; \forall\, i = 1, 2, ..., P$$

Step 5 Define the linear membership function $\mu^i(Z_i)\ \forall\ i = 1, 2, ..., P$.

where, $\mu^i(Z_i) = \begin{cases} 1, & Z_i \le L_i \\ 1 - \frac{Z_i - L_i}{U_i - L_i}, & L_i \le Z_i \le U_i \\ 0, & Z_i \ge U_i \end{cases}$

Step 6 Using the linear membership function, obtained in Step 5, convert the multi-objective linear programming problem, obtained in Step 2, into the following single objective crisp linear programming problem:

Maximize λ'

subject to

$$Z_i + \lambda'(U_i - L_i) \le U_i, \quad i = 1, 2, ..., P$$

$$AX \le, =, \ge b$$

$$X \ge 0,\ \lambda' \ge 0$$

Step 7 Solve the crisp linear programming problem, obtained in Step 6, to find the optimal compromise solution of the chosen multi-objective linear programming problem.

Step 8 Put the value of X, obtained from Step 7, in each objective function to find the optimal value of each objective function.

4.4 Proposed method

In this section, to overcome the limitations of the existing method [49] and the method proposed in Chapter 3, pointed out in Section 4.2, a new method is proposed for solving such unbalanced fully fuzzy multi-objective capacitated minimum cost flow problems in which all the parameters as well as decision variables are represented by LR flat fuzzy numbers.

If the supply of product at i^{th} purely source node and i^{th} source node is $\tilde{a}_i$ and $\tilde{e}_i$ respectively and the demand of the product at j^{th} purely destination node and j^{th} destination node is $\tilde{b}_j$ and $\tilde{d}_j$ respectively then the fuzzy optimal compromise solution of the fully fuzzy multi-objective capacitated minimum cost flow problems can be obtained by using the following steps:

Step 1 Use Step 1 of the method, proposed in Section 3.7, to obtain a balanced fully fuzzy multi-objective capacitated minimum cost flow problem.

Step 2 Formulate the balanced fully fuzzy multi-objective capacitated minimum cost flow problem, obtained in Step 1, into the fully fuzzy multi-objective linear programming problem $(P_{4.6})$ [40].

$$\begin{array}{lll}
\text{Minimize} \sum\limits_{(i,j)\in A} (\tilde{c}^{\eta}_{ij} \otimes \tilde{x}_{ij}) & ;\eta = 1, 2, ..., P & \\
\text{subject to} & & \\
\sum\limits_{j:(i,j)\in A} \tilde{x}_{ij} = \tilde{a}_i & i \in N_{PS} & \\
\sum\limits_{j:(i,j)\in A} \tilde{x}_{ij} = \sum\limits_{j:(j,i)\in A} \tilde{x}_{ji} \oplus \tilde{e}_i & i \in N_S & \\
\sum\limits_{i:(i,j)\in A} \tilde{x}_{ij} = \tilde{b}_j & j \in N_{PD} & \\
\sum\limits_{i:(i,j)\in A} \tilde{x}_{ij} = \sum\limits_{i:(j,i)\in A} \tilde{x}_{ji} \oplus \tilde{d}_j & j \in N_D & (P_{4.6}) \\
\sum\limits_{j:(i,j)\in A} \tilde{x}_{ij} = \sum\limits_{j:(j,i)\in A} \tilde{x}_{ji} & i \in N_T & \\
\tilde{l}_{ij} \preceq \tilde{x}_{ij} \preceq \tilde{u}_{ij} & \forall\ (i,j) \in A &
\end{array}$$

$\tilde{x}_{ij}$ is a non-negative LR flat fuzzy number $\forall\ (i, j) \in A$

$\tilde{c}^{\eta}_{ij}$: The fuzzy penalty per unit of flow through arc (i, j) in the η^{th} objective function.

P: is total number of objective functions.

Step 3 Assuming $\tilde{c}_{ij} = (\underline{c}_{ij}, \overline{c}_{ij}, c^L_{ij}, c^R_{ij})_{LR}$, $\tilde{x}_{ij} = (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR}$, $\tilde{a}_i = (\underline{a}_i, \overline{a}_i, a^L_i, a^R_i)_{LR}$, $\tilde{e}_i = (\underline{e}_i, \overline{e}_i, e^L_i, e^R_i)_{LR}$, $\tilde{b}_j = (\underline{b}_j, \overline{b}_j, b^L_j, b^R_j)_{LR}$, $\tilde{d}_j = (\underline{d}_j, \overline{d}_j, d^L_j, d^R_j)_{LR}$, $\tilde{l}_{ij} = (\underline{l}_{ij}, \overline{l}_{ij}, l^L_{ij}, l^R_{ij})_{LR}$ and $\tilde{u}_{ij} = (\underline{u}_{ij}, \overline{u}_{ij}, u^L_{ij}, u^R_{ij})_{LR}$ the fully fuzzy multi-objective linear programming problem $(P_{4.6})$ can be written as:

Minimize $\sum\limits_{(i,j)\in A} (((\underline{c}_{ij})^\eta, (\overline{c}_{ij})^\eta, (c^L_{ij})^\eta, (c^R_{ij})^\eta)_{LR} \otimes (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR}) \qquad ; \eta = 1, 2, ..., P$

subject to

$$
\begin{array}{lll}
\sum\limits_{j:(i,j)\in A} (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} = (\underline{a}_i, \overline{a}_i, a^L_i, a^R_i)_{LR} & i \in N_{PS} & \\
\sum\limits_{j:(i,j)\in A} (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} = \sum\limits_{j:(j,i)\in A} (\underline{x}_{ji}, \overline{x}_{ji}, x^L_{ji}, x^R_{ji})_{LR} \oplus (\underline{e}_i, \overline{e}_i, e^L_i, e^R_i)_{LR} & i \in N_S & \\
\sum\limits_{i:(i,j)\in A} (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} = (\underline{b}_j, \overline{b}_j, b^L_j, b^R_j)_{LR} & j \in N_{PD} & \\
\sum\limits_{i:(i,j)\in A} (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} = \sum\limits_{i:(j,i)\in A} (\underline{x}_{ji}, \overline{x}_{ji}, x^L_{ji}, x^R_{ji})_{LR} \oplus (\underline{d}_i, \overline{d}_i, d^L_i, d^R_i)_{LR} & j \in N_D & (P_{4.7}) \\
\sum\limits_{j:(i,j)\in A} (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} = \sum\limits_{j:(j,i)\in A} (\underline{x}_{ji}, \overline{x}_{ji}, x^L_{ji}, x^R_{ji})_{LR} & i \in N_T & \\
(\underline{l}_{ij}, \overline{l}_{ij}, l^L_{ij}, l^R_{ij})_{LR} \preceq (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} \preceq (\underline{u}_{ij}, \overline{u}_{ij}, u^L_{ij}, u^R_{ij})_{LR} & \forall\ (i,j) \in A &
\end{array}
$$

$(\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR}$ is a non-negative LR flat fuzzy number $\forall\ (i,j) \in A$

Step 4 Using the arithmetic operations of LR flat fuzzy numbers, presented in Section 2.1.2, the fully fuzzy multi-objective linear programming problem $(P_{4.7})$ can be written as:

Minimize $\sum\limits_{(i,j)\in A} ((\underline{c}_{ij})^\eta \underline{x}_{ij}, (\overline{c}_{ij})^\eta \overline{x}_{ij}, (\underline{c}_{ij})^\eta x^L_{ij} + (c^L_{ij})^\eta \underline{x}_{ij} - (c^L_{ij})^\eta x^L_{ij}, (\overline{c}_{ij})^\eta x^R_{ij} + (c^R_{ij})^\eta \overline{x}_{ij} + (c^R_{ij})^\eta x^R_{ij})_{LR} \qquad ; \eta = 1, 2, ..., P$

subject to

$$
\begin{array}{ll}
(\sum\limits_{j:(i,j)\in A} \underline{x}_{ij}, \sum\limits_{j:(i,j)\in A} \overline{x}_{ij}, \sum\limits_{j:(i,j)\in A} x^L_{ij}, \sum\limits_{j:(i,j)\in A} x^R_{ij})_{LR} = (\underline{a}_i, \overline{a}_i, a^L_i, a^R_i)_{LR} & i \in N_{PS} \\
(\sum\limits_{j:(i,j)\in A} \underline{x}_{ij}, \sum\limits_{j:(i,j)\in A} \overline{x}_{ij}, \sum\limits_{j:(i,j)\in A} x^L_{ij}, \sum\limits_{j:(i,j)\in A} x^R_{ij})_{LR} = (\sum\limits_{j:(j,i)\in A} \underline{x}_{ji} + \underline{e}_i, \sum\limits_{j:(j,i)\in A} \overline{x}_{ji} + \overline{e}_i, \sum\limits_{j:(j,i)\in A} x^L_{ji} & \\
+ e^L_i, \sum\limits_{j:(j,i)\in A} x^R_{ji} + e^R_i)_{LR}) & i \in N_S \\
(\sum\limits_{i:(i,j)\in A} \underline{x}_{ij}, \sum\limits_{i:(i,j)\in A} \overline{x}_{ij}, \sum\limits_{i:(i,j)\in A} x^L_{ij}, \sum\limits_{i:(i,j)\in A} x^R_{ij})_{LR} = (\underline{b}_j, \overline{b}_j, b^L_j, b^R_j)_{LR} & j \in N_{PD}
\end{array}
$$

$$\Big(\sum_{i:(i,j)\in A}\underline{x}_{ij},\ \sum_{i:(i,j)\in A}\overline{x}_{ij},\ \sum_{i:(i,j)\in A}x^{L}_{ij},\ \sum_{i:(i,j)\in A}x^{R}_{ij}\Big)_{LR}=\Big(\sum_{i:(j,i)\in A}\underline{x}_{ji}+\underline{d}_{j},\ \sum_{i:(j,i)\in A}\overline{x}_{ji}+\overline{d}_{j},\ \sum_{i:(j,i)\in A}x^{L}_{ji}$$
$$+d^{L}_{j},\ \sum_{i:(j,i)\in A}x^{R}_{ji}+d^{R}_{j}\Big)_{LR} \qquad j\in N_D \quad (P_{4.8})$$

$$\Big(\sum_{j:(i,j)\in A}\underline{x}_{ij},\ \sum_{j:(i,j)\in A}\overline{x}_{ij},\ \sum_{j:(i,j)\in A}x^{L}_{ij},\ \sum_{j:(i,j)\in A}x^{R}_{ij}\Big)_{LR}=\Big(\sum_{j:(j,i)\in A}\underline{x}_{ji},\ \sum_{j:(j,i)\in A}\overline{x}_{ji},\ \sum_{j:(j,i)\in A}x^{L}_{ji},\ \sum_{j:(j,i)\in A}$$
$$x^{R}_{ji}\Big)_{LR} \qquad i\in N_T$$

$$(\underline{l}_{ij},\overline{l}_{ij},l^{L}_{ij},l^{R}_{ij})_{LR}\preceq(\underline{x}_{ij},\overline{x}_{ij},x^{L}_{ij},x^{R}_{ij})_{LR}\preceq(\underline{u}_{ij},\overline{u}_{ij},u^{L}_{ij},u^{R}_{ij})_{LR} \qquad \forall\ (i,j)\in A$$

$(\underline{x}_{ij},\overline{x}_{ij},x^{L}_{ij},x^{R}_{ij})_{LR}$ is a non-negative LR flat fuzzy number $\forall\ (i,j)\in A$

Step 5 Using Definition 2.10, Definition 2.11, the fully fuzzy multi-objective linear programming problem $(P_{4.8})$ can be converted into fuzzy multi-objective linear programming problem $(P_{4.9})$:

Minimize $\sum\limits_{(i,j)\in A}((\underline{c}_{ij})^{\eta}\underline{x}_{ij},(\overline{c}_{ij})^{\eta}\overline{x}_{ij},(\underline{c}_{ij})^{\eta}x^{L}_{ij}+(c^{L}_{ij})^{\eta}\underline{x}_{ij}-(c^{L}_{ij})^{\eta}x^{L}_{ij},(\overline{c}_{ij})^{\eta}x^{R}_{ij}+(c^{R}_{ij})^{\eta}\overline{x}_{ij}+$
$(c^{R}_{ij})^{\eta}x^{R}_{ij})_{LR} \qquad ;\eta=1,2,...,P$

subject to

$$\begin{array}{lll}
\sum\limits_{j:(i,j)\in A}\underline{x}_{ij}=\underline{a}_{i} & i\in N_{PS} & \\
\sum\limits_{j:(i,j)\in A}\overline{x}_{ij}=\overline{a}_{i} & i\in N_{PS} & \\
\sum\limits_{j:(i,j)\in A}x^{L}_{ij}=a^{L}_{i} & i\in N_{PS} & \\
\sum\limits_{j:(i,j)\in A}x^{R}_{ij}=a^{R}_{i} & i\in N_{PS} & \\
\sum\limits_{j:(i,j)\in A}\underline{x}_{ij}=\sum\limits_{j:(j,i)\in A}\underline{x}_{ji}+\underline{e}_{i} & i\in N_{S} & \\
\sum\limits_{j:(i,j)\in A}\overline{x}_{ij}=\sum\limits_{j:(j,i)\in A}\overline{x}_{ji}+\overline{e}_{i} & i\in N_{S} & \\
\sum\limits_{j:(i,j)\in A}x^{L}_{ij}=\sum\limits_{j:(j,i)\in A}x^{L}_{ji}+e^{L}_{i} & i\in N_{S} & \\
\sum\limits_{j:(i,j)\in A}x^{R}_{ij}=\sum\limits_{j:(j,i)\in A}x^{R}_{ji}+e^{R}_{i} & i\in N_{S} & \\
\sum\limits_{i:(i,j)\in A}\underline{x}_{ij}=\underline{b}_{j} & j\in N_{PD} & \\
\sum\limits_{i:(i,j)\in A}\overline{x}_{ij}=\overline{b}_{j} & j\in N_{PD} & \\
\sum\limits_{i:(i,j)\in A}x^{L}_{ij}=b^{L}_{j} & j\in N_{PD} & (P_{4.9})
\end{array}$$

$$\begin{array}{ll}
\sum\limits_{i:(i,j)\in A} x^R_{ij} = b^R_j & j \in N_{PD} \\
\sum\limits_{i:(i,j)\in A} \underline{x}_{ij} = \sum\limits_{i:(j,i)\in A} \underline{x}_{ji} + \underline{d}_j & j \in N_D \\
\sum\limits_{i:(i,j)\in A} \overline{x}_{ij} = \sum\limits_{i:(j,i)\in A} \overline{x}_{ji} + \overline{d}_j & j \in N_D \\
\sum\limits_{i:(i,j)\in A} x^L_{ij} = \sum\limits_{i:(j,i)\in A} x^L_{ji} + d^L_j & j \in N_D \\
\sum\limits_{i:(i,j)\in A} x^R_{ij} = \sum\limits_{i:(j,i)\in A} x^R_{ji} + d^R_j & j \in N_D \\
\sum\limits_{j:(i,j)\in A} \underline{x}_{ij} = \sum\limits_{j:(j,i)\in A} \underline{x}_{ji} & i \in N_T \\
\sum\limits_{j:(i,j)\in A} \overline{x}_{ij} = \sum\limits_{j:(j,i)\in A} \overline{x}_{ji} & i \in N_T \\
\sum\limits_{j:(i,j)\in A} x^L_{ij} = \sum\limits_{j:(j,i)\in A} x^L_{ji} & i \in N_T \\
\sum\limits_{j:(i,j)\in A} x^R_{ij} = \sum\limits_{j:(j,i)\in A} x^R_{ji} & i \in N_T \\
\multicolumn{2}{l}{\underline{x}_{ij} - x^L_{ij} \geq 0, \overline{x}_{ij} - \underline{x}_{ij} \geq 0, x^L_{ij}, x^R_{ij} \geq 0 \ \ \forall\ (i,j) \in A}
\end{array}$$

$$\left.(\underline{l}_{ij}, \overline{l}_{ij}, l^L_{ij}, l^R_{ij})_{LR} \preceq (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} \preceq (\underline{u}_{ij}, \overline{u}_{ij}, u^L_{ij}, u^R_{ij})_{LR}\ \forall\ (i,j) \in A \ \right\} \quad (C_{4.1})$$

Step 6 As discussed in Step 6 of the method, proposed in Section 3.7, the fuzzy optimal compromise solution of the fuzzy multi-objective linear programming problem $(P_{4.9})$ can be obtained by solving the following crisp multi-objective linear programming problem:

Minimize $\sum\limits_{(i,j)\in A} \Re((\underline{c}_{ij})^\eta \underline{x}_{ij}, (\overline{c}_{ij})^\eta \overline{x}_{ij}, (\underline{c}_{ij})^\eta x^L_{ij} + (c^L_{ij})^\eta \underline{x}_{ij} - (c^L_{ij})^\eta x^L_{ij}, (\overline{c}_{ij})^\eta x^R_{ij} + (c^R_{ij})^\eta \overline{x}_{ij} + (c^R_{ij})^\eta x^R_{ij})_{LR}$ $\quad ; \eta = 1, 2, ..., P$

subject to $\quad (P_{4.10})$

$\Re(\underline{l}_{ij}, \overline{l}_{ij}, l^L_{ij}, l^R_{ij})_{LR} \leq \Re(\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR} \leq \Re(\underline{u}_{ij}, \overline{u}_{ij}, u^L_{ij}, u^R_{ij})_{LR}\ \forall\ (i,j) \in A$

as well as all the constraints of problem $(P_{4.9})$ except $(C_{4.1})$

Step 7 Using Section 2.4, the crisp linear programming problem $(P_{4.10})$ can be written as:

Minimize $\sum\limits_{(i,j)\in A} [\gamma(\int_0^1(((\underline{c}_{ij})^\eta \underline{x}_{ij})\lambda + ((\overline{c}_{ij})^\eta \overline{x}_{ij})(1-\lambda))d\rho) + (1-\gamma)(\lambda \int_0^1(((\underline{c}_{ij})^\eta \underline{x}_{ij}) - ((\underline{c}_{ij})^\eta x^L_{ij} + (c^L)^\eta_{ij} \underline{x}_{ij} - (c^L)^\eta_{ij} x^L_{ij}) L^{-1}(\rho))d\rho + (1-\lambda) \int_0^1((\overline{c}_{ij})^\eta \overline{x}_{ij} + ((\overline{c}_{ij})^\eta x^R_{ij} + (c^R)^\eta_{ij} \overline{x}_{ij} +$

$(c^R)^{\eta}_{ij}x^R_{ij})R^{-1}(\rho))d\rho)]$

subject to $\qquad (P_{4.11})$

$$\Big(\gamma(\int_0^1(\underline{l}_{ij}\lambda+\bar{l}_{ij}(1-\lambda))d\rho)+(1-\gamma)(\lambda\int_0^1(\underline{l}_{ij}-l^L_{ij}L^{-1}(\rho))d\rho+(1-\lambda)\int_0^1(\bar{l}_{ij}+l^R_{ij}R^{-1}(\rho))d\rho)\Big)$$

$$\leq \Big(\gamma(\int_0^1(\underline{x}_{ij}\lambda+\overline{x}_{ij}(1-\lambda))d\rho)+(1-\gamma)(\lambda\int_0^1(\underline{x}_{ij}-x^L_{ij}L^{-1}(\rho))d\rho+(1-\lambda)\int_0^1(\overline{x}_{ij}+ x^R_{ij}R^{-1}(\rho))d\rho)\Big) \leq \Big(\gamma(\int_0^1(\underline{u}_{ij}\lambda+\overline{u}_{ij}(1-\lambda))d\rho)+(1-\gamma)(\lambda\int_0^1(\underline{u}_{ij}-u^L_{ij}L^{-1}(\rho))d\rho+ (1-\lambda)\int_0^1(\overline{u}_{ij}+u^R_{ij}R^{-1}(\rho))d\rho)\Big)$$

as well as all the constraints of problem $(P_{4.9})$ except $(C_{4.1})$

Step 8 Solve the crisp multi-objective linear programming problem, obtained in Step 7, to find the optimal compromise solution $\{\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij}\}$ of the crisp multi-objective linear programming problem.

Step 9 Put the values of $\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij}$ in $\tilde{x}_{ij}=(\underline{x}_{ij},\overline{x}_{ij},x^L_{ij},x^R_{ij})_{LR}$ to find the fuzzy optimal compromise solution $\{\tilde{x}_{ij}\}$.

Step 10 Find the fuzzy optimal value of each objective function by putting the values of $\tilde{x}_{ij}$ in $\sum\limits_{(i,j)\in A}(\tilde{c}^{\eta}_{ij}\otimes\tilde{x}_{ij})\ \ ;\eta=1,2,...,P.$

Remark 4.1 In Step 6 of the existing method [49], discussed in Section 4.1 Liou and Wang [82] ranking approach is used for comparing fuzzy numbers. But, Garcia and Lamata [36] have proposed some modification in Liou and Wang [82] ranking approach. So, it is not genuine to apply Liou and Wang [82] ranking approach for comparing fuzzy numbers. Therefore, in Step 7, of the proposed method Garcia and Lamata ranking approach, discussed in Section 2.4, is used.

4.5 Advantages of the proposed method over the existing method and the method proposed in the previous chapter

The main advantage of the proposed method over the existing method [49] and the method proposed in Chapter 3 is that all the problems which can be solved by using the existing method [49] and the method proposed in Chapter 3 can also be solved by using the proposed method. However, as discussed in Section 4.2, there exist several problems which can be solved by the proposed method but can not be solved by using the existing method [49] and the method proposed in Chapter 3. To illustrate the proposed method and also to show its advantage the fully fuzzy multi-objective capacitated minimum cost flow problem, chosen in Example 4.5, which can not be solved by using the existing method [49] and the method proposed in Chapter 3 is solved by using the proposed method.

4.5.1 Fuzzy optimal compromise solution of the chosen problem

The fully fuzzy multi-objective capacitated minimum cost flow problem, chosen in Example 4.5, can be solved by the using the following steps of the proposed method:

Step 1 Total fuzzy supply $= (200, 250, 100, 50)_{LR}$ and total fuzzy demand $= (150, 250, 100, 50)_{LR}$. Since total fuzzy supply $\neq$ total fuzzy demand, so it is an unbalanced fully fuzzy multi-objective capacitated minimum cost flow problem.

Now, as described in the proposed method (using Case (c) of Step 1), the unbalanced fully fuzzy multi-objective capacitated minimum cost flow problem can be converted into a balanced fully fuzzy multi-objective capacitated minimum cost flow problem, by introducing a purely dummy source node (5) with fuzzy supply $(0, 50, 0, 0)_{LR}$ and a purely dummy destination node (6) with fuzzy demand $(50, 50, 0, 0)_{LR}$, so that total fuzzy supply = total fuzzy demand i.e., $(200, 250, 100, 50)_{LR} \oplus (0, 50, 0, 0)_{LR} = (150, 250, 100, 50)_{LR} \oplus (50, 50, 0, 0)_{LR}$.

Assume the fuzzy transportation cost and fuzzy transportation time for the one unit quantity of the product from the introduced purely dummy source node (5) to all intermediate nodes (1 and 3), existing purely destination node (4) and introduced purely dummy destination node (6) as zero LR flat fuzzy number. Similarly, assume the fuzzy transportation cost and fuzzy transportation time for the one unit quantity of the product from all intermediate nodes (1 and 3), existing purely source node (2) and introduced purely dummy source node (5) to the introduced purely dummy destination node (6) as zero LR flat fuzzy number i.e., $\tilde{c}^1_{51} = \tilde{c}^1_{53} = \tilde{c}^1_{54} = \tilde{c}^1_{56} = \tilde{c}^1_{36} = \tilde{c}^1_{26} = \tilde{c}^1_{16} = \tilde{c}^2_{51} = \tilde{c}^2_{53} = \tilde{c}^2_{54} = \tilde{c}^2_{56} = \tilde{c}^2_{36} = \tilde{c}^2_{26} = \tilde{c}^2_{16} = (0,0,0,0)_{LR}$.

Step 2 The balanced fully fuzzy multi-objective capacitated minimum cost flow problem, obtained from Step 1, can be formulated into the following fully fuzzy multi-objective linear programming problem:

Minimize $((10,12,3,7)_{LR} \otimes \tilde{x}_{21} \oplus (12,15,2,5)_{LR} \otimes \tilde{x}_{23} \oplus (9,11,2,19)_{LR} \otimes \tilde{x}_{24} \oplus (15,19,7,4)_{LR} \otimes \tilde{x}_{13} \oplus (10,15,5,5)_{LR} \otimes \tilde{x}_{34} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{16} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{26} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{36} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{51} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{53} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{54} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{56})$

Minimize $((10,12,3,7)_{LR} \otimes \tilde{x}_{21} \oplus (15,19,2,5)_{LR} \otimes \tilde{x}_{23} \oplus (12,14,7,4)_{LR} \otimes \tilde{x}_{24} \oplus (9,13,5,5)_{LR} \otimes \tilde{x}_{13} \oplus (9,11,2,19)_{LR} \otimes \tilde{x}_{34} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{16} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{26} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{36} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{51} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{53} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{54} \oplus (0,0,0,0)_{LR} \otimes \tilde{x}_{56})$

subject to

$$\tilde{x}_{13} \oplus \tilde{x}_{16} = \tilde{x}_{21} \oplus \tilde{x}_{51}$$

$$\tilde{x}_{21} \oplus \tilde{x}_{23} \oplus \tilde{x}_{24} \oplus \tilde{x}_{26} = (200,250,100,50)_{LR}$$

$$\tilde{x}_{13} \oplus \tilde{x}_{23} \oplus \tilde{x}_{53} = \tilde{x}_{34} \oplus \tilde{x}_{36} \oplus (100,150,80,50)_{LR}$$

$\tilde{x}_{24} \oplus \tilde{x}_{34} \oplus \tilde{x}_{54} = (50, 100, 20, 0)_{LR}$

$\tilde{x}_{51} \oplus \tilde{x}_{53} \oplus \tilde{x}_{54} \oplus \tilde{x}_{56} = (0, 50, 0, 0)_{LR}$

$\tilde{x}_{16} \oplus \tilde{x}_{26} \oplus \tilde{x}_{36} \oplus \tilde{x}_{56} = (50, 50, 0, 0)_{LR}$

$\tilde{x}_{21} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{21} \preceq (70, 90, 20, 10)_{LR}$, $\tilde{x}_{23} \succeq (12, 15, 2, 5)_{LR}$

$\tilde{x}_{23} \preceq (255, 270, 3, 2)_{LR}$, $\tilde{x}_{24} \succeq (5, 20, 3, 2)_{LR}$, $\tilde{x}_{24} \preceq (301, 351, 2, 5)_{LR}$

$\tilde{x}_{13} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{13} \preceq (80, 110, 10, 20)_{LR}$, $\tilde{x}_{34} \succeq (0, 0, 0, 0)_{LR}$

$\tilde{x}_{34} \preceq (65, 70, 15, 10)_{LR}$, $\tilde{x}_{16} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{26} \succeq (0, 0, 0, 0)_{LR}$

$\tilde{x}_{36} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{51} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{54} \succeq (0, 0, 0, 0)_{LR}$

$\tilde{x}_{53} \succeq (0, 0, 0, 0)_{LR}$, $\tilde{x}_{56} \succeq (0, 0, 0, 0)_{LR}$

$\tilde{x}_{ij}$ is a non-negative LR flat fuzzy number $\forall\ (i, j) \in A$

Step 3 Using Step 3 to Step 7 of the method, proposed in Section 4.4, the fully fuzzy multi-objective linear programming problem, obtained in Step 2, can be converted into the following crisp multi-objective linear programming problem:

Minimize $\frac{1}{30}(105\underline{x}_{21} + 150\underline{x}_{23} + 105\underline{x}_{24} + 120\underline{x}_{13} + 75\underline{x}_{34} + 236\overline{x}_{21} + 265\overline{x}_{23} + 317\overline{x}_{24} + 317\overline{x}_{13} + 265\overline{x}_{34} - 105x^L_{21} - 150x^L_{23} - 105x^L_{24} - 120x^L_{13} - 75x^L_{34} + 152x^R_{21} + 160x^R_{23} + 240x^R_{24} + 184x^R_{13} + 160x^R_{34})$

Minimize $\frac{1}{30}(105\underline{x}_{21} + 195\underline{x}_{23} + 75\underline{x}_{24} + 60\underline{x}_{13} + 105\underline{x}_{34} + 236\overline{x}_{21} + 325\overline{x}_{23} + 242\overline{x}_{24} + 235\overline{x}_{13} + 317\overline{x}_{34} - 105x^L_{21} - 195x^L_{23} - 175x^L_{24} - 60x^L_{13} - 105x^L_{34} + 152x^R_{21} + 192x^R_{23} + 144x^R_{24} + 144x^R_{13} + 240x^R_{34})$

subject to

$\underline{x}_{13} + \underline{x}_{16} - \underline{x}_{21} - \underline{x}_{51} = 0$, $\underline{x}_{21} + \underline{x}_{23} + \underline{x}_{24} + \underline{x}_{26} = 200$

$\underline{x}_{13} + \underline{x}_{23} + \underline{x}_{53} - \underline{x}_{34} - \underline{x}_{36} = 100$, $\underline{x}_{24} + \underline{x}_{34} + \underline{x}_{54} = 50$

$\underline{x}_{51} + \underline{x}_{53} + \underline{x}_{54} + \underline{x}_{56} = 0$, $\underline{x}_{16} + \underline{x}_{26} + \underline{x}_{36} + \underline{x}_{56} = 50$

$\overline{x}_{13} + \overline{x}_{16} - \overline{x}_{21} - \overline{x}_{51} = 0$, $\overline{x}_{21} + \overline{x}_{23} + \overline{x}_{24} + \overline{x}_{26} = 250$

$$\overline{x}_{13}+\overline{x}_{23}+\overline{x}_{53}-\overline{x}_{34}-\overline{x}_{36}=150, \qquad \overline{x}_{24}+\overline{x}_{34}+\overline{x}_{54}=100$$

$$\overline{x}_{51}+\overline{x}_{53}+\overline{x}_{54}+\overline{x}_{56}=50, \qquad \overline{x}_{16}+\overline{x}_{26}+\overline{x}_{36}+\overline{x}_{56}=50$$

$$x^L_{13}+x^L_{16}-x^L_{21}-x^L_{51}=0, \qquad x^L_{21}+x^L_{23}+x^L_{24}+x^L_{26}=100$$

$$x^L_{13}+x^L_{23}+x^L_{53}-x^L_{34}-x^L_{36}=80, \qquad x^L_{24}+x^L_{34}+x^L_{54}=20$$

$$x^L_{51}+x^L_{53}+x^L_{54}+x^L_{56}=0, \qquad x^L_{16}+x^L_{26}+x^L_{36}+x^L_{56}=0$$

$$x^R_{13}+x^R_{16}-x^R_{21}-x^R_{51}=0, \qquad x^R_{21}+x^R_{23}+x^R_{24}+x^R_{26}=50$$

$$x^R_{13}+x^R_{23}+x^R_{53}-x^R_{34}-x^R_{36}=50, \qquad x^R_{24}+x^R_{34}+x^R_{54}=0$$

$$x^R_{51}+x^R_{53}+x^R_{54}+x^R_{56}=0, \qquad x^R_{16}+x^R_{26}+x^R_{36}+x^R_{56}=0$$

$$15\underline{x}_{21}+15\overline{x}_{21}-15x^L_{21}+8x^R_{21}\leq 2180, \quad 15\underline{x}_{23}+15\overline{x}_{23}-15x^L_{23}+8x^R_{23}\leq 7846$$

$$15\underline{x}_{24}+15\overline{x}_{24}-15x^L_{24}+8x^R_{24}\leq 9790, \quad 15\underline{x}_{13}+15\overline{x}_{13}-15x^L_{13}+8x^R_{13}\leq 2860$$

$$15\underline{x}_{34}+15\overline{x}_{34}-15x^L_{34}+8x^R_{34}\leq 1880, \quad 15\underline{x}_{16}+15\overline{x}_{16}-15x^L_{16}+8x^R_{16}\geq 0$$

$$15\underline{x}_{36}+15\overline{x}_{36}-15x^L_{36}+8x^R_{36}\geq 0, \qquad 15\underline{x}_{53}+15\overline{x}_{53}-15x^L_{53}+8x^R_{53}\geq 0$$

$$15\underline{x}_{56}+15\overline{x}_{56}-15x^L_{56}+8x^R_{56}\geq 0, \qquad 15\underline{x}_{26}+15\overline{x}_{26}-15x^L_{26}+8x^R_{26}\geq 0$$

$$15\underline{x}_{51}+15\overline{x}_{51}-15x^L_{51}+8x^R_{51}\geq 0, \qquad 15\underline{x}_{54}+15\overline{x}_{54}-15x^L_{54}+8x^R_{54}\geq 0$$

$$15\underline{x}_{21}+15\overline{x}_{21}-15x^L_{21}+8x^R_{21}\geq 0, \qquad 15\underline{x}_{23}+15\overline{x}_{23}-15x^L_{23}+8x^R_{23}\geq 346$$

$$15\underline{x}_{24}+15\overline{x}_{24}-15x^L_{24}+8x^R_{24}\geq 415, \quad 15\underline{x}_{13}+15\overline{x}_{13}-15x^L_{13}+8x^R_{13}\geq 0$$

$$15\underline{x}_{34}+15\overline{x}_{34}-15x^L_{34}+8x^R_{34}\geq 0$$

$$\overline{x}_{21}-\underline{x}_{21}\geq 0, \quad \overline{x}_{23}-\underline{x}_{23}\geq 0, \quad \overline{x}_{24}-\underline{x}_{24}\geq 0, \quad \overline{x}_{26}-\underline{x}_{26}\geq 0, \quad \overline{x}_{13}-\underline{x}_{13}\geq 0$$

$$\overline{x}_{34}-\underline{x}_{34}\geq 0, \quad \overline{x}_{36}-\underline{x}_{36}\geq 0, \quad \overline{x}_{16}-\underline{x}_{16}\geq 0, \quad \overline{x}_{51}-\underline{x}_{51}\geq 0, \quad \overline{x}_{53}-\underline{x}_{53}\geq 0$$

$$\overline{x}_{54}-\underline{x}_{54}\geq 0, \quad \overline{x}_{56}-\underline{x}_{56}\geq 0, \quad \underline{x}_{21}-x^L_{21}\geq 0, \quad \underline{x}_{23}-x^L_{23}\geq 0, \quad \underline{x}_{24}-x^L_{24}\geq 0$$

$$\underline{x}_{26}-x^L_{26}\geq 0, \quad \underline{x}_{13}-x^L_{13}\geq 0, \quad \underline{x}_{34}-x^L_{34}\geq 0, \quad \underline{x}_{36}-x^L_{36}\geq 0, \quad \underline{x}_{16}-x^L_{16}\geq 0$$

$$\underline{x}_{51}-x^L_{51}\geq 0, \quad \underline{x}_{53}-x^L_{53}\geq 0, \quad \underline{x}_{54}-x^L_{54}\geq 0, \quad \underline{x}_{56}-x^L_{56}\geq 0$$

$$x^L_{21},\ x^L_{23},\ x^L_{24},\ x^L_{26},\ x^L_{13},\ x^L_{34},\ x^L_{36},\ x^L_{16},\ x^L_{51},\ x^L_{53},\ x^L_{54},\ x^L_{56},\ x^R_{21},\ x^R_{23},\ x^R_{24},\ x^R_{26},\ x^R_{13},\ x^R_{34},$$

$$x^R_{36},\ x^R_{16},\ x^R_{51},\ x^R_{53},\ x^R_{54},\ x^R_{56}\geq 0$$

Step 4 Using fuzzy programming technique, discussed in Section 4.3, the optimal compromise solution of the crisp multi-objective linear programming problem, obtained in Step 3, is $\underline{x}_{23} = 100, \underline{x}_{24} = 50, \underline{x}_{26} = 50, \overline{x}_{23} = 125, \overline{x}_{24} = 75, \overline{x}_{26} = 50, \overline{x}_{53} = 25, \overline{x}_{54} = 25, x^L_{23} = 80, x^L_{24} = 20, x^R_{23} = 50$ and the remaining values of $\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij}$ are zero respectively.

Step 5 Putting the values of $\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}$ and x^R_{ij} in $\tilde{x}_{ij} = (\underline{x}_{ij}, \overline{x}_{ij}, x^L_{ij}, x^R_{ij})_{LR}$, the fuzzy optimal compromise solution is $\tilde{x}_{23} = (100, 125, 80, 50)_{LR}$, $\tilde{x}_{24} = (50, 75, 20, 0)_{LR}$, $\tilde{x}_{26} = (50, 50, 0, 0)_{LR}$, $\tilde{x}_{53} = (0, 25, 0, 0)_{LR}$, $\tilde{x}_{54} = (0, 25, 0, 0)_{LR}$ and remaining values of $\tilde{x}_{ij}$ are $(0, 0, 0, 0)_{LR}$.

Step 6 Putting the values of $\tilde{x}_{21}, \tilde{x}_{23}, \tilde{x}_{24}, \tilde{x}_{26}, \tilde{x}_{13}, \tilde{x}_{34}, \tilde{x}_{36}, \tilde{x}_{16}, \tilde{x}_{51}, \tilde{x}_{53}, \tilde{x}_{54}, \tilde{x}_{56}$ in $\Big((10, 12, 3, 7)_{LR} \otimes \tilde{x}_{21} \oplus (12, 15, 2, 5)_{LR} \otimes \tilde{x}_{23} \oplus (9, 11, 2, 19)_{LR} \otimes \tilde{x}_{24} \oplus (15, 19, 7, 4)_{LR} \otimes \tilde{x}_{13} \oplus (10, 15, 5, 5)_{LR} \otimes \tilde{x}_{34} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{16} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{26} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{36} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{51} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{53} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{54} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{56}\Big)$ and $\Big((10, 12, 3, 7)_{LR} \otimes \tilde{x}_{21} \oplus (15, 19, 2, 5)_{LR} \otimes \tilde{x}_{23} \oplus (12, 14, 7, 4)_{LR} \otimes \tilde{x}_{24} \oplus (9, 13, 5, 5)_{LR} \otimes \tilde{x}_{13} \oplus (9, 11, 2, 19)_{LR} \otimes \tilde{x}_{34} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{16} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{26} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{36} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{51} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{53} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{54} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{56}\Big)$, the minimum total fuzzy transportation cost and the minimum total fuzzy passing time are $(1650, 2700, 1240, 3050)_{LR}$ and $(2100, 3425, 1690, 2125)_{LR}$ respectively.

4.5.2 Physical interpretation of the results

In this section, the minimum total fuzzy transportation cost obtained by using the proposed method, is physically interpreted. Similarly, the obtained minimum total fuzzy passing time and fuzzy optimal compromise solution can also be physically interpreted.

Using the proposed method the minimum total fuzzy transportation cost is $(1650, 2700, 1240, 3050)_{LR}$ which can be physically interpreted as follows:

(1) The least amount of minimum total transportation cost is 410.

(2) The most possible amount of minimum total transportation cost lies between 1650 and 2700.

(3) The greatest amount of minimum total transportation cost is 5750 i.e., the minimum total transportation cost will always be greater than 410 and less than 5750 and maximum chances are that the minimum total transportation cost will lie between 1650 and 2700.

The variation in minimum total transportation cost with respect to chances is shown in Figure 4.2.

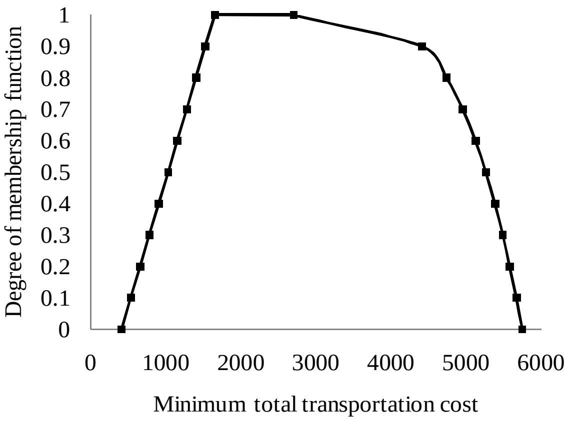

Figure 4.2 Membership function of *LR* flat fuzzy number representing the minimum total fuzzy transportation cost

4.6 Comparative study

To show the advantages of proposed method over the existing method [49] and the method proposed in Chapter 3, the results of an existing fully fuzzy multi-objective un-capacitated transportation problem [49, Example 4.2] and the results of problems, chosen in Example 4.1, Example 4.2, Example 4.3, Example 3.2, Example

4.4 and Example 4.5, obtained by using the existing method [49], method proposed in Chapter 3 and the method proposed in this chapter are shown in Table 4.9.

Table 4.9 Comparison of results obtained by using the existing method, method proposed in Chapter 3 and method proposed in this chapter

Example	Existing method [49]	Method proposed in Chapter 3	Method proposed in this chapter
4.2 [49]	$(111.25, 167.80, 41.90, 142)_{LR}$ $(151.75, 200.75, 47, 140.25)_{LR}$	Not applicable	$(113.90, 169.20, 43.50, 141.99)_{LR}$ $(149.99, 198.99, 47, 140.25)_{LR}$
4.1	Not applicable	$(630, 800, 230, 639.5)_{LR}$	$(630, 800, 230, 639.5)_{LR}$
4.2	Not applicable	Not applicable	$(671.62, 867.94, 205, 578.69)_{LR}$ $(494.69, 672.34, 158, 369.69)_{LR}$
4.3	Not applicable	$(1650, 2600, 1270, 4550)_{LR}$	$(1650, 2600, 1270, 4550)_{LR}$
3.2	Not applicable	$(1650, 2600, 1270, 4661.37)_{LR}$	$(1650, 2600, 1270, 4661.37)_{LR}$
4.4	Not applicable	Not applicable	$(1650, 2700, 1240, 3050)_{LR}$ $(2100, 3425, 1690, 2125)_{LR}$
4.5	Not applicable	Not applicable	$(1650, 2700, 1240, 3050)_{LR}$ $(2100, 3425, 1690, 2125)_{LR}$

The results, presented in Table 4.9, can be explained as follows:

(1) The existing method [49] can be used for solving fully fuzzy single and multi-objective un-capacitated transportation problems. However, the same method can neither be used for solving fully fuzzy single and multi-objective capacitated transportation problems nor for solving fully fuzzy single and multi-objective minimum cost flow problems. Since, the existing problem [49, Example 4.2] is a fully fuzzy multi-objective un-capacitated transportation problem, so it can be solved by using the existing method [49]. However, the problems chosen in Example 4.1, Example 4.2, Example 4.3, Example 3.2, Example 4.4 and Example 4.5 are fully fuzzy single objective capacitated transportation problem, fully fuzzy multi-objective capacitated transportation problem, fully fuzzy single objective un-capacitated minimum cost flow problem, fully fuzzy single objective capacitated minimum cost flow problem, fully fuzzy multi-objective un-capacitated minimum cost flow problem and fully fuzzy multi-

objective capacitated minimum cost flow problem respectively. So, none of these problems can be solved by using the existing method [49].

(2) The method proposed in Chapter 3 can be used for solving fully fuzzy single objective un-capacitated and capacitated transportation problems as well as fully fuzzy single objective un-capacitated and capacitated minimum cost flow problems. However, the same method can neither be used for solving fully fuzzy multi-objective un-capacitated and capacitated transportation problems nor for solving fully fuzzy multi-objective un-capacitated and capacitated minimum cost flow problems. Since, the problem chosen in Example 4.1, Example 4.3 and Example 3.2 are fully fuzzy single objective capacitated transportation problem, fully fuzzy single objective un-capacitated minimum cost flow problem and fully fuzzy single objective capacitated minimum cost flow problem respectively. So, these problems can be solved by using the method proposed in Chapter 3. However, the existing problem [49, Example 4.2] and the problems chosen in Example 4.2, Example 4.4 and Example 4.5 are fully fuzzy multi-objective problems so none of these problems can be solved by using the method proposed in Chapter 3.

(3) Since, the method proposed in this chapter can be used for solving fully fuzzy single and multi-objective un-capacitated and capacitated transportation problems as well as fully fuzzy single and multi-objective un-capacitated and capacitated minimum cost flow problems. So, the existing problem [49, Example 4.2] as well as the problems, chosen in Example 4.1, Example 4.2, Example 4.3, Example 3.2, Example 4.4 and Example 4.5 can be solved by using the method proposed in this chapter.

4.7 Conclusions

On the basis of present study, it can be concluded that it is better to use the method, proposed in this chapter, as compared to the existing methods [38, 49] and the method proposed in Chapter 3.

Chapter 5

A NEW METHOD FOR SOLVING FULLY FUZZY MULTI - OBJECTIVE CAPACITATED SOLID MINIMUM COST FLOW PROBLEMS

In single and multi-objective capacitated minimum cost flow problems it is assumed that there is only one conveyance which can be used for transporting the product. However, in real life problems more than one conveyances are used for transporting the product. To the best of my knowledge till now there is no method in the literature for solving such fully fuzzy single and multi-objective capacitated minimum cost flow problems in which more than one conveyances are used for transporting the product. Since, in the literature [51] such transportation problems in which more than one conveyances are used for transporting the product are named as solid transportation problems so, in this chapter, such type of fully fuzzy single and multi-objective capacitated minimum cost flow problems in which more than one conveyances are used for transporting the product are named as fully fuzzy single and multi-objective capacitated solid minimum cost flow problems and a new method is proposed for solving these problems. The proposed method is illustrated

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5.1 Proposed linear programming formulation of fully fuzzy multi-objective capacitated solid minimum cost flow problems

In this section, fuzzy linear programming formulation of balanced fully fuzzy multi-objective capacitated solid minimum cost flow problems is proposed.

Let $\tilde{a}_i$ and $\tilde{e}_i$ be the fuzzy supply of a product at i^{th} purely source node and at i^{th} source node, $\tilde{b}_j$ and $\tilde{d}_j$ be the fuzzy demand of the product at j^{th} purely destination node and at the j^{th} destination node, $\tilde{f}_k$ be the fuzzy capacity of the k^{th} conveyance i.e., the amount of product which can be carried by the k^{th} conveyance, $\tilde{c}^{\eta}_{ijk}$ be the fuzzy penalty per unit of flow from i^{th} node to j^{th} node by means of the k^{th} conveyance in the η^{th} objective function $\eta = 1, 2, ..., P$, where P is total number of objectives, $\tilde{x}_{ijk}$ be the fuzzy quantity of the product that should be transported from i^{th} node to j^{th} node by means of the k^{th} conveyance in order to minimize P objective functions, $\tilde{l}_{ijk}$ be the minimum fuzzy amount that can flow from i^{th} node to j^{th} node by means of the k^{th} conveyance and $\tilde{u}_{ijk}$ be the maximum fuzzy amount that can flow from i^{th} node to j^{th} node by means of the k^{th} conveyance. Then, any balanced fully fuzzy multi-objective capacitated solid minimum cost flow problem i.e., $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j = \sum\limits_{k \in S_C} \tilde{f}_k$, can be formulated into the following fuzzy multi-objective linear programming problem:

$$\text{Minimize} \sum_{(i,j) \in A} \sum_{k \in S_C} (\tilde{c}^{\eta}_{ijk} \otimes \tilde{x}_{ijk}) \qquad ; \eta = 1, 2, ..., P$$

subject to

$$\sum_{j:(i,j) \in A} \sum_{k \in S_C} \tilde{x}_{ijk} = \tilde{a}_i \qquad i \in N_{PS}$$

$$\begin{aligned}
&\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}\tilde{x}_{ijk} = \sum_{j:(j,i)\in A}\ \sum_{k\in S_C}\tilde{x}_{jik} \oplus \tilde{e}_i && i\in N_S \\
&\sum_{i:(i,j)\in A}\ \sum_{k\in S_C}\tilde{x}_{ijk} = \tilde{b}_j && j\in N_{PD} \\
&\sum_{i:(i,j)\in A}\ \sum_{k\in S_C}\tilde{x}_{ijk} = \sum_{i:(j,i)\in A}\ \sum_{k\in S_C}\tilde{x}_{jik} \oplus \tilde{d}_j && j\in N_D \qquad (P_{5.1}) \\
&\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}\tilde{x}_{ijk} = \sum_{j:(j,i)\in A}\ \sum_{k\in S_C}\tilde{x}_{jik} && i\in N_T \\
&\sum_{(i,j)\in A}\tilde{x}_{ijk} = \tilde{f}_k && k\in S_C \\
&\tilde{l}_{ijk} \preceq \tilde{x}_{ijk} \preceq \tilde{u}_{ijk} \qquad \forall\ (i,j)\in A,\ k\in S_C
\end{aligned}$$

$\tilde{x}_{ijk}$ is a non-negative LR flat fuzzy number $\forall\ (i,j)\in A,\ k\in S_C$

where, A is set of arcs joining different nodes and S_C is the set of all available conveyances.

Remark 5.1 If $\sum_{i\in N_{PS}}\tilde{a}_i \oplus \sum_{i\in N_S}\tilde{e}_i = \sum_{j\in N_{PD}}\tilde{b}_j \oplus \sum_{j\in N_D}\tilde{d}_j = \sum_{k\in S_C}\tilde{f}_k$ then a fully fuzzy multi-objective capacitated solid minimum cost flow problem is said to be a balanced fully fuzzy multi-objective capacitated solid minimum cost flow problem. Otherwise, it is said to be an unbalanced fully fuzzy multi-objective capacitated solid minimum cost flow problem.

5.2 Limitations of method proposed in the previous chapter

The method, proposed in the Chapter 4, can be used for solving fully fuzzy single and multi-objective capacitated transportation problems as well as fully fuzzy single and multi-objective capacitated minimum cost flow problems but can not be used for solving fully fuzzy single and multi-objective un-capacitated solid transportation problems, fully fuzzy single and multi-objective capacitated solid transportation problems, fully fuzzy single and multi-objective un-capacitated solid minimum cost flow problems and fully fuzzy single and multi-objective capacitated solid minimum cost flow problems e.g., fully fuzzy multi-objective un-capacitated solid transportation problem, fully fuzzy multi-objective capacitated solid transportation problem and fully fuzzy multi-objective capacitated solid minimum cost flow problem chosen in Example 5.1, Example 5.2 and Example 5.3 respectively can not be solved by using the method proposed in Chapter 4.

Example 5.1 A company has two sources S_1, S_2, three destinations D_1, D_2, D_3 and two conveyances F_1, F_2. The fuzzy supply of the product at sources S_1, S_2 is $(80, 100, 10, 20)_{LR}$ and $(70, 70, 10, 20)_{LR}$ respectively, the fuzzy demand of the product at destinations D_1, D_2, D_3 is $(30, 40, 20, 10)_{LR}$, $(50, 50, 10, 10)_{LR}$ and $(40, 60, 10, 10)_{LR}$ respectively and fuzzy capacity of the conveyances F_1, F_2 is $(80, 80, 10, 20)_{LR}$ and $(70, 70, 10, 20)_{LR}$ respectively. The fuzzy transportation cost $(\tilde{c}^1_{ijk})$ and fuzzy transportation time $(\tilde{c}^2_{ijk})$ required for transporting one unit quantity of product from i^{th} source to j^{th} destination by means of k^{th} conveyance are listed in Table 5.1. The company wants to determine the fuzzy quantity of the product to be transported from i^{th} source to j^{th} destination by means of k^{th} conveyance in order to minimize the total fuzzy transportation cost and minimize total fuzzy transportation time.

Table 5.1 Fuzzy transportation cost $(\tilde{c}^1_{ijk})$ and fuzzy transportation time $(\tilde{c}^2_{ijk})$

i	j	k	$\tilde{c}^1_{ijk}$	$\tilde{c}^2_{ijk}$
1	1	1	$(30, 30, 10, 10)_{LR}$	70
1	1	2	70	60
1	2	1	60	50
1	2	2	60	$(30, 30, 10, 10)_{LR}$
1	3	1	50	60
1	3	2	30	30
2	1	1	$(20, 20, 10, 10)_{LR}$	$(20, 30, 10, 10)_{LR}$
2	1	2	40	$(20, 20, 10, 10)_{LR}$
2	2	1	20	10
2	2	2	50	40
2	3	1	40	20
2	3	2	50	$(30, 40, 20, 20)_{LR}$

where, $L(x) = R(x) = \text{maximum } \{0, 1 - x^4\}$

Example 5.2 A company has two sources S_1, S_2, three destinations D_1, D_2, D_3 and two conveyances F_1, F_2. The fuzzy supply of the product at sources S_1, S_2, the fuzzy demand of the product at destinations D_1, D_2, D_3 and fuzzy capacity of the conveyances F_1, F_2 are same as in Example 5.1. The fuzzy transportation cost $(\tilde{c}^1_{ijk})$ and transportation time $(\tilde{c}^2_{ijk})$ required for transporting one unit quantity of product from i^{th} source to j^{th} destination by means of k^{th} conveyance, minimum fuzzy quantity of the product $(\tilde{l}_{ijk})$ and maximum fuzzy quantity of the product $(\tilde{u}_{ijk})$ from i^{th} source to j^{th} destination by means of k^{th} conveyance are listed in Table 5.2. The company wants to determine the fuzzy quantities of the product to be transported from i^{th} source to j^{th} destination by means of k^{th} conveyance in order to minimize the total fuzzy transportation cost and minimize total fuzzy transportation time.

Table 5.2 Fuzzy transportation cost $(\tilde{c}^1_{ijk})$, fuzzy transportation time $(\tilde{c}^2_{ijk})$, minimum fuzzy quantity of the product $(\tilde{l}_{ijk})$ and maximum fuzzy quantity of the product $(\tilde{l}_{ijk})$

i	j	k	$\tilde{c}^1_{ijk}$	$\tilde{c}^2_{ijk}$	$\tilde{l}_{ijk}$	$\tilde{u}_{ijk}$
1	1	1	$(30, 30, 10, 10)_{LR}$	70	$(0, 0, 0, 0)_{LR}$	$(10, 20, 10, 10)_{LR}$
1	1	2	70	60	$(0, 0, 0, 0)_{LR}$	$(30, 40, 10, 20)_{LR}$
1	2	1	60	50	$(0, 0, 0, 0)_{LR}$	$(30, 40, 10, 10)_{LR}$
1	2	2	60	$(30, 30, 10, 10)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(40, 50, 20, 10)_{LR}$
1	3	1	50	60	$(0, 0, 0, 0)_{LR}$	$(40, 60, 20, 10)_{LR}$
1	3	2	30	30	$(2, 3, 1, 2)_{LR}$	$(40, 60, 20, 20)_{LR}$
2	1	1	$(20, 20, 10, 10)_{LR}$	$(20, 30, 10, 10)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(10, 20, 10, 10)_{LR}$
2	1	2	40	$(20, 20, 10, 10)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(30, 40, 10, 20)_{LR}$
2	2	1	20	10	$(1, 3, 1, 1)_{LR}$	$(40, 50, 30, 20)_{LR}$
2	2	2	50	40	$(0, 0, 0, 0)_{LR}$	$(40, 50, 20, 10)_{LR}$
2	3	1	40	20	$(0, 0, 0, 0)_{LR}$	$(40, 60, 20, 10)_{LR}$
2	3	2	50	$(30, 40, 20, 20)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(40, 60, 20, 20)_{LR}$

where, $L(x) = R(x) = \text{maximum } \{0, 1 - x^4\}$

Example 5.3 Find the fuzzy optimal compromise solution of the fully fuzzy capacitated solid minimum cost flow problem depicted in Figure 5.1. The data is listed in Table 5.3 and Table 5.4.

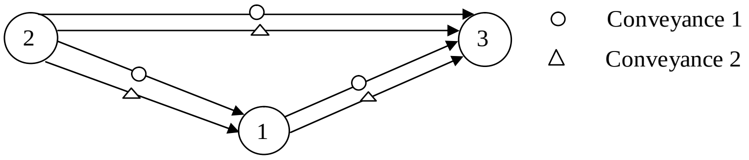

Figure 5.1 Network representing fully fuzzy capacitated solid minimum cost flow problem

Table 5.3 Fuzzy penalties $(\tilde{c}^1_{ijk})$, $(\tilde{c}^2_{ijk})$, $(\tilde{c}^3_{ijk})$, minimum fuzzy amount $(\tilde{l}_{ijk})$ and maximum fuzzy amount $(\tilde{u}_{ijk})$

i	j	k	$\tilde{c}^1_{ijk}$	$\tilde{c}^2_{ijk}$	$\tilde{c}^3_{ijk}$	$\tilde{l}_{ijk}$	$\tilde{u}_{ijk}$
1	3	1	$(8,10,2,2)_{LR}$	$(5,8,3,3)_{LR}$	$(6,9,2,3)_{LR}$	$(2,3,0,5)_{LR}$	$(10,20,10,0)_{LR}$
1	3	2	$(4,8,3,2)_{LR}$	$(8,10,2,2)_{LR}$	$(3,6,2,3)_{LR}$	$(7,10,5,3)_{LR}$	$(30,40,10,20)_{LR}$
2	1	1	$(8,10,4,4)_{LR}$	$(9,12,6,3)_{LR}$	$(5,8,3,3)_{LR}$	$(0,0,0,0)_{LR}$	$(30,40,10,10)_{LR}$
2	1	2	$(6,8,4,4)_{LR}$	$(8,10,4,4)_{LR}$	$(9,12,6,3)_{LR}$	$(0,0,0,0)_{LR}$	$(40,50,20,10)_{LR}$
2	3	1	$(9,12,6,3)_{LR}$	$(6,8,4,4)_{LR},$	$(2,4,2,2)_{LR}$	$(0,0,0,0)_{LR}$	$(40,60,20,10)_{LR}$
2	3	2	$(3,6,2,3)_{LR}$	$(6,9,2,3)_{LR}$	$(4,8,3,2)_{LR}$	$(2,3,1,2)_{LR}$	$(40,60,20,20)_{LR}$

Table 5.4 Fuzzy supply $(\tilde{a}_i/\tilde{e}_i)$, Fuzzy demand $(\tilde{b}_j/\tilde{d}_j)$ and fuzzy capacities $(\tilde{f}_k)$

Nodes	$\tilde{a}_i/\tilde{e}_i$	$\tilde{b}_j/\tilde{d}_j$	Conveyances	$\tilde{f}_k$
1	$(60,80,20,20)_{LR}$	–	1	$(60,80,20,10)_{LR}$
2	$(50,70,20,20)_{LR}$	–	2	$(50,70,20,40)_{LR}$
3	–	$(50,80,30,50)_{LR}$	–	–

where, $L(x) = \text{maximum } \{0, 1 - x^4\}$ $R(x) = \text{maximum } \{0, 1 - x\}$

5.3 Proposed method

In this section, to overcome the limitations of the method, proposed in Chapter 4, a new method is proposed for solving such fully fuzzy multi-objective capacitated solid minimum cost flow problems in which all the parameters as well as all the decision variables are represented by LR flat fuzzy numbers.

The steps of the proposed method are as follows:

Step 1 Find $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i$, $\sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j$ and $\sum\limits_{k \in S_C} \tilde{f}_k$. Let $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = (\underline{m}, \overline{m}, m^L, m^R)_{LR}$, $\sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j = (\underline{n}, \overline{n}, n^L, n^R)_{LR}$, $\sum\limits_{k \in S_C} \tilde{f}_k = (\underline{f}, \overline{f}, f^L, f^R)_{LR}$ and examine that the problem is balanced or unbalanced.

Case (1) If the problem is balanced, i.e., $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$, then Go to Step 4.

Case (2) If the problem is unbalanced i.e., $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j \neq \sum\limits_{k\in S_C} \tilde{f}_k$ or $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i \neq \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$ or $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{k\in S_C} \tilde{f}_k \neq \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j$ or $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i \neq \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j \neq \sum\limits_{k\in S_C} \tilde{f}_k$ then Go to Step 2.

Step 2 Check that $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j$ or $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i \neq \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j$.

Case (1) If $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j$ then Go to Step 3.

Case (2) If $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i \neq \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j$ then convert $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i \neq \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j$ into $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j$ as follows [65]:

Case (2a) If $\underline{m}-m^L \leq \underline{n}-n^L$, $m^L \leq n^L$, $\overline{m}-\underline{m} \leq \overline{n}-\underline{n}$ and $m^R \leq n^R$ then introduce a dummy purely source node with fuzzy supply $(\underline{n}-\underline{m}, \overline{n}-\overline{m}, n^L-m^L, n^R-m^R)_{LR}$ so that $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j$. Go to Step 3.

Case (2b) If $\underline{m}-m^L \geq \underline{n}-n^L$, $m^L \geq n^L$, $\overline{m}-\underline{m} \geq \overline{n}-\underline{n}$ and $m^R \geq n^R$ then introduce a dummy purely destination node with fuzzy demand $(\underline{m}-\underline{n}, \overline{m}-\overline{n}, m^L-n^L, m^R-n^R)_{LR}$ so that $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j$. Go to Step 3.

Case (2c) If neither Case (2a) nor Case (2b) is satisfied then introduce a dummy purely source node with fuzzy supply (maximum $\{0, (\underline{n}-n^L)-(\underline{m}-m^L)\}$ + maximum $\{0, (n^L-m^L)\}$, maximum $\{0, (\underline{n}-n^L)-(\underline{m}-m^L)\}$ + maximum $\{0, (n^L-m^L)\}$ + maximum $\{0, (\overline{n}-\underline{n})-(\overline{m}-\underline{m})\}$, maximum $\{0, (n^L-m^L)\}$, maximum $\{0, (n^R-m^R)\}$)$_{LR}$ and a dummy purely destination node with fuzzy demand (maximum $\{0, (\underline{m}-m^L)-(\underline{n}-n^L)\}$ + maximum $\{0, (m^L-n^L)\}$, maximum $\{0, (\underline{m}-m^L)-(\underline{n}-n^L)\}$ + maximum $\{0, (m^L-n^L)\}$ + maximum $\{0, (\overline{m}-\underline{m})-(\overline{n}-\underline{n})\}$, maximum $\{0, (m^L - n^L)\}$, maximum $\{0, (m^R - n^R)\})_{LR}$ so that $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j$. Go to Step 3.

Step 3 Using Step 2, $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j$. Let $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j = (\underline{g}, \overline{g}, g^L, g^R)_{LR}$ and $\sum\limits_{k \in S_C} \tilde{f}_k = (\underline{f}, \overline{f}, f^L, f^R)_{LR}$

Now check $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j = \sum\limits_{k \in S_C} \tilde{f}_k$ or $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j \neq \sum\limits_{k \in S_C} \tilde{f}_k$

Case (1) If $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j = \sum\limits_{k \in S_C} \tilde{f}_k$, then Go to Step 4.

Case (2) If $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j \neq \sum\limits_{k \in S_C} \tilde{f}_k$ then convert $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j \neq \sum\limits_{k \in S_C} \tilde{f}_k$ into $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j = \sum\limits_{k \in S_C} \tilde{f}_k$ as follows:

Case (2a) If $\underline{g} - g^L \leq \underline{f} - f^L$, $g^L \leq f^L$, $\overline{g} - \underline{g} \leq \overline{f} - \underline{f}$, and $g^R \leq f^R$ then check that in Step 2, a dummy purely source node is introduced or not and also check that a dummy purely destination node is introduced or not.

Case (i) If both the dummy purely source node and dummy purely destination node are introduced then increase both the fuzzy supply of the already introduced dummy purely source node and the fuzzy demand of the already introduced dummy purely destination node by the same fuzzy quantity $(\underline{f} - \underline{g}, \overline{f} - \overline{g}, f^L - g^L, f^R - g^R)_{LR}$ so that $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j = \sum\limits_{k \in S_C} \tilde{f}_k$. Go to Step 4.

Case (ii) If a dummy purely source node is introduced but no dummy purely destination node is introduced then increase the fuzzy supply of the already introduced dummy purely source node by the fuzzy quantity $(\underline{f} - \underline{g}, \overline{f} - \overline{g}, f^L - g^L, f^R - g^R)_{LR}$ and also introduce a dummy purely destination node with fuzzy demand $(\underline{f} - \underline{g}, \overline{f} - \overline{g}, f^L - g^L, f^R - g^R)_{LR}$ so that $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j = \sum\limits_{k \in S_C} \tilde{f}_k$. Go to Step 4.

Case (iii) If a dummy purely destination node is introduced but no dummy purely source node is introduced then increase the fuzzy demand of the already introduced dummy purely destination node by the fuzzy quantity $(\underline{f}-\underline{g}, \overline{f}-\overline{g}, f^L-g^L, f^R-g^R)_{LR}$ and also introduce a dummy purely source node with fuzzy supply $(\underline{f}-\underline{g}, \overline{f}-\overline{g}, f^L-g^L, f^R-g^R)_{LR}$ so that $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$. Go to Step 4.

Case (2b) If $\underline{g}-g^L \geq \underline{f}-f^L$, $g^L \geq f^L$, $\overline{g}-\underline{g} \geq \overline{f}-\underline{f}$, and $g^R \geq f^R$ then introduce a dummy conveyance with fuzzy capacity $(\underline{g}-\underline{f}, \overline{g}-\overline{f}, g^L-f^L, g^R-f^R)_{LR}$ so that $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$. Go to Step 4.

Case (2c) If neither Case (2a) nor Case (2b) is satisfied then check that in Step 2 a dummy purely source node is introduced or not and also check that a dummy purely destination node is introduced or not.

Case (i) If both the dummy purely source node and dummy purely destination node are introduced then increase both the fuzzy supply of the already introduced dummy purely source node and the fuzzy demand of the already introduced dummy purely destination node by the same fuzzy quantity $(\text{maximum}\{0, (\underline{f}-f^L)-(\underline{g}-g^L)\} + \text{maximum}\{0, (f^L-g^L)\}, \text{maximum}\{0, (\underline{f}-f^L)-(\underline{g}-g^L)\} + \text{maximum}\{0, (f^L-g^L)\} + \text{maximum}\{0, (\overline{f}-\underline{f})-(\overline{g}-\underline{g})\}, \text{maximum}\{0, (f^L-g^L)\}, \text{maximum}\{0, (f^R-g^R)\})_{LR}$ and also introduce a dummy purely conveyance with fuzzy capacity $(\text{maximum}\{0, (\underline{g}-g^L)-(\underline{f}-f^L)\} + \text{maximum}\{0, (g^L-f^L)\}, \text{maximum}\{0, (\underline{g}-g^L)-(\underline{f}-f^L)\} + \text{maximum}\{0, (g^L-f^L)\} + \text{maximum}\{0, (\overline{g}-\underline{g})-(\overline{f}-\underline{f})\}, \text{maximum}\{0, (g^L-f^L)\}, \text{maximum}\ \{0, (g^R-f^R)\})_{LR}$ so that $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$. Go to Step 4.

Case (ii) If a dummy purely source node is introduced but no dummy purely destination node is introduced then increase the fuzzy supply of the already introduced dummy purely source node by the fuzzy quantity (maximum$\{0,(\underline{f}-f^L)-(\underline{g}-g^L)\}+$ maximum$\{0,(f^L-g^L)\}$, maximum$\{0,(\underline{f}-f^L)-(\underline{g}-g^L)\}+$ maximum$\{0,(f^L-g^L)\}$ $+$ maximum$\{0,(\overline{f}-\underline{f})-(\overline{g}-\underline{g})\}$, maximum$\{0,(f^L-g^L)\}$, maximum$\{0,(f^R-g^R)\})_{LR}$ and also introduce a dummy purely destination node with fuzzy demand (maximum$\{0,(\underline{f}-f^L)-(\underline{g}-g^L)\}+$ maximum$\{0,(f^L-g^L)\}$, maximum$\{0,(\underline{f}-f^L)-(\underline{g}-g^L)\}+$ maximum$\{0,(f^L-g^L)\}+$ maximum$\{0,(\overline{f}-\underline{f})-(\overline{g}-\underline{g})\}$, maximum$\{0,(f^L-g^L)\}$, maximum $\{0,(f^R-g^R)\})_{LR}$. Also, introduce a dummy conveyance with fuzzy capacity (maximum$\{0,(\underline{g}-g^L)-(\underline{f}-f^L)\}+$ maximum$\{0,(g^L-f^L)\}$, maximum$\{0,(\underline{g}-g^L)-(\underline{f}-f^L)\}+$ maximum$\{0,(g^L-g^L)\}+$ maximum$\{0,(\overline{g}-\underline{g})-(\overline{f}-\underline{f})\}$, maximum$\{0,(g^L-f^L)\}$, maximum$\{0,(g^R-f^R)\})_{LR}$ so that $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i=\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{d}_j=\sum\limits_{k\in S_C}\tilde{f}_k$. Go to Step 4.

Case (iii) If a dummy purely destination node is introduced but no dummy purely source node is introduced then increase the fuzzy demand of the already introduced dummy purely destination node by the fuzzy quantity (maximum $\{0,(\underline{f}-f^L)-(\underline{g}-g^L)\}+$ maximum$\{0,(f^L-g^L)\}$, maximum$\{0,(\underline{f}-f^L)-(\underline{g}-g^L)\}$ $+$ maximum$\{0,(f^L-g^L)\}+$ maximum$\{0,(\overline{f}-\underline{f})-(\overline{g}-\underline{g})\}$, maximum$\{0,(f^L-g^L)\}$, maximum$\{0,(f^R-g^R)\})_{LR}$ and also introduce a dummy purely source node with fuzzy supply (maximum $\{0,(\underline{f}-f^L)-(\underline{g}-g^L)\}+$ maximum$\{0,(f^L-g^L)\}$, maximum$\{0,(\underline{f}-f^L)-(\underline{g}-g^L)\}+$ maximum$\{0,(f^L-g^L)\}+$ maximum$\{0,(\overline{f}-\underline{f})-(\overline{g}-\underline{g})\}$, maximum$\{0,(f^L-g^L)\}$, maximum$\{0,(f^R-g^R)\})_{LR}$. Also, introduce a dummy conveyance with fuzzy capacity (maximum$\{0,(\underline{g}-g^L)-(\underline{f}-f^L)\}+$ maximum$\{0,(g^L-f^L)\}$, maximum$\{0,(\underline{g}-g^L)-(\underline{f}-f^L)\}+$ maximum$\{0,(g^L-f^L)\}$

$+\,\text{maximum}\{0, (\overline{g}-\underline{g})-(\overline{f}-\underline{f})\}, \text{maximum}\{0, (g^L-f^L)\}, \text{maximum}\{0, (g^R-f^R)\})_{LR}$

so that $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$. Go to Step 4.

Step 4 The balanced fully fuzzy multi-objective solid minimum cost flow problem, obtained by using Step 1 to Step 3, can be formulated into the fuzzy linear programming problem $(P_{5.1})$ by assuming the following fuzzy penalty as zero LR flat fuzzy numbers:

(i) If any dummy purely source node is introduced then assume the fuzzy penalty for transporting one unit quantity of the product from the introduced dummy purely source node to all purely destination nodes and all intermediate nodes by all conveyance as zero LR flat fuzzy number.

(ii) If any dummy purely destination node is introduced then assume the fuzzy penalty for transporting one unit quantity of the product from all purely source nodes and intermediate nodes to the introduced dummy purely destination node by all conveyance as zero LR flat fuzzy number.

(iii) If any dummy conveyance is introduced then assume the fuzzy penalty for transporting one unit quantity of the product from all purely source nodes and intermediate nodes to all intermediate nodes and all purely destination nodes by introduced dummy conveyance as zero LR flat fuzzy number.

Step 5 Assuming $\tilde{c}^{\eta}_{ijk} = ((\underline{c})^{\eta}_{ijk}, (\overline{c})^{\eta}_{ijk}, (c^L)^{\eta}_{ijk}, (c^R)^{\eta}_{ijk})_{LR}$, $\tilde{x}_{ijk} = (\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}$, $\tilde{a}_i = (\underline{a}_i, \overline{a}_i, a^L_i, a^R_i)_{LR}$, $\tilde{e}_i = (\underline{e}_i, \overline{e}_i, e^L_i, e^R_i)_{LR}$, $\tilde{b}_j = (\underline{b}_j, \overline{b}_j, b^L_j, b^R_j)_{LR}$, $\tilde{d}_j = (\underline{d}_j, \overline{d}_j, d^L_j, d^R_j)_{LR}$, $\tilde{f}_k = (\underline{f}_k, \overline{f}_k, f^L_k, f^R_k)_{LR}$, $\tilde{l}_{ijk} = (\underline{l}_{ijk}, \overline{l}_{ijk}, l^L_{ijk}, l^R_{ijk})_{LR}$ and $\tilde{u}_{ijk} = (\underline{u}_{ijk}, \overline{u}_{ijk}, u^L_{ijk}, u^R_{ijk})_{LR}$ the fuzzy linear programming problem $(P_{5.1})$ can be written as:

Minimize $\sum_{(i,j)\in A}\sum_{k\in S_C}((\underline{c})^{\eta}_{ijk},(\overline{c})^{\eta}_{ijk},(c^L)^{\eta}_{ijk},(c^R)^{\eta}_{ijk})_{LR}\otimes(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR})$

$;\eta=1,2,...,P$

subject to

$\sum_{j:(i,j)\in A}\sum_{k\in S_C}(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR}=(\underline{a}_i,\overline{a}_i,a^L_i,a^R_i)_{LR}$ $\quad i\in N_{PS}$

$\sum_{j:(i,j)\in A}\sum_{k\in S_C}(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR}=\sum_{j:(j,i)\in A}\sum_{k\in S_C}(\underline{x}_{jik},\overline{x}_{jik},x^L_{jik},x^R_{jik})_{LR}\oplus(\underline{e}_i,\overline{e}_i,$
$e^L_i,e^R_i)_{LR}$ $\quad i\in N_S$

$\sum_{i:(i,j)\in A}\sum_{k\in S_C}(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR}=(\underline{b}_j,\overline{b}_j,b^L_j,b^R_j)_{LR}$ $\quad j\in N_{PD}$

$\sum_{i:(i,j)\in A}\sum_{k\in S_C}(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR}=\sum_{i:(j,i)\in A}\sum_{k\in S_C}(\underline{x}_{jik},\overline{x}_{jik},x^L_{jik},x^R_{jik})_{LR}\oplus(\underline{d}_j,\overline{d}_j,$
$d^L_j,d^R_j)_{LR}$ $\quad j\in N_D \qquad (P_{5.2})$

$\sum_{j:(i,j)\in A}\sum_{k\in S_C}(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR}=\sum_{j:(j,i)\in A}\sum_{k\in S_C}(\underline{x}_{jik},\overline{x}_{jik},x^L_{jik},x^R_{jik})_{LR}$ $\quad i\in N_T$

$\sum_{(i,j)\in A}(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR}=(\underline{f}_k,\overline{f}_k,f^L_k,f^R_k)_{LR}$ $\quad k\in S_C$

$(\underline{l}_{ijk},\overline{l}_{ijk},l^L_{ijk},l^R_{ijk})_{LR}\preceq(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR}\preceq(\underline{u}_{ijk},\overline{u}_{ijk},u^L_{ijk},u^R_{ijk})_{LR}$

$\forall\ (i,j)\in A,\ k\in S_C$

$(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR}$ is non-negative LR flat fuzzy number $\forall\ (i,j)\in A,\ k\in S_C$

Step 6 Using the arithmetic operations of LR flat fuzzy numbers, defined in Section 2.1.2, the fully fuzzy multi-objective linear programming problem $(P_{5.2})$ can be written as:

Minimize $\sum_{(i,j)\in A}\sum_{k\in S_C}((\underline{c})^{\eta}_{ijk}\underline{x}_{ijk},(\overline{c})^{\eta}_{ijk}\overline{x}_{ijk},(\underline{c})^{\eta}_{ijk}x^L_{ijk}+(c^L)^{\eta}_{ijk}\underline{x}_{ijk}-(c^L)^{\eta}_{ijk}x^L_{ijk},(\overline{c})^{\eta}_{ijk}$
$x^R_{ijk}+(c^R)^{\eta}_{ijk}\overline{x}_{ijk}+(c^R)^{\eta}_{ijk}x^R_{ijk})_{LR}$ $\quad ;\eta=1,2,...,P$

subject to

$(\sum_{j:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk},\sum_{j:(i,j)\in A}\sum_{k\in S_C}\overline{x}_{ijk},\sum_{j:(i,j)\in A}\sum_{k\in S_C}x^L_{ijk},\sum_{j:(i,j)\in A}\sum_{k\in S_C}x^R_{ijk})_{LR}=(\underline{a}_i,\overline{a}_i,a^L_i,$
$a^R_i)_{LR}$ $\quad i\in N_{PS}$

$(\sum_{j:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk},\sum_{j:(i,j)\in A}\sum_{k\in S_C}\overline{x}_{ijk},\sum_{j:(i,j)\in A}\sum_{k\in S_C}x^L_{ijk},\sum_{j:(i,j)\in A}\sum_{k\in S_C}x^R_{ijk})_{LR}=(\sum_{j:(j,i)\in A}$
$\sum_{k\in S_C}\underline{x}_{jik}+\underline{e}_i,\sum_{j:(j,i)\in A}\sum_{k\in S_C}\overline{x}_{jik}+\overline{e}_i,\sum_{j:(j,i)\in A}\sum_{k\in S_C}x^L_{jik}+e^L_i,\sum_{j:(j,i)\in A}\sum_{k\in S_C}x^R_{jik}+e^R_i)_{LR}$

$$i \in N_S$$

$$\Big(\sum_{i:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk},\ \sum_{i:(i,j)\in A}\sum_{k\in S_C}\overline{x}_{ijk},\ \sum_{i:(i,j)\in A}\sum_{k\in S_C}x^L_{ijk},\ \sum_{i:(i,j)\in A}\sum_{k\in S_C}x^R_{ijk}\Big)_{LR} = (\underline{b}_j, \overline{b}_j, b^L_j, b^R_j)_{LR} \qquad j \in N_{PD} \quad (P_{5.3})$$

$$\Big(\sum_{i:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk},\ \sum_{i:(i,j)\in A}\sum_{k\in S_C}\overline{x}_{ijk},\ \sum_{i:(i,j)\in A}\sum_{k\in S_C}x^L_{ijk},\ \sum_{i:(i,j)\in A}\sum_{k\in S_C}x^R_{ijk}\Big)_{LR} = \Big(\sum_{i:(j,i)\in A}\sum_{k\in S_C}\underline{x}_{jik} + \underline{d}_j,\ \sum_{i:(j,i)\in A}\sum_{k\in S_C}\overline{x}_{jik} + \overline{d}_j,\ \sum_{i:(j,i)\in A}\sum_{k\in S_C}x^L_{jik} + d^L_j,\ \sum_{i:(j,i)\in A}\sum_{k\in S_C}x^R_{jik} + d^R_j\Big)_{LR} \qquad j \in N_D$$

$$\Big(\sum_{j:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk},\ \sum_{j:(i,j)\in A}\sum_{k\in S_C}\overline{x}_{ijk},\ \sum_{j:(i,j)\in A}\sum_{k\in S_C}x^L_{ijk},\ \sum_{j:(i,j)\in A}\sum_{k\in S_C}x^R_{ijk}\Big)_{LR} = \Big(\sum_{j:(j,i)\in A}\sum_{k\in S_C}\underline{x}_{jik},\ \sum_{j:(j,i)\in A}\sum_{k\in S_C}\overline{x}_{jik},\ \sum_{j:(j,i)\in A}\sum_{k\in S_C}x^L_{jik},\ \sum_{j:(j,i)\in A}\sum_{k\in S_C}x^R_{jik}\Big)_{LR} \qquad i \in N_T$$

$$\Big(\sum_{(i,j)\in A}\underline{x}_{ijk},\ \sum_{(i,j)\in A}\overline{x}_{ijk},\ \sum_{(i,j)\in A}x^L_{ijk},\ \sum_{(i,j)\in A}x^R_{ijk}\Big)_{LR} = (\underline{f}_k, \overline{f}_k, f^L_k, f^R_k)_{LR} \qquad k \in S_C$$

$$(\underline{l}_{ijk}, \overline{l}_{ijk}, l^L_{ijk}, l^R_{ijk})_{LR} \preceq (\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR} \preceq (\underline{u}_{ijk}, \overline{u}_{ijk}, u^L_{ijk}, u^R_{ijk})_{LR}$$

$$\forall\ (i,j) \in A,\ k \in S_C$$

$(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L{}_{ijk}, x^R{}_{ijk})_{LR}$ is a non-negative LR flat fuzzy number $\forall\ (i,j) \in A,\ k \in S_C$

Step 7 Using the Definition 2.10, Definition 2.11, the fully fuzzy multi-objective linear programming problem $(P_{5.3})$, can be converted into the fully fuzzy multi-objective linear programming problem $(P_{5.4})$:

Minimize $$\sum_{(i,j)\in A}\sum_{k\in S_C}\Big((\underline{c})^{\eta}_{ijk}\underline{x}_{ijk},\ (\overline{c})^{\eta}_{ijk}\overline{x}_{ijk},\ (\underline{c})^{\eta}_{ijk}x^L_{ijk} + (c^L)^{\eta}_{ijk}\underline{x}_{ijk} - (c^L)^{\eta}_{ijk}x^L_{ijk},\ (\overline{c})^{\eta}_{ijk}x^R_{ijk} + (c^R)^{\eta}_{ijk}\overline{x}_{ijk} + (c^R)^{\eta}_{ijk}x^R_{ijk}\Big)_{LR} \quad ;\eta = 1,2,...,P$$

subject to

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk} = \underline{a}_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C}\overline{x}_{ijk} = \overline{a}_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C}x^L_{ijk} = a^L_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C}x^R_{ijk} = a^R_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk} = \sum_{j:(j,i)\in A}\sum_{k\in S_C}\underline{x}_{jik} + \underline{e}_i \qquad i \in N_S$$

$$
\begin{array}{lll}
\sum\limits_{j:(i,j)\in A}\sum\limits_{k\in S_C}\overline{x}_{ijk}=\sum\limits_{j:(j,i)\in A}\sum\limits_{k\in S_C}\overline{x}_{jik}+\overline{e}_i & i\in N_S & \\
\sum\limits_{j:(i,j)\in A}\sum\limits_{k\in S_C}x^L_{ijk}=\sum\limits_{j:(j,i)\in A}\sum\limits_{k\in S_C}x^L_{jik}+e^L_i & i\in N_S & \\
\sum\limits_{j:(i,j)\in A}\sum\limits_{k\in S_C}x^R_{ijk}=\sum\limits_{j:(j,i)\in A}\sum\limits_{k\in S_C}x^R_{jik}+e^R_i & i\in N_S & \\
\sum\limits_{i:(i,j)\in A}\sum\limits_{k\in S_C}\underline{x}_{ijk}=\underline{b}_j & j\in N_{PD} & \\
\sum\limits_{i:(i,j)\in A}\sum\limits_{k\in S_C}\overline{x}_{ijk}=\overline{b}_j & j\in N_{PD} & \\
\sum\limits_{i:(i,j)\in A}\sum\limits_{k\in S_C}x^L_{ijk}=b^L_j & j\in N_{PD} & \\
\sum\limits_{i:(i,j)\in A}\sum\limits_{k\in S_C}x^R_{ijk}=b^R_j & j\in N_{PD} & (P_{5.4})\\
\sum\limits_{i:(i,j)\in A}\sum\limits_{k\in S_C}\underline{x}_{ijk}=\sum\limits_{i:(j,i)\in A}\sum\limits_{k\in S_C}\underline{x}_{jik}+\underline{d}_j & j\in N_D & \\
\sum\limits_{i:(i,j)\in A}\sum\limits_{k\in S_C}\overline{x}_{ijk}=\sum\limits_{i:(j,i)\in A}\sum\limits_{k\in S_C}\overline{x}_{jik}+\overline{d}_j & j\in N_D & \\
\sum\limits_{i:(i,j)\in A}\sum\limits_{k\in S_C}x^L_{ijk}=\sum\limits_{i:(j,i)\in A}\sum\limits_{k\in S_C}x^L_{jik}+d^L_j & j\in N_D & \\
\sum\limits_{i:(i,j)\in A}\sum\limits_{k\in S_C}x^R_{ijk}=\sum\limits_{i:(j,i)\in A}\sum\limits_{k\in S_C}x^R_{jik}+d^R_j & j\in N_D & \\
\sum\limits_{j:(i,j)\in A}\sum\limits_{k\in S_C}\underline{x}_{ijk}=\sum\limits_{j:(j,i)\in A}\sum\limits_{k\in S_C}\underline{x}_{jik} & i\in N_T & \\
\sum\limits_{j:(i,j)\in A}\sum\limits_{k\in S_C}\overline{x}_{ijk}=\sum\limits_{j:(j,i)\in A}\sum\limits_{k\in S_C}\overline{x}_{jik} & i\in N_T & \\
\sum\limits_{j:(i,j)\in A}\sum\limits_{k\in S_C}x^L_{ijk}=\sum\limits_{j:(j,i)\in A}\sum\limits_{k\in S_C}x^L_{jik} & i\in N_T & \\
\sum\limits_{j:(i,j)\in A}\sum\limits_{k\in S_C}x^R_{ijk}=\sum\limits_{j:(j,i)\in A}\sum\limits_{k\in S_C}x^R_{jik} & i\in N_T & \\
\sum\limits_{(i,j)\in A}\underline{x}_{ijk}=\underline{f}_k & k\in S_C & \\
\sum\limits_{(i,j)\in A}\overline{x}_{ijk}=\overline{f}_k & k\in S_C & \\
\sum\limits_{(i,j)\in A}x^L_{ijk}=x^L_k & k\in S_C & \\
\sum\limits_{(i,j)\in A}x^R_{ijk}=x^R_k & k\in S_C & \\
\multicolumn{2}{c}{\underline{x}_{ijk}-x^L_{ijk},\ \overline{x}_{ijk}-\underline{x}_{ijk},\ x^L_{ijk},x^R_{ijk}\geq 0\ \ \forall\ (i,j)\in A,\ k\in S_C} &
\end{array}
$$

$$
\left.\begin{array}{c}
(\underline{l}_{ijk},\overline{l}_{ijk},l^L_{ijk},l^R_{ijk})_{LR}\preceq(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR}\preceq(\underline{u}_{ijk},\overline{u}_{ijk},u^L_{ijk},u^R_{ijk})_{LR}\\
\forall\ (i,j)\in A,\ k\in S_C
\end{array}\right\}(C_{5.1})
$$

Step 8 As discussed in Step 6 of the method, proposed in Section 3.7, the fuzzy optimal solution of the fuzzy multi-objective linear programming problem $(P_{5.4})$ can be obtained by solving the following crisp multi-objective linear programming problem:

$$\begin{gathered}\text{Minimize} \sum_{(i,j)\in A}\sum_{k\in S_C}\Re((\underline{c})^{\eta}_{ijk}\underline{x}_{ijk}, (\overline{c})^{\eta}_{ijk}\overline{x}_{ijk}, (\underline{c})^{\eta}_{ijk}x^{L}_{ijk} + (c^{L})^{\eta}_{ijk}\underline{x}_{ijk} - (c^{L})^{\eta}_{ijk}x^{L}_{ijk}, (\overline{c})^{\eta}_{ijk} \\ x^{R}_{ijk} + (c^{R})^{\eta}_{ijk}\overline{x}_{ijk} + (c^{R})^{\eta}_{ijk}x^{R}_{ijk})_{LR} \qquad ;\eta = 1, 2, ..., P\end{gathered}$$

subject to $(P_{5.5})$

$$\Re(\underline{l}_{ijk}, \overline{l}_{ijk}, l^{L}_{ijk}, l^{R}_{ijk})_{LR} \leq \Re(\underline{x}_{ijk}, \overline{x}_{ijk}, x^{L}_{ijk}, x^{R}_{ijk})_{LR} \leq \Re(\underline{u}_{ijk}, \overline{u}_{ijk}, u^{L}_{ijk}, u^{R}_{ijk})_{LR}$$

$$\forall\ (i, j) \in A,\ k \in S_C$$

as well as all the constraints of problem $(P_{5.4})$ except $(C_{5.1})$

Step 9 Using Section 2.4, the crisp linear programming problem $(P_{5.5})$ can be written as:

$$\begin{gathered}\text{Minimize} \sum_{(i,j)\in A}\sum_{k\in S_C}[\gamma(\textstyle\int_0^1(((\underline{c}_{ijk})^{\eta}\underline{x}_{ijk})\lambda+((\overline{c}_{ijk})^{\eta}\overline{x}_{ijk})(1-\lambda))d\rho)+(1-\gamma)(\lambda\int_0^1(((\underline{c}_{ijk})^{\eta} \\ \underline{x}_{ijk}) - ((\underline{c}_{ijk})^{\eta}x^{L}_{ijk} + (c^{L})^{\eta}_{ijk}\underline{x}_{ijk} - (c^{L})^{\eta}_{ijk}x^{L}_{ijk})L^{-1}(\rho))d\rho + (1-\lambda)\textstyle\int_0^1((\overline{c}_{ijk})^{\eta}\overline{x}_{ijk} + \\ ((\overline{c}_{ijk})^{\eta}x^{R}_{ijk} + (c^{R})^{\eta}_{ijk}\overline{x}_{ijk} + (c^{R})^{\eta}_{ijk}x^{R}_{ijk})R^{-1}(\rho))d\rho)] \qquad ;\eta = 1, 2, ..., P\end{gathered}$$

subject to $(P_{5.6})$

$$\begin{gathered}\Big(\gamma(\textstyle\int_0^1(\underline{l}_{ijk}\lambda + \overline{l}_{ijk}(1-\lambda))d\rho) + (1-\gamma)(\lambda\int_0^1(\underline{l}_{ijk} - l^{L}_{ijk}L^{-1}(\rho))d\rho + (1-\lambda)\int_0^1(\overline{l}_{ijk} + \\ l^{R}_{ijk}R^{-1}(\rho))d\rho)\Big) \leq \Big(\gamma(\textstyle\int_0^1(\underline{x}_{ijk}\lambda+\overline{x}_{ijk}(1-\lambda))d\rho)+(1-\gamma)(\lambda\int_0^1(\underline{x}_{ijk}-x^{L}_{ijk}L^{-1}(\rho))d\rho+ \\ (1-\lambda)\textstyle\int_0^1(\overline{x}_{ijk}+x^{R}_{ijk}R^{-1}(\rho))d\rho)\Big) \leq \Big(\gamma(\int_0^1(\underline{u}_{ijk}\lambda+\overline{u}_{ijk}(1-\lambda))d\rho)+(1-\gamma)(\lambda\int_0^1(\underline{u}_{ijk}- \\ u^{L}_{ijk}L^{-1}(\rho))d\rho + (1-\lambda)\textstyle\int_0^1(\overline{u}_{ijk} + u^{R}_{ijk}R^{-1}(\rho))d\rho)\Big)\end{gathered}$$

as well as all the constraints of problem $(P_{5.4})$ except $(C_{5.1})$

Step 10 Solve the crisp multi-objective linear programming problem $(P_{5.6})$, obtained in Step 9, by using any of the existing approach to find the optimal compromise solution $\{\underline{x}_{ijk}, \overline{x}_{ijk}, x^{L}_{ijk}, x^{R}_{ijk}\}$ of the crisp multi-objective linear programming problem.

Step 11 Put the values of $\underline{x}_{ijk}, \overline{x}_{ijk}, x^{L}_{ijk}, x^{R}_{ijk}$ in $\tilde{x}_{ijk} = (\underline{x}_{ijk}, \overline{x}_{ijk}, x^{L}_{ijk}, x^{R}_{ijk})_{LR}$ to find the fuzzy optimal compromise solution $\{\tilde{x}_{ijk}\}$.

Step 12 Find the fuzzy optimal value of each objective function by putting the values of $\tilde{x}_{ijk}$ in $\sum\limits_{(i,j)\in A}\sum\limits_{k\in S_C}(\tilde{c}^{\eta}_{ijk}\otimes\tilde{x}_{ijk})\ \ ;\eta=1,2,\ldots,P$.

5.4 Advantages of proposed method over the method proposed in previous chapter

The main advantage of the proposed method over the method proposed in Chapter 4 is that all the problems which can be solved by using the method proposed in Chapter 4 can also be solved by using the method proposed in this chapter. However, as discussed in Section 5.2, there exist several problems which can be solved by using the proposed method but can not be solved by using the method proposed in Chapter 4. To illustrate the proposed method and also to show its advantages the fully fuzzy multi-objective capacitated solid minimum cost flow problem, chosen in Example 5.3 which can not be solved by using the method proposed in Chapter 4, is solved by using the method proposed in this chapter.

5.4.1 Fuzzy optimal compromise solution of the chosen problem

The fuzzy optimal compromise solution of fully fuzzy multi-objective capacitated solid minimum cost flow problem, chosen in Example 5.3, can be obtained as follows:

Step 1 Total fuzzy supply $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i=(110,150,40,40)_{LR}$, total fuzzy demand $\sum\limits_{j\in N_D}\tilde{b}_j\oplus\sum\limits_{j\in N_{PD}}\tilde{d}_j=(50,80,30,50)_{LR}$ and total fuzzy capacity $\sum\limits_{k\in S_C}\tilde{f}_k=(110,150,40,50)_{LR}$. Since $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i\neq\sum\limits_{j\in N_D}\tilde{b}_j\oplus\sum\limits_{j\in N_{PD}}\tilde{d}_j\neq\sum\limits_{k\in S_C}\tilde{f}_k$, so it is an unbalanced fully fuzzy multi-objective capacitated solid minimum cost flow problem.

Step 2 Comparing $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i = (110,150,40,40)_{LR}$ by $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i = (\underline{m},\overline{m},m^L,m^R)_{LR}$ and $\sum\limits_{j\in N_D}\tilde{b}_j \oplus \sum\limits_{j\in N_{PD}}\tilde{d}_j = (50,80,30,50)_{LR}$ by $\sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j = (\underline{n},\overline{n},n^L,n^R)_{LR}$ the values of $\underline{m},\overline{m},m^L,m^R,\underline{n},\overline{n},n^L$ and n^R are 110, 150, 40, 40, 50, 80, 30 and 50 respectively.

Since, $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i \neq \sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j$ and neither the condition $\underline{m}-m^L \leq \underline{n}-n^L$, $m^L \leq n^L$, $\overline{m}-\underline{m} \leq \overline{n}-\underline{n}$, $m^R \leq n^R$ nor the condition $\underline{m}-m^L \geq \underline{n}-n^L$, $m^L \geq n^L$, $\overline{m}-\underline{m} \geq \overline{n}-\underline{n}$, $m^R \geq n^R$ is satisfying so, as described in Case (2c) of Step 2 of the proposed method, there is need to introduce a dummy purely source node (4) with fuzzy supply $\tilde{a}_4 = (0,0,0,10)_{LR}$ and a dummy purely destination node (5) with fuzzy demand $\tilde{b}_5 = (60,70,10,0)_{LR}$ so that $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i = \sum\limits_{j\in N_D}\tilde{b}_j \oplus \sum\limits_{j\in N_{PD}}\tilde{d}_j$.

Step 3 Since, $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i = \sum\limits_{j\in N_D}\tilde{b}_j \oplus \sum\limits_{j\in N_{PD}}\tilde{d}_j = (110,150,40,50)_{LR} = \sum\limits_{k\in S_C}\tilde{f}_k$, so the fully fuzzy multi-objective capacitated solid minimum cost flow problem, obtained in Step 2, is a balanced fully fuzzy multi-objective capacitated solid minimum cost flow problem.

Step 4 Since, a dummy purely source node (4) and a dummy purely destination node (5) are introduced. So, as described in Step 4 of the proposed method, by assuming $\tilde{c}^1_{4jk} = \tilde{c}^1_{i5k} = \tilde{c}^2_{4jk} = \tilde{c}^2_{i5k} = \tilde{c}^3_{4jk} = \tilde{c}^3_{i5k} = (0,0,0,0)_{LR}\ \forall\ i = 1,2,4;\quad j = 1,3,5;\quad k = 1,2$, the fuzzy linear programming formulation of the balanced fully fuzzy multi-objective capacitated solid minimum cost flow problem, obtained from Step 3, can be written as:

Minimize $((8,10,2,2)_{LR}\otimes\tilde{x}_{131}\oplus(4,8,3,2)_{LR}\otimes\tilde{x}_{132}\oplus(8,10,4,4)_{LR}\otimes\tilde{x}_{211}\oplus(6,8,4,4)_{LR}\otimes\tilde{x}_{212}\oplus(9,12,6,3)_{LR}\otimes\tilde{x}_{231}\oplus(3,6,2,3)_{LR}\otimes\tilde{x}_{232}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{411}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{412}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{431}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{432}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{251}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{252}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{451}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{452}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{151}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{152})$

$$\text{Minimize}(5,8,3,3)_{LR}\otimes\tilde{x}_{131}\oplus(8,10,2,2)_{LR}\otimes\tilde{x}_{132}\oplus(9,12,6,3)_{LR}\otimes\tilde{x}_{211}\oplus(8,10,4,4)_{LR}\otimes\tilde{x}_{212}\oplus(6,8,4,4)_{LR}\otimes\tilde{x}_{231}\oplus(6,9,2,3)_{LR}\otimes\tilde{x}_{232}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{411}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{412}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{431}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{432}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{251}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{252}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{451}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{452}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{151}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{152}))$$

$$\text{Minimize }(6,9,2,3)_{LR}\otimes\tilde{x}_{131}\oplus(3,6,2,3)_{LR}\otimes\tilde{x}_{132}\oplus(5,8,3,3)_{LR}\otimes\tilde{x}_{211}\oplus(9,12,6,3)_{LR}\otimes\tilde{x}_{212}\oplus(2,4,2,2)_{LR}\otimes\tilde{x}_{231}\oplus(4,8,3,2)_{LR}\otimes\tilde{x}_{232}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{411}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{412}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{431}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{432}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{251}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{252}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{451}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{452}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{151}\oplus(0,0,0,0)_{LR}\otimes\tilde{x}_{152})$$

subject to

$$\sum_{k=1}^{2}(\tilde{x}_{13k}\oplus\tilde{x}_{15k})=\sum_{k=1}^{2}(\tilde{x}_{41k}\oplus\tilde{x}_{21k})\oplus(60,80,20,20)_{LR}$$
$$\sum_{k=1}^{2}(\tilde{x}_{21k}\oplus\tilde{x}_{23k}\oplus\tilde{x}_{25k})=(50,70,20,20)_{LR}$$
$$\sum_{k=1}^{2}(\tilde{x}_{13k}\oplus\tilde{x}_{23k}\oplus\tilde{x}_{43k})=(50,80,30,50)_{LR}$$
$$\sum_{k=1}^{2}(\tilde{x}_{41k}\oplus\tilde{x}_{43k}\oplus\tilde{x}_{45k})=(0,0,0,10)_{LR}$$
$$\sum_{k=1}^{2}(\tilde{x}_{15k}\oplus\tilde{x}_{25k}\oplus\tilde{x}_{45k})=(60,70,10,0)_{LR}$$
$$\tilde{x}_{131}\oplus\tilde{x}_{151}\oplus\tilde{x}_{211}\oplus\tilde{x}_{231}\oplus\tilde{x}_{251}\oplus\tilde{x}_{411}\oplus\tilde{x}_{431}\oplus\tilde{x}_{451}=(60,80,20,10)_{LR}$$
$$\tilde{x}_{132}\oplus\tilde{x}_{152}\oplus\tilde{x}_{212}\oplus\tilde{x}_{232}\oplus\tilde{x}_{252}\oplus\tilde{x}_{412}\oplus\tilde{x}_{432}\oplus\tilde{x}_{452}=(50,70,20,40)_{LR}$$

$\tilde{x}_{131}\succeq(2,3,0,5)_{LR}$, $\tilde{x}_{131}\preceq(10,20,10,0)_{LR}$, $\tilde{x}_{132}\succeq(7,10,5,3)_{LR}$

$\tilde{x}_{132}\preceq(30,40,10,20)_{LR}$, $\tilde{x}_{211}\succeq(0,0,0,0)_{LR}$, $\tilde{x}_{211}\preceq(30,40,10,10)_{LR}$

$\tilde{x}_{212}\succeq(0,0,0,0)_{LR}$, $\tilde{x}_{212}\preceq(40,50,20,10)_{LR}$, $\tilde{x}_{231}\succeq(0,0,0,0)_{LR}$

$\tilde{x}_{231}\preceq(40,60,20,10)_{LR}$, $\tilde{x}_{232}\succeq(2,3,1,2)_{LR}$, $\tilde{x}_{232}\succeq(40,60,20,20)_{LR}$

$\tilde{x}_{ijk}$ are non-negative LR flat fuzzy numbers $\forall\ i=1,2,4;\quad j=1,3,5;\quad k=1,2.$

Step 5 Using Step 6 to Step 9, of the proposed method, the fuzzy multi-objective linear programming problem, obtained in Step 4, can be converted into the following crisp multi-objective linear programming problem:

Minimize $\frac{1}{30}(104\underline{x}_{131} - 48x^L_{131} + 180\overline{x}_{131} + 180x^R_{131} + 36\underline{x}_{132} - 8x^L_{132} + 150\overline{x}_{132} + 150x^R_{132} +$
$88\underline{x}_{211} - 32x^L_{211} + 210\overline{x}_{211} + 210x^R_{211} + 58\underline{x}_{212} - 16x^L_{212} + 180\overline{x}_{212} + 180x^R_{212} + 87\underline{x}_{231} -$
$24x^L_{231} + 225\overline{x}_{231} + 225x^R_{231} + 29\underline{x}_{232} - 8x^L_{232} + 135\overline{x}_{232} + 135x^R_{232})$

Minimize $\frac{1}{30}(51\underline{x}_{131} - 16x^L_{131} + 165\overline{x}_{131} + 165x^R_{131} + 104\underline{x}_{132} - 48x^L_{132} + 180\overline{x}_{132} +$
$180x^R_{132} + 87\underline{x}_{211} - 24x^L_{211} + 225\overline{x}_{211} + 225x^R_{211} + 88\underline{x}_{212} - 32x^L_{212} + 210\overline{x}_{212} + 210x^R_{212} +$
$58\underline{x}_{231} - 16x^L_{231} + 180\overline{x}_{231} + 180x^R_{231} + 74\underline{x}_{232} - 32x^L_{232} + 180\overline{x}_{232} + 180x^R_{232})$

Minimize $\frac{1}{30}(74\underline{x}_{131} - 32x^L_{131} + 180\overline{x}_{131} + 180x^R_{131} + 29\underline{x}_{132} - 8x^L_{132} + 135\overline{x}_{132} + 135x^R_{132} +$
$51\underline{x}_{211} - 16x^L_{211} + 165\overline{x}_{211} + 165x^R_{211} + 87\underline{x}_{212} - 24x^L_{212} + 225\overline{x}_{212} + 225x^R_{212} + 14\underline{x}_{231} -$
$0x^L_{231} + 90\overline{x}_{231} + 90x^R_{231} + 36\underline{x}_{232} - 8x^L_{232} + 150\overline{x}_{232} + 150x^R_{232})$

subject to

$$\sum_{k=1}^{2}(\underline{x}_{13k} + \underline{x}_{15k}) = \sum_{k=1}^{2}(\underline{x}_{41k} + \underline{x}_{21k}) + 60$$
$$\sum_{k=1}^{2}(\overline{x}_{13k} + \overline{x}_{15k}) = \sum_{k=1}^{2}(\overline{x}_{41k} + \overline{x}_{21k}) + 80$$
$$\sum_{k=1}^{2}(x^L_{13k} + x^L_{15k}) = \sum_{k=1}^{2}(x^L_{41k} + x^L_{21k}) + 20$$
$$\sum_{k=1}^{2}(x^R_{13k} + x^R_{15k}) = \sum_{k=1}^{2}(x^R_{41k} + x^R_{21k}) + 20$$
$$\sum_{k=1}^{2}(\underline{x}_{21k} + \underline{x}_{23k} + \underline{x}_{25k}) = 50, \qquad \sum_{k=1}^{2}(\overline{x}_{21k} + \overline{x}_{23k} + \overline{x}_{25k}) = 70$$
$$\sum_{k=1}^{2}(x^L_{21k} + x^L_{23k} + x^L_{25k}) = 20, \qquad \sum_{k=1}^{2}(x^R_{21k} + x^R_{23k} + x^R_{25k}) = 20$$
$$\sum_{k=1}^{2}(\underline{x}_{13k} + \underline{x}_{23k} + \underline{x}_{43k}) = 50, \qquad \sum_{k=1}^{2}(\overline{x}_{13k} + \overline{x}_{23k} + \overline{x}_{43k}) = 80$$
$$\sum_{k=1}^{2}(x^L_{13k} + x^L_{23k} + x^L_{43k}) = 30, \qquad \sum_{k=1}^{2}(x^R_{13k} + x^R_{23k} + x^R_{43k}) = 50$$
$$\sum_{k=1}^{2}(\underline{x}_{41k} + \underline{x}_{43k} + \underline{x}_{45k}) = 0, \qquad \sum_{k=1}^{2}(\overline{x}_{41k} + \overline{x}_{43k} + \overline{x}_{45k}) = 0$$
$$\sum_{k=1}^{2}(x^L_{41k} + x^L_{43k} + x^L_{45k}) = 0, \qquad \sum_{k=1}^{2}(x^R_{41k} + x^R_{43k} + x^R_{45k}) = 10$$
$$\sum_{k=1}^{2}(\underline{x}_{15k} + \underline{x}_{25k} + \underline{x}_{45k}) = 60, \qquad \sum_{k=1}^{2}(\overline{x}_{15k} + \overline{x}_{25k} + \overline{x}_{45k}) = 70$$
$$\sum_{k=1}^{2}(x^L_{15k} + x^L_{25k} + x^L_{45k}) = 10, \qquad \sum_{k=1}^{2}(x^R_{15k} + x^R_{25k} + x^R_{45k}) = 0$$

$\underline{x}_{131}+\underline{x}_{151}+\underline{x}_{211}+\underline{x}_{231}+\underline{x}_{251}+\underline{x}_{411}+\underline{x}_{431}+\underline{x}_{451}=60$

$\overline{x}_{131}+\overline{x}_{151}+\overline{x}_{211}+\overline{x}_{231}+\overline{x}_{251}+\overline{x}_{411}+\overline{x}_{431}+\overline{x}_{451}=80$

$x^L_{131}+x^L_{151}+x^L_{211}+x^L_{231}+x^L_{251}+x^L_{411}+x^L_{431}+x^L_{451}=20$

$x^R_{131}+x^R_{151}+x^R_{211}+x^R_{231}+x^R_{251}+x^R_{411}+x^R_{431}+x^R_{451}=10$

$\underline{x}_{132}+\underline{x}_{152}+\underline{x}_{212}+\underline{x}_{232}+\underline{x}_{252}+\underline{x}_{412}+\underline{x}_{432}+\underline{x}_{452}=50$

$\overline{x}_{132}+\overline{x}_{152}+\overline{x}_{212}+\overline{x}_{232}+\overline{x}_{252}+\overline{x}_{412}+\overline{x}_{432}+\overline{x}_{452}=70$

$x^L_{132}+x^L_{152}+x^L_{212}+x^L_{232}+x^L_{252}+x^L_{412}+x^L_{432}+x^L_{452}=20$

$x^R_{132}+x^R_{152}+x^R_{212}+x^R_{232}+x^R_{252}+x^R_{412}+x^R_{432}+x^R_{452}=40$

$15\underline{x}_{131}+15\overline{x}_{131}-8x^L_{131}+15x^R_{131}\leq 370,\quad 15\underline{x}_{132}+15\overline{x}_{132}-8x^L_{132}+15x^R_{132}\leq 1270$

$15\underline{x}_{211}+15\overline{x}_{211}-8x^L_{24}+15x^R_{211}\leq 1120,\quad 15\underline{x}_{212}+15\overline{x}_{13}-8x^L_{212}+15x^R_{212}\leq 1340$

$15\underline{x}_{231}+15\overline{x}_{231}-8x^L_{34}+15x^R_{231}\leq 1490,\quad 15\underline{x}_{232}+15\overline{x}_{232}-8x^L_{232}+15x^R_{232}\leq 1640$

$15\underline{x}_{131}+15\overline{x}_{131}-8x^L_{131}+15x^R_{131}\geq 150,\quad 15\underline{x}_{132}+15\overline{x}_{132}-8x^L_{132}+15x^R_{132}\geq 260$

$15\underline{x}_{211}+15\overline{x}_{211}-8x^L_{24}+15x^R_{211}\geq 0,\qquad 15\underline{x}_{212}+15\overline{x}_{13}-8x^L_{212}+15x^R_{212}\geq 0$

$15\underline{x}_{231}+15\overline{x}_{231}-8x^L_{34}+15x^R_{231}\geq 0,\qquad 15\underline{x}_{232}+15\overline{x}_{232}-8x^L_{232}+15x^R_{232}\geq 97$

$\underline{x}_{ijk}-x^L_{ijk},\overline{x}_{ijk}-\underline{x}_{ijk},x^L_{ijk},x^R_{ijk}\geq\ 0\ \ \forall\ i=1,2,4;\quad j=1,3,5;\quad k=1,2.$

Step 6 Using fuzzy programming technique [129] the optimal compromise solution of the crisp multi-objective linear programming problem, obtained in Step 5, is $\underline{x}_{131}=6.99,\overline{x}_{131}=6.99,x^L_{131}=6.99,\underline{x}_{132}=3,\overline{x}_{132}=23,x^L_{132}=3,x^R_{132}=20,\underline{x}_{231}=19.41,\overline{x}_{231}=29.41,x^L_{231}=3,\underline{x}_{232}=20.58,\overline{x}_{232}=20.59,x^L_{232}=16.99,x^R_{232}=20,\underline{x}_{151}=33.59,\overline{x}_{151}=33.59,x^L_{151}=10,\underline{x}_{152}=16.41,\overline{x}_{152}=16.41,\overline{x}_{251}=10,\underline{x}_{252}=10,\overline{x}_{252}=10,x^R_{431}=10$ and the remaining values of $\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk}$ are zero.

Step 7 Putting the values of $\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk}$ and x^R_{ijk} in $\tilde{x}_{ij}=(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR}$, the fuzzy optimal compromise solution is $\tilde{x}_{131}=(6.99,6.99,6.99,0)_{LR}$, $\tilde{x}_{132}=(3,23,3,20)_{LR}$, $\tilde{x}_{231}=(19.41,29.41,3,0)_{LR}$, $\tilde{x}_{232}=(20.58,20.59,16.99,20)_{LR}$, $\tilde{x}_{151}=(33.59,$

$33.59, 10, 0)_{LR}$, $\tilde{x}_{152} = (16.41, 16.41, 0, 0)_{LR}$, $\tilde{x}_{251} = (0, 10, 0, 0)_{LR}$, $\tilde{x}_{252} = (10, 10, 0, 0)_{LR}$, $\tilde{x}_{431} = (0, 0, 0, 10)_{LR}$ and remaining values of $\tilde{x}_{ijk} = (0, 0, 0, 0)_{LR}$.

Step 8 Putting the values of $\tilde{x}_{131}, \tilde{x}_{132}, \tilde{x}_{211}, \tilde{x}_{212}, \tilde{x}_{231}, \tilde{x}_{232}, \tilde{x}_{411}, \tilde{x}_{412}, \tilde{x}_{431}, \tilde{x}_{432}, \tilde{x}_{451}, \tilde{x}_{452}, \tilde{x}_{251}, \tilde{x}_{252}, \tilde{x}_{151}, \tilde{x}_{152}$ in $\Big((8, 10, 2, 2)_{LR} \otimes \tilde{x}_{131} \oplus (4, 8, 3, 2)_{LR} \otimes \tilde{x}_{132} \oplus (8, 10, 4, 4)_{LR} \otimes \tilde{x}_{211} \oplus (6, 8, 4, 4)_{LR} \otimes \tilde{x}_{212} \oplus (9, 12, 6, 3)_{LR} \otimes \tilde{x}_{231} \oplus (3, 6, 2, 3)_{LR} \otimes \tilde{x}_{232} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{411} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{412} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{431} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{432} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{251} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{252} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{451} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{452} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{151} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{152}\Big)$, $\Big((5, 8, 3, 3)_{LR} \otimes \tilde{x}_{131} \oplus (8, 10, 2, 2)_{LR} \otimes \tilde{x}_{132} \oplus (9, 12, 6, 3)_{LR} \otimes \tilde{x}_{211} \oplus (8, 10, 4, 4)_{LR} \otimes \tilde{x}_{212} \oplus (6, 8, 4, 4)_{LR} \otimes \tilde{x}_{231} \oplus (6, 9, 2, 3)_{LR} \otimes \tilde{x}_{232} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{411} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{412} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{431} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{432} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{251} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{252} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{451} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{452} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{151} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{152}\Big)$ and $\Big((6, 9, 2, 3)_{LR} \otimes \tilde{x}_{131} \oplus (3, 6, 2, 3)_{LR} \otimes \tilde{x}_{132} \oplus (5, 8, 3, 3)_{LR} \otimes \tilde{x}_{211} \oplus (9, 12, 6, 3)_{LR} \otimes \tilde{x}_{212} \oplus (2, 4, 2, 2)_{LR} \otimes \tilde{x}_{231} \oplus (4, 8, 3, 2)_{LR} \otimes \tilde{x}_{232} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{411} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{412} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{431} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{432} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{251} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{252} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{451} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{452} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{151} \oplus (0, 0, 0, 0)_{LR} \otimes \tilde{x}_{152}\Big)$ the fuzzy optimal values of first, second and third objectives are $(304.45, 730.48, 251.63, 589.96)_{LR}$, $(298.99, 706.57, 251.81, 726.40)_{LR}$ and $(172.16, 483.63, 168.57, 569.69)_{LR}$ respectively.

5.4.2 Physical interpretation of the results

In this section, the fuzzy optimal value of first objective, obtained by using the proposed method, is physically interpreted. Similarly, the obtained fuzzy optimal compromise solution, fuzzy optimal value of second objective function and the fuzzy optimal value of third objective function can also be physically interpreted.

Using the proposed method the fuzzy optimal value of the first objective is $(304.45, 730.48, 251.63, 589.96)_{LR}$ which can be physically interpreted as follows:

(1) The least amount of the fuzzy optimal value of the first objective is 52.82.

(2) The most possible amount of the fuzzy optimal value of the first objective lies between 304.45 and 730.48.

(3) The greatest amount of the fuzzy optimal value of the first objective is 1320.44 i.e., the fuzzy optimal value of the first objective will always be greater than 52.82 and less than 1320.44 and maximum chances are that the fuzzy optimal value of the first objective will lie between 304.45 and 730.48.

The variation in the optimal value of the first objective with respect to chances are shown in Figure 5.2.

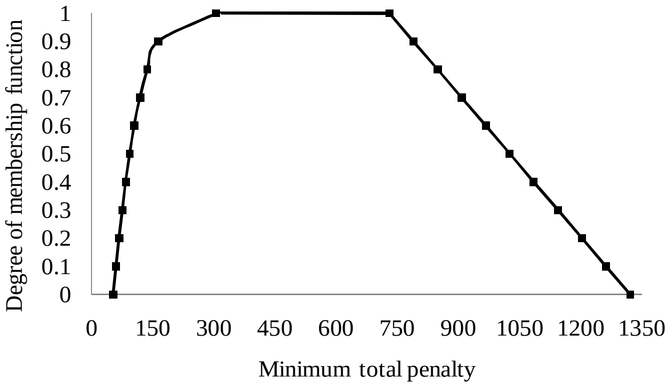

Figure 5.2 Membership function of *LR* flat fuzzy number representing the minimum total fuzzy penalty

5.5 Comparative study

To show the advantage of the proposed method over the method, proposed in Chapter 4, the results of some problems, obtained by using the method proposed in Chapter 4 and the method proposed in this chapter, are compared in Table 5.5.

Table 5.5 Comparison of results obtained by using the method proposed in Chapter 4 and method proposed in this chapter

Example	Method proposed in Chapter 4	Method proposed in this chapter
4.5	$(1650, 2700, 1240, 3050)_{LR}$ $(2100, 3425, 1690, 2125)_{LR}$	$(1650, 2700, 1240, 3050)_{LR}$ $(2100, 3425, 1690, 2125)_{LR}$
5.1	Not applicable	$(1987.99, 1987.99, 100, 1000)_{LR}$ $(1412.01, 1512.01, 100, 800)_{LR}$
5.2	Not applicable	$(1648.49, 1648.49, 100, 1000)_{LR}$ $(1241.83, 1341.83, 103.33, 803.33)_{LR}$
5.3	Not applicable	$(304.45, 730.48, 251.63, 589.96)_{LR}$ $(298.99, 706.57, 251.81, 726.40)_{LR}$ $(172.16, 483.63, 168.57, 569.69)_{LR}$

The results, presented in Table 5.5, can be explained as follows:

(1) The method, proposed in Chapter 4, can be used for solving fully fuzzy single and multi-objective capacitated transportation problems as well as fully fuzzy single and multi-objective un-capacitated and capacitated minimum cost flow problems. However, the same method can neither be used for solving fully fuzzy single and multi-objective un-capacitated and capacitated solid transportation problems nor for solving fully fuzzy single and multi-objective un-capacitated and capacitated solid minimum cost flow problems. Since, the problem chosen in Example 4.5 is a fully fuzzy multi-objective capacitated minimum cost flow problem so it can be solved by using the method proposed in Chapter 4. However, the problems chosen in Example 5.1, Example 5.2 and Example 5.3 are fully fuzzy multi-objective un-capacitated solid transportation problem, fully fuzzy multi-objective capacitated solid transportation problem and fully fuzzy multi-objective capacitated solid minimum cost flow problem respectively so none of these problems can be solved by using the method proposed in Chapter 4.

(2) Since, the method, proposed in this chapter, can be used for solving fully

fuzzy single and multi-objective transportation problems, fully fuzzy single and multi-objective solid transportation problems, fully fuzzy single and multi-objective minimum cost flow problems, fully fuzzy single and multi-objective solid minimum cost flow problems. So, all the problems, chosen in Example 4.5, Example 5.1, Example 5.2 and Example 5.3, can be solved by using the method proposed in this chapter.

5.6 Conclusions

On the basis of present study, it can be concluded that it is better to use the method, proposed in this chapter, as compared to the existing methods [38, 49] as well as the methods proposed in Chapter 3 and Chapter 4.

Chapter 6

A New Method For Solving Intuitionistic Fully Fuzzy Multi-Objective Capacitated Solid Minimum Cost Flow Problems

In real life, a person may assume that an object belongs to a set but it is possible that he (she) is not sure about it. In other words, there may be hesitation or confusion that whether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or confusion. A possible solution is to use intuitionistic fuzzy set [9]. In the methods, proposed in previous chapters, an existing ranking approach for comparing fuzzy numbers is used for converting fully fuzzy linear programming problem into crisp linear programming problem. In this chapter, it is pointed out that it is not genuine to use any of the existing ranking approaches for comparing intuitionistic fuzzy numbers for converting intuitionistic fully fuzzy linear programming problem into crisp linear programming problem and a new ranking approach is proposed for comparing intuitionistic fuzzy numbers. Also, with the help of proposed ranking approach, a new method for solving intuitionistic

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fully fuzzy single objective capacitated solid minimum cost flow problems as well as a new method for solving intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problems is proposed. The proposed methods are illustrated with the help of numerical examples.

6.1 Preliminaries

In this section, some basic definitions and arithmetic operations are presented [101].

6.1.1 Basic definitions

In this section, some basic definitions are presented [101].

Definition 6.1 An intuitionistic fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x))|\ x \in X\}$ on the universal set X is characterized by a truth membership function $\mu_{\tilde{A}}, \mu_{\tilde{A}} : X \to [0, 1]$ and a false membership function $\nu_{\tilde{A}}, \nu_{\tilde{A}} : X \to [0, 1]$. The values $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ represents the degree of membership and degree of non-membership for $x \in X$ and always satisfies the condition $\mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1\ \forall\ x \in X$. The value $(1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x))$ represents the degree of hesitation for $x \in X$.

Definition 6.2 Let $\tilde{A}$ be an intuitionistic fuzzy set. Then, $\tilde{A}_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha, \nu_{\tilde{A}}(x) \leq (1 - \alpha)\}$ is said to be an α-cut of $\tilde{A}$.

Definition 6.3 An intuitionistic fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X\}$ is called intuitionistic fuzzy-normal, if there exist at least two points $x_0, x_1 \in X$ such that $\mu_{\tilde{A}}(x_0) = 1, \nu_{\tilde{A}}(x_1) = 1$.

Definition 6.4 An intuitionistic fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X\}$ is called intuitionistic fuzzy-convex, if $\forall x_1, x_2 \in X, \lambda \in [0, 1]$

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$$

$$\nu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \leq \max(\nu_{\tilde{A}}(x_1), \nu_{\tilde{A}}(x_2))$$

Definition 6.5 An intuitionistic fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X\}$ defined on the universal set X is called intuitionistic fuzzy number if

(i) $\tilde{A}$ is intuitionistic fuzzy-normal

(ii) $\tilde{A}$ is intuitionistic fuzzy-convex

(iii) $\mu_{\tilde{A}}$ is upper semicontinuous and $\nu_{\tilde{A}}$ is lower semicontinuous

(iv) $\tilde{A} = \{x \in X \mid \nu_{\tilde{A}}(x) < 1\}$ is bounded

Definition 6.6 An intuitionistic fuzzy number $\tilde{A}$, defined on the universal set of real numbers $\mathbb{R}$, denoted as $\tilde{A} = \{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR}\}$, where $\underline{a'} - a'^L \leq \underline{a} - a^L \leq \underline{a'} \leq \underline{a} \leq \overline{a} \leq \overline{a'} \leq \overline{a} + a^R \leq \overline{a'} + a'^R$ is said to be an LR flat intuitionistic fuzzy number if degree of membership $\mu_{\tilde{A}}(x)$ and degree of non-membership $\nu_{\tilde{A}}(x)$ are given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{\underline{a}-x}{a^L}), & x \leq \underline{a},\ a^L > 0 \\ R(\frac{x-\overline{a}}{a^R}), & x \geq \overline{a},\ a^R > 0 \\ 1, & \underline{a} \leq x \leq \overline{a} \end{cases}$$

$$\nu_{\tilde{A}}(x) = \begin{cases} 1 - L(\frac{\underline{a'}-x}{a'^L}), & x \leq \underline{a'},\ a'^L > 0 \\ 1 - R(\frac{x-\overline{a'}}{a'^R}), & x \geq \overline{a'},\ a'^R > 0 \\ 0, & \underline{a'} \leq x \leq \overline{a'} \end{cases}$$

Definition 6.7 Two LR flat intuitionistic fuzzy numbers $\tilde{A} = \{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR}\}$ and $\tilde{B} = \{(\underline{b}, \overline{b}, b^L, b^R)_{LR}; (\underline{b'}, \overline{b'}, b'^L, b'^R)_{LR}\}$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ if and only if $\underline{a} = \underline{b}, \overline{a} = \overline{b}, a^L = b^L, a^R = b^R, \underline{a'} = \underline{b'}, \overline{a'} = \overline{b'}, a'^L = b'^L$ and $a'^R = b'^R$.

Definition 6.8 An LR flat intuitionistic fuzzy number $\tilde{A} = \{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR}\}$ is said to be a zero LR flat intuitionistic fuzzy number if and only if $\underline{a} = 0,\ \overline{a} = 0,\ a^L = 0,\ a^R = 0,\ \underline{a'} = 0,\ \overline{a'} = 0,\ a'^L = 0$ and $a'^R = 0$.

Definition 6.9 An LR flat intuitionistic fuzzy number $\tilde{A} = \{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR}\}$ is said to be non-negative LR flat intuitionistic fuzzy number if and only if $\underline{a'} - a'^L \geq 0$.

Remark 6.1 If $\underline{a} = \overline{a} = \underline{a'} = \overline{a'} = a$ (say) then an LR flat intuitionistic fuzzy number $\{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR}\}$ is said to be an LR intuitionistic fuzzy number and is denoted as $\{(a, a^L, a^R)_{LR}; (a, a'^L, a'^R)_{LR}\}$.

Remark 6.2 If $\underline{a} = \overline{a} = \underline{a'} = \overline{a'} = a$ (say) and $L(x) = R(x) = \text{maximum}\{0, 1 - x\}$ then an LR flat intuitionistic fuzzy number $\{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR}\}$ is said to be triangular intuitionistic fuzzy number and is denoted as $\{(a_1, a, a_2); (a'_1, a, a'_2)\}$ or $(a'_1, a_1, a, a_2, a'_2)$.
where, $a'_1 = a - a'^L$, $a_1 = a - a^L$, $a_2 = a + a^R$, $a'_2 = a + a'^R$

Remark 6.3 If $\underline{a} \neq \overline{a}$, $\underline{a'} \neq \overline{a'}$ and $L(x) = R(x) = \text{maximum}\{0, 1 - x\}$ then an LR flat intuitionistic fuzzy number $\{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR}\}$ is said to be trapezoidal intuitionistic fuzzy number and is denoted as $\{(a_1, a_2, a_3, a_4); (a'_1, a'_2, a'_3, a'_4)\}$ or $(a'_1, a_1, a'_2, a_2, a_3, a'_3, a_4, a'_4)$.
where, $a_1 = \underline{a} - a^L$, $a_2 = \underline{a}$, $a_3 = \overline{a}$, $a_4 = \overline{a} + a^R$, $a'_1 = \underline{a'} - a'^L$, $a'_2 = \underline{a'}$, $a'_3 = \overline{a'}$, $a'_4 = \overline{a'} + a'^R$

6.1.2 Arithmetic operations between LR flat intuitionistic fuzzy numbers

In this section, some arithmetic operations between LR flat intuitionistic fuzzy numbers, defined on universal set of real numbers $\mathbb{R}$, are presented.

(i) Let $\tilde{A} = \{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR}\}$ and $\tilde{B} = \{(\underline{b}, \overline{b}, b^L, b^R)_{LR}; (\underline{b'}, \overline{b'}, b'^L, b'^R)_{LR}\}$ be two LR flat intuitionistic fuzzy numbers. Then,

$$\tilde{A} \oplus \tilde{B} = \{(\underline{a}+\underline{b}, \overline{a}+\overline{b}, a^L+b^L, a^R+b^R)_{LR}; (\underline{a'}+\underline{b'}, \overline{a'}+\overline{b'}, a'^L+b'^L, a'^R+b'^R)_{LR}\}$$

(ii) Let $\tilde{A} = \{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR}\}$ be an LR flat intuitionistic fuzzy number. Then,

$$\lambda\tilde{A} = \begin{cases} \{(\lambda\underline{a}, \lambda\overline{a}, \lambda a^L, \lambda a^R)_{LR}; (\lambda\underline{a'}, \lambda\overline{a'}, \lambda a'^L, \lambda a'^R)_{LR}\}, & \lambda \geq 0 \\ \{(\lambda\overline{a}, \lambda\underline{a}, -\lambda a^R, -\lambda a^L,)_{RL}; (\lambda\overline{a'}, \lambda\underline{a'}, -\lambda a'^R, -\lambda a'^L)_{RL}\}, & \lambda \leq 0 \end{cases}$$

(iii) Let $\tilde{A} = \{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR}\}$ and $\tilde{B} = \{(\underline{b}, \overline{b}, b^L, b^R)_{LR}; (\underline{b'}, \overline{b'}, b'^L, b'^R)_{LR}\}$ be two non-negative LR flat intuitionistic fuzzy numbers. Then,

$$\tilde{A} \otimes \tilde{B} = \{(\underline{a}\ \underline{b}, \overline{a}\ \overline{b}, \underline{a}b^L + \underline{b}a^L - a^Lb^L, \overline{a}b^R + \overline{b}a^R + a^Rb^R)_{LR}; (\underline{a'}\ \underline{b'}, \overline{a'}\ \overline{b'}, \underline{a'}b'^L + \underline{b'}a'^L - a'^Lb'^L, \overline{a'}b'^R + \overline{b'}a'^R + a'^Rb'^R)_{LR}\}$$

6.2 Limitations of the existing methods as well as the methods proposed in previous chapters

The existing methods [38, 49] and the methods proposed in previous chapters can not be used for solving intuitionistic fully fuzzy single and multi-objective transportation problems, intuitionistic fully fuzzy single and multi-objective solid transportation problems, intuitionistic fully fuzzy single and multi-objective minimal cost flow problems and intuitionistic fully fuzzy single and multi-objective solid minimal cost flow problems e.g., the intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problem chosen in Example 6.1 and the intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem chosen in Example 6.2 can not be solved by using the existing methods [38, 49] and the methods proposed in previous chapters.

Example 6.1 Find the intuitionistic fuzzy optimal solution of the intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problem depicted in Figure 6.1. The data is listed in Table 6.1, Table 6.2, Table 6.3 and Table 6.4.

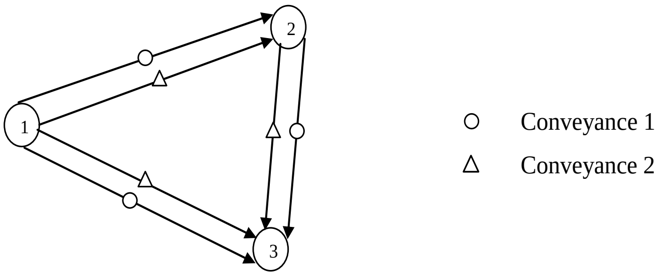

Figure 6.1 Network representing intuitionistic fully fuzzy capacitated solid minimum cost flow problem

Table 6.1 Intuitionistic fuzzy cost ($\tilde{c}_{ijk}$)

i	j	k	$\tilde{c}_{ijk}$
1	3	1	$\{(400, 4000, 360, 36000)_{LR}; (200, 22000, 180, 38000)_{LR}\}$
1	3	2	$\{(200, 2000, 180, 18000)_{LR}; (100, 11000, 90, 19000)_{LR}\}$
1	2	1	$\{(100, 1000, 90, 9000)_{LR}; (50, 5500, 45, 9500)_{LR}\}$
1	2	2	$\{(50, 200, 48, 1800)_{LR}; (10, 1100, 9, 1900)_{LR}\}$
2	3	1	$\{(300, 3000, 270, 27000)_{LR}; (150, 16500, 135, 28500)_{LR}\}$
2	3	2	$\{(500, 5000, 450, 45000)_{LR}; (250, 27500, 225, 47500)_{LR}\}$

Table 6.2 Minimum intuitionistic fuzzy amount ($\tilde{l}_{ijk}$) and maximum intuitionistic fuzzy amount ($\tilde{u}_{ijk}$)

i	j	k	$\tilde{l}_{ijk}$	$\tilde{u}_{ijk}$
1	3	1	$\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}$	$\{(40, 50, 20, 20)_{LR}; (30, 60, 20, 40)_{LR}\}$
1	3	2	$\{(6, 7, 3, 2)_{LR}; (4, 8, 3, 3)_{LR}\}$	$\{(150, 220, 100, 40)_{LR}; (100, 240, 80, 60)_{LR}\}$
1	2	1	$\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}$	$\{(65, 70, 35, 10)_{LR}; (60, 75, 45, 10)_{LR}\}$
1	2	2	$\{(2, 3, 1, 1)_{LR}; (1, 3, 1, 2)_{LR}\}$	$\{(100, 150, 80, 150)_{LR}; (60, 170, 60, 180)_{LR}\}$
2	3	1	$\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}$	$\{(150, 220, 100, 40)_{LR}; (100, 240, 80, 60)_{LR}\}$
2	3	2	$\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}$	$\{(50, 55, 42, 10)_{LR}; (13, 60, 9, 11)_{LR}\}$

Table 6.3 Intuitionistic fuzzy supply ($\tilde{a}_i/\tilde{e}_i$) and intuitionistic fuzzy demand ($\tilde{b}_j/\tilde{d}_j$)

Node	$\tilde{a}_i/\tilde{e}_i$	$\tilde{b}_j/\tilde{d}_j$
1	$\{(200, 250, 100, 50)_{LR}; (150, 270, 100, 80)_{LR}\}$	–
2	–	$\{(100, 150, 80, 50)_{LR}; (60, 170, 60, 80)_{LR}\}$
3	–	$\{(50, 100, 20, 0)_{LR}; (40, 100, 40, 50)_{LR}\}$

Table 6.4 Intuitionistic fuzzy capacity ($\tilde{f}_k$)

Conveyance	$\tilde{f}_k$
1	$\{(0, 50, 0, 0)_{LR}; (0, 50, 0, 50)_{LR}\}$
2	$\{(200, 250, 100, 50)_{LR}; (150, 270, 100, 80)_{LR}\}$

where, $L(x) = \text{maximum}\{0, 1 - x\},\ R(x) = \text{maximum }\{0, 1 - x\}$

Example 6.2 Consider an intuitionistic fully fuzzy bi-objective capacitated solid minimum cost flow problem with the same network flow structure and data as in Example 6.1. The first objective is same as in Example 6.1 and the second objective is to minimize the total intuitionistic fuzzy passing time. The intuitionistic fuzzy passing time required for transporting one unit quantity of product from i^{th} source to j^{th} destination by means of k^{th} conveyance is listed in Table 6.5.

Table 6.5 Intuitionistic fuzzy passing time ($\tilde{c}^2_{ijk}$)

i	j	k	$\tilde{c}^2_{ijk}$
1	3	1	$\{(500, 5000, 450, 45000)_{LR}; (250, 27500, 225, 47500)_{LR}\}$
1	3	2	$\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}$
1	2	1	$\{(150, 950, 141, 8050)_{LR}; (45, 5000, 41, 11000)_{LR}\}$
1	2	2	$\{(45, 150, 42, 1950)_{LR}; (11, 1000, 9, 1900)_{LR}\}$
2	3	1	$\{(400, 4000, 360, 36000)_{LR}; (200, 22000, 180, 38000)_{LR}\}$
2	3	2	$\{(200, 2000, 180, 18000)_{LR}; (100, 11000, 90, 19000)_{LR}\}$

where, $L(x) = \text{maximum}\{0, 1-x\},\ R(x) = \text{maximum } \{0, 1-x\}$

6.3 Need of proposing new ranking approach for comparing intuitionistic fuzzy numbers

To overcome the limitations of the existing methods [38, 49] and the methods proposed in previous chapters, pointed out in Section 6.2, it may be tried to modify the method, proposed in Chapter 5 with the help of existing methods [31, 47, 80, 91, 98–101, 122, 125] for comparing intuitionistic fuzzy numbers. However, it is not genuine to do so due to the following limitations and shortcomings of these methods.

6.3.1 Limitations of the existing methods

In this section, the limitations of the existing methods [31, 47, 80, 91, 99, 100, 122, 125] for comparing intuitionistic fuzzy numbers are pointed out.

(1) The existing methods [47, 91, 122, 125] can be used only for comparing intuitionistic fuzzy set. However, none of the existing methods [47, 91, 122, 125] can used for comparing intuitionistic fuzzy numbers.

(2) Nayagam and Sivaraman [99] pointed out the shortcomings of the existing method [100] and proposed a method for comparing such triangular intuitionistic fuzzy numbers $\{(a, b, c), (e, f, g)\}$ for which either the conditions $e \geq b$ and $f \geq c$ or the conditions $f \leq a$ and $g \leq b$ are satisfied. However, the existing method [99] can not be used for comparing such triangular intuitionistic fuzzy numbers for which neither the conditions $e \geq b$ and $f \geq c$ nor the conditions $f \leq a$ and $g \leq b$ are satisfied.

(3) Dubey and Mehra [31] pointed out the shortcomings of the existing methods [80] and proposed a method for comparing such triangular intuitionistic fuzzy sets $\tilde{a} = \{(\underline{a}^{\mu}, a, \overline{a}^{\mu}; w_{\tilde{a}}), (\underline{a}^{\nu}, a, \overline{a}^{\nu}; u_{\tilde{a}})\}$ for which the condition $V_{\mu}(\tilde{a}) \leq V_{\nu}(\tilde{a})$ is satisfied, where $V_{\mu}(\tilde{a}) = \frac{(\underline{a}^{\mu}+4a+\overline{a}^{\mu})w_{\tilde{a}}}{6}$ and $V_{\nu}(\tilde{a}) = \frac{(\underline{a}^{\nu}+4a+\overline{a}^{\nu})(1-u_{\tilde{a}})}{6}$. However, the existing method [99] can not be used for comparing such triangular intuitionistic fuzzy sets for which the condition $V_{\mu}(\tilde{a}) \leq V_{\nu}(\tilde{a})$ is not satisfied.

6.3.2 Shortcomings of the existing methods

In this section, the shortcomings of the existing methods [80, 98, 100, 101] for comparing intuitionistic fuzzy numbers are pointed out.

(1) Li [80] proposed the following method for comparing triangular intuitionistic fuzzy sets. Let $\tilde{A} = \langle (a_1, a_2, a_3); w_1, u_1 \rangle$ and $\tilde{B} = \langle (a_1, a_2, a_3); w_2, u_2 \rangle$ be two triangular intuitionistic fuzzy sets. Then,

(i) $\tilde{A} \prec \tilde{B}$ if $L_T^{\lambda}(\tilde{A}) < L_T^{\lambda}(\tilde{B})$

(ii) $\tilde{A} \succ \tilde{B}$ if $L_T^\lambda(\tilde{A}) > L_T^\lambda(\tilde{B})$

(iii) $\tilde{A} \sim \tilde{B}$ if $L_T^\lambda(\tilde{A}) = L_T^\lambda(\tilde{B})$

where, $L_T^\lambda(\tilde{A}) = \frac{V^\lambda(\tilde{A})}{1+A^\lambda(\tilde{A})}$, $V^\lambda(\tilde{A}) = V_\mu(\tilde{A}) + \lambda(V_\nu(\tilde{A}) - V_\mu(\tilde{A}))$,

$A^\lambda(\tilde{A}) = A_\mu(\tilde{A}) - \lambda(A_\nu(\tilde{A}) - A_\mu(\tilde{A}))$, $V_\mu(\tilde{A}) = \frac{w_1}{6}(a_1 + 4a_2 + a_3)$,

$V_\nu(\tilde{A}) = \frac{1-u_1}{6}(a_1 + 4a_2 + a_3)$, $A_\mu(\tilde{A}) = \frac{w_1}{3}(a_3 - a_1)$, $A_\nu(\tilde{A}) = \frac{1-u_1}{3}(a_3 - a_1)$

It is not genuine to apply this method due to the following reasons:

It is obvious from the existing ranking approach [80] that if $(a_1 + 4a_2 + a_3) \neq 0$ then the comparison of $\tilde{A}$ and $\tilde{B}$ will depend upon the values of w_1, u_1, w_2 and u_2 and if $(a_1+4a_2+a_3) = 0$ then $\tilde{A} \sim \tilde{B}$ for all values of w_1, u_1, w_2 and u_2 i.e., according to existing approach [80] , in first case comparison of triangular intuitionistic fuzzy sets depends upon degree of membership and non-membership of intuitionistic fuzzy sets while in second case comparison of triangular intuitionistic fuzzy sets does not depend upon degree of membership and non-membership of intuitionistic fuzzy sets which is a contradiction.

Example 6.3 Let $\tilde{A} = \langle(1, 1, 1); w_1, u_1\rangle$ and $\tilde{B} = \langle(1, 1, 1); w_2, u_2\rangle$ be two triangular intuitionistic fuzzy sets. Then, according to existing ranking approach [80] values of $L_T^\lambda(\tilde{A})$ and $L_T^\lambda(\tilde{B})$ will depend upon the values of w_1, u_1, w_2 and u_2 i.e., the ordering of $\tilde{A}$ and $\tilde{B}$ will depend upon the values of w_1, u_1, w_2 and u_2.

Example 6.4 Let $\tilde{A} = \langle(-8, 1, 4); w_1, u_1\rangle$ and $\tilde{B} = \langle(-8, 1, 4); w_2, u_2\rangle$ be two triangular intuitionistic fuzzy sets. Then, according to existing ranking approach [80], $L_T^\lambda(\tilde{A}) = L_T^\lambda(\tilde{B}) = 0 \Rightarrow \tilde{A} \sim \tilde{B}$ i.e., in this case the ordering of $\tilde{A}$ and $\tilde{B}$ is independent from the values of w_1, u_1, w_2 and u_2.

(2) Nayagam and Sivaraman [98] proposed the following method for comparing interval valued intuitionistic fuzzy sets:

Let $\tilde{A} = \{x : [a_1, b_1], [c_1, d_1] \mid x \in X\}$ and $\tilde{B} = \{x : [a_2, b_2], [c_2, d_2] \mid x \in X\}$ be two interval valued intuitionistic fuzzy sets defined on a universal set X. Then

(i) $\tilde{A} \prec \tilde{B}$ if $LG(\tilde{A}) < LG(\tilde{B})$

(ii) $\tilde{A} \succ \tilde{B}$ if $LG(\tilde{A}) > LG(\tilde{B})$

(iii) $\tilde{A} \sim \tilde{B}$ if $LG(\tilde{A}) = LG(\tilde{B})$

where, $LG(\tilde{A}) = \frac{(a_1+b_1)(1-\delta)+\delta(2-(c_1+d_1))}{2}$, $LG(\tilde{B}) = \frac{(a_2+b_2)(1-\delta)+\delta(2-(c_2+d_2))}{2}$ and $\delta \in [0, 1]$.

Nayagam and Sivaraman [98] used the same method for comparing trapezoidal intuitionistic fuzzy numbers $\tilde{A} = \{(a_1, a_1, b_1, b_1), (c_1, c_1, d_1, d_1)\}$ and $\tilde{B} = \{(a_2, a_2, b_2, b_2), (c_2, c_2, d_2, d_2)\}$. However, it is not genuine to use this method for comparing trapezoidal intuitionistic fuzzy numbers due to the following reasons:

In the intuitionistic fuzzy set $\tilde{A} = \{x : [a_1, b_1], [c_1, d_1] \mid x \in X\}$, a_1, b_1 and c_1, d_1 represents the infimum and supremum values of membership degree and non-membership degree corresponding to a point x. While, in the trapezoidal intuitionistic fuzzy number $\tilde{A} = \{(a_1, a_1, b_1, b_1), (c_1, c_1, d_1, d_1)\}$, a_1 and b_1 represents that points of universal set X corresponding to which the membership degree is 1. Similarly, c_1 and d_1 represents that points of universal set X corresponding to which the non-membership degree is 1.

(3) Since, intuitionistic fuzzy numbers are the generalization of fuzzy numbers. So, the approach which can be used for comparing intuitionistic fuzzy numbers can

also be used for comparing fuzzy numbers. To show the shortcomings of the existing approach [101], two fuzzy numbers are compared by using the existing approaches [36, 82, 101] and it is shown that ordering of fuzzy numbers obtained by using Nehi's approach [101] and Liou and Wang approach [82] is same and is contradicting the ordering of fuzzy numbers obtained by using the Garcia and Lamata approach [36]. Since, Garcia and Lamata [36] pointed out that their approach is better than Liou and Wang's approach [82]. So, it is not genuine to use the existing approach [101] for comparing intuitionistic fuzzy numbers.

Example: 6.5 Let $\tilde{A} = (4, 4, 3, 1)_{LR}$ and $\tilde{B} = (3, 3, 1, 3)_{LR}$ ($L(x) =$ maximum $\{0, 1-x\}$, $R(x) =$ maximum $\{0, 1-x\}$) be two LR flat fuzzy numbers. The intuitionistic representation of these LR flat fuzzy numbers is $\tilde{A} = \{(4, 4, 3, 1)_{LR}\ ; (4, 4, 3, 1)_{LR}\}$ and $\tilde{B} = \{(3, 3, 1, 3)_{LR}\ ; (3, 3, 1, 3)_{LR}\}$.

The ordering of these numbers obtained by using the existing ranking approaches [36, 82, 101] is shown in Table 6.6.

Table 6.6 Comparison of the results

Approach	Ordering
Liou and Wang [82]	$\tilde{A} \sim \tilde{B}$
Nehi [101]	$\tilde{A} \sim \tilde{B}$
Garcia and Lamata [36]	$\tilde{A} \succ \tilde{B}$

It is obvious from the results, shown in Table 6.6, that ordering of fuzzy numbers $\tilde{A}$ and $\tilde{B}$, obtained by using the existing approach [101], is same as obtained by the existing approach [82] while it is different from the ordering obtained by using the existing approach [36].

6.4 Proposed ranking approach for comparing LR flat intuitionistic fuzzy numbers

Garcia and Lamata [36] pointed out that although LR flat fuzzy number $\tilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is defined by four points but in the existing formula [82], $\Re(\tilde{A}) = \Big(\lambda \int_0^1 (\underline{a} - a^L L^{-1}(\rho))d\rho + (1-\lambda) \int_0^1 (\overline{a} + a^R R^{-1}(\rho))d\rho\Big)$ where, $\lambda \in [0,1]$ the central points $\underline{a}$ and $\overline{a}$ are taken into account in an indirect way and proposed the following modified formula, $\Re(\tilde{A}) = \gamma\Big[\int_0^1 (\underline{a}\lambda + \overline{a}(1-\lambda))d\rho\Big] + (1-\gamma)\Big[\lambda \int_0^1 (\underline{a} - a^L L^{-1}(\rho))d\rho + (1-\lambda) \int_0^1 (\overline{a} + a^R R^{-1}(\rho))d\rho\Big]$. Since, in the existing formula [101], $C_\mu^{k'}(\tilde{A}) = \frac{k'+1}{2} \int\limits_0^1 r^{k'}[\underline{a} - a^L L^{-1}(r) + \overline{a} + a^R R^{-1}(r)]dr$, $C_\nu^{k'}(\tilde{A}) = \frac{k'+1}{2} \int\limits_0^1 r^{k'}[\underline{a'} - a'^L L^{-1}(r) + \overline{a'} + a'^R R^{-1}(r)]dr$, where, $k' \in [0, \infty)$ the central points $\underline{a}, \overline{a}$ and $\underline{a'}, \overline{a'}$ are also taken into account in an indirect way so, to overcome the shortcomings of existing approach [101], pointed out in Section 6.3.2, a new ranking approach, by modifying the existing ranking approach [101], is proposed for comparing LR flat intuitionistic fuzzy numbers $\tilde{A} = \{(\underline{a}, \overline{a}, a^L, a^R)_{LR}; (\underline{a'}, \overline{a'}, a'^L, a'^R)_{LR})\}$ and $\tilde{B} = \{(\underline{b}, \overline{b}, b^L, b^R)_{LR}; (\underline{b'}, \overline{b'}, b'^L, b'^R)_{LR})\}$.

The steps of the proposed ranking approach are as follows:

Step 1 Calculate $M_\mu^{\beta,k'}(\tilde{A}) = \beta\Big[\frac{k'+1}{2} \int\limits_0^1 r^{k'}(\underline{a} + \overline{a})dr\Big] + (1-\beta)\Big[\frac{k'+1}{2} \int\limits_0^1 r^{k'}[\underline{a} - a^L L^{-1}(r) + \overline{a} + a^R R^{-1}(r)]dr\Big]$, $M_\mu^{\beta,k'}(\tilde{B}) = \beta\Big[\frac{k'+1}{2} \int\limits_0^1 r^{k'}(\underline{b} + \overline{b})dr\Big] + (1-\beta)\Big[\frac{k'+1}{2} \int\limits_0^1 r^{k'}[\underline{b} - b^L L^{-1}(r) + \overline{b} + b^R R^{-1}(r)]dr\Big]$ and check that $M_\mu^{\beta,k'}(\tilde{A}) > M_\mu^{\beta,k'}(\tilde{B})$ or $M_\mu^{\beta,k'}(\tilde{A}) < M_\mu^{\beta,k'}(\tilde{B})$ or $M_\mu^{\beta,k'}(\tilde{A}) = M_\mu^{\beta,k'}(\tilde{B})$.

Case (i) If $M_\mu^{\beta,k'}(\tilde{A}) > M_\mu^{\beta,k'}(\tilde{B})$ then $\tilde{A} \succ \tilde{B}$ i.e., minimum$(\tilde{A}, \tilde{B}) = \tilde{B}$

Case (ii) If $M_\mu^{\beta,k'}(\tilde{A}) < M_\mu^{\beta,k'}(\tilde{B})$ then $\tilde{A} \prec \tilde{B}$ i.e., minimum$(\tilde{A}, \tilde{B}) = \tilde{A}$

Case (iii) If $M_\mu^{\beta,k'}(\tilde{A}) = M_\mu^{\beta,k'}(\tilde{B})$ then Go to Step 2.

Step 2 Calculate $M_{\nu}^{\beta,k'}(\tilde{A}) = \beta\Big[\frac{k'+1}{2}\int\limits_{0}^{1} r^{k'}(\underline{a'}+\overline{a'})dr\Big] + (1-\beta)\Big[\frac{k'+1}{2}\int\limits_{0}^{1} r^{k'}[\underline{a'} - {a'}^{L}L^{-1}(r) + \overline{a}' + {a'}^{R}R^{-1}(r)]dr\Big]$, $M_{\nu}^{\beta,k'}(\tilde{B}) = \beta\Big[\frac{k'+1}{2}\int\limits_{0}^{1} r^{k'}(\underline{b'}+\overline{b'})dr\Big] + (1-\beta)\Big[\frac{k'+1}{2}\int\limits_{0}^{1} r^{k'}[\underline{b'} - {b'}^{L}L^{-1}(r) + \overline{b}' + {b'}^{R}R^{-1}(r)]dr\Big]$ and check that $-M_{\nu}^{\beta,k'}(\tilde{A}) > -M_{\nu}^{\beta,k'}(\tilde{B})$ or $-M_{\nu}^{\beta,k'}(\tilde{A}) < -M_{\nu}^{\beta,k'}(\tilde{B})$ or $M_{\nu}^{\beta,k'}(\tilde{A}) = M_{\nu}^{\beta,k'}(\tilde{B})$.

Case (i) If $-M_{\nu}^{\beta,k'}(\tilde{A}) > -M_{\nu}^{\beta,k'}(\tilde{B})$ then $\tilde{A} \succ \tilde{B}$ i.e., minimum$(\tilde{A},\tilde{B}) = \tilde{B}$

Case (ii) If $-M_{\nu}^{\beta,k'}(\tilde{A}) < -M_{\nu}^{\beta,k'}(\tilde{B})$ then $\tilde{A} \prec \tilde{B}$ i.e., minimum$(\tilde{A},\tilde{B}) = \tilde{A}$

Case (iii) If $M_{\nu}^{\beta,k'}(\tilde{A}) = M_{\nu}^{\beta,k'}(\tilde{B})$ then $\tilde{A} \sim \tilde{B}$

Remark 6.4 For $\beta = \frac{1}{3}$ and $k' = 0$ the index for membership and nonmembership functions $M_{\mu}^{1/3,0}(\tilde{A}_i)$ and $M_{\nu}^{1/3,0}(\tilde{A}_i)$, are as follows:

$$M_{\mu}^{1/3,0}(\tilde{A}_i) = \frac{1}{3}\Big(\frac{3\underline{a}+3\overline{a}-a^{L}+a^{R}}{2}\Big) \text{ and } M_{\nu}^{1/3,0}(\tilde{A}_i) = \frac{1}{3}\Big(\frac{3\underline{a'}+3\overline{a'}-{a'}^{L}+{a'}^{R}}{2}\Big)$$

6.5 Proposed linear programming formulation of balanced intuitionistic fully fuzzy single and multi-objective capacitated solid minimum cost flow problems

In this section, linear programming formulation of balanced intuitionistic fully fuzzy single and multi-objective capacitated solid minimum cost flow problems is proposed.

6.5.1 Linear programming formulation of balanced intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problems

Let $\tilde{a}_i$ and $\tilde{e}_i$ be the intuitionistic fuzzy supply of the product at i^{th} purely source node and at i^{th} source node, $\tilde{b}_j$ and $\tilde{d}_j$ be the intuitionistic fuzzy demand of the product at j^{th} purely destination node and at the j^{th} destination node, $\tilde{f}_k$ be the intuitionistic fuzzy capacity of the k^{th} conveyance i.e., the amount of product which can be carried by the k^{th} conveyance, $\tilde{c}_{ijk}$ be the intuitionistic fuzzy cost per unit of flow from i^{th} source to j^{th} destination by means of the k^{th} conveyance, $\tilde{x}_{ijk}$ be the intuitionistic fuzzy quantity of the product that should be transported from i^{th} node to j^{th} node by means of the k^{th} conveyance, $\tilde{l}_{ijk}$ be the minimum intuitionistic fuzzy amount that can flow from i^{th} node to j^{th} node by means of the k^{th} conveyance and $\tilde{u}_{ijk}$ be the maximum intuitionistic fuzzy amount that can flow from i^{th} node to j^{th} node by means of the k^{th} conveyance. Then, any balanced intuitionistic fully fuzzy capacitated solid minimum cost flow problem i.e., $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$, can be formulated into the following intuitionistic fuzzy multi-objective linear programming problem:

$$
\begin{array}{lll}
\text{Minimize} \sum\limits_{(i,j)\in A} \sum\limits_{k\in S_C} (\tilde{c}_{ijk} \otimes \tilde{x}_{ijk}) & & \\
\quad \text{subject to} & & \\
\quad \sum\limits_{j:(i,j)\in A} \sum\limits_{k\in S_C} \tilde{x}_{ijk} = \tilde{a}_i & i \in N_{PS} & \\
\quad \sum\limits_{j:(i,j)\in A} \sum\limits_{k\in S_C} \tilde{x}_{ijk} = \sum\limits_{j:(j,i)\in A} \sum\limits_{k\in S_C} \tilde{x}_{jik} \oplus \tilde{e}_i & i \in N_S & \\
\quad \sum\limits_{i:(i,j)\in A} \sum\limits_{k\in S_C} \tilde{x}_{ijk} = \tilde{b}_j & j \in N_{PD} & \\
\quad \sum\limits_{i:(i,j)\in A} \sum\limits_{k\in S_C} \tilde{x}_{ijk} = \sum\limits_{i:(j,i)\in A} \sum\limits_{k\in S_C} \tilde{x}_{jik} \oplus \tilde{d}_j & j \in N_D & (P_{6.1}) \\
\quad \sum\limits_{j:(i,j)\in A} \sum\limits_{k\in S_C} \tilde{x}_{ijk} = \sum\limits_{j:(j,i)\in A} \sum\limits_{k\in S_C} \tilde{x}_{jik} & i \in N_T & \\
\quad \sum\limits_{(i,j)\in A} \tilde{x}_{ijk} = \tilde{f}_k & k \in S_C & \\
\quad \tilde{l}_{ijk} \preceq \tilde{x}_{ijk} \preceq \tilde{u}_{ijk} \qquad \forall\ (i,j) \in A,\ k \in S_C & &
\end{array}
$$

$\tilde{x}_{ijk}$ is a non-negative LR flat intuitionistic fuzzy number $\forall\ (i,j) \in A,\ k \in S_C$

where, A is set of arcs joining different nodes and S_C is the set of all available conveyances.

6.5.2 Linear programming formulation of balanced intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problems

If there are more than one objectives (say P) than any balanced intuitionistic fully fuzzy capacitated solid minimum cost flow problem can be formulated into intuitionistic fully fuzzy multi-objective linear programming problem ($P_{6.2}$).

$$\begin{aligned}
&\text{Minimize} \sum_{(i,j)\in A}\sum_{k\in S_C}(\tilde{c}^{\eta}_{ijk}\otimes\tilde{x}_{ijk}) && ;\eta=1,2,...,P\\
&\quad\text{subject to}\\
&\quad\sum_{j:(i,j)\in A}\sum_{k\in S_C}\tilde{x}_{ijk}=\tilde{a}_i && i\in N_{PS}\\
&\quad\sum_{j:(i,j)\in A}\sum_{k\in S_C}\tilde{x}_{ijk}=\sum_{j:(j,i)\in A}\sum_{k\in S_C}\tilde{x}_{jik}\oplus\tilde{e}_i && i\in N_S\\
&\quad\sum_{i:(i,j)\in A}\sum_{k\in S_C}\tilde{x}_{ijk}=\tilde{b}_j && j\in N_{PD}\\
&\quad\sum_{i:(i,j)\in A}\sum_{k\in S_C}\tilde{x}_{ijk}=\sum_{i:(j,i)\in A}\sum_{k\in S_C}\tilde{x}_{jik}\oplus\tilde{d}_j && j\in N_D \qquad (P_{6.2})\\
&\quad\sum_{j:(i,j)\in A}\sum_{k\in S_C}\tilde{x}_{ijk}=\sum_{j:(j,i)\in A}\sum_{k\in S_C}\tilde{x}_{jik} && i\in N_T\\
&\quad\sum_{(i,j)\in A}\tilde{x}_{ijk}=\tilde{f}_k && k\in S_C\\
&\quad\tilde{l}_{ijk}\preceq\tilde{x}_{ijk}\preceq\tilde{u}_{ijk} \quad \forall\ (i,j)\in A,\ k\in S_C
\end{aligned}$$

$\tilde{x}_{ijk}$ is a non-negative LR flat intuitionistic fuzzy number $\forall\ (i,j)\in A,\ k\in S_C$

$\tilde{c}^{\eta}_{ijk}$: The intuitionistic fuzzy penalty per unit of flow from i^{th} source to j^{th} destination by means of the k^{th} conveyance in the η^{th} objective function.

P: total number of objectives.

Remark 6.5 If $\sum_{i\in N_{PS}}\tilde{a}_i\oplus\sum_{i\in N_S}\tilde{e}_i=\sum_{j\in N_{PD}}\tilde{b}_j\oplus\sum_{j\in N_D}\tilde{d}_j=\sum_{k\in S_C}\tilde{f}_k$ then an intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem is said to be a balanced intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem otherwise it is said to be an unbalanced intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem.

6.6 Proposed methods

In this section, to overcome the limitations of the methods, discussed in Section 6.2, a new method for solving intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problems as well as a new method for solving intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problems is proposed.

6.6.1 Proposed method for solving intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problems

In this section, a new method is proposed for finding the intuitionistic fuzzy optimal solution of such intuitionistic fully fuzzy single-objective capacitated solid minimum cost flow problems in which all the parameters are represented by LR flat intuitionistic fuzzy numbers.

The steps of the proposed method are as follows:

Step 1 Find $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i$, $\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{d}_j$ and $\sum\limits_{k\in S_C}\tilde{f}_k$. Let $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i=\{(\underline{m},\overline{m},m^L,m^R)_{LR};(\underline{m'},\overline{m'},m'^L,m'^R)_{LR}\}$, $\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{d}_j=\{(\underline{n},\overline{n},n^L,n^R)_{LR};(\underline{n'},\overline{n'},n'^L,n'^R)_{LR}\}$ and $\sum\limits_{k\in S_C}\tilde{f}_k=\{(\underline{f},\overline{f},f^L,f^R)_{LR};(\underline{f'},\overline{f'},f'^L,f'^R)_{LR}\}$. Examine that the problem is balanced or unbalanced.

Case (1) If the problem is balanced, i.e., $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i=\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{d}_j=\sum\limits_{k\in S_C}\tilde{f}_k$, then Go to Step 4.

Case (2) If the problem is unbalanced i.e., $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i=\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{d}_j\neq\sum\limits_{k\in S_C}\tilde{f}_k$ or $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i\neq\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{d}_j=\sum\limits_{k\in S_C}\tilde{f}_k$ or $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i=\sum\limits_{k\in S_C}\tilde{f}_k\neq\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{d}_j$ or $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i\neq\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{d}_j\neq\sum\limits_{k\in S_C}\tilde{f}_k$ then Go to Step 2.

Step 2 Check that $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i = \sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j$ or $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i \neq \sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j$.

Case (1) If $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i = \sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j$ then Go to Step 3.

Case (2) If $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i \neq \sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j$ then convert $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i \neq \sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j$ into $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i = \sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j$ as follows:

Case (2a) If $\underline{m'} - m'^L \leq \underline{n'} - n'^L$, $(\underline{m} - m^L) - (\underline{m'} - m'^L) \leq (\underline{n} - n^L) - (\underline{n'} - n'^L)$, $\underline{m'} - (\underline{m} - m^L) \leq \underline{n'} - (\underline{n} - n^L)$, $\underline{m} - \underline{m'} \leq \underline{n} - \underline{n'}$, $\overline{m} - \underline{m} \leq \overline{n} - \underline{n}$, $\overline{m'} - \overline{m} \leq \overline{n'} - \overline{n}$, $(\overline{m} + m^R) - \overline{m'} \leq (\overline{n} + n^R) - \overline{n'}$ and $(\overline{m'} + m'^R) - (\overline{m} + m^R) \leq (\overline{n'} + n'^R) - (\overline{n} + n^R)$ then introduce a dummy purely source node with intuitionistic fuzzy supply $\{(\underline{n} - \underline{m}, \overline{n} - \overline{m}, n^L - m^L, n^R - m^R)_{LR}; (\underline{n'} - \underline{m'}, \overline{n'} - \overline{m'}, n'^L - m'^L, n'^R - m'^R)_{LR}\}$ so that $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i = \sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j$. Go to Step 3.

Case (2b) If $\underline{m'} - m'^L \geq \underline{n'} - n'^L$, $(\underline{m} - m^L) - (\underline{m'} - m'^L) \geq (\underline{n} - n^L) - (\underline{n'} - n'^L)$, $\underline{m'} - (\underline{m} - m^L) \geq \underline{n'} - (\underline{n} - n^L)$, $\underline{m} - \underline{m'} \geq \underline{n} - \underline{n'}$, $\overline{m} - \underline{m} \geq \overline{n} - \underline{n}$, $\overline{m'} - \overline{m} \geq \overline{n'} - \overline{n}$, $(\overline{m} + m^R) - \overline{m'} \geq (\overline{n} + n^R) - \overline{n'}$ and $(\overline{m'} + m'^R) - (\overline{m} + m^R) \geq (\overline{n'} + n'^R) - (\overline{n} + n^R)$ then introduce a dummy purely destination node with intuitionistic fuzzy demand $\{(\underline{m} - \underline{n}, \overline{m} - \overline{n}, m^L - n^L, m^R - n^R)_{LR}; (\underline{m'} - \underline{n'}, \overline{m'} - \overline{n'}, m'^L - n'^L, m'^R - n'^R)_{LR}\}$ so that $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i = \sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j$. Go to Step 3.

Case (2c) If neither Case (2a) nor Case (2b) is satisfied then introduce a dummy purely source node with intuitionistic fuzzy supply $\{(\underline{C}, \overline{C}, C^L, C^R)_{LR};\ (\underline{C'}, \overline{C'}, C'^L, C'^R)_{LR}\}$ and also introduce a dummy destination node with intuitionistic fuzzy demand $\{(\underline{D}, \overline{D}, D^L, D^R)_{LR}; (\underline{D'}, \overline{D'}, D'^L, D'^R)_{LR}\}$ so that $\sum\limits_{i\in N_{PS}}\tilde{a}_i \oplus \sum\limits_{i\in N_S}\tilde{e}_i = \sum\limits_{j\in N_{PD}}\tilde{b}_j \oplus \sum\limits_{j\in N_D}\tilde{d}_j$. Go to Step 3.

where,

$\underline{C'} = \text{maximum}(0, [(\underline{n'} - n'^L) - (\underline{m'} - m'^L)]) + \text{maximum}(0, \{[(\underline{n} - n^L) - (\underline{n'} - n'^L)] -$

$[(\underline{m} - m^L) - (\underline{m'} - m'^L)]\})+\text{maximum}(0, [(\underline{n'} - (\underline{n} - n^L)) - (\underline{m'} - (\underline{m} - m^L)])$

$\underline{C} = \underline{C'}+\text{maximum}(0, [(\underline{n} - \underline{n'}) - (\underline{m} - \underline{m'})])$

$\overline{C} = \underline{C}+\text{maximum}(0, [(\overline{n} - \underline{n}) - (\overline{m} - \underline{m})])$

$\overline{C'} = \overline{C}+\text{maximum}(0, [(\overline{n'} - \overline{n}) - (\overline{m'} - \overline{m})])$

$C'^L = \text{maximum}(0, [(\underline{n} - n^L - \underline{n'} + n'^L) - (\underline{m} - m^L - \underline{m'} + m'^L)])+\text{maximum}(0, [(\underline{n'} - \underline{n} + n^L) - (\underline{m'} - \underline{m} + m^L)])$

$C^L = \text{maximum}(0, [(\underline{n'} - \underline{n} + n^L) - (\underline{m'} - \underline{m} + m^L)])+\text{maximum}(0, [(\underline{n} - \underline{n'}) - (\underline{m} - \underline{m'})])$

$C^R = \text{maximum}(0, [(\overline{n'} - \overline{n}) - (\overline{m'} - \overline{m})])+\text{maximum}(0, [(\overline{n} + n^R - \overline{n'}) - (\overline{m} + m^R - \overline{m'})])$

$C'^R = \text{maximum}(0, [(\overline{n} + n^R - \overline{n'}) - (\overline{m} + m^R - \overline{m'})])+\text{maximum}(0, [(\overline{n'} + n'^R - \overline{n} - n^R) - (\overline{m'} + m'^R - \overline{m} - m^R)])$ and

$\underline{D'} = \text{maximum}(0, [(\underline{m'} - m'^L) - (\underline{n'} - n'^L)]+\text{maximum}(0, [(\underline{m} - m^L) - (\underline{m'} - m'^L)] - [(\underline{n} - n^L) - (\underline{n'} - n'^L)])+\text{maximum}(0, [(\underline{m'} - (\underline{m} - m^L)) - (\underline{n'} - (\underline{n} - n^L)])$

$\underline{D} = \underline{D'}+\text{maximum}(0, [(\underline{m} - \underline{m'}) - (\underline{n} - \underline{n'})])$

$\overline{D} = \underline{D}+\text{maximum}(0, [(\overline{m} - \underline{m}) - (\overline{n} - \underline{n})])$

$\overline{D'} = \overline{D}+\text{maximum}(0, [(\overline{m'} - \overline{m}) - (\overline{n'} - \overline{n})])$

$D'^L = \text{maximum}(0, [(\underline{m} - m^L - \underline{m'} + m'^L) - (\underline{n} - n^L - \underline{n'} + n'^L)])+\text{maximum}(0, [(\underline{m'} - \underline{m} + m^L) - (\underline{n'} - \underline{n} + n^L)])$

$D^L = \text{maximum}(0, [(\underline{m'} - \underline{m} + m^L) - (\underline{n'} - \underline{n} + n^L)])+\text{maximum}(0, [(\underline{m} - \underline{m'}) - (\underline{n} - \underline{n'})])$

$D^R = \text{maximum}(0, [(\overline{m'} - \overline{m}) - (\overline{n'} - \overline{n})])+\text{maximum}(0, [(\overline{m} + m^R - \overline{m'}) - (\overline{n} + n^R - \overline{n'})])$

$D'^R = \text{maximum}(0, [(\overline{m} + m^R - \overline{m'}) - (\overline{n} + n^R - \overline{n'})])+\text{maximum}(0, [(\overline{m'} + m'^R - \overline{m} - m^R) - (\overline{n'} + n'^R - \overline{n} - n^R)])$

Step 3 Using Step 2, $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j$. Let $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{e}_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{d}_j = \{(\underline{g}, \overline{g}, g^L, g^R)_{LR}; (\underline{g'}, \overline{g'}, g'^L, g'^R)_{LR}\}$ and $\sum\limits_{k \in S_C} \tilde{f}_k = \{(\underline{f}, \overline{f}, f^L, f^R)_{LR}; (\underline{f'}, \overline{f'}, f'^L, f'^R)_{LR}\}$

Now check $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$ or $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j \neq \sum\limits_{k\in S_C} \tilde{f}_k$

Case (1) If $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$, then Go to Step 4.

Case (2) If $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j \neq \sum\limits_{k\in S_C} \tilde{f}_k$ then convert $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j \neq \sum\limits_{k\in S_C} \tilde{f}_k$ into $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$ as follows:

Case (2a) If $\underline{g'} - g'^L \le \underline{f'} - f'^L$, $(\underline{g} - g^L) - (\underline{g'} - g'^L) \le (\underline{f} - f^L) - (\underline{f'} - f'^L)$, $\underline{g'} - (\underline{g} - g^L) \le \underline{f'} - (\underline{f} - f^L)$, $\underline{g} - \underline{g'} \le \underline{f} - \underline{f'}$, $\overline{g} - \underline{g} \le \overline{f} - \underline{f}$, $\overline{g'} - \overline{g} \le \overline{f'} - \overline{f}$, $(\overline{g} + g^R) - \overline{g'} \le (\overline{f} + f^R) - \overline{f'}$ and $(\overline{g'} + g'^R) - (\overline{g} + g^R) \le (\overline{f'} + f'^R) - (\overline{f} + f^R)$ then check that in Step 2, a dummy purely source node is introduced or not and also check that a dummy purely destination node is introduced or not.

Case (i) If both the dummy purely source node and dummy purely destination node are introduced then increase both the intuitionistic fuzzy supply of the already introduced dummy purely source node and the intuitionistic fuzzy demand of the already introduced dummy purely destination node by the same intuitionistic fuzzy quantity $\{(\underline{f} - \underline{g}, \overline{f} - \overline{g}, f^L - g^L, f^R - g^R)_{LR}; (\underline{f'} - \underline{g'}, \overline{f'} - \overline{g'}, f'^L - g'^L, f'^R - g'^R)_{LR}\}$ so that $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$. Go to Step 4.

Case (ii) If a dummy purely source node is introduced but no dummy purely destination node is introduced then increase the intuitionistic fuzzy supply of the already introduced dummy purely source node by the intuitionistic fuzzy quantity $\{(\underline{f} - \underline{g}, \overline{f} - \overline{g}, f^L - g^L, f^R - g^R)_{LR}; (\underline{f'} - \underline{g'}, \overline{f'} - \overline{g'}, f'^L - g'^L, f'^R - g'^R)_{LR}\}$ and also introduce a dummy purely destination node with intuitionistic fuzzy demand $\{(\underline{f} - \underline{g}, \overline{f} - \overline{g}, f^L - g^L, f^R - g^R)_{LR}; (\underline{f'} - \underline{g'}, \overline{f'} - \overline{g'}, f'^L - g'^L, f'^R - g'^R)_{LR}\}$ so that $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \sum\limits_{j\in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j\in N_D} \tilde{d}_j = \sum\limits_{k\in S_C} \tilde{f}_k$. Go to Step 4.

Case (iii) If a dummy purely destination node is introduced but no dummy purely source node is introduced then increase the intuitionistic fuzzy demand of the already introduced dummy purely destination node by the intuitionistic fuzzy quantity $\{(\underline{f}-\underline{g},\overline{f}-\overline{g},f^L-g^L,f^R-g^R)_{LR};(\underline{f'}-\underline{g'},\overline{f'}-\overline{g'},f'^L-g'^L,f'^R-g'^R)_{LR}\}$ and also introduce a dummy purely source node with intuitionistic fuzzy supply $\{(\underline{f}-\underline{g},\overline{f}-\overline{g},f^L-g^L,f^R-g^R)_{LR};(\underline{f'}-\underline{g'},\overline{f'}-\overline{g'},f'^L-g'^L,f'^R-g'^R)_{LR}\}$ so that $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i=\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{d}_j=\sum\limits_{k\in S_C}\tilde{f}_k$. Go to Step 4.

Case (2b) If $\underline{g'}-g'^L\geq\underline{f'}-f'^L$, $(\underline{g}-g^L)-(\underline{g'}-g'^L)\geq(\underline{f}-f^L)-(\underline{f'}-f'^L)$, $\underline{g'}-(\underline{g}-g^L)\geq\underline{f'}-(\underline{f}-f^L)$, $\underline{g}-\underline{g'}\geq\underline{f}-\underline{f'}$, $\overline{g}-\underline{g}\geq\overline{f}-\underline{f}$, $\overline{g'}-\overline{g}\geq\overline{f'}-\overline{f}$, $(\overline{g}+g^R)-\overline{g'}\geq(\overline{f}+f^R)-\overline{f'}$ and $(\overline{g'}+g'^R)-(\overline{g}+m^R)\geq(\overline{f'}+f'^R)-(\overline{f}+f^R)$ then introduce a dummy conveyance with intuitionistic fuzzy capacity $\{(\underline{g}-\underline{f},\overline{g}-\overline{f},g^L-f^L,g^R-f^R)_{LR};(\underline{g'}-\underline{f'},\overline{g'}-\overline{f'},g'^L-f'^L,g'^R-f'^R)_{LR}\}$ so that $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{e}_i=\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{d}_j=\sum\limits_{k\in S_C}\tilde{f}_k$. Go to Step 4.

Case (2c) If neither Case (2a) nor Case (2b) is satisfied then check that in Step 2 a dummy purely source node is introduced or not and also check that a dummy purely destination node is introduced or not.

Case (i) If both the dummy purely source node and dummy purely destination node are introduced then increase both the intuitionistic fuzzy supply of the already introduced dummy purely source node and the intuitionistic fuzzy demand of the already introduced dummy purely destination node by the same intuitionistic fuzzy quantity $\{(\underline{I},\overline{I},I^L,I^R)_{LR};(\underline{I'},\overline{I'},I'^L,I'^R)_{LR}\}$ and also introduce a dummy purely conveyance with intuitionistic fuzzy capacity $\{(\underline{J},\overline{J},J^L,J^R)_{LR};(\underline{J'},\overline{J'},J'^L,J'^R)_{LR}\}$ so that $\sum\limits_{i\in N_{PS}}\tilde{a}_i\oplus\sum\limits_{i\in N_S}\tilde{a}'_i=\sum\limits_{j\in N_{PD}}\tilde{b}_j\oplus\sum\limits_{j\in N_D}\tilde{b}'_j=\sum\limits_{k\in S_C}\tilde{e}_k$. Go to Step 4.

where,

$$\underline{I'} = \text{maximum}(0, [(\underline{f'} - f'^L) - (\underline{g'} - g'^L)] + \text{maximum}(0, [(\underline{f} - f^L) - (\underline{f'} - f'^L)] - [(\underline{g} - g^L) - (\underline{g'} - g'^L)]) + \text{maximum}(0, [(\underline{f'} - (\underline{f} - f^L)) - (\underline{g'} - (\underline{g} - g^L)])$$

$$\underline{I} = \underline{I'} + \text{maximum}(0, [(\underline{f} - \underline{f'}) - (\underline{g} - \underline{g'})])$$

$$\overline{I} = \underline{I} + \text{maximum}(0, [(\overline{f} - \underline{f}) - (\overline{g} - \underline{g})])$$

$$\overline{I'} = \overline{I} + \text{maximum}(0, [(\overline{f'} - \overline{f}) - (\overline{g'} - \overline{g})])$$

$$I'^L = \text{maximum}(0, [(\underline{f} - f^L - \underline{f'} + f'^L) - (\underline{g} - g^L - \underline{g'} + g'^L)]) + \text{maximum}(0, [(\underline{f'} - \underline{f} + f^L) - (\underline{g'} - \underline{g} + g^L)])$$

$$I^L = \text{maximum}(0, [(\underline{f'} - \underline{f} + f^L) - (\underline{g'} - \underline{g} + g^L)]) + \text{maximum}(0, [(\underline{f} - \underline{f'}) - (\underline{g} - \underline{g'})])$$

$$I^R = \text{maximum}(0, [(\overline{f'} - \overline{f}) - (\overline{g'} - \overline{g})]) + \text{maximum}(0, [(\overline{f} + f^R - \overline{f'}) - (\overline{g} + g^R - \overline{g'})])$$

$$I'^R = \text{maximum}(0, [(\overline{f} + f^R - \overline{f'}) - (\overline{g} + g^R - \overline{g'})]) + \text{maximum}(0, [(\overline{f'} + f'^R - \overline{f} - f^R) - (\overline{g'} + g'^R - \overline{g} - g^R)]) \text{ and}$$

$$\underline{J'} = \text{maximum}(0, [(\underline{g'} - g'^L) - (\underline{f'} - f'^L)] + \text{maximum}(0, [(\underline{g} - g^L) - (\underline{g'} - g'^L)] - [(\underline{f} - f^L) - (\underline{f'} - f'^L)]) + \text{maximum}(0, [(\underline{g'} - (\underline{g} - g^L)) - (\underline{f'} - (\underline{f} - f^L)])$$

$$\underline{J} = \underline{J'} + \text{maximum}(0, [(\underline{g} - \underline{g'}) - (\underline{f} - \underline{f'})])$$

$$\overline{J} = \underline{J} + \text{maximum}(0, [(\overline{g} - \underline{g}) - (\overline{f} - \underline{f})])$$

$$\overline{J'} = \overline{J} + \text{maximum}(0, [(\overline{g'} - \overline{g}) - (\overline{f'} - \overline{f})])$$

$$J'^L = \text{maximum}(0, [(\underline{g} - g^L - \underline{g'} + g'^L) - (\underline{f} - f^L - \underline{f'} + f'^L)]) + \text{maximum}(0, [(\underline{g'} - \underline{g} + g^L) - (\underline{f'} - \underline{f} + f^L)])$$

$$J^L = \text{maximum}(0, [(\underline{g'} - \underline{g} + g^L) - (\underline{f'} - \underline{f} + f^L)]) + \text{maximum}(0, [(\underline{g} - \underline{g'}) - (\underline{f} - \underline{f'})])$$

$$J^R = \text{maximum}(0, [(\overline{g'} - \overline{g}) - (\overline{f'} - \overline{f})]) + \text{maximum}(0, [(\overline{g} + g^R - \overline{g'}) - (\overline{f} + f^R - \overline{f'})])$$

$$J'^R = \text{maximum}(0, [(\overline{g} + g^R - \overline{g'}) - (\overline{f} + f^R - \overline{f'})]) + \text{maximum}(0, [(\overline{g'} + g'^R - \overline{g} - g^R) - (\overline{f'} + f'^R - \overline{f} - f^R)])$$

Case (ii) If a dummy purely source node is introduced but no dummy purely destination node is introduced then increase the intuitionistic fuzzy supply of the already introduced dummy purely source node by the intuitionistic fuzzy quantity $\{(\underline{I}, \overline{I}, I^L, I^R)_{LR}; (\underline{I'}, \overline{I'}, I'^L, I'^R)_{LR}\}$ and introduce a dummy purely destination node with intuitionistic fuzzy demand $\{(\underline{I}, \overline{I}, I^L, I^R)_{LR}; (\underline{I'}, \overline{I'}, I'^L, I'^R)_{LR}\}$. Also, introduce a dummy conveyance with intuitionistic fuzzy capacity $\{(\underline{J}, \overline{J}, J^L, J^R)_{LR}; (\underline{J'}, \overline{J'}, J'^L, J'^R)_{LR}\}$ so that $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{a}'_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{b}'_j = \sum\limits_{k \in S_C} \tilde{e}_k$. Go to Step 4.

Case (iii) If a dummy purely destination node is introduced but no dummy purely source node is introduced then increase the intuitionistic fuzzy demand of the already introduced dummy purely destination node by the intuitionistic fuzzy quantity $\{(\underline{I}, \overline{I}, I^L, I^R)_{LR};\ (\underline{I'}, \overline{I'}, I'^L, I'^R)_{LR}\}$ and introduce a dummy purely source node with intuitionistic fuzzy supply $\{(\underline{I}, \overline{I}, I^L, I^R)_{LR}; (\underline{I'}, \overline{I'}, I'^L, I'^R)_{LR}\}$. Also, introduce a dummy conveyance with intuitionistic fuzzy capacity $\{(\underline{J}, \overline{J}, J^L, J^R)_{LR}; (\underline{J'}, \overline{J'}, J'^L, J'^R)_{LR}\}$ so that $\sum\limits_{i \in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i \in N_S} \tilde{a}'_i = \sum\limits_{j \in N_{PD}} \tilde{b}_j \oplus \sum\limits_{j \in N_D} \tilde{b}'_j = \sum\limits_{k \in S_C} \tilde{e}_k$. Go to Step 4.

Step 4 The balanced intuitionistic fully fuzzy single objective solid minimum cost flow problem, obtained by using Step 1 to Step 3, can be formulated into the intuitionistic fuzzy linear programming problem $(P_{6.1})$ by assuming the following intuitionistic fuzzy transportation cost as zero LR flat intuitionistic fuzzy numbers:

(i) If any dummy purely source node is introduced then assume the intuitionistic fuzzy transportation cost for transporting one unit quantity of the product from the introduced dummy purely source node to all purely destination nodes and all intermediate nodes by all conveyance as zero LR flat intuitionistic fuzzy number.

(ii) If any dummy purely destination node is introduced then assume the intuitionistic fuzzy transportation cost for transporting one unit quantity of the product from all purely source nodes and all intermediate nodes to the introduced dummy purely destination node by all conveyance as zero LR flat intuitionistic fuzzy number.

(iii) If any dummy conveyance is introduced then assume the intuitionistic fuzzy transportation cost for transporting one unit quantity of the product from all purely source nodes and intermediate nodes to all intermediate nodes and all purely destination nodes by introduced dummy conveyance as zero LR flat intuitionistic fuzzy number.

Step 5 Assuming $\tilde{c}_{ijk} = \{(\underline{c}_{ijk}, \overline{c}_{ijk}, c^L_{ijk}, c^R_{ijk})_{LR}; (\underline{c'}_{ijk}, \overline{c'}_{ijk}, {c'}^L_{ijk}, {c'}^R_{ijk})_{LR}\}$, $\tilde{x}_{ijk} = \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}^L_{ijk}, {x'}^R_{ijk})_{LR}\}$, $\tilde{a}_i = \{(\underline{a}_i, \overline{a}_i, a^L_i, a^R_i)_{LR}; (\underline{a'}_i, \overline{a'}_i, {a'}^L_i, {a'}^R_i)_{LR}\}$, $\tilde{e}_i = \{(\underline{e}_i, \overline{e}_i, e^L_i, e^R_i)_{LR}; (\underline{e'}_i, \overline{e'}_i, {e'}^L_i, {e'}^R_i)_{LR}\}$, $\tilde{b}_j = \{(\underline{b}_j, \overline{b}_j, b^L_j, b^R_j)_{LR};\ (\underline{b'}_j, \overline{b'}_j, {b'}^L_j, {b'}^R_j)_{LR}\}$, $\tilde{d}_j = \{(\underline{d}_j, \overline{d}_j, d^L_j, d^R_j)_{LR};\ (\underline{d'}_j, \overline{d'}_j, {d'}^L_j, {d'}^R_j)_{LR}\}$, $\tilde{f}_k = \{(\underline{f}_k, \overline{f}_k, f^L_k, f^R_k)_{LR}; (\underline{f'}_k, \overline{f'}_k, {f'}^L_k, {f'}^R_k)_{LR}\}$, $\tilde{l}_{ijk} = \{(\underline{l}_{ijk}, \overline{l}_{ijk}, l^L_{ijk}, l^R_{ijk})_{LR}; (\underline{l'}_{ijk}, \overline{l'}_{ijk}, {l'}^L_{ijk}, {l'}^R_{ijk})_{LR}\}$ and $\tilde{u}_{ijk} = \{(\underline{u}_{ijk}, \overline{u}_{ijk}, u^L_{ijk}, u^R_{ijk})_{LR}; (\underline{u'}_{ijk}, \overline{u'}_{ijk}, {u'}^L_{ijk}, {u'}^R_{ijk})_{LR}\}$ the intuitionistic fuzzy linear programming problem $(P_{6.1})$ can be written as:

Minimize $\sum\limits_{(i,j)\in A} \sum\limits_{k \in S_C} (\{(\underline{c}_{ijk}, \overline{c}_{ijk}, c^L_{ijk}, c^R_{ijk})_{LR}; (\underline{c'}_{ijk}, \overline{c'}_{ijk}, {c'}^L_{ijk}, {c'}^R_{ijk})_{LR}\} \otimes \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}^L_{ijk}, {x'}^R_{ijk})_{LR}\})$

subject to

$$\sum_{j:(i,j)\in A} \sum_{k \in S_C} \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}^L_{ijk}, {x'}^R_{ijk})_{LR}\} = \{(\underline{a}_i, \overline{a}_i, a^L_i, a^R_i)_{LR}; (\underline{a'}_i, \overline{a'}_i, {a'}^L_i, {a'}^R_i)_{LR}\} \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \sum_{k \in S_C} \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}^L_{ijk}, {x'}^R_{ijk})_{LR}\} = \sum_{j:(j,i)\in A} \sum_{k \in S_C} \{(\underline{x}_{jik}, \overline{x}_{jik}, x^L_{jik}, x^R_{jik})_{LR}; (\underline{x'}_{jik}, \overline{x'}_{jik}, {x'}^L_{jik}, {x'}^R_{jik})_{LR}\} \oplus \{(\underline{e}_i, \overline{e}_i, e^L_i, e^R_i)_{LR}; (\underline{e'}_i, \overline{e'}_i, {e'}^L_i, {e'}^R_i)_{LR}\} \qquad i \in N_S$$

$$\sum_{i:(i,j)\in A}\ \sum_{k\in S_C}\{(\underline{x}_{ijk},\overline{x}_{ijk},x^{L}_{ijk},x^{R}_{ijk})_{LR};(\underline{x'}_{ijk},\overline{x'}_{ijk},x'^{L}_{ijk},x'^{R}_{ijk})_{LR}\}=\{(\underline{b}_{j},\overline{b}_{j},b^{L}_{j},b^{R}_{j})_{LR};$$
$$(\underline{b'}_{j},\overline{b'}_{j},b'^{L}_{j},b'^{R}_{j})_{LR}\}\qquad j\in N_{PD}$$

$$\sum_{i:(i,j)\in A}\ \sum_{k\in S_C}\{(\underline{x}_{ijk},\overline{x}_{ijk},x^{L}_{ijk},x^{R}_{ijk})_{LR};(\underline{x'}_{ijk},\overline{x'}_{ijk},x'^{L}_{ijk},x'^{R}_{ijk})_{LR}\}=\sum_{i:(j,i)\in A}\ \sum_{k\in S_C}\{(\underline{x}_{jik},$$
$$\overline{x}_{jik},x^{L}_{jik},x^{R}_{jik})_{LR};(\underline{x'}_{jik},\overline{x'}_{jik},x'^{L}_{jik},x'^{R}_{jik})_{LR}\}\oplus\{(\underline{d}_{j},\overline{d}_{j},d^{L}_{j},d^{R}_{j})_{LR};(\underline{d'}_{j},\overline{d'}_{j},d'^{L}_{j},d'^{R}_{j})_{LR}\}$$
$$j\in N_{D}\quad (P_{6.3})$$

$$\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}\{(\underline{x}_{ijk},\overline{x}_{ijk},x^{L}_{ijk},x^{R}_{ijk})_{LR};(\underline{x'}_{ijk},\overline{x'}_{ijk},x'^{L}_{ijk},x'^{R}_{ijk})_{LR}\}=\sum_{j:(j,i)\in A}\ \sum_{k\in S_C}\{(\underline{x}_{jik},$$
$$\overline{x}_{jik},x^{L}_{jik},x^{R}_{jik})_{LR};(\underline{x'}_{jik},\overline{x'}_{jik},x'^{L}_{jik},x'^{R}_{jik})_{LR}\}\qquad i\in N_{T}$$

$$\sum_{(i,j)\in A}\{(\underline{x}_{ijk},\overline{x}_{ijk},x^{L}_{ijk},x^{R}_{ijk})_{LR};\ (\underline{x'}_{ijk},\overline{x'}_{ijk},x'^{L}_{ijk},x'^{R}_{ijk})_{LR}\}=\{(\underline{f}_{k},\overline{f}_{k},f^{L}_{k},f^{R}_{k})_{LR};\ (\underline{f'}_{k},$$
$$\overline{f'}_{k},f'^{L}_{k},f'^{R}_{k})_{LR}\}\qquad k\in S_C$$

$$\{(\underline{l}_{ijk},\overline{l}_{ijk},l^{L}_{ijk},l^{R}_{ijk})_{LR};(\underline{l'}_{ijk},\overline{l'}_{ijk},l'^{L}_{ijk},l'^{R}_{ijk})_{LR}\}\preceq\{(\underline{x}_{ijk},\overline{x}_{ijk},x^{L}_{ijk},x^{R}_{ijk})_{LR};(\underline{x'}_{ijk},\overline{x'}_{ijk}$$
$$x'^{L}_{ijk},x'^{R}_{ijk})_{LR}\}\preceq\{(\underline{u}_{ijk},\overline{u}_{ijk},u^{L}_{ijk},u^{R}_{ijk})_{LR};(\underline{u'}_{ijk},\overline{u'}_{ijk},u'^{L}_{ijk},u'^{R}_{ijk})_{LR}\}$$
$$\forall\ (i,j)\in A,\ k\in S_C$$

$\{(\underline{x}_{ijk},\overline{x}_{ijk},x^{L}_{ijk},x^{R}_{ijk})_{LR};(\underline{x'}_{ijk},\overline{x'}_{ijk},x'^{L}_{ijk},x'^{R}_{ijk})_{LR}\}$ is non-negative LR flat intuitionistic fuzzy number $\forall\ (i,j)\in A,\ k\in S_C$

Step 6 Using the arithmetic operations of LR flat intuitionistic fuzzy numbers, defined in Section 6.1.2, the balanced intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problem $(P_{6.3})$ can be written as:

Minimize $\displaystyle\sum_{(i,j)\in A}\sum_{k\in S_C}\{(\underline{c}_{ijk}\ \underline{x}_{ijk},\overline{c}_{ijk}\ \overline{x}_{ijk},\underline{c}_{ijk}x^{L}_{ijk}+c^{L}_{ijk}\underline{x}_{ijk}-c^{L}_{ijk}x^{L}_{ijk},\overline{c}_{ijk}x^{R}_{ijk}+c^{R}_{ijk}\overline{x}_{ijk}+$
$c^{R}_{ijk}x^{R}_{ijk})_{LR};(\underline{c'}_{ijk}\ \underline{x'}_{ijk},\overline{c'}_{ijk}\ \overline{x'}_{ijk},\underline{c'}_{ijk}x'^{L}_{ijk}+c'^{L}_{ijk}\underline{x'}_{ijk}-c'^{L}_{ijk}x'^{L}_{ijk},\overline{c'}_{ijk}x'^{R}_{ijk}+c'^{R}_{ijk}\overline{x'}_{ijk}+$
$c'^{R}_{ijk}x'^{R}_{ijk})_{LR}\}$

subject to

$$\{(\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}\underline{x}_{ijk},\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}\overline{x}_{ijk},\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}x^{L}_{ijk},\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}x^{R}_{ijk})_{LR}\ ;(\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}$$
$$\underline{x'}_{ijk},\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}\overline{x'}_{ijk},\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}x'^{L}_{ijk},\sum_{j:(i,j)\in A}\ \sum_{k\in S_C}x'^{R}_{ijk})_{LR}\}=\{(\underline{a}_{i},\overline{a}_{i},a^{L}_{i},a^{R}_{i})_{LR};(\underline{a'}_{i},$$
$$\overline{a'}_{i},a'^{L}_{i},a'^{R}_{i})_{LR}\}\qquad i\in N_{PS}$$

$$\{(\sum_{j:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk}, \sum_{j:(i,j)\in A}\sum_{k\in S_C}\overline{x}_{ijk}, \sum_{j:(i,j)\in A}\sum_{k\in S_C}x_{ijk}^L, \sum_{j:(i,j)\in A}\sum_{k\in S_C}x_{ijk}^R)_{LR}\,; (\sum_{j:(i,j)\in A}\sum_{k\in S_C}\underline{x'}_{ijk}, \sum_{j:(i,j)\in A}\sum_{k\in S_C}\overline{x'}_{ijk}, \sum_{j:(i,j)\in A}\sum_{k\in S_C}{x'}_{ijk}^L, \sum_{j:(i,j)\in A}\sum_{k\in S_C}{x'}_{ijk}^R)_{LR}\} = \{(\sum_{j:(j,i)\in A}\sum_{k\in S_C}\underline{x}_{jik}, \sum_{j:(j,i)\in A}\sum_{k\in S_C}\overline{x}_{jik}, \sum_{j:(j,i)\in A}\sum_{k\in S_C}x_{jik}^L, \sum_{j:(j,i)\in A}\sum_{k\in S_C}x_{jik}^R)_{LR}\,; (\sum_{j:(j,i)\in A}\sum_{k\in S_C}\underline{x'}_{jik}, \sum_{j:(j,i)\in A}\sum_{k\in S_C}\overline{x'}_{jik}, \sum_{j:(j,i)\in A}\sum_{k\in S_C}{x'}_{jik}^L, \sum_{j:(j,i)\in A}\sum_{k\in S_C}{x'}_{jik}^R)_{LR}\} \oplus \{(\underline{e}_i, \overline{e}_i, e_i^L, e_i^R)_{LR}; (\underline{e'}_i, \overline{e'}_i, {e'}_i^L, {e'}_i^R)_{LR}\} \qquad i \in N_S$$

$$\{(\sum_{i:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk}, \sum_{i:(i,j)\in A}\sum_{k\in S_C}\overline{x}_{ijk}, \sum_{i:(i,j)\in A}\sum_{k\in S_C}x_{ijk}^L, \sum_{i:(i,j)\in A}\sum_{k\in S_C}x_{ijk}^R)_{LR}\,; (\sum_{i:(i,j)\in A}\sum_{k\in S_C}\underline{x'}_{ijk}, \sum_{i:(i,j)\in A}\sum_{k\in S_C}\overline{x'}_{ijk}, \sum_{i:(i,j)\in A}\sum_{k\in S_C}{x'}_{ijk}^L, \sum_{i:(i,j)\in A}\sum_{k\in S_C}{x'}_{ijk}^R)_{LR}\} = \{(\underline{b}_j, \overline{b}_j, b_j^L, b_j^R)_{LR}; (\underline{b'}_j, \overline{b'}_j, {b'}_j^L, {b'}_j^R)_{LR}\} \qquad j \in N_{PD} \quad (P_{6.4})$$

$$\{(\sum_{i:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk}, \sum_{i:(i,j)\in A}\sum_{k\in S_C}\overline{x}_{ijk}, \sum_{i:(i,j)\in A}\sum_{k\in S_C}x_{ijk}^L, \sum_{i:(i,j)\in A}\sum_{k\in S_C}x_{ijk}^R)_{LR}\,; (\sum_{i:(i,j)\in A}\sum_{k\in S_C}\underline{x'}_{ijk}, \sum_{i:(i,j)\in A}\sum_{k\in S_C}\overline{x'}_{ijk}, \sum_{i:(i,j)\in A}\sum_{k\in S_C}{x'}_{ijk}^L, \sum_{i:(i,j)\in A}\sum_{k\in S_C}{x'}_{ijk}^R)_{LR}\} = \{(\sum_{i:(j,i)\in A}\sum_{k\in S_C}\underline{x}_{jik}, \sum_{i:(j,i)\in A}\sum_{k\in S_C}\overline{x}_{jik}, \sum_{i:(j,i)\in A}\sum_{k\in S_C}x_{jik}^L, \sum_{i:(j,i)\in A}\sum_{k\in S_C}x_{jik}^R)_{LR}\,; (\sum_{i:(j,i)\in A}\sum_{k\in S_C}\underline{x'}_{jik}, \sum_{i:(j,i)\in A}\sum_{k\in S_C}\overline{x'}_{jik}, \sum_{i:(j,i)\in A}\sum_{k\in S_C}{x'}_{jik}^L, \sum_{i:(j,i)\in A}\sum_{k\in S_C}{x'}_{jik}^R)_{LR}\} \oplus \{(\underline{d}_j, \overline{d}_j, d_j^L, d_j^R)_{LR}; (\underline{d'}_j, \overline{d'}_j, {d'}_j^L, {d'}_j^R)_{LR}\} \qquad j \in N_D$$

$$\{(\sum_{j:(i,j)\in A}\sum_{k\in S_C}\underline{x}_{ijk}, \sum_{j:(i,j)\in A}\sum_{k\in S_C}\overline{x}_{ijk}, \sum_{j:(i,j)\in A}\sum_{k\in S_C}x_{ijk}^L, \sum_{j:(i,j)\in A}\sum_{k\in S_C}x_{ijk}^R)_{LR}\,; (\sum_{j:(i,j)\in A}\sum_{k\in S_C}\underline{x'}_{ijk}, \sum_{j:(i,j)\in A}\sum_{k\in S_C}\overline{x'}_{ijk}, \sum_{j:(i,j)\in A}\sum_{k\in S_C}{x'}_{ijk}^L, \sum_{j:(i,j)\in A}\sum_{k\in S_C}{x'}_{ijk}^R)_{LR}\} = \{(\sum_{j:(j,i)\in A}\sum_{k\in S_C}\underline{x}_{jik}, \sum_{j:(j,i)\in A}\sum_{k\in S_C}\overline{x}_{jik}, \sum_{j:(j,i)\in A}\sum_{k\in S_C}x_{jik}^L, \sum_{j:(j,i)\in A}\sum_{k\in S_C}x_{jik}^R)_{LR}\,; (\sum_{j:(j,i)\in A}\sum_{k\in S_C}\underline{x'}_{jik}, \sum_{j:(j,i)\in A}\sum_{k\in S_C}\overline{x'}_{jik}, \sum_{j:(j,i)\in A}\sum_{k\in S_C}{x'}_{jik}^L, \sum_{j:(j,i)\in A}\sum_{k\in S_C}{x'}_{jik}^R)_{LR}\} \qquad i \in N_T$$

$$\{(\sum_{(i,j)\in A}\underline{x}_{ijk}, \sum_{(i,j)\in A}\overline{x}_{ijk}, \sum_{(i,j)\in A}x_{ijk}^L, \sum_{(i,j)\in A}x_{ijk}^R)_{LR}\,; (\sum_{(i,j)\in A}\underline{x'}_{ijk}, \sum_{(i,j)\in A}\overline{x'}_{ijk}, \sum_{(i,j)\in A}{x'}_{ijk}^L, \sum_{(i,j)\in A}{x'}_{ijk}^R)_{LR}\} = \{(\underline{f}_k, \overline{f}_k, f_k^L, f_k^R)_{LR}\,; \underline{f'}_k, \overline{f'}_k, {f'}_k^L, {f'}_k^R)_{LR})\} \qquad k \in S_C$$

$$\{(\underline{l}_{ijk}, \overline{l}_{ijk}, l_{ijk}^L, l_{ijk}^R)_{LR}; (\underline{l'}_{ijk}, \overline{l'}_{ijk}, {l'}_{ijk}^L, {l'}_{ijk}^R)_{LR}\} \preceq \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x_{ijk}^L, x_{ijk}^R)_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}_{ijk}^L, {x'}_{ijk}^R)_{LR}\} \preceq \{(\underline{u}_{ijk}, \overline{u}_{ijk}, u_{ijk}^L, u_{ijk}^R)_{LR}; (\underline{u'}_{ijk}, \overline{u'}_{ijk}, {u'}_{ijk}^L, {u'}_{ijk}^R)_{LR}\}$$

$$\forall\ (i,j) \in A,\ k \in S_C$$

$\{(\underline{x}_{ijk}, \overline{x}_{ijk}, x_{ijk}^L, x_{ijk}^R)_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}_{ijk}^L, {x'}_{ijk}^R)_{LR}\}$ is non-negative LR flat intuitionistic fuzzy number $\forall\ (i,j) \in A,\ k \in S_C$

Step 7 Using Definition 6.7 and Definition 6.9, the intuitionistic fuzzy single objective linear programming problem ($P_{6.4}$) can be written as:

$$\text{Minimize} \sum_{(i,j)\in A} \sum_{k\in S_C} \{(\underline{c}_{ijk}\ \underline{x}_{ijk}, \overline{c}_{ijk}\ \overline{x}_{ijk}, \underline{c}_{ijk}x^L_{ijk} + c^L_{ijk}\underline{x}_{ijk} - c^L_{ijk}x^L_{ijk}, \overline{c}_{ijk}x^R_{ijk} + c^R_{ijk}\overline{x}_{ijk} +$$

$$c^R_{ijk}x^R_{ijk})_{LR}; (\underline{c'}_{ijk}\ \underline{x'}_{ijk}, \overline{c'}_{ijk}\ \overline{x'}_{ijk}, \underline{c'}_{ijk}{x'}^L_{ijk} + {c'}^L_{ijk}\underline{x'}_{ijk} - {c'}^L_{ijk}{x'}^L_{ijk}, \overline{c'}_{ijk}{x'}^R_{ijk} + {c'}^R_{ijk}\overline{x'}_{ijk} +$$

$${c'}^R_{ijk}{x'}^R_{ijk})_{LR}\}$$

subject to

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} \underline{x}_{ijk} = \underline{a}_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} \overline{x}_{ijk} = \overline{a}_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} x^L_{ijk} = a^L_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} x^R_{ijk} = a^R_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} \underline{x'}_{ijk} = \underline{a'}_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} \overline{x'}_{ijk} = \overline{a'}_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} {x'}^L_{ijk} = {a'}^L_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} {x'}^R_{ijk} = {a'}^R_i \qquad i \in N_{PS}$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} \underline{x}_{ijk} = \sum_{j:(j,i)\in A} \sum_{k\in S_C} \underline{x}_{jik} + \underline{e}_i \qquad i \in N_S$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} \overline{x}_{ijk} = \sum_{j:(j,i)\in A} \sum_{k\in S_C} \overline{x}_{jik} + \overline{e}_i \qquad i \in N_S$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} x^L_{ijk} = \sum_{j:(j,i)\in A} \sum_{k\in S_C} x^L_{jik} + e^L_i \qquad i \in N_S$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} x^R_{ijk} = \sum_{j:(j,i)\in A} \sum_{k\in S_C} x^R_{jik} + e^R_i \qquad i \in N_S$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} \underline{x'}_{ijk} = \sum_{j:(j,i)\in A} \sum_{k\in S_C} \underline{x'}_{jik} + \underline{e'}_i \qquad i \in N_S$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} \overline{x'}_{ijk} = \sum_{j:(j,i)\in A} \sum_{k\in S_C} \overline{x'}_{jik} + \overline{e'}_i \qquad i \in N_S$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} {x'}^L_{ijk} = \sum_{j:(j,i)\in A} \sum_{k\in S_C} {x'}^L_{jik} + {e'}^L_i \qquad i \in N_S$$

$$\sum_{j:(i,j)\in A} \sum_{k\in S_C} {x'}^R_{ijk} = \sum_{j:(j,i)\in A} \sum_{k\in S_C} {x'}^R_{jik} + {e'}^R_i \qquad i \in N_S$$

$$\sum_{i:(i,j)\in A} \sum_{k\in S_C} \underline{x}_{ijk} = \underline{b}_j \qquad j \in N_{PD}$$

$$\sum_{i:(i,j)\in A} \sum_{k\in S_C} \overline{x}_{ijk} = \overline{b}_j \qquad j \in N_{PD}$$

$$\sum_{i:(i,j)\in A} \sum_{k\in S_C} x^L_{ijk} = b^L_j \qquad j \in N_{PD}$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} x^R_{ijk} = b^R_j \qquad j\in N_{PD}$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} \underline{x'}_{ijk} = \underline{b'}_j \qquad j\in N_{PD}$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} \overline{x'}_{ijk} = \overline{b'}_j \qquad j\in N_{PD}$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} x'^L_{ijk} = b'^L_j \qquad j\in N_{PD}$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} x'^R_{ijk} = b'^R_j \qquad j\in N_{PD} \tag{$P_{6.5}$}$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} \underline{x}_{ijk} = \sum_{i:(j,i)\in A}\sum_{k\in S_C} \underline{x}_{jik} + \underline{d}_j \qquad j\in N_D$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} \overline{x}_{ijk} = \sum_{i:(j,i)\in A}\sum_{k\in S_C} \overline{x}_{jik} + \overline{d}_j \qquad j\in N_D$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} x^L_{ijk} = \sum_{i:(j,i)\in A}\sum_{k\in S_C} x^L_{jik} + d^L_j \qquad j\in N_D$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} x^R_{ijk} = \sum_{i:(j,i)\in A}\sum_{k\in S_C} x^R_{jik} + d^R_j \qquad j\in N_D$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} \underline{x'}_{ijk} = \sum_{i:(j,i)\in A}\sum_{k\in S_C} \underline{x'}_{jik} + \underline{d'}_j \qquad j\in N_D$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} \overline{x'}_{ijk} = \sum_{i:(j,i)\in A}\sum_{k\in S_C} \overline{x'}_{jik} + \overline{d'}_j \qquad j\in N_D$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} x'^L_{ijk} = \sum_{i:(j,i)\in A}\sum_{k\in S_C} x'^L_{jik} + d'^L_j \qquad j\in N_D$$

$$\sum_{i:(i,j)\in A}\sum_{k\in S_C} x'^R_{ijk} = \sum_{i:(j,i)\in A}\sum_{k\in S_C} x'^R_{jik} + d'^R_j \qquad j\in N_D$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C} \underline{x}_{ijk} = \sum_{j:(j,i)\in A}\sum_{k\in S_C} \underline{x}_{jik} \qquad i\in N_T$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C} \overline{x}_{ijk} = \sum_{j:(j,i)\in A}\sum_{k\in S_C} \overline{x}_{jik} \qquad i\in N_T$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C} x^L_{ijk} = \sum_{j:(j,i)\in A}\sum_{k\in S_C} x^L_{jik} \qquad i\in N_T$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C} x^R_{ijk} = \sum_{j:(j,i)\in A}\sum_{k\in S_C} x^R_{jik} \qquad i\in N_T$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C} \underline{x'}_{ijk} = \sum_{j:(j,i)\in A}\sum_{k\in S_C} \underline{x'}_{jik} \qquad i\in N_T$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C} \overline{x'}_{ijk} = \sum_{j:(j,i)\in A}\sum_{k\in S_C} \overline{x'}_{jik} \qquad i\in N_T$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C} x'^L_{ijk} = \sum_{j:(j,i)\in A}\sum_{k\in S_C} x'^L_{jik} \qquad i\in N_T$$

$$\sum_{j:(i,j)\in A}\sum_{k\in S_C} x'^R_{ijk} = \sum_{j:(j,i)\in A}\sum_{k\in S_C} x'^R_{jik} \qquad i\in N_T$$

$$\sum_{(i,j)\in A} \underline{x}_{ijk} = \underline{f}_k \qquad k\in S_C$$

$$\sum_{(i,j)\in A} \overline{x}_{ijk} = \overline{f}_k \qquad k\in S_C$$

$$\sum_{(i,j)\in A} x^L_{ijk} = f^L_k \qquad k\in S_C$$

$$\sum_{(i,j)\in A} x^R_{ijk} = f^R_k \qquad k \in S_C$$

$$\sum_{(i,j)\in A} \underline{x'}_{ijk} = \underline{f'}_k \qquad k \in S_C$$

$$\sum_{(i,j)\in A} \overline{x'}_{ijk} = \overline{f'}_k \qquad k \in S_C$$

$$\sum_{(i,j)\in A} {x'}^L_{ijk} = {f'}^L_k \qquad k \in S_C$$

$$\sum_{(i,j)\in A} {x'}^R_{ijk}={f'}^R_k \qquad k \in S_C$$

$$\underline{x'}_{ijk} - {x'}^L_{ijk} \geq 0,\ (\underline{x}_{ijk} - x^L_{ijk}) - (\underline{x'}_{ijk} - {x'}^L_{ijk}) \geq 0,\ \underline{x'}_{ijk} - (\underline{x}_{ijk} - x^L_{ijk}) \geq 0,$$

$$\underline{x}_{ijk} - \underline{x'}_{ijk} \geq 0,\ \overline{x}_{ijk} - \underline{x}_{ijk} \geq 0,\ \overline{x'}_{ijk} - \overline{x}_{ijk} \geq 0, (\overline{x}_{ijk} + x^R_{ijk}) - \overline{x'}_{ijk} \geq 0,$$

$$(\overline{x'}_{ijk} + {x'}^R_{ijk}) - (\overline{x}_{ijk} + x^R_{ijk}) \geq 0 \quad \forall\ (i,j) \in A,\ k \in S_C$$

$$\left.\begin{array}{c} \{(\underline{l}_{ijk}, \overline{l}_{ijk}, l^L_{ijk}, l^R_{ijk})_{LR}; (\underline{l'}_{ijk}, \overline{l'}_{ijk}, {l'}^L_{ijk}, {l'}^R_{ijk})_{LR}\} \preceq \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}; \\ (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}^L_{ijk}, {x'}^R_{ijk})_{LR}\} \preceq \{(\underline{u}_{ijk}, \overline{u}_{ijk}, u^L_{ijk}, u^R_{ijk})_{LR}; (\underline{u'}_{ijk}, \overline{u'}_{ijk}, {u'}^L_{ijk}, \\ {u'}^R_{ijk})_{LR}\}, \qquad \forall\ (i,j) \in A,\ k \in S_C \end{array}\right\} \ (C_{6.1})$$

Step 8 Suppose the intuitionistic fuzzy linear programming problem $(P_{6.5})$ has f basic feasible solutions and $\{((\underline{x}_{ijk})^w, (\overline{x}_{ijk})^w, (x^L_{ijk})^w, (x^R_{ijk})^w)_{LR}; ((\underline{x'}_{ijk})^w, (\overline{x'}_{ijk})^w, ({x'}^L_{ijk})^w$ $, ({x'}^R_{ijk})^w)_{LR}\}$ be the w^{th} basic feasible solution then the goal is to find such a basic feasible solution corresponding to which the value of the objective function is minimum i.e, $\underset{1\leq w\leq f}{\text{minimum}}(\sum_{(i,j)\in A}\sum_{k\in S_C}\{(\underline{c}_{ijk}\ \underline{x}_{ijk}, \overline{c}_{ijk}\ \overline{x}_{ijk}, \underline{c}_{ijk}x^L_{ijk}+\underline{x}_{ijk}c^L_{ijk}-c^L_{ijk}x^L_{ijk}, \overline{c}_{ijk}x^R_{ijk}+ \overline{x}_{ijk}c^R_{ijk} + c^R_{ijk}x^R_{ijk})_{LR}; (\underline{c'}_{ijk}\ \underline{x'}_{ijk}, \overline{c'}_{ijk}\ \overline{x'}_{ijk}, \underline{c'}_{ijk}{x'}^L_{ijk} + \underline{x'}_{ijk}{c'}^L_{ijk} - {c'}^L_{ijk}{x'}^L_{ijk}, \overline{c'}_{ijk}{x'}^R_{ijk} + \overline{x'}_{ijk}{c'}^R_{ijk} + {c'}^R_{ijk}{x'}^R_{ijk})_{LR}\})$ which can be obtained by using the ranking approach proposed in Section 6.4 i.e. the intuitionistic fuzzy optimal solution of the intuitionistic fuzzy linear programming problem $(P_{6.5})$, can be obtained by solving the following crisp single-objective linear programming problem:

Minimize $\sum_{(i,j)\in A}\sum_{k\in S_C} M^{\beta,k'}_{\mu}(\{(\underline{c}_{ijk}\ \underline{x}_{ijk}, \overline{c}_{ijk}\ \overline{x}_{ijk}, \underline{c}_{ijk}x^L_{ijk} + \underline{x}_{ijk}c^L_{ijk} - c^L_{ijk}x^L_{ijk}, \overline{c}_{ijk}x^R_{ijk} + \overline{x}_{ijk}c^R_{ijk} + c^R_{ijk}x^R_{ijk})_{LR}; (\underline{c'}_{ijk}\ \underline{x'}_{ijk}, \overline{c'}_{ijk}\ \overline{x'}_{ijk}, \underline{c'}_{ijk}{x'}^L_{ijk} + \underline{x'}_{ijk}{c'}^L_{ijk} - {c'}^L_{ijk}{x'}^L_{ijk}, \overline{c'}_{ijk}{x'}^R_{ijk} + \overline{x'}_{ijk}{c'}^R_{ijk} + {c'}^R_{ijk}{x'}^R_{ijk})_{LR}\})$

subject to $\qquad (P_{6.6})$

$M_{\mu}^{\beta,k'}\{(\underline{l}_{ijk},\overline{l}_{ijk},l^L_{ijk},l^R_{ijk})_{LR};(\underline{l'}_{ijk},\overline{l'}_{ijk},{l'}^L_{ijk},{l'}^R_{ijk})_{LR}\} \leq M_{\mu}^{\beta,k'}\{(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR};$ $(\underline{x'}_{ijk},\overline{x'}_{ijk},{x'}^L_{ijk},{x'}^R_{ijk})_{LR}\} \leq M_{\mu}^{\beta,k'}\{(\underline{u}_{ijk},\overline{u}_{ijk},u^L_{ijk},u^R_{ijk})_{LR};(\underline{u'}_{ijk},\overline{u'}_{ijk},{u'}^L_{ijk},{u'}^R_{ijk})_{LR}\}$

as well as all the constraints of problem $(P_{6.5})$ except $(C_{6.1})$

Case (i) If there does not exist any alternative optimal solution then put the values of $\underline{x}_{ijk},\ \overline{x}_{ijk},\ x^L_{ijk},\ x^R_{ijk},\ \underline{x'}_{ijk},\ \overline{x'}_{ijk},\ {x'}^L_{ijk}$ and ${x'}^R_{ijk}$ in $\tilde{x}_{ijk}=\{(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR};$ $(\underline{x'}_{ijk},\overline{x'}_{ijk},{x'}^L_{ijk},{x'}^R_{ijk})_{LR}\}$ to find the intuitionistic fuzzy optimal solution $\{\tilde{x}_{ijk}\}$ and find the intuitionistic fuzzy optimal value $\sum\limits_{(i,j)\in A}\ \sum\limits_{k\in S_C}(\tilde{c}_{ijk}\otimes\tilde{x}_{ijk})$ by putting the values of $\tilde{x}_{ijk}$.

Case (ii) If alternative solution exist then Go to Step 9.

Step 9 Solve the crisp linear programming problem $(P_{6.7})$ to find the optimal solution $\{\underline{x}_{ijk},\ \overline{x}_{ijk},\ x^L_{ijk},\ x^R_{ijk},\ \underline{x'}_{ijk},\ \overline{x'}_{ijk},\ {x'}^L_{ijk},\ {x'}^R_{ijk}\}$.

Minimize $\sum\limits_{(i,j)\in A}\ \sum\limits_{k\in S_C}M_{\nu}^{\beta,k'}(\{(\underline{c}_{ijk}\ \underline{x}_{ijk},\overline{c}_{ijk}\ \overline{x}_{ijk},\underline{c}_{ijk}x^L_{ijk}+\underline{x}_{ijk}c^L_{ijk}-c^L_{ijk}x^L_{ijk},\overline{c}_{ijk}x^R_{ijk}+$ $\overline{x}_{ijk}c^R_{ijk}+c^R_{ijk}x^R_{ijk})_{LR};(\underline{c'}_{ijk}\ \underline{x'}_{ijk},\overline{c'}_{ijk}\ \overline{x'}_{ijk},\underline{c'}_{ijk}{x'}^L_{ijk}+\underline{x'}_{ijk}{c'}^L_{ijk}-{c'}^L_{ijk}{x'}^L_{ijk},\overline{c'}_{ijk}{x'}^R_{ijk}+$ $\overline{x'}_{ijk}{c'}^R_{ijk}+{c'}^R_{ijk}{x'}^R_{ijk})_{LR}\})$

subject to $(P_{6.7})$

$M_{\nu}^{\beta,k'}\{(\underline{l}_{ijk},\overline{l}_{ijk},l^L_{ijk},l^R_{ijk})_{LR};(\underline{l'}_{ijk},\overline{l'}_{ijk},{l'}^L_{ijk},{l'}^R_{ijk})_{LR}\} \leq M_{\nu}^{\beta,k'}\{(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR};$ $(\underline{x'}_{ijk},\overline{x'}_{ijk},{x'}^L_{ijk},{x'}^R_{ijk})_{LR}\} \leq M_{\nu}^{\beta,k'}\{(\underline{u}_{ijk},\overline{u}_{ijk},u^L_{ijk},u^R_{ijk})_{LR};(\underline{u'}_{ijk},\overline{u'}_{ijk},{u'}^L_{ijk},{u'}^R_{ijk})_{LR}\}$

$\sum\limits_{(i,j)\in A}\sum\limits_{k\in S_C}M_{\mu}^{\beta,k'}(\{(\underline{c}_{ijk}\ \underline{x}_{ijk},\overline{c}_{ijk}\ \overline{x}_{ijk},\underline{c}_{ijk}x^L_{ijk}+\underline{x}_{ijk}c^L_{ijk}-c^L_{ijk}x^L_{ijk},\overline{c}_{ijk}x^R_{ijk}+\overline{x}_{ijk}c^R_{ijk}+$ $c^R_{ijk}x^R_{ijk})_{LR};(\underline{c'}_{ijk}\ \underline{x'}_{ijk},\overline{c'}_{ijk}\ \overline{x'}_{ijk},\underline{c'}_{ijk}{x'}^L_{ijk}+\underline{x'}_{ijk}{c'}^L_{ijk}-{c'}^L_{ijk}{x'}^L_{ijk},\overline{c'}_{ijk}{x'}^R_{ijk}+\overline{x'}_{ijk}{c'}^R_{ijk}+$ ${c'}^R_{ijk}{x'}^R_{ijk})_{LR}\})=a$

as well as all the constraints of problem $(P_{6.5})$ except $(C_{6.1})$

where, a is the optimal value of the crisp linear programming problem $(P_{6.6})$.

Step 10 Put the values of $\underline{x}_{ijk},\ \overline{x}_{ijk},\ x^L_{ijk},\ x^R_{ijk},\ \underline{x'}_{ijk},\ \overline{x'}_{ijk},\ {x'}^L_{ijk}$ and ${x'}^R_{ijk}$ in $\tilde{x}_{ijk}=\{(\underline{x}_{ijk},\overline{x}_{ijk},x^L_{ijk},x^R_{ijk})_{LR};(\underline{x'}_{ijk},\overline{x'}_{ijk},{x'}^L_{ijk},{x'}^R_{ijk})_{LR}\}$ to find the intuitionistic fuzzy optimal solution $\{\tilde{x}_{ijk}\}$.

Step 11 Put the values of $\tilde{x}_{ijk}$, obtained from Step 10, in $\sum\limits_{(i,j)\in A}\sum\limits_{k\in S_C}(\tilde{c}_{ijk}\otimes\tilde{x}_{ijk})$, to find the minimum total intuitionistic fuzzy transportation cost.

6.6.2 Proposed method for solving intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problems

In this section, a new method is proposed for solving such intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problems in which all the parameters are represented by LR flat intuitionistic fuzzy numbers.

Step 1 Use Step 1 to Step 3 of the method, proposed in Section 6.6.1, for obtaining the balanced intuitionistic fully fuzzy multi-objective solid minimum cost flow problem.

Step 2 Formulate the chosen intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem into the intuitionistic fuzzy multi-objective linear programming problem ($P_{6.2}$) by assuming the following intuitionistic fuzzy penalty as zero LR flat intuitionistic fuzzy numbers:

(i) If any dummy purely source node is introduced then assume the intuitionistic fuzzy penalty for transporting one unit quantity of the product from the introduced dummy purely source node to all purely destination nodes and all intermediate nodes by all conveyance as zero LR flat intuitionistic fuzzy number.

(ii) If any dummy purely destination node is introduced then assume the intuitionistic fuzzy penalty for transporting one unit quantity of the product from all purely source nodes and all intermediate nodes to the introduced dummy purely destination node by all conveyance as zero LR flat intuitionistic fuzzy number.

(iii) If any dummy conveyance is introduced then assume the intuitionistic fuzzy penalty for transporting one unit quantity of the product from all purely source nodes and intermediate nodes to all intermediate nodes and all purely destination nodes by introduced dummy conveyance as zero LR flat intuitionistic fuzzy number.

Step 3 Use the method, proposed in Section 6.6.1, to find P intuitionistic fuzzy optimal solutions $\tilde{X}^1, \tilde{X}^2, \ldots, \tilde{X}^P$ of the problems $(P'_{6.1})$, $(P'_{6.2}), \ldots, (P'_{6.P})$ respectively.

$$\begin{aligned}
&\text{Minimize } \sum_{(i,j)\in A} \sum_{k\in S_C} (\tilde{c}^1_{ijk} \otimes \tilde{x}_{ijk}) \\
&\text{subject to} && (P'_{6.1}) \\
&\text{constraints of problem } (P_{6.1})
\end{aligned}$$

$$\begin{aligned}
&\text{Minimize } \sum_{(i,j)\in A} \sum_{k\in S_C} (\tilde{c}^2_{ijk} \otimes \tilde{x}_{ijk}) \\
&\text{subject to} && (P'_{6.2}) \\
&\text{constraints of problem } (P_{6.1})
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
&\text{Minimize } \sum_{(i,j)\in A} \sum_{k\in S_C} (\tilde{c}^P_{ijk} \otimes \tilde{x}_{ijk}) \\
&\text{subject to} && (P'_{6.P}) \\
&\text{constraints of problem } (P_{6.1})
\end{aligned}$$

Step 4 Find the value of each objective function corresponding to the each intuitionistic fuzzy optimal solution obtained in Step 3. Let the value of i^{th} objective function $\tilde{Z}_i$ corresponding to j^{th} intuitionistic fuzzy optimal solution $\tilde{X}^j$ is denoted by $\tilde{Z}_i(\tilde{X}^j)$.

Step 5 Find $U_i = \underset{1\leq j\leq P}{\text{maximum}}\{\Re(\tilde{Z}_i(\tilde{X}^j))\} \ \ \forall\ i = 1, 2, ..., P$

$$L_i = \underset{1\leq j\leq P}{\text{minimum}}\{\Re(\tilde{Z}_i(\tilde{X}^j))\} \ \ \forall\ i = 1, 2, ..., P$$

where, $\Re(\tilde{Z}_i(\tilde{X}^j)) = M_{\mu}^{\beta,k'}(\tilde{Z}_i(\tilde{X}^j))$ if no alternative solution exist for the i^{th} fuzzy linear programming problem $(P'_{6.i})$ otherwise $\Re(\tilde{Z}_i(\tilde{X}^j)) = M_{\nu}^{\beta,k'}(\tilde{Z}_i(\tilde{X}^j))$.

Step 6 Define the linear membership function $\mu^i(\Re(\tilde{Z}_i))\ \forall\ i = 1, 2, ..., P$.

$$\mu^i(\Re(\tilde{Z}_i)) = \begin{cases} 1, & \Re(\tilde{Z}_i) \leq L_i \\ 1 - \frac{\Re(\tilde{Z}_i)-L_i}{U_i-L_i}, & L_i \leq \Re(\tilde{Z}_i) \leq U_i \\ 0, & \Re(\tilde{Z}_i) \geq U_i \end{cases}$$

Step 7 Using the linear membership function, obtained in Step 6, the intuitionistic fuzzy multi-objective linear programming problem $(P_{6.2})$ can be converted into the following single objective crisp linear programming problem:

Maximize λ'

subject to $(P_{6.8})$

$\Re(\tilde{Z}_i) + \lambda'(U_i - L_i) \leq U_i, \quad i = 1, 2, ..., P,$

$\lambda' \geq 0$

as well as constraints of the problem $(P_{6.6})$

Step 8 Solve the crisp linear programming problem $(P_{6.8})$ to find the optimal values of $\underline{x}_{ijk},\ \overline{x}_{ijk},\ x^L_{ijk},\ x^R_{ijk},\ \underline{x'}_{ijk},\ \overline{x'}_{ijk},\ x'^L_{ijk},\ x'^R_{ijk}$.

Step 9 Put the value of $\underline{x}_{ijk},\ \overline{x}_{ijk},\ x^L_{ijk},\ x^R_{ijk},\ \underline{x'}_{ijk},\ \overline{x'}_{ijk},\ x'^L_{ijk}$ and x'^R_{ijk} in $\tilde{x}_{ijk} = \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, x'^L_{ijk}, x'^R_{ijk})_{LR}\}$ to find the intuitionistic fuzzy optimal solution $\{\tilde{x}_{ijk}\}$.

Step 10 Find the intuitionistic fuzzy optimal values of each function by putting the value of $\tilde{x}_{ijk}$ in $\sum\limits_{(i,j)\in A} \sum\limits_{k\in S_C} (\tilde{c}^{\eta}_{ijk} \otimes \tilde{x}_{ijk}) \ \ ; \eta = 1, 2, \ldots, P.$

6.7 Advantages of the proposed methods

The main advantage of the proposed methods over the existing methods [38,49] and the methods, proposed in previous chapters, is that all the problems which can be solved by the existing methods [38, 49] can also be solved by the methods proposed in this chapter. However, as discussed in Section 6.2, there exist several problems which can be solved by the methods proposed in this chapter but can neither be solved by the existing methods [38, 49] nor the methods proposed in previous chapters. To show the advantages of the proposed methods the intuitionistic fully fuzzy single and multi-objective capacitated solid minimum cost flow problems chosen in Example 6.1 and Example 6.2 which can not be solved by using the existing methods [38, 49] and the methods, proposed in previous chapters, are solved by proposed methods.

6.7.1 Intuitionistic fuzzy optimal solution of the chosen intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problem

The intuitionistic fuzzy optimal solution of intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problem, chosen in Example 6.1, can be obtained as follows:

Step 1 Total intuitionistic fuzzy supply $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i = \{(200,250,100,50)_{LR};(150,270,100,80)_{LR}\}$, total intuitionistic fuzzy demand $\sum\limits_{j\in N_D} \tilde{b}_j \oplus \sum\limits_{j\in N_{PD}} \tilde{d}_j = \{(150,250,100,50)_{LR};(100,270,100,130)_{LR}\}$ and total fuzzy capacity $\sum\limits_{k\in S_C} \tilde{f}_k = \{(200,300,100,50)_{LR};(150,320,100,130)_{LR}\}$. Since $\sum\limits_{i\in N_{PS}} \tilde{a}_i \oplus \sum\limits_{i\in N_S} \tilde{e}_i \neq \sum\limits_{j\in N_D} \tilde{b}_j \oplus \sum\limits_{j\in N_{PD}} \tilde{d}_j \neq \sum\limits_{k\in S_C} \tilde{f}_k$, so it is an unbalanced intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problem.

Step 2 Comparing $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i = \{(200,250,100,50)_{LR};(150,270,100,80)_{LR}\}$ by $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i = \{(\underline{m},\overline{m},m^L,m^R)_{LR};(\underline{m'},\overline{m'},m'^L,m'^R)_{LR}\}$ and $\sum_{j\in N_D} \tilde{b}_j \oplus \sum_{j\in N_{PD}} \tilde{d}_j$ $=\{(150,250,100,50)_{LR};(100,270,100,130)_{LR}\}$ by $\sum_{j\in N_{PD}} \tilde{b}_j \oplus \sum_{j\in N_D} \tilde{d}_j = \{(\underline{n},\overline{n},n^L,n^R)_{LR};(\underline{n'},\overline{n'},n'^L,n'^R)_{LR}\}$ the values of $\underline{m},\overline{m},m^L,m^R,\underline{m'},\overline{m'},m'^L,m'^R,\underline{n},\overline{n},n^L,n^R,\underline{n'},$ $\overline{n'},n'^L$ and n'^R are 200, 250, 100, 50, 150, 270, 100, 80, 150, 250, 100, 50, 100, 270, 100 and 130 respectively.

Since, $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i \neq \sum_{j\in N_{PD}} \tilde{b}_j \oplus \sum_{j\in N_{PD}} \tilde{d}_j$ and neither the condition $\underline{m'} - m'^L \leq \underline{n'} - n'^L$, $(\underline{m} - m^L) - (\underline{m'} - m'^L) \leq (\underline{n} - n^L) - (\underline{n'} - n'^L)$, $\underline{m'} - (\underline{m} - m^L) \leq \underline{n'} - (\underline{n} - n^L)$, $\underline{m} - \underline{m'} \leq \underline{n} - \underline{n'}$, $\overline{m} - \underline{m} \leq \overline{n} - \underline{n}$, $\overline{m'} - \overline{m} \leq \overline{n'} - \overline{n}$, $(\overline{m} + m^R) - \overline{m'} \leq (\overline{n} + n^R) - \overline{n'}$ and $(\overline{m'} + m'^R) - (\overline{m} + m^R) \leq (\overline{n'} + n'^R) - (\overline{n} + n^R)$ nor the condition $\underline{m'} - m'^L \geq \underline{n'} - n'^L$, $(\underline{m} - m^L) - (\underline{m'} - m'^L) \geq (\underline{n} - n^L) - (\underline{n'} - n'^L)$, $\underline{m'} - (\underline{m} - m^L) \geq \underline{n'} - (\underline{n} - n^L)$, $\underline{m} - \underline{m'} \geq \underline{n} - \underline{n'}$, $\overline{m} - \underline{m} \geq \overline{n} - \underline{n}$, $\overline{m'} - \overline{m} \geq \overline{n'} - \overline{n}$, $(\overline{m} + m^R) - \overline{m'} \geq (\overline{n} + n^R) - \overline{n'}$ and $(\overline{m'} + m'^R) - (\overline{m} + m^R) \geq (\overline{n'} + n'^R) - (\overline{n} + n^R)$ is satisfying so, as described in Case (2c) of Step 2 of the method, proposed in Section 6.6.1, there is need to introduce a dummy purely source node 4 with intuitionistic fuzzy supply $\tilde{a}_4 = \{(0,50,0,0)_{LR};(0,50,0,50)_{LR}\}$ and a dummy purely destination node 5 with intuitionistic fuzzy demand $\tilde{b}_5 = \{(50,50,0,0)_{LR};(50,50,0,0)_{LR}\}$ so that $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i = \sum_{j\in N_D} \tilde{b}_j \oplus \sum_{j\in N_{PD}} \tilde{d}_j$.

Step 3 Since, $\sum_{i\in N_{PS}} \tilde{a}_i \oplus \sum_{i\in N_S} \tilde{e}_i = \sum_{j\in N_D} \tilde{b}_j \oplus \sum_{j\in N_{PD}} \tilde{d}_j = \{(200,300,100,50)_{LR};(150,320,100,130)_{LR}\} = \sum_{k\in S_C} \tilde{e}_k$, so the intuitionistic fully fuzzy single objective solid minimum cost flow problem, obtained in Step 2, is a balanced intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problem.

Step 4 Since, a dummy purely source node (4) and a dummy purely destination node (5) are introduced. So, as described in Step 2 of the method, proposed in Section 6.6.1, by assuming $\tilde{c}_{4jk} = \tilde{c}_{i5k} = \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\}$ ∀ $i = 1,2,4;\quad j = 2,3,5;\quad k = 1,2$, the Intuitionistic fuzzy linear programming formulation of the balanced intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problem, obtained from Step 3, can be written as:

Minimize $(\{(100,1000,90,9000)_{LR}; (50,5500,45,9500)_{LR}\} \otimes \tilde{x}_{121} \oplus \{(400,4000,360, 36000)_{LR}; (200,22000,180,38000)_{LR}\} \otimes \tilde{x}_{131} \oplus \{(300,3000,270,27000)_{LR}; (150,16500, 135,28500)_{LR}\} \otimes \tilde{x}_{231} \oplus \{(50,200,48,1800)_{LR}; (10,1100,9,1900)_{LR}\} \otimes \tilde{x}_{122} \oplus \{(200, 2000,180,18000)_{LR}; (100,11000,90,19000)_{LR}\} \otimes \tilde{x}_{132} \oplus \{(500,5000,450,45000)_{LR}; (250,27500,225,47500)_{LR}\} \otimes \tilde{x}_{232} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{421} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{431} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{451} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{151} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{251} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{422} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{432} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{452} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{152} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{252})$

subject to

$\sum_{k=1}^{2} (\tilde{x}_{12k} \oplus \tilde{x}_{13k} \oplus \tilde{x}_{15k}) = \{(200,250,100,50)_{LR}; (150,270,100,80)_{LR}\}$

$\sum_{k=1}^{2} (\tilde{x}_{12k} \oplus \tilde{x}_{42k}) = \sum_{k=1}^{2} (\tilde{x}_{23k} \oplus \tilde{x}_{25k}) \oplus \{(100,150,80,50)_{LR}; (60,170,60,80)_{LR}\}$

$\sum_{k=1}^{2} (\tilde{x}_{13k} \oplus \tilde{x}_{23k} \oplus \tilde{x}_{43k}) = \{(50,100,20,0)_{LR}; (40,100,40,50)_{LR}\}$

$\sum_{k=1}^{2} (\tilde{x}_{42k} \oplus \tilde{x}_{43k} \oplus \tilde{x}_{45k}) = \{(0,50,0,0)_{LR}; (0,50,0,50)_{LR}\}$

$\sum_{k=1}^{2} (\tilde{x}_{15k} \oplus \tilde{x}_{25k} \oplus \tilde{x}_{45k}) = \{(50,50,0,0)_{LR}; (50,50,0,0)_{LR}\}$

$\tilde{x}_{121} \oplus \tilde{x}_{131} \oplus \tilde{x}_{151} \oplus \tilde{x}_{231} \oplus \tilde{x}_{251} \oplus \tilde{x}_{421} \oplus \tilde{x}_{431} \oplus \tilde{x}_{451} = \{(0,50,0,0)_{LR}; (0,50,0,50)_{LR}\}$

$\tilde{x}_{122} \oplus \tilde{x}_{132} \oplus \tilde{x}_{152} \oplus \tilde{x}_{232} \oplus \tilde{x}_{252} \oplus \tilde{x}_{422} \oplus \tilde{x}_{432} \oplus \tilde{x}_{452} = \{(200,250,100,50)_{LR}; (150, 270,100,80)_{LR}\}$

$\{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \preceq \tilde{x}_{121} \preceq \{(65,70,35,10)_{LR}; (60,75,45,10)_{LR}\}$

$\{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \preceq \tilde{x}_{131} \preceq \{(40,50,20,20)_{LR}; (30,60,20,40)_{LR}\}$

$\{(0,0,0,0)_{LR};(0,0,0,0)_{LR}\} \preceq \tilde{x}_{231} \preceq \{(150,220,100,40)_{LR};(100,240,80,60)_{LR}\}$

$\{(2,3,1,1)_{LR};(1,3,1,2)_{LR}\} \preceq \tilde{x}_{122} \preceq \{(100,150,80,150)_{LR};(60,170,60,180)_{LR}\}$

$\{(6,7,3,2)_{LR};(4,8,3,3)_{LR}\} \preceq \tilde{x}_{132} \preceq \{(150,220,100,40)_{LR};(100,240,80,60)_{LR}\}$

$\{(0,0,0,0)_{LR};(0,0,0,0)_{LR}\} \preceq \tilde{x}_{232} \preceq \{(50,55,42,10)_{LR};(13,60,9,11)_{LR}\}$

$\tilde{x}_{ijk}$ are non-negative LR flat intuitionistic fuzzy numbers $\forall\ i = 1,2,4;\ j = 2,3,5\ ;\ k = 1,2.$

Step 5 Using Step 6 and Step 7, of the method, proposed in Section 6.6.1, the intuitionistic fuzzy single-objective linear programming problem, obtained in Step 4, can be converted into the following single-objective intuitionistic fuzzy linear programming problem:

Minimize $\{(100\underline{x}_{121} + 400\underline{x}_{131} + 300\underline{x}_{231} + 50\underline{x}_{122} + 200\underline{x}_{132} + 500\underline{x}_{232}, 1000\overline{x}_{121} + 4000\overline{x}_{131} + 3000\overline{x}_{231} + 200\overline{x}_{122} + 2000\overline{x}_{132} + 5000\overline{x}_{232}, 90\underline{x}_{121} + 10x^L_{121} + 360\underline{x}_{131} + 40x^L_{131} + 270\underline{x}_{231} + 30x^L_{231} + 48\underline{x}_{122} + 2x^L_{122} + 180\underline{x}_{132} + 20x^L_{132} + 450\underline{x}_{232} + 50x^L_{232}, 9000\overline{x}_{121} + 10000x^R_{121} + 36000\overline{x}_{131} + 40000x^R_{131} + 27000\overline{x}_{231} + 30000x^R_{231} + 1800\overline{x}_{122} + 2000x^R_{122} + 18000\overline{x}_{132} + 20000x^R_{132} + 45000\overline{x}_{232} + 50000x^R_{232})_{LR};\ (50\underline{x'}_{121} + 200\underline{x'}_{131} + 150\underline{x'}_{231} + 10\underline{x'}_{122} + 100\underline{x'}_{132} + 250\underline{x'}_{232}, 5500\overline{x'}_{121} + 22000\overline{x'}_{131} + 16500\overline{x'}_{231} + 1100\overline{x'}_{122} + 11000\overline{x'}_{132} + 27500\overline{x'}_{232}, 45\underline{x'}_{121} + 5{x'}^L_{121} + 180\underline{x'}_{131} + 20{x'}^L_{131} + 135\underline{x'}_{231} + 15{x'}^L_{231} + 9\underline{x'}_{122} + {x'}^L_{122} + 90\underline{x'}_{132} + 10{x'}^L_{132} + 225\underline{x'}_{232} + 25{x'}^L_{232}, 9500\overline{x'}_{121} + 15000{x'}^R_{121} + 38000\overline{x'}_{131} + 60000{x'}^R_{131} + 28500\overline{x'}_{231} + 45000{x'}^R_{231} + 2900\overline{x'}_{122} + 3000{x'}^R_{122} + 19000\overline{x'}_{132} + 30000{x'}^R_{132} + 47500\overline{x'}_{232} + 75000{x'}^R_{232})_{LR}\}$

subject to

$\sum\limits_{k=1}^{2}(\underline{x}_{12k} + \underline{x}_{13k} + \underline{x}_{15k}) = 200,\qquad \sum\limits_{k=1}^{2}(\overline{x}_{12k} + \overline{x}_{13k} + \overline{x}_{15k}) = 250$

$\sum\limits_{k=1}^{2}(x^L_{12k} + x^L_{13k} + x^L_{15k}) = 100,\qquad \sum\limits_{k=1}^{2}(x^R_{12k} + x^R_{13k} + x^R_{15k}) = 50$

$\sum\limits_{k=1}^{2}(\underline{x'}_{12k} + \underline{x'}_{13k} + \underline{x'}_{15k}) = 150,\qquad \sum\limits_{k=1}^{2}(\overline{x'}_{12k} + \overline{x'}_{13k} + \overline{x'}_{15k}) = 270$

$$\sum_{k=1}^{2}(x'^{L}_{12k}+x'^{L}_{13k}+x'^{L}_{15k})=100, \qquad \sum_{k=1}^{2}(x'^{R}_{12k}+x'^{R}_{13k}+x'^{R}_{15k})=80$$

$$\sum_{k=1}^{2}(\underline{x}_{12k}+\underline{x}_{42k})=\sum_{k=1}^{2}(\underline{x}_{23k}+\underline{x}_{25k})+100$$

$$\sum_{k=1}^{2}(\overline{x}_{12k}+\overline{x}_{42k})=\sum_{k=1}^{2}(\overline{x}_{23k}+\overline{x}_{25k})+150$$

$$\sum_{k=1}^{2}(x^{L}_{12k}+x^{L}_{42k})=\sum_{k=1}^{2}(x^{L}_{23k}+x^{L}_{25k})+80$$

$$\sum_{k=1}^{2}(x^{R}_{12k}+x^{R}_{42k})=\sum_{k=1}^{2}(x^{R}_{23k}+x^{R}_{25k})+50$$

$$\sum_{k=1}^{2}(\underline{x'}_{12k}+\underline{x'}_{42k})=\sum_{k=1}^{2}(\underline{x'}_{23k}+\underline{x'}_{25k})+60$$

$$\sum_{k=1}^{2}(\overline{x'}_{12k}+\overline{x'}_{42k})=\sum_{k=1}^{2}(\overline{x'}_{23k}+\overline{x'}_{25k})+170$$

$$\sum_{k=1}^{2}(x'^{L}_{12k}+x'^{L}_{42k})=\sum_{k=1}^{2}(x'^{L}_{23k}+x'^{L}_{25k})+60$$

$$\sum_{k=1}^{2}(x'^{R}_{12k}+x'^{R}_{42k})=\sum_{k=1}^{2}(x'^{R}_{23k}+x'^{R}_{25k})+80$$

$$\sum_{k=1}^{2}(\underline{x}_{13k}+\underline{x}_{23k}+\underline{x}_{43k})=50, \qquad \sum_{k=1}^{2}(\overline{x}_{13k}+\overline{x}_{23k}+\overline{x}_{43k})=100$$

$$\sum_{k=1}^{2}(x^{L}_{13k}+x^{L}_{23k}+x^{L}_{43k})=20, \qquad \sum_{k=1}^{2}(x^{R}_{13k}+x^{R}_{23k}+x^{R}_{43k})=0$$

$$\sum_{k=1}^{2}(\underline{x'}_{13k}+\underline{x'}_{23k}+\underline{x'}_{43k})=40, \qquad \sum_{k=1}^{2}(\overline{x'}_{13k}+\overline{x'}_{23k}+\overline{x'}_{43k})=100$$

$$\sum_{k=1}^{2}(x'^{L}_{13k}+x'^{L}_{23k}+x'^{L}_{43k})=40, \qquad \sum_{k=1}^{2}(x'^{R}_{13k}+x'^{R}_{23k}+x'^{R}_{43k})=50$$

$$\sum_{k=1}^{2}(\underline{x}_{42k}+\underline{x}_{43k}+\underline{x}_{45k})=0, \qquad \sum_{k=1}^{2}(\overline{x}_{42k}+\overline{x}_{43k}+\overline{x}_{45k})=50$$

$$\sum_{k=1}^{2}(x^{L}_{42k}+x^{L}_{43k}+x^{L}_{45k})=0, \qquad \sum_{k=1}^{2}(x^{R}_{42k}+x^{R}_{43k}+x^{R}_{45k})=0$$

$$\sum_{k=1}^{2}(\underline{x'}_{42k}+\underline{x'}_{43k}+\underline{x'}_{45k})=0, \qquad \sum_{k=1}^{2}(\overline{x'}_{42k}+\overline{x'}_{43k}+\overline{x'}_{45k})=50$$

$$\sum_{k=1}^{2}(x'^{L}_{42k}+x'^{L}_{43k}+x'^{L}_{45k})=0, \qquad \sum_{k=1}^{2}(x'^{R}_{42k}+x'^{R}_{43k}+x'^{R}_{45k})=50$$

$$\sum_{k=1}^{2}(\underline{x}_{15k}+\underline{x}_{25k}+\underline{x}_{45k})=50, \qquad \sum_{k=1}^{2}(\overline{x}_{15k}+\overline{x}_{25k}+\overline{x}_{45k})=50$$

$$\sum_{k=1}^{2}(x^{L}_{15k}+x^{L}_{25k}+x^{L}_{45k})=0, \qquad \sum_{k=1}^{2}(x^{R}_{15k}+x^{R}_{25k}+x^{R}_{45k})=0$$

$$\sum_{k=1}^{2}(\underline{x'}_{15k}+\underline{x'}_{25k}+\underline{x'}_{45k})=50, \qquad \sum_{k=1}^{2}(\overline{x'}_{15k}+\overline{x'}_{25k}+\overline{x'}_{45k})=50$$

$$\sum_{k=1}^{2}(x'^{L}_{15k}+x'^{L}_{25k}+x'^{L}_{45k})=0, \qquad \sum_{k=1}^{2}(x'^{R}_{15k}+x'^{R}_{25k}+x'^{R}_{45k})=0 \tag{$P_{6.9}$}$$

$$\underline{x}_{121}+\underline{x}_{131}+\underline{x}_{151}+\underline{x}_{231}+\underline{x}_{251}+\underline{x}_{421}+\underline{x}_{431}+\underline{x}_{451}=0$$

$$\overline{x}_{121}+\overline{x}_{131}+\overline{x}_{151}+\overline{x}_{231}+\overline{x}_{251}+\overline{x}_{421}+\overline{x}_{431}+\overline{x}_{451}=50$$

$$x^L_{121}+x^L_{131}+x^L_{151}+x^L_{231}+x^L_{251}+x^L_{421}+x^L_{431}+x^L_{451}=0$$

$$x^R_{121}+x^R_{131}+x^R_{151}+x^R_{231}+x^R_{251}+x^R_{421}+x^R_{431}+x^R_{451}=0$$

$$\underline{x'}_{121}+\underline{x'}_{131}+\underline{x'}_{151}+\underline{x'}_{231}+\underline{x'}_{251}+\underline{x'}_{421}+\underline{x'}_{431}+\underline{x'}_{451}=0$$

$$\overline{x'}_{121}+\overline{x'}_{131}+\overline{x'}_{151}+\overline{x'}_{231}+\overline{x'}_{251}+\overline{x'}_{421}+\overline{x'}_{431}+\overline{x'}_{451}=50$$

$$x'^L_{121}+x'^L_{131}+x'^L_{151}+x'^L_{231}+x'^L_{251}+x'^L_{421}+x'^L_{431}+x'^L_{451}=0$$

$$x'^R_{121}+x'^R_{131}+x'^R_{151}+x'^R_{231}+x'^R_{251}+x'^R_{421}+x'^R_{431}+x'^R_{451}=50$$

$$\underline{x}_{122}+\underline{x}_{132}+\underline{x}_{152}+\underline{x}_{232}+\underline{x}_{252}+\underline{x}_{422}+\underline{x}_{432}+\underline{x}_{452}=200$$

$$\overline{x}_{122}+\overline{x}_{132}+\overline{x}_{152}+\overline{x}_{232}+\overline{x}_{252}+\overline{x}_{422}+\overline{x}_{432}+\overline{x}_{452}=250$$

$$x^L_{122}+x^L_{132}+x^L_{152}+x^L_{232}+x^L_{252}+x^L_{422}+x^L_{432}+x^L_{452}=100$$

$$x^R_{122}+x^R_{132}+x^R_{152}+x^R_{232}+x^R_{252}+x^R_{422}+x^R_{432}+x^R_{452}=50$$

$$\underline{x'}_{122}+\underline{x'}_{132}+\underline{x'}_{152}+\underline{x'}_{232}+\underline{x'}_{252}+\underline{x'}_{422}+\underline{x'}_{432}+\underline{x'}_{452}=150$$

$$\overline{x'}_{122}+\overline{x'}_{132}+\overline{x'}_{152}+\overline{x'}_{232}+\overline{x'}_{252}+\overline{x'}_{422}+\overline{x'}_{432}+\overline{x'}_{452}=270$$

$$x'^L_{122}+x'^L_{132}+x'^L_{152}+x'^L_{232}+x'^L_{252}+x'^L_{422}+x'^L_{432}+x'^L_{452}=100$$

$$x'^R_{122}+x'^R_{132}+x'^R_{152}+x'^R_{232}+x'^R_{252}+x'^R_{422}+x'^R_{432}+x'^R_{452}=80$$

$$\underline{x'}_{ijk}-x'^L_{ijk}\geq 0,\ (\underline{x}_{ijk}-x^L_{ijk})-(\underline{x'}_{ijk}-x'^L_{ijk})\geq 0,\ \underline{x'}_{ijk}-(\underline{x}_{ijk}-x^L_{ijk})\geq 0,$$

$$\underline{x}_{ijk}-\underline{x'}_{ijk}\geq 0,\ \overline{x}_{ijk}-\underline{x}_{ijk}\geq 0,\ \overline{x'}_{ijk}-\overline{x}_{ijk}\geq 0,\ (\overline{x}_{ijk}+x^R_{ijk})-\overline{x'}_{ijk}\geq 0,$$

$$(\overline{x'}_{ijk}+x'^R_{ijk})-(\overline{x}_{ijk}+x^R_{ijk})\geq 0\ \ \forall\ i=1,2,4;\ \ j=2,3,5;\ \ k=1,2$$

$$\left.\begin{array}{c}
\{(0,0,0,0)_{LR};(0,0,0,0)_{LR}\}\preceq\tilde{x}_{121}\preceq\{(65,70,35,10)_{LR};(60,75,45,10)_{LR}\}\\
\{(0,0,0,0)_{LR};(0,0,0,0)_{LR}\}\preceq\tilde{x}_{131}\preceq\{(40,50,20,20)_{LR};(30,60,20,40)_{LR}\}\\
\{(0,0,0,0)_{LR};(0,0,0,0)_{LR}\}\preceq\tilde{x}_{231}\preceq\{(150,220,100,40)_{LR};(100,240,80,60)_{LR}\}\\
\{(2,3,1,1)_{LR};(1,3,1,2)_{LR}\}\preceq\tilde{x}_{122}\preceq\{(100,150,80,150)_{LR};(60,170,60,180)_{LR}\}\\
\{(6,7,3,2)_{LR};(4,8,3,3)_{LR}\}\preceq\tilde{x}_{132}\preceq\{(150,220,100,40)_{LR};(100,240,80,60)_{LR}\}\\
\{(0,0,0,0)_{LR};(0,0,0,0)_{LR}\}\preceq\tilde{x}_{232}\preceq\{(50,55,42,10)_{LR};(13,60,9,11)_{LR}\}
\end{array}\right\}\ (C_{6.2})$$

Step 6 Using Step 8 of the method, proposed in Section 6.6.1 and assuming $\beta=\frac{1}{3}, k'=0$, the intuitionistic fuzzy optimal solution of the intuitionistic fuzzy linear programming problem, obtained in Step 5, can be obtained by solving the following crisp linear programming problem:

Minimize $\frac{1}{6}(210\underline{x}_{121}+12000\overline{x}_{121}+840\underline{x}_{131}+48000\overline{x}_{131}+630\underline{x}_{231}+36000\overline{x}_{231}+102\underline{x}_{122}+$ $2400\overline{x}_{122}+420\underline{x}_{132}+24000\overline{x}_{132}+1050\underline{x}_{232}+60\overline{x}_{232}-10x^L_{121}-40x^L_{131}-30x^L_{231}-2x^L_{122}-$ $20x^L_{132}-50x^L_{232}+10000x^R_{121}+40000x^R_{131}+30000x^R_{231}+2000x^R_{122}+20000x^R_{132}+50000x^R_{232})$

Subject to $(P_{6.10})$

$3\underline{x}_{121}+3\overline{x}_{121}-x^L_{121}+x^R_{121}\geq 0,\quad 3\underline{x}_{121}+3\overline{x}_{121}-x^L_{121}+x^R_{121}\leq 380$

$3\underline{x}131+3\overline{x}_{131}-x^L_{131}+x^R_{131}\geq 0,\quad 3\underline{x}_{131}+3\overline{x}_{131}-x^L_{131}+x^R_{131}\leq 270$

$3\underline{x}_{231}+3\overline{x}_{231}-x^L_{231}+x^R_{231}\geq 0,\quad 3\underline{x}_{231}+3\overline{x}_{231}-x^L_{231}+x^R_{231}\leq 1050$

$3\underline{x}_{122}+3\overline{x}_{122}-x^L_{122}+x^R_{122}\geq 15,\quad 3\underline{x}_{122}+3\overline{x}_{122}-x^L_{122}+x^R_{122}\leq 820$

$3\underline{x}_{132}+3\overline{x}_{132}-x^L_{132}+x^R_{132}\geq 38,\quad 3\underline{x}_{132}+3\overline{x}_{132}-x^L_{132}+x^R_{132}\leq 1050$

$3\underline{x}_{232}+3\overline{x}_{232}-x^L_{232}+x^R_{232}\geq 0,\quad 3\underline{x}_{232}+3\overline{x}_{232}-x^L_{232}+x^R_{232}\leq 283$

as well as all the constraints of problem $(P_{6.9})$ except $(C_{6.2})$

Step 7 Since, on solving the crisp linear programming problem $(P_{6.10})$, alternative optimal solutions are existing i.e. Case (ii) of Step 8 of the method, proposed in Section 6.6.1 is satisfied and the optimal value of the crisp linear programming problem $(P_{6.10})$ is $\frac{1690640}{6}$ so by using the Step 9 of the method, proposed in Section 6.6.1, the intuitionistic fuzzy optimal solution of the chosen intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problem can be obtained by solving the following crisp linear programming problem:

Minimize $\frac{1}{6}(105\underline{x'}_{121}+15000\overline{x'}_{121}+420\underline{x'}_{131}+60000\overline{x'}_{131}+315\underline{x'}_{231}+45000\overline{x'}_{231}+$ $21\underline{x'}_{122}+4000\overline{x'}_{122}+210\underline{x'}_{132}+30000\overline{x'}_{132}+525\underline{x'}_{232}+75000\overline{x'}_{232}-5x'^L_{121}-20x'^L_{131}-$ $15x'^L_{231}-x'^L_{122}-10x'^L_{132}-25x'^L_{232}+15000x'^R_{121}+60000x'^R_{131}+45000x'^R_{231}+3000x'^R_{122}+$ $36000x'^R_{132}+75000x'^R_{232})$

Subject to

$3\underline{x'}_{121} + 3\overline{x'}_{121} - {x'}^L_{121} + {x'}^R_{121} \geq 0, \quad 3\underline{x'}_{121} + 3\overline{x'}_{121} - {x'}^L_{121} + {x'}^R_{121} \leq 370$

$3\underline{x'}131 + 3\overline{x'}_{131} - {x'}^L_{131} + {x'}^R_{131} \geq 0, \quad 3\underline{x'}_{131} + 3\overline{x'}_{131} - {x'}^L_{131} + {x'}^R_{131} \leq 290$

$3\underline{x'}_{231} + 3\overline{x'}_{231} - {x'}^L_{231} + {x'}^R_{231} \geq 0, \quad 3\underline{x'}_{231} + 3\overline{x'}_{231} - {x'}^L_{231} + {x'}^R_{231} \leq 1000$

$3\underline{x'}_{122} + 3\overline{x'}_{122} - {x'}^L_{122} + {x'}^R_{122} \geq 13, \quad 3\underline{x'}_{122} + 3\overline{x'}_{122} - {x'}^L_{122} + {x'}^R_{122} \leq 810$

$3\underline{x'}_{132} + 3\overline{x'}_{132} - {x'}^L_{132} + {x'}^R_{132} \geq 36, \quad 3\underline{x'}_{132} + 3\overline{x'}_{132} - {x'}^L_{132} + {x'}^R_{132} \leq 1000$

$3\underline{x'}_{232} + 3\overline{x'}_{232} - {x'}^L_{232} + {x'}^R_{232} \geq 0, \quad 3\underline{x'}_{232} + 3\overline{x'}_{232} - {x'}^L_{232} + {x'}^R_{232} \leq 221$

$210\underline{x}_{121} + 12000\overline{x}_{121} + 840\underline{x}_{131} + 48000\overline{x}_{131} + 630\underline{x}_{231} + 36000\overline{x}_{231} + 102\underline{x}_{122} + 2400\overline{x}_{122} + 420\underline{x}_{132} + 24000\overline{x}_{132} + 1050\underline{x}_{232} + 60\overline{x}_{232} - 10x^L_{121} - 40x^L_{131} - 30x^L_{231} - 2x^L_{122} - 20x^L_{132} - 50x^L_{232} + 10000x^R_{121} + 40000x^R_{131} + 30000x^R_{231} + 2000x^R_{122} + 20000x^R_{132} + 50000x^R_{232} = 1690640$

as well as all the constraints of problem $(P_{6.9})$ except $(C_{6.2})$

Step 8 Solving the crisp linear programming problem, obtained in Step 7, the optimal values of $\underline{x}_{122}, \overline{x}_{122}, x^L_{122}, x^R_{122}, \underline{x'}_{122}, \overline{x'}_{122}, {x'}^L_{122}, {x'}^R_{122}, \underline{x}_{132}, \overline{x}_{132}, x^L_{132}, \underline{x'}_{132}, \overline{x'}_{132}, {x'}^L_{132}, \underline{x}_{152}, \overline{x}_{152}, \underline{x'}_{152}, \overline{x'}_{152}, \overline{x}_{431}, \overline{x'}_{431}$ and ${x'}^R_{431}$ are $100, 150, 80, 50, 60, 170, 60, 80, 50, 50, 20, 40, 50, 40, 50, 50, 50, 50, 50, 50$ and 50 respectively.

Step 9 Putting the values of $\underline{x}_{ijk}$, $\overline{x}_{ijk}$, x^L_{ijk}, x^R_{ijk}, $\underline{x'}_{ijk}$, $\overline{x'}_{ijk}$, ${x'}^L_{ijk}$ and ${x'}^R_{ijk}$ in $\tilde{x}_{ijk} = \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}^L_{ijk}, {x'}^R_{ijk})_{LR}\}$ the intuitionistic fuzzy optimal solution is $\tilde{x}_{122} = \{(100, 150, 80, 50)_{LR}; (60, 170, 60, 80)_{LR}\}$, $\tilde{x}_{132} = \{(50, 50, 20, 0)_{LR}; (40, 50, 40, 0)_{LR}\}$, $\tilde{x}_{152} = \{(50, 50, 0, 0)_{LR}; (50, 50, 0, 0)_{LR}\}$, $\tilde{x}_{431} = \{(0, 50, 0, 0)_{LR}; (0, 50, 0, 50)_{LR}\}$ and the remaining $\tilde{x}_{ijk}$ are $\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}$.

Step 10 Putting the values of $\tilde{x}_{ijk} = \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}^L_{ijk}, {x'}^R_{ijk})_{LR}\}$ in $(\{(100, 1000, 90, 9000)_{LR};\ (50, 5500, 45, 9500)_{LR}\} \otimes \tilde{x}_{121} \oplus\ \{(400, 4000, 360, 36000)_{LR}; (200, 22000, 180, 38000)_{LR}\} \otimes \tilde{x}_{131} \oplus \{(300, 3000, 270, 27000)_{LR}; (150, 16500, 135, 28500)_{LR}\} \otimes \tilde{x}_{231} \oplus \{(50, 200, 48, 1800)_{LR}; (10, 1100, 9, 1900)_{LR}\} \otimes \tilde{x}_{122} \oplus \{(200,$

$2000, 180, 18000)_{LR}; (100, 11000, 90, 19000)_{LR}\} \otimes \tilde{x}_{132} \oplus \{(500, 5000, 450, 45000)_{LR}; (250, 27500, 225, 47500)_{LR}\} \otimes \tilde{x}_{232} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{421} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{431} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{451} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{151} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{251} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{422} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{432} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{452} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{152} \oplus \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \otimes \tilde{x}_{252})$ the minimum total intuitionistic fuzzy transportation cost is $\{(15000, 130000, 14360, 13870000)_{LR}; (4600, 737000, 4600, 1513000)_{LR}\}$.

6.7.2 Intuitionistic fuzzy optimal compromise solution of the chosen intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem

The intuitionistic fuzzy optimal compromise solution of intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem, chosen in Example 6.2, can be obtained as follows:

Step 1 Using Step 1 of the method, proposed in Section 6.6.2, the chosen unbalanced intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem is converted into a balanced intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem.

Step 2 Since, a dummy purely source node (4) and a dummy purely destination node (5) are introduced. So, as described in Step 2 of the method, proposed in Section 6.6.2, by assuming $\tilde{c}^1_{4jk} = \tilde{c}^1_{i5k} = \tilde{c}^2_{4jk} = \tilde{c}^2_{i5k} = \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\}$ $\forall\ i = 1,2,4;\quad j = 2,3,5;\quad k = 1,2$, the intuitionistic fuzzy linear programming formulation of the balanced intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem, obtained from Step 1, can be written as:

Minimize $\tilde{Z}_1 = (\{(100, 1000, 90, 9000)_{LR}; (50, 5500, 45, 9500)_{LR}\} \otimes \tilde{x}_{121} \oplus \{(400, 4000, 360, 36000)_{LR}; (200, 22000, 180, 38000)_{LR}\} \otimes \tilde{x}_{131} \oplus \{(300, 3000, 270, 27000)_{LR}; (150, 16500, 135, 28500)_{LR}\} \otimes \tilde{x}_{231} \oplus \{(50, 200, 48, 1800)_{LR}; (10, 1100, 9, 1900)_{LR}\} \otimes \tilde{x}_{122} \oplus \{(200, 2000, 180, 18000)_{LR}; (100, 11000, 90, 19000)_{LR}\} \otimes \tilde{x}_{132} \oplus \{(500, 5000, 450, 45000)_{LR}; (250, 27500, 225, 47500)_{LR}\} \otimes \tilde{x}_{232} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{421} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{431} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{451} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{151} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{251} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{422} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{432} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{452} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{152} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{252})$

Minimize $\tilde{Z}_2 = (\{(150, 950, 141, 8050)_{LR}; (45, 5000, 41, 11000)_{LR}\} \otimes \tilde{x}_{121} \oplus \{(500, 5000, 450, 45000)_{LR}; (250, 27500, 225, 47500)_{LR}\} \otimes \tilde{x}_{131} \oplus \{(400, 4000, 360, 36000)_{LR}; (200, 22000, 180, 38000)_{LR}\} \otimes \tilde{x}_{231} \oplus \{(45, 150, 42, 1950)_{LR}; (11, 1000, 9, 1900)_{LR}\} \otimes \tilde{x}_{122} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{132} \oplus \{(200, 2000, 180, 18000)_{LR}; (100, 10000, 90, 20000)_{LR}\} \otimes \tilde{x}_{232} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{421} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{431} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{451} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{151} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{251} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{422} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{432} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{452} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{152} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{252})$

subject to

all the constraints of problem ($P_{6.9}$)

Step 3 Using the method, proposed in Section 6.6.1, the intuitionistic fuzzy optimal solutions of the problems $P_{6.11}$ and $P_{6.12}$ are $\tilde{X}^1$ and $\tilde{X}^2$ respectively.

Minimize $\tilde{Z}_1 = (\{(100, 1000, 90, 9000)_{LR}; (50, 5500, 45, 9500)_{LR}\} \otimes \tilde{x}_{121} \oplus \{(400, 4000, 360, 36000)_{LR}; (200, 22000, 180, 38000)_{LR}\} \otimes \tilde{x}_{131} \oplus \{(300, 3000, 270, 27000)_{LR}; (150,$

$$16500, 135, 28500)_{LR}\} \otimes \tilde{x}_{231} \oplus \{(50, 200, 48, 1800)_{LR}; (10, 1100, 9, 1900)_{LR}\} \otimes \tilde{x}_{122} \oplus \{(200, 2000, 180, 18000)_{LR}; (100, 11000, 90, 19000)_{LR}\} \otimes \tilde{x}_{132} \oplus \{(500, 5000, 450, 45000)_{LR}; (250, 27500, 225, 47500)_{LR}\} \otimes \tilde{x}_{232} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{421} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{431} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{451} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{151} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{251} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{422} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{432} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{452} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{152} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{252})$$

subject to $(P_{6.11})$

all the constraints of problem $(P_{6.9})$

and

Minimize $\tilde{Z}_2 = (\{(150, 950, 141, 8050)_{LR}; (45, 5000, 41, 11000)_{LR}\} \otimes \tilde{x}_{121} \oplus \{(500, 5000, 450, 45000)_{LR}; (250, 27500, 225, 47500)_{LR}\} \otimes \tilde{x}_{131} \oplus \{(400, 4000, 360, 36000)_{LR}; (200, 22000, 180, 38000)_{LR}\} \otimes \tilde{x}_{231} \oplus \{(45, 150, 42, 1950)_{LR}; (11, 1000, 9, 1900)_{LR}\} \otimes \tilde{x}_{122} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{132} \oplus \{(200, 2000, 180, 18000)_{LR}; (100, 10000, 90, 20000)_{LR}\} \otimes \tilde{x}_{232} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{421} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{431} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{451} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{151} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{251} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{422} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{432} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{452} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{152} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{252})$

subject to $(P_{6.12})$

all the constraints of problem $(P_{6.9})$

$$\tilde{X}^1 = \begin{cases} \tilde{x}_{122} = \{(100, 150, 80, 50)_{LR}; (60, 170, 60, 80)_{LR}\}, \\ \tilde{x}_{132} = \{(50, 50, 20, 0)_{LR}; (40, 50, 40, 0)_{LR}\}, \\ \tilde{x}_{431} = \{(0, 50, 0, 0)_{LR}; (0, 50, 0, 50)_{LR}\}, \\ \tilde{x}_{152} = \{(50, 50, 0, 0)_{LR}; (50, 50, 0, 0)_{LR}\} \\ \text{and the remaining } \ \tilde{x}_{ijk} \ \text{ are } \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \end{cases}$$

and

$$\tilde{X}^2 = \begin{cases} \tilde{x}_{122} = \{(100,100,80,50)_{LR}; (60,120,60,30)_{LR}\}, \\ \tilde{x}_{132} = \{(50,100,20,0)_{LR}; (40,100,40,50)_{LR}\}, \\ \tilde{x}_{421} = \{(0,50,0,0)_{LR}; (0,50,0,50)_{LR}\}, \\ \tilde{x}_{152} = \{(50,50,0,0)_{LR}; (50,50,0,0)_{LR}\} \\ \text{and the remaining } \ \tilde{x}_{ijk} \ \text{ are } \{(0,0,0,0)_{LR}; (0,0,0,0)_{LR}\} \end{cases}$$

Step 4 Using Step 4 of the method, proposed in Section 6.6.2, values of $\tilde{Z}_1(\tilde{X}_1)$, $\tilde{Z}_1(\tilde{X}_2)$, $\tilde{Z}_2(\tilde{X}_1)$ and $\tilde{Z}_2(\tilde{X}_2)$ are $\{(15000,130000,14360,13870000)_{LR}; (4600,737000,4600,1513000)_{LR}\}$, $\{(15000,220000,14360,2080000)_{LR}; (4600,1232000,4600,3718000)_{LR}\}$, $\{(4500,22500,4440,397500)_{LR}; (660,170000,660,555000)_{LR}\}$ and $\{(4500,15000,4440,300000)_{LR}; (660,120000,660,315000)_{LR}\}$ respectively.

Step 5 Using Step 5 of the method, proposed in Section 6.6.2, values of U_1, U_2, L_1 and L_2 are $7423200, 1066320, 3733200$ and 676320 respectively.

Step 6 Using Step 6 of the methods, proposed in Section 6.6.2, the linear membership function $\mu^1(M_\nu^{\beta,k'}(\tilde{Z}_1))$ and $\mu^2(M_\nu^{\beta,k'}(\tilde{Z}_2))$ are as follows:

$$\mu^1(M_\nu^{\beta,k'}(\tilde{Z}_1)) = \begin{cases} 1, & M_\nu^{\beta,k'}(\tilde{Z}_1) \leq 3733200 \\ 1 - \frac{M_\nu^{\beta,k'}(\tilde{Z}_1) - 3733200}{3690000}, & 3733200 \leq M_\nu^{\beta,k'}(\tilde{Z}_1) \leq 7423200 \\ 0, & M_\nu^{\beta,k'}(\tilde{Z}_1) \geq 7423200 \end{cases}$$

$$\mu^2(M_\nu^{\beta,k'}(\tilde{Z}_2)) = \begin{cases} 1, & M_\nu^{\beta,k'}(\tilde{Z}_2) \leq 676320 \\ 1 - \frac{M_\nu^{\beta,k'}(\tilde{Z}_2) - 676320}{390000}, & 676320 \leq M_\nu^{\beta,k'}(\tilde{Z}_2) \leq 1066320 \\ 0, & M_\nu^{\beta,k'}(\tilde{Z}_2) \geq 1066320 \end{cases}$$

Step 7 Using the linear membership functions, obtained from Step 6, the intuitionistic fuzzy multi-objective linear programming problem, obtained in Step 2, can be converted into the following single objective linear programming problem:

Maximize λ'

subject to

$M_\nu^{\beta,k'}(\tilde{Z}_1) + 3690000\lambda' \leq 7423200$

$M_{\nu}^{\beta,k'}(\tilde{Z}_2) + 390000\lambda' \leq 1066320$

$\lambda' \geq 0$

as well as all the constraints of problem $(P_{6.10})$

Step 8 Solving the crisp linear programming problem, obtained in Step 7, the optimal values of $\underline{x}_{122}, \overline{x}_{122}, x^L_{122}, x^R_{122}, \underline{x'}_{122}, \overline{x'}_{122}, {x'}^L_{122}, {x'}^R_{122}, \underline{x}_{132}, \overline{x}_{132}, x^L_{132}, \underline{x'}_{132}, \overline{x'}_{132}, {x'}^L_{132}, {x'}^R_{132}, \underline{x}_{152}, \overline{x}_{152}, \underline{x'}_{152}, \overline{x'}_{152}, \overline{x}_{431}, \overline{x'}_{431}, \overline{x}_{421}, \overline{x'}_{421}$ and ${x'}^R_{421}$ are $100, 139.61, 80, 50, 60, 159.61, 60, 30, 50, 60.39, 20, 40, 60.39, 40, 50, 50, 50, 50, 50, 39.61, 39.61, 10.39, 10.39$ and 50 respectively.

Step 9 Putting the values of $\underline{x}_{ijk}$, $\overline{x}_{ijk}$, x^L_{ijk}, x^R_{ijk}, $\underline{x'}_{ijk}$, $\overline{x'}_{ijk}$, ${x'}^L_{ijk}$ and ${x'}^R_{ijk}$ in $\tilde{x}_{ijk} = \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}^L_{ijk}, {x'}^R_{ijk})_{LR}\}$ the intuitionistic fuzzy optimal solution is $\tilde{x}_{122} = \{(100, 139.61, 80, 50)_{LR}; (60, 159.61, 60, 30)_{LR}\}$, $\tilde{x}_{132} = \{(50, 60.39, 20, 0)_{LR}; (40, 60.39, 40, 50)_{LR}\}$, $\tilde{x}_{152} = \{(50, 50, 0, 0)_{LR}; (50, 50, 0, 0)_{LR}\}$, $\tilde{x}_{431} = \{(0, 39.61, 0, 0)_{LR};\ (0, 39.61, 0, 0)_{LR}\}$, $\tilde{x}_{421} = \{(0, 10.39, 0, 0)_{LR};\ (0, 10.39, 0, 50)_{LR}\}$ and the remaining $\tilde{x}_{ijk}$ are $\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}$.

Step 10 Putting the values of $\tilde{x}_{ijk} = \{(\underline{x}_{ijk}, \overline{x}_{ijk}, x^L_{ijk}, x^R_{ijk})_{LR}; (\underline{x'}_{ijk}, \overline{x'}_{ijk}, {x'}^L_{ijk}, {x'}^R_{ijk})_{LR}\}$ in $\Big(\{(100, 1000, 90, 9000)_{LR};\ (50, 5500, 45, 9500)_{LR}\} \otimes \tilde{x}_{121} \oplus \{(400, 4000, 360, 36000)_{LR}; (200, 22000, 180, 38000)_{LR}\} \otimes \tilde{x}_{131} \oplus \{(300, 3000, 270, 27000)_{LR}; (150, 16500, 135, 28500)_{LR}\} \otimes \tilde{x}_{231} \oplus \{(50, 200, 48, 1800)_{LR}; (10, 1100, 9, 1900)_{LR}\} \otimes \tilde{x}_{122} \oplus \{(200, 2000, 180, 18000)_{LR}; (100, 11000, 90, 19000)_{LR}\} \otimes \tilde{x}_{132} \oplus \{(500, 5000, 450, 45000)_{LR}; (250, 27500, 225, 47500)_{LR}\} \otimes \tilde{x}_{232} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{421} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{431} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{451} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{151} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{251} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{422} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{432} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{452} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{152} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{252}\Big)$ and $\Big(\{(150, 950, 141,$

$8050)_{LR}; (45, 5000, 41, 11000)_{LR}\}\otimes\tilde{x}_{121}\oplus\{(500, 5000, 450, 45000)_{LR}; (250, 27500, 225, 47500)_{LR}\} \otimes \tilde{x}_{131} \oplus \{(400, 4000, 360, 36000)_{LR}; (200, 22000, 180, 38000)_{LR}\} \otimes \tilde{x}_{231} \oplus \{(45, 150, 42, 1950)_{LR}; (11, 1000, 9, 1900)_{LR}\} \otimes \tilde{x}_{122} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{132}\oplus\{(200, 2000, 180, 18000)_{LR}; (100, 10000, 90, 20000)_{LR}\}\otimes\tilde{x}_{232}\oplus\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}\otimes\tilde{x}_{421}\oplus\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}\otimes\tilde{x}_{431}\oplus\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}\otimes \tilde{x}_{451}\oplus\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}\otimes\tilde{x}_{151}\oplus\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}\otimes\tilde{x}_{251}\oplus\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}\otimes\tilde{x}_{422}\oplus\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\}\otimes\tilde{x}_{432}\oplus\{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{452} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{152} \oplus \{(0, 0, 0, 0)_{LR}; (0, 0, 0, 0)_{LR}\} \otimes \tilde{x}_{252}\Big)$ the minimum total intuitionistic fuzzy transportation cost and minimum total intuitionistic fuzzy passing time are $\{(15000, 148704.48, 14360, 1438340.34)_{LR}; (4600, 839874.65, 4600, 3040692.58)_{LR}\}$ and $\{(4500, 20941.29, 4440, 377236.81)_{LR}; (660, 159608.62, 660, 390256.38)_{LR}\}$ respectively.

6.8 Comparative study

To show the advantage of the proposed method over the existing methods [38, 49] and methods proposed in previous chapters (Chapter 3, Chapter 4 and Chapter 5) the results of some existing and chosen problems, obtained by using the existing methods [38, 49],methods proposed in previous chapters and the method proposed in this chapter are compared in Table 6.7.

Table 6.7 Comparison of results obtained by using existing methods, method proposed in Chapter 3, Chapter 4, Chapter 5 and method proposed in this chapter

Example	Existing method [38]	Existing method [49]	Method proposed in Chapter 3	Method proposed in Chapter 4	Method proposed in Chapter 5	Method proposed in this chapter
3.5 [39]	$(1924000, 19033100$ $, 7299800)_{LR}$	Not applicable	$(1924000, 19033100, 7299800)_{LR}$	$(1924000, 19033100, 7299800)_{LR}$	$(1924000, 19033100, 7299800)_{LR}$	$(1924000, 19033100, 7299800)_{LR}$
3.1	No feasible solution	Not applicable	$(1650, 3550, 1270, 4750)_{LR}$	$(1650, 3550, 1270, 4750)_{LR}$	$(1650, 3550, 1270, 4750)_{LR}$	$(1650, 3550, 1270, 4750)_{LR}$
3.2	No feasible solution	Not applicable	$(1650, 2600, 1270, 4661.37)_{LR}$	$(1650, 2600, 1270, 4661.37)_{LR}$	$(1650, 2600, 1270, 4661.37)_{LR}$	$(1650, 2600, 1270, 4661.37)_{LR}$
4.2 [49]	Not applicable	$(111.25, 167.80, 41.90, 142)_{LR}$ $(151.75, 200.75, 47, 140.25)_{LR}$	Not applicable	$(113.90, 169.20, 43.50, 141.99)_{LR}$ $(149.99, 198.99, 47, 140.25)_{LR}$	$(113.90, 169.20, 43.50, 141.99)_{LR}$ $(149.99, 198.99, 47, 140.25)_{LR}$	$(113.90, 169.20, 43.50, 141.99)_{LR}$ $(149.99, 198.99, 47, 140.25)_{LR}$
4.1	Not applicable	Not applicable	$(630, 800, 230, 639.5)_{LR}$	$(630, 800, 230, 639.5)_{LR}$	$(630, 800, 230, 639.5)_{LR}$	$(630, 800, 230, 639.5)_{LR}$
4.2	Not applicable	Not applicable	Not applicable	$(671.62, 867.93, 205, 578.69)_{LR}$ $(494.69, 672.34, 158, 369.69)_{LR}$	$(671.62, 867.93, 205, 578.69)_{LR}$ $(494.69, 672.34, 158, 369.69)_{LR}$	$(671.62, 86.935, 205, 578.69)_{LR}$ $(494.69, 672.34, 158, 369.69)_{LR}$
4.3	Not applicable	Not applicable	$(1650, 2600, 1270, 4550)_{LR}$	$(1650, 2600, 1270, 4550)_{LR}$	$(1650, 2600, 1270, 4550)_{LR}$	$(1650, 2600, 1270, 4550)_{LR}$
4.4	Not applicable	Not applicable	Not applicable	$(1650, 2700, 1240, 3050)_{LR}$ $(2100, 3425, 1690, 2125)_{LR}$	$(1650, 2700, 1240, 3050)_{LR}$ $(2100, 3425, 1690, 2125)_{LR}$	$(1650, 2700, 1240, 3050)_{LR}$ $(2100, 3425, 1690, 2125)_{LR}$
4.5	Not applicable	Not applicable	Not applicable	$(1650, 2700, 1240, 3050)_{LR}$ $(2100, 3425, 1690, 2125)_{LR}$	$(1650, 2700, 1240, 3050)_{LR}$ $(2100, 3425, 1690, 2125)_{LR}$	$(1650, 2700, 1240, 3050)_{LR}$ $(2100, 3425, 1690, 2125)_{LR}$
5.1	Not applicable	Not applicable	Not applicable	Not applicable	$(1987.991, 1987.991, 100, 1000)_{LR}$ $(1412.008, 1512.008, 100, 800)_{LR}$	$(1987.991, 1987.991, 100, 1000)_{LR}$ $(1412.008, 1512.008, 100, 800)_{LR}$
5.2	Not applicable	Not applicable	Not applicable	Not applicable	$(1648.49, 1648.49, 100, 1000)_{LR}$ $(1241.83, 1341.83, 103.33, 803.33)_{LR}$	$(1648.49, 1648.49, 100, 1000)_{LR}$ $(1241.83, 1341.83, 103.33, 803.33)_{LR}$
5.3	Not applicable	Not applicable	Not applicable	Not applicable	$(304.45, 730.48, 251.63, 589.96)_{LR}$ $(298.99, 706.57, 251.81, 726.40)_{LR}$ $(172.16, 483.63, 168.57, 569.69)_{LR}$	$(304.45, 730.48, 251.63, 589.96)_{LR}$ $(298.99, 706.57, 251.81, 726.40)_{LR}$ $(172.16, 483.63, 168.57, 569.69)_{LR}$
6.1	Not applicable	Not applicable	Not applicable	Not applicable	Not applicable	$\{(15000, 130000, 14360, 13870000)_{LR}$ $; (4600, 737000, 4600, 1513000)_{LR}\}$
6.2	Not applicable	Not applicable	Not applicable	Not applicable	Not applicable	$\{(15000, 148704.48, 14360, 1438340.34)_{LR}$ $; (4600, 839874.65, 4600, 3040692.58)_{LR}\}$ $\{(4500, 20941.29, 4440, 377236.81)_{LR}$ $; (660, 159608.62, 660, 390256.38)_{LR}\}$

The results, presented in Table 6.7, can be explained as follows:

(1) The existing method [38] can be used only for solving such fully fuzzy capacitated minimum cost flow problems for which the restrictions $\underline{l}_{ij} \leq \underline{x}_{ij} \leq \underline{u}_{ij}$, $\overline{l}_{ij} \leq \overline{x}_{ij} \leq \overline{u}_{ij}$, $l^L_{ij} \leq x^L_{ij} \leq u^L_{ij}$ and $l^R_{ij} \leq x^R_{ij} \leq u^R_{ij}$ are satisfied. However, no feasible solution is obtained by using the existing method [38] on solving such fully fuzzy capacitated minimum cost flow problems for which these restrictions are not satisfied. Since, for the values of $\tilde{l}_{ij}$ and $\tilde{u}_{ij}$ chosen in the existing problem [39, Example 3.5, pp.2498] the restrictions $l_{ij} \leq x_{ij} \leq u_{ij}$, $l^L_{ij} \leq x^L_{ij} \leq u^L_{ij}$ and $l^R_{ij} \leq x^R_{ij} \leq u^R_{ij}$ are satisfying so as discussed in Section 3.5, it can be solved by using the existing method [38]. However, for the values of $\tilde{l}_{ij}$ and $\tilde{u}_{ij}$ chosen in Example 3.1 and Example 3.2 the restrictions $\underline{l}_{ij} \leq \underline{x}_{ij} \leq \underline{u}_{ij}$, $\overline{l}_{ij} \leq \overline{x}_{ij} \leq \overline{u}_{ij}$, $l^L_{ij} \leq x^L_{ij} \leq u^L_{ij}$ and $l^R_{ij} \leq x^R_{ij} \leq u^R_{ij}$ are not satisfying so as discussed in Section 3.5 by using the existing method [38] no feasible solution is obtained for these problems. Also, the remaining problems are neither fully fuzzy single objective un-capacitated transportation problems and fully fuzzy single objective un-capacitated minimum cost flow problems nor such fully fuzzy single objective capacitated transportation problems and fully fuzzy single objective capacitated minimum cost flow problems, so none of the remaining problems can be solved by using the existing method [38].

(2) The method, proposed in Chapter 3, can be used only for solving fully fuzzy single objective un-capacitated and capacitated transportation problems as well as fully fuzzy single objective un-capacitated and capacitated minimum cost flow problems. Since, the existing problem [38, Example 3.5] and the problems, chosen in Example 3.1, Example 3.2, Example 4.1, Example 4.3, are fully

fuzzy single objective capacitated minimum cost flow problem and fully fuzzy single objective capacitated minimum cost flow problem, fully fuzzy single objective capacitated minimum cost flow problem, fully fuzzy single objective capacitated transportation problem, fully fuzzy single objective un-capacitated transportation problem respectively. So these problems can be solved by using the method proposed in Chapter 3. However, the remaining problems are neither fully fuzzy single objective un-capacitated and capacitated transportation problems nor fully fuzzy single objective un-capacitated and capacitated minimum cost flow problems. So, none of the remaining problem can be solved by the the method proposed in Chapter 3.

(3) The method, proposed in Chapter 4, can be used for solving fully fuzzy single and multi-objective un-capacitated and capacitated transportation problems and fully fuzzy single and multi-objective un-capacitated and capacitated minimum cost flow problems. Since, the existing problem [49, Example 4.2], the problems chosen in Example 4.2, Example 4.4 and Example 4.5 are fully fuzzy multi-objective un-capacitated transportation problem, fully fuzzy multi-objective capacitated transportation problem, fully fuzzy multi-objective un-capacitated minimum cost flow problem and fully fuzzy multi-objective capacitated minimum cost flow problem respectively. So, all these problems can be solved by the method proposed in Chapter 4, Also, the method proposed in Chapter 4 is the generalization of the method proposed in Chapter 3. So, all the problems which can be solved by the method proposed in Chapter 3 can also be solved by the method proposed in Chapter 4. However, the remaining problems are neither fully fuzzy single and multi-objective un-

capacitated and capacitated transportation problems nor fully fuzzy single and multi-objective un-capacitated and capacitated minimum cost flow problems. So, none of the remaining problem can be solved by the the method proposed in Chapter 4.

(4) The method, proposed in Chapter 5, can be used for solving fully fuzzy single and multi-objective un-capacitated and capacitated solid transportation problems and fully fuzzy single and multi-objective un-capacitated and capacitated solid minimum cost flow problems. Since, the problems chosen in Example 5.1, Example 5.2 and Example 5.3 are fully fuzzy multi-objective un-capacitated solid transportation problem, fully fuzzy multi-objective capacitated solid transportation problem and fully fuzzy multi-objective capacitated solid minimum cost flow problem respectively. So these problems can be solved by the method proposed in Chapter 5. Also, the method proposed in Chapter 5 is the generalization of the method proposed in Chapter 4. So, all the problems which can be solved by the method proposed in Chapter 4 can also be solved by the method proposed in Chapter 5. However, in the problems, chosen in Example 6.1 and Example 6.2 all the parameters are represented by intuitionistic fuzzy numbers so these problems can not be solved by using the method proposed in Chapter 5.

(5) The method, proposed this chapter, can be used for solving intuitionistic fully fuzzy single and multi-objective un-capacitated and capacitated solid transportation problems and intuitionistic fully fuzzy single and multi-objective un-capacitated and capacitated solid minimum cost flow problems. Since, the

problems chosen in Example 6.1 and Example 6.2 are intuitionistic fully fuzzy single objective capacitated solid minimum cost flow problem and intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem respectively. So, these problems can be solved by the methods proposed in this chapter. Also, the method proposed in this chapter is the generalization of the method proposed in Chapter 5. So, all the problems which can be solved by the method proposed, in Chapter 5, can also be solved by the method proposed in this chapter.

6.9 Conclusions

On the basis of the present study, it can be concluded that it is better to use the proposed ranking approach for comparing intuitionistic fuzzy numbers as compared to existing ranking approaches. Also, it can be concluded that it is not possible to find the intuitionistic fuzzy optimal solution of intuitionistic fully fuzzy single and multi-objective capacitated solid minimum cost flow problems by using any of the existing methods [38,49] as well as methods proposed in Chapter 3, Chapter 4 and Chapter 5. Only the methods proposed in this chapter can be used for the same.

Chapter 7

FUTURE SCOPE

The following may be treated as future directions:

(1) To collect the data of a real life intuitionistic fully fuzzy multi-objective capacitated solid minimum cost flow problem and to apply the methods proposed in Chapter 6 for solving it.

(2) To apply the method for comparing intuitionistic fuzzy numbers, proposed in Chapter 6, for solving other problems such as intuitionistic fuzzy shortest path problems, intuitionistic fuzzy maximum flow problems and intuitionistic fuzzy assignment problems etc.

(3) To develop a ranking approach for comparing interval valued LR flat intuitionistic fuzzy numbers and use it to modify the method proposed in Chapter 6 for solving interval valued intuitionistic fully fuzzy single and multi-objective capacitated solid minimum cost flow problems.

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