

# ON LATTICE PATHS AND $q$ -SERIES

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the award of the degree of

**Masters of Science**  
**in**  
**Mathematics and Computing**

submitted by

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under  
the guidance of

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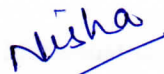
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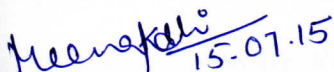
## CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled “**On Lattice Paths and  $q$ -Series**” in partial fulfillment of the requirements for the award of degree of Master of Science in Mathematics and Computing, School of Mathematics, Thapar University, Patiala is an authentic record of my work carried out under the supervision of **Dr. Meenakshi Rana**.

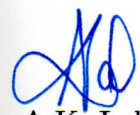
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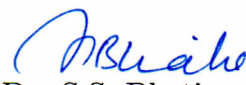
  
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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

  
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## ABSTRACT

In this thesis, we study one combinatorial tool, particularly, lattice paths. In subsequent chapters we interpret lattice paths as the generating functions for certain  $q$ -series.

The lattice paths are considered as a standard combinatorial tool which were introduced by Agarwal and Bressoud (Lattice paths and multiple basic hypergeometric series, Pacific J. Math. 136(2) (1989), 209-228) to understand the enumeration of  $q$ -series and gave a new insights in the field of combinatorics.

In **Chapter 1** we give some basic definitions, notations and preliminary results.

In **Chapter 2** we study lattice paths as the generating functions for many  $q$ -series.

In **Chapter 3** we provide some generalized  $q$ -series and study the combinatorial interpretations of the given  $q$ -series using the lattice paths.

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# Chapter 1

## Introduction

### 1.1 Introduction

#### 1.1.1 Definition of Partition

**Definition 1.1.1** A *partition* of a positive integer  $n$  is a finite non-increasing sequence of positive integers  $a_1, a_2, a_3, \dots, a_r$  such that

$$\sum_{i=1}^r a_i = n$$

where  $a_i$  are called parts or summands of partitions. The number of partition of  $n$  is denoted by  $p(n)$ .

**Example 1.**

$n$	$p(n)$	Relevant Partitions
1	1	1
2	2	2, 1 + 1
3	3	3, 2 + 1, 1 + 1 + 1
4	5	4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1
5	7	5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1

**Remark 1.**  $p(n) = 0$  for  $n < 0$  as we cannot write a negative number into sum of positive integers. Also  $p(0) = 1$ .

**Remark 2.** We use an exponent to denote that the part is repeated a certain number of times in partition. So a particular partition  $1 + 1 + 1$  of 3 can be written as  $1^3$  or  $111$ .

**Remark 3.** Two sums that differ only in the order of their summands are considered the same partition. If order matters, then the partitions are termed as composition or ordered partitions are compositions.

### 1.2 Representation of Partitions

There are two common diagrammatic methods to represent partitions: as Ferrers diagrams, named after Norman Macleod Ferrers, and as Young diagrams, named after the British mathematician Alfred Young.

### 1.2.1 Ferrers Diagram

**Definition 1.2.1** *The Ferrers diagram is a graphical representation of a particular partition  $\pi = t_1 + t_2 + \dots + t_i$  of  $n$ .*

The Ferrers diagram of a partition  $t_1, t_2, \dots, t_i$  of  $n$  is a set of  $i$ -rows if equispaced dots aligned on the left, where  $j$ -row has  $t_j$ -dots ( $1 \leq j \leq i$ ).

**Example 2.** The partition  $6 + 4 + 3 + 1$  of the positive integer 14 can be represented by the following diagram:

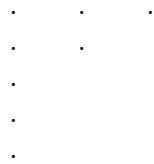


**Definition 1.2.2** *The **conjugate** of a partition is obtained by interchanging Ferrers diagram's rows with its columns. The conjugate of a partition  $\pi$  is denoted by  $\pi^c$ .*

**Example 3.** The Ferrers graph of the partition  $\pi=5 + 2 + 1$  of 8 is:



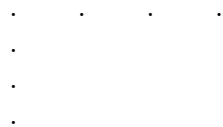
The conjugate of above partition is :



so,  $\pi^c=3 + 2 + 1 + 1 + 1$  of 8.

**Definition 1.2.3** *A partition is said to be **self conjugate** if it is identical with its conjugate. So,  $\pi = \pi^c$ .*

**Example 4.** Consider the partition  $\pi = 4 + 1 + 1 + 1$  of 7, then Ferrers graph is

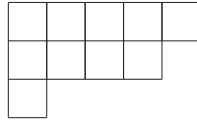


Now if read this graph vertically we get the same partition  $\pi^c=4 + 1 + 1 + 1$ , so this partition is self conjugate partition. In other words when  $\pi=\pi^c$ , then that particular partition is called a self conjugate partition.

### 1.2.2 Young Diagram

An alternative visual representation of an integer partition is its Young diagram. Rather than representing a partition with dots, the Young diagram uses boxes or squares.

**Example 5.** The Young diagram for partition  $5 + 4 + 1$  is given below:



### 1.3 Various Partitions and their Generating Functions

The partition function  $p(n)$  represents the number of possible partitions of a natural number  $n$ . The value of  $p(n)$  for a particular positive integer  $n$  can be calculated with the help of its generating function given below:

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)}$$

On expanding each term on the right-hand side as geometric series of above function, we can rewrite it as

$$(1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + x^6 + \dots)(1 + x^3 + x^6 + x^9 + \dots) \dots$$

The exponent of the term  $x^n$  in the product counts the number of ways to write  $n$  where

$$n = a_1 + 2a_2 + 3a_3 + \dots = (1 + 1 + \dots + 1) + (2 + 2 + \dots + 2) + (3 + 3 + \dots + 3) + \dots$$

where each number  $i$  appears  $a_i$  times which is precisely the definition of a partition of  $n$  and  $a_i \geq 0$ .

**Notation.**  $(a; q)_n$  is  $q$ -rising factorial defined by

$$(a; q)_n = \prod_{i=0}^{\infty} \frac{(1 - aq^i)}{(1 - aq^{i+n})}$$

If  $n$  is a positive integer, then

$$(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \dots (1 - aq^{n-1}),$$

$$(a; q)_{\infty} = (1 - a)(1 - aq)(1 - aq^2) \dots$$

and when  $a = q$ ,

$$(q; q)_n = \prod_{i=1}^n (1 - q^i),$$

also

$$(a, q)_0 = 1.$$

Let  $D(n)$  denote the number of partitions of  $n$  into distinct parts, then generating function for this is given by

$$\sum_{n=0}^{\infty} D(n)q^n = \prod_{n=1}^{\infty} (1 + q^n) = (-q; q)_{\infty}.$$

**Example 6.** For  $n = 5$ ,  $D(5) = 3$  and relevant partitions are  $5, 4 + 1, 3 + 2$ .

Let  $O(n)$  denote the number of partitions of  $n$  into odd parts, then generating function for this is given by

$$\sum_{n=0}^{\infty} O(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}} = \frac{1}{(q; q^2)_{\infty}}.$$

**Example 7.** For  $n = 5$ ,  $O(5) = 3$  and relevant partitions are  $5, 3 + 1 + 1, 1 + 1 + 1 + 1 + 1$ .

The above two generating functions give rise to a famous identity known as Euler's Identity.

**Theorem 1.3.1 (Euler's Identity)** *The number of partitions of  $n$  into distinct parts equals the number of partitions of  $n$  into odd parts.*

**Example 8.** Let  $n = 5$ .

partitions of 5	partitions relevant to $O(5)$	partitions relevant to $D(5)$
5	*	*
4 + 1		*
3 + 2		*
3 + 1 + 1	*	
2 + 2 + 1		
2 + 1 + 1 + 1		
1 + 1 + 1 + 1 + 1	*	

Hence  $D(5)=O(5)=3$ .

**Proof:** Using generating functions:

$$\begin{aligned} \sum_{n=0}^{\infty} D(n)q^n &= \prod_{n=1}^{\infty} (1 + q^n) \\ &= \prod_{n=1}^{\infty} \frac{(1 + q^n)(1 - q^n)}{(1 - q^n)} \\ &= \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}} \\ &= \sum_{n=0}^{\infty} O(n)q^n. \end{aligned}$$

### Euler's Bijective Proof:

A partition of  $n$  into distinct parts can be written as

$$n = d_1 + d_2 + d_3 + \cdots + d_k \quad (1.3.1)$$

where  $d_i \neq d_j$  where  $1 \leq i \leq j \leq k$  and each integer  $d_i$  can be uniquely expressed as a power of 2 times an odd number. Thus,

$$n = 2^{a_1}b_1 + 2^{a_2}b_2 + 2^{a_3}b_3 + \cdots + 2^{a_k}b_k$$

where each  $b_i$  is an odd positive integer and  $a_i \geq 0$ . If we now group together the odd numbers we get an expression like this:

$$\begin{aligned} n &= (2^{\alpha_1} + 2^{\alpha_2} + \cdots).1 + (2^{\beta_1} + 2^{\beta_2} + \cdots).3 + (2^{\gamma_1} + 2^{\gamma_2} + \cdots).5 + \cdots \\ &= \mu_1.1 + \mu_3.3 + \mu_5.5 \cdots \end{aligned}$$

In each series  $(2^{\alpha_1} + 2^{\alpha_2} + \cdots)$ , the  $\alpha_i$ 's are distinct. Thus the sum is the binary expansion of some  $\mu_j$ . We now see the partition of  $n$  into odd parts that corresponds, under this bijection, to the given partition (1.3.1) into distinct parts.

**Example 9.** Consider the partition of 5 into distinct parts  $5 = 3 + 2$ ,

$$\begin{aligned} 5 &= 3 + 2 \\ &= 2^0.3 + 2^1.1 \\ &= 2^1(1) + 2^0(3) \\ &= 1 + 1 + 3 \end{aligned}$$

so we now have a partition of 5 into odd parts.

## 1.4 Some Elementary Theorems

**Theorem 1.4.1** *The number of partitions of  $n$  with largest part  $l$  equals the number of partitions of  $n$  into  $l$  parts.*

**Theorem 1.4.2** *The number of partitions of  $n$  with parts  $\leq l$  equals to the number of partitions of  $n$  into at most  $l$  parts.*

**Theorem 1.4.3** *The number of partitions of  $n$  into distinct odd parts is equal to the number of self-conjugate partitions of  $n$ .*

## 1.5 $n$ -Color Partition

**Definition 1.5.1** *An  $n$ -color partition (also called a partition with “ $n$  copies of  $n$ ”) of a positive integer  $\nu$  is a partition in which a part of size  $n$ , can come in  $n$ -different colors denoted by subscripts  $n_1, n_2, \dots, n_n$  and the parts satisfy the order  $1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < \cdots$*

**Example 10.**

Value of $\nu$	Color partition	Ordinary Partition
$\nu = 1$	$1_1$	1
$\nu = 2$	$2_1, 2_2, 1_1 1_1$	2, 11
$\nu = 3$	$3_1, 3_2, 3_3, 2_1 1_1, 2_2 1_1, 1_1 1_1 1_1$	3, 21, 111

**Generating Function of  $n$ -Color Partition**

Let  $P(m)$  denote the  $n$ -color partition of  $m$ . Then generating function is given as :

$$\sum_{m=0}^{\infty} P(m)q^m = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^n}.$$

**Definition 1.5.2** The *weighted difference* of any pair of parts  $m_i, n_j$  is defined by  $m - n - i - j$  and denoted by  $((m_i - n_j))$  where  $(m \geq n)$ .

**Example 11.** If  $\pi = 5_3 + 3_1 + 1_1$ , then  
 $((5_3 - 3_1)) = 5 - 3 - 3 - 1 = -2$   
 $((3_1 - 1_1)) = 3 - 1 - 1 - 1 = 0$   
 $((5_3 - 1_1)) = 5 - 3 - 1 - 1 = 0.$

**Definition 1.5.3** An  $n$ -color partition obtained from  $\pi$  by replacing each of its parts by its conjugate will be called the **conjugate** of  $\pi$  and will be denoted by  $\pi^c$ . We call  $(a_i)_{a_i - b_i + 1}$  the conjugate of  $(a_i)_{b_i}$  where  $(a_i)_{b_i}$  is a part of  $n$ -color partition of  $\pi$ .

**Example 12.** For  $\nu=8, \pi=5_2+3_1$ , then  $\pi^c = 5_4 + 3_3$ .

**Definition 1.5.4** When  $\pi = \pi^c$ , then  $\pi$  is called a **self-conjugate**  $n$ -color partition.

Using the  $n$ -color partitions from [1], Agarwal interpreted the following identities from [9];

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q; q)_n (q; q^2)_n} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{10n})(1 - q^{10n-4})(1 - q^{10n-6}) \quad [[9], I(46), p.156] \quad (1.5.1)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n (q; q^2)_n} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{14n})(1 - q^{14n-6})(1 - q^{14n-8}) \quad [[9], I(61), p.158] \quad (1.5.2)$$

as given below;

**Theorem 1.5.1** Let  $A_1(\nu)$  be the number of  $n$ -color partitions of  $\nu$  such that each pair of summands has a positive weighted difference. Then  $A_1(\nu)$  is equal to the number of ordinary partitions of  $\nu$  into parts which are not congruent to  $0, \pm 4 \pmod{10}$ . So

$$\sum_{\nu=0}^{\infty} A_1(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q; q)_n(q; q^2)_n}.$$

**Theorem 1.5.2** Let  $A_2(\nu)$  be the number of  $n$ -color partitions of  $\nu$  such that each pair of summands has a non-negative weighted difference. Then  $A_2(\nu)$  is equal to the number of ordinary partitions of  $\nu$  into parts which are not congruent to  $0, \pm 6 \pmod{14}$ . So

$$\sum_{\nu=0}^{\infty} A_2(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n(q; q^2)_n}.$$

Theorem 1.5.1 and 1.5.2 were generalized by Agarwal and Andrews [6] and the analytic counter part of their combinatorial identity was given by Agarwal, Andrews and Bressoud [7].

## 1.6 $(n + t)$ -Color Partition

**Definition 1.6.1** A partition with “ $n + t$  copies of  $n$ ”,  $t \geq 0$  is a partition in which a part of size  $n$ , ( $n \geq 0$ ), can come in  $n + t$  different colors denoted by the subscripts,  $n_1, n_2, n_3, \dots, n_{n+t}$ .

For example, the relevant partitions of 2 with “ $n + 1$  copies of  $n$ ” are,

$$\begin{aligned} &2_1, \quad 2_10_1, \quad 1_11_1, \quad 1_11_10_1, \\ &2_2, \quad 2_20_1, \quad 1_21_1, \quad 1_21_10_1, \\ &2_3, \quad 2_30_1, \quad 1_21_2, \quad 1_21_20_1. \end{aligned}$$

Note that zeros are permitted if and only if  $t \geq 1$ . Also in no partition zeros are permitted to repeat.

Using  $n$ -color partitions and  $(n + t)$ -color partitions Agarwal in [2] interpreted combinatorially some  $q$ -identities from Slater’s compendium [9]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_{2n}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{10n-2})(1 - q^{10n-8})(1 - q^{20n-14}) \times (1 - q^{20n-5})(1 - q^{10n}) \quad [2, (79) - (98)] \quad (1.6.1)$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q)_{2n}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{8n-1})(1 - q^{8n-7})(1 - q^{16n-10}) \times (1 - q^{16n-6})(1 - q^{8n}) \quad [2, (39) - (83)] \quad (1.6.2)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_{2n}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{10n-1})(1 - q^{10n-9})(1 - q^{20n-9}) \times (1 - q^{20n-12})(1 - q^{10n}) \quad [2, (79)] \quad (1.6.3)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_{2n+1}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{10n-3})(1 - q^{10n-7})(1 - q^{20n-16}) \times (1 - q^{20n-4})(1 - q^{10n}) \quad [2, (94)]$$

(1.6.4)

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q; q)_{2n+1}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n+1})(1 - q^{20n+8})(1 - q^{20n+12})} \quad [2, (96)]$$

(1.6.5)

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q)_{2n+1}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{8n-3})(1 - q^{8n-5})(1 - q^{16n-14}) \times (1 - q^{16n-2})(1 - q^{8n}) \quad [2, (38) - (86)]$$

(1.6.6)

given as;

**Theorem 1.6.1** Let  $A_1(\nu)$  denote the number of  $n$ -color partitions of  $\nu$  such that

- (1.1) even parts appear with even subscripts and odd with odd, and
- (1.2) each pair of parts has nonnegative even weighted difference.

Let  $B_1(\nu)$  denote the number of ordinary partitions of  $\nu$  into parts which are either odd or congruent to  $\pm 4 \pmod{20}$ . Then

$$A_1(\nu) = B_1(\nu), \quad \text{for all } \nu.$$

**Theorem 1.6.2** Let  $A_2(\nu)$  denote the number of  $n$ -color partitions of  $\nu$  such that

- (1.1) even parts appear with even subscripts and odd with odd subscripts greater than 1, and
- (1.2) each pair of parts has nonnegative even weighted difference.

Let  $B_2(\nu)$  denote the number of ordinary partitions of  $\nu$  into parts which are not congruent to  $0, \pm 1, \pm 6, \pm 7, 8 \pmod{16}$ . Then

$$A_2(\nu) = B_2(\nu), \quad \text{for all } \nu.$$

**Theorem 1.6.3** Let  $A_3(\nu)$  denote the number of  $n$ -color partitions of  $\nu$  such that

- (1.1) even parts appear with even subscripts and odd with odd subscripts greater than 1, and
- (1.2) the weighted difference of each pair of parts is even and  $\geq -2$ .

Let  $B_3(\nu)$  denote the number of ordinary partitions of  $\nu$  into parts which are not congruent to  $0, \pm 1, \pm 8, \pm 9, 10 \pmod{20}$ . Then

$$A_3(\nu) = B_3(\nu), \quad \text{for all } \nu.$$

**Theorem 1.6.4** Let  $A_4(\nu)$  denote the number of  $n$ -color partitions of  $\nu$  with “ $n+1$  copies of  $n$ ” such that

- (1.1) even parts appear with odd subscripts and odd with even,
- (1.2) each pair of parts has nonnegative even weighted difference,
- (1.3) for some  $i, i_{i+1}$  is a part, and
- (1.4) the parts are nonnegative.

Let  $B_4(\nu)$  denote the number of ordinary partitions of  $\nu$  into parts which are not congruent to  $0, \pm 3, \pm 4, \pm 7, 10 \pmod{20}$ . Then

$$A_4(\nu) = B_4(\nu), \quad \text{for all } \nu.$$

**Theorem 1.6.5** Let  $A_5(\nu)$  denote the number of partitions of  $\nu$  with “ $n + 2$  copies of  $n$ ” such that  
(1.1) even parts appear with even subscripts and odd with odd,  
(1.2) each pair of parts has nonnegative even weighted difference,  
(1.3) for some  $i, i_{i+2}$  is a part, and  
(1.4) the parts are nonnegative.

Let  $B_5(\nu)$  denote the number of ordinary partitions of  $\nu$  into parts which are either odd or  $\equiv \pm 8 \pmod{20}$ . Then

$$A_5(\nu) = B_5(\nu), \quad \text{for all } \nu.$$

**Theorem 1.6.6** Let  $A_6(\nu)$  denote the number of partitions of  $\nu$  with “ $n + 2$  copies of  $n$ ” such that  
(1.1) even parts appear with even subscripts and odd with odd  $> 1$ ,  
(1.2) each pair of parts has nonnegative even weighted difference,  
(1.3) for some  $i, i_{i+2}$  is a part, and  
(1.4) the parts are nonnegative.

Let  $B_6(\nu)$  denote the number of ordinary partitions of  $\nu$  into parts which are not congruent to  $0, \pm 2, \pm 3, \pm 5, 8 \pmod{16}$ . Then

$$A_6(\nu) = B_6(\nu), \quad \text{for all } \nu.$$

## 1.7 Lattice Paths

The theory of partitions have a very close relationship with the theory of lattice paths. Agarwal and Bressoud [5] studied a new class of weighted lattice paths and used them to interpret several  $q$ -series combinatorially. To explain it more let us recall the following description of lattice paths from [5] which we shall be considering in this thesis.

All lengths will be of finite length lying in the the first quadrant. They will begin on the  $y$ -axis and terminate on the  $x$ -axis. Only three moves are allowed at each step:

northeast: from  $(i, j)$  to  $(i + 1, j + 1)$ ,

southeast: from  $(i, j)$  to  $(i + 1, j - 1)$ , only allowed if  $j > 0$ ,

horizontal: from  $(i, 0)$  to  $(i + 1, 0)$ , only allowed along  $x$ -axis.

All our lattice paths are either empty or terminate with a southeast step: from  $(i, 1)$  to  $(i + 1, 0)$ .

The following terminology will be used in describing lattice paths:

**PEAK:** Either a vertex on the  $y$ -axis which is followed by a southeast step.

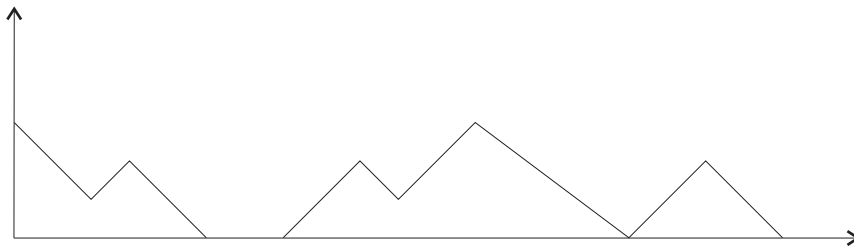
**VALLEY:** A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.

**MOUNTAIN:** A section of the path which starts on either the  $x$ - or  $y$ -axis, which ends on the  $x$ -axis, and which does not touch the  $x$ -axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.

**PLAIN:** A section of the path consisting of only horizontal steps which starts either on the  $y$ -axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.

The **HEIGHT** of a vertex is its  $y$ -coordinate. The weight of a vertex is its  $x$ -coordinate. The **WEIGHT OF A PATH** is the sum of the weights of its peaks.

**Example 13.** The following path has 5 peaks, 3 valleys, 3 mountains and 1 plain.



Agarwal and Bressoud [5] interpreted the left hand side of (1. 5. 1) and (1. 5. 2) as generating functions for certain weighted lattice path functions. Further Agarwal in [3] interpreted (1. 6. 1)-(1. 6. 6) using lattice paths. These results are given in Chapter 2.

## Chapter 2

# Lattice Paths and $q$ -Series

In Chapter 1, Agarwal [1] interpreted combinatorially some  $q$ -identities from Slater's compendium by using  $n$ -color partitions and give a different combinatorial meaning to the same  $q$ -series using lattice paths. This lead to new combinatorial identities involving lattice paths.

### 2.1 Combinatorial Interpretations of $q$ -Series using Lattice Paths

Agarwal and Bressoud [5] has interpreted the left hand side of (1. 5. 1) and (1. 5. 2) as the generating function for certain weighted lattice path functions. Their results may be stated as follows:

**Theorem 2.1.1** *Let  $B_1(\nu)$  denote the number of lattice paths of weight  $\nu$  which start at  $(0, 0)$ , have no valley and there is a plain of minimal length one between any two mountains. Then*

$$\sum_{\nu=0}^{\infty} B_1(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q; q)_n(q; q^2)_n}.$$

**Theorem 2.1.2** *Let  $B_2(\nu)$  denote the number of lattice paths of weight  $\nu$  which start at  $(0, 0)$ , have no valley above height 0. Then*

$$\sum_{\nu=0}^{\infty} B_2(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n(q; q^2)_n}.$$

The above two theorems in view of (1. 5. 1) and (1. 5. 2) lead to the following identities;

**Theorem 2.1.3**  $A_1(\nu) = B_1(\nu)$ , for all  $\nu$ , where  $A_1(\nu)$  is as defined in Theorem (1. 5. 1) of Chapter 1.

**Theorem 2.1.4**  $A_2(\nu) = B_2(\nu)$ , for all  $\nu$ , where  $A_2(\nu)$  is as defined in Theorem (1. 5. 2) of Chapter 1.

In [2], Agarwal gave  $n$ -color partition-theoretic interpretations of (1.6.1)-(1.6.6) in Chapter 1 and he further extended his results to lattice paths in [3] given below:

**Theorem 2.1.5** *Let  $C_1(\nu)$  denote the number of lattice paths of weight  $\nu$  which start at  $(0, 0)$ , have no valley above height 0, no plain with odd length. Then*

$$\sum_{\nu=0}^{\infty} C_1(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_{2n}}, \quad \text{for all } \nu.$$

**Theorem 2.1.6** *Let  $C_2(\nu)$  denote the number of lattice paths of weight  $\nu$  which start at  $(0, 0)$ , have no valley above height 0, no plain with odd length and for which the minimal height of each peak is 2. Then*

$$\sum_{\nu=0}^{\infty} C_2(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q)_{2n}}, \quad \text{for all } \nu.$$

**Theorem 2.1.7** *Let  $C_3(\nu)$  denote the number of lattice paths of weight  $\nu$  which start at  $(0, 0)$ , have no valley above height 0, no plain with odd length except in the beginning of the path where there is always a plain with odd length. Then*

$$\sum_{\nu=0}^{\infty} C_3(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_{2n}}, \quad \text{for all } \nu.$$

**Theorem 2.1.8** *Let  $C_4(\nu)$  denote the number of lattice paths of weight  $\nu$  which start at  $(0, 1)$ , have no valley above height 0, no plain with odd length. Then*

$$\sum_{\nu=0}^{\infty} C_4(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_{2n+1}}, \quad \text{for all } \nu.$$

**Theorem 2.1.9** *Let  $C_5(\nu)$  denote the number of lattice paths of weight  $\nu$  which start from  $(0, 2)$ , have no valley above height 0, no plain with odd length. Then*

$$\sum_{\nu=0}^{\infty} C_5(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q; q)_{2n+1}}, \quad \text{for all } \nu.$$

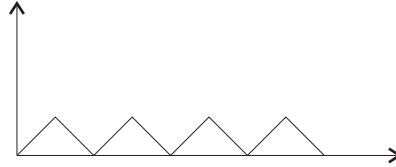
**Theorem 2.1.10** *Let  $C_6(\nu)$  denote the number of lattice paths of weight  $\nu$  which start from  $(0, 2)$ , have no valley above height 0, no plain with odd length and for which the minimal height of each peak is 2. Then*

$$\sum_{\nu=0}^{\infty} C_6(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q)_{2n+1}}, \quad \text{for all } \nu.$$

## 2.2 Proofs of Theorems (2. 1. 5)-(2. 1. 10)

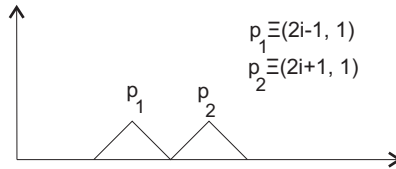
### Proof of Theorem 2. 1. 5

In  $\frac{q^{m^2}}{(q;q)_{2m}} = \frac{q^{m^2}}{(q^2;q^2)_m(q;q^2)_m}$ , the factor  $q^{m^2}$  generates the lattice path of  $m$  peaks starting at  $(0,0)$  and terminating at  $(2m,0)$ . For  $m = 4$ , the path begins as



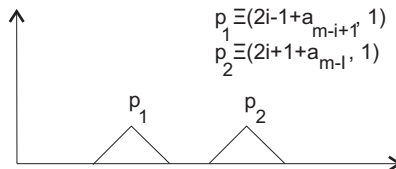
Graph A

In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



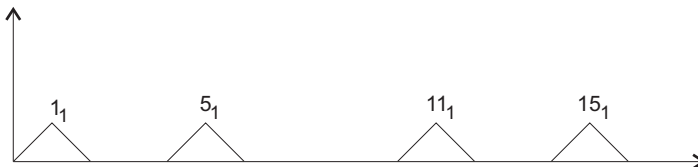
Graph B

The factor  $\frac{1}{(q^2;q^2)_m}$  generates  $m$  nonnegative even parts, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.



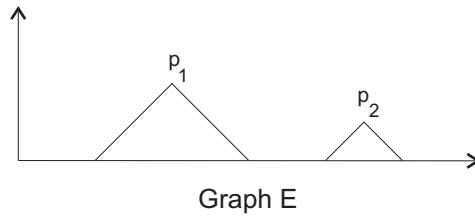
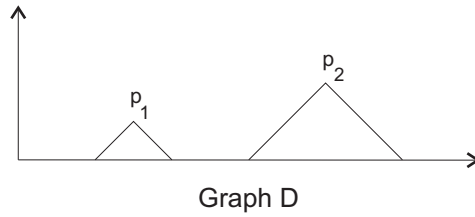
Graph C

If  $a_1 = 8, a_2 = 6, a_3 = 2, a_4 = 0$ , then graph becomes

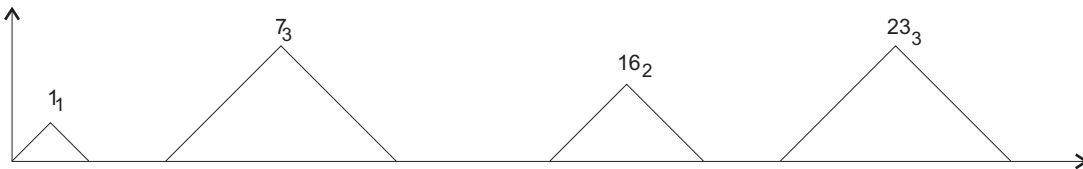


The factor  $\frac{1}{(q;q^2)_m}$  generates nonnegative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+1} + 1$ . Each

increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, Graph C now changes to Graph D or Graph E, depending on whether  $b_{m-i} > b_{m-i+1}$  or  $b_{m-i} < b_{m-i+1}$ . The new graph looks like graph D or graph E, respectively.



If  $b_1 = 2, b_2 = 1, b_3 = 2, b_4 = 0$ , then graph becomes

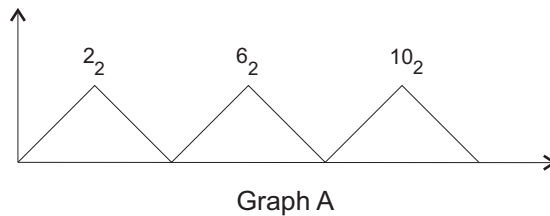


Every lattice path starting at  $(0, 0)$  with all valleys at height 0 and no plain with odd length is uniquely generated in this manner. This proves that the left hand side of equation (1. 6. 1) enumerates  $C_1$ .

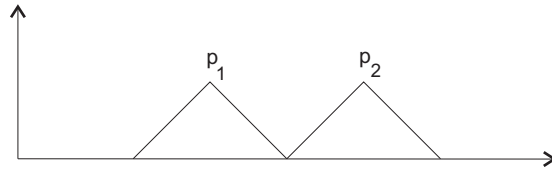
This completes the proof.

### Proof of Theorem 2. 1. 6

In  $\frac{q^{2m^2}}{(q;q)_{2m}} = \frac{q^{2m^2}}{(q^2;q^2)_m (q;q^2)_m}$ , the factor  $q^{2m^2}$  generates the lattice path of  $m$  peaks each of height 2 starting at  $(0, 0)$  and terminating at  $(4m, 0)$ . For  $m = 3$ , the path begins as

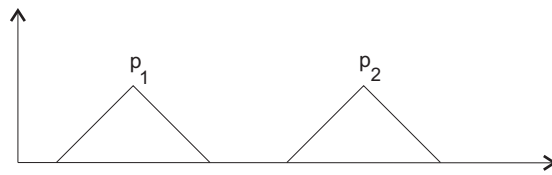


In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



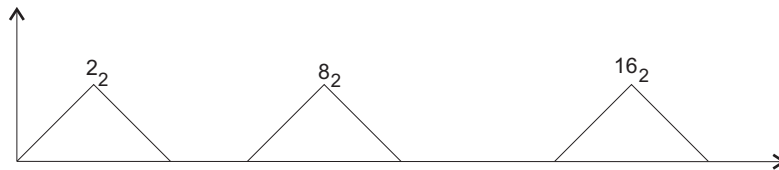
Graph B

The factor  $\frac{1}{(q^2; q^2)_m}$  generates  $m$  nonnegative even parts, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.

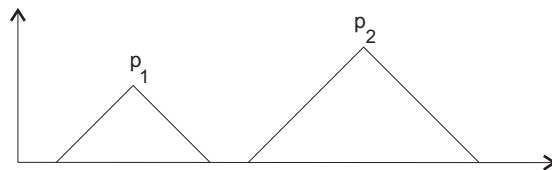


Graph C

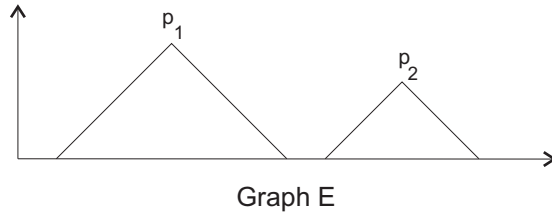
If  $a_1 = 6, a_2 = 2, a_3 = 0$ , then the graph becomes



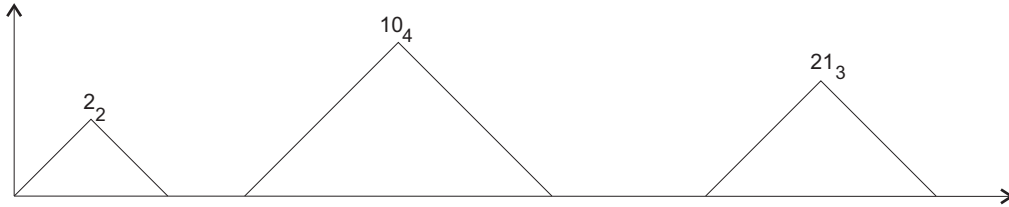
The factor  $\frac{1}{(q; q^2)_m}$  generates nonnegative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+1} + 2$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, Graph C now changes to Graph D or Graph E, depending on whether  $b_{m-i} > b_{m-i+1}$  or  $b_{m-i} < b_{m-i+1}$ . The new graph looks like graph D or graph E, respectively.



Graph D



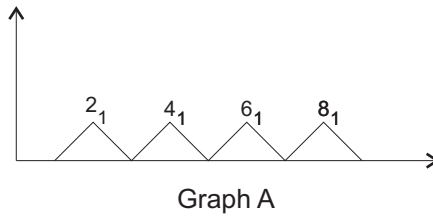
If  $b_1 = 1, b_2 = 2, b_3 = 0$ , then graph becomes



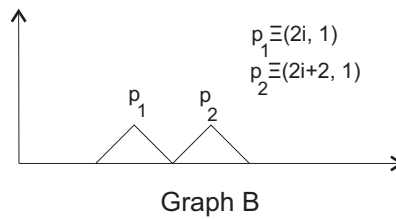
Every lattice path starting at  $(0,0)$  with all valleys at height 0 and no plain with odd length and for which minimal height of each peak is 2 is uniquely generated in this manner. This proves that the left hand side of equation (6. 1. 2) enumerates  $C_2$ . This completes the proof.

**Proof of Theorem 2. 1. 7**

In  $\frac{q^{m(m+1)}}{(q;q)_{2m}} = \frac{q^{m(m+1)}}{(q^2;q^2)_m(q;q^2)_m}$ , the factor  $q^{m(m+1)}$  generates the lattice path of  $m$  peaks each of height 1 in which  $q^m$  puts a horizontal step in front of the first peak i. e. path starts from  $(1,0)$  and terminating at  $(2m + 1, 0)$ . For  $m = 4$ , the path begins as

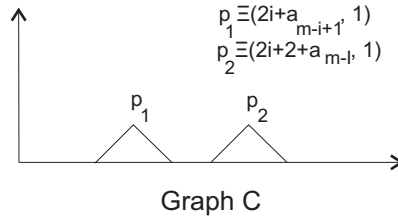


In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.

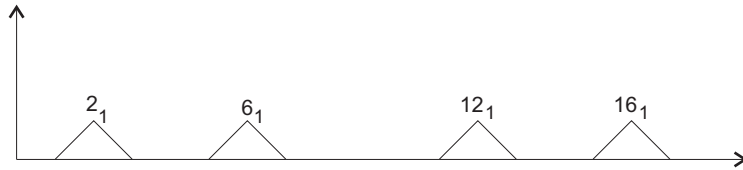


The factor  $\frac{1}{(q^2;q^2)_m}$  generates  $m$  nonnegative even parts, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is

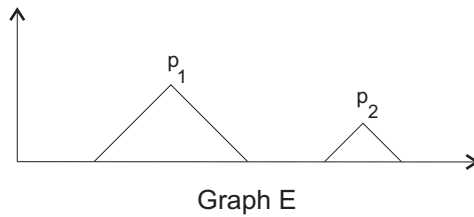
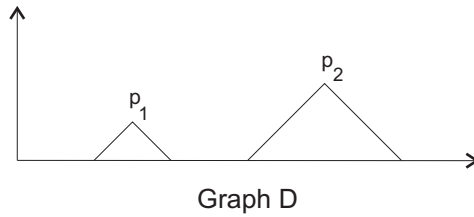
increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.



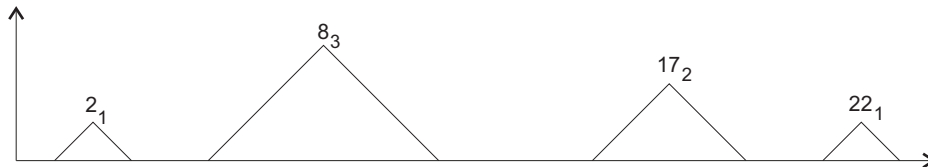
If  $a_1 = 8, a_2 = 6, a_3 = 2, a_4 = 0$ , then the graph becomes



The factor  $\frac{1}{(q; q^2)_m}$  generates nonnegative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+1} + 1$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, Graph C now changes to Graph D or Graph E, depending on whether  $b_{m-i} > b_{m-i+1}$  or  $b_{m-i} < b_{m-i+1}$ . The new graph looks like graph D or graph E, respectively.



If  $b_1 = 0, b_2 = 1, b_3 = 2, b_4 = 0$ , then graph becomes

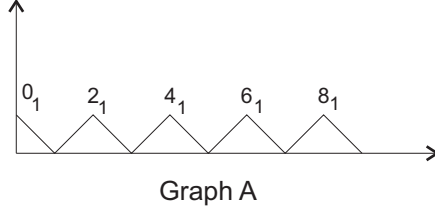


Every lattice path starting at  $(0,0)$  with all valleys at height 0 and no plain with odd length except in the beginning of the path where there is always a plain with odd length is uniquely generated in this manner. This proves that the left hand side of equation (1.6.3) enumerates  $C_3$ .

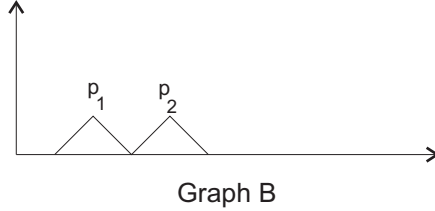
This completes the proof.

**Proof of Theorem 2. 1. 8**

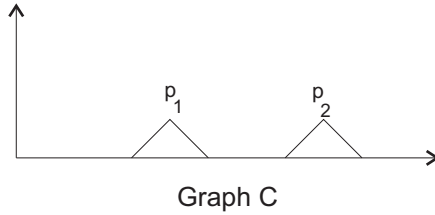
In  $\frac{q^{m(m+1)}}{(q;q)_{2m+1}} = \frac{q^{m(m+1)}}{(q^2;q^2)_m(q;q^2)_{m+1}}$ , the factor  $q^{m(m+1)}$  generates the lattice path of  $m + 1$  peaks starting at  $(0, 1)$  to  $(1, 0)$  at the front of lattice path. For  $m = 4$ , the path begins as



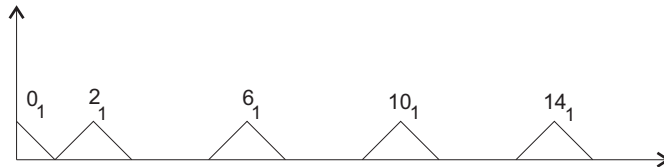
In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



The factor  $\frac{1}{(q^2;q^2)_m}$  generates  $m$  nonnegative even parts, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.

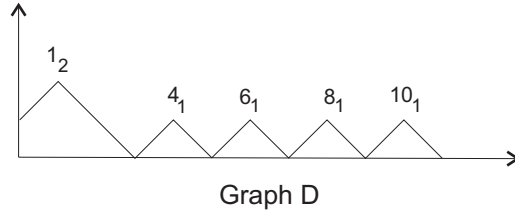


If  $a_1 = 6, a_2 = 4, a_3 = 2, a_4 = 0$ , then the graph becomes



The factor  $\frac{1}{(q;q^2)_{m+1}}$  generates nonnegative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m + 1$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1), b_{m+1} \times (2m + 1)$ . This is encoded by having the  $i$ th peak grow to height

$b_{m-i+2} + 1$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, we get Graph D.

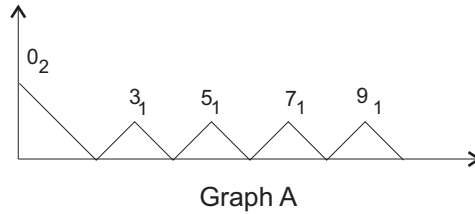


Every lattice path starting at  $(0, 1)$  with all valleys at height 0 and no plain with odd length is uniquely generated in this manner. This proves that the left hand side of equation (1. 6. 4) enumerates  $C_4$ .

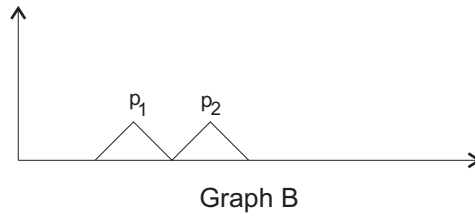
This completes the proof.

**Proof of Theorem 2. 1. 9**

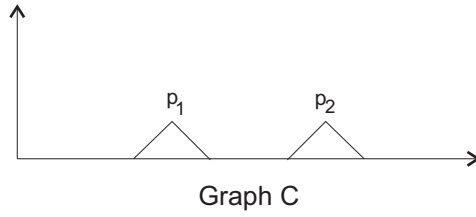
In  $\frac{q^{m(m+2)}}{(q; q)_{2m+1}} = \frac{q^{m(m+2)}}{(q^2; q^2)_m (q; q^2)_{m+1}}$ , the factor  $q^{m(m+2)}$  generates the lattice path of  $m + 1$  peaks starting at  $(0, 2)$  to  $(1, 1)$  and from  $(1, 1)$  to  $(2, 0)$  in front of lattice path. For  $m = 4$ , the path begins as



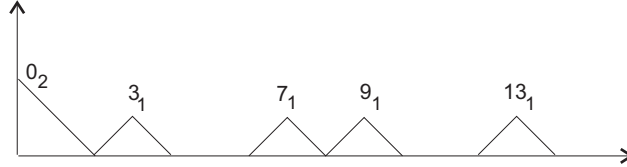
In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



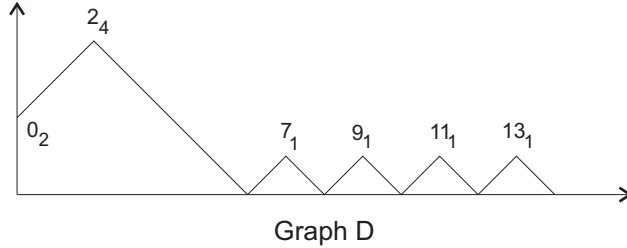
The factor  $\frac{1}{(q^2; q^2)_m}$  generates  $m$  nonnegative even parts, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.



If  $a_1 = 4, a_2 = 2, a_3 = 2, a_4 = 0$ , then the graph becomes



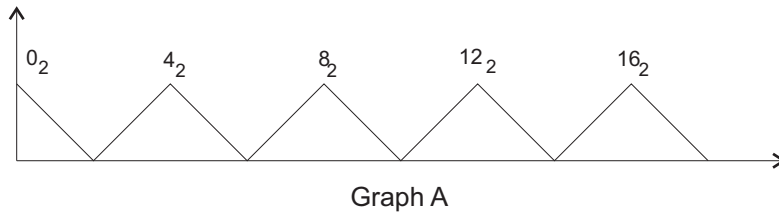
The factor  $\frac{1}{(q; q^2)_{m+1}}$  generates nonnegative multiples of  $(2i - 1), 1 \leq i \leq m + 1$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1), b_{m+1} \times (2m + 1)$ . This is encoded by having the 1st peak grow to height  $b_{m+1} + 2$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, we get Graph D.



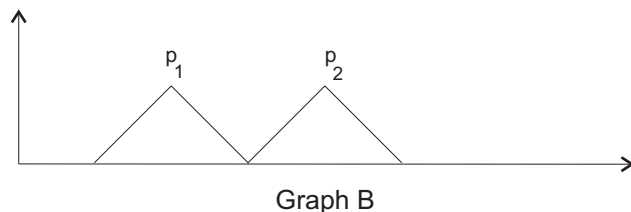
Every lattice path starting from  $(0, 2)$ , have no valley above height 0, no plain with odd length is uniquely generated in this manner. This proves that the left hand side of equation (1. 6. 5) enumerates  $C_5$ .  
This completes the proof.

**Proof of Theorem 2. 1. 10**

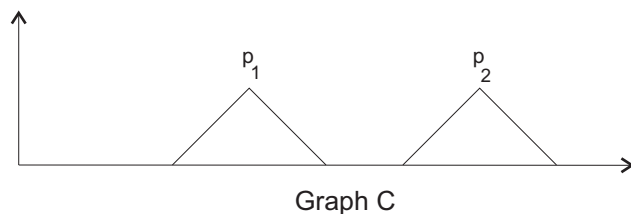
In  $\frac{q^{2m(m+1)}}{(q; q)_{2m+1}} = \frac{q^{2m(m+1)}}{(q^2; q^2)_m (q; q^2)_{m+1}}$ , the factor  $q^{2m^2+2m}$  generates the lattice path of  $m + 1$  peaks starting from  $(0, 2)$  to  $(1, 1)$  and from  $(1, 1)$  to  $(2, 0)$  in the front of lattice path of peaks each of height 2. For  $m = 4$ , the path begins as



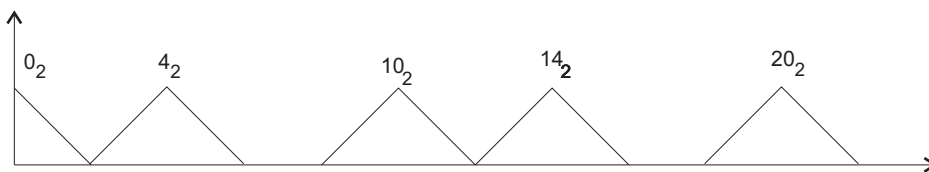
In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



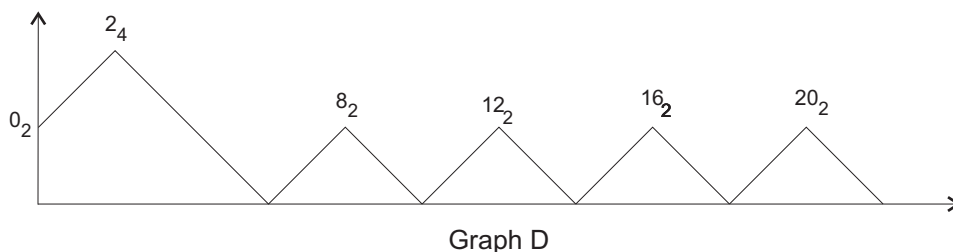
The factor  $\frac{1}{(q^2; q^2)_m}$  generates  $m$  nonnegative even parts, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.



If  $a_1 = 4, a_2 = 2, a_3 = 2, a_4 = 0$ , then the graph becomes



The factor  $\frac{1}{(q; q^2)_{m+1}}$  generates nonnegative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1), b_{m+1} \times (2m + 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+2} + 2$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, we get Graph D.



Every lattice path starting from  $(0, 2)$ , have no valley above height 0, no plain with odd length and for which the minimal height of each peak is 2 is uniquely generated in this manner. This proves that the left hand side of equation (1. 6. 6) enumerates  $C_6$ . This completes the proofs of the Theorems (2. 1. 5)-(2. 1. 10).

## 2.3 Some More $q$ -Series Identities

Recently Megha and Agarwal [10] interpreted some more  $q$ -series identities given below using  $n$ -color partitions.

$$\sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q; q^2)_n (q^4; q^4)_n} = \frac{(-q^3, -q^5, -q^7; q^{10})_{\infty}}{(q^4, q^6; q^{10})_{\infty}} \quad (2.3.1)$$

$$\sum_{n=0}^{\infty} \frac{q^{3n^2-2n}}{(q; q^2)_n (q^4; q^4)_n} = \frac{(-q, -q^5, -q^9; q^{10})_{\infty}}{(q^2, q^8; q^{10})_{\infty}} \quad (2.3.2)$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q^2)_n (q^4; q^4)_n} = \frac{(-q^3, -q^7, -q^{11}; q^{14})_{\infty}}{(q^2, q^6, q^8, q^{12}; q^{14})_{\infty}} \quad (2.3.3)$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_n (q^4; q^4)_n} = \frac{(-q^5, -q^7, -q^9; q^{14})_{\infty}}{(q^4, q^6, q^8, q^{10}; q^{14})_{\infty}} \quad (2.3.4)$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1} (q^4; q^4)_n} = \frac{(-q, -q^7, -q^{13}; q^{14})_{\infty}}{(q^2, q^4, q^{10}, q^{12}; q^{14})_{\infty}} \quad (2.3.5)$$

They further extended their results using lattice paths. Agarwal and Megha [8] provide the combinatorial interpretations of left hand side of (2.3.1)-(2.3.5) using lattice paths given in [5].

**Theorem 2.3.1** *Let  $A_1(\nu)$  denote the number of lattice paths of weight  $\nu$  which start from  $(0, 0)$ , have no valley above height 0, the lengths of plains, if any, are  $\equiv 0 \pmod{4}$  and the height of each peak is greater than 2. Then*

$$\sum_{\nu=0}^{\infty} A_1(\nu) q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q; q^2)_n (q^4; q^4)_n}, \quad \text{for all } \nu.$$

**Theorem 2.3.2** *Let  $A_2(\nu)$  denote the number of lattice paths of weight  $\nu$  which start from  $(0, 0)$ , have no valley above height 0, the lengths of plains are  $\equiv 0 \pmod{4}$  and there is a plain of length  $\geq 4$  between any two peaks. Then*

$$\sum_{\nu=0}^{\infty} A_2(\nu) q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{3n^2-2n}}{(q; q^2)_n (q^4; q^4)_n}, \quad \text{for all } \nu.$$

**Theorem 2.3.3** *Let  $A_3(\nu)$  denote the number of lattice paths of weight  $\nu$  which start from  $(0, 0)$ , have no valley above height 0, the lengths of plains, if any, are  $\equiv 0 \pmod{4}$  and the height of each peak is greater than 1. Then*

$$\sum_{\nu=0}^{\infty} A_3(\nu) q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q^2)_n (q^4; q^4)_n}, \quad \text{for all } \nu.$$

**Theorem 2.3.4** Let  $A_4(\nu)$  denote the number of lattice paths of weight  $\nu$  which start from  $(0,0)$ , have no valley above height 0, there is a plain of length  $\equiv 2(\text{mod}4)$  in the beginning of the path and the lengths of the other plains, if any, are  $\equiv 0(\text{mod}4)$  and the height of each peak is greater than 1. Then

$$\sum_{\nu=0}^{\infty} A_4(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_n (q^4; q^4)_n}, \quad \text{for all } \nu.$$

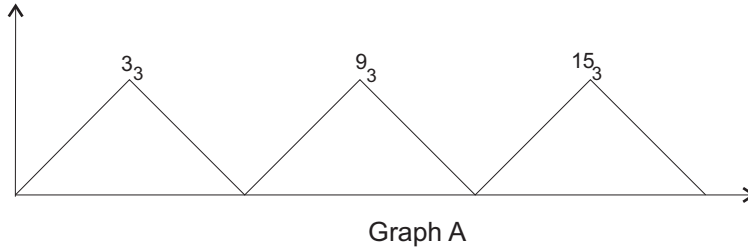
**Theorem 2.3.5** Let  $A_5(\nu)$  denote the number of lattice paths of weight  $\nu$  which start from  $(0,2)$ , have no valley above height 0, the lengths of the plains, if any, are  $\equiv 0(\text{mod}4)$  and the height of each peak is greater than 1. Then

$$\sum_{\nu=0}^{\infty} A_5(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1} (q^4; q^4)_n}, \quad \text{for all } \nu.$$

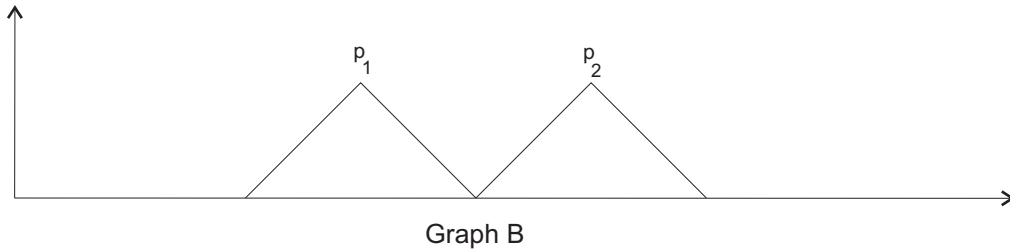
## 2.4 Proofs of Theorems given in Section (2. 3)

### Proof of Theorem 2. 3. 1

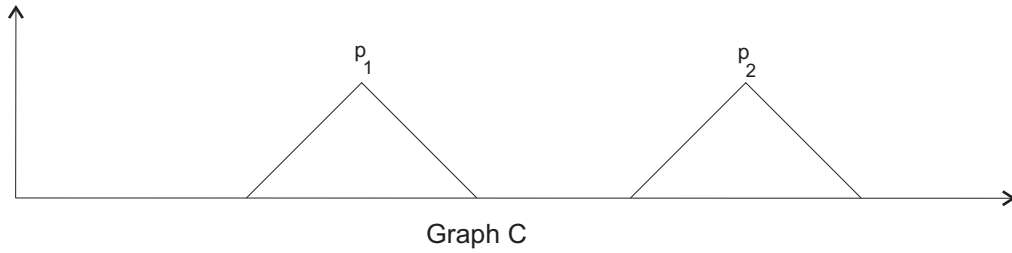
In  $\frac{q^{3m^2}}{(q; q^2)_m (q^4; q^4)_m}$ , the factor  $q^{3m^2}$  generates the lattice path of  $m$  peaks each of height 3 starting at  $(0,0)$  and terminating at  $(6m,0)$ . For  $m = 3$ , the path begins as



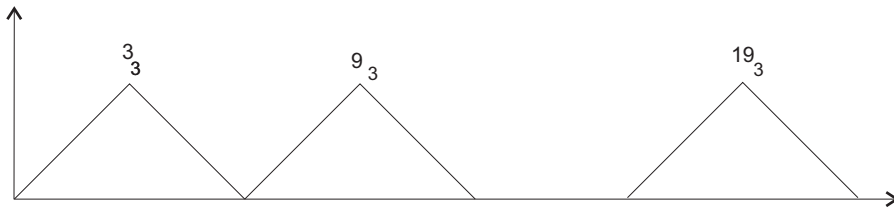
In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



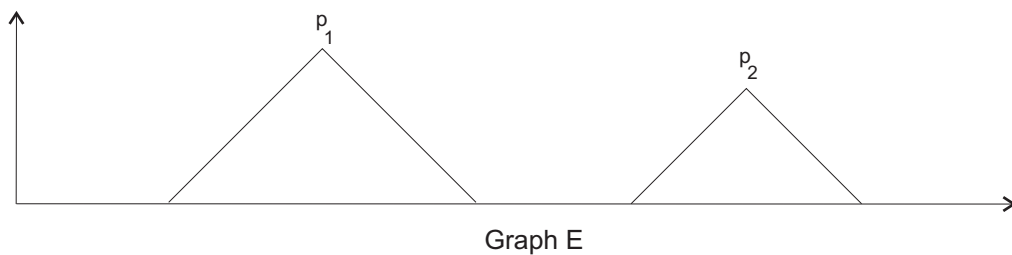
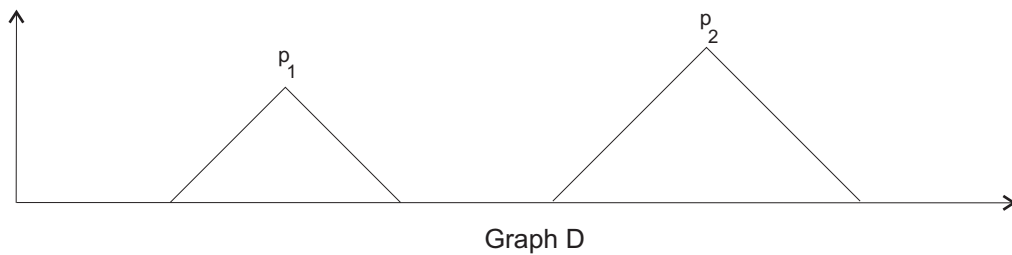
The factor  $\frac{1}{(q^4; q^4)_m}$  generates  $m$  nonnegative multiples of 4, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.



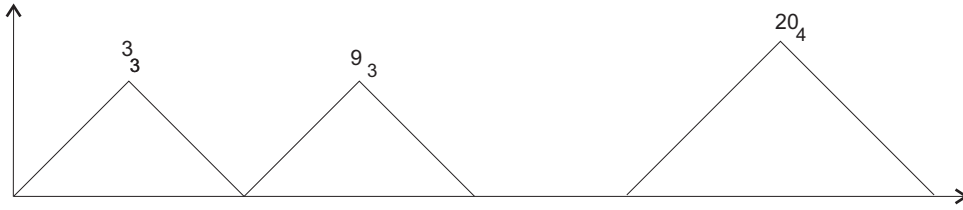
If  $a_1 = 4, a_2 = 0, a_3 = 0$ , then graph becomes



The factor  $\frac{1}{(q; q^2)_m}$  generates nonnegative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+1} + 3$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, Graph C now changes to Graph D or Graph E, depending on whether  $b_{m-i} > b_{m-i+1}$  or  $b_{m-i} < b_{m-i+1}$ . The new graph looks like graph D or graph E, respectively.



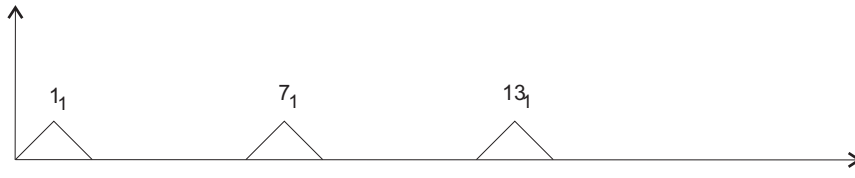
If  $b_1 = 1, b_2 = 0, b_3 = 0$ , then graph becomes



Every lattice path starting from  $(0, 0)$ , have no valley above height 0, the lengths of plains, if any, are  $\equiv 0 \pmod{4}$  and the height of each peak is greater than 2 is uniquely generated in this manner. This proves that the left hand side of equation (2. 3. 1) enumerates  $A_1$ . This completes the proof.

### Proof of Theorem 2. 3. 2

In  $\frac{q^{3m^2-2m}}{(q; q^2)_m (q^4; q^4)_m}$ , the factor  $q^{3m^2-2m}$  generates the lattice path of  $m$  peaks each of height 1 starting at  $(0, 0)$  and with a plain of length 4 between  $i$ th and  $(i+1)$ th peak where  $1 \leq i \leq m - 1$ . For  $m = 3$ , the path begins as



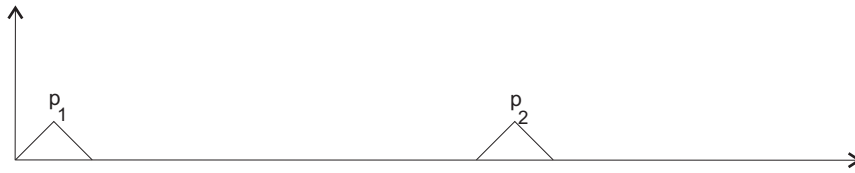
Graph A

In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



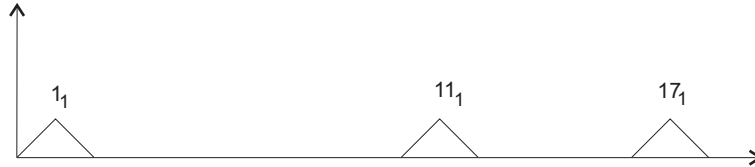
Graph B

The factor  $\frac{1}{(q^4; q^4)_m}$  generates  $m$  nonnegative multiples of 4, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.

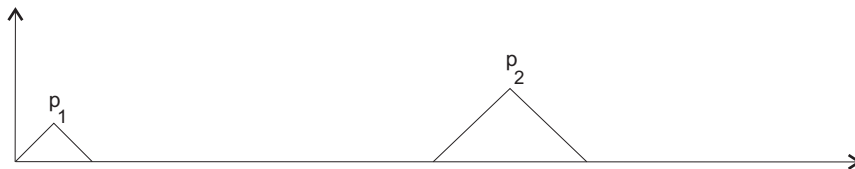


Graph C

If  $a_1 = 4, a_2 = 4, a_3 = 0$ , then graph becomes



The factor  $\frac{1}{(q; q^2)_m}$  generates nonnegative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+1} + 1$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, Graph C now changes to Graph D or Graph E, depending on whether  $b_{m-i} > b_{m-i+1}$  or  $b_{m-i} < b_{m-i+1}$ . The new graph looks like graph D or graph E, respectively.

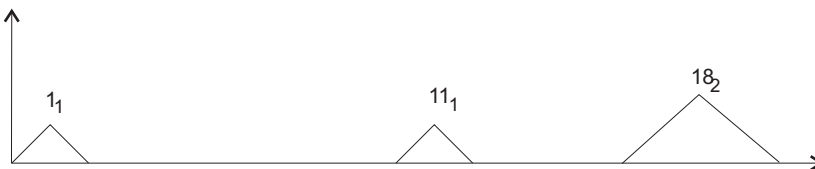


Graph D



Graph E

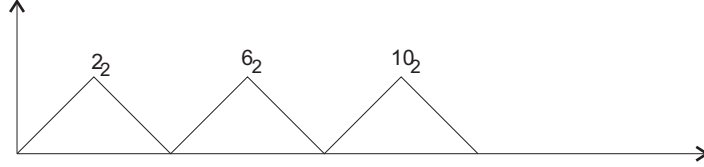
If  $b_1 = 1, b_2 = 0, b_3 = 0$ , then graph becomes



Every lattice path starting from  $(0, 0)$ , have no valley above height 0, the lengths of plains are  $\equiv 0 \pmod{4}$  and there is a plain of length  $\geq 4$  between any two peaks is uniquely generated in this manner. This proves that the left hand side of equation (2. 3. 2) enumerates  $A_2$ . This completes the proof.

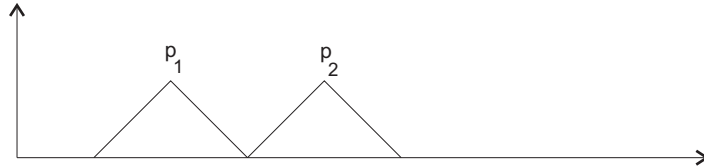
**Proof of Theorem 2. 3. 3**

In  $\frac{q^{2m^2}}{(q;q^2)_m(q^4;q^4)_m}$ , the factor  $q^{2m^2}$  generates the lattice path of  $m$  peaks each of height 2 starting at  $(0,0)$  and terminating at  $(4m,0)$ . For  $m = 3$ , the path begins as



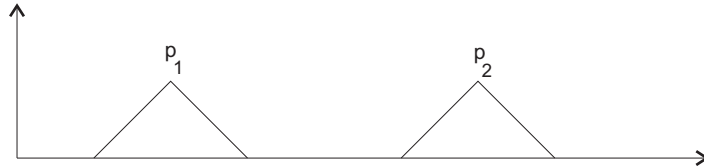
Graph A

In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



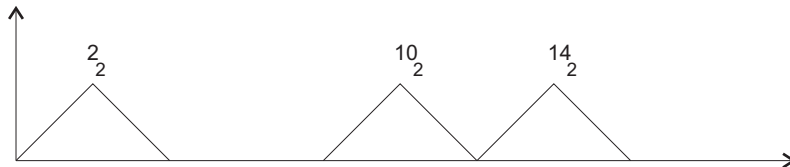
Graph B

The factor  $\frac{1}{(q^4;q^4)_m}$  generates  $m$  nonnegative multiples of 4, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.



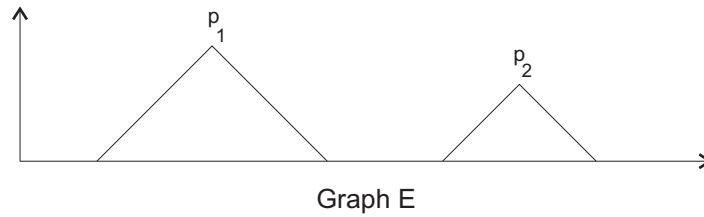
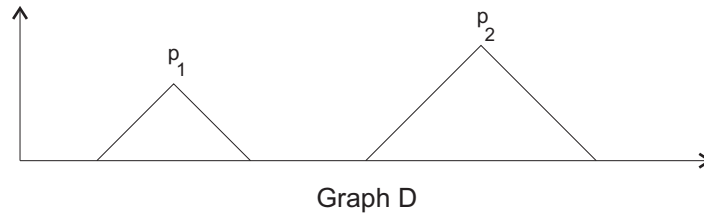
Graph C

If  $a_1 = 4, a_2 = 4, a_3 = 0$ , then graph becomes

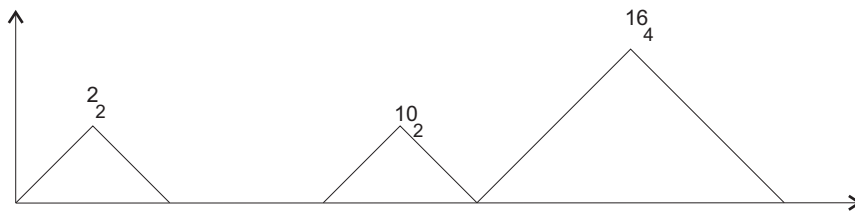


The factor  $\frac{1}{(q;q^2)_m}$  generates nonnegative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+1} + 2$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, Graph C now changes to Graph D or Graph E, depending on whether

$b_{m-i} > b_{m-i+1}$  or  $b_{m-i} < b_{m-i+1}$ . The new graph looks like graph D or graph E, respectively.



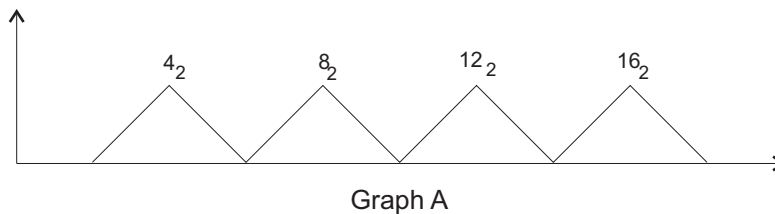
If  $b_1 = 2, b_2 = 0, b_3 = 0$ , then graph becomes



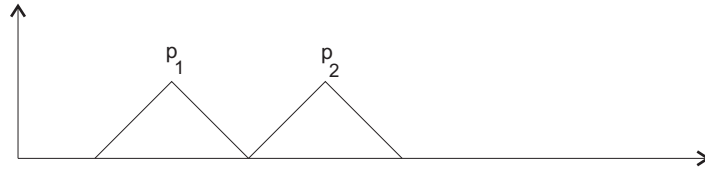
Every lattice path starting from  $(0, 0)$ , have no valley above height 0, the lengths of plains, if any, are  $\equiv 0 \pmod{4}$  and the height of each peak is greater than 1 is uniquely generated in this manner. This proves that the left hand side of equation (2. 3. 3) enumerates  $A_3$ . This completes the proof.

**Proof of Theorem 2. 3. 4**

In  $\frac{q^{2m(m+1)}}{(q;q^2)_m(q^4;q^4)_m}$ , the factor  $q^{2m(m+1)}$  generates the lattice path of  $m$  peaks each of height 2 in which  $q^{2m}$  puts a plain of length 2 in front of the first peak. For  $m = 4$ , the path begins as

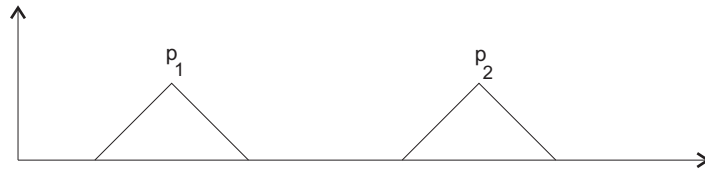


In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



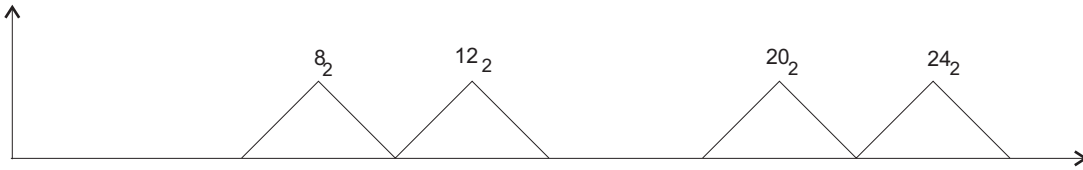
Graph B

The factor  $\frac{1}{(q^4; q^4)_m}$  generates  $m$  nonnegative multiples of 4, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.

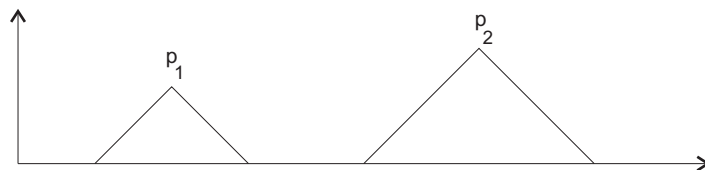


Graph C

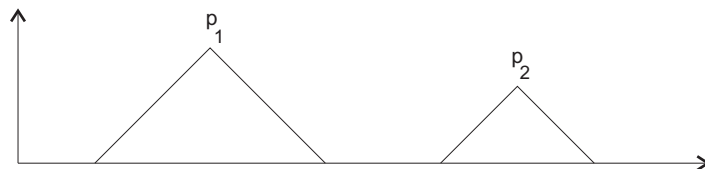
If  $a_1 = 8, a_2 = 8, a_3 = 4, a_4 = 4$ , then graph becomes



The factor  $\frac{1}{(q; q^2)_m}$  generates nonnegative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+1} + 2$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, Graph C now changes to Graph D or Graph E, depending on whether  $b_{m-i} > b_{m-i+1}$  or  $b_{m-i} < b_{m-i+1}$ . The new graph looks like graph D or graph E, respectively.

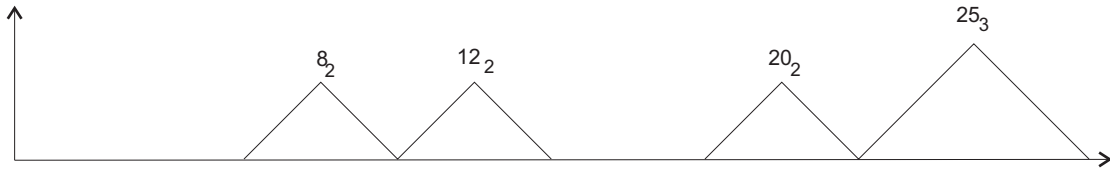


Graph D



Graph E

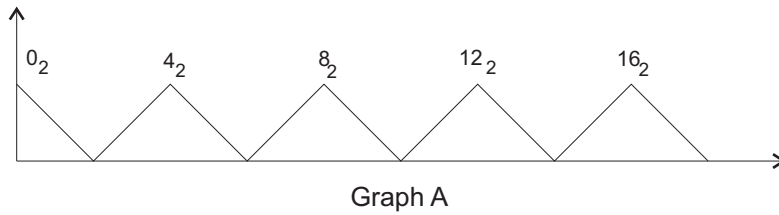
If  $b_1 = 1, b_2 = 0, b_3 = 0, b_4 = 0$ , then graph becomes



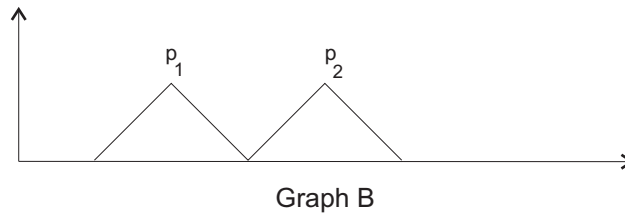
Every lattice path starting from  $(0, 0)$ , have no valley above height 0, there is a plain of length  $\equiv 2 \pmod{4}$  in the beginning of the path and the lengths of the other plains, if any, are  $\equiv 0 \pmod{4}$  and the height of each peak is greater than 1 is uniquely generated in this manner. This proves that the left hand side of equation (2. 3. 4) enumerates  $A_4$ . This completes the proof.

### Proof of Theorem 2. 3. 5

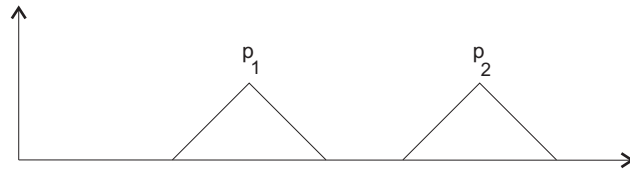
In  $\frac{q^{2m(m+1)}}{(q; q^2)_{m+1}(q^4; q^4)_m}$ , the factor  $q^{2m(m+1)}$  generates the lattice path of  $m + 1$  peaks. The extra factor  $q^{2m}$  puts two southeast steps: starting from  $(0, 2)$  to  $(1, 1)$  and from  $(1, 1)$  to  $(2, 0)$  in front of the lattice paths of peaks each of having height 2. For  $m = 4$ , the path begins as



In the graph A we consider two successive peaks, say  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.

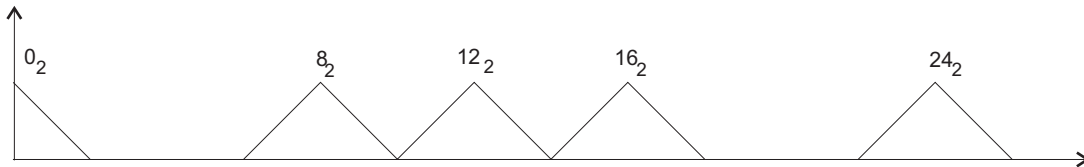


The factor  $\frac{1}{(q^4; q^4)_m}$  generates  $m$  nonnegative multiples of 4, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$  and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.

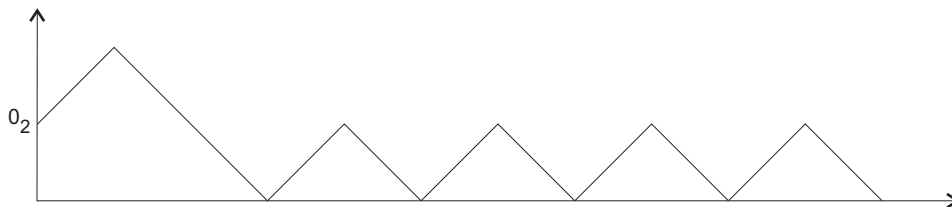


Graph C

If  $a_1 = 8, a_2 = 4, a_3 = 4, a_4 = 4$ , then graph becomes



The factor  $\frac{1}{(q; q^2)_{m+1}}$  generates nonnegative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m + 1$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1), b_{m+1} \times (2m + 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+2} + 2$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, we get Graph D.



Graph D

Every lattice path starting from  $(0, 2)$ , have no valley above height 0, the lengths of the plains, if any, are  $\equiv 0 \pmod{4}$  and the height of each peak is greater than 1 is uniquely generated in this manner. This proves that the left hand side of equation (2. 3. 5) enumerates  $A_5$ . This completes the proofs of the Theorems (2. 3. 1)-(2. 3. 5).

## Chapter 3

# Lattice Paths and Generalized $q$ -Series

To unify the  $q$ -series (1. 5. 1) and (1. 5. 2) given in Chapter 2, Agarwal in [4] gave three generalized  $q$ -series:

$$\sum_{\nu=0}^{\infty} B_1^k(\nu)q^\nu = \sum_{m=0}^{\infty} \frac{q^{m[1+\frac{(k+3)(m-1)}{2}]}}{(q; q)_m(q; q^2)_m} \quad (3.0.1)$$

$$\sum_{\nu=0}^{\infty} B_2^k(\nu)q^\nu = \sum_{m=0}^{\infty} \frac{q^{\frac{m(m+1)(k+3)}{2}}}{(q; q)_m(q; q^2)_{m+1}} \quad (3.0.2)$$

$$\sum_{\nu=0}^{\infty} B_3^k(\nu)q^\nu = \sum_{m=0}^{\infty} \frac{q^{m[1+\frac{(k+3)(m+1)}{2}]}}{(q; q)_m(q; q^2)_{m+1}} \quad (3.0.3)$$

and interpreted them using lattice paths in [4]. The combinatorial interpretations of (3.0.1)-(3.0.3) are given in next Section.

### 3.1 Main Theorems

**Theorem 3.1.1** For  $k \geq -1$ , let  $A_1^k(\nu)$  denote the number of lattice paths of weight  $\nu$  which start at a point on the  $x$ -axis have no valley above height 0 if  $k = -1$  and no valleys at all if  $k \geq 0$  and there is a plain of minimal length  $k + 1$  ( $k \geq 0$ ) between any two mountains. Then

$$A_1^k(\nu) = B_1^k(\nu), \quad \text{for all } \nu.$$

**Theorem 3.1.2** For  $k \geq -1$ , let  $A_2^k(\nu)$  denote the number of lattice paths of weight  $\nu$  which start from  $(0, 1)$  have no valley above height 0 if  $k = -1$  and no valleys at all if  $k > -1$  and there is a plain with minimum length  $k + 1$  ( $k \geq 0$ ) between any two mountains. Then

$$A_2^k(\nu) = B_2^k(\nu), \quad \text{for all } \nu.$$

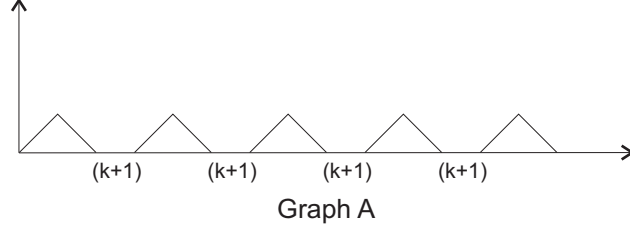
**Theorem 3.1.3** For  $k \geq -1$ , let  $A_3^k(\nu)$  denote the number of lattice paths of weight  $\nu$  which start from  $(0, 2)$  have no valley above height 0 if  $k = -1$  and no valleys at all if  $k > -1$  and there is a plain with minimal length  $k + 1$  ( $k \geq 0$ ) between any two mountains. Then

$$A_3^k(\nu) = B_3^k(\nu), \quad \text{for all } \nu.$$

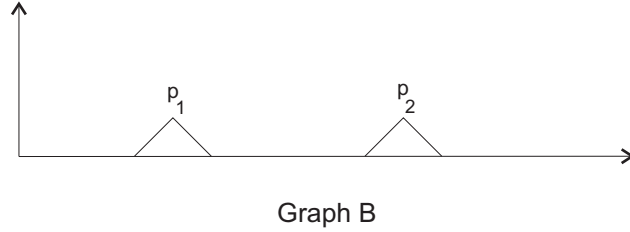
## 3.2 Proofs of Main Theorems

### Proof of Theorem 3. 1. 1

In  $\frac{q^{m[1+\frac{(k+3)(m-1)}{2}]}}{(q;q)_m(q;q^2)_m}$ , the factor  $q^{m[1+\frac{(k+3)(m-1)}{2}]}$  generates a lattice path from  $(0,0)$  to  $(2m + (m - 1)(k + 1), 0)$  having  $m$  peaks each of height 1 and a plain of length  $k + 1$  between any two successive peaks. Thus the path begins as



In the above graph we consider two successive peaks, say,  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



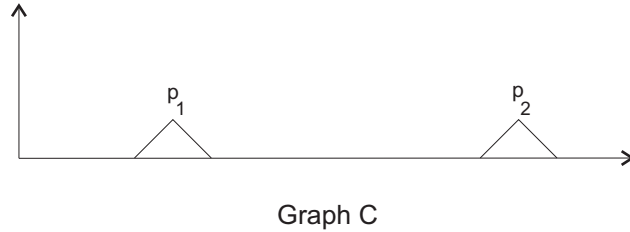
Clearly in Graph B

$$p_1 = ((2i - 1) + (i - 1)(k + 1), 1), \text{ and}$$

$$p_2 = ((2i + 1) + i(k + 1), 1).$$

The factor  $\frac{1}{(q;q)_m}$  generates  $m$  non-negative integers, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the first mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+1)$ st mountain,  $1 \leq i \leq m - 1$ . Thus, the  $x$ -coordinate of the  $i$ th peak is increased by  $a_m + (a_{m-1} - a_m) + \dots + (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$ , and the  $x$ -coordinate of the  $(i+1)$ th peak is increased by  $a_{m-i}$ . Graph B now becomes Graph C.

In Graph C,

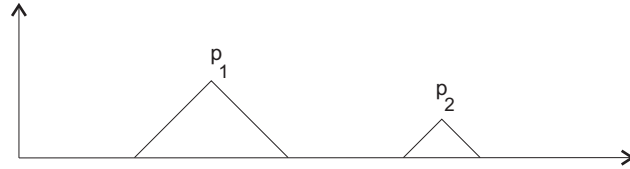
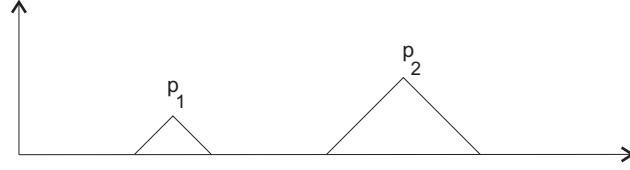


$$p_1 = ((2i - 1) + (i - 1)(k + 1) + a_{m-i+1}, 1), \text{ and}$$

$$p_2 = ((2i + 1) + i(k + 1) + a_{m-i}, 1).$$

The factor  $\frac{1}{(q; q^2)_m}$  generates non-negative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+1} + 1$ . Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, Graph C now changes to Graph D or Graph E, depending on whether  $b_{m-i} > b_{m-i+1}$  or  $b_{m-i} < b_{m-i+1}$ . The new graph looks like graph D or graph E, respectively.

In Graph D (or Graph E),



$$p_1 = ((2i - 1) + (i - 1)(k + 1) + a_{m-i+1} + 2(b_m + \dots + b_{m-i+2}) + b_{m-i+1},$$

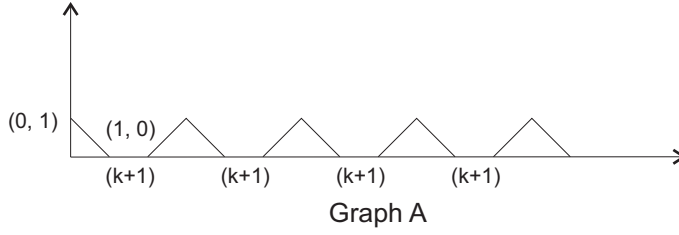
$$b_{m-i+1} + 1), \text{ and}$$

$$p_2 = ((2i + 1) + i(k + 1) + a_{m-i} + 2(b_m + \dots + b_{m-i+1}) + b_{m-i}, b_{m-i} + 1).$$

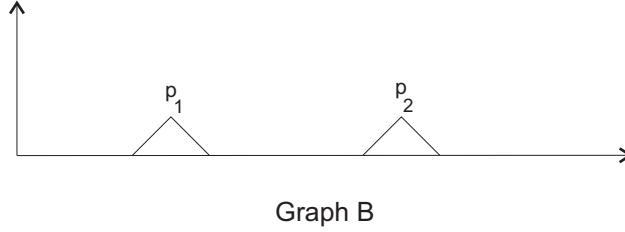
We see that every lattice path enumerated by  $A_1^k(\nu)$  is uniquely generated in this manner. This proves that right hand side of equation (3. 0. 1) also generates  $A_1^k(\nu)$ .

### Proof of Theorem 3. 1. 2

In  $\frac{q^{\frac{m(m+1)(k+3)}{2}}}{(q; q)_m (q; q^2)_{m+1}}$ , the extra factor of  $(k + 2)m$  puts a southeast step from  $(0, 1)$  to  $(1, 0)$  followed by a plain of length  $k + 1$  at the front of lattice path. So the factor  $q^{\frac{m(m+1)(k+3)}{2}}$  generates a lattice path with  $(m + 1)$  peaks starting from  $(0, 1)$  and ending at  $(1 + (k + 3)m, 0)$  and with a plain of length  $k + 1$  between two successive peaks.



In the above graph we consider two successive peaks, say,  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.

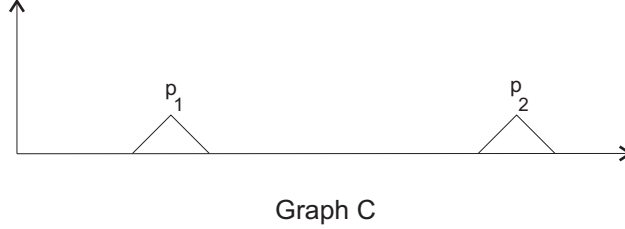


Clearly in Graph B

$$p_1 = (2b_{m+1} + (2i - 1) + (i - 1)(k + 1), 1), \text{ and}$$

$$p_2 = (2b_{m+1} + (2i) + i(k + 1), 1).$$

The factor  $\frac{1}{(q; q)_m}$  generates  $m$  non-negative integers, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the second mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m - i + 2)$ nd mountain,  $1 \leq i \leq m - 1$ . Graph B now becomes Graph C.

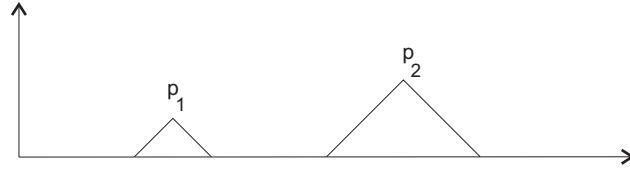


In Graph C,

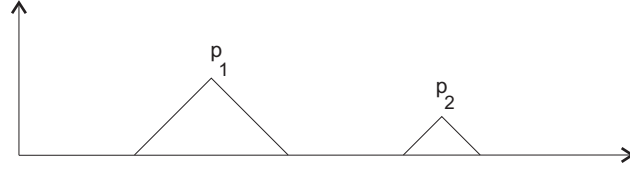
$$p_1 = ((2i - 1) + (i - 1)(k + 1) + a_{m-i+1}, 1), \text{ and}$$

$$p_2 = ((2i + 1) + i(k + 1) + a_{m-i}, 1).$$

The factor  $\frac{1}{(q; q^2)_m}$  generates non-negative multiples of  $(2i - 1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m - 1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+2} + 1$ ,  $2 \leq i \leq m + 1$ . Also the extra factor of  $(1 - q^{2m+1})$  introduces a non-negative multiple of  $(2m + 1)$ , say  $b_{m+1} * (2m + 1)$ . This is encoded by having the first peak grow to height  $b_{m+1} + 1$  in the northeast direction. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, Graph C now changes to Graph D or Graph E, depending on whether  $b_{m-i} > b_{m-i+1}$  or  $b_{m-i} < b_{m-i+1}$ . The new graph looks like graph D or graph E, respectively.



Graph D



Graph E

Considering these changes in the Graph D (or Graph E) we see that the  $i$ th and  $(i+1)$ st peaks now become  $(i+1)$ st and  $(i+2)$ nd peaks, respectively.

In Graph D (or Graph E),

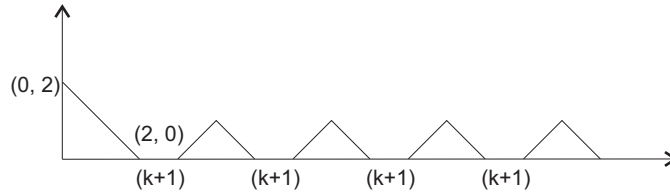
$$p_1 = (2b_{m+1} + 2(i-1) + (i-1)(k+1) + a_{m-i+1} + a_{m-i+1} + 2(b_m + \dots + b_{m-i+2}) + b_{m-i+1}, b_{m-i} + 1), \text{ and}$$

$$p_2 = (2b_{m+1} + 2i + i(k+1) + a_{m-i} + 2(b_m + \dots + b_{m-i+1}) + b_{m-i}, b_{m-i} + 1).$$

We see that every lattice path enumerated by  $A_2^k(\nu)$  is uniquely generated in this manner. This proves that right hand side of equation (3. 0. 2) also generates  $A_2^k(\nu)$ .

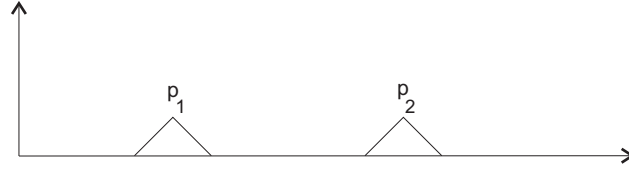
### Proof of Theorem 3. 1. 3

In  $\frac{q^{m[1 + \frac{(k+3)(m+1)}{2}]}}{(q; q)_m (q; q^2)_{m+1}}$ , the extra factor of  $q^m$  puts two southeast step from  $(0, 2)$  to  $(1, 1)$  and  $(1, 1)$  to  $(2, 0)$ . So the path begins with  $(m+1)$  peaks starting from  $(0, 2)$  and ending at  $(2 + (k+3)m, 0)$  and with a plain of length  $k+1$  between every two successive peaks.



Graph A

In the above graph we consider two successive peaks, say,  $i$ th and  $(i+1)$ th and denote them by  $p_1$  and  $p_2$ , respectively.



Graph B

Clearly in Graph B

$$p_1 = (2b_{m+1} + 2(i-1) + 1 + (i-1)(k+1), 1), \text{ and}$$

$$p_2 = (2b_{m+1} + (2i) + 1 + i(k+1), 1).$$

The factor  $\frac{1}{(q; q^2)_m}$  generates  $m$  non-negative integers, say  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ , which are encoded by inserting  $a_m$  horizontal steps in front of the second mountain and  $a_i - a_{i+1}$  horizontal steps in front of the  $(m-i+2)$ nd mountain,  $1 \leq i \leq m-1$ . Graph B now becomes Graph C.



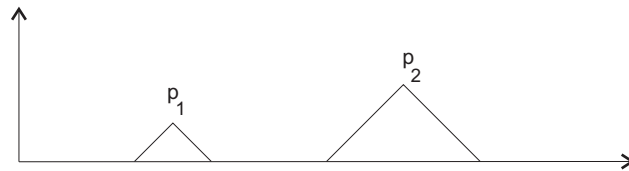
Graph C

In Graph C,

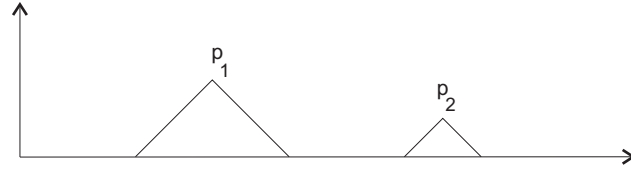
$$p_1 = (2b_{m+1} + 2(i-1) + 1 + (i-1)(k+1) + a_{m-i+1}, 1), \text{ and}$$

$$p_2 = (2b_{m+1} + 2i + 1 + i(k+1) + a_{m-i}, 1).$$

The factor  $\frac{1}{(q; q^2)_m}$  generates non-negative multiples of  $(2i-1)$ ,  $1 \leq i \leq m$ , say  $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m-1)$ . This is encoded by having the  $i$ th peak grow to height  $b_{m-i+2} + 1$ ,  $2 \leq i \leq m+1$ . Also the extra factor of  $(1 - q^{2m+1})$  introduces a non-negative multiple of  $(2m+1)$ , say  $b_{m+1} * (2m+1)$ . This is encoded by having the first peak grow to height  $b_{m+1} + 2$  in the northeast direction. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two, Graph C now changes to Graph D or Graph E, depending on whether  $b_{m-i} > b_{m-i+1}$  or  $b_{m-i} < b_{m-i+1}$ . The new graph looks like graph D or graph E, respectively.



Graph D



Graph E

Considering these changes in the Graph D (or Graph E) we see that the  $i$ th and  $(i+1)$ st peaks now become  $(i+1)$ st and  $(i+2)$ nd peaks, respectively.

In Graph D (or Graph E),

$$p_1 = (2b_{m+1} + 2(i-1) + 1 + (i-1)(k+1) + a_{m-i+1} + 2(b_m + \cdots + b_{m-i+2}) + b_{m-i} + 1, b_{m-i+1} + 1), \text{ and}$$

$$p_2 = (2b_{m+1} + 2i + i(k+1) + a_{m-i} + 2(b_m + \cdots + b_{m-i+2}) + b_{m-i}, b_{m-i} + 1).$$

We see that every lattice path enumerated by  $A_3^k(\nu)$  is uniquely generated in this manner. This proves that right hand side of equation (3.0.3) also generates  $A_3^k(\nu)$ .

This completes the proofs of the Theorems (3. 1. 1)-(3. 1. 3).

### 3.3 Conclusion

The study of this thesis concludes that the lattice path plays an important role in partition theory as a combinatorial tool. There are chances of interpreting more  $q$ -series using this tool. The lattice paths and other tools have applications in different areas such as Orthogonal Polynomial, Combinatorics, Particle Physics and Statistical Mechanics.

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