Equilibrium Structures of the Rotating Stars and Stars in Binary Systems

A thesis in the partial fulfillment of the requirements for the award of the degree

of

Master of Science

in

Mathematics and Computing

Submitted by

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July-2017

CERTIFICATE

This is to certify that the thesis entitled **"Equilibrium Structures of the rotating stars and stars in binary systems"** submitted by **Ms. Aashmeen Kaur Chauhan** in the partial fulfillment of the requirements for the degree of **Master of Science** in **Mathematics and Computing** from **Thapar University, Patiala** is a bonafide piece of work carried out under the guidance and supervision of Dr. Ankush Pathania, Assistant Professor, School of Mathematics, Thapar University, Patiala and no part of this project has been submitted for award of any degree in this or any other university.

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DECLARATION

I hereby certify that the work which is being presented in the dissertation entitled, "**Equilibrium Structures of the rotating stars and stars in binary systems**" in the partial fulfilment of the requirement for the award of the Degree of **Masters of Mathematics and Computing**, submitted to the **School of Mathematics** of **Thapar University, Patiala**, is an authentic record of my own work carried under the supervision of **Dr. Ankush Pathania**. It refers others researcher's work which are duly listed in the reference section. The matter contained in this dissertation has not been submitted, neither in part nor in full to any other degree to any other university or institute except reported in text and references.

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Dedicated to

my parents

and

younger brother

ACKNOWLEDGEMENT

First and foremost I take the privilege to offer my deepest sense of gratitude to Dr. Ankush Pathania, Assistant Professor, Thapar University, Patiala, for his commendable support and constant motivation throughout this report. With deep humility, I thank him for all the insightful conversations and his valuable comments. His guidance has helped me to improve my knowledge and perspective towards the work. I will always be indebted.

I am thankful to Dr. A. K. Lal, Associate Professor and HOD, Thapar University, Patiala, for constantly encouraging and helping me to put my best in my research project and Mr. Tarun Sachdeva, for his motivational approach.

My sincerest thanks to all the faculty members and staff of School of Mathematics at Thapar University, Patiala, who have bestowed their guidance at appropriate time without which it would have been very difficult to proceed with my work. I further express heartfelt gratitude to my parents and friends who have constantly helped me to keep my morale high through my work.

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Abstract

In this dissertation, we have studied the equilibrium structures of rotating stars and stars in binary systems (rotationally and/or tidally distorted stars) that are distorted by both rotational and tidal forces.

While studying the equilibrium structures of primary component of synchronous binaries, Kopal (1972) obtained a series expression for computing the radius of the Roche equipotential surfaces by retaining terms up to r_0^{10} in r_0. This series expression was then used in his work to find the series expressions for computing volume of Roche Equipotential Surfaces. Mohan and Saxena (1983) extended this analysis to study rotating stars, synchronous and non synchronous binaries and used the series expressions for radius, volume, surface area and other parameters by retaining terms up to r_0^{10} in r_0.

Now as these series solutions of radius, volume and surface area of the Roche equipotential surfaces are the only approximate analytical solutions available in literature that is why they are of great significance. However, the numerical results obtained from these series expressions are not very accurate.

Keeping these factors in view, in the present work we have used the series expressions of the radius of Roche equipotential surfaces as given by Pathania and Medupe (2012) in which terms up to r_0^{14} in r_0 has been retained and has used it subsequently to obtain the series expressions for volume, surface area and various other parameters by retaining terms up to r_0^{14} in r_0. The objective of the present work is to check the variations in the numerical results on expanding the series expressions further.

Thesis consists of 3 chapters. Chapter I is introductory in nature. In this chapter, brief literature survey, Roche equipotential and methodology of Mohan and Saxena (1983) to study the equilibrium structures of rotationally and/or tidally distorted (from here after we will call rotationally and/or for tidally distorted stars as RTD stars) stars has been discussed.

In chapter II, we have used the extended series expression of radius of the Roche Equipotential surfaces as given by Pathania and Medupe (2012) to obtain extended series expressions for volume, surface area and other parameters. The methodology of Mohan and Saxena (1983) along with the results of Kopal (1972) has been used to obtain these series expressions.

In chapter III we have used these extended series expressions to find the equilibrium structures of RTD polytropic models of stars. The numerical results so obtained have been compared with the results of Mohan and Saxena (1983). Finally conclusions based on the present study has also been drawn in this chapter.

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Chapter – 1

Introduction

1.1 Overview

Structure and oscillations of the stars constitute one of the major problems of astrophysics. The analytical studies of the problems of equilibrium structures of rotating stars and stars in the binary systems has practical importance in astrophysics as they help in better understanding the nature of the structures of the stars, as well as their certain other observed phenomenon. Such studies are also expected to help in better understanding the problems of stellar stability and stellar variability.

This chapter is introductory in nature. In section 1.1, literature review is explored briefly. In Section 1.2 an approach of Kippenhahn and Thomas (1970) is discussed in detail. Approach of Mohan & Saxena (1983) to determine the equilibrium structures of rotationally and tidally distorted stars has been discussed in section 1.3.

1.2 Literature Review

Research over the past decades suggests that more than 80% of all stars in the universe are a part of multiple star systems and the most common multiple star systems are the binary star systems (Observations shows that more than 50% of all stellar systems are the binary systems). Binary system consists of two stars in which one rotating star revolves round the other rotating star or both revolve round a common centre. The brighter star is primary and dimmer is called secondary.

A star is usually assumed to be undistorted spherical gaseous sphere in most of the theoretical studies related to equilibrium structure of stars. However, practically it is not true because if a star which is assumed to be a gaseous sphere is rotating then its equilibrium structure will be distorted by rotational forces. Further, if that star is a member of binary system then equilibrium structure of that star will be affected by both rotational & tidal forces. So mathematical models of such stars will be rotationally & tidally distorted gaseous spheres.

Theoretical studies of the problems related to the equilibrium structure and stability of rotating and self gravitating objects begin with the work of Newton. He used the law of gravitation to explain figures of celestial bodies, Later on authors like Clairaut, Jacobi, Laplace, Legendre, Maclaurin & Poincare contributed ideas for the development of theory of rotating bodies.

Edward Arthur Milne (1923) developed a methodology to construct first detailed model of slowly rotating stars in pure radiative equilibrium. Later on, this method was further generalised to distorted Polytropes by Chandrasekhar (1933, 1939). Most of the work in literature on the problems of equilibrium structures of rotating stars can be found in Clement (1970), Kippenhahn & Weigert (1990), Strimatter (1989) and Tassoul (1978).

The study of equilibrium structures of rotating stars in binary systems that are distorted by both rotational and tidal forces (or RTD stars) is quite complex. However, authors like Chen & Chau (1979), Deupree & Karakas (2005), James (1984), Landin et al (2009), Kopal, (1978), Linnell (1981), Naylor and Anand (1972), Singh &Singh (1983), Song et al (2009), Tassaul & Tassaul (1982), Mohan et al (1990), La et al (2006) and Pathania and Medupe (2014) have also studied the various problems of RTD stars.

The concept of Roche equipotentials has been frequently used in literature to study the equilibrium structures of rotating stars and stars in binary systems. In the theory of Roche equipotential exact analytical solution for determining the radius of Roche equipotential surfaces distorted by both rotational and tidal forces is not available. However, Kopal (1972) has given a approximate series solution for the radius of Roche equipotential surfaces that has been used by various authors like Mohan & Singh (1982), Mohan & Saxena (1983), Mohan et al (1990), Lal et al (2006) and Pathania et al (2012) for studying various problems of rotating stars and stars in binary systems.

In the series expression for the radius of Roche equipotential surfaces, Kopal (1972) has retained terms up to r_0^{10} in r_0. As such numerical results using these series expressions are not very accurate. So in the present study we will consider series expression of radius of Roche equipotential surfaces as given by Pathania & Medupe (2012) in which terms up to r_0^{14} in r_0 has been retained. Using this series expression we will compute equilibrium structures of rotating stars & stars in binary system and will check whether there is any effect on the results on considering more terms in the series expressions of radius of Roche equipotential surfaces & various other parameters.

1.3 Kippenhahn and Thomas approach

Corresponding to actual equipotential surfaces distorted by rotational and tidal forces, Kippennhahn and Thomas (1970) developed an approach of topologically equivalent spherical surfaces for analysing the effects of rotational and tidal distortions on equilibrium structures of gaseous spheres. Quantities like $\bar{f}, \bar{g}$ are defined by them on equivalently spherical surfaces which indicate some averages of quantities f, g on actual equipotential surfaces. If ψ denote total potential of a RTD stellar model at an arbitrary point P(x, y, z) then $\psi(x, y, z)$=constant represents equipotential surfaces. Let Volume enclosed by equipotential surfaces ($\psi = \text{constant}$) be denoted by V_ψ and Surface area be denoted by S_ψ. For any function f(x, y, z) mean value ($\bar{f}$) of this function over equipotential surface ψ = constant is defined as

$$\bar{f} = \frac{1}{s_\psi} \int_{\psi=\text{Constant}} f d\sigma \tag{1.0}$$

where $d\sigma$ is surface element of equipotential surfaces (ψ=constant). Kippenhahn and Thomas also defined variable r_ψ by relation

$$V_\psi = \frac{4}{3}\pi r_\psi^3 \tag{1.1a}$$

$$S_\psi = \int_{\psi=\text{Constant}} d\sigma \tag{1.1b}$$

Here S_ψ is not taken equal to $4\pi r_\psi^2$ in general. A function $g(x,y,z)$ is described by Kippenhahn and Thomas by the relation

$$g = \frac{d\psi}{dn} \tag{1.1c}$$

where g is sphere's force of gravity. The distance between surfaces in neighbourhood $\psi = \text{constant}$ and $\psi+d\psi$ = constant is represented by dn and this distance can be different for different points of surfaces. The mean values like $\bar{g}$ & $\overline{g^{-1}}$ are defined as

$$\bar{g} = \frac{1}{S_\psi} \int_{\psi=\text{Constant}} \frac{d\psi}{dn} d\sigma \tag{1.2a}$$

$$\overline{g^{-1}} = \frac{1}{S_\psi} \int_{\psi=\text{Constant}} \left(\frac{d\psi}{dn}\right)^{-1} d\sigma \tag{1.2b}$$

These functions represent the value of $\bar{g}$ and $\overline{g^{-1}}$ which are functions of ψ on topologically equivalent spherical surfaces. The volume dV_ψ between these neighbouring surfaces $\psi = \text{constant}$ and $\psi + d\psi = \text{constant}$ is given by

$$dV_\psi = \int\limits_{\psi=\text{Constant}} dnd\sigma = \int\limits_{\psi=\text{Constant}} (\frac{d\psi}{dn})^{-1} dn = S_\psi \overline{g^{-1}} d\psi \tag{1.3}$$

Now the various non dimensional parameters as defined by Kippenhahn and Thomas (1970) are

$$u = \frac{S_\psi}{4\pi r_\psi^3} \tag{1.4}$$

$$v = \frac{\bar{g} r_\psi^2}{GM_\psi} \tag{1.5}$$

$$w = \frac{\overline{g^{-1}} GM_\psi}{r_\psi^2} \tag{1.6}$$

where M_ψ is the mass enclosed by equipotential surface $\psi = \text{constant}$ Equipotential surfaces ($\psi = \text{Constant}$) are thus considered topologically equivalent to a sphere whose radius is defined by r_ψ and various functions are defined by previous relations.

Equations (1.1) to (1.6) are mathematical expressions implemented by Kippenhahn and Thomas (1970) for describing gravitational fields on the RTD gaseous spheres. Since Equipotential Surfaces are also equipressure and equidensity surfaces in the case of hydrostatic equilibrium, therefore P_ψ and density ρ_ψ is also constant on equipotential surface $\psi = \text{constant}$. By using these concepts, Kippenhahn and Thomas (1970) derived equations of equilibrium structures of RTD stellar models in following way.

By using eq (1.1a) mass between the surfaces in equipotential $\psi = \text{constant}$ and $\psi + d\psi = \text{constant}$ is given by

$$dM_\psi = dV_\psi \rho_\psi = 4\pi r_\psi^2 \rho_\psi dr_\psi \tag{1.7}$$

Therefore, we get-

$$\frac{dM_\psi}{dr_\psi} = 4\pi r_\psi^2 \rho_\psi \tag{1.8}$$

Now by the use of equation (1.3), (1.7) we have

$$d\psi = \frac{d\psi}{dV_\psi} dV_\psi = (\frac{dV_\psi}{d\psi})^{-1} \frac{dM_\psi}{\rho_\psi} = \frac{dM_\psi}{S_\psi \overline{g^{-1}} \rho_\psi} \tag{1.9}$$

And using (1.3), (1.4), (1.5), we get

$$d\psi = \frac{GM_\psi dM_\psi}{4\pi r_\psi^2 \rho_\psi uw} \tag{1.10}$$

In hydrostatic equilibrium $\frac{dP_\psi}{d\psi} = -\rho_\psi$, so using equations (1.4), (1.5), & (1.6) we get

$$\frac{dP_\psi}{dM_\psi} = -\frac{GM_\psi}{4\pi r_\psi^4} f_p \tag{1.11}$$

where

$$f_p = \frac{1}{uw} = \frac{4\pi r_\psi^4}{GM_\psi} \frac{1}{S_\psi \overline{g^{-1}}} \tag{1.12}$$

Here f_p is a function of ψ . Equipotential surfaces can now be determined from the value of $\psi = \text{constant}$ and consequently all other values such as $S_\psi, r_\psi, \bar{g} \,\&\, \overline{g^{-1}}$ for equipotential surfaces can be derived easily from their geometry. Mass M_ψ of equipotential surfaces can be determined by the integration of equation (1.8).

Let 'ε' represents nuclear energy generation rate which depends upon the Density (ρ_ψ) and Temperature (T_ψ) of equilibrium gaseous spheres. So Energy (L_ψ) that passes per second through equipotential surfaces $\psi = \text{constant}$ is given by

$$\frac{dL_\psi}{dM_\psi} = \varepsilon \tag{1.13}$$

Using equation (1.8), (1.13) can be written as

$$\frac{dL_\psi}{dM_\psi} = 4\pi r_\psi^2 \rho_\psi \varepsilon \tag{1.14}$$

When energy is transported by radiation then the equation of energy transport becomes

$$F_\psi = -\frac{4acT_\psi^3}{3\kappa}\frac{d\psi}{dn}\frac{dT_\psi}{dM_\psi}\frac{4\pi r_\psi^4 uw}{GM_\psi} \quad (1.15)$$

where F_ψ is the Radiative flux on the Equipotential surface ($\psi = \text{Constant}$). On integrating the F_ψ over equipotential surface $\psi = \text{Constant}$ we get

$$L_\psi = \int_{\psi=\text{constant}} F_\psi d\sigma$$

$$= -\frac{4acT_\psi^3}{3\kappa}\frac{dT_\psi^3}{dM_\psi} uw \frac{4\pi r_\psi^4}{GM_\psi}\int_{\psi=\text{Constant}} (\frac{d\psi}{dn})\, d\,\sigma$$

$$= -\frac{64\pi^2 acT_\psi^3 r_\psi^4}{3\kappa} u^2 vw \frac{dT_\psi}{dM_\psi} \quad (1.16)$$

Equation (1.16) can also be written as

$$\frac{dT_\psi}{dM_\psi} = -\frac{3\kappa L_\psi}{64\pi^2 acT_\psi^3 r_\psi^4} f_T \quad (1.17)$$

Using equation in (1.17), we get

$$\frac{dT_\psi}{dr_\psi} = -\frac{3\kappa\rho_\psi L_\psi}{16\pi acT_\psi^3 r_\psi^2} f_T \quad (1.18)$$

Here f_T is

$$f_T = \frac{1}{u^2 vw}$$

So for rotationally and tidally distorted gaseous spheres, the four basic equations (1.8), (1.11), (1.13) & (1.17) can be collectively written as

$$\frac{dM_\psi}{dr_\psi} = 4\pi r_\psi^2 \rho_\psi \quad (1.19a)$$

$$\frac{dP_\psi}{dM_\psi} = -\frac{GM_\psi}{4\pi r_\psi^4} f_p \quad (1.19b)$$

$$\frac{dL_\psi}{dM_\psi} = \varepsilon \tag{1.19c}$$

&

$$\frac{dT_\psi}{dM_\psi} = -\frac{3\kappa L_\psi}{64\pi^2 ac T_\psi^3 r_\psi^4} f_T \tag{1.19d}$$

where

$$f_P = \frac{1}{uw} \;\&\; f_T = \frac{1}{u^2 vw}$$

These equations reduce to normal equations, that determine the equilibrium structures of undistorted gaseous spheres when values of u, v & w are each taken to be equal to 1. Now the boundary conditions which the above equations has to satisfy are

At the centre $r_\psi = 0$

$$M_\psi = 0\ ,\ L_\psi = 0 \tag{1.19e}$$

At the free surface $r_\psi = R_\psi$

$$M_\psi = M_0, L_\psi = L_{\psi S}$$

$$P_\psi = 0, \rho_\psi = 0, T_\psi = 0 \text{ or } P_\psi = P_{\psi S}, T_\psi = T_{\psi S}, \rho_\psi = \rho_{\psi s} \tag{1.19f}$$

Here M_0 is total mass of the gaseous structure on the outermost equipotential surfaces and $L_{\psi S}, P_{\psi S,} T_{\psi S}, \rho_{\psi S}$ represents the values of $L_\psi, P_{\psi,} T_\psi, \rho_\psi$ respectively on the outermost surface.

1.4 Roche equipotential Surfaces

In Closed Binary systems Equipotential surfaces are usually explained with the help of a model named as ‘Roche Model’. It is a model in which the total mass of the star is assumed to be contained in the centre which is enclosed by an envelope where density changes inversely by the square of the distance from its centre.

In a Close binary system let us suppose m_1 and m_2 be masses of two components parted by Distance R. Let us assume the position of two bodies in a binary system is in cartesian coordinates rectangular system. Origin is considered at the centre of Primary Component with mass m_1.The mass of primary is assumed much larger than that of the secondary component $(m_1 \geq m_2)$ which is considered as point mass. The line passing through the centre of mass of two stars is taken as x axis and perpendicular to the plane is taken as z axis.

Let $r_2^2 = (R-x)^2 + y^2 + z^2$, $r_1^2 = x^2 + y^2 + z^2$ represent the distance to any point $P(x, y, z)$ from the centre of masses of bodies m_1 and m_2 having centres O & O_1 respectively. The centre of mass of the system is $C\left(\frac{m_2 R}{m_1 + m_2}, 0, 0\right)$.

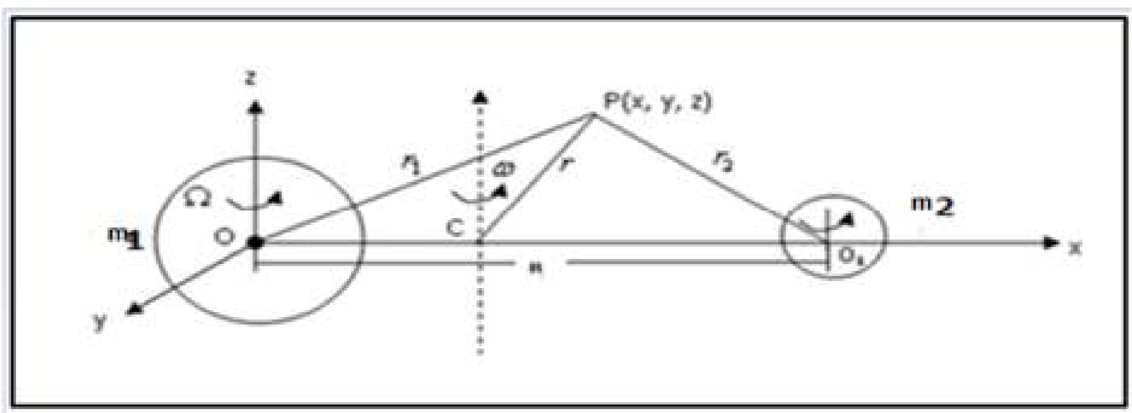

Fig I - Binary Components

Let angular velocity of the rotation of these components about an axis perpendicular to xy plane is Ω and angular velocity of revolution of systems is ω. So for such a system as described in fig I the total Potential at a point P(x,y,z) due to all forces is given by

$$\psi = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2} + \frac{\omega^2}{2}\left\{\left(x - \frac{m_2 R}{m_1 + m_2}\right)^2 + y^2\right\} \tag{1.20}$$

Here $\frac{Gm_1}{r_1}$ is potential arising because of the mass m_1 of the primary component, $\frac{Gm_2}{r_2}$ is potential arising from mass m_2 and third term in (1.20) represents the potential due to the centrifugal force. We have assumed that angular velocity of rotation (Ω) and revolution (ω) are same while deriving (1.20). Also if angular velocity of rotation is assumed to be equal to keplerian angular velocity that is

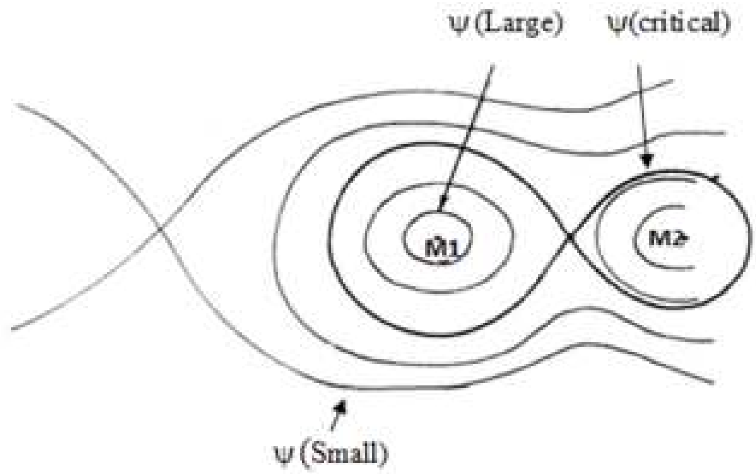

Fig II - 2D-Roche Model at equipotential surfaces

$$\Omega^2 = \Omega_k^2 = G\frac{m_1 + m_2}{R^3} \qquad (1.21)$$

then we get a relation of the type $n = \frac{q+1}{2}$ where $q = \frac{m_1}{m_2}$ is tidal parameter that represents distortions due to tidal forces & n is rotational parameter that represents distortions due to rotational forces. Now in non dimensional form, equation (1.20) can be written as

$$\psi^* = \frac{1}{r_1} + q\left\{\frac{1}{\sqrt{1-2\lambda r_1 + r_1^2}} - \lambda r\right\} + nr_1^2(1-v^2) \qquad (1.22)$$

where $\psi^* = \frac{\psi R}{Gm_1} - \frac{m_2^2}{2m_1(m_1+m_2)}$. Also $\lambda = r\sin\theta\cos\phi, \mu = r\sin\phi\cos\theta, \gamma = r\cos\theta$ where r, θ, ϕ represents spherical polar co-ordinates at point $P(x, y, z)$. If we put q=0 in (1.22) then it represents total potential of a rotating star. Now if we put $\psi^* = \text{constant}$ in (1.22) then it represents equipotential surfaces also called Roche equipotential surfaces. Roche equipotential surfaces will be in the form of two separate spherical ovals around each centre of masses for large values of ψ. With the diminishing values of ψ, these spherical ovals become elongated along the line joining the centres of the gravity of system. Then for certain value of ψ these oval surfaces get united at a point on x axis to form a dumbbell like structure. This critical value of ψ is called Roche Limit.

Kopal (1972) defined a non dimensional variables r_0 as

$$r_0 = \frac{1}{\psi^* - q} \tag{1.23}$$

Kopal has also given a series expression for determining radius of Roche equipotential surfaces as

$$r = r_0\left[1 + C_3 r_0^3 + C_4 r_0^4 + C_5 r_0^5 + C_6 r_0^6 + C_7 r_0^7 + C_8 r_0^8 + C_9 r_0^9\right] \tag{1.24}$$

where

$$\begin{aligned} C_3 &= qP_2 + n(1-\nu^2) \\ C_4 &= qP_3 \\ C_5 &= qP_4 \\ C_6 &= qP_5 + 3C_3^2 \\ C_7 &= qP_6 + 7qC_3P_3 \\ C_8 &= qP_7 + 8qC_3P_4 + 4q^2P_3^2 \\ C_9 &= qP_8 + 9qC_3P_5 + 9q^2P_3P_4 \end{aligned} \tag{1.25}$$

where $P_j = P_j(\lambda)$ in equation (1.24) are Legendre polynomials. In this series expression terms up to r_0^{10} in r_0 and up to second order of smallness in n and q has been retained.

Volume enclosed by the equipotential surfaces can be computed by using relation

$$V_\psi = \frac{2}{3}\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \frac{r^3}{\mu}\, d\lambda\, d\nu \tag{1.26}$$

Kopal derived the series expression for volume in terms of r_0, n & q as

$$V_\psi = \frac{4}{3}\pi D^3 r_0^3[1 + 2nr_0^3 + (\frac{12}{5}q^2 + \frac{8}{5}nq + \frac{32}{5}n^2)r_0^6 + \frac{15}{7}q^2 r_0^8 + 2q^2 r_0^{10}] \tag{1.27}$$

Using the same approach as adopted by Kopal (1972), Mohan and Singh (1982) obtained the explicit expression for surface area S_ψ and several other parameters $r_\psi, \bar{g}, \overline{g^{-1}}$ for equipotential surfaces as

$$S_\psi = 2\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \frac{r^2}{\mu}\, d\lambda d\nu \tag{1.28a}$$

$$S_\psi = 4\pi D^2 r_0^2[1+\frac{4n}{3}r_0^3+(\frac{7}{5}q^2+\frac{14}{5}nq+\frac{56}{15}n^2)r_0^6+\frac{9}{7}q^2 r_0^8+\frac{11}{9}q^2 r_0^{10}] \quad (1.28b)$$

$$r_\psi = (\frac{3V_\psi}{4\pi})^{\frac{1}{3}} \quad (1.29a)$$

$$r_\psi = Dr_0[1+\frac{2n}{3}r_0^3+(\frac{4}{5}q^2+\frac{8}{15}nq+\frac{76}{45}n^2)r_0^6+\frac{5}{7}q^2 r_0^8+\frac{2}{3}q^2 r_0^{10}] \quad (1.29b)$$

$$\bar{g} = \frac{2}{S_\psi}\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}}(\frac{d\psi}{dn})\frac{r^2}{\mu}d\lambda d\nu \quad (1.30)$$

$$\bar{g} = \frac{GM_\psi}{D^2 r_0^2}[1-\frac{8n}{3}r_0^3-(3q^2+2nq+\frac{40}{9}n^2)r_0^6-\frac{51}{14}q^2 r_0^8-\frac{13}{3}q^2 r_0^{10}]$$

$$\overline{g^{-1}} = \frac{2}{S_\psi}\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}}(\frac{d\psi}{dn})^{-1}\frac{r^2}{\mu}d\lambda d\nu$$

$$(1.31)$$

$$\overline{g^{-1}} = \frac{D^2 r_0^2}{GM_\psi}[1+\frac{8n}{3}r_0^3+(\frac{31}{5}q^2+\frac{62}{15}nq+\frac{584}{45}n^2)r_0^6+\frac{101}{14}q^2 r_0^8+\frac{25}{3}q^2 r_0^{10}+]$$

By inverting the equation (1.29b), we get

$$r_0 = r_\psi^*[1-\frac{2n}{3}r_\psi^{*3}-(\frac{4}{5}q^2+\frac{8}{15}nq+\frac{76}{45}n^2)r_\psi^{*6}-\frac{5}{7}q^2 r_\psi^{*8}-\frac{2}{3}q^2 r_\psi^{*10}] \quad (1.32)$$

Here $r_\psi^* = \frac{r_\psi}{D}$ which is in nondimensional form.

1.5 Mohan & Saxena approach to determine the equilibrium structures of rotationally and tidally distorted stars.

For calculating the internal structures of RTD gaseous spheres , stellar structure equations (1.19a-1.19d) has to be integrated numerically subject to the boundary conditions (1.19e & 1.19f). Since evaluation of actual RTD equipotential surface is quite complicated, so Kippenhahn and Thomas (1970), proposed that for evaluating distortion parameters $u, v, w. f_P, f_T$, the actual equipotential surfaces can be replaced by Roche equipotential surfaces and this approximations is reasonably valid on actual structures of stars. (Chandrasekhar 1933)

Once RTD star is approximated by Roche equipotentials, the results of Kopal (1972) and Mohan and Saxena (1983) can be used for evaluating series expressions for distortion parameters u, v, w, f_P, f_T on equipotential surfaces described in stellar structure equations (1.19a) & (1.19d). Using equations (1.4 - 1.6), (1.19a –1.19d), & (1.27-1.32) the series expressions of distortion parameters $u, v, w. f_P, f_T$ as derived by Mohan & Saxena (1983) on equipotential surfaces are given by

$$u = 1 - (\frac{1}{5}q^2 + \frac{2}{15}nq + \frac{4}{45}n^2) r_\psi^{*6} - \frac{1}{7}q^2 r_\psi^{*8} - \frac{1}{9}q^2 r_\psi^{*10} \quad (1.33a)$$

$$v = 1 - \frac{4n}{3} r_\psi^{*3} - (\frac{7}{5}q^2 + \frac{14}{15}nq + \frac{68}{45}n^2) r_\psi^{*6} - \frac{31}{14}q^2 r_\psi^{*8} - 3q^2 r_\psi^{*10} \quad (1.33b)$$

$$w = 1 + \frac{4n}{3} r_\psi^{*3} + (\frac{23}{5}q^2 + \frac{16}{15}nq + \frac{212}{45}n^2) r_\psi^{*6} + \frac{81}{14}q^2 r_\psi^{*8} + 7q^2 r_\psi^{*10} \quad (1.33c)$$

$$f_P = 1 - \frac{4n}{3} r_\psi^{*3} - (\frac{22}{5}q^2 + \frac{44}{15}nq + \frac{128}{45}n^2) r_\psi^{*6} - \frac{79}{14}q^2 r_\psi^{*8} - \frac{62}{9}q^2 r_\psi^{*10} \quad (1.33d)$$

and

$$f_T = 1 - (\frac{14}{5}q^2 + \frac{28}{15}nq + \frac{56}{45}n^2) r_\psi^{*6} - \frac{46}{14}q^2 r_\psi^{*8} - \frac{34}{9}q^2 r_\psi^{*10} \quad (1.33e)$$

Here $r_\psi^* = \frac{r_\psi}{D}$ is non dimensional value of r_ψ. Also in these series expressions terms up to r_ψ^{10} in r_ψ & up to second order of smallness in n & q has been retained.

Values of M_ψ, P_ψ, L_ψ for RTD equipotential surfaces can be derived by solving system of differential equations. (1.19a - 1.19d) subject to the boundary conditions (1.19e- 1.19f) while using distortion parameters f_P & f_T as given in equation (1.33d - 1.33e).

While in the case when thermal properties are not considered important and only hydrostatic equilibrium is considered for RTD gaseous spheres, then equilibrium structure of RTD gaseous sphere can be obtained by integrating equations (1.19a) & (1.19b) subject to boundary conditions

At centre $r_\psi = 0$

$M_\psi = 0$ (1.34a)

and on the free surface

$M_\psi = M_0,\ P_\psi = 0,\ \rho_\psi = 0$

or $\rho_\psi = \rho_{\psi S}$, $P_\psi = P_{\psi S}$ (1.34b)

In case star is distorted by rotational forces alone we may set q = 0 in (1.33) and can still use this approach to determine equilibrium structures of the rotationally distorted stellar models. For determining equilibrium structure of synchronous binaries we take 2n=q+1 & for non synchronous binaries $2n \neq q+1$ is considered.

Mohan and Saxena (1983) considered it more convinient to use r_0 in place of r_ψ or M_ψ as independent variables. So using equation (1.32) which connects variables r_ψ with r_0 differential equations that governs the equilibrium structures of RTD stellar models as given by Saxena (1984) are

$$\frac{dM_\psi}{dr_0} = 4\pi D^3 \rho_\psi r_0^2 f_1 \qquad (1.35a)$$

$$\frac{dP_\psi}{dr_0} = -\frac{GM_\psi}{Dr_0^2}\rho_\psi f_2 \qquad (1.35b)$$

$$\frac{dL_\psi}{dr_0} = 4\pi\varepsilon D^3 \rho_\psi r_0^2 f_1 \qquad (1.35c)$$

and

$$\frac{dT_\psi}{dr_0} = -\frac{3\kappa L_\psi}{16\pi DacT_\psi^3}\frac{\rho_\psi}{r_0^2} f_3 \qquad (1.35d)$$

where $f_1 = \frac{r^2}{r_0^2}\frac{dr}{dr_0}$, $f_2 = f_P \frac{r_0^2}{r^2}\frac{dr}{dr_0}$, $f_3 = f_T \frac{r_0^2}{r^2}\frac{dr}{dr_0}$ are functions of $n, q \,\&\, r_0$ whose series expressions are given by

$$f_1 = 1 + 4nr_0^3 + (\frac{36}{5}q^2 + \frac{24}{5}nq + \frac{96}{5}n^2)r_0^6 + \frac{55}{7}q^2 r_0^8 + \frac{26}{3}q^2 r_0^{10} \qquad (1.36)$$

$$f_2 = 1 - (\frac{2}{5}q^2 + \frac{4}{15}nq + \frac{16}{15}n^2)r_0^6 - \frac{9}{14}q^2r_0^8 - \frac{8}{9}q^2r_0^{10} \tag{1.37}$$

and

$$f_3 = 1 + \frac{4n}{3}r_0^3 + (\frac{6}{5}q^2 + \frac{4}{5}nq + \frac{224}{45}n^2)r_0^6 + \frac{24}{14}q^2r_0^8 + \frac{20}{9}q^2r_0^{10} \tag{1.38}$$

Here terms up to r_0^{10} in r_0 and second order of smallness in n & q has been retained. The boundary conditions that system of differential equations (1.35) has to satisfy are

At centre $r_0 = 0$

$$M_\psi = 0, L_\psi = 0 \tag{1.39a}$$

and on free space $r_0 = r_{0S}$, $M_\psi = M_0, L_\psi = L_{\psi S}$

$$P_\psi = 0, \rho_\psi = 0, T_\psi = 0 \text{ or } P_\psi = P_{\psi S}, \rho_\psi = \rho_{\psi S}, T_\psi = T_{\psi S} \tag{1.39b}$$

where r_{0S} is value of r_0 at the free surface.

CHAPTER –II

Equilibrium structures of the rotating stars and stars in binary systems using extended series expressions.

2.1 Overview

Mohan & Saxena (1983) has determined the equilibrium structures of the primary component of binary system taking into account Kippenhahn & Thomas averaging technique (1970) along with the certain results of Roche Equipotential as that given by Kopal (1972). In the present chapter we have used the extended series expressions of radius of Roche Equipotential Surfaces as given by Pathania and Medupe (2012) that consider terms upto r_0^{14} in r_0. Using this expression we have obtained the extended series expressions of volume, surface area and various other parameters -containing terms up to r_0^{14} in r_0- in section 2.2. These series expressions are further used in section (2.3) to obtain extended series expressions for various distortion parameters as discussed in Kippenhahn and Thomas(1970) approach.

2.2 Extended series expressions for Volume, Surface area & various other parameters.

The expression for radius of Roche Equipotential surfaces as given by Pathania & Medupe (2012) that consider terms upto r_0^{14} in r_0 is given by

$$r = r_0\left[1 + C_3r_0^3 + C_4r_0^4 + C_5r_0^5 + C_6r_0^6 + C_7r_0^7 + C_8r_0^8 + C_9r_0^9 + C_{10}r_0^{10} + C_{11}r_0^{11} + C_{12}r_0^{12} + C_{13}r_0^{13}\right] \tag{2.1}$$

where $r_0 = \dfrac{1}{\psi^* - q}$

$$\begin{aligned}
C_3 &= qP_2 + n(1-\nu^2)\\
C_4 &= qP_3\\
C_5 &= qP_4\\
C_6 &= qP_5 + 3(qP_2 + n(1-\nu^2))^2\\
C_7 &= qP_6 + 7qC_3P_3\\
C_8 &= qP_7 + 8qC_3P_4 + 4q^2P_3^2\\
C_9 &= qP_8 + 9qC_3P_5 + 9q^2P_3P_4\\
C_{10} &= qP_9 + 10C_3qP_6 + 5q^2\{P_4^2 + 2P_3P_5\}\\
C_{11} &= qP_{10} + 11C_3qP_7 + 11q^2\{P_3P_6 + P_4P_5\}\\
C_{12} &= qP_{11} + 12C_3qP_8 + 6q^2P_5^2 + 12q^2\{P_3P_7 + P_4P_6\}\\
C_{13} &= qP_{12} + 13C_3qP_9 + 13q^2\{P_3P_8 + P_4P_7 + P_5P_6\}
\end{aligned} \tag{2.2}$$

Using equation (2.2), equation (2.1) can now be written as

$$r = r_0 \begin{bmatrix} 1 + (qP_2 + n(1-\nu^2))r_0^3 + (qP_3)r_0^4 + (qP_4)r_0^5 + (qP_5 + 3C_3^2)r_0^6 + (qP_6 + 7qC_3P_3)r_0^7 \\ +(qP_7 + 8qC_3P_4 + 4q^2P_3^2)r_0^8 + (qP_8 + 9qC_3P_5 + 9q^2P_3P_4)r_0^9 + (qP_9 + 10C_3qP_6 + \\ 5q^2\{P_4^2 + 2P_3P_5\})r_0^{10} + (qP_{10} + 11C_3qP_7 + 11q^2\{P_3P_6 + P_4P_5\})r_0^{11} + (qP_{11} + 12C_3q \\ P_8 + 6q^2P_5^2 + 12q^2\{P_3P_7 + P_4P_6\})r_0^{12} + (qP_{12} + 13C_3qP_9 + 13q^2\{P_3P_8 + P_4P_7 + P_5P_6\})r_0^{13} \end{bmatrix}$$

. (2.3)

Now by following the approach of Kopal (1972) and Mohan and Singh (1982) extended series expressions of Volume V_ψ , surface area S_ψ and radial distance r_ψ of a point on equipotential surfaces ψ^*= constant for primary component of the binary systems are given by

$$V_\psi = \frac{4}{3}\pi r_0^3(1 + 2nr_0^3 + \frac{4}{5}(8n^2 + 2nq + 3q^2)r_0^6 + \frac{15}{7}q^2r_0^8 + 2q^2r_0^{10} + \frac{21}{11}q^2r_0^{12} + \frac{3}{13}q^2r_0^{14}) \tag{2.4}$$

$$S_\psi = 4\pi r_0^2(1 + \frac{4}{3}nr_0^3 + \frac{7}{15}(8n^2 + 2nq + 3q^2)r_0^6 + \frac{9}{7}q^2r_0^8 + \frac{11}{9}q^2r_0^{10} + \frac{13}{11}q^2r_0^{12} + \frac{1}{13}q^2r_0^{14}) \tag{2.5}$$

and

$$r_\psi = r_0(1 + \frac{2}{3}nr_0^3 + (\frac{76}{45}n^2 + \frac{8}{15}nq + \frac{4}{5}q^2)r_0^6 + \frac{5}{7}q^2r_0^8 + \frac{2}{3}q^2r_0^{10} + \frac{7}{11}q^2r_0^{12} + \frac{1}{13}q^2r_0^{14}) \tag{2.6}$$

Inverting relation (2.6) we get

$$r_0 = r_\psi^*(1 - \frac{2n}{3}r_\psi^3 + (\frac{4n^2}{45} - \frac{8nq}{15} - \frac{4q^2}{5})r_\psi^6 - \frac{5q^2}{7}r_\psi^8 - \frac{2q^2}{3}r_\psi^{10} - \frac{7q^2}{11}r_\psi^{12} - \frac{1}{13}r_\psi^{14}) \tag{2.7}$$

where $r_\psi^* = \frac{r_\psi}{D}$ and here r_ψ^* is in non dimensional form. Similarly using equation (1.30) & (1.31) (from chapter I), the extended explicit expressions for $\bar{g}$ & $\overline{g^{-1}}$ are

$$\bar{g} = \frac{1}{r_0^2}(1 - \frac{8n}{3}r_0^3 + (-\frac{28n^2}{9} - \frac{4nq}{3} - 2q^2)r_0^6 - \frac{15q^2}{7}r_0^8 - \frac{7q^2}{3}r_0^{10} - \frac{28q^2}{11}r_0^{12} - \frac{156q^2}{25}r_0^{13} - \frac{22q^2}{13}r_0^{14})$$

(2.8)

and

$$\overline{g^{-1}} = r_0^2(1+\frac{8}{3}nr_0^3+(\frac{524}{45}n^2+\frac{52}{15}nq+\frac{26}{5}q^2)r_0^6+\frac{40}{7}q^2r_0^8+\frac{19}{3}q^2r_0^{10}+7q^2r_0^{12}+(-\frac{q^2}{13})r_0^{14}) \qquad (2.9)$$

In all these extended series expressions terms upto r_0^{14} in r_0 and upto second order of smallness in n & q has been retained.

2.3 Extended series expressions for various distortion parameters used in determining equilibrium structures of RTD stars.

Now using the extended series expressions as obtained in section (2.2). The extended series expressions for the various distortion parameters as discussed in section (1.5) of chapter I are given by

$$u = 1+(-\frac{4}{45}n^2-\frac{2}{15}nq-\frac{1}{5}q^2)r_\psi^6-\frac{1}{7}q^2r_\psi^8-\frac{1}{9}q^2r_\psi^{10}-\frac{1}{11}q^2r_\psi^{12}-\frac{1}{13}q^2r_\psi^{14} \qquad (2.10)$$

$$v = 1-\frac{4}{3}nr_\psi^3+(-\frac{8}{45}n^2-\frac{4}{45}nq-\frac{2}{5}q^2)r_\psi^6-\frac{5}{7}q^2r_\psi^8-q^2r_\psi^{10}-\frac{14}{11}q^2r_\psi^{12}-\frac{156}{25}q^2r_\psi^{13}-\frac{20}{13}q^2r_\psi^{14} \qquad (2.11)$$

$$w = 1+\frac{4}{3}nr_\psi^3+(\frac{152}{45}n^2+\frac{12}{5}nq+\frac{18}{5}q^2)r_\psi^6+\frac{30}{7}q^2r_\psi^8+5q^2r_\psi^{10}+\frac{63}{11}q^2r_\psi^{12}-\frac{3}{13}q^2r_\psi^{14} \qquad (2.12)$$

$$f_P = 1-\frac{4}{3}nr_\psi^3+(-\frac{68}{45}n^2-\frac{34}{15}nq-\frac{17}{5}q^2)r_\psi^6-\frac{29}{7}q^2r_\psi^8-\frac{44}{9}q^2r_\psi^{10}-\frac{62}{11}q^2r_\psi^{12}+\frac{4}{13}q^2r_\psi^{14} \qquad (2.13)$$

$$f_T = 1+(-\frac{56}{45}n^2-\frac{28}{15}nq-\frac{14}{5}q^2)r_\psi^6-\frac{23}{7}q^2r_\psi^8-\frac{34}{9}q^2r_\psi^{10}-\frac{47}{11}q^2r_\psi^{12}+\frac{156}{25}q^2r_\psi^{13}+\frac{25}{13}q^2r_\psi^{14} \qquad (2.14)$$

In all the above extended series expressions terms up to r_0^{14} in r_0 & up to second order of smallness in n and q has been retained.

The values of P_ψ and ρ_ψ on the various equipotential surfaces of RTD stars can now be obtained by solving system of differential equations given by the equation (1.19a) to (1.19d) subject to boundary conditions (1.19e) – (1.19f) where values of f_P & f_T are given by equation (2.13), (2.14).

For computational purposes, Mohan & Saxena (1983) find it easier to do calculations with r_0 than with independent variable r_ψ. So using equation (2.7) in equation (1.19a) - (1.19d) we get system of differential equations that governs the equilibrium structures of RTD stars as

$$\frac{dM_\psi}{dr_0} = 4\pi D^3 \rho_\psi r_0^2 f_1 \tag{2.15a}$$

$$\frac{dP_\psi}{dr_0} = -\frac{GM_\psi}{Dr_0^2}\rho_\psi f_2 \tag{2.15b}$$

$$\frac{dL_\psi}{dr_0} = 4\pi\varepsilon D^3 \rho_\psi r_0^2 f_1 \tag{2.15c}$$

&

$$\frac{dT_\psi}{dr_0} = -\frac{3\kappa L_\psi \rho_\psi}{16\pi DacT_\psi^3 r_0^2} f_3 \tag{2.15d}$$

where $r_0 = \dfrac{1}{\psi^* - q}$

Also f_1, f_2, f_3 are functions of distortion parameters n, q, r_0 given by $f_1 = \dfrac{r^2}{r_0^2}\dfrac{dr}{dr_0}$, $f_2 = f_P \dfrac{r_0^2}{r^2}\dfrac{dr}{dr_0}$, $f_3 = f_T \dfrac{r_0^2}{r^2}\dfrac{dr}{dr_0}$ such that their extended series expressions are

$$f_1 = 1 + 4nr_0^3 + (\frac{96}{5}n^2 + \frac{24}{5}nq + \frac{36}{6}q^2)r_0^6 + \frac{55}{7}q^2 r_0^8 + \frac{26}{3}q^2 r_0^{10} + \frac{105}{11}q^2 r_0^{12} + \frac{17}{13}q^2 r_0^{14} \tag{2.16a}$$

$$f_2 = 1 + (\frac{4}{15}n^2 + \frac{2}{5}nq + \frac{3}{5}q^2)r_0^6 + \frac{6}{7}q^2 r_0^8 + \frac{10}{9}q^2 r_0^{10} + \frac{15}{11}q^2 r_0^{12} + \frac{17}{13}q^2 r_0^{14} \tag{2.16b}$$

$$f_3 = 1 + \frac{4}{3}nr_0^3 + (\frac{224}{45}n^2 + \frac{4}{5}nq + \frac{6}{5}q^2)r_0^6 + \frac{12}{7}q^2 r_0^8 + \frac{20}{9}q^2 r_0^{10} + \frac{30}{11}q^2 r_0^{12} + \frac{156}{25}q^2 r_0^{13} + \frac{38}{13}q^2 r_0^{14} \tag{2.16c}$$

and

$$r_0 = r_\psi(1 - \frac{2n}{3}r_\psi^3 + (\frac{4n^2}{45} - \frac{8nq}{15} - \frac{4q^2}{5})r_\psi^6 - \frac{5q^2}{7}r_\psi^8 - \frac{2q^2}{3}r_\psi^{10} - \frac{7q^2}{11}r_\psi^{12} - \frac{1}{13}r_\psi^{14}) \tag{2.16d}$$

where r_ψ is the non dimensional value of radius of the topologically equivalent spherical surface. The boundary condition (19e-19f) now becomes-

At the centre, $r_0=0$

$$M_\psi = 0, L_\psi = 0 \tag{2.17a}$$

and at free surface, $r_0 = r_{0s}$:

$$M_\psi = M_0, L_\psi = L_{\psi S} \tag{2.17b}$$

$$P_\psi = 0 \text{ or } P_{\psi S}, T_\psi = 0 \text{ or } T_{\psi S}, \rho_\psi = 0 \text{ or } \rho_{\psi s} \tag{2.17c}$$

where M_0 is total mass of structure and $L_{\psi S}, P_{\psi S}, T_{\psi S}, \rho_{\psi s}$ are the values of $L_\psi, P_\psi, T_\psi, \rho_\psi$ respectively, on the outermost equipotential surfaces ψ^* = constant. Also r_{0s} is value of r_0 at surface such that

$$r_{0s} = \frac{1}{\psi_S^* - q} \tag{2.18}$$

where ψ_s^* a non dimensional value of total potential ψ_s on outermost equipotential surface of RTD stellar model.

CHAPTER-III

Equilibrium structures of the polytropic models of the rotating stars and stars in binary systems

3.1 Overview

Polytropic model of a star is a model in which amount of heat supplied (dq) is directly proportional to the instantaneous change in temperature (dt), that is, ($\frac{dq}{dt}$ = constant). In a polytropic model the pressure (P) and density ρ at any arbitrary point in a model is defined by the relation

$$P = P_c\theta^{N+1} \;\&\; \rho = \rho_c\theta^{N} \tag{3.1}$$

where P_c and ρ_c are pressure and density respectively at the centre of the model. θ is a parameter whose value lies between 0 & 1 and depends upon the distance of the selected point from the centre. N is the polytropic index and its value for practical analysis is taken between 1 to 5. It is a measure of the central concentration of the model. If N = 0, structure is homogeneous and density is uniformly distributed throughout the model while for N = 5 central condensation becomes very high and radius extends up to infinity.

Polytropic models of stars has been frequently used in literature to study the equilibrium structures of rotating stars & stars in binary systems. In this chapter we have used the approach of Mohan & Saxena (1983) to determine the equilibrium structures of polytropic models of rotating stars & stars in binary systems, taking into account the extended series expressions of various parameters. In section 3.2 we have adopted the Mohan & Saxena (1983) approach to obtain the differential equations that determine the equilibrium structures of rotationally and/or for tidally distorted polytropic models of stars. The extended series expressions for volume, surface area & other parameters of RTD polytropic models of stars are then obtained in section 3.3. In section 3.4 numerical computations has been done to obtain volume, surface area and other parameters of certain RTD polytropic models of stars. Conclusion of the present study has been next drawn in section 3.5.

3.2 Equilibrium structures of the primary component of RTD polytropic models of stars.

Let P_ψ be the pressure and ρ_ψ be the density of the equipotential surface ψ^* = constant that is distorted by both rotational and tidal forces. Then the values of pressure and density on the equivalent surface of a topologically equivalent spherical model will also be P_ψ and ρ_ψ respectively. Let this distorted model behave like polytropic model. Then P_ψ and ρ_ψ be connected through the relation of type.

$$\rho_\psi = \rho_{c\psi}\theta_\psi^N \tag{3.2}$$

&

$$P_\psi = P_{c\psi}\theta_\psi^{N+1} \tag{3.3}$$

Here $P_{c\psi}$ & $\rho_{c\psi}$ are the values of P_ψ & ρ_ψ respectively at centre. θ_ψ is the average of the value of polytropic parameter θ at various points of equipotential surfaces ψ^* = constant. Now on combining equation (1.35a) and (1.35b) of hydrostatic equilibrium with polytropic equation (3.2) & (3.3) we get the second order non linear differential equation as-

$$\frac{1}{r_0^2}\frac{d}{dr_0}\left(\frac{r_0^2}{f_2}\frac{d\theta_\psi}{dr_0}\right) = -\frac{R^2}{\ell^2} f_1\theta_\psi^N \tag{3.4}$$

where f_1 and f_2 are distortion parameters as given in (2.16a) & (2.16b) in which series expressions has been retained up to r_0^{14} in r_0 and

$$\ell^2 = \frac{(N+1)P_{c\psi}}{4\pi G\rho_{c\psi}^2} \tag{3.5}$$

Also the boundary conditions that (3.4) has to satisfy are

At the centre $r_0 = 0$, $\theta_\psi = 1$, $\frac{d\theta_\psi}{dr_0} = 0$ (3.6a)

& on the surface $r_0 = r_{0s}$, $\theta_\psi = 0$ (3.6b)

where r_{0s} is the value of r_0 at surface.

Also in equation (3.5) parameter ℓ is in dimensions of length. Let $R = \ell\xi_u$ where ξ_u is non dimensional variable & is value of ξ at outermost surface of undistorted Polytropic model. Then equation (3.4) that determines the equilibrium structures of RTD Polytropic models of stars in non dimensional form becomes

$$\frac{d}{dr_0}[A(r_0 n,q)\frac{d\theta_\psi}{dr_0}] = -\frac{\xi_u^2}{K^2}\theta_\psi^N r_0^2 B(r_0,n,q) \tag{3.7}$$

where

$$A(r_0,n,q) = \frac{r_0^2}{f_2} = r_0^2[1+(-\frac{3}{5}q^2-\frac{4}{15}n^2-\frac{2}{5}nq)]r_0^6-\frac{6}{7}q^2r_0^6-\frac{10}{9}q^2r_0^8-\frac{15}{11}q^2r_0^{12}-\frac{17}{13}q^2r_0^{14}$$

$$B(r_0,n,q) = f_1 = 1+4nr_0^3+(\frac{96}{5}n^2+\frac{24}{5}nq+\frac{36}{6}q^2)r_0^6+\frac{55}{7}q^2r_0^8+\frac{26}{3}q^2r_0^{10}+\frac{105}{11}q^2r_0^{12}+\frac{17}{13}q^2r_0^{14}$$

and $r_0 = \dfrac{1}{\psi^* - q}$

In the series expressions of A & B, terms up to r_0^{14} in r_0 and up to second order of smallness in n & q has been retained. In (3.7), K is a dimensionless constant and is ratio of 'Rψ' undistorted radius of primary to 'D' separation between primary and secondary. Also we have

$$\frac{D}{\ell} = \frac{D\xi_u}{\ell\xi_u} = \frac{D}{R_\psi}\xi_u = \frac{1}{K}\xi_u \tag{3.8}$$

So equation (3.7) along with the boundary conditions (3.6) now determines the equilibrium structures of RTD polytropic models of stars.

3.3 Calculations of Volume and Surface area for Primary Component of RTD Polytropic models of stars.

Using the approach of Kopal (1972) & Mohan and Saxena (1982) as discussed in section (2.2) of chapter (2) the extended series expressions for volume & surface area of Rotationally and Tidally distorted polytropic models are given by

$$V_\psi = \frac{4}{3}\pi(\frac{\ell\xi_u}{K})^3 r_{0s}^3[1 + 2nr_{0s}^3 + \frac{4}{5}(8n^2 + 2nq + 3q^2)r_{0s}^6 + \frac{15}{7}q^2 r_{0s}^8 + 2q^2 r_{0s}^{10} + \frac{21}{11}q^2 r_{0s}^{12} + \frac{3}{13}q^2 r_{0s}^{14}]$$

(3.9)

$$S_\psi = 4\pi(\frac{\ell\xi_u}{K})^2 r_{0s}^2[1 + \frac{4}{3}nr_{0s}^3 + \frac{7}{15}(8n^2 + 2nq + 3q^2)r_{0s}^6 + \frac{9}{7}q^2 r_{0s}^8 + \frac{11}{9}q^2 r_{0s}^{10} + \frac{13}{11}q^2 r_{0s}^{12} + \frac{1}{13}q^2 r_{0s}^{14}].$$

(3.10)

Also

$$r = r_{0s}(\frac{\ell\xi_u}{K})\begin{bmatrix} 1 + (qP_2 + n(1-\nu^2))r_0^3 + (qP_3)r_0^4 + (qP_4)r_0^5 + (qP_5 + 3C_3^2)r_0^6 + (qP_6 + 7qC_3P_3)r_0^7 \\ +(qP_7 + 8qC_3P_4 + 4q^2P_3^2)r_0^8 + (qP_8 + 9qC_3P_5 + 9q^2P_3P_4)r_0^9 + (qP_9 + 10C_3qP_6 + \\ 5q^2\{P_4^2 + 2P_3P_5\})r_0^{10} + (qP_{10} + 11C_3qP_7 + 11q^2\{P_3P_6 + P_4P_5\})r_0^{11} + (qP_{11} + 12C_3q \\ P_8 + 6q^2P_5^2 + 12q^2\{P_3P_7 + P_4P_6\})r_0^{12} + (qP_{12} + 13C_3qP_9 + 13q^2\{P_3P_8 + P_4P_7 + P_5P_6\})r_0^{13} \end{bmatrix}$$

(3.11)

Here $P_j = P_j(\lambda)$ is Legendre polynomial. The equation (3.11) can be used to obtain shapes of outermost surface of these RTD Polytropic models of stars. In all these series expressions terms up to r_0^{14} in r_0 and up to second order of smallness in n and q has been retained.

3. 4 Numerical Computations

Equation (3.7) has been integrated numerically under boundary conditions (3.6) to obtain inner structure, Volume & surface area of certain RTD Polytropic models for specific values of parameters N, ξ_u , n, q & K . The value of K is taken as 1 for uniformly rotating Polytropic models of stars and 0.5 for synchronous (angular velocity of rotation and revolution are same) and non synchronous (angular velocity of rotation and revolution are not same) Polytropic model of stars .

Equations (3.7) has been integrated numerically using Runge Kutta 4rth order method for specified values of parameters. Since there is singularity at centre so series solution is obtained (as obtained by Chandrasekhar (1939)) for integrating the points near centre. The obtained series solution is

$$\theta_\psi = 1-\frac{\xi_u^2}{6}r_0^2+\frac{N\xi_u^4}{120}r_0^4-\frac{2\xi_u^2 n}{15}r_0^5-\frac{N(8N-5)}{15120}\xi_u^6+\frac{\xi_u^4 nN}{70}r_0^7+(-\frac{5}{18}n^2\xi_u^2-\frac{nq\xi_u^2}{12}-\frac{q^2\xi_u^2}{8}$$
$$-\frac{29N-26}{1088640}N^2\xi_u^8)r_0^8+(\frac{-nN^2}{6300}\xi_u^6-\frac{1}{4050}\xi_u^4 nN(N-1)-\frac{nN}{8100}(8N-5)\xi_u^6)r_0^9+(\frac{142}{24750}n^2N$$
$$\xi_u^4-\frac{\xi_u^2q^2}{10}+\frac{11\xi_u^4 Nqn}{8250}+\frac{11}{5500}\xi_u^4 Nq^2+\frac{1}{110}(\frac{16}{5}n^2+\frac{4}{5}nq+\frac{6}{5}q^2)\xi_u^2)r_0^{10} \qquad (3.12)$$

We take starting value at r_0=0.005 from the series (3.12) to start the integration using Runge Kutta 4th order method. With a step length of 0.005, numerical integration was carried further till θ_ψ becomes zero. Volume & Surface area has been then computed using equations (3.9) & (3.10).

The numerical results thus obtained are tabulated in table (1-7) . In table (1-4) the values of θ_ψ, P_ψ and ρ_ψ has been calculated for standard model (N = 3.0). The numerical values obtained for Volume and Surface area has been listed in table (5-7) for certain polytropic models with Polytropic index 1.5, 3.0 and 4.0.

The previous values of Volume and surface area (in which series expressions has been expanded up to r_0^{10} in r_0) as obtained by Mohan & Saxena (1983) has also been listed in these tables for comparison

3.5 Analysis of results

Results in table (1-4) show that there is not any appreciable change (<1%) in the present values of pressure and density as compared to the previous values of Mohan & Saxena (1983)

Again from results in tables (5-7) we can conclude that the present values of volume and surface area do not show much variation (<1%) when compared to the values computed by Mohan & Saxena (1983).

So overall we can conclude that while determining the equilibrium structures of RTD polytropic models of stars on extending the series expressions for radius, volume, surface area and various other parameters there is not any appreciable effect on the results. However, since in our present study we have retained terms in all the series expressions up to r_0^{14} in r_0 and up to 2nd order of smallness in n & q. So it will be worthwhile to check the effects on the results on retaining terms upto third order of smallness in n & q.

TABLE 1 – Uniformly Rotating Star

POLYTROPIC INDEX N =3.0 ξ_u = 6.896850			
n= 0.1	q = 0.0	k=1.0	r_{0s} = 0.9825577 (0.9825577)
x	θ	θ^{N+1}	θ^N
0.0	1.000000 (1.000000)	1.000000 (1.000000)	1.000000 (1.000000)
0.1	0.925995 (0.925995)	0.735249 (0.735249)	0.794009 (0.794009)
0.2	0.753098 (0.753098)	0.321667 (0.321667)	0.427125 (0.427125)
0.3	0.564422 (0.564422)	0.101488 (0.101488)	0.179809 (0.179809)
0.4	0.404662 (0.404662)	0.026815 (0.026815)	0.066264 (0.066264)
0.5	0.281941 (0.281941)	0.006319 (0.006319)	0.022412 (0.022412)
0.6	0.190234 (0.190234)	0.001310 (0.001310)	0.006884 (0.006884)
0.7	0.121412 (0.121412)	0.000217 (0.000217)	0.001790 (0.001790)
0.8	0.068824 (0.068824)	0.000022 (0.000022)	0.000326 (0.000326)
0.9	0.027692 (0.027692)	0.000001 (0.000001)	0.000021 (0.000021)
1.0	0.000000 (0.000000)	0.000000 (0.000000)	0.000000 (0.000000)

Note: 1) Values in the parenthesis are the values obtained by Mohan and Saxena (1983).

2) Here $x = \frac{r_0}{r_{0s}}$.

3) Here Pressure P_ψ & density ρ_ψ are in units of P_c & ρ_c respectively.

TABLE 2 – Non synchronous Binary

POLYTROPIC INDEX N =3.0 ξ_u = 6.896850			
n= 0.1 q = 0.2 k=0.5 r_{0s} = 0.4983183 (0.4988662)			
x	θ	θ^{N+1}	θ^{N}
0.0	1.000000 (1.000000)	1.000000 (1.000000)	1.000000 (1.000000)
0.1	0.926000 (0.926000)	0.735264 (0.735264)	0.794022 (0.794022)
0.2	0.753202 (0.753202)	0.321844 (0.321844)	0.427301 (0.427301)
0.3	0.564883 (0.564883)	0.101820 (0.101820)	0.180250 (0.180250)
0.4	0.405742 (0.405742)	0.027102 (0.027102)	0.066578 (0.066578)
0.5	0.283766 (0.283766)	0.006484 (0.006484)	0.022850 (0.022850)
0.6	0.192797 (0.192797)	0.001382 (0.001382)	0.007166 (0.007166)
0.7	0.124630 (0.124630)	0.000241 (0.000241)	0.001936 (0.001936)
0.8	0.072590 (0.072590)	0.000028 (0.000028)	0.000382 (0.000382)
0.9	0.031909 (0.031909)	0.000001 (0.000001)	0.000032 (0.000032)
1.0	0.000000 (0.000000)	0.000000 (0.000000)	0.000000 (0.000000)

Note: 1) Values in the parenthesis are the values obtained by Mohan and Saxena (1983).

2) Here $x = \frac{r_0}{r_{0s}}$.

3) Here Pressure P_ψ & density ρ_ψ are in units of P_c & ρ_c respectively.

TABLE 3 - Synchronous Binary

POLYTROPIC INDEX N =3.0 ξ_u = 6.896850			
n= 0.55	**q = 0.1**	**k=0.5**	r_{0s} **= 0.4939982 (0.4939995)**
x	θ	θ^{N+1}	θ^{N}
0.0	1.000000 (1.000000)	1.000000 (1.000000)	1.000000 (1.000000)
0.1	0.925996 (0.925996)	0.735254 (0.735254)	0.794014 (0.794014)
0.2	0.753135 (0.753135)	0.321730 (0.321730)	0.427188 (0.427188)
0.3	0.564587 (0.564587)	0.101607 (0.101607)	0.179966 (0.179966)
0.4	0.405049 (0.405049)	0.026917 (0.026917)	0.066454 (0.066454)
0.5	0.282597 (0.282597)	0.066378 (0.066378)	0.022569 (0.022569)
0.6	0.191158 (0.191158)	0.001335 (0.001335)	0.006985 (0.006985)
0.7	0.122576 (0.122576)	0.000226 (0.000226)	0.001842 (0.001842)
0.8	0.070190 (0.070190)	0.000024 (0.000024)	0.000346 (0.000346)
0.9	0.029226 (0.029226)	0.000001 (0.000001)	0.000025 (0.000025)
1.0	0.000000 (0.00000)	0.000000 (0.000000)	0.000000 (0.000000)

Note: 1) Values in the parenthesis are the values obtained by Mohan and Saxena (1983).

2) Here $x = \frac{r_0}{r_{0s}}$.

3) Here Pressure P_ψ & density ρ_ψ are in units of Pc & ρ_c respectively.

TABLE 4 – Synchronous Binary

POLYTROPIC INDEX N =3.0 ξ_u = 6.896850			
n= 0.6 **q = 0.2** **k=0.5** r_{0s} **= 0.4933582 (0.4933638)**			
x	θ	θ^{N+1}	θ^N
0.0	1.000000 (1.000000)	1.000000 (1.00000)	1.000000 (1.000000)
0.1	0.925996 (0.925996)	0.735253 (0.735253)	0.794013 (0.794013)
0.2	0.753128 (0.753128)	0.321718 (0.321718)	0.427175 (0.427175)
0.3	0.564553 (0.564553)	0.101583 (0.101583)	0.179935 (0.179935)
0.4	0.404971 (0.404971)	0.026897 (0.026897)	0.066416 (0.066416)
0.5	0.282464 (0.282464)	0.006366 (0.006366)	0.022537 (0.022537)
0.6	0.190968 (0.190968)	0.001330 (0.001330)	0.006964 (0.006964)
0.7	0.122332 (0.122332)	0.000224 (0.000224)	0.001831 (0.001831)
0.8	0.069896 (0.069896)	0.000024 (0.000024)	0.000341 (0.000341)
0.9	0.028883 (0.028884)	0.000001 (0.000001)	0.000024 (0.000024)
1.0	0.000000 (0.000000)	0.000000 (0.000000)	0.000000 (0.000000)

Note: 1) Values in the parenthesis are the values obtained by Mohan and Saxena (1983).

2) Here $x = \frac{r_0}{r_{0s}}$.

3) Here Pressure P_ψ & density ρ_ψ are in units of Pc & ρ_c respectively.

TABLE .5 - Volume and Surface area of polytropic models of stars.

POLYTROPIC INDEX N=3.0, ξ_u =6.896850						
Uniformly rotating stars						
Model No.	**n**	**q**	**K**	r_{0s}	V_ψ	S_ψ
1	0.02	0.0	1.0	0.996591 (0.996591)	1.584591 (1.584591)	6.102094 (6.102094)
2	0.05	0.0	1.0	0.991395 (0.991395)	1.876303 (1.876303)	6.308662 6.308662
3	0.1	0.0	1.0	0.982557 (0.982557)	2.301471 (2.301471)	6.694403 (6.694403)
4	0.15	0.0	1.0	0.973571 (0.973571)	2.655756 (2.655756)	7.116517 (7.116517)
5	0.20	0.0	1.0	0.964505 (0.964505)	2.946045 (2.946045)	7.559558 (7.559558)
Non synchronous binaries						
6	0.02	0.20	0.5	0.499721 (0.499728)	1.384239 (1.383703)	5.997848 (5.997650)
7	0.05	0.20	0.5	0.499399 (0.499405)	1.396041 (1.395507)	6.020993 (6.020791)
8	0.10	0.20	0.5	0.498859 (0.498866)	1.415413 (1.414891)	6.060443 (6.060255)
9	0.15	0.20	0.5	0.498318 (0.498324)	1.434421 (1.433905)	6.100961 (6.100776)
10	0.20	0.20	0.5	0.497780 (0.497780)	1.453063 (1.452556)	6.142499 (6.142320)
Synchronous binaries						
11	0.525	0.01	0.5	0.494322 (0.494322)	1.561576 (1.561565)	6.424565 (6.424565)
12	0.55	0.1	0.5	0.493998 (0.493999)	1.571102 (1.570988)	6.452216 (6.452179)
13	0.6	0.2	0.5	0.493358 (0.493363)	1.589460 (1.589017)	6.506959 (6.506834)

Note: 1) Values in the parenthesis are the values obtained by Mohan and Saxena (1983).

TABLE 6 - Volume and Surface area of polytropic models of stars.

POLYTROPIC INDEX N =1.5 ξ_u = 3.653750						
Uniformly rotating stars						
Model No.	**n**	**q**	**K**	r_{0s}	$V_\psi * 10^{-2}$	$S_\psi * 10^{-2}$
1	0.02	0.0	1.0	0.993744 (0.993744)	2.33093 (2.33093)	1.702409 (1.702404)
2	0.05	0.0	1.0	0.984301 (0.984301)	2.70130 (2.70129)	1.742468 (1.742465)
3	0.1	0.0	1.0	0.968633 (0.968633)	3.175960 (3.175952)	1.813274 (1.813272)
4	0.15	0.0	1.0	0.953311 (0.953311)	3.50576 (3.50576)	1.884909 (1.884909)
5	0.20	0.0	1.0	0.938527 (0.938527)	3.725119 (3.725119)	1.954242 (1.954242)
Non synchronous binaries						
6	0.02	0.20	0.5	0.499507 (0.499513)	2.055456 (2.054656)	1.681881 (1.681821)
7	0.05	0.20	0.5	0.498915 (0.498921)	2.06950 (2.06871)	1.686508 (1.686447)
8	0.10	0.20	0.5	0.497927 (0.497933)	2.09215 (2.09238)	1.694364 (1.694305)
9	0.15	0.20	0.5	0.496937 (0.496943)	2.11388 (2.11382)	1.702385 (1.702382)
10	0.20	0.20	0.5	0.495946 (0.495945)	2.13470 (2.13469)	1.710558 (1.710560)
Synchronous binaries						
11	0.525	0.01	0.5	0.489686 (0.489686)	2.24512 (2.24511)	1.764307 (1.764304)
12	0.55	0.1	0.5	0.489122 (0.489123)	2.25420 (2.25320)	1.769513 (1.774230)
13	0.6	0.2	0.5	0.488016 (0.488020)	2.27132 (2.27071)	1.779721 (1.779680)

Note: 1) Values in the parenthesis are the values obtained by Mohan and Saxena (1983).

TABLE 7 - Volume and Surface area of polytropic models of stars.

POLYTROPIC INDEX N =4.0 $\xi_u = 14.97155$						
Uniformly rotating stars						
Model No.	**n**	**q**	**K**	r_{0s}	$V_\psi * 10^{-2}$	$S_\psi * 10^{-2}$
1	0.02	0.0	1.0	0.998637 (0.993744)	16.334510 (16.209340)	2.887809 (2.875484)
2	0.05	0.0	1.0	0.996509 (0.996509)	19.645590 (19.645590)	3.007183 (3.007177)
3	0.1	0.0	1.0	0.992758 (0.992758)	24.871910 (24.87183)	3.237449 (3.237445)
4	0.15	0.0	1.0	0.988769 (0.988769)	29.719750 (29.71960)	3.502388 (3.502379)
5	0.20	0.0	1.0	0.984567 (0.984567)	34.180510 (34.180330)	3.796802 (3.796791)
Non synchronous binaries						
6	0.02	0.20	0.5	0.499865 (0.499871)	14.172220 (14.176666)	2.827996 (2.827895)
7	0.05	0.20	0.5	0.499734 (0.499741)	14.310140 (14.304650)	2.841142 (2.841048)
8	0.10	0.20	0.5	0.499515 (0.499521)	14.538640 (14.533160)	2.803595 (2.806349)
9	0.15	0.20	0.5	0.499294 (0.499300)	14.765550 (14.760090)	2.886732 (2.886633)
10	0.20	0.20	0.5	0.499071 (0.499077)	14.990710 (14.98533)	2.910527 (2.91043)
Synchronous binaries						
11	0.525	0.01	0.5	0.497684 (0.497684)	16.366440 (16.366330)	3.075355 (3.075343)
12	0.55	0.1	0.5	0.497524 (0.497525)	16.489520 (16.488050)	3.091469 (3.091426)
13	0.6	0.2	0.5	0.497211 (0.497211)	16.729840 (16.72940)	3.123620 (3.123454)

Note: 1) Values in the parenthesis are the values obtained by Mohan and Saxena (1983).

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