# EFFICIENT METHODS FOR SOLVING SOME DECISION MAKING PROBLEMS UNDER FUZZY ENVIRONMENT AND ITS EXTENSIONS

Thesis submitted in partial fulfillment of the requirements for the award of the degree of

**DOCTOR OF PHILOSOPHY**

**IN**

**SCHOOL OF MATHEMATICS**

**by**


**AKANKSHA SINGH**
(Regd. No. 901611011)


**THAPAR INSTITUTE**
**OF ENGINEERING & TECHNOLOGY**
**(Deemed to be University)**


**SCHOOL OF MATHEMATICS**

**THAPAR INSTITUTE OF ENGINEERING & TECHNOLOGY**

**Patiala - 147 004 (PUNJAB), INDIA**


**February - 2021**

# CERTIFICATE

This is to certify that the thesis entitled, "**Efficient methods for solving some decision making problems under fuzzy environment and its extensions**" submitted by **Akanksha Singh** in the fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar Institute of Engineering & Technology, Patiala, is a record of candidate's own work carried out by her under our supervision and guidance.

The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

Attestation by supervisors

**Dr. Amit Kumar**
**Associate Professor**
School of Mathematics
Thapar Institute of Engineering &
Technology
Patiala-147004, India

**Dr. S. S. Appadoo**
**Professor & Head**
Department of Supply Chain Management
Asper School of Business
University of Manitoba
Winnipeg, MB
Canada,
R3T-5V4

# DECLARATION

It is certified that the thesis is entirely my own. The ideas and references cited herein have been duly acknowledged.

Akanksha Singh
(Regn. No. 901611011)

Attestation by supervisors

S.S.

| | |
|---|---|
| **Dr. Amit Kumar** | **Dr. S. S. Appadoo** |
| **Associate Professor** | **Professor & Head** |
| School of Mathematics | Department of Supply Chain Management |
| Thapar Institute of Engineering & Technology | Asper School of Business |
| Patiala-147004, India | University of Manitoba |
| | Winnipeg, MB |
| | Canada, |
| | R3T-5V4 |

***DEDICATED***

***TO***

***MY FAMILY,***

***MY SUPERVISORS***

***&***

***SUPREME***

# ACKNOWLEDGEMENTS

***"TAMSO MA JYOTIRGMAYA"***

I believe when you focus on being a blessing, God makes sure that you are always blessed in abundance. I would like to thank God for one of His blessings, in the form of courage, motivation and providing me a way to fulfill my most cherished dream of pursuing PhD.

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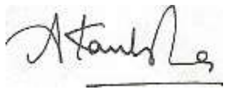

Patiala
February 15, 2021

Akanksha Singh

*"It's A Tribute to My Father Mr R.K. Singh (03.07.1955 - 28.11.2020)"*

# Table of Contents

# List of Abbreviations

| | |
|---|---|
| CCTr | Charnes and Cooper's Transformation |
| CLFPPr | Crisp linear fractional programming problem |
| CLFPPrs | Crisp linear fractional programming problems |
| CLPPr | Crisp linear programming problem |
| CN | Connection number |
| CNs | Connection numbers |
| CoCf | Correlation coefficient |
| CoCfs | Correlation coefficients |
| DHF | Dual hesitant fuzzy |
| DHFS | Dual hesitant fuzzy set |
| DHFSs | Dual hesitant fuzzy sets |
| DHFSS | Dual hesitant fuzzy soft set |
| DHFSSs | Dual hesitant fuzzy soft sets |
| DiM | Distance measure |
| DiMs | Distance measures |
| DM | Decision making |
| DMPr | Decision making problem |
| DMPrs | Decision making problems |
| FLPPr | Fuzzy linear programming problem |
| FLPPrs | Fuzzy linear programming problems |
| FNs | Fuzzy numbers |
| FS | Fuzzy set |
| FSs | Fuzzy sets |

| | |
|---|---|
| FSS | Fuzzy soft set |
| HFS | Hesitant fuzzy set |
| HFSs | Hesitant fuzzy sets |
| HFSS | Hesitant fuzzy soft set |
| IF | Intuitionistic fuzzy |
| IFDM | Intuitionistic fuzzy decision matrix |
| IFLPPr | Intuitionistic fuzzy linear programming problem |
| IFLPPrs | Intuitionistic fuzzy linear programming problems |
| IFMADMM | Intuitionistic fuzzy multi-attribute decision making method |
| IFMADMPr | Intuitionistic fuzzy multi-attribute decision making problem |
| IFMADMPrs | Intuitionistic fuzzy multi-attribute decision making problems |
| IFN | Intuitionistic fuzzy number |
| IFNs | Intuitionistic fuzzy numbers |
| IFS | Intuitionistic fuzzy set |
| IFSs | Intuitionistic fuzzy sets |
| IFSS | Intuitionistic fuzzy soft set |
| IFV | Intuitionistic fuzzy value |
| INSDM | Interval-valued neutrosophic decision matrix |
| INSMCDMPr | Interval-valued neutrosophic multi-criteria decision making problem |
| INSMCDMPrs | Interval-valued neutrosophic multi-criteria decision making problems |
| IVF | Interval-valued fuzzy |
| IVFS | Interval-valued fuzzy set |
| IVFSs | Interval-valued fuzzy sets |
| IVFSS | Interval-valued fuzzy soft set |
| IVIF | Interval-valued intuitionistic fuzzy |

| | |
|---|---|
| IVIFDM | Interval-valued intuitionistic fuzzy decision matrix |
| IVIFMADMM | Interval-valued intuitionistic fuzzy multi-attribute decision making method |
| IVIFMADMPr | Interval-valued intuitionistic fuzzy multi-attribute decision making problem |
| IVIFMADMPrs | Interval-valued intuitionistic fuzzy multi-attribute decision making problems |
| IVIFRV | Interval-valued intuitionistic fuzzy rating value |
| IVIFS | Interval-valued intuitionistic fuzzy set |
| IVIFSs | Interval-valued intuitionistic fuzzy sets |
| IVIFSDM | Interval-valued intuitionistic fuzzy soft decision matrix |
| IVIFSLFPPr | Interval-valued intuitionistic fuzzy soft linear fractional programming problem |
| IVIFSMADMPr | Interval-valued intuitionistic fuzzy soft multi-attribute decision making problem |
| IVIFSMADMPrs | Interval-valued intuitionistic fuzzy soft multi-attribute decision making problems |
| IVIFSS | Interval-valued intuitionistic fuzzy soft set |
| IVIFSSs | Interval-valued intuitionistic fuzzy soft sets |
| IVNS | Interval-valued Neutrosophic set |
| IVNSs | Interval-valued Neutrosophic sets |
| IVPF | Interval valued Pythagorean fuzzy |
| IVPFNs | Interval valued Pythagorean fuzzy numbers |
| IVPFS | Interval-valued Pythagorean fuzzy set |
| IVPFSs | Interval-valued Pythagorean fuzzy sets |

| | |
|---|---|
| LFPPr | Linear fractional programming problem |
| LPPr | Linear programming problem |
| LPPrs | Linear programming problems |
| MADM | Multi-attribute decision making |
| MADMPr | Multi-attribute decision making problem |
| MADMPrs | Multi-attribute decision making problems |
| MCDMPr | Multi-criteria decision making problem |
| MCDMPrs | Multi-criteria decision making problems |
| NIS | Negative ideal scheme |
| NLP | Non-linear programming |
| NLPM | Non-linear programming method |
| NN | Neutrosophic number |
| NNs | Neutrosophic numbers |
| NSLPPr | Neutrosophic linear programming problem |
| NSLPPrs | Neutrosophic linear programming problems |
| NSTrC | Neutrosophic transportation cost |
| NSTrPr | Neutrosophic transportation problem |
| NSTrPrs | Neutrosophic transportation problems |
| PFSs | Pythagorean fuzzy sets |
| PIS | Positive ideal scheme |
| PO | Preference order |
| RM | Ranking method |
| RN | Real number |
| RNs | Real numbers |
| RV | Rating value |

| | |
|---|---|
| SM | Similarity measure |
| SMs | Similarity measures |
| SS | Soft set |
| SVNNs | Single valued neutrosophic numbers |
| SVNS | Single valued neutrosophic set |
| SVNSs | Single valued neutrosophic sets |
| SVTrNN | Single valued trapezoidal neutrosophic number |
| SVTrNNs | Single valued trapezoidal neutrosophic numbers |
| TrC | Transportation cost |
| TrFN | Trapezoidal fuzzy number |
| TrNN | Trapezoidal neutrosophic number |
| TrNNs | Trapezoidal neutrosophic numbers |
| TrPr | Transportation problem |
| TrPrs | Transportation problems |

# Abstract


In daily life problems, a process is followed by an individual or group of persons to finalize a decision. This process is called DM and the problems are called DMPrs. DMPrs can be mainly classified into the two following categories:

**(a)** **Single/multi-attribute DMPrs:** This category contains those DMPrs in which a finite number of alternatives are known and the aim is to rank these alternatives e.g.,

(i) To rank 50 students of a class on the basis of the marks, secured in Mathematics, is a single attribute DMPr.

(ii) To rank 50 students of a class on the basis of marks, secured in Mathematics, Physics and Chemistry, is a MADMPr.

**(b)** **Single/multi-objective DMPrs:** This category contains those DMPrs in which the aim is to find a way which will maximize/minimize one or more functions subject to various restrictions e.g.,

(i) To find the quantity of the product that should be supplied from various sources to various destinations in such a manner that the total TrC is minimum is a single objective DMPr.

(ii) To find the quantity of the product that should be supplied from various sources to various destinations in such a manner that the total TrC as well as the total transportation risk is minimum is a multi-objective DMPr.

One of the important steps of DM is to collect the information/data regarding the problem. It is pertinent to mention that it is not always possible to represent the collected data/information as a RN e.g.,

(i) The cost to hire a cab between two fixed places cannot be represented by a RN as it varies from time to time depending on the traffic/weather-conditions/route etc.

(ii) The rating of a movie review cannot be presented by a RN instead it can be expressed in linguistic terms such as poor, average, good, excellent etc.

In the literature, different ways have been introduced to handle these types of data. One of the way, used by several researchers, to handle the same is to express the data as FS [189] and its extensions [9, 13, 52, 80, 127, 167, 168].

In the last few years, several researchers have proposed various methods for solving DMPrs under fuzzy environment and its extensions. These methods can be classified into different categories. Some of these categories are as follows:

(i) **Methods for solving DMPrs under IF environment:**

This category contains all those methods for solving DMPrs in which some or all the collected information/data is expressed as IFS [9].

(ii) **Methods for solving DMPrs under IVIF environment:**

This category contains all those methods for solving DMPrs in which some or all the collected information/data is expressed as IVIFS [13].

(iii) **Methods for solving DMPrs under DHF soft environment:**

This category contains all those methods for solving DMPrs in which some or all the collected information/data is expressed as DHFSS [52].

(iv) **Methods for solving DMPrs under IVIF soft environment:**

This category contains all those methods for solving DMPrs in which some or all the collected information/data is expressed as IVIFSS [80].

(v) **Methods for solving DMPrs under neutrosophic environment:**

This category contains all those methods for solving DMPrs in which some or all the collected information/data is expressed as SVNS [168].

(vi) **Methods for solving DMPrs under interval-valued neutrosophic environment:** This category contains all those methods for solving DMPrs in which some or all those collected information/data is expressed as IVNS [167].

(vii) **Methods for solving DMPrs under IVPF environment:** This category contains all those methods for solving DMPrs in which some or all those collected information/data is expressed as IVPFS [127].

In the last few years, various approaches have been proposed for solving MADMPrs under various extensions of fuzzy environment. After a deep study, some limitations and/or shortcomings have been observed in the existing methods [1, 8, 51, 56, 98, 99, 157] for solving DMPrs under various extensions of fuzzy environment. Keeping the same in mind, the aim of this thesis is

(i) To point out as well as to overcome the limitations of the existing method [99].

(ii) To point out as well as to overcome the limitations of the existing method [98].

(iii) To point out as well as to resolve the shortcomings of the existing method [8].

(iv) To point out as well as to resolve the shortcomings of the existing method [51].

(v) To point out as well as to overcome the shortcomings of the existing method [1].

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(vii) To point out as well as to overcome the shortcomings of the existing method [56].

# List of published/communicated Papers

(1) **Singh A.,** Kumar A., Appadoo S.S. (2017) Modified Approach for optimization of real life transportation problem in neutrosophic environment. Mathematical Problems in Engineering, Article ID 2139791, 9 pages. **(Impact factor: 1.009)**

(2) **Singh A.,** Kumar A., Appadoo S.S. (2018) Mehar ranking method for comparing connection numbers and its application in decision making. Journal of Intelligent & Fuzzy Systems, vol. 35, pp. 5523-5528. **(Impact factor: 1.851)**

(3) **Singh A.,** Kumar A., Appadoo S.S. (2019) A novel method for solving the fully neutrosophic linear programming problems: Suggested modification. Journal of Intelligent & Fuzzy Systems, vol. 37, pp. 885-895. **(Impact factor: 1.851)**

(4) **Singh A.** (2018) Modified method for solving non-linear programming for multi-criteria decision making problems under interval neutrosophic set environment. Mathematical Sciences International Research Journal, vol. 7, pp. 41-52.

(5) **Singh A.,** Appadoo S.S., Modified non-linear programming methodology for multi-attribute decision-making problem with interval-valued intuitionistic fuzzy soft sets information (Communicated in Journal of Intelligent & Fuzzy Systems).

(6) **Singh A.,** Kumar A., Appadoo S.S., Mehar interval-valued intuitionistic fuzzy multi-atribute decision-making method without using the concept of connection number (Communicated in Journal of Intelligent & Fuzzy Systems).

(7) **Singh A.,** Kumar A., Appadoo S.S., Modified expressions to evaluate the correlation coefficient between two dual hesitant fuzzy soft sets and their application in decision making (Communicated in Engineering Applications of Artificial Intelligence).

# List of papers presented in Symposium/Conferences/Congress

(1) **Singh A.,** Modified non-linear programming method for multi-criteria decision making problems under interval neutrosophic set environment, International Symposium on Operations Research and Game Theory: Modeling and Computation organized by Indian Statistical Institute, Delhi, India during January 9-11, 2018.

(2) **Singh A.,** Modified method for solving non-linear programming for multi-criteria decision making problems under interval neutrosophic set environment, International Conference on Advances in Mathematics, Engineering & Technology- 2018 organized by Carmel College for Women, Nuvem, Goa, India in collaboration with International Multidisciplinary Research Foundation Institute for Education and Research during December 28-29, 2018.

(3) **Singh A.,** Kumar A., Appadoo S.S., A note on "An improved score function for ranking neutrosophic sets and its application to decision-making process", 9th International Congress on Industrial and Applied Mathematics organized by International Council for Industrial and Applied Mathematics during July 15-19, 2019 Valencia, Spain.

# List of workshops attended

(1) Lecture series on Partial Differential Equations and its Applications organized by Thapar Mathematical Society, Thapar Institute of Engineering & Technology, Patiala during March 29-30, 2019.

(2) Workshop on Applications of Optimization Techniques organized by Thapar Mathematical Society in collaboration with Scimatics, Thapar Institute of Engineering & Technology, Patiala during March 30-31, 2018.

(3) One day workshop on Python for Asset Pricing-I organized by Indian Institute of Technology Ropar on November 18, 2017.

(4) International Workshop on Fuzzy Logic Research Applications (ITWFLA-2017) organized by Department of CSE, Indira Gandhi Delhi Technical University for Women (IGDTUW), New Delhi, India during September 16-20, 2017.

# Chapter 1

# Introduction

## 1.1 Introduction

In daily life problems, a process is followed by an individual or group of persons to finalize a decision. This process is called DM and the problems are called DMPrs. DMPrs can be mainly classified into the two following categories:

**(a)** **Single/multi-attribute DMPrs:** This category contains those DMPrs in which a finite number of alternatives are known and the aim is to rank these alternatives e.g.,

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One of the important steps of DM is to collect the information/data regarding the problem. It is pertinent to mention that it is not always possible to represent the collected data/information as a RN e.g.,

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In the last few years, several researchers have proposed various methods for solving DMPrs under fuzzy environment and its extensions. These methods can be classified into different categories. Some of these categories are as follows:

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## 1.2 Objectives of the Thesis

In the last few years, various approaches have been proposed for solving MADMPrs under various extensions of fuzzy environment. After a deep study, some limitations and/or shortcomings have been observed in the existing methods [1, 8, 51, 56, 98, 99, 157] for solving DMPrs under various extensions of fuzzy environment. Keeping the same in mind, the aim of this thesis is

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(vii) To point out as well as to overcome the shortcomings of the existing method [56].

## 1.3 Chapter wise summary of the Thesis

The chapter wise summary of the thesis is as follows:

**Chapter 1** is an introductory in nature.

**Chapter 2**

**Mehar RM for comparing CNs and its application in DM**

Kumar and Garg [99] considered some IFMADMPrs to show that the existing IFMADMM [18] fails to rank the alternatives of the considered IFMADMPrs. Kumar and Garg also pointed out that this limitation of the existing IFMADMM can be resolved with the help of a CN [28]. Since, to do the same there was need to transform each IFN [9] of the IFDM into a CN. But, there was no method in the literature to transform an IFN into a CN. Therefore, Kumar and Garg firstly proposed a method to transform an IFN into a CN. Then, using this method, Kumar and Garg proposed a method to solve IFMADMPrs. In this chapter, it is shown that the RM, used in Step 5 of Kumar and Garg's method for comparing CNs, fails to compare two distinct CNs. Hence, Kumar and Garg's method fails to rank the alternatives of IFMADMPrs. Furthermore, to overcome the limitation of Kumar and Garg's method, a new RM (named as Mehar RM) is proposed for comparing CNs.

**Chapter 3**

**Mehar IVIFMADMM without using the concept of CN**

Kumar and Garg [98] pointed out that several researchers [18, 24, 43, 76, 133, 164, 170, 172] have used the CN [28] for solving MADMPrs under crisp environment, fuzzy environment, IVF environment and IF environment. However, till now no one have used the same for solving MADMPrs under IVIF environment. Since to fill this gap, there was need to propose a method for transforming an IVIFS into a CN as well as a RM for comparing CNs. Therefore, Kumar and Garg, firstly, proposed the methods for the same. Then, using these methods, Kumar and Garg proposed an IVIFMADMM for solving IVIFMADMPrs. In this chapter, an IVIFMADMPr is solved by Kumar and Garg's IVIFMADMM and shown that Kumar and Garg's method fails to rank the alternatives of the considered problem. To overcome this limitation of Kumar and Garg's IVIFMADMM, a new method (named as Mehar IVIFMADMM) is proposed for solving IVIFMADMPrs without transforming the elements of IVIFDM into CNs. Also, the PO for the alternatives of the considered IVIFMADMPr is obtained by the proposed Mehar IVIFMADMM. Furthermore, the advantages of applying the proposed Mehar IVIFMADMM over Kumar and Garg's IVIFMADMM are discussed.

**Chapter 4**

**Modified expressions to evaluate the CoCf between two DHFSSs and their application in DM**

Arora and Garg [8] proposed two expressions for evaluating the weighted CoCfs between two DHFSSs [52]. Arora and Garg claimed that their proposed expressions can be used for finding the solution for several real-life MCDMPrs under DHFSS environment. To validate the claim, Arora and Garg solved three real-life problems (finding the best candidate, medical diagnosis problem and pattern recognition). In future, other researchers may use Arora and Garg's expressions for solving same type of real-life problems or some other type of real-life problems. However, after a deep study, it is observed that the Arora and Garg have used some mathematical incorrect assumptions to obtain their proposed expressions i.e., Arora and Garg's expressions are not valid in its present form. Therefore, if one will apply these expressions then the obtained results may or may not be exact. Keeping the same in mind, Arora and Garg's expressions have been modified. Furthermore, using the modified expressions, the exact results of the real-life problems, considered by Arora and Garg, have been obtained.

## Chapter 5

### Modified NLP methodology for MADMPrs with IVIFSSs information

Garg and Arora [51] claimed that there is no method in the literature to solve IVIFSMADMPrs and hence, proposed a NLPM for solving IVIFSMADMPrs. Since, it is only method for solving IVIFSMADMPrs so the other researchers may be attracted to use this method for solving real-life IVIFSMADMPrs. However, after a deep study, it is observed that some mathematical incorrect assumptions have been considered in this method. Therefore, it is scientifically incorrect to use this method for solving real-life IVIFSMADMPrs. Keeping the same in mind, Garg and Arora's method is modified.

## Chapter 6

### A novel method for solving fully NSLPPrs: Suggested modifications

Abdel-Basset et al. [1] claimed that although several methods have been proposed in the literature to find the solution of different types of FLPPr/IFLPPrs (LPPrs in which some/all the parameters are represented as FNs/ IFNs) [5, 17, 19, 37, 38, 44, 70, 71, 100, 104, 105, 112, 122, 134, 142, 149, 156, 159, 171, 197, 202]. However, there is no method in the literature for solving such NSLPPrs in which some/all the parameters are represented as TrNNs. To fill this gap, Abdel-Basset et al. proposed methods for solving different types of NSLPPrs. In Abdel-Basset et al.'s methods, firstly, a NSLPPr is transformed into a CLPPr by replacing each parameter of the NSLPPr, represented by a TrNN, with its equivalent defuzzified crisp value. Then, the optimal solution of the transformed CLPPr is used to find the optimal solution and optimal value of the considered NSLPPr. Abdel-Basset et al. also pointed out that as a TrFN is a special case of TrNN. Therefore, the FLPPrs, can be solved by the existing methods [38, 44, 100, 134], can also be solved by their proposed methods. Abdel-Basset et al. also solved the same FLPPrs by their proposed methods as well as by the existing methods [38, 44, 100, 134] and shown that the results, obtained on applying by their proposed methods, are better than the results obtained on applying the existing methods [38, 44, 100, 134]. In this chapter, it is shown that for the ranking function, used by Abdel-Basset et al., to transform a TrNN into its equivalent crisp value, the linearity property is not satisfying. Whereas, Abdel-Basset et al. have used the linearity property in their proposed methods to transform a NSLPPr into its equivalent CLPPr. Therefore, Abdel-Basset et al.'s methods are not valid in its present form. Furthermore, the required modifications in Abdel-Basset et al.'s methods are suggested.

**Chapter 7**

**Modified approach for optimization of real-life TrPr in neutrosophic environment**

In daily life problems, there is a need to transport the product from various sources to different destinations. To find a way to transport the product in such a manner so that the total TrC is minimum is called the optimal way and the problem is called cost minimization TrPrs [72]. Different methods have been proposed in the literature to find the optimal way of such cost minimization TrPrs in which cost for transporting unit quantity of the product, availability of the product at the sources and demand of the product at the destinations are represented as a RNs. However, to assume these parameters as RNs is not always valid according to real-life situations e.g., the TrC depends upon the circumstances like price of petrol/diesel, weather, travel time, traffic jam etc. Similarly, the availability of crops varies according to the monsoon, fertilizers, chemicals etc., the demand of the various clothes depend on the season, fashion trends, discount offers etc. Furthermore, the opinions of the experts about these parameters cannot always be represented as a RNs, e.g., generally experts provide their opinion about these parameters in terms of linguistic variables like high, very high, low, very low etc.

One of the way, widely adopted in the literature to deal with such situations, is to represent these parameters as FNs [189] and its extensions [20]. Thamaraiselvi and Santhi

[157] pointed out that neutrosophic set [168], one of the extensions of FS, is used in different research areas. However, till now no one have used the neutrosophic set in TrPrs. While, several researchers have used FNs for representing various parameters of TrPrs [23, 45, 62, 82, 89, 90, 124, 129, 143]. Therefore, Thamaraiselvi and Santhi proposed the approaches for solving NSTrPr of Type-I (TrPrs in which the cost for transporting unit quantity of the product is represented as TrNN, whereas the availability and the demand are represented as RNs) and NSTrPr of Type-II (TrPrs in which the cost for transporting unit quantity of the product, availability of the product and demand of the product are represented as TrNNs). Since, NSTrPrs is new area of research so others may be attracted to extend these approaches for solving other types of NSTrPrs like neutrosophic solid TrPrs, neutrosophic time minimization TrPrs, neutrosophic transshipment problems etc. However, after a deep study of these existing approaches, it is noticed that a mathematical incorrect assumption has been used in these existing approaches. Therefore, there is need to modify these existing approaches. Keeping the same in mind, in this chapter, these existing approaches are modified. Furthermore, the exact results of some existing TrPrs are obtained by the modified approaches.

**Chapter 8**

**Modified NLPM for MCDMPrs under IVNS environment**

Garg and Nancy [56] claimed that there is no method in the literature to solve INSMCDMPrs and hence, proposed a NLPM for solving INSMCDMPrs. Since, it is only method for solving INSMCDMPrs so the other researchers may be attracted to use this method for solving real-life INSMCDMPrs. However, after a deep study, it is observed that some mathematical incorrect assumptions have been considered in this method. Therefore, it is scientifically incorrect to use this method for solving real-life INSMCDMPrs. Keeping the same in mind, the method, proposed by Garg and Nancy, is modified.

# Chapter 9

## Future Scope

In this chapter, some open research problems are discussed.

# Chapter 2

# Mehar RM for comparing CNs and its application in DM*

Kumar and Garg [99] considered some IFMADMPrs to show that the existing IFMADMM [18] fails to rank the alternatives of the considered IFMADMPrs. Kumar and Garg also pointed out that this limitation of the existing IFMADMM [18] can be resolved with the help of a CN [28]. Since, to do the same there was need to transform each IFN [9] of the IFDM into a CN. But, there was no method in the literature to transform an IFN into a CN. Therefore, Kumar and Garg firstly proposed a method to transform an IFN into a CN. Then, using this method, Kumar and Garg proposed a method to solve IFMADMPrs. In this chapter, it is shown that the RM, used in Step 5 of Kumar and Garg's method for comparing CNs, fails to compare two distinct CNs. Hence, Kumar and Garg's method fails to rank the alternatives of IFMADMPrs. Furthermore, to overcome the limitation of Kumar and Garg's method, a new RM (named as Mehar RM) is proposed for comparing CNs.

## 2.1 Preliminaries

In this section, some basic definitions are presented.

**Definition 2.1 [189]** A set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle \mid x \in X,\ 0 \le \mu_{\tilde{A}}(x) \le 1\}$, defined on the universal set $X$, is said to be a FS, where $\mu_{\tilde{A}}(x)$ represents the degree of membership of the element $x$ in $\tilde{A}$.

**Definition 2.2 [9]** A set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x),\ \nu_{\tilde{A}}(x)\rangle \mid x \in X,\ 0 \le \mu_{\tilde{A}}(x) \le 1, 0 \le \nu_{\tilde{A}}(x) \le 1,$ $\mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \le 1\}$, defined on the universal set $X$, is said to be an IFS, where, $\mu_{\tilde{A}}(x)$ and

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$\nu_{\tilde{A}}(x)$ represents the degree of membership and degree of non-membership respectively of the element $x$ in $\tilde{A}$. The pair $\langle \mu_{\tilde{A}}, \nu_{\tilde{A}} \rangle$ is called an IFN or an IFV, where $\mu_{\tilde{A}} \in [0,1]$, $\nu_{\tilde{A}} \in [0,1]$, $\mu_{\tilde{A}} + \nu_{\tilde{A}} \leq 1$ .

**Definition 2.3 [28]** If two sets $A$ and $B$, having same number of characteristics (say $N$), are put together to form a pair $H$ with respect to the problem $W$, then the pair $H$ is called a set pair and the number $\mu(H,W) = \left(\frac{S}{N}\right) + \left(\frac{P}{N}\right)$ is called a CN of the set pair $H$, where $S$ represents the number of identity characteristics and $P$ represents the number of contrary characteristics. Furthermore, $\left(\frac{S}{N}\right)$ and $\left(\frac{P}{N}\right)$ are called identical degree and the contrary degree of these two sets respectively. Assuming $\left(\frac{S}{N}\right) = a$ and $\left(\frac{P}{N}\right) = c$, the CN $\mu(H,W) = \left(\frac{S}{N}\right) + \left(\frac{P}{N}\right)j$ can also be written as $\mu(H,W) = a + cj$.

**2.2 Kumar and Garg's IFMADMM**

Kumar and Garg [99] proposed the following IFMADMM for solving IFMADMPrs.

**Step 1:** Check that all the attribute of the IFDM $\widetilde{D}$ are of same type or not.

**Case (i)** If all the attributes are of same type then go to Step 2.

**Case (ii)** If some attributes are benefit attributes and others are cost attributes then normalize the IFDM $\widetilde{D}$ by transforming the cost attributes into benefit attributes, by the following way:

If the $p^{th}$ attribute is cost attribute then replace all the elements $\langle \mu_{ip}, \nu_{ip} \rangle$ of the $p^{th}$ column of the IFDM $\widetilde{D}$ with $\langle \nu_{ip}, \mu_{ip} \rangle$.

**Step 2:** Find the PIS $\langle {\mu_t}^+, {\nu_t}^+ \rangle = \langle max_{1\leq i\leq m} \{\mu_{it}\}, min_{1\leq i\leq m} \{\nu_{it}\} \rangle$, $t = 1$ to $n$ as well as the NIS $\langle {\mu_t}^-, {\nu_t}^- \rangle = \langle min_{1\leq i\leq m} \{\mu_{it}\}, max_{1\leq i\leq m} \{\nu_{it}\} \rangle$, $t = 1$ to $n$.

**Step 3:** Find the CN, $a_{it} + c_{it}j$, $i = 1$ to $m$, $t = 1$ to $n$, where, $a_{it} + c_{it}j = ({a_{it}}^+ \times {c_{it}}^-) + ({c_{it}}^+ \times {a_{it}}^-)j$

$= \left[\left(\frac{\mu_{it}}{\mu_t^{+}} \times \frac{\nu_t^{+}}{\nu_{it}}\right) \times \left(\frac{\mu_{it}-\mu_t^{-}}{\mu_{it}} \times \frac{\nu_t^{-}-\nu_{it}}{\nu_t^{-}}\right)\right] + \left[\left(\frac{\mu_t^{+}-\mu_{it}}{\mu_t^{+}} \times \frac{\nu_{it}-\nu_t^{+}}{\nu_{it}}\right) \times \left(\frac{\mu_t^{-}}{\mu_{it}} \times \frac{\nu_{it}}{\nu_t^{-}}\right)\right] j.$

**Step 4:** Find the CN, $\mu_i = a_i + c_i j = \sum_{t=1}^{n}(w_t a_{it}) + \sum_{t=1}^{n}(w_t c_{it}) j$, $i = 1$ to $m$, where, $w_t$ is the crisp weight of the $t^{th}$ attribute.

**Step 5:** Find $T(\mu_i) = \frac{a_i}{a_i + c_i}$, $i = 1$ to $m$ and check that $T(\mu_p) > T(\mu_q)$ or $T(\mu_p) < T(\mu_q)$ or $T(\mu_p) = T(\mu_q)$.

**Case (i)** If $T(\mu_p) > T(\mu_q)$, then the PO for the alternatives $A_p$ and $A_q$ is $A_p > A_q$.

**Case (ii)** If $T(\mu_p) < T(\mu_q)$, then the PO for the alternatives $A_p$ and $A_q$ is $A_p < A_q$.

**Case (iii)** If $T(\mu_p) = T(\mu_q)$, then the PO for the alternatives $A_p$ and $A_q$ is $A_p = A_q$.

### 2.3 Limitations of Kumar and Garg's IFMADMM

In this section, an IFMADMPr, having four alternatives and two benefit attributes, is solved by Kumar and Garg's IFMADMM [99] to show that,

(i) Kumar and Garg's IFMADMM [99] fails to rank two alternatives of the considered IFMADMPr.

(ii) The obtained PO of the remaining two alternatives is not correct.

Let

$$\tilde{D} = \langle \tilde{\alpha}_{ij} \rangle = \left(\langle \mu_{ij}, \nu_{ij} \rangle\right)_{4\times 2} = \begin{array}{c c} & \begin{array}{cc} G_1 & G_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} \langle 0.1,0.2\rangle & \langle 0.3,0.4\rangle \\ \langle 0.2,0.4\rangle & \langle 0.3,0.4\rangle \\ \langle 0.3,0.6\rangle & \langle 0.3,0.4\rangle \\ \langle 0.4,0.8\rangle & \langle 0.3,0.4\rangle \end{bmatrix} \end{array}$$

represents the IFDM of an IFMADMPr, where

(i) $A_1, A_2, A_3$ and $A_4$ are the alternatives.

(ii) $G_1$ and $G_2$ are the benefit attributes.

(iii) $\tilde{\alpha}_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle$ is the RV of the $i^{th}$ alternative over the $j^{th}$ attribute.

Since, the RV of all the alternatives over the second attribute $G_2$ is same, therefore, the PO of the alternatives will depend only upon the RV of all the alternatives over the attribute $G_1$. Furthermore, as the RV of all the alternatives over the attribute $G_1$ are different. Therefore, the PO of any of these alternatives cannot be same i.e., $A_i \neq A_j$ for $i \neq j$.

While, in this section, it is shown that if Kumar and Garg's IFMADMM [99] is applied to find the PO of the alternatives of the considered IFMADMPr. Then,

(i) Kumar and Garg's method [99] fails to find the ranking of the alternatives $A_1$ and $A_4$.

(ii) The ranking of the alternatives $A_2$ and $A_3$, obtained by Kumar and Garg's method [99], is $A_2 = A_3$, which is mathematically incorrect.

Using Kumar and Garg's IFMADMM [99], discussed in Section 2.2, the ranking of the alternatives for the considered IFMADMPr can be obtained as follows:

**Step 1:** Since both the attributes are benefit attributes. Therefore, according to Case (i) of Step 1 of Kumar and Garg's IFMADMM [99], there is no need to normalize the given IFDM $\widetilde{D}$.

**Step 2:** Using Step 2 of Kumar and Garg's IFMADMM [99],

(i) $\langle {\mu_1}^+, {\nu_1}^+ \rangle = \langle max\{0.1,0.2,0.3,0.4\}, min\{0.2,0.4,0.6,0.8\} \rangle = \langle 0.4,0.2 \rangle$.

(ii) $\langle {\mu_2}^+, {\nu_2}^+ \rangle = \langle max\{0.3,0.3,0.3,0.3\}, min\{0.4,0.4,0.4,0.4\} \rangle = \langle 0.3,0.4 \rangle$.

(iii) $\langle {\mu_1}^-, {\nu_1}^- \rangle = \langle min\{0.1,0.2,0.3,0.4\}, max\{0.2,0.4,0.6,0.8\} \rangle = \langle 0.1,0.8 \rangle$.

(iv) $\langle {\mu_2}^-, {\nu_2}^- \rangle = \langle min\{0.3,0.3,0.3,0.3\}, max\{0.4,0.4,0.4,0.4\} \rangle = \langle 0.3,0.4 \rangle$.

**Step 3:** Using Step 3 of Kumar and Garg's IFMADMM [99],

(i) The CN corresponding to the IFN $\langle 0.1,0.2 \rangle$ is $0 + 0j$.

(ii) The CN corresponding to the IFN $\langle 0.2,0.4 \rangle$ is $\frac{1}{16} + \frac{1}{16}j$.

(iii) The CN corresponding to the IFN $\langle 0.3,0.6 \rangle$ is $\frac{1}{24} + \frac{1}{24}j$.

(iv) The CN corresponding to the IFN $\langle 0.4, 0.8\rangle$ is $0 + 0j$.

(v) The CN corresponding to the IFN $\langle 0.3, 0.4\rangle$ is $0 + 0j$.

**Step 4:** If the weights of the attributes $G_1$ and $G_2$ are $w_1 = 0.5$ and $w_2 = 0.5$, respectively then using Step 4 of Kumar and Garg's IFMADMM [99],

(i) $\mu_1 = a_1 + c_1 j = 0.5(0 + 0) + 0.5(0 + 0)j = 0 + 0j$.

(ii) $\mu_2 = a_2 + c_2 j = 0.5\left(\frac{1}{16} + 0\right) + 0.5\left(\frac{1}{16} + 0\right)j = \frac{0.5}{16} + \frac{0.5}{16}j$.

(iii) $\mu_3 = a_3 + c_3 j = 0.5\left(\frac{1}{24} + 0\right) + 0.5\left(\frac{1}{24} + 0\right)j = \frac{0.5}{24} + \frac{0.5}{24}j$.

(iv) $\mu_4 = a_4 + c_4 j = 0.5(0 + 0) + 0.5(0 + 0)j = 0 + 0j$.

**Step 5:** Using Step 5 of Kumar and Garg's IFMADMM [99],

(i) $T(\mu_1) = \frac{a_1}{a_1 + c_1} = \frac{0}{0+0} = \frac{0}{0}$.

(ii) $T(\mu_2) = \frac{a_2}{a_2 + c_2} = \frac{\frac{0.5}{16}}{\frac{0.5}{16} + \frac{0.5}{16}} = \frac{\frac{0.5}{16}}{\frac{0.5}{16}(1+1)} = \frac{1}{2}$.

(iii) $T(\mu_3) = \frac{a_3}{a_3 + c_3} = \frac{\frac{0.5}{24}}{\frac{0.5}{24} + \frac{0.5}{24}} = \frac{\frac{0.5}{24}}{\frac{0.5}{24}(1+1)} = \frac{1}{2}$.

(iv) $T(\mu_4) = \frac{a_4}{a_4 + c_4} = \frac{0}{0+0} = \frac{0}{0}$.

Since $T(\mu_2) = T(\mu_3)$, so according to Step 5 of Kumar and Garg's IFMADMM [99], $A_2 = A_3$, which is mathematically incorrect as RV of $A_2$ over the attribute $G_1$ is not equal to RV of alternative $A_3$ over the attribute $G_1$. Furthermore, the values of $T(\mu_1)$ and $T(\mu_4)$ are indeterminate values. Therefore, it is not possible to conclude any result about the ordering of $A_1$ and $A_4$.

## 2.4 Reasons for the occurrence of the limitations

Kumar and Garg's IFMADMM [99] fails to find the PO of the alternatives for the IFMADMPr, considered in Section 2.3, due to the following reasons.

It is obvious from Step 5 of Kumar and Garg's IFMADMM [99] that Kumar and Garg [99] have used the following method for comparing two CNs, $\mu_1 = a_1 + c_1 j$ and $\mu_2 = a_2 + c_2 j$.

Find $T(\mu_1) = \frac{a_1}{a_1+c_1}$ and $T(\mu_2) = \frac{a_2}{a_2+c_2}$ and check that $T(\mu_1) > T(\mu_2)$ or $T(\mu_1) < T(\mu_2)$ or $T(\mu_1) = T(\mu_2)$.

**Case (i)** If $T(\mu_1) > T(\mu_2)$ then $\mu_1 > \mu_2$.

**Case (ii)** If $T(\mu_1) < T(\mu_2)$ then $\mu_1 < \mu_2$.

**Case (iii)** If $T(\mu_1) = T(\mu_2)$ then $\mu_1 = \mu_2$.

However, the following clearly indicates that it is not appropriate to use this method.

(i) If $\mu_1 = 0 + 0j$ and/or $\mu_2 = 0 + 0j$ then $T(\mu_1) = \frac{0}{0}$ and/or $T(\mu_2) = \frac{0}{0}$ i.e., $T(\mu_1)$ and/or $T(\mu_2)$ will be indeterminate values. So, none of the conditions $T(\mu_1) > T(\mu_2)$, $T(\mu_1) < T(\mu_2)$ and $T(\mu_1) = T(\mu_2)$ will be satisfied.

(ii) If $a_1 = c_1$ and $a_2 = c_2$ then always the relation, $T(\mu_1) = T(\mu_2) = \frac{1}{2}$ will be obtained.

### 2.5 Proposed Mehar RM for comparing CNs and its validity

It is obvious from Section 2.4 that to overcome the limitations of Kumar and Garg's IFMADMM [99], there is need to propose a RM for comparing CNs. Keeping the same in mind in this section, a new RM (named as Mehar RM) is proposed for comparing CNs, $\mu_1 = a_1 + c_1 j$ and $\mu_2 = a_2 + c_2 j$.

The steps of the proposed Mehar RM are as follows:

**Step 1:** Check that $a_1 - c_1 > a_2 - c_2$ or $a_1 - c_1 < a_2 - c_2$ or $a_1 - c_1 = a_2 - c_2$

**Case (i)** If $a_1 - c_1 > a_2 - c_2$ then $\mu_1 > \mu_2$.

**Case (ii)** If $a_1 - c_1 < a_2 - c_2$ then $\mu_1 < \mu_2$.

**Case (iii)** If $a_1 - c_1 = a_2 - c_2$ then go to Step 2.

**Step 2:** Check that $a_1 + c_1 > a_2 + c_2$ or $a_1 + c_1 < a_2 + c_2$ or $a_1 + c_1 = a_2 + c_2$

**Case (i)** If $a_1 + c_1 > a_2 + c_2$ then $\mu_1 > \mu_2$.

**Case (ii)** If $a_1 + c_1 < a_2 + c_2$ then $\mu_1 < \mu_2$.

**Case (iii)** If $a_1 + c_1 = a_2 + c_2$ then $\mu_1 = \mu_2$.

To prove that the proposed Mehar RM is valid, it is sufficient to show that on comparing two CNs, $\mu_1 = a_1 + c_1 j$ and $\mu_2 = a_2 + c_2 j$, with the help of proposed Mehar RM, the relation $\mu_1 = \mu_2$ will hold only if $a_1 = a_2$ and $c_1 = c_2$.

According to the proposed Mehar RM, $\mu_1 = \mu_2 \Rightarrow a_1 - c_1 = a_2 - c_2$ as well as $a_1 + c_1 = a_2 + c_2$. It is obvious that both equations will be satisfied only if $a_1 = a_2$ and $c_1 = c_2$ i.e. $\mu_1 = \mu_2 \Rightarrow a_1 = a_2$ and $c_1 = c_2$.

### 2.6 Exact PO of the alternatives for the considered IFMADMPr

It is obvious from the results, discussed in Section 2.4 that the Kumar and Garg's IFMADMM [99] fails to rank the alternatives due to using the inappropriate RM for comparing CNs. If in Step 5 of Kumar and Garg's IFMADMM [99], the proposed Mehar RM will be used instead of the existing RM. Then, neither Kumar and Garg's IFMADMM [99] will fail to rank the alternatives nor incorrect ranking of the alternatives will be obtained on using Kumar and Garg's IFMADMM [99].

To validate this claim the IFMADMPr, considered in Section 2.3, is solved by Kumar and Garg's IFMADMM [99] with the suggested modification.

Using Kumar and Garg's IFMADMM [99] with the suggested modification, the exact PO of the alternatives for the considered IFMADMPr can be obtained as follows:

**Step 1:** Use Step 1 to Step 4 of Kumar and Garg's IFMADMM [99], discussed in Section 2.2,

(i) $\mu_1 = a_1 + c_1 j = w_1(0) + w_2(0)j = 0 + 0j$.

(ii) $\mu_2 = a_2 + c_2 j = w_1\left(\frac{1}{16}\right) + w_2\left(\frac{1}{16}\right)j$.

(iii) $\mu_3 = a_3 + c_3 j = w_1\left(\frac{1}{24}\right) + w_2\left(\frac{1}{24}\right)j$.

(iv) $\mu_4 = a_4 + c_4 j = w_1(0) + w_2(0)j = 0 + 0j$.

Now according to Step 1 of the proposed Mehar RM, $a_1 - c_1 = 0$, $a_2 - c_2 = (w_1 - w_2)\frac{1}{16}$, $a_3 - c_3 = (w_1 - w_2)\frac{1}{24}$ and $a_4 - c_4 = 0$.

**Case (i)** If $w_1 = w_2$, then $a_1 - c_1 = a_2 - c_2 = a_3 - c_3 = a_4 - c_4 = 0$. Therefore, according to Case (ii) of Step 1 of the proposed Mehar RM there is need to go to Step 2.

**Case (ii)** If $w_1 > w_2$ then $a_1 - c_1 = a_4 - c_4 = 0$ but $a_2 - c_2 > a_3 - c_3$. Therefore, according to Case (ii) of Step 2 of the proposed Mehar RM $A_2 > A_3$. But, to find the relation between $A_1$ and $A_4$ there is a need to go to Step 2.

**Case (iii)** If $w_2 > w_1$ then $a_1 - c_1 = a_4 - c_4 = 0$ but $a_3 - c_3 > a_2 - c_2$. Therefore, according to Case (ii) of Step 2 of the proposed Mehar RM $A_3 > A_2$. But, to find the relation between $A_1$ and $A_4$ there is a need to go to Step 2.

**Step 2:** $a_1 + c_1 = 0$, $a_2 + c_2 = \frac{1}{16}$, $a_3 + c_3 = \frac{1}{24}$ and $a_4 + c_4 = 0$. Since $a_1 + c_1 = a_4 + c_4$ and $a_2 + c_2 > a_3 + c_3$ for all values of $w_1$ and $w_2$. Therefore, if $w_1 = w_2$ then $(A_1 = A_4) < A_3 < A_2$.

**2.7 Conclusions**

The limitations of Kumar and Garg's IFMADMM [99] are pointed out. Also, it is shown that these limitations are occurring due to using an inappropriate RM for comparing CNs. Furthermore, to overcome the limitations of Kumar and Garg's IFMADMM [99], a new RM (named as Mehar RM) is proposed for comparing CNs.

# Chapter 3

# Mehar IVIFMADMM without using the concept of CN*

Kumar and Garg [98] pointed out that several researchers [18, 24, 43, 76, 133, 164, 170, 172] have used the CN [28] for solving MADMPrs under crisp environment, fuzzy environment, IVF environment and IF environment. However, till now no one have used the same for solving MADMPrs under IVIF environment. Since to fill this gap, there was need to propose a method for transforming an IVIFS into a CN as well as a RM for comparing CNs. Therefore, Kumar and Garg, firstly, proposed the methods for the same. Then, using these methods, Kumar and Garg proposed an IVIFMADMM for solving IVIFMADMPrs. In this chapter, an IVIFMADMPr is solved by Kumar and Garg's IVIFMADMM and shown that Kumar and Garg's method fails to rank the alternatives of the considered problem. To overcome this limitation of Kumar and Garg's IVIFMADMM, a new method (named as Mehar IVIFMADMM) is proposed for solving IVIFMADMPrs without transforming the elements of IVIFDM into CNs. Also, the PO for the alternatives of the considered IVIFMADMPr is obtained by the proposed Mehar IVIFMADMM. Furthermore, the advantages of applying the proposed Mehar IVIFMADMM over Kumar and Garg's IVIFMADMM are discussed.

**3.1 IVIFS [13]**

A set $\tilde{A} = \{\langle x, [\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)], [\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)]\rangle \mid x \in X,\ 0 \le \mu_{\tilde{A}}^L(x) \le \mu_{\tilde{A}}^U(x) \le 1,\ 0 \le \nu_{\tilde{A}}^L(x) \le \nu_{\tilde{A}}^U(x) \le 1,\ \mu_{\tilde{A}}^U(x) + \nu_{\tilde{A}}^U(x) \le 1\}$, defined on the universal set $X$, is said to be an

* **The contents of this chapter have been communicated for the possible publication in Journal of Intelligent & Fuzzy Systems.**

IVIFS, where, $\left[\mu_{\tilde{A}}^{L}(x), \mu_{\tilde{A}}^{U}(x)\right]$ and $\left[\nu_{\tilde{A}}^{L}(x), \nu_{\tilde{A}}^{U}(x)\right]$ represents the intervals of degree of membership and degree of non-membership respectively of the element $x$ in $\tilde{A}$.

### 3.2 Kumar and Garg's IVIFMADMM

To point out the limitation of Kumar and Garg's IVIFMADMM [98], there is need to discuss Kumar and Garg's IVIFMADMM [98]. Therefore, the same method is discussed in this section. Kumar and Garg [98] proposed the following IVIFMADMM for solving IVIFMADMPrs.

**Step 1:** Normalize the IVIFDM, $\widetilde{D} = \langle \tilde{\alpha}_{ij} \rangle = \left(\left\langle\left[\mu_{ij}^{L}, \mu_{ij}^{U}\right],\left[\nu_{ij}^{L}, \nu_{ij}^{U}\right]\right\rangle\right)_{m\times n}$, where, the IVIFS $\tilde{\alpha}_{ij} = \left\langle\left[\mu_{ij}^{L}, \mu_{ij}^{U}\right],\left[\nu_{ij}^{L}, \nu_{ij}^{U}\right]\right\rangle$ represents the RV of the $i^{th}$ alternative over the $j^{th}$ attribute.

**Step 2:** Find the PIS

$\langle[\mu_t^{L+}, \mu_t^{U+}], [\nu_t^{L+}, \nu_t^{U+}]\rangle =$

$\langle[max_{1\le i\le m}\{\mu_{it}^{L}\}, max_{1\le i\le m}\{\mu_{it}^{U}\}], [min_{1\le i\le m}\{\nu_{it}^{L}\}, min_{1\le i\le m}\{\nu_{it}^{U}\}]\rangle$, $t = 1$ to $n$ as well as the NIS

$\langle[\mu_t^{L-}, \mu_t^{U-}], [\nu_t^{L-}, \nu_t^{U-}]\rangle =$

$\langle[min_{1\le i\le m}\{\mu_{it}^{L}\}, min_{1\le i\le m}\{\mu_{it}^{U}\}], [max_{1\le i\le m}\{\nu_{it}^{L}\}, max_{1\le i\le m}\{\nu_{it}^{U}\}]\rangle$, $t = 1$ to $n$.

**Step 3:** Find the CN, $a_{it} + c_{it}j$, $i = 1$ to $m$, $t = 1$ to $n$, where, $a_{it} + c_{it}j = (a_{it}{}^{+} \times c_{it}{}^{-}) + (c_{it}{}^{+} \times a_{it}{}^{-})j$

$$= \left[\left(\frac{(\mu_{kt}^{L}+\mu_{kt}^{U})}{(\mu_t^{L+}+\mu_t^{U+})} \times \frac{(\nu_t^{L+}+\nu_t^{U+})}{(\nu_{kt}^{L}+\nu_{kt}^{U})}\right) \times \left(\frac{(\mu_{kt}^{L}+\mu_{kt}^{U})-(\mu_t^{L-}+\mu_t^{U-})}{(\mu_{kt}^{L}+\mu_{kt}^{U})} \times \frac{(\nu_t^{L-}+\nu_t^{U-})-(\nu_{kt}^{L}+\nu_{kt}^{U})}{(\nu_t^{L-}+\nu_t^{U-})}\right)\right] + \left[\left(\frac{(\mu_t^{L+}+\mu_t^{U+})-(\mu_{kt}^{L}+\mu_{kt}^{U})}{(\mu_t^{L+}+\mu_t^{U+})} \times \frac{(\nu_{kt}^{L}+\nu_{kt}^{U})-(\nu_t^{L+}+\nu_t^{U+})}{(\nu_{kt}^{L}+\nu_{kt}^{U})}\right) \times \left(\frac{(\mu_t^{L-}+\mu_t^{U-})}{(\mu_{kt}^{L}+\mu_{kt}^{U})} \times \frac{(\nu_{kt}^{L}+\nu_{kt}^{U})}{(\nu_t^{L-}+\nu_t^{U-})}\right)\right] j.$$

**Step 4:** Find the CN, $\mu_i = a_i + c_i j = \sum_{t=1}^{n} w_t(a_{it} + c_{it}j)$, $i = 1$ to $m$.

**Step 5:** Find $T(\mu_i) = \frac{a_i}{a_i + c_i}$, $i = 1$ to $m$ and check that $T(\mu_p) > T(\mu_q)$ or $T(\mu_p) < T(\mu_q)$ or $T(\mu_p) = T(\mu_q)$.

**Case (i)** If $T(\mu_p) > T(\mu_q)$, then the PO for the alternatives $A_p$ and $A_q$ is $A_p > A_q$.

**Case (ii)** If $T(\mu_p) < T(\mu_q)$, then the PO for the alternatives $A_p$ and $A_q$ is $A_p < A_q$.

**Case (iii)** If $T(\mu_p) = T(\mu_q)$, then the PO for the alternatives $A_p$ and $A_q$ is $A_p = A_q$.

### 3.3 Limitation of Kumar and Garg's IVIFMADMM

In this section, to point out the limitation of Kumar and Garg's IVIFMADMM [98], an IVIFMADMPr is considered and shown that Kumar and Garg's IVIFMADMM [98] fails to rank the alternatives of the considered IVIFMADMPr.

**Step 1:** Let $\widetilde{D} = \langle \tilde{\alpha}_{ij} \rangle = \left(\langle [\mu_{ij}^L, \mu_{ij}^U], [\nu_{ij}^L, \nu_{ij}^U] \rangle\right)_{2\times 2}$

$$= \begin{array}{c c} & \begin{array}{cc} G_1 & G_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} \left\langle \left[\frac{1}{8}, \frac{3}{8}\right], \left[\frac{1}{16}, \frac{7}{16}\right] \right\rangle & \left\langle \left[\frac{1}{16}, \frac{3}{16}\right], \left[\frac{1}{16}, \frac{3}{16}\right] \right\rangle \\ \left\langle \left[\frac{1}{9}, \frac{2}{9}\right], \left[\frac{1}{18}, \frac{5}{18}\right] \right\rangle & \left\langle \left[\frac{1}{45}, \frac{8}{45}\right], \left[\frac{1}{20}, \frac{3}{20}\right] \right\rangle \end{bmatrix} \end{array}$$

represent the normalized IVIFDM of the considered IVIFMADMPr, where

(i) $A_1$ and $A_2$ are alternatives.

(ii) $G_1$ and $G_2$ are benefit attributes.

(iii) $\tilde{\alpha}_{ij} = \langle [\mu_{ij}^L, \mu_{ij}^U], [\nu_{ij}^L, \nu_{ij}^U] \rangle$ is the RV of the $i^{th}$ alternative over the $j^{th}$ attribute.

**Step 2:** Using Step 2 of Kumar and Garg's IVIFMADMM [98],

(i) $\langle [\mu_1^{L+}, \mu_1^{U+}], [\nu_1^{L+}, \nu_1^{U+}] \rangle$

$$= \left\langle \left[max\left\{\frac{1}{8}, \frac{1}{9}\right\}, max\left\{\frac{3}{8}, \frac{2}{9}\right\}\right], \left[min\left\{\frac{1}{16}, \frac{1}{18}\right\}, min\left\{\frac{7}{16}, \frac{5}{18}\right\}\right] \right\rangle = \left\langle \left[\frac{1}{8}, \frac{3}{8}\right], \left[\frac{1}{18}, \frac{5}{18}\right] \right\rangle.$$

(ii) $\langle [\mu_2^{L+}, \mu_2^{U+}], [\nu_2^{L+}, \nu_2^{U+}] \rangle =$

$$\left\langle \left[max\left\{\frac{1}{16}, \frac{1}{45}\right\}, max\left\{\frac{3}{16}, \frac{8}{45}\right\}\right], \left[min\left\{\frac{1}{16}, \frac{1}{20}\right\}, min\left\{\frac{3}{16}, \frac{3}{20}\right\}\right] \right\rangle = \left\langle \left[\frac{1}{16}, \frac{3}{16}\right], \left[\frac{1}{20}, \frac{3}{20}\right] \right\rangle.$$

(iii) $\langle [\mu_1^{L-}, \mu_1^{U-}], [\nu_1^{L-}, \nu_1^{U-}] \rangle$

$$= \left\langle \left[min\left\{\frac{1}{8}, \frac{1}{9}\right\}, min\left\{\frac{3}{8}, \frac{2}{9}\right\}\right], \left[max\left\{\frac{1}{16}, \frac{1}{18}\right\}, max\left\{\frac{7}{16}, \frac{5}{18}\right\}\right] \right\rangle = \left\langle \left[\frac{1}{9}, \frac{2}{9}\right], \left[\frac{1}{16}, \frac{7}{16}\right] \right\rangle.$$

(iv) $\langle[\mu_2^{L-},\mu_2^{U-}],[\nu_2^{L-},\nu_2^{U-}]\rangle$

$$=\left\langle\left[min\left\{\frac{1}{16},\frac{1}{45}\right\},min\left\{\frac{3}{16},\frac{8}{45}\right\}\right],\left[max\left\{\frac{1}{16},\frac{1}{20}\right\},max\left\{\frac{3}{16},\frac{3}{20}\right\}\right]\right\rangle=\left\langle\left[\frac{1}{45},\frac{8}{45}\right],\left[\frac{1}{16},\frac{3}{16}\right]\right\rangle.$$

**Step 3:** Using Step 3 of Kumar and Garg's IVIFMADMM [98],

(i) The CN corresponding to the IVIFS $\left\langle\left[\frac{1}{8},\frac{3}{8}\right],\left[\frac{1}{16},\frac{7}{16}\right]\right\rangle$ is $0+0j$.

(ii) The CN corresponding to the IVIFS $\left\langle\left[\frac{1}{9},\frac{2}{9}\right],\left[\frac{1}{18},\frac{5}{18}\right]\right\rangle$ is $0+0j$.

(iii) The CN corresponding to the IVIFS $\left\langle\left[\frac{1}{16},\frac{3}{16}\right],\left[\frac{1}{16},\frac{3}{16}\right]\right\rangle$ is $0+0j$.

(iv) The CN corresponding to the IVIFS $\left\langle\left[\frac{1}{45},\frac{8}{45}\right],\left[\frac{1}{20},\frac{3}{20}\right]\right\rangle$ is $0+0j$.

**Step 4:** If it is assumed that the weights of the attributes $G_1$ and $G_2$ are $w_1=0.5$ and $w_2=0.5$ respectively. Then using Step 4 of Kumar and Garg's IVIFMADMM [98],

(i) $\mu_1=a_1+c_1j=0.5(0+0j)+0.5(0+0j)=0+0j.$

(ii) $\mu_2=a_2+c_2j=0.5(0+0j)+0.5(0+0j)=0+0j.$

**Step 5:** Using Step 5 of Kumar and Garg's IVIFMADMM [98],

(i) $T(\mu_1)=\frac{a_1}{a_1+c_1}=\frac{0}{0+0}=\frac{0}{0}.$

(ii) $T(\mu_2)=\frac{a_2}{a_2+c_2}=\frac{0}{0+0}=\frac{0}{0}.$

Since the values of $T(\mu_1)$ and $T(\mu_2)$ cannot be compared as these are indeterminate values. Therefore, it is not possible to conclude any result about the ordering of $A_1$ and $A_2$.

### 3.4 Methodology used by Kumar and Garg's for transforming an IVIFS into a CN

Although, in Kumar and Garg's IVIFMADMM [98], discussed in Section 3.2, firstly an IVIFS has been transformed into an IFS. Then, using the existing method [99], the obtained IFS has been transformed into a CN. But, the same cannot be observed after going through the steps of Kumar and Garg's IVIFMADMM [98], discussed in Section 3.2. Therefore, in this section, the methodology, used by Kumar and Garg [98] in their proposed IVIFMADMM for transforming an IVIFS into a CN, is discussed.

Kumar and Garg [98] have used the following methodology in Step 3 of their proposed IVIFMADMM [98], discussed in Section 3.2, to transform an IVIFS into a CN.

**Step 1:** Transform each element $\langle[\mu_{ij}^L, \mu_{ij}^U], [\nu_{ij}^L, \nu_{ij}^U]\rangle$ of the normalized IVIFDM $\widetilde{D} = (\langle[\mu_{ij}^L, \mu_{ij}^U], [\nu_{ij}^L, \nu_{ij}^U]\rangle)_{m\times n}$ into the IFS $\left\langle\frac{(\mu_{ij}^L+\mu_{ij}^U)}{2}, \frac{(\nu_{ij}^L+\nu_{ij}^U)}{2}\right\rangle$.

**Step 2:** Transform IVIF PIS

$\langle[\mu_t^{L+}, \mu_t^{U+}], [\nu_t^{L+}, \nu_t^{U+}]\rangle = \left\langle\left[\max_{1\le i\le m}\{\mu_{it}^L\}, \max_{1\le i\le m}\{\mu_{it}^U\}\right], \left[\min_{1\le i\le m}\{\nu_{it}^L\}, \min_{1\le i\le m}\{\nu_{it}^U\}\right]\right\rangle, \quad t = 1 \text{ to } n,$

into the IF PIS, $\langle{\mu_t}^+, {\nu_t}^+\rangle = \left\langle\frac{\max_{1\le i\le m}\{\mu_{it}^L\} + \max_{1\le i\le m}\{\mu_{it}^U\}}{2}, \frac{\min_{1\le i\le m}\{\nu_{it}^L\} + \min_{1\le i\le m}\{\nu_{it}^U\}}{2}\right\rangle$, $t = 1$ to $n$, as well as transform IVIF NIS

$\langle[\mu_t^{L-}, \mu_t^{U-}], [\nu_t^{L-}, \nu_t^{U-}]\rangle = \left\langle\left[\min_{1\le i\le m}\{\mu_{it}^L\}, \min_{1\le i\le m}\{\mu_{it}^U\}\right], \left[\max_{1\le i\le m}\{\nu_{it}^L\}, \max_{1\le i\le m}\{\nu_{it}^U\}\right]\right\rangle, \quad t = 1 \text{ to } n,$

into the IF NIS $\langle{\mu_t}^-, {\nu_t}^-\rangle = \left\langle\frac{\min_{1\le i\le m}\{\mu_{it}^L\} + \min_{1\le i\le m}\{\mu_{it}^U\}}{2}, \frac{\max_{1\le i\le m}\{\nu_{it}^L\} + \max_{1\le i\le m}\{\nu_{it}^U\}}{2}\right\rangle$, $t = 1$ to $n$.

**Step 3:** Using the existing method [99], transform each IFS $\left\langle\frac{(\mu_{ij}^L+\mu_{ij}^U)}{2}, \frac{(\nu_{ij}^L+\nu_{ij}^U)}{2}\right\rangle$ into the CN, $a_{it} + c_{it}j$, $i = 1$ to $m$, $t = 1$ to $n$, where, $a_{it} + c_{it}j = \ ({a_{it}}^+ \times\ {c_{it}}^-) + ({c_{it}}^+ \times {a_{it}}^-)j$

$$= \left[\left(\frac{\mu_{it}}{{\mu_t}^+} \times \frac{{\nu_t}^+}{\nu_{it}}\right) \times \left(\frac{\mu_{it}-{\mu_t}^-}{\mu_{it}} \times \frac{{\nu_t}^- - \nu_{it}}{{\nu_t}^-}\right)\right] + \left[\left(\frac{{\mu_t}^+ - \mu_{it}}{{\mu_t}^+} \times \frac{\nu_{it} - {\nu_t}^+}{\nu_{it}}\right) \times \left(\frac{{\mu_t}^-}{\mu_{it}} \times \frac{\nu_{it}}{{\nu_t}^-}\right)\right] j.$$

### 3.5 Proposed Mehar method for solving IVIFMADMPr without using the concept of CN

It is obvious from Section 3.4 that Kumar and Garg [98] have firstly transformed an IVIFS into an IFS and then the obtained IFS has been transformed into a CN.

In this section, with the help of Kumar and Garg's IVIFMADMM [98], a new method (named as Mehar IVIFMADMM) is proposed without using the concept of CN.

The steps of the proposed Mehar IVIFMADMM are as follows:

**Step 1:** Use Step 1 of Kumar and Garg's IVIFMADMM [98], discussed in Section 3.4, to transform the IVIFDM, $\widetilde{D} = \left(\left\langle\left[\mu_{ij}^L, \mu_{ij}^U\right], \left[\nu_{ij}^L, \nu_{ij}^U\right]\right\rangle\right)_{m\times n}$ into the IFDM $\widetilde{M} = \left\langle\frac{\left(\mu_{ij}^L+\mu_{ij}^U\right)}{2}, \frac{\left(\nu_{ij}^L+\nu_{ij}^U\right)}{2}\right\rangle$.

**Step 2:** Find the IFS, $\langle\mu_i, \nu_i\rangle = \sum_{t=1}^{n} w_t \times \langle\mu_{it}, \nu_{it}\rangle = \langle 1 - \prod_{t=1}^{n}(1-\mu_{it})^{w_i}, \prod_{t=1}^{n}(\nu_{it})^{w_i}\rangle$, $i = 1$ to $m$.

**Step 3:** Using the following steps of the existing method [175], check that $\langle\mu_p, \nu_p\rangle > \langle\mu_q, \nu_q\rangle$ or $\langle\mu_p, \nu_p\rangle < \langle\mu_q, \nu_q\rangle$ or $\langle\mu_p, \nu_p\rangle = \langle\mu_q, \nu_q\rangle$.

**Step 3(a):** Check that $\mu_p - \nu_p > \mu_q - \nu_q$ or $\mu_p - \nu_p < \mu_q - \nu_q$ or $\mu_p - \nu_p = \mu_q - \nu_q$

**Case (i)** If $\mu_p - \nu_p > \mu_q - \nu_q$ then $\langle\mu_p, \nu_p\rangle > \langle\mu_q, \nu_q\rangle$.

**Case (ii)** If $\mu_p - \nu_p < \mu_q - \nu_q$ then $\langle\mu_p, \nu_p\rangle < \langle\mu_q, \nu_q\rangle$.

**Case (iii)** If $\mu_p - \nu_p = \mu_q - \nu_q$ then go to Step 3(b).

**Step 3(b):** Check that $\mu_p + \nu_p > \mu_q + \nu_q$ or $\mu_p + \nu_p < \mu_q + \nu_q$ or $\mu_p + \nu_p = \mu_q + \nu_q$

**Case (i)** If $\mu_p + \nu_p > \mu_q + \nu_q$ then $\langle\mu_p, \nu_p\rangle > \langle\mu_q, \nu_q\rangle$.

**Case (ii)** If $\mu_p + \nu_p < \mu_q + \nu_q$ then $\langle\mu_p, \nu_p\rangle < \langle\mu_q, \nu_q\rangle$.

**Case (iii)** If $\mu_p + \nu_p = \mu_q + \nu_q$ then $\langle\mu_p, \nu_p\rangle = \langle\mu_q, \nu_q\rangle$.

**Step 4:** Check that which relation out of the relations $\langle\mu_p, \nu_p\rangle > \langle\mu_q, \nu_q\rangle$ or $\langle\mu_p, \nu_p\rangle < \langle\mu_q, \nu_q\rangle$ or $\langle\mu_p, \nu_p\rangle = \langle\mu_q, \nu_q\rangle$ is obtained.

**Case (i)** If $\langle\mu_p, \nu_p\rangle > \langle\mu_q, \nu_q\rangle$ then $A_p > A_q$.

**Case (ii)** If $\langle\mu_p, \nu_p\rangle < \langle\mu_q, \nu_q\rangle$ then $A_p < A_q$.

**Case (iii)** If $\langle\mu_p, \nu_p\rangle = \langle\mu_q, \nu_q\rangle$ then $A_p = A_q$.

**Remark:** In Kumar and Garg's IVIFMADMM [98], it is assumed that on considering the expression $T(\mu_i) = \frac{a_i}{a_i+c_i}$, the connection number $\mu_i = a_i + c_i j$ can be transformed into its equivalent real number. While, this assumption is not correct as if $a_i = c_i = 0$. Then, $T(\mu_i)$ will be an indeterminate value instead of a real number. Since, the proposed Mehar method is independent from any such relation on considering which an indeterminate value is obtained. Therefore, in general, the proposed Mehar IVIFMADMM can solve problems that cannot be solved using Kumar and Garg's IVIFMADMM.

### 3.6 Exact PO for the alternatives of the considered IVIFMADMPr

In Section 3.3, an IVIFMADMPr is solved by Kumar and Garg's IVIFMADMM [98] and shown that this method fails to rank the alternatives $A_1$ and $A_2$.

In this section, the same IVIFMADMPr is solved by the proposed Mehar IVIFMADMM.

Using the proposed Mehar IVIFMADMM, the PO for the alternatives of the considered IVIFMADMPr can be obtained as follows:

**Step 1:** Using Step 1 of the proposed Mehar IVIFMADMM, the IVIFDM

$$\tilde{D} = \left(\langle[\mu_{ij}^L, \mu_{ij}^U], [\nu_{ij}^L, \nu_{ij}^U]\rangle\right)_{2\times 2} = \begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cc} \quad G_1 \quad & \quad G_2 \quad \end{array} \\ \begin{bmatrix} \left\langle\left[\frac{1}{8},\frac{3}{8}\right],\left[\frac{1}{16},\frac{7}{16}\right]\right\rangle & \left\langle\left[\frac{1}{16},\frac{3}{16}\right],\left[\frac{1}{16},\frac{3}{16}\right]\right\rangle \\ \left\langle\left[\frac{1}{9},\frac{2}{9}\right],\left[\frac{1}{18},\frac{5}{18}\right]\right\rangle & \left\langle\left[\frac{1}{45},\frac{8}{45}\right],\left[\frac{1}{20},\frac{3}{20}\right]\right\rangle \end{bmatrix} \end{array}$$

can be transformed into the IFDM

$$\tilde{M} = \left(\left\langle\frac{\left(\mu_{ij}^L+\mu_{ij}^U\right)}{2}, \frac{\left(\nu_{ij}^L+\nu_{ij}^U\right)}{2}\right\rangle\right)_{2\times 2} = \begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cc} \quad G_1 \quad & \quad G_2 \quad \end{array} \\ \begin{bmatrix} \left\langle\frac{\left(\frac{1}{8}+\frac{3}{8}\right)}{2},\frac{\left(\frac{1}{16}+\frac{7}{16}\right)}{2}\right\rangle & \left\langle\frac{\left(\frac{1}{16}+\frac{3}{16}\right)}{2},\frac{\left(\frac{1}{16}+\frac{3}{16}\right)}{2}\right\rangle \\ \left\langle\frac{\left(\frac{1}{9}+\frac{2}{9}\right)}{2},\frac{\left(\frac{1}{18}+\frac{5}{18}\right)}{2}\right\rangle & \left\langle\frac{\left(\frac{1}{45}+\frac{8}{45}\right)}{2},\frac{\left(\frac{1}{20}+\frac{3}{20}\right)}{2}\right\rangle \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cc} G_1 & G_2 \end{array} \\ \begin{bmatrix} \left\langle\frac{1}{4},\frac{1}{4}\right\rangle & \left\langle\frac{1}{8},\frac{1}{8}\right\rangle \\ \left\langle\frac{1}{6},\frac{1}{6}\right\rangle & \left\langle\frac{1}{10},\frac{1}{10}\right\rangle \end{bmatrix} \end{array}.$$

**Step 2:** Using Step 2 of the proposed Mehar IVIFMADMM, $\langle\mu_1, \nu_1\rangle = w_1\left\langle\frac{1}{4},\frac{1}{4}\right\rangle + w_2\left\langle\frac{1}{8},\frac{1}{8}\right\rangle = 0.5\left\langle\frac{1}{4},\frac{1}{4}\right\rangle + 0.5\left\langle\frac{1}{8},\frac{1}{8}\right\rangle = \langle 0.1899, 0.1768\rangle$ and $\langle\mu_2, \nu_2\rangle = w_1\left\langle\frac{1}{6},\frac{1}{6}\right\rangle + w_2\left\langle\frac{1}{10},\frac{1}{10}\right\rangle = 0.5\left\langle\frac{1}{6},\frac{1}{6}\right\rangle + 0.5\left\langle\frac{1}{10},\frac{1}{10}\right\rangle = \langle 0.1339, 0.1291\rangle$.

**Step 3:** Using Step 3(a) of the proposed Mehar IVIFMADMM, $\mu_1 - \nu_1 = 0.0131$ and $\mu_2 - \nu_2 = 0.00488$. Since $\mu_1 - \nu_1 > \mu_2 - \nu_2$, therefore, according to Case (i) of Step 3(a) of the proposed Mehar IVIFMADMM $\langle\mu_1, \nu_1\rangle > \langle\mu_2, \nu_2\rangle$.

**Step 4:** Since $\langle\mu_1, \nu_1\rangle > \langle\mu_2, \nu_2\rangle$ so according to Case (i) of Step 4 of the proposed Mehar IVIFMADMM, $A_1 > A_2$.

## 3.7 Advantages of the proposed Mehar IVIFMADMM over Kumar and Garg's IVIFMADMM

It is better to apply the proposed Mehar IVIFMADMM as compared to Kumar and Garg's IVIFMADMM [98] due to the following reasons:

(i) It is obvious from Section 3.3 and Section 3.6 that there can exist IVIFMADMPrs which cannot be solved by Kumar and Garg's IVIFMADMM [98] but can be solved by the proposed Mehar IVIFMADMM.

(ii) It is obvious from Section 3.4 that in Kumar and Garg's IVIFMADMM [98], firstly, each element of the IVIFDM, represented by an IVIFS, is transformed into an IFS. Then, the obtained IFS is transformed into a CN. While, it is obvious from Section 3.5 that in the proposed Mehar IVIFMADMM, the obtained IFS are not transformed into a CN. Therefore, less computational efforts are required to solve an IVIFMADMPr by the proposed Mehar IVIFMADMM as compared to Kumar and Garg's IVIFMADMM [98].

## 3.8 Conclusions

It is shown that Kumar and Garg's IVIFMADMM [98] fails to rank the alternatives of IVIFMADMPrs. Also, a new IVIFMADMM (named as Mehar IVIFMADMM) is proposed without using the concept of CN. Furthermore, it is shown that less computational efforts are required to apply the proposed Mehar IVIFMADMM as compared to Kumar and Garg's IVIFMADMM [98].

# Chapter 4

# Modified expressions to evaluate the CoCf between two DHFSSs and their application in DM*

Arora and Garg [8] proposed two expressions for evaluating the weighted CoCfs between two DHFSSs [52]. Arora and Garg claimed that their proposed expressions can be used for finding the solution for several real-life MCDMPrs under DHFSS environment. To validate the claim, Arora and Garg solved three real-life problems (finding the best candidate, medical diagnosis problem and pattern recognition). In future, other researchers may use Arora and Garg's expressions for solving same type of real-life problems or some other type of real-life problems. However, after a deep study, it is observed that the Arora and Garg have used some mathematical incorrect assumptions to obtain their proposed expressions i.e., Arora and Garg's expressions are not valid in its present form. Therefore, if one will apply these expressions then the obtained results may or may not be exact. Keeping the same in mind, Arora and Garg's expressions have been modified. Furthermore, using the modified expressions, the exact results of the real-life problems, considered by Arora and Garg, have been obtained.

## 4.1 Preliminaries

In this section, some basic definitions are presented.

**Definition 4.1 [40]** Let $X$ be an initial universe of objects. A set $\tilde{A}$ on $X$ defined as $\tilde{A} = \{\langle x,\ \mu_{\tilde{A}}(x^{(s)})\rangle | x \in X\}$ is called a HFS, where $\mu_{\tilde{A}}(x^{(s)})$ is a mapping defined by

* **The contents of this chapter have been communicated for the possible publication in Engineering Applications of Artificial Intelligence.**

$$\mu_{\tilde{A}}\left(x^{(s)}\right): X \rightarrow [0,1]$$

where, $\mu_{\tilde{A}}\left(x^{(s)}\right)$ is a set of some different values in [0,1] and s represents the number of possible membership degrees of the element $x \in X$ to $\tilde{A}$.

**Definition 4.2 [52]** A set $\tilde{A}$ on $X$ defined as $\tilde{A} = \left\{\langle x,\ \mu_{\tilde{A}}\left(x^{(s)}\right), \nu_{\tilde{A}}\left(x^{(t)}\right)\rangle | x \in X\right\}$ is called a DHFS, where, $\mu_{\tilde{A}}\left(x^{(s)}\right), \nu_{\tilde{A}}\left(x^{(t)}\right)$ is a mapping defined by

$$\mu_{\tilde{A}}\left(x^{(s)}\right), \nu_{\tilde{A}}\left(x^{(t)}\right): X \rightarrow [0,1],$$

where $\mu_{\tilde{A}}\left(x^{(s)}\right), \nu_{\tilde{A}}\left(x^{(t)}\right)$ is a set of some different values in [0,1], s represent the number of possible membership degrees and $t$ represent the number of possible non membership degrees of the element $x \in X$ to $\tilde{A}$.

**Definition 4.3 [80]** Let $X$ be an initial universe of objects and $E$ the set of parameters in relation to objects in $X$ and $A \subseteq E$. Parameters are often attributes, characteristics, or properties of objects. Let $\mathcal{P}(X)$ denote the power set of $X$. Then, the pair $\left(\tilde{F}, A\right)$ is called a SS over $X$, where $\tilde{F}$ is a mapping defined by

$$\tilde{F}: A \rightarrow \mathcal{P}(X).$$

**Definition 4.4 [80]** Let $\mathcal{F}(X)$ be the set of all fuzzy subsets in $X$. Then, the pair $\left(\tilde{F}, A\right)$ is called an FSS over $X$, where $\tilde{F}$ is a mapping defined by

$$\tilde{F}: A \rightarrow \mathcal{F}(X)\ .$$

**Definition 4.5[80]** Let $\mathcal{H}(X)$ be the set of all HFSs in $X$. Then, the pair $\left(\tilde{F}, A\right)$ is called a HFSS over $X$, where $\tilde{F}$ is a mapping defined by

$$\tilde{F}: A \rightarrow \mathcal{H}(X).$$

**Definition 4.6 [52]** Let $\mathfrak{D}\mathcal{H}(X)$ be the set of all DHFSs in $X$. Then, the pair $\left(\tilde{F}, A\right)$ is called a DHFSS set over $X$, where $\tilde{F}$ is a mapping defined by

$$\tilde{F}: A \rightarrow \mathfrak{D}\mathcal{H}(X).$$

### 4.2 A brief review about Arora and Garg's expressions

Arora and Garg [8] pointed out that although there exist expressions to evaluate

(i) The CoCf between two FSs [77].

(ii) The CoCf between two IFSs [58, 154].

(iii) The CoCf between two IVIFSs [21].

(iv) The CoCf between two PFSs [47].

(v) The CoCf between two intuitionistic multiplicative sets [50].

(vi) The CoCf between two HFSs [26, 178].

(vii) The CoCf between two DHFSs [183].

But, there does not exist any expression to evaluate the CoCf between two DHFSSs [183]. To fill this gap, Arora and Garg [8] proposed the expression (4.1) as well as expression (4.2) to evaluate the weighted CoCf between two DHFSSs $(\tilde{F},E)=\left\{\langle\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right),\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\rangle\right\}$ and $(\tilde{G},E)=\left\{\langle\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right),\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\rangle\right\}$, where $i=1,2,\dots,n$, $j=1,2,\dots,m$, and $s$, $t$ represents the number of values in $\mu_{\tilde{F}(e_j)}$ and $\nu_{\tilde{F}(e_j)}$ respectively.

$$\rho_3\left((\tilde{F},E),(\tilde{G},E)\right)=$$

$$\frac{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)\right)}{\left\{\begin{array}{c}\sqrt{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu^2_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu^2_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\right)\right)}\times\\ \sqrt{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu^2_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu^2_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)\right)}\end{array}\right\}}\qquad(4.1)$$

$\rho_4\left(\left(\tilde{F},E\right),\left(\tilde{G},E\right)\right) =$

$$\frac{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)\right)}{max\left\{\begin{array}{c}\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu^2_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu^2_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\right)\right),\\ \sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu^2_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu^2_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)\right)\end{array}\right\}} \tag{4.2}$$

where,

(i) $\xi_i$ represents the normalized weight ($\xi_i \geq 0$ $and$ $\sum_{i=1}^{n}\xi_i = 1$) of the $i^{th}$ expert.

(ii) $\eta_j$ represents the normalized weight $\left(\eta_j \geq 0 \; and \; \sum_{j=1}^{m}\eta_j = 1\right)$ of the $j^{th}$ criteria.

(iii) $n$ represents the number of experts.

(iv) $m$ represents the number of criteria.

(v) $\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)$ and $\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)$ are two sets of some values in $[0,1]$. Out of these two, $\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)$ represents the set of all the possible membership degree and $\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)$ represents the set of all the possible non-membership degree.

(vi) $k_i$ represents the number of values in $\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)$.

(vii) $l_i$ represents the number of values in $\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)$.

Arora and Garg [8] claimed that if $\xi_i = \frac{1}{n}$ for all $i$ and $\eta_j = \frac{1}{m}$ for all $j$. Then, the expressions (4.1) and (4.2) will be transformed into the expressions (4.3) and (4.4) respectively.

$\rho_1\left(\left(\tilde{F},E\right),\left(\tilde{G},E\right)\right) =$

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)\right)}{\left\{\begin{array}{c}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu^2_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu^2_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\right)\right)}\times\\ \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu^2_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu^2_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)\right)}\end{array}\right\}} \tag{4.3}$$

$$\rho_2\left(\left(\tilde{F},E\right),\left(\tilde{G},E\right)\right) =$$

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)\right)}{max\left\{\begin{array}{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu_{\tilde{F}(e_j)}^2\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu_{\tilde{F}(e_j)}^2\left(x_i^{(t)}\right)\right)\right),\\ \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left(\mu_{\tilde{G}(e_j)}^2\left(x_i^{(s)}\right)\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\left(\nu_{\tilde{G}(e_j)}^2\left(x_i^{(t)}\right)\right)\right)\end{array}\right\}} \quad (4.4)$$

To demonstrate the need of the proposed expressions, Arora and Garg [8] solved the following three real-life problems by the proposed expressions (4.1) and (4.2).

### 4.2.1 First real-life problem

There is need to recruit an Assistant Professor out of the three candidates $A_1$, $A_2$ and $A_3$ in the department of Mathematics in a Central Government University. For the same purpose a penal of three experts $x_1$, $x_2$ and $x_3$ have been constituted. These experts have to select one candidate on the basis of the four criteria, (i) $e_1$: qualification (ii) $e_2$: teaching experience (iii) $e_3$: number of publication (iv) $e_4$: teaching ability.

Arora and Garg [8] claimed that if it is assumed that

(i) The weights assigned to the first, second and third expert are $0.3, 0.5$ and $0.2$ respectively.

(ii) The weights assigned to the first, second, third and fourth criteria are $0.1, 0.3$, $0.4$ and $0.2$ respectively.

(iii) The $(i,j)^{th}$ element of Table 4.1, Table 4.2 and Table 4.3 represented by a DHFSS, represents the rating value of the $i^{th}$ candidate $A_i$ $(i = 1,2 \text{ and } 3)$ over the $j^{th}$ criteria $e_j$ $(j = 1,2,3 \text{ and } 4)$.

(iv) The $(i,j)^{th}$ element of Table 4.4, represented by a DHFSS, represents the rating value of the reference set $B$.

Then, one may conclude that on applying the expression (4.1) and (4.2) the obtained best candidate for the post of Assistant Professor is $A_2$.

**Table 4.1. Rating values of the alternative $A_1$**

$$
\begin{array}{cc}
 & \begin{array}{cccc} \qquad e_1 \qquad & \qquad e_2 \qquad & \qquad e_3 \qquad & \qquad e_4 \qquad \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} &
\begin{bmatrix}
\{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\} & \{\{0.6,0.5,0.4\},\{0.4,0.3,0.2\}\} & \{\{0.3,0.2,0.1\},\{0.7,0.6,0.6\}\} & \{\{0.8,0.7,0.5\},\{0.2,0.1\}\} \\
\{\{0.6,0.5,0.4\},\{0.3,0.2,0.1\}\} & \{\{0.5,0.4,0.3\},\{0.5,0.3,0.3\}\} & \{\{0.7,0.7,0.5\},\{0.3,0.2,0.2\}\} & \{\{0.4,0.3,0.2\},\{0.5\}\} \\
\{\{0.8,0.7,0.7\},\{0.2,0.1\}\} & \{\{0.8,0.8,0.7\},\{0.2,0.2,0.1\}\} & \{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\} & \{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\}
\end{bmatrix}.
\end{array}
$$

**Table 4.2 Rating values of the alternative $A_2$**

$$
\begin{array}{cc}
 & \begin{array}{cccc} \qquad e_1 \qquad & \qquad e_2 \qquad & \qquad e_3 \qquad & \qquad e_4 \qquad \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} &
\begin{bmatrix}
\{\{0.4,0.4,0.3\},\{0.6,0.5,0.4\}\} & \{\{0.6,0.6,0.5\},\{0.4,0.3,0.2\}\} & \{\{0.8,0.7,0.6\},\{0.2,0.1,0.1\}\} & \{\{0.7,0.5,0.4\},\{0.2,0.2,0.1\}\} \\
\{\{0.7,0.6,0.4\},\{0.3,0.2\}\} & \{\{0.6,0.4,0.3\},\{0.3,0.1\}\} & \{\{0.6,0.4,0.3\},\{0.4,0.1\}\} & \{\{0.5,0.4,0.2\},\{0.4,0.3,0.1\}\} \\
\{\{0.9,0.7,0.6\},\{0.1\}\} & \{\{0.7,0.5,0.4\},\{0.3,0.2\}\} & \{\{0.4,0.3,0.3\},\{0.6,0.5\}\} & \{\{0.7,0.6,0.4\},\{0.2,0.1\}\}
\end{bmatrix}.
\end{array}
$$

**Table 4.3 Rating values of the alternative $A_3$**

$$
\begin{array}{cc}
 & \begin{array}{cccc} \qquad e_1 \qquad & \qquad e_2 \qquad & \qquad e_3 \qquad & \qquad e_4 \qquad \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} &
\begin{bmatrix}
\{\{0.8,0.7,0.6\},\{0.2,0.1\}\} & \{\{0.5,0.4\},\{0.2,0.1\}\} & \{\{0.9,0.8,0.6\},\{0.1\}\} & \{\{0.9,0.7,0.5\},\{0.1\}\} \\
\{\{0.6,0.5,0.5\},\{0.4,0.2\}\} & \{\{0.8,0.7,0.5\},\{0.2,0.1\}\} & \{\{0.6,0.4,0.3\},\{0.3,0.1\}\} & \{\{0.3,0.2,0.1\},\{0.5,0.3,0.2\}\} \\
\{\{0.7,0.6,0.4\},\{0.2\}\} & \{\{0.6,0.4,0.3\},\{0.3\}\} & \{\{0.4,0.3,0.3\},\{0.2,0.1\}\} & \{\{0.6,0.3,0.2\},\{0.2,0.1\}\}
\end{bmatrix}.
\end{array}
$$

**Table 4.4 Rating values of the reference set $B$**

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \end{array}$$

$$\begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline x_1 & \{\{0.3,0.2,0.1\},\{0.2,0.1\}\} & \{\{0.4\},\{0.3,0.2\}\} & \{\{0.6,0.4,0.3\},\{0.2\}\} & \{\{0.4\},\{0.5,0.1\}\} \\ x_2 & \{\{0.6,0.5,0.2\},\{0.4,0.3\}\} & \{\{0.5,0.2\},\{0.3,0.1\}\} & \{\{0.7,0.3\},\{0.3,0.1\}\} & \{\{0.2,0.1\},\{0.3\}\} \\ x_3 & \{\{0.6,0.4,0.3\},\{0.2\}\} & \{\{0.9,0.7,0.5\},\{0.1\}\} & \{\{0.4,0.2\},\{0.5,0.3\}\} & \{\{0.8\},\{0.1\}\} \end{array}.$$

### 4.2.2 Second real-life problem

A patient have the five symptoms, namely temperature $(e_1)$, headache $(e_2)$, stomach pain $(e_3)$, cough $(e_4)$ and chest pain $(e_5)$, of the following three diseases (i) viral fever $(C_1)$ (ii) malaria $(C_2)$ (iii) typhoid $(C_3)$. A panel of four doctors $d_1$, $d_2$ $d_3$ and $d_4$ have been constituted to verify the actual disease of the patient.

Arora and Garg [8] claimed that if it is assumed that

(i) The weights assigned to the first, second, third and fourth doctor are $0.5, 0.2, 0.2$ and $0.1$ respectively.

(ii) The weights assigned to the first, second, third, fourth and fifth symptom are $0.4, 0.1, 0.2, 0.2$ and $0.1$ respectively.

(iii) The $(i, j)^{th}$ element of Table 4.5, Table 4.6 and Table 4.7 represented by a DHFSS, represents the rating values of each diagnosis, over different symptoms by the doctors.

(iv) The $(i, j)^{th}$ element of Table 4.8, represented by a DHFSS, represents the rating value of the reference set $B$.

Then, on applying the expression $(4.1)$ one may conclude that the patient is suffering from the disease $C_3$. While, on applying the expression $(4.2)$ one may conclude the patient is suffering from the disease $C_2$.

**Table 4.5 Rating values of the alternative $C_1$**

$$
\begin{array}{cc}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} &
\left[\begin{array}{ccccc}
\{\{0.6,0.4,0.3\},\{0.3,0.1\}\} & \{\{0.8,0.7,0.5\},\{0.1\}\} & \{\{0.4,0.3\},\{0.2\}\} & \{\{0.8,0.7,0.5\},\{0.2,0.1\}\} & \{\{0.8,0.7\},\{0.2,0.1\}\} \\
\{\{0.9,0.8,0.7\},\{0.1\}\} & \{\{0.7,0.6,0.4\},\{0.3,0.2\}\} & \{\{0.5,0.3,0.1\},\{0.3\}\} & \{\{0.5,0.4,0.3\},\{0.5,0.3\}\} & \{\{0.5,0.3,0.1\},\{0.4,0.3\}\} \\
\{\{0.5,0.4,0.2\},\{0.5,0.3\}\} & \{\{0.5,0.4,0.3\},\{0.4,0.2,0.1\}\} & \{\{0.8,0.6,0.5\},\{0.2,0.1\}\} & \{\{0.6,0.3,0.2\},\{0.4,0.1\}\} & \{\{0.2,0.1\},\{0.8\}\} \\
\{\{0.3,0.2\},\{0.4,0.2\}\} & \{\{0.7,0.6\},\{0.3,0.2,0.1\}\} & \{\{0.2,0.1\},\{0.5,0.4\}\} & \{\{0.2,0.1\},\{0.7,0.5\}\} & \{\{0.3,0.1\},\{0.6,0.4\}\}
\end{array}\right].
\end{array}
$$

**Table 4.6 Rating values of the alternative $C_2$**

$$
\begin{array}{cc}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} &
\left[\begin{array}{ccccc}
\{\{0.8,0.7\},\{0.2,0.1\}\} & \{\{0.5,0.4,0.3\},\{0.4,0.3\}\} & \{\{0.6,0.4\},\{0.4,0.2\}\} & \{\{0.3,0.2\},\{0.6,0.5\}\} & \{\{0.6,0.5,0.3\},\{0.2\}\} \\
\{\{0.4,0.3\},\{0.6,0.2\}\} & \{\{0.7,0.6\},\{0.3,0.2\}\} & \{\{0.7,0.6\},\{0.3,0.2\}\} & \{\{0.7,0.6,0.4\},\{0.2,0.1\}\} & \{\{0.7,0.3\},\{0.3,0.1\}\} \\
\{\{0.5,0.4\},\{0.3,0.2\}\} & \{\{0.6,0.4\},\{0.3\}\} & \{\{0.6,0.5\},\{0.3\}\} & \{\{0.6,0.5\},\{0.3,0.2\}\} & \{\{0.4,0.2\},\{0.5,0.3\}\} \\
\{\{0.8,0.7\},\{0.2,0.1\}\} & \{\{0.8,0.7\},\{0.2,0.1\}\} & \{\{0.7,0.6\},\{0.2\}\} & \{\{0.4,0.3\},\{0.2,0.1\}\} & \{\{0.2,0.1\},\{0.3\}\}
\end{array}\right].
\end{array}
$$

**Table 4.7 Rating values of the alternative $C_3$**

$$
\begin{array}{cc}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} &
\left[\begin{array}{ccccc}
\{\{0.6,0.5\},\{0.2,0.1\}\} & \{\{0.6,0.4\},\{0.3,0.2\}\} & \{\{0.7,0.4,0.2\},\{0.2,0.1\}\} & \{\{0.7,0.6\},\{0.3,0.2\}\} & \{\{0.6,0.4,0.3\},\{0.3,0.1\}\} \\
\{\{0.5,0.4\},\{0.2\}\} & \{\{0.7,0.6,0.4\},\{0.3,0.2\}\} & \{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\} & \{\{0.9,0.7,0.6\},\{0.1\}\} & \{\{0.7,0.5,0.4\},\{0.3,0.2\}\} \\
\{\{0.6,0.5\},\{0.1\}\} & \{\{0.4,0.3\},\{0.6,0.5\}\} & \{\{0.6,0.5,0.4\},\{0.3,0.2,0.1\}\} & \{\{0.8,0.7,0.6\},\{0.2,0.1\}\} & \{\{0.5,0.4\},\{0.4,0.3\}\} \\
\{\{0.5,0.4\},\{0.2,0.1\}\} & \{\{0.8,0.7\},\{0.2\}\} & \{\{0.4,0.4,0.3\},\{0.6\}\} & \{\{0.6,0.5\},\{0.4,0.2\}\} & \{\{0.6,0.5\},\{0.2,0.1\}\}
\end{array}\right].
\end{array}
$$

**Table 4.8 Rating value of the reference set $B$**

$$
\begin{array}{c c}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} &
\begin{bmatrix}
\{\{0.9\},\{0.1\}\} & \{\{0.6,0.5\},\{0.4,0.3,0.1\}\} & \{\{0.4,0.3,0.2\},\{0.5\}\} & \{\{0.8,0.7\},\{0.1\}\} & \{\{0.4,0.3\},\{0.6,0.5\}\} \\
\{\{0.6,0.4\},\{0.4,0.2\}\} & \{\{0.8\},\{0.2\}\} & \{\{0.7,0.5\},\{0.3,0.2\}\} & \{\{0.3,0.2,0.1\},\{0.5,0.3\}\} & \{\{0.7,0.5\},\{0.3,0.1\}\} \\
\{\{0.7\},\{0.2,0.1\}\} & \{\{0.4,0.3\},\{0.2,0.1\}\} & \{\{0.8,0.7\},\{0.2,0.1\}\} & \{\{0.6,0.3\},\{0.2\}\} & \{\{0.6\},\{0.4,0.2\}\} \\
\{\{0.8\},\{0.2\}\} & \{\{0.9,0.6\},\{0.1\}\} & \{\{0.6,0.4\},\{0.3\}\} & \{\{0.5,0.4\},\{0.2,0.1\}\} & \{\{0.6,0.4\},\{0.2\}\}
\end{bmatrix}
\end{array}.
$$

### 4.2.3 Third real-life problem

There is need to identify the best unknown pattern among $A_1$, $A_2$ and $A_3$, corresponding to a known pattern $B$ by considering the three different criteria $e_1$, $e_2$ and $e_3$. Three experts are assigned for the same purpose.

Arora and Garg [8] claimed that if it is assumed that

(i) The weights assigned to the first, second and third expert are $0.5, 0.3$ and $0.2$ respectively.

(ii) The weights assigned to the first, second and third criteria are $0.4, 0.3$ and $0.3$ respectively.

(iii) The $(i,j)^{th}$ element of Table 4.9, Table 4.10 and Table 4.11 represented a the DHFSS, represents the rating values provided by the decision makers over the criteria for the patterns $A_1$, $A_2$ and $A_3$.

(iv) The $(i,j)^{th}$ element of Table 4.12, represented by a DHFSS, represents the rating values of the known pattern $B$.

Then, one may conclude that on applying both the expressions (4.1) and (4.2) the obtained most desirable pattern is $A_2$.

**Table 4.9 Rating values of the pattern $A_1$**

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} \{\{0.7,0.6\},\{0.2\}\} & \{\{0.8,0.7\},\{0.1\}\} & \{\{0.2\},\{0.4,0.3\}\} \\ \{\{0.9\},\{0.1\}\} & \{\{0.7\},\{0.3,0.2\}\} & \{\{0.6\},\{0.2,0.1\}\} \\ \{\{0.6,0.5\},\{0.2,0.1\}\} & \{\{0.4\},\{0.5,0.3\}\} & \{\{0.7,0.6\},\{0.1\}\} \end{bmatrix} \end{array}.$$

**Table 4.10 Rating values of the pattern $A_2$**

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} \{\{0.7,0.6\},\{0.2,0.1\}\} & \{\{0.5\},\{0.4,0.3\}\} & \{\{0.3,0.2\},\{0.6\}\} \\ \{\{0.4,0.3\},\{0.5\}\} & \{\{0.8\},\{0.2,0.1\}\} & \{\{0.8,0.7\},\{0.2,0.1\}\} \\ \{\{0.5\},\{0.3,0.2\}\} & \{\{0.7,0.6\},\{0.2\}\} & \{\{0.6\},\{0.2\}\} \end{bmatrix} \end{array}.$$

**Table 4.11 Rating values of the pattern $A_3$**

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} \{\{0.2,0.1\},\{0.7\}\} & \{\{0.7,0.6\},\{0.3,0.1\}\} & \{\{0.7\},\{0.1\}\} \\ \{\{0.7,0.6\},\{0.2,0.1\}\} & \{\{0.4\},\{0.5\}\} & \{\{0.6\},\{0.2,0.1\}\} \\ \{\{0.6\},\{0.3,0.1\}\} & \{\{0.1\},\{0.7\}\} & \{\{0.6,0.5\},\{0.2\}\} \end{bmatrix} \end{array}.$$

**Table 4.12 Rating values of the known pattern *B***

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} \{\{0.7,0.6\},\{0.2\}\} & \{\{0.2\},\{0.8,0.7\}\} & \{\{0.6,0.4\},\{0.2\}\} \\ \{\{0.6\},\{0.3,0.1\}\} & \{\{0.3,0.2\},\{0.7\}\} & \{\{0.7\},\{0.1\}\} \\ \{\{0.5,0.3\},\{0.4\}\} & \{\{0.5\},\{0.3,0.2\}\} & \{\{0.3,0.1\},\{0.6\}\} \end{bmatrix} \end{array}.$$

## 4.3 Origin of Arora and Garg's expressions

In this chapter, it is claimed that the expressions (4.1) and (4.2) are not valid in its present form. To prove that this claim is valid, there is need to discuss the origin of the expressions (4.1) and (4.2). Therefore, the same is discussed in this section.

### 4.3.1 Origin of the first expression

It can be easily verified that the Arora and Garg [8] have obtained the expression (4.1) in the following manner:

$\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)$

$=$

$\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sum_{s=1}^{k_i}\xi_i\eta_j\frac{1}{k_i}\left(\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)\right)+$

$\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sum_{t=1}^{l_i}\xi_i\eta_j\frac{1}{l_i}\left(\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)\right)$

$=\left(\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\right)\times\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)\right)+$

$\left(\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\right)\times\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)\right)$

Assuming, $X_1=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)$,

$Y_1=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)$,

$X_2=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)$ and $Y_2=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)$.

$\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)=$

$(X_1Y_1+X_2Y_2)\le\sqrt{X_1^2+X_2^2}\sqrt{Y_1^2+Y_2^2}$

$\le\left(\sqrt{\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\right)^2+\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\right)^2}\right)\times$

$\left(\sqrt{\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)^2+\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi}_i\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)^2}\right)$

$\le\left(\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{\xi}_i\sqrt{\eta_j}\right)^2\left(\left(\sum_{s=1}^{k_i}\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2+\left(\sum_{t=1}^{l_i}\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2\right)}\right)\right)\times$

$\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{\xi}_i\sqrt{\eta_j}\right)^2\left(\left(\sum_{s=1}^{k_i}\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2+\left(\sum_{t=1}^{l_i}\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2\right)}\right)$

$$\leq \left( \left( \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i \eta_j \left( \left( \sum_{s=1}^{k_i} \frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}} \right)^2 + \left( \sum_{t=1}^{l_i} \frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}} \right)^2 \right)} \right) \right) \times$$

$$\left( \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i \eta_j \left( \left( \sum_{s=1}^{k_i} \frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}} \right)^2 + \left( \sum_{t=1}^{l_i} \frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}} \right)^2 \right)} \right)$$

$$\leq \left( \left( \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i \eta_j \left( \frac{1}{k_i} \sum_{s=1}^{k_i} \mu_{\tilde{F}(e_j)}^2\left(x_i^{(s)}\right) + \frac{1}{l_i} \sum_{t=1}^{l_i} \nu_{\tilde{F}(e_j)}^2\left(x_i^{(t)}\right) \right)} \right) \right) \times$$

$$\left( \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i \eta_j \left( \frac{1}{k_i} \sum_{s=1}^{k_i} \mu_{\tilde{G}(e_j)}^2\left(x_i^{(s)}\right) + \frac{1}{l_i} \sum_{t=1}^{l_i} \nu_{\tilde{G}(e_j)}^2\left(x_i^{(t)}\right) \right)} \right)$$

$$\Rightarrow \sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} \eta_j \left( \frac{1}{k_i} \sum_{s=1}^{k_i} \mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right) \mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right) + \frac{1}{l_i} \sum_{t=1}^{l_i} \nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right) \nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right) \right)$$

$$\leq$$

$$\left\{ \left( \left( \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i \eta_j \left( \frac{1}{k_i} \sum_{s=1}^{k_i} \mu_{\tilde{F}(e_j)}^2\left(x_i^{(s)}\right) + \frac{1}{l_i} \sum_{t=1}^{l_i} \nu_{\tilde{F}(e_j)}^2\left(x_i^{(t)}\right) \right)} \right) \right) \times \right.$$

$$\left. \left( \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i \eta_j \left( \frac{1}{k_i} \sum_{s=1}^{k_i} \mu_{\tilde{G}(e_j)}^2\left(x_i^{(s)}\right) + \frac{1}{l_i} \sum_{t=1}^{l_i} \nu_{\tilde{G}(e_j)}^2\left(x_i^{(t)}\right) \right)} \right) \right\}$$

$$\Rightarrow \frac{\sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} \eta_j \left( \frac{1}{k_i} \sum_{s=1}^{k_i} \mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right) \mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right) + \frac{1}{l_i} \sum_{t=1}^{l_i} \nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right) \nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right) \right)}{\left\{ \begin{array}{c} \left( \left( \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i \eta_j \left( \frac{1}{k_i} \sum_{s=1}^{k_i} \mu_{\tilde{F}(e_j)}^2\left(x_i^{(s)}\right) + \frac{1}{l_i} \sum_{t=1}^{l_i} \nu_{\tilde{F}(e_j)}^2\left(x_i^{(t)}\right) \right)} \right) \right) \times \\ \left( \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i \eta_j \left( \frac{1}{k_i} \sum_{s=1}^{k_i} \mu_{\tilde{G}(e_j)}^2\left(x_i^{(s)}\right) + \frac{1}{l_i} \sum_{t=1}^{l_i} \nu_{\tilde{G}(e_j)}^2\left(x_i^{(t)}\right) \right)} \right) \end{array} \right\}} \leq 1.$$

### 4.3.2 Origin of the second expression

It can be easily verified that the Arora and Garg [8] have obtained the expression (4.2) in the following manner:

$$\sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} \eta_j \left( \frac{1}{k_i} \sum_{s=1}^{k_i} \mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right) \mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right) + \frac{1}{l_i} \sum_{t=1}^{l_i} \nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right) \nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right) \right) =$$

$\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sum_{s=1}^{k_i}\xi_i\eta_j\frac{1}{k_i}\left(\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)\right)+$

$\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sum_{t=1}^{l_i}\xi_i\eta_j\frac{1}{l_i}\left(\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)\right)=$

$\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\times\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)+$

$\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\times\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)$

Assuming, $X_1=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)$,

$Y_1=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)$,

$X_2=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)$ and $Y_2=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)$

$\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)=$

$X_1Y_1+X_2Y_2\leq max\left\{\left({X_1}^2+{X_2}^2\right),\left({Y_1}^2+{Y_2}^2\right)\right\}$

$$\leq max\left\{\begin{array}{l}\left[\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\right)^2+\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\right)^2\right],\\ \left[\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)^2+\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)^2\right]\end{array}\right\}$$

$$\leq max\left\{\begin{array}{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\eta_j\left(\frac{1}{k_i}\left(\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\right)^2+\frac{1}{l_i}\left(\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\right)^2\right),\\ \sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\eta_j\left(\frac{1}{k_i}\left(\sum_{s=1}^{k_i}\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)^2+\frac{1}{l_i}\left(\sum_{t=1}^{l_i}\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)^2\right)\end{array}\right\}$$

$$\leq max\left\{\begin{array}{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}^2\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}^2\left(x_i^{(t)}\right)\right),\\ \sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{G}(e_j)}^2\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{G}(e_j)}^2\left(x_i^{(t)}\right)\right)\end{array}\right\}$$

$\Rightarrow\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)$

$$\leq max\left\{\begin{array}{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}^2\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}^2\left(x_i^{(t)}\right)\right),\\ \sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{G}(e_j)}^2\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{G}(e_j)}^2\left(x_i^{(t)}\right)\right)\end{array}\right\}$$

$$\Rightarrow \frac{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)}{max\left\{\begin{array}{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}^2\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}^2\left(x_i^{(t)}\right)\right),\\ \sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{G}(e_j)}^2\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{G}(e_j)}^2\left(x_i^{(t)}\right)\right)\end{array}\right\}} \leq 1.$$

### 4.4 Mathematical incorrect assumptions

It can be easily verified from Section 4.3.1 and Section 4.3.2 that to obtain the expressions (4.1) and (4.2), Arora and Garg [8] have assumed the relations (i) to (vi) will be satisfied. While, it is obvious from Example 4.1 that the relations (i) to (vi) are not satisfying. Hence, the expressions (4.1) and (4.2), proposed by Arora and Garg [8], are not valid.

(i) $\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sum_{s=1}^{k_i}\frac{\xi_i\eta_j}{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)=\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\right)\times\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)\right)$

(ii) $\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sum_{t=1}^{l_i}\frac{\xi_i\eta_j}{l_i}\nu_{(\tilde{F}e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)=\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\right)\times\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)$

(iii) $\left(\sum_{s=1}^{k}\frac{1}{\sqrt{k}}\mu_{\tilde{A}(e_1)}\left(x_1^{(s)}\right)\right)^2=\frac{1}{k}\sum_{s=1}^{k}\mu_{\tilde{A}(e_1)}^2\left(x_1^{(s)}\right)$

(iv) $\left(\sum_{t=1}^{l}\frac{1}{\sqrt{l}}\nu_{\tilde{A}(e_1)}\left(x_1^{(t)}\right)\right)^2=\frac{1}{l}\sum_{t=1}^{l}\nu_{\tilde{A}(e_1)}^2\left(x_1^{(t)}\right)$

(v) $\left(\sum_{s=1}^{k}\frac{1}{\sqrt{k}}\mu_{\tilde{B}(e_1)}\left(x_1^{(s)}\right)\right)^2=\frac{1}{k}\sum_{s=1}^{k}\mu_{\tilde{B}(e_1)}^2\left(x_1^{(s)}\right)$

(vi) $\left(\sum_{t=1}^{l}\frac{1}{\sqrt{l}}\nu_{\tilde{B}(e_1)}\left(x_1^{(t)}\right)\right)^2=\frac{1}{l}\sum_{t=1}^{l}\nu_{\tilde{B}(e_1)}^2\left(x_1^{(t)}\right).$

**Example 4.1:** Let

$$(\tilde{A},E)=\begin{array}{cc} & \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} \{\{0.1,0.2,0.5\},\{0.3\}\} & \{\{0.2,0.4,0.6\},\{0.4,0.5,0.8\}\} \\ \{\{0.2,0.3,0.5\},\{0.3,0.6,0.9\}\} & \{\{0.2,0.3,0.7\},\{0.1,0.9\}\} \end{bmatrix} \end{array} \text{ and}$$

$$(\tilde{B},E) = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{cc} \qquad e_1 \qquad\qquad & \qquad e_2 \end{array} \\ \begin{bmatrix} \{\{0.1,0.2,0.4\},\{0.6,0.8,0.9\}\} & \{\{0.2,0.4,0.1\},\{0.8,0.9,0.6\}\} \\ \{\{0.6,0.3,0.5\},\{0.9,0.2,0.3\}\} & \{\{0.5\},\{0.9\}\} \end{bmatrix} \end{array}$$ be two

DHFSSs.

Furthermore, let $\xi_1 = \frac{1}{3},\ \xi_2 = \frac{2}{3}$ and $\eta_1 = \frac{3}{5}, \eta_2 = \frac{3}{5}$.

Then, it can be easily verified that

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\sum_{s=1}^{k_i}\frac{\xi_i\eta_j}{k_i}\left(\mu_{\tilde{A}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{B}(e_j)}\left(x_i^{(s)}\right)\right)\right) = 0.1429,$$

$$\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{A}(e_j)}\left(x_i^{(s)}\right)\right)\times\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{B}(e_j)}\left(x_i^{(s)}\right)\right) = 1.2980$$

It is obvious that

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\sum_{s=1}^{k_i}\frac{\xi_i\eta_j}{k_i}\mu_{\tilde{A}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{B}(e_j)}\left(x_i^{(s)}\right)\right) \neq \left(\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{A}(e_j)}\left(x_i^{(s)}\right)\right)\times$$

$$\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{s=1}^{k_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{k_i}}\mu_{\tilde{B}(e_j)}\left(x_i^{(s)}\right)\right).$$

Also, it can be verified that

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\sum_{t=1}^{l_i}\frac{\xi_i\eta_j}{l_i}\left(\nu_{\tilde{A}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{B}(e_j)}\left(x_i^{(t)}\right)\right)\right) = 0.3096,$$

$$\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{A}(e_j)}\left(x_i^{(t)}\right)\right)\times\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{B}(e_j)}\left(x_i^{(t)}\right)\right) = 3.1130\ .$$

It is obvious that

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\sum_{t=1}^{l_i}\frac{\xi_i\eta_j}{l_i}\nu_{\tilde{A}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{B}(e_j)}\left(x_i^{(t)}\right)\right) \neq \left(\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{A}(e_j)}\left(x_i^{(t)}\right)\right)\times$$

$$\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{t=1}^{l_i}\frac{\sqrt{\xi_i}\sqrt{\eta_j}}{\sqrt{l_i}}\nu_{\tilde{B}(e_j)}\left(x_i^{(t)}\right)\right).$$

Furthermore, it can be easily verified that

$$\left(\sum_{s=1}^{k}\frac{1}{\sqrt{k}}\mu_{\tilde{A}(e_1)}\left(x_1^{(s)}\right)\right)^2 = 1.6333, \qquad \frac{1}{k}\sum_{s=1}^{k}\mu_{\tilde{A}(e_1)}^2\left(x_1^{(s)}\right) = 0.0700,$$

$$\left(\sum_{t=1}^{l} \frac{1}{\sqrt{l}} \nu_{\tilde{A}(e_1)}\left(x_1^{(t)}\right)\right)^2 = 1.7633, \qquad \frac{1}{l}\sum_{t=1}^{l} \nu^2_{\tilde{A}(e_1)}\left(x_1^{(t)}\right) = 0.6033,$$

$$\left(\sum_{s=1}^{k} \frac{1}{\sqrt{k}} \mu_{\tilde{B}(e_1)}\left(x_1^{(s)}\right)\right)^2 = 0.2133, \qquad \frac{1}{k}\sum_{s=1}^{k} \mu^2_{\tilde{B}(e_1)}\left(x_1^{(s)}\right) = 0.1000,$$

$$\left(\sum_{t=1}^{l} \frac{1}{\sqrt{l}} \nu_{\tilde{B}(e_1)}\left(x_1^{(t)}\right)\right)^2 = 0.0900, \qquad \frac{1}{l}\sum_{t=1}^{l} \nu^2_{\tilde{B}(e_1)}\left(x_1^{(t)}\right) = 0.0900.$$

It is obvious that

$$\left(\sum_{s=1}^{k} \frac{1}{\sqrt{k}} \mu_{\tilde{A}(e_1)}\left(x_1^{(s)}\right)\right)^2 \neq \frac{1}{k}\sum_{s=1}^{k} \mu^2_{\tilde{A}(e_1)}\left(x_1^{(s)}\right),$$

$$\left(\sum_{t=1}^{l} \frac{1}{\sqrt{l}} \nu_{\tilde{A}(e_1)}\left(x_1^{(t)}\right)\right)^2 \neq \frac{1}{l}\sum_{t=1}^{l} \nu^2_{\tilde{A}(e_1)}\left(x_1^{(t)}\right).$$

$$\left(\sum_{s=1}^{k} \frac{1}{\sqrt{k}} \mu_{\tilde{B}(e_1)}\left(x_1^{(s)}\right)\right)^2 \neq \frac{1}{k}\sum_{s=1}^{k} \mu^2_{\tilde{B}(e_1)}\left(x_1^{(s)}\right),$$

$$\left(\sum_{t=1}^{l} \frac{1}{\sqrt{l}} \nu_{\tilde{B}(e_1)}\left(x_1^{(t)}\right)\right)^2 = \frac{1}{l}\sum_{t=1}^{l} \nu^2_{\tilde{B}(e_1)}\left(x_1^{(t)}\right).$$

**4.5 Modified expressions and their origin**

The expression $(4.5)$ and $(4.6)$ represents the modified form of the expression $(4.1)$ and $(4.2)$ respectively. In this section, the origin of the modified expressions is discussed.

$$\rho_3 = \frac{\sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} \eta_j \left(\frac{1}{k_i}\sum_{s=1}^{k_i} \mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right) + \frac{1}{l_i}\sum_{t=1}^{l_i} \nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)}{\sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} \eta_j \left(\left(\sqrt{\sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2 \sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2}\right) + \left(\sqrt{\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2 \sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2}\right)\right)} \tag{4.5}$$

$\rho_4 =$

$$\frac{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)}{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(max\left\{\sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2+\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2,\sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2+\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2\right\}\right)}$$

(4.6)

### 4.5.1 Origin of the first modified expression

The modified expression (4.5) has been obtained as follows:

$$\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)$$

$$=\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\left(\sum_{s=1}^{k_i}\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)+\left(\sum_{t=1}^{l_i}\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)\right).$$

Assuming,

$$X^{(s)}=\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}},\quad Y^{(s)}=\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}},$$

$$X^{(t)}=\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\text{ and }\quad Y^{(t)}=\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}$$

$$\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)=$$

$$\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\sum_{s=1}^{k_i}X^{(s)}\,Y^{(s)}+\sum_{t=1}^{l_i}X^{(t)}\,Y^{(t)}\right)$$

$$\leq\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\sqrt{\left(\sum_{s=1}^{k_i}(X^{(s)})^2\times\sum_{s=1}^{k_i}(Y^{(s)})^2\right)}+\sqrt{\sum_{t=1}^{l_i}(X^{(t)})^2\times\sum_{t=1}^{l_i}(Y^{(t)})^2}\right)$$

$\leq$

$$\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\left(\left(\sqrt{\sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2\times\sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2}\right)+\right.\right.$$

$$\left.\left.\left(\sqrt{\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2\times\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2}\right)\right)\right)$$

$\Rightarrow$

$$\frac{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)}{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\left(\sqrt{\sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2\sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2}\right)+\left(\sqrt{\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2}\right)\right)}$$

$\leq 1.$

### 4.5.2 Origin of the second modified expression

The modified expression (4.6) has been obtained as follows:

$$\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)$$

$$=\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\left(\sum_{s=1}^{k_i}\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)+\left(\sum_{t=1}^{l_i}\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)\right)$$

Assuming,

$X^{(s)}=\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}$, $Y^{(s)}=\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}$, $X^{(t)}=\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}$ and $Y^{(t)}=\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}$

$$C_{WDHFSS}\left((\tilde{F},E),(\tilde{G},E)\right)=\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\sum_{s=1}^{k_i}X^{(s)}\,Y^{(s)}+\sum_{t=1}^{l_i}X^{(t)}\,Y^{(t)}\right)$$

$$\leq \sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(max\left\{\sum_{s=1}^{k_i}\left(X^{(s)}\right)^2+\sum_{t=1}^{l_i}\left(X^{(t)}\right)^2,\sum_{s=1}^{k_i}\left(Y^{(s)}\right)^2+\sum_{t=1}^{l_i}\left(Y^{(t)}\right)^2\right\}\right)\leq$$

$$\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(max\left\{\sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2+\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2,\sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2+\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2\right\}\right)$$

$$\Rightarrow \frac{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)+\frac{1}{l_i}\sum_{t=1}^{l_i}\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)\right)}{\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m}\eta_j\left(max\left\{\begin{array}{l}\sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{F}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2+\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{F}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2,\\ \sum_{s=1}^{k_i}\left(\frac{\mu_{\tilde{G}(e_j)}\left(x_i^{(s)}\right)}{\sqrt{k_i}}\right)^2+\sum_{t=1}^{l_i}\left(\frac{\nu_{\tilde{G}(e_j)}\left(x_i^{(t)}\right)}{\sqrt{l_i}}\right)^2\end{array}\right\}\right)}\leq 1.$$

### 4.6 Exact results of the existing real-life problems

Arora and Garg [8] used the expressions (4.1) and (4.2) to find the solutions of three real-life problems, discussed in Section 4.2. However, as discussed in Section 4.4 that the expressions (4.1) and (4.2) are not valid. Therefore, the results of the real-life problems, obtained by Arora and Garg [8], are also not exact. The results of all the three real-life problems, obtained by the existing expressions (4.1) and (4.2) as well as obtained by the modified expressions (4.5) and (4.6), are shown in Table 4.13.

**Table 4.13 Results of real-life problems**

| Existing real-life problem [8] | Existing expressions (4.1) and (4.2) [8] | Modified expressions (4.5) and (4.6) |
|---|---|---|
| First real-life problem<br>(Best candidate) | $\rho_3((A_1,E),(B,E)) = 0.8390$<br>$\rho_3((A_2,E),(B,E)) = 0.8926$<br>$\rho_3((A_3,E),(B,E)) = 0.8356$<br>$\rho_4((A_1,E),(B,E)) = 0.6878$<br>$\rho_4((A_2,E),(B,E)) = 0.7946$<br>$\rho_4((A_3,E),(B,E)) = 0.7509$ | $\rho_3((A_1,E),(B,E)) = 0.5347$<br>$\rho_3((A_2,E),(B,E)) = 0.9420$<br>$\rho_3((A_3,E),(B,E)) = 0.9314$<br>$\rho_4((A_1,E),(B,E)) = 0.6248$<br>$\rho_4((A_2,E),(B,E)) = 0.7292$<br>$\rho_4((A_3,E),(B,E)) = 0.6392$ |

| | | |
|---|---|---|
| Medical diagnosis | $\rho_3\big((C_1,E),(B,E)\big) = 0.8202$<br>$\rho_3\big((C_2,E),(B,E)\big) = 0.8936$<br>$\rho_3\big((C_3,E),(B,E)\big) = 0.8976$<br>$\rho_4\big((C_1,E),(B,E)\big) = 0.6836$<br>$\rho_4\big((C_2,E),(B,E)\big) = 0.7952$<br>$\rho_4\big((C_3,E),(B,E)\big) = 0.7476$ | $\rho_3\big((C_1,E),(B,E)\big) = 0.9572$<br>$\rho_3\big((C_2,E),(B,E)\big) = 0.9918$<br>$\rho_3\big((C_3,E),(B,E)\big) = 0.9788$<br>$\rho_4\big((C_1,E),(B,E)\big) = 0.6229$<br>$\rho_4\big((C_2,E),(B,E)\big) = 0.7654$<br>$\rho_4\big((C_3,E),(B,E)\big) = 0.6975$ |
| Pattern recognition | $\rho_3\big((A_1,E),(B,E)\big) = 0.7604$<br>$\rho_3\big((A_2,E),(B,E)\big) = 0.8051$<br>$\rho_3\big((A_3,E),(B,E)\big) = 0.7328$<br>$\rho_4\big((A_1,E),(B,E)\big) = 0.7466$<br>$\rho_4\big((A_2,E),(B,E)\big) = 0.8031$<br>$\rho_4\big((A_3,E),(B,E)\big) = 0.7203$ | $\rho_3\big((A_1,E),(B,E)\big) = 0.9925$<br>$\rho_3\big((A_2,E),(B,E)\big) = 0.9868$<br>$\rho_3\big((A_3,E),(B,E)\big) = 0.9817$<br>$\rho_4\big((A_1,E),(B,E)\big) = 0.6936$<br>$\rho_4\big((A_2,E),(B,E)\big) = 0.7316$<br>$\rho_4\big((A_3,E),(B,E)\big) = 0.6535$ |

It is obvious from the results, shown in Table 4.13, that

According to the existing expression (4.1), the obtained PO of the alternatives is $A_3 < A_1 < A_2$. Hence, the most preferred candidate is $A_2$. While, according to the modified expression (4.5), the obtained PO of the alternatives is $A_1 < A_3 < A_2$. Hence, the most preferred candidate is $A_2$. Furthermore, according to the existing expression (4.2), the obtained PO of the alternatives is $A_1 < A_3 < A_2$. Hence, the most preferred candidate is $A_2$. While, according to the modified expression (4.6), the obtained PO of the alternatives is $A_1 < A_3 < A_2$. Hence, the most preferred candidate is $A_2$.

According to the existing expression (4.1), the obtained RV of the disease is $C_1 < C_2 < C_3$. Hence, the patient is suffering from the disease $C_3$. While, according to the modified expression (4.5), the obtained RV of the disease is $C_1 < C_3 < C_2$. Hence, the patient is suffering from the disease $C_2$. Furthermore, according to the existing expression (4.2), the obtained RV of the disease is $C_1 < C_3 < C_2$. Hence, the patient is suffering from the disease $C_2$. While, according to the modified expression (4.6), the obtained RV of the disease is $C_1 < C_3 < C_2$. Hence, the patient is suffering from the disease $C_2$.

According to the existing expression (4.1), the obtained PO of the alternatives is $A_3 < A_1 < A_2$. Hence, the most preferred pattern is $A_2$. While, according to the modified expression (4.5), the obtained PO of the alternatives is $A_3 < A_2 < A_1$. Hence, the most preferred pattern is $A_1$. Furthermore, according to the existing expression (4.2), the obtained PO of the alternatives is $A_3 < A_1 < A_2$. Hence, the most preferred pattern is $A_2$. While, according to the modified expression (4.6), the obtained PO of the alternatives is $A_3 < A_1 < A_2$. Hence, the most preferred pattern is $A_2$.

### 4.7 Conclusions

It is pointed out that the existing expressions [8] to evaluate the CoCf between two DHFSSs are not valid. Also, valid expressions are proposed for the same.

# Chapter 5

# Modified NLP methodology for MADMPrs with IVIFSSs information*

Garg and Arora [51] claimed that there is no method in the literature to solve IVIFSMADMPrs and hence, proposed a NLPM for solving IVIFSMADMPrs. Since, it is only method for solving IVIFSMADMPrs so the other researchers may be attracted to use this method for solving real-life IVIFSMADMPrs. However, after a deep study, it is observed that some mathematical incorrect assumptions have been considered in this method. Therefore, it is scientifically incorrect to use this method for solving real-life IVIFSMADMPrs. Keeping the same in mind, Garg and Arora's method is modified.

### 5.1 Preliminaries

In this section, some basic definitions are presented.

**Definition 5.1 [80]** Let $X$ be an initial universe of objects, $E$ the set of parameters in relation to objects in $X$ and $A \subseteq E$. Parameters are often attributes, characteristics, or properties of objects. Let $\mathcal{JF}(X)$ be the set of all IFSs in $X$. Then, the pair $(\tilde{F}, A)$ is called an IFSS over $X$, where, $\tilde{F}$ is a mapping defined by

$$\tilde{F}: A \to \mathcal{JF}(X).$$

**Definition 5.2 [80]** Let $\mathcal{JVF}(X)$ be the set of all IVFSs in $X$. Then, the pair $(\tilde{F}, A)$ is called an IVFSS over $X$, where, $\tilde{F}$ is a mapping defined by

$$\tilde{F}: A \to \mathcal{JVF}(X).$$

* **The contents of this chapter have been communicated for the possible publication in Journal of Intelligent & Fuzzy Systems.**

**Definition 5.3 [80]** Let $\mathcal{JVJFS}(X)$ be the set of IVIFSs in $X$. Then, the pair $(\tilde{F}, A)$ is called an IVIFSS over $X$, where, $\tilde{F}$ is a mapping defined by

$$\tilde{F}: A \rightarrow \mathcal{JVJFS}(X).$$

**5.2 A brief review of Garg and Arora's method**

The aim of this chapter is to point out the mathematical incorrect assumptions considered in the existing method [51] as well as to propose a modified method. Since, to do the same there is need to discuss the existing method [51]. Therefore, in this section, the existing method [51] is presented in a brief manner.

The steps of the existing method [51] are as follows:

**Step 1:** Write the CLFPPr $(P_{5.1})$ and the CLFPPr $(P_{5.2})$ with the help of the IVIFSDM $\widetilde{M} = \left(\left\langle\left[\left(\mu_{ij}^{(k)}\right)^L, \left(\mu_{ij}^{(k)}\right)^U\right], \left[\left(\nu_{ij}^{(k)}\right)^L, \left(\nu_{ij}^{(k)}\right)^U\right]\right\rangle\right)_{n\times m}$ corresponding to the $k^{th}$ alternative of the considered IVIFSMADMPr.

$$\left(R^{(k)}\right)^L = min\left\{\frac{\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\omega_i\left(\mu_{ij}^{(k)}\right)^L\right)+\sum_{j=1}^{m}\eta_j\left(\sum_{i=1}^{n}\rho_i\left(1-\left(\nu_{ij}^{(k)}\right)^U\right)\right)}{\sum_{j=1}^{m}\sum_{i=1}^{n}(\xi_j\omega_i+\eta_j\rho_i)}\right\}$$

Subject to $\quad (P_{5.1})$

$$\begin{cases}\omega_i^L \leq \omega_i \leq \omega_i^U & ; \quad i=1,2,\dots,n; \quad k=1,2,\dots,l,\\ \rho_i^L \leq \rho_i \leq \rho_i^U & ; \quad i=1,2,\dots,n; \quad k=1,2,\dots,l,\\ \xi_j^L \leq \xi_j \leq \xi_j^U & ; \quad j=1,2,\dots,m; \quad k=1,2,\dots,l,\\ \eta_j^L \leq \eta_j \leq \eta_j^U & ; \quad j=1,2,\dots,m; \quad k=1,2,\dots,l.\end{cases}$$

$$\left(R^{(k)}\right)^U = max\left\{\frac{\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\omega_i\left(\mu_{ij}^{(k)}\right)^U\right)+\sum_{j=1}^{m}\eta_j\left(\sum_{i=1}^{n}\rho_i\left(1-\left(\nu_{ij}^{(k)}\right)^L\right)\right)}{\sum_{j=1}^{m}\sum_{i=1}^{n}(\xi_j\omega_i+\eta_j\rho_i)}\right\}$$

Subject to $\quad (P_{5.2})$

Constraints of the CLFPPr $(P_{5.1})$

where,

(i) The IVIFSS $\left\langle\left[\left(\mu_{ij}^{(k)}\right)^{L},\left(\mu_{ij}^{(k)}\right)^{U}\right],\left[\left(\nu_{ij}^{(k)}\right)^{L},\left(\nu_{ij}^{(k)}\right)^{U}\right]\right\rangle$ represents the RV for the $k^{th}$ alternative provided by the $i^{th}$ expert over the $j^{th}$ attribute.

(ii) The IVIFS $\langle[\omega_i^L,\omega_i^U],[\rho_i^L,\rho_i^U]\rangle$ represents the weight of the $i^{th}$ expert such that $0\le\omega_i^L\le\omega_i^U\le 1,\ 0\le\rho_i^L\le\rho_i^U\le 1$ and $\omega_i^U+\rho_i^U\le 1$.

(iii) The IVIFS $\langle[\xi_j^L,\xi_j^U],[\eta_j^L,\eta_j^U]\rangle$ represents the weight of the $j^{th}$ attribute such that $0\le\xi_j^L\le\xi_j^U\le 1, 0\le\eta_j^L\le\eta_j^U\le 1$ and $\xi_j^U+\eta_j^U\le 1$.

(iv) $\omega_i$ represents the membership degree of the weight of the $i^{th}$ expert which is given as an interval i.e., $\omega_i\in[\omega_i^L,\omega_i^U]$.

(v) $\rho_i$ represents the non-membership degree of the weight of the $i^{th}$ expert which is given as an interval i.e., $\rho_i\in[\rho_i^L,\rho_i^U]$.

(vi) $\xi_j$ represents the membership degree of the weight of the $j^{th}$ attribute which is given as an interval i.e., $\xi_j\in\left[\xi_j^L,\xi_j^U\right]$.

(vii) $\eta_j$ represents the non-membership degree of the weight of the $j^{th}$ attribute which is given as an interval i.e., $\eta_j\in\left[\eta_j^L,\eta_j^U\right]$.

(viii) $l$ represents the number of alternatives.

(ix) $n$ represents the number of experts.

(x) $m$ represents the number of parameters.

**Step 2:** Using CCTtr [59], the CLFPPr ($P_{5.1}$) and the CLFPPr ($P_{5.2}$) can be transformed into its equivalent the CLPPr ($P_{5.3}$) [51, Section 3.5, Eq. 12, p. 2036] and the CLPPr ($P_{5.4}$) [51, Section 3.5, Eq. 13, p. 2037] respectively.

$$\left(R^{(k)}\right)^{L}=min\left\{\sum_{j=1}^{m}\sum_{i=1}^{n}t_{ij}\left(\mu_{ij}^{(k)}\right)^{L}+y_{ij}\left(1-\left(\nu_{ij}^{(k)}\right)^{U}\right)\right\}$$

Subject to $\qquad (P_{5.3})$

$$\begin{cases} z\xi_j^L \omega_i^L \le t_{ij} \le z\xi_j^U \omega_i^U\ ; \quad i = 1,2,\dots,n; \ \ j = 1,2,\dots,m; \quad k = 1,2,\dots,l, \\ z\eta_j^L \rho_i^L \le y_{ij} \le z\eta_j^U \rho_i^U\ ; \ \ i = 1,2,\dots,n; \quad j = 1,2,\dots,m; \quad k = 1,2,\dots,l, \\ \sum_{j=1}^{m}\sum_{i=1}^{n}\left(t_{ij} + y_{ij}\right) = 1, \\ z \ge 0. \end{cases}$$

$$\left(R^{(k)}\right)^U = max\left\{\sum_{j=1}^{m}\sum_{i=1}^{n} t_{ij}\left(\mu_{ij}^{(k)}\right)^U + y_{ij}\left(1-\left(\nu_{ij}^{(k)}\right)^L\right)\right\}$$

Subject to $(P_{5.4})$

Constraints of the CLPPr $(P_{5.3})$.

**Step 3:** Using the optimal values $\left(R^{(k)}\right)^L$ and $\left(R^{(k)}\right)^U$ of the CLPPr $(P_{5.3})$ and the CLPPr $(P_{5.4})$, obtain $\left(R^{(k)}\right) = \left[\left(R^{(k)}\right)^L, \left(R^{(k)}\right)^U\right]$ $(k = 1,2,\dots,l)$.

**Step 4:** Using the values of $\left(R^{(k)}\right) = \left[\left(R^{(k)}\right)^L, \left(R^{(k)}\right)^U\right]$ $(k = 1,2,\dots,l)$, obtained in Step 3, construct a $l \times l$ matrix $P = \left[p^{(k\nu)}\right]_{l\times l}$ $(k, \nu = 1,2,\dots,l)$, where,

$$p^{(k\nu)} = \left\{\begin{matrix} max\left\{1 - max\left(\frac{\left(R^{(\nu)}\right)^U - \left(R^{(k)}\right)^L}{\left(R^{(k)}\right)^U - \left(R^{(k)}\right)^L + \left(R^{(\nu)}\right)^U - \left(R^{(\nu)}\right)^L}, 0\right), 0\right\}; & if & k \ne \nu \\ \frac{1}{2} & ;\ \ if & k = \nu \end{matrix}\right\}.$$

**Step 5:** Find the value of $\theta^{(k)} = \frac{\sum_{\nu=1}^{l} p^{(k\nu)} + \frac{l}{2} - 1}{l(l-1)}$, $(k, \nu = 1,2,\dots,l)$ and check that $\theta^{(k)} > \theta^{(\nu)}$ or $\theta^{(k)} < \theta^{(\nu)}$ or $\theta^{(k)} = \theta^{(\nu)}$.

**Case (i)** If $\theta^{(k)} > \theta^{(\nu)}$ then $A^{(k)} > A^{(\nu)}$.

**Case (ii)** If $\theta^{(k)} < \theta^{(\nu)}$ then $A^{(k)} < A^{(\nu)}$.

**Case (iii)** If $\theta^{(k)} = \theta^{(\nu)}$ then $A^{(k)} = A^{(\nu)}$.

### 5.3 Mathematical incorrect assumption considered in Garg and Arora's method

The objective of the CLFPPr $(P_{5.1})$ and the CLFPPr $(P_{5.2})$ is to find such values of $\omega_i, \rho_i, \xi_j, \eta_j$ $(i = 1,2\dots,n;\ j = 1,2,\dots,m)$ where $0 \le \omega_i, \rho_i, \xi_j, \eta_j \le 1$ corresponding to which the value of the expression $\frac{\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\omega_i\left(\mu_{ij}^{(k)}\right)^L\right) + \sum_{j=1}^{m}\eta_j\left(\sum_{i=1}^{n}\rho_i\left(1-\left(\nu_{ij}^{(k)}\right)^U\right)\right)}{\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\xi_j\omega_i + \eta_j\rho_i\right)}$ will be minimum and the value of the expression $\frac{\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\omega_i\left(\mu_{ij}^{(k)}\right)^U\right)+\sum_{j=1}^{m}\eta_j\left(\sum_{i=1}^{n}\rho_i\left(1-\left(\nu_{ij}^{(k)}\right)^L\right)\right)}{\sum_{j=1}^{m}\sum_{i=1}^{n}(\xi_j\omega_i+\eta_j\rho_i)}$ will be maximum.

To achieve this objective, Garg and Arora [51, Section 3.5, p. 2035] have solved the CLFPPr $(P_{5.1})$ and the CLFPPr $(P_{5.2})$ independently by transforming the CLFPPr $(P_{5.1})$ and the CLFPPr $(P_{5.2})$ into the CLPPr $(P_{5.3})$ and the CLPPr $(P_{5.4})$ respectively. However, it is mathematically incorrect to solve the CLPPr $(P_{5.3})$ and the CLPPr $(P_{5.4})$ independently due to the following reasons:

On solving the CLPPr $(P_{5.3})$ and the CLPPr $(P_{5.4})$ independently the obtained values of $t_{ij}, y_{ij}$ $(i = 1,2\dots,n; j = 1,2,\dots,m)$ will not necessarily be equal. While, as $t_{ij}, y_{ij}$ $(i = 1,2\dots,n; j = 1,2,\dots,m)$ are RNs so the values of $t_{ij}, y_{ij}$ $(i = 1,2\dots,n; j = 1,2,\dots,m)$, obtained on solving the CLPPr $(P_{5.3})$ and the CLPPr $(P_{5.4})$, should be equal.

For example, to find the solution of the existing problem [51, Section 5, p. 2040], the CLFPPr $(P_{5.5})$ and the CLFPPr $(P_{5.6})$ are solved independently by transforming the CLFPPr $(P_{5.5})$ and the CLFPPr $(P_{5.6})$ into the CLPPr $(P_{5.7})$ [51, Section 5, Eq. 30, p. 2041] and the CLPPr $(P_{5.8})$ [51, Section 5, Eq. 31, p. 2042] with the help of CCTr [59].

$$\left(R^{(1)}\right)^L = min\left\{\frac{\begin{array}{c}0.4\xi_1\omega_1+0.4\xi_2\omega_1+0.1\xi_3\omega_1+0.6\xi_1\omega_2+0.6\xi_2\omega_2+0.4\xi_3\omega_2+\\0.3\xi_1\omega_3+0.5\xi_2\omega_3+0.5\xi_3\omega_3+0.7\xi_1\omega_4+0.6\xi_2\omega_4+0.3\xi_3\omega_4+\\0.6\eta_1\rho_1+0.6\eta_2\rho_1+0.4\eta_3\rho_1+0.7\eta_1\rho_2+0.7\eta_2\rho_2+0.8\eta_3\rho_2+\\0.6\eta_1\rho_3+0.6\eta_2\rho_3+0.7\eta_3\rho_3+0.8\eta_1\rho_4+0.7\eta_2\rho_4+0.8\eta_3\rho_4\end{array}}{\begin{array}{c}\xi_1\omega_1+\xi_1\omega_2+\xi_1\omega_3+\xi_1\omega_4+\eta_1\rho_1+\eta_1\rho_2+\eta_1\rho_3+\eta_1\rho_4+\xi_2\omega_1+\xi_2\omega_2+\xi_2\omega_3+\xi_2\omega_4+\\\eta_2\rho_1+\eta_2\rho_2+\eta_2\rho_3+\eta_2\rho_4+\xi_3\omega_1+\xi_3\omega_2+\xi_3\omega_3+\xi_3\omega_4+\eta_3\rho_1+\eta_3\rho_2+\eta_3\rho_3+\eta_3\rho_4\end{array}}\right\}$$

Subject to $\qquad (P_{5.5})$

$$\begin{cases} 0.15 \le \omega_1 \le 0.45 \ ; \ 0.2 \le \rho_1 \le 0.4 \ ; \ 0.1 \le \xi_1 \le 0.4 \ ; \ 0.2 \le \eta_1 \le 0.55; \\ 0.3 \le \omega_2 \le 0.4 \ ; \ 0.4 \le \rho_2 \le 0.5 \ ; \ 0.2 \le \xi_2 \le 0.5 \ ; \ 0.15 \le \eta_2 \le 0.45; \\ 0.6 \le \omega_3 \le 0.7 \ ; \ 0.1 \le \rho_3 \le 0.2 \ ; \ 0.25 \le \xi_3 \le 0.6 \ ; \ 0.15 \le \eta_3 \le 0.38; \\ 0.5 \le \omega_4 \le 0.7 \ ; \ 0.1 \le \rho_4 \le 0.3. \end{cases}$$

$$\left(R^{(1)}\right)^U = max\left\{\frac{\begin{array}{c}0.5\xi_1\omega_1+0.6\xi_2\omega_1+0.3\xi_3\omega_1+0.7\xi_1\omega_2+0.7\xi_2\omega_2+0.7\xi_3\omega_2+\\ 0.6\xi_1\omega_3+0.6\xi_2\omega_3+0.6\xi_3\omega_3+0.8\xi_1\omega_4+0.7\xi_2\omega_4+0.4\xi_3\omega_4+\\ 0.7\eta_1\rho_1+0.8\eta_2\rho_1+0.5\eta_3\rho_1+0.8\eta_1\rho_2+0.8\eta_2\rho_2+0.9\eta_3\rho_2+\\ 0.7\eta_1\rho_3+0.7\eta_2\rho_3+0.9\eta_3\rho_3+0.9\eta_1\rho_4+0.9\eta_2\rho_4+0.9\eta_3\rho_4\end{array}}{\begin{array}{c}\xi_1\omega_1+\xi_1\omega_2+\xi_1\omega_3+\xi_1\omega_4+\eta_1\rho_1+\eta_1\rho_2+\eta_1\rho_3+\eta_1\rho_4+\xi_2\omega_1+\xi_2\omega_2+\xi_2\omega_3+\xi_2\omega_4+\\ \eta_2\rho_1+\eta_2\rho_2+\eta_2\rho_3+\eta_2\rho_4+\xi_3\omega_1+\xi_3\omega_2+\xi_3\omega_3+\xi_3\omega_4+\eta_3\rho_1+\eta_3\rho_2+\eta_3\rho_3+\eta_3\rho_4\end{array}}\right\}$$

Subject to $(P_{5.6})$

Constraints of the CLFPPr $(P_{5.5})$.

$$\left(R^{(1)}\right)^L = min(0.4t_{11} + 0.4t_{12} + 0.1t_{13} + 0.6t_{21} + 0.6t_{22} + 0.4\, t_{23} + 0.3t_{31} + 0.5t_{32} + 0.5t_{33} + 0.7t_{41} + 0.6\, t_{42} + 0.3t_{43} + 0.6y_{11} + 0.6y_{12} + 0.4y_{13} + 0.7y_{21} + 0.7y_{22} + 0.8y_{23} + 0.6y_{31} + 0.6y_{32} + 0.7y_{33} + 0.8y_{41} + 0.7\, y_{42} + 0.8y_{43})$$

Subject to $(P_{5.7})$

$$\begin{cases} 0.015z \le t_{11} \le 0.180z \ ; \ 0.030z \le t_{12} \le 0.225z \ ; \ 0.037z \le t_{13} \le 0.270z; \\ 0.030z \le t_{21} \le 0.160z \ ; 0.060z \le t_{22} \le 0.200z \ ; 0.075z \le t_{23} \le 0.240z; \\ 0.060z \le t_{31} \le 0.280z \ ; \ 0.120z \le t_{32} \le 0.350z \ ; \ 0.150z \le t_{33} \le 0.420z; \\ 0.050z \le t_{41} \le 0.280z \ ; 0.100z \le t_{42} \le 0.350z \ ; \ 0.125z \le t_{43} \le 0.420z; \\ 0.040z \le y_{11} \le 0.220z \ ; 0.030z \le y_{12} \le 0.180z \ ; 0.030z \le y_{13} \le 0.152z; \\ 0.080z \le y_{21} \le 0.275z \ ; 0.060z \le y_{22} \le 0.225z \ ; 0.060z \le y_{23} \le 0.190z; \\ 0.020z \le y_{31} \le 0.110z \ ; 0.015z \le y_{32} \le 0.090z \ ; \ 0.015z \le y_{33} \le 0.076z; \\ 0.020z \le y_{41} \le 0.165z \ ; 0.015z \le y_{42} \le 0.135z \ ; 0.015z \le y_{43} \le 0.114z \\ \sum_{j=1}^{3}\sum_{i=1}^{4}\left(t_{ij} + y_{ij}\right) = 1 \quad ; \quad z \ge 0. \end{cases}$$

$$\left(R^{(1)}\right)^U = max\ (0.5t_{11} + 0.6t_{12} + 0.3t_{13} + 0.7t_{21} + 0.7t_{22} + 0.7\, t_{23} + 0.6t_{31} + 0.6t_{32} + 0.6t_{33} + 0.8t_{41} + 0.7t_{42} + 0.4t_{43} + 0.7y_{11} + 0.8y_{12} + 0.5y_{13} + 0.8y_{21} + 0.8\, y_{22} + 0.9y_{23} + 0.7y_{31} + 0.7y_{32} + 0.9y_{33} + 0.9y_{41} + 0.9y_{42} + 0.9y_{43})$$

Subject to $(P_{5.8})$

Constraints of the CLPPr $(P_{5.7})$.

The optimal values of $t_{11}, t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33}, t_{41}, t_{42}, t_{43}, y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, y_{31}, y_{32}, y_{33}, y_{41}, y_{42}$

and $y_{43}$, obtained on solving the CLPPr $(P_{5.7})$ [51, Section 5, Eq. 30, p. 2041] and the CLPPr $(P_{5.8})$ [51, Section 5, Eq. 31, p. 2042], are shown in Table 5.1.

**Table 5.1 Optimal values of the variables**

| Min $\left(R^{(1)}\right)^L$ | Max $\left(R^{(1)}\right)^U$ |
|---|---|
| $t_{11}=\frac{180}{2647}$, $y_{11}=\frac{40}{2647}$ | $t_{11}=\frac{6}{1019}$, $y_{11}=\frac{16}{1019}$ |
| $t_{12}=\frac{225}{2647}$, $y_{12}=\frac{30}{2647}$ | $t_{12}=\frac{12}{1019}$, $y_{12}=\frac{72}{1019}$ |
| $t_{13}=\frac{270}{2647}$, $y_{13}=\frac{152}{2647}$ | $t_{13}=\frac{15}{1019}$, $y_{13}=\frac{12}{1019}$ |
| $t_{21}=\frac{30}{2647}$, $y_{21}=\frac{80}{2647}$ | $t_{21}=\frac{12}{1019}$, $y_{21}=\frac{110}{1019}$ |
| $t_{22}=\frac{60}{2647}$, $y_{22}=\frac{60}{2647}$ | $t_{22}=\frac{24}{1019}$, $y_{22}=\frac{90}{1019}$ |
| $t_{23}=\frac{240}{2647}$, $y_{23}=\frac{60}{2647}$ | $t_{23}=\frac{30}{1019}$, $y_{23}=\frac{76}{1019}$ |
| $t_{31}=\frac{280}{2647}$, $y_{31}=\frac{20}{2647}$ | $t_{31}=\frac{24}{1019}$, $y_{31}=\frac{8}{1019}$ |
| $t_{32}=\frac{120}{2647}$, $y_{32}=\frac{15}{2647}$ | $t_{32}=\frac{48}{1019}$, $y_{32}=\frac{6}{1019}$ |
| $t_{33}=\frac{150}{2647}$, $y_{33}=\frac{15}{2647}$ | $t_{33}=\frac{60}{1019}$, $y_{33}=\frac{152}{5095}$ |
| $t_{41}=\frac{50}{2647}$, $y_{41}=\frac{20}{2647}$ | $t_{41}=\frac{12}{1019}$, $y_{41}=\frac{66}{1019}$ |
| $t_{42}=\frac{100}{2647}$, $y_{42}=\frac{15}{2647}$ | $t_{42}=\frac{40}{1019}$, $y_{42}=\frac{54}{1019}$ |
| $t_{43}=\frac{420}{2647}$, $y_{43}=\frac{15}{2647}$ | $t_{43}=\frac{50}{1019}$, $y_{43}=\frac{228}{5095}$ |
| $z=\frac{100}{2647}$ | $z=\frac{400}{1019}$ |
| Value of the objective function= $\frac{5489}{13235}$ | Value of the objective function= $\frac{7671}{10190}$ |

It is obvious from the results, shown in Table 5.1, that the values of the variables $t_{11}, t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33}, t_{41}, t_{42}, t_{43}, y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, y_{31}, y_{32}, y_{33}, y_{41}, y_{42}$ and $y_{43}$, obtained on solving the existing CLPPr $(P_{5.7})$ [51, Section 5, Eq. 30, p. 2041] and the existing CLPPr $(P_{5.8})$ [51, Section 5, Eq. 31, p. 2042], are not equal, which is mathematically incorrect.

## 5.4 Suggested modifications

In the existing method [51, Section 3.5, p. 2035], two different CLPPr ($P_{5.3}$) [51, Section 3.5, Eq. 12, p. 2036] and the CLPPr ($P_{5.4}$) [51, Section 3.5, Eq. 13, p. 2037] are solved. Due to the same reason, for each variable two distinct optimal values are obtained. If to achieve the objective i.e., to maximize $\frac{\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\omega_i\left(\mu_{ij}^{(k)}\right)^U\right)+\sum_{j=1}^{m}\eta_j\left(\sum_{i=1}^{n}\rho_i\left(1-\left(\nu_{ij}^{(k)}\right)^L\right)\right)}{\sum_{j=1}^{m}\sum_{i=1}^{n}(\xi_j\omega_i+\eta_j\rho_i)}$ and to minimize $\frac{\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\omega_i\left(\mu_{ij}^{(k)}\right)^L\right)+\sum_{j=1}^{m}\eta_j\left(\sum_{i=1}^{n}\rho_i\left(1-\left(\nu_{ij}^{(k)}\right)^U\right)\right)}{\sum_{j=1}^{m}\sum_{i=1}^{n}(\xi_j\omega_i+\eta_j\rho_i)}$, the CLFPPr ($P_{5.9}$) is solved instead of the CLFPPr ($P_{5.1}$) and the CLFPPr ($P_{5.2}$) independently. Then, a unique optimal value will be obtained for each variable i.e., the flaws of the existing method [51, Section 3.5, p. 2035], pointed in Section 5.3, will be resolved.

$$max\left(R^{(k)}\right) = \left\{\frac{\left(\begin{array}{c}\left\{\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\omega_i\left(\mu_{ij}^{(k)}\right)^U\right)+\sum_{j=1}^{m}\eta_j\left(\sum_{i=1}^{n}\rho_i\left(1-\left(\nu_{ij}^{(k)}\right)^L\right)\right)\right\}-\\ \left\{\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\omega_i\left(\mu_{ij}^{(k)}\right)^L\right)+\sum_{j=1}^{m}\eta_j\left(\sum_{i=1}^{n}\rho_i\left(1-\left(\nu_{ij}^{(k)}\right)^U\right)\right)\right\}\end{array}\right)}{\sum_{j=1}^{m}\sum_{i=1}^{n}(\xi_j\omega_i+\eta_j\rho_i)}\right\}$$

$$=\left\{\frac{\sum_{j=1}^{m}\left(\xi_j\left(\sum_{i=1}^{n}\omega_i\left(\left(\mu_{ij}^{(k)}\right)^U-\left(\mu_{ij}^{(k)}\right)^L\right)\right)\right)+\sum_{j=1}^{m}\left(\eta_j\left(\sum_{i=1}^{n}\rho_i\left(\left(1-\left(\nu_{ij}^{(k)}\right)^L\right)-\left(1-\left(\nu_{ij}^{(k)}\right)^U\right)\right)\right)\right)}{\sum_{j=1}^{m}\sum_{i=1}^{n}(\xi_j\omega_i+\eta_j\rho_i)}\right\}$$

Subject to $\qquad\qquad (P_{5.9})$

$$\begin{cases}\omega_i^L \le \omega_i \le \omega_i^U & ; \quad i=1,2,\dots,n; \quad k=1,2,\dots,l,\\ \rho_i^L \le \rho_i \le \rho_i^U & ; \quad i=1,2,\dots,n; \quad k=1,2,\dots,l,\\ \xi_j^L \le \xi_j \le \xi_j^U & ; \quad j=1,2,\dots,m; \quad k=1,2,\dots,l,\\ \eta_j^L \le \eta_j \le \eta_j^U & ; \quad j=1,2,\dots,m; \quad k=1,2,\dots,l.\end{cases}$$

The CLFPPr ($P_{5.9}$) can be solved as follows:

**Step 1:** Using CCTr [59], the CLFPPr ($P_{5.9}$) can be transformed into its equivalent the CLPPr ($P_{5.10}$).

$$max\left(R^{(k)}\right)=\sum_{j=1}^{m}\sum_{i=1}^{n}\left\{t_{ij}\left(\left(\mu_{ij}^{(k)}\right)^U-\left(\mu_{ij}^{(k)}\right)^L\right)+y_{ij}\left(\left(1-\left(\nu_{ij}^{(k)}\right)^L\right)\right)-\left(1-\left(\nu_{ij}^{(k)}\right)^U\right)\right\}$$

Subject to $\quad (P_{5.10})$

$$\begin{cases} z\xi_j^L\omega_i^L \le t_{ij} \le z\xi_j^U\omega_i^U\,; \quad i=1,2,\dots,n; \ \ j=1,2,\dots,m; \quad k=1,2,\dots,l, \\ z\eta_j^L\rho_i^L \le y_{ij} \le z\eta_j^U\rho_i^U\,; \quad i=1,2,\dots,n; \quad j=1,2,\dots,m; \quad k=1,2,\dots,l, \\ \sum_{j=1}^{m}\sum_{i=1}^{n}\left(t_{ij}+y_{ij}\right)=1, \\ z \ge 0. \end{cases}$$

**Step 2:** Find the optimal solution $\{t_{ij}, y_{ij};\ i=1,2,\dots,n;\ j=1,2,\dots,m\}$ of the CLPPr $(P_{5.10})$.

**Step 3:** Using the optimal solution, obtained in Step 2, find

$\left(R^{(k)}\right)^L = \left\{\sum_{j=1}^{m}\sum_{i=1}^{n} t_{ij}\left(\mu_{ij}^{(k)}\right)^L + y_{ij}\left(1-\left(\nu_{ij}^{(k)}\right)^U\right)\right\}$ and

$\left(R^{(k)}\right)^U = \sum_{j=1}^{m}\sum_{i=1}^{n} t_{ij}\left(\mu_{ij}^{(k)}\right)^U + y_{ij}\left(1-\left(\nu_{ij}^{(k)}\right)^L\right).$

**Step 4:** Construct a $l \times l$ matrix $P=\left[p^{(k\nu)}\right]_{l\times l}$ $(k,\nu=1,2,\dots,l)$, where,

$$p^{(k\nu)} = \left\{\begin{matrix} max\left\{1-max\left(\frac{\left(R^{(\nu)}\right)^U-\left(R^{(k)}\right)^L}{\left(R^{(k)}\right)^U-\left(R^{(k)}\right)^L+\left(R^{(\nu)}\right)^U-\left(R^{(\nu)}\right)^L},0\right),0\right\}; & if \quad k\neq\nu \\ \frac{1}{2} \qquad\qquad\qquad\qquad\qquad\qquad ; & if \qquad k=\nu \end{matrix}\right\}.$$

**Step 5:** Find the value of $\theta^{(k)} = \frac{\sum_{\nu=1}^{l} p^{(k\nu)}+\frac{l}{2}-1}{l(l-1)}$, $(k,\nu=1,2,\dots,l)$ and check that $\theta^{(k)} > \theta^{(\nu)}$ or $\theta^{(k)} < \theta^{(\nu)}$ or $\theta^{(k)} = \theta^{(\nu)}$.

**Case (i)** If $\theta^{(k)} > \theta^{(\nu)}$ then $A^{(k)} > A^{(\nu)}$.

**Case (ii)** If $\theta^{(k)} < \theta^{(\nu)}$ then $A^{(k)} < A^{(\nu)}$.

**Case (iii)** If $\theta^{(k)} = \theta^{(\nu)}$ then $A^{(k)} = A^{(\nu)}$.

### 5.5 Exact solution of the existing real-life problem

Garg and Arora [51, Section 5, p. 2040] solved a real-life problem to illustrate their proposed method. However, as discussed in Section 5.2 that Garg and Arora [51, Section 5, p. 2035] have used some mathematical incorrect assumptions in their proposed method, therefore the results of the real-life problem, obtained by Garg and Arora [51, Section 5, p. 2040], are not exact. In this section, the exact result of the same real-life problem is obtained by the modified method.

Using the modified method, the exact results of the existing real-life problem [51, Section 5, p. 2040] can be obtained as follows:

**Step 1:** Using Step 1 of the modified method, the CLFPPr $(P_{5.11})$, the CLFPPr $(P_{5.12})$ and the CLFPPr $(P_{5.13})$ can be obtained.

$$max\left(R^{(1)}\right) = \left\{ \frac{\begin{array}{c} 0.1\xi_1\omega_1+0.2\xi_2\omega_1+0.2\xi_3\omega_1+0.1\xi_1\omega_2+0.1\xi_2\omega_2+0.3\xi_3\omega_2+ \\ 0.3\xi_1\omega_3+0.1\xi_2\omega_3+0.1\xi_3\omega_3+0.1\xi_1\omega_4+0.1\xi_2\omega_4+0.1\xi_3\omega_4+ \\ 0.1\eta_1\rho_1+0.2\eta_2\rho_1+0.1\eta_3\rho_1+0.1\eta_1\rho_2+0.1\eta_2\rho_2+0.1\eta_3\rho_2+ \\ 0.1\eta_1\rho_3+0.1\eta_2\rho_3+0.2\eta_3\rho_3+0.1\eta_1\rho_4+0.2\eta_2\rho_4+0.1\eta_3\rho_4 \end{array}}{\begin{array}{c} \xi_1\omega_1+\xi_1\omega_2+\xi_1\omega_3+\xi_1\omega_4+\eta_1\rho_1+\eta_1\rho_2+\eta_1\rho_3+\eta_1\rho_4+\xi_2\omega_1+\xi_2\omega_2+\xi_2\omega_3+\xi_2\omega_4+ \\ \eta_2\rho_1+\eta_2\rho_2+\eta_2\rho_3+\eta_2\rho_4+\xi_3\omega_1+\xi_3\omega_2+\xi_3\omega_3+\xi_3\omega_4+\eta_3\rho_1+\eta_3\rho_2+\eta_3\rho_3+\eta_3\rho_4 \end{array}} \right\}$$

Subject to $(P_{5.11})$

Constraints of the CLFPPr $(P_{5.5})$.

$$max\left(R^{(2)}\right) = \left\{ \frac{\begin{array}{c} 0.1\xi_1\omega_1+0.2\xi_2\omega_1+0.1\xi_3\omega_1+0.2\xi_1\omega_2+0.1\xi_2\omega_2+0.1\xi_3\omega_2+ \\ 0.2\xi_1\omega_3+0.1\xi_2\omega_3+0.1\xi_3\omega_3+0.1\xi_1\omega_4+0.1\xi_2\omega_4+0.2\xi_3\omega_4+ \\ 0.1\eta_1\rho_1+0.1\eta_2\rho_1+0.2\eta_3\rho_1+0.1\eta_1\rho_2+0.1\eta_2\rho_2+0.1\eta_3\rho_2+ \\ 0.1\eta_1\rho_3+0.1\eta_2\rho_3+0.1\eta_3\rho_3+0.1\eta_1\rho_4+0.1\eta_2\rho_4+0.1\eta_3\rho_4 \end{array}}{\begin{array}{c} \xi_1\omega_1+\xi_1\omega_2+\xi_1\omega_3+\xi_1\omega_4+\eta_1\rho_1+\eta_1\rho_2+\eta_1\rho_3+\eta_1\rho_4+\xi_2\omega_1+\xi_2\omega_2+\xi_2\omega_3+\xi_2\omega_4+ \\ \eta_2\rho_1+\eta_2\rho_2+\eta_2\rho_3+\eta_2\rho_4+\xi_3\omega_1+\xi_3\omega_2+\xi_3\omega_3+\xi_3\omega_4+\eta_3\rho_1+\eta_3\rho_2+\eta_3\rho_3+\eta_3\rho_4 \end{array}} \right\}$$

Subject to $(P_{5.12})$

Constraints of the CLFPPr $(P_{5.5})$.

$$max\left(R^{(3)}\right) = \left\{ \frac{\begin{array}{c} 0.1\xi_1\omega_1+0.1\xi_2\omega_1+0.1\xi_3\omega_1+0.1\xi_1\omega_2+0.1\xi_2\omega_2+0.1\xi_3\omega_2+ \\ 0.1\xi_1\omega_3+0.1\xi_2\omega_3+0.1\xi_3\omega_3+0.1\xi_1\omega_4+0.2\xi_2\omega_4+0.1\xi_3\omega_4+ \\ 0.1\eta_1\rho_1+0.2\eta_2\rho_1+0.2\eta_3\rho_1+0.1\eta_1\rho_2+0.2\eta_2\rho_2+0.1\eta_3\rho_2+ \\ 0.1\eta_1\rho_3+0.1\eta_2\rho_3+0.1\eta_3\rho_3+0.1\eta_1\rho_4+0.2\eta_2\rho_4+0.1\eta_3\rho_4 \end{array}}{\begin{array}{c} \xi_1\omega_1+\xi_1\omega_2+\xi_1\omega_3+\xi_1\omega_4+\eta_1\rho_1+\eta_1\rho_2+\eta_1\rho_3+\eta_1\rho_4+\xi_2\omega_1+\xi_2\omega_2+\xi_2\omega_3+\xi_2\omega_4+ \\ \eta_2\rho_1+\eta_2\rho_2+\eta_2\rho_3+\eta_2\rho_4+\xi_3\omega_1+\xi_3\omega_2+\xi_3\omega_3+\xi_3\omega_4+\eta_3\rho_1+\eta_3\rho_2+\eta_3\rho_3+\eta_3\rho_4 \end{array}} \right\}$$

Subject to $(P_{5.13})$

Constraints of the CLFPPr $(P_{5.5})$.

**Step 2:** Using Step 2 of the modified method, the CLFPPr $(P_{5.11})$,$(P_{5.12})$ and $(P_{5.13})$ can be transformed into the CLPPr $(P_{5.14})$, $(P_{5.15})$ and $(P_{5.16})$ respectively.

$max(R^{(1)}) = \{0.1t_{11} + 0.2t_{12} + 0.2t_{13} + 0.1t_{21} + 0.1t_{22} + 0.3\,t_{23} + 0.3t_{31} + 0.1t_{32} +$

$0.1t_{33} + 0.1t_{41} + 0.1t_{42} + 0.1t_{43} + 0.1y_{11} + 0.2y_{12} + 0.1y_{13} + 0.1y_{21} + 0.1y_{22} +$

$0.1y_{23} + 0.1y_{31} + 0.1y_{32} + 0.2y_{33} + 0.1y_{41} + 0.2y_{42} + 0.1y_{43}\}$

Subject to

$$\begin{cases} 0.015z \le t_{11} \le 0.180z\ ;\ 0.030z \le t_{12} \le 0.225z\ ;\ 0.037z \le t_{13} \le 0.270z; \\ 0.030z \le t_{21} \le 0.160z\,; 0.060z \le t_{22} \le 0.200z\ ; 0.075z \le t_{23} \le 0.240z; \\ 0.060z \le t_{31} \le 0.280z\ ;\ 0.120z \le t_{32} \le 0.350z\ ;\ 0.150z \le t_{33} \le 0.420z; \\ 0.050z \le t_{41} \le 0.280z\,; 0.100z \le t_{42} \le 0.350z\,;\ 0.125z \le t_{43} \le 0.420z; \\ 0.040z \le y_{11} \le 0.220z\,; 0.030z \le y_{12} \le 0.180z\,; 0.030z \le y_{13} \le 0.152z; \\ 0.080z \le y_{21} \le 0.275z\,; 0.060z \le y_{22} \le 0.225z\ ; 0.060z \le y_{23} \le 0.190z; \\ 0.020z \le y_{31} \le 0.110z\,; 0.015z \le y_{32} \le 0.090z\,;\ 0.015z \le y_{33} \le 0.076z; \\ 0.020z \le y_{41} \le 0.165z\,; 0.015z \le y_{42} \le 0.135z\,; 0.015z \le y_{43} \le 0.114z \\ \sum_{j=1}^{3}\sum_{i=1}^{4}(t_{ij} + y_{ij}) = 1 \quad ; \quad z \ge 0. \end{cases} \qquad (P_{5.14})$$

$max(R^{(2)}) = \{0.1t_{11} + 0.2t_{12} + 0.1t_{13} + 0.2t_{21} + 0.1t_{22} + 0.1t_{23} + 0.2t_{31} + 0.1t_{32} +$

$0.1t_{33} + 0.1t_{41} + 0.1t_{42} + 0.2t_{43} + 0.1y_{11} + 0.1y_{12} + 0.2y_{13} + 0.1y_{21} + 0.1y_{22} +$

$0.1y_{23} + 0.1y_{31} + 0.1y_{32} + 0.1y_{33} + 0.1y_{41} + 0.1y_{42} + 0.1y_{43}\}$

Subject to $(P_{5.15})$

Constraints of the CLPPr $(P_{5.14})$.

$max(R^{(3)}) = \{0.1t_{11} + 0.1t_{12} + 0.1t_{13} + 0.1t_{21} + 0.1t_{22} + 0.1t_{23} + 0.1t_{31} +$

$0.1t_{32} + 0.1t_{33} + 0.1t_{41} + 0.2t_{42} + 0.1t_{43} + 0.1y_{11} + 0.2y_{12} + 0.2y_{13} +$

$0.1y_{21} + 0.2y_{22} + 0.1y_{23} + 0.1y_{31} + 0.1y_{32} + 0.1y_{33} + 0.1y_{41} + 0.2y_{42} +$

$0.1y_{43}\}$

Subject to $(P_{5.16})$

Constraints of the CLPPr $(P_{5.14})$.

**Step 3:** The optimal solutions $\{t_{ij}, y_{ij};\ \ i = 1,2,\dots,n; j = 1,2,\dots,m\}$ of the CLPPr $(P_{5.14})$, $(P_{5.15})$ and $(P_{5.16})$ are,

$\{t_{11} = \frac{15}{2396}, t_{12} = \frac{225}{2396}, t_{13} = \frac{135}{1198}, t_{21} = \frac{15}{1198}, t_{22} = \frac{15}{599}, t_{23} = \frac{60}{599}, t_{31} = \frac{70}{599}, t_{32} =$

$\frac{30}{599}, t_{33} = \frac{75}{1198}, t_{41} = \frac{25}{1198}, t_{42} = \frac{25}{599}, t_{43} = \frac{125}{2396}, y_{11} = \frac{10}{599}, y_{12} = \frac{45}{599}, y_{13} = \frac{15}{1198}, y_{21} =$

$\frac{20}{599}, y_{22} = \frac{15}{599}, y_{23} = \frac{15}{599}, y_{31} = \frac{5}{599}, y_{32} = \frac{15}{2396}, y_{33} = \frac{19}{599}, y_{41} = \frac{5}{599}, y_{42} = \frac{135}{2396}$ and $y_{43} = \frac{15}{2396}\Big\}$,

$\Big\{t_{11} = \frac{30}{4429}, t_{12} = \frac{450}{4429}, t_{13} = \frac{75}{4429}, t_{21} = \frac{320}{4429}, t_{22} = \frac{120}{4429}, t_{23} = \frac{150}{4429}, t_{31} = \frac{560}{4429}, t_{32} = \frac{240}{4429}, t_{33} = \frac{300}{4429}, t_{41} = \frac{100}{4429}, t_{42} = \frac{200}{4429}, t_{43} = \frac{840}{4429}, y_{11} = \frac{80}{4429}, y_{12} = \frac{60}{4429}, y_{13} = \frac{304}{4429}, y_{21} = \frac{160}{4429}, y_{22} = \frac{120}{4429}, y_{23} = \frac{120}{4429}, y_{31} = \frac{40}{4429}, y_{32} = \frac{30}{4429}, y_{33} = \frac{30}{4429}, y_{41} = \frac{40}{4429}, y_{42} = \frac{30}{4429}$ and $y_{43} = \frac{30}{4429}\Big\}$

and

$\Big\{t_{11} = \frac{10}{1373}, t_{12} = \frac{20}{1373}, t_{13} = \frac{25}{1373}, t_{21} = \frac{20}{1373}, t_{22} = \frac{40}{1373}, t_{23} = \frac{50}{1373}, t_{31} = \frac{40}{1373}, t_{32} = \frac{80}{1373}, t_{33} = \frac{100}{1373}, t_{41} = \frac{100}{4119}, t_{42} = \frac{700}{4119}, t_{43} = \frac{250}{4119}, y_{11} = \frac{80}{4119}, y_{12} = \frac{120}{1373}, y_{13} = \frac{304}{4119}, y_{21} = \frac{160}{4119}, y_{22} = \frac{150}{1373}, y_{23} = \frac{40}{1373}, y_{31} = \frac{40}{4119}, y_{32} = \frac{10}{1373}, y_{33} = \frac{10}{1373}, y_{41} = \frac{40}{4119}, y_{42} = \frac{90}{1373}$ and $y_{43} = \frac{10}{1373}\Big\}$ respectively.

**Step 4:** Using the optimal solutions $\{t_{ij}, y_{ij};\ i = 1,2,\ldots,n; j = 1,2,\ldots,m\}$, obtained in Step 3, $\left(R^{(1)}\right)^L = \frac{1389}{2995}$, $\left(R^{(1)}\right)^U = \frac{7717}{11980}$; $\left(R^{(2)}\right)^L = \frac{2418}{4429}$, $\left(R^{(2)}\right)^U = \frac{31083}{44290}$; $\left(R^{(3)}\right)^L = \frac{12407}{20595}$, $\left(R^{(3)}\right)^U = \frac{10339}{13730}$.

**Step 5:** Using Step 5 of the modified method,

$$P = \begin{bmatrix} 0.5 & 0.53647 & 0.12608 \\ 0.70792 & 0.5 & 0.32428 \\ 0.87392 & 0.67571 & 0.5 \end{bmatrix}.$$

**Step 6:** Using Step 6 of the modified method $\theta^{(1)} = 0.277091666$, $\theta^{(2)} = 0.3387$ and $\theta^{(3)} = 0.42493833$ respectively. Furthermore, since, the ranking order obtained is $\theta^{(3)} > \theta^{(2)} > \theta^{(1)}$ so the PO of the alternatives is $A^{(3)} > A^{(2)} > A^{(1)}$.

## 5.6 Conclusions

It is shown that the existing method [51] is not valid in its present form. Also, the modified version of the existing method [51] is proposed. Furthermore, to illustrate the proposed method the existing IVIFSMADMPr [51] is solved by the modified method.

# Chapter 6

# A novel method for solving fully NSLPPrs: Suggested modifications*

Abdel-Basset et al. [1] claimed that although several methods have been proposed in the literature to find the solution of different types of FLPPr/IFLPPrs (LPPrs in which some/all the parameters are represented as FNs/ IFNs) [5, 17, 19, 37, 38, 44, 70, 71, 100, 104, 105, 112, 122, 134, 142, 149, 156, 159, 171, 197, 202]. However, there is no method in the literature for solving such NSLPPrs in which some/all the parameters are represented as TrNNs. To fill this gap, Abdel-Basset et al. proposed methods for solving different types of NSLPPrs. In Abdel-Basset et al.'s methods, firstly, a NSLPPr is transformed into a CLPPr by replacing each parameter of the NSLPPr, represented by a TrNN, with its equivalent defuzzified crisp value. Then, the optimal solution of the transformed CLPPr is used to find the optimal solution and optimal value of the considered NSLPPr. Abdel-Basset et al. also pointed out that as a TrFN is a special case of TrNN. Therefore, the FLPPrs, can be solved by the existing methods [38, 44, 100, 134], can also be solved by their proposed methods. Abdel-Basset et al. also solved the same FLPPrs by their proposed methods as well as by the existing methods [38, 44, 100, 134] and shown that the results, obtained on applying by their proposed methods, are better than the results obtained on applying the existing methods [38, 44, 100, 134]. In this chapter, it is shown that for the ranking function, used by Abdel-Basset et al., to transform a TrNN into its equivalent crisp value, the linearity property is not satisfying. Whereas, Abdel-Basset et al. have used the linearity property in their proposed

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methods to transform a NSLPPr into its equivalent CLPPr. Therefore, Abdel-Basset et al.'s methods are not valid in its present form. Furthermore, the required modifications in Abdel-Basset et al.'s methods are suggested.

### 6.1 Preliminaries

In this section, some basic definitions are presented.

**Definition 6.1 [168]** A set $\tilde{A}^N = \{\langle x, T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x), F_{\tilde{A}^N}(x)\rangle | x \in X,\ 0 \leq T_{\tilde{A}^N}(x) \leq 1,\ 0 \leq I_{\tilde{A}^N}(x) \leq 1,\ 0 \leq F_{\tilde{A}^N} \leq 1,\ T_{\tilde{A}^N}(x) + I_{\tilde{A}^N}(x) + F_{\tilde{A}^N}(x) \leq 3\}$, defined on the universal set $X$, is said to be a SVNS, where, $T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x)$ and $F_{\tilde{A}^N}(x)$ represents the degree of truth-membership, the degree of indeterminacy-membership and degree of falsity-membership respectively of the element $x$ in $\tilde{A}^N$.

**Definition 6.2 [185]** A SVNS $\tilde{A}^N$ through $X$ is said to be a SVTrNN, if $T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x)$ and $F_{\tilde{A}^N}(x)$ are defined as,

$$T_{\tilde{A}^N}(x) = \begin{cases} w_{\tilde{A}^N}\left(\dfrac{x-a_1}{a_2-a_1}\right), & \text{for } a_1 \leq x < a_2 \\ w_{\tilde{A}^N}, & \text{for } a_2 \leq x \leq a_3 \\ w_{\tilde{A}^N}\left(\dfrac{a_4-x}{a_4-a_3}\right), & \text{for } a_3 < x \leq a_4 \\ 0, & \text{otherwise,} \end{cases}$$

$$I_{\tilde{A}^N}(x) = \begin{cases} \dfrac{a_2-x+u_{\tilde{A}^N}(x-a_1)}{a_2-a_1}, & \text{for } a_1 \leq x < a_2 \\ u_{\tilde{A}^N}, & \text{for } a_2 \leq x \leq a_3 \\ \dfrac{x-a_3+u_{\tilde{A}^N}(a_4-x)}{a_4-a_3}, & \text{for } a_3 < x \leq a_4 \\ 1, & \text{otherwise,} \end{cases}$$

$$F_{\tilde{A}^N}(x) = \begin{cases} \frac{a_2-x+y_{\tilde{A}^N}(x-a_1)}{a_2-a_1}, & \text{for } a_1 \leq x < a_2 \\ y_{\tilde{A}^N}, & \text{for } a_2 \leq x \leq a_3 \\ \frac{x-a_3+y_{\tilde{A}^N}(a_4-x)}{a_4-a_3}, & \text{for } a_3 < x \leq a_4 \\ 1, & \text{otherwise,} \end{cases}.$$

A SVTrNN is represented as, $\tilde{A}^N = \langle (a_1, a_2, a_3, a_4); w_{\tilde{A}^N}, u_{\tilde{A}^N}, y_{\tilde{A}^N} \rangle$ where $w_{\tilde{A}^N}, u_{\tilde{A}^N}$ and $y_{\tilde{A}^N}$ denote the maximum truth-membership degree, minimum-indeterminacy membership degree, and minimum falsity-membership degree, respectively.

**Definition 6.3 [185]** Let $\tilde{z}_1 = \langle (a_1, a_2, a_3, a_4); w_{\tilde{z}_1}, u_{\tilde{z}_1}, y_{\tilde{z}_1} \rangle$ and $\tilde{z}_2 = \langle (b_1, b_2, b_3, b_4); w_{\tilde{z}_2}, u_{\tilde{z}_2}, y_{\tilde{z}_2} \rangle$ be two SVTrNNs. Then,

(i) $\tilde{z}_1 + \tilde{z}_2 =$

$$\left\langle (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); min\left(w_{\tilde{z}_1}, w_{\tilde{z}_2}\right), max\left(u_{\tilde{z}_1}, u_{\tilde{z}_2}\right), max\left(y_{\tilde{z}_1}, y_{\tilde{z}_2}\right) \right\rangle$$

(ii) $\tilde{z}_1 - \tilde{z}_2 =$

$$\left\langle (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); min\left(w_{\tilde{z}_1}, w_{\tilde{z}_2}\right), max\left(u_{\tilde{z}_1}, u_{\tilde{z}_2}\right), max\left(y_{\tilde{z}_1}, y_{\tilde{z}_2}\right) \right\rangle$$

(iii) $\tilde{z}_1 . \tilde{z}_2 =$

$$\begin{cases} \left\langle (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4); min\left(w_{\tilde{z}_1}, w_{\tilde{z}_2}\right), max\left(u_{\tilde{z}_1}, u_{\tilde{z}_2}\right), max\left(y_{\tilde{z}_1}, y_{\tilde{z}_2}\right), \text{ if } a_4 > 0,\ b_4 > 0 \right\rangle \\ \left\langle (a_1 b_4, a_2 b_3, a_3 b_2, a_4 b_1); min\left(w_{\tilde{z}_1}, w_{\tilde{z}_2}\right), max\left(u_{\tilde{z}_1}, u_{\tilde{z}_2}\right), max\left(y_{\tilde{z}_1}, y_{\tilde{z}_2}\right), \text{ if } a_4 < 0,\ b_4 > 0 \right\rangle \\ \left\langle (a_4 b_4, a_3 b_3, a_2 b_2, a_1 b_1); min\left(w_{\tilde{z}_1}, w_{\tilde{z}_2}\right), max\left(u_{\tilde{z}_1}, u_{\tilde{z}_2}\right), max\left(y_{\tilde{z}_1}, y_{\tilde{z}_2}\right), \text{ if } a_4 < 0,\ b_4 < 0 \right\rangle \end{cases}$$

(iv) $$\frac{\tilde{z}_1}{\tilde{z}_2} = \begin{cases} \left\langle \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right); min\left(w_{\tilde{z}_1}, w_{\tilde{z}_2}\right), max\left(u_{\tilde{z}_1}, u_{\tilde{z}_2}\right), max\left(y_{\tilde{z}_1}, y_{\tilde{z}_2}\right), \text{ if } a_4 > 0,\ b_4 > 0 \right\rangle \\ \left\langle \left(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}\right); min\left(w_{\tilde{z}_1}, w_{\tilde{z}_2}\right), max\left(u_{\tilde{z}_1}, u_{\tilde{z}_2}\right), max\left(y_{\tilde{z}_1}, y_{\tilde{z}_2}\right), \text{ if } a_4 < 0,\ b_4 > 0 \right\rangle \\ \left\langle \left(\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4}\right); min\left(w_{\tilde{z}_1}, w_{\tilde{z}_2}\right), max\left(u_{\tilde{z}_1}, u_{\tilde{z}_2}\right), max\left(y_{\tilde{z}_1}, y_{\tilde{z}_2}\right), \text{ if } a_4 < 0,\ b_4 < 0 \right\rangle \end{cases}$$

(v) $$\lambda \tilde{z}_1 = \begin{cases} \langle (\lambda a_1, \lambda a_2, \lambda\, a_3, \lambda a_4); w_{\tilde{z}_1}, u_{\tilde{z}_1}, y_{\tilde{z}_1} \rangle, \text{ if } \lambda > 0 \\ \langle (\lambda a_4, \lambda a_3, \lambda\, a_2, \lambda a_1); w_{\tilde{z}_1}, u_{\tilde{z}_1}, y_{\tilde{z}_1} \rangle, \text{ if } \lambda < 0 \end{cases}$$

(vi) $\tilde{z}_1^{-1} = \left\langle \left(\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}\right); w_{\tilde{z}_1}, u_{\tilde{z}_1}, y_{\tilde{z}_1} \right\rangle,\ \tilde{z}_1 \neq 0.$

## 6.2 Existing method for comparing two TrNNs

Abdel-Basset et al. [1] have used the following method for comparing two TrNNs $\tilde{A}_1 = \langle a_1^l, a_1^{m_1}, a_1^{m_2}, a_1^u; T_{\tilde{A}_1}, I_{\tilde{A}_1}, F_{\tilde{A}_1} \rangle$ and $\tilde{A}_2 = \langle a_2^l, a_2^{m_1}, a_2^{m_2}, a_2^u; T_{\tilde{A}_2}, I_{\tilde{A}_2}, F_{\tilde{A}_2} \rangle$,

Check that the considered NSLPPr is a maximization problem or a minimization problem.

**Case (i)** If the considered NSLPPr is a maximization problem, then

(i) $\tilde{A}_1 > \tilde{A}_2$ if $R(\tilde{A}_1) > R(\tilde{A}_2)$

(ii) $\tilde{A}_1 < \tilde{A}_2$ if $R(\tilde{A}_1) < R(\tilde{A}_2)$

(iii) $\tilde{A}_1 = \tilde{A}_2$ if $R(\tilde{A}_1) = R(\tilde{A}_2)$

where,

$R(\tilde{A}_1) = \left(\frac{a_1^l + a_1^u + 2(a_1^{m_1} + a_1^{m_2})}{2}\right) + \left(T_{\tilde{A}_1} - I_{\tilde{A}_1} - F_{\tilde{A}_1}\right)$ and

$R(\tilde{A}_2) = \left(\frac{a_2^l + a_2^u + 2(a_2^{m_1} + a_2^{m_2})}{2}\right) + \left(T_{\tilde{A}_2} - I_{\tilde{A}_2} - F_{\tilde{A}_2}\right)$.

**Case (ii)** If the considered NSLPPr is a minimization problem then

(i) $\tilde{A}_1 > \tilde{A}_2$ if $R(\tilde{A}_1) < R(\tilde{A}_2)$

(ii) $\tilde{A}_1 < \tilde{A}_2$ if $R(\tilde{A}_1) > R(\tilde{A}_2)$

(iii) $\tilde{A}_1 = \tilde{A}_2$ if $R(\tilde{A}_1) = (\tilde{A}_2)$

where,

$R(\tilde{A}_1) = \left(\frac{a_1^l + a_1^u - 3(a_1^{m_1} + a_1^{m_2})}{2}\right) + \left(T_{\tilde{A}_1} - I_{\tilde{A}_1} - F_{\tilde{A}_1}\right)$ and

$R(\tilde{A}_2) = \left(\frac{a_2^l + a_2^u - 3(a_2^{m_1} + a_2^{m_2})}{2}\right) + \left(T_{\tilde{A}_2} - I_{\tilde{A}_2} - F_{\tilde{A}_2}\right)$.

### 6.3 Abdel-Basset et al.'s method

In this section, the Abdel-Basset et al.'s method [1] for solving different type of NSLPPrs is discussed.

#### 6.3.1 NSLPPr of first type

Abdel-Basset et al. [1] proposed the following method for solving NSLPPr $(P_{6.1})$.

$Maximize/Minimize\left[\sum_{j=1}^{n} \tilde{c}_j x_j\right]$

Subject to $(P_{6.1})$

$\sum_{j=1}^{n} a_{ij} x_j \leq, =, \geq b_i,\ i = 1,2,\ldots,m;$

$x_j \geq 0,\ j = 1,2,\ldots,n,$

where, $\tilde{c}_j = \langle c_j^l, c_j^{m_1}, c_j^{m_2}, c_j^u; T_{\tilde{c}_j}, I_{\tilde{c}_j}, F_{\tilde{c}_j}\rangle$ is a TrNN.

**Step 1:** Transform the NSLPPr $(P_{6.1})$ into its equivalent CLPPr $(P_{6.2})$.

$Maximize/Minimize\left[\sum_{j=1}^{n} R(\tilde{c}_j)x_j\right]$

Subject to $(P_{6.2})$

Constraints of the NSLPPr $(P_{6.1})$.

**Step 2:** Find the optimal solution $\{x_j\}$ of the CLPPr $(P_{6.2})$.

**Step 3:** Using the optimal solution $\{x_j\}$, obtained in Step 2, and using the relation $\sum_{j=1}^{n} \langle c_j^l, c_j^{m_1}, c_j^{m_2}, c_j^u; T_{\tilde{c}_j}, I_{\tilde{c}_j}, F_{\tilde{c}_j}\rangle \times x_j = \left\langle \left(\sum_{j=1}^{n} c_j^l\right) \times x_j, \left(\sum_{j=1}^{n} c_j^{m_1}\right) \times x_j, \left(\sum_{j=1}^{n} c_j^{m_2}\right) \times x_j, \left(\sum_{j=1}^{n} c_j^u\right) \times x_j; min_{1\le j\le n}\left\{T_{\tilde{c}_j}\right\}, max_{1\le j\le n}\left\{I_{\tilde{c}_j}\right\}, max_{1\le j\le n}\left\{F_{\tilde{c}_j}\right\}\right\rangle$, find the optimal value $\sum_{j=1}^{n} \langle c_j^l, c_j^{m_1}, c_j^{m_2}, c_j^u; T_{\tilde{c}_j}, I_{\tilde{c}_j}, F_{\tilde{c}_j}\rangle \times x_j$ of the NSLPPr $(P_{6.1})$.

### 6.3.2 NSLPPr of second type

Abdel-Basset et al. [1] proposed the following method for solving the NSLPPr $(P_{6.3})$.

$Maximize/Minimize\left[\sum_{j=1}^{n} c_j x_j\right]$

Subject to

$\sum_{j=1}^{n} \tilde{a}_{ij}x_j \le, =, \ge \tilde{b}_i,\ i = 1,2,\dots,m;$ $(P_{6.3})$

$x_j \ge 0,\ j = 1,2,\dots,n,$

where, $\tilde{a}_{ij} = \langle a_{ij}^l, a_{ij}^{m_1}, a_{ij}^{m_2}, a_{ij}^u; T_{\tilde{a}_{ij}}, I_{\tilde{a}_{ij}}, F_{\tilde{a}_{ij}}\rangle$ and $\tilde{b}_i = \langle b_i^l, b_i^{m_1}, b_i^{m_2}, b_i^u; T_{\tilde{b}_i}, I_{\tilde{b}_i}, F_{\tilde{b}_i}\rangle$ are TrNNs.

**Step 1:** Transform the NSLPPr $(P_{6.3})$ into its equivalent CLPPr $(P_{6.4})$.

$Maximize/Minimize\left[\sum_{j=1}^{n} c_j x_j\right]$

Subject to $(P_{6.4})$

$\sum_{j=1}^{n} R(\tilde{a}_{ij})x_j \le, =, \ge R(\tilde{b}_i),\ i = 1,2,\dots,m;$

$x_j \ge 0,\ j = 1,2,\dots,n.$

**Step 2:** Find the optimal solution $\{x_j\}$ of the CLPPr $(P_{6.4})$.

**Step 3:** Using the optimal solution $\{x_j\}$, obtained in Step 2, the optimal value of the considered NSLPPr $(P_{6.3})$ is $\sum_{j=1}^{n} c_j x_j$.

### 6.3.3 NSLPPr of third type

Abdel-Basset et al. [1] proposed the following method for solving NSLPPr $(P_{6.5})$.

$Maximize/Minimize\left[\sum_{j=1}^{n} \tilde{c}_j x_j\right]$

Subject to $\qquad (P_{6.5})$

$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq, =, \geq \tilde{b}_i,\ i = 1,2,\ldots,m;$

$x_j \geq 0,\ j = 1,2,\ldots,n,$

where,

$\tilde{c}_j = \langle c_j^l, c_j^{m_1}, c_j^{m_2}, c_j^u; T_{\tilde{c}_j}, I_{\tilde{c}_j}, F_{\tilde{c}_j}\rangle$, $\tilde{a}_{ij} = \langle a_{ij}^l, a_{ij}^{m_1}, a_{ij}^{m_2}, a_{ij}^u; T_{\tilde{a}_{ij}}, I_{\tilde{a}_{ij}}, F_{\tilde{a}_{ij}}\rangle$ and $\tilde{b}_i = \langle b_i^l, b_i^{m_1}, b_i^{m_2}, b_i^u; T_{\tilde{b}_i}, I_{\tilde{b}_i}, F_{\tilde{b}_i}\rangle$ are TrNNs.

**Step 1:** Transform the NSLPPr $(P_{6.5})$ into its equivalent CLPPr $(P_{6.6})$.

$Maximize/Minimize\left[\sum_{j=1}^{n} R(\tilde{c}_j) x_j\right]$

Subject to $\qquad (P_{6.6})$

$\sum_{j=1}^{n} R(\tilde{a}_{ij}) x_j \leq, =, \geq R(\tilde{b}_i),\ i = 1,2,\ldots,m;$

$x_j \geq 0,\ j = 1,2,\ldots,n.$

**Step 2:** Find the optimal solution $\{x_j\}$ of the CLPPr $(P_{6.4})$.

**Step 3:** Using the optimal solution $\{x_j\}$, obtained in Step 2, the optimal value of the considered NSLPPr $(P_{6.3})$ is $\sum_{j=1}^{n} \tilde{c}_j x_j$.

## 6.4 Origin of Abdel-Basset et al.'s method

In this section, the origin of Abdel-Basset et al.'s method [1] is discussed.

### 6.4.1 NSLPPr of first type

Abdel-Basset et al. [1] have used the following methodology to transform the NSLPPr $(P_{6.1})$ into a CLPPr $(P_{6.2})$.

**Step 1:** Using the method for comparing TrNNs, discussed in Section 6.2, the NSLPPr $(P_{6.1})$ can be transformed into its equivalent CLPPr $(P_{6.7})$.

$Maximize/Minimize\left[R\left(\sum_{j=1}^{n} \tilde{c}_j x_j\right)\right]$

Subject to $\qquad (P_{6.7})$

$\sum_{j=1}^{n} a_{ij}x_j \leq, =, \geq b_i\,,\ i = 1,2, \dots, m;$

$x_j \geq 0,\ j = 1,2, \dots, n.$

**Step 2:** Using the property $R\left(\sum_{j=1}^{n} \tilde{c}_j x_j\right) = \sum_{j=1}^{n}\left[R\left(\tilde{c}_j\right)\right] x_j$, the CLPPr $(P_{6.7})$ can be transformed into its equivalent CLPPr $(P_{6.2})$.

**6.4.2 NSLPPr of second type**

Abdel-Basset et al. [1] have used the following methodology to transform the NSLPPr $(P_{6.3})$ into the CLPPr $(P_{6.4})$.

**Step 1:** Using the method for comparing TrNNs, discussed in Section 6.2, the NSLPPr $(P_{6.3})$ can be transformed into the CLPPr $(P_{6.8})$.

$Maximize/Minimize\left[\sum_{j=1}^{n} c_j x_j\right]$

Subject to $\qquad (P_{6.8})$

$R\left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j\right) \leq, =, \geq R\left(\tilde{b}_i\right),\ i = 1,2, \dots, m;$

$x_j \geq 0,\ j = 1,2, \dots, n.$

**Step 2:** Using the linearity property, $R\left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j\right) = \sum_{j=1}^{n} R\left(\tilde{a}_{ij}\right) x_j$, the CLPPr $(P_{6.8})$ can be transformed into the CLPPr $(P_{6.4})$.

**6.4.3 NSLPPr of third type**

Abdel-Basset et al. [1] have used the following methodology to transform the NSLPPr $(P_{6.5})$ into the CLPPr $(P_{6.6})$.

**Step 1:** Using the method for comparing TrNNs, discussed in Section 6.2, the NSLPPr $(P_{6.5})$ can be transformed into the CLPPr $(P_{6.9})$.

$Maximize/Minimize\left[R\left(\sum_{j=1}^{n}\tilde{c}_j x_j\right)\right]$

Subject to $(P_{6.9})$

$R\left(\sum_{j=1}^{n}\tilde{a}_{ij}x_j\right) \leq, =, \geq R\left(\tilde{b}_i\right),\ i = 1,2,\dots,m;$

$x_j \geq 0,\ j = 1,2,\dots,n.$

**Step 2:** Using the linearity property, $R\left(\sum_{j=1}^{n}\tilde{a}_{ij}x_j\right) = \sum_{j=1}^{n}R\left(\tilde{a}_{ij}\right)x_j$, as well as using the relation $R(a) = a$, the CLPPr $(P_{6.9})$ can be transformed into the CLPPr $(P_{6.6})$.

**6.5 Mathematical incorrect assumptions**

The following mathematical incorrect assumptions have been considered by Abdel-Basset et al. [1].

It is obvious from Section 6.4 that

(i) Abdel-Basset et al. [1] have used the property $\left[R\left(\sum_{j=1}^{n}\tilde{c}_j x_j\right)\right] = \left[\left(\sum_{j=1}^{n}R\left(\tilde{c}_j\right)x_j\right)\right]$ to transform the objective function $\left[R\left(\sum_{j=1}^{n}\tilde{c}_j x_j\right)\right]$ of the CLPPr $(P_{6.7})$ into the objective function $\left[\left(\sum_{j=1}^{n}R\left(\tilde{c}_j\right)x_j\right)\right]$ of the CLPPr $(P_{6.2})$.

(ii) Abdel-Basset et al. [1] have used the property $\left[R\left(\sum_{j=1}^{n}\tilde{a}_{ij}x_j\right)\right] = \left[\left(\sum_{j=1}^{n}R\left(\tilde{a}_{ij}\right)x_j\right)\right]$ to transform the constraint $R\left(\sum_{j=1}^{n}\tilde{a}_{ij}x_j\right) \leq, =, \geq R\left(\tilde{b}_i\right),\ i = 1,2,\dots,m;$ of the CLPPr $(P_{6.8})$ into the constraint $\sum_{j=1}^{n}R\left(\tilde{a}_{ij}\right)x_j \leq, =, \geq R\left(\tilde{b}_i\right),\ i = 1,2,\dots,m;$ of the CLPPr $(P_{6.4})$.

However, the following clearly indicates that if $\tilde{A}_1 = \langle a_1^l, a_1^{m_1}, a_1^{m_2}, a_1^u; T_{\tilde{A}_1}, I_{\tilde{A}_1}, F_{\tilde{A}_1}\rangle$ and $\tilde{A}_2 = \langle a_2^l, a_2^{m_1}, a_2^{m_2}, a_2^u; T_{\tilde{A}_2}, I_{\tilde{A}_2}, F_{\tilde{A}_2}\rangle$ are two TrNNs then $R\left(\tilde{A}_1 \oplus \tilde{A}_2\right) \neq R\left(\tilde{A}_1\right) + R\left(\tilde{A}_2\right)$.

$$R\left(\tilde{A}_1 \oplus \tilde{A}_2\right) = R\begin{pmatrix}\left(a_1^l + a_2^l\right), \left(a_1^{m_1} + a_2^{m_1}\right), \left(a_1^{m_2} + a_2^{m_2}\right), \left(a_1^u + a_2^u\right); \\ min\left\{T_{\tilde{A}_1}, T_{\tilde{A}_2}\right\}, max\left\{I_{\tilde{A}_1}, I_{\tilde{A}_2}\right\}, max\left\{F_{\tilde{A}_1}, F_{\tilde{A}_2}\right\}\end{pmatrix}$$

$= \frac{a_1^l+a_2^l+a_1^u+a_2^u+2(a_1^{m_1}+a_2^{m_1}+a_1^{m_2}+a_2^{m_2})}{2} + \left(min\{T_{\tilde{A}_1}, T_{\tilde{A}_2}\} - max\{I_{\tilde{A}_1}, I_{\tilde{A}_2}\} - max\{F_{\tilde{A}_1}, F_{\tilde{A}_2}\}\right)$ (6.1)

while,

$R(\tilde{A}_1) + R(\tilde{A}_2) = \frac{a_1^l+2(a_1^{m_1}+a_1^{m_2})+a_1^u}{2} + (T_{\tilde{A}_1} - I_{\tilde{A}_1} - F_{\tilde{A}_1}) + \frac{a_2^l+2(a_2^{m_1}+a_2^{m_2})+a_2^u}{2} + (T_{\tilde{A}_2} - I_{\tilde{A}_2} - F_{\tilde{A}_2})$ (6.2)

It is obvious from (6.1) and (6.2) that $R(\tilde{A}_1 \oplus \tilde{A}_2) \neq R(\tilde{A}_1) + R(\tilde{A}_2)$.

(iii) Abdel-Basset et al. [1] have assumed that if $'a'$ is a RN then $R(a) = a$ and have used this relation to transform the constraint $R\left(\sum_{j=1}^{n} a_{ij}x_j\right) \leq, =, \geq R(\tilde{b}_i)$ of the CLPPr ($P_{6.9}$) into the constraint $\sum_{j=1}^{n} R(a_{ij})x_j \leq, =, \geq R(\tilde{b}_i)$ of the CLPPr ($P_{6.6}$).

However, the following clearly indicates that $R(a) \neq a$.

Abdel-Basset et al. [1] have pointed out that if $T_{\tilde{A}} = 1$, $I_{\tilde{A}} = 0$ and $F_{\tilde{A}} = 0$ then the TrNN $\tilde{a} = \langle a^l, a^{m_1}, a^{m_2}, a^u; T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}\rangle$ will be transformed into a TrFN $\tilde{a} = \langle a^l, a^{m_1}, a^{m_2}, a^u; 1,0,0\rangle$ and hence, in this case,

(i) The expression $R(\tilde{a}) = \left(\frac{a^l+a^u+2(a^{m_1}+a^{m_2})}{2}\right) + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ is equivalent to the expression $R(\tilde{a}) = \left(\frac{a^l+a^u+2(a^{m_1}+a^{m_2})}{2}\right) + 1$.

(ii) The expression $R(\tilde{a}) = \left(\frac{a^l+a^u-3(a^{m_1}+a^{m_2})}{2}\right) + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ is equivalent to $R(\tilde{a}) = \left(\frac{a^l+a^u-3(a^{m_1}+a^{m_2})}{2}\right) + 1$.

Furthermore, it is well known fact that if $a^l = a^u = a^{m_1} = a^{m_2}$ then the TrFN $\tilde{A} = \langle a^l, a^{m_1}, a^{m_2}, a^u; 1,0,0\rangle$ will be transformed into a RN $A = (a, a, a, a; 1,0,0)$ and hence, in this case,

(i) The expression $R(\tilde{A}) = \left(\frac{a^l+a^u+2(a^{m_1}+a^{m_2})}{2}\right) + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ is equivalent to the expression $R(A) = 3a + 1 \neq a$.

(ii) The expression $R(\tilde{A})=\left(\frac{a^l+a^u-3(a^{m_1}+a^{m_2})}{2}\right)+(T_{\tilde{a}}-I_{\tilde{a}}-F_{\tilde{a}})$ is equivalent to the expression $R(A)=-2a+1\neq a$.

(iv) The TrNN $\tilde{A}=\langle a^l,a^{m_1},a^{m_2},a^u;T_{\tilde{A}},I_{\tilde{A}},F_{\tilde{A}}\rangle$ can also be represented as $\tilde{A}=\langle a^{m_1},a^{m_2},\alpha,\beta;T_{\tilde{A}},I_{\tilde{A}},F_{\tilde{A}}\rangle$, where, $\alpha=a^{m_1}-a^l$ and $\beta=a^u-a^{m_2}$.

It is pertinent to mention that to find $R\langle a^{m_1},a^{m_2},\alpha,\beta;T_{\tilde{A}},I_{\tilde{A}},F_{\tilde{A}}\rangle$, firstly, there is need to transform $\langle a^{m_1},a^{m_2},\alpha,\beta;T_{\tilde{A}},I_{\tilde{A}},F_{\tilde{A}}\rangle$ into the representation $\langle a^{m_1}-\alpha,a^{m_1},a^{m_2},a^{m_2}+\beta;T_{\tilde{A}},I_{\tilde{A}},F_{\tilde{A}}\rangle$. However, the following clearly indicates that the value of $R\langle a^{m_1},a^{m_2},\alpha,\beta;T_{\tilde{A}},I_{\tilde{A}},F_{\tilde{A}}\rangle$, in the existing NSLPPr [1, Section 6.1, Ex 1], has been obtained by considering $a^{m_1}$ as $a^{l_1}$, $a^{m_2}$ as $a^{m_1}$, $\alpha$ as $a^{m_2}$ and $\beta$ as $a^u$, which is mathematically incorrect.

In the existing NSLPPr [1, Section 6.1, Ex 1], the TrNN $(13,15,2,2)$ has been replaced by the crisp number 19. This crisp number 19 has been obtained by considering $a^l=13$, $a^{m_1}=15$, $a^{m_2}=2$, $a^u=2$ in the expression $R\langle a^l,a^{m_1},a^{m_2},a^u\rangle=\left(\frac{a^l+a^u+2(a^{m_1}+a^{m_2})}{2}\right)+1$.

While, in actual case, to find a crisp number corresponding to the TrNN $(13,15,2,2)$ is $43$, which is obtained as follows:

Since, the TrNN $(13,15,2,2)$ is written in the representation $\langle a^l,a^{m_1},a^{m_2},a^u;T_{\tilde{A}},I_{\tilde{A}},F_{\tilde{A}}\rangle$. So, firstly, there is need to represent it into the representation $\langle a^{m_1}-\alpha,a^{m_1},a^{m_2},a^{m_2}+\beta;T_{\tilde{A}},I_{\tilde{A}},F_{\tilde{A}}\rangle$. In this representation, the TrNN $(13,15,2,2)$ can be rewritten as $(13-2,13,15,15+2)=(11,13,15,17)$. Now, as $a^l=11$, $a^{m_1}=13$, $a^{m_2}=15$ and $a^u=17$, therefore, using the expression $R\langle a^l,a^{m_1},a^{m_2},a^u\rangle=\left(\frac{a^l+a^u+2(a^{m_1}+a^{m_2})}{2}\right)+1$, $R(13,15,2,2)=R(11,13,15,17)=$

$\frac{11+17+2(13+15)}{2}+1=43$ i.e., the actual crisp number corresponding to TrNN (13,15,2,2) is 43 instead of 19.

**6.6. Suggested modifications**

It is obvious from Section 6.5 that several mathematical incorrect assumptions have been considered in Abdel-Basset et al.'s method [1]. Therefore, it is scientifically incorrect to use Abdel-Basset et al.'s method [1] in its present form.

In this section, the required modifications in Abdel-Basset et al.'s method [1] are suggested.

Since, $R\left(\sum_{i=1}^{m}\langle a_i^l, a_i^{m_1}, a_i^{m_2}, a_i^u; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle\right) = \sum_{i=1}^{m} R\langle a_i^l, a_i^{m_1}, a_i^{m_2}, a_i^u; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle - \sum_{i=1}^{m} T_{\tilde{a}_i} + \sum_{i=1}^{m} I_{\tilde{a}_i} + \sum_{i=1}^{m} F_{\tilde{a}_i} + min_{1\le j\le n}\left\{T_{\tilde{c}_j}\right\} - max_{1\le j\le n}\left\{I_{\tilde{c}_j}\right\} - max_{1\le j\le n}\left\{F_{\tilde{c}_j}\right\}$ instead of $R\left(\sum_{i=1}^{m}\langle a_i^l, a_i^{m_1}, a_i^{m_2}, a_i^u; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle\right) = \sum_{i=1}^{m} R\,\langle a_i^l, a_i^{m_1}, a_i^{m_2}, a_i^u; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle$. Therefore,

(i) The exact CLPPr corresponding to the NSLPPr $(P_{6.1})$ is $(P_{6.10})$ instead of the CLPPr $(P_{6.2})$. Therefore, the optimal solution of the NSLPPr $(P_{6.1})$ should be obtained by solving the CLPPr $(P_{6.10})$ instead by solving the CLPPr $(P_{6.2})$.

$Maximize/Minimize$

$$\left[\sum_{j=1}^{n} R(\tilde{c}_j x_j) - \sum_{j=1}^{n} T_{\tilde{c}_j x_j} + \sum_{j=1}^{n} I_{\tilde{c}_j x_j} + \sum_{j=1}^{n} F_{\tilde{c}_j x_j} + min_{1\le j\le n}\left\{T_{\tilde{c}_j x_j}\right\} - max_{1\le j\le n}\left\{I_{\tilde{c}_j x_j}\right\} - max_{1\le j\le n}\left\{F_{\tilde{c}_j x_j}\right\}\right]$$

Subject to $\qquad (P_{6.10})$

$\sum_{j=1}^{n} a_{ij} x_j \le, =, \ge b_i,\ i = 1,2,\dots,m;$

$x_j \ge 0,\ j = 1,2,\dots,n.$

(ii) The exact CLPPr corresponding to the NSLPPr $(P_{6.3})$ is $(P_{6.11})$ instead of the CLPPr $(P_{6.4})$. Therefore, the optimal solution of the NSLPPr $(P_{6.3})$ should be obtained by solving the CLPPr $(P_{6.11})$ instead by solving the CLPPr $(P_{6.4})$.

$Maximize/Minimize\left[\sum_{j=1}^{n} c_j x_j\right]$

Subject to $(P_{6.11})$

$\sum_{i=1}^{m} R\left(\tilde{a}_{ij} x_j\right) - \sum_{j=1}^{n} T_{\tilde{a}_{ij} x_j} + \sum_{j=1}^{n} I_{\tilde{a}_{ij} x_j} + \sum_{j=1}^{n} F_{\tilde{a}_{ij} x_j} + min_{1 \le j \le n}\left\{T_{\tilde{a}_{ij} x_j}\right\} -$

$\max\limits_{1 \le j \le n}\left\{I_{\tilde{a}_{ij} x_j}\right\} - max_{1 \le j \le n}\left\{F_{\tilde{a}_{ij} x_j}\right\} \le, =, \ge R\left(\tilde{b}_i\right),\ i = 1,2, \dots, m;$

$x_j \ge 0,\ j = 1,2, \dots, n.$

Furthermore, it is obvious from Section 6.5 that $R(a) = 3a + 1$ (in case of maximization problem) and $R(a) = -2a + 1$ (in case of minimization problem).

(i) Therefore, the exact CLPPr corresponding to the NSLPPr $(P_{6.5})$ (in case of maximization problem) is $(P_{6.12})$ instead of the CLPPr $(P_{6.8})$. Hence, the optimal solution of the NSLPPr $(P_{6.5})$ (in case of maximization problem) should be obtained by solving the CLPPr $(P_{6.12})$ instead by solving the CLPPr $(P_{6.8})$.

$Maximize$

$\left[\sum_{j=1}^{n} R\left(\tilde{c}_j x_j\right) - \sum_{j=1}^{n} T_{\tilde{c}_j x_j} + \sum_{j=1}^{n} I_{\tilde{c}_j x_j} + \sum_{j=1}^{n} F_{\tilde{c}_j x_j} + min_{1 \le j \le n}\left\{T_{\tilde{c}_j x_j}\right\} -\right.$

$\left. max_{1 \le j \le n}\left\{I_{\tilde{c}_j x_j}\right\} - max_{1 \le j \le n}\left\{F_{\tilde{c}_j x_j}\right\}\right]$

Subject to $(P_{6.12})$

$3\left[\sum_{j=1}^{n} R\left(\tilde{a}_{ij}\right) x_j\right] + 1 \le, =\ge R\left(\tilde{b}_i\right),\ i = 1,2, \dots, m$

$x_j \ge 0,\ j = 1,2, \dots, n.$

(ii) The exact CLPPr corresponding to the NSLPPr $(P_{6.5})$ (in case of minimization problem) is $(P_{6.13})$ instead of the CLPPr $(P_{6.9})$. Hence, the optimal solution of the NSLPPr $(P_{6.5})$ (in case of minimization problem) should be obtained by solving the CLPPr $(P_{6.13})$ instead by solving the CLPPr $(P_{6.9})$.

$Maximize\left[\sum_{j=1}^{n} R\left(\tilde{c}_j x_j\right) - \sum_{j=1}^{n} T_{\tilde{c}_j x_j} + \sum_{j=1}^{n} I_{\tilde{c}_j x_j} + \sum_{j=1}^{n} F_{\tilde{c}_j x_j} + min_{1 \le j \le n}\left\{T_{\tilde{c}_j x_j}\right\} -\right.$

$\left. max_{1 \le j \le n}\left\{I_{\tilde{c}_j x_j}\right\} - max_{1 \le j \le n}\left\{F_{\tilde{c}_j x_j}\right\}\right]$

Subject to $(P_{6.13})$

$-2\left[\sum_{j=1}^{n} R\left(\tilde{a}_{ij}\right)x_j\right] + 1 \leq, =\geq R\left(\tilde{b}_i\right),\ i = 1,2,\ldots,m$

$x_j \geq 0,\ j = 1,2,\ldots,n.$

### 6.7 Correct solution of the existing NSLPPrs

Abdel-Basset et al. [1] solved some NSLPPrs as well as a NSLPPr of a real-life problem to illustrate their proposed method. However, as discussed in earlier sections that several mathematical incorrect assumptions have been considered for the same. Therefore, the solutions, of the NSLPPrs, obtained by Abdel-Basset et al. [1], are not correct. The correct solutions of the NSLPPrs, considered by Abdel-Basset et al. [1], are obtained in this section.

#### 6.7.1 Correct solution of the first NSLPPr

Abdel-Basset et al. [1] solved the NSLPPr $(P_{6.14})$ to illustrate their proposed method.

$Maximize[(13,15,2,2)x_1 + (12,14,3,3)x_2 + (15,17,2,2)x_3]$

Subject to $(P_{6.14})$

$12x_1 + 13x_2 + 12x_3 \leq (475,505,6,6),$

$14x_1 + 13x_3 \leq (460,480,8,8),$

$12x_1 + 15x_2 \leq (465,495,5,5),$

$x_1, x_2, x_3 \geq 0.$

The correct solution of the NSLPPr $(P_{6.14})$ can be obtained as follows:

**Step 1:** Since, in the NSLPPr $(P_{6.14})$, the TrNNs have been represented in the form $\langle a^{m_1}, a^{m_2}, \alpha, \beta\rangle$, where, $\alpha = a^{m_1} - a^l$ and $\beta = a^u - a^{m_2}$. Therefore, firstly, there is need to replace each TrNN $\langle a^{m_1}, a^{m_2}, \alpha, \beta\rangle$ with its another representation $(a^{m_1} - \alpha, a^{m_1}, a^{m_2}, a^{m_2} + \beta)$ i.e., $\langle a^l, a^{m_1}, a^{m_2}, a^u\rangle$. Following the same the NSLPPr $(P_{6.14})$ can be transformed into NSLPPr $(P_{6.15})$.

$Maximize[(11,13,15,17)x_1 + (9,12,14,17)x_2 + (13,15,17,19)x_3]$

Subject to $(P_{6.15})$

$12x_1 + 13x_2 + 12x_3 \leq (469,475,505,511),$

$14x_1 + 13x_3 \leq (452,460,480,488),$

$12x_1 + 15x_2 \leq (460,465,495,500),$

$x_1, x_2, x_3 \geq 0.$

**Step 2:** To find the solution of the NSLPPr ($P_{6.15}$) is equivalent to find the solution of the CLPPr ($P_{6.16}$)

Maximize $\left[R\left((11,13,15,17)x_1 + (9,12,14,17)x_2 + (13,15,17,19)x_3\right)\right]$

Subject to $\qquad (P_{6.16})$

$R(12x_1 + 13x_2 + 12x_3) \leq R(469,475,505,511),$

$R(14x_1 + 13x_3) \leq R(452,460,480,488),$

$R(12x_1 + 15x_2) \leq R(460,465,495,500),$

$x_1, x_2, x_3 \geq 0.$

**Step 3:** Using the expression

$R\left(\sum_{i=1}^{m}\left(a_i^l, a_i^{m_1}, a_i^{m_2}, a_i^u; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\right)\right) = \sum_{i=1}^{m} R\left(a_i^l, a_i^{m_1}, a_i^{m_2}, a_i^u; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\right) - \sum_{i=1}^{m} T_{\tilde{a}_i} +$ $\sum_{i=1}^{m} I_{\tilde{a}_i} + \sum_{i=1}^{m} F_{\tilde{a}_i} + min_{1\leq j\leq n}\left\{T_{\tilde{c}_j}\right\} - max_{1\leq j\leq n}\left\{I_{\tilde{c}_j}\right\} - max_{1\leq j\leq n}\left\{F_{\tilde{c}_j}\right\}$ and the expression $R(a) = 3a + 1$, the CLPPr ($P_{6.16}$) can be transformed into its equivalent CLPPr ($P_{6.17}$).

$Maximize\left[R\left((11,13,15,17)x_1\right) + R\left((9,12,14,17)x_2\right) + R\left((13,15,17,19)x_3\right) - 3 + 0 + 0 + 1 - 0 - 0\right]$

Subject to $\qquad (P_{6.17})$

$3[12x_1 + 13x_2 + 12x_3] + 1 \leq R(469,475,505,511),$

$3[14x_1 + 13x_3] + 1 \leq R(452,460,480,488),$

$3[12x_1 + 15x_2] + 1 \leq R(460,465,495,500),$

$x_1, x_2, x_3 \geq 0.$

**Step 4:** Using the expression, $R(a^l, a^{m_1}, a^{m_2}, a^u; T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}) = \left(\frac{a^l+a^u+2(a^{m_1}+a^{m_2})}{2}\right) + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ with $T_{\tilde{a}} = 1$, $F_{\tilde{a}} = 0$, $I_{\tilde{a}} = 0$, the CLPPr $(P_{6.17})$ can be transformed into the CLPPr $(P_{6.18})$.

$$Maximize\left[\left(\frac{11+17+2(13+15)}{2} + 1 - 0 - 0\right)x_1 + \left(\frac{9+17+2(12+14)}{2} + 1 - 0 - 0\right)x_2 + \left(\frac{13+19+2(15+17)}{2} + 1 - 0 - 0\right)x_3 - 3 - 0 - 0 + 1 - 0 - 0\right]$$

Subject to $(P_{6.18})$

$$3[12x_1 + 13x_2 + 12x_3] + 1 \leq \frac{469+511+2(475+505)}{2} + 1 - 0 - 0,$$

$$3[14x_1 + 13x_3] + 1 \leq \frac{452+488+2(460+480)}{2} + 1 - 0 - 0,$$

$$3[12x_1 + 15x_2] + 1 \leq \frac{460+500+2(465+495)}{2} + 1 - 0 - 0,$$

$$x_1, x_2, x_3 \geq 0.$$

**Step 5:** On solving the CLPPr $(P_{6.18})$, the obtained optimal solution is $x_1 = 0,\ x_2 = 0\ , x_3 = \frac{245}{18}$.

**Step 6:** Using the optimal solution, obtained in Step 5, the optimal value of the NSLPPr $(P_{6.14})$ is $(11,13,15,17)x_1 + (9,12,14,17)x_2 + (13,15,17,19)x_3$

$$= (11,13,15,17)(0) + (9,12,14,17)(0) + (13,15,17,19)\left(\frac{245}{18}\right)$$

$$= 13\left(\frac{245}{18}\right) + 15\left(\frac{245}{18}\right) + 17\left(\frac{245}{18}\right) + 19\left(\frac{245}{18}\right)$$

$$= \frac{7840}{9}.$$

### 6.7.2 Correct solution of the second NSLPPr

Abdel-Basset et al. [1] solved the NSLPPr $(P_{6.19})$ to illustrate their proposed method.

$$Maximize[25x_1 + 48x_2]$$

Subject to $(P_{6.19})$

$$(14,15,17,18)x_1 + (25,30,34,38)x_2 \leq (44980,45000,45030,45070),$$

$(21,24,26,33)x_1 + (4,6,8,11)x_2 \leq (23980,24000,24050,24060)$,

$(17,21,22,26)x_1 + (12,14,19,22)x_2 \leq (27990,28000,28030,28040)$,

$x_1, x_2 \geq 0$.

The correct solution of the NSLPPr ($P_{6.19}$) can be obtained as follows:

**Step 1:** Since, in the NSLPPr the TrNNs have been represented in the form $\langle a^l, a^{m_1}, a^{m_2}, a^u \rangle$. Therefore, there is no need to change it.

**Step 2:** To find the solution of the NSLPPr ($P_{6.19}$) is equivalent to find the solution of the CLPPr ($P_{6.20}$).

$Maximize[25x_1 + 48x_2\ ]$

Subject to $\qquad (P_{6.20})$

$R[(14,15,17,18)x_1 + (25,30,34,38)x_2] \leq R(44980,45000,45030,45070)$,

$R[(21,24,26,33)x_1 + (4,6,8,11)x_2] \leq R(23980,24000,24050,24060)$,

$R[(17,21,22,26)x_1 + (12,14,19,22)x_2] \leq R(27990,28000,28030,28040)$,

$x_1, x_2 \geq 0$.

**Step 3:** Using the expression

$R\left(\sum_{i=1}^{m}\left(a_i^l, a_i^{m_1}, a_i^{m_2}, a_i^u; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\right)\right) = \sum_{i=1}^{m} R\left(a_i^l, a_i^{m_1}, a_i^{m_2}, a_i^u; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\right) - \sum_{i=1}^{m} T_{\tilde{a}_i} + \sum_{i=1}^{m} I_{\tilde{a}_i} + \sum_{i=1}^{m} F_{\tilde{a}_i} + min_{1 \leq j \leq n}\left\{T_{\tilde{c}_j}\right\} - max_{1 \leq j \leq n}\left\{I_{\tilde{c}_j}\right\} - max_{1 \leq j \leq n}\left\{F_{\tilde{c}_j}\right\}$ and the expression $R(a) = 3a + 1$, the CLPPr ($P_{6.20}$) can be transformed into its equivalent CLPPr ($P_{6.21}$).

$Maximize[25x_1 + 48x_2\ ]$

Subject to $\qquad (P_{6.21})$

$R[(14,15,17,18)x_1] + R[(25,30,34,38)x_2] - 2 - 0 - 0 + 1 - 0 - 0 \leq$

$R(44980,45000,45030,45070)$,

$R[(21,24,26,33)x_1] + R[(4,6,8,11)x_2] - 2 - 0 - 0 + 1 - 0 - 0 \leq$

$R(23980,24000,24050,24060)$,

$R[(17,21,22,26)x_1] + R[(12,14,19,22)x_2] - 2 - 0 - 0 + 1 - 0 - 0 \leq$

$R(27990,28000,28030,28040)$,

$x_1, x_2 \geq 0$.

**Step 4:** Using the expression, $R(a^l, a^{m_1}, a^{m_2}, a^u; T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}) = \left(\frac{a^l+a^u+2(a^{m_1}+a^{m_2})}{2}\right) + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ with $T_{\tilde{a}} = 1$, $F_{\tilde{a}} = 0$, $I_{\tilde{a}} = 0$, the CLPPr $(P_{6.21})$ can be transformed into the CLPPr $(P_{6.22})$.

$Maximize[25x_1 + 48x_2\ ]$

Subject to $(P_{6.22})$

$\left(\frac{14+18+2(15+17)}{2} + 1 - 0 - 0\right)x_1 + \left(\frac{25+38+2(30+34)}{2} + 1 - 0 - 0\right)x_2 - 2 - 0 - 0 + 1 - 0 - 0 \leq \left(\frac{44980+45070+2(45000+45030)}{2} + 1 - 0 - 0\right)$

$\left(\frac{21+33+2(24+26)}{2} + 1 - 0 - 0\right)x_1 + \left(\frac{4+11+2(6+8)}{2} + 1 - 0 - 0\right)x_2 - 2 - 0 - 0 + 1 - 0 - 0 \leq \left(\frac{23980+24060+2(24000+24050)}{2} + 1 - 0 - 0\right)$,

$\left(\frac{17+26+2(21+22)}{2} + 1 - 0 - 0\right)x_1 + \left(\frac{12+22+2(14+19)}{2} + 1 - 0 - 0\right)x_2 - 2 - 0 - 0 + 1 - 0 - 0 \leq \left(\frac{27990+28040+2(28000+28030)}{2} + 1 - 0 - 0\right)$,

$x_1, x_2 \geq 0$.

**Step 5:** On solving the CLPPr $(P_{6.22})$, the obtained optimal solution is $x_1 = \frac{579043}{1810}$, $x_2 = \frac{943916}{763}$.

**Step 6:** Using the optimal solution, obtained in Step 5, the optimal value of the NSLPPr $(P_{6.19})$ is $25x_1 + 48x_2 = 25\left(\frac{579043}{1810}\right) + 48\left(\frac{943916}{763}\right) = \frac{10174256}{151}$.

### 6.7.3 Correct solution of the third NSLPPr

Abdel-Basset et al. [1] solved the NSLPPr $(P_{6.23})$ to illustrate their proposed method.

$Minimize\ (6x_1 + 10x_2)$

Subject to $(P_{6.23})$

$2x_1 + 5x_2 \geq (5,8,3,13)$,

$3x_1 + 4x_2 \geq (6,0,4,16)$,

$x_1, x_2 \geq 0$.

The correct solution of the NSLPPr $(P_{6.23})$ can be obtained as follows:

**Step 1:** Since, in the NSLPPr the TrNNs have been represented in the form $\langle a^l, a^{m_1}, a^{m_2}, a^u \rangle$. Therefore, there is no need to change it.

**Step 2:** To find the solution of the NSLPPr $(P_{6.23})$ is equivalent to find the solution of the CLPPr $(P_{6.24})$.

$Minimize[6x_1 + 10x_2\ ]$

Subject to $(P_{6.24})$

$R[2x_1 + 5x_2] \geq R(5,8,3,13)$,

$R[3x_1 + 4x_2] \geq R(6,0,4,16)$,

$x_1, x_2 \geq 0$.

**Step 3:** Using the expression

$R\left(\sum_{i=1}^{m}\left(a_i^L, a_i^{m_1}, a_i^{m_2}, a_i^U; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\right)\right) = \sum_{i=1}^{m} R\left(a_i^L, a_i^{m_1}, a_i^{m_2}, a_i^U; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\right) - \sum_{i=1}^{m} T_{\tilde{a}_i} + \sum_{i=1}^{m} I_{\tilde{a}_i} + \sum_{i=1}^{m} F_{\tilde{a}_i} + min_{1\leq j\leq n}\left\{T_{\tilde{c}_j}\right\} - max_{1\leq j\leq n}\left\{I_{\tilde{c}_j}\right\} - max_{1\leq j\leq n}\left\{F_{\tilde{c}_j}\right\}$ and the expression $R(a) = -2a + 1$, the CLPPr $(P_{6.24})$ can be transformed into its equivalent CLPPr $(P_{6.25})$.

$Minimize[6x_1 + 10x_2\ ]$

Subject to $(P_{6.25})$

$-2[2x_1 + 5x_2] + 1 \geq \frac{5+13-3(8+3)}{2} + 1$

$-2[3x_1 + 4x_2] + 1 \geq \frac{6+16-3(0+4)}{2} + 1$,

$x_1, x_2 \geq 0$.

**Step 4:** On solving the CLPPr $(P_{6.25})$, the obtained optimal solution is $x_1 = 0$ and $x_2 = \frac{3}{4}$.

**Step 5:** Using the optimal solution, obtained in Step 4, the optimal value of the NSLPPr $(P_{6.23})$ is $6x_1 + 10x_2 = 6(0) + 10\left(\frac{3}{4}\right) = \frac{15}{2}$.

### 6.7.4 Correct solution of the real-life NSLPPr

Abdel-Basset et al. [1] solved the NSLPPr $(P_{6.26})$ to obtain the solution of a real-life problem.

$Maximize[(6,8,9,12)x_1 + (9,10,12,14)x_2 + (12,13,15,17)x_3 + (8,9,11,13)x_4]$

Subject to $\qquad (P_{6.26})$

$0.5x_1 + 1.5x_2 + 1.5x_3 + x_4 \leq (1200,1300,1500,1700),$

$3x_1 + x_2 + 2x_3 + 3x_4 \leq (2200,2250,2350,2400),$

$2x_1 + 4x_2 + x_3 + 2x_4 \leq (2200,2400,2600,2800),$

$0.5x_1 + x_2 + 0.5x_3 + 0.5x_4 \leq (1000,1100,1200,1300),$

$x_1 \geq (120,130,150,170),$

$x_2 \geq (70,80,100,120),$

$x_3 \geq (270,280,300,320),$

$x_4 \geq (370,380,400,420),$

$x_1, x_2, x_3, x_4 \geq 0.$

The correct solution of the NSLPPr $(P_{6.26})$ can be obtained as follows:

**Step 1:** Since, in the NSLPPr $(P_{6.26})$, the TrNNs have been represented in the form $\langle a^l, a^{m_1}, a^{m_2}, a^u \rangle$. Therefore, there is no need to change it.

**Step 2:** To find the solution of the NSLPPr $(P_{6.26})$ is equivalent to find the solution of the CLPPr $(P_{6.27})$

$Maximize\big[R\big((6,8,9,12)x_1 + (9,10,12,14)x_2 + (12,13,15,17)x_3 + (8,9,11,13)x_4\big)\big]$

Subject to $\qquad (P_{6.27})$

$R(0.5x_1 + 1.5x_2 + 1.5x_3 + x_4) \leq R(1200,1300,1500,1700),$

$R(3x_1 + x_2 + 2x_3 + 3x_4) \leq R(2200,2250,2350,2400),$

$R(2x_1 + 4x_2 + x_3 + 2x_4) \leq R(2200{,}2400{,}2600{,}2800)$,

$R(0.5x_1 + x_2 + 0.5x_3 + 0.5x_4) \leq R(1000{,}1100{,}1200{,}1300)$,

$x_1 \geq R(120{,}130{,}150{,}170)$,

$x_2 \geq R(70{,}80{,}100{,}120)$,

$x_3 \geq R(270{,}280{,}300{,}320)$,

$x_4 \geq R(370{,}380{,}400{,}420)$,

$x_1, x_2, x_3, x_4 \geq 0$.

**Step 3:** Using the expression

$R\left(\sum_{i=1}^{m}\left(a_i^l, a_i^{m_1}, a_i^{m_2}, a_i^u; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\right)\right) = \sum_{i=1}^{m} R\left(a_i^l, a_i^{m_1}, a_i^{m_2}, a_i^u; T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\right) - \sum_{i=1}^{m} T_{\tilde{a}_i} + \sum_{i=1}^{m} I_{\tilde{a}_i} + \sum_{i=1}^{m} F_{\tilde{a}_i} + min_{1\leq j\leq n}\left\{T_{\tilde{c}_j}\right\} - max_{1\leq j\leq n}\left\{I_{\tilde{c}_j}\right\} - max_{1\leq j\leq n}\left\{F_{\tilde{c}_j}\right\}$ and the expression $R(a) = 3a + 1$, the CLPPr $(P_{6.27})$ can be transformed into its equivalent CLPPr $(P_{6.28})$.

$Maximize[R(6{,}8{,}9{,}12)x_1 + R(9{,}10{,}12{,}14)x_2 + R(12{,}13{,}15{,}17)x_3 + R(8{,}9{,}11{,}13)x_4] - 4 + 0 + 0 + 1 - 0 - 0$

Subject to $(P_{6.28})$

$3[0.5x_1 + 1.5x_2 + 1.5x_3 + x_4] + 1 \leq R(1200{,}1300{,}1500{,}1700)$,

$3[3x_1 + x_2 + 2x_3 + 3x_4] + 1 \leq R(2200{,}2250{,}2350{,}2400)$,

$3[2x_1 + 4x_2 + x_3 + 2x_4] + 1 \leq R(2200{,}2400{,}2600{,}2800)$,

$3[0.5x_1 + x_2 + 0.5x_3 + 0.5x_4] + 1 \leq R(1000{,}1100{,}1200{,}1300)$,

$x_1 \geq R(120{,}130{,}150{,}170)$,

$x_2 \geq R(70{,}80{,}100{,}120)$,

$x_3 \geq R(270{,}280{,}300{,}320)$,

$x_4 \geq R(370{,}380{,}400{,}420)$,

$x_1, x_2, x_3, x_4 \geq 0$.

**Step 4:** Using the expression, $R(a^l, a^{m_1}, a^{m_2}, a^u; T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}) = \left(\frac{a^l+a^u+2(a^{m_1}+a^{m_2})}{2}\right) + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ with $T_{\tilde{a}} = 1$, $F_{\tilde{a}} = 0$, $I_{\tilde{a}} = 0$, the CLPPr $(P_{6.28})$ can be transformed into the CLPPr $(P_{6.29})$.

$$Maximize \left[\left(\frac{6+12+2(8+9)}{2} + 1 - 0 - 0\right)x_1 + \left(\frac{9+14+2(10+12)}{2} + 1 - 0 - 0\right)x_2 + \left(\frac{12+17+2(13+15)}{2} + 1 - 0 - 0\right)x_3 + \left(\frac{8+13+2(9+11)}{2} + 1 - 0 - 0\right)x_2 - 4 - 0 - 0 + 1 - 0 - 0\right]$$

Subject to $(P_{6.29})$

$$3[0.5x_1 + 1.5x_2 + 1.5x_3 + x_4] + 1 \leq \left(\frac{1200+1700+2(1300+1500)}{2} + 1 - 0 - 0\right),$$

$$3[3x_1 + x_2 + 2x_3 + 3x_4] + 1 \leq \left(\frac{2200+2400+2(2250+2350)}{2} + 1 - 0 - 0\right),$$

$$3[2x_1 + 4x_2 + x_3 + 2x_4] + 1 \leq \left(\frac{2200+2800+2(2400+2600)}{2} + 1 - 0 - 0\right),$$

$$3[0.5x_1 + x_2 + 0.5x_3 + 0.5x_4] + 1 \leq \left(\frac{1000+1300+2(1100+1200)}{2} + 1 - 0 - 0\right),$$

$x_1 \geq 426,$

$x_2 \geq 343,$

$x_3 \geq 876,$

$x_4 \geq 1176,$

$x_1, x_2, x_3, x_4 \geq 0.$

**Step 5:** On solving the CLPPr $(P_{6.18})$, the obtained optimal solution is $x_1 = \frac{3700}{21}$, $x_2 = 0$, $x_3 = \frac{6200}{7}$, $x_4 = 0$.

**Step 6:** Using the optimal solution, obtained in Step 5, the optimal value of the NSLPPr $(P_{6.29})$ is $(6,8,9,12)x_1 + (9,10,12,14)x_2 + (12,13,15,17)x_3 + (8,9,11,13)x_4$

$$= (6,8,9,12)\left(\frac{3700}{21}\right) + (9,10,12,14)(0) + (12,13,15,17)\left(\frac{6200}{7}\right) + (8,9,11,13)(0)$$

$$= \left[6\left(\frac{3700}{21}\right) + 8\left(\frac{3700}{21}\right) + 9\left(\frac{3700}{21}\right) + 12\left(\frac{3700}{21}\right)\right] + \left[12\left(\frac{6200}{7}\right) + 13\left(\frac{6200}{7}\right) + 15\left(\frac{6200}{7}\right) + 17\left(\frac{6200}{7}\right)\right]$$

$$= \frac{1189700}{21}.$$

**6.8 Conclusions**

The mathematical incorrect assumptions, used in Abdel-Basset et al.'s method [1], are pointed out. Also, the required modifications in Abdel-Basset et al.'s method [1] are suggested. Furthermore, the correct results of the NSLPPrs, solved by Abdel-Basset et al. [1] to illustrate their proposed method, are obtained.

# Chapter 7

# Modified approach for optimization of real-life TrPr in neutrosophic environment*

In daily life problems, there is a need to transport the product from various sources to different destinations. To find a way to transport the product in such a manner so that the total TrC is minimum is called the optimal way and the problem is called cost minimization TrPrs [72]. Different methods have been proposed in the literature to find the optimal way of such cost minimization TrPrs in which cost for transporting unit quantity of the product, availability of the product at the sources and demand of the product at the destinations are represented as a RNs. However, to assume these parameters as RNs is not always valid according to real-life situations e.g., the TrC depends upon the circumstances like price of petrol/diesel, weather, travel time, traffic jam etc. Similarly, the availability of crops varies according to the monsoon, fertilizers, chemicals etc., the demand of the various clothes depend on the season, fashion trends, discount offers etc. Furthermore, the opinions of the experts about these parameters cannot always be represented as a RNs, e.g., generally experts provide their opinion about these parameters in terms of linguistic variables like high, very high, low, very low etc.

One of the way, widely adopted in the literature to deal with such situations, is to represent these parameters as FNs [189] and its extensions [20]. Thamaraiselvi and Santhi [157] pointed out that NS [168], one of the extensions of FS [189], is used in different research areas. However, till now no one have used the NS in TrPrs. While, several

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researchers have used FNs for representing various parameters of TrPrs [23, 45, 62, 82, 89, 90, 124, 129, 143]. Therefore, Thamaraiselvi and Santhi proposed the approaches for solving NSTrPr of Type-I (TrPrs in which the cost for transporting unit quantity of the product is represented as TrNN, whereas the availability and the demand are represented as RNs) and NSTrPr of Type-II (TrPrs in which the cost for transporting unit quantity of the product, availability of the product and demand of the product are represented as TrNNs). Since, NSTrPrs is new area of research so others may be attracted to extend these approaches for solving other types of NSTrPrs like neutrosophic solid TrPrs, neutrosophic time minimization TrPrs, neutrosophic transshipment problems etc. However, after a deep study of these existing approaches, it is noticed that a mathematical incorrect assumption has been used in these existing approaches. Therefore, there is need to modify these existing approaches. Keeping the same in mind, in this chapter, these existing approaches are modified. Furthermore, the exact results of some existing TrPrs are obtained by the modified approaches.

### 7.1 Existing method for comparing SVTrNNs

Thamaraiselvi and Santhi [157] have used the following method for comparing two SVNNs. Let $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ and $\tilde{b} = \langle (b_1, b_2, b_3, b_4); w_{\tilde{b}}, u_{\tilde{b}}, y_{\tilde{b}} \rangle$ be two SVNNs.

**Step 1:** Find the score function,

$S(\tilde{a}) = \frac{1}{16}([a_1 + a_2 + a_3 + a_4] \times [\mu_{\tilde{a}} + (1 - \nu_{\tilde{a}}) + (1 - \lambda_{\tilde{a}})])$ as well as $S(\tilde{b}) = \frac{1}{16}([b_1 + b_2 + b_3 + b_4] \times [\mu_{\tilde{b}} + (1 - \nu_{\tilde{b}}) + (1 - \lambda_{\tilde{b}})])$ and check that $S(\tilde{a}) < S(\tilde{b})$ or $S(\tilde{a}) > S(\tilde{b})$ or $S(\tilde{a}) = S(\tilde{b})$.

**Case (i)** If $S(\tilde{a}) < S(\tilde{b})$ then $\tilde{a} < \tilde{b}$.

**Case (ii)** If $S(\tilde{a}) > S(\tilde{b})$ then $\tilde{a} > \tilde{b}$.

**Case (iii)** If $S(\tilde{a}) = S(\tilde{b})$ then go to Step 2.

**Step 2:** Find the accuracy function,

$A(\tilde{a}) = \frac{1}{16}([a_1 + a_2 + a_3 + a_4] \times [\mu_{\tilde{a}} + (1 - \nu_{\tilde{a}}) + (1 + \lambda_{\tilde{a}})])$ as well as $A(\tilde{b}) = \frac{1}{16}([b_1 + b_2 + b_3 + b_4] \times [\mu_{\tilde{b}} + (1 - \nu_{\tilde{b}}) + (1 + \lambda_{\tilde{b}})])$ and check that $A(\tilde{a}) < A(\tilde{b})$ or $A(\tilde{a}) > A(\tilde{b})$ or $A(\tilde{a}) = A(\tilde{b})$.

**Case (i)** If $A(\tilde{a}) > A(\tilde{b})$ then $\tilde{a} > \tilde{b}$.

**Case (ii)** If $A(\tilde{a}) < A(\tilde{b})$ then $\tilde{a} < \tilde{b}$.

**Case (iii)** If $A(\tilde{a}) = A(\tilde{b})$ then $\tilde{a} = \tilde{b}$.

### 7.2 A brief review of Thamaraiselvi and Santhi approaches

To point out the mathematical incorrect assumptions in the approaches, proposed by Thamaraiselvi and Santhi [157], there is a need to describe these approaches. Therefore, in this section, a brief review of the approaches, proposed by the Thamaraiselvi and Santhi [157], for solving both types of NSTrPrs are discussed in a brief manner.

#### 7.2.1 Thamaraiselvi and Santhi approach for solving NSTrPr of Type-I

Using the approach, proposed by Thamaraiselvi and Santhi [157], the optimal solution of a NSTrPr of Type - I can be obtained as follows:

**Step 1:** Formulate the NSTrPr as a NSLPPr $(P_{7.1})$.

$Minimize \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}^{N} x_{ij}\right)$

Subject to

$\sum_{j=1}^{n} x_{ij} = a_i, \; i = 1, 2, \ldots, m,$

$\sum_{i=1}^{m} x_{ij} = b_j, \; j = 1, 2, \ldots, n,$ $(P_{7.1})$

$x_{ij} \geq 0, \; \forall \; i, j.$

where,

(i) $x_{ij}$ is the number of units of the product transported from $i^{th}$ source to $j^{th}$ destination.

(ii) $\tilde{c}_{ij}^{N}$ is the neutrosophic cost of one unit quantity transported from $i^{th}$ source to $j^{th}$ destination.

(iii) $a_i$ is the total availability of the product at the source $i$.

(iv) $b_j$ is the total demand of the product at the destination $j$.

(v) $m$ is the total number of sources.

(vi) $n$ is the total number of destinations.

**Step 2:** Transform the NSLPPr $(P_{7.1})$ into its equivalent CLPPr $(P_{7.2})$.

$$Minimize\ S\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}^{N}\,x_{ij}\right)$$

Subject to $(P_{7.2})$

Constraints of the CLPPr $(P_{7.1})$

where,

$$S(\tilde{a}^{\,N}) = S(<(a_1, a_2, a_3, a_4), \mu_{\tilde{a}}, \nu_{\tilde{a}}, \lambda_{\tilde{a}}>)$$

$$= \frac{1}{16}[a_1 + a_2 + a_3 + a_4][\mu_{\tilde{a}} + (1-\nu_{\tilde{a}}) + (1-\lambda_{\tilde{a}})]\ . \tag{7.1}$$

**Step 3:** Transform the CLPPr $(P_{7.2})$ into its equivalent CLPPr $(P_{7.3})$.

$$Minimize\ \left(\sum_{i=1}^{m}\ \sum_{j=1}^{n}\ S(\tilde{c}_{ij}^{N})x_{ij}\right)$$

Subject to $(P_{7.3})$

Constraints of the CLPPr $(P_{7.1})$.

**Step 4:** Represent the CLPPr $(P_{7.3})$ into tabular form shown in Table 7.1.

**Table 7.1 Tabular representation of the transformed crisp TrPr**

| Destinations $\longrightarrow$ / Sources $\downarrow$ | $D_1$ | $D_2$ | $D_3$ | $\cdots$ | $D_n$ | Supply |
|---|---|---|---|---|---|---|
| $S_1$ | $S(\tilde{c}_{11}^{N})$ | $S(\tilde{c}_{12}^{N})$ | $S(\tilde{c}_{13}^{N})$ | $\cdots$ | $S(\tilde{c}_{1n}^{N})$ | $a_1$ |
| $S_2$ | $S(\tilde{c}_{21}^{N})$ | $S(\tilde{c}_{22}^{N})$ | $S(\tilde{c}_{23}^{N})$ | $\cdots$ | $S(\tilde{c}_{2n}^{N})$ | $a_2$ |
| $S_3$ | $S(\tilde{c}_{31}^{N})$ | $S(\tilde{c}_{32}^{N})$ | $S(\tilde{c}_{33}^{N})$ | $\cdots$ | $S(\tilde{c}_{3n}^{N})$ | $a_3$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $S_m$ | $S(\tilde{c}_{m1}^{N})$ | $S(\tilde{c}_{m2}^{N})$ | $S(\tilde{c}_{m3}^{N})$ | $\cdots$ | $S(\tilde{c}_{mn}^{N})$ | $a_m$ |
| Demand | $b_1$ | $b_2$ | $b_3$ | $\cdots$ | $b_n$ | |

**Step 5:** Find the crisp optimal solutions $\{x_{ij}\}$ of crisp TrPr represented by Table 7.1.

**Step 6:** Find the total minimum NSTrC by putting the optimal solution $\{x_{ij}\}$, obtained from Step 5, in $\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}^{N}\,x_{ij}$.

### 7.2.2 Thamaraiselvi and Santhi approach for solving NSTrPr of Type-II

Using the approach, proposed by Thamaraiselvi and Santhi [157], the optimal solution of a NSTrPr of Type - II can be obtained as follows:

**Step 1:** Formulate the NSTrPr as a NSLPPr $(P_{7.4})$.

$Minimize\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}^{N}\tilde{x}_{ij}^{N}\right)$

Subject to $\qquad (P_{7.4})$

$\sum_{j=1}^{n}\tilde{x}_{ij}^{N} = \tilde{a}_{i}^{N}, \; i = 1, 2 \ldots, m,$

$\sum_{i=1}^{m}\tilde{x}_{ij}^{N} = \tilde{b}_{j}^{N}, \; j = 1, 2 \ldots, n,$

$\tilde{x}_{ij}^{N} \geq 0 \;, \; \forall \; i \;, j$

**Step 2:** Transform the NSLPPr $(P_{7.4})$ into its equivalent NSLPPr $(P_{7.5})$.

Minimize $S\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}^{N}\, \tilde{x}_{ij}^{N}\right)$

Subject to $(P_{7.5})$

Constraints of the CLPPr $(P_{7.4})$.

**Step 3:** Transform the NSLPPr $(P_{7.5})$ into its equivalent NSLPPr $(P_{7.6})$.

$Minimize\ \sum_{i=1}^{m}\ \sum_{j=1}^{n} S(\tilde{c}_{ij}^{N})\, \tilde{x}_{ij}^{N}$

Subject to $(P_{7.6})$

Constraints of CLPPr $(P_{7.4})$.

**Step 4:** Represent the NSLPPr $(P_{7.6})$ into tabular form as shown in Table 7.2.

**Table 7.2 Tabular representation of the NSLPPr of Type – II**

| Destinations $\rightarrow$ / $\downarrow$ Sources | $D_1$ | $D_2$ | $D_3$ | ... | $D_n$ | Supply |
|---|---|---|---|---|---|---|
| $S_1$ | $S(\tilde{c}_{11}^{N})$ | $S(\tilde{c}_{12}^{N})$ | $S(\tilde{c}_{13}^{N})$ | ... | $S(\tilde{c}_{1n}^{N})$ | $\tilde{a}_1^N$ |
| $S_2$ | $S(\tilde{c}_{21}^{N})$ | $S(\tilde{c}_{22}^{N})$ | $S(\tilde{c}_{23}^{N})$ | ... | $S(\tilde{c}_{2n}^{N})$ | $\tilde{a}_2^N$ |
| $S_3$ | $S(\tilde{c}_{31}^{N})$ | $S(\tilde{c}_{32}^{N})$ | $S(\tilde{c}_{33}^{N})$ | ... | $S(\tilde{c}_{3n}^{N})$ | $\tilde{a}_3^N$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $S_m$ | $S(\tilde{c}_{m1}^{N})$ | $S(\tilde{c}_{m2}^{N})$ | $S(\tilde{c}_{m3}^{N})$ | ... | $S(\tilde{c}_{mn}^{N})$ | $\tilde{a}_m^N$ |
| Demand | $\tilde{b}_1^N$ | $\tilde{b}_2^N$ | $\tilde{b}_3^N$ | ... | $\tilde{b}_n^N$ | |

**Step 5:** Find the neutrosophic optimal solution $\{\tilde{x}_{ij}^{N}\}$ of the TrPr represented by Table 7.2.

**Step 6:** Find the total minimum NSTrC by putting the optimal solution $\{\tilde{x}_{ij}^{N}\}$, obtained in Step 5, in $\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}^{N}\, \tilde{x}_{ij}^{N}$.

## 7.3. Mathematical incorrect assumption considered in Thamaraiselvi and Santhi approaches

In this section, the mathematical incorrect assumption, considered in Thamaraiselvi and Santhi approaches [157], is pointed out.

It is obvious from Step 2 and Step 3 of the first approach, presented in Section 7.2.1, that Thamaraiselvi and Santhi [157] have assumed that the NSLPPr $(P_{7.2})$ can be transformed into the NSLPPr $(P_{7.3})$. Similarly, it is obvious from Step 2 and Step 3 of the second approach, presented in Section 7.2.2, that Thamaraiselvi and Santhi [157] have assumed that the NSLPPr $(P_{7.5})$ can be transformed into NSLPPr $(P_{7.6})$.

It is pertinent to mention that to transform the NSLPPr $(P_{7.2})$ into the NSLPPr $(P_{7.3})$ as well as the NSLPPr $(P_{7.5})$ into the NSLPPr $(P_{7.6})$, Thamaraiselvi and Santhi [157] have considered that, $\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}^{N}x_{ij}\right)=\sum_{i=1}^{m}\sum_{j=1}^{n}S\left(\tilde{c}_{ij}^{N}\right)x_{ij}$ i.e., Thamaraiselvi and Santhi [157] have considered the assumption that if $\tilde{a}^{\,N}=<(a_1,a_2,a_3,a_4),\mu_{\tilde{a}}\,,\nu_{\tilde{a}},\lambda_{\tilde{a}}>$ and $\tilde{b}^{\,N}=<(b_1,b_2,b_3,b_4),\mu_{\tilde{a}}\,,\nu_{\tilde{a}},\lambda_{\tilde{a}}>$ are two TrNNs, then

$$S\left(\tilde{a}^{\,N}+\tilde{b}^{\,N}\right)=S(\tilde{a}^{\,N})+S\left(\tilde{b}^{\,N}\right). \tag{7.2}$$

However, the following example clearly indicates that

$$S\left(\tilde{a}^{\,N}+\tilde{b}^{\,N}\right)\neq S(\tilde{a}^{\,N})+S\left(\tilde{b}^{\,N}\right). \tag{7.3}$$

Let $\tilde{a}^{\,N}=\langle(2,4,6,8)\,;0.1,0.2,0.3\rangle$ and $\tilde{b}^{\,N}=\langle(3,6,9,12)\,;0.4,0.5,0.6\rangle$ be two TrNNs then according to existing result [157, Def. 10, p. 4]

$$\begin{aligned}\tilde{a}^{\,N}+\tilde{b}^{\,N}&=\langle(2,4,6,8)\,;0.1,0.2,0.3\rangle+\langle(3,6,9,12)\,;0.4,0.5,0.6\rangle\\&=\langle(5,10,15,20)\,;0.1,0.5,0.6\rangle\end{aligned} \tag{7.4}$$

Furthermore, using the existing result [157, Def. 11, p. 4]

$$S\left(\tilde{a}^{\,N}+\tilde{b}^{\,N}\right)=\frac{1}{16}(5+10+15+20)(0.1+0.5+0.4)=3.125\,. \tag{7.5}$$

While,

$S(\tilde{a}^{\,N}) + S(\tilde{b}^{\,N}) = \left\{\frac{1}{16}(2+4+6+8)(0.1+0.8+0.7)\right\} + \left\{\frac{1}{16}(3+6+9+12)(0.4+\right.$

$\left.0.5+\ 0.4)\right\} = 2 + 2.4375$

$= 4.4375$ (7.6)

It is obvious that

$S(\tilde{a}^{\,N}\ +\ \tilde{b}^{\,N}) \neq S(\tilde{a}^{\,N}) + S(\tilde{b}^{\,N}).$ (7.7)

Hence, the approaches for solving NSTrPr, proposed by Thamaraiselvi and Santhi [157], are not valid.

**7.4 An important result**

It is obvious from Section 7.3 that to modify the existing approaches [157], there is need to find the exact relation between $S(\sum_{i=1}^{n} a_i)$ and $\sum_{i=1}^{n} S(a_i)$. Therefore, in this section, the exact relation between $S(\sum_{i=1}^{n} a_i)$ and $\sum_{i=1}^{n} S(a_i)$ is obtained.

Let $\tilde{a}^{\,N} = <(a_1, a_2, a_3, a_4), \mu_{\tilde{a}}, \nu_{\tilde{a}}, \lambda_{\tilde{a}}>$ and $\tilde{b}^{\,N} = <(b_1, b_2, b_3, b_4), \mu_{\tilde{a}}, \nu_{\tilde{a}}, \lambda_{\tilde{a}}>$ be two TrNNs. Then, using the existing arithmetic operations [157, Def.10, p. 4]

$\sum_{i=1}^{n} \tilde{a}^{\,N} = \langle(\sum_{i=1}^{n} a_{i1}\ , \sum_{i=1}^{n} a_{i2}\, , \sum_{i=1}^{n} a_{i3}\, , \sum_{i=1}^{n} a_{i4}); min_{1\le i\le n}(\mu_{a_i})\ ,\ max_{1\le i\le n}(\nu_{a_i}),$

$max_{1\le i\le n}(\lambda_{a_i})\rangle$ and using the existing results [157, Def. 12, p. 4],

$S\ (\sum_{i=1}^{n} \tilde{a}^{\,N}) =$

$S\langle(\sum_{i=1}^{n} a_{i1}\ , \sum_{i=1}^{n} a_{i2}\, , \sum_{i=1}^{n} a_{i3}\, , \sum_{i=1}^{n} a_{i4}); min_{1\le i\le n}(\mu_{a_i})\, , max_{1\le i\le n}(\nu_{a_i}),\ max_{1\le i\le n}(\lambda_{a_i})\rangle$

$= \frac{1}{16}(\sum_{i=1}^{n} a_{i1} + \sum_{i=1}^{n} a_{i2} + \sum_{i=1}^{n} a_{i3} + \sum_{i=1}^{n} a_{i4})\big[min_{1\le i\le n}(\mu_{a_i}) + (1 - max_{1\le i\le n}(\nu_{a_i})) +$

$(1 - max_{1\le i\le n}(\lambda_{a_i}))\big]$ (7.8)

Furthermore, using the existing results [157, Def. 12, p. 4],

$S(a_i) = \frac{1}{16}(a_{i1} + a_{i2} + a_{i3} + a_{i4})\big[\mu_{a_i} + (1 - \nu_{a_i}) + (1 - \lambda_{a_i})\big]$ (7.9)

From the eq. (7.1) and the eq. (7.2) ,

$S\left(\sum_{i=1}^{n} \tilde{a}^{\,N}\right) =$

$$\left( min_{1\le i\le n}(\mu_{a_i}) + max_{1\le i\le n}(1-\nu_{a_i}) + max_{1\le i\le n}\left(1-\lambda_{a_i}\right) \right) \sum_{i=1}^{n}\left(\frac{S(\tilde{a}_i^N)}{\mu_{a_i}+\left(1-\nu_{a_i}\right)+\left(1-\lambda_{a_i}\right)}\right) \tag{7.10}$$

**7.5. Modified approaches for solving NSTrPrs**

In this section, to resolve the flaws of the existing approaches, pointed out in Section 7.3, the existing approaches [157] are modified.

**7.5.1 Modified approach for solving NSTrPrs of Type - I**

In this section, a modified approach has been proposed to solve the NSTrPrs of Type - I. The steps of the modified approach are as follows:

**Step 1:** Use Step1 and Step 2 of the existing approach [157] to obtain the problem$(P_{7.2})$.

**Step 2:** Using the relation, obtained in Section 7.4, transform the problem $(P_{7.2})$ into its equivalent NSLPPr $(P_{7.7})$.

$Minimize\left(\sum_{i=1}^{m}\sum_{j=1}^{n} M\,(\tilde{c}_{ij}^N)x_{ij}\right)$

Subject to $\qquad (P_{7.7})$

Constraints of the CLPPr $(P_{7.1})$

where,

$M(\,\tilde{c}_{ij}^N) =$

$$\left( min_{1\le i\le n}(\mu_{\tilde{c}_{ij}^N}) + max_{1\le i\le n}(1-\nu_{\tilde{c}_{ij}^N}) + max_{1\le i\le n}(1-\lambda_{\tilde{c}_{ij}^N})\right) \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{S\left(\tilde{c}_{ij}^N\,x_{ij}\right)}{\mu_{\tilde{c}_{ij}^N}+(1-\nu_{\tilde{c}_{ij}^N})+(1-\lambda_{\tilde{c}_{ij}^N})}\right) \tag{7.11}$$

**Step 3:** Represent the NSLPPr $(P_{7.7})$ into tabular form as shown in Table 7.3.

**Table 7.3 Tabular representation of the transformed crisp TrPr**

| Destinations → / Sources ↓ | $D_1$ | $D_2$ | $D_3$ | … | $D_n$ | Supply |
|---|---|---|---|---|---|---|
| $S_1$ | $M(\tilde{c}_{11}^{N})$ | $M(\tilde{c}_{12}^{N})$ | $M(\tilde{c}_{13}^{N})$ | … | $M(\tilde{c}_{1n}^{N})$ | $a_1$ |
| $S_2$ | $M(\tilde{c}_{21}^{N})$ | $M(\tilde{c}_{22}^{N})$ | $M(\tilde{c}_{23}^{N})$ | … | $M(\tilde{c}_{2n}^{N})$ | $a_2$ |
| $S_3$ | $M(\tilde{c}_{31}^{N})$ | $M(\tilde{c}_{32}^{N})$ | $M(\tilde{c}_{33}^{N})$ | … | $M(\tilde{c}_{3n}^{N})$ | $a_3$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $S_m$ | $M(\tilde{c}_{m1}^{N})$ | $M(\tilde{c}_{m2}^{N})$ | $M(\tilde{c}_{m3}^{N})$ | … | $M(\tilde{c}_{mn}^{N})$ | $a_m$ |
| Demand | $b_1$ | $b_2$ | $b_3$ | … | $b_n$ | |

**Step 4:** Find the crisp optimal solution $\{x_{ij}\}$ of the crisp TrPr presented by Table 7.3.

**Step 5:** Find the total minimum NSTrC by putting the optimal solution $\{x_{ij}\}$, obtained from Step 4, in $\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}^{N}\,x_{ij}$ .

### 7.5.2 Modified approach for solving NSTrPrs of Type-II

In this section, a modified approach has been proposed to solve the NSTrPrs of Type - II.

The steps of the modified approach are as follows:

**Step 1**: Use Step 1 and Step 2 of the existing approach [157] to obtain the NSLPPr $(P_{7.5})$ .

**Step 2:** Using the relation, obtained in Section 7.4, transform the NSLPPr $(P_{7.5})$ into NSLPPr $(P_{7.8})$

$Minimize\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\, M(\tilde{c}_{ij}^{N})\,\tilde{x}_{ij}^{N}\right)$

Subject to $(P_{7.8})$

Constraints of the CLPPr $(P_{7.4})$.

**Step 3:** Represent the NSLPPr $(P_{7.8})$ into tabular form as shown in Table 7.4.

**Table 7.4 Tabular representation of the NSTrPr of Type – II**

| Destinations $\longrightarrow$ / Sources $\downarrow$ | $D_1$ | $D_2$ | $D_3$ | … | $D_n$ | Supply |
|---|---|---|---|---|---|---|
| $S_1$ | $M(\tilde{c}_{11}^{N})$ | $M(\tilde{c}_{12}^{N})$ | $M(\tilde{c}_{13}^{N})$ | … | $M(\tilde{c}_{1n}^{N})$ | $\tilde{a}_1^N$ |
| $S_2$ | $M(\tilde{c}_{21}^{N})$ | $M(\tilde{c}_{22}^{N})$ | $M(\tilde{c}_{23}^{N})$ | … | $M(\tilde{c}_{2n}^{N})$ | $\tilde{a}_2^N$ |
| $S_3$ | $M(\tilde{c}_{31}^{N})$ | $M(\tilde{c}_{32}^{N})$ | $M(\tilde{c}_{33}^{N})$ | … | $M(\tilde{c}_{3n}^{N})$ | $\tilde{a}_3^N$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $S_m$ | $M(\tilde{c}_{m1}^{N})$ | $M(\tilde{c}_{m2}^{N})$ | $M(\tilde{c}_{m3}^{N})$ | … | $M(\tilde{c}_{mn}^{N})$ | $\tilde{a}_m^N$ |
| Demand | $\tilde{b}_1^N$ | $\tilde{b}_2^N$ | $\tilde{b}_3^N$ | … | $\tilde{b}_n^N$ | |

**Step 4:** Find the neutrosophic optimal solution $\{\tilde{x}_{ij}^N\}$ of NSTrPr presented by Table 7.4.

**Step 5:** Find the total minimum NSTrC by putting the optimal solution $\{\tilde{x}_{ij}^N\}$, obtained in Step 4 in, $\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}^N \tilde{x}_{ij}^N$ .

### 7.6 Exact solution of the numerical problems

Thamaraiselvi and Santhi [157] solved a NSTrPr of Type – I and NSTrPr of Type – II to illustrate their proposed approaches. However as discussed in Section 7.3 that Thamaraiselvi and Santhi [157] have used some mathematical incorrect assumptions in their proposed approaches. Therefore, the optimal solution of these problems, obtained by Thamaraiselvi and Santhi [157], are not exact. In this section, the exact solution of these problems is obtained by modified approaches.

### 7.6.1 Exact solution of the NSTrPr of Type – I

Thamaraiselvi and Santhi [157] solved the NSTrPr of Type – I, represented by Table 7.5, by their proposed approach.

**Table 7.5 Input data for the NSTrPr**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $S_1$ | $\langle(3,5,6,8);$ $0.6,0.5,0.4\rangle$ | $\langle(5,8,10,14);$ $0.3,0.6,0.6\rangle$ | $\langle(12,15,19,22);$ $0.6,0.4,0.5\rangle$ | $\langle(14,17,21,28);$ $0.8,0.2,0.6\rangle$ | 26 |
| $S_2$ | $\langle(0,1,3,6);$ $0.7,0.5,0.3\rangle$ | $\langle(5,7,9,11);$ $0.9,0.7,0.5\rangle$ | $\langle(15,17,19,22);$ $0.4,0.8,0.4\rangle$ | $\langle(9,11,14,16);$ $0.5,0.4,0.7\rangle$ | 24 |
| $S_3$ | $\langle(4,8,11,15);$ $0.6,0.3,0.2\rangle$ | $\langle(1,3,4,6);$ $0.6,0.3,0.5\rangle$ | $\langle(5,7,8,10);$ $0.5,0.4,0.7\rangle$ | $\langle(5,9,14,19);$ $0.3,0.7,0.6\rangle$ | 30 |
| Demand | 17 | 23 | 28 | 12 | |

In this section, the exact solution of this problem is obtained by the modified approach.

Using the modified approach, the exact solution of NSTrPr of Type – I, represented by in Table 7.5, can be obtained as follows:

**Step 1:** Using Step 1 to Step 3 of the modified approach, the NSTrPr, represented in Table 7.5, can be transformed into the crisp TrPr (represented by Table 7.6).

**Table 7.6 Transformed crisp TrPr**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $S_1$ | 3 | 4 | 8 | 9 | 26 |
| $S_2$ | 1 | 4 | 8 | 6 | 24 |
| $S_3$ | 4 | 2 | 3 | 5 | 30 |
| Demand | 17 | 23 | 28 | 12 | |

**Step 2:** On solving the crisp TrPr (Table 7.6), the obtained optimal solution is $x_{12} = 23$, $x_{14} = 3,\ x_{21} = 17,\ x_{24} = 7,\ \ x_{33} = 28,\ x_{34} = 2$ . (7.12)

**Step 3:** Using the obtained optimal solution, the minimum total NSTrC is,

$\sum_{i=1}^{3}\sum_{j=1}^{4} x_{ij}\ \tilde{c}_{ij}^{N}) = 23\langle(5,8,10,14); 0.3,0.8,0.7\rangle + 3\langle(14,17,21,28); 0.3,0.8,0.7\rangle +$

$17\langle(0,1,3,6); 0.3,0.8,0.7\rangle + 7\langle(9,11,14,16); 0.3,0.8,0.7\rangle + 28\langle(5,7,8,10); 0.3,0.8,0.7\rangle +$

$2\langle(5,9,14,19); 0.3,0.8,0.7\rangle$

$= \langle(370,543,694,938); 0.3,0.8,0.7\rangle$. (7.13)

### 7.6.2 Exact solution of the NSTrPr of Type – II

Thamaraiselvi and Santhi [157] solved the NSTrPr of Type – II, represented by Table 7.7, by their proposed approach.

**Table 7.7 Input data for the NSTrPr**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $S_1$ | $\langle(3,5,6,8); 0.6,0.5,0.4\rangle$ | $\langle(5,8,10,14); 0.3,0.6,0.6\rangle$ | $\langle(12,15,19,22); 0.6,0.4,0.5\rangle$ | $\langle(14,17,21,28); 0.8,0.2,0.6\rangle$ | $\langle(22,26,28,32); 0.7,0.3,0.4\rangle$ |
| $S_2$ | $\langle(0,1,3,6); 0.7,0.5,0.3\rangle$ | $\langle(5,7,9,11); 0.9,0.7,0.5\rangle$ | $\langle(15,17,19,22); 0.4,0.8,0.4\rangle$ | $\langle(9,11,14,16); 0.5,0.4,0.7\rangle$ | $\langle(17,22,27,31); 0.6,0.4,0.5\rangle$ |
| $S_3$ | $\langle(4,8,11,15); 0.6,0.3,0.2\rangle$ | $\langle(1,3,4,6); 0.6,0.3,0.5\rangle$ | $\langle(5,7,8,10); 0.5,0.4,0.7\rangle$ | $\langle(5,9,14,19); 0.3,0.7,0.6\rangle$ | $\langle(21,28,32,37); 0.8,0.2,0.4\rangle$ |
| Demand | $\langle(13,16,18,21); 0.5,0.5,0.6\rangle$ | $\langle(17,21,24,28); 0.8,0.2,0.4\rangle$ | $\langle(24,29,32,35); 0.9,0.5,0.3\rangle$ | $\langle(6,10,13,15); 0.7,0.3,0.4\rangle$ | |

In this section, the exact solution of this problem is obtained by the modified approach.

Using the modified approach, the exact solution of the NSTrPr of Type – II, represented by Table 7.7, can be obtained as follows:

**Step 1:** Using Step 1 to Step 3 of the modified approach, the NSTrPr, represented by Table 7.7, can be transformed into NSTrPr (represented by Table 7.8).

**Table 7.8 NSTrPr with crisp cost**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $S_1$ | 1 | 2 | 3 | 4 | $\langle(22,26,28,32); 0.7,0.3,0.4\rangle$ |
| $S_2$ | 1 | 2 | 4 | 3 | $\langle(17,22,27,31); 0.6,0.4,0.5\rangle$ |
| $S_3$ | 2 | 1 | 2 | 2 | $\langle(21,28,32,37); 0.8,0.2,0.4\rangle$ |
| Demand | $\langle(13,16,18,21); 0.5,0.5,0.6\rangle$ | $\langle(17,21,24,28); 0.8,0.2,0.4\rangle$ | $\langle(24,29,32,35); 0.9,0.5,0.3\rangle$ | $\langle(6,10,13,15); 0.7,0.3,0.4\rangle$ | |

**Step 2:** On solving the NSTrPr with crisp cost (Table 7.8), the obtained optimal solution is

$\tilde{x}_{11}^{N} = \langle(-6,2,7,11); 0.7,0.3,0.4\rangle$ , $\tilde{x}_{12}^{N} = \langle(17,21,24,28); 0.8,0.2,0.4\rangle$ ,

$\tilde{x}_{21}^{N} = \langle(2,9,16,27); 0.5,0.5,0.6\rangle$ , $\tilde{x}_{24}^{N} = \langle(-10,6,18,29); 0.5,0.5,0.6\rangle$ ,

$\tilde{x}_{33}^{N} = \langle(24,29,32,35); 0.9,0.5,0.3\rangle$, $\tilde{x}_{34}^{N} = \langle(-14,-4,3,13); 0.8,0.5,0.4\rangle$ (7.14)

**Step 3:** Using the obtained optimal solution, the minimum total NSTrC is

$\sum_{i=1}^{3}\sum_{j=1}^{4}\tilde{x}_{ij}^{N}\ \tilde{c}_{ij}^{N} =$

$\langle(-6,2,7,11); 0.7,0.3,0.4\rangle \times \langle(3,5,6,8); 0.6,0.5,0.4\rangle + \langle(17,21,24,28); 0.8,0.2,0.4\rangle \times$

$\langle(5,8,10,14); 0.3,0.6,0.6\rangle + \langle(2,9,16,27); 0.5,0.5,0.6\rangle \times \langle(0,1,3,6); 0.7,0.5,0.3\rangle +$

$\langle(-10,6,18,29); 0.5,0.5,0.6\rangle \times \langle(9,11,14,16); 0.5,0.4,0.7\rangle + \langle(24,29,32,35); 0.9,0.5,0.3\rangle \times$

$\langle(5,7,8,10); 0.5,0.4,0.7\rangle + \langle(-14,-4,3,13); 0.8,0.5,0.4\rangle \times \langle(5,9,14,19); 0.3,0.7,0.6\rangle$

$= \langle(27,420,880,1703); 0.3,0.7,0.7\rangle$. (7.15)

**7.7 Conclusions**

It is pointed out that it is not genuine to use the existing approaches [157] as in these approaches mathematical incorrect assumption has been used. Therefore, the modified approaches are proposed. Furthermore, the exact solution of two existing NSTrPrs are obtained by the modified approaches.

# Chapter 8

# Modified NLPM for MCDMPrs under IVNS environment*

Garg and Nancy [56] claimed that there is no method in the literature to solve INSMCDMPrs and hence, proposed a NLPM for solving INSMCDMPrs. Since, it is only method for solving INSMCDMPrs so the other researchers may be attracted to use this method for solving real-life INSMCDMPrs. However, after a deep study, it is observed that some mathematical incorrect assumptions have been considered in this method. Therefore, it is scientifically incorrect to use this method for solving real-life INSMCDMPrs. Keeping the same in mind, the method, proposed by Garg and Nancy, is modified.

**8.1 IVNS [167]**

A set $\tilde{A}^N = \{\langle x, [T^L_{\tilde{A}^N}(x), T^U_{\tilde{A}^N}(x)], [I^L_{\tilde{A}^N}(x), I^U_{\tilde{A}^N}(x)], [F^L_{\tilde{A}^N}(x), F^U_{\tilde{A}^N}(x)]\rangle | x \in X,\ 0 \leq T^L_{\tilde{A}^N}(x) \leq T^U_{\tilde{A}^N}(x) \leq 1,\ 0 \leq I^L_{\tilde{A}^N}(x) \leq I^U_{\tilde{A}^N}(x) \leq 1,\ 0 \leq F^L_{\tilde{A}^N}(x) \leq F^U_{\tilde{A}^N}(x) \leq 1,\ T^U_{\tilde{A}^N}(x) + I^U_{\tilde{A}^N}(x) + F^U_{\tilde{A}^N}(x) \leq 3\}$, defined on the universal set $X$, is said to be an IVNS, where, $[T^L_{\tilde{A}^N}(x), T^U_{\tilde{A}^N}(x)]$, $[I^L_{\tilde{A}^N}(x), I^U_{\tilde{A}^N}(x)]$ and $[F^L_{\tilde{A}^N}(x), F^U_{\tilde{A}^N}(x)]$ represents the intervals of the degree of truth-membership, the degree of indeterminacy-membership and the degree of falsity-membership respectively of the element $x$ in $\tilde{A}^N$.

**8. 2 A brief review of Nancy and Garg's method**

The aim of this chapter is to point out the mathematical incorrect assumptions considered in the existing method [56] as well as to propose a modified method. Since, to achieve this

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aim, there is need to discuss the existing method [56]. Therefore, in this section, the existing method [56] is presented in a brief manner.

The steps of the existing method [56] are as follows:

**Step 1:** Write the CLFPPrs ($P_{8.1}$) [56, Section 3.5, Eq. 16] and ($P_{8.2}$) [56, Section 3.5, Eq. 18], with the help of the INSDM $\widetilde{D} = \left(\langle[\mu_{ij}^L, \mu_{ij}^U], [\rho_{ij}^L, \rho_{ij}^U], [\nu_{ij}^L, \nu_{ij}^U]\rangle\right)_{m\times n}$, corresponding to the $i^{th}$ alternative of the considered INSMCDMPr.

$$K_i^L = min\left\{\frac{\sum_{j=1}^{n}\left\{\omega_j\,\mu_{ij}^L + \xi_j\left(1-\rho_{ij}^U\right) + \eta_j\left(1-\nu_{ij}^U\right)\right\}}{\sum_{j=1}^{n}(\omega_j + \xi_j + \nu_j)}\right\}$$

Subject to ($P_{8.1}$)

$$\begin{cases}\omega_j^L \le \omega_j \le \omega_j^U \quad ; \quad j = 1,2,\dots,n,\\ \xi_j^L \le \xi_j \le \xi_j^U \quad ; \quad j = 1,2,\dots,n,\\ \eta_j^L \le \eta_j \le \eta_j^U \quad ; \quad j = 1,2,\dots,n.\end{cases}$$

$$K_i^U = max\left\{\frac{\sum_{j=1}^{n}\left\{\omega_j\,\mu_{ij}^U + \xi_j\left(1-\rho_{ij}^L\right) + \eta_j\left(1-\nu_{ij}^L\right)\right\}}{\sum_{j=1}^{n}(\omega_j + \xi_j + \nu_j)}\right\}$$

Subject to ($P_{8.2}$)

Constraints of the CLFPPr ($P_{8.1}$).

Here

(i) The IVNS, $\langle[\mu_{ij}^L, \mu_{ij}^U], [\rho_{ij}^L, \rho_{ij}^U], [\nu_{ij}^L, \nu_{ij}^U]\rangle$ represents the RV of the $i^{th}$ alternative over the $j^{th}$ criteria.

(ii) The IVNS $\langle[\omega_j^L, \omega_j^U], [\xi_j^L, \xi_j^U], [\eta_j^L, \eta_j^U]\rangle$ represents the normalized weight $\left(w_j \ge 0, \text{and } \sum_{j=1}^{n} w_j = 1\right)$ of the $j^{th}$ criteria.

(iii) $\omega_j$ represents the membership degree of the weight of the $j^{th}$ criteria which is given as an interval i.e., $\omega_j \in [\omega_j^L, \omega_j^U]$.

(iv) $\xi_j$ represents the indeterminacy degree of the weight of the $j^{th}$ criteria which is given as an interval i.e., $\xi_j \in \left[\xi_j^L, \xi_j^U\right]$.

(v) $\eta_j$ represents the falsity degree of the weight of the $j^{th}$ criteria which is given as an interval i.e., $\eta_j \in \left[\eta_j^L, \eta_j^U\right]$.

(vi) $m$ represents the number of alternatives.

(vii) $n$ represents the number of criteria.

**Step 2:** Using CCTr [61], the CLFPPr ($P_{8.1}$) and the CLFPPr ($P_{8.2}$) can be transformed into its equivalent CLFPPr ($P_{8.3}$) [56, Section 3.5, Eq. 19] and the CLFPPr ($P_{8.4}$) [56, Section 3.5, Eq. 20] respectively.

$K_i^L = min\left\{\sum_{j=1}^{n}\left\{t_j\, \mu_{ij}^L + r_j\left(1-\rho_{ij}^U\right) + y_j\left(1-\nu_{ij}^U\right)\right\}\right\}$

Subject to ($P_{8.3}$)

$$\begin{cases} z\omega_j^L \le t_j \le z\omega_j^U\,; & j = 1,2,\dots,n, \\ z\xi_j^L \le r_j \le z\xi_j^U\,; & j = 1,2,\dots,n, \\ z\eta_j^L \le y_j \le z\eta_j^U\,; & j = 1,2,\dots,n, \\ \sum_{j=1}^{n}\left(t_j + r_j + y_j\right) = 1, & \\ z \ge 0. & \end{cases}$$

$K_i^U = max\left\{\sum_{j=1}^{n}\left\{t_j\, \mu_{ij}^U + r_j\left(1-\rho_{ij}^L\right) + y_j\left(1-\nu_{ij}^L\right)\right\}\right\}$

Subject to ($P_{8.4}$)

Constraints of the CLPPr ($P_{8.4}$).

**Step 3:** Using the optimal values $K_i^L$ and $K_i^U$ of the CLPPr ($P_{8.3}$) and the CLPPr ($P_{8.4}$), we obtain $K_i = [K_i^L, K_i^U]$ $(i = 1,2,\dots,m)$.

**Step 4:** Using the values of $K_i = [K_i^L, K_i^U]$ $(i = 1,2,\dots,m)$, obtained in Step 3, we construct a $m \times m$ matrix $P = [p_{ik}]_{m\times m}$, where,

$$p_{ik} = \begin{Bmatrix} max\left\{1 - max\left(\frac{K_k^U - K_i^L}{K_i^U - K_i^L + K_k^U - K_k^L}, 0\right), 0\right\}; & if\ \ i \ne k \\ \frac{1}{2} & ;\ if\ \ i = k \end{Bmatrix}$$

**Step 5:** Find the value of $\theta_i = \frac{\sum_{j=1}^{n} p_{ik} + \frac{n}{2} - 1}{n(n-1)}$, $(i = 1,2,\dots,m;\ k = 1,2,\dots,m)$ and we check that $\theta_i > \theta_k$ or $\theta_i < \theta_k$ or $\theta_i = \theta_k$.

**Case (iv)** If $\theta_i > \theta_k$ then $A_i > A_k$.

**Case (v)** If $\theta_i < \theta_k$ then $A_i < A_k$.

**Case (vi)** If $\theta_i = \theta_k$ then $A_i = A_k$.

**8.3 Mathematical incorrect assumptions considered in the existing method**

The objective of the CLFPPr ($P_{8.1}$) and the CLFPPr ($P_{8.2}$) is to find such values of $\omega_j, \xi_j, \eta_j$ $(j = 1,2,\dots,n)$ where $0 \le \omega_j, \xi_j, \eta_j \le 1$ corresponding to which the value of the expression $\frac{\sum_{j=1}^{n}\left\{\omega_j\, \mu_{ij}^{L} + \xi_j\left(1-\rho_{ij}^{U}\right) + \eta_j\left(1-\nu_{ij}^{U}\right)\right\}}{\sum_{j=1}^{n}(\omega_j + \xi_j + \nu_j)}$ will be minimum and the value of the expression $\frac{\sum_{j=1}^{n}\left\{\omega_j\, \mu_{ij}^{U} + \xi_j\left(1-\rho_{ij}^{L}\right) + \eta_j\left(1-\nu_{ij}^{L}\right)\right\}}{\sum_{j=1}^{n}(\omega_j + \xi_j + \nu_j)}$ will be maximum.

To achieve this objective, Garg and Nancy [56] have solved the CLFPPr ($P_{8.1}$) and the CLFPPr ($P_{8.2}$) independently by transforming the CLFPPr ($P_{8.1}$) and the CLFPPr ($P_{8.2}$) into the CLPPr ($P_{8.3}$) and the CLPPr ($P_{8.4}$) respectively. However, it is mathematically incorrect to solve the CLPPr ($P_{8.3}$) and the CLPPr ($P_{8.4}$) independently due to the following reasons:

On solving the CLPPr ($P_{8.3}$) and the CLPPr ($P_{8.4}$) independently, the obtained values of $t_j, r_j, y_j$ $(j = 1,2,\dots,n)$ will not necessarily be equal. While, as $t_j, r_j, y_j$ $(j = 1,2,\dots,n)$ are RNs so the values of $t_j, r_j, y_j$ $(j = 1,2,\dots,n)$, obtained on solving the CLPPr ($P_{8.3}$) and the CLPPr ($P_{8.4}$), should be equal.

For example, to find the solution of the existing problem [56], the CLFPPr ($P_{8.5}$) and the CLFPPr ($P_{8.6}$) are solved independently by transforming the CLFPPr ($P_{8.5}$) and the CLFPPr

$(P_{8.6})$ into the CLPPr $(P_{8.7})$ [56, Section 5.1, Eq. 37] and the CLPPr $(P_{8.8})$ [56, Section 5.1, Eq. 38] with the help of CCTr [59].

$$K_1^L = min\left\{\frac{\begin{array}{c}0.7\ \omega_1 + 0.6\ \omega_2 + 0.8\ \omega_3 + 0.7\ \omega_4 + \\ 0.3\ \xi_1 + 0.5\ \xi_2 + 0.4\ \xi_3 + 0.6\ \xi_4 + \\ 0.8\ \eta_1 + 0.7\ \eta_2 + 0.8\ \eta_3 + 0.8\ \eta_4\end{array}}{\omega_1+\omega_2+\omega_3+\omega_4+\xi_1+\xi_2+\xi_3+\xi_4+\eta_1+\eta_2+\eta_3+\eta_4}\right\}$$

Subject to $(P_{8.5})$

$$\begin{cases} 0.10 \le \omega_1 \le 0.3;\ 0.10 \le \xi_1 \le 0.2; \\ 0.2 \le \eta_1 \le 0.4; \\ 0.20 \le \omega_2 \le 0.5;\ 0.1 \le \xi_2 \le 0.2; \\ 0.15 \le \eta_2 \le 0.25; \\ 0.25 \le \omega_3 \le 0.4;\ 0.2 \le \xi_3 \le 0.3; \\ 0.15 \le \eta_3 \le 0.3; \\ 0.15 \le \omega_4 \le 0.3;\ 0.1 \le \xi_4 \le 0.3; \\ 0.3 \le \eta_4 \le 0.4. \end{cases}$$

$$K_1^U = max\left\{\frac{\begin{array}{c}0.8\ \omega_1 + 0.8\ \omega_2 + 0.8\ \omega_3 + 0.9\ \omega_4 + \\ 0.5\ \xi_1 + 0.6\ \xi_2 + 0.6\ \xi_3 + 0.7\ \xi_4 + \\ 0.9\ \eta_1 + 0.7\ \eta_2 + 0.9\ \eta_3 + 0.8\ \eta_4\end{array}}{\omega_1+\omega_2+\omega_3+\omega_4+\xi_1+\xi_2+\xi_3+\xi_4+\eta_1+\eta_2+\eta_3+\eta_4}\right\}$$

Subject to $(P_{8.6})$

Constraints of the CLFPPr $(P_{8.5})$.

$$K_1^L = min(0.7t_1 + 0.6t_2 + 0.8\,t_3 + 0.7t_4 + 0.3\,r_1 + 0.5\,r_2 + 0.4r_3 + 0.6r_4 + 0.8\,y_1 + 0.7\,y_2 + 0.8y_3 + 0.8\,y_4)$$

Subject to $(P_{8.7})$

$$\begin{cases} 0.10z \le t_1 \le 0.3z\,;\ 0.10z \le r_1 \le 0.2z; \\ 0.2z \le y_1 \le 0.4z; \\ 0.20z \le t_2 \le 0.5z;\ 0.1z \le r_2 \le 0.2z; \\ 0.15z \le y_2 \le 0.25z; \\ 0.25z \le t_3 \le 0.4z;\ 0.2z \le r_3 \le 0.3z; \\ 0.15z \le y_3 \le 0.3z; \\ 0.15z \le t_4 \le 0.3z;\ 0.1z \le r_4 \le 0.3z; \\ 0.3z \le y_4 \le 0.4z; \\ \sum_{j=1}^{n}\left(t_j + r_j + y_j\right) = 1; \\ z \ge 0. \end{cases}$$

$K_1^U = max(0.8t_1 + 0.8t_2 + 0.8\, t_3 + 0.9t_4 + 0.5\, r_1 + 0.6\, r_2 + 0.6r_3 + 0.7r_4 + 0.9\, y_1 + 0.7\, y_2 + 0.9y_3 + 0.8\, y_4)$

Subject to $(P_{8.8})$

Constraints of the CLPPr $(P_{8.7})$.

The optimal values of $t_1, t_2, t_3, t_4, r_1, r_2, r_3, r_4, y_1, y_2, y_3$ and $y_4$, obtained on solving the CLPPr $(P_{8.7})$ and the CLPPr $(P_{8.8})$, are shown in Table 8.1.

**Table 8.1 Optimal values of the variables**

| Variables → | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $z$ | Value of the objective function |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Min $K_1^L$ | $\frac{1}{28}$ | $\frac{5}{28}$ | $\frac{5}{56}$ | $\frac{3}{56}$ | $\frac{1}{14}$ | $\frac{1}{14}$ | $\frac{3}{28}$ | $\frac{3}{28}$ | $\frac{1}{14}$ | $\frac{3}{56}$ | $\frac{3}{56}$ | $\frac{3}{28}$ | $\frac{5}{14}$ | $\frac{22}{35}$ |
| Max $K_1^U$ | $\frac{6}{65}$ | $\frac{2}{13}$ | $\frac{8}{65}$ | $\frac{6}{65}$ | $\frac{2}{65}$ | $\frac{2}{65}$ | $\frac{4}{65}$ | $\frac{2}{65}$ | $\frac{8}{65}$ | $\frac{3}{65}$ | $\frac{6}{65}$ | $\frac{8}{65}$ | $\frac{4}{13}$ | $\frac{517}{650}$ |

It is obvious from the result, shown in Table 8.1, that the values of the variables $t_1, t_2, t_3, t_4, r_1, r_2, r_3, r_4, y_1, y_2, y_3$ and $y_4$, obtained on solving the existing CLPPr $(P_{8.7})$ [56, Section 5.1, Eq 37] and the existing CLPPr $(P_{8.8})$ [56, Section 5.1, Eq. 38], are not equal, which is mathematically incorrect.

**8.4 Impact of the mathematical incorrect assumptions**

It is obvious from Section 8.2 that on applying the existing method [56] the obtained optimal value will be an interval but the values of the variables will not be RNs. This is like a situation that one is saying that the profit is rupees 20 (assumed) but it is not possible to find a strategy corresponding to which this profit will be achieved. In actual case, on applying the existing method [56], there doesn’t exist any such optimal solution corresponding to which the profit is rupees 20. Hence, it is not appropriate to apply the existing method [56] for solving real-life problems.

**8.5 Suggested modifications**

The shortcomings of the existing method [56], pointed out in Section 8.2, can be resolved if to achieve the objective i.e., to maximize $\frac{\sum_{j=1}^{n}\left\{\omega_j\ \mu_{ij}^{U}+\ \xi_j\left(1-\rho_{ij}^{L}\right)+\ \eta_j\left(1-\nu_{ij}^{L}\right)\right\}}{\sum_{j=1}^{n}(\omega_j+\xi_j+\nu_j)}$ and to minimize $\frac{\sum_{j=1}^{n}\left\{\omega_j\,\mu_{ij}^{L}+\xi_j\left(1-\rho_{ij}^{U}\right)+\eta_j\left(1-\nu_{ij}^{U}\right)\right\}}{\sum_{j=1}^{n}(\omega_j+\xi_j+\nu_j)}$, the CLFPPr $(P_{8.9})$ is solved instead of the CLFPPr $(P_{8.1})$ and the CLFPPr $(P_{8.2})$ independently.

$$K_i=max\left\{\frac{\sum_{j=1}^{n}\left\{\begin{matrix}\omega_j\ \mu_{ij}^{U}+\ \xi_j\left(1-\rho_{ij}^{L}\right)+\\ \eta_j\left(1-\nu_{ij}^{L}\right)\end{matrix}\right\}-\sum_{j=1}^{n}\left\{\begin{matrix}\omega_j\,\mu_{ij}^{L}+\xi_j\left(1-\rho_{ij}^{U}\right)+\\ \eta_j\left(1-\nu_{ij}^{U}\right)\end{matrix}\right\}}{\sum_{j=1}^{n}(\omega_j+\xi_j+\nu_j)}\right\}$$

$$=max\left\{\frac{\sum_{j=1}^{n}\left\{\begin{matrix}\omega_j\left(\mu_{ij}^{U}-\mu_{ij}^{L}\right)+\ \xi_j\left(\left(1-\rho_{ij}^{L}\right)-\left(1-\rho_{ij}^{U}\right)\right)+\\ \eta_j\left(\left(1-\nu_{ij}^{L}\right)-\left(1-\nu_{ij}^{U}\right)\right)\end{matrix}\right\}}{\sum_{j=1}^{n}(\omega_j+\xi_j+\nu_j)}\right\}$$

Subject to $(P_{8.9})$

$$\begin{cases}\omega_j^{L}\le\omega_j\le\omega_j^{U} & ;\quad j=1,2,\dots,n,\\ \xi_j^{L}\le\xi_j\le\xi_j^{U} & ;\quad j=1,2,\dots,n,.\\ \eta_j^{L}\le\eta_j\le\eta_j^{U} & ;\quad j=1,2,\dots,n.\end{cases}$$

The CLFPPr $(P_{8.9})$ can be solved as follows:

**Step 1:** Using CCTr [59], the CLFPPr $(P_{8.9})$ can be transformed into its equivalent CLPPr $(P_{8.10})$.

$$K_i=max\left\{\sum_{j=1}^{n}\left\{t_j\left(\mu_{ij}^{U}-\mu_{ij}^{L}\right)+\ r_j\ \left(\left(1-\rho_{ij}^{L}\right)-\left(1-\rho_{ij}^{U}\right)\right)+y_j\left(\left(1-\nu_{ij}^{L}\right)-\left(1-\nu_{ij}^{U}\right)\right)\right\}\right\}$$

Subject to $(P_{8.10})$

$$\begin{cases} z\omega_j^L \le t_j \le z\omega_j^U\ ; & j = 1,2,\dots,n, \\ z\xi_j^L \le r_j \le z\xi_j^U\ ; & j = 1,2,\dots,n, \\ z\eta_j^L \le y_j \le z\eta_j^U\ ; & j = 1,2,\dots,n,\cdot \\ \sum_{j=1}^{n}\left(t_j + r_j + y_j\right) = 1, & \\ z \ge 0. & \end{cases}$$

**Step 2:** Find the optimal solution $\{t_j, r_j, y_j;\ j = 1,2,\dots,n\}$ of the CLPPr $(P_{8.10})$.

**Step 3:** Using the optimal solution, obtained in Step 2, find

$K_i^L = \sum_{j=1}^{n}\{t_j\,\mu_{ij}^L + r_j\left(1-\rho_{ij}^U\right) + y_j\left(1-\nu_{ij}^U\right)\}$ and

$K_i^U = \sum_{j=1}^{n}\{t_j\,\mu_{ij}^U + r_j\left(1-\rho_{ij}^L\right) + y_j\left(1-\nu_{ij}^L\right)\}$.

**Step 4:** Construct a $m \times m$ matrix $P = [p_{ik}]_{m\times m}$, where,

$$p_{ik} = \begin{Bmatrix} max\left\{1 - max\left(\frac{K_k^U - K_i^L}{K_i^U - K_i^L + K_k^U - K_k^L}, 0\right), 0\right\};\ if\ \ i \ne k \\ \frac{1}{2} \qquad\qquad\qquad\qquad\qquad\qquad ;\ if\ \ i = k \end{Bmatrix}.$$

**Step 5:** Find the value of $\theta_i = \frac{\sum_{j=1}^{n} p_{ik} + \frac{n}{2} - 1}{n(n-1)}$, $(i = 1,2,\dots,m;\ k = 1,2,\dots,m)$ and check that $\theta_i > \theta_k$ or $\theta_i < \theta_k$ or $\theta_i = \theta_k$.

**Case (i)** If $\theta_i > \theta_k$ then $A_i > A_k$.

**Case (ii)** If $\theta_i < \theta_k$ then $A_i < A_k$.

**Case (iii)** If $\theta_i = \theta_k$ then $A_i = A_k$.

## 8.6 Exact solution of the existing problem

Garg and Nancy [56] solved a real-life problem to illustrate their proposed method. However as discussed in Section 8.2 that Garg and Nancy [56] have used some mathematical incorrect assumptions in their proposed method, therefore the results of the real-life problem, obtained by Garg and Nancy [56], are not exact. In this section, the exact result of the same problem is obtained by the modified method.

Using the modified method, the exact results of the real-life problem [56] can be obtained as follows:

**Step 1:** Using Step 1 of the modified method, the CLFPPr ($P_{8.11}$), the CLFPPr ($P_{8.12}$), the CLFPPr ($P_{8.13}$), the CLFPPr ($P_{8.14}$) and the CLFPPr ($P_{8.15}$) can be obtained.

$$K_1 = max\left\{\frac{\begin{matrix}0.1\ \omega_1 + 0.2\ \omega_2 + 0\ \omega_3 + 0.2\ \omega_4 + \\ 0.2\ \xi_1 + 0.1\ \xi_2 + 0.2\ \xi_3 + 0.1\ \xi_4 + \\ 0.1\ \eta_1 + 0\ \eta_2 + 0.1\ \eta_3 + 0\ \eta_4\end{matrix}}{\omega_1+\omega_2+\omega_3+\omega_4+\xi_1+\xi_2+\xi_3+\xi_4+\eta_1+\eta_2+\eta_3+\eta_4}\right\}$$

Subject to $(P_{8.11})$

Constraints of the CLFPPr ($P_{8.5}$).

$$K_2 = max\left\{\frac{\begin{matrix}0.2\ \omega_1 + 0.2\ \omega_2 + 0\ \omega_3 + 0.2\ \omega_4 + \\ 0.2\ \xi_1 + 0.2\ \xi_2 + 0.1\ \xi_3 + 0\ \xi_4 + \\ 0.2\ \eta_1 + 0.2\ \eta_2 + 0.1\ \eta_3 + 0.2\ \eta_4\end{matrix}}{\omega_1+\omega_2+\omega_3+\omega_4+\xi_1+\xi_2+\xi_3+\xi_4+\eta_1+\eta_2+\eta_3+\eta_4}\right\}$$

Subject to $(P_{8.12})$

Constraints of the CLFPPr ($P_{8.5}$).

$$K_3 = max\left\{\frac{\begin{matrix}0.2\ \omega_1 + 0.1\ \omega_2 + 0.1\ \omega_3 + 0.1\ \omega_4 + \\ 0\ \xi_1 + 0.2\ \xi_2 + 0.1\ \xi_3 + 0.1\ \xi_4 + \\ 0.2\ \eta_1 + 0.1\ \eta_2 + 0.1\ \eta_3 + 0.1\ \eta_4\end{matrix}}{\omega_1+\omega_2+\omega_3+\omega_4+\xi_1+\xi_2+\xi_3+\xi_4+\eta_1+\eta_2+\eta_3+\eta_4}\right\}$$

Subject to $(P_{8.13})$

Constraints of the CLFPPr ($P_{8.5}$).

$$K_4 = max\left\{\frac{\begin{matrix}0.1\ \omega_1 + 0.1\ \omega_2 + 0.1\ \omega_3 + 0.1\ \omega_4 + \\ 0.1\ \xi_1 + 0.1\ \xi_2 + 0.1\ \xi_3 + 0.1\ \xi_4 + \\ 0\ \eta_1 + 0.1\ \eta_2 + 0.1\ \eta_3 + 0.1\ \eta_4\end{matrix}}{\omega_1+\omega_2+\omega_3+\omega_4+\xi_1+\xi_2+\xi_3+\xi_4+\eta_1+\eta_2+\eta_3+\eta_4}\right\}$$

Subject to $(P_{8.14})$

Constraints of the CLFPPr ($P_{8.5}$).

$$K_5 = max\left\{\frac{\begin{matrix}0.1\ \omega_1 + 0.1\ \omega_2 + 0.1\ \omega_3 + 0.2\ \omega_4 + \\ 0.1\ \xi_1 + 0.1\ \xi_2 + 0.1\ \xi_3 + 0\ \xi_4 + \\ 0.1\ \eta_1 + 0.1\ \eta_2 + 0.1\ \eta_3 + 0.1\ \eta_4\end{matrix}}{\omega_1+\omega_2+\omega_3+\omega_4+\xi_1+\xi_2+\xi_3+\xi_4+\eta_1+\eta_2+\eta_3+\eta_4}\right\}$$

Subject to $(P_{8.15})$

Constraints of the CLFPPr ($P_{8.5}$).

**Step 2:** Using Step 2 of the modified method the CLFPPr $(P_{8.11})$,$(P_{8.12})$, $(P_{8.13})$, $(P_{8.14})$ and $(P_{8.15})$ can be transformed into the CLPPr $(P_{8.16})$, $(P_{8.17})$, $(P_{8.18})$, $(P_{8.19})$ and $(P_{8.20})$ respectively.

$K_1 = max\,\{0.1t_1 + 0.2t_2 + 0t_3 + 0.2t_4 + 0.2r_1 + 0.1r_2 + 0.2r_3 + 0.1r_4 + 0.1\,y_1 + 0y_2 + 0.1y_3 + 0y_4\}$

Subject to $\qquad (P_{8.16})$

$$\begin{cases} 0.10z \le t_1 \le 0.3z\,; 0.10z \le r_1 \le 0.2z; \\ 0.2z \le y_1 \le 0.4z; \\ 0.20z \le t_2 \le 0.5z\,; 0.1z \le r_2 \le 0.2z; \\ 0.15z \le y_2 \le 0.25z; \\ 0.25z \le t_3 \le 0.4z\,; 0.2z \le r_3 \le 0.3z; \\ 0.15z \le y_3 \le 0.3z; \\ 0.15z \le t_4 \le 0.3z\,; 0.1z \le r_4 \le 0.3z; \\ 0.3z \le y_4 \le 0.4z; \\ \sum_{j=1}^{n}\left(t_j + r_j + y_j\right) = 1; \\ z \ge 0. \end{cases}$$

$K_2 = max\,\{0.2t_1 + 0.2t_2 + 0\,t_3 + 0.2t_4 + 0.2r_1 + 0.2\,r_2 + 0.1r_3 + 0r_4 + 0.2y_1 + 0.2\,y_2 + 0.1y_3 + 0.2\,y_4\}$

Subject to $\qquad (P_{8.17})$

Constraints of the CLPPr $(P_{8.16})$.

$K_3 = max\,\{0.2t_1 + 0.1\,t_2 + 0.1t_3 + 0.1t_4 + 0r_1 + 0.2r_2 + 0.1\,r_3 + 0.1r_4 + 0.2\,y_1 + 0.1y_2 + 0.1y_3 + 0.1y_4\}$

Subject to $\qquad (P_{8.18})$

Constraints of the CLPPr $(P_{8.16})$.

$K_4 = max\,\{0.1t_1 + 0.1t_2 + 0.1t_3 + 0.1t_4 + 0.1r_1 + 0.1\,r_2 + 0.1\,r_3 + 0.1\,r_4 + 0\,y_1 + 0.1y_2 + 0.1y_3 + 0.1y_4\}$

Subject to $\qquad (P_{8.19})$

Constraints of the CLPPr $(P_{8.16})$.

$K_5 = max\,\{0.1t_1 + \; 0.1t_2 + 0.1t_3 + 0.2t_4 + 0.1r_1 \; + 0.1r_2 + 0.1r_3 + 0r_4 + 0.1\,y_1 +$

$0\,.1y_2 + 0.1y_3 + 0.1y_4\}$

Subject to $(P_{8.20})$

Constraints of the CLPPr $(P_{8.16})$.

**Step 3:** The optimal solutions $\{t_j, r_j, y_j;\; j = 1,2, \dots, n\}$ of the CLPPr $(P_{8.16})$, $(P_{8.17})$, $(P_{8.18})$, $(P_{8.19})$ and $(P_{8.20})$ are,

$\left\{t_1 = \frac{2}{53}, t_2 = \frac{10}{53}, t_3 = \frac{5}{53}, t_4 = \frac{6}{53}, r_1 = \frac{4}{53}, r_2 = \frac{2}{53}, r_3 = \frac{6}{53}, r_4 = \frac{2}{53}, y_1 = \frac{4}{53}, y_2 = \frac{3}{53}, y_3 = \frac{3}{53}, y_4 = \frac{6}{53}\right\}$,

$\left\{t_1 = \frac{6}{65}, t_2 = \frac{2}{13}, t_3 = \frac{1}{13}, t_4 = \frac{6}{65}, r_1 = \frac{4}{65}, r_2 = \frac{4}{65}, r_3 = \frac{4}{65}, r_4 = \frac{2}{65}, y_1 = \frac{8}{65}, y_2 = \frac{1}{13}, y_3 = \frac{3}{65}, y_4 = \frac{8}{65}\right\}$,

$\left\{t_1 = \frac{3}{25}, t_2 = \frac{2}{25}, t_3 = \frac{1}{10}, t_4 = \frac{3}{50}, r_1 = \frac{1}{25}, r_2 = \frac{2}{25}, r_3 = \frac{2}{25}, r_4 = \frac{1}{25}, y_1 = \frac{4}{25}, y_2 = \frac{3}{50}, y_3 = \frac{3}{50}, y_4 = \frac{3}{25}\right\}$,

$\left\{t_1 = \frac{6}{73}, t_2 = \frac{10}{73}, t_3 = \frac{8}{73}, t_4 = \frac{6}{73}, r_1 = \frac{4}{73}, r_2 = \frac{4}{73}, r_3 = \frac{6}{73}, r_4 = \frac{6}{73}, y_1 = \frac{4}{73}, y_2 = \frac{5}{73}, y_3 = \frac{6}{73}, y_4 = \frac{8}{73}\right\}$ and

$\left\{t_1 = \frac{2}{43}, t_2 = \frac{4}{43}, t_3 = \frac{5}{43}, t_4 = \frac{6}{43}, r_1 = \frac{2}{43}, r_2 = \frac{2}{43}, r_3 = \frac{4}{43}, r_4 = \frac{2}{43}, y_1 = \frac{4}{43}, y_2 = \frac{3}{43}, y_3 = \frac{3}{43}, y_4 = \frac{6}{43}\right\}$ respectively.

**Step 4:** Using the optimal solution $\{t_j, r_j, y_j;\; j = 1,2, \dots, n\}$, obtained in Step 3, $K_1^L = \frac{339}{530}$, $K_1^U = \frac{202}{265}$; $K_2^L = \frac{191}{325}$, $K_2^U = \frac{491}{650}$; $K_3^L = \frac{141}{250}$, $K_3^U = \frac{87}{125}$; $K_4^L = \frac{413}{730}$, $K_4^U = \frac{241}{365}$; $K_5^L = \frac{267}{430}$, $K_5^U = \frac{157}{215}$.

**Step 5:** Using Step 5 of the modified method,

$$P = \begin{bmatrix} 0.5 & 0.60127 & 0.74536 & 0.78284 & 0.76806 \\ 0.39872 & 0.5 & 0.63860 & 0.72319 & 0.57499 \\ 0.22139 & 0.36139 & 0.5 & 0.57499 & 0.31110 \\ 0.09509 & 0.27680 & 0.42501 & 0.5 & 0.19301 \\ 0.3906 & 0.51459 & 0.68889 & 0.73532 & 0.5 \end{bmatrix}.$$

**Step 6:** Using Step 6 of the modified method, $\theta_1 = 0.2448765$, $\theta_2 = 0.216775$, $\theta_3 = 0.1734435$, $\theta_4 = 0.1494955$ and $\theta_5 = 0.2164725$ respectively. Furthermore, since, the ranking order is $\theta_1 > \theta_2 > \theta_5 > \theta_3 > \theta_4$, so the PO of the alternatives is $A_1 > A_2 > A_5 > A_3 > A_4$.

## 8.7 Conclusions

The mathematical incorrect assumptions considered in the existing method [56] are pointed out. Also, the impact of these mathematical incorrect assumptions on the solutions of real-life problems is discussed. Furthermore, the required modifications in the existing method [56], to resolve its flaws, are suggested.

# Chapter 9

# Future Scope

The following problems may be considered as challenging open research problems.

1. Zhang [194] proposed the following method for comparing IVPFNs, $\alpha_1 = \langle [a_1, b_1], [c_1, d_1] \rangle$ and $\alpha_2 = \langle [a_2, b_2], [c_2, d_2] \rangle$, where $0 \leq a_1 \leq b_1 \leq 1$, $0 \leq c_1 \leq d_1 \leq 1$, $0 \leq a_2 \leq b_2 \leq 1$, $0 \leq c_2 \leq d_2 \leq 1$, $b_1^2 + d_1^2 \leq 1$ and $b_2^2 + d_2^2 \leq 1$.

   "A set $\tilde{A} = \left\{ \langle x, \left[\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)\right], \left[\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)\right] \rangle \mid x \in X,\ 0 \leq \mu_{\tilde{A}}^L(x) \leq \mu_{\tilde{A}}^U(x) \leq 1, 0 \leq \nu_{\tilde{A}}^L(x) \leq \nu_{\tilde{A}}^U(x) \leq 1,\ \left(\mu_{\tilde{A}}^U(x)\right)^2 + \left(\nu_{\tilde{A}}^U(x)\right)^2 \leq 1 \right\}$, defined on the universal set $X$, is said to be an IVPFS [127], where, $\left[\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)\right]$ and $\left[\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)\right]$ represents the intervals of degree of membership and degree of non-membership respectively of the element $x$ in $\tilde{A}$."

   **Step 1:** Find $S(\alpha_1) = \frac{a_1^2 + b_1^2 - c_1^2 - d_1^2}{2}$ and $S(\alpha_2) = \frac{a_2^2 + b_2^2 - c_2^2 - d_2^2}{2}$ and check that $S(\alpha_1) > S(\alpha_2)$ or $S(\alpha_1) < S(\alpha_2)$ or $S(\alpha_1) = S(\alpha_2)$.

   **Case (i)** If $S(\alpha_1) > S(\alpha_2)$ then $\alpha_1 > \alpha_2$.

   **Case (ii)** If $S(\alpha_1) < S(\alpha_2)$ then $\alpha_1 < \alpha_2$.

   **Case (iii)** If $S(\alpha_1) = S(\alpha_2)$ then go to Step 2.

   **Step 2:** Find $H(\alpha_1) = \frac{a_1^2 + b_1^2 + c_1^2 + d_1^2}{2}$ and $H(\alpha_2) = \frac{a_2^2 + b_2^2 + c_2^2 + d_2^2}{2}$ and check that $H(\alpha_1) > H(\alpha_2)$ or $H(\alpha_1) < H(\alpha_2)$ or $H(\alpha_1) = H(\alpha_2)$.

   **Case (i)** If $H(\alpha_1) > H(\alpha_2)$ then $\alpha_1 > \alpha_2$.

   **Case (ii)** If $H(\alpha_1) < H(\alpha_2)$ then $\alpha_1 < \alpha_2$.

   **Case (iii)** If $H(\alpha_1) = H(\alpha_2)$ then $\alpha_1 = \alpha_2$.

Garg [48] pointed out that there can exist two different IVPFNs, $\alpha_1 = \langle [a_1, b_1], [c_1, d_1] \rangle$ and $\alpha_2 = \langle [a_2, b_2], [c_2, d_2] \rangle$, where $0 \le a_1 \le b_1 \le 1$, $0 \le c_1 \le d_1 \le 1$, $0 \le a_2 \le b_2 \le 1$, $0 \le c_2 \le d_2 \le 1$, $b_1^2 + d_1^2 \le 1$ and $b_2^2 + d_2^2 \le 1$ such that $S(\alpha_1) = S(\alpha_2)$ as well as $H(\alpha_1) = H(\alpha_2)$.

For example, if $\alpha_1 = \langle [0.2, 0.3], [0,0] \rangle$ and $\alpha_2 = \langle [0, \sqrt{0.13}], [0,0] \rangle$ are two IVPFNs, then $S(\alpha_1) = S(\alpha_2) = 0.65$ and $H(\alpha_1) = H(\alpha_2) = 0.65$. While, $\alpha_1 \neq \alpha_2$.

Therefore, it is inappropriate to use Zhang's method [194] for comparing IVPFNs.

To handle this situation, Garg [48, Section 3, p. 536] proposed the following method for comparing IVPFNs.

**Step 1:** Find $S(\alpha_1) = \frac{a_1^2 + b_1^2 - c_1^2 - d_1^2}{2}$ and $S(\alpha_2) = \frac{a_2^2 + b_2^2 - c_2^2 - d_2^2}{2}$ and check that $S(\alpha_1) > S(\alpha_2)$ or $S(\alpha_1) < S(\alpha_2)$ or $S(\alpha_1) = S(\alpha_2)$.

**Case (i)** If $S(\alpha_1) > S(\alpha_2)$ then $\alpha_1 > \alpha_2$.

**Case (ii)** If $S(\alpha_1) < S(\alpha_2)$ then $\alpha_1 < \alpha_2$.

**Case (iii)** If $S(\alpha_1) = S(\alpha_2)$ then go to Step 2.

**Step 2:** Find $H(\alpha_1) = \frac{a_1^2 + b_1^2 + c_1^2 + d_1^2}{2}$ and $H(\alpha_2) = \frac{a_2^2 + b_2^2 + c_2^2 + d_2^2}{2}$ and check that $H(\alpha_1) > H(\alpha_2)$ or $H(\alpha_1) < H(\alpha_2)$ or $H(\alpha_1) = H(\alpha_2)$.

**Case (i)** If $H(\alpha_1) > H(\alpha_2)$ then $\alpha_1 > \alpha_2$.

**Case (ii)** If $H(\alpha_1) < H(\alpha_2)$ then $\alpha_1 < \alpha_2$.

**Case (iii)** If $H(\alpha_1) = H(\alpha_2)$ then go to Step 3.

**Step 3:** Find $M(\alpha_1) = \frac{a_1^2 - \sqrt{1 - a_1^2 - c_1^2} + b_1^2 - \sqrt{1 - b_1^2 - d_1^2}}{2}$ and $M(\alpha_2) = \frac{a_2^2 - \sqrt{1 - a_2^2 - c_2^2} + b_2^2 - \sqrt{1 - b_2^2 - d_2^2}}{2}$ and check that $M(\alpha_1) > M(\alpha_2)$ or $M(\alpha_1) < M(\alpha_2)$ or $M(\alpha_1) = M(\alpha_2)$.

**Case (i)** If $M(\alpha_1) > M(\alpha_2)$ then $\alpha_1 > \alpha_2$.

**Case (ii)** If $M(\alpha_1) < M(\alpha_2)$ then $\alpha_1 < \alpha_2$.

**Case (iii)** If $M(\alpha_1) = M(\alpha_2)$ then $\alpha_1 = \alpha_2$.

Garg [49] also proposed the following method for ranking two IVPFNs, $\alpha_1 = \langle [a_1, b_1], [c_1, d_1] \rangle$ and $\alpha_2 = \langle [a_2, b_2], [c_2, d_2] \rangle$.

Find, $K(\alpha_1) = \frac{a_1^2 + b_1^2\sqrt{1 - a_1^2 - c_1^2} + b_1^2 + a_1^2\sqrt{1 - b_1^2 - d_1^2}}{2}$ as well as $K(\alpha_2) = \frac{a_2^2 + b_2^2\sqrt{1 - a_2^2 - c_2^2} + b_2^2 + a_2^2\sqrt{1 - b_2^2 - d_2^2}}{2}$ and check that $K(\alpha_1) < K(\alpha_2)$ or $K(\alpha_1) > K(\alpha_2)$ or $K(\alpha_1) = K(\alpha_2)$.

**Case (i)** If $K(\alpha_1) < K(\alpha_2)$ then $\alpha_1 < \alpha_2$.

**Case (ii)** If $K(\alpha_1) > K(\alpha_2)$ then $\alpha_1 > \alpha_2$.

**Case (iii)** If $K(\alpha_1) = K(\alpha_2)$ then $\alpha_1 = \alpha_2$.

However, the following example clearly indicates that the methods for comparing two IVPFNs, proposed by Garg [48, Section 3, p. 536], are also not valid.

(i) Let $\alpha_1 = \langle [\sqrt{0.02}, \sqrt{0.07}], [0.2, \sqrt{0.05}] \rangle$ and $\alpha_2 = \langle [0.2, \sqrt{0.05}], [\sqrt{0.02}, \sqrt{0.07}] \rangle$ be two IVPFNs. It is obvious that $\alpha_1 \neq \alpha_2$. But, as $S(\alpha_1) = S(\alpha_2) = 0$, $H(\alpha_1) = H(\alpha_2) = 0.09$ and $M(\alpha_1) = M(\alpha_2) = -0.908809561$. Therefore, according to the first RM, proposed by Garg [50], $\alpha_1 = \alpha_2$, which is mathematical incorrect. This clearly indicates that the flaw, pointed out by Garg [48] in the existing method [194], is also occurring in Garg's method [48]. Therefore, the first RM for comparing IVPFNs, proposed by Garg [48] is also not valid.

(ii) It is obvious that infinite number of IVPFNs can be obtained by changing the values of $c$ and $d$ in $\alpha = \langle [0,0], [c, d] \rangle$, where, $0 \leq c \leq 1$ and $0 \leq d \leq 1$. Also, for all these infinite numbers of IVPFNs, the value of $K(\alpha) = \frac{a^2 + b^2\sqrt{1 - a^2 - c^2} + b^2 + a^2\sqrt{1 - b^2 - d^2}}{2}$ will be zero. This clearly indicates according to the second RM, proposed by Garg [49], all these infinite IVPFNs are equal. While, it is obvious that these infinite IVPFNs are not equal. For example, choosing $c = 0.4$, $d = 0.6$ and $c = 0.6$, $d = 0.8$, two

IVPFNs $\alpha_1 = \langle [0,0], [0.4,0.6] \rangle$ and $\alpha_2 = \langle [0,0], [0.6,0.8] \rangle$ are obtained, it is obvious that $\alpha_1 \neq \alpha_2$. While, $K(\alpha_1) = K(\alpha_2) = 0$. Therefore, the second RM, proposed by Garg [49], is not valid.

Hence, to propose a valid accuracy function for comparing IVPFNs may be considered as a challenging open research problem.

2. Sahin [137] proposed the following method for comparing two SVNSs, $A_1 = \langle a_1, b_1, c_1 \rangle$ and $A_2 = \langle a_2, b_2, c_2 \rangle$.

Find $K(A_1) = \frac{1+a_1-2b_1-c_1}{2}$ and $K(A_2) = \frac{1+a_2-2b_2-c_2}{2}$ and check that $K(A_1) > K(A_2)$ or $K(A_1) < K(A_2)$ or $K(A_1) = K(A_2)$.

**Case (i)** If $K(A_1) > K(A_2)$ then $A_1 > A_2$.

**Case (ii)** If $K(A_1) < K(A_2)$ then $A_1 < A_2$.

**Case (iii)** If $K(A_1) = K(A_2)$ then $A_1 = A_2$.

Nancy and Garg [123, Section 2, Def. 2.6, Ex. 2.1, p. 379] considered two different SVNSs, $A_1 = \langle 0.5,0.2,0.6 \rangle$ and $A_2 = \langle 0.2,0.2,0.3 \rangle$ to show that on considering the existing method [137], the relation $A_1 = A_2$ is obtained. While, it is obvious that $A_1 \neq A_2$. On the basis of this numerical example, Nancy and Garg [123, Section 2, Def. 2.6, Ex. 2.1, p. 379] claimed that the existing method [137] for comparing SVNSs is not valid.

Sahin [137] also proposed the following method for comparing of two IVNSs, $A_1 = \langle [a_1^L, a_1^U], [b_1^L, b_1^U], [c_1^L, c_1^U] \rangle$ and $A_2 = \langle [a_2^L, a_2^U], [b_2^L, b_2^U], [c_2^L, c_2^U] \rangle$.

Find $L(A_1) = \frac{2+a_1^L+a_1^U-2b_1^L-2b_1^U-c_1^L-c_1^U}{4}$ and $L(A_2) = \frac{2+a_2^L+a_2^U-2b_2^L-2b_2^U-c_2^L-c_2^U}{4}$, and check that $L(A_1) > L(A_2)$ or $L(A_1) < L(A_2)$ or $L(A_1) = L(A_2)$.

**Case (i)** If $L(A_1) > L(A_2)$ then $A_1 > A_2$.

**Case (ii)** If $L(A_1) < L(A_2)$ then $A_1 < A_2$.

**Case (iii)** If $L(A_1) = L(A_2)$ then $A_1 = A_2$.

It is pertinent to mention that the SVNS, $A_1 = \langle 0.5,0.2,0.6\rangle$ and $A_2 = \langle 0.2,0.2,0.3\rangle$ can also be represented as IVNS, $A_1 = \langle [0.5,0.5],[0.2,0.2],[0.6,0.6]\rangle$ and $A_2 = \langle [0.2,0.2],[0.2,0.2],[0.3,0.3]\rangle$. It can be verified that on considering the existing method [137], the relation $A_1 = A_2$ is obtained. While, it is obvious that $A_1 \neq A_2$. Hence, the existing method [137] for comparing IVNSs is also not valid.

To resolve these shortcomings of the existing methods [137], Nancy and Garg [123, Section 3, Def. 3.1, p. 379] proposed the following method for comparing two SVNSs, $A_1 = \langle a_1, b_1, c_1\rangle$ and $A_2 = \langle a_2, b_2, c_2\rangle$.

Find $N(A_1) = \frac{1+(a_1-2b_1-c_1)(2-a_1-c_1)}{2}$ and $N(A_2) = \frac{1+(a_2-2b_2-c_2)(2-a_2-c_2)}{2}$ and check that $N(A_1) > N(A_2)$ or $N(A_1) < N(A_2)$ or $N(A_1) = N(A_2)$.

**Case (i)** If $N(A_1) > N(A_2)$ then $A_1 > A_2$.

**Case (ii)** If $N(A_1) < N(A_2)$ then $A_1 < A_2$.

**Case (iii)** If $N(A_1) = N(A_2)$ then $A_1 = A_2$.

Furthermore, Nancy and Garg [123, Section 3, Def. 3.2, p. 381] proposed the following method for comparing two IVNSs, $A_1 = \langle [a_1^L, a_1^U],[b_1^L, b_1^U],[c_1^L c_1^U]\rangle$ and $A_2 = \langle [a_2^L, a_2^U],[b_2^L, b_2^U],[c_2^L c_2^U]\rangle$. Find $M(A_1) = \frac{4+(a_1^L+a_1^U-c_1^L-c_1^U-2b_1^L-2b_1^U)(4-a_1^L-a_1^U-c_1^L-c_1^U)}{8}$ and $M(A_2) = \frac{4+(a_2^L+a_2^U-c_2^L-c_2^U-2b_2^L-2b_2^U)(4-a_2^L-a_2^U-c_2^L-c_2^U)}{8}$, and check that $M(A_1) > M(A_2)$ or $M(A_1) < M(A_2)$ or $M(A_1) = M(A_2)$.

**Case (i)** If $M(A_1) > M(A_2)$ then $A_1 > A_2$.

**Case (ii)** If $M(A_1) < M(A_2)$ then $A_1 < A_2$.

**Case (iii)** If $M(A_1) = M(A_2)$ then $A_1 = A_2$.

However, the following example clearly indicates that the existing methods for comparing SVNSs and IVNSs are not valid.

(i) Let $A_1 = \langle 0.8,0.1,0.6 \rangle$ and $A_2 = \langle 0.8,0.2,0.4 \rangle$ be two SVNSs. Then, according to the existing method [123, Section 3, Def. 3.1, p. 379], $N(A_1) = N(A_2) = 0.5$. Therefore, according to the existing method [123, Section 3, Def. 3.1, p. 379], $A_1 = A_2$. While, it is obvious that $A_1 \neq A_2$. So, the existing method [123, Section 3, Def. 3.1, p. 379] for comparing two SVNSs is not valid.

(ii) Let $A_1 = \langle [0.1,0.7],[0.05,0.15],[0.1,0.3] \rangle$ and $A_2 = \langle [0.2,0.8],[0.05,0.15],[0.2,0.4] \rangle$ be any two IVNSs. Then, according to the existing method [123, Section 3, Def. 3.2, p. 381], $M(A_1) = M(A_2) = 0.5$. Therefore, according to the existing method [123, Section 3, Def. 3.2, p. 381], $A_1 = A_2$. While, it is obvious that $A_1 \neq A_2$. So, the existing method [123, Section 3, Def. 3.2, p. 381] for comparing two IVNSs is not valid.

Hence, to propose the valid methods for comparing two SVNSs and IVNSs may be considered as challenging future research problems.

3. Garg and Kumar [53] discussed a brief review of the existing CoCf [21, 27, 47, 58, 74, 75, 101, 165, 173, 177]. Furthermore, Garg and Kumar [53, Section 4, Eq. 13, p. 8] used the existing expression (9.1) [192] to evaluate the CoCf of three known patterns, $A_1 = \{\langle x_1, 0.4,0.5 \rangle, \langle x_2, 0.7,0.1 \rangle, \langle x_3, 0.3,0.3 \rangle\}$, $A_2 = \{\langle x_1, 0.5,0.4 \rangle, \langle x_2, 0.7,0.2 \rangle, \langle x_3, 0.4,0.3 \rangle\}$ and $A_3 = \{\langle x_1, 0.4,0.5 \rangle, \langle x_2, 0.7,0.1 \rangle, \langle x_3, 0.4,0.3 \rangle\}$ with the unknown pattern, $B = \{\langle x_1, 0.1,0.1 \rangle, \langle x_2, 1.0,0.0 \rangle, \langle x_3, 0.0,1.0 \rangle\}$ and shown that the obtained CoCf between $A_1$ and $B$, the obtained CoCf between $A_2$ and $B$ as well as the obtained CoCf between $A_3$ and $B$ are equal. Therefore, the expression (9.1) [192] cannot be used to identify a suitable pattern, for the unknown pattern $B$, from the known patterns $A_1, A_2$ and $A_3$.

$$ZL(A,B) = \frac{\sum_{t=1}^{n}\left(u_A(x_t)u_B(x_t)+v_A(x_t)v_B(x_t)+\pi_A(x_t)\pi_B(x_t)\right)}{\sqrt{\sum_{t=1}^{n}\left(u_A^2(x_t)+v_A^2(x_t)+\pi_A^2(x_t)\right).\sum_{t=1}^{n}\left(u_B^2(x_t)+v_B^2(x_t)+\pi_B^2(x_t)\right)}} \tag{9.1}$$

Garg and Kumar [53, Section 4, Eq. 14, p. 9] used the existing expression (9.2) [154] to evaluate the CoCf between the IFSs, $A = \{\langle x_1, 0.1,0.2\rangle, \langle x_2, 0.2,0.1\rangle, \langle x_3, 0.29,0.0\rangle\}$ and $B = \{\langle x_1, 0.1,0.3\rangle, \langle x_2, 0.2,0.2\rangle, \langle x_3, 0.29,0.1\rangle\}$ and shown that the obtained CoCf between $A$ and $B$ is 1, which is mathematically incorrect as it indicates that the IFSs $A$ and $B$ are equal. Whereas, it is obvious that both the IFSs are not equal. Therefore, the existing expression (9.2) [154] cannot be used to obtain CoCf between $A$ and $B$.

$$r(A,B) = \frac{r_1(A,B)+r_2(A,B)+r_3(A,B)}{3} \tag{9.2}$$

where,

$$r_1(A,B) = \frac{\sum_{t=1}^{n}(u_A(x_t)-\overline{u_A})(u_B(x_t)-\overline{u_B})}{\sqrt{\sum_{t=1}^{n}(u_A(x_t)-\overline{u_A})^2 . \sum_{t=1}^{n}(u_B(x_t)-\overline{u_B})^2}},$$

$$r_2(A,B) = \frac{\sum_{t=1}^{n}(v_A(x_t)-\overline{v_A})(v_B(x_t)-\overline{v_B})}{\sqrt{\sum_{t=1}^{n}(v_A(x_t)-\overline{v_A})^2 . \sum_{t=1}^{n}(v_B(x_t)-\overline{v_B})^2}},$$

$$r_3(A,B) = \frac{\sum_{t=1}^{n}(\pi_A(x_t)-\overline{\pi_A})(\pi_B(x_t)-\overline{\pi_B})}{\sqrt{\sum_{t=1}^{n}(\pi_A(x_t)-\overline{\pi_A})^2 . \sum_{t=1}^{n}(\pi_B(x_t)-\overline{\pi_B})^2}},$$

$$\overline{u_A} = \frac{1}{n}\sum_{t=1}^{n} u_A(x_t),\ \overline{u_B} = \frac{1}{n}\sum_{t=1}^{n} u_B(x_t),$$

$$\overline{v_A} = \frac{1}{n}\sum_{t=1}^{n} v_A(x_t),\ \overline{v_B} = \frac{1}{n}\sum_{t=1}^{n} v_B(x_t),$$

$$\overline{\pi_A} = \frac{1}{n}\sum_{t=1}^{n} \pi_A(x_t),\ \overline{\pi_B} = \frac{1}{n}\sum_{t=1}^{n} \pi_B(x_t).$$

Garg and Kumar [53, Section 4, Eq. 15, p. 10] used the existing expression (9.3) [177] to evaluate the CoCf between the three known patterns, $A_1 = \{\langle x_1, 0.4,0.5\rangle, \langle x_2, 0.7,0.1\rangle, \langle x_3, 0.3,0.3\rangle\}$, $A_2 = \{\langle x_1, 0.5,0.4\rangle, \langle x_2, 0.7,0.2\rangle, \langle x_3, 0.4,0.3\rangle\}$ and $A_3 = \{\langle x_1, 0.4,0.5\rangle, \langle x_2, 0.7,0.1\rangle, \langle x_3, 0.4,0.3\rangle\}$ with the unknown pattern, $B = \{\langle x_1, 0.1,0.1\rangle, \langle x_2, 1.0,0.0\rangle, \langle x_3, 0.0,1.0\rangle\}$ and shown that the obtained CoCf between $A_1$ and $B$, the obtained CoCf between $A_2$ and $B$ as well as the obtained CoCf between $A_3$ and

$B$ are equal. Therefore, the expression (9.3) [177] cannot be used to identify a suitable pattern, for the unknown pattern $B$, from the known patterns $A_1, A_2$ and $A_3$.

$$Xu_1(A,B) = \frac{\sum_{t=1}^{n}\left(u_A(x_t)u_B(x_t)+v_A(x_t)v_B(x_t)+\pi_A(x_t)\pi_B(x_t)\right)}{max\left\{\sum_{t=1}^{n}\left(u_A^2(x_t)+v_A^2(x_t)+\pi_A^2(x_t)\right).\ \sum_{t=1}^{n}\left(u_B^2(x_t)+v_B^2(x_t)+\pi_B^2(x_t)\right)\right\}} \tag{9.3}$$

To overcome this limitation of the existing expressions $(9.1) - (9.3)$, Garg and Kumar [53], firstly, proposed the expression (9.4) [53, Section 3, Eq. 6, p. 4] to transform an an IFN $\langle \mu_p(x_t), \nu_p(x_t)\rangle$ into a CN $a_p(x_t) + b_p(x_t)i + c_p(x_t)j$ [28]. Then, using the proposed the expression (9.4), Garg and Kumar [53] proposed the expression (9.5) [53, Section 3, Eq. 9, p. 5], the expression (9.6) [53, Section 3, Eq. 10, p. 7] for evaluating the CoCf and the expression (9.7) [53, Section 3, Eq. 11, p. 7], the expression (9.8) [53, Section 3, Eq. 12, p. 7], for evaluating the weighted CoCf between two IFSs $A = \{\langle \mu_1(x_t), \nu_1(x_t)\rangle\}$ and $B = \{\langle \mu_2(x_t), \nu_2(x_t)\rangle,\}$ $t = 1,2,\ldots,n$.

$$a_p(x_t) = \mu_p(x_t)\left(1-\nu_p(x_t)\right), \qquad b_p(x_t) = 1-\mu_p(x_t)\left(1-\nu_p(x_t)\right)-\nu_p(x_t)\left(1-\mu_p(x_t)\right) \text{ and } c_p(x_t) = \nu_p(x_t)\left(1-\mu_p(x_t)\right) \tag{9.4}$$

$$K_1(A,B) = \frac{\sum_{t=1}^{n}\left(a_1(x_t)a_2(x_t)+b_1(x_t)b_2(x_t)+c_1(x_t)c_2(x_t)\right)}{\sqrt{\sum_{t=1}^{n}\left(a_1^2(x_t)+b_1^2(x_t)+c_1^2(x_t)\right).\ \sum_{t=1}^{n}\left(a_2^2(x_t)+b_2^2(x_t)+c_2^2(x_t)\right)}} \tag{9.5}$$

$$K_2(A,B) = \frac{\sum_{t=1}^{n}\left(a_1(x_t)a_2(x_t)+b_1(x_t)b_2(x_t)+c_1(x_t)c_2(x_t)\right)}{max\left\{\sum_{t=1}^{n}\left(a_1^2(x_t)+b_1^2(x_t)+c_1^2(x_t)\right),\ \sum_{t=1}^{n}\left(a_2^2(x_t)+b_2^2(x_t)+c_2^2(x_t)\right)\right\}} \tag{9.6}$$

$$K_3(A,B) = \frac{\sum_{t=1}^{n} w_t\left(a_1(x_t)a_2(x_t)+b_1(x_t)b_2(x_t)+c_1(x_t)c_2(x_t)\right)}{\sqrt{\sum_{t=1}^{n} w_t\left(a_1^2(x_t)+b_1^2(x_t)+c_1^2(x_t)\right).\ \sum_{t=1}^{n} w_t\left(a_2^2(x_t)+b_2^2(x_t)+c_2^2(x_t)\right)}} \tag{9.7}$$

$$K_4(A,B) = \frac{\sum_{t=1}^{n} w_t\left(a_1(x_t)a_2(x_t)+b_1(x_t)b_2(x_t)+c_1(x_t)c_2(x_t)\right)}{max\left\{\sum_{t=1}^{n} w_t\left(a_1^2(x_t)+b_1^2(x_t)+c_1^2(x_t)\right),\ \sum_{t=1}^{n} w_t\left(a_2^2(x_t)+b_2^2(x_t)+c_2^2(x_t)\right)\right\}} \tag{9.8}$$

However, the following clearly indicates that the shortcoming, pointed out by Garg and Kumar [53] in existing methods [154, 177, 192], is also occurring in Garg and Kumar's method [53].

Let us consider two known patterns $A_1 = \{\langle x_1, 0.1,0.4\rangle, \langle x_2, 0.4,0.3\rangle, \langle x_3, 0.25,0.35\rangle\}$, $A_2 = \{\langle x_1, 0.4,0.1\rangle, \langle x_2, 0.3,0.4\rangle, \langle x_3, 0.35,0.25\rangle\}$ and an unknown pattern $B = \{\langle x_1, 0.3,0.3\rangle, \langle x_2, 0.2,0.2\rangle, \langle x_3, 0.1,0.1\rangle\}$, represented by IFSs. Also, let the weights of a relative importance be $(0.40, 0.45,0.15)$.

To apply the CoCfs $(9.5) - (9.8)$ [53], proposed by Garg and Kumar [53], firstly, there is need to transform each element of $A_1, A_2$ and $B$ into a CN.

Using the expression (9.4), proposed by Garg and Kumar [53] for transforming an IFN into a CN, the elements $\langle 0.1,0.4\rangle$, $\langle 0.4,0.3\rangle$, $\langle 0.25,0.35\rangle$, $\langle 0.4,0.1\rangle$, $\langle 0.3,0.4\rangle$, $\langle 0.35,0.25\rangle$, $\langle 0.3,0.3\rangle$, $\langle 0.2,0.2\rangle$ and $\langle 0.1,0.1\rangle$ can be transformed into its equivalent CNs $\langle 0.06,0.58,0.36\rangle$, $\langle 0.28,0.54,0.18\rangle$, $\langle 0.1625,0.575,0.2625\rangle$, $\langle 0.36,0.58,0.06\rangle$, $\langle 0.18,0.54,0.28\rangle$, $\langle 0.2625,0.575,0.1625\rangle$, $\langle 0.21,0.58,0.21\rangle$, $\langle 0.16,0.68,0.16\rangle$ and $\langle 0.09,0.82,0.09\rangle$ respectively. Therefore, $A_1$, $A_2$ and $B$ in terms of CNs can be rewritten as

$A_1 = \{\langle x_1, 0.06,0.58,0.36\rangle, \langle x_2, 0.28,0.54,0.18\rangle, \langle x_3, 0.1625,0.575,0.2625\rangle\}$,

$A_2 = \{\langle x_1, 0.36,0.58,0.06\rangle, \langle x_2, 0.18,0.54,0.28\rangle, \langle x_3, 0.2625,0.575,0.1625\rangle\}$,

and,

$B = \{\langle x_1, 0.21,0.58,0.21\rangle, \langle x_2, 0.16,0.68,0.16\rangle, \langle x_3, 0.09,0.82,0.09\rangle\}$.

Now,

(i) Using the existing expression (9.5), proposed by Garg and Kumar [53] for evaluating the CoCf between IFSs, $K_1(A_1, B) = 0.946359402$ and $K_1(A_2, B) = 0.946359402$. Since $K_1(A_1, B) = K_1(A_2, B)$ so it is not possible to identify the suitable classifier for the unknown pattern $B$ from the known patterns $A_1$ and $A_2$.

Hence, the flaw, pointed out by Garg and Kumar [53] in the existing CoCfs $(9.1) - (9.3)$, is also occurring in Garg and Kumar's expression (9.5) [53].

(ii) Using the existing expression (9.6), proposed by Garg and Kumar [53] for evaluating the CoCf between the IFS,

$K_2(A_1, B) = 0.84530981$ and $K_2(A_2, B) = 0.84530981$.

Since $K_2(A_1, B) = K_2(A_2, B)$ so it is not possible to identify the suitable classifier for the unknown pattern $B$ from the known patterns $A_1$ and $A_2$.

Hence, the flaw, pointed out by Garg and Kumar [53] in the existing CoCf $(9.1) - (9.3)$, is also occurring in Garg and Kumar's expression (9.6) [53].

(iii) Using the existing expression (9.7), proposed by Garg and Kumar [53] for evaluating the CoCf between the IFSs,

$K_3(A_1, B) = 0.951828261$ and $K_3(A_2, B) = 0.951828261$.

Since $K_3(A_1, B) = K_3(A_2, B)$ so it is not possible to identify the suitable classifier for the unknown pattern $B$ from the known patterns $A_1$ and $A_2$.

Hence, the flaw, pointed out by Garg and Kumar [53] in the existing CoCfs $(9.1) - (9.3)$, is also occurring in Garg and Kumar's expression (9.7) [53].

(iv) Using the existing expression (9.8), proposed by Garg and Kumar [53] for evaluating the CoCf between the IFSs,

$K_4(A_1, B) = 0.881829449$ and $K_4(A_2, B) = 0.881829449$.

Since $K_4(A_1, B) = K_4(A_2, B)$ so it is not possible to identify the suitable classifier for the unknown pattern $B$ from the known patterns $A_1$ and $A_2$.

Hence, the flaw, pointed out by Garg and Kumar [53] in the existing CoCfs $(9.1) - (9.3)$, is also occurring in Garg and Kumar's expression (9.8) [53].

4. In the last few years, several methods have been proposed in the literature to find the ranking of the alternatives for such MADMPrs in which RV of each alternative over each criterion is represented as an IFS. Garg and Kumar [54] pointed out that the following concept can be used to find the ranking of alternatives such that,

"Construct the IFS $P_i$ whose elements are the elements of the $i^{th}$ row of the IFDM. Then, find the distance of each $P_i$ from the fixed IFS (goal) and use these distances to rank the alternatives."

But, on applying the existing DiMs $(9.9) - (9.13)$ [39, 78, 153, 169] to measure the distance of a fixed IFS with other distinct IFSs, the obtained distances are equal. Therefore, it is inappropriate to use the existing DiMs $(9.9) - (9.13)$ [39, 78, 153, 169] for the same purpose.

(i) To validate this claim, Garg and Kumar [54, Section 2, Def. 2, Eq. 1, Eq. 4 and Eq. 5] used the DiM $(9.9)$ [169], the DiM $(9.10)$ [153] and the DiM $(9.11)$ [78] to evaluate the distance of two known patterns $A = \{\langle x_1, 0.5,0.3\rangle, \langle x_2, 0.5,0.2\rangle, \langle x_3, 0.9,0.0\rangle, \langle x_4, 0.5,0.4\rangle, \langle x_5, 0.7,0.1\rangle\}$ and $B = \{\langle x_1, 0.7,0.2\rangle, \langle x_2, 0.5,0.4\rangle, \langle x_3, 0.9,0.1\rangle, \langle x_4, 0.6,0.3\rangle, \langle x_5, 0.8,0.0\rangle\}$ with the unknown pattern $P = \{\langle x_1, 0.7,0.1\rangle, \langle x_2, 0.6,0.3\rangle, \langle x_3, 0.7,0.1\rangle, \langle x_4, 0.5,0.4\rangle, \langle x_5, 0.4,0.5\rangle\}$ and shown that the obtained distance between $A$ and $P$ as well as the obtained distance between $B$ and $P$ are equal. Therefore, none of the DiM $(9.9) - (9.11)$ [78, 153, 169] can be used to identify a suitable pattern for the unknown pattern $P$ from the known patterns $A$ and $B$.

$$d'_{IFS}(A,B) = \frac{1}{2n}\sum_{t=1}^{n}\{|\mu_A(x_t) - \mu_B(x_t)| + |\nu_A(x_t) - \nu_B(x_t)| + |\pi_A(x_t) - \pi_B(x_t)|\} \tag{9.9}$$

$$d^H_{IFS}(A,B) = \frac{1}{n}\sum_{t=1}^{n} max\{|\mu_A(x_t) - \mu_B(x_t)|, |\nu_A(x_t) - \nu_B(x_t)|, |\pi_A(x_t) - \pi_B(x_t)|\} \tag{9.10}$$

$$d^S_{IFS} = 1 - S(A,B) \tag{9.11}$$

where,

$S(A,B)=\frac{\sum_{t=1}^{n}\left(1-\frac{1}{2}\{|\mu_A(x_t)-\mu_B(x_t)|+|\nu_A(x_t)-\nu_B(x_t)|\}\right)}{n}$, $\pi_A(x_t)=1-\mu_A(x_t)-\nu_A(x_t)$, $\pi_B(x_t)=1-\mu_B(x_t)-\nu_B(x_t)$.

(ii) Garg and Kumar [54, Section 2, Def. 2, Eq. 2 and Eq. 3] also used the DiM (9.12) [169] and the DiM (9.13) [39] to evaluate the distance of two unknown patterns, $A=\{\langle x_1,0.9,0.1\rangle,\langle x_2,0.8,0.1\rangle,\langle x_3,0.8,0.1\rangle,\langle x_4,0.5,0.3\rangle,\langle x_5,0.7,0.2\rangle\}$ and $B=\{\langle x_1,0.7,0.2\rangle,\langle x_2,0.5,0.4\rangle,\langle x_3,0.9,0.1\rangle,\langle x_4,0.6,0.3\rangle,\langle x_5,0.8,0.0\rangle\}$ with the known pattern $P=\{\langle x_1,0.5,0.3\rangle,\langle x_2,0.6,0.2\rangle,\langle x_3,0.5,0.3\rangle,\langle x_4,0.4,0.5\rangle,\langle x_5,0.7,0.2\rangle\}$ and shown that the obtained distance between $A$ and $P$ as well as the obtained distance between $B$ and $P$ are equal. Therefore, none of the DiMs (9.12) − (9.13) [39, 169] can be used to identify a suitable pattern for the unknown pattern $P$ from the known patterns $A$ and $B$.

$$d_{IFS}^{H}(A,B)=\left(\frac{1}{2n}\sum_{t=1}^{n}\{|\mu_A(x_t)-\mu_B(x_t)|^2+|\nu_A(x_t)-\nu_B(x_t)|^2+|\pi_A(x_t)-\pi_B(x_t)|^2\}\right)^{\frac{1}{2}} \quad (9.12)$$

$$d_{IFS}^{N}(A,B)=\frac{1}{2n}\sum_{t=1}^{n}\Big[|\mu_A(x_t)-\mu_B(x_t)|+\big||\mu_A(x_t)-\nu_A(x_t)|-|\mu_B(x_t)-\nu_B(x_t)|\big|+\big||\mu_A(x_t)-\pi_A(x_t)|-|\mu_B(x_t)-\pi_B(x_t)|\big|\Big] \quad (9.13)$$

(iii) Furthermore, Garg and Kumar [54] considered a real-life MADMPr under IF environment and shown that if the existing DiMs (9.9) − (9.11) [78, 153, 169] are used to solve this problem then the obtained results are not appropriate.

To overcome this shortcoming of the existing DiMs (9.9) − (9.13) [39, 78, 153, 169], Garg and Kumar [54] proposed the DiM (9.14) [54, Section 3, Def. 6, Eq. 9], the DiM (9.15) [54, Section 3, Def. 6, Eq. 10], the DiM (9.16) [54, Section 3, Def. 6, Eq. 11], the DiM (9.17) [54, Section 3, Def. 6, Eq. 12] and the DiM (9.18) [54, Section 3, Proposition 6, Eq. 13], the DiM $(9.19)$ [54, Section 3, Proposition 6, Eq. 14], the DiM $(9.20)$ [54, Section 3, Proposition 6, Eq. 15], the DiM $(9.21)$ [54, Section 3, Proposition 6, Eq. 16], the DiM $(9.22)$ [54, Section 3, Proposition 7, Eq. 17], the DiM $(9.23)$ [54, Section 3, Proposition 7, Eq. 18], the DiM $(9.24)$ [54, Section 3, Proposition 7, Eq. 19], the DiM $(9.25)$ [54, Section 3, Proposition 7, Eq. 20] for evaluating the distance and the weighted distance respectively between two IFSs $A = \{\langle \mu_A(x_t), \nu_A(x_t) \rangle\}$ and $B = \{\langle \mu_B(x_t), \nu_B(x_t) \rangle\}$ $t = 1,2, \ldots, n$.

All the DiMs $(9.14) - (9.25)$ [54] are based upon CNs $a_A(x_t) + b_A(x_t)i + c_A(x_t)j$ and $a_B(x_t) + b_B(x_t)i + c_B(x_t)j$ [28] which are obtained corresponding to the IFNs $\langle \mu_A(x_t), \nu_A(x_t) \rangle$ and $\langle \mu_B(x_t), \nu_B(x_t) \rangle$ respectively by considering the existing expressions (9.26) $-$ (9.31) [98],

$$d_1(A,B) = \frac{1}{3}\sum_{t=1}^{n}\{|a_A(x_t) - a_B(x_t)| + |b_A(x_t) - b_B(x_t)| + |c_A(x_t) - c_B(x_t)|\} \tag{9.14}$$

$$d_2(A,B) = \frac{1}{3n}\sum_{t=1}^{n}\{|a_A(x_t) - a_B(x_t)| + |b_A(x_t) - b_B(x_t)| + |c_A(x_t) - c_B(x_t)|\} \tag{9.15}$$

$$d_3(A,B) = \left[\frac{1}{3}\sum_{t=1}^{n}\{|a_A(x_t) - a_B(x_t)|^2 + |b_A(x_t) - b_B(x_t)|^2 + |c_A(x_t) - c_B(x_t)|^2\}\right]^{\frac{1}{2}} \tag{9.16}$$

$$d_4(A,B) = \left[\frac{1}{3n}\sum_{t=1}^{n}\{|a_A(x_t) - a_B(x_t)|^2 + |b_A(x_t) - b_B(x_t)|^2 + |c_A(x_t) - c_B(x_t)|^2\}\right]^{\frac{1}{2}} \tag{9.17}$$

$$d_5(A,B) = \frac{1}{3}\sum_{t=1}^{n}\omega_t\{|a_A(x_t) - a_B(x_t)| + |b_A(x_t) - b_B(x_t)| + |c_A(x_t) - c_B(x_t)|\} \tag{9.18}$$

$$d_6(A,B) = \left[\frac{1}{3}\sum_{t=1}^{n}\omega_t\{|a_A(x_t) - a_B(x_t)|^2 + |b_A(x_t) - b_B(x_t)|^2 + |c_A(x_t) - c_B(x_t)|^2\}\right]^{\frac{1}{2}} \quad (9.19)$$

$$d_7(A,B) = \frac{1}{3n}\sum_{t=1}^{n}\omega_t\{|a_A(x_t) - a_B(x_t)| + |b_A(x_t) - b_B(x_t)| + |c_A(x_t) - c_B(x_t)|\} \quad (9.20)$$

$$d_8(A,B) = \left[\frac{1}{3n}\sum_{t=1}^{n}\omega_t\{|a_A(x_t) - a_B(x_t)|^2 + |b_A(x_t) - b_B(x_t)|^2 + |c_A(x_t) - c_B(x_t)|^2\}\right]^{\frac{1}{2}} \quad (9.21)$$

$$d_1^H(A,B) = \frac{1}{3n}\sum_{t=1}^{n} max\{|a_A(x_t) - a_B(x_t)|, |b_A(x_t) - b_B(x_t)|, |c_A(x_t) - c_B(x_t)|\} \quad (9.22)$$

$$d_2^H(A,B) = \frac{1}{3}\sum_{t=1}^{n}\omega_t max\{|a_A(x_t) - a_B(x_t)|, |b_A(x_t) - b_B(x_t)|, |c_A(x_t) - c_B(x_t)|\} \quad (9.23)$$

$$d_3^H(A,B) = \left[\frac{1}{3n}\sum_{t=1}^{n} max\{|a_A(x_t) - a_B(x_t)|^2, |b_A(x_t) - b_B(x_t)|^2, |c_A(x_t) - c_B(x_t)|^2\}\right]^{\frac{1}{2}} \quad (9.24)$$

$$d_4^H(A,B) = \left[\frac{1}{3}\sum_{t=1}^{n}\omega_t max\{|a_A(x_t) - a_B(x_t)|^2, |b_A(x_t) - b_B(x_t)|^2, |c_A(x_t) - c_B(x_t)|^2\}\right]^{\frac{1}{2}} \quad (9.25)$$

$$a_A(x_t) = \mu_A(x_t)\big(1 - \nu_A(x_t)\big) \quad (9.26)$$

$$b_A(x_t) = 1 - \mu_A(x_t)\big(1 - \nu_A(x_t)\big) - \nu_A(x_t)\big(1 - \mu_A(x_t)\big) \quad (9.27)$$

$$c_A(x_t) = \nu_A(x_t)\big(1 - \mu_A(x_t)\big) \quad (9.28)$$

$$a_B(x_t) = \mu_B(x_t)\big(1 - \nu_B(x_t)\big) \quad (9.29)$$

$$b_B(x_t) = 1 - \mu_B(x_t)\big(1 - \nu_B(x_t)\big) - \nu_B(x_t)\big(1 - \mu_B(x_t)\big) \quad (9.30)$$

$$c_B(x_t) = \nu_B(x_t)\big(1 - \mu_B(x_t)\big) \quad (9.31)$$

To prove the validity of these proposed DiMs, Garg and Kumar [54] considered the same IFSs as well as real-life problem, considered to show the invalidity of the existing DiMs [39, 78, 153, 169] and shown that on applying the DiMs, proposed by them, the distance of the fixed IFS with the other considered distinct IFSs are distinct as well as the results of the considered MADMPr are appropriate.

In future, other researchers may adopt the same methodology to point out the shortcomings of other DiMs as well as to prove the validity of the proposed DiMs. However, the following clearly indicates that Garg and Kumar's method is also not valid.

Garg and Kumar [54] have shown that the existing DiMs $(9.9)-(9.13)$ [39, 78, 153, 169] fails to identify a suitable classifier for the unknown pattern $P=\{\langle x_1,0.7,0.1\rangle,\langle x_2,0.6,0.3\rangle,\langle x_3,0.7,0.1\rangle,\langle x_4,0.5,0.4\rangle,\langle x_5,0.4,0.5\rangle\}$ from two known patterns $A=\{\langle x_1,0.5,0.3\rangle,\langle x_2,0.5,0.2\rangle,\langle x_3,0.9,0.0\rangle,\langle x_4,0.5,0.4\rangle,\langle x_5,0.7,0.1\rangle\}$, and $B=\{\langle x_1,0.7,0.2\rangle,\langle x_2,0.5,0.4\rangle,\langle x_3,0.9,0.1\rangle,\langle x_4,0.6,0.3\rangle,\langle x_5,0.8,0.0\rangle\}$. Therefore, it is inappropriate to use the existing DiMs $(9.9)-(9.13)$ [39, 78, 153, 169].

On the same direction, two known patterns $A=\{\langle x_1,0.11,0.53\rangle,\langle x_2,0.3,0.65\rangle,\langle x_3,0.21,0.47\rangle\}$, $B=\{\langle x_1,0.53,0.11\rangle,\langle x_2,0.65,0.3\rangle,\langle x_3,0.47,0.21\rangle\}$ and an unknown pattern $P=\{\langle x_1,0.14,0.14\rangle,\langle x_2,0.27,0.27\rangle,\langle x_3,0.45,0.45\rangle\}$, represented by IFSs, are considered, and shown that the DiMs $(9.14)-(9.25)$, proposed by Garg and Kumar [54], also fails to identify that either $A$ or $B$ is a suitable classifier for the unknown pattern $P$ i.e., it is also inappropriate to use Garg and Kumar's DiMs $(9.14)-(9.25)$ [54].

To apply the DiMs $(9.14)-(9.25)$, proposed by Garg and Kumar [54], firstly, there is need to transform each element of $A$, $B$ and $P$ into a CN.

Using the expressions $(9.26)-(9.28)$ or $(9.29)-(9.31)$ the IFNs $\langle 0.11,0.53\rangle,\langle 0.3,0.65\rangle,\langle 0.21,0.47\rangle$, $\langle 0.53,0.11\rangle,\langle 0.65,0.3\rangle,\langle 0.47,0.21\rangle$,

$\langle 0.14, 0.14\rangle, \langle 0.27, 0.27\rangle$ and $\langle 0.45, 0.45\rangle$ can be transformed into its equivalent CNs $\langle x_1, 0.0517 + 0.4766i + 0.4717j\rangle, \langle x_2, 0.105 + 0.44i + 0.455j\rangle, \langle x_3, 0.1113 + 0.5174i + 0.3713j\rangle$, $\langle x_1, 0.4717 + 0.4766i + 0.0517j\rangle, \langle x_2, 0.455 + 0.44i + 0.105j\rangle, \langle x_3, 0.3713 + 0.5174i + 0.1113j\rangle$, $\langle x_1, 0.1204 + 0.7592i + 0.1204j\rangle, \langle x_2, 0.1971 + 0.6058i + 0.1971j\rangle, \langle x_3, 0.2475 + 0.505i + 0.2475j\rangle$ respectively. Therefore, $A$, $B$ and $P$ in terms of CNs can be rewritten as

$A = \{\langle x_1, 0.0517 + 0.4766i + 0.4717j\rangle, \langle x_2, 0.105 + 0.44i + 0.455j\rangle, \langle x_3, 0.1113 + 0.5174i + 0.3713j\rangle\}$,

$B = \{\langle x_1, 0.4717 + 0.4766i + 0.0517j\rangle, \langle x_2, 0.455 + 0.44i + 0.105j\rangle, \langle x_3, 0.3713 + 0.5174i + 0.1113j\rangle\}$ and $P = \{\langle x_1, 0.1204 + 0.7592i + 0.1204j\rangle, \langle x_2, 0.1971 + 0.6058i + 0.1971j\rangle, \langle x_3, 0.2475 + 0.505i + 0.2475j\rangle\}$.

The distances of $A$ and $B$ from $P$, obtained on using the expressions $(9.14) - (9.25)$ [54], are shown in Table 9.1.

**Table 9.1 Distances of *A* and *B* from *P***

| | $(P, A)$ | $(P, B)$ |
|---|---|---|
| $d_1$ | 0.496933333 | 0.496933333 |
| $d_2$ | 0.165644444 | 0.165644444 |
| $d_3$ | 0.338875213 | 0.338875213 |
| $d_4$ | 0.195649695 | 0.195649695 |
| $d_5$ | 0.159103333 | 0.159103333 |
| $d_6$ | 0.186996469 | 0.186996469 |
| $d_7$ | 0.053034444 | 0.053034444 |
| $d_8$ | 0.107962462 | 0.107962462 |
| $d_H^1$ | 0.082822222 | 0.082822222 |
| $d_H^2$ | 0.079551666 | 0.079551666 |

| | | |
|---|---|---|
| $d_H^3$ | 0.152196707 | 0.152196707 |
| $d_H^4$ | 0.146002899 | 0.146002899 |

It is obvious from the results, shown in Table 9.1, that

(i) $d_1(P,A) = d_1(P,B) = 0.496933333$.

(ii) $d_2(P,A) = d_2(P,B) = 0.165644444$.

(iii) $d_3(P,A) = d_3(P,B) = 0.338875213$.

(iv) $d_4(P,A) = d_4(P,B) = 0.195649695$.

(v) $d_5(P,A) = d_5(P,B) = 0.159103333$.

(vi) $d_6(P,A) = d_6(P,B) = 0.186996469$.

(vii) $d_7(P,A) = d_7(P,B) = 0.053034444$.

(viii) $d_8(P,A) = d_8(P,B) = 0.107962462$.

(ix) $d_1^H(P,A) = d_1^H(P,B) = 0.082822222$.

(x) $d_2^H(P,A) = d_2^H(P,B) = 0.079551666$.

(xi) $d_3^H(P,A) = d_3^H(P,B) = 0.152196707$.

(xii) $d_4^H(P,A) = d_4^H(P,B) = 0.146002899$.

Hence, if the existing DiMs $(9.9) - (9.13)$ [39, 78, 153, 169], are inappropriate then the DiMs $(9.14) - (9.25)$, proposed by Garg and Kumar [54], are also inappropriate. To modify Garg and Kumar's method [54] may be considered as an open research problem.

5. In the last few years, several methods have been proposed in the literature to find the ranking of the alternatives for such MADMPrs in which RV of each alternative over each criterion is represented as IFS. Garg and Kumar [55] pointed out that the following concept can be used to find the ranking of the alternative such that,

"Construct the IFS $P_i$ whose elements are the elements of the $i^{th}$ row of the IFDM. Then, find the relative strength of each $P_i$ from the fixed IFS (goal) and use this relative strength to rank the alternatives."

But, on applying the existing SMs $(9.32) - (9.35)$ [78, 174] to measure the relative strength of a fixed IFS with other distinct IFSs, the obtained relative strength are equal. Therefore, it is inappropriate to use the existing SMs $(9.32) - (9.35)$ [78, 174] for the same purpose. To validate this claim,

(i) Garg and Kumar [55, Section 4, Eq. 8] used the SM (9.32) [174] to evaluate the relative strength of two unknown patterns $A = \{\langle x_1, 0.5,0.3\rangle, \langle x_2, 0.5,0.2\rangle, \langle x_3, 0.9,0.0\rangle, \langle x_4, 0.5,0.4\rangle, \langle x_5, 0.7,0.1\rangle\}$ and $B = \{\langle x_1, 0.7,0.2\rangle, \langle x_2, 0.5,0.4\rangle, \langle x_3, 0.9,0.1\rangle, \langle x_4, 0.6,0.3\rangle, \langle x_5, 0.8,0.0\rangle\}$ with the known pattern $P = \{\langle x_1, 0.7,0.1\rangle, \langle x_2, 0.6,0.3\rangle, \langle x_3, 0.7,0.1\rangle, \langle x_4, 0.5,0.4\rangle, \langle x_5, 0.4,0.5\rangle\}$ and shown that the obtained relative strength between $A$ and $P$ as well as the obtained relative strength between $B$ and $P$ are equal. Therefore, the SM (9.32) [174] cannot be used to identify a suitable patterns for the unknown pattern $A$ and $B$ from the known pattern $P$.

$$s_1'(A,B) = 1 - \left[\frac{1}{2n}\sum_{t=1}^{n}\{|\mu_A(x_t) - \mu_B(x_t)| + |\nu_A(x_t) - \nu_B(x_t)| + |\pi_A(x_t) - \pi_B(x_t)|\}\right] \tag{9.32}$$

(ii) Garg and Kumar [55, Section 4, Eq. 9, Eq. 10] used the SM (9.33) and (9.34) [174] to evaluate the relative strength of three unknown patterns $A = \{\langle x_1, 0.4,0.5\rangle, \langle x_2, 0.7,0.1\rangle, \langle x_3, 0.3,0.3\rangle\}$, $B = \{\langle x_1, 0.5,0.4\rangle, \langle x_2, 0.7,0.2\rangle, \langle x_3, 0.4,0.3\rangle\}$ and $C = \{\langle x_1, 0.4,0.5\rangle, \langle x_2, 0.7,0.1\rangle, \langle x_3, 0.4,0.3\rangle\}$ with the known pattern $P = \{\langle x_1, 0.1,0.1\rangle, \langle x_2, 1.0,0.0\rangle, \langle x_3, 0.0,1.0\rangle\}$ and shown that the obtained relative strength between $A$ and $P$, the obtained relative strength between $B$ and $P$ and the obtained relative strength between $C$ and $P$ are equal. Therefore, the SM (9.33) and (9.34) [174] cannot be used to identify a suitable pattern for the unknown patterns $A$, $B$ and $C$ from the known pattern $P$.

$$s_2'(A,B) = \frac{\sum_{t=1}^{n}\left(min(\mu_A(x_t),\mu_B(x_t))+min(\nu_A(x_t),\nu_B(x_t))+min(\pi_A(x_t),\pi_B(x_t))\right)}{\sum_{t=1}^{n}\left(max(\mu_A(x_t),\mu_B(x_t))+max(\nu_A(x_t),\nu_B(x_t))+max(\pi_A(x_t),\pi_B(x_t))\right)}. \quad (9.33)$$

$$s_3'(A,B) = \frac{\sum_{t=1}^{n}\left((\mu_A(x_t),\mu_B(x_t))+(\nu_A(x_t),\nu_B(x_t))+(\pi_A(x_t),\pi_B(x_t))\right)}{max\left\{\sum_{t=1}^{n}\left(\mu_A^2(x_t)+\nu_A^2(x_t)+\pi_A^2(x_t)\right),\sum_{t=1}^{n}\left(\mu_B^2(x_t)+\nu_B^2(x_t)+\pi_B^2(x_t)\right)\right\}}. \quad (9.34)$$

(iii) Garg and Kumar [55, Section 4, Eq. 11] used the SM (9.35) [78] to evaluate the relative strength of two unknown patterns, $A = \{\langle x_1, 0.4,0.5\rangle, \langle x_2, 0.4,0.4\rangle, \langle x_3, 0.7,0.0\rangle, \langle x_4, 0.4,0.6\rangle, \langle x_5, 0.6,0.3\rangle\}$ and $B = \{\langle x_1, 0.6,0.4\rangle, \langle x_2, 0.4,0.6\rangle, \langle x_3, 0.9,0.1\rangle, \langle x_4, 0.5,0.5\rangle, \langle x_5, 0.7,0.2\rangle\}$ with the known pattern $P = \{\langle x_1, 0.7,0.1\rangle, \langle x_2, 0.6,0.3\rangle, \langle x_3, 0.7,0.1\rangle, \langle x_4, 0.5,0.4\rangle, \langle x_5, 0.4,0.5\rangle\}$ and shown that the obtained relative strength between $A$ and $P$ as well as the obtained relative strength between $B$ and $P$ are equal. Therefore, SM (9.35) [78] cannot be used to identify a suitable pattern for the unknown patterns $A$ and $B$ from the known pattern $P$.

$$s_4'(A,B) = \frac{\sum_{t=1}^{n}\left(1-\frac{1}{2}\{|\mu_A(x_t)-\mu_B(x_t)|+|\nu_A(x_t)-\nu_B(x_t)|\}\right)}{n}. \quad (9.35)$$

(iv) Furthermore, Garg and Kumar [55] considered a real-life MADMPr under IF environment and shown that if the existing SM (9.32) – (9.35) [78, 174] are used to solve this problem then the obtained results are not appropriate.

To overcome this shortcoming of the existing SMs (9.32) – (9.35) [78, 174], Garg and Kumar [55] proposed the SM (9.36) [55, Section 3.1, Eq. 4], SM (9.37) [55, Section 3.2, Eq. 5], SM (9.38) [55, Section 3.2, Eq. 6], SM (9.39) [55, Section 3.2, Eq. 7], for evaluating the relative strength and the weighted relative strength respectively between two IFSs $A=\{\langle\mu_A(x_t),\nu_A(x_t)\rangle\}$ and $B=\{\langle\mu_B(x_t),\nu_B(x_t)\rangle\}$ $t=1,2,\dots,n$.

All the SMs (9.36) − (9.39) [55] are based upon CNs $a_A(x_t)+b_A(x_t)i+c_A(x_t)j$ and $a_B(x_t)+b_B(x_t)i+c_B(x_t)j$ [28] which are obtained corresponding to the IFNs $\langle\mu_A(x_t),\nu_A(x_t)\rangle$ and $\langle\mu_B(x_t),\nu_B(x_t)\rangle$ respectively by considering the existing expressions (9.40) − (9.45) [98],

$$s_1(A,B)=\frac{1}{n}\sum_{t=1}^{n}\left(1-\frac{1}{3}\{|a_A(x_t)-a_B(x_t)|+|b_A(x_t)-b_B(x_t)|+|c_A(x_t)-c_B(x_t)|\}\right) \tag{9.36}$$

$$s_2(A,B)=\frac{\sum_{t=1}^{n}\left(min\left(a_A(x_t),a_B(x_t)\right)+min\left(b_A(x_t),b_B(x_t)\right)+min\left(c_A(x_t),c_B(x_t)\right)\right)}{\sum_{t=1}^{n}\left(max\left(a_A(x_t),a_B(x_t)\right)+max\left(b_A(x_t),b_B(x_t)\right)+max\left(c_A(x_t),c_B(x_t)\right)\right)} \tag{9.37}$$

$$s_1^w(A,B)=$$

$$\sum_{t=1}^{n}w_t\left(1-\frac{1}{3}\{|a_A(x_t)-a_B(x_t)|+|b_A(x_t)-b_B(x_t)|+|c_A(x_t)-c_B(x_t)|\}\right) \tag{9.38}$$

$$s_2^w(A,B)=\frac{\sum_{t=1}^{n}w_t\left(min\left(a_A(x_t),a_B(x_t)\right)+min\left(b_A(x_t),b_B(x_t)\right)+min\left(c_A(x_t),c_B(x_t)\right)\right)}{\sum_{t=1}^{n}w_t\left(max\left(a_A(x_t),a_B(x_t)\right)+max\left(b_A(x_t),b_B(x_t)\right)+max\left(c_A(x_t),c_B(x_t)\right)\right)} \tag{9.39}$$

$$a_A(x_t)=\mu_A(x_t)\left(1-\nu_A(x_t)\right) \tag{9.40}$$

$$b_A(x_t)=1-\mu_A(x_t)\left(1-\nu_A(x_t)\right)-\nu_A(x_t)\left(1-\mu_A(x_t)\right) \tag{9.41}$$

$$c_A(x_t)=\nu_A(x_t)\left(1-\mu_A(x_t)\right) \tag{9.42}$$

$$a_B(x_t)=\mu_B(x_t)\left(1-\nu_B(x_t)\right) \tag{9.43}$$

$$b_B(x_t)=1-\mu_B(x_t)\left(1-\nu_B(x_t)\right)-\nu_B(x_t)\left(1-\mu_B(x_t)\right) \tag{9.44}$$

$$c_B(x_t)=\nu_B(x_t)\left(1-\mu_B(x_t)\right) \tag{9.45}$$

To prove the validity of these proposed SMs, Garg and Kumar [55] considered the same IFSs as well as real-life problem, considered to show the invalidity of the existing SMs [78,174] and shown that on applying the SMs, proposed by them, the relative strength of the fixed IFS with the other considered distinct IFSs are distinct as well as the results of the considered MADMPr are appropriate.

In future, other researchers may adopt the same methodology to point out the shortcomings of other SMs as well as to prove the validity of the proposed SMs. However, the following clearly indicates that Garg and Kumar's method is also not valid.

Garg and Kumar [55] have shown that the existing SMs $(9.32)-(9.35)$ [78,174] fails to identify a suitable classifier for the,

(i) Known pattern $P=\{\langle x_1,0.7,0.1\rangle,\langle x_2,0.6,0.3\rangle,\langle x_3,0.7,0.1\rangle,\langle x_4,0.5,0.4\rangle,\langle x_5,0.4,0.5\rangle\}$ from two unknown patterns $A=\{\langle x_1,0.5,0.3\rangle,\langle x_2,0.5,0.2\rangle,\langle x_3,0.9,0.0\rangle,\langle x_4,0.5,0.4\rangle,\langle x_5,0.7,0.1\rangle\}$, and $B=\{\langle x_1,0.7,0.2\rangle,\langle x_2,0.5,0.4\rangle,\langle x_3,0.9,0.1\rangle,\langle x_4,0.6,0.3\rangle,\langle x_5,0.8,0.0\rangle\}$.

(ii) Known pattern $P=\{\langle x_1,0.1,0.1\rangle,\langle x_2,1.0,0.0\rangle,\langle x_3,0.0,1.0\rangle\}$ from three unknown patterns $A=\{\langle x_1,0.4,0.5\rangle,\langle x_2,0.7,0.1\rangle,\langle x_3,0.3,0.3\rangle\}$, $B=\{\langle x_1,0.5,0.4\rangle,\langle x_2,0.7,0.2\rangle,\langle x_3,0.4,0.3\rangle\}$ and $C=\{\langle x_1,0.4,0.5\rangle,\langle x_2,0.7,0.1\rangle,\langle x_3,0.4,0.3\rangle\}$.

(iii) Known pattern $P=\{\langle x_1,0.7,0.1\rangle,\langle x_2,0.6,0.3\rangle,\langle x_3,0.7,0.1\rangle,\langle x_4,0.5,0.4\rangle,\langle x_5,0.4,0.5\rangle\}$ from two unknown patterns $A=\{\langle x_1,0.4,0.5\rangle,\langle x_2,0.4,0.4\rangle,\langle x_3,0.7,0.0\rangle,\langle x_4,0.4,0.6\rangle,\langle x_5,0.6,0.3\rangle\}$ and $B=\{\langle x_1,0.6,0.4\rangle,\langle x_2,0.4,0.6\rangle,\langle x_3,0.9,0.1\rangle,\langle x_4,0.5,0.5\rangle,\langle x_5,0.7,0.2\rangle\}$ with the known pattern. Therefore, it is inappropriate to use the existing SMs $(9.32)-(9.35)$ [78,174].

On the same direction, two unknown patterns $A=\{\langle x_1,0.11,0.53\rangle,\langle x_2,0.3,0.65\rangle,\langle x_3,0.21,0.47\rangle\}$, $B=\{\langle x_1,0.53,0.11\rangle,\langle x_2,0.65,0.3\rangle,\langle x_3,0.47,0.21\rangle\}$ and an known pattern $P=\{\langle x_1,0.14,0.14\rangle,\langle x_2,0.27,0.27\rangle,\langle x_3,0.45,0.45\rangle\}$, represented by IFSs, are considered, and shown that the SMs $(9.36)-(9.39)$, proposed by Garg and Kumar [55], also fails to identify that either $A$ or $B$ is a suitable classifier for the known pattern $P$ i.e., it is also inappropriate to use Garg and Kumar's SMs $(9.36)-(9.39)$ [55].

To apply the SMs $(9.36)-(9.39)$, proposed by Garg and Kumar [55], firstly, there is need to transform each element of $A$, $B$ and $P$ into a CN.

Using the expressions $(9.40)-(9.42)$ or $(9.43)-(9.45)$ the IFNs $\langle x_1, 0.11, 0.53\rangle, \langle x_2, 0.3, 0.65\rangle, \langle x_3, 0.21, 0.47\rangle$; $\langle x_1, 0.53, 0.11\rangle, \langle x_2, 0.65, 0.3\rangle, \langle x_3, 0.47, 0.21\rangle$; $\langle x_1, 0.14, 0.14\rangle, \langle x_2, 0.27, 0.27\rangle$ and $\langle x_3, 0.45, 0.45\rangle$ can be transformed into its equivalent CNs $\langle x_1, 0.0517+0.4766i+0.4717j\rangle, \langle x_2, 0.105+0.44i+0.455j\rangle, \langle x_3, 0.1113+0.5174i+0.3713j\rangle$; $\langle x_1, 0.4717+0.4766i+0.0517j\rangle, \langle x_2, 0.455+0.44i+0.105j\rangle, \langle x_3, 0.3713+0.5174i+0.1113j\rangle$; $\langle x_1, 0.1204+0.7592i+0.1204j\rangle, \langle x_2, 0.1971+0.6058i+0.1971j\rangle, \langle x_3, 0.2475+0.505i+0.2475j\rangle$ respectively. Therefore, $A$, $B$ and $P$ in terms of CNs can be rewritten as

$A=\{\langle x_1, 0.0517+0.4766i+0.4717j\rangle, \langle x_2, 0.105+0.44i+0.455j\rangle, \langle x_3, 0.1113+0.5174i+0.3713j\rangle\}$,

$B=\{\langle x_1, 0.4717+0.4766i+0.0517j\rangle, \langle x_2, 0.455+0.44i+0.105j\rangle, \langle x_3, 0.3713+0.5174i+0.1113j\rangle\}$ and

$P=\{\langle x_1, 0.1204+0.7592i+0.1204j\rangle, \langle x_2, 0.1971+0.6058i+0.1971j\rangle, \langle x_3, 0.2475+0.505i+0.2475j\rangle\}$.

The relative strength of $A$ and $B$ from $P$, obtained on using the expressions $(9.36)-(9.39)$ [55], are shown in Table 9.3.

**Table 9.3 Relative strength of $A$ and $B$ from $P$**

| | $(P,A)$ | $(P,B)$ |
|---|---|---|
| $s_1$ | 0.834355555 | 0.834355555 |
| $s_2$ | 0.601965077 | 0.601965077 |
| $s_1^w$ | 0.834355555 | 0.834355555 |
| $s_2^w$ | 0.601965077 | 0.601965077 |

It is obvious from the results, shown in Table 9.3, that

(i) $s_1(P,A) = s_1(P,B) = 0.834355555$

(ii) $s_2(P,A) = s_2(P,B) = 0.601965077$

(iii) $s_1^w(P,A) = s_1^w(P,B) = 0.834355555$

(iv) $s_2^w(P,A) = s_2^w(P,B) = 0.601965077$ .

Hence, if the existing SMs$(9.32) - (9.35)$ [78, 174] are inappropriate then the SMs $(9.36) - (9.39)$, proposed by Garg and Kumar [55], are also inappropriate. To modify Garg and Kumar's method [55] may be considered as an open research problem.

6. Ye [187] proposed the following operational laws of the NNs.

Let $z_1 = a_1 + b_1 I$ and $z_2 = a_2 + b_2 I$ be two NNs, where $a_1, b_1, a_2, b_2$ are RNs and $I = [I^L, I^U]$ is a closed interval. Then,

(i) $z_1 + z_2 = a_1 + a_2 + (b_1 + b_2)I = [a_1 + a_2 + b_1 I^L + b_2 I^L,\ a_1 + a_2 + b_1 I^U + b_2 I^U]$

(ii) $z_1 - z_2 = a_1 - a_2 + (b_1 - b_2)I = [a_1 - a_2 + b_1 I^L - b_2 I^L,\ a_1 - a_2 + b_1 I^U - b_2 I^U]$

(iii) $z_1 \times z_2 = a_1a_2 + (a_1b_2 + a_2b_1)I + b_1b_2I^2$

$$= \left[ min \begin{pmatrix} (a_1 + b_1I^L)(a_2 + b_2I^L), \\ (a_1 + b_1I^L)(a_2 + b_2I^U), \\ (a_1 + b_1I^U)(a_2 + b_2I^L), \\ (a_1 + b_1I^U)(a_2 + b_2I^U) \end{pmatrix}, max \begin{pmatrix} (a_1 + b_1I^L)(a_2 + b_2I^L), \\ (a_1 + b_1I^L)(a_2 + b_2I^U), \\ (a_1 + b_1I^U)(a_2 + b_2I^L), \\ (a_1 + b_1I^U)(a_2 + b_2I^U) \end{pmatrix} \right]$$

(iv) $\frac{z_1}{z_2} = \frac{a_1+b_1I}{a_2+b_2I} = \frac{[a_1+b_1I^L, a_1+b_1I^U]}{[a_2+b_2I^L, a_2+b_2I^U]}$

$= \left[min\left(\frac{a_1+b_1I^L}{a_2+b_2I^U}, \frac{a_1+b_1I^L}{a_2+b_2I^L}, \frac{a_1+b_1I^U}{a_2+b_2I^U}, \frac{a_1+b_1I^U}{a_2+b_2I^L}\right), max\left(\frac{a_1+b_1I^L}{a_2+b_2I^U}, \frac{a_1+b_1I^L}{a_2+b_2I^L}, \frac{a_1+b_1I^U}{a_2+b_2I^U},\right.\right.$

$\left.\left.\frac{a_1+b_1I^U}{a_2+b_2I^L}\right)\right]$.

The aim of this note is to point out the limitations of the existing basic operational laws of NNs. For this purpose, there is a need to discuss the origin of these operational laws. Therefore, the same is discussed in this section. These operational laws have been obtained in the following manner:

**Step 1:** Using the arithmetic operations, $b_1[I^L, I^U] = [b_1I^L, b_1I^U]$ and $b_2[I^L, I^U] = [b_2I^L, b_2I^U]$, the NNs, $z_1 = a_1 + b_1I = a_1 + b_1[I^L, I^U]$ and $z_2 = a_2 + b_2I = a_2 + b_2[I^L, I^U]$ can be rewritten as $z_1 = a_1 + [b_1I^L, b_1I^U]$ and $z_2 = a_2 + [b_2I^L, b_2I^U]$ respectively.

**Step 2:** Using the arithmetic operation, $u + [v, w] = [u + v, u + w]$, the NNs, $z_1 = a_1 + [b_1I^L, b_1I^U]$ and $z_2 = a_2 + [b_2I^L, b_2I^U]$ can be rewritten as $z_1 = [a_1 + b_1I^L, a_1 + b_1I^U]$ and $z_2 = [a_2 + b_2I^L, a_2 + b_2I^U]$ respectively.

**Step 2(a):** Using the arithmetic operation, $[u_1, u_2] + [v_1, v_2] = [u_1 + v_1, u_2 + v_2]$, the first basic operational law, $z_1 + z_2 = [a_1 + a_2 + b_1I^L + b_2I^L, a_1 + a_2 + b_1I^U + b_2I^U]$ is obtained.

**Step 2(b):** Using the arithmetic operation, $[u_1, u_2] - [v_1, v_2] = [u_1 - v_1, u_2 - v_2]$, the second basic operational law, $z_1 - z_2 = [a_1 - a_2 + b_1I^L - b_2I^L, a_1 - a_2 + b_1I^U - b_2I^U]$ is obtained.

**Step 2(c):** Using the arithmetic operation,

$[u_1, u_2] \times [v_1, v_2] = \begin{bmatrix} min\big((u_1v_1), (u_1v_2), (u_2v_1), (u_2v_2)\big) \\ max\big((u_1v_1), (u_1v_2), (u_2v_1), (u_2v_2)\big) \end{bmatrix}$, the third basic operational law,

$$z_1 \times z_2 = \begin{bmatrix} min\begin{pmatrix} (a_1 + b_1I^L)(a_2 + b_2I^L), & (a_1 + b_1I^L)(a_2 + b_2I^U) \\ (a_1 + b_1I^U)(a_2 + b_2I^L), & (a_1 + b_1I^U)(a_2 + b_2I^U) \end{pmatrix}, \\ max\begin{pmatrix} (a_1 + b_1I^L)(a_2 + b_2I^L), & (a_1 + b_1I^L)(a_2 + b_2I^U) \\ (a_1 + b_1I^U)(a_2 + b_2I^L), & (a_1 + b_1I^U)(a_2 + b_2I^U) \end{pmatrix} \end{bmatrix}$$ is obtained.

**Step 2(d):** Using the arithmetic operation,

$\frac{[u_1,u_2]}{[v_1,v_2]} = \left[min\left(\frac{u_1}{v_1}, \frac{u_1}{v_2}, \frac{u_2}{v_1}, \frac{u_2}{v_2}\right), max\left(\frac{u_1}{v_1}, \frac{u_1}{v_2}, \frac{u_2}{v_1}, \frac{u_2}{v_2}\right)\right]$, the fourth basic operational law,

$\frac{z_1}{z_2} = \left[min\left(\frac{a_1+b_1I^L}{a_2+b_2I^U}, \frac{a_1+b_1I^L}{a_2+b_2I^L}, \frac{a_1+b_1I^U}{a_2+b_2I^U}, \frac{a_1+b_1I^U}{a_2+b_2I^L}\right), max\left(\frac{a_1+b_1I^L}{a_2+b_2I^U}, \frac{a_1+b_1I^L}{a_2+b_2I^L}, \frac{a_1+b_1I^U}{a_2+b_2I^U}, \frac{a_1+b_1I^U}{a_2+b_2I^L}\right)\right]$ is obtained.

It is obvious from Step 1 of the method, discussed above, that to obtain all the basic operational laws, the assumptions $b_1[I^L, I^U] = [b_1I^L, b_1I^U]$ and $b_2[I^L, I^U] = [b_2I^L, b_2I^U]$ are considered. However, these assumptions are valid only if $b_1$ and $b_2$ are non-negative RNs. While, it is well known fact that in a NN $z = a + bI,\ b$ is a real number i.e., $b$ may be non-negative or $b$ may be negative. Therefore, the existing basic operational laws of NNs, discussed above, are valid only for such NNs $z_1 = a_1 + b_1I$ and $z_2 = a_2 + b_2I$ in which $b_1$ and $b_2$ will be non-negative RNs.

To propose the generalized operational laws of NNs may be considered as a challenging open research problem.

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