

**Exact Solutions of Some Partial Differential Equations Using  $\left(\frac{G'}{G}\right)$ -  
Expansion Method**

*Thesis submitted in partial fulfillment of the requirements  
for the award of the degree of  
Masters of Science  
in*

**Mathematics and Computing**

*Submitted by*

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*the guidance of*

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*to the*



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**DEDICATED  
TO  
GOD, MY PARENTS, MY TEACHERS AND MY FRIENDS**

Certificate

Acknowledgement

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## CERTIFICATE

I hereby certify that the work which is being presented in this thesis entitled "Exact Solutions of Some Partial Differential Equations Using  $\left(\frac{G'}{G}\right)$ -Expansion Method" which is being submitted for the award of degree of Master of Sciences, School of Mathematics and Computer Applications, Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of **Dr. Ram Jiwari**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

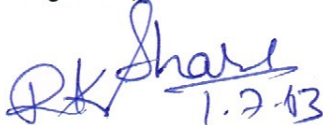
  
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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.



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## ABSTRACT

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In this thesis an attempt has been made to find the exact solutions of some nonlinear partial differential equations by using  $\left(\frac{G'}{G}\right)$ -expansion method. The chapter wise summary of the thesis is as follows

In chapter 2, we have applied the  $\left(\frac{G'}{G}\right)$ -expansion method to find the traveling wave solutions of the Ramani equation

$$u_{xxxxx} + 15u_x u_{xxx} + 15u_{xx} u_{xxx} + 45u_x^2 u_{xx} - 5u_{xxx} - 15u_x u_{xt} - 15u_t u_{xx} - 5u_{tt} = 0$$

The Ramani equation was first introduced by Hirota and Ramani [33]. Ji [34] obtained some travelling soliton solutions of this equation by using Exp-function method [34]. This equation is completely integrable by the inverse scattering method. Equation is studied in [33, 35, 36] where new kind of solutions were obtained. Hirota-Ramani equation is widely used in various branches of physics, and such as plasma physics, fluid physics, and quantum field theory. It also describes a variety of wave phenomena in plasma and solid state [33].

In chapter 3, a generalized  $\left(\frac{G'}{G}\right)$ -expansion method is used to seek more general exact solutions of the fifth-order Sawada-Kotera equation

$$u_{xxxxx} + u_t + 45u_x u^2 + 15u_x u_{xx} + uu_{xxx} = 0$$

As a result, the traveling wave solutions with three arbitrary functions are obtained including hyperbolic function solutions, trigonometric function solutions and rational solutions. The method appears to be easier and faster by means of some mathematical software.

Taking full advantage of the generalized solitary solutions, we can recover some known solutions obtained by the most existing methods.

The equation Sawada-Kotera (SK) was first introduced by Sawada and Kotera [38]. The SK equation is widely used in various branches of physics, and such as plasma physics, fluid physics.

### 1.1 Differential Equation

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics and other disciplines. Differential equations arise in many areas of science and technology. Specifically, whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and time (expressed as derivatives) is known. In

other words differential equation is an equation  $f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0$ , which express a relation between independent and dependent variables and their derivatives of any order.

#### 1.1.1 Partial differential Equation

Partial differential equations form the basis of many mathematical models of physical, chemical and biological phenomena and more recently their use has spread into economics, financial forecasting, image processing and many other fields. These equations provide a mathematical description of many natural phenomena. Partial differential equations are one of the basic fields in the area of the applied analysis and it is difficult to imagine any area of

applications where its impact is not felt. Most physical situations involving the rates of change can be modeled with the help differential equations. Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variable in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables.

During the last half of nineteenth century, a large number of mathematicians became active in the investigation of numerous physical problems modeled by partial differential equations in order to see the physical behavior of the problems. Modern approaches seek methods applicable to non linear partial differential equations as well as linear ones. Analytic and numerical solutions of partial differential equations play an important role in the proper understanding of various features of many phenomena and processes in different areas of sciences and engineering. The strong desire of finding analytic solutions to nonlinear partial differential equations leading nonlinear phenomenon for technological enhancement and for research purpose made tremendous growth. These solutions often with several important physical parameters prove its usefulness in discussing and examining the sensitivity of physical phenomena they describe. Currently, there are many numerical and exact techniques are available in the literature to find the solutions of nonlinear partial differential equations. Among them, tanh method, lie-classical method, exp- method, painleve integrability method, variable coefficient method, homogenous balanced method, sine-cosine method, one method and  $\left(\frac{G'}{G}\right)$  - expansion method etc. are some exact techniques while finite differences methods, finite

elements method, finite volume method, spectral methods, differential quadrature methods [39-52] etc. are some numerical techniques.

## 1.2 Tanh Method

Hyperbolic tangent (tanh) method is a powerful technique to symbolically compute traveling waves solutions of one-dimensional nonlinear wave and evolution equations. In particular, the method is well suited for problems where dispersion, convection, and reaction diffusion phenomena play an important role. The technique is outlined for the computation of closed form tanh-solutions for nonlinear partial differential equations and ordinary differential equations. Tanh- method, one of most direct and effective algebraic method developed many years back for finding exact solutions of nonlinear equations. Recently, much work has been concentrated on the various extensions and applications of the method. The pioneer work Malfiet [1, 2] introduced the powerful tanh method for a reliable treatment of the nonlinear wave equations. The useful tanh method is widely used by many researchers. Later, the extended tanh method, developed by Wazwaz [3, 4], is a direct and effective algebraic method for handling nonlinear equations.

### 1.2.1 Methodology of Tanh Method

Wazwaz has summarized the use of the extended tanh method. Consider a partial differential equation

$$P(u, u_t, u_x, u_{xx}, \dots) = 0, \quad (1)$$

The equation (1) can be converted to the following ODE

$$Q(U, U', U'', U''', \dots) = 0, \quad (2)$$

by means of a wave variable  $\xi = x - \beta t$  so that  $u(x, t) = U(\xi)$  and using the following change of variables (in the derivatives)

$$\frac{\partial}{\partial t} = -\beta \frac{d}{d\xi}, \frac{\partial}{\partial x} = \frac{d}{d\xi}, \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2}, \dots \quad (3)$$

Eq. (2) is then integrated as long as all terms contain derivatives where integration constants are considered zeros. Introducing a new independent variable

$$Y = \tanh(\xi), \quad (4)$$

Leads to a change in the derivatives

$$\begin{aligned} \frac{d}{d\xi} &= (1 - Y^2) \frac{d}{dY}, \\ \frac{d^2}{d\xi^2} &= (1 - Y^2) \left\{ -2Y \frac{d}{dY} + (1 - Y^2) \frac{d^2}{dY^2} \right\}, \\ \frac{d^3}{d\xi^3} &= (1 - Y^2) \left\{ (6Y^2 - 2) \frac{d}{dY} - 6Y(1 - Y^2) \frac{d^2}{dY^2} + (1 - Y^2) \frac{d^3}{dY^3} \right\}, \end{aligned} \quad (5)$$

and the remaining derivatives were derived similarly. The extended tanh -method admits the use of finite expansion

$$U(\xi) = S(Y) = a_0 + \sum_{k=1}^m [a_k Y^k + a_{-k} Y^{-k}] \quad (6)$$

where  $m$  is a positive integer which will be determined. The parameter  $m$  is usually obtained by balancing the highest order derivatives with the nonlinear terms in (2). Substituting (6) into (2) results in algebraic equations in powers of  $Y$ , that will lead to the determination of the parameters  $a_k$ , ( $k = 0, 1, 2, 3, \dots, m$ ),  $a_{-k}$ , ( $k = 1, 2, 3, \dots, m$ ) and  $\beta$ .

### 1.3 Painlevé Integrability

Weiss, Tabor and Carnevale [5] have introduced the Painlevé test for PDEs and have shown that there exists a close relationship between PP (Painlevé Property) and integrability. In the case of ordinary differential equations, the Painlevé test of the solution in the complex plane has played an important role in deciding between integrable and non-integrable dynamical systems. More specifically, one could classify an ODE or a system of ODEs in the complex domain to be of Painlevé type or has Painlevé property (PP) if the only movable singularities of all its solutions are poles. Fundamental contribution connecting Painlevé property and integrability in the case of ODE has been made by Kovalevskaya (1889) [6], Yoshida (1983) [7], Erconlani and Siggia (1986) [8].

The Painlevé integrability conditions of both single and double component non autonomous NLS equations with the time-dependent dispersion, nonlinearity and harmonic potential, using Painlevé analysis. In the single-component case, the condition obtained is consistent with that obtained by the lax pair. However, its physical meanings have not yet been discussed in a deep way. In the absence of the harmonic potential, one of the fundamental integrable forms of the coupled NLS equations is developed by M. Abdollahzadeh. [9] and other integrable forms for the two-component [10] and multi-component NLS equations [11,12] have been further reported in the literature. The procedure obtaining the Painlevé integrability condition in this case is tedious but the result is physically straightforward, namely for each component the competition between the kinetic and potential energies balance the competitions between all possible combinations of the dispersion and nonlinearity including intra- and intercomponents

Among the various approaches followed to study the integrability of nonlinear evolution equations, the Painlevé analysis has proved to be one of the most successful and widely applied tools. If a nonlinear equation passes the Painlevé test, we call it a Painlevé integrable model. The Painlevé integrable models are prime candidates for being completely integrable, thus the singular manifold method of Weiss always allows the recovery of the Lax pair and Darboux transformation (DT), and so also the Backlund transformation, from a truncated Painlevé expansion. It is very tedious to study whether a given PDE passes the Painlevé test, thus the application of computer algebra can be very helpful in such calculations. Various researchers have developed computer programs for the Painlevé test of nonlinear equations. It is much more difficult to prove the Painlevé property for nonlinear models with parameter coefficients, which occur in many branches of physics.

### 1.3.1 Methodology of Painlevé Integrability

Consider a system of nonlinear PDEs with parameter coefficients, say in two independent variables  $x$  and  $t$

$$H_s(u^{(i)}, u_x^{(i)}, u_t^{(i)}, u_{xt}^{(i)}, u_{xx}^{(i)}, \dots) = 0, s = 1, 2, \dots, m, \quad (7)$$

where  $u^{(i)} = u^{(i)}(x, t) (i = 1, 2, \dots, m)$  are dependent variables, the subscripts denote partial derivatives,  $H_s (s = 1, \dots, m)$  are polynomials about  $u^{(i)}$  and their derivatives, maybe after a preliminary change of variables. Eq. (7) are said to pass the Painlevé test, if all solutions of Eq. (7) can be expressed as Laurent series,

$$u^{(i)} = \sum_{j=0}^{\infty} u_j^{(i)} \phi(x, t)^{(j+\alpha_i)}, \quad i = 1, \dots, m$$

with sufficient number of arbitrary functions as the order of (7),  $u_j^{(i)}$  are analytic functions,  $\alpha_i$  are negative integers. In order to simplify the involved computations, we apply the Kruskal's gauge for the singular manifold:  $\phi(x,t) = x - \psi(t)$ ,  $u_j^{(i)} = u_j^{(i)}(t)$ . The algorithm of the Painlevé test for Eq. (7) is made of following three steps:

### Step a. Leading order analysis

To determine leading order exponents  $\alpha_i$  and coefficients  $u_0^{(i)}$  ( $i = 1, \dots, m$ ), letting  $u^{(i)} = u_0^{(i)} \phi^{\alpha_i}$ ,  $i = 1, \dots, m$  and inserting them into (7), then balancing the minimal power terms, one can obtain all possible  $(\alpha_i, u_0^{(i)})$  for all parameters constraints.

### Step b. To find the resonances

Substituting the following truncated Painlevé expansion:

$$u^{(i)} = u_0^{(i)} \phi^{\alpha_i} + u_r^{(i)} \phi^{\alpha_i+r}, \quad i = 1, \dots, m$$

Into (7) and collecting the terms with the lowest powers of  $\phi$ , we get

$$Q(r) \cdot (u^{(1)}, \dots, u^{(m)})^T = 0,$$

where  $Q$  is an  $m \times m$  matrix, whose elements depend on  $r$ . The resonances are the roots of  $\det(Q(r)) = 0$ . If all resonances are found to be integers, then one has to proceed the third step. Otherwise, we should investigate under what parameters constraints such that all resonances are integers. Then for each set of parameters constraints, the Painlevé test will be performed from the Step a.

### Step c. To verify compatibility conditions

The compatibility conditions should be verified for every non-negative integer resonance. To this end, we insert the truncated expansions

$$u^{(i)} = \sum_{j=0}^{r \max} u_j^{(i)} \phi^{j+\alpha_i}, \quad i=1, \dots, m$$

into (7). If there are compatibility conditions which can not be satisfied, one should need investigate under what parameters constraints such that the compatibility conditions are satisfied identically. Subsequently, the Painleve' test of the original equations can be performed for all possible parameters constraints one by one.

### 1.4. Lie-Classical Method

The conservation laws (or first integrals) of mechanical systems are always of mathematical importance. The modern methods in finding conservation laws are mainly the Lie method. In 1918, using the invariance of Hamilton action under infinitesimal transformations, Noether studied conservation laws of mechanical systems and presented the famous Noether's theorem. In 1979, G. W. Bluman et al. [13] introduced the method of the extended group which Lie used to study the invariance of differential equations at the end of the 19th century. The author investigated the method and posed the Lie symmetries which make the differential equations of motion invariant. In the past twenty years or so, great progress has been made in seeking modern methods for finding conservation laws, i.e. Lie method.

A local symmetry maps solutions of the PDE system into one-parameter families of solutions. However, there can exist solutions that map into themselves, i.e., are invariant, under the action of a local symmetry of the PDE system. Such solutions are called invariant solutions (similarity solutions) and include the well-known self-similar solutions (automodel solutions)

that result from scaling symmetries. The method of finding invariant solutions is commonly referred to as the classical method. To start with classical method, the classical method essentially consists of finding symmetry reductions of PDEs with the help of determining equations obtained under the condition of invariance of the system of PDEs. More specifically, when a given system of PDEs is subjected to invariance under one-parameter Lie group of transformations, one arrives at an over determined linear system of differential equations for the group infinitesimals. These infinitesimals of the transformations help us obtain the reductions of the system.

One of the important applications of the classical method due to Lie is the separation of variables for linear PDEs. [14]. It is certain that functionally separable solutions and group-invariant solutions form two classes of important exact solutions of PDEs. They usually reflect some basic and significant properties of PDEs. This method has been applied successfully to find some interesting exact solutions and symmetry reductions of certain nonlinear PDEs.

#### 1.4.1 Methodology of Lie Classical Method

Consider a system of nonlinear partial differential equations,  $u$  and  $n$  independent variables  $x = (x_1, x_2, x_3, \dots, x_n)$  given by

$$F_\mu(x, u, \partial u, \partial^2 u, \dots, \partial^k u) = 0, \mu = 1, 2, 3, \dots, N \quad (8)$$

Step 1: Let the one-parameter Lie group of point transformations leaves invariant the system of PDEs.

Step 2: Apply the prolonged operator  $X^{(k)}$  to each equation of the system (8) and require that

$$X^{(k)} F^\mu |_{F^\mu=0} = 0, \mu, \nu = 1, 2, \dots, N \quad (9)$$

The meaning of the condition (9) is that  $X^{(k)}$  vanishes on the solution set of the originally given system (8). Precisely, this condition assures that  $u(x)$  is solution of (8) whenever  $u^*(x^*)$  is one.

Step 3: From the invariance condition, a system of linear PDEs for  $\xi$  and  $\eta$  that constitutes a set of determining equations for the infinitesimal generator  $X$  admitted by the given system of PDEs (8) is obtained.

Step 4: The solutions of the determining equations will lead to the explicit forms of  $\xi$  and  $\eta$ .

Step 5: Construct the corresponding characteristics equations and obtain  $u$  in terms of  $n-1$  new independent variables.

Step 6: Rewrite the system (8) in these new coordinates to get the reduced form of the system.

### 1.5 Exp-Method

In mathematics many nonlinear partial differential equations widely described important phenomena in many branches of sciences such as physical, chemical, economical and biological. Most of these equations have been studied to find the exact or approximate solutions by using different methods. He and Wu [15] are proposed a straightforward and concise method called exp-function method. The exp-function method used recently [16] to find the exact solution to the system of nonlinear partial differential equations. Zhang [17] is used the exp-function method for solving Maccari's system. Demiray in [18] is used the exponential rational function approach to present a travelling wave solution to the KdV-Burgers equation, but some researchers think that the exponential function method in rational form is a special case [15]. Zhu extended the method to solve differential difference equations. Zhou *et al.* suggested a modified version of the Exp-function method for nonlinear equation with high order of nonlinearity. The main advantage

of the generalized Exp function method is that it can be applied to nonlinear equations including those in which the odd and even order derivative terms are co-exist.

### 1.5.1 Methodology of Exp-Method

Suppose the general nonlinear partial differential equation in two independent variables  $x$  and  $t$  is:

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx} \dots) = 0, \quad (10)$$

where  $u = u(x, t)$  is an unknown function and  $P$  is a polynomial in  $u = u(x, t)$  which has various partial derivatives, and the highest order derivatives and nonlinear terms are involved in it. The important steps of the exp-function method are discussed as follows

Consider the traveling wave transformation:

$$u(x, t) = u(\xi), \quad \xi = x + st, \quad (11)$$

where  $s$  is the wave speed and  $\xi$  is the combination of two independent variables  $x$  and  $t$ . After applying the transformation (11) on the equation (10), we get an ordinary differential equation as follows

$$Q(u, su', u', s^2u'', su'', u'', \dots) = 0 \quad (12)$$

where primes denote the ordinary derivative with respect to  $\xi$ . We assume that the wave solution of equation (12) in the form:

$$u(\xi) = \frac{\sum_{n=-c}^d a_n \exp(n\xi)}{\sum_{m=-p}^q b_m \exp(m\xi)} = \frac{a_{-c} \exp(-c\xi) + \dots + a_d \exp(d\xi)}{b_{-p} \exp(-p\xi) + \dots + b_q \exp(q\xi)}, \quad (13)$$

where  $c, d, p$  and  $q$  are positive integers that could be determine subsequently,  $a_n$  and  $b_m$  are unknown constants. The equation (13) can be re-written as

$$u(\xi) = \frac{a_c \exp(c\xi) + \dots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \dots + b_{-q} \exp(-q\xi)}.$$

(14)

In order to determine the values of  $c$  and  $p$ , we balance the highest order linear term with the highest order nonlinear term in equation (12). Similarly, to determine the values of  $d$  and  $q$ , we balance the lowest order linear term with the lowest order nonlinear term in equation (12). Putting the values of  $c$ ,  $d$ ,  $p$  and  $q$  into equation (14) and then substituting equation (14) into equation (12) and simplifying, we obtain;

$$\sum_k C_k \exp(\pm k\xi) = 0, k = 0, 1, 2, 3, \dots \quad (15)$$

Then setting each coefficient  $C_k = 0$ , yields a set of algebraic equations for  $a_c$ 's and  $b_p$ 's. The unknown  $a_c$ 's and  $b_p$ 's can be obtained by solving the algebraic equations. Substituting these values into equation (14), we obtain traveling wave solutions of the equation (10).

## 1.6 Homogenous Balanced Method

The Homogeneous balance (HB) method is used to search for Backlund transformations and similarity reductions for nonlinear partial differential equations. The existence of the Lax pairs, Backlund transformation (BT) and infinite number of symmetries are three intriguing properties which had been shown to be among the most intrinsic features of soliton systems such as Korteweg–de Vries (KdV), sine-Gordon (SG) and nonlinear Schrodinger (NLS) equations. On one hand, not only the Lax pairs are the base of solving an integrable model by using the inverse scattering method.

From 70's, a vast variety of the simple and direct methods to find analytic solutions of PDE have been developed. Recently, the homogeneous balance (HB) method has drawn lots of interests in seeking the solitary wave solution and other kinds of solutions, the solitary wave solution of some nonlinear PDEs have been obtained by this method and the relation between the HB method and the CK method or Backlund transformation has been discussed. The homogeneous balance (HB) method is also called quasi-solution technique, which is useful for finding solitary wave solutions of nonlinear partial differential equations. Bai [19] developed the extended homogeneous balance method and gave the multiple soliton solutions of the high-order Broer–Kaup equations.

### 1.6.1 Methodology of Homogeneous Balance Method

Considering a non-linear evolution equation, with a physical field  $u$ , and two independent variables  $x, t$  as

$$G(u, u_t, u_x, u_{tx}, u_{tt}, u_{xx} \dots) = 0, \quad (16)$$

#### Step 1 :

Assume

$$u_1 = u_0 + c \frac{\partial^{n+m} \ln(w)}{\partial x^n \partial t^m}, \quad (17)$$

where  $w = w(t, x)$ ,  $n \in \mathbb{Z}^+$   $u_0$  is a initial value, and  $u_1$  is new solution to eq.(16). If we use Fan's HBM method, we should change non-linear terms into linear ones, while we needn't do such by means of the simplified form Eq. (17), and make the procedure easier. Then, we can

determine the  $c, n$  and  $m$  here by balancing the highest degree linear term with the nonlinear term.

### Step 2:

Substituted eq. (17) into eq. (16), we derive a series about  $w$ . Then, making the coefficient of the lowest degree zero, we can gain Backlund transformation.

### 1.7 $\left(\frac{G'}{G}\right)$ -Expansion Method

$\left(\frac{G'}{G}\right)$ -expansion method is a new approach to obtain the analytic solutions of partial differential equations. Firstly, the method was introduced by Wang *et al.* [20] in 2008. Later, the generalized  $\left(\frac{G'}{G}\right)$ -expansion method, the modified  $\left(\frac{G'}{G}\right)$ -expansion method, the extended  $\left(\frac{G'}{G}\right)$ -expansion method and the improved  $\left(\frac{G'}{G}\right)$ -expansion method have been proposed on the basis of  $\left(\frac{G'}{G}\right)$ -expansion method. Presently, a large number of researchers are working on this method to solve nonlinear partial differential equations. The main advantage of  $\left(\frac{G'}{G}\right)$ -expansion method is that it gives more general solutions with some free parameters and it handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset. The method was soon been applied to other non-linear problems by several authors [21-24].

Noher *et al.* [25] obtained abundant traveling wave solutions of the Caudray-Dodd-Gibbon equation by Using  $\left(\frac{G'}{G}\right)$ -expansion method. Abazari [26] used the same method for constructing exact solutions of three nonlinear evolution equations. Fung *et al.* applied  $\left(\frac{G'}{G}\right)$ -expansion method to seek solutions of the Kolmogorov-Petrovskii-Piskunov equation. Neirmeh and Alibeigi [27] constructed the traveling wave solutions of the (3+1)- dimensional Kadomtsev-Petviashvili equation via the  $\left(\frac{G'}{G}\right)$ -expansion method . To spread out the range of applicability and to show the efficiency of the  $\left(\frac{G'}{G}\right)$ -expansion method, further research is being carried out by many researchers. For example, Zhang *et al.* [20] extended the  $\left(\frac{G'}{G}\right)$ -expansion method and called the improved  $\left(\frac{G'}{G}\right)$ -expansion method. Consequently, several researchers studied various nonlinear PDEs to generate traveling wave solutions via the improved  $\left(\frac{G'}{G}\right)$ - expansion method.

Roozi and Mahmeiani [29] studied the (2+1)-dimensional Kadomtsev –Petviashvili equation to construct exact solutions via the same method. Hayek [30] expands the method called extended  $\left(\frac{G'}{G}\right)$ -expansion method to construct exact solutions of the KdV Burgers equation with power-law nonlinearity while Guo and Zhou [30] enlarge the method to obtain analytical solutions for some NLEEs and so on. Hamad *et al.* [31] implemented the same method to the higher dimensional potential YTSI equation. Nofel *et al.* [32] used improved

$\left(\frac{G'}{G}\right)$ -expansion method for finding traveling wave solutions of the fifth-order KdV equation and so on.

Recently, Chen has presented extended multiple  $\left(\frac{G'}{G}\right)$ -expansion method for analytic solutions of nonlinear evolution equation in mathematical physics. Naher *et al.* [25] have used improved  $\left(\frac{G'}{G}\right)$ -expansion method for finding solutions of combined KdV-MKdV equation.

### 1.7.1 Methodology of $\left(\frac{G'}{G}\right)$ -Expansion Method

In this section we describe the  $\left(\frac{G'}{G}\right)$ -Expansion Method for finding the solution of non linear evolution equation. Suppose that a nonlinear equation say in two independent variable  $x$  and  $t$  is given by

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \dots) = 0, \quad (18)$$

where  $u = u(x, t)$  is an unknown function,  $P$  is a polynomial and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. In following we gives the main steps of  $\left(\frac{G'}{G}\right)$ -Expansion Method.

**Step 1:** Suppose that  $u = u(x, t)$ . The traveling wave variable allows us reducing to an ODE for  $u = u(\xi)$

$$P(u, u', u'', \dots) = 0, \quad (19)$$

where prime denotes the derivative with respect to  $\xi$ .

**Step 2:** Suppose the solution of equation can be expressed by a polynomial in  $\left(\frac{G'}{G}\right)$  as follows.

$$u(\xi) = \sum_{i=0}^N c_i \left(\frac{G'(\xi)}{G(\xi)}\right) \quad (20)$$

where  $c_i$  are real constants with  $c_i \neq 0$  to be determined,  $N$  is a positive integer to be determined. The function  $G(\xi)$  is the solution of the auxiliary linear ordinary differential equation.

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (21)$$

where  $\lambda$  and  $\mu$  are real constants to be determined.

**Step 3:** Substituting (20) into (19) and using second order LODE (21). Separate all terms with same order of  $\left(\frac{G'}{G}\right)$  together, the left hand side of (19) is converted into another polynomial in

$\left(\frac{G'}{G}\right)$ . Equating each coefficient of polynomial to zero. Then we get algebraic equation for  $c_i, \dots, \lambda$  and  $\mu$ .

**Step 4:** Since the following general solution of equation (21) has been well known for us, then substituting  $c_i, c$  and general solution of equation (21) into (20). We have more traveling wave solution of non linear partial differential equation (18).

## 1.8 Organization of the Thesis

In this thesis, an attempt has been made to solve some partial differential equations by using  $\frac{G'}{G}$ -expansion method. The chapter wise summary of the thesis is as follows.

In chapter 2, we considered one dimensional Ramani equation

$$u_{xxxxx} + 15u_x u_{xxx} + 15u_{xx} u_{xxx} + 45u_x^2 u_{xx} - 5u_{xxx} - 15u_x u_{xt} - 15u_t u_{xx} - 5u_{tt} = 0$$

This equation is completely integrable by the inverse scattering method. Hirota-Ramani equation is widely used in various branches of physics, and such as plasma physics, fluid physics, and quantum field theory. It also describes a variety of wave phenomena in plasma and solid state.

In chapter 3, we considered one dimensional Sawada-Kotera equation

$$u_{xxxx} + u_t + 45u_x u^2 + 15u_x u_{xx} + uu_{xxx} = 0$$

The equation Sawada-Kotera was first proposed by Sawada and Kotera and the equation has too many applications in various branches of physics, and such as plasma physics, fluid physics.

In this chapter,  $\frac{G'}{G}$ -expansion method is proposed for an exact solution of Sawada-Kotera equation. The results show that, the proposed method is more powerful, effective and accurate.

## Chapter 2

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### Traveling Wave Solutions of Ramani Equation

#### 2.1 Introduction:

In this chapter, we have applied the  $\left(\frac{G'}{G}\right)$ -expansion method to find the traveling wave solutions of the Ramani equation. The Ramani equation was first introduced by Hirota and Ramani in [33]. Ji obtained some travelling soliton solutions of this equation by using Exp-function method [34]. This equation is completely integrable by the inverse scattering method. Equation is studied in [33,35,36] where new kind of solutions were obtained. Hirota-Ramani equation is widely used in various branches of physics, and such as plasma physics, fluid physics, and quantum field theory. It also describes a variety of wave phenomena in plasma and solid state [33]. The Ramani equation is

$$u_{xxxxx} + 15u_x u_{xxxx} + 15u_{xx} u_{xxx} + 45u_x^2 u_{xx} - 5u_{xxx} - 15u_x u_{xt} - 15u_t u_{xx} - 5u_{tt} = 0 \quad (22)$$

#### 2.2 $\left(\frac{G'}{G}\right)$ -Expansion Method for Ramani Equation

In order to obtain the travelling wave solutions of eq. (22), we suppose that

$$u(x, t) = u(\xi), \quad \xi = x - ct \quad (23)$$

$c$  is a constant that to be determined later. By using the wave variable (23), Eq. (22) is converted into an ODE

$$u'''''' + 15u' u'''' + 15u'' u''' + 45(u')^2 u'' + 5cu'''' + 15cu' u'' + 15cu'' u' - 5c^2 u' = 0 \quad (24)$$

Balancing  $u''u''', u'u^{(4)}$  with  $u^{(6)}$  then gives  $m = 1$ . Therefore, we can write the solution of equation (24) in the form

$$u(\xi) = a_1 \left( \frac{G'}{G} \right)^1 + a_0 \quad (25)$$

where  $a_1 \neq 0$  and  $G = G(\xi)$ , The function  $G(\xi)$  is the solution of the auxiliary linear ordinary differential equation.

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \quad (26)$$

$$G''(\xi) = -\lambda G'(\xi) - \mu G(\xi)$$

Differentiate equation (24), we get

$$u'(\xi) = a_1 \left( \frac{G(G'') - (G')^2}{G^2} \right)$$

$$u'(\xi) = a_1 \left( \frac{G(-\mu G - \lambda G') - (G')^2}{G^2} \right)$$

$$u'(\xi) = -a_1 \left( \frac{G'}{G} \right)^2 - a_1 \lambda \left( \frac{G'}{G} \right) - a_1 \mu \quad (27)$$

$$u''(\xi) = -2a_1 \left( \frac{G(-\mu G - \lambda G') - (G')^2}{G^2} \right) \left( \frac{G'}{G} \right) - a_1 \lambda \left( \frac{G(-\mu G - \lambda G') - (G')^2}{G^2} \right)$$

$$u''(\xi) = 2a_1 \mu \left( \frac{G'}{G} \right) + 2a_1 \lambda \left( \frac{G'}{G} \right)^2 + 2a_1 \left( \frac{G'}{G} \right)^3 + a_1 \mu \lambda + a_1 \lambda^2 \left( \frac{G'}{G} \right) + a_1 \lambda \left( \frac{G'}{G} \right)^2$$

$$u''(\xi) = 2a_1 \left( \frac{G'}{G} \right)^3 + 3a_1 \lambda \left( \frac{G'}{G} \right)^2 + (2a_1 \mu + a_1 \lambda^2) \left( \frac{G'}{G} \right) + a_1 \mu \lambda \quad (28)$$

$$u''''(\xi) = (2a_1)(3)\left(\frac{G'}{G}\right)^2\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) + (3a_1\lambda)(2)\left(\frac{G'}{G}\right)$$

$$\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) + (2a_1\mu + a_1\lambda^2)\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right)$$

$$u'''' = -6a_1\left(\frac{G'}{G}\right)^4 - 12a_1\lambda\left(\frac{G'}{G}\right)^3 - (8a_1\mu + 7a_1\lambda^2)\left(\frac{G'}{G}\right)^2 - (8a_1\lambda\mu + a_1\lambda^3)\left(\frac{G'}{G}\right) \quad (29)$$

$$- (2a_1\mu^2 + a_1\lambda^2\mu)$$

$$u''''''(\xi) = -24a_1\left(\frac{G'}{G}\right)^3\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) - 36a_1\lambda\left(\frac{G'}{G}\right)^2\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right)$$

$$- (14a_1\lambda^2 + 16a_1\mu)\left(\frac{G'}{G}\right)\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) - (8a_1\lambda\mu + a_1\lambda^3)\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right)$$

$$u''''''(\xi) = 24a_1\left(\frac{G'}{G}\right)^5 + (60a_1\lambda)\left(\frac{G'}{G}\right)^4 + (40a_1\mu + 50a_1\lambda^2)\left(\frac{G'}{G}\right)^3 + (36a_1\lambda + 24a_1\lambda\mu + 15a_1\lambda^3)\left(\frac{G'}{G}\right)^2$$

$$+ (16a_1\mu^2 + 22a_1\lambda^2\mu + a_1\lambda^4)\left(\frac{G'}{G}\right) + 8a_1\lambda\mu^2 + a_1\mu\lambda^3 \quad (30)$$

$$u''''''''(\xi) = 120a_1\left(\frac{G'}{G}\right)^4\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) + 240a_1\lambda\left(\frac{G'}{G}\right)^3\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right)$$

$$+ (120a_1\mu + 150a_1\lambda^2)\left(\frac{G'}{G}\right)^2\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) - (48a_1\lambda\mu + 72a_1\lambda + 30a_1\lambda^3)\left(\frac{G'}{G}\right)$$

$$\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) + (a_1\lambda^4 + 22a_1\lambda^2\mu + 16a_1\mu^2)$$

$$\begin{aligned}
u''''''(\xi) = & -120a_1\left(\frac{G'}{G}\right)^6 - 360a_1\lambda\left(\frac{G'}{G}\right)^5 - (240a_1\mu + 390a_1\lambda^2)\left(\frac{G'}{G}\right)^4 - (72a_1\lambda + 408a_1\lambda\mu \\
& + 180a_1\lambda^3)\left(\frac{G'}{G}\right)^3 - (136a_1\mu^2 + 220a_1\lambda^2\mu + 72a_1\lambda^2 + 31a_1\lambda^4)\left(\frac{G'}{G}\right)^2 - (72a_1\lambda\mu + a_1\lambda^5 + 52a_1\lambda^3\mu \\
& + 64a_1\lambda\mu^2)\left(\frac{G'}{G}\right) - (22a_1\lambda^2\mu^2 + a_1\lambda^4\mu + 16a_1\mu^3)
\end{aligned} \tag{31}$$

$$\begin{aligned}
u''''''(\xi) = & 720a_1\left(\frac{G'}{G}\right)^5\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) - 1800a_1\lambda\left(\frac{G'}{G}\right)^4\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) \\
& - (960a_1\mu + 1560a_1\lambda^2)\left(\frac{G'}{G}\right)^3\left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) - (1224a_1\lambda\mu + 216a_1\lambda + 540a_1\lambda^3)\left(\frac{G'}{G}\right)^2 \\
& \left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) - (62a_1\lambda^4 + 440a_1\lambda^2\mu + 272a_1\mu^2 + 144a_1\lambda^2)\left(\frac{G'}{G}\right)\left(-\mu - \lambda\left(\frac{G'}{G}\right) - \left(\frac{G'}{G}\right)^2\right) \\
& - (72a_1\lambda\mu + a_1\lambda^5 + 52a_1\lambda^3\mu + 64a_1\lambda\mu^2)\left(-\mu - \lambda\left(\frac{G'}{G}\right) - \left(\frac{G'}{G}\right)^2\right)
\end{aligned}$$

$$\begin{aligned}
u''''''(\xi) = & 720a_1\left(\frac{G'}{G}\right)^7 + 2520a_1\lambda\left(\frac{G'}{G}\right)^6 + (1680a_1\mu + 2190a_1\lambda^2)\left(\frac{G'}{G}\right)^5 + (216a_1\lambda + 3984a_1\lambda\mu \\
& + 930a_1\lambda^3)\left(\frac{G'}{G}\right)^4 + (1232a_1\mu^2 + 2054a_1\lambda^2\mu + 360a_1\lambda^2 + 602a_1\lambda^4)\left(\frac{G'}{G}\right)^3 - (288a_1\lambda\mu + 63a_1\lambda^5 \\
& + 1132a_1\lambda^3\mu + 1560a_1\lambda\mu^2 + 144a_1\lambda^3)\left(\frac{G'}{G}\right)^2 + (272a_1\mu^3 + 504a_1\lambda^2\mu^2 + 104a_1\lambda^4\mu + 216a_1\lambda^2\mu + a_1\lambda^6) \\
& \left(\frac{G'}{G}\right) + (72a_1\lambda\mu^2 + a_1\lambda^5\mu + 64a_1\lambda\mu^3 + 52a_1\lambda^3\mu^2)
\end{aligned} \tag{32}$$

Substituting equations (27)-(32) into equation (24), setting the coefficients of  $\left(\frac{G'}{G}\right)^i$ ;

$i = 0, 1, \dots, 8$  to zero, we obtain a system of algebraic equations for  $a_0, a_1, a_2, c, \lambda$  and  $\mu$  as follows:

$$\left(\frac{G'}{G}\right)^0 : -5a_1c^2\lambda\mu + 5a_1c\lambda^3\mu + a_1\lambda^5\mu + 40a_1c\lambda\mu^2 - 30a_1^2c\lambda\mu^2 + 52a_1\lambda^3\mu^2 - 30a_1^2\lambda^3\mu^2 + 136a_1\lambda\mu^3 - 150a_1^2\lambda\mu^3 + 45a_1^3\lambda\mu^3 = 0,$$

$$\left(\frac{G'}{G}\right)^1 : -5a_1c^2\lambda^2 + 5a_1c\lambda^4 + a_1\lambda^6 - 10a_1c^2\mu + 110a_1c\lambda^2\mu - 60a_1^2c\lambda^2\mu + 114a_1\lambda^4\mu - 60a_1^2\lambda^4\mu + 80a_1c\mu^2 - 60a_1^2c\mu^2 - 720a_1\lambda^2\mu^2 - 630a_1^2\lambda^2\mu^2 + 135a_1^3\lambda^2\mu^2 + 272a_1\mu^3 - 300a_1^2\mu^3 + 90a_1^3\mu^3 = 0,$$

$$\left(\frac{G'}{G}\right)^2 : -15a_1c^2\lambda + 75a_1c\lambda^3 - 30a_1^2c\lambda^3 + 63a_1\lambda^5 - 30a_1^2\lambda^5 + 300a_1c\lambda\mu - 180a_1^2c\lambda\mu + 1176a_1\lambda^3\mu - 870a_1^2\lambda^3\mu + 135a_1^3\lambda^3\mu + 1848a_1\lambda\mu^2 - 1710a_1^2\lambda\mu^2 + 405a_1^3\lambda\mu^2 = 0,$$

$$\left(\frac{G'}{G}\right)^3 : -10a_1c^2 + 250a_1c\lambda^2 - 120a_1^2c\lambda^2 + 602a_1\lambda^4 - 390a_1^2k\lambda^4 + 45a_1^3\lambda^4 + 200a_1c\mu - 120a_1^2c\mu - 3584a_1\lambda^2\mu - 2880a_1^2\lambda^2\mu + 540a_1^3\lambda^2\mu + 1232a_1\mu^2 - 1140a_1^2\mu^2 + 270a_1^3\mu^2 = 0,$$

$$\left(\frac{G'}{G}\right)^4 : 300a_1c\lambda - 150a_1^2c\lambda + 2100a_1\lambda^3 - 1500a_1^2\lambda^3 + 225a_1^3\lambda^3 + 4200a_1\lambda\mu - 3450a_1^2\lambda\mu + 675a_1^3\lambda\mu = 0,$$

$$\left(\frac{G'}{G}\right)^5 : 120a_1c - 60a_1^2c + 3360a_1\lambda^2 - 2490a_1^2\lambda^2 + 405a_1^3\lambda^2 + 1680a_1\mu - 1380a_1^2\mu + 270a_1^3\mu = 0,$$

$$\left(\frac{G'}{G}\right)^6 : 2520a_1\lambda - 1890a_1^2\lambda + 315a_1^3\lambda = 0,$$

$$\left(\frac{G'}{G}\right)^7 : 720a_1 - 540a_1^2 + 90a_1^3 = 0.$$

Solving these systems of algebraic equations by Maple, we have

**Case 1:**

$$a_0 = a_0, a_1 = 2, c = \frac{5\lambda^2 - 20\mu + 3\sqrt{5}\sqrt{\lambda^4 - 8\lambda^2\mu + 16\mu^2}}{10}, \quad (33)$$

Substituting (33) into (25), we have three types of traveling wave solutions of (22) which are given as follows

(i) When  $\lambda^2 - 4\mu > 0$ , we obtain the hyperbolic function traveling wave solutions

$$u_1(\xi) = 2 \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \frac{\left( C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) \right)}{\left( C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) \right)} \right) + a_0 - \frac{\lambda}{2}$$

where

$$\xi = x - \left( \frac{5\lambda^2 - 20\mu + 3\sqrt{5}\sqrt{\lambda^4 - 8\lambda^2\mu + 16\mu^2}}{10} \right) t,$$

$C_1$  and  $C_2$  are arbitrary constants.

(ii) When  $\lambda^2 - 4\mu < 0$ , we obtain the trigonometric function traveling wave solutions

$$u_2(\xi) = 2 \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} \right) \right) + a_0 - \frac{\lambda}{2}$$

where

$$\xi = x - \left( \frac{5\lambda^2 - 20\mu + 3\sqrt{5}\sqrt{\lambda^4 - 8\lambda^2\mu + 16\mu^2}}{10} \right) t,$$

$C_1$  and  $C_2$  are arbitrary constants.

(iii) When  $\lambda^2 - 4\mu = 0$ , we obtain the rational function solutions

$$u_3(\xi) = 2 \left( \frac{C_2}{C_1 + C_2 x} \right) + a_0 - \lambda,$$

$C_1$  and  $C_2$  are arbitrary constants.

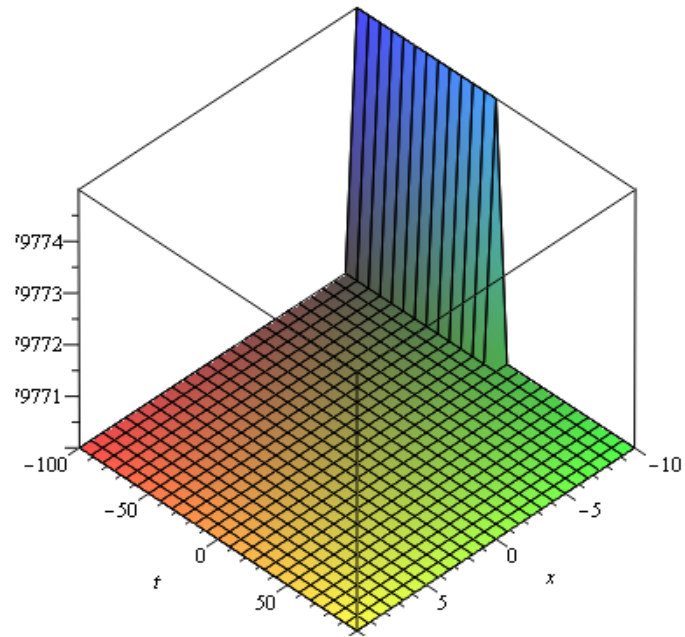


Figure 1: Graphical representation of hyperbolic function solutions for case 1 when  $a_0 = 1, a_1 = 2, c = 5.85, \lambda = 3, \mu = 1, C_1 = 1, C_2 = 1$ .

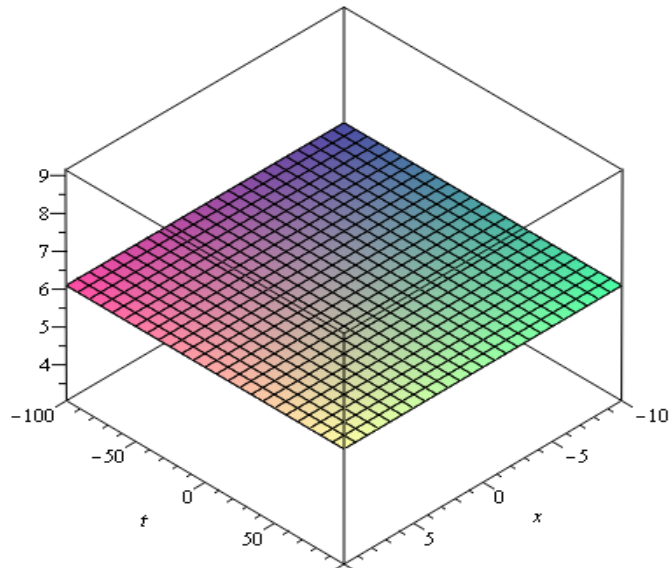


Figure 2: Graphical representation of trigonometric function solutions for case 1 when  $a_0 = 2, a_1 = 2, c = 0.512, \lambda = 1, \mu = 1, C_1 = 1, C_2 = 1$ .

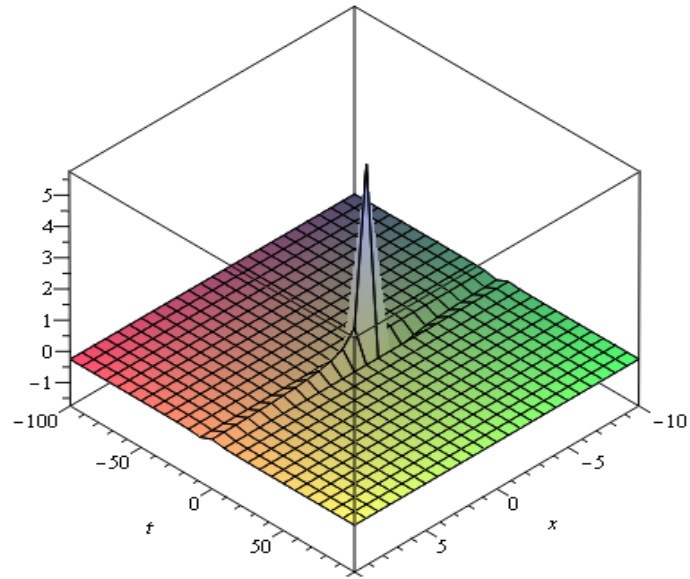


Figure 3: Graphical representation of rational function solutions for case 1 when

$$a_0 = 1, a_1 = 3, c = -3.15, \lambda = \frac{1}{2}, \mu = \frac{1}{16}, C_1 = 1, C_2 = 1.$$

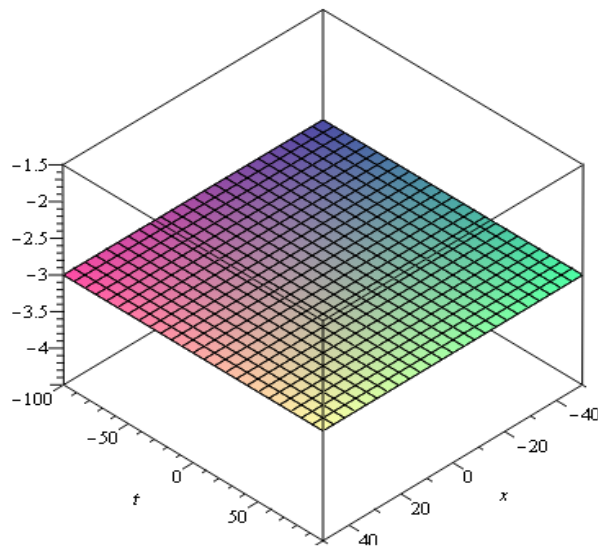


Figure 4: Graphical representation of hyperbolic function solutions for case 1 when

$$a_0 = -1, a_1 = 2, \lambda = 4, \mu = 3, c = 4.683, C_1 = 1, C_2 = 1.$$

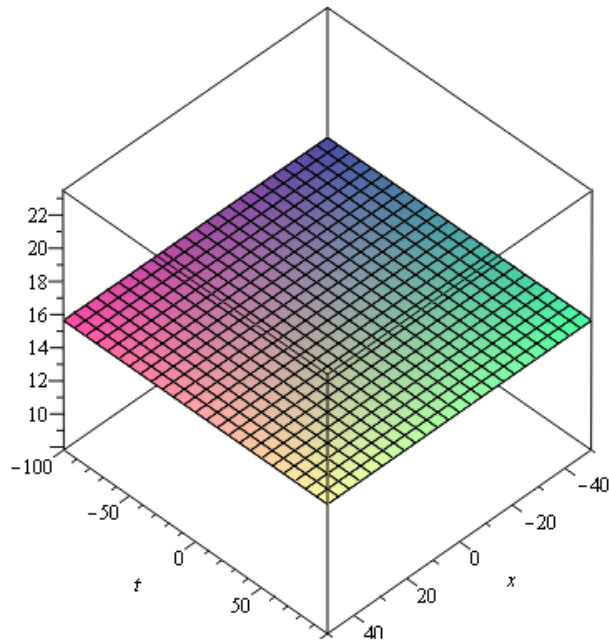


Figure 5: Graphical representation of trigonometric function solutions for case 1 when  $a_0 = 10, a_1 = 2, c = 1.195, \lambda = 3, \mu = 4, C_1 = 1, C_2 = 1$ .

## Chapter 3

### Traveling Wave Solutions for the Fifth-Order Sawada-Kotera Equation

#### 3.1 Introduction

A generalized  $\left(\frac{G'}{G}\right)$ -expansion method is used to seek more general exact solutions of the fifth-order Sawada-Kotera equation

$$u_{xxxxx} + u_t + 45u_x u^2 + 15u_x u_{xx} + uu_{xxx} = 0 \quad (34)$$

The traveling wave solutions of this equation occur in three functions including hyperbolic function solutions, trigonometric function solutions and rational solutions. The method appears to be easier and faster by means of some mathematical software.

#### 3.2 $\left(\frac{G'}{G}\right)$ -Expansion Method for Sawada-Kotera Equation

In order to obtain the travelling wave solutions of Eq. (34), we suppose that

$$u(x, t) = u(\xi), \quad \xi = x - ct \quad (35)$$

$c$  is a constant that to be determined later.

By using the wave variable (35), Eq. (34) is converted into an ODE

$$u'''''' - cu' + 45u' u^2 + 15u' u'' + 15uu''' = 0 \quad (36)$$

Integrating (36) with respect to  $\xi$  once, we obtain

$$u'''' - cu + 45uu^2 + 15u'' + 15uu' = g \quad (37)$$

where ,  $g$  is the integration constant that can be determined later.

Suppose that the solution of the ODE (37) can be expressed by a polynomial in  $\left(\frac{G'}{G}\right)$  as

follows:

$$u(\xi) = \sum_{i=0}^m a_i \left(\frac{G'}{G}\right)^i \quad (38)$$

where  $a_i$  are constants,  $G = G(\xi)$  satisfies the second order LODE in the form:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \quad (39)$$

where  $\lambda$  and  $\mu$  are constants.

Balancing the order of  $u''$  and  $u''''$  in Eq.(37), we get

that  $3m = m + 4 \Rightarrow m = 2$ , so Eq.(38) can be rewritten as

$$u(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0 \quad (40)$$

$a_0, a_1, a_2$  are constants to be determined later. Then we can obtain

$$u'(\xi) = a_1 \left(\frac{G(G'') - (G')^2}{G^2}\right) + 2a_2 \left(\frac{G(G'') - (G')^2}{G^2}\right) \left(\frac{G'}{G}\right)$$

$$u'(\xi) = a_1 \left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) + 2a_2 \left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) \left(\frac{G'}{G}\right)$$

$$u'(\xi) = -2a_2 \left(\frac{G'}{G}\right)^3 - (a_1 + 2a_2\lambda) \left(\frac{G'}{G}\right)^2 - (a_1\lambda + 2a_2\mu) \left(\frac{G'}{G}\right) - a_1\mu$$

Differentiate again and again, we get

$$u''(\xi) = -6a_2 \left(\frac{G'}{G}\right)^2 \left(\frac{GG'' - (G')^2}{G^2}\right) - 2(a_1 + 2a_2\lambda) \left(\frac{GG'' - (G')^2}{G^2}\right) \left(\frac{G'}{G}\right) - (a_1\lambda + 2a_2\mu) \left(\frac{GG'' - (G')^2}{G^2}\right)$$

$$u''(\xi) = -6a_2 \left(\frac{G'}{G}\right)^2 \left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) - 2(a_1 + 2a_2\lambda) \left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) \left(\frac{G'}{G}\right) - (a_1\lambda + 2a_2\mu) \left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right)$$

$$u''(\xi) = 6a_2\mu \left(\frac{G'}{G}\right)^2 + 6a_2\lambda \left(\frac{G'}{G}\right)^3 + 6a_2 \left(\frac{G'}{G}\right)^4 + 2a_1\mu \left(\frac{G'}{G}\right) + 2a_1\lambda \left(\frac{G'}{G}\right)^2 + 2a_1 \left(\frac{G'}{G}\right)^3 + 4a_2\lambda\mu \left(\frac{G'}{G}\right) + 4a_2\lambda^2 \left(\frac{G'}{G}\right)^2 + 4a_2\lambda \left(\frac{G'}{G}\right)^3 + a_1\mu\lambda + a_1\lambda^2 \left(\frac{G'}{G}\right) + a_1\lambda \left(\frac{G'}{G}\right)^2 + 2a_2\mu^2 + 2a_2\mu\lambda \left(\frac{G'}{G}\right) + 2a_2\mu \left(\frac{G'}{G}\right)^2$$

$$u''(\xi) = 6a_2 \left(\frac{G'}{G}\right)^4 + (10a_2\lambda + 2a_1) \left(\frac{G'}{G}\right)^3 + (4a_2\lambda^2 + 8a_2\mu + 3a_1\lambda) \left(\frac{G'}{G}\right)^2 + (2a_1\mu + a_1\lambda^2 + 6a_2\lambda\mu) \left(\frac{G'}{G}\right) + a_1\mu\lambda + 2a_2\mu^2$$

$$u'''(\xi) = 6a_2(4) \left(\frac{G'}{G}\right)^3 \left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) + (10a_2\lambda + 2a_1)(3) \left(\frac{G'}{G}\right)^2 \left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) + (4a_2\lambda^2 + 8a_2\mu + 3a_1\lambda)(2) \left(\frac{G'}{G}\right) \left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right) + (2a_1\mu + a_1\lambda^2 + 6a_2\lambda\mu) \left(\frac{G(-\mu G - \lambda G') - (G')^2}{G^2}\right)$$

$$\begin{aligned}
u'''' &= -24a_2 \left( \frac{G'}{G} \right)^5 - (24a_2\lambda + 30a_2\lambda + 6a_1) \left( \frac{G'}{G} \right)^4 - (24a_2\mu + 30a_2\lambda^2 + 6a_1\lambda + 8a_2\lambda^2 \\
&+ 16a_2\mu + 6a_1\lambda) \left( \frac{G'}{G} \right)^3 - (30a_2\mu\lambda + 6a_1\mu + 8a_2\lambda^3 + 16a_2\lambda\mu + 6a_1\lambda^2 + 2a_1\mu + a_1\lambda^2 \\
&+ 6a_2\lambda\mu) \left( \frac{G'}{G} \right)^2 - (8a_2\mu\lambda^2 + 16a_2\mu^2 + 6a_1\lambda\mu + 2a_1\mu\lambda + a_1\lambda^3 + 6a_2\lambda^2\mu) \left( \frac{G'}{G} \right) \\
&- 6a_2\lambda\mu^2 - 2a_1\mu^2 - a_1\lambda^2\mu
\end{aligned}$$

$$\begin{aligned}
u'''' &= -24a_2 \left( \frac{G'}{G} \right)^5 - (6a_1 + 54a_2\lambda) \left( \frac{G'}{G} \right)^4 - (40a_2\mu + 38a_2\lambda^2 + 12a_1\lambda) \left( \frac{G'}{G} \right)^3 - (52a_2\lambda\mu + 8a_2\lambda^3 \\
&+ 7a_1\lambda^2 + 8a_1\mu) \left( \frac{G'}{G} \right)^2 - (8a_1\lambda\mu + 14a_2\lambda^2\mu + 16a_2\mu^2 + a_1\lambda^3) \left( \frac{G'}{G} \right) - (6a_2\lambda\mu^2 + 2a_1\mu^2 + a_1\lambda^2\mu)
\end{aligned}$$

$$\begin{aligned}
u''''(\xi) &= -120a_2 \left( \frac{G'}{G} \right)^4 \left( \frac{GG'' - (G')^2}{G^2} \right) - (216a_2\lambda + 24a_1) \left( \frac{G'}{G} \right)^3 \left( \frac{GG'' - (G')^2}{G^2} \right) - (114a_2\lambda^2 \\
&+ 120a_2\mu + 36a_1\lambda) \left( \frac{G'}{G} \right)^2 \left( \frac{GG'' - (G')^2}{G^2} \right) - (16a_1\mu + 14a_1\lambda^2 + 16a_2\lambda^3 + 104a_2\lambda\mu) \\
&\left( \frac{G'}{G} \right) \left( \frac{GG'' - (G')^2}{G^2} \right) - (14a_2\lambda^2\mu + 8a_1\lambda\mu + 16a_2\mu^2 + a_1\lambda^3) \left( \frac{GG'' - (G')^2}{G^2} \right)
\end{aligned}$$

$$\begin{aligned}
u''''(\xi) &= 120a_2 \left( \frac{G'}{G} \right)^6 + (336a_2\lambda + 24a_1) \left( \frac{G'}{G} \right)^5 + (330a_2\lambda^2 + 240a_2\mu + 60a_1\lambda) \left( \frac{G'}{G} \right)^4 + (440a_2\lambda\mu + \\
&40a_1\mu + 130a_2\lambda^3 + 50a_1\lambda^2) \left( \frac{G'}{G} \right)^3 + (136a_2\mu^2 + 60a_1\lambda\mu + 232a_2\mu\lambda^2 + 16a_2\lambda^4 + 15a_1\lambda^3) \left( \frac{G'}{G} \right)^2 + \\
&(16a_1\mu^2 + 30a_2\lambda^3\mu + 22a_1\lambda^2\mu + 120a_2\lambda\mu^2 + a_1\lambda^4) \left( \frac{G'}{G} \right) + 14a_2\lambda^2\mu^2 + 8a_1\lambda\mu^2 + 16a_2\mu^3 + a_1\mu\lambda^3
\end{aligned}$$

On substituting Eq.(3.7) into the ODE (3.4) and collecting all the terms with the same power of  $\left(\frac{G'}{G}\right)$  together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$\left(\frac{G'}{G}\right)^0 : 16a_2\mu^3 + 15a_0a_1\lambda\mu + 14a_2\lambda^2\mu^2 + 8a_1\lambda\mu^2 - ca_0 + 15a_0^3 + a_1\lambda^3\mu - g + 30a_0a_2\mu^2 = 0$$

$$\left(\frac{G'}{G}\right)^1 : 30a_2\lambda^3\mu + 30a_0a_1\mu + 30a_1a_2\mu^2 + 45a_0^2a_1 - 90a_0a_2\lambda\mu + 15a_0a_1\lambda^2 + 22a_1\lambda^2\mu - ca_1 + 16a_1\mu^2 + a_1\lambda^4 + 15a_1^2\lambda\mu + 120a_2\lambda\mu^2 = 0$$

$$\left(\frac{G'}{G}\right)^2 : 120a_0a_2\mu + 15a_1^2\lambda^2 + 16a_2\lambda^4 + 30a_1^2\mu + 45a_0^2a_2 + 136a_2\mu^2 + 105a_1a_2\lambda\mu + 232a_2\lambda^2\mu + 15a_1\lambda^3 + 60a_1\lambda\mu + 60a_0a_2\lambda^2 + 45a_0a_1^2 + 45a_0a_1\lambda - ca_2 + 30a_2^2\mu^2 = 0$$

$$\left(\frac{G'}{G}\right)^3 : 75a_1a_2\lambda^2 + 130a_2\lambda^3 + 90a_2^2\lambda\mu + 90a_0a_1a_2 + 30a_0a_1 + 150a_1a_2\mu + 15a_1^3 + 440a_2\lambda\mu + 40a_1\mu + 45a_1^2\lambda + 50a_1\lambda^2 + 150a_0a_2\lambda = 0$$

$$\left(\frac{G'}{G}\right)^4 : 330a_2\lambda^2 + 30a_1^2 + 90a_0a_2 + 120a_2^2\mu + 60a_2^2\lambda^2 + 60a_1\lambda + 45a_1^2a_2 + 195a_1a_2\lambda + 240a_2\mu + 45a_0a_2^2\lambda = 0$$

$$\left(\frac{G'}{G}\right)^5 : 45a_1a_2^2 + 150a_2^2\lambda + 336a_2\lambda + 24a_1 + 120a_1a_2 = 0$$

$$\left(\frac{G'}{G}\right)^6 : 15a_2^3 + 90a_2^2 + 120a_2 = 0$$

Solving the algebraic equations above, we can get the results into two cases:

**Case 1:**

$$c = 120a_0\mu + \frac{1}{16}a_1^4 + \frac{11}{2}a_1^2\mu + 45a_0^2 + 76\mu^2 + \frac{15}{4}a_0a_1^2, g = -32\mu^3 - 13a_0a_1^2\mu - 11a_1^2\mu^2$$

$$-120a_0^2\mu - \frac{1}{16}a_0a_1^4 - 30a_0^3 - 136a_0\mu^2 - \frac{15}{4}a_0^2a_1^2 - \frac{1}{8}a_1^4\mu, \mu = \mu, \lambda = -\frac{1}{2}a_1,$$

$$a_0 = a_0, a_1 = a_1, a_2 = -2$$

(41)

where  $a_0, \lambda, \mu$  are arbitrary constants.

Substituting (41) into (40), we get that

$$u(\xi) = -2\left(\frac{G'}{G}\right)^2 - 2\lambda\left(\frac{G'}{G}\right) + a_0 \quad (42)$$

$$\xi = x + (120a_0\mu + 45a_0^2 + 22\lambda^2\mu + 15a_0\lambda^2 + \lambda^4 + 76\mu^2)t$$

where  $a_0, \lambda, \mu$  are arbitrary constants.

Substituting the general solutions of Eq.(39) into (42), we can obtain three types of traveling wave solutions of the fifth-order Sawada-Kotera equation (34) as follows:

(i) When  $\lambda^2 - 4\mu > 0$ , we obtain the hyperbolic function traveling wave solutions

$$u_1(\xi) = \frac{1}{2}\lambda^2 - \frac{1}{2}(\lambda^2 - 4\mu) \cdot \left( \frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi} \right)^2 + a_0,$$

where

$$\xi = x + (120a_0\mu + 45a_0^2 + 22\lambda^2\mu + 15a_0\lambda^2 + \lambda^4 + 76\mu^2)t.$$

$a_0, C_1, C_2$  are arbitrary constants.

(ii) When  $\lambda^2 - 4\mu < 0$ , we obtain the trigonometric function traveling wave solutions

$$u_2(\xi) = \frac{1}{2}\lambda^2 - \frac{1}{2}(4\mu - \lambda^2) \cdot \left( \frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi} \right)^2 + a_0,$$

where

$$\xi = x + (120a_0\mu + 45a_0^2 + 22\lambda^2\mu + 15a_0\lambda^2 + \lambda^4 + 76\mu^2)t.$$

$a_0, C_1, C_2$  are arbitrary constants.

(iii) When  $\lambda^2 - 4\mu = 0$ , we obtain the rational function solutions

$$u_3(\xi) = \frac{1}{2}\lambda^2 - \frac{2C_2^2}{(C_1 + C_2\xi)^2} + a_0,$$

where

$$\xi = x + (120a_0\mu + 45a_0^2 + 22\lambda^2\mu + 15a_0\lambda^2 + \lambda^4 + 76\mu^2)t.$$

$a_0, C_1, C_2$  are arbitrary constants.

## Case 2:

$$c = \frac{9}{32}a_0a_1^2 + \frac{9}{4}a_0^2 + \frac{9}{1024}a_1^4, g = -\frac{3}{16384}a_1^6 - \frac{9}{1024}a_0a_1^4 - \frac{9}{64}a_0^2a_1^2 - \frac{3}{4}a_0^3, \mu = -\frac{1}{128}a_1^2 - \frac{3}{8}a_0,$$

$$\lambda = -\frac{1}{4}a_1, a_0 = a_0, a_1 = a_1, a_2 = -4$$

(43)

where  $a_0, \lambda, \mu$  are arbitrary constants.

Substituting (43) into (40), we get that

$$\begin{aligned}
u(\xi) &= -4\left(\frac{G'}{G}\right)^2 - 4\lambda\left(\frac{G'}{G}\right) - \frac{1}{3}(\lambda^2 + 8\mu) \\
\xi &= x + (-8\lambda^2\mu + 16\mu^2 + \lambda^4)t
\end{aligned} \tag{44}$$

where  $\lambda, \mu$  are arbitrary constants.

Substituting the general solutions of eq. (39) into (44), we can obtain three types of traveling wave solutions of the fifth-order Sawada-Kotera equation (34) as follows:

(i) When  $\lambda^2 - 4\mu > 0$ , we obtain the hyperbolic function traveling wave solutions

$$u_1(\xi) = \lambda^2 - (\lambda^2 - 4\mu) \cdot \left( \frac{C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right)^2 - \frac{1}{3}(\lambda^2 + 8\mu),$$

where

$$\xi = x + (-8\lambda^2\mu + 16\mu^2 + \lambda^4)t$$

$C_1, C_2$  are arbitrary constants.

(ii) When  $\lambda^2 - 4\mu < 0$ , we obtain the trigonometric function traveling wave solutions

$$u_2(\xi) = \lambda^2 - (4\mu - \lambda^2) \cdot \left( \frac{-C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{C_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right)^2 - \frac{1}{3}(\lambda^2 + 8\mu),$$

where

$$\xi = x + (-8\lambda^2\mu + 16\mu^2 + \lambda^4)t$$

$C_1, C_2$  are arbitrary constants.

(iii) When  $\lambda^2 - 4\mu = 0$ , we obtain the rational function solutions

$$u_3(\xi) = \lambda^2 - \frac{4C_2^2}{(C_1 + C_2\xi)^2} - \frac{1}{3}(\lambda^2 + 8\mu),$$

where

$$\xi = x + (-8\lambda^2\mu + 16\mu^2 + \lambda^4)t$$

$C_1, C_2$  are arbitrary constants.

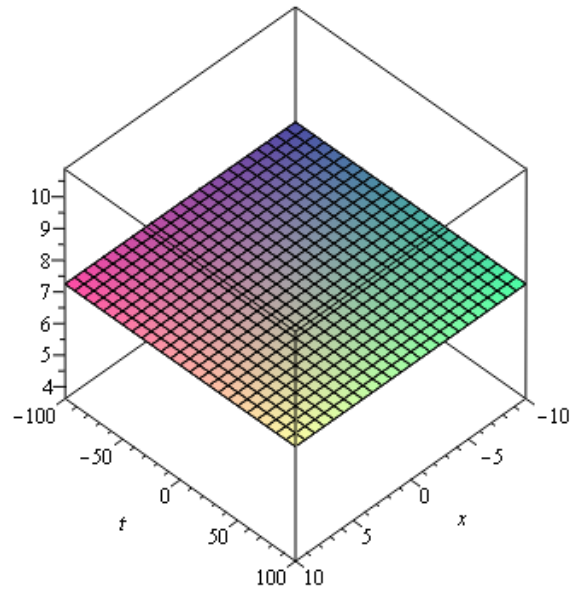


Figure 1: Graphical representation of hyperbolic function solutions for case 1 when  $a_0 = 1, a_1 = -6, a_2 = -2, \lambda = 3, \mu = 2, c = 1201, C_1 = 1, C_2 = 1$ .

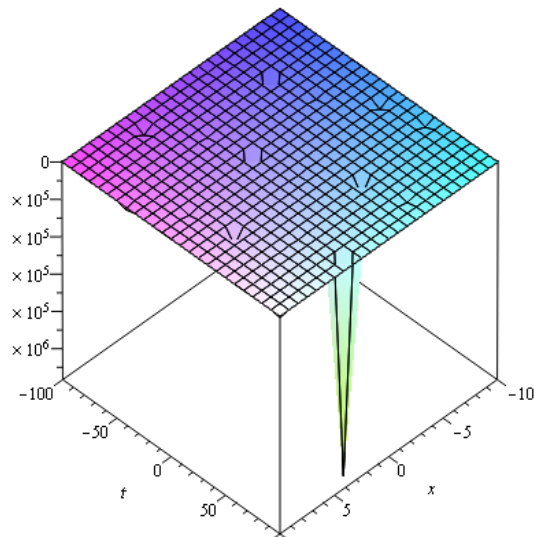


Figure 2: Graphical representation of trigonometric function solutions for Case 1

when  $a_0 = 1, a_1 = 1, a_2 = -2, \lambda = -\frac{1}{2}, \mu = 1, C_1 = 1, C_2 = 1, C = 913.06$ .

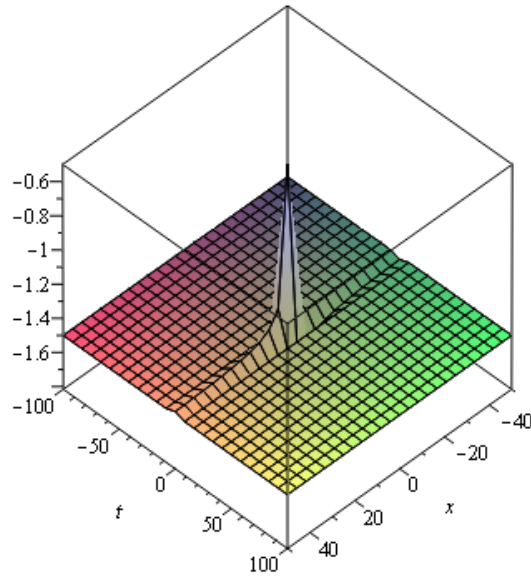


Figure 3: Graphical representation of rational function solutions for case 1 when  $C_1 = 1, C_2 = 1, \lambda = 3$ .

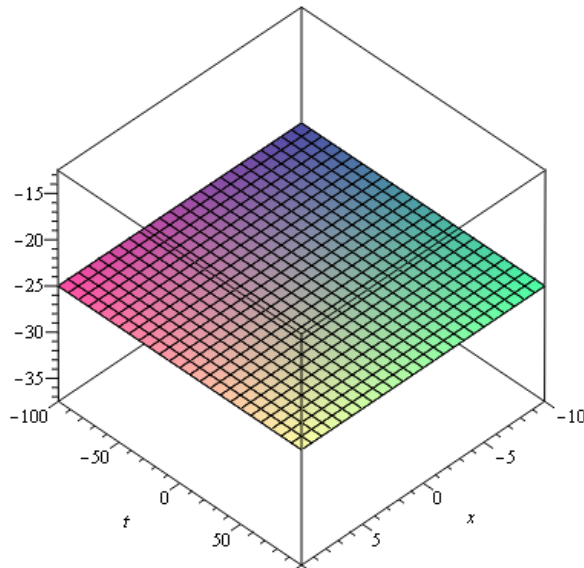


Figure 4: Graphical representation of hyperbolic function solutions for case 1 when  $a_0 = -1, a_1 = -8, a_2 = -2, \lambda = 4, \mu = 3, c = 1441, C_1 = 1, C_2 = 1$ .

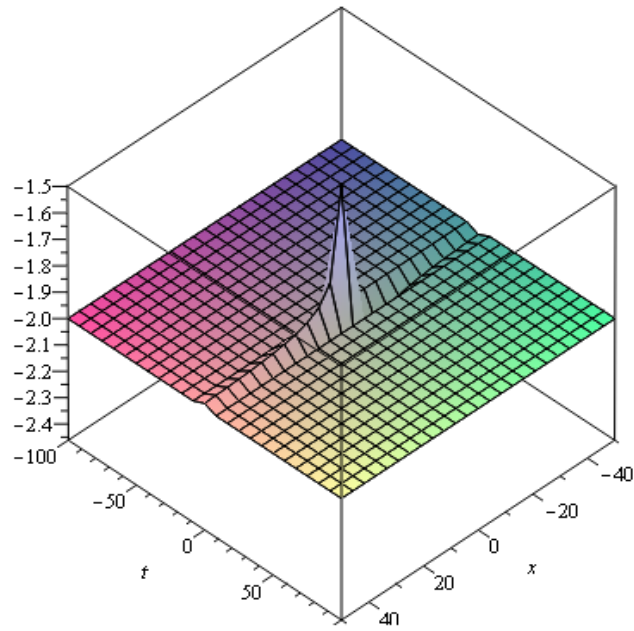


Figure 5: Graphical representation of rational function solutions for case 1 when  $C_1 = -2, C_2 = -1, \lambda = 4, c = 2205$ .

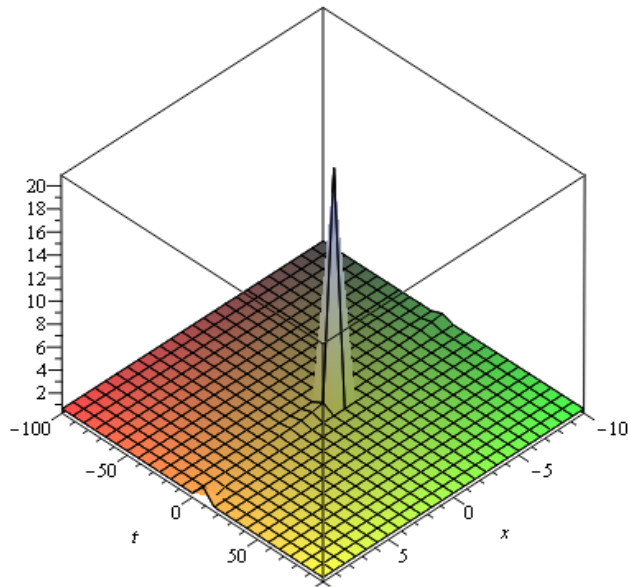


Figure 6: Graphical representation of hyperbolic function solutions for Case 2 when  $a_0 = -1, a_1 = 2, a_2 = -4, \lambda = -\frac{1}{2}, \mu = -\frac{1}{128}, c = 1.265, C_1 = 1, C_2 = 1$ .

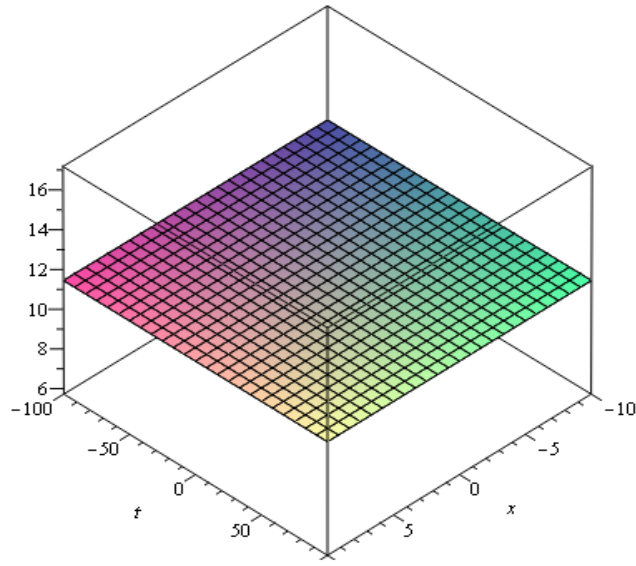


Figure 7: Graphical representation of trigonometric function solutions for case 2 when  $a_0 = -1, a_1 = \sqrt{32}, a_2 = -4, \lambda = -\frac{\sqrt{32}}{4}, \mu = \frac{1}{8}, c = 2.25, C_1 = 1, C_2 = 1$ .

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