ON MULTIPLICITY OF PARTS IN n-COLOUR COMPOSITIONS

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CERTIFICATE

This is to certify that the work done in this dissertation entitled "On Multiplicity of Parts In n-Colour Compositions" in accordance with the requirements for the award of the degree of **Master of Science in Mathematics and Computing** from Thapar Institute of Engineering and Technology during the year **2018-2019** is a legitimate work carried out by **Watanjeet Singh** under the guidance of **Dr. Meenakshi Rana.**
The matter embodied in this dissertation is candidate's own work and has not been submitted by this or any other university in partial or full form of award of such a degree.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Abstract

The first chapter introduces the concepts of partitions, compositions and n-colour compositions. It also includes standard results on compositions and n-colour compositions. Thus, this chapter provides the required toolkit to solve higher level problems.
The second chapter mainly focus on multiplicity of parts in random composition of an integer. Here we construct two variable generating function for a particular part size in composition and discuss mean and variance of the multiplicity of part size in composition.
In third chapter we extend the results of chapter 2 for n-colour compositions. Here we discuss expected value of multiplicity of a part size in randomly choosen n-colour composition.

Contents

Chapter 1

Introduction

A partition of an integer n is a representation of n as a sum of finite non-increasing sequence of positive integers, that is there exists $a_0 \geq a_1 \geq a_2 \geq ... \geq a_r$ such that

$$\sum_{i=0}^{r} a_i = n.$$

The positive integers a_0,a_1,...,a_r are called the part sizes of an integer n. For example, the partitions of 5 are

$$1+1+1+1+1,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 5.$$

The corresponding partition function which counts the total number of partitions of an integer is denoted by $p(n)$ and is called the unrestricted partition function. Hence the total partitions of $n = 5$ are 7, that is

$$p(5) = 7.$$

The generating function $f(q)$ for the sequence a_0,a_1,a_2,a_3,... is the power series

$$f(q) = \sum_{i=0}^{\infty} a_i q^i.$$

Euler showed that this infinite series can be expressed as an infinite product. The coefficients of the power series represents the partition of an integer n.

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

This result can be also represented in q-rising factorial, where q-rising factorial is defined as

$$(a, q)_0=1,$$

and

$$(a, q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \text{ for } a \geq 0.$$

Now put $a = q$ in the above equation, then we have

$$(q, q)_n = (1-q)(1-q^2)...(1-q^n).$$

Thus we can write

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q, q)_\infty},$$

where |q| < 1 ensures the convergence and $p(0) = 1$.

q-binomial theorem

Let α be any complex number, then

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n$$

where $(\alpha)_n$ is known as rising factorial defined as, $(\alpha)_0 = 1$ and

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1).$$

1.1 Compositions

The partitions in which the order of part size matters are known as compositions. These are also known as ordered partitions.
For example, the total ordered partitions or compositions of an integer 4 are

4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1.

Thus total compositions of 4 are 8, that is $C(4) = 8$.

Theorem 1.1.1. *The number of compositions of an integer n with exactly l parts is equal to $\binom{n-1}{n-l}$ or $\binom{n-1}{l-1}$.*

Proof. The total number of compositions of an integer n into l parts is obtained from the coefficients q^n in the expansion of

$$(q + q^2 + q^3 + ...)^l.$$

Here we write the function as the product of l factors and collect the terms $q^{a_1}, q^{a_2}, ..., q^{a_l}$, where

$$a_1 + a_2 + ... + a_l = n.$$

We get a particular term $q^{a_1+a_2+...+a_l}$ of the product, where $(a_1, a_2, ..., a_l)$ is one of the compositions of n into l parts. Thus, the generating function of composition of an integer n into l parts is given by

$$\sum_{n=0}^{\infty} C_l(n)q^n = (q + q^2 + q^3 + ...)^l.$$

Then the result follows as

$$\begin{aligned}\sum_{n\geq l} C_l(n)q^n &= (q^1 + q^2 + q^3 + q^4 + ...)^l \\ &= \big[q(1 + q + q^2 + ...)\big]^l \\ &= q^l\big(1 + q + q^2 + ...\big)^l \\ &= q^l\big(\frac{1}{1-q}\big)^l \\ &= q^l(1-q)^{-l} \\ &= q^l \sum_{r=0}^{\infty} \frac{(l)_r}{r!} q^r\end{aligned}$$

using q-binomial theorem

$$\begin{aligned}\sum_{n\geq l} C_l(n)q^n &= q^l \sum_{r=0}^{\infty} \binom{l+r-1}{r} q^r \\ &= \sum_{r=0}^{\infty} \binom{l+r-1}{r} q^{r+l}\end{aligned}$$

now substitute $r + l = n$; which implies $r = n - l$. Thus, we get

$$\sum_{n\geq l} C_l(n)q^n = \sum_{n=l}^{\infty} \binom{n-1}{n-l} q^n$$

now we equate the coefficient of q^n on both sides , and we have

$$C_l(n) = \binom{n-1}{n-l} = \binom{n-1}{l-1}.$$

□

Theorem 1.1.2. *The total compositions of n are 2^{n-1}.*

Proof. Let $C(n)$ be the total number of compositions of n. Thus, we can write

$$C(n) = \sum_{l=1}^{\infty} (\text{ Number of compositions of } n \text{ into } l \text{ parts})$$

$$\begin{aligned} C(n) &= \sum_{l=1}^{\infty} C_l(n) \\ &= \sum_{l=1}^{\infty} \binom{n-1}{l-1} \\ &= \sum_{l=1}^{n} \binom{n-1}{l-1} \end{aligned}$$

Because, $\binom{n}{l} = 0 \; \forall \; l > n$

$$C(n) = \binom{n-1}{0} + \binom{n-1}{1} + ... + \binom{n-1}{n-1}$$

Thus,

$$C(n) = 2^{n-1}.$$

□

Theorem 1.1.3. *The number of compositions of integer n where each part size is less than or equal to s is determined by the coefficients of the generating function*

$$C(q; s) = \frac{q - q^{s+1}}{1 - 2q + q^{s+1}}.$$

Proof. The number of compositions of n into l parts where each part is less than or equal to s is given by $C_{l,n}(s)$ and its generating function is

$$\begin{aligned} C_l(q;s) &= (q^1 + q^2 + q^3 + ... + q^s)^l \\ &= q^l(1 + q^1 + q^2 + ... + q^{s-1})^l \\ &= q^l\left(\frac{1-q^s}{1-q}\right)^l \end{aligned}$$

thus,

$$C_l(q;s) = \left(\frac{q-q^{s+1}}{1-q}\right)^l.$$

Now if we remove the condition of l parts and allow all possible parts, we get

$$C(q;s) = \sum_{l=1}^{\infty} C_l(q;s) = \sum_{l=1}^{\infty}\left(\frac{q-q^{s+1}}{1-q}\right)^l \quad = \sum_{l=0}^{\infty}\left(\frac{q-q^{s+1}}{1-q}\right)^l - 1$$

now using the result

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

we get

$$C(q;s) = \frac{1}{1-\left(\frac{q-q^{s+1}}{1-q}\right)} - 1$$

or

$$C(q;s) = \frac{q-q^{s+1}}{1-2q+q^{s+1}}.$$

□

From above result we can find a relation between compositions and Fibonacci numbers. Let us recall the Fibonacci numbers.

The Fibonacci numbers forms a sequence in which each number is a sum of two preceeding numbers, and the sequence starts from 0 and 1. These are denoted by F_n.
We define $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-2} + F_{n-1}.$$

Generating function of Fibonaccci sequence

The ordinary generating function of the Fibonacci sequence is

$$\sum_{n=0}^{\infty} F_n q^n = \frac{q}{1-q-q^2}.$$

The proof follows as, let

$$\begin{aligned}
A(q) &= \sum_{n=0}^{\infty} F_n q^n \\
&= F_0 + F_1 q + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) q^n \\
&= q + q \sum_{n=2}^{\infty} F_{n-1} q^{n-1} + q^2 \sum_{n=2}^{\infty} F_{n-2} q^{n-2} \\
&= q + qA(q) + q^2 A(q)
\end{aligned}$$

or

$$A(q) = \frac{q}{1-q-q^2}.$$

Theorem 1.1.4. *Let $C_n(2)$ be the number of compositions of n into parts in which each part is ≤ 2 and F_n be the Fibonacci number. Then*

$$C_n(2) = F_{n+1}.$$

Proof. The proof of the result follows immediately from the generating functions. By *Theorem* 1.1.3

$$\begin{aligned}
C(q;s) &= \frac{q-q^{s+1}}{1-2q+q^{s+1}} \\
\sum_{n=1}^{\infty} C_n(2) q^n &= \frac{q-q^{2+1}}{1-2q+q^{2+1}} \\
&= \frac{q(1-q^2)}{1-q-q+q^3} \\
&= \frac{q(1-q)(1+q)}{(1-q)-q(1-q^2)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q(1+q)}{1-q-q^2} \\
&= \frac{1}{1-q-q^2} - 1 \\
&= \sum_{n=0}^{\infty} F_{n+1} q^n - 1 \\
&= \sum_{n=1}^{\infty} F_{n+1} q^n + F_1 q^0 - 1 \\
&= \sum_{n=1}^{\infty} F_{n+1} q^n + 1 - 1 \\
\sum_{n=1}^{\infty} C_n(2) q^n &= \sum_{n=1}^{\infty} F_{n+1} q^n.
\end{aligned}
$$

Now comparing coefficients of q^n on both sides, we have

$$C_n(2) = F_{n+1}.$$

This result can be explained with the help of an example, take $n = 4$ then the total compositions of 4 where each part is less than or equal to 2 are

2+2, 1+2+1, 2+1+1, 1+1+2, 1+1+1+1

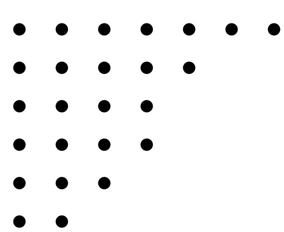
and hence $C_4(2) = 5 = F_5$. □

1.2 Graphical representations

Ordinary Partitions

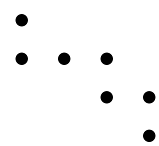
Ordinary partitions can be represented graphically using Ferrers graph. It represent partitions as dots in the plane in such a way that i^{th} row has same number of dots as the i^{th} part in the partition. This can be illustrated with an example.

Let $n = 25$ and partition of n be 7+5+4+4+3+2, then its ferrers graph representation is



Compositions

Compositions can be represented by zig-zag graphs. The first row has dots equal to the first part size of the composition and the next row have the dots starting right from the last column of the first row. This can be illustrated with an example. Let $n = 7$ and its composition be (1321), then its zig-zag represention is



1.3 n-colour compositions

We first define an n-colour partition of a positive integer. An n-colour partition of an integer ν is the partition in which a part size of n can come in n different colours. These colours are denoted by the subscripts such as n_1, n_2, n_3, ..., n_n and these parts satisfy an order

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < ...,.$$

For example, the n-colour partitions of 3 are

$$3_1,\ 3_2,\ 3_3,\ 2_1 1_1,\ 2_2 1_1,\ 1_1 1_1 1_1.$$

Thus there are 6 n-colour partition of the integer 3. Let $P(\nu)$ be the total number of n-colour partitions of an integer ν. Then we have a generating function for n-colour partitions as

$$\sum_{\nu=0}^{\infty} p(\nu) q^{\nu} = \prod_{n=1}^{\infty} (1 - q^n)^{-n}.$$

An ordered n-colour partition is known as n-colour composition of an integer. That is the order of summands is taken into consideration.
For example, the n-colour compositions of 3 are

$$3_1, 3_2, 3_3, 2_1 1_1, 2_2 1_1, 1_1 2_1, 1_1 2_2, 1_1 1_1 1_1$$

The total number of n-colour compositions of integer ν is denoted by $C(\nu)$ and the total n-colour compositions of ν into l parts is denoted by $C(l, \nu)$. We now state some of the important generating functions discussed in [2] on n-colour compositions that will be further used to introduce new results.

Theorem 1.3.1. *Let $C(l;q)$ and $C(q)$ be the generating functions for $C(l,\nu)$ and $C(\nu)$ respectively, then*

$$C(l;q) = \frac{q^l}{(1-q)^{2l}}$$
$$C(q) = \frac{q}{1-3q+q^2}$$
$$C(\nu) = F_{2\nu}$$

where $F_{2\nu}$ is the $(2\nu)^{th}$ Fibonacci number.

Proof. We know that an n-colour composition of an integer ν is a composition where each part size n can appear in n different colours. This means that n-colour composition is the weighted composition with weights 1,2,3...
Thus, we get the generating function

$$\begin{aligned} C(l;q) &= \sum_{\nu=1}^{\infty} C(l,v)q^v \\ &= (q+2q^2+3q^3+...)^l \\ C(l;q) &= \frac{q^l}{(1-q)^{2l}}. \end{aligned}$$

Now the total n-colour compositions of an integer will be obtained if we remove the condition of number of part sizes, that is

$$\begin{aligned} C(q) &= \sum_{l=1}^{\infty} C(l;q) \\ &= \sum_{l=1}^{\infty} \frac{q^l}{(1-q)^{2l}} \\ &= \frac{q}{1-3q+q^2}. \end{aligned}$$

This RHS in above expression is also the generating function for $F_{2\nu}$, that is the $(2\nu)^{th}$ Fibonacci number. So,

$$C(\nu) = F_{2\nu}.$$

□

1.4 Multiplicity of parts

Consider a random composition of a positive integer n, say $a = (a_1, a_2, ..., a_l)$ into l parts such that

$$\sum_{i=1}^{l} a_i = n.$$

Here a_i parts are in no particular order. For example, the compositions of 3 are

3, 2+1, 1+2, 1+1+1.

Now if we read compositions of 3 as 1.3 then it means that part 3 repeat once or part 3 has multiplicity 1. Similarly for other compositions 1.2+1.1 means that parts 2 and 1 have multiplicity 1. If we write choose composition 3.1, it means that part 1 has multiplicity 3.

Hitczenko posed a question in [8] : What is the probability of a randomly choosen part size in a random composition of an integer n has multiplicity m as $n \to \infty$ (asymptotically)?

In Chapter 2 and 3, we study some results related to multiplicity of a part size and its related generating functions. In particular we have studied the mean and variance on the multiplicity of a part size in randomly choosen composition of an integer, and then extended the work to n-colour compositions.

Chapter 2

On expected value and variance of multiplicity of part size in composition

We start with an overview of the multiplicity of parts in compositions. Let $a = (a_1, a_2, ..., a_l)$ be a composition of an integer n. We already know that values of a_i's are called part sizes. Now the multiplicity of a part size a_i is defined as the number of times it is repeating in a particular composition of an integer n, that is if in composition a of n, the part size a_i is repeated m times then we say that a_i has multiplicity m.

This chapter discussed in [3] aims to obtain a formula to calculate the expected value and variance of the multiplicity of a given part size in a random composition of an integer. The random compositions are widely used as the theoretical models for some applications.

Let us denote the set of all compositions of an integer n by Ω_n. We know that random composition of an integer is the composition being choosen from Ω_n according to uniform probability measure. We denote the multiplicity of j part size in a randomly choosen composition a of an integer n by $X_{n,j}$.

2.1 Generatingfunctionology notation

We first introduce generatingfunctionology notation discussed in [5] which will be useful in proving the results.
Consider $f(x)$ be a series with powers in x. Then the notation $[x^n]f(x)$ denotes the coefficient of x^n in the series $f(x)$.
For an example,

$$[x^n]e^x = \frac{1}{n!}$$

some properties of this notation are,

$$[x^n]x^\alpha f(x) = [x^{n-\alpha}]f(x)$$

and,

$$[ax^n]f(x) = \frac{1}{a}[x^n]f(x).$$

For example,

$$[x^n]x^2e^x = [x^{n-2}]e^x$$

and also,

$$\left[\frac{x^n}{n!}\right]e^x = 1$$

2.2 Construction of a two variable generating function

Let $G_j(q, u)$ denote the two variable generating function where $[q^n u^m]G_j(q, u)$ denote the number of compositions of n where part size j has multiplicity m discussed in [10]. Now, we construct the generating function.
We know that generating function for composition with l parts (need not to be distinct) is

$$(q + q^2 + ... + uq^j + q^{j+1} + ...)^l.$$

Here variable u is attached to q^j so that multiplicities of part size j can be counted. Now as

$$uq^j = (u + 1 - 1)q^j = (u - 1)q^j + q^j.$$

Thus we have the generating function,

$$(q + q^2 + ... + q^j + (u - 1)q^j + q^{j+1} + ...)^l,$$

this further implies,

$$\left(\frac{q}{1-q} + (u - 1)q^j\right)^l.$$

Now removing the restriction of l parts we get,

$$\begin{aligned}
G_j(q,u) &= \sum_{l=1}^{\infty}\left(\frac{q}{1-q}+(u-1)q^j\right)^l \\
&= \left(\frac{q}{1-q}+(u-1)q^j\right)+\left(\frac{q}{1-q}+(u-1)q^j\right)^2+\ldots+\left(\frac{q}{1-q}+(u-1)q^j\right)^k+\ldots \\
&= \frac{\left(\frac{q}{1-q}+(u-1)q^j\right)}{1-\left(\frac{q}{1-q}+(u-1)q^j\right)} \\
&= \frac{q+(u-1)q^j(1-q)}{(1-q)-(q+(u-1)q^j(1-q))}.
\end{aligned}$$

Thus we have,

$$G_j(q,u) = \frac{1-q}{1-2q+(1-u)q^j(1-q)} - 1.$$

2.3 Statement and proof of the main result

There are two results based on the multiplicity of part size j of random composition of an integer n. They are

Theorem 2.3.1. *Assume $n \geq 2$, and $1 \leq j \leq n-1$, then the expected value and the variance of $X_{n,j}$ satisfies the equalities.*

$$E[X_{n,j}] = \frac{n+3-j}{2^{j+1}}$$

and

$$Var[X_{n,j}] = \frac{3j^2-2nj+5n-14j+11}{4^{j+1}}.$$

Proof. Let $g_j(n,m)$ be the total count that part size j of an integer n has multiplicity m. As $G_j(q,u)$ is the two variable generating function. Therefore,

$$G_j(q,u) = \sum_{n\geq 1}\sum_{m\geq 0} g_j(n,m)q^n u^m. \tag{2.1}$$

We have proved that,

$$G_j(q,u) = \frac{1-q}{1-2q+(1-u)q^j(1-q)} - 1;$$

and also from the generatingfunctionology notation. If $A(x) = \sum_n a_n x^n$, then $a_n = [x^n]A(x)$. Thus following this notation we denote

$$G_{n,j}(u) = [q^n]G_j(q,u).$$

Expected value proof

From [5], we know that

$$E[X_{n,j}] = \frac{G'_{n,j}(1)}{G_{n,j}(1)}. \tag{2.2}$$

For any integer $n \geq 2$ and $j \leq n-1$, we can write

$$G_{n,j}(u) = [q^n]\Big\{(1-q)H_j(q,u)\Big\}. \tag{2.3}$$

Now let

$$[q^n]H_j(q,u) = F_{n,j}(u),$$

then

$$F_{n,j}(u) - F_{n-1,j}(u) = [q^n]H_j(q,u) - [q^{n-1}]H_j(q,u),$$

this implies

$$F_{n,j}(u) - F_{n-1,j}(u) = [q^n]\Big\{(1-q)H_j(q,u)\Big\}.$$

Thus from (2.3) we have

$$G_{n,j}(u) = F_{n,j}(u) - F_{n-1,j}(u),$$

and

$$H_j(q,u) = \frac{1}{1-(2q-(1-u)q^j(1-q))}.$$

Using geometric series expansion, we have

$$\begin{aligned}\frac{1}{1-(2q-(1-u)q^j(1-q))} &= \big(1-(2q-q^j(1-q)(1-u))\big)^{-1}\\ &= \sum_{r\geq 0}(2q-q^j(1-q)(1-u))^r.\end{aligned}$$

Now using the binomial theorem, we get:

$$\sum_{r\geq 0}(2q-q^j(1-q)(1-u))^r = \sum_{r\geq 0}\sum_{s=0}^{r}\binom{r}{s}\big(-q^j(1-q)(1-u)\big)^{r-s}(2q)^s$$
$$= \sum_{r\geq 0}\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}q^{j(r-s)}(1-q)^{r-s}(1-u)^{r-s}2^s q^s.$$

Now again applying binomial theorem, we get:

$$H_j(q,u) = \sum_{r\geq 0}\sum_{s=0}^{r}\sum_{p=0}^{r-s}(-1)^{2(r-s)-p}2^s\binom{r}{s}\binom{r-s}{p}(1-u)^{r-s}q^{j(r-s)+r-p}.$$

We implement a substitution ie: put $j(r-s)+r-p=n$. Thus we have $r=\dfrac{n+(js+p)}{j+1}$. So,

$$0\leq s\leq r \iff 0\leq s\leq \tfrac{n+(js+p)}{j+1},$$

and

$$0\leq p\leq r-s \iff 0\leq p\leq \tfrac{n+(js+p)}{j+1}-s.$$

To obtain the desired results we need to modify these two inequalities. Since

$$\begin{aligned}
&0\leq p\leq \frac{n+(js+p)}{j+1}-s\\
\iff &0\leq p\leq \frac{n+js+p-js-s}{j+1}\\
\iff &0\leq p\leq \frac{n+p-s}{j+1}\\
\iff &0\leq p(j+1)\leq n+p-s\\
\iff &0\leq p\leq \frac{n-s}{j}.
\end{aligned}$$

Now similarly for the first inequality we have

$$\begin{aligned}
&0\leq s\leq \frac{n+(js+p)}{j+1}\\
\iff &0\leq s\leq \frac{n+js+\frac{n-s}{j}}{j+1}\\
\iff &0\leq s\leq \frac{n-s}{j}+s\\
\iff &0\leq s\leq n.
\end{aligned}$$

Hence, we can rewrite the expression for $H_j(q,u)$ as follows:

$$H_j(q,u) = \sum_n \sum_{s=0}^{n} \sum_{p=0}^{\left[\frac{n-s}{j}\right]} I_{\{(j+1)|(n+p+js)\}}(-1)^{2(r-s)-p}2^s \binom{r_n}{s}\binom{r_n-s}{p}(1-u)^{r_n-s}q^n,$$

where

$$r_n = \frac{n+p+js}{j+1}. \tag{2.4}$$

Since $F_{n,j}(u) = [q^n]H_j(q,u)$,

this yields

$$F_{n,j}(u) = \sum_{s=0}^{n} \sum_{p=0}^{\left[\frac{n-s}{j}\right]} I_{\{(j+1)|(n+p+js)\}}(-1)^{2(r-s)-p}2^s \binom{r_n}{s}\binom{r_n-s}{p}(1-u)^{r_n-s}. \tag{2.5}$$

Now we need to calculate $F_{n,j}(1)$ so that the value of $G_{n,j}(1)$ is obtained. Put $u = 1$ in the equation (2.5), then the term that do not vanish are only those for which $r_n - s = 0$. Now, since $r_n - s = 0$, then p must be zero as $p \leq r_n - s$. Therefore, we have $r_n = s$ and $p = 0$. Substituting these values in (2.4), we get $s = n$. Thus from (2.5) we get,

$$F_{n,j}(u) = 2^n,$$

and

$$F_{n,j}(1) - F_{n-1,j}(1) = 2^n - 2^{n-1}.$$

From (2.3) we have,

$$G_{n,j}(1) = 2^n - 2^{n-1} = 2^{n-1}. \tag{2.6}$$

This is true, since $G_{n,j}(1)$ is the total number of composition of n where part size j can have any multiplicity. We now want to find $G'_{n,j}(1)$, so we need to differentiate $G_{n,j}(u)$. And from (2.3) we can have $G'_{n,j}(u) = F'_{n,j}(u) - F'_{n-1,j}(u)$. Thus

$$F'_{n,j}(u) = \sum_{s=0}^{n-1} \sum_{p=0}^{\left[\frac{n-s}{j}\right]} I_{(j+1)|(n+p+js)}(-1)^{2(r-s)-p+1}(r_n-s)2^s \binom{r_n}{s}\binom{r_n-s}{p}(1-u)^{r_n-s-1}. \tag{2.7}$$

Now, put $u = 1$ in above equation, so that $F'_{n,j}(1)$ can be obtained. Following the same steps, the terms that do not vanish in $F'_{n,j}(1)$ are the terms for which

$$r_n - s - 1 = 0. \tag{2.8}$$

From equation (2.7), $\binom{r_n - s}{p} = \binom{1}{p}$, and $\binom{1}{p} \neq 0$ if and only if $p = 0$ or $p = 1$. These both values will be considered. If $p = 0$, then from (2.4) we have

$$r_n = s + 1 = \frac{n + js}{j + 1}.$$

This implies

$$s = n - j - 1,$$

thus,

$$\binom{r_n}{s} = \binom{s+1}{s} = s + 1 = n - j.$$

This results in term $-2^{n-j-1}(n - j)$.

Now we consider the case when $p = 1$, then from (2.4) we have

$$r_n = \frac{n + 1 + js}{j + 1} = s + 1,$$

which implies $s = n - j$ and $\binom{r_n}{s} = \binom{s+1}{s} = s + 1 = n - j + 1$.

The term associated with it is $2^{n-j}(n - j + 1)$. This implies

$$F'_{n,j}(1) = -2^{n-j-1}(n - j) + 2^{n-j}(n - j + 1) = 2^{n-j-1}(n - j + 2).$$

Thus

$$G'_{n,j}(1) = F'_{n,j}(1) - F'_{n-1,j}(1) = 2^{n-j-2}(n - j + 3).$$

Now combining it with eqn. (2.2) and (2.6), we have

$$E[X_{n,j}] = \frac{n - j + 3}{2^{j+1}}.$$

Variance proof

Now we discuss the variance of the $X_{n,j}$, we recall from [5] that

$$Var[X_{n,j}] = \frac{G''_{n,j}(1) + G'_{n,j}(1)}{G_{n,j}(1)} - \left(\frac{G'_{n,j}(1)}{G_{n,j}(1)}\right)^2. \quad (2.9)$$

We differentiate eqn. (2.7) to obtain $F''_{n,j}(u)$

$$F''_{n,j}(u) = \sum_{s=0}^{n-1} \sum_{p=0}^{\left[\frac{n-s}{j}\right]} I_{(j+1)|(n+p+js)}(-1)^{2(r-s)-p+1}(r_n-s)(r_n-s-1)2^s \binom{r_n}{s}\binom{r_n-s}{p}(1-u)^{r_n-s-2}.$$

Repeating the same procedure we put $u = 1$ and collect all non-zero terms in above equation. We see that only the non-zero terms must have $r_n - s - 2 = 0$. This means $\binom{2}{p} = 0$ for any value of $p > 2$. Thus we consider $p = 0, 1$ and 2.

When $p = 0$, we get the term $2^{n-2j-2}(n-2j-1)(n-2j)$;

When $p = 1$, we get the term $-2^{n-2j}(n-2j)(n-2j+1)$;

When $p = 2$, we get the term $2^{n-2j}(n-2j+1)(n-2j+2)$.

Combining all we get,

$$F''_{n,j}(1) = 2^{n-2j-2}(n-2j-1)(n-2j) - 2^{n-2j}(n-2j)(n-2j+1) + 2^{n-2j}(n-2j+1)(n-2j+2)$$
$$F''_{n,j}(1) = 2^{n-2j-2}((n-2j)(n-2j+7)+8).$$

Hence,

$$G''_{n,j}(1) = F''_{n,j}(1) - F''_{n-1,j}(1) = 2^{n-2j-3}\bigg((n-2j)^2 + 9(n-2j) + 14\bigg);$$

Now simplifying (2.9) we get,

$$Var[X_{n,j}] = \frac{2^{n-2j-3}\bigg((n-2j)^2 + 9(n-2j) + 14\bigg) + 2^{n-j-2}(n-j+3)}{2^{n-1}} - \left[\frac{2^{n-j-2}(n-j+3)}{2^{n-1}}\right]^2$$

so

$$Var[X_{n,j}] = \frac{3j^2 - 2nj + 5n - 14j + 11}{2^{2j+2}}.$$

This completes the proof. □

Chapter 3

On expected value of multiplicity of part size in n-colour composition

3.1 Introduction

In this chapter our main focus is on the extension of works of compositions to n-colour compositions. We firstly define ω_ν be the set of all n-colour compositions of an integer ν. Let s be a randomly choosen n-colour composition from ω_ν and $M_m(s)$ be the set of part sizes of s that have multiplicity m.

Then we have a result which is an extension of Lemma discussed in [10] and Section (2.2) of Chapter 2.

Lemma 3.1.1. *For a randomly choosen n-colour compositions of ν, the*

$$P\big(j \in M_m(s)\big) = \frac{1}{F_{2\nu}}[q^\nu u^m]\frac{(1-q)^2}{1-3q+q^2+(1-u)jq^j(1-q)^2}$$

where $P\big(j \in M_m(s)\big)$ is the probability that part size j has multiplicity m.

Proof. Let $G_j(q,u)$ be the two variable generating function where $[q^\nu u^m]G_j(q,u)$ is the number of n-colour composition of ν in which part size j has multiplicity m. We firstly construct this generating function.
We know that generating function for n-colour compositions with l parts (need not to be distinct) is

$$(q + 2q^2 + 3q^3 + ... + ujq^j + ...)^l.$$

Here variable u is attached to q^j so that multiplicities of part size j can be counted.
This further implies,

$$(q+2q^2+3q^3+\ldots+ujq^j+\ldots)^l=\left(\frac{q}{(1-q)^2}+(u-1)jq^j\right)^l,$$

thus

$$\begin{aligned}G_j(q,u)&=\sum_{l=1}^{\infty}\left(\frac{q}{(1-q)^2}+(u-1)jq^j\right)^l\\&=\frac{q+(u-1)jq^j(1-q)^2}{1-3q+q^2-(u-1)jq^j(1-q)^2}\\&=\frac{(1-q)^2}{1-3q+q^2+(1-u)jq^j(1-q)^2}-1.\end{aligned}$$

Since there are total $F_{2\nu}$ n-colour compositions of ν, thus we have

$$P\big(j\in M_m(s)\big)=\frac{1}{F_{2\nu}}[q^\nu u^m]\frac{(1-q)^2}{1-3q+q^2+(1-u)jq^j(1-q)^2}.$$

□

3.2 Main Results

We now develop some new expressions which can be used to calculate the expected value of multiplicity of parts in n-colour compositions. We extend the work done in [3] and (Sec 2.3) that is discussed in Chapter 2 to n-colour compositions.

Theorem 3.2.1. *For $\nu\geq 2$, the expected value of $X_{\nu,j}$, the multiplicity of part size j in a random n-colour composition of an integer is given by*

$$E[X_{\nu,j}]=\frac{F'_{\nu,j}(1)+F'_{\nu-2,j}(1)-2F'_{\nu-1,j}(1)}{F_{\nu,j}(1)+F_{\nu-2,j}(1)-2F_{\nu-1,j}(1)},$$

where,

$$F_{\nu,j}(u)=\sum_{s=0}^{\nu}\sum_{p=0}^{\left[\frac{2(\nu-s)}{j}\right]}\sum_{t=0}^{s}I_{\{j+2|(\nu+js+p+t\}}(-1)^{3r_\nu-2s-p-t}(3)^t\binom{r_\nu}{s}\binom{2(r_\nu-s)}{p}\binom{s}{t}(1-u)^{r_\nu-s}j^{r_\nu-s}.$$

Proof. Let $g_j(\nu,m)$ be the total count that part size j of an integer ν has multiplicity m. Also $G_j(q,u)$ is a two variable generating function.
Therefore,

$$G_j(q,u)=\sum_{\nu\geq 1}\sum_{m\geq 0}g_j(\nu,m)q^\nu u^m.$$

We have shown that

$$G_j(q,u) = \frac{(1-q)^2}{1-3q+q^2+(1-u)jq^j(1-q)^2} - 1. \tag{3.1}$$

In similar way, we denote

$$G_{\nu,j}(u) = [q^\nu]G_j(q,u),$$

then $E[X_{\nu,j}]$ is the expected value, where $X_{\nu,j}$ is the multiplicity of part size j in a random n-colour composition of an integer ν, and from [9] we have

$$E[X_{\nu,j}] = \frac{G'_{\nu,j}(1)}{G_{\nu,j}(1)}. \tag{3.2}$$

For $\nu \geq 2$, we can write

$$\begin{aligned} G_{\nu,j}(u) &= [q^\nu]\big\{(1-q)^2 H_j(q,u)\big\} \\ &= [q^\nu]H_j(q,u) + q^2[q^\nu]H_j(q,u) - 2q[q^\nu]H_j(q,u) \\ &= [q^\nu]H_j(q,u) + [q^{\nu-2}]H_j(q,u) - 2[q^{\nu-1}]H_j(q,u). \end{aligned}$$

We write,

$$F_{\nu,j}(u) = [q^\nu]H_j(q,u), \tag{3.3}$$

so

$$G_{\nu,j}(u) = F_{\nu,j}(u) + F_{\nu-2,j}(u) - 2F_{\nu-1,j}(u), \tag{3.4}$$

and

$$H_j(q,u) = \frac{1}{1-3q+q^2+(1-u)jq^j(1-q)^2}. \tag{3.5}$$

Using the geometric series expansion and then thrice the binomial theorem, we obtain:

$$H_j(q,u) = \sum_{r\geq 0}\sum_{s=0}^{r}\sum_{p=0}^{2(r-s)}\sum_{t=0}^{s}\binom{r}{s}\binom{2(r-s)}{p}\binom{s}{t}(-1)^{3r-2s-p-t}(1-u)^{r-s}j^{r-s}q^{j(r-s)+2r-p-t}(3)^t.$$

We substitute $\nu = j(r-s)+2r-p-t$ and simplify the inequalities,

$$\begin{aligned} &0 \leq s \leq r, \\ &0 \leq p \leq 2(r-s), \\ &0 \leq t \leq s. \end{aligned}$$

The more precise inequalities are

$$\begin{aligned} &0 \le s \le \nu, \\ &0 \le p \le \frac{2(\nu - s)}{j}, \\ &0 \le t \le s. \end{aligned}$$

Therefore, we have

$$H_j(q,u) = \sum_{\nu}\sum_{s=0}^{\nu}\sum_{p=0}^{\left[\frac{2(\nu-s)}{j}\right]}\sum_{t=0}^{s} I_{\{j+2|(\nu+js+p+t\}}(-1)^{3r_\nu-2s-p-t}(3)^t\binom{r_\nu}{s}\binom{2(r_\nu-s)}{p}\binom{s}{t}(1-u)^{r_\nu-s}j^{r_\nu-s}q^\nu. \tag{3.6}$$

Using (3.3), we have

$$F_{\nu,j}(u) = \sum_{s=0}^{\nu}\sum_{p=0}^{\left[\frac{2(\nu-s)}{j}\right]}\sum_{t=0}^{s} I_{\{j+2|(\nu+js+p+t\}}(-1)^{3r_\nu-2s-p-t}(3)^t\binom{r_\nu}{s}\binom{2(r_\nu-s)}{p}\binom{s}{t}(1-u)^{r_\nu-s}j^{r_\nu-s}, \tag{3.7}$$

and

$$F'_{\nu,j}(u) = \sum_{s=0}^{\nu}\sum_{p=0}^{\left[\frac{2(\nu-s)}{j}\right]}\sum_{t=0}^{s} I_{\{j+2|(\nu+js+p+t\}}(-1)^{3r_\nu-2s-p-t}(3)^t\binom{r_\nu}{s}\binom{2(r_\nu-s)}{p}\binom{s}{t}(r_\nu-s)(1-u)^{r_\nu-s-1}j^{r_\nu-s}. \tag{3.8}$$

Differentiating (3.4), we have

$$G_{\nu,j}(u) = F'_{\nu,j}(u) + F'_{\nu-2,j}(u) - 2F'_{\nu-1,j}(u).$$

If we put $u = 1$, then from (3.2), we have

$$E[X_{\nu,j}] = \frac{F'_{\nu,j}(1) + F'_{\nu-2,j}(1) - 2F'_{\nu-1,j}(1)}{F_{\nu,j}(1) + F_{\nu-2,j}(1) - 2F_{\nu-1,j}(1)},$$

where $F_{\nu,j}(u)$ and $F'_{\nu,j}(u)$ are given by eqn. (3.7) and (3.8) respectively.
This expression gives the expected value of the multiplicity of part size j in a random composition of an integer ν. □

3.3 Future scope of research

The Integer Partitions have been studied for a long time in context of combinatorial, analytical and statistical aspects, but its probabilistic aspect was studied by Erdös and Lehner [9]. From the probabilistic prespective, they considered $P(n)$, the set of all partitions of an integer n, as the probability space with uniform probability measure. Erdos and Lehner, studied the limiting distribution of the total number of parts in the partition.

Corteel, Pittel, Savage, and Wilf [11] provided an argument to the following question. Consider a sampling procedure with two steps. Firstly, choose a partition a of n uniformly at random. Then from that partition a, pick one part size out of all different part sizes in a uniformly at random. Then what is the unconditional probability that randomly choosen part size has a multiplicity m ? This case can be discussed in context to n-colour compositions. The multiplicity of parts in a random n-colour composition of a large integer can also be discussed.

The generating functions discussed in Lemma 1 of this chapter can be discussed in more detail with main focus on finding the singularties of the resulting function. The main results in this chapter can be generalised more so that there is a better understanding to the concept.

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