SOME METHODS FOR FINDING MULTIPLICATIVE INVERSE OF FUZZY AND INTUITIONISTIC FUZZY MATRICES

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Submitted by
Akriti Jindal
Reg. No.-301403001

Under the guidance of
Dr. Amit Kumar

School of Mathematics
Thapar University
Patiala-147004(PUNJAB)
INDIA

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CERTIFICATE

I hereby certify that the dissertation entitled, **"Some methods for finding multiplicative inverse of fuzzy and intuitionistic fuzzy matrices"**, which is being submitted by **Miss. Akriti Jindal** (Roll No. 301403001), in the partial fulfillment of the requirement for the award of the degree of Master of Science in the School of Mathematics, Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of **Dr. Amit Kumar** during the period from January 2016 to June 2016.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University/ Institute for the award of any degree.

Akriti Jindal 23/06/16.

Akriti Jindal

(301403001)

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

23/06/16

Dr. Amit Kumar
Associate Professor
School of Mathematics
Thapar University, Patiala

Countersigned by:

Dr. A.K. Lal
Associate Professor and Head
School of Mathematics
Thapar University, Patiala

Dr. S.S. Bhatia
Dean of Academic Affairs
Thapar University, Patiala

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Patiala
June 2016

Akriti Jindal

ABSTRACT

In this thesis, an existing method to find multiplicative inverse of fuzzy matrices (matrices whose elements are triangular fuzzy numbers) is presented as well as the flaws of this method are pointed out. Also, another existing method which was proposed to resolve these flaws, is presented. Furthermore, a new method to find multiplicative inverse of symmetric intuitionistic fuzzy matrices (matrices whose elements are symmetric triangular intuitionistic fuzzy numbers) as well as a new method to find multiplicative inverse of non-symmetric intuitionistic fuzzy matrices (matrices whose elements are non-symmetric triangular intuitionistic fuzzy numbers) are proposed.

Contents

Chapter 1

INTRODUCTION

It is well known that, if $A = (a_{ij})_{n\times n}$ is a non-singular matrix and all the values of a_{ij} are precise. Then, its multiplicative inverse $X = (x_{jk})_{n\times n}$, can be obtained by solving the system of linear equations (1.1)

$$\sum_{j=1}^{n} a_{ij}x_{jk} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases} \tag{1.1}$$

However, if there exist impreciseness about some/all values of a_{ij} then the multiplicative inverse X cannot be obtained by solving the system of linear equations (1.1). In the literature [8], it is pointed out that the imprecise data may be represented as a fuzzy number. The matrices in which some/all the elements are represented by fuzzy numbers are known as fuzzy matrix of fuzzy numbers and the system of linear equations in which some/all the parameters are represented by fuzzy numbers are known as system of fuzzy linear equations.

1.1 Preliminaries

In this section, some basic definitions related to fuzzy set theory as well as arithmetic operations of fuzzy numbers are presented.

1.1.1 Some basic definitions

In this section some basic definitions are presented.

Definition 1.1. [4] Let X be a classical set of objects. Then, the set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x) : x \in X\}$, where $\mu_{\tilde{A}} : X \to [0,1]$, is called a fuzzy set in X. The function $\mu_{\tilde{A}}$ is called the membership function.

Definition 1.2. [4] Let $\tilde{A}$ be a fuzzy set in X and $\alpha \in (0,1]$. The α-level set of the fuzzy set $\tilde{A}$ is the crisp set, denoted as $\tilde{A}_\alpha$ and is defined as $\tilde{A}_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$.

Definition 1.3. [4] The support of fuzzy set $\tilde{A}$, represented $Supp(\tilde{A})$, within a universal set X is a crisp set that contains all the elements of X that have a non-zero membership value in $\tilde{A}$ i.e., $Supp(\tilde{A}) = \{x \in X | \mu_{\tilde{A}}(x) > 0\}$.

Definition 1.4. [4] The height of a fuzzy set $\tilde{A}$, represented by $h(\tilde{A})$, is the largest membership grade obtained by any element in that set i.e., $h(\tilde{A}) = Supp\{\mu_{\tilde{A}}(x) | x \in X\}$.

Definition 1.5. [4] A fuzzy set $\tilde{A}$ is called normal when $h(\tilde{A}) = 1$.

Definition 1.6. [4] A fuzzy set $\tilde{A}$ in X is said to be fuzzy number if it satisfies the following four properties:

(i) $\tilde{A}$ is normal, i.e., there exists an $x \in R$(set of real numbers) such that $\mu_{\tilde{A}}(x) = 1$;

(ii) The membership function $\mu_{\tilde{A}}$ is quasi-concave, i.e., $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$ for all $\lambda \in [0,1]$;

(iii) The membership function $\mu_{\tilde{A}}$ is upper semi continuous, i.e., $\{x \in R : \mu_{\tilde{A}}(x) \geq \alpha\}$ is a closed subset of R for all $\alpha \in [0,1]$;

(iv) The 0-level set $\tilde{A}_0$ is compact(closed and bounded in R).

Definition 1.7. [5] A fuzzy number $\tilde{A} = (a, \alpha, \beta)$ is said to be a triangular fuzzy number if its membership function $\mu_{\tilde{A}}$ is given by,

$$\mu_{\tilde{A}}(a) = \begin{cases} \dfrac{x-(a-\alpha)}{\alpha} & , a-\alpha \leq x < a \\ 1 & , x = a \\ \dfrac{(a+\beta)-x}{\beta} & , a < x \leq a+\beta \\ 0 & , \text{otherwise} \end{cases}$$

If $\alpha = \beta$, then the triangular fuzzy number $\tilde{A} = (a, \alpha, \beta)$ is said to be symmetric triangular fuzzy number and is denoted as $\tilde{A} = (a, \alpha)$.

1.1.2 Arithmetic operations

In this section, some arithmetic operations of triangular fuzzy numbers are presented.[3] Let $\tilde{A}_1 = (a_1, \alpha_1, \beta_1)$ and $\tilde{A}_2 = (a_2, \alpha_2, \beta_2)$ be two triangular fuzzy numbers. Then,

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$.

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 = (a_1a_2, a_1\alpha_2 + a_2\alpha_1, a_1\beta_2 + a_2\beta_1)$.

(iii) $\tilde{A}_1 = \tilde{A}_2$ if and only if $a_1 = a_2$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$.

1.2 Literature review

Basaran [3] pointed out that if $\tilde{A} = (a_{ij})_{n\times n}$ is a matrix of fuzzy numbers then its multiplicative inverse $\tilde{X} = (x_{jk})_{n\times n}$ can't be obtained by solving the system of fuzzy linear equations (1.2) which is obtained by replacing the crisp values a_{ij}, x_{jk}, 1 and 0 of system of linear equations (1.1) with fuzzy numbers $\tilde{a}_{ij}$, $\tilde{x}_{jk}$, $\tilde{1}$ and $\tilde{0}$ respectively.

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_{jk} = \begin{cases} \tilde{1} & \text{if } i = k, \\ \tilde{0} & \text{if } i \neq k. \end{cases} \tag{1.2}$$

To handle this problem, Basaran [3] suggested that the spreads of the fuzzy numbers $\tilde{1}$ and $\tilde{0}$ should be considered as non-zero spreads δ and γ instead of 0 and 0 respectively. Furthermore, Basaran [3] shown that the multiplicative inverse of such a fuzzy matrix of fuzzy numbers whose some/all the elements are represented by symmetric triangular fuzzy numbers can be obtained by solving the system of fuzzy linear equations (1.3).

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_{jk} \approx \tilde{l}_{ik} \tag{1.3}$$

$$\text{where} \quad \tilde{l}_{ik} = \begin{cases} (1, \delta) & \text{if } i = k, \\ (0, \gamma) & \text{if } i \neq k. \end{cases}$$

Mosleh and Otadi [6] pointed out some drawbacks in the Basaran's method [3] and proposed a modified method to find the the multiplicative inverse of fuzzy matrices considered by Basaran [3].

In this thesis, Basaran's method [3], the drawbacks of Basaran's method [3], pointed out by Mosleh and Otadi [6] and the modified method, proposed by Mosleh and Otadi [6], are discussed in a detailed manner. Furthermore, need of intuitionistic fuzzy set [1,2] (generalization of fuzzy set), is discussed and new methods are proposed to find:

1. Multiplicative inverse of matrices having elements as symmetric intuitionistic triangular fuzzy numbers.

2. Multiplicative inverse of matrices having elements as non-symmetric intuitionistic triangular fuzzy numbers.

The chapter wise summary is as follows:

Chapter 1: Introduction

In this chapter some basic definitions (membership function, fuzzy set, fuzzy number, triangular fuzzy number etc.), arithmetic operations of fuzzy numbers as well as a brief review of the methods for finding multiplicative inverse of matrices whose elements are triangular fuzzy numbers is presented.

Chapter 2: Basaran's Method For Calculating Multiplicative Inverse Of Matrices Having Elements As Symmetric Triangular Fuzzy Numbers.

In this chapter, Basaran [3] proposed the following method to calculate the multiplicative inverse matrix $\tilde{X} = (\tilde{x}_{jk})_{n\times n}$ where $\tilde{x}_{jk}$ is a symmetric triangular fuzzy number, of a fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ where $\tilde{a}_{ij}$ is a symmetric triangular fuzzy number.

Chapter 3: Mosleh And Otadi Method For Calculating Multiplicative Inverse Of Matrices Having Elements As Symmetric Triangular Fuzzy Numbers.

In this chapter, Mosleh and Otadi [6] pointed out that the method, proposed by Basaran [3], is not valid and proposed a method by modifying the Basaran's method [3]. In this chapter, the mathematically incorrect assumptions, pointed out by Mosleh and Otadi [6] in Basaran's method [3], are discussed. Furthermore, the modified method, proposed by Mosleh and Otadi [6], is presented in a detailed manner.

Chapter 4: Proposed Mehar Method-I For Calculating Multiplicative Inverse Of Matrices Having Elements As Symmetric Triangular Intuitionistic Fuzzy Numbers.

In this chapter, Atanassov [1] introduced the concept of intuitionistic fuzzy set which is one of the important generalizations of fuzzy set theory. The major advantage of intuitionistic fuzzy set over fuzzy set is that intuitionistic fuzzy set separates the degree of membership (acceptance level) and the degree of non-membership (non-acceptance level) of an element in the set. Thus, this theory is found to be highly useful to deal with vagueness. Many authors have used fuzzy and intuitionistic fuzzy set theory for solving real life optimization problems such as planning, scheduling, transportation, manufacturing etc.

To the best of my knowledge, there is no method in the literature for calculating multiplicative inverse of such a matrix whose elements are intuitionistic fuzzy numbers. In this chapter, with the help of the method, discussed in previous chapter, a new method (named as Mehar method-I) is proposed for calculating the multiplicative inverse of such fuzzy matrices whose elements are symmetric triangular intuitionistic fuzzy numbers. To illustrate the proposed Mehar method-I the multiplicative inverse of a 2×2 intuitionistic fuzzy matrix is calculated.

Chapter 5: Proposed Mehar Method-II For Calculating Multiplicative Inverse Of Matrices Having Elements As Non-Symmetric Triangular Intuitionistic Fuzzy Numbers.

The Mehar method-I, proposed in previous chapter, can be used only to find the multiplicative inverse of such matrices whose elements are symmetric triangular intuitionistic fuzzy numbers. However, this method cannot be used to find multiplicative inverse of those matrices whose elements are non-symmetric triangular intuitionistic fuzzy numbers. In this chapter, a new method (named as Mehar method-II) is proposed to find the multiplicative inverse of these type of matrices.

Chapter 2

BASARAN'S METHOD FOR CALCULATING MULTIPLICATIVE INVERSE OF MATRICES HAVING ELEMENTS AS SYMMETRIC TRIANGULAR FUZZY NUMBERS

Basaran [3] proposed a method for calculating multiplicative inverse of such a matrix whose elements are symmetric triangular fuzzy numbers. In this chapter, the same method is presented in a detailed manner.

2.1 Basaran's method

Basaran [3] proposed the following method to calculate the multiplicative inverse matrix $\tilde{X} = (\tilde{x}_{jk})_{n\times n}$ where $\tilde{x}_{jk}$ is a symmetric triangular fuzzy number, of a fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ where $\tilde{a}_{ij}$ is a symmetric triangular fuzzy number.

Step 1: Since $\tilde{X}$ is assumed as multiplicative inverse of matrix $\tilde{A}$ so $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$, where $\tilde{I}$ is a $n \times n$ fuzzy matrix whose diagonal elements are symmetric triangular fuzzy numbers $(1, \delta)$ and remaining elements are symmetric triangular fuzzy numbers $(0, \gamma)$.

Step 2: The equation $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$ is equivalent to system of fuzzy linear equations

(2.1).

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_{jk} \approx \tilde{l}_{ik} \tag{2.1}$$

$$\text{where} \quad \tilde{l}_{ik} = \begin{cases} (1,\delta) & \text{if } i = k, \\ (0,\gamma) & \text{if } i \neq k. \end{cases}$$

Step 3: Assuming $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}), \tilde{x}_{jk} = (x_{jk}, \omega_{jk})$, the system of fuzzy linear equations (2.1) can be transformed into system of fuzzy linear equations (2.2).

$$\sum_{j=1}^{n} (a_{ij}, \alpha_{ij}) \otimes (x_{jk}, \omega_{jk}) \approx \begin{cases} (1,\delta) & \text{if } i = k, \\ (0,\gamma) & \text{if } i \neq k. \end{cases} \tag{2.2}$$

Step 4: Using the product of triangular fuzzy numbers, defined in Section 1.1.2, the system of fuzzy linear equations (2.2) can be transformed into system of fuzzy linear equations (2.3).

$$\sum_{j=1}^{n} (a_{ij}x_{jk}, a_{ij}\omega_{jk} + x_{jk}\alpha_{ij}) \approx \begin{cases} (1,\delta) & \text{if } i = k, \\ (0,\gamma) & \text{if } i \neq k. \end{cases} \tag{2.3}$$

Step 5: Using the property $\sum_{i=1}^{n}(a_i, \alpha_i) = (\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} \alpha_i)$, the system of fuzzy linear equations (2.3) can be transformed into system of fuzzy linear equations (2.4).

$$(\sum_{j=1}^{n} a_{ij}x_{jk}, \sum_{j=1}^{n} a_{ij}\omega_{jk} + x_{jk}\alpha_{ij}) \approx \begin{cases} (1,\delta) & \text{if } i = k, \\ (0,\gamma) & \text{if } i \neq k. \end{cases} \tag{2.4}$$

Step 6: Using the property $(a, \alpha) = (b, \beta) \Longrightarrow a = b$ and $\alpha = \beta$, the system of fuzzy linear equations (2.4) can be transformed into crisp system of linear equations (2.5) to (2.8).

$$\sum_{j=1}^{n} a_{ij}x_{jk} = 1, \quad \text{if } i = k, \tag{2.5}$$

$$\sum_{j=1}^{n} a_{ij}x_{jk} = 0, \quad \text{if } i \neq k, \tag{2.6}$$

$$\sum_{j=1}^{n} (a_{ij}\omega_{jk} + x_{jk}\alpha_{ij}) = \delta, \quad \text{if } i = k, \tag{2.7}$$

$$\sum_{j=1}^{n} (a_{ij}\omega_{jk} + x_{jk}\alpha_{ij}) = \gamma, \quad \text{if } i \neq k. \tag{2.8}$$

Step 7: Solve the crisp system of linear equations (2.5) and (2.6) to find the values of x_{jk}.

Step 8: Put the values of x_{jk}, obtained in Step 7, in crisp system of linear equations (2.7) and (2.8) and then solve these crisp system of linear equations by assuming some arbitrary values of δ and γ.

Step 9: Using the value of x_{jk}, obtained in Step 7, and values of ω_{jk}, obtained in Step 8, the multiplicative inverse of fuzzy matrix $\tilde{A}$ is $\tilde{X} = (x_{jk}, |\omega_{jk}|)_{n\times n}$.

2.2 Illustrative example

In this section, the Basaran method [3], discussed in Section 2.1, is illustrated with the help of a numerical example.

Using the Basaran's method [3] the multiplicative inverse of the fuzzy matrix $\tilde{A} = \begin{pmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{pmatrix}$ can be obtained as follows.

Step 1: If $\tilde{X}$ is assumed as multiplicative inverse of matrix $\tilde{A}$ then $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$ where $\tilde{I}$ is a 2×2 fuzzy matrix whose diagonal elements are $(1,\delta)$ and and remaining elements are $(0,\gamma)$ i.e., $\tilde{I} = \begin{pmatrix} (1,\delta) & (0,\gamma) \\ (0,\gamma) & (1,\delta) \end{pmatrix}$

Step 2: The equation $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$ is equivalent to system of fuzzy linear equations (2.9).

$$\begin{pmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{pmatrix} \otimes \begin{pmatrix} (x_{11},\omega_{11}) & (x_{12},\omega_{12}) \\ (x_{21},\omega_{21}) & (x_{22},\omega_{22}) \end{pmatrix} \approx \begin{pmatrix} (1,\delta) & (0,\gamma) \\ (0,\gamma) & (1,\delta) \end{pmatrix} \tag{2.9}$$

Step 3: The system of fuzzy linear equations (2.9) can be transformed into system of fuzzy linear equations (2.10).

$$\begin{aligned}
((10,4) \otimes (x_{11},\omega_{11})) \oplus ((8,3) \otimes (x_{21},\omega_{21})) &= (1,\delta), \\
((10,4) \otimes (x_{12},\omega_{12})) \oplus ((8,3) \otimes (x_{22},\omega_{22})) &= (0,\gamma), \\
((6,2) \otimes (x_{11},\omega_{11})) \oplus ((4,3) \otimes (x_{21},\omega_{21})) &= (0,\gamma), \\
((6,2) \otimes (x_{12},\omega_{12})) \oplus ((4,3) \otimes (x_{22},\omega_{22})) &= (1,\delta).
\end{aligned} \tag{2.10}$$

Step 4: Using product of triangular fuzzy numbers, defined in Section 1.2, the system of fuzzy linear equations (2.10) can be transformed into system of fuzzy linear equations (2.11).

$$\begin{aligned}
(10x_{11}, 10\omega_{11} + 4x_{11}) \oplus (8x_{21}, 8\omega_{21} + 3x_{21}) &= (1, \delta),\\
(10x_{12}, 10\omega_{12} + 4x_{12}) \oplus (8x_{22}, 8\omega_{22} + 3x_{22}) &= (0, \gamma),\\
(6x_{11}, 6\omega_{11} + 2x_{11}) \oplus (4x_{21}, 4\omega_{21} + 3x_{21}) &= (0, \gamma),\\
(6x_{12}, 6\omega_{12} + 2x_{12}) \oplus (4x_{22}, 4\omega_{22} + 3x_{22}) &= (1, \delta).
\end{aligned} \tag{2.11}$$

Step 5: Using the property $\sum_{i=1}^{n}(a_i, \alpha_i) = (\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} \alpha_i)$, the system of fuzzy linear equations (2.11) can be transformed into system of fuzzy linear equations (2.12).

$$\begin{aligned}
(10x_{11} + 8x_{21}, 10\omega_{11} + 4x_{11} + 8\omega_{21} + 3x_{21}) &= (1, \delta),\\
(10x_{12} + 8x_{22}, 10\omega_{12} + 4x_{12} + 8\omega_{22} + 3x_{22}) &= (0, \gamma),\\
(6x_{11} + 4x_{21}, 6\omega_{11} + 2x_{11} + 4\omega_{21} + 3x_{21}) &= (0, \gamma),\\
(6x_{12} + 4x_{22}, 6\omega_{12} + 2x_{12} + 4\omega_{22} + 3x_{22}) &= (1, \delta).
\end{aligned} \tag{2.12}$$

Step 6: Using the property $(a, \alpha) = (b, \beta) \implies a = b$ and $\alpha = \beta$, the system of fuzzy linear equations (2.12) can be transformed into crisp system of linear equations (2.13) and (2.14).

Center parts:

$$\begin{aligned}
10x_{11} + 8x_{21} &= 1,\\
10x_{12} + 8x_{22} &= 0,\\
6x_{11} + 4x_{21} &= 0,\\
6x_{12} + 4x_{22} &= 1.
\end{aligned} \tag{2.13}$$

Spread parts:

$$\begin{aligned}
10\omega_{11} + 4x_{11} + 8\omega_{21} + 3x_{21} &= \delta,\\
10\omega_{12} + 4x_{12} + 8\omega_{22} + 3x_{22} &= \gamma,\\
6\omega_{11} + 2x_{11} + 4\omega_{21} + 3x_{21} &= \gamma,\\
6\omega_{12} + 2x_{12} + 4\omega_{22} + 3x_{22} &= \delta.
\end{aligned} \tag{2.14}$$

Step 7: Solving the crisp system of linear equations (2.13), the obtained values of x_{11}, x_{12}, x_{21} and x_{22} are -0.5, 1, 0.75 and -1.25 respectively.

Step 8: Solving the crisp system of linear equations (2.14) after putting the values of x_{jk} where $j = 1, 2$ and $k = 1, 2$, obtained in Step 7 the obtained values are:

$$\begin{aligned} w_{11} &= \gamma - \frac{\delta}{2} - 1.13, \\ w_{21} &= \frac{3\delta}{4} - \frac{5\gamma}{4} - 1.38, \\ w_{12} &= \delta + 1.88 - \frac{\gamma}{2}, \\ w_{22} &= \frac{3\gamma}{4} - \frac{5\delta}{4} - 2.32. \end{aligned}$$

Assuming $\delta = 0.5$ and $\gamma = 0.5$, the obtained values are $|w_{11}| = 0.88$, $|w_{12}| = 2.13$, $|w_{21}| = 1.63$, $|w_{22}| = 2.57$.

Step 9: Using the values of x_{11}, x_{12}, x_{21} and x_{22}, obtained in Step 7,and the values of $|w_{11}|$, $|w_{12}|$, $|w_{21}|$ and $|w_{22}|$, obtained in Step 8, the obtained multiplicative inverse of fuzzy matrix $\tilde{A} = \begin{pmatrix} (10, 4) & (8, 3) \\ (6, 2) & (4, 3) \end{pmatrix}$ is

$$\tilde{X} = \begin{pmatrix} (-0.5, 0.88) & (1, 2.13) \\ (0.75, 1.63) & (-1.25, 2.57) \end{pmatrix}$$

2.3 Conclusion

The existing method [3] for finding the multiplicative inverse of such matrices whose elements are symmetric triangular fuzzy numbers, is presented. Furthermore, this existing method [3] is illustrated with the help of a numerical example.

Chapter 3

MOSLEH AND OTADI METHOD FOR CALCULATING MULTIPLICATIVE INVERSE OF MATRICES HAVING ELEMENTS AS SYMMETRIC TRIANGULAR FUZZY NUMBERS

Mosleh and Otadi [6] pointed out that the method, proposed by Basaran [3], is not valid and proposed a method by modifying the Basaran's method [3]. In this chapter, the mathematically incorrect assumptions, pointed out by Mosleh and Otadi [6] in Basaran's method [3], are discussed. Furthermore, the modified method, proposed by Mosleh and Otadi [6], is presented in a detailed manner.

3.1 Flaws in Basaran's method

Mosleh and Otadi [6] pointed out that Basaran's [3] has used the multiplication $(a_{ij}, \alpha_{ij}) \otimes (x_{jk}, \omega_{jk}) = (a_{ij}x_{jk}, a_{ij}\omega_{jk} + \alpha_{ij}x_{jk})$ which is valid only if $a_{ij} - \alpha_{ij} \geq 0$ as well as $x_{jk} - \omega_{jk} \geq 0$. However, for the elements of fuzzy matrix $\tilde{A}$ and elements of its multiplicative inverse $\tilde{X}$, this property will not be necessarily satisfied.

For example, for the elements of matrix $\tilde{A} = \begin{pmatrix} (-6,3) & (-4,2) \\ (-4,1) & (-3,1) \end{pmatrix}$ this property is not satisfying. Therefore, the multiplicative inverse of matrix $\tilde{A}$ cannot be obtained by Basaran's method [3]. Furthermore, the multiplicative inverse of

$\tilde{A} = \begin{pmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{pmatrix}$, obtained by Basaran method [3], is not valid as to obtain the same the elements of multiplicative inverse matrix are assumed as non-negative symmetric triangular fuzzy numbers i.e., it is assumed that for the elements (x_{jk}, ω_{jk}) of inverse matrix the property $x_{jk} - \omega_{jk} \geq 0$ will be satisfied. However, the elements of multiplicative inverse matrix may also be negative symmetric triangular fuzzy numbers i.e., $x_{jk} - \omega_{jk} < 0$. Hence, it is not genuine to use the Basaran's method [3] for calculating the multiplicative inverse of a fuzzy matrix.

3.2 Mosleh and Otadi method

Mosleh and Otadi [6] proposed the following method to find the multiplicative inverse of fuzzy matrix $\tilde{A}$.

Step 1: Solve the crisp system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_{jk} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

, obtained in Step 6 of Basaran's method [3], to find the values of x_{jk}.

Step 2: Let S_1 be the set of those values of x_{jk} which are greater than equal to zero and S_2 be the set of those values of x_{jk} which are less than zero. Similarly, let T_1 be the set of those values of a_{ij} which are greater than equal to zero and T_2 be the set of those values of a_{ij} which are less than zero. Then, the system of fuzzy linear equations (2.1) can be transformed into system of fuzzy linear equations (3.1).

$$\sum_{\substack{x_{jk} \in S_1 \\ a_{ij} \in T_1}} \tilde{a}_{ij} \otimes \tilde{x}_{jk} + \sum_{\substack{x_{jk} \in S_2 \\ a_{ij} \in T_1}} \tilde{a}_{ij} \otimes \tilde{x}_{jk} + \sum_{\substack{x_{jk} \in S_1 \\ a_{ij} \in T_2}} \tilde{a}_{ij} \otimes \tilde{x}_{jk} + \sum_{\substack{x_{jk} \in S_2 \\ a_{ij} \in T_2}} \tilde{a}_{ij} \otimes \tilde{x}_{jk} = \begin{cases} (1, \delta) & \text{if } i = k, \\ (0, \gamma) & \text{if } i \neq k. \end{cases} \tag{3.1}$$

Step 3: Assuming $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij})$, $\tilde{x}_{jk} = (x_{jk}, \omega_{jk})$, the system of fuzzy linear equations (3.1) can be transformed into system of fuzzy linear equations (3.2).

$$\begin{aligned} \sum_{\substack{x_{jk} \in S_1 \\ a_{ij} \in T_1}} (a_{ij}, \alpha_{ij}) \otimes (x_{jk}, \omega_{jk}) &+ \sum_{\substack{x_{jk} \in S_2 \\ a_{ij} \in T_1}} (a_{ij}, \alpha_{ij}) \otimes (x_{jk}, \omega_{jk}) + \sum_{\substack{x_{jk} \in S_1 \\ a_{ij} \in T_2}} (a_{ij}, \alpha_{ij}) \otimes (x_{jk}, \omega_{jk}) \\ &+ \sum_{\substack{x_{jk} \in S_2 \\ a_{ij} \in T_2}} (a_{ij}, \alpha_{ij}) \otimes (x_{jk}, \omega_{jk}) = \begin{cases} (1, \delta) & \text{if } i = k, \\ (0, \gamma) & \text{if } i \neq k. \end{cases} \end{aligned} \tag{3.2}$$

Step 4: The system of fuzzy linear equations (3.2) can be transformed into system of fuzzy linear equations (3.3).

$$\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_1}}(e_{ik},\lambda_{ik})+\sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_1}}(f_{ik},\mu_{ik})+\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_2}}(g_{ik},\nu_{ik})+\sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_2}}(h_{ik},\eta_{ik})=\begin{cases}(1,\delta) & \text{if } i=k,\\ (0,\gamma) & \text{if } i\neq k.\end{cases} \tag{3.3}$$

where

$$\begin{aligned}(e_{ik},\lambda_{ik}) &= (a_{ij}x_{jk}, a_{ij}\omega_{jk}+x_{jk}\alpha_{ij}),\\ (f_{ik},\mu_{ik}) &= (a_{ij}x_{jk}, a_{ij}\omega_{jk}-x_{jk}\alpha_{ij}),\\ (g_{ik},\nu_{ik}) &= (a_{ij}x_{jk}, x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}),\\ (h_{ik},\eta_{ik}) &= (a_{ij}x_{jk}, -x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}).\end{aligned}$$

Step 5: Using the property $\sum_{i=1}^{n}(a_i,\alpha_i)=(\sum_{i=1}^{n}a_i,\sum_{i=1}^{n}\alpha_i)$, the system of fuzzy linear equations (3.3) can be transformed into system of fuzzy linear equations (3.4).

$$\begin{aligned}(\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_1}}e_{ik},\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_1}}\lambda_{ik}) &+ (\sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_1}}f_{ik},\sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_1}}\mu_{ik})+(\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_2}}g_{ik},\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_2}}\nu_{ik})\\ &+ (\sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_2}}h_{ik},\sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_2}}\eta_{ik})=\begin{cases}(1,\delta) & \text{if } i=k,\\ (0,\gamma) & \text{if } i\neq k.\end{cases}\end{aligned} \tag{3.4}$$

Step 6: Using the property $(a,\alpha)=(b,\beta)\implies a=b$ and $\alpha=\beta$, the system of fuzzy linear equations (3.4) can be transformed into crisp system of linear equations (3.5).

$$\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_1}}\lambda_{ik}+\sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_1}}\mu_{ik}+\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_2}}\nu_{ik}+\sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_2}}\eta_{ik}=\begin{cases}\delta & \text{if } i=k,\\ \gamma & \text{if } i\neq k.\end{cases} \tag{3.5}$$

Step 7: Put the values of x_{jk}, obtained in Step 1, in crisp system of linear equations (3.5) and then solve these crisp system of linear equations by assuming some arbitrary values of δ and γ.

Step 8: Using the value of x_{jk}, obtained in Step 1, and values of ω_{jk}, obtained in Step 7, the multiplicative inverse of fuzzy matrix $\tilde{A}$ is $\tilde{X}=(x_{jk},|\omega_{jk}|)_{n\times n}$.

3.3 Illustrative examples

In this section, the Mosleh and Otadi method [6], discussed in Section 2.1, is illustrated with the help of numerical examples.

1. Using the Mosleh and Otadi method [6], the multiplicative inverse of the fuzzy matrix $\tilde{A} = \begin{pmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{pmatrix}$ can be obtained as follows.

   Step 1: If $\tilde{X}$ is assumed as multiplicative inverse of matrix $\tilde{A}$ then $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$ where $\tilde{I}$ is a 2×2 fuzzy matrix whose diagonal elements are $(1, \delta)$ and and remaining elements are $(0, \gamma)$ i.e. $\tilde{I} = \begin{pmatrix} (1,\delta) & (0,\gamma) \\ (0,\gamma) & (1,\delta) \end{pmatrix}$.

   Step 2: The equation $\tilde{A} \otimes \tilde{x} \approx \tilde{I}$ is equivalent to system of fuzzy linear equations (3.6).

$$\begin{pmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{pmatrix} \otimes \begin{pmatrix} (x_{11},\omega_{11}) & (x_{12},\omega_{12}) \\ (x_{21},\omega_{21}) & (x_{22},\omega_{22}) \end{pmatrix} \approx \begin{pmatrix} (1,\delta) & (0,\gamma) \\ (0,\gamma) & (1,\delta) \end{pmatrix} \quad (3.6)$$

   Step 3: The system of fuzzy linear equations (3.6) can be transformed into system of fuzzy linear equations (3.7).

$$\begin{aligned} ((10,4) \otimes (x_{11},\omega_{11})) \oplus ((8,3) \otimes (x_{21},\omega_{21})) &= (1,\delta), \\ ((10,4) \otimes (x_{12},\omega_{12})) \oplus ((8,3) \otimes (x_{22},\omega_{22})) &= (0,\gamma), \\ ((6,2) \otimes (x_{11},\omega_{11})) \oplus ((4,3) \otimes (x_{21},\omega_{21})) &= (0,\gamma), \\ ((6,2) \otimes (x_{12},\omega_{12})) \oplus ((4,3) \otimes (x_{22},\omega_{22})) &= (1,\delta). \end{aligned} \quad (3.7)$$

   Step 4: Using Step 4 to Step 6 of the Mosleh and Otadi method, discussed in Section 3.2, the system of fuzzy linear equations (3.7) can be transformed into system of crisp linear equations (3.8).

$$\begin{aligned} 10x_{11} + 8x_{21} &= 1, \\ 10x_{12} + 8x_{22} &= 0, \\ 6x_{11} + 4x_{21} &= 0, \\ 6x_{12} + 4x_{22} &= 1. \end{aligned} \quad (3.8)$$

Solving the system of crisp linear equations (3.8), the obtained values of x_{11}, x_{12}, x_{21} and x_{22} are -0.5, 1, 0.75 and -1.25 respectively.

Step 5: Now, substituting the obtained values obtained of x_{jk} where $j,k = 1,2$, the system of fuzzy linear equations (3.7) can be transformed into system of fuzzy linear equations (3.9).

$$\begin{aligned}
((10,4)\otimes(-0.5,\omega_{11}))\oplus((8,3)\otimes(0.75,\omega_{21})) &= (1,\delta),\\
((10,4)\otimes(1,\omega_{12}))\oplus((8,3)\otimes(-1.25,\omega_{22})) &= (0,\gamma),\\
((6,2)\otimes(-0.5,\omega_{11}))\oplus((4,3)\otimes(0.75,\omega_{21})) &= (0,\gamma),\\
((6,2)\otimes(1,\omega_{12}))\oplus((4,3)\otimes(-1.25,\omega_{22})) &= (1,\delta).
\end{aligned} \tag{3.9}$$

Step 6: Using the multiplication [5]

$$(a_{ij},\alpha_{ij})\otimes(x_{jk},\omega_{jk}) \begin{cases} (a_{ij}x_{jk}, a_{ij}\omega_{jk}+x_{jk}\alpha_{ij}) & \text{if } a_{ij}\geq 0, x_{jk}\geq 0,\\ (a_{ij}x_{jk}, a_{ij}\omega_{jk}-x_{jk}\alpha_{ij}) & \text{if } a_{ij}\geq 0, x_{jk}\leq 0,\\ (a_{ij}x_{jk}, x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}) & \text{if } a_{ij}\leq 0, x_{jk}\geq 0,\\ (a_{ij}x_{jk}, -x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}) & \text{if } a_{ij}\leq 0, x_{jk}\leq 0. \end{cases}$$

the system of fuzzy linear equations (3.9) can be transformed into system of fuzzy linear equations (3.10).

$$\begin{aligned}
(-5,10\omega_{11}+2)\oplus(6,8\omega_{21}+2.25) &= (1,\delta),\\
(10,10\omega_{12}+4)\oplus(-10,8\omega_{22}+3.75) &= (0,\gamma),\\
(-3,6\omega_{11}-1)\oplus(3,4\omega_{21}+2.25) &= (0,\gamma),\\
(6,6\omega_{12}+2)\oplus(-5,4\omega_{22}+3.75) &= (1,\delta).
\end{aligned} \tag{3.10}$$

Step 7: Using the property $(a,\alpha)\oplus(b,\beta) = (a+b,\alpha+\beta)$ the system of fuzzy linear equations (3.10) can be transformed into system of fuzzy linear equations (3.11).

$$\begin{aligned}
(-5+6,10\omega_{11}+2+8\omega_{21}+2.25) &= (1,\delta),\\
(10-10,10\omega_{12}+4+8\omega_{22}+3.75) &= (0,\gamma),\\
(-3+3,6\omega_{11}+1+4\omega_{21}+2.25) &= (0,\gamma),\\
(6-5,6\omega_{12}+2+4\omega_{22}+3.75) &= (1,\delta).
\end{aligned} \tag{3.11}$$

Step 8: Using the property $(a, \alpha) = (b, \beta) \implies a = b$ and $\alpha = \beta$, the system of fuzzy linear equations (3.11) can be transformed into crisp system of linear equations (3.12).

$$\begin{aligned} 10\omega_{11} + 2 + 8\omega_{21} + 2.25 &= \delta, \\ 10\omega_{12} + 4 + 8\omega_{22} + 3.75 &= \gamma, \\ 6\omega_{11} + 1 + 4\omega_{21} + 2.25 &= \gamma, \\ 6\omega_{12} + 2 + 4\omega_{22} + 3.75 &= \delta. \end{aligned} \tag{3.12}$$

Solving system of linear equations (3.12), the obtained values are

$$\begin{aligned} \omega_{11} &= \gamma - \frac{\delta}{2} - \frac{9}{8}, \\ \omega_{12} &= \delta - \frac{\gamma}{2} - \frac{15}{8}, \\ \omega_{21} &= \frac{7}{8} - \frac{5}{4}\gamma + \frac{3}{4}\delta, \\ \omega_{22} &= \frac{3}{4}\gamma - \frac{5}{4}\delta + \frac{11}{8}. \end{aligned}$$

Step 9: Assuming $\delta = 0.5$ and $\gamma = 0.5$, the obtained values are $|\omega_{11}| = 0.875$, $|\omega_{12}| = 1.625$, $|\omega_{21}| = 0.625$, $|\omega_{22}| = 1.125$.

Step 10: Using the values of x_{11}, x_{12}, x_{21} and x_{22}, obtained in Step 4, and the values of $|w_{11}|$, $|w_{12}|$, $|w_{21}|$ and $|w_{22}|$, obtained in Step 8, the obtained multiplicative inverse of fuzzy matrix $\tilde{A} = \begin{pmatrix} (10,4) & (8,3) \\ (6,2) & (4,3) \end{pmatrix}$ is

$\tilde{X} = \begin{pmatrix} (-0.5, 0.875) & (1, 1.625) \\ (0.75, 0.625) & (-1.25, 1.125) \end{pmatrix}$

2. Using the Mosleh and Otadi method [6], the multiplicative inverse of the fuzzy matrix $\tilde{A} = \begin{pmatrix} (-6,3) & (-4,2) \\ (-4,1) & (-3,1) \end{pmatrix}$ can be obtained as follows.

Step 1: If $\tilde{X}$ is assumed as multiplicative inverse of matrix $\tilde{A}$ then $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$ where $\tilde{I}$ is a 2×2 fuzzy matrix whose diagonal elements are $(1, \delta)$ and and remaining elements are $(0, \gamma)$ i.e., $\tilde{I} = \begin{pmatrix} (1,\delta) & (0,\gamma) \\ (0,\gamma) & (1,\delta) \end{pmatrix}$.

Step 2: The equation $\tilde{A} \otimes \tilde{x} \approx \tilde{I}$ is equivalent to

$$\begin{pmatrix} (-6,3) & (-4,2) \\ (-4,1) & (-3,1) \end{pmatrix} \otimes \begin{pmatrix} (x_{11},\omega_{11}) & (x_{12},\omega_{12}) \\ (x_{21},\omega_{21}) & (x_{22},\omega_{22}) \end{pmatrix} \approx \begin{pmatrix} (1,\delta) & (0,\gamma) \\ (0,\gamma) & (1,\delta) \end{pmatrix} \tag{3.13}$$

Step 3: The system of fuzzy linear equations (3.13) can be transformed into system of fuzzy linear equations (3.14).

$$\begin{aligned} ((-6,3) \otimes (x_{11},\omega_{11})) \oplus ((-4,2) \otimes (x_{21},\omega_{21})) &= (1,\delta), \\ ((-6,3) \otimes (x_{12},\omega_{12})) \oplus ((-4,2) \otimes (x_{22},\omega_{22})) &= (0,\gamma), \\ ((-4,1) \otimes (x_{11},\omega_{11})) \oplus ((-3,1) \otimes (x_{21},\omega_{21})) &= (0,\gamma), \\ ((-4,1) \otimes (x_{12},\omega_{12})) \oplus ((-3,1) \otimes (x_{22},\omega_{22})) &= (1,\delta). \end{aligned} \tag{3.14}$$

Step 4: Using Step 4 to Step 6 of the Mosleh and Otadi method [6], discussed in Section 3.2, the system of fuzzy linear equations (3.14) can be transformed into system of crisp linear equations (3.15).

$$\begin{aligned} -6x_{11} + (-4)x_{21} &= 1, \\ -6x_{12} + (-4)x_{22} &= 0, \\ -4x_{11} + (-3)x_{21} &= 0, \\ -4x_{12} + (-3)x_{22} &= 1. \end{aligned} \tag{3.15}$$

Solving the system of crisp linear equations (3.15), the obtained values of x_{11}, x_{12}, x_{21} and x_{22} are -1.5, 2, 2 and -3 respctively.

Step 5: Now, substituting the obtained values of x_{jk} where $j,k = 1,2$, the system of fuzzy linear equations (3.14) can be transformed into system of fuzzy linear equations (3.16)

$$\begin{aligned} ((-6,3) \otimes (-1.5,\omega_{11})) \oplus ((-4,2) \otimes (2,\omega_{21})) &= (1,\delta), \\ ((-6,3) \otimes (2,\omega_{12})) \oplus ((-4,2) \otimes (-3,\omega_{22})) &= (0,\gamma), \\ ((-4,1) \otimes (-1.5,\omega_{11})) \oplus ((-3,1) \otimes (2,\omega_{21})) &= (0,\gamma), \\ ((-4,1) \otimes (2,\omega_{12})) \oplus ((-3,1) \otimes (-3,\omega_{22})) &= (1,\delta). \end{aligned} \tag{3.16}$$

Step 6: Using the multiplication [5]

$$(a_{ij},\alpha_{ij})\otimes(x_{jk},\omega_{jk})\begin{cases}(a_{ij}x_{jk},a_{ij}\omega_{jk}+x_{jk}\alpha_{ij}) & \text{if } a_{ij}\geq 0, x_{jk}\geq 0,\\ (a_{ij}x_{jk},a_{ij}\omega_{jk}-x_{jk}\alpha_{ij}) & \text{if } a_{ij}\geq 0, x_{jk}\leq 0,\\ (a_{ij}x_{jk},x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}) & \text{if } a_{ij}\leq 0, x_{jk}\geq 0,\\ (a_{ij}x_{jk},-x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}) & \text{if } a_{ij}\leq 0, x_{jk}\leq 0.\end{cases}$$

,the system of fuzzy linear equations (3.16) can be transformed into system of fuzzy linear equations (3.17)

$$\begin{aligned}(9,4.5+6\omega_{11})\oplus(-8,4+4\omega_{21})&=(1,\delta),\\(-12,6+6\omega_{12})\oplus(12,6+4\omega_{22})&=(0,\gamma),\\(6,1.5+4\omega_{11})\oplus(-6,2+3\omega_{21})&=(0,\gamma),\\(-8,2+4\omega_{12})\oplus(9,3+3\omega_{22})&=(1,\delta).\end{aligned}\tag{3.17}$$

Step 7: Using the property $(a,\alpha)\oplus(b,\beta)=(a+b,\alpha+\beta)$ the system of fuzzy linear equations (3.17) can be transformed into system of fuzzy linear equations (3.18).

$$\begin{aligned}(9-8,4.5+6\omega_{11}+4+4\omega_{21})&=(1,\delta),\\(-12+12,6+6\omega_{12}+6+4\omega_{22})&=(0,\gamma),\\(6-6,1.5+4\omega_{11}+2+3\omega_{21})&=(0,\gamma),\\(-8+9,2+4\omega_{12}+3+3\omega_{22})&=(1,\delta).\end{aligned}\tag{3.18}$$

Step 8: Using the property $(a,\alpha)=(b,\beta)\Longrightarrow a=b$ and $\alpha=\beta$, the system of fuzzy linear equations (3.18) can be transformed into crisp system of linear equations (3.19).

$$\begin{aligned}4.5+6\omega_{11}+4+4\omega_{21}&=\delta,\\6+6\omega_{12}+6+4\omega_{22}&=\gamma,\\1.5+4\omega_{11}+2+3\omega_{21}&=\gamma,\\2+4\omega_{12}+3+3\omega_{22}&=\delta.\end{aligned}\tag{3.19}$$

Solving system of linear equations (3.19), the obtained values are

$$\begin{aligned}
\omega_{11} &= \frac{3}{2}\delta - 2\gamma - 5.75, \\
\omega_{12} &= 3\delta - 2\gamma + 9, \\
\omega_{21} &= \frac{13}{2} + 3\gamma - 2\delta, \\
\omega_{22} &= \frac{13}{4}\gamma - \frac{9}{2}\delta - \frac{33}{2}.
\end{aligned}$$

Step 9: Assuming $\delta = 0.5$ and $\gamma = 0.5$, the obtained values are $|\omega_{11}| = 6$, $|\omega_{12}| = 8.25$, $|\omega_{21}| = 7$, $|\omega_{22}| = 9.5$.

Step 10: Using the values of x_{11}, x_{12}, x_{21} and x_{22}, obtained in Step 4,and the values of $|w_{11}|$, $|w_{12}|$, $|w_{21}|$ and $|w_{22}|$, obtained in Step 8, the obtained multiplicative inverse of fuzzy matrix $\tilde{A} = \begin{pmatrix} (-6,3) & (-4,2) \\ (-4,1) & (-3,1) \end{pmatrix}$ is

$$\tilde{X} = \begin{pmatrix} (-1.5,6) & (2,8.25) \\ (2,7) & (-3,9.5) \end{pmatrix}.$$

3.4 Conclusion

The flaws in the existing method [3], pointed out by Mosleh and Otadi [6] are discussed. Also, the method, proposed by Mosleh and Otadi [6], for finding the multiplicative inverse of such matrices whose elements are symmetric triangular fuzzy numbers, is discussed. Furthermore, the method, proposed by Mosleh and Otadi [6], is illustrated with the help of numerical examples.

Chapter 4

PROPOSED MEHAR METHOD-I FOR CALCULATING MULTIPLICATIVE INVERSE OF MATRICES HAVING ELEMENTS AS SYMMETRIC TRIANGULAR INTUITIONISTIC FUZZY NUMBERS

Atanassov [1] introduced the concept of intuitionistic fuzzy set which is one of the important generalizations of fuzzy set theory. The major advantage of intuitionistic fuzzy set over fuzzy set is that intuitionistic fuzzy set separates the degree of membership (acceptance level) and the degree of non-membership (non-acceptance level) of an element in the set. Thus, this theory is found to be highly useful to deal with vagueness. Many authors have used fuzzy and intuitionistic fuzzy set theory for solving real life optimization problems such as planning, scheduling, transportation, manufacturing etc.

To the best of my knowledge, there is no method in the literature for calculating multiplicative inverse of such a matrix whose elements are intuitionistic fuzzy numbers. In this chapter, with the help of the method, discussed in previous chapter, a new method (named as Mehar method-I) is proposed for calculating the multiplicative inverse of such fuzzy matrices whose elements are symmetric triangular intuitionistic fuzzy numbers. To illustrate the proposed Mehar method-I the multiplicative inverse of a 2×2 intuitionistic fuzzy matrix is calculated.

4.1 Preliminaries

In this section, some basic definitions related to intuitionistic fuzzy set theory are presented [7].

4.1.1 Some basic definitions

In this section, some basic definitions related to intuitionistic fuzzy set theory are presented.

Definition 4.1. Let X be the universe of discourse. Then an intuitionistic fuzzy set $\tilde{A}^I$ in X is defined by a set of ordered triples

$$\tilde{A}^I = \{(x, \mu_{A^I}(x), \nu_{A^I}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x) : X \to [0, 1]$$

are functions such that $0 \leq \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leq 1, \forall x \in X$.
Here, $\mu_{\tilde{A}^I}$ represents the degree of membership function and $\nu_{\tilde{A}^I}$ represents the degree of non-membership function. $h(x) = 1 - (\mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x))$ is called the degree of hesitation for $x \in X$ being in $\tilde{A}$.

Definition 4.2. An intuitionistic fuzzy subset $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in X\}$ of the real line $\mathbb{R}$ is called an intuitionistic fuzzy number if the following hold:

1. There exists $r \in R$ such that $\mu_{\tilde{A}^I}(r) = 1$ and $\nu_{\tilde{A}^I}(r) = 0$ (r is called the mean value of $\tilde{A}^I$).

2. $\mu_{\tilde{A}^I}(x)$ and $\nu_{\tilde{A}^I}(x)$ are piecewise continuous mapping from $\mathbb{R}$ to the closed interval [0,1] and the relation $0 \leq \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leq 1, \forall x \in X$ holds.
   The membership and non-membership functions of $\tilde{A}$ are of the following form:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} g_1(x) & , r - \alpha \leq x < r, \\ 1 & , x = r, \\ h_1(x) & , r < x \leq r + \beta, \\ 0 & , \text{otherwise.} \end{cases}$$

where $g_1(x)$, $h_1(x)$ are piecewise continuous; strictly increasing and strictly decreasing function in $[r-\alpha, r)$ and $(r, r+\beta]$ respectively.

$$\nu_{\tilde{A}^I}(x) = \begin{cases} g_2(x) & , r-\alpha^{'} \leq x < r; 0 \leq g_1(x)+g_2(x) \leq 1, \\ 0 & , x = r, \\ h_2(x) & , r < x \leq r+\beta^{'}; 0 \leq h_1(x)+h_2(x) \leq 1, \\ 1 & , \text{otherwise.} \end{cases}$$

Here r is the mean value of $\tilde{A}^I$; α and β are the left and right spreads of membership function $\mu_{\tilde{A}^I}$ respectively, $\alpha^{'}$ and $\beta^{'}$ are the left and right spreads of membership function $\nu_{\tilde{A}}$ respectively. The intuitionistic fuzzy number $\tilde{A}^I$ is represented by $\tilde{A}^I = (r, \alpha, \beta; r, \alpha^{'}, \beta^{'})$.

Definition 4.3. A triangular intuitionistic fuzzy number $\tilde{A}^I = (a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'})$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the membership function $\mu_{\tilde{A}^I}$ and nonmembership function $\nu_{\tilde{A}^I}$ is defined by

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & , a_1 \leq x < a_2, \\ 1 & , x = a_2, \\ \dfrac{a_3-x}{a_3-a_2} & , a_2 < x \leq a_3, \\ 0 & , \text{otherwise.} \end{cases}$$

and

$$\nu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{a_2-x}{a_2-a_1^{'}} & , a_1^{'} \leq x < a_2, \\ 0 & , x = a_2, \\ \dfrac{x-a_2}{a_3^{'}-a_2} & , a_2 < x \leq a_3^{'}, \\ 1 & , otherwise. \end{cases}$$

respectively, where $a_1^{'} \leq a_1 < a_2 < a_3 \leq a_3^{'}$. The triangular intuitionistic fuzzy number $\tilde{A}^I = (a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'})$ also denoted by $\tilde{A}^I = (a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'})$ or $\tilde{A}^I = (a_2, \alpha, \beta; a_2, \alpha^{'}, \beta^{'})$ where $\alpha = a_2 - a_1$, $\beta = a_3 - a_2$, $\alpha^{'} = a_2 - a_1^{'}$ and $\beta^{'} = a_3^{'} - a_2$.

Definition 4.4. A triangular intuitionistic fuzzy number $\tilde{A}^I = (a, \alpha, \beta; a, \alpha^{'}, \beta^{'})$ whose left spread α is equal to right spread β of a membership function as well as left spread α' is equal to right spread β' of a non-membership function is called symmetric triangular intuitionistic fuzzy number and is denoted by $\tilde{A}^I = (a, \alpha; a, \alpha')$.

4.2 Proposed Mehar method-I

In this section, a new method (named as Mehar method-I) is proposed for calculating the multiplicative inverse of such intuitionistic fuzzy matrices whose elements are symmetric triangular intuitionistic fuzzy numbers.

The steps of the proposed Mehar method are as follows:

Step 1: Since $\tilde{X}$ is assumed as multiplicative inverse of intuitionistic fuzzy matrix $\tilde{A}$ so $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$ where $\tilde{I}$ is $n \times n$ intuitionistic fuzzy matrix whose diagonal elements are $(1, \delta; 1, \delta')$ and remaining elements are $(0, \gamma; 0, \gamma')$.

Step 2: The equation $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$ is equivalent to system of intuitionistic fuzzy linear equations (4.1).

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_{jk} \approx \tilde{l}_{ik} \tag{4.1}$$

$$\text{where,} \quad \tilde{l}_{ik} = \begin{cases} (1, \delta; 1, \delta') & \text{if } i = k, \\ (0, \gamma; 0, \gamma') & \text{if } i \neq k. \end{cases}$$

Step 3: Assuming $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}; a_{ij}, \alpha'_{ij}), \tilde{x}_{jk} = (x_{jk}, \omega_{jk}; x_{jk}, \omega'_{jk})$, the system of intuitionistic fuzzy linear equations (4.1) can be transformed into system of intuitionistic fuzzy linear equations (4.2).

$$\sum_{j=1}^{n} (a_{ij}, \alpha_{ij}; a_{ij}, \alpha'_{ij}) \otimes (x_{jk}, \omega_{jk}; x_{jk}, \omega'_{jk}) \approx \begin{cases} (1, \delta; 1, \delta') & \text{if } i = k, \\ (0, \gamma; 0, \gamma') & \text{if } i \neq k. \end{cases} \tag{4.2}$$

Step 4: Using the property of intuitionistic fuzzy number $(a_1, \alpha_1; a_1, \beta_1) \otimes (a_2, \alpha_2; a_2, \beta_2) \approx ((a_1, \alpha_1) \otimes (a_2, \alpha_2); (a_1, \beta_1) \otimes (a_2, \beta_2))$, the system of intuitionistic fuzzy linear equations (4.2) can be transformed into system of intuitionistic fuzzy linear equations (4.3).

$$\sum_{j=1}^{n} ((a_{ij}, \alpha_{ij}) \otimes (x_{jk}, \omega_{jk}); (a_{ij}, \alpha'_{ij}) \otimes (x_{jk}, \omega'_{jk})) \approx \begin{cases} (1, \delta; 1, \delta') & \text{if } i = k, \\ (0, \gamma; 0, \gamma') & \text{if } i \neq k. \end{cases} \tag{4.3}$$

Step 5: Using the property of intuitionistic fuzzy number $(a_1, \alpha_1; a_1, \beta_1) = (a_2, \alpha_2; a_2, \beta_2)$ $\implies (a_1, \alpha_1) = (a_2, \alpha_2)$ and $(a_1, \beta_1) = (a_2, \beta_2)$, the system of intuitionistic fuzzy linear equations (4.3) can be transformed into system of fuzzy linear equations (4.4) and (4.5).

$$\sum_{j=1}^{n}((a_{ij}, \alpha_{ij}) \otimes (x_{jk}, \omega_{jk})) = \begin{cases} (1, \delta) & \text{if } i = k, \\ (0, \gamma) & \text{if } i \neq k. \end{cases} \tag{4.4}$$

and

$$\sum_{j=1}^{n}((a_{ij}, \alpha'_{ij}) \otimes (x_{jk}, \omega'_{jk})) = \begin{cases} (1, \delta') & \text{if } i = k, \\ (0, \gamma') & \text{if } i \neq k. \end{cases} \tag{4.5}$$

Step 6: Now, apply Mosleh and Otadi method [6], discussed in Section 2.2, on system of fuzzy linear equations (4.4) and (4.5) individually, to obtain the values of x_{jk}, $|\omega_{jk}|$ and $|\omega'_{jk}|$, where $j = 1, \ldots, n$ and $k = 1, \ldots, n$.

Step 7: Using the values of x_{jk}, $|\omega_{jk}|$ and $|\omega'_{jk}|$, obtained in Step 7, the multiplicative inverse of the intuitionistic fuzzy matrix $\tilde{A}$ is $\tilde{X} = (x_{jk}, |\omega_{jk}|; x_{jk}, |\omega'_{jk}|)_{n\times n}$

4.3 Illustrative example

In this section, Mehar method-I, proposed in Section 4.1, is illustrated with the help of a numerical example.

Using the proposed Mehar method-I, the multiplicative inverse of the intuitionistic fuzzy matrix $\tilde{A} = \begin{pmatrix} (-2, 1; -2, 2) & (3, 2; 3, 10) \\ (-5, 6; -5, 8) & (-6, 3; -6, 6) \end{pmatrix}$ can be obtained as follows.

Step 1: If $\tilde{X}$ is assumed as multiplicative inverse of matrix $\tilde{A}$ then $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$ where $\tilde{I}$ is a 2×2 intuitionistic fuzzy matrix whose diagonal elements are $(1, \delta; 1, \delta')$ and and remaining elements are $(0, \gamma; 0, \gamma')$ i.e., $\tilde{I} = \begin{pmatrix} (1, \delta; 1, \delta') & (0, \gamma; 0, \gamma') \\ (0, \gamma; 0, \gamma') & (1, \delta; 1, \delta') \end{pmatrix}$

Step 2: The equation $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$ is equivalent to

$$\begin{pmatrix} (-2, 1; -2, 2) & (3, 2; 3, 10) \\ (-5, 6; -5, 4) & (-6, 3; -6, 6) \end{pmatrix} \otimes \begin{pmatrix} (x_{11}, \omega_{11}; x_{11}, \omega'_{11}) & (x_{12}, \omega_{12}; x_{12}, \omega'_{12}) \\ (x_{21}, \omega_{21}; x_{21}, \omega'_{21}) & (x_{22}, \omega_{22}; x_{22}, \omega'_{22}) \end{pmatrix}$$

$$\approx \begin{pmatrix} (1, \delta; 1, \delta') & (0, \gamma; 0, \gamma') \\ (0, \gamma; 0, \gamma') & (1, \delta; 1, \delta') \end{pmatrix} \tag{4.6}$$

Step 3: The system of intuitionistic fuzzy linear equations (4.6) can be transformed into system of intuitionistic fuzzy linear equations (4.7).

$$\begin{aligned}
&((-2,1;-2,2)\otimes(x_{11},\omega_{11};x_{11},\omega_{11}^{'}))\oplus((3,2;3,10)\otimes(x_{21},\omega_{21};x_{21},\omega_{21}^{'}))=(1,\delta;1,\delta^{'}),\\
&((-2,1;-2,2)\otimes(x_{12},\omega_{12};x_{12},\omega_{12}^{'}))\oplus((3,2;3,10)\otimes(x_{22},\omega_{22};x_{22},\omega_{22}^{'}))=(0,\gamma;0,\gamma^{'}),\\
&((-5,6;-5,8)\otimes(x_{11},\omega_{11};x_{11},\omega_{11}^{'}))\oplus((-6,3;-6,6)\otimes(x_{21},\omega_{21};x_{21},\omega_{21}^{'}))=(0,\gamma;0,\gamma^{'}),\\
&((-5,6;-5,8)\otimes(x_{12},\omega_{12};x_{12},\omega_{12}^{'}))\oplus((-6,3;-6,6)\otimes(x_{22},\omega_{22};x_{22},\omega_{22}^{'}))=(1,\delta;1,\delta^{'}).
\end{aligned}\tag{4.7}$$

Step 4: Using the property $(a_1,\alpha_1;a_1,\beta_1)\otimes(a_2,\alpha_2;a_2,\beta_2)\Longrightarrow((a_1,\alpha_1)\otimes(a_2,\alpha_2);(a_1,\beta_1)\otimes(a_2,\beta_2))$, the system of intuitionistic fuzzy linear equations (4.7) can be transformed into system of intuitionistic fuzzy linear equations (4.8).

$$\begin{aligned}
&((-2,1)\otimes(x_{11},\omega_{11});(-2,2)\otimes(x_{11},\omega_{11}^{'}))\oplus((3,2)\otimes(x_{21},\omega_{21});(3,10)\otimes(x_{21},\omega_{21}^{'}))=(1,\delta;1,\delta^{'}),\\
&((-2,1)\otimes(x_{12},\omega_{12});(-2,2)\otimes(x_{12},\omega_{12}^{'}))\oplus((3,2)\otimes(x_{22},\omega_{22});(3,10)\otimes(x_{22},\omega_{22}^{'}))=(0,\gamma;0,\gamma^{'}),\\
&((-5,6)\otimes(x_{11},\omega_{11});(-5,4)\otimes(x_{11},\omega_{11}^{'}))\oplus((-6,3)\otimes(x_{21},\omega_{21});(-6,6)\otimes(x_{21},\omega_{21}^{'}))=(0,\gamma;0,\gamma^{'}),\\
&((-5,6)\otimes(x_{12},\omega_{12});(-5,4)\otimes(x_{12},\omega_{12}^{'}))\oplus((-6,3)\otimes(x_{22},\omega_{22});(-6,6)\otimes(x_{22},\omega_{22}^{'}))=(1,\delta;1,\delta^{'}).
\end{aligned}\tag{4.8}$$

Step 5: Using the property $(a_1,\alpha_1;a_1,\beta_1)\oplus(a_2,\alpha_2;a_2,\beta_2)\Longrightarrow((a_1,\alpha_1)\oplus(a_2,\alpha_2);(a_1,\beta_1)\oplus(a_2,\beta_2))$, the system of intuitionistic fuzzy linear equations (4.8) can be transformed into system of intuitionistic fuzzy linear equations (4.9).

$$\begin{aligned}
&(((-2,1)\otimes(x_{11},\omega_{11}))\oplus((3,2)\otimes(x_{21},\omega_{21}));(((-2,2)\otimes(x_{11},\omega_{11}^{'}))\oplus((3,10)\otimes(x_{21},\omega_{21}^{'})))=(1,\delta;1,\delta^{'}),\\
&(((-2,1)\otimes(x_{12},\omega_{12})\oplus(3,2)\otimes(x_{22},\omega_{22}));(((-2,2)\otimes(x_{12},\omega_{12}^{'}))\oplus(3,10)\otimes(x_{22},\omega_{22}^{'})))=(0,\gamma;0,\gamma^{'}),\\
&(((-5,6)\otimes(x_{11},\omega_{11})\oplus(-6,3)\otimes(x_{21},\omega_{21}));(((-5,8)\otimes(x_{11},\omega_{11}^{'}))\oplus(-6,6)\otimes(x_{21},\omega_{21}^{'})))=(0,\gamma;0,\gamma^{'}),\\
&(((-5,6)\otimes(x_{12},\omega_{12})\oplus(-6,3)\otimes(x_{22},\omega_{22}));(((-5,8)\otimes(x_{12},\omega_{12}^{'}))\oplus(-6,6)\otimes(x_{22},\omega_{22}^{'})))=(1,\delta;1,\delta^{'}).
\end{aligned}\tag{4.9}$$

Step 6: Using the property $(a_1,\alpha_1;a_1,\beta_1)=(a_2,\alpha_2;a_2,\beta_2)\Longrightarrow(a_1,\alpha_1)=(a_2,\alpha_2)$ and $(a_1,\beta_1)=(a_2,\beta_2)$, the system of intuitionistic fuzzy linear equations (4.9) can be transformed into system of fuzzy linear equations (4.10) and (4.11)

$$\begin{aligned}
((-2,1)\otimes(x_{11},\omega_{11}))\oplus((3,2)\otimes(x_{21},\omega_{21})) &= (1,\delta),\\
((-2,1)\otimes(x_{12},\omega_{12}))\oplus((3,2)\otimes(x_{22},\omega_{22})) &= (0,\gamma),\\
((-5,6)\otimes(x_{11},\omega_{11}))\oplus((-6,3)\otimes(x_{21},\omega_{21})) &= (0,\gamma),\\
((-5,6)\otimes(x_{12},\omega_{12}))\oplus((-6,3)\otimes(x_{22},\omega_{22})) &= (1,\delta).
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
((-2,2)\otimes(x_{11},\omega_{11}^{'}))\oplus((3,10)\otimes(x_{21},\omega_{21}^{'})) &= (1,\delta^{'}),\\
((-2,2)\otimes(x_{12},\omega_{12}^{'}))\oplus((3,10)\otimes(x_{22},\omega_{22}^{'})) &= (0,\gamma^{'}),\\
((-5,8)\otimes(x_{11},\omega_{11}^{'}))\oplus((-6,6)\otimes(x_{21},\omega_{21}^{'})) &= (0,\gamma^{'}),\\
((-5,8)\otimes(x_{12},\omega_{12}^{'}))\oplus((-6,6)\otimes(x_{22},\omega_{22}^{'})) &= (1,\delta^{'}).
\end{aligned} \tag{4.11}$$

Step 7: Using Step 4 to Step 6 of the Mosleh and Otadi method [6], discussed in Section 3.2, system of fuzzy linear equations (4.10) can be transformed into system of crisp linear equations (4.12)

$$\begin{aligned}
-2x_{11}+3x_{21} &= 1,\\
-2x_{12}+3x_{22} &= 0,\\
-5x_{11}-6x_{21} &= 0,\\
-5x_{12}-6x_{22} &= 1.
\end{aligned} \tag{4.12}$$

Solving the system of crisp linear equations (4.12), the obtained values of x_{11}, x_{12}, x_{21} and x_{22} are -0.222, -0.111, 0.185 and -0.0741 respectively.

Step 8: Now, substituting the values obtained of x_{jk} where $j,k=1,2$, the system of fuzzy linear equations (4.10) and (4.11) can be transformed into system of fuzzy linear equations as (4.13) and (4.14)

$$\begin{aligned}
((-2,1)\otimes(-0.222,\omega_{11}))\oplus((3,2)\otimes(0.185,\omega_{21})) &= (1,\delta),\\
((-2,1)\otimes(-0.111,\omega_{12}))\oplus((3,2)\otimes(-0.0741,\omega_{22})) &= (0,\gamma),\\
((-5,6)\otimes(-0.222,\omega_{11}))\oplus((-6,3)\otimes(0.185,\omega_{21})) &= (0,\gamma),\\
((-5,6)\otimes(-0.111,\omega_{12}))\oplus((-6,3)\otimes(-0.0741,\omega_{22})) &= (1,\delta).
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
((-2,2)\otimes(-0.222,\omega_{11}^{'}))\oplus((3,10)\otimes(0.185,\omega_{21}^{'})) &= (1,\delta^{'}),\\
((-2,2)\otimes(-0.111,\omega_{12}^{'}))\oplus((3,10)\otimes(-0.0741,\omega_{22}^{'})) &= (0,\gamma^{'}),\\
((-5,8)\otimes(-0.222,\omega_{11}^{'}))\oplus((-6,6)\otimes(0.185,\omega_{21}^{'})) &= (0,\gamma^{'}),\\
((-5,8)\otimes(-0.111,\omega_{12}^{'}))\oplus((-6,6)\otimes(-0.0741,\omega_{22}^{'})) &= (1,\delta^{'}).
\end{aligned} \tag{4.14}$$

Step 9: Using the multiplication [5]

$$(a_{ij},\alpha_{ij})\otimes(x_{jk},\omega_{jk})\begin{cases}(a_{ij}x_{jk}, a_{ij}\omega_{jk}+x_{jk}\alpha_{ij}) & \text{if } a_{ij}\geq 0, x_{jk}\geq 0,\\ (a_{ij}x_{jk}, a_{ij}\omega_{jk}-x_{jk}\alpha_{ij}) & \text{if } a_{ij}\geq 0, x_{jk}\leq 0,\\ (a_{ij}x_{jk}, x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}) & \text{if } a_{ij}\leq 0, x_{jk}\geq 0,\\ (a_{ij}x_{jk}, -x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}) & \text{if } a_{ij}\leq 0, x_{jk}\leq 0.\end{cases}$$

the system of fuzzy linear equations (4.13) and (4.14) can be transformed into system of fuzzy linear equations (4.15) and (4.16) respectively.

$$\begin{aligned}(0.444, 0.222+2\omega_{11})\oplus(0.555, 3\omega_{21}+0.37) &= (1,\delta),\\ (0.222, 0.111+2\omega_{12})\oplus(-0.2223, 3\omega_{22}+0.1482) &= (0,\gamma),\\ (1.11, 1.332+5\omega_{11})\oplus(-1.11, 0.555+6\omega_{21}) &= (0,\gamma),\\ (0.555, 0.666+5\omega_{12})\oplus(0.444, 0.2223+6\omega 22) &= (1,\delta).\end{aligned}\tag{4.15}$$

and

$$\begin{aligned}(0.444, 0.444+2\omega_{11}^{'})\oplus(0.555, 3\omega_{21}^{'}+1.85) &= (1,\delta),\\ (0.222, 0.222+2\omega_{12}^{'})\oplus(-0.2223, 3\omega_{22}^{'}+0.741) &= (0,\gamma),\\ (1.11, 1.776+5\omega_{11}^{'})\oplus(-1.11, 1.111+6\omega_{21}^{'}) &= (0,\gamma),\\ (0.555, 0.8888+5\omega_{12}^{'})\oplus(0.444, 0.4446+6\omega_{22}^{'}) &= (1,\delta).\end{aligned}\tag{4.16}$$

Step 10: Using the property $(a,\alpha)\oplus(b,\beta)=(a+b,\alpha+\beta)$, the system of fuzzy linear equations (4.15) and (4.16) can be transformed into system of fuzzy linear equations (4.17) and (4.18) respectively.

$$\begin{aligned}(0.444+.555, 0.222+2\omega_{11}+3\omega_{21}+0.37)=(1,\delta),\\ (0.222-0.2223, 0.111+2\omega_{12}+3\omega_{22}+0.1482)=(0,\gamma),\\ (1.11-1.11, 1.332+5\omega_{11}+0.555+6\omega_{21})=(0,\gamma),\\ (0.555+0.444, 0.666+5\omega_{12}+0.2223+6\omega 22)=(1,\delta).\end{aligned}\tag{4.17}$$

and

$$\begin{aligned}(0.444+.555, 0.444+2\omega_{11}^{'}+3\omega_{21}^{'}+1.85)=(1,\delta^{'}),\\ (0.222-0.2223, 0.222+2\omega_{12}^{'}+3\omega_{22}^{'}+0.741)=(0,\gamma^{'}),\\ (1.11-1.11, 1.776+5\omega_{11}^{'}+1.111+6\omega_{21}^{'})=(0,\gamma^{'}),\\ (0.555+0.444, 0.8888+5\omega_{12}^{'}+0.4446+6\omega_{22}^{'})=(1,\delta^{)\prime}.\end{aligned}\tag{4.18}$$

Step 11: Using the property $(a, \alpha) = (b, \beta) \implies a = b$ and $\alpha = \beta$, the system of fuzzy linear equations (4.17) and (4.18) can be transformed into crisp system of linear equations (4.19) and (4.20) respectively.

$$\begin{aligned} 0.222 + 2\omega_{11} + 3\omega_{21} + 0.37 &= \delta, \\ 0.111 + 2\omega_{12} + 3\omega_{22} + 0.1482 &= \gamma, \\ 1.332 + 5\omega_{11} + 0.555 + 6\omega_{21} &= \gamma, \\ 0.666 + 5\omega_{12} + 0.2223 + 6\omega_{22} &= \delta. \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} 0.444 + 2\omega_{11}^{'} + 3\omega_{21}^{'} + 1.85 &= \delta^{'}, \\ 0.222 + 2\omega_{12}^{'} + 3\omega_{22}^{'} + 0.741 &= \gamma^{'}, \\ 1.776 + 5\omega_{11}^{'} + 1.111 + 6\omega_{21}^{'} &= \gamma^{'}, \\ 0.8888 + 5\omega_{12}^{'} + 0.4446 + 6\omega_{22}^{'} &= \delta^{'}. \end{aligned} \tag{4.20}$$

Solving system of linear equations (4.19), the obtained values are

$$\begin{aligned} \omega_{11} &= -2\delta + \gamma - 0.703, \\ \omega_{12} &= \delta - 2\gamma - 0.3703, \\ \omega_{21} &= \frac{5}{3}\delta - \frac{2}{3}\gamma - 0.2713, \\ \omega_{22} &= \frac{5}{3}\gamma - \frac{2}{3}\delta + 0.1604. \end{aligned}$$

Similarly solving system of linear equations (4.20), the obtained values are

$$\begin{aligned} \omega_{11}^{'} &= \frac{5}{3}\delta^{'} - \frac{2}{3}\gamma^{'} - 1.8993, \\ \omega_{12}^{'} &= -2\delta^{'} + \gamma^{'} + 1.702, \\ \omega_{21}^{'} &= \frac{5}{3}\delta^{'} - \frac{2}{3}\gamma^{'} - 0.7164, \\ \omega_{22}^{'} &= \frac{5}{3}\gamma^{'} - \frac{2}{3}\delta^{'} - 0.593. \end{aligned}$$

Step 12: Assuming $\delta = 0.5$ and $\gamma = 0.5$, the obtained values are $|\omega_{11}| = 1.203$, $|\omega_{12}| = 0.8703$, $|\omega_{21}| = 0.7713$, $|\omega_{22}| = 0.6604$ and $|\omega_{11}^{'}| = 1.202$, $|\omega_{12}^{'}| = 0.093$, $|\omega_{21}^{'}| = 1.3993$, $|\omega_{22}^{'}| = 0.2164$.

Step 13: Using the values of x_{11}, x_{12}, x_{21} and x_{22}, obtained in Step 7, and the values of $|w_{11}|$, $|w_{12}|$, $|w_{21}|$, $|w_{22}|$, $|w'_{11}|$, $|w'_{12}|$, $|w'_{21}|$, $|w'_{22}|$, obtained in Step 12, the obtained multiplicative inverse of intuitionistic fuzzy matrix $\tilde{A} = \begin{pmatrix} (-2,1;-2,2) & (3,2;3,10) \\ (-5,6;-5,4) & (-6,3;-6,6) \end{pmatrix}$ is

$$\tilde{X} = \begin{pmatrix} (-0.222, 1.203; -0.222, 2.09) & (-0.111, 0.8703; -0.111, 0.5374) \\ (0.185, 0.7713; 0.185, 1.991) & (-0.0741, 0.6604; -0.0741, 0.5127) \end{pmatrix}$$

4.4 Conclusion

It is pointed out that there is no method in the literature for finding the multiplicative inverse of such matrices whose elements are symmetric triangular intuitionistic fuzzy numbers and hence, a new method (named as Mehar method-I), with the help of existing method [6], is proposed for the same. To illustrate the proposed Mehar method-I the multiplicative inverse of a 2×2 matrix whose elements are symmetric triangular intuitionistic fuzzy numbers is calculated with the help of proposed Mehar method-I.

Chapter 5

PROPOSED MEHAR METHOD-II FOR CALCULATING MULTIPLICATIVE INVERSE OF MATRICES HAVING ELEMENTS AS NON-SYMMETRIC TRIANGULAR INTUITIONISTIC FUZZY NUMBERS

The Mehar method-I, proposed in previous chapter, can be used only to find the multiplicative inverse of such matrices whose elements are symmetric triangular intuitionistic fuzzy numbers. However, this method cannot be used to find multiplicative inverse of those matrices whose elements are non-symmetric triangular intuitionistic fuzzy numbers. In this chapter, a new method (named as Mehar method-II) is proposed to find the multiplicative inverse of these type of matrices.

5.1 Proposed Mehar method-II

In this section, a new method (named as Mehar method-II) is proposed to calculate the multiplicative inverse of such fuzzy matrices whose elements are non-symmetric triangular intuitionistic fuzzy numbers.

The steps of the proposed Mehar method-II are as follows:

Step 1: Since $\tilde{X}$ is assumed as multiplicative inverse of intuitionistic fuzzy matrix $\tilde{A}$ so $\tilde{A} \otimes \tilde{X} \approx \tilde{I}$ where $\tilde{I}$ is $n \times n$ intuitionistic fuzzy matrix whose diagonal elements are non-symmetric triangular intuitionistic fuzzy numbers $(1, \delta, \gamma; 1, \delta', \gamma')$

and remaining elements are non-symmetric triangular intuitionistic fuzzy numbers $(0,\mu,\nu;0,\mu',\nu')$.

Step 2: The equation $\tilde{A}\otimes\tilde{X}\approx\tilde{I}$ is equivalent to system of intuitionistic fuzzy linear equations (5.1).

$$\sum_{j=1}^{n}\tilde{a}_{ij}\otimes\tilde{x}_{jk}\approx\tilde{l}_{ik} \tag{5.1}$$

$$\text{where}\quad \tilde{l}_{ik}=\begin{cases}(1,\delta,\gamma;1,\delta',\gamma') & \text{if } i=k,\\ (0,\mu,\nu;0,\mu',\nu') & \text{if } i\neq k.\end{cases}$$

Step 3: Assuming $\tilde{a}_{ij}=(a_{ij},\alpha_{ij},\beta_{ij};a_{ij},\alpha'_{ij},\beta'_{ij})$, $\tilde{x}_{jk}=(x_{jk},\omega_{jk},\lambda_{jk};x_{jk},\omega'_{jk},\lambda'_{jk})$, the system of intuitionistic fuzzy linear equations (5.1) can be transformed into system of intuitionistic fuzzy linear equations (5.2).

$$\sum_{j=1}^{n}(a_{ij},\alpha_{ij},\beta_{ij};a_{ij},\alpha'_{ij},\beta'_{ij})\otimes(x_{jk},\omega_{jk},\lambda_{jk};x_{jk},\omega'_{jk},\lambda'_{jk})\approx\begin{cases}(1,\delta,\gamma;1,\delta',\gamma') & \text{if } i=k,\\ (0,\mu,\nu;0,\mu',\nu') & \text{if } i\neq k.\end{cases} \tag{5.2}$$

Step 4: Using the property $(a_1,\alpha_1,\gamma_1;a_1,\beta_1,\omega_1)\otimes(a_2,\alpha_2,\gamma_2;a_2,\beta_2,\omega_2)\Longrightarrow((a_1,\alpha_1,\gamma_1)\otimes(a_2,\alpha_2,\gamma_2);(a_1,\beta_1,\omega_1)\otimes(a_2,\beta_2,\omega_2))$, the system of intuitionistic fuzzy linear equations (5.2) can be transformed into system of intuitionistic fuzzy linear equations (5.3).

$$\sum_{j=1}^{n}((a_{ij},\alpha_{ij},\beta_{ij})\otimes(x_{jk},\omega_{jk},\lambda_{jk});(a_{ij},\alpha'_{ij},\beta'_{ij})\otimes(x_{jk},\omega'_{jk},\lambda'_{jk}))\approx\begin{cases}(1,\delta,\gamma;1,\delta',\gamma') & \text{if } i=k,\\ (0,\mu,\nu;0,\mu',\nu') & \text{if } i\neq k..\end{cases} \tag{5.3}$$

Step 5: Using the property $(a_1,\alpha_1,\gamma_1;a_1,\beta_1,\omega_1)=(a_2,\alpha_2,\gamma_2;a_2,\beta_2,\omega_2)\Longrightarrow(a_1,\alpha_1,\gamma_1)=(a_2,\alpha_2,\gamma_2)$ and $(a_1,\beta_1,\omega_1)=(a_2,\beta_2,\omega_2)$, the system of intuitionistic fuzzy linear equations (5.3) can be transformed into system of fuzzy linear equations (5.4) and (5.5).

$$\sum_{j=1}^{n}(a_{ij},\alpha_{ij},\beta_{ij})\otimes(x_{jk},\omega_{jk},\lambda_{jk})=\begin{cases}(1,\delta,\gamma) & \text{if } i=k,\\ (0,\mu,\nu) & \text{if } i\neq k.\end{cases} \tag{5.4}$$

and

$$\sum_{j=1}^{n}((a_{ij},\alpha'_{ij},\beta'_{ij})\otimes(x_{jk},\omega'_{jk},\lambda'_{jk}))=\begin{cases}(1,\delta',\gamma') & \text{if } i=k,\\ (0,\mu',\nu') & \text{if } i\neq k.\end{cases} \tag{5.5}$$

Step 6: Since, the system (5.4) and (5.5) are system of fuzzy linear equations, so using the property of product of two fuzzy numbers (the center parts are multiplied directly) and using the property $(a_1, b_1, c_1) = (a_2, b_2, c_2) \implies a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2$, the center part of system of fuzzy linear equations (5.4) can be transformed into crisp system of linear equations (5.6)

$$a_{ij}x_{jk} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases} \tag{5.6}$$

Solve the crisp system of linear equations (5.6) to find the values of x_{jk}.

Step 7: Let S_1 be the set of those values of x_{jk} which are greater than equal to zero and S_2 be the set of those values of x_{jk} which are less than zero. Similarly, let T_1 be the set of those values of a_{ij} which are greater than equal to zero and T_2 be the set of those values of a_{ij} which are less than zero, the system of fuzzy linear equations (5.4) and (5.5) can be transformed into system of fuzzy linear equations (5.7) and (5.8) respectively.

$$\begin{aligned} &\sum_{\substack{x_{jk}\in S_1 \\ a_{ij}\in T_1}} (a_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_{jk}, \omega_{jk}, \lambda_{jk}) + \sum_{\substack{x_{jk}\in S_2 \\ a_{ij}\in T_1}} (a_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_{jk}, \omega_{jk}, \lambda_{jk}) \\ + &\sum_{\substack{x_{jk}\in S_1 \\ a_{ij}\in T_2}} (a_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_{jk}, \omega_{jk}, \lambda_{jk} + \sum_{\substack{x_{jk}\in S_2 \\ a_{ij}\in T_2}} (a_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_{jk}, \omega_{jk}, \lambda_{jk}) \\ = &\begin{cases} (1, \delta, \gamma) & \text{if } i = k, \\ (0, \mu, \nu) & \text{if } i \neq k. \end{cases} \end{aligned} \tag{5.7}$$

Similarly

$$\begin{aligned} &\sum_{\substack{x_{jk}\in S_1 \\ a_{ij}\in T_1}} (a_{ij}, \alpha'_{ij}, \beta'_{ij}) \otimes (x_{jk}, \omega'_{jk}, \lambda'_{jk}) + \sum_{\substack{x_{jk}\in S_2 \\ a_{ij}\in T_1}} (a_{ij}, \alpha'_{ij}, \beta'_{ij}) \otimes (x_{jk}, \omega'_{jk}, \lambda'_{jk}) \\ + &\sum_{\substack{x_{jk}\in S_1 \\ a_{ij}\in T_2}} (a_{ij}, \alpha'_{ij}, \beta'_{ij}) \otimes (x_{jk}, \omega'_{jk}, \lambda'_{jk}) + \sum_{\substack{x_{jk}\in S_2 \\ a_{ij}\in T_2}} (a_{ij}, \alpha'_{ij}, \beta'_{ij}) \otimes (x_{jk}, \omega'_{jk}, \lambda'_{jk}) \\ = &\begin{cases} (1, \delta', \gamma') & \text{if } i = k, \\ (0, \mu', \nu') & \text{if } i \neq k. \end{cases} \end{aligned} \tag{5.8}$$

Step 8: The system of fuzzy linear equations (5.7) can be transformed into system of fuzzy linear equations (5.9).

$$\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_1}}(e_{ik},\phi_{ik},\rho_{ik}) + \sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_1}}(f_{ik},\psi_{ik},\sigma_{ik}) + \sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_2}}(g_{ik},\chi_{ik},\tau_{ik}) + \sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_2}}(h_{ik},\eta_{ik},\xi_{ik}) = \begin{cases}(1,\delta,\gamma) & \text{if } i=k,\\ (0,\mu,\nu) & \text{if } i\neq k.\end{cases} \tag{5.9}$$

where

$$\begin{aligned}
(e_{ik},\phi_{ik},\rho_{ik}) &= (a_{ij}x_{jk}, a_{ij}\omega_{jk}+x_{jk}\alpha_{ij}, a_{ij}\lambda_{jk}+x_{jk}\beta_{ij}),\\
(f_{ik},\psi_{ik},\sigma_{ik}) &= (a_{ij}x_{jk}, a_{ij}\omega_{jk}-x_{jk}\beta_{ij}, a_{ij}\lambda_{jk}-x_{jk}\alpha_{ij}),\\
(g_{ik},\chi_{ik},\tau_{ik}) &= (a_{ij}x_{jk}, x_{jk}\alpha_{ij}-a_{ij}\lambda_{jk}, x_{jk}\beta_{ij}-a_{ij}\omega_{jk}),\\
(h_{ik},\eta_{ik},\xi_{ik}) &= (a_{ij}x_{jk}, -x_{jk}\beta_{ij}-a_{ij}\lambda_{jk}, -x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}).
\end{aligned}$$

Similarly, the system of fuzzy linear equations (5.8) can be transformed into system of fuzzy linear equations (5.10).

$$\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_1}}(e_{ik},\phi'_{ik},\rho'_{ik}) + \sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_1}}(f_{ik},\psi'_{ik},\sigma'_{ik}) + \sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_2}}(g_{ik},\chi'_{ik},\tau'_{ik}) + \sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_2}}(h_{ik},\eta'_{ik},\xi'_{ik}) = \begin{cases}(1,\delta',\gamma') & \text{if } i=k,\\ (0,\mu',\nu') & \text{if } i\neq k.\end{cases} \tag{5.10}$$

where

$$\begin{aligned}
(e_{ik},\phi'_{ik},\rho'_{ik}) &= (a_{ij}x_{jk}, a_{ij}\omega_{jk}+x_{jk}\alpha_{ij}, a_{ij}\lambda_{jk}+x_{jk}\beta_{ij}),\\
(f_{ik},\psi'_{ik},\sigma'_{ik}) &= (a_{ij}x_{jk}, a_{ij}\omega_{jk}-x_{jk}\beta_{ij}, a_{ij}\lambda_{jk}-x_{jk}\alpha_{ij}),\\
(g_{ik},\chi'_{ik},\tau'_{ik}) &= (a_{ij}x_{jk}, x_{jk}\alpha_{ij}-a_{ij}\lambda_{jk}, x_{jk}\beta_{ij}-a_{ij}\omega_{jk}),\\
(h_{ik},\eta'_{ik},\xi'_{ik}) &= (a_{ij}x_{jk}, -x_{jk}\beta_{ij}-a_{ij}\lambda_{jk}, -x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}).
\end{aligned}$$

Step 9: Using the property $\sum_{i=1}^{n}(a_i,\alpha_i,\beta) = (\sum_{i=1}^{n} a_i, \sum_{i=1}^{n}\alpha_i, \sum_{i=1}^{n}\beta_i)$, the system of fuzzy linear equations (5.9) can be transformed into system of fuzzy linear equations (5.11).

$$\Big(\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_1}} e_{ik}, \sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_1}} \phi_{ik}, \sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_1}} \rho_{ik}\Big) + \Big(\sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_1}} f_{ik}, \sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_1}} \psi_{ik}, \sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_1}} \sigma_{ik}\Big)$$

$$+ \Big(\sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_2}} g_{ik}, \sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_2}} \chi_{ik}, \sum_{\substack{x_{jk}\in S_1\\ a_{ij}\in T_2}} \tau_{ik}\Big) + \Big(\sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_2}} h_{ik}, \sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_2}} \eta_{ik}, \sum_{\substack{x_{jk}\in S_2\\ a_{ij}\in T_2}} \xi_{ik}\Big) = \begin{cases}(1,\delta,\gamma) & \text{if } i=k,\\ (0,\mu,\nu) & \text{if } i\neq k.\end{cases} \tag{5.11}$$

Similarly, the system of fuzzy linear equations (5.10) can be transformed into system of fuzzy linear equations (5.12).

$$(\sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_1}} e_{ik}, \sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_1}} \phi'_{ik}, \sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_1}} \rho'_{ik}) + (\sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_1}} f_{ik}, \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_1}} \psi'_{ik}, \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_1}} \sigma'_{ik})$$

$$+ \ (\sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_2}} g_{ik}, \sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_2}} \chi'_{ik}, \sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_2}} \tau'_{ik}) + (\sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_2}} h_{ik}, \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_2}} \eta'_{ik}, \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_2}} \xi'_{ik}) = \begin{cases}(1,\delta',\gamma') & \text{if } i=k,\\ (0,\mu',\nu') & \text{if } i\neq k.\end{cases} \quad (5.12)$$

Step 10: Using the property $(a,\alpha,\gamma)=(b,\beta,\omega) \implies a=b$, $\alpha=\beta$ and $\gamma=\omega$, the system of fuzzy linear equations (5.11) can be transformed into crisp system of linear equations (5.13) and (5.14).

$$\sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_1}} \phi_{ik} + \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_1}} \psi_{ik} + \sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_2}} \chi_{ik} + \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_2}} \eta_{ik} = \begin{cases}\delta & \text{if } i=k,\\ \mu & \text{if } i\neq k.\end{cases} \quad (5.13)$$

and

$$\sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_1}} \rho_{ik} + \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_1}} \sigma_{ik} + \sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_2}} \tau_{ik} + \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_2}} \xi_{ik} = \begin{cases}\gamma & \text{if } i=k,\\ \nu & \text{if } i\neq k.\end{cases} \quad (5.14)$$

Similarly, the system of fuzzy linear equations (5.12) can be transformed into crisp system of linear equations (5.15) and (5.16).

$$\sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_1}} \phi'_{ik} + \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_1}} \psi'_{ik} + \sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_2}} \chi'_{ik} + \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_2}} \eta'_{ik} = \begin{cases}\delta' & \text{if } i=k,\\ \mu' & \text{if } i\neq k.\end{cases} \quad (5.15)$$

and

$$\sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_1}} \rho'_{ik} + \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_1}} \sigma'_{ik} + \sum_{\substack{x_{jk}\in S_1\\a_{ij}\in T_2}} \tau'_{ik} + \sum_{\substack{x_{jk}\in S_2\\a_{ij}\in T_2}} \xi'_{ik} = \begin{cases}\gamma' & \text{if } i=k,\\ \nu' & \text{if } i\neq k.\end{cases} \quad (5.16)$$

Step 12: Put the values of x_{jk}, obtained in Step 9, in crisp system of linear equations (5.13) to (5.16) and then solve these crisp system of linear equations, by assuming some arbitrary values of δ, γ, ψ, χ, δ', γ', ψ' and χ', to find the values of ω, λ, ω', λ'.

Step 13: Using the value of x_{jk}, obtained in Step 9, and values of ω, λ, ω' and λ', obtained in Step 12, the multiplicative inverse of fuzzy matrix $\tilde{A}$ is $\tilde{X} = (x_{jk}, |\omega_{jk}|, |\lambda_{jk}|; x_{jk}, |\omega'_{jk}|, |\lambda'_{jk}|)_{n\times n}$.

5.2 Illustrative example

In this section Mehar method-II, proposed in Section 5.1, is illustrated with the help of a numerical example.

Using the proposed Mehar method-II the multiplicative inverse of the intuitionistic fuzzy matrix $\tilde{A} = \begin{pmatrix} (2,2,3;2,5,4) & (-3,8,4;-3,10,7) \\ (-7,5,2;-7,8,6) & (5,1,6;5,2,7) \end{pmatrix}$ can be obtained as follows.

Step 1: If $\tilde{X}$ is assumed as multiplicative inverse of matrix $\tilde{A}$ then $\tilde{A}\otimes\tilde{X} \approx \tilde{I}$ where $\tilde{I}$ is a 2×2 intuitionistic fuzzy matrix whose diagonal elements are $(1,\delta,\gamma;1,\delta^{'},\gamma^{'})$ and and remaining elements are $(0,\mu,\nu;0,\mu^{'},\nu^{'})$ i.e.,

$$\tilde{I} = \begin{pmatrix} (1,\delta,\gamma;1,\delta^{'},\gamma^{'}) & (0,\mu,\nu;0,\mu^{'},\nu^{'}) \\ (0,\mu,\nu;0,\mu^{'},\nu^{'}) & (1,\delta,\gamma;1,\delta^{'},\gamma^{'}) \end{pmatrix}$$

Step 2: The equation $\tilde{A}\otimes\tilde{X} \approx \tilde{I}$ is equivalent to

$$\begin{pmatrix} (2,2,3;2,5,4) & (-3,8,4;-3,10,7) \\ (-7,5,2;-7,8,6) & (5,1,6;5,2,7) \end{pmatrix} \otimes \begin{pmatrix} (x_{11},\omega_{11},\lambda_{11};x_{11},\omega^{'}_{11},\lambda^{'}_{11}) & (x_{12},\omega_{12},\lambda_{12};x_{12},\omega^{'}_{12},\lambda^{'}_{12}) \\ (x_{21},\omega_{21},\lambda_{21};x_{21},\omega^{'}_{21},\lambda^{'}_{21}) & (x_{22},\omega_{22},\lambda_{22};x_{22},\omega^{'}_{22},\lambda^{'}_{22}) \end{pmatrix}$$

$$\approx \begin{pmatrix} (1,\delta,\gamma;1,\delta^{'},\gamma^{'}) & \text{if } i=k, \\ (0,\mu,\nu;0,\mu^{'},\nu^{'}) & \text{if } i\neq k. \end{pmatrix} \tag{5.17}$$

Step 3: The system of intuitionistic fuzzy linear equations (5.17) can be transformed into system of intuitionistic fuzzy linear equations (5.18).

$$\begin{aligned}
&((2,2,3;2,5,4)\otimes(x_{11},\omega_{11},\lambda_{11});(x_{11},\omega^{'}_{11},\lambda^{'}_{11}))\oplus((-3,8,4;-3,10,7)\otimes(x_{21},\omega_{21},\lambda_{21};x_{21},\omega^{'}_{21},\lambda^{'}_{21})) = (1,\delta,\gamma;1,\delta^{'},\gamma^{'}),\\
&((2,2,3;2,5,4)\otimes(x_{12},\omega_{12},\lambda_{12});(x_{12},\omega^{'}_{12},\lambda^{'}_{12}))\oplus((-3,8,4;-3,10,7)\otimes(x_{22},\omega_{22},\lambda_{22};x_{22},\omega^{'}_{22},\lambda^{'}_{22})) = (0,\mu,\nu;0,\mu^{'},\nu^{'}),\\
&((-7,5,2;-7,8,6)\otimes(x_{11},\omega_{11},\lambda_{11});(x_{11},\omega^{'}_{11},\lambda^{'}_{11}))\oplus((5,1,6;5,2,7)\otimes(x_{21},\omega_{21},\lambda_{21};x_{21},\omega^{'}_{21},\lambda^{'}_{21})) = (0,\mu,\nu;0,\mu^{'},\nu^{'}),\\
&((-7,5,2;-7,8,6)\otimes(x_{12},\omega_{12},\lambda_{12});(x_{12},\omega^{'}_{12},\lambda^{'}_{12}))\oplus((5,1,6;5,2,7)\otimes(x_{22},\omega_{22},\lambda_{22};x_{22},\omega^{'}_{22},\lambda^{'}_{22})) = (1,\delta,\gamma;1,\delta^{'},\gamma^{'}).
\end{aligned} \tag{5.18}$$

Step 4: Using the property $(a_1,\alpha_1,\gamma_1;a_1,\beta_1,\omega_1)\otimes(a_2,\alpha_2,\gamma_2;a_2,\beta_2,\omega_2) \Longrightarrow ((a_1,\alpha_1,\gamma_1)\otimes(a_2,\alpha_2,\gamma_2);(a_1,\beta_1,\omega_1)\otimes(a_2,\beta_2,\omega_2))$, the system of intuitionistic fuzzy linear equations (5.18) can be transformed into system of intuitionistic fuzzy linear equations (5.19).

$$\begin{aligned}
&((2,2,3)\otimes(x_{11},\omega_{11},\lambda_{11});(2,5,4)\otimes(x_{11},\omega^{'}_{11},\lambda^{'}_{11}))\oplus((-3,8,4)\otimes(x_{21},\omega_{21},\lambda_{21});(3,10,7)\otimes(x_{21},\omega^{'}_{21},\lambda^{'}_{21})) = (1,\delta,\gamma;1,\delta^{'},\gamma^{'}),\\
&((2,2,3)\otimes(x_{12},\omega_{12},\lambda_{12});(2,5,4)\otimes(x_{12},\omega^{'}_{12},\lambda^{'}_{12}))\oplus((-3,8,4)\otimes(x_{22},\omega_{22},\lambda_{22});(3,10,7)\otimes(x_{22},\omega^{'}_{22},\lambda^{'}_{22})) = (0,\mu,\nu;0,\mu^{'},\nu^{'}),\\
&((-7,5,2)\otimes(x_{11},\omega_{11},\lambda_{11});(-7,8,6)\otimes(x_{11},\omega^{'}_{11},\lambda^{'}_{11}))\oplus((5,1,6)\otimes(x_{21},\omega_{21},\lambda_{21});(5,2,7)\otimes(x_{21},\omega^{'}_{21},\lambda^{'}_{21})) = (0,\mu,\nu;0,\mu^{'},\nu^{'}),\\
&((-7,5,2)\otimes(x_{12},\omega_{12},\lambda_{12});(-7,8,6)\otimes(x_{12},\omega^{'}_{12},\lambda^{'}_{11}))\oplus((5,1,6)\otimes(x_{22},\omega_{22},\lambda_{22});(5,2,7)\otimes(x_{22},\omega^{'}_{22},\lambda_{22})) = (1,\delta,\gamma;1,\delta^{'},\gamma^{'}).
\end{aligned} \tag{5.19}$$

Step 5: Using the property $(a_1,\alpha_1,\gamma_1;a_1,\beta_1,\omega_1)\oplus(a_2,\alpha_2\gamma_2;a_2,\beta_2,\omega_2)\Longrightarrow((a_1,\alpha_1,\gamma_1)\oplus(a_2,\alpha_2,\gamma_2);(a_1,\beta_1,\omega_1)\oplus(a_2,\beta_2,\omega_2))$, the system of intuitionistic fuzzy linear equations (5.19) can be transformed into system of intuitionistic fuzzy linear equations (5.20)

$$\begin{aligned}
&(((2,2,3)\otimes(x_{11},\omega_{11},\lambda_{11}))\oplus((-3,8,4)\otimes(x_{21},\omega_{21},\lambda_{21}));(((2,5,4)\otimes(x_{11},\omega_{11}^{'},\lambda_{11}^{'}))\oplus((3,10,7)\otimes(x_{21},\omega_{21}^{'},\lambda_{21}^{'})))\\
&\qquad=(1,\delta,\gamma;1,\delta^{'},\gamma^{'}),\\
&(((2,2,3)\otimes(x_{12},\omega_{12},\lambda_{12})\oplus(-3,8,4)\otimes(x_{22},\omega_{22},\lambda_{22}));(((2,5,4)\otimes(x_{12},\omega_{12}^{'},\lambda_{12}^{'}))\oplus(3,10,7)\otimes(x_{22},\omega_{22}^{'},\lambda_{22}^{'})))\\
&\qquad=(0,\mu,\nu;0,\mu^{'},\nu^{'}),\\
&(((-7,5,2)\otimes(x_{11},\omega_{11},\lambda_{11})\oplus(5,1,6)\otimes(x_{21},\omega_{21},\lambda_{21}));(((-7,8,6)\otimes(x_{11},\omega_{11}^{'},\lambda_{11}^{'}))\oplus(5,2,7)\otimes(x_{21},\omega_{21}^{'},\lambda_{21}^{'})))\\
&\qquad=(0,\mu,\nu;0,\mu^{'},\nu^{'}),\quad(5.20)\\
&(((-7,5,2)\otimes(x_{12},\omega_{12},\lambda_{12})\oplus(5,1,6)\otimes(x_{22},\omega_{22},\lambda_{22}));(((-7,8,6)\otimes(x_{12},\omega_{12}^{'},\lambda_{11}^{'}))\oplus(5,2,7)\otimes(x_{22},\omega_{22}^{'},\lambda_{22})))\\
&\qquad=(1,\delta,\gamma;1,\delta^{'},\gamma^{'}).
\end{aligned}$$

Step 6: Using the property $(a_1,\alpha_1,\gamma_1;a_1,\beta_1,\omega_1)=(a_2,\alpha_2,\gamma_2;a_2,\beta_2,\omega_2)\Longrightarrow(a_1,\alpha_1,\gamma_1)=(a_2,\alpha_2,\gamma_2)$ and $(a_1,\beta_1,\omega_1)=(a_2,\beta_2,\omega_2)$, the system of intuitionistic fuzzy linear equations (5.20) can be transformed into system of fuzzy linear equations (5.21) and (5.22)

$$\begin{aligned}
((2,2,3)\otimes(x_{11},\omega_{11},\lambda_{11}))\oplus((-3,8,4)\otimes(x_{21},\omega_{21},\lambda_{21})) &= (1,\delta,\gamma),\\
((2,2,3)\otimes(x_{12},\omega_{12},\lambda_{12})\oplus(-3,8,4)\otimes(x_{22},\omega_{22},\lambda_{22})) &= (0,\mu,\nu),\\
((-7,5,2)\otimes(x_{11},\omega_{11},\lambda_{11})\oplus(5,1,6)\otimes(x_{21},\omega_{21},\lambda_{21})) &= (0,\mu,\nu),\quad(5.21)\\
((-7,5,2)\otimes(x_{12},\omega_{12},\lambda_{12})\oplus(5,1,6)\otimes(x_{22},\omega_{22},\lambda_{22})) &= (1,\delta,\gamma).
\end{aligned}$$

and

$$\begin{aligned}
((2,5,4)\otimes(x_{11},\omega_{11}^{'},\lambda_{11}^{'}))\oplus((3,10,7)\otimes(x_{21},\omega_{21}^{'},\lambda_{21}^{'})) &= (1,\delta^{'},\gamma^{'}),\\
((2,5,4)\otimes(x_{12},\omega_{12}^{'},\lambda_{12}^{'}))\oplus(3,10,7)\otimes(x_{22},\omega_{22}^{'},\lambda_{22}^{'})) &= (0,\mu^{'},\nu^{'}),\\
((-7,8,6)\otimes(x_{11},\omega_{11}^{'},\lambda_{11}^{'}))\oplus(5,2,7)\otimes(x_{21},\omega_{21}^{'},\lambda_{21}^{'})) &= (0,\mu^{'},\nu^{'}),\quad(5.22)\\
((-7,8,6)\otimes(x_{12},\omega_{12}^{'},\lambda_{11}^{'}))\oplus(5,2,7)\otimes(x_{22},\omega_{22}^{'},\lambda_{22})) &= (1,\delta^{'},\gamma^{'}).
\end{aligned}$$

Step 7: Using product of triangular fuzzy numbers, (center parts are multiplied directly), and using the property $(a,\alpha,\gamma)=(b,\beta,\omega)\Longrightarrow a=b$, $\alpha=\beta$ and $\gamma=\omega$, the center part of system of fuzzy linear equations (5.21) can be transformed into system of crisp linear equations (5.23).

$$\begin{aligned} 2x_{11}-3x_{21}&=1,\\ 2x_{12}-3x_{22}&=0,\\ -7x_{11}+5x_{21}&=0,\\ -7x_{12}+5x_{22}&=1. \end{aligned} \tag{5.23}$$

Solving the system of crisp linear equations (5.23), the obtained values of x_{11}, x_{12}, x_{21} and x_{22} are -0.4545, -0.2727, -0.63636 and -0.1818 respectively.

Step 8: Now, substituting the values obtained of x_{jk} where $j,k=1,2$, the system of fuzzy linear equations (5.21) and (5.22) can be transformed into system of fuzzy linear equations as (5.24) and (5.25) respectively.

$$\begin{aligned} ((2,2,3)\otimes(-0.4545,\omega_{11},\lambda_{11}))\oplus((-3,8,4)\otimes(-0.63636,\omega_{21},\lambda_{21})) &= (1,\delta,\gamma),\\ ((2,2,3)\otimes(-0.2727,\omega_{12},\lambda_{12})\oplus(-3,8,4)\otimes(-0.1818,\omega_{22},\lambda_{22})) &= (0,\mu,\nu),\\ ((-7,5,2)\otimes(-0.4545,\omega_{11},\lambda_{11})\oplus(5,1,6)\otimes(-0.63636,\omega_{21},\lambda_{21})) &= (0,\mu,\nu),\\ ((-7,5,2)\otimes(-0.2727,\omega_{12},\lambda_{12})\oplus(5,1,6)\otimes(-0.1818,\omega_{22},\lambda_{22})) &= (1,\delta,\gamma). \end{aligned} \tag{5.24}$$

and

$$\begin{aligned} ((2,5,4)\otimes(-0.4545,\omega_{11}^{'},\lambda_{11}^{'}))\oplus((3,10,7)\otimes(-0.63636,\omega_{21}^{'},\lambda_{21}^{'})) &= (1,\delta^{'},\gamma^{'}),\\ ((2,5,4)\otimes(-0.2727,\omega_{12}^{'},\lambda_{12}^{'}))\oplus(3,10,7)\otimes(-0.1818,\omega_{22}^{'},\lambda_{22}^{'})) &= (0,\mu^{'},\nu^{'}),\\ ((-7,8,6)\otimes(-0.4545,\omega_{11}^{'},\lambda_{11}^{'}))\oplus(5,2,7)\otimes(-0.63636,\omega_{21}^{'},\lambda_{21}^{'})) &= (0,\mu^{'},\nu^{'}),\\ ((-7,8,6)\otimes(-0.2727,\omega_{12}^{'},\lambda_{11}^{'}))\oplus(5,2,7)\otimes(-0.1818,\omega_{22}^{'},\lambda_{22})) &= (1,\delta^{'},\gamma^{'}). \end{aligned} \tag{5.25}$$

Step 9: Using the multiplication [5]

$$(a_{ij},\alpha_{ij},\beta_{ij})\otimes(x_{jk},\omega_{jk},\lambda_{jk}) \begin{cases} (a_{ij}x_{jk},a_{ij}\omega_{jk}+x_{jk}\alpha_{ij},a_{ij}\lambda_{jk}+x_{jk}\beta_{ij}) & \text{if } a_{ij}\geq 0, x_{jk}\geq 0,\\ (a_{ij}x_{jk},a_{ij}\omega_{jk}-x_{jk}\beta_{ij},a_{ij}\lambda_{jk}-x_{jk}\alpha_{ij}) & \text{if } a_{ij}\geq 0, x_{jk}\leq 0,\\ (a_{ij}x_{jk},x_{jk}\alpha_{ij}-a_{ij}\lambda_{jk},x_{jk}\beta_{ij}-a_{ij}\omega_{jk}) & \text{if } a_{ij}\leq 0, x_{jk}\geq 0,\\ (a_{ij}x_{jk},-x_{jk}\beta_{ij}-a_{ij}\lambda_{jk},-x_{jk}\alpha_{ij}-a_{ij}\omega_{jk}) & \text{if } a_{ij}\leq 0, x_{jk}\leq 0. \end{cases}$$

,the system of fuzzy linear equations (5.24), (5.25) can be transformed into system of fuzzy linear equations (5.26), (5.27) respectively.

$$\begin{aligned} (-0.909,2\omega_{11}+1.3635,2\lambda_{11}+1.3635)\oplus(1.909,2.5456+3\lambda_{21},5.0912+3\omega_{21}) &= (1,\delta,\gamma),\\ (-0.5454,2\omega_{12}+0.8181,2\lambda_{12}+0.8181)\oplus(0.5454,0.7272+3\lambda_{22},1.4544+3\omega_{22}) &= (0,\mu,\nu),\\ (3.1815,0.909+7\lambda_{11},2.2725+7\omega_{11})\oplus(-3.1818,5\omega_{21}+3.8184,5\lambda_{21}+0.6364) &= (0,\mu,\nu),\\ (1.9089,0.5454+7\lambda_{12},1.3635+7\omega_{12})\oplus(-0.909,5\omega_{22}+1.0908,5\lambda_{22}+0.1818) &= (1,\delta,\gamma). \end{aligned} \tag{5.26}$$

and

$$
\begin{aligned}
(-0.909, 2\omega_{11}^{'} + 1.818, 2\lambda_{11}^{'} + 1.818) \oplus (1.909, 4.4548 + 3\lambda_{21}^{'}, 6.364 + 3\omega_{21}^{'}) &= (1, \delta^{'}, \gamma^{'}), \\
(-0.5454, 2\omega_{12}^{'} + 1.0908, 2\lambda_{12}^{'} + 1.0908) \oplus (0.5454, 1.2726 + 3\lambda_{22}^{'}, 1.818 + 3\omega_{22}^{'}) &= (0, \mu^{'}, \nu^{'}), \\
(3.1815, 2.727 + 7\lambda_{11}^{'}, 3.636 + 7\omega_{11}^{'}) \oplus (-3.1818, 5\omega_{21}^{'} + 4.4548, 5\lambda_{21}^{'} + 1.2728) &= (0, \mu^{'}, \nu^{'}), \\
(1.9089, 1.6362 + 7\lambda_{12}^{'}, 2.1816 + 7\omega_{12}^{'}) \oplus (-0.909, 5\omega_{22}^{'} + 1.2726, 5\lambda_{22} + 0.3636) &= (1, \delta^{'}, \gamma^{'}).
\end{aligned} \tag{5.27}
$$

Step 10: Using the property $(a, \alpha, \gamma) \oplus (b, \beta, \omega) = (a+b, \alpha+\beta, \gamma+\omega)$, the system of fuzzy linear equations (5.26) and (5.27) can be transformed into system of fuzzy linear equations (5.28) and (5.29) respectively.

$$
\begin{aligned}
(-0.909 + 1.909, 2\omega_{11} + 1.3635 + 2.5456 + 3\lambda_{21}, 2\lambda_{11} + 1.3635 + 5.0912 + 3\omega_{21}) &= (1, \delta, \gamma), \\
(-0.5454 + 0.5454, 2\omega_{12} + 0.8181 + 0.7272 + 3\lambda_{22}, 2\lambda_{12} + 0.8181 + 1.4544 + 3\omega_{22}) &= (0, \mu, \nu), \\
(3.1815 + (-3.1818), 0.909 + 7\lambda_{11} + 5\omega_{21} + 3.8184, 2.2725 + 7\omega_{11} + 5\lambda_{21} + 0.6364) &= (0, \mu, \nu), \\
(1.9089 + (-0.909), 0.5454 + 7\lambda_{12} + 5\omega_{22} + 1.0908, 1.3635 + 7\omega_{12} + 5\lambda_{22} + 0.1818) &= (1, \delta, \gamma).
\end{aligned} \tag{5.28}
$$

and

$$
\begin{aligned}
(-0.909 + 1.909, 2\omega_{11}^{'} + 1.818 + 4.4548 + 3\lambda_{21}^{'}, 2\lambda_{11}^{'} + 1.818 + 6.364 + 3\omega_{21}^{'}) &= (1, \delta^{'}, \gamma^{'}), \\
(-0.5454 + 0.5454, 2\omega_{12}^{'} + 1.0908 + 1.2726 + 3\lambda_{22}^{'}, 2\lambda_{12}^{'} + 1.0908 + 1.818 + 3\omega_{22}^{'}) &= (0, \mu^{'}, \nu^{'}), \\
(3.1815 + (-3.1818), 2.727 + 7\lambda_{11}^{'} + 5\omega_{21}^{'} + 4.4548, 3.636 + 7\omega_{11}^{'} + 5\lambda_{21}^{'} + 1.2728) &= (0, \mu^{'}, \nu^{'}), \\
(1.9089 + (-0.909), 7\lambda_{12}^{'} + 5\omega_{22}^{'} + 1.2726, 2.1816 + 7\omega_{12}^{'} + 5\lambda_{22}^{'} + 0.3636) &= (1, \delta^{'}, \gamma^{'}).
\end{aligned} \tag{5.29}
$$

Step 11: Using the property $(a, \alpha, \gamma) = (b, \beta, \omega) \implies a = b$, $\alpha = \beta$ and $\gamma = \omega$, the system of fuzzy linear equations (5.28) can be transformed into crisp system of linear equations (5.30) and (5.31) respectively.

$$
\begin{aligned}
2\omega_{11} + 1.3635 + 2.5456 + 3\lambda_{21} &= \delta, \\
2\omega_{12} + 0.8181 + 0.7272 + 3\lambda_{22} &= \mu, \\
0.909 + 7\lambda_{11} + 5\omega_{21} + 3.8184 &= \mu, \\
0.5454 + 7\lambda_{12} + 5\omega_{22} + 1.0908 &= \delta,
\end{aligned} \tag{5.30}
$$

and

$$
\begin{aligned}
2\lambda_{11} + 1.3635 + 5.0912 + 3\omega_{21} &= \gamma, \\
2\lambda_{12} + 0.8181 + 1.4544 + 3\omega_{22} &= \nu, \\
2.2725 + 7\omega_{11} + 5\lambda_{21} + 0.6364 &= \nu, \\
1.3635 + 7\omega_{12} + 5\lambda_{22} + 0.1818 &= \gamma.
\end{aligned} \tag{5.31}
$$

Similarly, the system of fuzzy linear equations (5.29) can be transformed into crisp system of linear equations (5.32) and (5.33) respectively.

$$\begin{aligned} 2\omega_{11}^{'} + 1.818 + 4.4548 + 3\lambda_{21}^{'} &= \delta^{'}, \\ 2\omega_{12}^{'} + 1.0908 + 1.2726 + 3\lambda_{22}^{'} &= \mu^{'}, \\ 2.727 + 7\lambda_{11}^{'} + 5\omega_{21}^{'} + 4.4548 &= \mu^{'}, \\ 7\lambda_{12}^{'} + 5\omega_{22}^{'} + 1.2726 &= \delta^{'}. \end{aligned} \tag{5.32}$$

and

$$\begin{aligned} 2\lambda_{11}^{'} + 1.818 + 6.364 + 3\omega_{21}^{'} &= \gamma^{'}, \\ 2\lambda_{12}^{'} + 1.0908 + 1.818 + 3\omega_{22}^{'} &= \nu^{'}, \\ 3.636 + 7\omega_{11}^{'} + 5\lambda_{21}^{'} + 1.2728 &= \nu^{'}, \\ 2.1816 + 7\omega_{12}^{'} + 5\lambda_{22}^{'} + 0.3636 &= \gamma^{'}. \end{aligned} \tag{5.33}$$

Solving system of crisp linear equations (5.30) and (5.31), the obtained values are

$$\begin{aligned} \omega_{11} &= 0.98365 - \frac{5}{11}\delta + \frac{3}{11}\mu, \\ \omega_{12} &= \frac{3}{11}\gamma - \frac{5}{11}\mu + 0.28095, \\ \omega_{21} &= \frac{7}{11}\gamma - \frac{2}{11}\mu - 3.24802, \\ \omega_{22} &= \frac{7}{11}\nu - \frac{2}{11}\delta - 1.14862. \end{aligned} \tag{5.34}$$

and

$$\begin{aligned} \lambda_{11} &= \frac{3}{11}\mu - \frac{5}{11}\gamma + 1.6447, \\ \lambda_{12} &= \frac{3}{11}\mu - \frac{5}{11}\gamma + 0.5867, \\ \lambda_{21} &= \frac{7}{11}\delta - \frac{2}{11}\nu - 1.9588, \\ \lambda_{22} &= \frac{7}{11}\nu - \frac{2}{11}\delta - 0.7024. \end{aligned} \tag{5.35}$$

Similarly, solving the system of crisp linear equations (5.32) and (5.33), the obtained values are

$$\begin{aligned}
\omega^{'}_{11} &= 1.51255 - \frac{5}{11}\delta^{'} + \frac{3}{11}\mu^{'},\\
\omega^{'}_{12} &= \frac{3}{11}\gamma^{'} - \frac{5}{11}\mu^{'} + 0.3396,\\
\omega^{'}_{21} &= \frac{7}{11}\gamma^{'} - \frac{2}{11}\mu^{'} - 3.90092,\\
\omega^{'}_{22} &= \frac{7}{11}\nu^{'} - \frac{2}{11}\delta^{'} - 1.3222.
\end{aligned}$$

and

$$\begin{aligned}
\lambda^{'}_{11} &= \frac{3}{11}\mu^{'} - \frac{5}{11}\gamma^{'} + 1.7604,\\
\lambda^{'}_{12} &= \frac{3}{11}\mu^{'} - \frac{5}{11}\gamma^{'} + 0.5289,\\
\lambda^{'}_{21} &= \frac{7}{11}\delta^{'} - \frac{2}{11}\nu^{'} - 3.0993,\\
\lambda^{'}_{22} &= \frac{7}{11}\nu^{'} - \frac{2}{11}\delta^{'} - 1.0412.
\end{aligned}$$

Step 12: Assuming $\delta = 0.5$, $\gamma = 0.5$, $\mu = 0.5$ and $\nu = 0.5$, the obtained values are $|\omega_{11}| = 0.8927$, $|\omega_{12}| = 0.19004$, $|\omega_{21}| = 3.0207$, $|\omega_{22}| = 0.9213$, $|\lambda_{11}| = 1.5532$, $|\lambda_{12}| = 0.4958$, $|\lambda_{21}| = 1.7315$, $|\lambda_{22}| = 0.4751$.

Similarly, assuming $\delta^{'} = 0.5$, $\gamma^{'} = 0.5$, $\mu^{'} = 0.5$ and $\nu^{'} = 0.5$, the obtained values are $|\omega^{'}_{11}| = 1.4216$, $|\omega^{'}_{12}| = 0.2487$, $|\omega^{'}_{21}| = 3.6736$, $|\omega^{'}_{22}| = 1.0949$, $|\lambda^{'}_{11}| = 1.6695$, $|\lambda^{'}_{12}| = 0.43799$, $|\lambda^{'}_{21}| = 2.872$, $|\lambda^{'}_{22}| = 0.8139$.

Step 13: Using the values of x_{11}, x_{12}, x_{21} and x_{22}, obtained in Step 7, and the values of $|w_{11}|$, $|w_{12}|$, $|w_{21}|$, $|w_{22}|$, $|w^{'}_{11}|$, $|w^{'}_{12}|$, $|w^{'}_{21}|$, $|w^{'}_{22}|$, $|\lambda_{11}|$, $|\lambda_{12}|$, $|\lambda_{21}|$, $|\lambda_{22}|$, $|\lambda^{'}_{11}|$, $|\lambda^{'}_{12}|$, $|\lambda^{'}_{21}|$, $|\lambda^{'}_{22}|$ obtained in Step 12, the obtained multiplicative inverse of intuitionistic fuzzy matrix $\tilde{A} = \begin{pmatrix} (2,2,3;2,5,4) & (-3,8,4;-3,10,7) \\ (-7,5,2;-7,8,6) & (5,1,6;5,2,7) \end{pmatrix}$ is

$$\tilde{X} = \begin{pmatrix} (-0.4545, 0.8927, 1.5532; -0.4545, 1.4216, 1.6695) & (-0.2727, 0.19004, 0.4958; -0.2727, 0.2487, 0.43799) \\ (-0.63636, 3.0207, 1.7315; -0.63636, 3.6736, 2.872) & (-0.1818, 0.9213, 0.4751; -0.1818, 1.0949, 0.8139) \end{pmatrix}$$

5.3 Conclusion

A new method (named as Mehar method-II) is proposed to find the multiplicative inverse of such matrices whose elements are non-symmetric triangular intuitionistic fuzzy numbers. To illustrate the proposed Mehar method-II, the multiplicative inverse of a 2×2 matrix whose elements are non-symmetric triangular intuitionistic fuzzy numbers is calculated.

Chapter 6

FUTURE SCOPE

After studying Chapter 2, 3, 4 and 5, it may be easily concluded that the existing and proposed methods can be used only to find the multiplicative inverse of such matrices in which some/all elements are symmetric/non-symmetric triangular fuzzy/intuitionistic fuzzy numbers. However, neither the existing methods nor the proposed methods can be used to find the the multiplicative inverse of such matrices in which some/all elements are symmetric/non-symmetric trapezoidal fuzzy/intuitionistic fuzzy numbers. In future, the methods, proposed in this thesis, may be extended to find the multiplicative inverse of these type of matrices.

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