On L^1-Convergence of Trigonometric Sine Series with special coefficients

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Submitted by

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Certificate

It is certified that the content contained in this thesis entitled "**On L^1−Convergence of trigonometric sine series with special coefficients**" in partial fulfillment of the requirements for the award of degree of Master of Science in Mathematics and Computing to the School of Mathematics, Thapar Institute of Engineering and Technology (TIET), Patiala is an authentic record of my own work studied under the supervision of Dr. Jatinderdeep Kaur.

The matter embodied in this thesis has not been submitted by me for the award of any other degree of this or any other University/Institute.

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This is to certify that above statement made by the candidate is correct and true to the best of my knowledge.

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Kirtika Singla

Abstract

The present dissertation entitled "**On L^1-Convergence of trigonometric sine series with special coefficients**" contains a brief account of study carried out by me on L^1-convergence of Trigonometric Sine Sums under the supervision of **Dr. Jatinderdeep Kaur,** Associate Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

In the literature so far available, very few work has been done concerning the convergence of trigonometric sine series in L^1-norm. Keeping this in view, The L^1-convergence of trigonometric sine series under different conditions on coefficients have been studied. Also, several authors introduced modified trigonometric cosine and sine sums.

The first chapter is introductory. In this chapter, apart from setting up the notations and terminology to be used in sequel, we have presented some known results. The purpose of chapter II is to study the L^1-convergence of trigonometric sine series using class $\widetilde{S}_1$.

In the chapter III, the result on L^1-convergence of trigonometric sine series has been obtained using Ram and Kumari Modified sine sums $\left(\sum_{k=1}^{n}\sum_{j=k}^{n}\triangle\left(\frac{a_j}{j}\right)k\sin kx\right)$ under the classes $\widetilde{B}V\bigcap\widetilde{C}$.

In chapter IV, L^1-convergence of r times differentiable trigonometric sine series has been studied using the classes $\widetilde{B}V_r(r=0,1,2,3,\dots)$ *and* $\widetilde{C}_r(r=0,1,2,3,\dots)$.

In the end, references of various publications cited in the present dissertation have been reported.

Contents

Chapter 1

Introduction

1.1 Introduction

The present thesis comprises certain results studied by the author on L^1-*Convergence of trigonometric sine series with special coefficients*". It is well known that the convergence of trigonometric series in L^1-metric to a function $f \in L^1$, implies that it is a Fourier series of the function f. Riesz([1],Vol.II,Ch. VIII §22) gave a counter example to show that in L^1-metric, the converse of the above result does not hold good. This encouraged various researchers to study the L^1-convergence of trigonometric series $\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\right)$ with special coefficients.

L^1-convergence of trigonometric series with special coefficients have been studied by number of researchers. The work was initiated by Young [15] by taking class of convex sequences $(\Delta^2 b_n \geqslant 0)$ and Kolmogorov [7] by taking class of quasi convex sequences $\left(\sum_{n=1}^{\infty} n|\Delta^2 b_n| < \infty\right)$.

In 1973, Telyakovskii [14] studied class S which was introduced by Sidon [11] in 1939 for L^1-convergence of trigonometric cosine series. First Later in 1984, Telyakovskii [13] introduced new class $\widetilde{S}$ for study of L^1-convergence of trigonometric sine series. The results achieved by these authors were further generalized and extended by Móricz [9], Garrett and Stanojević [3], Kano [8], Sidon [11].

In the literature so far available, we found that several authors introduced trigonometric sums "*as these sums approximate their limits better than the classical trigonometric sums in the sense that these sums converge in* L^1*-metric to the sum of trigonomet-*

ric series whereas classical series itself may not". Rees and Stanojević [10], Kumari and Ram [5] introduced new modified trigonometric sums and studied their L^1-convergence under various classes of coefficient sequences.

In the present thesis, number of results have been studied by the author, some of which are directly associated with the work of above mentioned authors.

To provide adequate background for later chapters, a summary of basic concepts, notations and a brief chapter wise resume of the results contained in the thesis have been given in the introductory chapter. However, some of the notations and definitions will be repeated occasionally in various chapters for the sake of convenience.

1.2 Notations & Definitions

Let $\{b_n\}$ be a sequence. Then we write

$$\begin{aligned} \Delta b_n &= b_n - b_{n+1} \\ \Delta^2 b_n &= \Delta\left(\Delta b_n\right) \\ &= \Delta\left(b_n - b_{n+1}\right) \\ &= \Delta b_n - \Delta b_{n+1} \\ &= b_n - 2b_{n+1} + b_{n+2} \end{aligned}$$

Abel's transformation([1],Vol. I, p.1). If $b_0, b_1, b_2, \dots$ and $w_0, w_1, w_2, \dots$ are any real numbers and assume that

$$W_n = w_0 + w_1 + w_2 + \cdots + w_n.$$

Then for value of n,

$$\sum_{k=1}^{n} b_k w_k = \sum_{k=1}^{n-1} \Delta b_k W_k + b_n W_n - b_1 W_0.$$

Null Sequence A sequence $\{b_k\}$ is null sequence if

$$b_k \to 0 \ \ as \ \ k \to \infty.$$

Convex Null sequence A sequence $\{b_k\}$ is called convex null sequence if

$$\Delta^2 b_k > 0 \quad \forall\ k \in \mathbb{N}.$$

Quasi-Convex null sequence([1], Vol.II, p.202) A null sequence $\{b_k\}$ is called quasi-convex null sequence if

$$\sum_{k=1}^{\infty} (k+1)\,|\Delta^2 b_k| < \infty.$$

Class S([11], [14]) A sequence $\{b_k\}$ belongs to class S if $b_k \rightarrow 0 \ \ as \ \ k \rightarrow \infty$ and there exist a monotonically decreasing sequence $\{B_k\}$ such that

$$\sum_{k=1}^{\infty} B_k < \infty;$$

and

$$|\Delta b_k| \leq B_k, \forall\ k = 1, 2, 3, \ldots$$

Example 1.2.1. Consider a sequence

$$b_k = \frac{1}{k}; \quad k = 1, 2, 3, \ldots$$

then, there exist a monotone decreasing sequence

$$B_k = \frac{1}{k^2},$$

the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.(by p-test)
We are left to show

$$|\Delta b_k| = \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \leq B_k = \frac{1}{k^2}$$

We will show it by contradiction, consider

$$\begin{aligned} \frac{1}{k(k+1)} &> \frac{1}{k^2} \\ \frac{1}{k+1} &> \frac{1}{k} \\ k+1 &< k \end{aligned}$$

which contradicts our consideration. Hence $\{b_k\} \in$ class S.

Class $\widetilde{S}$ [13] A null sequence $\{b_k\}$ belongs to the class $\widetilde{S}$ if there exists a non-increasing sequence B_k of numbers such that

$$\begin{aligned} |\Delta a_k| &\leq B_k, \quad \forall \quad k = 1, 2, 3, \ldots \\ \sum_{k=1}^{\infty} k B_k &< \infty; \end{aligned}$$

where

$$a_k = \frac{b_k}{k}, \ \Delta a_k = a_k - a_{k+1}.$$

Class $\widetilde{S}_r$ [2] A null sequence $\{b_k\}$ belongs to the class $\widetilde{S}_r$ (r=0,1,2,...) if there exists a non-increasing sequence B_k of numbers such that

$$\begin{aligned} |\Delta a_k| &\leq B_k, \quad \forall \quad k = 1, 2, 3, \ldots \\ \sum_{k=1}^{\infty} k^{r+1} B_k &< \infty, r = 0, 1, 2, \ldots; \end{aligned}$$

where

$$a_k = \frac{b_k}{k}, \ \Delta a_k = a_k - a_{k+1}$$

Class BV([1], Vol.I, p.3) A sequence $\{b_k\}$ is said to be of bounded variation if

$$\sum_{k=1}^{\infty} |\Delta b_k| < \infty.$$

Example 1.2.2. The series $\sum_{k=1}^{\infty} \Delta (b_k) = \sum_{k=1}^{\infty} \Delta \left(\frac{1}{k}\right)$ is convergent.
Hence, $b_k = \frac{1}{k} \in$ class BV.

Class $\widetilde{BV}$ [9] A null sequence $\{b_k\}$ belongs to the class $\widetilde{BV}$ if

$$\sum_{k=1}^{\infty} k \, |\Delta a_k| < \infty; \text{ where } a_k = \tfrac{b_k}{k}.$$

Class $\widetilde{B}V_r$ [2] A null sequence $\{b_k\}$ belongs to the class $\widetilde{B}V_r$ (r=0,1,2,3,...) if

$$\sum_{k=1}^{\infty} k^{r+1}|\Delta a_k| < \infty; \text{ where } a_k = \tfrac{b_k}{k}.$$

Dirichlet kernel([1], Vol.I, p.85) The Dirichlet's kernel $D_k(x)$ is defined by

$$D_k(x) = \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos kt.$$

Multiply both sides by $2\sin\frac{x}{2}$,

$$2\sin\frac{x}{2}D_k(x) = \sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + \cdots + 2\sin\frac{x}{2}\cos kt$$

On solving,

$$2\sin\frac{x}{2}D_k(x) = \sin\left(k+\frac{1}{2}\right)x$$

$$D_k(x) = \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

The uniform estimate of $D_k(x)$ is

$$|D_k(x)| \leq k + \frac{1}{2}, \quad for \quad any \quad x$$

Moreover, if $x \neq 0(\ mod\ 2\pi)$, then

$$|D_k(x)| \leq \frac{\pi}{2x}, \quad for \quad 0 < |x| \leq \pi,$$

and the estimate for Lebesgue constant is

$$L_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_k(x)|\,dx \approx \frac{4}{\pi^2}\log k.$$

Conjugate Dirichlet Kernel([1], Vol.I, p.85) The conjugate Dirichlet kernel $\widetilde{D_k}(x)$ is defined by

$$\widetilde{D_k}(x) = \sin x + \sin 2x + \cdots + \sin kt$$

Multiply both sides by $2\sin\frac{x}{2}$,

$$2\sin\frac{x}{2}\widetilde{D_k}(x) = 2\sin\frac{x}{2}\sin x + 2\sin\frac{x}{2}\sin 2x + \cdots + 2\sin\frac{x}{2}\cos kt$$

On solving,

$$\widetilde{D_k}(x) = \frac{\cos\frac{x}{2} + \cos\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

Moreover, if $x \neq 0(\ mod\ 2\pi)$, then

$$|\widetilde{D_k}(x)| \leq \frac{\pi}{x}, \quad for 0 < |x| \leq \pi,$$

and the estimate for Lebesgue constant is

$$\widetilde{L}_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\widetilde{D_k}(x)\right| dx \approx \log k.$$

Class $\mathcal{C}$ [3] A null sequence $\{b_k\}$ belong to the class $\mathcal{C}$ if for every $\epsilon > 0$, there exists $\delta > 0$ independent of n, such that

$$\int_0^\delta \left|\sum_{k=n}^{\infty} \Delta b_k D_k(x)\right| dx \leq \epsilon \quad \text{for all } n \in \mathbb{N};$$

where, $D_k(x)$ denotes the Dirichlet kernel.

Class $\widetilde{C}$ [9] A null sequence $\{b_k\}$ belongs to the class $\widetilde{C}$ if for every $\epsilon > 0$ there exist $\delta > 0$, independent of n, such that $\forall\ n$,

$$\int_0^\delta \left|\sum_{k=n}^{\infty} \Delta a_k D_k^{'}(x)\right| dx \leq \epsilon.$$

Here, $D_k^{'}(x)$ denotes the first derivative of Dirichlet kernel. $\left(D_k(x) = \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}\right).$

Class $\widetilde{C}_r$ [2] A null sequence $\{b_k\}$ belongs to the class $\widetilde{C}_r$ (r=0,1,2,...), if for every $\epsilon > 0$, there exists $\delta > 0$, independent of n, such that for all n

$$\int_0^\delta \left|\sum_{k=n}^{\infty} \Delta a_k D_k^{r+1}(x)\right| dx \leq \epsilon; \quad where \quad a_k = \frac{b_k}{k}.$$

Here, $D_k^{r+1}(x)$ denotes the $(r+1)^{th}$ derivative of Dirichlet kernel.

The Classes L^p If f is a measurable function on E, then $|f|^p$ is also p,$-\infty < p < \infty$, $p \neq 0$.Designated by $L^p(E)$, the class of all p-integrable functions over E, *i.e.*,

$$L^p(E) = \left\{f : \int_E |f|^p < \infty\right\}.$$

Example 1.2.3. Let E=[0,16] and $f : E \to \mathbb{R}$ be a function defined by $f(x) = (x)^{\frac{-1}{4}}$. Then $f \in L^p(E)$ where p=1,2,3.

Fourier Series A trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1.2.1}$$

the coefficients a_0, a_k and b_k of which are determined by Fourier formulas;

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ dx; \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ \cos kx\ dx \quad (k = 1, 2, 3, \dots); \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ \sin kx\ dx \quad (k = 1, 2, 3, \dots). \end{aligned}$$

derived from the function $f(x)$, is called Fourier series of the function $f(x)$. We then write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1.2.2}$$

O-o Relation [1] Let $\{a_k\}$ and $\{b_k\}$ be two sequences belongs to $\mathbb{R}$. Then $\{a_k\}$ be of order $\{b_k\}$ *i.e.* $a_k = o(b_k)$ *if*

$$\lim_{k \to \infty} \frac{a_k}{b_k} = 0;$$

and if $\frac{a_k}{b_k}$ is bounded, then

$$a_k = O(b_k).$$

Example 1.2.4. Consider,

$$a_k = \frac{1}{k+1} \quad and \quad b_k = 2$$

It can be easily seen that

$$\frac{a_k}{b_k} \to 0 \quad as \quad k \to \infty;$$

therefore,

$$a_k = o(2) \ \ as \ \ k \to \infty.$$

Also,

$$0 \leqslant \frac{a_k}{b_k} < 1 \ \ \forall k \Rightarrow a_k = O(1).$$

L^1-Convergence Let $\{f_n\}$ be a sequence of an integrable functions. Then $\{f_n\}$ is said to be convergent in L^1-norm if for $\epsilon > 0$, there exist a positive integer N *s.t.*

$$\int_{-\pi}^{\pi} |f_n(x) - f(x)| dx < \epsilon \quad \forall \ \ n \geqslant N$$

1.3 Modified Trigonometric Sums

In 1976, Rees and Stanojević [10] introduced new modified cosine sums as

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta a_j)\cos(jx) \tag{1.3.1}$$

and Garrett and Stanojević [4] proved the following result regarding L^1-convergence of modified cosine sum (1.3.1)

Theorem 1.3.1. *[4] If $\{a_k\}$ is a null sequence and $\sum_{k=1}^{n} |\Delta a_k| < \infty$. Then $g_n(x)$ converges to $f(x)$ in $L^1 - metric\ iff$ $\{a_k\}$ belongs to class $\mathcal{C}$.*

Further, Ram and Kumari([6], [5]) introduced new modified sine and cosine sums as

$$g_n(x) = \sum_{j=1}^{n} \left[\sum_{k=j}^{n} \Delta\left(\frac{a_k}{k}\right)\right] j\sin(jx) \tag{1.3.2}$$

and

$$f_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n} \left[\sum_{k=j}^{n} \Delta\left(\frac{a_k}{k}\right)\right] j\cos(jx) \tag{1.3.3}$$

and have studied their L^1-convergence.

They proved the following results:

Theorem 1.3.2. *[5] Let* $\{a_k\}$ *belong to the class* S. *If* $\lim_{n\to\infty} |a_{n+1}| \log n = 0$. *Then* $\|f(x) - f_n(x)\|_{L^1} = o(1)$, $n \to \infty$.

Theorem 1.3.3. *[6] Let* $\{a_k\}$ *belong to the class* R. *then* $\|u(x) - u_n(x)\|_{L^1} = o(1)$, $n \to \infty$, *where* u_n *represents either* g_n *or* f_n.

In the chapter II, we have studied the L^1-convergence of Fourier sine series using modified Ram and Kumari modified sine sum under the class $\widetilde{S}_1$ of coefficient sequences.

The main aim of the chapter III is to study the L^1-convergence of sine series using modified Ram and Kumari modified sine sum in different way under the class $\widetilde{BV}$ and $\widetilde{C}$ of coefficient sequence.

In chapter IV, the results of chapter III have been extended by considering classes $\widetilde{BV}_r$ and $\widetilde{C}_r$ (r=0,1,2,...).

Chapter 2

L^1-convergence of Sine Series with coefficients belonging to class $\widetilde{S}_1$

2.1 Introduction

Consider the sine trigonometric series

$$\sum_{j=1}^{\infty} a_j \sin jx \tag{2.1.1}$$

Let the partial sum of (2.2.1) is denoted by $S_n(x)$ and $g(x) = \lim_{n\to\infty} S_n(x)$.
Ram and Kumari introduced modified sine sum as

$$g_n(x) = \sum_{k=1}^{n} \left[\sum_{j=k}^{n} \Delta \left(\frac{a_j}{j} \right) \right] k \sin(kx)$$

Definition 2.1.1. [2] A null sequence $\{a_j\}$ belongs to the class $\widetilde{S}_1$, if there exists a non-increasing sequence B_j of numbers such that

$$\begin{aligned} |\Delta b_j| &\leq B_j, \ \ \forall \ \ j = 1, 2, 3, \ldots \\ \sum_{j=1}^{\infty} j^2 B_j &< \infty; \end{aligned}$$

where

$$b_j = \frac{a_j}{j}, \Delta b_j = b_j - b_{j+1}.$$

2.2 Lemmas

In this section, two Lemmas are presented which were proved by Sheng [12].

Lemma 2.2.1. *[12] Let $n \geq 1$, and let r be a positive integer, $x \in [\epsilon, \pi]$. Then*

$$|D_n^r(x)| \leq \frac{C_\epsilon n^r}{x}; \tag{2.2.1}$$

where C_ϵ is a positive constant depending on ϵ and $0 < \epsilon < \pi$.

Proof. Consider, the Dirichlet kernel

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kx)$$

is the Dirichlet Kernel.
Now using the fact that cosine is an even function and sine is an odd function above can be rewritten as:-

$$\begin{aligned}
2D_n(x) &= 1 + 2\sum_{k=1}^{n} \cos(kx) \\
&= \sum_{k=-n}^{n} \cos(kx) + \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \sum_{k=-n}^{n} \sin(kx) \\
&= \frac{1}{\sin\frac{x}{2}} \left[\sum_{k=-n}^{n} \left(\sin\frac{x}{2}\cos(kx) + \cos\frac{x}{2} sinkx \right) \right] \\
2D_n(x) &= \frac{1}{\sin\frac{x}{2}} \left[\sum_{k=-n}^{n} \sin\left(k + \frac{1}{2}\right) x \right] \\
D_n(x) &= \frac{\sin\left(n + \frac{1}{2}\right) x}{2\sin\frac{x}{2}}
\end{aligned}$$

By using the method of Mathematical Induction; (2.2.1) holds for $r = 0$; as

$$|D_n(x)| = \frac{\left|\sin\left(n + \frac{1}{2}\right) x\right|}{2\left|\sin\frac{x}{2}\right|} \leq \frac{C}{x}.$$

Assume that equation (2.2.1) holds for $r = m$

$$|D_n^m(x)| \leq \frac{C_\epsilon n^m}{x}$$

Next,we prove it for $r = m + 1$.
By some calculation of $|D_n^{m+1}(x)|$,we get

$$\left|D_n^{m+1}(x)\right| = \frac{1}{\sin\frac{x}{2}} \sum_{k=0}^{m+1} \left(n + \frac{1}{2}\right)^k \sin\left[\left(n + \frac{1}{2}\right)x + \frac{k\pi}{2}\right] \leq \frac{C_\epsilon n^{m+1}}{x}.$$

□

Lemma 2.2.2. *[12] $\|D_n^r(x)\|_{L^1} = O\left(n^r \log n\right)$, $r \in 0, 1, 2, \ldots$ where $D_n^r(x)$ represents the r^{th} derivative of the Dirichlet Kernel.*

Proof. It follows from

$$D_n^r(x) = O(n^{r+1})$$

and Lemma 2.2.1 that

$$\begin{aligned}\|D_n^r(x)\| = 2\left[\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi}\right]|D_n^r(x)|dx &= O(n^r) + 2n^r \int_{\frac{\pi}{n}}^{\pi} \frac{|\sin(nx + \frac{r\pi}{2})|}{x} + \sum_{k=0}^{n-1} \int_{\frac{\pi}{n}}^{\pi} n^k x^{k-1-r} dx \\ &= O(n^r) + \frac{4}{\pi} n^r \lg n = O(n^r \log n).\end{aligned}$$

□

2.3 Main Result

The main result of this chapter read as

Theorem 2.3.1. *Let $\{a_j\}$ be the sequence of numbers belonging to the class $\widetilde{S}_1$, then $\|g_n(x) - g(x)\| = o(1)$; $n \to \infty$.*
where $g_n(x)$ denotes the Ram-Kumari modified sine sum.

Proof. Consider

$$\begin{aligned}
g_n(x) &= \sum_{j=1}^{n}\left[\sum_{k=j}^{n}\Delta\left(\frac{a_k}{k}\right)\right]j\sin(jx)\\
&= \sum_{j=1}^{n}\left[\frac{a_j}{j}-\frac{a_j+1}{j+1}+\frac{a_j+1}{j+1}+\cdots-\cdots+\frac{a_n}{n}-\frac{a_n+1}{n+1}\right]j\sin(jx).\\
&= \sum_{j=1}^{n}\left[\frac{a_j}{j}-\frac{a_n+1}{n+1}\right]j\sin(jx)\\
&= \sum_{j=1}^{n}\left[\frac{a_j}{j}j\sin(jx)-\frac{a_n+1}{n+1}j\sin(jx)\right]\\
&= \sum_{j=1}^{n}\frac{a_j}{j}j\sin(jx)+\frac{a_n+1}{n+1}D_n^{'}(x).\\
g_n(x) &= \sum_{j=1}^{n}b_j j\sin(jx)+b_{n+1}D_n^{'}(x).
\end{aligned}$$

Applying Abel's Transformation

$$\begin{aligned}
g_n(x) &= -\sum_{j=1}^{n-1}\Delta b_j D_j^{'}(x)-b_nD_n^{'}(x)+b_{n+1}D_n^{'}(x).\\
&= -\sum_{j=1}^{n-1}\Delta b_j D_j^{'}(x)-\left(b_n-b_{n+1}\right)D_n^{'}(x).\\
&= -\sum_{j=1}^{n-1}\Delta b_j D_j^{'}(x)-\Delta b_n D_n^{'}(x).\\
&= -\left[\sum_{j=1}^{n}\Delta\left(b_j\right)D_j^{'}(x)\right].
\end{aligned}$$

Taking Absolute summation of the above terms and using Lemma 2.2.2

$$\left|\sum_{j=1}^{n} \Delta b_j D_j^{'}(x)\right| \leq \sum_{j=1}^{n} |\Delta b_j||D_j^{'}(x)|.$$
$$\leq C\sum_{j=1}^{n} j|\Delta b_j|.$$
$$\leq C\sum_{j=1}^{n} j|\Delta B_j|.$$

Where C is an absolute constant and $x \in [\epsilon, \pi]$.

$$= O\left(\sum_{j=1}^{n} j\Delta B_j\right)$$

By given hypothesis,

$$\lim_{n\to\infty} g_n(x) = g(x) \ \ exists \ \ in \ \ [\epsilon, \pi]$$

Next, consider

$$\begin{aligned} g(x) - g_n(x) &= \sum_{j=1}^{\infty} a_j \sin(jx) - \sum_{j=1}^{n} a_j \sin(jx) - \frac{a_{n+1}}{n+1}D_n^{'}(x) \\ &= \sum_{j=n+1}^{\infty} \frac{a_j}{j} j\sin(jx) - \frac{a_{n+1}}{n+1}D_n^{'}(x) \\ &= \sum_{j=n+1}^{\infty} b_j\left(j\sin(jx)\right) - b_{n+1}D_n^{'}(x) \end{aligned}$$

Apply Abel's Transformation on first term we get,

$$\begin{aligned} &= -\sum_{j=n+1}^{\infty} \Delta b_j D_j^{'}(x) - b_{n+1}D_n^{'}(x) + b_{n+1}D_n^{'}(x) \\ &= -\sum_{j=n+1}^{\infty} \Delta b_j D_j^{'}(x) \end{aligned}$$

Next, consider

$$\begin{aligned}
\|g_n(x) - g(x)\| &= \left\|\sum_{j=n+1}^{\infty} \Delta b_j D_j^{'}(x)\right\| \\
&\leq \sum_{j=n+1}^{\infty} \left\|\Delta b_j D_j^{'}(x)\right\| \\
&= \sum_{j=n+1}^{\infty} |\Delta b_j| \left\|D_j^{'}(x)\right\| \\
&\leq \sum_{j=n+1}^{\infty} B_j O(j \log j) \\
&= O\left(\sum_{j=n+1}^{\infty} j^2 B_j\right) \\
&= o(1) \quad as \quad n \to \infty.
\end{aligned}$$

□

Corollary 2.3.2. *Let* $\{a_j\}$ *be the sequence of numbers belonging to the class* $\widetilde{S}_1$*, and if*

$$\lim_{n\to\infty} a_{n+1} \log n = 0$$

then $\|S_n - g\| = o(1)$*;* $n \to \infty$*.*

Proof.

$$\begin{aligned}
\|S_n - g\| &\leq \|S_n - g_n\| + \|g_n - g\| \\
&\leq \|S_n - g_n\| + o(1) \\
&= \left\|-\sum_{j=1}^{n} \Delta b_j D_j^{'}(x) - b_{n+1} D_n^{'}(x) + \sum_{j=1}^{n} \Delta b_j D_j^{'}(x)\right\| \\
&\leq b_{n+1} \int_0^{\pi} \left|D_n^{'}(x)\right| dx \\
&= \frac{n}{n+1} a_{n+1} \log n \quad (by \quad lemma \quad 2.2.2) \\
&= o(1);\ n \to \infty.
\end{aligned}$$

□

Chapter 3

L^1-convergence of Sine Series with coefficients belonging to class $\widetilde{B}V \cap \widetilde{C}$

3.1 Introduction

Consider the trigonometric sine series as

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{3.1.1}$$

where $a_0, a_1, a_2, \ldots$ are the real coefficients.The partial sum S_n, of series (3.1.1) can be written as

$$S_n = \sum_{k=1}^{n} a_k \sin kx = -\sum_{k=1}^{n} b_k \left(\cos kx\right)^{'}. \tag{3.1.2}$$

where $'$ denotes the differentiation and $b_k{=}\frac{a_k}{k}$. Also,

$$g(x) = \lim_{n\to\infty} S_n(x) = \sum_{k=1}^{\infty} a_k \sin kx, \quad (say). \tag{3.1.3}$$

In 1984, Teljakovskii [13] introduced a class $\widetilde{S}$ as follows:

Definition 3.1.1. [13] A null sequence $\{a_k\}$ belongs to the class $\widetilde{S}$, if there exists a non-increasing sequence B_k of numbers such that

$$\begin{aligned} |\Delta b_k| &\leq B_k, \quad \forall \;\; k = 1, 2, 3, \ldots \\ \sum_{k=1}^{\infty} k B_k &< \infty; \end{aligned}$$

where

$$b_k = \frac{a_k}{k}, \;\; \Delta b_k = b_k - b_{k+1};$$

and he proved the following result.

Theorem 3.1.2. *[13] If $\{a_k\} \in \widetilde{S}$, then series (2.1.1) is the Fourier series of some function $g \in L^1(0, \pi)$.*

Example 3.1.3. Consider a sequence

$$a_k = \frac{1}{k}; \;\; k = 1, 2, 3, \ldots$$

$$b_k = \frac{1}{k^2}$$

And

$$\Delta b_k = \frac{1 + 2k}{k^2 (k + 1)^2}$$

Also, let

$$B_k = \frac{1}{k^3}$$

be the decreasing sequence such that
Clearly,

$$|\Delta b_k| \leq B_k \;\; \forall \;\; k = 1, 2, 3, \ldots$$

.

$$\sum_{k=1}^{\infty} k B_k = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

(by p-series test). Thus, $a_k \in$ class $\widetilde{S}$.

In 1989, Móricz [9] introduced new classes $\widetilde{B}V$ and $\widetilde{C}$ of the coefficient sequences of sine series as follows:

Definition 3.1.4. [9] A null sequence $\{a_k\}$ belongs to the class $\widetilde{B}V$ if

$$\sum_{k=1}^{\infty} k\,|\Delta b_k| < \infty;$$

where

$$b_k = \frac{a_k}{k}.$$

Example 3.1.5. Consider

$$\begin{aligned}\Delta b_k &= \frac{1}{k^2} - \frac{1}{k(k+1)}\\ k\Delta b_k &= \frac{1}{k} - \frac{1}{k+1}\end{aligned}$$

Put k=1,2,3,. . . ,n.

$$\begin{aligned}\Delta b_1 &= 1 - \frac{1}{2}\\ 2\Delta b_2 &= \frac{1}{2} - \frac{1}{3}\\ &\quad .\\ &\quad .\\ &\quad .\\ n\Delta b_n &= \frac{1}{n} - \frac{1}{n+1}\end{aligned}$$

Adding all above, we get

$$S_n(x) = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = 1$$

Thus series $\sum k\,|\Delta b_k|$ is convergent. Hence $b_k \in$ class$\widetilde{B}V$

Definition 3.1.6. [9] A null sequence $\{a_k\}$ belongs to the class $\widetilde{C}$, if for every $\epsilon > 0$ there exist $\delta > 0$ independent of n, such that $\forall\ n$,

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta b_k D_k^{'}(x) \right| dx \leq \epsilon$$

Here, $D_k^{'}(x)$ denotes the first derivative of Dirichlet kernel.

$$\left(D_k(x) = \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}} \right).$$

The following result was proved by Móricz [9]

Theorem 3.1.7. *[9] If* $\{a_k\} \in \widetilde{B}V$*, then* $\|u_n(x) - f\| \to 0$ *as* $n \to \infty$ *iff* $\{a_k\} \in \widetilde{C}$*, where* $u_n(x) = S_n(x) + b_{n+1}D_n^{'}(x)$*.*

The classes $\widetilde{B}V$ and $\widetilde{C}$ are more appropriated classes than the classes BV and **C** for sine series. Further, in this paper, Móricz [9] has proved that $\widetilde{S}\subset\widetilde{B}V\bigcap\widetilde{C}$.

Now, the aim of this chapter is to present the result of L^1-convergence of Trigonometric sine series using Ram and Kumari modified Sine Sum.

$$\left(g_n(x) = \sum_{j=1}^{n} \left[\sum_{k=j}^{n} \Delta\left(\frac{a_k}{k}\right) \right] jsin(jx) \right)$$

under more appropriate class $\widetilde{B}V$ and $\widetilde{C}$ of coefficient sequences.

3.2 Lemmas

Lemma 3.2.1. *[12] Let $n \geq 1$, and let r be a positive integer, $x \in [\epsilon, \pi]$. Then*

$$|D_n^r(x)| \leq \frac{C_\epsilon n^r}{x}; \tag{3.2.1}$$

where C_ϵ is a positive constant depending on ϵ and $0 < \epsilon < \pi$.

Lemma 3.2.2. *[12] $\|D_n^r(x)\|_{L^1} = O\left(n^r logn\right)$, $r \in 0, 1, 2, \ldots$ where $D_n^r(x)$ represents the r^{th} derivative of the Dirichlet Kernel.*

3.3 Main Result

The main result of this chapter reads as follows:

Theorem 3.3.1. *Let $\{a_k\}$ be the sequence of numbers belonging to the class $\widetilde{B}V \cap \widetilde{C}$, then $\|g_n(x) - g(x)\| = o(1)$; $n \to \infty$.*

Proof. Consider

$$\begin{aligned} g_n(x) &= \sum_{j=1}^{n} \left[\sum_{k=j}^{n} \Delta \left(\frac{a_k}{k} \right) \right] j \sin(jx) \\ &= \sum_{j=1}^{n} \left[\frac{a_j}{j} - \frac{a_j+1}{j+1} + \frac{a_j+1}{j+1} + \cdots - \cdots + \frac{a_n}{n} - \frac{a_n+1}{n+1} \right] j \sin(jx). \\ &= \sum_{j=1}^{n} \left[\frac{a_j}{j} - \frac{a_n+1}{n+1} \right] j \sin(jx) \\ &= \sum_{j=1}^{n} \left[\frac{a_j}{j} j sin(jx) - \frac{a_n+1}{n+1} j \sin(jx) \right] \end{aligned}$$

$$\begin{aligned} g_n(x) &= \sum_{j=1}^{n} a_j \sin(jx) - \frac{a_n+1}{n+1}\sum_{j=1}^{n} j\sin(jx) \\ &= \sum_{j=1}^{n} a_j \sin(jx) + \frac{a_n+1}{n+1} D_n^{'}(x). \\ &= \sum_{j=1}^{n} \frac{a_j}{j} j\sin(jx) + \frac{a_n+1}{n+1} D_n^{'}(x). \\ &= \sum_{j=1}^{n} b_j j\sin(jx) + b_{n+1} D_n^{'}(x). \end{aligned}$$

Applying Abel's Transformation on the first term; we get

$$\begin{aligned} g_n(x) &= -\sum_{j=1}^{n-1} \Delta b_j D_j^{'}(x) - b_n D_n^{'}(x) + b_{n+1} D_n^{'}(x). \\ &= -\sum_{j=1}^{n-1} \Delta b_j D_j^{'}(x) - \Big[b_n - b_{n+1}\Big] D_n^{'}(x). \\ &= -\sum_{j=1}^{n-1} \Delta b_j D_j^{'}(x) - \Delta b_n D_n^{'}(x). \\ &= -\left[\sum_{j=1}^{n} \Delta\left(b_j\right) D_j^{'}(x)\right]. \end{aligned}$$

Taking absolute summation of the above terms and using Lemma 3.2.2

$$\begin{aligned} \left|\sum_{j=1}^{n} \Delta b_j D_j^{'}(x)\right| &\leq \sum_{j=1}^{n} |\Delta b_j||D_j^{'}(x)|. \\ &\leq C\sum_{j=1}^{n} j|\Delta b_j|. \end{aligned}$$

Where C is an absolute constant and $x \in [\epsilon, \pi]$.

$$g_n(x) = O\left(\sum_{j=1}^{n} j\Delta b_j\right).$$

By given hypothesis,

$$\lim_{n\to\infty} g_n(x) = g(x) \;\; exists \;\; in \;\; [\epsilon, \pi]$$

Next, consider

$$g(x) - g_n(x) = -\sum_{j=n+1}^{\infty} \Delta b_j D_j^{'}(x)$$

Since $\{a_k\} \in$ class $\widetilde{C}$,

$\therefore$ Given any $\epsilon > 0$, let there exist positive δ, such that

$$\int_0^{\delta} \left| \sum_{j=n+1}^{\infty} \Delta b_j D_j^{'}(x) \right| dx < \frac{\epsilon}{2} \ for \ all \ n \geq 0$$

Thus,

$$\begin{aligned} \|g_n - g\| &\leq \int_0^{\delta} \left| \sum_{j=n+1}^{\infty} \Delta b_j D_j^{'}(x) \right| dx + \int_{\delta}^{\pi} \left| \sum_{j=n+1}^{\infty} \Delta b_j D_j^{'}(x) \right| dx \\ &\leq \frac{\epsilon}{2} + \sum_{j=n+1}^{\infty} \Delta b_j \int_{\delta}^{\pi} \left| D_j^{'}(x) \right| dx. \\ &\leq \frac{\epsilon}{2} + C \sum_{j=n+1}^{\infty} j \left| \Delta b_j \right| \int_{\delta}^{\pi} \frac{dx}{x}. \\ &= \frac{\epsilon}{2} + C\delta^{-2} \sum_{j=n+1}^{\infty} j |\Delta b_j| \\ &< \epsilon. \end{aligned}$$

□

Corollary 3.3.2. *Let $\{a_k\}$ be the sequence of numbers belonging to the class $\widetilde{B}V \cap \widetilde{C}$, and if*

$$\lim_{n\to\infty} a_n logn = 0,$$

then $\|S_n - g\| = o(1)$; $n \to \infty$.

Proof.

$$\begin{aligned} \|S_n - g\| &\leq \|S_n - g_n\| + \|g_n - g\| \\ &\leq \|S_n - g_n\| + o(1) \ \ (by \ \ above \ \ theorem) \\ &\leq \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} D_n^{'}(x) \right| dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{n}{n+1} a_{n+1} \log n \quad (by \quad Lemma \quad 3.2.2) \\
&= o(1), \quad n \to \infty
\end{aligned}
$$

□

Chapter 4

L^1-convergence of r-times differentiable Trigonometric Sine Sum

4.1 Introduction

Consider the trigonometric sine series as

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{4.1.1}$$

Ram and Kumari [5] introduced new modified sine sum as

$$\left(g_n(x) = \sum_{j=1}^{n} \left[\sum_{k=j}^{n} \Delta \left(\frac{a_k}{k} \right) \right] j sin(jx) \right)$$

The main aim of this chapter is to study the L^1 convergence of $r-times$ differentiable trigonometric sine series under extended classes $\widetilde{B}V_r$ and $\widetilde{C}_r$; (r=0,1,2,...) defined as follows:

Definition 4.1.1. [2] A null sequence $\{a_k\}$ belongs to the class $\widetilde{B}V_r$ (r=0,1,2,...) if

$$\sum_{k=1}^{\infty} k^{r+1} |\Delta b_k| < \infty$$

Example 4.1.2. Consider,

$$a_k = \sum_{j=k}^{\infty} \frac{1}{j^{r+2}}$$

$$\begin{aligned} a_k = kb_k = k\sum_{j=k}^{\infty} \Delta b_j &= \sum_{j=k}^{\infty} \frac{1}{j^{r+2}} \\ = \sum_{j=k}^{\infty} j\Delta b_j &= \sum_{j=k}^{\infty} \frac{1}{j^{r+2}} \\ = \sum_{j=k}^{\infty} \Delta b_j &= \sum_{j=k}^{\infty} \frac{1}{j^{r+3}} \end{aligned}$$

Thus,

$$\Delta b_j = \frac{1}{j^{r+3}} \quad r = 1, 2, 3, \ldots and \quad j = 1, 2, 3, \ldots.$$

$$\sum_{j=1}^{\infty} j^{r+1}|\Delta b_j| = \frac{1}{j^2} < \infty \quad (by \quad p - test)$$

This implies $\{a_k\} \in \widetilde{B}V_r$.

Definition 4.1.3. [2] A null sequence $\{a_k\}$ belongs to the class $\widetilde{C}_r$ (r=0,1,2,...) if for every $\epsilon > 0$, there exists $\delta > 0$, independent of n, such that for all n

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx \leq \epsilon.$$

Here, $D_k^{r+1}(x)$ denotes the $(r+1)^{th}$ derivative of Dirichlet Kernel.

Example 4.1.4. Consider,

$$\Delta b_j = \frac{1}{j^{r+3}} \quad r = 1, 2, 3, \ldots and j = 1, 2, 3, \ldots.$$

$$\begin{aligned}
\int_0^\pi \left| \sum_{k=n}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx &= \sum_{k=n}^{\infty} \frac{1}{k^{r+3}} \int_0^\pi |D_k^{r+1}(x)| dx \\
&= \sum_{k=n}^{\infty} \frac{1}{k^{r+3}} O(k^{r+1} \log k) \\
&= O\left(\sum_{k=n}^{\infty} \frac{k^{r+1}}{k^{r+3}} \log k \right) \\
&= O\left(\sum_{k=n}^{\infty} \frac{\log k}{k^2} \right)
\end{aligned}$$

Therefore,$\{a_k\} \in \widetilde{C}_r$

4.2 Lemma

Lemma 4.2.1. *[12] $\|D_n^r(x)\|_{L^1} = O\left(n^r logn\right)$, $r \in 0, 1, 2, \ldots$ where $D_n^r(x)$ represents the r^{th} derivative of the Dirichlet kernel.*

4.3 Main Result

The main result of this chapter reads as follows:

Theorem 4.3.1. *Let $\{a_k\}$ be the sequence of numbers belonging to the class $\widetilde{B}V_r \cap \widetilde{C}_r$, (r=0,1,2,...). Then $\|g_n^r(x) - g^r(x)\| = o(1)$; $n \to \infty$.*
Here, $g^r(x)$ is the rth derivative of $g(x)$,where $r = 0, 1, 2, \ldots$

Proof. Consider,

$$\begin{aligned} g_n(x) &= \sum_{j=1}^{n}\left[\sum_{k=j}^{n}\Delta\left(\frac{a_k}{k}\right)\right] j\sin(jx). \\ &= \sum_{j=1}^{n}\left[\frac{a_j}{j}-\frac{a_{n+1}}{n+1}\right] j\sin(jx). \\ &= \sum_{j=1}^{n} a_j\sin(jx)-\frac{a_{n+1}}{n+1}\sum_{j=1}^{n} j\sin(jx). \end{aligned}$$

Take $r - times$ differentiation of $g_n(x)$ we get,

$$\begin{aligned} g_n^r(x) &= S_n^r(x)-\frac{a_{n+1}}{n+1}\sum_{j=1}^{n} j^{r+1}\sin\left(jx+\frac{r\pi}{2}\right) \\ g_n^r(x) &= \sum_{j=1}^{n} a_j j^r\sin\left(jx+\frac{r\pi}{2}\right)-\frac{a_{n+1}}{n+1}\sum_{j=1}^{n} j^{r+1}\sin\left(jx+\frac{r\pi}{2}\right) \\ &= -\sum_{j=1}^{n} b_j j^{r+1}\cos\left(jx+(r+1)\frac{\pi}{2}\right)+\frac{a_{n+1}}{n+1}\sum_{j=1}^{n} j^{r+1}\cos\left(jx+(r+1)\frac{\pi}{2}\right) \end{aligned}$$

Applying Abel's Transformation

$$\begin{aligned} &= -\sum_{j=1}^{n-1}\Delta b_j D_j^{r+1}(x)-b_nD_n^{r+1}(x)+b_{n+1}D_j^{r+1}(x) \\ &= -\sum_{j=1}^{n-1}\Delta b_j D_j^{r+1}(x)-\Delta b_nD_n^{r+1}(x) \\ &= -\sum_{j=1}^{n}\Delta b_j D_j^{r+1}(x) \end{aligned}$$

Taking absolute summation of the above terms and using Lemma 4.2.1

$$\begin{aligned} \left|\sum_{j=1}^{n}\Delta b_j D_j^{r+1}(x)\right| &\leq \sum_{j=1}^{n}|\Delta b_j||D_j^{r+1}(x)|. \\ &\leq \sum_{j=1}^{n} Cj^{r+1}|\Delta b_j|. \end{aligned}$$

Where C is an absolute constant and $x \in [\epsilon, \pi]$.

$$= O\left(\sum_{j=1}^{n} j^{r+1}|\Delta b_j|\right)$$

By given hypothesis,

$$\lim_{n\to\infty} g_n^r(x) = g^r(x) \;\; exists \;\; in \;\; [\epsilon, \pi]$$

Next,consider

$$g^r(x) - g_n^r(x) = -\sum_{j=n+1}^{\infty} \Delta b_j D_j^{r+1}(x)$$

Given any $\epsilon > 0$, let there exist positive δ, such that

$$\int_0^{\delta} \left|\sum_{j=n+1}^{\infty} \Delta b_j D_j^{r+1}(x)\right| dx < \frac{\epsilon}{2} \; for \; all \; n \geq 0.$$

Then

$$\begin{aligned}
\|g^r(x) - g_n^r(x)\| &= \int_0^{\pi} \left|\sum_{j=n+1}^{\infty} \Delta b_j D_j^{r+1}(x)\right| dx \\
&= \int_0^{\delta} \left|\sum_{j=n+1}^{\infty} \Delta b_j D_j^{r+1}(x)\right| dx + \int_{\delta}^{\pi} \left|\sum_{j=n+1}^{\infty} \Delta b_j D_j^{r+1}(x)\right| dx \\
&\leq \frac{\epsilon}{2} + \sum_{j=n+1}^{\infty} \Delta b_j \int_{\delta}^{\pi} \left|D_j^{r+1}(x)\right| dx. \\
&\leq \frac{\epsilon}{2} + \sum_{j=n+1}^{\infty} C j^{r+1} \left|\Delta b_j\right| \int_{\delta}^{\pi} \frac{dx}{x^{r+2}}. \\
&= \frac{\epsilon}{2} + C\delta^{-(r+1)} \sum_{j=n+1}^{\infty} j^{r+1}|\Delta b_j| \\
&< \epsilon.
\end{aligned}$$

Therefore, $\|g^r(x) - g_n^r(x)\|_{L^1} = o(1)$ as $n \to \infty$.

□

Corollary 4.3.2. *Let $\{a_k\}$ be the sequence of numbers belonging to the class $\widetilde{B}V_r \cap \widetilde{C}_r$, and if*

$$\lim_{n\to\infty} n^r a_n logn = 0$$

then $\|S_n^r(x) - g^r(x)\| = o(1)$*;* $n \to \infty$*; r=0,1,2,....*

Proof.

$$\begin{aligned}
\|S_n^r(x) - g^r(x)\| &\leq \|S_n^r(x) - g_n^r(x)\| + \|g_n^r(x) - g^r(x)\| \\
&\leq \|S_n^r(x) - g_n^r(x)\| + o(1) \\
&\leq \int_0^\pi \left| \frac{a_{n+1}}{n+1} D_n^{r+1}(x) \right| dx \\
&= \frac{|a_{n+1}|}{n+1} n^{r+1} \log n \quad (by \quad Lemma \quad 4.2.1) \\
&= o(1), \; n \to \infty.
\end{aligned}$$

□

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