Function Spaces and Weak Formulation of Partial Differential Equations

Thesis submitted in partial fulfillment of the requirements

for the award of the degree of

Masters of Science

in

Mathematics and Computing

Submitted by

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July 2016

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CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **'Function Spaces and Weak Formulation of Partial Differential Equations"** *in partial fulfillment of the requirements for the award of degree of Master of Science at School of Mathematics, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Paramjeet Singh.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ACKNOWLEDGEMENT

The thesis entitled "**Function Spaces and Weak Formulation of Partial Differential Equations**" contains my research work carried out at the School of Mathematics at Thapar University in Patiala. First of all, I would like to thank the almighty for granting perseverance. Every project is successful largely due to the effort of wonderful people who have always given their valuable advice or lent a helping hand. I would like to express my gratitute to my supervisor Dr. Paramjeet Singh, Assistant Professor, School of Mathematics, Thapar University, Patiala, for his guidance, motivation, immense knowledge and engagement through the learning process of this master thesis. I am thankful to him for introducing me to the topic as well for the support on the way. I am grateful to my mentor for enlightening me the first glance of research.

Besides my supervisor, I am thankful to Dr. A. K. Lal, Head, SOM, Thapar University, Patiala, for providing all the necessary facilities in the department. His support and sincere attitude towards department have always been encouraging and helped me pave my way for the dissertation.

I owe my gratitude to my family, who have supported me spiritually throughout entire process, both by keeping me harmonious and helping me putting pieces together. Without their support and endless blessings, it is impossible for me to complete my education seamlessly. I will be grateful forever for your love.

I am indebted to Mr. Santosh Kumar, Senior Research Fellow at the School of Mathematics, Thapar University, Patiala who deserves the credit for providing much needed assistance with administrative tasks and keeping the work running smoothly. He always have been there for me whenever I was in trouble or had a question about my research or writing.

I am also very thankful to Ms. Dipty Sharma, Research Fellow at the School of Mathematics, Thapar University, Patiala who helped me while writing this thesis.

Last but not the least, I am heartfelt to my friends for the stimulating discussions, for the times we work together before deadlines, and for all the moments we have fun together. I greatly value their friendship and deeply appreciate their belief in me. Thank you.

(Rajvinder Kaur)

ABSTRACT

Functional analysis plays an increasing role in the applied sciences and mathematics. It is an abstract branch of mathematics that is developed from classical analysis. Proper functional analytic setting is important for the study of initial and boundary value problems. It is also important for the construction of effective numerical schemes. With the discovery of distributions, the role of functional analysis in partial differential equations has become more important. Numerical methods like finite element and finite volume methods depend on the results of functional analysis both for the construction and error analysis of the schemes. The accuracy of several approximations to partial differential equations very much depends on the smoothness of the analytical solution to the equation under consideration. So, smoothness of data becomes very important in the analysis. For this motive, we will consider some classes of functions with some specific differentiability and integrability properties, called function spaces.

Chapter 1 starts with an introduction to function spaces with examples and we define the different norms which are used in analysis of partial differential equations. In addition to this, we have included the different inequalities which will be used in forthcoming chapters.

In Chapter 2, we discuss the theory of distributions and also discussed the need of distributions. Several properties like differentiability, measure as distribution and functions as distributions have discussed. The concept of weak derivatives and convolution of distributions have included.

Finally, in Chapter 3, we study the Sobolev spaces and norm defined on theses spaces. Then we study some extension theorems, imbedding theorems, compactness theorems, and trace theory. Then motivation for weak solution and its existence and uniqueness through Lax-Milgram Theorem. Some test examples have been studied for weak formulation from elliptic partial differential equations.

Contents

Notation and conventions

(i) $\mathbb{N}$ denotes the set of non-negative integers.

(ii) $\mathbb{R}^n$ denotes the n-dimensional Euclidean space over $\mathbb{R}$.

(iii) $\Omega \subset \mathbb{R}^n$ will always be an open and non empty subset.

(iv) $\bar{\Omega}$ stands for the closure of Ω in $\mathbb{R}^n$.

(v) $\partial\Omega$ stands for the boundary of Ω.

(vi) $\Omega' \subset\subset \Omega$ means that Ω' is a relatively compact open subset of Ω.

(vii) $C(\Omega)$ is the space of continuous functions on Ω.

(viii) $C(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$.

(ix) $C^k(\Omega)$ is the space of k-times continuously differentiable functions on Ω.

(x) $C^k(\bar{\Omega})$ is the space of functions in $C^k(\Omega)$ which together with all derivatives possess continuous extensions to $\bar{\Omega}$.

(xi) $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$.

(xii) $C^\infty(\bar{\Omega}) = \bigcap_{k=0}^{\infty} C^k(\bar{\Omega})$.

(xiii) $D(\Omega)$ is the space of functions in $C^\infty(\Omega)$ with compact support in Ω.

(xiv) $D = D(\mathbb{R}^n)$.

(xv) $D'(\Omega)$ is the space of distributions on Ω.

(xvi) $D' = D'(\mathbb{R}^n)$.

(xvii) $\mathcal{E}(\Omega) = C^\infty(\Omega)$.

(xviii) $\mathcal{E} = \mathcal{E}(\mathbb{R}^n)$.

(xix) $\mathcal{E}'(\Omega)$ is the space of distributions with compact support in Ω.

(xx) $\mathcal{E}' = \mathcal{E}'(\mathbb{R}^n)$.

(xxi) $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions in $\mathbb{R}^n$.

(xxii) $W^{m,p}(\Omega)$ is the Sobolev space of order m for $1 \leqslant p \leqslant \infty$ with norm $\|.\|_{m,p,\Omega}$ and semi-norm $|.|_{m,p,\Omega}$.

(xxiii) $W_0^{m,p}(\Omega)$ is the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.

(xxiv) $W^{m,2}(\Omega) = H^m(\Omega)$ with norm $\|.\|_{m,\Omega}$ and semi-norm $|.|_{m,\Omega}$.

(xxv) $W_0^{m,2}(\Omega) = H_0^m(\Omega)$.

(xxvi) $W^{0,p}(\Omega) = L^p(\Omega)$ with norm $|.|_{0,p,\Omega}$.

Chapter 1

Function Spaces

1.1 Introduction

In analysis we can investigate functions in two ways. First way is classical. In this way we investigate properties of functions individually by calculating their values, derivatives, integrals etc. On the other hand second way is general and modern. In this way we consider a set in which functions are appeared as elements of the set. We analyze the geometric and algebraic properties of such a set. First approach is used in classical analysis while the second approach is used in functional analysis. From the second point of view we can define the function space. A function space is a set of functions with specific differentiability and integrability properties. In this chapter we will study various types of function spaces. For the sake of notational convenience, we introduce the concept of multi-index.

Definition 1. (Multi-index notation) Let $\mathbb{N}$ be the set of non negative integers. An n-tuple

$$\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$$

is called multi-index of length $|\alpha| = \alpha_1 + .. + \alpha_n$.

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} .... \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} ... \partial {x_n}^{\alpha_n}}.$$

Example 1. *Let* $n = 3$, *and* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_j \in \mathbb{N}$, $j = 1, 2, 3$.
Then for a function v *of three variables* x_1, x_2, x_3, *we have*

$$\begin{aligned}\sum_{|\alpha|=3} D^{\alpha} v &= \frac{\partial^3 v}{\partial {x_1}^3} + \frac{\partial^3 v}{\partial {x_1}^2 \partial x_2} + \frac{\partial^3 v}{\partial {x_1}^2 \partial x_3} + \frac{\partial^3 v}{\partial x_1 \partial {x_2}^2} + \frac{\partial^3 v}{\partial x_1 \partial {x_3}^2} \\ &+ \frac{\partial^3 v}{\partial {x_2}^3} + \frac{\partial^3 v}{\partial x_1 \partial x_2 \partial x_3} + \frac{\partial^3 v}{\partial {x_2}^2 \partial x_3} + \frac{\partial^3 v}{\partial x_2 \partial {x_3}^2} + \frac{\partial^3 v}{\partial {x_3}^3}.\end{aligned}$$

This example shows the importance of multi-index notation. Instead of writing ten terms on the right-hand side, we can compress the information by writing the single expression shown on the left.

Definition 2. (Space $C^k(\Omega)$) Let Ω be an open set in $\mathbb{R}^n$ and let $k \in \mathbb{N}$. Then $C^k(\Omega)$ be the set of all continuous real-valued functions v defined on Ω such that $D^{\alpha} v$ is continuous on Ω for all $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| \leqslant k, i.e.$

$$C^k(\Omega) := \{v : \Omega \to \mathbb{R} | v \text{ is k-times continuously differentiable}\}.$$

Definition 3. (Space $C^k(\bar{\Omega})$) Let Ω is a bounded open set in $\mathbb{R}^n$. Then $C^k(\bar{\Omega})$ be the set of all v in $C^k(\Omega)$ such that $D^{\alpha} v$ can be extended from Ω to a continuous function on $\bar{\Omega}$, where $\bar{\Omega}$ denotes the closure of the set Ω.

Example 2. *Consider* $\Omega = (0, 1) \subset \mathbb{R}$. *The function*

$$v(x) = \frac{1}{x} \in C^k(\Omega), \ \textit{for each } k \geqslant 0.$$

Now $\bar{\Omega} = [0, 1]$, *clearly* v *is not continuous at x=0. Therefore,* $v \notin C^k(\bar{\Omega})$ *for any* $k \geqslant 0$.

1.1.1 Norm on $C^k(\bar{\Omega})$

Norm on $C^k(\bar{\Omega})$ is defined as

$$\|v\|_{C^k(\bar{\Omega})} := \sum_{|\alpha| \leqslant k} \sup_{x \in \Omega} |D^{\alpha} v(x)|.$$

In particular if $k = 0$, we shall write $C(\bar{\Omega})$ instead of $C^0(\bar{\Omega})$ which denotes the set of all continuous functions defined on $\bar{\Omega}$. In this case norm is defined as

$$\|v\|_{C(\bar{\Omega})} = \sup_{x\in\Omega} |v(x)|$$

If $k = 1$, norm is defined as

$$\begin{aligned} \|v\|_{C^1(\bar{\Omega})} &= \sum_{|\alpha|\leqslant 1} \sup_{x\in\Omega} |D^{\alpha} v(x)| \\ &= \sup_{x\in\Omega} |v(x)| + \sum_{j=1}^{n} \sup_{x\in\Omega} |\frac{\partial v}{\partial x_j}(x)|. \end{aligned}$$

Definition 4. (**Support of a function**) Let $v \in C(\Omega)$ i.e. v be a continuous function defined on Ω. The support of v is written as $\text{supp}(v)$ and is defined as the closure (in Ω) of the set where v is non-zero, i.e.,

$$\text{supp}(v) := \overline{\{x \in \Omega \mid v(x) \neq 0\}}$$

Remark 1. Let $v \in C(\Omega)$,

(i) $v = 0 \iff \text{supp}(v) = \phi$.

(ii) $\text{supp}(v)$ is closed subset of Ω.

Example 3. *Let v be the function defined on $\mathbb{R}^n$ by*

$$v(x) = \begin{cases} e^{\frac{-1}{1-|x|^2}}, & |x| < 1; \\ 0, & \textit{Otherwise}, \end{cases}$$

where $|x| = ({x_1}^2 + ... + {x_n}^2)^{1/2}$. Clearly, the support of v is the closed unit ball

$$\{x \in \mathbb{R}^n : |x| \leqslant 1\}.$$

Definition 5. (Space $C^\infty(\Omega)$) Let Ω be an open set in $\mathbb{R}^n$. $C^\infty(\Omega)$ denotes the space of infinitely continuously differentiable functions, i.e.

$$C^\infty(\Omega) := \bigcap_{k\geqslant 0} C^k(\Omega).$$

Elements of $C^\infty(\Omega)$ are called smooth functions.

Example 4. *Let $v : \mathbb{R} \to \mathbb{R}$ be defined by*

$$v(x) = \begin{cases} e^{-x^{-2}} \quad , \quad x > 0, \\ 0 \quad , \quad x \leqslant 0. \end{cases}$$

Here, we only need to check smoothness at $x = 0$. For $x \leqslant 0$ all the derivatives will become zero but for $x > 0$ the derivatives are finite linear combinations of terms of the form $x^{-m}e^{-x^{-2}}$, where m is an integer greater than or equal to zero. By using the l'Hôpital's Rule these terms will become zero. So, $v \in C^\infty(\mathbb{R})$ i.e. v is a smooth function.

Definition 6. (**Test functions**) Let Ω be any open set in $\mathbb{R}^n$. The space of smooth functions with compact support is said to be the space of test functions. It is denoted by $D(\Omega)$ i.e.,

$$D(\Omega) := \{v \in C^\infty(\Omega) | \text{supp(v) is compact}\}.$$

Remark: $D(\Omega)$ can also be written as $C_c^\infty(\Omega)$.

1.1.2 Non triviality of the space $D(\Omega)$

For $k \in \mathbb{N}$, we have the inclusions

$$D(\Omega) \subset C_c^{k+1} \subset C_c^k(\Omega).$$

We know that zero function belong to $D(\Omega)$. But the question arises whether there exist any non-zero functions in $D(\Omega)$ or not. For the answer of this question we will study Bump functions.

Example 5. *(**Bump function**) The function* $\psi : \mathbb{R} \to \mathbb{R}$ *given by*

$$\psi(x) = \begin{cases} e^{\frac{-1}{1-x^2}}, & |x| < 1; \\ 0, & Otherwise. \end{cases}$$

This is an example of bump function in one dimension. Since a function defined on $\mathbb{R}$ *has compact support iff its has bounded support. Therefore, this function has compact support which is [-1,1]. Also it is smooth function. So,* $\psi \in D(\Omega)$.

1.2 The L^p spaces

Let Ω denotes the open set and $\mathcal{F}$ be the collection of all measurable extended real valued functions on Ω which are finite a.e on Ω. The functions $f, g \in \mathcal{F}$ are said to be equivalent and is denoted by $f \cong g$ if

$$f(x) = g(x) \text{ for almost all } x \in \Omega.$$

Thus we get a equivalence relation, i.e. it is reflexive, symmetric and transitive. Therefore it induces a partition of $\mathcal{F}$ into a disjoint collection of equivalence classes, which is denoted by $\mathcal{F}/\cong$. There is a natural linear structure on $\mathcal{F}/\cong$. For any two functions $f, g \in \mathcal{F}$, their equivalence classes $[f]$,$[g]$ and real numbers α and β, we define the linear combination $\alpha[f] + \beta[g]$ to be the equivalence class of the functions in $\mathcal{F}$ that take the value $\alpha f(x) + \beta g(x)$ at points x in Ω at which both f and g are finite [1].
The zero element of this linear space is the equivalence class of functions that vanishes a.e on Ω.

Definition 7. Let Ω be a open set and $1 \leqslant p < \infty$ be a real number. The space $L^p(\Omega)$ is the collection of equivalence classes $[f]$ for which

$$\int_\Omega |f|^p dx < \infty.$$

In the late nineteenth century it was noticed that whether real valued functions of one or more real variables were the basic elements of classical analysis but it is also useful to consider real valued functions that have as their domain linear space of functions, such functions are called **functionals**. Now we extend the concept of absolute value from the real numbers to general linear spaces, which is known as norm.

Definition 8. (**Normed linear space**) Let X be a linear space. A real valued functional $\|.\|$ on X is said to be norm on X if it satisfies the following properties:

$$
\begin{array}{ll}
(i) & \|f\| \geqslant 0 \quad and \quad \|f\| = 0 \iff f = 0, \quad \forall f \in X. \\
(ii) & \|f+g\| \leqslant \|f\| + \|g\|, \quad \forall f, g \in X. \\
(iii) & \|\alpha f\| = |\alpha| \|f\|, \quad \forall f \in X, \alpha \in \mathbb{R}.
\end{array}
$$

A linear space together with a norm is called a normed linear space. If X is a normed linear space with the norm $\|.\|$, then a function f in X is said to be unit function if $\|f\| = 1$. For any $f \neq 0 \in X$ the function $f/\|f\|$ is a unit function. Clearly it is a scalar multiple of f which we call the **normalization** of f.

Example 6. *Let $f \in L^1(\Omega)$, where $L^1(\Omega)$ consists of equivalence classes of integrable functions. Define*

$$\|f\|_1 = \int_\Omega |f|.$$

Then $\|.\|_1$ defines a norm on $L^1(\Omega)$.
Let $f, g \in L^1(\Omega)$, therefore f and g are finite a.e on Ω.
We deduce from the triangle inequality of real numbers that

$$|f+g| \leqslant |f| + |g| \text{ a.e on } \Omega \,.$$

Therefore

$$
\begin{aligned}
\|f+g\|_1 &= \int_\Omega |f+g| \\
&\leqslant \int_\Omega (|f| + |g|) \\
&= \int_\Omega |f| + \int_\Omega |g| \\
&= \|f\|_1 + \|g\|_1.
\end{aligned}
$$

Hence we have $\|f+g\|_1 \leqslant \|f\|_1 + \|g\|_1$.
Also, $\|f\|_1$ is always $\geqslant 0$ for all $f \in L^1(\Omega)$.

Now, if $f \in L^1(\Omega)$ and $\|f\| = 0$ then we have $f = 0$ a.e on Ω.
Therefore $[f]$ is the zero element of the linear space $L^1(\Omega)$ i.e $f = 0$.

Example 7. *Consider the space $L^2(\Omega)$ and let $f \in L^2(\Omega)$.*
Then the norm is defined as

$$\|f\|_{L_2(\Omega)} := \left(\int_\Omega |f|^2\right)^{1/2}$$

Theorem 1. *(The Cauchy-Schwarz inequality) Let $u, v \in L_2(\Omega)$. Then*

$$|(u,v)| \leqslant \|u\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)}.$$

Proof. Let $\alpha \in \mathbb{R}$, then

$$\begin{aligned} 0 \leqslant \|u+\alpha v\|^2_{L_2(\Omega)} &= (u+\alpha v, u+\alpha v) \\ &= (u,u) + (u,\alpha v) + (\alpha v, u) + (\alpha v, \alpha v) \\ &= \|u\|^2_{L_2(\Omega)} + 2\alpha(u,v) + \alpha^2 \|v\|^2_{L_2(\Omega)}. \end{aligned}$$

The right-hand side is a quadratic polynomial in α with real coefficients, and it is non-negative for all $\alpha \in \mathbb{R}$, therefore its discriminant is non-positive, i.e.

$$|2(u,v)|^2 - 4\|u\|^2_{L_2(\Omega)} \|v\|^2_{L_2(\Omega)} \leqslant 0,$$

and hence we get the required inequality. □

Theorem 2. *(The triangle inequality) Let $u, v \in L_2(\Omega)$, then $u+v \in L_2(\Omega)$, and*

$$\|u+v\|_{L_2(\Omega)} \leqslant \|u\|_{L_2(\Omega)} + \|v\|_{L_2(\Omega)}.$$

Proof. Here we use the Cauchy-Schwarz inequality.

$$\begin{aligned} \|u+v\|^2_{L_2(\Omega)} &= (u+v, u+v) = \|u\|^2_{L_2(\Omega)} + 2(u,v) + \|v\|^2_{L_2(\Omega)} \\ &\leqslant (\|u\|_{L_2(\Omega)} + \|v\|_{L_2(\Omega)})^2. \end{aligned}$$

By taking the square root on both sides we get the required inequality. □

Definition 9. (**Essential supremum of a function**) Let v be a function defined on Ω. The smallest positive number M such that $|v| \leqslant M$ for almost every x in Ω is said to be essential supremum of $|v|$. We write

$$M = ess.\ supp_{\,x \in \Omega} |v(x)|$$

Definition 10. (Space $L_\infty(\Omega)$) The space $L_\infty(\Omega)$ consists of functions which have finite essential supremum on Ω. Here norm is defined as

$$\|v\|_{L_\infty(\Omega)} = M = ess.\ supp_{\,x \in \Omega} |v(x)|.$$

So, we have defined the $L^1(\Omega)$ and $L^\infty(\Omega)$ spaces and their corresponding norms. Now we define norm on $L^p(\Omega)$ for $1 < p < \infty$.

Definition 11. Let Ω be an open set and $1 < p < \infty$. Let f be a function in $L^p(\Omega)$. Then we will show that norm on $L_p(\Omega)$ is defined as

$$\|f\|_{L_p(\Omega)} := \left(\int_\Omega |f|^p \right)^{1/p}.$$

Here $\|f\| \geqslant 0 \forall f \in L^p(\Omega)$.
Also $f \in L^p(\Omega)$ and $\|f\|_p = 0$ then we have $f = 0$ a.e on Ω.
Therefore $[f]$ is the zero element of the linear space $L^p(\Omega)$ i.e $f = 0$.
For triangle inequality we need,

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p \text{ for all } f,\ g\ in\ L^p(\Omega).$$

But this is not obvious. It is called Minkowski's Inequality and further we will prove it.

Young's Inequality: Let p and q be positive real numbers satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, for every positive real numbers a and b,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. We know that a real valued function f defined on an interval I is convex if for every $\alpha,\beta \in I$ and for $\lambda,\mu \in [0,1]$ with $\lambda+\mu = 1$,

$$f(\lambda\alpha+\mu\beta) \leqslant \lambda f(\alpha)+\mu f(\beta).$$

Also, we know that the function $f(t) = e^t, t > 0$ is convex. Thus, for every $\alpha,\beta \in I$ and for $\lambda,\mu \in [0,1]$ with $\lambda+\mu = 1$,

$$e^{\lambda\alpha+\mu\beta} \leqslant \lambda e^{\alpha}+\mu e^{\beta},$$

equivalently,

$$e^{\lambda\alpha}e^{\mu\beta} \leqslant \lambda e^{\alpha}+\mu e^{\beta}.$$

By taking $\lambda = 1/p$ and $\mu = 1/q$ and α and β such that $a = e^{\alpha/p}$, $b = e^{\beta/q}$,
we obtain the required inequality. □

Hölder's inequality:For any two functions $v \in L_p(\Omega), w \in L_q(\Omega)$ with $\frac{1}{p}+\frac{1}{q} = 1$,

$$\left|\int_\Omega v(t)w(t)dt\right| \leqslant \|v\|_{L_p(\Omega)}\|w\|_{L_q(\Omega)}.$$

Proof. For $v \in L^p(\Omega)$, let

$$\|v\|_p = \left(\int_\Omega |v(t)|^p dt\right)^{1/p}.$$

Note that

$$\|v\|_p = 0 \iff v = 0.$$

Thus, for $v,w \in L^p(\Omega)$, if at least one of v and w is 0, then the inequality holds trivially. So, we assume that both v and w are nonzero vectors. For each t, take

$$a = \frac{|v(t)|}{\|v\|_p},$$

$$b = \frac{|w(t)|}{\|w\|_q}.$$

Now, substitute a and b in Young's inequality we have

$$\frac{|v(t)w(t)|}{\|v\|_p\|w\|_q} \leqslant \frac{|v(t)|^p}{p\|v\|_p^p}+\frac{|w(t)|^q}{q\|w\|_q^q}.$$

Taking integration on both sides, we get

$$\int_\Omega \frac{|v(t)w(t)|}{\|v\|_p\|w\|_q}dt \leqslant \int_\Omega \frac{|v(t)|^p}{p\|v\|_p^p}dt+\int_\Omega \frac{|w(t)|^q}{q\|w\|_q^q}dt = \frac{1}{p}+\frac{1}{q} = 1.$$

From this, the required inequality follows. □

Minkowski's Inequality: For $1 \leqslant p < \infty$, and for any two functions $f, g \in L_p(\Omega)$,

$$\|f+g\|_{L_p(\Omega)} \leqslant \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}.$$

Proof.

$$\begin{aligned} \|f+g\|_p^p &= \int_\Omega |f+g|^p \\ &= \int_\Omega |f+g||f+g|^{p-1} \\ &\leqslant \int_\Omega (|f|+|g|)(|f+g|)^{p-1} \\ &= \int_\Omega (|f||f+g|^{p-1} + |g||f+g|^{p-1}). \end{aligned}$$

Applying Hölder's inequality, we have

$$\begin{aligned} \int_\Omega |f||f+g|^{p-1} &\leqslant \|f\|_p \left(\int_\Omega |f+g|^{(p-1)q} \right)^{1/q} \\ &= \|f\|_p \|f+g\|_p^{p/q}. \end{aligned}$$

Similarly,

$$\int_\Omega |g||f+g|^{p-1} \leqslant \|g\|_p \|f+g\|_p^{p/q}.$$

Hence,

$$\|f+g\|_p^p \leqslant (\|f\|_p + \|g\|_p)\|f+g\|_p^{p/q}.$$

From this, we obtain

$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p.$$

□

1.2.1 L^p is complete: Riesz-Fischer theorem

Definition 12. Let $\{f_n\}$ be a sequence in normed linear space X. Then $\{f_n\}$ is said to converge to f in X if

$$\lim_{n\to\infty} \|f - f_n\| = 0.$$

We write

$$\{f_n\} \to f \text{ in } X.$$

Definition 13. Let $\{f_n\}$ be a sequence in normed linear space X. Then $\{f_n\}$ is said to be Cauchy in X if for each $\varepsilon > 0$, there exist a natural number N such that

$$\|f_n - f_m\| < \varepsilon \quad \text{for all } m,\ n \geqslant N.$$

A normed linear space X is said to be complete provided every Cauchy sequence in X converges to a function in X. A complete normed space is called Banach space.

Lemma 1. *Let X be a normed linear space. Then every convergent sequence in X is Cauchy. Moreover, a Cauchy sequence in X converges if it has a convergent subsequence.*

Proof. Let $\{f_n\}$ be a convergent sequence in X and it converges to f i.e.,

$$lim_{n\to\infty}\|f_n - f\| = 0.$$

By the triangle inequality we have,

$$\|f_n - f_m\| \leqslant \|f_n - f\| + \|f_m - f\| \text{ for all } m, n.$$

Therefore $\{f_n\}$ is Cauchy.
Now let $\{f_n\}$ be a Cauchy sequence in X which has subsequence $\{f_{n_k}\} \to f$ in X.
Let $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, therefore we can choose a natural number N such that

$$\|f_n - f_m\|_p < \varepsilon/2$$

for all $n, m \geqslant N$.
Since $\{f_{n_k}\}$ converges to f, so we may choose k such that $n_k > N$ and $\|f_{n_k} - f\|_p < \varepsilon/2$.
Then, by triangle inequality for the norm we have,

$$\begin{aligned} \|f_n - f\|_p &= \|[f_n - f_{n_k}] + [f_{n_k} - f]\|_p \\ &\leqslant \|f_n - f_{n_k}\|_p + \|f_{n_k} - f\|_p \\ &< \varepsilon \text{ for } n \geqslant N. \end{aligned}$$

Therefore $\{f_n\} \to f$ in X.

□

Definition 14. Let X be a normed linear space. A sequence $\{f_n\}$ in X is said to be rapidly Cauchy if there exist a convergent series of positive numbers $\sum_{k=1}^{\infty} \varepsilon_k$ for which

$$\|f_{k+1} - f_k\| \leqslant \varepsilon_k^2 \text{ for all } k.$$

It is useful to observe that if $\{f_n\}$ is a sequence in a normed linear space and the sequence of non-negative numbers $\{a_k\}$ has the property that

$$\|f_{k+1} - f_k\| \leqslant a_k \text{ for all } k,$$

then

$$f_{n+k} - f_n = \sum_{j=n}^{n+k-1} [f_{j+1} - f_j] \text{ for all } n, k, \tag{1.1}$$

$$\|f_{n+k} - f_n\| \leqslant \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\| \leqslant \sum_{j=n}^{\infty} a_j \text{ for all } n, k. \tag{1.2}$$

Lemma 2. *let X be a normed linear space. Then every rapidly Cauchy sequence in X is Cauchy. Furthermore, every Cauchy sequence has a rapidly Cauchy subsequence.*

Proof. Let $\{f_n\}$ be a rapidly Cauchy sequence in X and $\sum_{k=1}^{\infty} \varepsilon_k$ a convergent series of non-negative numbers for which

$$\|f_{k+1} - f_k\| \leqslant \varepsilon_k^2 \text{ for all } k. \tag{1.3}$$

From (1.2) we have

$$\|f_{n+k} - f_n\| \leqslant \sum_{j=n}^{\infty} \varepsilon_j^2 \text{ for all } n, k. \tag{1.4}$$

Since the series $\sum_{k=1}^{\infty} \varepsilon_k$ converges, therefore the series $\sum_{k=1}^{\infty} \varepsilon_k^2$ is also convergent.
Now from (1.4) we have that $\{f_n\}$ is a Cauchy sequence in X. We may inductively choose a strictly increasing sequence of natural numbers $\{n_k\}$ for which

$$\|f_{n_{k+1}} - f_{n_k}\| \leqslant (1/2)^k \text{ for all } k.$$

The subsequence $\{f_{n_k}\}$ is rapidly Cauchy since the geometric series with ratio $1/\sqrt{(2)}$ converges. □

Remark 2. Let E be a measurable set and $1 \leqslant p \leqslant \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.

The Riesz-Fischer theorem: Let E be a measurable set and $1 \leqslant p \leqslant \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in $L^p(E)$, a subsequence of $\{f_n\}$ converges pointwise a.e. on E to f.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $L^p(E)$. According to Lemma(2), there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$, that is rapidly Cauchy. The preceding remark tells us that $\{f_{n_k}\}$ converges to a function f in $L^p(E)$ both with respect to the $L^p(E)$ norm and pointwise a.e. on E. According to the lemma(1) the whole Cauchy sequence converges to f with respect to the $L^p(E)$ norm. □

Chapter 2

Distributions

2.1 Introduction

When we study the solution of partial differential equation in the classical sense, we know that the solution must be differentiable at least as many times as the order of the equation and it must satisfy the equation everywhere in space and time [2]. However, such point of view is very restrictive and we have several equations which fail to possess such solutions. Let us consider the following example.

Example 8. *Consider the following partial differential equation*

$$u_t + uu_x = 0,\ x \in \mathbb{R},\ t > 0, \tag{2.1}$$

where the subscripts denote differentiation with respect to the corresponding independent variable. This equation is known as Burger's equation and is closely related to a class of partial differential equations known as hyperbolic conservation laws. Let $u(x,t)$ be a smooth solution of given equation which satisfies an initial condition of the form

$$u(x,0) = u_0(x),\ x \in \mathbb{R}, \tag{2.2}$$

where $u_0(x)$ is a given function of x. Let us now define a curve $x = x(t)$ in the $x-t$ plane by means of the ordinary differential equation

$$\frac{dx}{dt}(t) = u(x(t),t). \tag{2.3}$$

Along such a curve we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + uu_x = 0$$

since u satisfies (2.1). *Hence along each such curve, u=a constant. It then follows from* (2.3) *that such curves are straight lines. These are called 'characteristic curves' and the curve through the point* $(x_0, 0)$ *on the real line will have the form*

$$x = x_0 + ct,\ c = u_0(x_0) \tag{2.4}$$

and all along this curve $u(x,t) = u_0(x_0)$. *Now consider a smooth initial function* u_0. *Here the two characteristic curves meet at the same point(say P) and hence the value of u at P is not even well defined. Thus except for a short time, we can not even expect the function to be continuous. If we only wish to study classical solutions, such equations can not be tackled. So, there is need of generalizing the notion of a solution of partial differential equations. In other words, we will study the larger class of objects, called Distributions, on which we can define a generalized derivative and wherein the usual rules of calculus will hold. Distributions are also known as generalized functions. To study the definition of distribution, we first need to understand linear functional and convergence of test functions.*

Definition 15. (**Linear functional**) A linear functional f is a mapping from a vector space into a field i.e.

$$f : V \to F$$

such that

(i) $f(x+y) = f(x) + f(y)$,

(ii) $f(\alpha x) = \alpha f(x)$,

for all $x, y \in V$ *and* $\alpha \in F$.

Example 9. *Let* $V = C[a,b]$ *then* f *is defined by*

$$f(x) = \int_a^b x(t)dt, \quad x \in C[a,b].$$

Here f *is a linear functional.*

Example 10. *Consider the Hilbert space l^2 and choose $a = (\alpha_i) \in l^2$, then*

$$f(x) = \sum_{i=1}^{\infty} x_i \alpha_i, \quad x = (x_i) \in l^2.$$

Here f is linear functional.

Definition 16. **(Convergence of test functions)** Let $\{\phi_n\}$ be a sequence of functions in $D(\Omega)$. Then $\phi_n \to \phi$ $(as\ n \to \infty)$ in $D(\Omega)$ if
(i) there exist a compact set $K \subset \Omega$ such that $\text{supp}(\phi) \subset K$ and $\text{supp}(\phi_n) \subset K$ $\forall n$, and
(ii)$\partial^\alpha \phi_n \to \partial^\alpha \phi$ uniformly on K for all $\alpha \in \mathbb{N}^n$.

Definition 17. A linear functional T on $D(\Omega)$ is said to be **distribution** on Ω if whenever $\phi_n \to 0$ in $D(\Omega)$ then $T(\phi_n) \to 0$ in F, equivalently we can say that if $\phi_n \to \phi$ in $D(\Omega)$ then $T(\phi_n) \to T(\phi)$ in F.

Remark 3. The space of all distributions on Ω is denoted by $D'(\Omega)$.

Definition 18. A function $u : \Omega \to \mathbb{R}$ is said to be **locally integrable** if for every compact subset K of Ω, we have

$$\int_K |u| < \infty.$$

For example every continuous function is locally integrable.

The locally integrable functions generates distribution. For this see the following examples.

Example 11. *Consider the locally integrable function(on $\mathbb{R}^2$) as r^{-1} where $r = |x|$. If B is the ball of radius ε centered at the origin, then*

$$\int_B \frac{1}{r} = \int_0^\varepsilon \int_0^{2\pi} \frac{1}{r} r d\theta dr,$$

which is finite. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a locally integrable function.

Then a functional $T_u : D(\mathbb{R}^n) \to \mathbb{C}$ defined by

$$T_u(\phi) = \int_{\mathbb{R}^n} u(x)\phi(x)dx,\ \phi(x) \in D(\mathbb{R}^n),$$

is a distribution.

If f and g are two locally integrable functions such that $f = g$ a.e. then it is obvious that $T_f = T_g$. In particular if $f = 0$ a.e., it defines the zero distribution. In fact, the converse is also true. If $T_f = 0$ then $f = 0$ a.e., given that f is locally integrable.

Example 12. *Let $x \in \mathbb{R}^n$ and the functional δ_x is defined as*

$$\delta_x(\phi) = \phi(x), \ \ \phi \in D(\mathbb{R}^n).$$

Then δ_x is distribution.
In particular, if $x = 0$ we write this distribution as δ and it is the well known "Dirac' delta distribution", introduced by P.A.M. Dirac early in this century. His symbolic calculus of the δ–function made no mathematical sense until correctly formulated in the framework of the theory of distributions developed essentially and L. Schwartz.

Theorem 3. ***(Legendre's lemma)*** *Let $v : \mathbb{R}^n \to \mathbb{C}$ be a locally integrable function. Then*

$$T_v(\phi) = \int_{\mathbb{R}^n} v(x)\phi(x)dx = 0, \ \forall\ \phi \in D(\mathbb{R}^n), \qquad (1)$$

if and only if $v(x) = 0$ almost everywhere in $D(\mathbb{R}^n)$.

Proof. Let $v(x) = 0$. Then clearly $T_v = 0$.
Conversely assume that $T_v = 0$.
To prove: $v(x) = 0$ almost everywhere(a.e).
In this proof we will first extend the relation (1) from $D(\mathbb{R}^n)$ to $C(\mathbb{R}^n)$ of all continuous functions with compact support and then to the set $M(\mathbb{R}^n)$ of all bounded measurable functions with compact support.
Let $c > 0$. Let the set K_c denotes the c-neighbourhood of a compact set $K \subset \Omega$. We know that the set K_c consists of points having distance from K less than c.
By uniform-boundedness theorem, for every given $\varepsilon > 0, c > 0$ and a function $f \in C(\mathbb{R}^n)$ having support K there exist a function $\phi_1 \in D(\mathbb{R}^n)$ such that

$$supp(\phi_1) \subset K_c \ and \ |f(x) - \phi_1(x)| \leqslant \varepsilon, \ x \in K_c.$$

Therefore, we have

$$\left| \int_{\mathbb{R}^n} v(x)[\phi_1(x) - f(x)]dx \right| \leqslant \varepsilon \int_{K_c} |v(x)|dx,$$

Now by using (1) the above inequality is reduced to

$$\left|\int_{\mathbb{R}^n} v(x)f(x)dx\right| \leqslant \varepsilon \int_{K_c} |v(x)|dx.$$

For fixed c if we assume $\varepsilon \to 0$, we have

$$\int_{\mathbb{R}^n} v(x)\phi_1(x)dx = 0.$$

Now let h be a arbitrary function in $M(\mathbb{R}^n)$.
According to the Lebesgue theory of integration, there exist a sequence of functions $h_m \in C(\mathbb{R}^n)$ converging to h a.e as $m \to \infty$. Then we have

$$\int_{\mathbb{R}^n} v(x)h(x)dx = lim_{m\to\infty} \int_{\mathbb{R}^n} v(x)h_m(x)dx = 0.$$

Therefore for any $h \in M(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} v(x)h(x)dx = 0.$$

Finally, let us represent a complex valued function $v(x)$ defined by

$$v(x) = |v(x)|e^{\theta(x)},$$

and define a function $h_a(x)$ by

$$h_a(x) = \begin{cases} e^{-i\theta(x)}, & |x| \leqslant a \ \ and \ \ v(x) = 0; \\ 0, & |x| > a \ \ or \ \ v(x) = 0. \end{cases}$$

where a is the positive number.
Therefore, for any $a > 0$ we have

$$0 = \int_{\mathbb{R}^n} v(x)h_a(x)dx = \int_{|x|\leqslant a} |v(x)|dx.$$

As $|v(x) \geqslant 0|$ and $\int_{|x|\leqslant a} |v(x)|dx = 0$.
So we must have $v(x) = 0$ a.e on the ball $|x| \leqslant a$. Since the number $a > 0$ is arbitrary, so we conclude that $v(x) = 0$ a.e on $\mathbb{R}^n$. □

Theorem 4. *Let Ω be an open subset of $\mathbb{R}^n$. Then the following statements are equivalent:*

(i) T *is distribution on* Ω.

(ii) For every compact subset K of Ω *there exist constants c and* α *depend on K such that*

$$|T(\phi)| \leqslant c\|\phi\|_\alpha, \qquad (*)$$

for all $\phi \in D(\Omega)$ *and supp(*ϕ*)*$\subset K$, *where* $\|\phi\|_\alpha$ *is the maximum absolute value of* ϕ *over* Ω *and all its derivatives upto order* α.

Proof. (i) $\Longrightarrow$ (ii)
Let T is distribution on Ω i.e. $T \in D'(\Omega)$.
Let D_k denotes the subspace of functions in $D(\Omega)$ which have support in K and let $\phi_m \to 0$ in $D(\Omega)$. Then $\phi_m \to 0$ in $D(\Omega)$.
Also, $T(\phi_m) \to 0$ because T is distribution. Now, the induced topology on D_k is metrizable and is generated by the semi-norms $\|.\|_\alpha$. Thus T restricted to D_k is continuous for any compact set $K \subset \Omega$ and (*) just expresses this fact.

(ii) $\Longrightarrow$ (i)
Let $\phi_m \to 0$ in $D(\Omega)$. Then $\|\phi_m\|_\alpha \to 0$ for all α.
By using the given condition in (ii), $T(\phi_m) \to 0$.
Hence $T \in D'(\Omega)$. □

2.1.1 Calculus with distributions

Functions as distributions

Suppose u is a locally integrable complex valued function in Ω, i.e. for every compact subset K of Ω, we have

$$\int_K |u| < \infty.$$

Define

$$T_u(\phi) = \int_\Omega \phi(x)u(x)dx, \qquad \phi \in D(\Omega).$$

Since

$$|T_u(\phi)| \leqslant \left(\int_K |u|\right) \|\phi\|_0, \qquad \phi \in D_K.$$

Now, by theorem (4) $T_u \in D'(\Omega)$.

Measures as distributions

If v is a complex Borel measure on Ω, or if v is a positive measure on Ω with $v(K) < \infty$ for every compact set $K \subset \Omega$, the equation

$$T_v(\phi) = \int_{\Omega} \phi dv, \qquad \phi \in D(\Omega),$$

defines a distribution T_v in Ω.

Differentiation of distributions

If α is a multi-index and $T \in D'(\Omega)$, then

$$(D^{\alpha}T)(\phi) = (-1)^{|\alpha|}T(D^{\alpha}\phi), \qquad \phi \in D(\Omega)$$

defines a linear functional $D^{\alpha}T$ on $D(\Omega)$. If

$$|T\phi| \leqslant c\|\phi\|_n,$$

for all $\phi \in D_K$, then

$$|(D^{\alpha}T)(\phi)| \leqslant c\|D^{\alpha}\phi\|_n \leqslant c\|\phi\|_{N+|\alpha|}.$$

Therefore, by theorem (4) $D^{\alpha}T \in D'(\Omega)$.

Remark 4. For every distribution T and for all multi-induces α and β we have,

$$D^{\alpha}D^{\beta}T = D^{\alpha+\beta}T = D^{\beta}D^{\alpha}T.$$

Proof.

$$\begin{aligned}
(D^{\alpha}D^{\beta}T)(\phi) &= (-1)^{|\alpha|}(D^{\beta}T)(D^{\alpha}\phi) \\
&= (-1)^{|\alpha|+|\beta|}T(D^{\beta}D^{\alpha}\phi) \\
&= (-1)^{|\alpha+\beta|}T(D^{\alpha+\beta}\phi) \\
&= (D^{\alpha+\beta}T)(\phi).
\end{aligned}$$

□

Example 13. *Consider the Dirac distribution* δ *on* $\mathbb{R}$

$$\frac{d\delta}{dx}(\phi) = -\phi'(0)$$

which is, upto a sign, the doublet distribution.

Example 14. *Consider the Heaviside function on* $\mathbb{R}$

$$H(x) = \begin{cases} 1, & x \geqslant 0; \\ 0, & x < 0. \end{cases}$$

Clearly this function is locally integrable. Therefore it defines the distribution. Let us denote this distribution by T_H*. Let* $\phi \in D(\mathbb{R})$*. Then*

$$\begin{aligned} \frac{dT_H}{dx}(\phi) &= -T_H\left(\frac{d\phi}{dx}\right) \\ &= -\int_0^\infty \frac{d\phi}{dx} = \phi(0) = \delta(\phi). \end{aligned}$$

Thus, we have

$$\frac{dT_H}{dx} = \delta.$$

2.1.2 Weak derivatives

We denote by $L^1_{loc}(\mathbb{R})$ *the space of locally integrable functions* $f : \mathbb{R} \to \mathbb{R}$*. These are the Lebesgue measurable functions which are integrable over every bounded interval. Let* $C_c^\infty(\mathbb{R})$ *be the space of all continuous functions with compact support.*

Definition 19. Given an integer $k \geqslant 1$, the distributional derivative of order k of $f \in L^1_{loc}$ is the linear functional

$$T_{D^k(f)}(\phi) = (-1)^k \int_{\mathbb{R}} f(x) D^k \phi(x) dx.$$

If there exist a locally integrable function g such that $T_{D^k(f)} = T_g$, namely

$$\int_{\mathbb{R}} g(x)\phi(x)dx = (-1)^k \int_{\mathbb{R}} f(x) D^k \phi(x) dx \text{ for all } \phi \in C_c^\infty(\mathbb{R})$$

then we say that g is the weak derivative of order k of f.

Example 15. *Consider the function*

$$f(x)=\begin{cases}0, & x\leqslant 0;\\ x, & x>0.\end{cases}$$

It's distributional derivative is the map

$$T(\phi)=-\int_0^\infty x.\phi'(x)dx=\int_0^\infty \phi(x)dx=\int_{\mathbb{R}} H(x)\phi(x)dx$$

where

$$H(x)=\begin{cases}1, & x\geqslant 0;\\ 0, & x<0.\end{cases}$$

Here the Heaviside function H is the weak derivative of f.

Remark 5. (Leibniz's Formula) Let $\Omega\subset\mathbb{R}^n$ be an open set, $\phi\in C^{co}(\Omega)$, $T\in D'(\Omega)$. Then for any multi-index α,

$$D^{\alpha}(\phi T)=\sum_{\beta\leqslant\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}D^{\beta}\phi D^{\alpha-\beta}T.$$

Definition 20. (Support of a distribution) The support of a distribution is the complement of the largest open set on which the distribution vanishes.

Example 16. *If $T=T_f$, the distribution generated by a continuous function, then*

$$supp\ T_f=supp\ (f).$$

Example 17. *The support of the Dirac distribution and all its derivatives is the set* $\{0\}$.

Definition 21. Let $T\in D'(\Omega)$. The singular support of T is the complement of the largest open set on which T is C^{∞}.

Clearly the singular support of T, denoted by sing.supp(T), is closed in Ω and

$$\text{sing supp}(T)\subset \text{supp}(T).$$

2.1.3 Convolution of functions

We will now study an important and useful operation on functions which we will later extend to certain classes of distributions.
Let $f,g \in L^1(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, consider the integral,

$$h(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \tag{2.5}$$

This integral is well-defined since $F(x,y) = f(x-y)g(y)$ is measurable in the product space $\mathbb{R}^n_x \times \mathbb{R}^n_y$ and by Fubini's theorem and the property of translation invariance of the Lebesgue measure,

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} |F(x,y)|dxdy &= \int_{\mathbb{R}^n} |g(y)|dy \int_{\mathbb{R}^n} |f(x-y)|dx \\ &= \|g\|_{L^1(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)} < +\infty. \end{aligned}$$

Hence, again by Fubini's Theorem, the x-section of F, viz.

$$F_x^*(y) = f(x-y)g(y)$$

is in $L^{-1}(\mathbb{R}^n)$ and $h(x)$ is well-defined. Further $h \in L^1(\mathbb{R}^n)$ as a function of x and by the preceding computation,

$$\|h\|_{L^1(\mathbb{R}^n)} \leqslant \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}.$$

Definition 22. Let $f,g \in L^1(\mathbb{R}^n)$. Then the function $h \in L^1(\mathbb{R}^n)$ defined via the equation (2.5) is called convolution of f and g and is denoted by

$$h = f * g.$$

Theorem 5. *The convolution is a commutative and associative binary operation on $L^1(\mathbb{R}^n)$.*

Proof. Let $f,g \in L^1(\mathbb{R}^n)$. Then

$$(f*g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} f(z)g(x-z)dz = (g*f)(x)$$

using the change of variable $z = x - y$. This proves the commutativity.
If $f, g, h \in L^1(\mathbb{R}^n)$, then by use of the change of variable $z = t - y$ and by Fubini's theorem,

$$\begin{aligned}((f*g)*h)(x) &= \int_{\mathbb{R}^n_y} (f*g)(x-y)h(y)dy \\ &= \int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_z} f(x-y-z)g(z)h(y)dzdy \\ &= \int_{\mathbb{R}^n_t} f(x-t)\int_{\mathbb{R}^n_y} g(t-y)h(y)dydt \\ &= \int_{\mathbb{R}^n_t} f(x-t)(g*h)(t)dt \\ &= (f*(g*h))(x)\end{aligned}$$

which proves the associativity. □

Theorem 6. *Let* $1 < p < \infty$ *and* $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$. *The* $f*g$ *is well defined and further,* $f*g \in L^p(\mathbb{R}^n)$ *with*

$$\|f*g\|_{L^p(\mathbb{R}^n)} \leqslant \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}. \tag{2.6}$$

Proof. Let q be the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Let $h \in L^q(\mathbb{R}^n)$. Then $(x,y) \to f(x-y)g(y)h(x)$ is measurable and

$$\begin{aligned}\int_{\mathbb{R}^n_x}\int_{\mathbb{R}^n_y} |f(x-y)g(y)h(x)|dxdy &= \int_{\mathbb{R}^n_x} |h(x)|\int_{\mathbb{R}^n_y} |f(x-y)g(y)|dydx \\ &= \int_{\mathbb{R}^n_x} |h(x)|\int_{\mathbb{R}^n_t} |f(t)g(x-t)|dtdx \\ &= \int_{\mathbb{R}^n_x} |f(t)|dt\int_{\mathbb{R}^n_x} |h(x)||g(x-t)|dx \\ &\leqslant \|h\|_{L^q(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}\|f\|_{L^1(\mathbb{R}^n)} < +\infty\end{aligned}$$

where we have used Hölder's inequality and the fact that by the translation invariance of the Lebesgue measure $g(x)$ and $g(x-t)$ have the same L^p norm. Thus by Fubini's theorem

$$\int_{\mathbb{R}^n_y} h(x)f(x-y)g(y)dy$$

exists for almost for all x and since we can choose $h(x) \neq 0$ for all $x(e.g. h(x) = exp(-|x|^2), |x|^2 = \sum_{i=1}^n x_i^2)$ we deduce that $f * g$, defined by (2.5) is well defined. Also

$$h \to \int (f * g)h$$

is a continuous linear functional on $L^q(\mathbb{R}^n)$ with norm bounded by $\|g\|_{L^p(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)}$ which shows, by the Riesz Representation theorem, that $f * g \in L^p(\mathbb{R}^n)$ and the required inequality holds. □

Remark 6. The inequality (2.6) is a particular case of Young's inequality.
Let $1 \leqslant p, q, r < \infty$ such that

$$(1/p) + (1/q) = 1 + (1/r).$$

If $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$ then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

If f is a continuous function on $\mathbb{R}^n$ and g continuous with compact support then again the integral in (2.5) makes sense. Thus in this case also we use (2.5) to define convolution $f * g$. More generally, if $f_1, ..., f_k$ are continuous functions on $\mathbb{R}^n$ such that all but at most one of them has compact support, then again we can define $f_1 * ... * f_k$ by considering the functions as follows (for instance):

$$f_1 * (f_2 * ..(f_{k-1} * f_k)).$$

The actual order will be unimportant by commutativity and associativity. That between each parenthesis, at least one element has compact support is a consequence of the following result.

Theorem 7. *Let f be continuous and g continuous with compact support. Then*

$$\textit{supp } (f * g) \subset \textit{supp } (f) + \textit{supp } (g) \tag{2.7}$$

where for sets A and B in $\mathbb{R}^n$,

$$A + B = \{x + y | x \in A, \ y \in B\}. \tag{2.8}$$

Proof. Set A=supp(f), B=supp(g), B compact.
Then $A+B$ is closed, for if $x_k + y_k \in A+B, x_k + y_k \to z$, then (for a subsequence) $y_k \to y \in B$. Hence $x_k \to z - y$ and the limit must be in A. Thus $z - y = x \in A$ and so $z \in A+B$.
In order that $(f*g)(x) \neq 0$, we need clearly only consider the integral (2.5) over the set B=supp (g). Then necessarily, $x - y \in A$ = supp (f) for all y in a subset(of non -zero measure) of B. Hence $x \in$ supp (f) + supp (g) and so

$$\text{supp } (f*g) = \overline{\{x|(f*g)(x) \neq 0\}} \subset \text{supp } (f) + \text{supp } (g).$$

□

2.1.4 Convolution of distributions

Let u be a function on $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. We define

$$(\tau_x u)(y) = u(y-x) \tag{2.9}$$

the translation operator, and

$$\check{u}(y) = u(-y). \tag{2.10}$$

Starting from these relations, the following are easy to verify:

$$(\tau_x u)\check{} = \tau_{-x}(\check{u}) \tag{2.11}$$

$$\tau_x \tau_y = \tau_{x+y}. \tag{2.12}$$

Given two functions u and v on $\mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} (\tau_x u)v = \int_{\mathbb{R}^n} u(y-x)v(y)dy = \int_{\mathbb{R}^n} u(z)v(z+x)dz = \int_{\mathbb{R}^n} u(\tau_{-x}v). \tag{2.13}$$

In the same vein, if $T \in D'(\mathbb{R}^n)$ we define $\tau_x T \in D'(\mathbb{R}^n)$ by

$$(\tau_x T)(\phi) = T(\tau_{-x}\phi) \text{ for all } \phi \in D'(\mathbb{R}^n) \tag{2.14}$$

which is exactly (2.13) if T is a function. It is easy to see that $\tau_x T$ is indeed a distribution.
If u and v are functions whose convolution can be defined, then we have

$$(u*v)(x) = \int_{\mathbb{R}^n} u(y)v(x-y)dy = \int_{\mathbb{R}^n} u(y)(\tau_x \check{v})(y)dy. \tag{2.15}$$

Thus we are led to the following:

Definition 23. Let $T \in D'(\mathbb{R}^n)$ and $\phi \in D(\mathbb{R}^n)$. Then the convolution of T and ϕ is a function $T * \phi$ defined by

$$(T * \phi)(x) = T(\tau_x \check{\phi}).$$

We now prove some of the basic properties of the convolution defined above.

Theorem 8. *Let $T \in D'(\mathbb{R}^n)$ and $\phi \in D(\mathbb{R}^n)$. Then:*

(i) for any $x \in \mathbb{R}^n$,

$$\tau_x(T * \phi) = (\tau_x T) * \phi = T * (\tau_x \phi)$$

(ii) if α is any multi-index,

$$D^\alpha(T * \phi) = (D^\alpha T) * \phi = T * (D^\alpha \phi)$$

in particular, $T * \phi \in \mathcal{E}(\mathbb{R}^n)$.

(iii) If $\psi \in D(\mathbb{R}^n)$, then

$$T * (\phi * \psi) = (T * \phi) * \psi.$$

Proof. (i) The proof of (i) is by a straightforward computation using above relationships.

$$\begin{aligned} \tau_x(T * \phi) &= (T * \phi)(y - x) = T(\tau_{y-x}\check{\phi}) \\ &= T(\tau_{-x}\tau_y\check{\phi}) = (\tau_x T)(\tau_y\check{\phi}) = (\tau_x T * \phi)(y) \\ &= T(\tau_y(\tau_x\phi)\check{\ }) = (T * \tau_x\phi)(y). \end{aligned}$$

(ii) It suffices to prove (ii) when $\alpha = (0, 0, ..., 1, ..., 0)$ with 1 at the *ith* place. By iterating this suitably, we can deduce the general case.

Let e_i be the ith (standard) basis vector of $\mathbb{R}^n$. Then

$$\begin{aligned}
\frac{\partial}{\partial x_i}(T*\phi)(x) &= lim_{h\to 0}\frac{(T*\phi)(x)-(T*\phi)(x-he_i)}{h} \\
&= lim_{h\to 0}\frac{1}{h}[(T*\phi)(x)-\tau_{he_i}(T*\phi)(x)] \\
&= lim_{h\to 0}\frac{1}{h}[(T*\phi)(x)-(T*(\tau_{he_i}\phi))(x)] \\
&= lim_{h\to 0}\frac{1}{h}(T*(\phi-\tau_{he_i}\phi))(x) \\
&= lim_{h\to 0}T\left(\tau_x\frac{(\phi-\tau_{he_i}\phi)\check{}}{h}\right) \\
&= T\left(\tau_x\left(\frac{\partial\phi}{\partial x_i}\right)\check{}\right) \\
&= \left(T*\frac{\partial\phi}{\partial x_i}\right)(x).
\end{aligned}$$

Since $\frac{\phi-\tau_{he_i}}{h}\to\frac{\partial\phi}{\partial x_i}$ in $D(\mathbb{R}^n)$.

Iterating this we get one part of (ii). To prove the other, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}(T*\phi)(x) &= lim_{h\to 0}\left(\left(\frac{T-\tau_{he_i}T}{h}\right)*\phi\right)(x) \\
&= lim_{h\to 0}\left(\frac{T-\tau he_iT}{h}\right)(\tau_x\check{\phi}) \\
&= \frac{\partial T}{\partial x_i}(\tau_x\check{\phi}) \\
&= \left(\frac{\partial T}{\partial x_i}*\phi\right)(x)
\end{aligned}$$

Here we use the fact that $\frac{T-\tau_{he_i}T}{h}$ converges to $\frac{\partial T}{\partial x_i}$ in $D'(\Omega)$.

(iii) Let $\phi,\psi\in D(\mathbb{R}^n)$. Now it needs to prove that

$$(T*(\phi*\psi))(0)=((T*\phi)*\psi)(0). \tag{2.16}$$

Then (iii) will follow since for any $x\in\mathbb{R}^n$ we can apply (2.16) with $\tau_{-x}\psi$ instead of ψ and then use (i). But

$$(T*(\phi*\psi))(0)=T((\phi*\psi)\check{}).$$

Now

$$\begin{aligned}(\phi * \psi)\check{}(x) &= (\phi * \psi)(-x) \\ &= \int \phi(-x-y)\psi(y)dy \\ &= \int \tau_y \check{\phi}(x)\check{\psi}(y)dy\end{aligned}$$

and it suffices to consider this integral over the compact set, supp($\check{\psi}$). This integral could be considered as the limit, as $\varepsilon \to 0$, of the Riemann sum

$$\varepsilon^n \sum_p \tau_{\varepsilon p}\check{\phi}(x)\check{\psi}(\varepsilon p)$$

where the sum extends over all integral lattice points p in $\mathbb{R}^n$. Again this is essentially only a finite sum since for any fixed number $\varepsilon > 0$, only a finite number of εp will lie in the set supp($\check{\psi}$), which is compact. Thus

$$(\phi * \psi)\check{}(.) = lim_{\varepsilon \to 0}\varepsilon^n \sum_p \ldots \sum_p \tau_{\varepsilon p}\check{\phi}(.)\check{\psi}(\varepsilon p)$$

in the topology of $D(\mathbb{R}^n)$ and so

$$\begin{aligned}(T * (\phi * \psi))(0) &= T((\phi * \psi)\check{}\ ) \\ &= lim_{\varepsilon \to 0}\varepsilon^n \sum_p T(\tau_{\varepsilon p}\check{\phi})\check{\psi}(\varepsilon p) \\ &= \int (T * \phi)(y)\psi(-y)dy \\ &= ((T * \phi) * \psi)(0).\end{aligned}$$

This completes the proof.

□

2.2 The Fourier Transform

Definition 24. Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f, denoted by $\widehat{f}$, a function defined on $\mathbb{R}^n$ by the formula

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{(-2\pi i x.\xi)} f(x)dx, \quad i = \sqrt{-1},$$

where $x.\xi = \sum_{j=1}^{n} x_j \xi_j$ is the usual euclidean inner-product in $\mathbb{R}^n$.

Since $f \in L^1(\mathbb{R}^n)$, it is immediate to see that $\widehat{f}(\xi)$ is well-defined for every $\xi \in \mathbb{R}^n$. In fact we have

$$|\widehat{f}(\xi)| \leqslant \|f\|_{L^1(\mathbb{R}^n)}. \tag{2.17}$$

Remark 7. The function $\widehat{f}$ is uniformly continuous function.

Definition 25. (The Schwartz space, $\mathcal{S}$) The Schwartz space, or the space of rapidly decreasing functions, $\mathcal{S}$, is given by

$$\mathcal{S} = \{f \in \mathcal{E}(\mathbb{R}^n) \mid \lim_{|x|\to\infty} |x^\beta D^\alpha f(x)| = 0 \text{ for all multi-indices } \alpha \text{ and } \beta\}. \tag{2.18}$$

Example 18. *Any function $\phi \in D$ is trivially in $\mathcal{S}$.*

Example 19. *Let $n = 1$ and $f(x) = e^{-x^2}$. Then $f \in \mathcal{S}$. Here we can easily check that $|x|^k e^{-x^2}$ tends to zero as $|x| \to \infty$.*

Remark 8. (The Weak Parseval Relation) Let $f, g \in \mathcal{S}$. Then

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) g(\xi) d\xi = \int_{\mathbb{R}^n} f(x)\widehat{g}(x) dx. \tag{2.19}$$

Remark 9. (Fourier Inversion Formula) Let $g \in \mathcal{S}$. Then

$$g(x) = \int_{\mathbb{R}^n} exp(2\pi i x.\xi)\widehat{g}(\xi) d\xi.$$

Remark 10. The Fourier transform is a (topological) isomorphism of $\mathcal{S}$ onto itself.

Remark 11. (Strong Parseval Relation) Let $f, g \in \mathcal{S}$, Then

$$\int_{\mathbb{R}^n} f\overline{g} = \int_{\mathbb{R}^n} \widehat{f}\,\overline{\widehat{g}}.$$

Chapter 3

Sobolev spaces and weak solutions to elliptic problems

In this chapter we will study some of the important properties of a class of function spaces, known as Sobolev spaces, named after the Russian mathematician S.L. Sobolev. This space will provide the proper functional setting for the study of the partial differential equations[3, 4].

Definition 26. Let $m > 0$ be an integer and let $1 \leqslant p \leqslant \infty$. The Sobolev space $W^{m,p}(\Omega)$ is defined as

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega) \quad \text{for all } |\alpha| \leqslant m\}. \tag{3.1}$$

In other words, $W^{m,p}(\Omega)$ is the collection of all functions in $L^p(\Omega)$ such that all distribution derivatives upto order m are also in $L^p(\Omega)$.
Clearly $W^{m,p}(\Omega)$ is a vector space. Also it is normed linear space with the norm

$$\|u\|_{m,p,\Omega} = \sum_{|\alpha| \leqslant m} \|D^\alpha u\|_{L^p(\Omega)} \tag{3.2}$$

or, equivalently, for $1 < p < \infty$,

$$\|u\|_{m,p,\Omega} = \left(\sum_{|\alpha| \leqslant m} \int_\Omega |D^\alpha u|^p \right)^{1/p} = \left(\sum_{|\alpha| \leqslant m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}. \tag{3.3}$$

Notations

1. The case $p = 2$ will play a special role in the sequel. These spaces will be denoted by $H^m(\Omega)$. Thus

$$H^m(\Omega) = W^{m,2}(\Omega) \tag{3.4}$$

and for $u \in H^m(\Omega)$, we denote its norm by $\|u\|_{m,\Omega}$, i.e.

$$\|u\|_{m,\Omega} = \|u\|_{m,2,\Omega}. \tag{3.5}$$

The space $H^m(\Omega)$ have a natural inner-product defined by

$$(u,v)_{m,\Omega} = \sum_{|\alpha|\leqslant m} \int_\Omega D^\alpha u D^\alpha v, \quad for\; u,v \in H^m(\Omega). \tag{3.6}$$

2. We will also often use the semi-norms which consists of the L^p-norms of the highest order derivatives. We denote these by $|.|_{m,p,\Omega}$. Thus for $u \in W^{m,p}(\Omega)$,

$$|u|_{m,p,\Omega} = \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\Omega)}.$$

3. We can naturally consider the space $L^p(\Omega)$ as a special case of the Sobolev class when $m = 0$ i.e. we do not bother about derivatives. In particular we denote the L^p-norm of a function by $|.|_{0,p,\Omega}$(since in this case the semi-norm and norm are the same). Again the $L^2(\Omega)-$ norm will be denoted by $|.|_{0,\Omega}$.

4. In case $\Omega = \mathbb{R}^n$ the space $H^m(\mathbb{R}^n)$ can also be defined via the Fourier transform. Let $u \in H^m(\mathbb{R}^n)$. Then by definition, $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leqslant m$. Hence the Fourier transform of $D^\alpha u$ is well defined and we have

$$(D^\alpha u)^\wedge = (2\pi i)^{|\alpha|}\, \xi^\alpha\, \hat{u},$$

and so $\xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leqslant m$. Conversely if $u \in L^2(\mathbb{R}^n)$ such that $\xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leqslant m$, we have $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leqslant m$ and so $u \in H^m(\mathbb{R}^n)$. We can express this in a more compact form using the following lemma.

Lemma 3. *There exist a positive constants M_1 and M_2 depending only on m and n such that*

$$M_1(1+|\xi|^2)^m \leqslant \sum_{|\alpha|\leqslant m} |\xi^\alpha|^2 \leqslant M_2(1+|\xi|^2)^m \tag{3.7}$$

for all $\xi \in \mathbb{R}^n$.

Proof. Recall that $|\xi|^2 = \xi_1^2 + ... + \xi_n^2$ and $|\xi^\alpha| = |\xi_1|^{\alpha_1}...|\xi_n|^{\alpha_n}$. By a simple induction argument on m we can see that same powers of ξ occur in $(1+|\xi|^2)^m$ and $\sum\limits_{|\alpha|\leqslant m} |\xi^\alpha|^2$, albeit with different coefficients, which depend only on n and m. Since the number of terms is finite and depends again only on n and m, the inequalities (3.7) follow. □

By virtue of the preceding lemma we can define that space $H^m(\mathbb{R}^n)$ as follows:

$$H^m(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) | (1+|\xi|^2)^{m/2}(\hat{u})(\xi) \in L^2(\mathbb{R}^n)\}. \tag{3.8}$$

Again, by the Plancherel Theorem, it follows that the norm $\|.\|_{m,\mathbb{R}^n}$ on $H^m(\mathbb{R}^n)$ is equivalent to the norm:

$$\|u\|_{H^m(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^m |\hat{u}(\xi)|^2 d\xi\right)^{1/2}. \tag{3.9}$$

The advantage of this definition is that it is immediate to generalize to all $s \geqslant 0$. If $s \geqslant 0$ we define $H^s(\mathbb{R}^n)$

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) | (1+|\xi|^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n)\} \tag{3.10}$$

with associated norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\right)^{1/2}. \tag{3.11}$$

We will investigate such spaces defined over other open sets of $\mathbb{R}^n$ later. Let us return to the spaces $W^{m,p}(\Omega)$. The map

$$u \in W^{1,p}(\Omega) \to \left(u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right) \in (L^p(\Omega)^{n+1} \tag{3.12}$$

is an isometry of $W^{1,p}(\Omega)$ into $(L^p(\Omega))^{n+1}$ if we provide the latter space with the norm

$$\|u\| = \sum_{i=1}^{n+1} |u_i|_{0,p,\Omega} \quad or \quad \|u\| = \left(\sum_{i=1}^{n+1} |u_i|_{0,p,\Omega}^p\right)^{1/p}$$

for $u = (u_i) \in (L^p(\Omega))^{n+1}$, depending on whether we use the formula (3.2) or (3.3) for the norm on $W^{1,p}(\Omega)$. This is a useful fact to remember and will be used in the proof of the following result.

Theorem 9. *For every $1 \leqslant p \leqslant \infty$, the space $W^{1,p}(\Omega)$ is a Banach space. If $1 < p < \infty$, it is reflexive and if $1 \leqslant p < \infty$, it is separable. In particular $H^1(\Omega)$ is a separable Hilbert space.*

Proof. Let $\{u_m\}$ be a Cauchy sequence in $W^{1,p}(\Omega)$. It follows from the definition of the norm that $\{u_m\}$ and $\left\{\frac{\partial u_m}{\partial x_i}\right\}, 1 \leqslant i \leqslant n$ are all Cauchy sequences in $L^p(\Omega)$. Let $u_m \to u$ and $\frac{\partial u_m}{\partial x_i} \to v_i, 1 \leqslant i \leqslant n$, in $L^p(\Omega)$. The completeness of the space $W^{1,p}(\Omega)$ will be proved if we show that $\frac{\partial u}{\partial x_i} = v_i$ in the sence of distributions so that, on one hand, $u \in W^{1,p}(\Omega)$ and, on the other $u_m \to u$ in $W^{1,p}(\Omega)$.
Let $\phi \in D(\Omega)$. We need to show that

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega v_i \phi. \tag{3.13}$$

Now, since $u_m \in W^{1,p}(\Omega)$, we know that

$$\int_\Omega u_m \frac{\partial \phi}{\partial x_i} = -\int_\Omega \frac{\partial u_m}{\partial x_i} \phi. \tag{3.14}$$

Since $\phi \in D(\Omega)$, $\phi \in L^q(\Omega)$ for all $1 \leqslant q \leqslant \infty$ and so we pass to the limit on both sides of (3.14) as $m \in \infty$ to obtain (3.13). Thus $W^{1,p}(\Omega)$ is complete and if $p = 2$, we get $H^1(\Omega)$ is a Hilbert space.
Now $(L^p(\Omega))^{n+1}$ is reflexive for $1 < p < \infty$ and separable for $1 \leqslant p < \infty$. Since $W^{1,p}(\Omega)$ is complete, its image under the isometry (3.12) is a closed subspace of $(L^p(\Omega))^{n+1}$ which inherits the corresponding properties. □

Remark 12. The results of this theorem can be proved by the same way for any integer $m \geqslant 2$.

Theorem 10. *Let $I \subset \mathbb{R}$ be an open interval and let $u \in W^{1,p}(I)$. Then u is absolutely continuous.*

Proof. Let $x_0 \in I$ and define

$$\overline{u}(x) = \int_{x_0}^{x} u'(t)dt, \tag{3.15}$$

which, by definition, is absolutely continuous. Hence its classical derivative exists a.e. and is equal a.e. to u'; this is also its distribution derivative. Hence, in the sense of distributions,

$$(u - \overline{u})' = 0$$

and so $u - \overline{u} = c$, a constant a.e. Thus $u = \overline{u} + c$ a.e. and the latter function is absolutely continuous.

□

We can deduce an important property of $W^{1,p}(I)$ from the preceding theorem, when I is a bounded interval. Let, for instance, $I = (0,1)$. Then if $u \in W^{1,p}(I)$, we can write

$$u(x) = u(0) + \int_0^x u'(t)dt. \tag{3.16}$$

Hence by Hölder's inequality, if q is the conjugate exponent of p, i.e., $p^{-1} + q^{-1} = 1$, we have

$$|u(0)| \leqslant |u(x)| + |u'|_{0,p,I}|x|^{1/q}.$$

Thus it follows (on integration) that

$$|u(0)| \leqslant C(|u|_{0,p,I} + |u'|_{0,p,I}) = C\|u\|_{1,p,I} \tag{3.17}$$

where $C > 0$ is a constant not depending on u.
Now using (3.16) and (3.17) we also deduce that for any $x \in I$

$$|u(x)| \leqslant C\|u\|_{1,p,I},\ C > 0,\ \text{independent of } u. \tag{3.18}$$

Let B be the unit ball in $W^{1,p}(I)$. Then

$$B = \{u \in W^{1,p}(I) | \|u\|_{1,p,I} \leqslant 1\} \tag{3.19}$$

It follows that if $i : W^{1,p}(I) \to C(I)$ is the inclusion map then $B = i(B)$ is a uniformly bounded set in $C(I)$. Again if $x, y \in I$, by (3.16), we have

$$|u(x) - u(y)| \leqslant |u'|_{0,p,I}|x-y|^{1/q} \leqslant \|u\|_{1,p,I}x - y^{1/q} \tag{3.20}$$

from which it follows that B is equicontinuous in $C(I)$. It follows from Ascoli-Arzela theorem that B is relatively compact in $C(I)$; in other words, the map $i : W^{1,p}(I) \to C(I)$ is a compact operator.
Finally, we introduce an important subspace of the space $W^{m,p}(\Omega)$. If $1 \leqslant p < \infty$, we know that $D(\Omega)$ is dense in $L^p(\Omega)$.Also, if $\phi \in D(\Omega)$, so does every derivative of ϕ and so $D(\Omega) \subset W_0^{m,p}(\Omega)$ is a closed subspace of $W^{m,p}(\Omega)$ and its elements can be approximated in the $W^{m,p}(\Omega)$ norm by C^∞ functions with compact support. In general this is a strict subspace of $W^{m,p}(\Omega)$, except when $\Omega = \mathbb{R}^n$.

Remark 13. Let $1 \leqslant p < \infty$. Then for any integer $m \geqslant 0$,

$$W^{m,p}(\mathbb{R}^n) = W_0^{m,p}(\mathbb{R}^n)$$

3.1 Approximations By Smooth Functions

Remark 14. Let $1 \leqslant p < \infty$, and let $u \in W^{1,p}(\Omega)$. Then there exists a sequence $\{u_m\}$ in $D(\mathbb{R}^n)$ such that $u_m \to u$ in $L^p(\Omega)$ and $\frac{\partial u_m}{\partial x_i}|_{\Omega'} \to \frac{\partial u}{\partial x_i}|_{\Omega'}$ in $L_p(\Omega')$ for every $1 \leqslant i \leqslant n$ and for every $\Omega' \subset\subset \Omega$.

Definition 27. Let Ω be an open set in $\mathbb{R}^n$. An extension operator P for $W^{1,p}(\Omega)$ is a bounded linear operator

$$P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

such that $Pu|_{\Omega} = u$ for every $u \in W^{1,p}(\Omega)$.

Remark 15. If Ω is an open set in $\mathbb{R}^n$ such that an extension operator $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ exists, then given $u \in W^{1,p}(\Omega)$ there exist a sequence $\{u_m\}$ in $D(\mathbb{R}^n)$ such that $u_m|_{\Omega} \to u$ in $W^{1,p}(\Omega)$.

Now, the above remark tells us that if Ω admits an extension operator, then elements of $W^{1,p}(\Omega)$ can be approximated by functions in $C^{\infty}(\Omega)$ which are restrictions of functions of $D(\mathbb{R}^n)$. However, a more difficult theorem of Meyers and Serrin says that the set of all functions in $C^{\infty}(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ when $1 \leqslant p < \infty$, for all open sets Ω.

None of the results extend to the case $p = \infty$. If $\Omega = \mathbb{R}^n$, it is known that the completion of continuous functions with compact support with respect to the $L^{\infty}-$ norm is the space of continuous functions which vanish at infinity. Thus while the function $u \equiv 1$ is in $W^{1,\infty}(\mathbb{R}^n)$, it cannot be approximated by elements of $D(\mathbb{R}^n)$. We give below another simple example.

Example 20. *Let Ω be an open interval, say $(-1,1)$. Consider the function*

$$u(x) = \begin{cases} 0, & x \leqslant 0; \\ x, & x \geqslant 0. \end{cases}$$

Then as u is absolutely continuous, its distribution derivative is given by

$$u(x) = \begin{cases} 0, & x < 0; \\ 1, & x > 0. \end{cases}$$

Then $\phi \in C^{\infty}(\Omega)$ such that

$$|\phi' - u'|_{0,\infty,\Omega} < \varepsilon.$$

Thus if $x < 0$, $|\phi'(x)| < \varepsilon$ and if $x > 0$, $|\phi'(x) - 1| < \varepsilon$ or $\phi'(x) > 1 - \varepsilon$.
Thus by continuity $\phi'(0) \leqslant \varepsilon$ and $\geqslant 1 - \varepsilon$ which is impossible if $\varepsilon < 1/2$. Thus u cannot be approximated in $W^{1,\infty}(\Omega)$ by smooth functions.

Remark 16. (Chain Rule) Let $G \in C^1(\mathbb{R})$ such that $G(0) = 0$ and $|G'(s)| \leqslant M$ for all $s \in \mathbb{R}$. Let $u \in W^{1,p}(\Omega)$. Then the function $G_o u \in W^{1,p}(\Omega)$ and

$$\frac{\partial}{\partial x_i}(G_o u) = (G'_o)\frac{\partial u}{\partial x_i}, \quad 1 \leqslant i \leqslant n.$$

Remark 17. Let $1 \leqslant p < \infty$, and let $u \in W^{1,p}(\Omega)$ such that u vanishes outside a compact set contained in Ω. Then $u \in W_0^{1,p}(\Omega)$.

Remark 18. Let G be a Lipschitz continuous function of $\mathbb{R}$ into itself such that $G(0) = 0$. Then if Ω is bounded, $1 < p < \infty$, and $u \in W_0^{1,p}(\Omega)$, we have $G_o u \in W_0^{1,p}(\Omega)$.

Corollary 1. *Let $u \in H_0^1(\Omega)$, Ω is a bounded open set in $\mathbb{R}^n$. Then $|u|, u^+$ and u^- belong to $H_0^1(\Omega)$ where*

$$u^+(x) = max\{u(x), 0\} \tag{3.21}$$

$$u^-(x) = max\{-u(x), 0\}. \tag{3.22}$$

Proof. We use the above remark with $p = 2$ and $G(t) = |t|$. Thus $|u| \in H_0^1(\Omega)$ for $u \in H_0^1(\Omega)$. Now

$$u^+ = \frac{|u| + u}{2}, u = \frac{|u| - u}{2}$$

and so $u^+, u^- \in H_0^1(\Omega)$.

□

3.2 Extension Theorems

We know that it is easy to prove properties of the Sobolev space $W^{m,p}(\Omega)$ by taking $\Omega = \mathbb{R}^n$. The corresponding results for a general Ω often follow easily if we have an extension operator Ω to $\mathbb{R}^n$. We have seen that such operators exist. But these extensions operators are not unique. A domain Ω may have many such operators. For this consider the following example.

Example 21. *Let $\Omega = (0,1) \subset \mathbb{R}$. Let $u \in W^{1,p}(\Omega)$, then u is absolutely continuous and also the value of u at any point is uniformly bounded by the $W^{1,p}-$ norm of u. (i.e. $|u(x)| \leqslant C\|u\|_{1,p,\Omega}$, where C is constant, $x \in \Omega$)*
Now consider the following function defined on $\mathbb{R}$:

$$\overline{u}(x) = \begin{cases} 0, & x \leqslant -1, x \geqslant 2 \\ (x+1)u(0), & -1 \leqslant x \leqslant 0 \\ u(x), & 0 \leqslant x \leqslant 1 \\ (2-x)u(1), & 1 \leqslant x \leqslant 2. \end{cases}$$

Then $\overline{u}(x)$ is an absolutely continuous function (and hence its distribution derivative is its classical derivative). Thus

$$\overline{u}'(x) = \begin{cases} 0, & x < -1, x > 2 \\ u(0), & -1 < x < 0 \\ u'(x), & 0 < x < 1 \\ -u(1), & 1 < x < 2. \end{cases}$$

Clearly $\overline{u}'(x) \in L^p(\mathbb{R}^n)$ and so $\overline{u}(x) \in W^{1,p}(\mathbb{R})$. Also $u \to \overline{u}$ is an extension operator from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R})$.
In the same way, we can define several other such operators. For instance

$$\overline{u}_1(x) = \begin{cases} 0, & x \leqslant -2, x \geqslant 3 \\ (1/2)(x+2)u(0), & -2 \leqslant x \leqslant 0 \\ u(x), & 0 \leqslant x \leqslant 1 \\ (1/2)(3-x)u(1), & 1 \leqslant x \leqslant 3. \end{cases}$$

defines yet another extension operator $u \to \overline{u}_1$ of $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R})$.

Theorem 11. *(Poincare's Inequality) Let Ω be a bounded open set in $\mathbb{R}^n$. Then there exist a positive constant $C = C(\Omega, p)$ such that*

$$|u|_{0,p,\Omega} \leqslant C|u|_{1,p,\Omega} \quad \text{for every } u \in W_0^{1,p}(\Omega). \tag{3.23}$$

In particular, $u \to |u|_{1,p,\Omega}$ defines a norm on $W_0^{1,p}(\Omega)$, which is equivalent to the norm $\|.\|_{1,p,\Omega}$. On $H_0^1(\Omega)$, the bilinear form

$$(u,v) \to \int_\Omega \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i},$$

defines an inner product giving rise to the norm $|.|_{1,\Omega}$, is equivalent to the norm $\|.\|_{1,\Omega}$.

Proof. Let $\Omega = (-a,a)^n$, $a > 0$. Let $u \in D(\Omega)$. Then

$$u(x) = \int_{-a}^{x_n} \frac{\partial u}{\partial x_n}(x',t)dt,\ x = (x',x_n),\ \text{since } u(x',-a) = 0. \tag{3.24}$$

Hence

$$\begin{aligned} |u(x)| &\leqslant \left(\int_{-a}^{x_n} \left|\frac{\partial u}{\partial x_n}(x',t)\right|^p dt\right)^{1/p} |x_n + a|_{1/q},\ \ (p^{-1}+q^{-1}=1) \\ \text{or, } |u(x)|^p &\leqslant x_n + a^{p/q} \int_{-a}^{a} \left|\frac{\partial u}{\partial x_n}(x',t)\right|^p dt \end{aligned}$$

integrating over x',

$$\begin{aligned} \int |u(x',x_n)|^p dx' &\leqslant 2a^{p/q} \int_\Omega \left|\frac{\partial u}{\partial x_n}\right|^p, \\ \text{and so } \int_\Omega u(x)^p\, dx &= (2a)^{p/q+1} \int_\Omega \left|\frac{\partial u}{\partial x_n}\right|^p, \end{aligned}$$

which proves (3.23) for $u \in D(\Omega)$. But $D(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ and the inequality follows for all $u \in W_0^{1,p}(\Omega)$. If Ω is not a 'box' and let $\Omega \subset \tilde{\Omega}$, box of the form $(-a,a)^n$ and extend $u \in W_0^{1,p}(\Omega)$ by zero to get $\tilde{u} \in W_0^{1,p}(\tilde{\Omega})$ and apply the result which is available for $\tilde{\Omega}$ and the inequality follows. □

3.3 Imbedding Theorems

Remark 19. (Sobolev's Inequality) Let $1 \leqslant p < n$. Then there exist a constant $C = C(p,n) > 0$, such that

$$|u|_{0,p*,\mathbb{R}^n} \leqslant C|u|_{1,p,\mathbb{R}^n}$$

for every $u \in W^{1,p}(\mathbb{R}^n)$. In particular, we have the continuous inclusion

$$W^{1,p}(\mathbb{R}^n) \to L^{p*}(\mathbb{R}^n).$$

Remark 20. Let $\Omega \subset \mathbb{R}^n$ be an open set. Then

$$W_0^{1,p}(\Omega) \subset L^q(\Omega) \text{ for all } q \in [n,\infty).$$

Remark 21. Let $p > n$. Then we have the continuous inclusion

$$W^{1,p}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n).$$

Further, there exist a constant $C = C(p,n) > 0$ such that

$$|u(x) - u(y)| \leqslant C|x-y|^{\alpha}|u|_{1,p,\mathbb{R}^n} \text{ a.e. in } \mathbb{R}^n \tag{3.25}$$

for every $u \in W^{1,p}(\mathbb{R}^n)$, where $\alpha = 1-(n/p)$. Again, if Ω is an open set in $\mathbb{R}^n$, the same conclusions hold for $W_0^{1,p}(\Omega)$.

Remark 22. Let $\Omega = \mathbb{R}^n_+$ or an open set of class C^1 with bounded boundary $\partial\Omega$. Then we have the continuous inclusions

(i) if $1 \leqslant p < n$, $\quad W^{1,p}(\Omega) \to L^{p*}(\Omega)$

(ii) if $p = n$, $\quad W^{1,n}(\Omega) \to L^q(\Omega)$ for all $q \in [n,\infty)$

(iii) if $p > n$, $\quad W^{1,p}(\Omega) \to L^{\infty}(\Omega)$

and further, in the latter case, u is Hölder continuous of exponent $\alpha = 1-(n/p)$. In particular,

$$W^{1,p}(\Omega) \subset C(\overline{\Omega}), \;\; p > n.$$

Remark 23. Let $m \geqslant 1$ be an integer and $1 \leqslant p < \infty$. Then

1. if $\frac{1}{p} - \frac{m}{n} > 0$, $W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$, $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$,
2. if $\frac{1}{p} - \frac{m}{n} = 0$, $W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$, for $q \in [p,\infty)$
3. if $\frac{1}{p} - \frac{m}{n} < 0$, $W^{m,p}(\mathbb{R}^n) \subset L^{\infty}(\mathbb{R}^n)$.

3.4 Compactness Theorems

Remark 24. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\Omega' \subset\subset \Omega$. Let $\mathcal{F}$ be a bounded set in $L^p(\Omega)$, $1 \leqslant p < \infty$. Assume that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

1. $\delta < dist\ (\Omega', \mathbb{R}^n \backslash \Omega)$.
2. For every $h \in \mathbb{R}^n$ with $|h| < \delta$

$$|\tau_{-h}f - f|_{0,p,\Omega'} < \varepsilon \text{ for every } f \in \mathcal{F},$$

where $\tau_y(f)(x) = f(x-y)$. Then $\mathcal{F}|_{\Omega'}$, the set of restrictions of elements of $\mathcal{F}$ to Ω', is relatively compact in $L^p(\Omega')$.

Remark 25. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $1 \leqslant p < \infty$. Let $\mathcal{F}$ be a bounded open set in $L^p(\Omega)$. Assume that

1. for every $\varepsilon > 0$ and for every $\Omega' \subset\subset \Omega$, there exist $\delta > 0$ such that $\delta < dist\ (\Omega', \mathbb{R}^n)$ and
$$|\tau_{-h} f - f|_{0,p,\Omega'} < \varepsilon \tag{3.26}$$
for all $h \in \mathbb{R}^n$ with $|h| < \delta$ and for all $f \in \mathcal{F}$,

2. for every $\varepsilon > 0$ there exist $\Omega' \subset\subset \Omega$ such that
$$|f|_{0,p,\Omega/\overline{\Omega}} < \varepsilon \quad \text{for every } f \in \mathcal{F}. \tag{3.27}$$

Theorem 12. *(Rellich-Kondrasov) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1. Then the following inclusions are compact.*

(i) If $p < n$, $W^{1,p}(\Omega) \to L^q(\Omega), \quad 1 \leqslant q < p*$,

(ii) if $p = n$, $W^{1,n}(\Omega) \to L^q(\Omega), \quad 1 \leqslant q < \infty$,

(iii) if $p > n$, $W^{1,p}(\Omega) \to C(\overline{\Omega})$.

Proof. When $p > n$, as observed earlier the functions of $W^{1,p}(\Omega)$ are Hölder continuous. If B is the unit ball in $W^{1,p}(\Omega)$, then the functions in B are uniformely bounded and equicontinuous in $C(\overline{\Omega})$. Thus B is relatively compact in $C(\overline{\Omega})$ by the Ascoli-Arzela Theorem.
Assume for the moment that the result is true for $p < n$. Notice that as $p \to n,\ p* \to \infty$. Hence, since Ω is bounded $W^{1,n}(\Omega) \subset W^{1,n-\varepsilon}(\Omega)$ for every $\varepsilon > 0$ and given any $q < \infty$ we can find $\varepsilon > 0$ such that $1 \leqslant q < (n-\varepsilon)^*$. Hence by using the case for $p = n - \varepsilon < n$, we deduce that $W^{1,n}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ for any $1 \leqslant q < \infty$.
Thus the theorem will be proved if we prove it for the case $p < n$. Let B be the unit ball in $W^{1,p}(\Omega)$. We now verify conditions (i) and (ii) of remark (25). Let $1 \leqslant q < p*$. Then choose α such that $0 < \alpha \leqslant 1$ and
$$\frac{1}{q} = \frac{\alpha}{1} + \frac{1-\alpha}{p*}.$$
Then if $u \in B$, $\Omega' \subset\subset \Omega$ and $h \in \mathbb{R}^n$ such that $|h| < dist(\Omega', \mathbb{R}^n)\Omega$,
$$\begin{aligned} |\tau_{-h} u - u|_{0,p,\Omega'} &= |\tau_{-h} u - u|^{\alpha}_{0,1,\Omega'} |\tau_{-h} u - u|^{1-\alpha}_{0,p*,\Omega'} \\ &\leqslant (|h|^{\alpha} |u|^{\alpha}_{1,1,\Omega}) 2 |u|_{0,p*,\Omega}{}^{1-\alpha} \\ &\leqslant C|h|^{\alpha}. \end{aligned}$$

We choose h small enough such that $C|h|^{\alpha} < \varepsilon$. This will verify (3.26)
now if $u \in B$ and $\Omega' \subset\subset \Omega$, it follows by Hölder's inequality that

$$\begin{aligned} |u|_{0,q,\Omega\backslash\overline{\Omega'}} &\leqslant |u|_{0,p*,\Omega\setminus\overline{\Omega'}}\, meas\ (\Omega\overline{\Omega'})^{1-(q/p*)} \\ &\leqslant C\, meas\ (\Omega\overline{\Omega'})^{1-(q/p*)} \end{aligned}$$

which can be made to be less than any given $\varepsilon > 0$ choosing $\Omega' \subset\subset \Omega$ to be 'as closely filling Ω' as needed. This verifies (3.27). Thus B is relatively compact in $L^q(\Omega)$ for $1 \leqslant q < p*$ and the theorem is proved. □

Theorem 13. *(Poincare-Wirtinger Inequality) There exist a constant $C > 0$ such that for every $u \in W^{1,p}(\Omega)$, $1 \leqslant p \leqslant \infty$,*

$$|u - \overline{u}|_{0,p*,\Omega} \leqslant C|u|_{1,p,\Omega} \tag{3.28}$$

where

$$\overline{u} = \frac{1}{meas(\Omega)} \int_{\Omega} u$$

is the average of u over Ω. Further, if $p < n$, then

$$|u - \overline{u}|_{0,p*,\Omega} \leqslant C|u|_{1,p,\Omega}.$$

Proof. If we set $u = \overline{u}$ .... with $V = L^p(\Omega)$..., we easily

□

3.5 Trace Theory

Theorem 14. *Let $\Omega = \mathbb{R}^n_+$. Then there exist a continuous linear map $\gamma_0 : H^1(\mathbb{R}^n_+) \to L^2(\mathbb{R}^{n-1})$ which is such that if v is continuous on $\overline{\mathbb{R}}^n_+$ then*

$$\gamma_0(v) = v|_{\mathbb{R}^{n-1}}. \tag{3.29}$$

Proof. Let $v \in D(\mathbb{R}^n)$. We denote $x \in \mathbb{R}^n$ by $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$.
Now

$$\begin{aligned} |v(x',0)|^2 &= -\int_0^{\infty} \frac{\partial}{\partial x_n}(|v(x',x_n)|^2)dx_n \\ &= -2\int_0^{\infty} v(x',x_n)\frac{\partial v}{\partial x_n}(x',x_n)dx_n \\ &\leqslant \int_0^{\infty} \left[(v(x',x_n))^2 + \left(\frac{\partial v}{\partial x_n}(x',x_n)\right)^2\right] dx_n. \end{aligned}$$

Integrating both sides of this relation with respect to x', we get

$$|v(x',0)|^2_{0,\mathbb{R}^{n-1}} \leqslant \|v\|_{1,\mathbb{R}^n_+}.$$

Thus the map $v \to v|_{\mathbb{R}^{n-1}}$ is continuous on $D(\mathbb{R}^n)$ with the $H^1(\mathbb{R}^n)$ topology. But the restriction of $D(\mathbb{R}^n)$ functions to $\mathbb{R}^n_+$ are dense in $H^1(\mathbb{R}^n_+)$. Thus there exist a unique continuous extension to $H^1(\mathbb{R}^n_+)$ of this map.

Remark 26. The range of the map γ_0 is the space $H^{1/2}(\mathbb{R}^{n-1})$.

Remark 27. (Green's formula) let $u,v \in H^1(\mathbb{R}^n_+)$. Then

$$\int_{\mathbb{R}^n_+} u\frac{\partial v}{\partial x_i} = -\int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_i} v \quad \text{if } 1 \leqslant i \leqslant (n-1) \tag{3.30}$$

$$\int_{\mathbb{R}^n_+} u\frac{\partial v}{\partial x_n} = -\int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_n} v - \int_{\mathbb{R}^{n-1}} \gamma_0(u)\gamma_0(v). \tag{3.31}$$

□

Remark 28. Let $v \in \ker(\gamma_0)$. Then its extension by zero outside $\mathbb{R}^n_+$, denoted $\tilde{v}$, is in $H^1(\mathbb{R}^n)$, and

$$\frac{\partial \tilde{v}}{\partial x_i} = \left(\frac{\partial v}{\partial x_i}\right)^{\sim}, \quad 1 \leqslant i \leqslant n. \tag{3.32}$$

Remark 29. Let $1 \leqslant p < \infty$ and $\overline{h} \in \mathbb{R}^n$. Then if $f \in L^p(\mathbb{R}^n)$

$$lim_{\overline{h}\to 0}|\tau_{\overline{h}}f - f|_{0,p,\mathbb{R}^n} = 0.$$

Remark 30. $\ker(\gamma_0) = H^1_0(\mathbb{R}^n_+)$.

Theorem 15. *(Trace Theorem) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^{m+1} with boundary Γ. Then there exists a trace map $\gamma = (\gamma_0, \gamma_1, ..., \gamma_{m-1})$ from $H^m(\Omega)$ into $(L^2(\Omega))^m$ such that*

1. *if $v \in C^\infty(\overline{\Omega})$, then $\gamma_0(v) = v|_\Gamma$, $\gamma_1(v) = \frac{\partial v}{\partial \nu}|_\Gamma$, ...and $\gamma_{m-1}(v) = \frac{\partial m-1}{\partial \nu^{m-1}}(v)|_\Gamma$, where ν is the unit exterior normal to the boundary Γ.*

2. *The range of γ is the space*

$$\prod_{j=0}^{m-1} H^{m-j-1/2}(\Gamma).$$

3. *The kernel of γ is $H^m_0(\Omega)$.*

Now, we shall focus on boundary value problems for elliptic partial differential equations [5]. Elliptic equations are typified by the Laplace equation

$$\Delta u = 0,$$

and its non-homogeneous counterpart, Poisson's equation

$$-\Delta u = f,$$

where we used the notation

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

for the Laplace operator.

More generally, let Ω be an bounded open set in $\mathbb{R}^n$ and consider the linear second-order partial differential equation

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x),\ x \in \Omega, \tag{3.33}$$

where the coefficients a_{ij}, b_i, c and f satisfy the following conditions:

$$\begin{aligned} a_{ij} &\in C^1(\overline{\Omega}), \quad i,j = 1,...,n; \\ b_i &\in C(\overline{\Omega}), \quad i,j = 1,...,n; \\ c &\in C(\overline{\Omega}), \quad f \in C(\overline{\Omega}), \end{aligned}$$

and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \tilde{c}\sum_{i=1}^{n}\xi^2, \quad \forall \xi = (\xi_1,...,\xi_n) \in \mathbb{R}^n,\ x \in \overline{\Omega}. \tag{3.34}$$

here $\tilde{c}$ is a positive constant independent of x and ξ. The condition (3.34) is usually referred to as uniform ellipticity and (3.33) is called an elliptic equation.

We begin by considering the homogeneous Dirichlet boundary value problem

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x),\ x \in \Omega, \tag{3.35}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{3.36}$$

where a_{ij}, b_i, c and f are as in (3.34).

A function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying (3.35) and (3.36) is called classical solution of this problem.

3.6 Motivation for definition of weak solutions

How should we define a weak or generalized solution? Assuming for the moment u is really a smooth solution, let us multiply the PDE, $Lu = f$ by a smooth test function $v \in C_c^\infty(\Omega)$ and integrate over Ω, to find

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{n} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} v dx \tag{3.37}$$

$$+ \int_\Omega c(x) uv dx = \int_\Omega f(x) v(x) dx(\Omega) \tag{3.38}$$

Where we have integrated by parts in the first term on the left-hand side. There are no boundary terms since $v = 0$ on $\partial\Omega$. By approximation we find the same identity holds with the smooth function v replaced by any $v \in H_0^1(\Omega)$, and the resulting identity makes sense if only $u \in H_0^1(\Omega.)$

Definition 28. Let $a_{ij} \in L_\infty(\Omega)$, $i = 1,...,n$, $b_i \in L_\infty(\Omega)$, $i = 1,..,n$, $c \in L_\infty(\Omega)$, and let $f \in L_2(\Omega)$. A function $u \in H_0^1(\Omega)$ satisfying

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{n} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} v dx \tag{3.39}$$

$$+ \int_\Omega c(x) uv dx = \int_\Omega f(x) v(x) dx \quad \forall v \in H_0^1(\Omega) \tag{3.40}$$

is called weak solution of (3.35) and (3.36). All partial derivatives in (3.40) should be understood as weak derivatives [6, 7].

3.7 Existence of weak solutions

Theorem 16. *(Lax and Milgram theorem) Suppose that V is a real Hilbert space equipped with norm $\|.\|_V$. Let $a(.,.)$ be a bilinear functional on $V \times V$ such that:*

(i) $\exists\, c_0 > 0\ \forall\, v \in V\ \ a(v,v) \geqslant c_0 \|v\|_V^2$,

(ii) $\exists\, c_1 > 0\ \forall\, v,w \in V\ \ |a(w,v)| \leqslant c_1 \|w\|_V \|v\|_V$, *and let $l(.)$ be a linear functional on V such that*

(iii) $\exists\, c_2 > 0\ \forall\, v \in V\ \ |l(v)| \leqslant c_2 \|v\|_V$.

Then, there exist a unique $u \in V$ such that

$$a(u,v) = l(v) \quad \forall v \in V.$$

Example 22. *Consider, the following Dirichlet-Neumann mixed boundary value problem:*

$$\begin{aligned} -\Delta u &= f \quad in \quad \Omega, \\ u &= 0 \quad on \quad \Gamma_1, \\ \frac{\partial u}{\partial v} &= g \quad on \quad \Gamma_2, \end{aligned}$$

where Γ_1 is a non-empty, relatively open subset of $\partial\Omega$ and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$. We shall suppose that $f \in L_2(\Omega)$ and that $g \in L_2(\Gamma_2)$. Now consider the special Sobolev space

$$H^1_{0,\Gamma_1}(\Omega) = \{v \in H^1(\Omega) : v = 0 \; on \; \Gamma_1\},$$

and define the weak formulation of the mixed problem as follows: find $u \in H^1_{0,\Gamma_1}(\Omega)$ such that

$$\alpha(u,v) = l(v)$$

for all v in $H^1_{0,\Gamma_1}(\Omega)$, where we put

$$\alpha(u,v) = \int_\Omega \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx$$

and

$$I(v) = \int_\Omega f(x)v(x)dx + \int_{\Gamma_2} g(s)v(s)ds.$$

Applying the Lax-Milgram theorem with $V = H^1_{0,\Gamma_1}(\Omega)$ the existence and uniqueness of a weak solution to this mixed problem easily follows.

3.8 Conclusions

In this thesis we have discussed function spaces and then some important inequalities which are useful in analysis of partial differential equations. Further we studied the need of distributions and then some important properties of distributions. We have also included extension theorems, compactness theorems, imbedding theorems and trace theory. Since there are many functions whose derivative does not exist in classical sense so we introduced the notion of weak derivative. The theory of Sobolev spaces is also included which are useful in weak formulation of partial differential equations.

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