Existence of Fixed Points for Some Mappings in Various Generalized Metric Spaces

A

Dissertation submitted in partial fulfillment of the requirements for

the award of the degree of

Masters of Science

In

Mathematics and Computing

Submitted by

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JULY, 2015

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CERTIFICATE

Certified that the dissertation entitled, **"Existence of Fixed Points for Some Mappings in Various Generalized Metric Spaces"**, which is being submitted by **Mr. Kapil Chawla (Roll No. 301303004)**, in the partial fulfillment of the requirements for the award of the degree of **Master of Science** in - "Mathematics and Computing", to the School of Mathematics, Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of **Dr. Jatinderdeep Kaur** during the period from January 2015 to July 2015.

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The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University/ Institute for the award of any degree. This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.

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ACKNOWLEDGEMENT

I express my deepest sense of gratitude to the almighty whose abundant blessings enabled me to do my dissertation work successfully. It is moment of pride to put on record the immense encouragement and valuable guidance I have recieved from my guide, **Dr. Jatinderdeep Kaur**, Assistant Professor, School of Mathematics, Thapar University, Patiala, Punjab.

I wish to express my sincere gratitude for her understanding and patience during our association without which it would not have been possible to have reached this stage. It is her confidence imbibing attitude, splendid discussion and endless endeavors through which I have gained a lot to build up my future and personality.

I am also very thankful to **Dr. A.K. Lal**, Head of School and to the entire faculty and staff members of school of Mathematics for their direct or indirect help, cooperation, love and affection.

Life at Thapar University, Patiala has been enjoyable with friends who have been always there for me, listening to me, rejoicing me, complaining and pondering my way throughout my study. I thank them all for their great company.

My deepest appreciation goes to my parents and brothers who always influenced my life with their strengthening support.

Above all, I thank Almighty God for providing me sufficient strength to carry out this thesis work.

Kapil Chawla

July, 2015 (Kapil Chawla)

ABSTRACT

The intent of this dissertation entitled **"Existence of Fixed Points for Some Mappings in Various Generalized Metric Spaces"** embodies a brief account of investigation carried out by authors on existence of fixed points of self mappings in various metric spaces under the supervision of **Dr. Jatinderdeep Kaur**, Assistant Professor, School of Mathematics, Thapar University, Patiala.

The aim of this work is to study and obtain some result on existence and uniqueness of fixed points. Fixed point theory is one of the most powerful tool in nonlinear analysis. Fixed point theory has wide ranging application in many area of mathematics. For example, in finding the solution of the system of linear equations, in proving the existence of solutions of ordinary and partial differential equation, integral equations, analysis and many other disciplines.

The whole work is divided into four chapters. Chapter I is introduction which includes brief account of definitions and results which will be required for the later chapters. In Chapter II, we have studied Fixed point theorem for $\alpha - \psi$ contractive mapping which guarantees the existence and uniqueness of fixed point in b-metric spaces. The aim of Chapter III is to study common fixed point results for weakly isotone increasing mapping in partially ordered b-metric-like space. Chapter IV is concerned with common fixed theorems using newly defined contractive type mappings for partially ordered b-metric-like space. Moreover, corollary is obtained in this chapter.

At the end, we have have presented some references of research papers and books cited in the dissertation.

LIST OF PUBLICATION

Research Paper Published In International/ National Journal

1. **Common fixed point results for weakly isotone increasing mappings in partially ordered b-metric like spaces**, *International Journal of Functional Analysis, Operator Theory and Applications 7(1), 2015, 19-56.* .

Papers Presented In International/ National conferences

1. **Common fixed point results for weakly isotone increasing mappings in partially ordered b-metric-like spaces**, *18th Punjab Science Congress, Desh Bhagat University, Mandi Gobindgarh, 2015.*
2. **Fixed Point Theorems for $\alpha - \psi$ contractive mappings**, *International Conference on Special Functions and their Applications Thapar University, Patiala, 2014 , 24.*
3. **Common fixed point results in b-metric-like spaces**, *18th Punjab Science Congress, Desh Bhagat University, Mandi Gobindgarh, 2015.*
4. **Common fixed point results for generalized contraction in b-metric-like spaces**, *18th Punjab Science Congress, Desh Bhagat University, Mandi Gobindgarh, 2015.*

LIST OF SYMBOLS

$\in$	*Belongs to*
$\exists$	*There exists*
$\forall$	*For all*
$\Rightarrow$	*Implies*
$\Leftrightarrow$	*Implies and is implied by, if and only if*
$\mathbb{N}$	*Set of natural numbers*
$\mathbb{Z}$	*Set of integers*
$\mathbb{Q}$	*Set of rational numbers*
$\mathbb{R}$	*Set of real numbers*
$\mathbb{R}_+$	*Set of positive real numbers*
$\mathbb{C}$	*Set of complex numbers*
$\mathbb{R}_u$	*Usual metric space*

Contents

1

Introduction

1.1 Introduction

A fixed point of a function is a point that is mapped to itself by the function. Let X be any non-empty set. Given a function $f : X \to X$, a fixed point of f is a point $x \in X$ such that $f(x) = x$, that is, a point which remains invariant under the mapping f.

For example, if a function f is defined on the real numbers by $f(x) = x^2 - 3x + 4$, then 2 is a fixed point of f. Not all functions have fixed points: for example, if f is a function defined on the real numbers as $f(x) = x + 1$, then it has no fixed points.

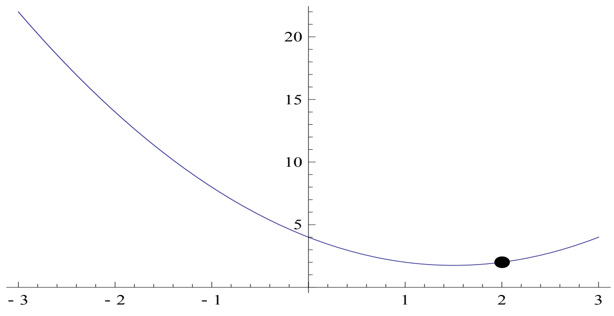

Fig. 1.1:

1.2 Brief survey of the literature

The first fixed point result in metric fixed point theory was proved by Polish mathematician Stefan Banach in 1922, popularly referred as Banach contraction principle. This principle states that ***"A contraction mapping of a complete metric space into itself has a unique fixed point"***. The simplicity and utility of this classical and celebrated theorem makes it a popular tool for proving the existence and uniqueness theorems in different branches of mathematical analysis.

The Banach Contraction Principle was the only main tool to establish the existence and uniqueness of fixed points until 1968. This principle has been considered as the key of metric fixed point theory, but it suffers from one drawback, i.e., ***it requires the mapping to be continuous at all points of its domain.***

In 1968, Kannan [19] introduced a contractive condition which possessed a unique fixed point like that of Banach. However, unlike the Banach condition, Kannan [19] proved that there are mappings that have a discontinuity in their domain but still have fixed point, although such mappings are continuous at their fixed point.

In literature, I found that there exists many generalization of the concept of metric space. In particular, Matthews [21] introduced the notion of partial metric space and proved that Banach contraction mapping theorem can be generalized to partial metric context for applications in program verification. After that, fixed point results in partial metric spaces have been studied by many authors. The concept of b-metric space was introduced and studied by Czerwik [12]. since then serval papers have been dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces. Amini-Harandi [8] introduced the notion of metric-like space, which is an interesting generalization of partial metric space. Recently, Mohammed Ali Algamdi [5] introduced a new generalization of metric-like space and partial metric space which studied some fixed point theorems on generalized space called as b-metric-like space. These results improved some well-known results in the literature.

In the present dissertation, some results on fixed point theory have been studied which generalized the well known results of above mentioned authors.

1.3 Definitions

In this subsection, we recall some basic notations, definitions and well known results. However, some of the definitions and notations will be repeated occasionally in various chapter for the sake of convenience.

Definition 1.3.1. *[16] A metric space on a non empty set X is a function $d : X \times X \to [0, +\infty)$ such that for $x, y, z \in X$ and the following conditions hold true:*

$(a1)$ $d(x, y) \geq 0$ *and* $d(x, y) = 0 \Leftrightarrow x = y$;

$(a2)$ $d(x, y) = d(y, x)$;

$(a3)$ $d(x, z) \leq d(x, y) + d(y, z)$;

The pair (X, d) is called a metric space.

Any function with these three properties is called a distance function or a metric.

Example The most important examples of metric spaces, from our standpoint, are the euclidean space $\mathbb{R}_u$. Especially $\mathbb{R}_1$(the real line) and $\mathbb{R}_2$ (the complex plane); the distance in $\mathbb{R}_u$ is defined by $d(x, y) = |x - y|$ $(x, y \in \mathbb{R}_u)$. Then d is a metric on $\mathbb{R}$ and $\mathbb{R}_u$ is known as usual metric space.

Definition 1.3.2. *[16] A metric space (X, d) is said to be complete if every cauchy sequence in X is convergent in X.*

Definition 1.3.3. *[16] Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction on X if there exists a real number α with $0 \leq \alpha < 1$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y) \quad for \quad all \quad x, y \in X, x \neq y.$$

The well known result in fixed point theory is Banach Contraction Principle which is stated as:

Theorem 1.3.1. *[16] Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then, T has a unique fixed point in X.*

In 1993, Stefan Czerwik [12] modified the triangular inequality and introduced a new concept of b-metric space is defined as follows:

Definition 1.3.4. *[12] A b-metric on a non empty set X is a function $d : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$ and a constant $K \geq 1$ the following three conditions hold true:*

$(b1)$ $\quad if\ d(x, y) = 0 \ \Leftrightarrow\ x = y;$

$(b2)$ $\quad d(x, y) = d(y, x);$

$(b3)$ $\quad d(x, z) \leq K(d(x, y) + d(y, z));$

The pair (X, d) is called a b-metric space.

Remark 1.3.2. *For $K = 1$; the b-metric space is itself a metric space. Clearly, b-metric space is a generalization of metric space.*

Further in 2009, Steve Matthews *et.al*[21] introduced a new concept of self distances in metric space and defined a new space called partial metric space as follows:

Definition 1.3.5. *[21] A partial metric on a non empty set X is a function $p : X \times X \to \mathbb{R}_+$, such that for $x, y, z \in X$ and the following conditions hold true:*

$(c1)$ $\quad x = y \Leftrightarrow\ p(x, x) = p(x, y) = p(y, y);$

$(c2)$ $\quad p(x, x) \leq p(x, y);$

$(c3)$ $\quad p(x, y) = p(y, x);$

$(c4)$ $\quad p(x, z) \leq p(x, y) + p(y, z) - p(y, y);$

The pair (X, p) is called a partial metric space.

Recently, A Amini Harandi [8] generalized the concept of partial metric space and introduced the concept of metric-like space which is defined as follows:

Definition 1.3.6. *[8] A mapping $\sigma : X \times X \to \mathbb{R}^+$, where X is a non empty set, is said to be* *metric-like on X if for any $x, y, z \in X$, the following three conditions hold true:*

$(d1)$ $\sigma(x, y) = 0 \Longrightarrow x = y$;

$(d2)$ $\sigma(x, y) = \sigma(y, x)$;

$(d3)$ $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$;

The pair (X, σ) is then called a metric-like space.

Remark 1.3.3. *Clearly, Every metric space is metric-like but converse need not be true.*

Further, In 2013, Satish Shukla [31] generalized metric space as follows:

Definition 1.3.7. *[31] A partial b-metric on a non empty set X is a function $b : X \times X \to \mathbb{R}_+$, such that for $x, y, z \in X$ and a constant $K \geq 1$ the following conditions hold true:*

$(e1)$ $x = y \Leftrightarrow$ $b(x, x) = b(x, y) = b(y, y)$;

$(e2)$ $b(x, x) \leq b(x, y)$;

$(e3)$ $b(x, y) = b(y, x)$;

$(e4)$ $b(x, z) \leq K(b(x, y) + b(y, z)) - b(y, y)$;

The pair (X, b) is called a partial b-metric space.

Remark 1.3.4. *One can easily note that every partial metric space is partial b-metric but the converse may not be true.*

Recently, Mohammed Ali Alghamdi *et.al* [5] introduced a new generalized metric space named as b-metric-like space as follows:

Definition 1.3.8. *[5] A b-metric-like on a non empty set X is a function $\mathfrak{D} : X \times X \to [0, +\infty)$ such that for all $p, q, r \in X$ and a constant $K \geq 1$ the following three conditions hold true:*

$(f1)$ *if* $\mathfrak{D}(p, q) = 0 \Rightarrow p = q$;

$(f2)$ $\mathfrak{D}(p, q) = \mathfrak{D}(q, p)$;

$(f3)$ $\mathfrak{D}(p, q) \leq K(\mathfrak{D}(p, r) + \mathfrak{D}(r, q))$;

The pair $(X, \mathfrak{D})$ is called a b-metric-like space.

Example [5] Let $X = [0, +\infty)$. Define the function $\mathfrak{D} : X^2 \to [0, +\infty)$ by $\mathfrak{D}(p, q) = (p+q)^2$. Then $(X, \mathfrak{D})$ is a b-metric-like space with constant $K = 2$. Clearly $(X, \mathfrak{D})$ is not a b-metric or metric-like space. Indeed, for all $p, q, r \in X$

$$\begin{aligned} \mathfrak{D}(p,q) &= (p+q)^2 \leq (p+r+r+q)^2 = (p+r)^2 + (r+q)^2 + 2(p+r)(r+q) \\ &\leq 2[(p+r)^2 + (r+q)^2] \\ &= 2(\mathfrak{D}(p,r) + \mathfrak{D}(r,q)) \end{aligned}$$

and so $(f3)$ holds. Clearly, $(f1)$ and $(f2)$ hold.

Remark 1.3.5. *It can be easily noted that b-metric-like space is till now most generalized metric space. The tree diagram representing the relations between various metric spaces is given below:-*

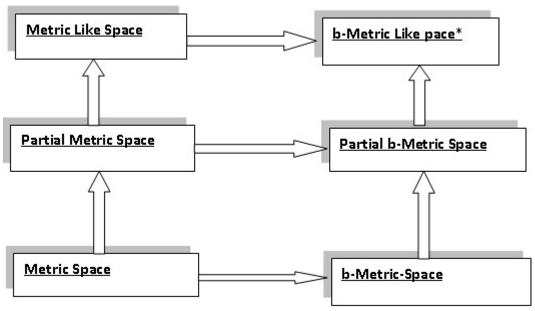

Fig. 1.2:

Now, a brief chapter wise summary of the results contains in this dissertation.
In chapter II, we have studied existence and uniqueness of fixed points in b-metric-like spaces using α-ψ Contractive Type mapping and have obtained well known results and corollaries. Further, in chapter III, some results and their consequences have been obtained using weakly isotone increasing mapping in partially ordered b-metric-like space.
The objective of chapter IV, is to study the fixed point theorems using newly defined Contractive type mappings for partially ordered b-metric-like space. Moreover, corollary is obtained in this chapter.

2

Fixed Point Theorem for α-ψ Contractive type mapping

2.1 Introduction

In 2011, Samet *et.al* [30] proved the fixed point theorem for α-ψ contractive type mapping in complete metric space. In this chapter, we have defined α-ψ contractive mapping in b-metric-like space and proved the fixed point theorem for α-ψ contractive type mapping in b-metric-like space.

Definition 2.1.1. *Let $(Y, \mathfrak{D})$ be a b-metric-like space and $T : Y \to Y$ be a given mapping. We say that T is an $\alpha - \psi$ contractive mapping if there exists two function $\alpha : Y \times Y \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)\mathfrak{D}(Tx, Ty) \leq \psi(\mathfrak{D}(x, y)), \tag{2.1.1}$$

for all $x, y \in Y$.

Definition 2.1.2. *Let $T : Y \to Y$ and $\alpha : Y \times Y \to [0, +\infty)$. We say that T is $\alpha-$ admissible if $x, y \in Y$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.*

[0] The content of this chapter has been presented in ***International Conference on Special Functions and their Applications,*** Thapar University, Patiala, 2014

Let Y be the set of all non-negative real numbers. Let us define the mapping $\alpha : Y \times Y \to [0, \infty)$ by

$$\mathfrak{D}(x,y) = \begin{cases} 1 & if \quad x \geq y \\ 0 & if \quad x < y \end{cases}$$

and define the mapping $T : Y \to Y$ by $Tx = x^2$ for all $x \in Y$. Then T is $\alpha-$admissible.

2.2 Lemma

Let Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n^{th} iterate of ψ.

Lemma 2.2.1. *[30] For every function $\psi : [0, +\infty) \to [0, +\infty)$ the following holds: if ψ is nondecreasing, then for each $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ implies $\psi(t) < t$*

2.3 Main Results

The first main result of this chapter reads as follows:

Theorem 2.3.1. *Let $(Y, \mathfrak{D})$ be a complete b-metric-like space and $T : Y \to Y$ be an $\alpha - \psi$ contractive mapping satisfying the following conditions:*
(1) T is $\alpha-$admissible;
(2) there exists $x_0 \in Y$ such that $\alpha(x_0, Tx_0) \geq 1$;
(3) T is continuous.
Then, T has a fixed point , that is there exists $x^* \in Y$ such that $Tx^* = x^*$.

Proof. Let $x_0 \in Y$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in Y by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$.
If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of T.
Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is $\alpha-$ admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$$

Proceeding in the same way, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \;\; for \; all \; n \in \mathbb{N} \tag{2.3.1}$$

Take $x = x_{n-1}$ and $y = x_n$,

$$\mathfrak{D}(x_n, x_{n+1}) = \mathfrak{D}(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n)\mathfrak{D}(Tx_{n-1}, Tx_n) \leq \psi(\mathfrak{D}(x_{n-1}, x_n)).$$

Inductively repeating the steps in above inequality, we obtain

$$\mathfrak{D}(x_n, x_{n+1}) \leq \psi^n(\mathfrak{D}(x_0, x_1)), \; for \; all \; n \in \mathbb{N}$$

Fix $\epsilon > 0$ and let $n(\epsilon) \in \mathbb{N}$ such that $\sum_{n \geq n(\epsilon)} \psi^n(\mathfrak{D}(x_0, x_1)) < \frac{\epsilon}{K}$, where $K \geq 1$. Let $n, m \in \mathbb{N}$ with $m > n > n(\epsilon)$, we get

$$\begin{aligned} \mathfrak{D}(x_n, x_m) &\leq \sum_{j=n}^{m-1} K\mathfrak{D}(x_j, x_{j+1}) \\ &\leq \sum_{j=n}^{m-1} K\psi^j(\mathfrak{D}(x_0, x_1)) \\ &\leq K \sum_{n \geq n(\epsilon)} \psi^j(\mathfrak{D}(x_0, x_1)) < \epsilon \end{aligned}$$

The above inequality implies the existence of cauchy sequence in b-metric-like space $(Y, \mathfrak{D})$. By completeness of Y, there exists x^* in Y such that $x_n \to x^*$ as $n \to +\infty$.
The continuity of T implies that $x_{n+1} = Tx_n \to Tx^* \;\; as \; n \to \infty$. Also, by the uniqueness of the limit we get $Tx^* = x^*$,i.e x^* is a fixed point of T. □

The second main result of this chapter read as:

Theorem 2.3.2. *Let $(Y, \mathfrak{D})$ be a complete b-metric-like space and a constant $K \geq 1$ and $T : Y \to Y$ be an $\alpha - \psi$ contractive mapping satisfying the following conditions:*
(1) T is $\alpha-$admissible.
(2) there exists $x_0 \in Y$ such that $\alpha(x_0, Tx_0) \geq 1$.
(3) if $\{x_n\}$ is a sequence in Y such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in Y$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1 \;\; for \;\; all \; n$
Then, T has a fixed point.

Proof. Following the proof of Theorem (2.3.1), we know that $\{x_n\}$ is a cauchy sequence in the complete b-metric-like space $(Y, \mathfrak{D})$. Then, there exists $x^* \in Y$ such that $x_n \to x^*$ as $n \to +\infty$. On the other hand, from (2.3.1) and hypothesis (3), we have, $\alpha(x_n, x^*) \geq 1$, for all $n \in \mathbb{N}$

$$\begin{aligned}
\mathfrak{D}(Tx^*, x^*) &\leq K\mathfrak{D}(Tx^*, Tx_n) + K\mathfrak{D}(x_{n+1}, x^*) \\
&\leq K\alpha(x_n, x^*)\mathfrak{D}(Tx_n, Tx^*) + K\ \mathfrak{D}(x_{n+1}, x^*) \\
&\leq K\psi(\mathfrak{D}(x_n, x^*)) + K\ \mathfrak{D}(x_{n+1}, x^*)
\end{aligned}$$

By the continuity of ψ, The conclusion of the results holds. □

2.4 Consequences

Corollary 2.4.1. *Let $(Y, \mathfrak{D})$ be a complete b-metric-like space and $T: Y \to Y$ be an $\alpha - \psi$ contractive mapping satisfying the following conditions:*
(1) T is $\alpha-$admissible;
(2) there exists $x_0 \in Y$ such that $\alpha(x_0, Tx_0) \geq 1$;
(3) T is continuous.
Then, T has a fixed point , that is there exists $x^* \in Y$ such that $Tx^* = x^*$.

Corollary 2.4.2. *Let $(Y, \mathfrak{D})$ be a complete b-metric-like space and a constant $K \geq 1$ and $T: Y \to Y$ be an $\alpha - \psi$ contractive mapping satisfying the following conditions:*
(1) T is $\alpha-$admissible.
(2) there exists $x_0 \in Y$ such that $\alpha(x_0, Tx_0) \geq 1$.
(3) if $\{x_n\}$ is a sequence in Y such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in Y$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1 \quad for \quad all\ n$
Then, T has a fixed point.

Example Let $Y = \mathbb{R}_+$ and b-metric-like space $\mathfrak{D} : Y \times Y \to \mathbb{R}_+$ be defined by

$$\mathfrak{D}(x, y) = (x + y)^2$$

Clearly, $(Y, \mathfrak{D}, 2)$ is a complete b-metric-like space. Let $T : Y \to Y$ be defined by

$$Tx = \begin{cases} \frac{x}{5} & if \quad x \in [0,1) \\ 0 & if \quad otherwise \end{cases}$$

Now, we define the mapping $\alpha : Y \times Y \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & if \quad either \quad x \in [0,1) \;\; or \;\; y \in [0,1) \\ 0 & if \quad otherwise \end{cases}$$

Clearly, T is an $\alpha - \psi$-contractive mapping with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. In fact, for all $x, y \in Y$, we have

Case(1): either $x \in [0,1)$ or $y \in [0,1)$

$$\begin{aligned} \alpha(x,y)\mathfrak{D}(Tx,Ty) &\leq \psi(\mathfrak{D}(x,y)) \\ \mathfrak{D}\left(\frac{x}{5}, \frac{y}{5}\right) &\leq \psi((x+y)^2) \\ \left(\frac{x+y}{5}\right)^2 &\leq \frac{(x+y)^2}{2} \end{aligned}$$

Obviously, T is continuous and so it remains to show that T is $\alpha-$admissible. In doing so, let $x, y \in Y$ such that $\alpha(x,y) \geq 1$. This implies that either $x \in [0,1)$ or $y \in [0,1)$ and by the definitions of T and α, we have $Tx = \frac{x}{5} \in [0,1)$, $Ty = \frac{y}{5} \in [0,1)$.

$\alpha(Tx, Ty) \geq 1$ for each case.

Then,T is $\alpha-$admissible.

Now, all the conditions are satisfied. Consequently, T has a fixed point.

3

Common Fixed Point Results for Weakly Isotone Increasing Mappings

3.1 Introduction

In this subsection, we recall some preliminaries, definitions and well known results.

Let $(x, \preccurlyeq)$ be a partially ordered set and let f, g be two self-maps on X. We will use the following terminology:

(a) elements $p, q \in X$ are called comparable if $p \leq q$ or $q \leq p$ holds;

(b) a subset S of X is said to be well ordered if every two elements of S are comparable;

(c) f is called nondecreasing w.r.t. $\preccurlyeq$ if $p \preccurlyeq q$ implies $fp \preccurlyeq fq$;

(d)([7]) the pair (f, g) is said to be weakly increasing if $fp \preccurlyeq gfp$ and $gp \preccurlyeq fgp$ for all $p \in X$;

(e)([25]) f is said to be g-weakly isotone increasing if for all $p \in X$ we have $fp \preccurlyeq gfp \preccurlyeq fgfp$;

If $f, g : X \to X$ are weakly increasing, then f is g-weakly isotone increasing. Also, in (e), if $f = g$, we say that f is weakly isotone increasing, In this case, for each $p \in X$, we have

[0] The content of this chapter has been published in ***International Journal of Functional Analysis, Operator Theory and Applications,*** 7(1), 2015, 19-56.

$fp \preccurlyeq ffp$.

Recently, M.L Roshan *et. al* proved the common fixed point theorem for weakly isotone increasing mapping in ordered b-metric space. The aim of this chapter to prove the common fixed point theorem for weakly isotone increasing mapping in partially ordered b-metric-like space.

3.2 Lemmas

The following lemmas are required in the proof of our main results.

Lemma 3.2.1. *[5] Let $(X, \mathfrak{D}, K)$ be a b-metric-like space and $\{p_n\}$ be a sequence in X such that $\lim_{n\to\infty} \mathfrak{D}(p_n, p) = 0$. Moreover $z \in X$, we have*

$$\frac{1}{K}\mathfrak{D}(p, z) \leq \liminf_{n\to\infty} \mathfrak{D}(p_n, z) \leq \limsup_{n\to\infty} \mathfrak{D}(p_n, z) \leq K\mathfrak{D}(p, z).$$

Lemma 3.2.2. *Let $(X, \mathfrak{D})$ be a b-metric-like space and let $\{p_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} \mathfrak{D}(p_n, p_{n+1}) = 0. \tag{3.2.1}$$

If $\{p_n\}$ is not a b-cauchy sequence, there exists $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for the following four sequences
$\mathfrak{D}(p_{m(k)}, p_{n(k)})$, $\mathfrak{D}(p_{m(k)}, p_{m(k)+1})$, $\mathfrak{D}(p_{m(k)+1}, p_{n(k)})$ and $\mathfrak{D}(p_{m(k)+1}, p_{n(k)+1})$
it holds:

$$\begin{aligned}
\epsilon &\leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq K\epsilon \\
\frac{\epsilon}{K} &\leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq K^2\epsilon \\
\frac{\epsilon}{K} &\leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)}) \leq K^2\epsilon \\
\frac{\epsilon}{K^2} &\leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq K^3\epsilon
\end{aligned}$$

Proof. If $\{p_n\}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ and sequences $\{m(j)\}$ and $\{n(j)\}$ of positive integers such that

$$n(j) > m(j) > j, \quad \mathfrak{D}(p_{m(j)}, p_{n(j)-1}) < \epsilon, \quad \mathfrak{D}(p_{m(j)}, p_{n(j)}) \geq \epsilon \tag{3.2.2}$$

for all positive integers j. Now, from (3.2.2) and using the triangle inequality we have

$$\epsilon \leq \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq K[\mathfrak{D}(p_{m(j)}, p_{n(j)-1}) + \mathfrak{D}(p_{n(j)-1}, p_{n(j)})] < K\epsilon + K\mathfrak{D}(p_{n(j)-1}, p_n(j)). \quad (3.2.3)$$

Taking the upper and lower limits as $j \to \infty$ in (3.2.3), and using (4.2.1) we obtain that

$$\epsilon \leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq K\epsilon \quad (3.2.4)$$

By again using the triangle inequality we have

$$\begin{aligned} \mathfrak{D}(p_{m(j)}, p_{n(j)}) &\leq K[\mathfrak{D}(p_{m(j)}, p_{n(j)+1}) + \mathfrak{D}(p_{n(j)+1}, p_{n(j)})] \\ &\leq K^2[\mathfrak{D}(p_{m(j)}, p_{n(j)}) + \mathfrak{D}(p_{n(j)+1}, p_{n(j)})] + K\mathfrak{D}(p_{n(j)+1}, n(j)) \end{aligned}$$

Taking the upper and lower limits as $j \to \infty$, then we get

$$\epsilon \leq K \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq K^3\epsilon$$

or,

$$\frac{\epsilon}{K} \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq K^2\epsilon$$

The remaining two conditions of the lemma can be proved in a similar way. □

3.3 Main Results

Let $(X, \preccurlyeq, \mathfrak{D})$ be an ordered b-metric-like space with $K > 1$ and $f, g : X \to X$ be two mappings. For all $p, q \in X$, let

$$M_s(p, q) = max\left\{\psi(\mathfrak{D}(p, q)), \psi(\mathfrak{D}(p, fp)), \psi(\mathfrak{D}(q, gq)), \psi\left(\frac{\mathfrak{D}(p, gq) + \mathfrak{D}(q, fp)}{6K}\right)\right\} \quad (3.3.1)$$

and

$$N_s(p, q) = min\{\psi(\mathfrak{D}^s(p, fp)), \psi(\mathfrak{D}^s(q, gq)), \psi(\mathfrak{D}^s(p, gq)), \psi(\mathfrak{D}^s(q, fp))\} \quad (3.3.2)$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous function with $\psi(t) < t$ for each $t > 0$ and $\psi(0) = 0$.

Theorem 3.3.1. *Let $(X, \preccurlyeq, \mathfrak{D})$ be a complete partially ordered b-metric-like space. Let $f, g : X \to X$ be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements $p, q \in X$, we have*

$$K^4\mathfrak{D}(fp, gq) \leq \frac{M_s(p,q) + N_s(p,q)}{2} \tag{3.3.3}$$

Then, the pair (f, g) has a common fixed point z in X if one of f or g is continuous. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. Let p_0 be an arbitrary point of X. Choose $p_1 \in X$ such that $fp_0 = p_1$ and $p_2 \in X$ such that $gp_1 = p_2$. Continuing in this way, construct a sequence $\{p_n\}$ defined by:

$$p_{2n+1} = fp_{2n} \;\; and \;\; p_{2n+2} = gp_{2n+1}$$

for all $n \geq 0$. As f is g-weakly isotone increasing, we have

$$p_1 = fp_0 \leq gfp_0 = gx_1 = x_2 \leq fgfp_0 = fp_2 = p_3$$

Repeating this process, we obtain $p_n \leq p_{n+1} \;\; \forall \;\; n \geq 1$.

We have proved the theorem in three steps.

Step1 : First prove that $\lim_{n\to\infty} \mathfrak{D}(p_n, p_{n+1}) = 0$.

Suppose $\mathfrak{D}(p_{j_0}, p_{j_0+1}) = 0$ for some j_0. Then $p_{j_0} = p_{j_0+1}$. In this case $j_0 = 2n$, $p_{2n} = p_{2n+1}$. we need to show that $p_{2n+1} = p_{2n+2}$

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) = K^4\mathfrak{D}(fp_{2n}, gp_{2n+1}) \leq \frac{M_s(p_{2n}, p_{2n+1}) + N_s(p_{2n}, p_{2n+1})}{2} \tag{3.3.4}$$

where

$$\begin{aligned} M_s(p_{2n}, p_{2n+1}) &= max \left\{ \begin{array}{r} \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, fp_{2n})), \psi(\mathfrak{D}(p_{2n+1}, gp_{2n+1})), \\ \psi\left(\dfrac{\mathfrak{D}(p_{2n}, gp_{2n+1}) + \mathfrak{D}(p_{2n+1}, fp_{2n})}{6K}\right) \end{array} \right\} \\ &= max \left\{ \begin{array}{r} \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \\ \psi\left(\dfrac{\mathfrak{D}(p_{2n}, p_{2n+2}) + \mathfrak{D}(p_{2n+1}, p_{2n+1})}{6K}\right) \end{array} \right\} \end{aligned}$$

$$
\begin{aligned}
&= \max\left\{\begin{array}{c}\psi(\mathfrak{D}(p_{2n+1},p_{2n+1})),\psi(\mathfrak{D}(p_{2n+1},p_{2n+1})),\psi(\mathfrak{D}(p_{2n+1},p_{2n+2})),\\ \psi\left(\frac{\mathfrak{D}(p_{2n+1},p_{2n+2})+\mathfrak{D}(p_{2n+1},p_{2n+1})}{6K}\right)\end{array}\right\}\\
&= \max\left\{\psi(\mathfrak{D}(p_{2n+1},p_{2n+1})),\psi(\mathfrak{D}(p_{2n+1},p_{2n+2})),\psi\left(\frac{\mathfrak{D}(p_{2n+1},p_{2n+2})+\mathfrak{D}(p_{2n+1},p_{2n+1})}{6K}\right)\right\}
\end{aligned}
$$

By $D3$, we have

$$
\begin{aligned}
\mathfrak{D}(p_{2n+1},p_{2n+1}) &\leq 2K\mathfrak{D}(p_{2n+1},p_{2n+2})\\
\mathfrak{D}(p_{2n+1},p_{2n+2}) &\leq K\mathfrak{D}(p_{2n+1},p_{2n+2})
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}(p_{2n+1},p_{2n+1})+\mathfrak{D}(p_{2n+1},p_{2n+2}) &\leq 3K\mathfrak{D}(p_{2n+1},p_{2n+2})\\
\frac{\mathfrak{D}(p_{2n+1},p_{2n+1})+\mathfrak{D}(p_{2n+1},p_{2n+2})}{6K} &\leq \frac{\mathfrak{D}(p_{2n+1},p_{2n+2})}{2}
\end{aligned}
$$

$$
M_s(p_{2n},p_{2n+1}) \leq max\left\{\psi(2K\mathfrak{D}(p_{2n+1},p_{2n+2})),\psi(\mathfrak{D}(p_{2n+1},p_{2n+2})),\psi\left(\frac{\mathfrak{D}(p_{2n+1},p_{2n+2})}{2}\right)\right\}
$$

Now,

$$
\begin{aligned}
N_s(p_{2n},p_{2n+1}) &= min\{\psi(\mathfrak{D}^s(p_{2n},fp_{2n})),\psi(\mathfrak{D}^s(p_{2n+1},gp_{2n+1})),\psi(\mathfrak{D}^s(p_{2n},gp_{2n+1})),\psi(\mathfrak{D}^s(p_{2n+1},fp_{2n}))\}\\
&= min\{\psi(\mathfrak{D}^s(p_{2n},p_{2n+1})),\psi(\mathfrak{D}^s(p_{2n+1},p_{2n+2})),\psi(\mathfrak{D}^s(p_{2n},p_{2n+2})),\psi(\mathfrak{D}^s(p_{2n+1},p_{2n+1}))\}
\end{aligned}
$$

If $N_s(p_{2n},p_{2n+1}) = \psi(\mathfrak{D}^s(p_{2n+1},p_{2n+1}))$

$$
\mathfrak{D}^s(p_{2n+1},p_{2n+1}) = |2\mathfrak{D}(p_{2n+1},p_{2n+1}) - \mathfrak{D}(p_{2n+1},p_{2n+1}) - \mathfrak{D}(p_{2n+1},p_{2n+1})|
$$

clearly, $N_s(p_{2n},p_{2n+1}) = 0$. then from (3.3.3), we have

$$
\begin{aligned}
K^4\mathfrak{D}(p_{2n+1},p_{2n+2}) &\leq \frac{M_s(p_{2n},p_{2n+1})+N_s(p_{2n},p_{2n+1})}{2}\\
K^4\mathfrak{D}(p_{2n+1},p_{2n+2}) &\leq \frac{M_s(p_{2n},p_{2n+1})+0}{2}
\end{aligned}
$$

$$
K^4\mathfrak{D}(p_{2n+1},p_{2n+2}) \leq \frac{M_s(p_{2n},p_{2n+1})}{2} \tag{3.3.5}
$$

where,

$$
M_s(p_{2n},p_{2n+1}) \leq max\left\{\psi(2K\mathfrak{D}(p_{2n+1},p_{2n+2})),\psi(\mathfrak{D}(p_{2n+1},p_{2n+2})),\psi\left(\frac{\mathfrak{D}(p_{2n+1},p_{2n+2})}{2}\right)\right\}
$$

If $M_s(p_{2n}, p_{2n+1}) = \psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2}))$, then from (3.3.5), we have

$$\begin{aligned} K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) &\leq \psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2})) \\ &< \frac{2K\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2} \\ K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) - K\mathfrak{D}(p_{2n+1}, p_{2n+2}) &< 0 \\ K(k^3 - 1)\mathfrak{D}(p_{2n+1}, p_{2n+2}) &< 0 \end{aligned}$$

a contradiction. If $M_s(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2}))$, then from (3.3.5), we have

$$\begin{aligned} K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) &\leq \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})) \\ &< \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2} \\ (k^4 - \frac{1}{2})\mathfrak{D}(p_{2n+1}, p_{2n+2}) &< 0 \end{aligned}$$

a contradiction. If $M_s(p_{2n}, p_{2n+1}) = \psi(\frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2})$, then from (3.3.5), we have

$$\begin{aligned} K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) &\leq \psi(\frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2}) \\ &< \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{4} \\ (k^4 - \frac{1}{4})\mathfrak{D}(p_{2n+1}, p_{2n+2}) &< 0. \end{aligned}$$

that is, $p_{2n+1} = p_{2n+2}$.

Similarly, if $j_0 = 2n + 1$, then $p_{2n+1} = p_{2n+2}$ gives $p_{2n+2} = p_{2n+3}$. Consequently, the sequence $\{p_k\}$ becomes constant for $j \geq j_0$ and $\{p_{j0}\}$ is a coincidence point of f and g. For this, let $j_0 = 2n$. Then, we know that $p_{2n} = p_{2n+1} = p_{2n+2}$. hence

$$p_{2n} = p_{2n+1} = fp_{2n} = p_{2n+2} = gp_{2n+1}$$

This means that $fp_{2n} = gp_{2n+1}$, Now since $p_{2n} = p_{2n+1}$, we have $fp_{2n} = gp_{2n}$

In the other case, when $k_0 = 2n+1$, Similarly, it can be easily shown that p_{2n+1} is a coincidence point of the pair (f, g).

Suppose now that $\mathfrak{D}(p_{j_0}, p_{j_0+1}) > 0$ for each j_0. We claim the inequality

$$\mathfrak{D}(p_{j_0+1}, p_{j_0+2}) \leq \mathfrak{D}(p_{j_0}, p_{j_0+1}) \tag{3.3.6}$$

holds for each j_0=1,2,....

Let $j_0 = 2n$ and for $n \geq 0$,

$$\mathfrak{D}(p_{2n+1}, p_{2n+2}) > \mathfrak{D}(p_{2n}, p_{2n+1}) > 0 \tag{3.3.7}$$

Then, as $p_{2n} \leq p_{2n+1}$, using (3.3.3) we obtain that

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) = K^4\mathfrak{D}(fp_{2n}, gp_{2n+1}) \leq \frac{M_s(p_{2n}, p_{2n+1}) + N_s(p_{2n}, p_{2n+1})}{2}$$

where

$$M_s(p_{2n}, p_{2n+1}) = max \left\{ \begin{array}{r} \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, fp_{2n})), \psi(\mathfrak{D}(p_{2n+1}, gp_{2n+1})), \\ \psi\left(\dfrac{\mathfrak{D}(p_{2n}, gp_{2n+1}) + \mathfrak{D}(p_{2n+1}, fp_{2n})}{6K}\right) \end{array} \right\}$$

$$M_s(p_{2n}, p_{2n+1}) = max \left\{ \begin{array}{l} \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \\ \psi\left(\dfrac{\mathfrak{D}(p_{2n}, p_{2n+2}) + \mathfrak{D}(p_{2n+1}, p_{2n+1})}{6K}\right) \end{array} \right\}$$

and

$N_s(p_{2n}, p_{2n+1}) = 0$

If $M_s(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2}))$, then from (3.3.5), we have

$$\begin{aligned} K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) &\leq \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})) \\ &< \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2} \\ K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) - \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2} &< 0 \\ (k^4 - \frac{1}{2})\mathfrak{D}(p_{2n+1}, p_{2n+2}) &< 0 \end{aligned}$$

a contradiction. If $M_s(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n}, p_{2n+1}))$, then from (3.3.5), we have

$$\begin{aligned} K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) &\leq \psi(\mathfrak{D}(p_{2n}, p_{2n+1})) \\ &< \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2} \\ (k^4 - \frac{1}{2})\mathfrak{D}(p_{2n+1}, p_{2n+2}) &< 0 \end{aligned}$$

a contradiction. If $M_s(p_{2n}, p_{2n+1}) = \psi(\frac{\mathfrak{D}(p_{2n},p_{2n+2})+\mathfrak{D}(p_{2n+1},p_{2n+1})}{6K})$, then from (3.3.5), we have

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \leq \psi(\frac{\mathfrak{D}(p_{2n}, p_{2n+2}) + \mathfrak{D}(p_{2n+1}, p_{2n+1})}{6K}) < \frac{\mathfrak{D}(p_{2n}, p_{2n+2}) + \mathfrak{D}(p_{2n+1}, p_{2n+1})}{12K}$$
$$\leq \frac{\mathfrak{D}(p_{2n}, p_{2n+1}) + \mathfrak{D}(p_{2n+1}, p_{2n+2}) + 2\mathfrak{D}(p_{2n+1}, p_{2n+2})}{12} < \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{3}$$
$$(k^4 - \frac{1}{3})\mathfrak{D}(p_{2n+1}, p_{2n+2}) < 0.$$

That is, $p_{2n+1} = p_{2n+2}$. Hence, (3.3.7) is false, that is, $\mathfrak{D}(p_{2n+1}, p_{2n+2}) \leq \mathfrak{D}(p_{2n}, p_{2n+1})$ holds for all n.

Therefore (3.3.6) is proved for $j_0 = 2n$. Similarly, it can be shown that

$$\mathfrak{D}(p_{2n+2}, p_{2n+3}) \leq \mathfrak{D}(p_{2n+1}, p_{2n+2})$$

Hence, $\{\mathfrak{D}(p_{j_0}, p_{j_0+1})\}$ is a nondecreasing sequence of nonnegative real numbers.

We claim that $\lim_{j_0\to\infty} \mathfrak{D}(p_{j_0}, p_{j_0+1})$=0

Assume that $\lim_{j_0\to\infty} \mathfrak{D}(p_{j_0}, p_{j_0+1}) = r$, where $r > 0$ we have

$$M_s(p_{2n}, p_{2n+1}) \leq max\left\{\psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi\left(\frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2}\right)\right\} \tag{3.3.8}$$

and $N_s(p_{2n}, p_{2n+1}) = 0$.

Now taking the upper limit as $n \to \infty$ in (3.3.8), we obtain

$$\limsup_{n\to\infty} M_s(p_{2n}, p_{2n+1}) \leq r$$

Taking the upper limit, we have

$$\begin{aligned} K^4\mathfrak{D}(p_{2n}, p_{2n+1}) &\leq \frac{M_s(p_{2n}, p_{2n+1}) + N_s(p_{2n}, p_{2n+1})}{2} \\ K^4 r &\leq \frac{r}{2} \\ (K^4 - \frac{1}{2})r &\leq 0 \end{aligned}$$

a contradiction. Hence,

$$r = \lim_{j_0 \to \infty} \mathfrak{D}(p_{j_0}, p_{j_0+1}) = 0$$

Step 2 : Next we show that $\{p_n\}$ is a b-cauchy sequence in X. That is, for every $\epsilon > 0$, there exists $J \in \mathbb{N}$ such that for all $m, n \geq J$, $\mathfrak{D}(p_m, p_n) < \epsilon$.

Assume to contrary, that $\{p_n\}$ is not a b-cauchy sequence . Then from Lemma4.2.2, there exists $\epsilon > 0$ for which we can find a subsequences $\{p_{m(j)}\}$ and $\{p_{n(j)}\}$ such that $n(j) \geq m(j) \geq j$ and:

(a) $m(j) = 2t$ and $n(j) = 2t' + 1$, where t and t' are non-negative integers,

(b) $\mathfrak{D}(p_{m(j)}, p_{n(j)}) \geq \epsilon$ and

(c) $n(j)$ is the smallest number such that the condition (b) holds; i.e $\mathfrak{D}(p_{m(j)}, p_{n(j)-1}) < \epsilon$

Then we have

$$\begin{aligned}
\epsilon &\leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq K\epsilon \\
\frac{\epsilon}{K} &\leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq K^2\epsilon \\
\frac{\epsilon}{K} &\leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)}) \leq K^2\epsilon \\
\frac{\epsilon}{K^2} &\leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq K^3\epsilon
\end{aligned}$$

Since $n(j) > m(j)$, we have $p_{m(j)} \leq p_{n(j)}$

$$\begin{aligned}
K^4\mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) &= K^4\mathfrak{D}(fp_{m(j)}, gp_{n(j)}) \\
&\leq \frac{M_s(p_{m(j)}, p_{n(j)}) + N_s(p_{m(j)}, p_{n(j)})}{2}
\end{aligned}$$

where

$$\begin{aligned}
M_s(p_{m(j)}, p_{n(j)}) &= max\left\{\begin{array}{r} \psi(\mathfrak{D}(p_{m(j)}, p_{n(j)})), \psi(\mathfrak{D}(p_{m(j)}, fp_{m(j)})), \psi(\mathfrak{D}(p_{n(j)}, gp_{n(j)})) \\ , \psi\left(\dfrac{\mathfrak{D}(p_{m(j)}, gp_{n(j)}) + \mathfrak{D}(p_{n(j)}, fp_{m(j)})}{6K}\right) \end{array}\right\} \\
&= max\left\{\begin{array}{r} \psi(\mathfrak{D}(p_{m(j)}, p_{n(j)})), \psi(\mathfrak{D}(p_{m(j)}, p_{m(j)+1})), \psi(\mathfrak{D}(p_{n(j)}, p_{n(j)+1})) \\ , \psi\left(\dfrac{\mathfrak{D}(p_{m(j)}, p_{n(j)+1}) + \mathfrak{D}(p_{n(j)}, p_{m(j)+1})}{6K}\right) \end{array}\right\}
\end{aligned}$$

$$< max\left\{ \begin{array}{r} \mathfrak{D}(p_{m(j)},p_{n(j)}),\mathfrak{D}(p_{m(j)},p_{m(j)+1}),\mathfrak{D}(p_{n(j)},p_{n(j)+1}) \\ ,\left(\frac{\mathfrak{D}(p_{m(j)},p_{n(j)+1})+\mathfrak{D}(p_{n(j)},p_{m(j)+1})}{6K}\right) \end{array} \right\}$$

Taking the upper limit as $j \to \infty$, then we have

$$\begin{aligned} \limsup_{j\to\infty} M_s(p_{m(j)},p_{n(j)}) &\leq max\left\{ \begin{array}{r} \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)},p_{n(j)}),\limsup_{j\to\infty} \mathfrak{D}(p_{m(j)},p_{m(j)+1}),\limsup_{j\to\infty} \mathfrak{D}(p_{n(j)},p_{n(j)+1}) \\ ,\left(\limsup_{j\to\infty} \frac{\mathfrak{D}(p_{m(j)},p_{n(j)+1})+\mathfrak{D}(p_{n(j)},p_{m(j)+1})}{6K}\right) \end{array} \right\} \\ &\leq max\left\{K\epsilon,0,0,\frac{K^2\epsilon+K^2\epsilon}{6K}\right\} = K\epsilon \end{aligned}$$

Similarly

$$\begin{aligned} N_s(p_{m(j)},p_{n(j)}) &= min\left\{ \begin{array}{r} \psi(\mathfrak{D}^s(p_{m(j)},fp_{m(j)})),\psi(\mathfrak{D}^s(p_{n(j)},gp_{n(j)})),\psi(\mathfrak{D}^s(p_{m(j)},gp_{n(j)})), \\ \psi(\mathfrak{D}^s(p_{n(j)},fp_{m(j)})) \end{array} \right\} \\ &= min\left\{ \begin{array}{r} \psi(\mathfrak{D}^s(p_{m(j)},p_{m(j)+1})),\psi(\mathfrak{D}^s(p_{n(j)},p_{n(j)+1})),\psi(\mathfrak{D}^s(p_{m(j)},p_{n(j)+1})), \\ \psi(\mathfrak{D}^s(p_{n(j)},p_{m(j)+1})) \end{array} \right\} \end{aligned}$$

Now,

$$\begin{aligned} \mathfrak{D}^s(p_{n(j)},p_{n(j)+1}) &\leq |2\mathfrak{D}(p_{n(j)},p_{n(j)+1})-\mathfrak{D}(p_{n(j)},p_{n(j)})-\mathfrak{D}(p_{n(j)+1},p_{n(j)+1})| \\ &\leq |2\mathfrak{D}(p_{n(j)},p_{n(j)+1})-(\mathfrak{D}(p_{n(j)},p_{n(j)})+\mathfrak{D}(p_{n(j)+1},p_{n(j)+1}))| \\ &\leq |\mathfrak{D}(p_{n(j)},p_{n(j)})+\mathfrak{D}(p_{n(j)+1},p_{n(j)+1})-2\mathfrak{D}(p_{n(j)},p_{n(j)+1})| \end{aligned}$$

By $D3$

$$\begin{aligned} \mathfrak{D}(p_{n(j)},p_{n(j)}) &\leq 2K\mathfrak{D}(p_{n(j)},p_{n(j)+1}) \\ \mathfrak{D}(p_{n(j)+1},p_{n(j)+1}) &\leq 2K\mathfrak{D}(p_{n(j)},p_{n(j)+1}) \end{aligned}$$

$$\mathfrak{D}(p_{n(j)},p_{n(j)})+\mathfrak{D}(p_{n(j)+1},p_{n(j)+1}) \leq 4K\mathfrak{D}(p_{n(j)},p_{n(j)+1})$$

$$\begin{aligned} \mathfrak{D}^s(p_{n(j)},p_{n(j)+1}) &\leq |(4K-2)(p_{n(j)},p_{n(j)+1})| \\ \limsup_{j\to\infty} \mathfrak{D}^s(p_{n(j)},p_{n(j)+1}) &\leq |(4K-2)\limsup_{j\to\infty}(p_{n(j)},p_{n(j)+1})| \end{aligned}$$

clearly, $N_s(p_n(j), p_m(j)) = 0$

Hence, by taking the upper limit as $j \to \infty$, we have

$$K^4 \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq \frac{\limsup_{j\to\infty} M_s(p_{m(j)}, p_{n(j)}) + \limsup_{j\to\infty} N_s(p_{m(j)}, p_{n(j)})}{2}$$

$$K^4 \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq \frac{K\epsilon}{2} \leq K\epsilon$$

Which implies that $\limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) < \frac{\epsilon}{K^3} < \frac{\epsilon}{K^2}$
a contradiction to (4) property proving above. Hence $\{p_n\}$ is a b-cauchy sequence.

Step 3 : In this step, We will show that f and g have a common fixed point.

Since $\{p_n\}$ is a b-cauchy sequence in the complete b-metric-like space X, there exists $z \in X$ such that

$$\lim_{n\to\infty} \mathfrak{D}(p_{2n}, z) = \lim_{n\to\infty} \mathfrak{D}(p_{2n+1}, z) = \lim_{n\to\infty} \mathfrak{D}(fp_{2n}, z) = 0 \tag{3.3.9}$$

By the triangle inequality, we have

$$\mathfrak{D}(fz, z) \leq K[\mathfrak{D}(fz, fp_{2n}) + \mathfrak{D}(fp_{2n}, z)] = K[\mathfrak{D}(fz, fp_{2n}) + \mathfrak{D}(p_{2n+1}, z)] \tag{3.3.10}$$

Suppose that f is continuous. Letting $n \to \infty$ in (3.3.10) and applying (3.3.9) we have

$$\mathfrak{D}(fz, z) \leq K[\lim_{n\to\infty} \mathfrak{D}(fz, fp_{2n}) + \lim_{n\to\infty} \mathfrak{D}(fp_{2n}, z)] = 0$$

which implies that $fz = z$.

Let $\mathfrak{D}(z, gz) > 0$. As z and gz are comparable by (3.3.3) we have

$$K^4\mathfrak{D}(z, gz) = K^4\mathfrak{D}(fz, gz) \leq \frac{M_s(z, z) + N_s(z, z)}{2} \tag{3.3.11}$$

where

$$M_s(z, z) = max\left\{\psi(\mathfrak{D}(z, z)), \psi(\mathfrak{D}(z, fz)), \psi(\mathfrak{D}(z, gz)), \psi(\frac{\mathfrak{D}(z, gz) + \mathfrak{D}(z, fz)}{6K})\right\}$$

By the triangle inequality, we have

$$\begin{aligned} \mathfrak{D}(z, z) &\leq 2K\mathfrak{D}(z, gz) \\ \mathfrak{D}(z, gz) &\leq K\mathfrak{D}(z, gz) \end{aligned}$$

$$\begin{aligned}\mathfrak{D}(z,z)+\mathfrak{D}(z,gz) &\leq 3K\mathfrak{D}(z,gz)\\ \frac{\mathfrak{D}(z,z)+\mathfrak{D}(z,gz)}{6K} &\leq \frac{(\mathfrak{D}(z,gz))}{2}\end{aligned}$$

$$M_s(z,z) \leq max\left\{\psi(2K\mathfrak{D}(z,gz)),\psi(2K\mathfrak{D}(z,gz)),\psi(\mathfrak{D}(z,gz)),\psi(\frac{(\mathfrak{D}(z,gz))}{2})\right\} < 2K\mathfrak{D}(z,gz)$$

Similarly,

$$N_s(z,z) = min\{\psi(\mathfrak{D}^s(z,fz)),\psi(\mathfrak{D}^s(z,gz)),\psi(\mathfrak{D}^s(z,gz)),\psi(\mathfrak{D}^s(z,fz))\}$$

$$\mathfrak{D}^s(z,z) = |2\mathfrak{D}(z,z)-\mathfrak{D}(z,z)-\mathfrak{D}(z,z)| = 0$$

Clearly, $N_s(z,z) = 0$

Hence, (3.3.11) gives $K^4\mathfrak{D}(z,gz) < K\mathfrak{D}(z,gz)$, which is contradiction. Thus, $\mathfrak{D}(z,gz) = 0$. Similarly, if g is continuous, the desired result is obtained. □

Theorem 3.3.2. *Let $(X,\preccurlyeq,\mathfrak{D})$ be a complete partially ordered b-metric-like space. Let $f,g : X \to X$ be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements $p,q \in X$, we have*

$$K^4\mathfrak{D}(fp,gq) \leq \frac{M_s(p,q)+N_s(p,q)}{2} \tag{3.3.12}$$

Then, the pair (f,g) has a common fixed point z in X if X is regular. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. Following the proof of theorem (3.3.1), there exists $z \in X$ such that

$$\lim_{n\to\infty}\mathfrak{D}(p_n,z) = 0$$

Now we prove that z is a common fixed point of f and g. Since $p_{2n+1} \to z$ as $n \to \infty$ from regularity of X, $p_{2n+1} \leqslant z$. Therefore, from (3.3.3), we have

$$K^4\mathfrak{D}(fz,gp_{2n+1}) \leq \frac{M_s(z,p_{2n+1})+N_s(z,p_{2n+1})}{2} \tag{3.3.13}$$

where

$$M_s(z,p_{2n+1}) = max\{\psi(\mathfrak{D}(z,p_{2n+1})), \psi(\mathfrak{D}(z,fz)), \psi(\mathfrak{D}(p_{2n+1},gp_{2n+1})), \psi(\frac{\mathfrak{D}(z,gp_{2n+1})+\mathfrak{D}(p_{2n+1},fz)}{6K})\}$$

Taking the limit as $n \to \infty$ in (3.3.13) and using lemma (4.2.1), we obtain that

$$\begin{aligned} K^3\mathfrak{D}(fz,z) &= K^4\frac{1}{K}\mathfrak{D}(fz,z) \leq K^4 \limsup_{n\to\infty}\mathfrak{D}(fz,gp_{2n+1}) \leq \frac{\limsup_{n\to\infty} M_s(z,p_{2n+1}) + \limsup_{n\to\infty} N_s(z,p_{2n+1})}{2} \\ &= \frac{max\left\{\begin{array}{r}\limsup_{n\to\infty}\mathfrak{D}(z,p_{2n+1}), \limsup_{n\to\infty}\mathfrak{D}(z,fz), \limsup_{n\to\infty}\mathfrak{D}(p_{2n+1},gp_{2n+1}), \\ \limsup_{n\to\infty}\frac{\mathfrak{D}(z,gp_{2n+1})+\mathfrak{D}(p_{2n+1},fz)}{6K}\end{array}\right\} + \limsup_{n\to\infty} N_s(z,p_{2n+1})}{2} \\ &\leq \frac{1}{2}max\{\mathfrak{D}(z,z), \mathfrak{D}(z,fz), \mathfrak{D}(z,z), \frac{\mathfrak{D}(z,fz)+\mathfrak{D}(z,z)}{6K}\} + \limsup_{n\to\infty} N_s(z,p_{2n+1}) \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} \mathfrak{D}(z,z) &\leq 2K\mathfrak{D}(z,fz) \\ \mathfrak{D}(z,fz) &\leq K\mathfrak{D}(z,fz) \end{aligned}$$

$$\begin{aligned} \mathfrak{D}(z,z)+\mathfrak{D}(z,fz) &\leq 3K\mathfrak{D}(z,fz) \\ \frac{\mathfrak{D}(z,z)+\mathfrak{D}(z,fz)}{6K} &\leq \frac{(\mathfrak{D}(z,fz))}{2} \end{aligned}$$

$$M_s(z,z) \leq max\left\{\psi(2K\mathfrak{D}(z,fz)), \psi(2K\mathfrak{D}(z,fz)), \psi(\mathfrak{D}(z,fz)), \psi(\frac{(\mathfrak{D}(z,fz))}{2})\right\} < 2K\mathfrak{D}(z,fz)$$

Similarly,

$$N_s(z,z) = min\{\psi(\mathfrak{D}^s(z,fz)), \psi(\mathfrak{D}^s(z,gz)), \psi(\mathfrak{D}^s(z,gz)), \psi(\mathfrak{D}^s(z,fz))\}$$

$$\mathfrak{D}^s(z,z) = |2\mathfrak{D}(z,z) - \mathfrak{D}(z,z) - \mathfrak{D}(z,z)| = 0$$

Clearly, $N_s(z,z) = 0$

Hence by the (3.3.13), we have

$$\begin{aligned} K^4\mathfrak{D}(z,fz) &< \frac{2K\mathfrak{D}(z,fz)+0}{2} \\ K^4\mathfrak{D}(z,fz) - \frac{2K\mathfrak{D}(z,fz)+0}{2} &< 0 \\ K(K^3-1)\mathfrak{D}(z,fz) &< 0 \end{aligned}$$

a contradiction. this implies that $fz = z$.

Similarly, it can be shown that z is a fixed point of g. □

Corollary 3.3.3. *Let $(X, \preccurlyeq, d)$ be a complete partially ordered b-metric space. Let $f, g : X \to X$ be two mappings such that f is g-weakly isotone increasing.Suppose that for every two comparable elements $x, y \in X$ and a constant $s > 1$, we have*

$$s^4 d(fx, gy) \leq M_s(x, y) \tag{3.3.14}$$

Then, the pair (f, g) has a common fixed point z in X if one of f or g is continuous. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Example: Let $X = [0, \infty)$ be equipped with the b-metric-like $\mathfrak{D}(p, q) = |p + q|^2$, $p, q \in X$ where $K = 2$ according to example (1.3) and define a relation $\preccurlyeq$ on X by $p \preccurlyeq q$ iff $q \leq p$, where $\leq$ is the usual ordering on $\mathbb{R}$, Define function $f, g : X \to X$ by

$$fp = \frac{p}{9} \quad and \quad gp = \frac{p}{7}$$

Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$. Then we have the following:

(1) $(X, \preccurlyeq, \mathfrak{D})$ is a complete partially ordered b-metric-like space.

(2) f is g-weakly isotone increasing with respect to $\preccurlyeq$.

(3) f and g are continuous.

(4) for every two comparable elements $p, q \in X$ the inequality (3.3.3) holds, where $M_s(p, q)$ and $N_s(p, q)$ is given by (3.3.1), (3.3.2) respectively.

Proof. **Step 1**: The proof of (1) is clear.

Step 2: To prove (2), for each $p \in X$, $fp = \frac{p}{9} < p$ and $gp = \frac{p}{7} < p$. Thus for each $p \in X$ we have $gfp = g(\frac{p}{9}) = \frac{p}{63} \leq fp$ and $fgfp = f(\frac{p}{63}) = \frac{p}{567} \leq gfp$, i.e., $fp \preccurlyeq gfp \preccurlyeq fgfp$. Thus f is g-weakly isotone increasing with respect to $\preccurlyeq$.

Step 3: To prove (3), it is easy to see that f and g are continuous.

Step 4: To prove (4), Assume $p, q \in X$ with $P \preccurlyeq q$, i.e., $q \leq p$.

$$K^4 \mathfrak{D}(fp, gq) \leq \frac{M_s(p, q) + N_s(p, q)}{2} \tag{3.3.15}$$

where

$$M_s(p,q) = max\left\{\psi(\mathfrak{D}(p,q)), \psi(\mathfrak{D}(p,fp)), \psi(\mathfrak{D}(q,gq)), \psi\left(\frac{\mathfrak{D}(p,gq)+\mathfrak{D}(q,fp)}{6K}\right)\right\}$$

$$M_s(p,q) = max\left\{\psi((p+q)^2), \psi((p+\frac{p}{9})^2), \psi((q+\frac{q}{7})^2), \psi\left(\frac{(p+\frac{q}{7})^2+(q+\frac{p}{9})^2}{6K}\right)\right\}$$

We have the following cases:

Case 1: If $\frac{q}{7} \leq \frac{p}{9}$ then we have

$$\begin{aligned} M_s(p,q) &\leq max\left\{\psi((p+\frac{7p}{9})^2), \psi((\frac{10p}{9})^2), \psi((\frac{8p}{9})^2), \psi\left(\frac{(p+\frac{p}{9})^2+(q+\frac{p}{9})^2}{6K}\right)\right\} \\ M_s(p,q) &\leq max\{\psi((\frac{16p}{9})^2), \psi((\frac{10p}{9})^2), \psi((\frac{8p}{9})^2), \psi(\frac{164p^2}{972})\} \end{aligned}$$

Clearly, $M_s(p,q) = \frac{1}{2}(\frac{16p}{9})^2$

and

$$\begin{aligned} N_s(p,q) &= min\{\psi(\mathfrak{D}^s(p,fp)), \psi(\mathfrak{D}^s(q,gq)), \psi(\mathfrak{D}^s(p,gq)), \psi(\mathfrak{D}^s(q,fp))\} \\ N_s(p,q) &= min\{\psi(\mathfrak{D}^s(p,\frac{p}{9})), \psi(\mathfrak{D}^s(q,\frac{q}{7})), \psi(\mathfrak{D}^s(p,\frac{q}{7})), \psi(\mathfrak{D}^s(q,\frac{p}{9}))\} \end{aligned}$$

$$\begin{aligned} \mathfrak{D}^s(p,\frac{q}{7}) &\leq \mathfrak{D}^s(p,p) \qquad\qquad (\because \frac{q}{7} \leq \frac{p}{9} \leq p) \\ &= |2\mathfrak{D}(p,p) - \mathfrak{D}(p,p) - \mathfrak{D}(p,p)| \\ &= 0 \end{aligned}$$

Clearly, $N_s(p,q) = 0$. Then by (3.3.15)

$$\begin{aligned} 16\mathfrak{D}(\frac{p}{9},\frac{q}{7}) &\leq 16(\frac{p}{9}+\frac{p}{9})^2 \leq 16(\frac{2p}{9})^2 \\ &= \frac{16\times 16\times p^2}{81\times 4} \end{aligned}$$

Clearly, $16\mathfrak{D}(\frac{p}{9},\frac{q}{7}) \leq \frac{M_s(p,q)+N_s(p,q)}{2}$

Case 2: If $\frac{p}{9} < \frac{q}{7}$ then we have

$$\begin{aligned} M_s(p,q) &\leq max\left\{\psi((q+\frac{9q}{7})^2), \psi((\frac{10q}{7})^2), \psi((\frac{8q}{7})^2), \psi\left(\frac{(\frac{9q}{7}+\frac{q}{7})^2+(q+\frac{q}{7})^2}{12}\right)\right\} \\ M_s(p,q) &\leq max\{\psi((\frac{16q}{7})^2), \psi((\frac{10q}{7})^2), \psi((\frac{8q}{7})^2), \psi(\frac{164q^2}{588})\} \end{aligned}$$

Clearly, $M_s(p,q) = \frac{1}{2}(\frac{16q}{7})^2$

and

$$\begin{aligned} N_s(p,q) &= min\{\psi(\mathfrak{D}^s(p,fp)), \psi(\mathfrak{D}^s(q,gq)), \psi(\mathfrak{D}^s(p,gq)), \psi(\mathfrak{D}^s(q,fp))\} \\ N_s(p,q) &= min\{\psi(\mathfrak{D}^s(p,\frac{p}{9})), \psi(\mathfrak{D}^s(q,\frac{q}{7})), \psi(\mathfrak{D}^s(p,\frac{q}{7})), \psi(\mathfrak{D}^s(q,\frac{p}{9}))\} \end{aligned}$$

$$\begin{aligned} \mathfrak{D}^s(q,\frac{p}{9}) &\leq \mathfrak{D}^s(q,q) \qquad\qquad (\because \frac{p}{9} \leq \frac{q}{7} \leq q) \\ &= |2\mathfrak{D}(q,q) - \mathfrak{D}(q,q) - \mathfrak{D}(q,q)| \\ &= 0 \end{aligned}$$

Clearly, $N_s(p,q) = 0$. Then by (3.3.15)

$$\begin{aligned} 16\mathfrak{D}(\frac{p}{9},\frac{q}{7}) &\leq 16(\frac{q}{7}+\frac{q}{7})^2 \leq 16(\frac{2p}{7})^2 \\ &= \frac{16 \times 16 \times p^2}{49 \times 4} \end{aligned}$$

Clearly, $16\mathfrak{D}(\frac{p}{9},\frac{q}{7}) \leq \frac{M_s(p,q)+N_s(p,q)}{2}$

By combining all cases together, we conclude that f, g and ψ satisfy all the hypothesis of theorem(3.3.1) and hence f and g have a common fixed point. Indeed, 0 is the unique fixed point of f and g. □

4

Common Fixed Point Result in Partially Ordered b-metric-like Space

4.1 Introduction

In this subsection, we recall some preliminaries, definitions and well known results.

Let $(x, \preccurlyeq)$ be a partially ordered set and let f, g be two self-maps on X. We will use the following terminology:

(a) elements $p, q \in X$ are called comparable if $p \leq q$ or $q \leq p$ holds;

(b) a subset S of X is said to be well ordered if every two elements of S are comparable;

(c) f is called nondecreasing w.r.t. $\preccurlyeq$ if $p \preccurlyeq q$ implies $fp \preccurlyeq fq$;

(d)([7]) the pair (f, g) is said to be weakly increasing if $fp \preccurlyeq gfp$ and $gp \preccurlyeq fgp$ for all $p \in X$;

(e)([25]) f is said to be g-weakly isotone increasing if for all $p \in X$ we have $fp \preccurlyeq gfp \preccurlyeq fgfp$;

If $f, g : X \to X$ are weakly increasing, then f is g-weakly isotone increasing. Also, in (e), if $f = g$, we say that f is weakly isotone increasing, In this case, for each $p \in X$, we have

[0] The part of this chapter has been published in ***International Journal of Functional Analysis, Operator Theory and Applications,*** 7(1), 2015, 19-56.

$fp \preccurlyeq ffp$.

Definition 4.1.1. *[28] Let $(X, \preccurlyeq)$ be a partially ordered set and d be a metric on X. We say that $(X, \preccurlyeq, d)$ is regular if the following conditions hold:*

(1) if a non-decreasing sequence $x_n \to x$, $n \to \infty$, then $x_n \preccurlyeq x$ for all n,

(2) if a non-increasing sequence $y_n \to y$, $n \to \infty$, then $y_n \succcurlyeq y$ for all n.

4.2 Lemmas

The following lemmas are required in the proof of our main results.

Lemma 4.2.1. *[5] Let $(X, \mathfrak{D}, K)$ be a b-metric-like space and $\{p_n\}$ be a sequence in X such that $\lim_{n\to\infty} \mathfrak{D}(p_n, p) = 0$. Moreover $z \in X$, we have*

$$\frac{1}{K}\mathfrak{D}(p, z) \leq \liminf_{n\to\infty} \mathfrak{D}(p_n, z) \leq \limsup_{n\to\infty} \mathfrak{D}(p_n, z) \leq K\mathfrak{D}(p, z).$$

Lemma 4.2.2. *Let $(X, \mathfrak{D})$ be a b-metric-like space and let $\{p_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} \mathfrak{D}(p_n, p_{n+1}) = 0. \tag{4.2.1}$$

If $\{p_n\}$ is not a b-cauchy sequence, there exists $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for the following four sequences

$\mathfrak{D}(p_{m(k)}, p_{n(k)})$, $\mathfrak{D}(p_{m(k)}, p_{m(k)+1})$, $\mathfrak{D}(p_{m(k)+1}, p_{n(k)})$ and $\mathfrak{D}(p_{m(k)+1}, p_{n(k)+1})$

it holds:

$$\begin{aligned}
\epsilon &\leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq K\epsilon \\
\frac{\epsilon}{K} &\leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq K^2\epsilon \\
\frac{\epsilon}{K} &\leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)}) \leq K^2\epsilon \\
\frac{\epsilon}{K^2} &\leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq K^3\epsilon
\end{aligned}$$

Proof. The proof of this lemma is already proved in Chapter 3. □

4.3 Main Result

Theorem 4.3.1. *Let* $(X, \preccurlyeq, \mathfrak{D})$ *be a complete partially ordered b-metric-like space with* $K > 1$*. Let* $f, g : X \to X$ *be two mappings such that* f *is g-weakly isotone increasing. Suppose that for every two comparable elements* $p, q \in X$ *, we have*

$$K^4 \mathfrak{D}(fp, gq) \leq \frac{N(p,q) + N_s(p,q)}{2} \tag{4.3.1}$$

Where

$$N(p,q) = max\{\psi(\mathfrak{D}(p,q)), \psi(\mathfrak{D}(p,fp)), \psi(\mathfrak{D}(q,gq)), \psi(\mathfrak{D}(p,gq)), \psi(\mathfrak{D}(q,fp))\} \tag{4.3.2}$$

and

$$N_s(p,q) = min\{\psi(\mathfrak{D}^s(p,fp)), \psi(\mathfrak{D}^s(q,gq)), \psi(\mathfrak{D}^s(p,gq)), \psi(\mathfrak{D}^s(q,fp))\} \tag{4.3.3}$$

and $\psi : [0, \infty) \to [0, \infty)$ *is a continuous function with* $\psi(t) < \frac{t}{2K}$ *for each* $t > 0$ *and* $\psi(0) = 0$*. Then, the pair* (f, g) *has a common fixed point* z *in* X *if one of* f *or* g *is continuous. Moreover, the set of common fixed points of* f *and* g *is well ordered if and only if* f *and* g *have one and only one common fixed point.*

Proof. Let p_0 be an arbitrary point of X. Choose $p_1 \in X$ such that $fp_0 = p_1$ and $p_2 \in X$ such that $gp_1 = p_2$. continuing in this way, construct a sequence $\{p_n\}$ defined by:

$$p_{2n+1} = fp_{2n} \;\; and \;\; p_{2n+2} = gp_{2n+1}$$

for all $n \geq 0$. As f is g-weakly isotone increasing, we have

$$p_1 = fp_0 \leq gfp_0 = gx_1 = x_2 \leq fgfp_0 = fp_2 = p_3$$

Repeating this process, we obtain $p_n \leq p_{n+1} \;\; \forall \;\; n \geq 1$.

We will prove the theorem in three steps.

Step 1: Firstly we prove that $\lim_{n\to\infty} \mathfrak{D}(p_n, p_{n+1}) = 0$.

Suppose $\mathfrak{D}(p_{j_0}, p_{j_0+1}) = 0$ for some j_0. Then $p_{j_0} = p_{j_0+1}$.In this case $p_0 = 2n$,$p_{2n} = p_{2n+1}$. we need to show that $p_{2n+1} = p_{2n+2}$. If $\mathfrak{D}(p_{2n+1}, p_{2n+2}) > 0$, then from (4.3.1) we have

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) = K^4\mathfrak{D}(fp_{2n}, gp_{2n+1}) \leq \frac{N(p_{2n}, p_{2n+1}) + N_s(p_{2n}, p_{2n+1})}{2} \tag{4.3.4}$$

where

$$
\begin{aligned}
N(p_{2n}, p_{2n+1}) &= max \left\{ \begin{array}{r} \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, fp_{2n})), \psi(\mathfrak{D}(p_{2n+1}, gp_{2n+1})), \psi(\mathfrak{D}(p_{2n}, gp_{2n+1})), \\ \psi(\mathfrak{D}(p_{2n+1}, fp_{2n})) \end{array} \right\} \\
&= max \left\{ \begin{array}{r} \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n}, p_{2n+2})), \\ \psi(\mathfrak{D}(p_{2n+1}, p_{2n+1})) \end{array} \right\} \\
&= max \left\{ \begin{array}{r} \psi(\mathfrak{D}(p_{2n+1}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \\ \psi(\mathfrak{D}(p_{2n+1}, p_{2n+1})) \end{array} \right\} \\
&= max \{\psi(\mathfrak{D}(p_{2n+1}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2}))\}
\end{aligned}
$$

By $D3$, we have

$$\mathfrak{D}(p_{2n+1}, p_{2n+1}) \leq 2K\mathfrak{D}(p_{2n+1}, p_{2n+2})$$

$$
\begin{aligned}
N(p_{2n}, p_{2n+1}) &\leq \{\psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2}))\} \\
&= \psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2}))
\end{aligned}
$$

Now,

$$
\begin{aligned}
N_s(p_{2n}, p_{2n+1}) &= min\{\psi(\mathfrak{D}^s(p_{2n}, fp_{2n})), \psi(\mathfrak{D}^s(p_{2n+1}, gp_{2n+1})), \psi(\mathfrak{D}^s(p_{2n}, gp_{2n+1})), \psi(\mathfrak{D}^s(p_{2n+1}, fp_{2n}))\} \\
&= min\{\psi(\mathfrak{D}^s(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}^s(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}^s(p_{2n}, p_{2n+2})), \psi(\mathfrak{D}^s(p_{2n+1}, p_{2n+1}))\}
\end{aligned}
$$

If $N_s(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}^s(p_{2n+1}, p_{2n+1}))$, then

$$\mathfrak{D}^s(p_{2n+1}, p_{2n+1}) = |2\mathfrak{D}(p_{2n+1}, p_{2n+1}) - \mathfrak{D}(p_{2n+1}, p_{2n+1}) - \mathfrak{D}(p_{2n+1}, p_{2n+1})|$$

clearly, $N_s(p_{2n}, p_{2n+1}) = 0$. then from (4.3.1), we have

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \leq \psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2})) < \frac{(2K\mathfrak{D}(p_{2n+1}, p_{2n+2}))}{4K}$$

Hence, $[2K^4 - 1]\mathfrak{D}(p_{2n+1}, p_{2n+2}) < 0$, which is a contradiction, so $p_{2n+1} = p_{2n+2}$.

Similarly, if $j_0 = 2n + 1$, then $p_{2n+1} = p_{2n+2}$ gives $p_{2n+2} = p_{2n+3}$. consequently, the sequence

$\{p_k\}$ becomes constant for $j \geq j_0$ and $\{p_{j0}\}$ is a coincidence point of f and g. For this, let $j_0 = 2n$. Then, we know that $p_{2n} = p_{2n+1} = p_{2n+2}$. hence

$$p_{2n} = p_{2n+1} = fp_{2n} = p_{2n+2} = gp_{2n+1}$$

This means that $fp_{2n} = gp_{2n+1}$, Now since $p_{2n} = p_{2n+1}$, we have $fp_{2n} = gp_{2n}$

In the other case, when $j_0 = 2n+1$, Similarly, it can easily be shown that p_{2n+1} is a coincidence point of the pair (f, g).

Suppose now that $\mathfrak{D}(p_{j_0}, p_{j_0+1}) > 0$ for each j_0. we claim the inequality

$$\mathfrak{D}(p_{j_0+1}, p_{j_0+2}) \leq \mathfrak{D}(p_{j_0}, p_{j_0+1}) \tag{4.3.5}$$

holds for each $j_0 = 1, 2, ...$

Let $j = 2n$ and for an $n \geq 0$,

$$\mathfrak{D}(p_{2n+1}, p_{2n+2}) > \mathfrak{D}(p_{2n}, p_{2n+1}) > 0 \tag{4.3.6}$$

Then, as $p_{2n} \leq p_{2n+1}$, using (4.3.1) we obtain that

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) = K^4\mathfrak{D}(fp_{2n}, gp_{2n+1}) \leq \frac{N(p_{2n}, p_{2n+1}) + N_s(p_{2n}, p_{2n+1})}{2} \tag{4.3.7}$$

where by the definition , clearly $N_s(p_{2n}, p_{2n+1}) = 0$ and

$$\begin{aligned} N(p_{2n}, p_{2n+1}) &= max \left\{ \begin{array}{r} \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, fp_{2n})), \psi(\mathfrak{D}(p_{2n+1}, gp_{2n+1})), \psi(\mathfrak{D}(p_{2n}, gp_{2n+1})), \\ \psi(\mathfrak{D}(p_{2n+1}, fp_{2n})) \end{array} \right\} \\ &= max\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+1}))\} \\ &\leq max\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n}, p_{2n+2})), \psi(2K\mathfrak{D}(p_{2n}, p_{2n+1}))\} \end{aligned}$$

If $N(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2}))$, then from (4.3.7), we have

$$\begin{aligned} K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) &\leq \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})) \\ &< \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{4K} \end{aligned}$$

a contradiction. If $N(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n}, p_{2n+2}))$, then from (4.3.7), we have

$$\begin{aligned} K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) &\leq \psi(\mathfrak{D}(p_{2n}, p_{2n+2})) \\ &< \frac{\mathfrak{D}(p_{2n}, p_{2n+2})}{4K} \end{aligned}$$
$$\leq \frac{1}{2K}K[\mathfrak{D}(p_{2n}, p_{2n+1}) + \mathfrak{D}(p_{2n+1}, p_{2n+2})] < \mathfrak{D}(p_{2n+1}, p_{2n+2}),$$

a contradiction. If $N(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n}, p_{2n+1}))$, then from (4.3.7), we have

$$\begin{aligned} K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) &\leq \psi(\mathfrak{D}(p_{2n}, p_{2n+1})) \\ &< \frac{\mathfrak{D}(p_{2n}, p_{2n+1})}{4K} < \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{4K} \end{aligned}$$

a contradiction. If $N(p_{2n}, p_{2n+1}) = \psi(2K\mathfrak{D}(p_{2n}, p_{2n+1}))$, then from (4.3.7), we have

$$\begin{aligned} K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) &\leq \psi(2K\mathfrak{D}(p_{2n}, p_{2n+1})) \\ &< \frac{2K\mathfrak{D}(p_{2n}, p_{2n+1})}{4K} < \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2} \end{aligned}$$

a contradiction.

Hence, (4.3.6) is false, that is, $\mathfrak{D}(p_{2n+1}, p_{2n+2}) \leq \mathfrak{D}(p_{2n}, p_{2n+1})$ holds for all n.

Therefore (4.3.5) is proved for $j_0 = 2n$. Similarly, it can be shown that

$$\mathfrak{D}(p_{2n+2}, p_{2n+3}) \leq \mathfrak{D}(p_{2n+1}, p_{2n+2})$$

Hence, $\{\mathfrak{D}(p_{j_0}, p_{j_0+1})\}$ is a nondecreasing sequence of nonnegative real numbers.

We claim that $\lim_{j_0\to\infty} \mathfrak{D}(p_{j_0}, p_{j_0+1})$=0

Assume that $\lim_{j_0\to\infty} \mathfrak{D}(p_{j_0}, p_{j_0+1}) = r$, where $r > 0$,then we have

$$\begin{aligned} N(p_{2n}, p_{2n+1}) &\leq \frac{1}{2K} max\{2K\mathfrak{D}(p_{2n}, p_{2n+1}), \mathfrak{D}(p_{2n+1}, p_{2n+2}), \mathfrak{D}(p_{2n}, p_{2n+2})\} &(4.3.8) \\ &\leq \frac{1}{2K} max\{2K\mathfrak{D}(p_{2n}, p_{2n+1}), \mathfrak{D}(p_{2n+1}, p_{2n+2}), K\mathfrak{D}(p_{2n}, p_{2n+1}) + K\mathfrak{D}(p_{2n+1}, p_{2n+2})\} &(4.3.9) \end{aligned}$$

and $N_s(p_{2n}, p_{2n+1}) = 0$.

Now taking the upper limit as $n \to \infty$ in (4.3.9), we obtain

$$\limsup_{n\to\infty} N(p_{2n}, p_{2n+1}) \leq \frac{1}{2K} max\{r, 2Kr\} = r \tag{4.3.10}$$

Taking the upper limit as $n \to \infty$ in (4.3.7) in (4.3.9) , we have $K^4 r \leq r$. Therefore $(K^4 - 1)r \leq 0$, a contradiction with $K > 1$. Hence,

$$r = \lim_{n\to\infty} \mathfrak{D}(p_{j_0}, p_{j_0+1}) = 0 \tag{4.3.11}$$

Step 2: we shown that $\{p_n\}$ is a b-cauchy sequence in X. That is, for every $\epsilon > 0$, there exists $J \in \mathbb{N}$ such that for all $m, n \geq j$, $\mathfrak{D}(p_m, p_n) < \epsilon$.

Assume to contrary, that $\{p_n\}$ is not a b-cauchy sequence . Then from Lemma(4.2.2), there exists $\epsilon > 0$ for which we can find a subsequences $\{p_{m(j)}\}$ and $\{p_{n(j)}\}$ such that $n(j) \geq m(j) \geq j$ and:

(a) m(j)=2t and n(j)=2t'+1, where t and t' are non-negative integers,

(b) $\mathfrak{D}(p_{m(j)}, p_{n(j)}) \geq \epsilon$ and

(c) $n(j)$ is the smallest number such that the condition (b) holds; i.e $\mathfrak{D}(p_{m(j)}, p_{n(j)-1}) < \epsilon$

Then we have

$$\begin{aligned}
\epsilon &\leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq K\epsilon \\
\frac{\epsilon}{K} &\leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq K^2\epsilon \\
\frac{\epsilon}{K} &\leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)}) \leq K^2\epsilon \\
\frac{\epsilon}{K^2} &\leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq K^3\epsilon
\end{aligned}$$

Since $n(j) > m(j)$, we have $p_{m(j)} \leq p_{n(j)}$

$$\begin{aligned}
K^4\mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) &= K^4\mathfrak{D}(fp_{m(j)}, gp_{n(j)}) \\
&\leq \frac{N(p_{m(j)}, p_{n(j)}) + N_s(p_{m(j)}, p_{n(j)})}{2}
\end{aligned}$$

where

$$\begin{aligned} N(p_{m(j)}, p_{n(j)}) &= max \left\{ \begin{array}{r} \psi(\mathfrak{D}(p_{m(j)}, p_{n(j)})), \psi(\mathfrak{D}(p_{m(j)}, fp_{m(j)})), \psi(\mathfrak{D}(p_{n(j)}, gp_{n(j)})), \psi(\mathfrak{D}(p_{m(j)}, gp_{n(j)})), \\ \psi(\mathfrak{D}(p_{n(j)}, fp_{m(j)})) \end{array} \right\} \\ &= max \left\{ \begin{array}{r} \psi(\mathfrak{D}(p_{m(j)}, p_{n(j)})), \psi(\mathfrak{D}(p_{m(j)}, p_{m(j)+1})), \psi(\mathfrak{D}(p_{n(j)}, p_{n(j)+1})), \\ \psi(\mathfrak{D}(p_{m(j)}, p_{n(j)+1})), \psi(\mathfrak{D}(p_{n(j)}, p_{m(j)+1})) \end{array} \right\} \\ &< \frac{1}{2K} max \left\{ \begin{array}{r} \mathfrak{D}(p_{m(j)}, p_{n(j)}), \mathfrak{D}(p_{m(j)}, p_{m(j)+1}), \mathfrak{D}(p_{n(j)}, p_{n(j)+1}), \psi(\mathfrak{D}(p_{m(j)}, p_{n(j)+1})), \\ \psi(\mathfrak{D}(p_{n(j)}, p_{m(j)+1})) \end{array} \right\} \end{aligned}$$

Taking the upper limit as $j \to \infty$, then we have

$$\begin{aligned} \limsup_{j\to\infty} N(p_{m(j)}, p_{n(j)}) &\leq \frac{1}{2K} max \left\{ \begin{array}{r} \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}), \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{m(j)+1}), \\ \limsup_{j\to\infty} \mathfrak{D}(p_{n(j)}, p_{n(j)+1}), \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}), \\ \limsup_{j\to\infty} \mathfrak{D}(p_{n(j)}, p_{m(j)+1}) \end{array} \right\} \\ &\leq \frac{1}{2K} max\{K\epsilon, 0, 0, K^2\epsilon, K^2\epsilon\} = \frac{K\epsilon}{2} \end{aligned}$$

Similarly

$$\begin{aligned} N_s(p_{m(j)}, p_{n(j)}) &= min \left\{ \begin{array}{r} \psi(\mathfrak{D}^s(p_{m(j)}, fp_{m(j)})), \psi(\mathfrak{D}^s(p_{n(j)}, gp_{n(j)})), \psi(\mathfrak{D}^s(p_{m(j)}, gp_{n(j)})), \\ \psi(\mathfrak{D}^s(p_{n(j)}, fp_{m(j)})) \end{array} \right\} \\ &= min \left\{ \begin{array}{r} \psi(\mathfrak{D}^s(p_{m(j)}, p_{m(j)+1})), \psi(\mathfrak{D}^s(p_{n(j)}, p_{n(j)+1})), \psi(\mathfrak{D}^s(p_{m(j)}, p_{n(j)+1})), \\ \psi(\mathfrak{D}^s(p_{n(j)}, p_{m(j)+1})) \end{array} \right\} \end{aligned}$$

Now,

$$\begin{aligned} \mathfrak{D}^s(p_{n(j)}, p_{n(j)+1}) &\leq |2\mathfrak{D}(p_{n(j)}, p_{n(j)+1}) - \mathfrak{D}(p_{n(j)}, p_{n(j)}) - \mathfrak{D}(p_{n(j)+1}, p_{n(j)+1})| \\ &\leq |2\mathfrak{D}(p_{n(j)}, p_{n(j)+1}) - (\mathfrak{D}(p_{n(j)}, p_{n(j)}) + \mathfrak{D}(p_{n(j)+1}, p_{n(j)+1}))| \\ &\leq |\mathfrak{D}(p_{n(j)}, p_{n(j)}) + \mathfrak{D}(p_{n(j)+1}, p_{n(j)+1}) - 2\mathfrak{D}(p_{n(j)}, p_{n(j)+1})| \end{aligned}$$

By $D3$

$$\begin{aligned} \mathfrak{D}(p_{n(j)}, p_{n(j)}) &\leq 2K\mathfrak{D}(p_{n(j)}, p_{n(j)+1}) \\ \mathfrak{D}(p_{n(j)+1}, p_{n(j)+1}) &\leq 2K\mathfrak{D}(p_{n(j)}, p_{n(j)+1}) \end{aligned}$$

$$\mathfrak{D}(p_{n(j)}, p_{n(j)}) + \mathfrak{D}(p_{n(j)+1}, p_{n(j)+1}) \leq 4K\mathfrak{D}(p_{n(j)}, p_{n(j)+1})$$

$$\begin{aligned} \mathfrak{D}^s(p_{n(j)}, p_{n(j)+1}) &\leq |(4K-2)(p_{n(j)}, p_{n(j)+1})| \\ \limsup_{j\to\infty} \mathfrak{D}^s(p_{n(j)}, p_{n(j)+1}) &\leq |(4K-2)\limsup_{j\to\infty}(p_{n(j)}, p_{n(j)+1})| \end{aligned}$$

clearly, $N_s(p_n(j), p_m(j)) = 0$

Hence, By taking the upper limit as $j \to \infty$, we have

$$K^4 \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq \frac{\limsup_{j\to\infty} N(p_{m(j)}, p_{n(j)}) + \limsup_{j\to\infty} N_s(p_{m(j)}, p_{n(j)})}{2}$$

$$K^4 \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq \frac{K\epsilon}{2} \leq K\epsilon$$

Which implies that $\limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) < \frac{\epsilon}{K^3} < \frac{\epsilon}{K^2}$

a contradiction to (4) property proving above. Hence $\{p_n\}$ is a b-cauchy sequence.

Step 3: In this step, We will show that f and g have a common fixed point.

Since $\{p_n\}$ is a b-cauchy sequence in the complete b-metric-like space X, there exists $z \in X$ such that

$$\lim_{n\to\infty} \mathfrak{D}(p_{2n}, z) = \lim_{n\to\infty} \mathfrak{D}(p_{2n+1}, z) = \lim_{n\to\infty} \mathfrak{D}(fp_{2n}, z) = 0 \tag{4.3.12}$$

By the triangle inequality, we have

$$\mathfrak{D}(fz, z) \leq K[\mathfrak{D}(fz, fp_{2n}) + \mathfrak{D}(fp_{2n}, z)] = K[\mathfrak{D}(fz, fp_{2n}) + \mathfrak{D}(p_{2n+1}, z)] \tag{4.3.13}$$

Suppose that f is continuous. Letting $n \to \infty$ in (4.3.13) and applying (4.3.12) we have

$$\mathfrak{D}(fz, z) \leq K[\lim_{n\to\infty} \mathfrak{D}(fz, fp_{2n}) + \lim_{n\to\infty} \mathfrak{D}(fp_{2n}, z)] = 0$$

which implies that $fz = z$.

Let $\mathfrak{D}(z, gz) > 0$. As z and z are comparable by (4.3.1) we have

$$K^4\mathfrak{D}(z, gz) = K^4\mathfrak{D}(fz, gz) \leq \frac{N(z,z) + N_s(z,z)}{2} \tag{4.3.14}$$

where

$$N(z,z) = max\{\psi(\mathfrak{D}(z,z)), \psi(\mathfrak{D}(z,fz)), \psi(\mathfrak{D}(z,gz)), \psi(\mathfrak{D}(z,gz)), \psi(\mathfrak{D}(z,fz))\}$$

By the triangle inequality, we have

$$\mathfrak{D}(z,z) \le 2K\mathfrak{D}(z,gz)$$

$$N(z,z) \le max\{\psi(2K\mathfrak{D}(z,gz)), \psi(2K\mathfrak{D}(z,gz)), \psi(\mathfrak{D}(z,gz)), \psi((\mathfrak{D}(z,gz))), \psi(2K\mathfrak{D}(z,gz))\} < \mathfrak{D}(z,gz)$$

Similarly,

$$N_s(z,z) = min\{\psi(\mathfrak{D}^s(z,fz)), \psi(\mathfrak{D}^s(z,gz)), \psi(\mathfrak{D}^s(z,gz)), \psi(\mathfrak{D}^s(z,fz))\}$$

$$\mathfrak{D}^s(z,z) = |2\mathfrak{D}(z,z) - \mathfrak{D}(z,z) - \mathfrak{D}(z,z)| = 0$$

Clearly, $N_s(z,z) = 0$

Hence, (4.3.14) gives $K^4\mathfrak{D}(z,gz) < \frac{\mathfrak{D}(z,gz)}{2}$, which is contradiction. Thus, $\mathfrak{D}(z,gz) = 0$.

Similarly, if g is continuous, the desired result is obtained. □

Theorem 4.3.2. *Let $(X, \preccurlyeq, \mathfrak{D})$ be a complete partially ordered b-metric-like space with $K > 1$. Let $f, g : X \to X$ be two mappings such that f is g-weakly isotone increasing. suppose that for every two comparable elements $p, q \in X$, we have*

$$K^4\mathfrak{D}(fp, gq) \le \frac{N(p,q) + N_s(p,q)}{2} \tag{4.3.15}$$

Where

$$N(p,q) = max\{\psi(\mathfrak{D}(p,q)), \psi(\mathfrak{D}(p,fp)), \psi(\mathfrak{D}(q,gq)), \psi(\mathfrak{D}(p,gq)), \psi(\mathfrak{D}(q,fp))\} \tag{4.3.16}$$

and

$$N_s(p,q) = min\{\psi(\mathfrak{D}^s(p,fp)), \psi(\mathfrak{D}^s(q,gq)), \psi(\mathfrak{D}^s(p,gq)), \psi(\mathfrak{D}^s(q,fp))\} \tag{4.3.17}$$

Then, the pair (f, g) has a common fixed point z in X if X is regular. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. Following the proof of theorem (4.3.1), there exists $z \in X$ such that

$$\lim_{n\to\infty} \mathfrak{D}(p_n, z) = 0$$

Now we prove that z is a common fixed point of f and g. Since $p_{2n+1} \to z$ as $n \to \infty$ from regularity of X, $p_{2n+1} \leqslant z$. Therefore, from (3.3.3), we have

$$K^4\mathfrak{D}(fz, gp_{2n+1}) \leq \frac{N(z, p_{2n+1}) + N_s(z, p_{2n+1})}{2} \tag{4.3.18}$$

where

$$N(z, p_{2n+1}) = max\{\psi(\mathfrak{D}(z, p_{2n+1})), \psi(\mathfrak{D}(z, fz)), \psi(\mathfrak{D}(p_{2n+1}, gp_{2n+1})), \psi(\mathfrak{D}(z, gp_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, fz))\}$$

Taking the limit as $n \to \infty$ in (4.3.18) and using lemma (4.2.1), we obtain that

$$\begin{aligned} K^3\mathfrak{D}(fz, z) &= K^4\frac{1}{K}\mathfrak{D}(fz, z) \leq K^4 \limsup_{n\to\infty} \mathfrak{D}(fz, gp_{2n+1}) \leq \frac{\limsup_{n\to\infty} N(z, p_{2n+1}) + \limsup_{n\to\infty} N_s(z, p_{2n+1})}{2} \\ &= \frac{max\left\{\begin{array}{c} \limsup_{n\to\infty}\mathfrak{D}(z, p_{2n+1}), \limsup_{n\to\infty}\mathfrak{D}(z, fz), \limsup_{n\to\infty}\mathfrak{D}(p_{2n+1}, gp_{2n+1}), \\ \limsup_{n\to\infty}\mathfrak{D}(z, gp_{2n+1}), \limsup_{n\to\infty}\mathfrak{D}(p_{2n+1}, fz) \end{array}\right\} + \limsup_{n\to\infty} N_s(z, p_{2n+1})}{4K} \\ &\leq \frac{1}{4K}(max\{\mathfrak{D}(z, z), \mathfrak{D}(z, fz), \mathfrak{D}(z, z), \mathfrak{D}(z, z), \mathfrak{D}(z, fz)\} + \limsup_{n\to\infty} N_s(z, p_{2n+1})) \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} \mathfrak{D}(z, z) &\leq 2K\mathfrak{D}(z, fz) \\ \mathfrak{D}(z, fz) &\leq K\mathfrak{D}(z, fz) \end{aligned}$$

$$N(z, z) \leq max\{\psi(2K\mathfrak{D}(z, fz)), \psi(K\mathfrak{D}(z, fz)), \psi(2K\mathfrak{D}(z, fz)), \psi((\mathfrak{D}(z, fz)))\} < 2K\mathfrak{D}(z, fz)$$

Similarly,

$$N_s(z, z) = min\{\psi(\mathfrak{D}^s(z, fz)), \psi(\mathfrak{D}^s(z, gz)), \psi(\mathfrak{D}^s(z, gz)), \psi(\mathfrak{D}^s(z, fz))\}$$

$$\mathfrak{D}^s(z, z) = |2\mathfrak{D}(z, z) - \mathfrak{D}(z, z) - \mathfrak{D}(z, z)| = 0$$

Clearly, $N_s(z, z) = 0$

Hence by the (4.3.18), we have

$$\begin{aligned} K^4\mathfrak{D}(z, fz) &< \frac{2K\mathfrak{D}(z, fz) + 0}{4K} \\ K^4\mathfrak{D}(z, fz) - \frac{\mathfrak{D}(z, fz) + 0}{2} &< 0 \\ (K^4 - \frac{1}{2})\mathfrak{D}(z, fz) &< 0 \end{aligned}$$

a contradiction. this implies that $fz = z$.

Similarly, it can be shown that z is a fixed point of g. $\square$

Corollary 4.3.3. *Let $(X, \preccurlyeq, d)$ be a complete partially ordered b-metric space. Let $f : X \to X$ be a mapping such that f is weakly isotone increasing. Suppose that for every two comparable elements $x, y \in X$ and a constant $s > 1$, we have*

$$s^4 d(fx, gy) \leq M_s(x, y)$$

Where

$$M_s(x, y) = max\left\{\psi(d(x, y)), \psi(d(x, fx)), \psi(d(y, fy)), \psi\left(\frac{d(x, fy) + d(y, fx)}{2K}\right)\right\}$$

Then f has a fixed point z in X if either:

(a) f is continuous, or

(b) X is regular.

Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

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