EFFICIENT METHODS FOR SOLVING SOME OPTIMIZATION PROBLEMS UNDER FUZZY ENVIRONMENT AND THEIR EXTENSIONS

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CERTIFICATE

This is to certify that the thesis entitled, "**Efficient methods for solving some optimization problems under fuzzy environment and their extensions**" submitted by **Tanveen Kaur Bhatia** in the fulfilment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar Institute of Engineering & Technology, Patiala, is a record of candidate's own work carried out by her under our supervision and guidance.

The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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DECLARATION

It is certified that the thesis is entirely my own. The ideas and references cited herein have been duly acknowledged.

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Patiala (**TANVEEN KAUR BHATIA**)

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List of abbreviations

CrIpP	Crisp integer programming problem
CrLFpP	Crisp linear fractional programming problem
CrLpP	Crisp linear programming problem
CrLpPs	Crisp linear programming problems
CrMoLFpP	Crisp multi-objective linear fractional programming problem
CrMoLpP	Crisp multi-objective linear programming problem
CrMpP	Crisp mathematical programming problem
CrMpPs	Crisp mathematical programming problems
CrSpP	Crisp shortest path problem
CrTp	Crisp transportation problem
FlFuLFpP	Fully fuzzy linear fractional programming problem
FlFuLFpPs	Fully fuzzy linear fractional programming problems
FlFuLFtP	Fully fuzzy linear fractional transportation problem
FlFuLFtPs	Fully fuzzy linear fractional transportation problems
FlFuLpP	Fully fuzzy linear programming problem
FlFuMpP	Fully fuzzy mathematical programming problem
FlItFuLpP	Fully intuitionistic fuzzy linear programming problem
FuLFpP	Fuzzy linear fractional programming problem
FuLFMiCfP	Fuzzy linear fractional minimal cost flow problem
FuLFMiCfPs	Fuzzy linear fractional minimal cost flow problems
FuLFtP	Fuzzy linear fractional transportation problem
FuLFtPs	Fuzzy linear fractional transportation problems

FuLpP	Fuzzy linear programming problem
FuMpP	Fuzzy mathematical programming problem
IvPyFuD	Interval-valued Pythagorean fuzzy distance
IvPyFuN	Interval-valued Pythagorean fuzzy number
IvPyFuNs	Interval-valued Pythagorean fuzzy numbers
IvPyFuSpP	Interval-valued Pythagorean fuzzy shortest path problem
IvPyFuSpPs	Interval-valued Pythagorean fuzzy shortest path problems
IvTFuD	Interval-valued triangular fuzzy distance
IvTFuLpP	Interval-valued triangular fuzzy linear programming problem
IvTFuN	Interval-valued triangular fuzzy number
IvTFuNs	Interval-valued triangular fuzzy numbers
IvTFuSpP	Interval-valued triangular fuzzy shortest path problem
IvTFuSpPs	Interval-valued triangular fuzzy shortest path problems
IvTrFuSpP	Interval-valued trapezoidal fuzzy shortest path problem
IvTrFuSpPs	Interval-valued trapezoidal fuzzy shortest path problems
IvTrFuD	Interval-valued trapezoidal fuzzy distance
IvTrFuN	Interval-valued trapezoidal fuzzy number
IvTrFuNs	Interval-valued trapezoidal fuzzy numbers
IvTrFuSpP	Interval-valued trapezoidal fuzzy shortest path problem
IvTrFuSpPs	Interval-valued trapezoidal fuzzy shortest path problems
LFMiCfP	Linear fractional minimal cost flow problem
LFpP	Linear fractional programming problem
LFpPs	Linear fractional programming problems

LFtP	Linear fractional transportation problem
LFtPs	Linear fractional transportation problems
LpP	Linear programming problem
LpPs	Linear programming problems
NoTFlItFuTpS	Normal triangular fully intuitionistic fuzzy transportation problems
NoTItFuN	Normal triangular intuitionistic fuzzy number
NoTItFuNs	Normal triangular intuitionistic fuzzy numbers
NoTrFlItFuTpS	Normal trapezoidal fully intuitionistic fuzzy transportation problems
NoTrItFuN	Normal trapezoidal intuitionistic fuzzy number
NoTrItFuNs	Normal trapezoidal intuitionistic fuzzy numbers
PyFuN	Pythagorean fuzzy number
PyFuNs	Pythagorean fuzzy numbers
PyFuTp	Pythagorean fuzzy transportation problem
PyFuTpS	Pythagorean fuzzy transportation problems
SpP	Shortest path problem
SpPs	Shortest path problems
SvNeLpP	Single-valued neutrosophic linear programming problem
SvNeLpPs	Single-valued neutrosophic linear programming problems
SvNeN	Single-valued neutrosophic number
SvNeNs	Single-valued neutrosophic numbers
SvTNeLFpP	Single-valued triangular neutrosophic linear fractional programming problem
SvTNeLFpPs	Single-valued triangular neutrosophic linear fractional programming problems

SvTNeIpP	Single-valued triangular neutrosophic integer programming problem
SvTNeIpPs	Single-valued triangular neutrosophic integer programming problems
SvTNeLpP	Single-valued triangular neutrosophic linear programming problem
SvTNeLpPs	Single-valued triangular neutrosophic linear programming problems
SvTNeN	Single-valued triangular neutrosophic number
SvTNeNs	Single-valued triangular neutrosophic numbers
SvTrNeLFpP	Single-valued trapezoidal neutrosophic linear fractional programming problem
SvTrNeLFpPs	Single-valued trapezoidal neutrosophic linear fractional programming problems
SvTrNeLpP	Single-valued trapezoidal neutrosophic linear programming problem
SvTrNeLpPs	Single-valued trapezoidal neutrosophic linear programming problems
SvTrNeN	Single-valued trapezoidal neutrosophic number
SvTrNeNs	Single-valued trapezoidal neutrosophic numbers
TFuN	Triangular fuzzy number
TFuNs	Triangular fuzzy numbers
TrFuN	Trapezoidal fuzzy number
TrFuNs	Trapezoidal fuzzy numbers
TsFuD	T-spherical fuzzy distance
TsFuN	T-spherical fuzzy number
TsFuNs	T-spherical fuzzy numbers
TsFuSpP	T-spherical fuzzy shortest path problem
TsFuSpPs	T-spherical fuzzy shortest path problems

ABSTRACT

After reviewing the literature, it may be concluded that

(i) Only the existing methods [9, 10, 110] are proposed for solving FlFuLFtPs.

(ii) There does not exist any method except Mahmoodirad et al.'s method [117] for solving FuLFMiCfPs.

(iii) Only the existing methods [102, 127, 163] are proposed for solving PyFuTpS (transportation problems in which the transportation cost for supplying one unit quantity of the product from a source to a destination is represented by a PyFuN. While, all other parameters are represented by a non-negative real number).

(iv) There does not exist any method except Ebrahimnejad et al.'s method [57] for solving IvTFuSpPs.

(v) There does not exist any method except Enayattabar et al.'s method [63] for solving IvPyFuSpPs.

(vi) Only the existing methods [162, 175] are proposed for solving TsFuSpPs.

(vii) Only the existing method [100] is proposed for solving SvNeLpPs by considering the attitude of the decision maker towards the risk.

In this thesis, it is pointed out that

(i) It is inappropriate to use the existing methods [9, 10, 110] for solving FlFuLFtPs. Also, an efficient method (named as Mehar method) is proposed for solving FlFuLFtPs.

(ii) It is inappropriate to use the existing method [117] for solving FuLFMiCfPs. Also, an efficient method (named as Mehar method) is proposed for solving FuLFMiCfPs.

(iii) It is inappropriate to use the existing methods [102, 127, 163] for solving PyFuTpS. Also, an efficient method (named as Mehar method) is proposed for solving PyFuTpS.

(iv) Much computational efforts are required to apply Ebrahimnejad et al.'s method [57] for solving IvTFuSpPs. Also, an efficient method (named as Mehar method) is proposed for solving IvTFuSpPs.

(v) It is inappropriate to use Enayattabar et al.'s method [63] for solving IvPyFuSpPs. Also, it is pointed out that the reasons for the inappropriateness in Enayattabar et al.'s method [63] are the inappropriateness of existing expression [63] to evaluate sum of IvPyFuNs as well as the inappropriateness of existing method [63] for comparing two IvPyFuNs. Then, an appropriate expression to evaluate sum of IvPyFuNs is proposed. Thereafter, the existing method [78] for comparing two interval-valued intuitionistic fuzzy numbers is extended for comparing two IvPyFuNs. Finally, using the proposed expression as well as the extended method for comparing IvPyFuNs, an efficient method (named as Mehar method) is proposed for solving IvPyFuSpPs.

(vi) It is inappropriate to use the existing methods [162, 175] for solving TsFuSpPs. Also, it is pointed out that the reasons for the inappropriateness in the existing methods [162, 175] are the inappropriateness of existing expression [162, 175] to evaluate sum of TsFuNs as well as the inappropriateness of existing methods [162, 175] for comparing two TsFuNs. Then, an appropriate expression to evaluate sum of TsFuNs is proposed. Thereafter, by aggregating the existing methods [5, 115] for comparing two TsFuNs, a new method is proposed for comparing two TsFuNs. Finally, using the proposed expression as well as the proposed method for comparing TsFuNs, an efficient method (named as Mehar method) is proposed for solving TsFuSpPs.

(vii) It is inappropriate to use the existing method [100] for solving SvNeLpPs. Also, an efficient method (named as Mehar method) is proposed for solving SvNeLpPs.

List of published / communicated papers

1. Tanveen Kaur Bhatia, Amit Kumar, M. K. Sharma, Mehar approach to solve fuzzy linear fractional transportation problems, Soft Computing 26 (2022) 11525-11551. **(SCIE Journal: Impact Factor: 3.732)**
2. Tanveen Kaur Bhatia, Amit Kumar, M. K. Sharma, S. S. Appadoo, Mehar approach to solve neutrosophic linear programming problems using possibilistic mean, Soft Computing 26 (2022) 8479-8495. **(SCIE Journal: Impact Factor: 3.732)**
3. Tanveen Kaur Bhatia, Amit Kumar, M. K. Sharma, S. S. Appadoo, Mehar approach to solve fuzzy linear fractional minimal cost flow problems, Journal of Intelligent & Fuzzy Systems 43 (2022) 1035-1051. **(SCIE Journal: Impact Factor: 1.737)**
4. Tanveen Kaur Bhatia, Amit Kumar, S. S. Appadoo, Yuvraj Gajpal, M. K. Sharma, Mehar approach for finding shortest path in supply chain network, Sustainability 13 (2021) 4016. **(SCIE Journal: Impact Factor: 3.889)**
5. Tanveen Kaur Bhatia, Amit Kumar, S. S. Appadoo, M. K. Sharma, A note on "A new ranking approach for solving fully fuzzy transportation problem in intuitionistic fuzzy environment" in Journal of Control, Automation and Electrical Systems. **(Accepted, Scopus Indexed Journal)**
6. Tanveen Kaur Bhatia, Amit Kumar, M. K. Sharma, S. S. Appadoo, Mehar approach for solving shortest path problems with interval-valued triangular fuzzy arc weights in International Journal of Fuzzy System Applications. **(Accepted, Scopus Indexed Journal)**
7. Tanveen Kaur Bhatia, Amit Kumar, M. K. Sharma, S. S. Appadoo, A mathematical programming approach to solve T-spherical fuzzy shortest path problems (Communicated in International Journal of Fuzzy Systems)

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List of attended Workshops / Seminars / Webinars / Conferences / Short term courses

1. Workshop entitled "MATLAB and DIFFERENTIAL EQUATIONS" organized by Thapar Mathematical Society, School of Mathematics, Thapar Institute of Engineering & Technology, Patiala, Punjab, India, during February 15, 2020.
2. Online short-term course entitled "Optimization Theory, Methods and Applications" organized by Department of Mathematics, IIT Roorkee, UK, India, during August 18-20, 2020.
3. Five days online short-term course entitled "Recent Trends in Artificial Neural Networks and Optimization" organized by National Institute of Technology Hamirpur, HP, India, during December 18-22, 2020.
4. Two days International Webinar entitled "Modern Nature Inspired Optimization Techniques" organized by Department of Mathematics, Indira Gandhi National Tribal University, Amarkantak, MP, India, during January 24-25, 2021.
5. Virtual International Conference entitled "Soft Computing, Optimization Theory and Applications" organized at BIT Mesra, Ranchi, India, during March 26-27, 2021 and delivered a contributed talk titled "A fuzzy logic based approach to solve interval multi-objective non-linear transportation problem: Suggested modifications".
6. International Workshop "Advances in Mathematics using Research Tools (AMURT-2022)" organized by Department of Mathematics, Akal University, Talwandi Sabo, Bathinda, Punjab, India, during March 25-26, 2022.

Chapter 1

Introduction

1.1 Introduction

The aim of each company/industry is to provide final product to customers at minimum possible cost. Since, various parties like suppliers, manufacturers, transporters, retailers, warehouses etc. are involved directly or indirectly, known as supply chain, to supply the product to customers. Therefore, to achieve the company's/industry's objective, a management (named as supply chain management) is required to minimize the costs of all the involved parties. Transportation cost play an important role to achieve company's/industry's objective as the cost of a final product can be minimized by minimizing the total transportation cost. Furthermore, as the total transportation cost depends upon the route followed for transporting the product between two places. Therefore, the travelled distance between two places is an important factor to reduce the total transportation cost.

Cost minimization transportation problem and distance minimization SpP are two important topics of Operations Research [4, 18, 33, 39, 90, 161]. The aim of a cost minimization transportation problem is to find the quantity of the product to be supplied from one place to another place so that the total transportation cost is minimum. While, the aim of a distance minimization SpP is to find the route having the shortest distance between two places.

In general, a cost minimization transportation problem is solved by considering the following assumptions.

(i) The product will be directly supplied from a source to a destination i.e., there will be no storage of the product at any place.

(ii) Single mode of conveyance (train or truck or ship or cargo plane etc.) will be used to transport the product from a source to a destination.

(iii) There is only one objective i.e., to minimize the total transportation cost.

(iv) All the parameters are known precisely.

While, these assumptions are not always valid in real-life problems due to the following reasons.

(i) The product is stored at different places (generally called warehouses/distribution centers).

(ii) More than one mode of conveyances (train, truck, ship and cargo plane etc.) are used to transport the product from a source to a destination.

(iii) There may be more than one objective like to maximize profit, to minimize cost, to minimize risk, to minimize delivery time etc.

(iv) The precise values of the parameters are not always known.

To handle (i), (ii) and (iii), different variants of transportation problems named as minimal cost flow problem [4], solid transportation problem [75] and fractional transportation problem [18] or multi-objective transportation problem [107] respectively are proposed in the literature. While, to handle (iv), in the literature [9, 10, 102, 110, 117, 127, 163], fuzzy set and their various extensions [28] are used to represent the imprecise values of the parameters of transportation problems and its variants.

Similarly, a distance minimization SpP is solved by considering the assumption that all the parameters are known precisely.

But, in reality this assumption does not hold true. Due to the same reason, in the literature [57, 63, 162, 175], fuzzy set and their various extensions [28] are used to represent the imprecise values of the parameters of SpPs.

1.2 Literature review

In this section, some recently proposed methods for solving

(i) Transportation problems and its variants under fuzzy environment and their various extensions are discussed in a brief manner (see Kacher and Singh [86], Chhibber et al. [36], Kaur et al. [93] and references therein for more details).

(ii) SpPs under fuzzy environment and their various extensions are discussed in a brief manner (see Broumi et al. [27] and references therein for more details).

(iii) Mathematical programming problems under neutrosophic environment [151] (see Ghanbari et al. [68], Kaur and Kumar [94], Nasseri et al. [129] and references therein for more details).

1.2.1 A brief review of some existing methods for solving transportation problems and its variants under fuzzy environment and their various extensions

Liu [110] proposed a method to solve triangular FlFuLFtPs. In Liu's method [110] firstly, a FuMpP having crisp decision variables is obtained corresponding to a FuLFtP. Then, the obtained FuMpP is transformed into two equivalent CrLpPs. Finally, the transformed CrLpPs are solved to obtain an optimal solution and the fuzzy optimal value of the triangular FlFuLFtP.

Anukokila et al. [9] proposed a method for comparing two TFuNs. Using the proposed comparing method, Anukokila et al. [9] proposed a method to solve triangular FlFuLFtPs. In Anukokila et al.'s method [9] firstly, a FuMpP having fuzzy decision variables is obtained corresponding to a triangular FlFuLFtP. Then, the obtained FuMpP is transformed into its equivalent CrMoLFpP. Thereafter, using the lexicographic approach [18], an efficient solution of the transformed CrMoLFpP is obtained. Finally, using the obtained efficient solution, a fuzzy optimal solution and the corresponding fuzzy optimal value of the triangular FlFuLFtP is obtained.

Mahmoodirad et al. [117] proposed a method to solve triangular/trapezoidal FuLFMiCfPs. In Mahmoodirad et al.'s method [117] firstly, a FuMpP having crisp decision variables is obtained corresponding to a triangular/trapezoidal FuLFMiCfP. Then, the obtained FuMpP is transformed into two equivalent CrLpPs. Finally, the transformed CrLpPs are solved to obtain an optimal solution and the fuzzy optimal value of the triangular/trapezoidal FuLFMiCfP.

Anukokila and Radhakrishnan [10] proposed a method, based on the existing method [153], to solve trapezoidal FlFuLFtPs. In Anukokila and Radhakrishnan's method [10] firstly, a FuMpP having fuzzy decision variables is obtained corresponding to a trapezoidal FlFuLFtP. Then, the obtained FuMpP is transformed into its equivalent CrMpP. Finally, the optimal solution of the transformed CrMpP is used to obtain a fuzzy optimal solution and the fuzzy optimal value of the trapezoidal FlFuLFtP.

Kumar et al. [102] proposed a method to solve PyFuTpS. In Kumar et al.'s method [102] firstly, using the score function, a PyFuTp is transformed into its equivalent CrTp. Then, the transformed CrTp is solved to obtain an optimal solution of the PyFuTp.

Umamageswari and Uthra [163] proposed a new score function for comparing two PyFuNs. Also, Umamageswari and Uthra [163] used their proposed score function in Kumar et al.'s method [102] for solving PyFuTpS.

Nagar et al. [127] pointed out some drawbacks of the existing score functions [114, 134, 177] for comparing two PyFuNs. Also, to resolve the drawbacks of the existing score functions, Nagar et al. [127] proposed a new score function for comparing two PyFuNs. Furthermore, Nagar et al. [127] used their proposed score function in Kumar et al.'s method [102] for solving PyFuTpS.

1.2.2 A brief review of some existing methods for solving SpPs under fuzzy environment and their various extensions

Enayattabar et al. [63] claimed that till now no one has used an IvPyFuN to represent imprecise parameters for SpPs. To fill this gap, Enayattabar et al. [63] proposed IvPyFuSpPs. Enayattabar et al. [63] also extended the existing crisp Dijkstra's algorithm [54] to interval-valued Pythagorean fuzzy Dijkstra's algorithm to find the shortest IvPyFuD from a source node to a destination node.

Ebrahimnejad et al. [57] claimed that till now no one has used an IvTFuN to represent imprecise parameters for SpPs. To fill this gap, Ebrahimnejad et al. [57] proposed IvTFuSpPs. Ebrahimnejad et al. [57] also proposed a method to solve IvTFuSpPs. In Ebrahimnejad et al.'s method [57] firstly, an IvTFuLpP is obtained corresponding to an IvTFuSpP. Then, the obtained IvTFuLpP is transformed into its equivalent CrMoLpP. Finally, the transformed CrMoLpP is solved using the lexicographic approach [18] to obtain the shortest path and the corresponding shortest IvTFuD.

Zedam et al. [175] and Ullah et al. [162] extended the existing crisp Dijkstra's algorithm [54] to solve such SpPs in which distance between every two nodes is represented by a TsFuN.

1.2.3 A brief review of some existing methods for solving mathematical programming problems under neutrosophic environment

Hussian et al. [79] proposed a method to solve SvTNeLpPs. In Hussian et al.'s method [79] firstly, a SvTNeLpP is transformed into its equivalent CrMoLpP. Then, the obtained CrMoLpP is transformed into its equivalent CrLpP. Finally, it is assumed that an optimal solution of the transformed CrLpP also represents an optimal solution of SvTNeLpP.

Hussian et al. [80] proposed a method to solve SvTNeLFpPs. In Hussian et al.'s method [80] firstly, a SvTNeLFpP is transformed into its equivalent CrMoLFpP. Then, the obtained CrMoLFpP is transformed into its equivalent CrMoLpP. After that, the obtained CrMoLpP is transformed into its equivalent CrLpP. Finally, it is assumed that an optimal solution of the transformed CrLpP also represents an optimal solution of SvTNeLFpP.

Abdel-Basset et al. [1] proposed a method for comparing two SvTrNeNs. Then, using the proposed comparing method, Abdel-Basset et al. [1] proposed a method to solve SvTrNeLpPs. In Abdel-Basset et al.'s method [1] firstly, a SvTrNeLpP is transformed into its equivalent CrLpP. Finally, it is assumed that an optimal solution of the transformed CrLpP also represents an optimal solution of SvTrNeLpP.

Singh et al. [148] pointed out that some mathematical incorrect results are considered in Abdel-Basset et al.'s method [1]. Hence, it is inappropriate to use Abdel-Basset et al.'s method [1] in its present form. Singh et al. [148] also suggested some modifications to resolve the inappropriateness of Abdel-Basset et al.'s method [1].

Abdel-Basset et al. [2] proposed a method to solve SvTNeLFpPs. In Abdel-Basset et al.'s method [2] firstly, a SvTNeLFpP is transformed into its equivalent CrMoLFpP. Then, the obtained CrMoLFpP is transformed into its equivalent CrMoLpP. After that, the obtained CrMoLpP is transformed into its equivalent CrLpP. Finally, it is assumed that an optimal solution of the transformed CrLpP also represents an optimal solution of SvTNeLFpP.

Nafei and Nasseri [125] proposed a method for comparing two SvTNeNs. Then, using the proposed comparing method, Nafei and Nasseri [125] proposed a method to solve SvTNeIpPs. In

Nafei and Nasseri’s method [125] firstly, a SvTNeIpP is transformed into its equivalent CrIpP. Finally, it is assumed that an optimal solution of the transformed CrIpP also represents an optimal solution of SvTNeIpP.

Das and Dash [41] pointed out that it is inappropriate to use Hussian et al.’s method [79] for solving SvTNeLpPs. Das and Dash [41] also suggested to use Nafei and Nasseri’s method [125] for solving SvTNeLpPs.

Das and Edalatpanah [42] pointed out that a mathematical incorrect result is considered in Nafei and Nasseri’s method [125]. Hence, it is inappropriate to use Nafei and Nasseri’s method [125]. Das and Edalatpanah [42] also proposed a method to solve SvTNeIpPs. In Das and Edalatpanah’s method [42] firstly, a SvTNeIpP is transformed into its equivalent CrIpP. Finally, it is assumed that an optimal solution of the transformed CrIpP also represents an optimal solution of SvTNeIpP.

Khatter [100] pointed out that although several methods are proposed in the literature to solve SvNeLpPs. However, all the methods for comparing SvNeNs, used in existing methods, are independent from the attitude of the decision maker towards the risk. To fill this gap, Khatter [100] proposed a method for comparing two SvNeNs by considering the attitude of the decision maker towards the risk. Then, using the proposed comparing method, Khatter [100] proposed a method to solve SvNeLpPs. In Khatter’s method [100], a SvNeLpP is transformed into its equivalent CrLpP. Finally, it is assumed that an optimal solution of the transformed CrLpP also represents an optimal solution of SvNeLpP.

Badr et al. [16] proposed a method for comparing two SvTrNeNs. Then, using the proposed comparing method, Badr et al. [16] generalized the crisp two-phase simplex algorithm for solving SvTrNeLpPs.

Das et al. [43] proposed a method to solve SvTNeLFpPs. In Das et al.'s method [43] firstly, a SvTNeLFpP is split into two equivalent SvNeLpPs. Then, the obtained SvNeLpPs are transformed into their equivalent CrLpPs. Finally, it is assumed that both optimal solutions of the transformed CrLpPs also represents an optimal solution of SvTNeLFpP.

Abdelfattah [3] proposed a method to solve SvTNeLpPs. In Abdelfattah's method [3] firstly, a SvTNeLpP is split into two CrLpPs. Then, the obtained CrLpPs are solved independently. Finally, it is assumed that both optimal solutions of the transformed CrLpPs also represents an optimal solution of SvTNeLpP.

Kar et al. [89] proposed a simplex algorithm for solving SvTNeLpPs, Badr et al. [15] proposed a simplex algorithm for solving SvTrNeLpPs and Rabie et al. [137] proposed a two-phase simplex algorithm for solving SvTrNeLpPs.

Das et al. [44] proposed a method to solve SvTrNeLpPs. In Das et al.'s method [44] firstly, a SvTrNeLpP is transformed into its equivalent CrMoLpP. Then, using a lexicographic approach, the transformed CrMoLpP is solved. Finally, it is assumed that an efficient solution of the transformed CrMoLpP also represents an optimal solution of SvTrNeLpP.

ElHadidi et al. [58] proposed a method for comparing two SvTrNeNs. Then, using the proposed comparing method, ElHadidi et al. [58] proposed a method to solve SvTrNeLpPs. In ElHadidi et al.'s method [58] firstly, a SvTrNeLpP is transformed into its equivalent CrLpP. Finally, it is assumed that an optimal solution of the transformed CrLpP also represents an optimal solution of SvTrNeLpP.

ElHadidi et al. [59] proposed a method to solve SvTrNeLFpPs. In ElHadidi et al.'s method [59] firstly, a SvTrNeLFpP is transformed into its equivalent CrMoLFpP. Then, the obtained CrMoLFpP is transformed into its equivalent CrMoLpP. After that, the obtained CrMoLpP is transformed into its equivalent CrLpP. Finally, it is assumed that an optimal solution of the transformed CrLpP also represents an optimal solution of SvTrNeLFpP.

1.3 Gaps in the present study

After a deep study of the literature, it is observed that

(i) Some mathematical incorrect results are considered in all the existing methods [9, 10, 110] for solving triangular/trapezoidal FlFuLFtPs. Therefore, it is inappropriate to use existing methods for solving triangular/trapezoidal FlFuLFtPs.

(ii) Some mathematical incorrect results are considered in the existing method [117] for solving triangular/trapezoidal FuLFMiCfPs. Therefore, it is inappropriate to use existing method for solving triangular/trapezoidal FuLFMiCfPs.

(iii) Some mathematical incorrect results are considered in all the existing methods [102, 127, 163] for solving PyFuTpS. Therefore, it is inappropriate to use existing methods for solving PyFuTpS.

(iv) Much computational efforts are required to apply Ebrahimnejad et al.'s method [57] for solving IvTFuSpPs.

(v) Some mathematical incorrect results are considered in the existing method [63] for solving IvPyFuSpPs. Therefore, it is inappropriate to use existing method for solving IvPyFuSpPs.

(vi) Some mathematical incorrect results are considered in all the existing methods [162, 175] for solving TsFuSpPs. Therefore, it is inappropriate to use existing methods for solving TsFuSpPs.

(vii) A mathematical incorrect result is considered in the existing method [100] for solving SvNeLpPs by considering the attitude of the decision maker towards the risk.

1.4 Objectives

The objectives of the present study is to propose an efficient method for solving

(i) FlFuLFtPs.

(ii) FuLFMiCfPs.

(iii) PyFuTpS.

(iv) IvTFuSpPs.

(v) IvPyFuSpPs.

(vi) TsFuSpPs.

(vii) SvNeLpPs

1.5 Brief review about the work

The chapter wise summary of thesis is as follows:

Chapter 2

Efficient Method for Solving Fully Fuzzy Linear Fractional Transportation Problems

After reviewing the literature, it may be concluded that only the existing methods [9, 10, 110] are proposed for solving FlFuLFtPs. In this chapter,

(i) It is pointed out that it is inappropriate to use the existing methods [9, 10, 110].

(ii) An efficient method is proposed for solving FlFuLFtPs.

Chapter 3

Efficient Method for Solving Fuzzy Linear Fractional Minimal Cost Flow Problems

After reviewing the literature, it may be concluded that there does not exist any method except Mahmoodirad et al.'s method [117] for solving FuLFMiCfPs. In this chapter,

(i) It is pointed out that it is inappropriate to use the existing method [117].

(ii) An efficient method is proposed for solving FuLFMiCfPs.

Chapter 4

Efficient Method for Solving Pythagorean Fuzzy Transportation Problems

In this chapter, it is pointed out that some mathematical incorrect results are considered in all the existing methods [102, 127, 163] for solving PyFuTpS (transportation problems in which the transportation cost for supplying one unit quantity of the product from a source to a destination is represented by a PyFuN. While, all other parameters are represented by a non-negative real number). Therefore, it is inappropriate to use any of the existing methods [102, 127, 163] for solving PyFuTpS. Also, a new method (named as Mehar method) is proposed for solving PyFuTpS. Finally, the proposed Mehar method is illustrated with the help of a numerical example.

Chapter 5

Efficient Methods for Solving Shortest Path Problems Under Fuzzy Environment and Their Extensions

In this chapter, limitations and/or shortcomings of some recently proposed methods for solving SpPs under fuzzy environment and their extensions [57, 63, 162, 175] are discussed. Also, to resolve the shortcomings and/or to overcome the limitations, efficient methods are proposed.

Chapter 6

Efficient Method for Solving Linear Programming Problems Under Neutrosophic Environment

Khatter [100] pointed out that although several methods are proposed in the literature to solve SvNeLpPs (LpPs in which all the parameters except decision variables are either represented by SvTNeNs or SvTrNeNs). However, all the methods for comparing SvNeNs, used in existing methods, are independent from the attitude of the decision maker towards the risk. To fill this gap, Khatter [100], proposed a method for comparing two SvNeNs by considering the attitude of the decision maker towards the risk. Then, using the proposed comparing method, Khatter [100] proposed a method for solving SvNeLpPs.

In this chapter,

(i) It is pointed out that a mathematical incorrect result is considered in Khatter's method [100].

(ii) It is pointed out that some mathematical incorrect results are considered in other existing methods for solving SvNeLpPs.

(iii) An efficient method (named as Mehar method) is proposed for solving SvNeLpPs.

Chapter 7

Future Scope

In this chapter, some open research problems are discussed.

Appendix A

Nishad and Abhishekh [130] proposed a method to solve NoTrFlItFuTpS/NoTFlItFuTpS (transportation problems in which each parameter is represented by a NoTrItFuN/NoTItFuN). Nishad and Abhishekh [130] claimed that their proposed method is superior than the existing methods [8, 103, 149]. In this appendix, it is shown that Nishad and Abhishekh [130] have considered a mathematical incorrect result in their proposed method. Therefore, it is inappropriate to use Nishad and Abhishekh's method [130] to solve NoTrFlItFuTpS/NoTFlItFuTpS.

Chapter 2

Efficient Method for Solving Fully Fuzzy Linear Fractional Transportation Problems[1]

After reviewing the literature, it may be concluded that only the existing methods [9, 10, 110] are proposed for solving FlFuLFtPs. In this chapter,

(i) It is pointed out that it is inappropriate to use the existing methods [9, 10, 110].

(ii) An efficient method is proposed for solving FlFuLFtPs.

2.1 Preliminaries

In this section, some basic definitions are discussed.

Definition 2.1.1 [21] Let X be a universal set. Then, the set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle : x \in X\}$, defined over the universal set X, is said to be a fuzzy set, where $\mu_{\tilde{A}}: X \to [0,1]$ is said to be the membership function and the value $\mu_{\tilde{A}}(x)$ is called the degree of membership for x belongs to the set $\tilde{A}$.

Definition 2.1.2 [21] Let $\tilde{A}$ be a fuzzy set defined over the universal set X and $\alpha \in (0,1]$. Then, the crisp set $A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$ is said to be the α-cut of the fuzzy set $\tilde{A}$.

Definition 2.1.3 [21] Let $\tilde{A}$ be a fuzzy set defined over the universal set X. Then, the crisp set $S(\tilde{A}) = \{x \in X : \mu_{\tilde{A}}(x) > 0\}$ is said to be the support of the fuzzy set $\tilde{A}$.

Definition 2.1.4 [21] Let $\tilde{A}$ be a fuzzy set defined over the universal set X. Then, the crisp number $h(\tilde{A}) = \underset{x \in X}{\text{supremum}} \{\mu_{\tilde{A}}(x)\}$ is said to be the height of the fuzzy set $\tilde{A}$. If $h(\tilde{A}) = 1$, then the fuzzy set $\tilde{A}$ is said to be a normal fuzzy set.

[1] The contents of this chapter are published in "Soft Computing 26 (2022) 11525-11551".

Definition 2.1.5 [21] A fuzzy set $\tilde{A}$ defined over the set of real numbers is said to be fuzzy number if it satisfies the following conditions

(i) $\tilde{A}$ is normal,

(ii) A_α is a closed interval for every $\alpha \in (0,1]$,

(iii) The support of $\tilde{A}$ is bounded.

Definition 2.1.6 [21] A fuzzy number $\tilde{A}$ is said to be TFuN if its membership function is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a_1, x > a_3, \\ \frac{x-a_1}{a_2-a_1}, & a_1 < x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 < x \leq a_3 \end{cases}.$$

It is represented as $\tilde{A} = (a_1, a_2, a_3)$.

Definition 2.1.7 [21] A fuzzy number $\tilde{A}$ is said to be TrFuN if its membership function is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a_1, x > a_4, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \frac{a_4-x}{a_4-a_3}, & a_3 < x \leq a_4 \end{cases}.$$

It is represented as $\tilde{A} = (a_1, a_2, a_3, a_4)$.

Definition 2.1.8 [180] A TFuN $\tilde{A} = (a_1, a_2, a_3)$ is said to be a non-negative TFuN if $a_1 \geq 0$.

Definition 2.1.9 [180] A TrFuN $\tilde{A} = (a_1, a_2, a_3, a_4)$ is said to be a non-negative TrFuN if $a_1 \geq 0$.

Definition 2.1.10 [180] A TFuN $\tilde{A} = (a_1, a_2, a_3)$ is said to be a positive TFuN if $a_1 > 0$.

Definition 2.1.11 [180] A TrFuN $\tilde{A} = (a_1, a_2, a_3, a_4)$ is said to be a positive TrFuN if $a_1 > 0$.

Definition 2.1.12 [180] A TFuN $\tilde{A} = (a_1, a_2, a_3)$ is said to be zero TFuN if $a_1 = a_2 = a_3 = 0$.

Definition 2.1.13 [180] A TrFuN $\tilde{A} = (a_1, a_2, a_3, a_4)$ is said to be zero TrFuN if $a_1 = a_2 = a_3 = a_4 = 0$.

Definition 2.1.14 [21] Let $\tilde{A} = (a_1, a_2, a_3)$ be a TFuN, then its α-cut is defined as $A_\alpha = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)]; \alpha \in (0,1]$.

Definition 2.1.15 [21] Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ be a TrFuN, then its α-cut is defined as $A_\alpha = [a_1 + \alpha(a_2 - a_1), a_4 - \alpha(a_4 - a_3)]; \alpha \in (0,1]$.

Definition 2.1.16 [180] Two TFuNs $\tilde{A}_1 = (a_{11}, a_{12}, a_{13})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23})$ are said to be equal i.e., $\tilde{A}_1 = \tilde{A}_2$ if $a_{1k} = a_{2k}; k = 1,2,3$.

Definition 2.1.17 [180] Two TrFuNs $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}, a_{14})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}, a_{24})$ are said to be equal i.e., $\tilde{A}_1 = \tilde{A}_2$ if $a_{1k} = a_{2k}; k = 1,2,3,4$.

Definition 2.1.18 [21] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23})$ be two TFuNs, then $\tilde{A}_1 + \tilde{A}_2 = (a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23})$.

Definition 2.1.19 [21] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}, a_{14})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}, a_{24})$ be two TrFuNs, then $\tilde{A}_1 + \tilde{A}_2 = (a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}, a_{14} + a_{24})$.

Definition 2.1.20 [21] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23})$ be two TFuNs, then $\tilde{A}_1 - \tilde{A}_2 = (a_{11} - a_{23}, a_{12} - a_{22}, a_{13} - a_{21})$.

Definition 2.1.21 [21] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}, a_{14})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}, a_{24})$ be two TrFuNs, then $\tilde{A}_1 - \tilde{A}_2 = (a_{11} - a_{24}, a_{12} - a_{23}, a_{13} - a_{22}, a_{14} - a_{21})$.

Definition 2.1.22 [94] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23})$ be two TFuNs, then

$$\tilde{A}_1\tilde{A}_2 = (minimum\{a_{11}a_{21}, a_{11}a_{23}, a_{13}a_{21}, a_{13}a_{23}\}, a_{12}a_{22}, maximum\{a_{11}a_{21}, a_{11}a_{23}, a_{13}a_{21}, a_{13}a_{23}\}).$$

Definition 2.1.23 [94] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}, a_{14})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}, a_{24})$ be two TrFuNs, then

$\tilde{A}_1\tilde{A}_2 =$

$$\begin{pmatrix} minimum\{a_{11}a_{21}, a_{11}a_{24}, a_{14}a_{21}, a_{14}a_{24}\}, minimum\{a_{12}a_{22}, a_{12}a_{23}, a_{13}a_{22}, a_{13}a_{23}\}, \\ maximum\{a_{12}a_{22}, a_{12}a_{23}, a_{13}a_{22}, a_{13}a_{23}\}, maximum\{a_{11}a_{21}, a_{11}a_{24}, a_{14}a_{21}, a_{14}a_{24}\} \end{pmatrix}.$$

Definition 2.1.24 [94] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23})$ be two non-negative TFuNs, then $\tilde{A}_1\tilde{A}_2 = (a_{11}a_{21}, a_{12}a_{22}, a_{13}a_{23})$.

Definition 2.1.25 [94] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}, a_{14})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}, a_{24})$ be two non-negative TrFuNs, then $\tilde{A}_1\tilde{A}_2 = (a_{11}a_{21}, a_{12}a_{22}, a_{13}a_{23}, a_{14}a_{24})$.

Definition 2.1.26 [94] Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ be non-negative TrFuN and $k \geq 0$, then $k\tilde{A} = (ka_1, ka_2, ka_3, ka_4)$.

Definition 2.1.27 [94] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13})$ be a TFuN and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23})$ be a positive TFuN, then $\frac{\tilde{A}_1}{\tilde{A}_2} = \left(minimum\left\{\frac{a_{11}}{a_{21}}, \frac{a_{11}}{a_{23}}, \frac{a_{13}}{a_{21}}, \frac{a_{13}}{a_{23}}\right\}, \frac{a_{12}}{a_{22}}, maximum\left\{\frac{a_{11}}{a_{21}}, \frac{a_{11}}{a_{23}}, \frac{a_{13}}{a_{21}}, \frac{a_{13}}{a_{23}}\right\}\right)$.

Definition 2.1.28 [94] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}, a_{14})$ be a TrFuN and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}, a_{24})$ be a positive TrFuN, then $\frac{\tilde{A}_1}{\tilde{A}_2} = \begin{pmatrix} minimum\left\{\frac{a_{11}}{a_{21}}, \frac{a_{11}}{a_{24}}, \frac{a_{14}}{a_{21}}, \frac{a_{14}}{a_{24}}\right\}, minimum\left\{\frac{a_{12}}{a_{22}}, \frac{a_{12}}{a_{23}}, \frac{a_{13}}{a_{22}}, \frac{a_{13}}{a_{23}}\right\}, \\ maximum\left\{\frac{a_{12}}{a_{22}}, \frac{a_{12}}{a_{23}}, \frac{a_{13}}{a_{22}}, \frac{a_{13}}{a_{23}}\right\}, maximum\left\{\frac{a_{11}}{a_{21}}, \frac{a_{11}}{a_{24}}, \frac{a_{14}}{a_{21}}, \frac{a_{14}}{a_{24}}\right\} \end{pmatrix}$.

Definition 2.1.29 [94] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13})$ be a non-negative TFuN and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23})$ be a positive TFuN, then $\frac{\tilde{A}_1}{\tilde{A}_2} = \left(\frac{a_{11}}{a_{23}}, \frac{a_{12}}{a_{22}}, \frac{a_{13}}{a_{21}}\right)$.

Definition 2.1.30 [94] Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}, a_{14})$ be a non-negative TrFuN and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}, a_{24})$ be a positive TrFuN, then $\frac{\tilde{A}_1}{\tilde{A}_2} = \left(\frac{a_{11}}{a_{24}}, \frac{a_{12}}{a_{23}}, \frac{a_{13}}{a_{22}}, \frac{a_{14}}{a_{21}}\right)$.

2.2 Existing methods for solving FlFuLFtPs

In the literature, several extensions of transportation problems are proposed. The LFtP [18] is one of these extensions. The aim of a LFtP is to determine a way for supplying the quantity of the product from various sources to various destinations so that the cost per unit profit is minimum or the profit per unit cost is maximum.

A LFtP can be classified into the following two categories:

(i) **Balanced LFtP:** If the total availability of the product is equal to the total demand of the product. Then, a LFtP is said to be a balanced LFtP.

(ii) **Unbalanced LFtP:** If the total availability of the product is not equal to the total demand of the product. Then, a LFtP is said to be an unbalanced LFtP.

Several methods are proposed in the literature to solve balanced and unbalanced LFtPs. One of these methods is to solve

(i) The LFpP $(P2.2.1)$ [18] corresponding to a balanced LFtP having m sources and n destinations.

LFpP $(\boldsymbol{P2.2.1})$

$$Minimize\ \left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij}x_{ij}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij}x_{ij}+\beta}\right)$$

Subject to

$$\sum_{j=1}^{n}x_{ij}=A_i, \qquad i=1,2,\ldots,m,$$

$$\sum_{i=1}^{m}x_{ij}=B_j, \qquad j=1,2,\ldots,n,$$

$$x_{ij}\geq 0\ \ \forall\ i,j.$$

where,

(a) The positive real number C_{ij} represents the cost for transporting one unit quantity of the product from the i^{th} source to the j^{th} destination.

(b) The positive real number D_{ij} represents the profit on transporting one unit quantity of the product from the i^{th} source to the j^{th} destination.

(c) The positive real number A_i represents the availability of the product at the i^{th} source.

(d) The positive real number B_j represents the demand of the product at the j^{th} destination.

(e) The non-negative real number x_{ij} represents the quantity of the product to be supplied from the i^{th} source to the j^{th} destination.

(f) The non-negative real numbers θ and β represents the fixed costs.

(ii) The LFpP $(P2.2.2)$ [18] corresponding to an unbalanced LFtP having m sources and n destinations.

LFpP $(\boldsymbol{P2.2.2})$

$$Minimize\ \left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij}x_{ij}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij}x_{ij}+\beta}\right)$$

Subject to

$$\sum_{j=1}^{n}x_{ij} \leq A_i, \qquad i = 1,2,\dots,m,$$

$$\sum_{i=1}^{m}x_{ij} \geq B_j, \qquad j = 1,2,\dots,n,$$

$$x_{ij} \geq 0 \ \ \forall\, i,j.$$

It is pertinent to mention that the LFpPs $(P2.2.1)$ and $(P2.2.2)$ are obtained by considering the assumption that the precise value of each parameter is known. However, it is not a realistic assumption as in real-life situations, some or all the parameters may not be known precisely due to unmanageable factors such as weather, social or economic conditions. Due to the same reasons, various ways are proposed in the literature to deal with imprecise parameters. One of the ways, used in the literature, is to represent an imprecise parameter by a fuzzy set [174].

In the last few years, several researchers have used fuzzy set to represent the imprecise parameters of LFtPs [9, 10, 82, 88, 110, 128, 138]. These LFtPs may be categorized as follows.

(i) **FuLFtPs of type-I**: LFtPs in which the cost and profit for supplying one unit quantity of the product from a source to a destination are represented by positive fuzzy numbers. While, the availability of the product at each source and the demand at each destination are represented by positive real numbers.

(ii) **FuLFtPs of type-II**: LFtPs in which the availability of the product at each source and the demand at each destination are represented by positive fuzzy numbers. While, the cost and profit for supplying one unit quantity of the product from a source to a destination are represented by positive real numbers.

(iii) **FlFuLFtPs**: LFtPs in which all the parameters i.e., the cost and profit for supplying one unit quantity of the product from a source to a destination, the availability of the product at each source and the demand at each destination are represented by positive fuzzy numbers.

Since, FuLFtPs of type-I and type-II are special cases of FlFuLFtPs. Therefore, if a method can be used to solve a FlFuLFtP. Then, the same method can also be used to solve a FuLFtP of type-I and a FuLFtP of type-II. However, the converse is not true. Keeping the same in mind, in this section, only a brief review of existing methods for solving FlFuLFtPs is discussed.

Liu [110] claimed that to find an optimal solution of an unbalanced triangular FlFuLFtP (LFtP in which each known parameter is represented by a positive TFuN) is equivalent to find an optimal solution of its equivalent FuLFpP $(P2.2.3)$. Hence, Liu [110] proposed a method to find an optimal solution of the FuLFpP $(P2.2.3)$.

FuLFpP $(\boldsymbol{P2.2.3})$

$Minimize\ \left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{C}_{ij}x_{ij}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{D}_{ij}x_{ij}+\beta}\right)$

Subject to

$\sum_{j=1}^{n}x_{ij} \leq \tilde{A}_i,$ $\qquad\qquad\qquad\qquad i = 1,2,\dots,m,$

$\sum_{i=1}^{m}x_{ij} \geq \tilde{B}_j,$ $\qquad\qquad\qquad\qquad j = 1,2,\dots,n,$

$x_{ij} \geq 0\ \ \forall\, i,j.$

where,

(i) m represents the number of sources.

(ii) n represents the number of destinations.

(iii) The positive TFuN $\tilde{C}_{ij}$ represents the cost for transporting one unit quantity of the product from the i^{th} source to the j^{th} destination.

(iv) The positive TFuN $\tilde{D}_{ij}$ represents the profit on transporting one unit quantity of the product from the i^{th} source to the j^{th} destination.

(v) The positive TFuN $\tilde{A}_i$ represents the availability of the product at the i^{th} source.

(vi) The positive TFuN $\tilde{B}_j$ represents the demand of the product at the j^{th} destination.

(vii) The non-negative real number x_{ij} represents the quantity of the product to be supplied from the i^{th} source to the j^{th} destination.

(viii) The non-negative real numbers θ and β represents the fixed costs.

Anukokila et al. [9] claimed that to find an optimal solution of a balanced triangular FlFuLFtP is equivalent to find an optimal solution of its equivalent FlFuLFpP $(P2.2.4)$. Hence, Anukokila et al. [9] proposed a method to find an optimal solution of the FlFuLFpP $(P2.2.4)$.

FlFuLFpP $(\boldsymbol{P2.2.4})$

$$Minimize\ \left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{C}_{ij}\tilde{x}_{ij}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{D}_{ij}\tilde{x}_{ij}+\beta}\right)$$

Subject to

$\sum_{j=1}^{n}\tilde{x}_{ij}=\tilde{A}_i,\qquad i=1,2,\ldots,m,$

$\sum_{i=1}^{m}\tilde{x}_{ij}=\tilde{B}_j,\qquad j=1,2,\ldots,n,$

$\tilde{x}_{ij}$ is a non-negative TFuN $\forall\ i,j$.

where,

The non-negative TFuN $\tilde{x}_{ij}$ represents the quantity of the product to be supplied from the i^{th} source to the j^{th} destination.

Anukokila and Radhakrishnan [10] claimed that to find an optimal solution of a balanced trapezoidal FlFuLFtP (LFtP in which each known parameter is represented by a positive TrFuN) is equivalent to find an optimal solution of its equivalent FlFuLFpP $(P2.2.5)$. Hence, Anukokila and Radhakrishnan [10] used the existing method [153] to find an optimal solution of the FlFuLFpP $(P2.2.5)$.

FlFuLFpP $(\boldsymbol{P2.2.5})$

$$Minimize\ \left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{C}_{ij}\tilde{x}_{ij}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{D}_{ij}\tilde{x}_{ij}+\beta}\right)$$

Subject to

$\sum_{j=1}^{n}\tilde{x}_{ij}=\tilde{A}_i,\qquad i=1,2,\ldots,m,$

$\sum_{i=1}^{m}\tilde{x}_{ij}=\tilde{B}_j,\qquad j=1,2,\ldots,n,$

$\tilde{x}_{ij}$ is a non-negative TrFuN $\forall\ i,j$.

where,

(i) The positive TrFuN $\tilde{C}_{ij}$ represents the cost for transporting one unit quantity of the product from the i^{th} source to the j^{th} destination.

(ii) The positive TrFuN $\widetilde{D}_{ij}$ represents the profit on transporting one unit quantity of the product from the i^{th} source to the j^{th} destination.

(iii) The positive TrFuN $\tilde{A}_i$ represents the availability of the product at the i^{th} source.

(iv) The positive TrFuN $\tilde{B}_j$ represents the demand of the product at the j^{th} destination.

(v) The non-negative TrFuN $\tilde{x}_{ij}$ represents the quantity of the product to be supplied from the i^{th} source to the j^{th} destination.

It is pertinent to mention that

(i) In the existing FuLFpP $(P2.2.3)$ [110], corresponding to a FlFuLFtP, the decision variables are considered as non-negative real numbers x_{ij}. While, in the existing FlFuLFpPs $(P2.2.4)$ [9] and $(P2.2.5)$ [10], corresponding to a FlFuLFtP, the decision variables are considered as non-negative fuzzy numbers $\tilde{x}_{ij}$.

(ii) Neither any limitation nor any flaw of the existing FuLFpP $(P2.2.3)$ [110] is discussed in the published papers [9, 10].

(iii) Neither any limitation nor any flaw of the existing method [110] is discussed in the published papers [9, 10].

(iv) Neither any limitation nor any flaw of the existing method [9] is discussed in the published paper [10].

Therefore, the following questions are natural.

(i) It is correct to consider the decision variables as non-negative real numbers or it is correct to consider the decision variables as non-negative fuzzy numbers.

(ii) Which of the existing methods [9, 10, 110] for solving FlFuLFtPs is best?

Keeping the same in mind, in this chapter, it is shown that

(i) It is correct to consider the decision variables as non-negative fuzzy numbers.

(ii) Some mathematical incorrect results are considered in all the existing methods [9, 10, 110] for solving FlFuLFtPs. Hence, it is inappropriate to use any of these existing methods for solving FlFuLFtPs.

2.3 Correct FlFuLFpP corresponding to a FlFuLFtP

In Section 2.2, it is pointed out that the following question may arise in the mind of a researcher working in this research area.

"It is correct to consider the decision variables as non-negative real numbers or it is correct to consider the decision variables as non-negative fuzzy numbers."

The following example clearly indicates that it is correct to consider the decision variables as non-negative fuzzy numbers.

Let us consider that Table 2.1 represents a balanced triangular FlFuLFtP.

Table 2.1 Balanced triangular FlFuLFtP

	D_1	D_2	Availability
S_1	$\tilde{c}_{11} = (5,7,9)$ $\tilde{d}_{11} = (5,9,13)$	$\tilde{c}_{12} = (7,9,11)$ $\tilde{d}_{12} = (6,8,10)$	$(42,44,46)$
S_2	$\tilde{c}_{21} = (4,6,8)$ $\tilde{d}_{21} = (2,6,9)$	$\tilde{c}_{22} = (6,8,10)$ $\tilde{d}_{22} = (5,8,11)$	$(24,28,30)$
Demand	$(43,43,45)$	$(23,29,31)$	

According to Liu's method [110], to solve the balanced triangular FlFuLFtP (represented by Table 2.1) is equivalent to solve the FuLFpP $(P2.3.1)$.

FuLFpP $(\boldsymbol{P2.3.1})$

$$Minimize\left(\frac{(5,7,9)x_{11}+(7,9,11)x_{12}+(4,6,8)x_{21}+(6,8,10)x_{22}}{(5,9,13)x_{11}+(6,8,10)x_{12}+(2,6,9)x_{21}+(5,8,11)x_{22}}\right)$$

Subject to

$x_{11}+x_{12}=(42,44,46)$,

$x_{21}+x_{22}=(24,28,30)$,

$x_{11}+x_{21}=(43,43,45)$,

$x_{12}+x_{22}=(23,29,31)$,

$x_{ij}\geq 0\ \forall\ i,j$.

If x_{11}, x_{12} and x_{21} are considered as basic variables. Then, using the basis matrix $B=\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 1\end{bmatrix}$ and the resource matrix $\tilde{b}=\begin{bmatrix}(42,44,46)\\(24,28,30)\\(43,43,45)\end{bmatrix}$, we have

$$\begin{bmatrix}x_{11}\\x_{12}\\x_{21}\end{bmatrix}=B^{-1}\tilde{b}=\begin{bmatrix}0 & -1 & 1\\ 1 & 1 & -1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}(42,44,46)\\(24,28,30)\\(43,43,45)\end{bmatrix}=\begin{bmatrix}(43,43,45)-(24,28,30)\\(42,44,46)+(24,28,30)-(43,43,45)\\(24,28,30)\end{bmatrix}=\begin{bmatrix}(13,15,21)\\(21,29,33)\\(24,28,30)\end{bmatrix}$$

i.e.,

$x_{11}=(13,15,21)$,

$x_{12}=(21,29,33)$

$x_{21}=(24,28,30)$.

It is obvious that the obtained values of the decision variables x_{11}, x_{12} and x_{21} are TFuNs. Therefore, the correct FuLFpP corresponding to the balanced triangular FlFuLFtP (represented by Table 2.1) is the FlFuLFpP $(P2.3.2)$ instead of the FuLFpP $(P2.3.1)$.

FlFuLFpP $(\boldsymbol{P2.3.2})$

$$Minimize\left(\frac{(5,7,9)\tilde{x}_{11}+(7,9,11)\tilde{x}_{12}+(4,6,8)\tilde{x}_{21}+(6,8,10)\tilde{x}_{22}}{(5,9,13)\tilde{x}_{11}+(6,8,10)\tilde{x}_{12}+(2,6,9)\tilde{x}_{21}+(5,8,11)\tilde{x}_{22}}\right)$$

Subject to

$\tilde{x}_{11}+\tilde{x}_{12}=(42,44,46)$,

$\tilde{x}_{21}+\tilde{x}_{22}=(24,28,30)$,

$\tilde{x}_{11}+\tilde{x}_{21}=(43,43,45)$,

$\tilde{x}_{12}+\tilde{x}_{22}=(23,29,31)$,

$\tilde{x}_{ij}$ is a non-negative TFuN $\forall\ i,j$.

2.4 Inappropriateness of existing methods for solving FlFuLFtPs

The aim of this section to point out that some mathematical incorrect results are considered in the existing methods [9, 10, 110] for solving FlFuLFtPs. Hence, it is inappropriate to use any of these existing methods. Since, to achieve this aim, there is a need to discuss these existing methods. Therefore, firstly, all these existing methods are discussed in a brief manner. Then, the mathematical incorrect results, considered in these existing methods, are pointed out.

2.4.1 A brief review of existing methods

In this section, the existing methods [9, 10, 110] are discussed in a brief manner.

2.4.1.1 Liu's method

Liu [110] proposed the following method to find an optimal solution of the FuLFpP $(P2.2.3)$.

Step 1: Transform the FuLFpP $(P2.2.3)$ into its equivalent CrMpPs $(P2.4.1.1.1)$ and $(P2.4.1.1.2)$.

CrMpP ($\boldsymbol{P2.4.1.1.1}$)

$$Z_\alpha^L = \min_{\substack{(C_{ij})_\alpha^L \le c_{ij} \le (C_{ij})_\alpha^U \\ (D_{ij})_\alpha^L \le d_{ij} \le (D_{ij})_\alpha^U \\ (A_i)_\alpha^L \le a_i \le (A_i)_\alpha^U \\ (B_j)_\alpha^L \le b_j \le (B_j)_\alpha^U \\ \forall\, i,j}} \left[\min \left(\frac{\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \theta}{\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} + \beta} \right) \right]$$

Subject to

$\sum_{j=1}^{n} x_{ij} \le a_i,$ $\qquad i = 1,2, \dots, m,$

$\sum_{i=1}^{m} x_{ij} \ge b_j,$ $\qquad j = 1,2, \dots, n,$

$x_{ij} \ge 0 \;\; \forall\, i, j$

where,

(i) $(C_{ij})_\alpha^L$ is the lower bound of the α −cut for the fuzzy number $\tilde{c}_{ij}$.

(ii) $(C_{ij})_\alpha^U$ is the upper bound of the α −cut for the fuzzy number $\tilde{c}_{ij}$.

(iii) $(D_{ij})_\alpha^L$ is the lower bound of the α −cut for the fuzzy number $\tilde{d}_{ij}$.

(iv) $(D_{ij})_\alpha^U$ is the upper bound of the α −cut for the fuzzy number $\tilde{d}_{ij}$.

(v) $(A_i)_\alpha^L$ is the lower bound of the α −cut for the fuzzy number $\tilde{a}_i$.

(vi) $(A_i)_\alpha^U$ is the upper bound of the α −cut for the fuzzy number $\tilde{a}_i$.

(vii) $(B_j)_\alpha^L$ is the lower bound of the α −cut for the fuzzy number $\tilde{b}_j$.

(viii) $(B_j)_\alpha^U$ is the upper bound of the α −cut for the fuzzy number $\tilde{b}_j$.

CrMpP $(\boldsymbol{P2.4.1.1.2})$

$$Z_\alpha^U = \max_{\substack{(C_{ij})_\alpha^L \le c_{ij} \le (C_{ij})_\alpha^U \\ (D_{ij})_\alpha^L \le d_{ij} \le (D_{ij})_\alpha^U \\ (A_i)_\alpha^L \le a_i \le (A_i)_\alpha^U \\ (B_j)_\alpha^L \le b_j \le (B_j)_\alpha^U \\ \forall\, i,j}} \left[\min\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}x_{ij}+\beta}\right)\right]$$

Subject to

$\sum_{j=1}^{n} x_{ij} \le a_i,$ $\quad i = 1,2,\dots,m,$

$\sum_{i=1}^{m} x_{ij} \ge b_j,$ $\quad j = 1,2,\dots,n,$

$x_{ij} \ge 0 \;\; \forall\, i,j.$

Step 2: Use the following steps to transform the CrMpP $(P2.4.1.1.1)$ into its equivalent CrLpP.

Step 2a: Transform the CrMpP $(P2.4.1.1.1)$ into its equivalent CrMpP $(P2.4.1.1.3)$.

CrMpP $(\boldsymbol{P2.4.1.1.3})$

$$Z_\alpha^L = \min\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}x_{ij}+\beta}\right)$$

Subject to

$\sum_{j=1}^{n} x_{ij} \le a_i,$ $\quad i = 1,2,\dots,m,$

$\sum_{i=1}^{m} x_{ij} \ge b_j,$ $\quad j = 1,2,\dots,n,$

$\left(C_{ij}\right)_\alpha^L \le c_{ij} \le \left(C_{ij}\right)_\alpha^U,$ $\quad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$\left(D_{ij}\right)_\alpha^L \le d_{ij} \le \left(D_{ij}\right)_\alpha^U,$ $\quad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$(A_i)_\alpha^L \le a_i \le (A_i)_\alpha^U,$ $\quad i = 1,2,\dots,m,$

$\left(B_j\right)_\alpha^L \le b_j \le \left(B_j\right)_\alpha^U$ $\quad j = 1,2,\dots,n,$

$x_{ij} \ge 0 \;\; \forall\, i,j.$

Step 2b: Transform the CrMpP $(P2.4.1.1.3)$ into its equivalent CrMpP $(P2.4.1.1.4)$.

CrMpP $(\boldsymbol{P2.4.1.1.4})$

$Z_\alpha^L = \min\ \left(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, y_{ij} + \theta t\right)$

Subject to

$-\sum_{j=1}^{n} y_{ij} + a_i t \geq 0,$ $\quad i = 1,2,\dots,m,$

$\sum_{i=1}^{m} y_{ij} - b_j t \geq 0,$ $\quad j = 1,2,\dots,n,$

$\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}\, y_{ij} + \beta t = 1,$ $\quad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$\left(C_{ij}\right)_\alpha^L \leq c_{ij} \leq \left(C_{ij}\right)_\alpha^U,$ $\quad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$\left(D_{ij}\right)_\alpha^L \leq d_{ij} \leq \left(D_{ij}\right)_\alpha^U,$ $\quad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$(A_i)_\alpha^L \leq a_i \leq (A_i)_\alpha^U,$ $\quad i = 1,2,\dots,m,$

$\left(B_j\right)_\alpha^L \leq b_j \leq \left(B_j\right)_\alpha^U$ $\quad j = 1,2,\dots,n,$

$t > 0, y_{ij} \geq 0,\ \forall\, i,j.$

Step 2c: Transform the CrMpP $(P2.4.1.1.4)$ into its equivalent CrMpP $(P2.4.1.1.5)$.

CrMpP $(\boldsymbol{P2.4.1.1.5})$

$Z_\alpha^L = \min\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left(C_{ij}\right)_\alpha^L y_{ij} + \theta t\right)$

Subject to

$-\sum_{j=1}^{n} y_{ij} + a_i t \geq 0,$ $\quad i = 1,2,\dots,m,$

$\sum_{i=1}^{m} y_{ij} - b_j t \geq 0,$ $\quad j = 1,2,\dots,n,$

$\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}\, y_{ij} + \beta t = 1,$ $\quad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$\left(D_{ij}\right)_\alpha^L \leq d_{ij} \leq \left(D_{ij}\right)_\alpha^U,$ $\quad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$(A_i)_\alpha^L \leq a_i \leq (A_i)_\alpha^U,$ $\quad i = 1,2,\dots,m,$

$\left(B_j\right)_\alpha^L \leq b_j \leq \left(B_j\right)_\alpha^U$ $\quad j = 1,2,\dots,n,$

$t > 0, y_{ij} \geq 0, \ \forall\, i, j.$

Step 2d: Transform the CrMpP $(P2.4.1.1.5)$ into its equivalent CrLpP $(P2.4.1.1.6)$.

CrLpP $(\boldsymbol{P2.4.1.1.6})$

$$Z_\alpha^L = \min\left(\sum_{i=1}^m \sum_{j=1}^n \left(C_{ij}\right)_\alpha^L y_{ij} + \theta t\right)$$

Subject to

$$-\sum_{j=1}^n y_{ij} + r_i \geq 0, \qquad i = 1,2, \dots, m,$$

$$\sum_{i=1}^m y_{ij} - s_j \geq 0, \qquad j = 1,2, \dots, n,$$

$$\sum_{i=1}^m \sum_{j=1}^n \xi_{ij} + \beta t = 1\,, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$$

$$\left(D_{ij}\right)_\alpha^L y_{ij} \leq \xi_{ij} \leq \left(D_{ij}\right)_\alpha^U y_{ij}, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$$

$$(A_i)_\alpha^L t \leq r_i \leq (A_i)_\alpha^U t, \qquad i = 1,2, \dots, m,$$

$$\left(B_j\right)_\alpha^L t \leq s_j \leq \left(B_j\right)_\alpha^U t, \qquad j = 1,2, \dots, n,$$

$$t > 0, y_{ij} \geq 0, \ \forall\, i, j.$$

Step 3: Use the following steps to transform the CrMpP $(P2.4.1.1.2)$ into its equivalent CrLpP.

Step 3a: Transform the CrMpP $(P2.4.1.1.2)$ into its equivalent CrMpP $(P2.4.1.1.7)$.

CrMpP $(\boldsymbol{P2.4.1.1.7})$

$$Z_\alpha^U = \max_{\substack{\left(C_{ij}\right)_\alpha^L \leq c_{ij} \leq \left(C_{ij}\right)_\alpha^U \\ \left(D_{ij}\right)_\alpha^L \leq d_{ij} \leq \left(D_{ij}\right)_\alpha^U \\ (A_i)_\alpha^L \leq a_i \leq (A_i)_\alpha^U \\ \left(B_j\right)_\alpha^L \leq b_j \leq \left(B_j\right)_\alpha^U \\ \forall\, i,j}} \left[\min\ \left(\sum_{i=1}^m \sum_{j=1}^n c_{ij}\, y_{ij} + \theta t\right)\right]$$

Subject to

$$-\sum_{j=1}^n y_{ij} + a_i t \geq 0, \qquad i = 1,2, \dots, m,$$

$$\sum_{i=1}^m y_{ij} - b_j t \geq 0, \qquad j = 1,2, \dots, n,$$

$\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}\, y_{ij} + \beta t = 1, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$

$t > 0, y_{ij} \geq 0, \ \forall\, i, j.$

Step 3b: Transform the CrMpP $(P2.4.1.1.7)$ into its equivalent CrMpP $(P2.4.1.1.8)$.

CrMpP $(\boldsymbol{P2.4.1.1.8})$

$Z_{\alpha}^{U} = \max \lambda$

Subject to

$-u_i + v_j + d_{ij}\lambda \leq c_{ij}, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$

$\sum_{i=1}^{m} a_i u_i - \sum_{j=1}^{n} b_j v_j + \beta\lambda \leq \theta, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$

$\left(C_{ij}\right)_{\alpha}^{L} \leq c_{ij} \leq \left(C_{ij}\right)_{\alpha}^{U}, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$

$\left(D_{ij}\right)_{\alpha}^{L} \leq d_{ij} \leq \left(D_{ij}\right)_{\alpha}^{U}, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$

$(A_i)_{\alpha}^{L} \leq a_i \leq (A_i)_{\alpha}^{U}, \qquad i = 1,2, \dots, m,$

$\left(B_j\right)_{\alpha}^{L} \leq b_j \leq \left(B_j\right)_{\alpha}^{U} \qquad j = 1,2, \dots, n,$

$u_i, v_j \geq 0, \ \ \forall\, i, j, \ \lambda$ unrestricted in sign.

Step 3c: Transform the CrMpP $(P2.4.1.1.8)$ into its equivalent CrMpP $(P2.4.1.1.9)$.

CrMpP $(\boldsymbol{P2.4.1.1.9})$

$Z_{\alpha}^{U} = \max \lambda$

Subject to

$-u_i + v_j + d_{ij}\lambda \leq \left(C_{ij}\right)_{\alpha}^{U}, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$

$\sum_{i=1}^{m} a_i u_i - \sum_{j=1}^{n} b_j v_j + \beta\lambda \leq \theta, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$

$\left(D_{ij}\right)_{\alpha}^{L} \leq d_{ij} \leq \left(D_{ij}\right)_{\alpha}^{U}, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$

$(A_i)_{\alpha}^{L} \leq a_i \leq (A_i)_{\alpha}^{U}, \qquad i = 1,2, \dots, m,$

$\left(B_j\right)_\alpha^L \le b_j \le \left(B_j\right)_\alpha^U \qquad j = 1,2,\dots,n,$

$u_i, v_j \ge 0,\ \ \forall\, i,j,\ \ \lambda > 0.$

Step 3d: Transform the CrMpP $(P2.4.1.1.9)$ into its equivalent CrLpP $(P2.4.1.1.10)$.

CrLpP $(\boldsymbol{P2.4.1.1.10})$

$Z_\alpha^U = \max \lambda$

Subject to

$-u_i + v_j + \delta_{ij} \le \left(C_{ij}\right)_\alpha^U, \qquad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$\sum_{i=1}^m p_i - \sum_{j=1}^n q_j + \beta\lambda \le \theta, \qquad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$\left(D_{ij}\right)_\alpha^L \lambda \le \delta_{ij} \le \left(D_{ij}\right)_\alpha^U \lambda \qquad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$(A_i)_\alpha^L u_i \le p_i \le (A_i)_\alpha^U u_i, \qquad i = 1,2,\dots,m,$

$\left(B_j\right)_\alpha^L v_j \le q_j \le \left(B_j\right)_\alpha^U v_j, \qquad j = 1,2,\dots,n,$

$u_i, v_j \ge 0,\ \ \forall\, i,j,\ \ \lambda > 0.$

Step 4: Find an optimal solution $\left\{\left(x_{ij}\right)_{\alpha=0}\right\}$ and the corresponding optimal value $(Z_{\alpha=0}^L)$ of the CrLpP $(P2.4.1.1.6)$.

Step 5: Find an optimal solution $\left\{\left(x_{ij}\right)_{\alpha=1}\right\}$ and the corresponding optimal value $(Z_{\alpha=1}^L)$ of the CrLpP $(P2.4.1.1.6)$.

Step 6: Find an optimal solution $\left\{\left(x_{ij}\right)_{\alpha=0}\right\}$ and the corresponding optimal value $(Z_{\alpha=0}^U)$ of the CrLpP $(P2.4.1.1.10)$.

Step 7: Find an optimal solution $\left\{\left(x_{ij}\right)_{\alpha=1}\right\}$ and the corresponding optimal value $(Z_{\alpha=1}^U)$ of the CrLpP $(P2.4.1.1.10)$.

Step 8: Using the optimal values, obtained in Step 4 to Step 7, the obtained fuzzy optimal value is

$(Z^L_{\alpha=0}, Z^L_{\alpha=1}, Z^U_{\alpha=1}, Z^U_{\alpha=0})$.

2.4.1.2 Anukokila et al.'s method

Anukokila et al. [9] have used a method for comparing two TFuNs. In this section, firstly, the comparing method, used by Anukokila et al. [9], is discussed. Then, Anukokila et al.'s method [9] is discussed.

2.4.1.2.1 Anukokila et al.'s comparing method

Anukokila et al. [9] proposed the following method for comparing two TFuNs $\tilde{A}_1 = (a_{11}, a_{12}, a_{13})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23})$.

Step 1: Check that $R_1(\tilde{A}_1) > R_1(\tilde{A}_2)$ or $R_1(\tilde{A}_1) < R_1(\tilde{A}_2)$ or $R_1(\tilde{A}_1) = R_1(\tilde{A}_2)$, where

$$R_1(\tilde{A}_l) = a_{l2}; l = 1,2 \tag{2.4.1.2.1.1}$$

Case (i): If $R_1(\tilde{A}_1) > R_1(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (ii): If $R_1(\tilde{A}_1) < R_1(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (iii): If $R_1(\tilde{A}_1) = R_1(\tilde{A}_2)$, then go to Step 2.

Step 2: Check that $R_2(\tilde{A}_1) > R_2(\tilde{A}_2)$ or $R_2(\tilde{A}_1) < R_2(\tilde{A}_2)$ or $R_2(\tilde{A}_1) = R_2(\tilde{A}_2)$, where

$$R_2(\tilde{A}_l) = a_{l3} - a_{l1}; l = 1,2 \tag{2.4.1.2.1.2}$$

Case (i): If $R_2(\tilde{A}_1) > R_2(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (ii): If $R_2(\tilde{A}_1) < R_2(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (iii): If $R_2(\tilde{A}_1) = R_2(\tilde{A}_2)$, then go to Step 3.

Step 3: Check that $R_3(\tilde{A}_1) > R_3(\tilde{A}_2)$ or $R_3(\tilde{A}_1) < R_3(\tilde{A}_2)$ or $R_3(\tilde{A}_1) = R_3(\tilde{A}_2)$, where

$$R_3(\tilde{A}_l) = a_{l3} + a_{l1}; l = 1,2 \tag{2.4.1.2.1.3}$$

Case (i): If $R_3(\tilde{A}_1) < R_3(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (ii): If $R_3(\tilde{A}_1) > R_3(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (iii): If $R_3(\tilde{A}_1) = R_3(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

2.4.1.2.2 Anukokila et al.'s method

Anukokila et al. [9] proposed the following method to find an optimal solution of the FlFuLFpP $(P2.2.4)$.

Step 1: Using Definition 2.1.18, transform the FlFuLFpP $(P2.2.4)$ into the FlFuLFpP $(P2.4.1.2.2.1)$.

FlFuLFpP $(\boldsymbol{P2.4.1.2.2.1})$

$$Minimize\left(\frac{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta\right)}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta,\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta,\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta\right)}\right)$$

Subject to

$$\left(\sum_{j=1}^{n}x_{ij1},\sum_{j=1}^{n}x_{ij2},\sum_{j=1}^{n}x_{ij3}\right) = (A_{i1},A_{i2},A_{i3}), \qquad i = 1,2,\dots,m,$$

$$\left(\sum_{j=1}^{n}x_{ij1},\sum_{j=1}^{n}x_{ij2},\sum_{j=1}^{n}x_{ij3}\right) = \left(B_{j1},B_{j2},B_{j3}\right), \qquad j = 1,2,\dots,n,$$

$\left(x_{ij1},x_{ij2},x_{ij3}\right)$ is a non-negative TFuN $\forall\, i,j$.

Step 2: Using Definition 2.1.8 and Definition 2.1.16, transform the FlFuLFpP $(P2.4.1.2.2.1)$ into the FlFuLFpP $(P2.4.1.2.2.2)$.

FlFuLFpP $(\boldsymbol{P2.4.1.2.2.2})$

$$Minimize\left(\frac{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta\right)}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta,\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta,\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta\right)}\right)$$

Subject to

$$\sum_{j=1}^{n}x_{ijk} = A_{ik}, \qquad i = 1,2,\dots,m, k = 1,2,3,$$

$$\sum_{i=1}^{m}x_{ijk} = B_{jk}, \qquad j = 1,2,\dots,n, k = 1,2,3,$$

$$x_{ij2} - x_{ij1} \geq 0 \ \ \forall\, i,j,$$

$x_{ij3} - x_{ij2} \geq 0 \;\; \forall\, i, j,$

$x_{ij1} \geq 0 \;\; \forall\, i, j.$

Step 3: Using the comparing method, transform the FlFuLFpP $(P2.4.1.2.2.2)$ into the CrMoLFpP $(P2.4.1.2.2.3)$.

CrMoLFpP $(\boldsymbol{P2.4.1.2.2.3})$

$$Minimize\left(R_1\left(\frac{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta, \sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta, \sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta\right)}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta, \sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta, \sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta\right)}\right)\right)$$

$$Maximize\left(R_2\left(\frac{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta, \sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta, \sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta\right)}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta, \sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta, \sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta\right)}\right)\right)$$

$$Minimize\left(R_3\left(\frac{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta, \sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta, \sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta\right)}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta, \sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta, \sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta\right)}\right)\right)$$

Subject to

Constraints of the FlFuLFpP $(P2.4.1.2.2.2)$.

Step 4: Using the relations $R_1\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right) = \frac{R_1(\tilde{A}_1)}{R_1(\tilde{A}_2)}$, $R_2\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right) = \frac{R_2(\tilde{A}_1)}{R_2(\tilde{A}_2)}$ and $R_3\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right) = \frac{R_3(\tilde{A}_1)}{R_3(\tilde{A}_2)}$, transform the CrMoLFpP $(P2.4.1.2.2.3)$ into the CrMoLFpP $(P2.4.1.2.2.4)$.

CrMoLFpP $(\boldsymbol{P2.4.1.2.2.4})$

$$Minimize\left(\frac{R_1\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta, \sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta, \sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta\right)}{R_1\left(\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta, \sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta, \sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta\right)} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta}\right)$$

$$Maximize\left(\frac{R_2\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta, \sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta, \sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta\right)}{R_2\left(\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta, \sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta, \sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta\right)} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(C_{ij3}x_{ij3}-C_{ij1}x_{ij1}\right)+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(D_{ij3}x_{ij3}-D_{ij1}x_{ij1}\right)+\beta}\right)$$

$$Minimize\left(\frac{R_3\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta\right)}{R_3\left(\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta,\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta,\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta\right)}=\right.$$

$$\left.\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij1}x_{ij1}+C_{ij3}x_{ij3})+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}(D_{ij1}x_{ij1}+D_{ij3}x_{ij3})+\beta}\right)$$

Subject to

Constraints of the FlFuLFpP $(P2.4.1.2.2.2)$.

Step 5: Find an optimal solution of the CrLFpP $(P2.4.1.2.2.5)$.

CrLFpP $(\boldsymbol{P2.4.1.2.2.5})$

$$Minimize\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta}\right)$$

Subject to

Constraints of the FlFuLFpP $(P2.4.1.2.2.2)$.

Step 6: Find an optimal solution of the CrLFpP $(P2.4.1.2.2.6)$.

CrLFpP $(\boldsymbol{P2.4.1.2.2.6})$

$$Maximize\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij3}x_{ij3}-C_{ij1}x_{ij1})+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}(D_{ij3}x_{ij3}-D_{ij1}x_{ij1})+\beta}\right)$$

Subject to

Constraints of the FlFuLFpP $(P2.4.1.2.2.2)$ with additional constraint $\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta}=$

p, where, p is the optimal value of the CrLFpP $(P2.4.1.2.2.5)$.

Step 7: Find an optimal solution of the CrLFpP $(P2.4.1.2.2.7)$.

CrLFpP $(\boldsymbol{P2.4.1.2.2.7})$

$$Minimize\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij1}x_{ij1}+C_{ij3}x_{ij3})+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}(D_{ij1}x_{ij1}+D_{ij3}x_{ij3})+\beta}\right)$$

Subject to

Constraints of the CrLFpP $(P2.4.1.2.2.6)$ with additional constraint $\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij3}x_{ij3}-C_{ij1}x_{ij1})+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}(D_{ij3}x_{ij3}-D_{ij1}x_{ij1})+\beta}=q$, where, q is the optimal value of the CrLFpP $(P2.4.1.2.2.6)$.

Step 8: Using the optimal solution $\{x_{ij1},x_{ij2},x_{ij3}\}$, obtained in Step 7, find a fuzzy optimal solution $\{(x_{ij1},x_{ij2},x_{ij3})\}$ and the value of the objective function of the FlFuLFpP $(P2.2.4)$ i.e., $\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij1},C_{ij2},C_{ij3})(x_{ij1},x_{ij2},x_{ij3})+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}(D_{ij1},D_{ij2},D_{ij3})(x_{ij1},x_{ij2},x_{ij3})+\beta}$.

2.4.1.3 Anukokila and Radhakrishnan's method

Anukokila and Radhakrishnan [10] used the existing method [153] to find an optimal solution of the FlFuLFpP $(P2.2.5)$. In this section, Anukokila and Radhakrishnan's method [10] is discussed in a brief manner.

Step 1: Using the following steps of Charnes and Cooper transformation method [34] under fuzzy environment, transform the FlFuLFpP $(P2.2.5)$ into a FlFuLpP.

Step 1a: Assuming $\tilde{t}=\frac{1}{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{D}_{ij}\tilde{x}_{ij}+\beta}$ and $\tilde{x}_{ij}=\frac{\tilde{y}_{ij}}{\tilde{t}}$, transform the FlFuLFpP $(P2.2.5)$ into the FlFuMpP $(P2.4.1.3.1)$.

FlFuMpP $(\boldsymbol{P2.4.1.3.1})$

$$Minimize\left(\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{C}_{ij}\tilde{y}_{ij}}{\tilde{t}}+\theta\right)\tilde{t}\right)$$

Subject to

$$\sum_{j=1}^{n}\frac{\tilde{y}_{ij}}{\tilde{t}}=\tilde{A}_i, \qquad i=1,2,\dots,m,$$

$$\sum_{i=1}^{m}\frac{\tilde{y}_{ij}}{\tilde{t}}=\tilde{B}_j, \qquad j=1,2,\dots,n,$$

$$\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{D}_{ij}\tilde{y}_{ij}}{\tilde{t}}+\beta\right)\tilde{t}=\tilde{1},$$

$\tilde{y}_{ij}, \tilde{t}$ are non-negative TrFuN $\forall\ i, j$.

Step 1b: Transform the FlFuMpP $(P2.4.1.3.1)$ into the FlFuMpP $(P2.4.1.3.2)$.

FlFuMpP $(\boldsymbol{P2.4.1.3.2})$

$$Minimize \left(\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{C}_{ij}\tilde{y}_{ij} + \theta\tilde{t}\right)\frac{\tilde{t}}{\tilde{t}}\right)$$

Subject to

$$\sum_{j=1}^{n}\left(\frac{\tilde{y}_{ij}}{\tilde{t}}\right)\tilde{t} = \tilde{A}_i\tilde{t}, \qquad i = 1,2,\dots,m,$$

$$\sum_{i=1}^{m}\left(\frac{\tilde{y}_{ij}}{\tilde{t}}\right)\tilde{t} = \tilde{B}_j\tilde{t}, \qquad j = 1,2,\dots,n,$$

$$\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\widetilde{D}_{ij}\tilde{y}_{ij} + \beta\tilde{t}\right)\frac{\tilde{t}}{\tilde{t}} = \tilde{1},$$

$\tilde{y}_{ij}, \tilde{t}$ are non-negative TrFuN $\forall\ i, j$.

Step 1c: Transform the FlFuMpP $(P2.4.1.3.2)$ into the FlFuLpP $(P2.4.1.3.3)$.

FlFuLpP $(\boldsymbol{P2.4.1.3.3})$

$$Minimize \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{C}_{ij}\tilde{y}_{ij} + \theta\tilde{t}\right)$$

Subject to

$$\sum_{j=1}^{n}\tilde{y}_{ij} = \tilde{A}_i\tilde{t}, \qquad i = 1,2,\dots,m,$$

$$\sum_{i=1}^{m}\tilde{y}_{ij} = \tilde{B}_j\tilde{t}, \qquad j = 1,2,\dots,n,$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\widetilde{D}_{ij}\tilde{y}_{ij} + \beta\tilde{t} = \tilde{1},$$

$\tilde{y}_{ij}, \tilde{t}$ are non-negative TrFuN $\forall\ i, j$.

Step 1d: Transform the FlFuLpP $(P2.4.1.3.3)$ into the FlFuLpP $(P2.4.1.3.4)$.

FlFuLpP $(\boldsymbol{P2.4.1.3.4})$

$$Minimize \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{C}_{ij}\tilde{y}_{ij} + \theta\tilde{t}\right)$$

Subject to

$\sum_{j=1}^{n} \tilde{y}_{ij} - \tilde{A}_i\tilde{t} = \tilde{A}_i\tilde{t} - \tilde{A}_i\tilde{t},$ $\quad i = 1,2, \dots, m,$

$\sum_{i=1}^{m} \tilde{y}_{ij} - \tilde{B}_j\tilde{t} = \tilde{B}_j\tilde{t} - \tilde{B}_j\tilde{t},$ $\quad j = 1,2, \dots, n,$

$\sum_{i=1}^{m}\sum_{j=1}^{n} \widetilde{D}_{ij}\tilde{y}_{ij} + \beta\tilde{t} = \tilde{1},$

$\tilde{y}_{ij}, \tilde{t}$ are non-negative TrFuN $\forall\ i, j$.

Step 1e: Transform the FlFuLpP $(P2.4.1.3.4)$ into the FlFuLpP $(P2.4.1.3.5)$.

FlFuLpP $(\boldsymbol{P2.4.1.3.5})$

$Minimize \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{C}_{ij}\tilde{y}_{ij} + \theta\tilde{t}\right)$

Subject to

$\sum_{j=1}^{n} \tilde{y}_{ij} - \tilde{A}_i\tilde{t} = \tilde{0},$ $\quad i = 1,2, \dots, m,$

$\sum_{i=1}^{m} \tilde{y}_{ij} - \tilde{B}_j\tilde{t} = \tilde{0},$ $\quad j = 1,2, \dots, n,$

$\sum_{i=1}^{m}\sum_{j=1}^{n} \widetilde{D}_{ij}\tilde{y}_{ij} + \beta\tilde{t} = \tilde{1},$

$\tilde{y}_{ij}, \tilde{t}$ are non-negative TrFuN $\forall\ i, j$.

Step 2: Find a fuzzy optimal solution $\{\tilde{y}_{ij}, \tilde{t}\}$ of the FlFuLpP $(P2.4.1.3.5)$.

Step 3: Using the obtained fuzzy optimal solution $\{\tilde{y}_{ij}, \tilde{t}\}$ and the relation $\tilde{x}_{ij} = \frac{\tilde{y}_{ij}}{\tilde{t}}$, find a fuzzy optimal solution $\{\tilde{x}_{ij}\}$ of the FlFuLFpP $(P2.2.5)$.

2.4.2 Inappropriateness of existing methods

In this section, inappropriateness of the existing methods [9, 10, 110] is pointed out.

2.4.2.1 Inappropriateness of Liu's method

It is inappropriate to use Liu's method [110] as the following mathematical incorrect results are considered in Liu's method [110].

(i) Liu [110] assumed that to find an optimal solution of an unbalanced triangular FlFuLFtP is equivalent to find an optimal solution of the FuLFpP $(P2.2.3)$. Hence, Liu [110] proposed a method to solve the FuLFpP $(P2.2.3)$. While, it is obvious from Section 2.3 that, in actual case, to find an optimal solution of an unbalanced triangular FlFuLFtP is equivalent to find an optimal solution of the FlFuLFpP $(P2.4.2.1.1)$. Therefore, Liu's [110] assumption is not valid and hence, it is inappropriate to use Liu's method [110] to find an optimal solution of an unbalanced triangular FlFuLFtP.

FlFuLFtP $(\boldsymbol{P2.4.2.1.1})$

$Minimize \left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij1},C_{ij2},C_{ij3})(x_{ij1},x_{ij2},x_{ij3})+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}(D_{ij1},D_{ij2},D_{ij3})(x_{ij1},x_{ij2},x_{ij3})+\beta}\right)$

Subject to

$\sum_{j=1}^{n}(x_{ij1},x_{ij2},x_{ij3}) \leq (A_{i1},A_{i2},A_{i3}),$ $\quad i = 1,2,\ldots,m,$

$\sum_{i=1}^{m}(x_{ij1},x_{ij2},x_{ij3}) \geq (B_{j1},B_{j2},B_{j3}),$ $\quad j = 1,2,\ldots,n,$

$(x_{ij1},x_{ij2},x_{ij3})$ is a non-negative TFuN $\forall\, i,j$.

(ii) It is obvious from Step 2c and Step 2d of Liu's method [110], discussed in Section 2.4.1.1, that Liu [110] assumed $c_{ij} = (C_{ij})_{\alpha}^{L}, a_i t = r_i, b_j t = s_j$ and $d_{ij}y_{ij} = \xi_{ij}$ to transform the CrMpP $(P2.4.1.1.4)$ into the CrLpP $(P2.4.1.1.6)$ i.e., Liu [110] assumed that the CrMpP $(P2.4.1.1.4)$ is equivalent to the CrLpP $(P2.4.1.1.6)$. However, the CrLpP $(P2.4.1.1.6)$ is not equivalent to the CrMpP $(P2.4.1.1.4)$.

The following validates this claim.

In the numerical example, considered by Liu [110] to illustrate his proposed method, Liu [110] used $c_{ij} = (C_{ij})_{\alpha}^{L}, a_i t = r_i, b_j t = s_j$ and $d_{ij}y_{ij} = \xi_{ij}$ to transform the CrMpP $(P2.4.2.1.2)$ into the CrLpP $(P2.4.2.1.3)$ i.e., Liu [110] assumed that the CrLpP $(P2.4.2.1.3)$ is equivalent to the CrMpP $(P2.4.2.1.2)$.

CrMpP $(\boldsymbol{P2.4.2.1.2})$

$Z_{\alpha}^{L} = \min\ (c_{11}y_{11} + c_{12}y_{12} + c_{13}y_{13} + c_{14}y_{14} + c_{21}y_{21} + c_{22}y_{22} + c_{23}y_{23} +$

$c_{24}y_{24} + c_{31}y_{31} + c_{32}y_{32} + c_{33}y_{33} + c_{34}y_{34} + 55t)$

Subject to

$-y_{11} - y_{12} - y_{13} - y_{14} + a_1 t \geq 0,$

$-y_{21} - y_{22} - y_{23} - y_{24} + a_2 t \geq 0,$

$-y_{31} - y_{32} - y_{33} - y_{34} + a_3 t \geq 0,$

$y_{11} + y_{21} + y_{31} - b_1 t \geq 0,$

$y_{12} + y_{22} + y_{32} - b_2 t \geq 0,$

$y_{13} + y_{23} + y_{33} - b_3 t \geq 0,$

$y_{14} + y_{24} + y_{34} - b_4 t \geq 0,$

$d_{11}y_{11} + d_{12}y_{12} + d_{13}y_{13} + d_{14}y_{14} + d_{21}y_{21} + d_{22}y_{22} + d_{23}y_{23} + d_{24}y_{24} +$

$d_{31}y_{31} + d_{32}y_{32} + d_{33}y_{33} + d_{34}y_{34} + 60t = 1,$

$(4 + \alpha) \leq c_{11} \leq (6 - \alpha),$

$c_{12} = 6, c_{13} = 2,$

$(2 + \alpha) \leq c_{14} \leq (4 - \alpha),$

$c_{21} = 3,$

$(5 + \alpha) \leq c_{22} \leq (7 - \alpha),$

$(7 + \alpha) \leq c_{23} \leq (10 - 2\alpha),$

$c_{24} = 2, c_{31} = 4, c_{32} = 1,$

$(8 + \alpha) \leq c_{33} \leq (10 - \alpha),$

$(9 + 2\alpha) \leq c_{34} \leq (12 - \alpha),$

$(2 + \alpha) \leq d_{11} \leq (5 - 2\alpha),$

$d_{12} = 5, d_{13} = 3,$

$(3 + \alpha) \leq d_{14} \leq (5 - \alpha),$

$d_{21} = 4,$

$(3 + \alpha) \leq d_{22} \leq (6 - 2\alpha),$

$(5 + 2\alpha) \leq d_{23} \leq (8 - \alpha),$

$d_{24} = 4, d_{31} = 3, d_{32} = 2,$

$(2 + \alpha) \leq d_{33} \leq (7 - \alpha),$

$(8 + \alpha) \leq d_{34} \leq (10 - \alpha),$

$(40 + 10\alpha) \leq a_1 \leq (70 - 20\alpha),$

$(50 + 10\alpha) \leq a_2 \leq (70 - 10\alpha),$

$(70 + 10\alpha) \leq a_3 \leq (90 - 10\alpha),$

$(10 + 20\alpha) \leq b_1 \leq (40 - 10\alpha),$

$(10 + 10\alpha) \leq b_2 \leq (30 - 10\alpha),$

$(30 + 10\alpha) \leq b_3 \leq (50 - 10\alpha),$

$(20 + 10\alpha) \leq b_4 \leq (40 - 10\alpha),$

$t > 0, y_{ij} \geq 0; i = 1,2,3, j = 1,2,3,4.$

CrLpP ($\boldsymbol{P2.4.2.1.3}$)

$Z_\alpha^L = \min\ [(4 + \alpha)y_{11} + 6y_{12} + 2y_{13} + (2 + \alpha)y_{14} + 3y_{21} + (5 + \alpha)y_{22} + (7 + \alpha)y_{23} + 2y_{24} + 4y_{31} + y_{32} + (8 + \alpha)y_{33} + (9 + \alpha)y_{34} + 55t]$

Subject to

$-y_{11} - y_{12} - y_{13} - y_{14} + r_1 \geq 0,$

$-y_{21} - y_{22} - y_{23} - y_{24} + r_2 \geq 0,$

$-y_{31} - y_{32} - y_{33} - y_{34} + r_3 \geq 0,$

$y_{11} + y_{21} + y_{31} - s_1 \geq 0,$

$y_{12} + y_{22} + y_{32} - s_2 \geq 0,$

$y_{13} + y_{23} + y_{33} - s_3 \geq 0,$

$y_{14} + y_{24} + y_{34} - s_4 \geq 0,$

$\xi_{11} + \xi_{12} + \xi_{13} + \xi_{14} + \xi_{21} + \xi_{22} + \xi_{23} + \xi_{24} + \xi_{31} + \xi_{32} + \xi_{33} + \xi_{34} + 60t = 1,$

$(2 + \alpha)y_{11} \leq \xi_{11} \leq (5 - 2\alpha)y_{11},$

$\xi_{12} = 5y_{12}, \xi_{13} = 3y_{13},$

$(3 + \alpha)y_{14} \leq \xi_{14} \leq (5 - \alpha)y_{14},$

$\xi_{21} = 4y_{21},$

$(3 + \alpha)y_{22} \leq \xi_{22} \leq (6 - 2\alpha)y_{22},$

$(5 + 2\alpha)y_{23} \leq \xi_{23} \leq (8 - \alpha)y_{23},$

$\xi_{24} = 4y_{24}, \xi_{31} = 3y_{31}, \xi_{32} = 2y_{32},$

$(2 + \alpha)y_{33} \leq \xi_{33} \leq (7 - \alpha)y_{33},$

$(8 + \alpha)y_{34} \leq \xi_{34} \leq (10 - \alpha)y_{34},$

$(40 + 10\alpha)t \leq r_1 \leq (70 - 20\alpha)t,$

$(50 + 10\alpha)t \leq r_2 \leq (70 - 10\alpha)t,$

$(70 + 10\alpha)t \leq r_3 \leq (90 - 10\alpha)t,$

$(10 + 20\alpha)t \leq s_1 \leq (40 - 10\alpha)t,$

$(10 + 10\alpha)t \leq s_2 \leq (30 - 10\alpha)t,$

$(30 + 10\alpha)t \leq s_3 \leq (50 - 10\alpha)t,$

$(20 + 10\alpha)t \leq s_4 \leq (40 - 10\alpha)t,$

$t > 0, y_{ij} \geq 0; i = 1,2,3, j = 1,2,3,4.$

While, in actual case, these problems are not equivalent as on solving

(a) The CrMpP $(P2.4.2.1.2)$, the obtained optimal solution, for $\alpha = 0$, is $a_1 = 40, a_2 = 70, a_3 = 90, b_1 = 10, b_2 = 10, b_3 = 30, b_4 = 20, c_{11} = 4.418270$, $c_{12} = 6, c_{13} = 2, c_{14} = 2.418270, c_{21} = 3, c_{22} = 5.418270, c_{23} = 7.418270$, $c_{24} = 2, c_{31} = 4, c_{32} = 1, c_{33} = 8.418270, c_{34} = 9.418270, d_{11} = 2.157112$, $d_{12} = 5, d_{13} = 3, d_{14} = 3.157112, d_{21} = 4, d_{22} = 3.157112, d_{23} = 5.157112$, $d_{24} = 4, d_{31} = 3, d_{32} = 2, d_{33} = 4.157112, d_{34} = 8.157112, y_{11} = y_{12} = y_{14} = y_{22} = y_{23} = y_{31} = y_{33} = y_{34} = 0, y_{13} = 0.049180327, y_{21} = 0.01639344, y_{24} = 0.098360, y_{32} = 0.1475409, t = 0.001639344$ and the corresponding optimal value is 0.5819672.

(b) The CrLpP $(P2.4.2.1.3)$, the obtained optimal solution, for $\alpha = 0$, is $r_1 = 0.086419, r_2 = 0.086419, r_3 = 0.111111, s_1 = 0.012346, s_2 = 0.012346, s_3 = 0.037037, s_4 = 0.030909, \xi_{11} = \xi_{12} = \xi_{22} = \xi_{23} = \xi_{31} = \xi_{33} = \xi_{34} = 0, \xi_{13} = 0.111111, \xi_{14} = 0.246913, \xi_{21} = 0.049383, \xi_{24} = 0.296296, \xi_{32} = 0.222222$, $y_{11} = y_{12} = y_{22} = y_{23} = y_{31} = y_{33} = y_{34} = 0, y_{13} = 0.037037, y_{14} = 0.049383, y_{21} = 0.012346, y_{24} = 0.074074, y_{32} = 0.111111, t = 0.001234$ and the corresponding optimal value is 0.5370370.

(iii) It is obvious from Step 3c and 3d of Liu's method [110], discussed in Section 2.4.1.1, that Liu [110] assumed $c_{ij} = \left(C_{ij}\right)_{\alpha}^{U}, \delta_{ij} = d_{ij}\lambda, p_i = a_i u_i$ and $q_j = b_j v_j$ to transform the CrMpP $(P2.4.1.1.8)$ into the CrLpP $(P2.4.1.1.10)$ i.e., Liu [110] assumed that the CrMpP $(P2.4.1.1.8)$ is equivalent to the CrLpP $(P2.4.1.1.10)$. However, the CrLpP $(P2.4.1.1.10)$ is not equivalent to the CrMpP $(P2.4.1.1.8)$.

The following validates this claim.

In the numerical example, considered by Liu [110] to illustrate his proposed method, Liu [110] used $c_{ij} = \left(C_{ij}\right)_{\alpha}^{U}, \delta_{ij} = d_{ij}\lambda, p_i = a_i u_i$ and $q_j = b_j v_j$ to transform the CrMpP $(P2.4.2.1.4)$ into the CrLpP $(P2.4.2.1.5)$ i.e., Liu [110] assumed that the CrLpP $(P2.4.2.1.5)$ is equivalent to the CrMpP $(P2.4.2.1.4)$.

CrMpP $(\boldsymbol{P2.4.2.1.4})$

$Z_{\alpha}^{U} = \max \lambda$

Subject to

$-u_1 + v_1 + d_{11}\lambda \leq c_{11},$

$-u_1 + v_2 + d_{12}\lambda \leq c_{12},$

$-u_1 + v_3 + d_{13}\lambda \leq c_{13},$

$-u_1 + v_4 + d_{14}\lambda \leq c_{14},$

$-u_2 + v_1 + d_{21}\lambda \leq c_{21},$

$-u_2 + v_2 + d_{22}\lambda \leq c_{22},$

$-u_2 + v_3 + d_{23}\lambda \leq c_{23},$

$-u_2 + v_4 + d_{24}\lambda \leq c_{24},$

$-u_3 + v_1 + d_{31}\lambda \leq c_{31},$

$-u_3 + v_2 + d_{32}\lambda \leq c_{32},$

$-u_3 + v_3 + d_{33}\lambda \leq c_{33},$

$-u_3 + v_4 + d_{34}\lambda \leq c_{34},$

$a_1u_1 + a_2u_2 + a_3u_3 - b_1v_1 - b_2v_2 - b_3v_3 - b_4v_4 + 60\lambda \leq 55,$

$(4+\alpha) \leq c_{11} \leq (6-\alpha),$

$c_{12} = 6, c_{13} = 2,$

$(2+\alpha) \leq c_{14} \leq (4-\alpha),$

$c_{21} = 3,$

$(5 + \alpha) \leq c_{22} \leq (7 - \alpha),$

$(7 + \alpha) \leq c_{23} \leq (10 - 2\alpha),$

$c_{24} = 2, c_{31} = 4, c_{32} = 1,$

$(8 + \alpha) \leq c_{33} \leq (10 - \alpha),$

$(9 + 2\alpha) \leq c_{34} \leq (12 - \alpha),$

$(2 + \alpha) \leq d_{11} \leq (5 - 2\alpha),$

$d_{12} = 5, d_{13} = 3,$

$(3 + \alpha) \leq d_{14} \leq (5 - \alpha),$

$d_{21} = 4,$

$(3 + \alpha) \leq d_{22} \leq (6 - 2\alpha),$

$(5 + 2\alpha) \leq d_{23} \leq (8 - \alpha),$

$d_{24} = 4, d_{31} = 3, d_{32} = 2,$

$(2 + \alpha) \leq d_{33} \leq (7 - \alpha),$

$(8 + \alpha) \leq d_{34} \leq (10 - \alpha),$

$(40 + 10\alpha) \leq a_1 \leq (70 - 20\alpha),$

$(50 + 10\alpha) \leq a_2 \leq (70 - 10\alpha),$

$(70 + 10\alpha) \leq a_3 \leq (90 - 10\alpha),$

$(10 + 20\alpha) \leq b_1 \leq (40 - 10\alpha),$

$(10 + 10\alpha) \leq b_2 \leq (30 - 10\alpha),$

$(30 + 10\alpha) \leq b_3 \leq (50 - 10\alpha),$

$(20 + 10\alpha) \leq b_4 \leq (40 - 10\alpha),$

CrLpP $(\boldsymbol{P2.4.2.1.5})$

$Z_{\alpha}^{U} = \max \lambda$

Subject to

$-u_1 + v_1 + \delta_{11} \leq (6-\alpha),$

$-u_1 + v_2 + \delta_{12} \leq 6,$

$-u_1 + v_3 + \delta_{13} \leq 2,$

$-u_1 + v_4 + \delta_{14} \leq (4-\alpha),$

$-u_2 + v_1 + \delta_{21} \leq 3,$

$-u_2 + v_2 + \delta_{22} \leq (7-\alpha),$

$-u_2 + v_3 + \delta_{23} \leq (10-2\alpha),$

$-u_2 + v_4 + \delta_{24} \leq 2,$

$-u_3 + v_1 + \delta_{31} \leq 4,$

$-u_3 + v_2 + \delta_{32} \leq 1,$

$-u_3 + v_3 + \delta_{33} \leq (10-\alpha),$

$-u_3 + v_4 + \delta_{34} \leq (12-\alpha),$

$p_1 + p_2 + p_3 - q_1 - q_2 - q_3 - q_4 + 60\lambda \leq 55,$

$(2+\alpha)\lambda \leq \delta_{11} \leq (5-2\alpha)\lambda,$

$\delta_{12} = 5\lambda, \delta_{13} = 3\lambda,$

$(3+\alpha)\lambda \leq \delta_{14} \leq (5-\alpha)\lambda,$

$\delta_{21} = 4\lambda,$

$(3+\alpha)\lambda \leq \delta_{22} \leq (6-2\alpha)\lambda,$

$(5+2\alpha)\lambda \leq \delta_{23} \leq (8-\alpha)\lambda,$

$\delta_{24} = 4\lambda, \delta_{31} = 3\lambda, \delta_{32} = 2\lambda,$

$(4+2\alpha)\lambda \le \delta_{33} \le (7-\alpha)\lambda,$

$(8+\alpha)\lambda \le \delta_{34} \le (10-\alpha)\lambda,$

$(40+10\alpha)u_1 \le p_1 \le (70-20\alpha)u_1,$

$(50+10\alpha)u_2 \le p_2 \le (70-10\alpha)u_2,$

$(70+10\alpha)u_3 \le p_3 \le (90-10\alpha)u_3,$

$(10+20\alpha)v_1 \le q_1 \le (40-10\alpha)v_1,$

$(10+10\alpha)v_2 \le q_2 \le (30-10\alpha)v_2,$

$(30+10\alpha)v_3 \le q_3 \le (50-10\alpha)v_3,$

$(20+10\alpha)v_4 \le q_4 \le (40-10\alpha)v_4,$

$\lambda > 0, u_i, v_j \ge 0; i = 1,2,3, j = 1,2,3,4.$

While, in actual case, these problems are not equivalent as on solving

(a) The CrMpP $(P2.4.2.1.4)$, no feasible solution is obtained for $\alpha = 0$.

(b) The CrLpP $(P2.4.2.1.5)$, the obtained optimal solution, for $\alpha = 0$, is $p_1 = 315.2167, p_2 = 147.719, p_3 = 74.5661, q_1 = 95.9091, q_2 = 8.60684, q_3 = 375.4279, q_4 = 55.9091, u_1 = 7.880418, u_2 = 2.95439, u_3 = 1.06523, v_1 = 2.39773, v_2 = 0.28689, v_3 = 7.508558, v_4 = 1.39773, \delta_{11} = 2.6694153,$ $\delta_{12} = 4.445840, \delta_{13} = 2.667504, \delta_{14} = 3.596439, \delta_{21} = 3.55667, \delta_{22} = 3.59644, \delta_{23} = 5.154876, \delta_{24} = 3.55667, \delta_{31} = 2.66750, \delta_{32} = 1.77834,$ $\delta_{33} = 3.55667, \delta_{34} = 8.816638, \lambda = 0.88916805$ and the corresponding optimal value is 0.8891680.

(iv) It is obvious from Liu's method [110], discussed in Section 2.4.1.1, that Liu [110] considered the FuLFpP $(P2.2.3)$ corresponding to the considered unbalanced triangular FlFuLFtP, in which each known parameter is represented by a TFuN whereas the decision variables are represented by non-negative real numbers. However, the decision variables obtained by Liu [110] are neither non-negative real numbers nor TFuNs.

The following validates this claim.

The optimal solution, obtained by Liu [110] to illustrate his proposed method, is as follows

(a) At $\alpha = 0$, the lower bound $Z^L_{\alpha=0} = 0.5370$ occurring at $x_{13} = 30, x_{14} = 40, x_{21} = 10, x_{24} = 60, x_{32} = 90, x_{11} = x_{12} = x_{22} = x_{23} = x_{31} = x_{33} = x_{34} = 0$.

(b) At $\alpha = 0$, the upper bound $Z^U_{\alpha=0} = 0.8684$ occurring at $x_{13} = 40$, $x_{21} = 10, x_{24} = 40, x_{31} = 30$, $x_{32} = 30$, $x_{33} = 10$, $x_{11} = x_{12} = x_{14} = x_{22} = x_{23} = x_{34} = 0$.

(c) At $\alpha = 1$, the upper bound and the lower bound $Z^U_{\alpha=0} = Z^L_{\alpha=0} = 0.6293$ occurring at $x_{13} = 40$, $x_{21} = 30$, $x_{24} = 30$, $x_{32} = 80$, $x_{11} = x_{12} = x_{14} = x_{22} = x_{23} = x_{31} = x_{33} = x_{34} = 0$.

The fuzzy optimal value obtained by Liu [110] is given by $(0.5370, 0.6293, 0.8684)$ corresponding to the decision variables $x_{11} = (0,0,0), x_{12} = (0,0,0), x_{13} = (30,40,40), x_{14} = (40,0,0), x_{21} = (10,30,10), x_{22} = (0,0,0), x_{23} = (0,0,0), x_{24} = (60,30,40), x_{31} = (0,0,30), x_{32} = (90,80,30), x_{33} = (0,0,10), x_{34} = (0,0,0)$.

Clearly, $x_{14}, x_{21}, x_{24}, x_{32}$ are not TFuNs. Thus, the decision variables obtained by Liu [110] are neither non-negative real numbers nor TFuNs.

2.4.2.2 Inappropriateness of Anukokila et al.'s method

The following mathematical incorrect results are considered in Anukokila et al.'s method [9].

Hence, it is inappropriate to use Anukokila et al.'s method [9] for solving FlFuLFtPs.

(i) It is obvious from Step 4 of Anukokila et al.'s method [9], discussed in Section 2.4.1.2.2, that the relations $R_2\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right)=\frac{R_2(\tilde{A}_1)}{R_2(\tilde{A}_2)}$ and $R_3\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right)=\frac{R_3(\tilde{A}_1)}{R_3(\tilde{A}_2)}$ are used to transform the CrMoLFpP $(P2.4.1.2.2.3)$ into the CrMoLFpP $(P2.4.1.2.2.4)$. While, the following example clearly indicates that $R_2\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right)\neq\frac{R_2(\tilde{A}_1)}{R_2(\tilde{A}_2)}$ and $R_3\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right)\neq\frac{R_3(\tilde{A}_1)}{R_3(\tilde{A}_2)}$. Therefore, the CrMoLFpP $(P2.4.1.2.2.3)$ cannot be transformed into the CrMoLFpP $(P2.4.1.2.2.4)$. Hence, it is inappropriate to use Anukokila et al.'s method [9] to find an optimal solution of the FlFuLFpP $(P2.2.4)$.

Let $\tilde{A}_1=(6,8,10)$ and $\tilde{A}_2=(2,4,6)$ be two TFuNs. Then, using Definition 2.1.29, $\frac{\tilde{A}_1}{\tilde{A}_2}=\frac{(6,8,10)}{(2,4,6)}=\left(\frac{6}{6},\frac{8}{4},\frac{10}{2}\right)=(1,2,5)$.

Using the existing expressions (2.4.1.2.1.2) and (2.4.1.2.1.3) [9],

$$R_2\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right)=R_2(1,2,5)=5-1=4 \tag{2.4.2.2.1}$$

$$R_3\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right)=R_3(1,2,5)=5+1=6 \tag{2.4.2.2.2}$$

$$\frac{R_2(\tilde{A}_1)}{R_2(\tilde{A}_2)}=\frac{R_2(6,8,10)}{R_2(2,4,6)}=\frac{10-6}{6-2}=\frac{4}{4}=1 \tag{2.4.2.2.3}$$

$$\frac{R_3(\tilde{A}_1)}{R_3(\tilde{A}_2)}=\frac{R_3(6,8,10)}{R_3(2,4,6)}=\frac{10+6}{6+2}=\frac{16}{8}=2 \tag{2.4.2.2.4}$$

It is obvious from (2.4.2.2.1) and (2.4.2.2.3) that $R_2\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right)\neq\frac{R_2(\tilde{A}_1)}{R_2(\tilde{A}_2)}$.

Also, it is obvious from (2.4.2.2.2) and (2.4.2.2.4) that $R_3\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right)\neq\frac{R_3(\tilde{A}_1)}{R_3(\tilde{A}_2)}$.

(ii) It is obvious that in the numerator of the second objective function of the CrMoLFpP $(P2.4.1.2.2.4)$, it is assumed that $R_2\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta\right)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left(C_{ij3}x_{ij3}-C_{ij1}x_{ij1}\right)+\theta$. Also, in the denominator of the second objective function of the CrMoLFpP $(P2.4.1.2.2.4)$, it is assumed that $R_2\left(\sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij1}x_{ij1} + \beta, \sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij2}x_{ij2} + \beta, \sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij3}x_{ij3} + \beta\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(D_{ij3}x_{ij3} - D_{ij1}x_{ij1}\right) + \beta$.

While, according to the comparing method, $R_2\left(\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij1}x_{ij1} + \theta, \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij2}x_{ij2} + \theta, \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij3}x_{ij3} + \theta\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(C_{ij3}x_{ij3} - C_{ij1}x_{ij1}\right)$ and $R_2\left(\sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij1}x_{ij1} + \beta, \sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij2}x_{ij2} + \beta, \sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij3}x_{ij3} + \beta\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(D_{ij3}x_{ij3} - D_{ij1}x_{ij1}\right)$.

(iii) It is obvious that in the numerator of the third objective function of the CrMoLFpP $(P2.4.1.2.2.4)$, it is assumed that $R_3\left(\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij1}x_{ij1} + \theta, \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij2}x_{ij2} + \theta, \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij3}x_{ij3} + \theta\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(C_{ij1}x_{ij1} + C_{ij3}x_{ij3}\right) + \theta$. Also, in the denominator of the third objective function of the CrMoLFpP $(P2.4.1.2.2.4)$, it is assumed that $R_3\left(\sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij1}x_{ij1} + \beta, \sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij2}x_{ij2} + \beta, \sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij3}x_{ij3} + \beta\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(D_{ij1}x_{ij1} + D_{ij3}x_{ij3}\right) + \beta$.

While, according to the comparing method, $R_3\left(\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij1}x_{ij1} + \theta, \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij2}x_{ij2} + \theta, \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij3}x_{ij3} + \theta\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(C_{ij1}x_{ij1} + C_{ij3}x_{ij3}\right) + 2\theta$ and $R_3\left(\sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij1}x_{ij1} + \beta, \sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij2}x_{ij2} + \beta, \sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij3}x_{ij3} + \beta\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(D_{ij1}x_{ij1} + D_{ij3}x_{ij3}\right) + 2\beta$.

2.4.2.3 Inappropriateness of Anukokila and Radhakrishnan’s method

The following mathematical incorrect results are considered in the method, discussed in Section 2.4.1.3, to transform the FlFuLFpP $(P2.2.5)$ into the FlFuLpP $(P2.4.1.3.5)$.

(i) It is obvious from Step 1c of Anukokila and Radhakrishnan’s method [10], discussed in Section 2.4.1.3, that to transform the FlFuMpP $(P2.4.1.3.2)$ into the FlFuLpP

$(P2.4.1.3.3)$, Anukokila and Radhakrishnan [10] have assumed that $\frac{\tilde{t}}{\tilde{t}} = \tilde{1}$. However, this result is not valid as it is obvious from Definition 2.1.30 that if $\tilde{t} = (t_1, t_2, t_3, t_4)$ is a positive TrFuN. Then, $\frac{\tilde{t}}{\tilde{t}} = \frac{(t_1,t_2,t_3,t_4)}{(t_1,t_2,t_3,t_4)} = \left(\frac{t_1}{t_4}, \frac{t_2}{t_3}, \frac{t_3}{t_2}, \frac{t_4}{t_1}\right) \neq (1,1,1,1)$.

(ii) It is obvious from Step 1e of Anukokila and Radhakrishnan's method [10], discussed in Section 2.4.1.3, that to transform the FlFuLpP $(P2.4.1.3.4)$ into the FlFuLpP $(P2.4.1.3.5)$, Anukokila and Radhakrishnan [10] have assumed that $\tilde{A}_i\tilde{t} - \tilde{A}_i\tilde{t} = \tilde{0}$ and $\tilde{B}_j\tilde{t} - \tilde{B}_j\tilde{t} = \tilde{0}$. However, this result is not valid as it is obvious from Definition 2.1.21 that if $\tilde{t} = (t_1, t_2, t_3, t_4)$ is a TrFuN. Then, $\tilde{t} - \tilde{t} = (t_1, t_2, t_3, t_4) - (t_1, t_2, t_3, t_4) = (t_1 - t_4, t_2 - t_3, t_3 - t_2, t_4 - t_1) \neq (0,0,0,0)$.

2.5 Inappropriateness of existing methods to solve FlFuLFpPs

In the last few years, several methods [45-50, 95, 153, 154] are proposed in the literature to solve FlFuLFpPs. So, one may claim that any of these existing methods can be used to solve

(i) The FlFuLFpP $(P2.2.5)$ to find a fuzzy optimal solution of a balanced FlFuLFtP.

(ii) The FlFuLFpP $(P2.4.2.1.1)$ to find a fuzzy optimal solution of an unbalanced FlFuLFtP.

However, it is inappropriate to use these existing methods as it can be easily verified that the following mathematical incorrect results are considered in these existing methods.

(i) In the existing methods [46, 48, 49, 95, 153, 154], Charnes and Cooper transformation method under fuzzy environment is used to transform the FlFuLFpP $(P2.2.5)$ into the FlFuLpP $(P2.4.1.3.5)$. However, as discussed in Section 2.4.2.3, that the FlFuLpP $(P2.4.1.3.5)$, obtained by Charnes and Cooper transformation method under fuzzy environment, will not be equivalent to the FlFuLFpP $(P2.2.5)$.

Hence, it is inappropriate to use the existing methods [46, 48, 49, 95, 153, 154] to find a fuzzy optimal solution of the FlFuLFpPs $(P2.2.5)$ and $(P2.4.2.1.1)$.

(ii) In the existing methods [45, 50], the relation $R\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right) = \frac{R(\tilde{A}_1)}{R(\tilde{A}_2)}$ is used. However, as discussed in Section 2.4.2.2, $R\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right) \neq \frac{R(\tilde{A}_1)}{R(\tilde{A}_2)}$. Hence, it is inappropriate to use the existing methods [45, 50] to find a fuzzy optimal solution of the FlFuLFpPs $(P2.2.5)$ and $(P2.4.2.1.1)$.

(iii) Although, in the existing method [47], neither Charnes and Cooper transformation method under fuzzy environment nor the relation $R\left(\frac{\tilde{A}_1}{\tilde{A}_2}\right) = \frac{R(\tilde{A}_1)}{R(\tilde{A}_2)}$ is used. However, in the existing method [47], it is assumed that a fuzzy inequality constraint $(a_1, a_2, a_3, a_4) \leq (b_1, b_2, b_3, b_4)$ can be transformed into four crisp constraints $a_k \leq b_k; k = 1,2,3,4$. However, this result is not valid due to the following reason.

If $(a_1, a_2, a_3, a_4) \leq (b_1, b_2, b_3, b_4)$, then there should exist a non-negative TrFuN (s_1, s_2, s_3, s_4) such that $(a_1, a_2, a_3, a_4) + (s_1, s_2, s_3, s_4) = (b_1, b_2, b_3, b_4)$. However, for the values of s_1, s_2, s_3, s_4, obtained by $a_k \leq b_k\,; k = 1,2,3,4 \Rightarrow a_k + s_k = b_k$; $k = 1,2,3,4$, where $s_k \geq 0$, the relation $0 \leq s_1 \leq s_2 \leq s_3 \leq s_4$ will not necessarily be satisfied i.e., (s_1, s_2, s_3, s_4) will not necessarily be a non-negative TrFuN. The following validates this claim.

Let $(a_1, a_2, a_3, a_4) = (7,10,12,14)$ and $(b_1, b_2, b_3, b_4) = (10,11,14,15)$ be two TrFuNs. Since, $a_k < b_k \; \forall \; k = 1,2,3,4$. So, $(7,10,12,14) < (10,11,14,15)$. However, for $s_1 = 10 - 7 = 3, s_2 = 11 - 10 = 1, s_3 = 14 - 12 = 2, s_4 = 15 - 14 = 1$, the relation $0 \leq s_1 \leq s_2 \leq s_3 \leq s_4$ is not satisfying i.e., $(s_1, s_2, s_3, s_4) = (3,1,2,1)$ is not a TrFuN.

Hence, it is inappropriate to use the existing method [47] to find a fuzzy optimal solution of the FlFuLFpP $(P2.4.2.1.1)$. However, as for all the constraints of the FlFuLFpP $(P2.2.5)$, the relation $(a_1, a_2, a_3, a_4) = (b_1, b_2, b_3, b_4)$ is satisfying i.e., (s_1, s_2, s_3, s_4) will always be a zero TrFuN. Therefore, the existing method [47] can be used to find a fuzzy optimal solution of the FlFuLFpP $(P2.2.5)$ i.e., a fuzzy optimal solution of balanced FlFuLFtPs.

2.6 Proposed Mehar method for solving FlFuLFtPs

It is obvious from point (d) of Section 2.5 that it is inappropriate to use the existing method [47] to find a fuzzy optimal solution of unbalanced FlFuLFtPs. However, it is appropriate to use the existing method [47] to find a fuzzy optimal solution of balanced FlFuLFtPs. Keeping the same in mind, in this section, by integrating the existing method [93] for transforming an unbalanced fully fuzzy transportation problem into its equivalent balanced fully fuzzy transportation problem and the existing method [47] for solving balanced FlFuLFtPs, a new method (named as Mehar method) is proposed to solve FlFuLFtPs.

The steps of the proposed Mehar method are as follows.

Step 1: Use the following step of the existing method [93] to transform an unbalanced FlFuLFtP into its equivalent balanced FlFuLFtP.

Find the total fuzzy availability $\sum_{i=1}^{m} \tilde{A}_i = (A_{i1}, A_{i2}, A_{i3}, A_{i4})$ and the total fuzzy demand $\sum_{j=1}^{n} \tilde{B}_j = (B_{j1}, B_{j2}, B_{j3}, B_{j4})$.

Check that $\sum_{i=1}^{m} \tilde{A}_i = \sum_{j=1}^{n} \tilde{B}_j$ or $\sum_{i=1}^{m} \tilde{A}_i \neq \sum_{j=1}^{n} \tilde{B}_j$.

Case (1) If $\sum_{i=1}^{m} \tilde{A}_i = \sum_{j=1}^{n} \tilde{B}_j$, then the FlFuLFtP is balanced. Go to Step 2.

Case (2) If $\sum_{i=1}^{m}\tilde{A}_i \neq \sum_{j=1}^{n}\tilde{B}_j$, then check that $A_{i1} \leq B_{j1}, A_{i2} - A_{i1} \leq B_{j2} - B_{j1}, A_{i3} - A_{i2} \leq B_{j3} - B_{j2}, A_{i4} - A_{i3} \leq B_{j4} - B_{j3}$ or $A_{i1} \geq B_{j1}, A_{i2} - A_{i1} \geq B_{j2} - B_{j1}, A_{i3} - A_{i2} \geq B_{j3} - B_{j2}, A_{i4} - A_{i3} \geq B_{j4} - B_{j3}$.

Case (2a) If $A_{i1} \leq B_{j1}, A_{i2} - A_{i1} \leq B_{j2} - B_{j1}, A_{i3} - A_{i2} \leq B_{j3} - B_{j2}, A_{i4} - A_{i3} \leq B_{j4} - B_{j3}$, then introduce a dummy source with fuzzy availability $(B_{j1} - A_{i1}, B_{j2} - A_{i2}, B_{j3} - A_{i3}, B_{j4} - A_{i4})$. Assume the cost for transporting one unit quantity of the product from the introduced dummy source to all destinations as zero TrFuN and then go to Step 2.

Case (2b) If $A_{i1} \geq B_{j1}, A_{i2} - A_{i1} \geq B_{j2} - B_{j1}, A_{i3} - A_{i2} \geq B_{j3} - B_{j2}, A_{i4} - A_{i3} \geq B_{j4} - B_{j3}$, then introduce a dummy destination with fuzzy demand $(A_{i1} - B_{j1}, A_{i2} - B_{j2}, A_{i3} - B_{j3}, A_{i4} - B_{j4})$. Assume the cost for transporting one unit quantity of the product from all the sources to the introduced dummy destination as zero TrFuN and then go to Step 2.

Case (2c) If neither Case (2a) nor Case (2b) is satisfied, then introduce a dummy source with fuzzy availability $(maximum\,\{0, (B_{j1} - A_{i1})\}, maximum\,\{0, (B_{j1} - A_{i1})\} + maximum\,\{0, (B_{j2} - B_{j1}) - (A_{i2} - A_{i1})\}, maximum\,\{0, (B_{j1} - A_{i1})\} + maximum\,\{0, (B_{j2} - B_{j1}) - (A_{i2} - A_{i1})\} + maximum\,\{0, (B_{j3} - B_{j2}) - (A_{i3} - A_{i2})\}, maximum\,\{0, (B_{j1} - A_{i1})\} + maximum\,\{0, (B_{j2} - B_{j1}) - (A_{i2} - A_{i1})\} + maximum\,\{0, (B_{j3} - B_{j2}) - (A_{i3} - A_{i2})\} + maximum\,\{0, (B_{j4} - B_{j3}) - (A_{i4} - A_{i3})\})$ and a dummy destination with fuzzy demand $(maximum\,\{0, (A_{i1} - B_{j1})\}, maximum\,\{0, (A_{i1} - B_{j1})\} + maximum\,\{0, (A_{i2} - A_{i1}) - (B_{j2} - B_{j1})\}, maximum\,\{0, (A_{i1} - B_{j1})\} +$

$maximum\ \{0, (A_{i2} - A_{i1}) - (B_{j2} - B_{j1})\} +\ maximum\ \{0, (A_{i3} - A_{i2}) - (B_{j3} - B_{j2})\}, maximum\ \{0, (A_{i1} - B_{j1})\} + maximum\ \{0, (A_{i2} - A_{i1}) - (B_{j2} - B_{j1})\} + maximum\ \{0, (A_{i3} - A_{i2}) - (B_{j3} - B_{j2})\} + maximum\ \{0, (A_{i4} - A_{i3}) - (B_{j4} - B_{j3})\})$. Assume the cost for transporting one unit quantity of the product from the introduced dummy source to all destinations and from all the sources to the introduced dummy destination as zero TrFuN and then go to Step 2.

Step 2: Write the FlFuLFpP $(P2.6.1)$ corresponding to the transformed balanced FlFuLFtP.

FlFuLFpP $(\boldsymbol{P2.6.1})$

$$Minimize\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij1},C_{ij2},C_{ij3},C_{ij4})(x_{ij1},x_{ij2},x_{ij3},x_{ij4})+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}(D_{ij1},D_{ij2},D_{ij3},D_{ij4})(x_{ij1},x_{ij2},x_{ij3},x_{ij4})+\beta}\right)$$

Subject to

$\sum_{j=1}^{n}(x_{ij1}, x_{ij2}, x_{ij3}, x_{ij4}) = (A_{i1}, A_{i2}, A_{i3}, A_{i4}),\qquad i = 1,2, \dots, m;\ m = p$ or $p + 1$,

$\sum_{i=1}^{m}(x_{ij1}, x_{ij2}, x_{ij3}, x_{ij4}) = (B_{j1}, B_{j2}, B_{j3}, B_{j4}),\qquad j = 1,2, \dots, n; n = q$ or $q + 1$,

$(x_{ij1}, x_{ij2}, x_{ij3}, x_{ij4})$ is a non-negative TrFuN $\forall\ i, j$.

Step 3: Use the following steps of the existing method [47], to transform the FlFuLFpP $(P2.6.1)$ into its equivalent CrMoLFpP.

Step 3(a): Using Definition 2.1.25, transform the FlFuLFpP $(P2.6.1)$ into its equivalent FlFuLFpP $(P2.6.2)$.

FlFuLFpP $(\boldsymbol{P2.6.2})$

$$Minimize\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij1}x_{ij1},C_{ij2}x_{ij2},C_{ij3}x_{ij3},C_{ij4}x_{ij4})+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}(D_{ij1}x_{ij1},D_{ij2}x_{ij2},D_{ij3}x_{ij3},D_{ij4}x_{ij4})+\beta}\right)$$

Subject to

Constraints of the FlFuLFpP $(P2.6.1)$.

Step 3(b): Using Definition 2.1.19, transform the FlFuLFpP $(P2.6.2)$ into its equivalent FlFuLFpP $(P2.6.3)$.

FlFuLFpP $(\boldsymbol{P2.6.3})$

$$Minimize\left(\frac{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta,\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij4}x_{ij4}+\theta\right)}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta,\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta,\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta,\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij4}x_{ij4}+\beta\right)}\right)$$

Subject to

$\left(\sum_{j=1}^{n}x_{ij1},\sum_{j=1}^{n}x_{ij2},\sum_{j=1}^{n}x_{ij3},\sum_{j=1}^{n}x_{ij4}\right)=(A_{i1},A_{i2},A_{i3},A_{i4}), i=$

$1,2,\dots,m; m=p$ or $p+1$,

$\left(\sum_{i=1}^{m}x_{ij1},\sum_{i=1}^{m}x_{ij2},\sum_{i=1}^{m}x_{ij3},\sum_{i=1}^{m}x_{ij4}\right)=\left(B_{j1},B_{j2},B_{j3},B_{j4}\right), j=$

$1,2,\dots,n; n=q$ or $q+1$,

$\left(x_{ij1},x_{ij2},x_{ij3},x_{ij4}\right)$ is a non-negative TrFuN $\forall\ i,j$.

Step 3(c): Using Definition 2.1.30, transform the FlFuLFpP $(P2.6.3)$ into its equivalent FlFuLFpP $(P2.6.4)$.

FlFuLFpP $(\boldsymbol{P2.6.4})$

$$Minimize\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij4}x_{ij4}+\beta},\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta},\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta},\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij4}x_{ij4}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta}\right)$$

Subject to

Constraints of the FlFuLFpP $(P2.6.3)$.

Step 3(d): Using Definition 2.1.9 and Definition 2.1.17, transform the FlFuLFpP $(P2.6.4)$ into its equivalent CrMoLFpP $(P2.6.5)$.

CrMoLFpP $(\boldsymbol{P2.6.5})$

$$Minimize\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij1}x_{ij1}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij4}x_{ij4}+\beta}\right)$$

$$Minimize\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij2}x_{ij2}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij3}x_{ij3}+\beta}\right)$$

$$Minimize\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij3}x_{ij3}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij2}x_{ij2}+\beta}\right)$$

$$Minimize\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij4}x_{ij4}+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}D_{ij1}x_{ij1}+\beta}\right)$$

Subject to

$\sum_{j=1}^{n}x_{ijk}=A_{ik}, i=1,2,\dots,m; m=p$ or $p+1; k=1,2,3,4.$

$\sum_{i=1}^{m}x_{ijk}=B_{jk}, j=1,2,\dots,n; n=q$ or $q+1; k=1,2,3,4.$

$0\le x_{ij1}\le x_{ij2}\le x_{ij3}\le x_{ij4}\ \forall\ i,j.$

Step 4: Using Das et al.'s method [47] or any other appropriate method [18], find an efficient solution $\{x_{ij1},x_{ij2},x_{ij3},x_{ij4}\}$ of the CrMoLFpP $(P2.6.5)$.

Step 5: Using the obtained efficient solution $\{x_{ij1},x_{ij2},x_{ij3},x_{ij4}\}$, find an efficient fuzzy solution $\{(x_{ij1},x_{ij2},x_{ij3},x_{ij4})\}$ of the FlFuLFpP $(P2.6.1)$.

Step 6: Using the obtained efficient fuzzy solution $\{(x_{ij1},x_{ij2},x_{ij3},x_{ij4})\}$, find the fuzzy optimal value of the FlFuLFpP $(P2.6.1)$ i.e., $\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij1},C_{ij2},C_{ij3},C_{ij4})(x_{ij1},x_{ij2},x_{ij3},x_{ij4})+\theta}{\sum_{i=1}^{m}\sum_{j=1}^{n}(D_{ij1},D_{ij2},D_{ij3},D_{ij4})(x_{ij1},x_{ij2},x_{ij3},x_{ij4})+\beta}$.

Remark 2.1: It is pertinent to mention that in the existing method [164], the fuzzy mathematical programming method [30] is used to transform the CrMoLFpP $(P2.6.5)$ into a CrMoLpP. While, in the literature [155], it is pointed out that it is inappropriate to use the existing method [30] to transform the CrMoLFpP $(P2.6.5)$ into a CrMoLpP. Therefore, it is inappropriate to use the existing method [164] in its present form. The proposed Mehar method is a generalization of Veeramani's method [164]. Step 2 to Step 3 of the proposed Mehar method are same as Veeramani's method [164]. While, in Step 4, the lexicographic approach [18] is used to solve the CrMoLFpP instead of the existing method [30].

2.7 Correct fuzzy optimal solutions of an existing FlFuLFtP

Liu [110] applied his proposed method to solve the FlFuLFtP having 3 sources and 4 destinations by considering

(i) $\tilde{A}_1 = (40,50,70), \tilde{A}_2 = (50,60,70), \tilde{A}_3 = (70,80,90)$.

(ii) $\tilde{B}_1 = (10,30,40), \tilde{B}_2 = (10,20,30), \tilde{B}_3 = (30,40,50), \tilde{B}_4 = (20,30,40)$.

(iii) $\tilde{C}_{11} = (4,5,6), \tilde{C}_{12} = (6,6,6), \tilde{C}_{13} = (2,2,2), \tilde{C}_{14} = (2,3,4), \tilde{C}_{21} = (3,3,3), \tilde{C}_{22} = (5,6,7), \tilde{C}_{23} = (7,8,10), \tilde{C}_{24} = (2,2,2), \tilde{C}_{31} = (4,4,4), \tilde{C}_{32} = (1,1,1), \tilde{C}_{33} = (8,9,10), \tilde{C}_{34} = (9,11,12)$.

(iv) $\widetilde{D}_{11} = (2,3,5), \widetilde{D}_{12} = (5,5,5), \widetilde{D}_{13} = (3,3,3), \widetilde{D}_{14} = (3,4,5), \widetilde{D}_{21} = (4,4,4), \widetilde{D}_{22} = (3,4,6), \widetilde{D}_{23} = (5,7,8), \widetilde{D}_{24} = (4,4,4), \widetilde{D}_{31} = (3,3,3), \widetilde{D}_{32} = (2,2,2), \widetilde{D}_{33} = (4,6,7), \widetilde{D}_{34} = (8,9,10)$.

(v) $\theta = 55, \beta = 60$.

However, as discussed in Section 2.4.2.1, that Liu's method [110] is not valid. Therefore, the optimal solution of the FlFuLFtP, obtained by Liu [110], is also not valid. In this section, correct fuzzy optimal solutions of the same problem are obtained by the proposed Mehar method.

Step 1: It is obvious that the total fuzzy availability $\sum_{i=1}^{3} \tilde{A}_i = (160,190,230)$ is not equal to the total fuzzy demand $\sum_{j=1}^{4} \tilde{B}_j = (70,120,160)$. Also, it is obvious that the condition, mentioned in Case (2c) of Step 1 of the proposed Mehar method, is satisfying. So, according to Step 1 of the proposed Mehar method, there is a need to introduce

(i) A dummy source having fuzzy supply $(0,20,20)$ with the assumption that the cost for transporting one unit quantity of the product from the introduced dummy source to all destinations as zero TFuN.

(ii) A dummy destination with fuzzy demand $(90,90,90)$ with the assumption that the cost for transporting one unit quantity of the product from all the sources to the introduced dummy destination as zero TFuN.

Step 2: Using Step 2 of the proposed Mehar method, the fuzzy optimal solution of the transformed balanced FlFuLFtP can be obtained by solving its equivalent FlFuLFpP $(P2.7.1)$.

FlFuLFpP $(\boldsymbol{P2.7.1})$

$Minimize\left(\frac{\tilde{P}}{\tilde{Q}}\right)$

Subject to

$(x_{111},x_{112},x_{113})+(x_{121},x_{122},x_{123})+(x_{131},x_{132},x_{133})+(x_{141},x_{142},x_{143})+(x_{151},x_{152},x_{153})=(40,50,70)$,

$(x_{211},x_{212},x_{213})+(x_{221},x_{222},x_{223})+(x_{231},x_{232},x_{233})+(x_{241},x_{242},x_{243})+(x_{251},x_{252},x_{253})=(50,60,70)$,

$(x_{311},x_{312},x_{313})+(x_{321},x_{322},x_{323})+(x_{331},x_{332},x_{333})+(x_{341},x_{342},x_{343})+(x_{351},x_{352},x_{353})=(70,80,90)$,

$(x_{411},x_{412},x_{413})+(x_{421},x_{422},x_{423})+(x_{431},x_{432},x_{433})+(x_{441},x_{442},x_{443})+(x_{451},x_{452},x_{453})=(0,20,20)$,

$(x_{111},x_{112},x_{113})+(x_{211},x_{212},x_{213})+(x_{311},x_{312},x_{313})+(x_{411},x_{412},x_{413})=(10,30,40)$,

$(x_{121},x_{122},x_{123})+(x_{221},x_{222},x_{223})+(x_{321},x_{322},x_{323})+(x_{421},x_{422},x_{423})=(10,20,30)$,

$(x_{131},x_{132},x_{133})+(x_{231},x_{232},x_{233})+(x_{331},x_{332},x_{333})+(x_{431},x_{432},x_{433})=(30,40,50)$,

$(x_{141},x_{142},x_{143})+(x_{241},x_{242},x_{243})+(x_{341},x_{342},x_{343})+(x_{441},x_{442},x_{443})=(20,30,40)$,

$(x_{151}, x_{152}, x_{153}) + (x_{251}, x_{252}, x_{253}) + (x_{351}, x_{352}, x_{353}) + (x_{451}, x_{452}, x_{453}) =$

$(90,90,90)$,

$\left(x_{ij1}, x_{ij2}, x_{ij3}\right)$ is a non-negative TFuN $\forall\ i = 1,2,3,4,\ \ j = 1,2,3,4,5$

where,

$\tilde{P} = (4,5,6)\tilde{x}_{11} + (6,6,6)\tilde{x}_{12} + (2,2,2)\tilde{x}_{13} + (2,3,4)\tilde{x}_{14} + (3,3,3)\tilde{x}_{21} + (5,6,7)\tilde{x}_{22} +$

$(7,8,10)\tilde{x}_{23} + (2,2,2)\tilde{x}_{24} + (4,4,4)\tilde{x}_{31} + (1,1,1)\tilde{x}_{32} + (8,9,10)\tilde{x}_{33} + (9,11,12)\tilde{x}_{34} +$

55,

$\tilde{Q} = (2,3,5)\tilde{x}_{11} + (5,5,5)\tilde{x}_{12} + (3,3,3)\tilde{x}_{13} + (3,4,5)\tilde{x}_{14} + (4,4,4)\tilde{x}_{21} + (3,4,6)\tilde{x}_{22} +$

$(5,7,8)\tilde{x}_{23} + (4,4,4)\tilde{x}_{24} + (3,3,3)\tilde{x}_{31} + (2,2,2)\tilde{x}_{32} + (4,6,7)\tilde{x}_{33} + (8,9,10)\tilde{x}_{34} + 60.$

Step 3: According to Step 3 of the proposed Mehar method, there is a need to use Das et al.'s method [47] to transform the FlFuLFpP $(P2.7.1)$ into its equivalent CrMoLFpP.

Step 3(a): The FlFuLFpP $(P2.7.1)$ can be transformed into its equivalent FlFuLFpP $(P2.7.2)$.

FlFuLFpP $(\boldsymbol{P2.7.2})$

$Minimize\left(\frac{\tilde{X}_1}{\tilde{X}_2}\right)$

Subject to

Constraints of the FlFuLFpP $(P2.7.1)$

where,

$\tilde{X}_1 = (4x_{111}, 5x_{112}, 6x_{113}) + (6x_{121}, 6x_{122}, 6x_{123}) + (2x_{131}, 2x_{132}, 2x_{133}) +$

$(2x_{141}, 3x_{142}, 4x_{143}) + (3x_{211}, 3x_{212}, 3x_{213}) + (5x_{221}, 6x_{222}, 7x_{223}) +$

$(7x_{231}, 8x_{232}, 10x_{233}) + (2x_{241}, 2x_{242}, 2x_{243}) + (4x_{311}, 4x_{312}, 4x_{313}) +$

$(x_{321}, x_{322}, x_{323}) + (8x_{331}, 9x_{332}, 10x_{333}) + (9x_{341}, 11x_{342}, 12x_{343}) + 55$

$\tilde{X}_2 = (2x_{111}, 3x_{112}, 5x_{113}) + (5x_{121}, 5x_{122}, 5x_{123}) + (3x_{131}, 3x_{132}, 3x_{133}) + (3x_{141}, 4x_{142}, 5x_{143}) + (4x_{211}, 4x_{212}, 4x_{213}) + (3x_{221}, 4x_{222}, 6x_{223}) + (5x_{231}, 7x_{232}, 8x_{233}) + (4x_{241}, 4x_{242}, 4x_{243}) + (3x_{311}, 3x_{312}, 3x_{313}) + (2x_{321}, 2x_{322}, 2x_{323}) + (4x_{331}, 6x_{332}, 7x_{333}) + (8x_{341}, 9x_{342}, 10x_{343})\tilde{x}_{34} + 60.$

Step 3(b): The FlFuLFpP $(P2.7.2)$ can be transformed into its equivalent FlFuLFpP $(P2.7.3)$.

FlFuLFpP $(\boldsymbol{P2.7.3})$

$Minimize\left(\frac{(X_{11}, X_{12}, X_{13})}{(X_{21}, X_{22}, X_{23})}\right)$

Subject to

$(x_{111} + x_{121} + x_{131} + x_{141} + x_{151}, x_{112} + x_{122} + x_{132} + x_{142} + x_{152}, x_{113} + x_{123} + x_{133} + x_{143} + x_{153}) = (40,50,70),$

$(x_{211} + x_{221} + x_{231} + x_{241} + x_{251}, x_{212} + x_{222} + x_{232} + x_{242} + x_{252}, x_{213} + x_{223} + x_{233} + x_{243} + x_{253}) = (50,60,70),$

$(x_{311} + x_{321} + x_{331} + x_{341} + x_{351}, x_{312} + x_{322} + x_{332} + x_{342} + x_{352}, x_{313} + x_{323} + x_{333} + x_{343} + x_{353}) = (70,80,90),$

$(x_{411} + x_{421} + x_{431} + x_{441} + x_{451}, x_{412} + x_{422} + x_{432} + x_{442} + x_{452}, x_{413} + x_{423} + x_{433} + x_{443} + x_{453}) = (0,20,20),$

$(x_{111} + x_{211} + x_{311} + x_{411}, x_{112} + x_{212} + x_{312} + x_{412}, x_{113} + x_{213} + x_{313} + x_{413}) = (10,30,40),$

$(x_{121} + x_{221} + x_{321} + x_{421}, x_{122} + x_{222} + x_{322} + x_{422}, x_{123} + x_{223} + x_{323} + x_{423}) = (10,20,30),$

$(x_{131} + x_{231} + x_{331} + x_{431}, x_{132} + x_{232} + x_{332} + x_{432}, x_{133} + x_{233} + x_{333} + x_{433}) = (30,40,50)$,

$(x_{141} + x_{241} + x_{341} + x_{441}, x_{142} + x_{242} + x_{342} + x_{442}, x_{143} + x_{243} + x_{343} + x_{443}) = (20,30,40)$,

$(x_{151} + x_{251} + x_{351} + x_{451}, x_{152} + x_{252} + x_{352} + x_{452}, x_{153} + x_{253} + x_{353} + x_{453}) = (90,90,90)$,

$(x_{ij1}, x_{ij2}, x_{ij3})$ is a non-negative TFuN $\forall\ i = 1,2,3,4,\ \ j = 1,2,3,4,5$.

where,

$X_{11} = 4x_{111} + 6x_{121} + 2x_{131} + 2x_{141} + 3x_{211} + 5x_{221} + 7x_{231} + 2x_{241} + 4x_{311} + x_{321} + 8x_{331} + 9x_{341} + 55$

$X_{12} = 5x_{112} + 6x_{122} + 2x_{132} + 3x_{142} + 3x_{212} + 6x_{222} + 8x_{232} + 2x_{242} + 4x_{312} + x_{322} + 9x_{332} + 11x_{342} + 55$

$X_{13} = 6x_{113} + 6x_{123} + 2x_{133} + 4x_{143} + 3x_{213} + 7x_{223} + 10x_{233} + 2x_{243} + 4x_{313} + x_{323} + 10x_{333} + 12x_{343} + 55$

$X_{21} = 2x_{111} + 5x_{121} + 3x_{131} + 3x_{141} + 4x_{211} + 3x_{221} + 5x_{231} + 4x_{241} + 3x_{311} + 2x_{321} + 4x_{331} + 8x_{341} + 60$

$X_{22} = 3x_{112} + 5x_{122} + 3x_{132} + 4x_{142} + 4x_{212} + 4x_{222} + 7x_{232} + 4x_{242} + 3x_{312} + 2x_{322} + 6x_{332} + 9x_{342} + 60$

$X_{23} = 5x_{113} + 5x_{123} + 3x_{133} + 5x_{143} + 4x_{213} + 6x_{223} + 8x_{233} + 4x_{243} + 3x_{313} + 2x_{323} + 7x_{333} + 10x_{343} + 60$.

Step 3(c): The FlFuLFpP $(P2.7.3)$ can be transformed into its equivalent FlFuLFpP $(P2.7.4)$.

FlFuLFpP $(\boldsymbol{P2.7.4})$

$Minimize\left(\frac{X_{11}}{X_{23}},\frac{X_{12}}{X_{22}},\frac{X_{13}}{X_{21}}\right)$

Subject to

Constraints of the FlFuLFpP $(P2.7.3)$.

Step 3(d): The FlFuLFpP $(P2.7.4)$ can be transformed into its equivalent CrMoLFpP $(P2.7.5)$.

CrMoLFpP $(\boldsymbol{P2.7.5})$

$Minimize\left(\frac{4x_{111}+6x_{121}+2x_{131}+2x_{141}+3x_{211}+5x_{221}+7x_{231}+2x_{241}+4x_{311}+x_{321}+8x_{331}+9x_{341}+55}{5x_{113}+5x_{123}+3x_{133}+5x_{143}+4x_{213}+6x_{223}+8x_{233}+4x_{243}+3x_{313}+2x_{323}+7x_{333}+10x_{343}+60}\right),$

$Minimize\left(\frac{5x_{112}+6x_{122}+2x_{132}+3x_{142}+3x_{212}+6x_{222}+8x_{232}+2x_{242}+4x_{312}+x_{322}+9x_{332}+11x_{342}+55}{3x_{112}+5x_{122}+3x_{132}+4x_{142}+4x_{212}+4x_{222}+7x_{232}+4x_{242}+3x_{312}+2x_{322}+6x_{332}+9x_{342}+60}\right),$

$Minimize\left(\frac{6x_{113}+6x_{123}+2x_{133}+4x_{143}+3x_{213}+7x_{223}+10x_{233}+2x_{243}+4x_{313}+x_{323}+10x_{333}+12x_{343}+55}{2x_{111}+5x_{121}+3x_{131}+3x_{141}+4x_{211}+3x_{221}+5x_{231}+4x_{241}+3x_{311}+2x_{321}+4x_{331}+8x_{341}+60}\right)$

Subject to

$x_{111}+x_{121}+x_{131}+x_{141}+x_{151}=40,$

$x_{112}+x_{122}+x_{132}+x_{142}+x_{152}=50,$

$x_{113}+x_{123}+x_{133}+x_{143}+x_{153}=70,$

$x_{211}+x_{221}+x_{231}+x_{241}+x_{251}=50,$

$x_{212}+x_{222}+x_{232}+x_{242}+x_{252}=60,$

$x_{213}+x_{223}+x_{233}+x_{243}+x_{253}=70,$

$x_{311}+x_{321}+x_{331}+x_{341}+x_{351}=70,$

$x_{312}+x_{322}+x_{332}+x_{342}+x_{352}=80,$

$x_{313}+x_{323}+x_{333}+x_{343}+x_{353}=90,$

$x_{411}+x_{421}+x_{431}+x_{441}+x_{451}=0,$

$x_{412}+x_{422}+x_{432}+x_{442}+x_{452}=20,$

$x_{413} + x_{423} + x_{433} + x_{443} + x_{453} = 20,$

$x_{111} + x_{211} + x_{311} + x_{411} = 10,$

$x_{112} + x_{212} + x_{312} + x_{412} = 30,$

$x_{113} + x_{213} + x_{313} + x_{413} = 40,$

$x_{121} + x_{221} + x_{321} + x_{421} = 10,$

$x_{122} + x_{222} + x_{322} + x_{422} = 20,$

$x_{123} + x_{223} + x_{323} + x_{423} = 30,$

$x_{131} + x_{231} + x_{331} + x_{431} = 30,$

$x_{132} + x_{232} + x_{332} + x_{432} = 40,$

$x_{133} + x_{233} + x_{333} + x_{433} = 50,$

$x_{141} + x_{241} + x_{341} + x_{441} = 20,$

$x_{142} + x_{242} + x_{342} + x_{442} = 30,$

$x_{143} + x_{243} + x_{343} + x_{443} = 40,$

$x_{151} + x_{251} + x_{351} + x_{451} = 90,$

$x_{152} + x_{252} + x_{352} + x_{452} = 90,$

$x_{153} + x_{253} + x_{353} + x_{453} = 90,$

$0 \leq x_{ij1} \leq x_{ij2} \leq x_{ij3}; i = 1,2,3,4, j = 1,2,3,4,5.$

Step 4: Using Step 4 of the proposed Mehar method, there is a need to apply an appropriate method to solve the CrMoLFpP $(P2.7.5)$.

Using the lexicographic approach [18], the obtained crisp efficient solutions are

(i) $x_{131} = x_{132} = x_{133} = 40, x_{152} = x_{153} = x_{211} = x_{212} = x_{213} = 10, x_{241} = x_{242} = x_{243} = 30, x_{252} = x_{253} = x_{321} = x_{322} = x_{323} = x_{412} = x_{413} = 20, x_{352} = x_{353} = 6, x_{111} = x_{112} = x_{113} = x_{121} = x_{122} = x_{123} = x_{141} = x_{142} = x_{143} = x_{151} = x_{221} =$

$x_{222} = x_{223} = x_{231} = x_{232} = x_{233} = x_{251} = x_{311} = x_{312} = x_{313} = x_{331} = x_{332} = x_{333} = x_{341} = x_{342} = x_{343} = x_{351} = x_{411} = x_{421} = x_{422} = x_{423} = x_{431} = x_{432} = x_{433} = x_{441} = x_{442} = x_{443} = x_{451} = x_{452} = x_{453} = 0.$

(ii) $x_{132} = x_{133} = 40, x_{152} = x_{153} = x_{212} = x_{213} = 10, x_{211} = 5.54, x_{242} = x_{243} = 30, x_{252} = x_{253} = x_{322} = x_{323} = x_{412} = x_{413} = 20, x_{352} = x_{353} = 60, x_{111} = x_{112} = x_{113} = x_{121} = x_{122} = x_{123} = x_{131} = x_{141} = x_{142} = x_{143} = x_{151} = x_{221} = x_{222} = x_{223} = x_{231} = x_{232} = x_{233} = x_{241} = x_{251} = x_{311} = x_{312} = x_{313} = x_{321} = x_{331} = x_{332} = x_{333} = x_{341} = x_{342} = x_{343} = x_{351} = x_{411} = x_{421} = x_{422} = x_{423} = x_{431} = x_{432} = x_{433} = x_{441} = x_{442} = x_{443} = x_{451} = x_{452} = x_{453} = 0.$

(iii) $x_{113} = 13.82, x_{131} = x_{132} = 30, x_{133} = 46.18, x_{151} = x_{152} = x_{153} = x_{211} = x_{212} = x_{321} = x_{322} = x_{323} = 10, x_{213} = 26.18, x_{233} = 3.82, x_{241} = x_{242} = x_{243} = x_{251} = x_{252} = x_{253} = x_{343} = x_{423} = 20, x_{351} = x_{352} = x_{353} = 60, x_{111} = x_{112} = x_{121} = x_{122} = x_{123} = x_{141} = x_{142} = x_{143} = x_{221} = x_{222} = x_{223} = x_{231} = x_{232} = x_{311} = x_{312} = x_{313} = x_{331} = x_{332} = x_{333} = x_{341} = x_{342} = x_{411} = x_{412} = x_{413} = x_{421} = x_{422} = x_{431} = x_{432} = x_{433} = x_{441} = x_{442} = x_{443} = x_{451} = x_{452} = x_{453} = 0.$

(iv) $x_{131} = x_{132} = 30, x_{133} = 50.001, x_{143} = x_{151} = x_{152} = x_{153} = x_{211} = x_{321} = x_{322} = 10, x_{212} = x_{213} = 20.13, x_{241} = x_{242} = x_{251} = x_{252} = x_{253} = 20, x_{243} = x_{323} = 29.86, x_{342} = x_{343} = 0.136, x_{351} = x_{352} = x_{353} = 60, x_{413} = 19.86, x_{423} = 0.135, x_{111} = x_{112} = x_{113} = x_{121} = x_{122} = x_{123} = x_{141} = x_{142} = x_{221} = x_{222} = x_{223} = x_{231} = x_{232} = x_{233} = x_{311} = x_{312} = x_{313} = x_{331} = x_{332} = x_{333} = x_{341} = x_{411} = x_{412} = x_{421} = x_{422} = x_{431} = x_{432} = x_{433} = x_{441} = x_{442} = x_{443} = x_{451} = x_{452} = x_{453} = 0.$

(v) $x_{113} = x_{131} = x_{132} = x_{133} = 30.01, x_{151} = x_{152} = x_{153} = x_{211} = x_{212} = x_{213} = x_{321} = x_{322} = x_{323} = 10, x_{233} = x_{343} = 6.14, x_{241} = x_{251} = x_{252} = x_{253} = x_{423} = 20, x_{242} = x_{243} = 33.88, x_{333} = 13.87, x_{351} = x_{352} = x_{353} = 60, x_{111} = x_{112} = x_{121} = x_{122} = x_{123} = x_{141} = x_{142} = x_{143} = x_{221} = x_{222} = x_{223} = x_{231} = x_{232} = x_{311} = x_{312} = x_{313} = x_{331} = x_{332} = x_{341} = x_{342} = x_{411} = x_{412} = x_{413} = x_{421} = x_{422} = x_{431} = x_{432} = x_{433} = x_{441} = x_{442} = x_{443} = x_{451} = x_{452} = x_{453} = 0.$

Step 5: Using the crisp efficient solutions, the fuzzy efficient solutions of the FlFuLFpP $(P2.7.1)$ are

(i) $(x_{111}, x_{112}, x_{113}) = (0,0,0), (x_{121}, x_{122}, x_{123}) = (0,0,0), (x_{131}, x_{132}, x_{133}) = (40,40,40), (x_{141}, x_{142}, x_{143}) = (0,0,0), (x_{151}, x_{152}, x_{153}) = (0,10,10), (x_{211}, x_{212}, x_{213}) = (10,10,10), (x_{221}, x_{222}, x_{223}) = (0,0,0), (x_{231}, x_{232}, x_{233}) = (0,0,0), (x_{241}, x_{242}, x_{243}) = (30,30,30), (x_{251}, x_{252}, x_{253}) = (0,20,20), (x_{311}, x_{312}, x_{313}) = (0,0,0), (x_{321}, x_{322}, x_{323}) = (20,20,20), (x_{331}, x_{332}, x_{333}) = (0,0,0), (x_{341}, x_{342}, x_{343}) = (0,0,0), (x_{351}, x_{352}, x_{353}) = (0,6,6), (x_{411}, x_{412}, x_{413}) = (0,20,20), (x_{421}, x_{422}, x_{423}) = (0,0,0), (x_{431}, x_{432}, x_{433}) = (0,0,0), (x_{441}, x_{442}, x_{443}) = (0,0,0), (x_{451}, x_{452}, x_{453}) = (0,0,0).$

(ii) $(x_{111}, x_{112}, x_{113}) = (0,0,0), (x_{121}, x_{122}, x_{123}) = (0,0,0), (x_{131}, x_{132}, x_{133}) = (0,40,40), (x_{141}, x_{142}, x_{143}) = (0,0,0), (x_{151}, x_{152}, x_{153}) = (0,10,10), (x_{211}, x_{212}, x_{213}) = (5.54,10,10), (x_{221}, x_{222}, x_{223}) = (0,0,0), (x_{231}, x_{232}, x_{233}) = (0,0,0), (x_{241}, x_{242}, x_{243}) = (0,30,30), (x_{251}, x_{252}, x_{253}) = (0,20,20), (x_{311}, x_{312}, x_{313}) = (0,0,0), (x_{321}, x_{322}, x_{323}) = (0,20,20), (x_{331}, x_{332}, x_{333}) = (0,0,0), (x_{341}, x_{342}, x_{343}) = (0,0,0), (x_{351}, x_{352}, x_{353}) = (0,60,60),$

$(x_{411}, x_{412}, x_{413}) = (0,20,20), (x_{421}, x_{422}, x_{423}) = (0,0,0), (x_{431}, x_{432}, x_{433}) = (0,0,0)$, $(x_{441}, x_{442}, x_{443}) = (0,0,0), (x_{451}, x_{452}, x_{453}) = (0,0,0)$.

(iii) $(x_{111}, x_{112}, x_{113}) = (0,0,13.82), (x_{121}, x_{122}, x_{123}) = (0,0,0), (x_{131}, x_{132}, x_{133}) = (30,30,46.18), (x_{141}, x_{142}, x_{143}) = (0,0,0), (x_{151}, x_{152}, x_{153}) = (10,10,10), (x_{211}, x_{212}, x_{213}) = (10,10,26.18), (x_{221}, x_{222}, x_{223}) = (0,0,0), (x_{231}, x_{232}, x_{233}) = (0,0,3.82), (x_{241}, x_{242}, x_{243}) = (20,20,20), (x_{251}, x_{252}, x_{253}) = (20,20,20), (x_{311}, x_{312}, x_{313}) = (0,0,0), (x_{321}, x_{322}, x_{323}) = (10,10,10), (x_{331}, x_{332}, x_{333}) = (0,0,0), (x_{341}, x_{342}, x_{343}) = (0,0,20), (x_{351}, x_{352}, x_{353}) = (60,60,60), (x_{411}, x_{412}, x_{413}) = (0,0,0), (x_{421}, x_{422}, x_{423}) = (0,0,20), (x_{431}, x_{432}, x_{433}) = (0,0,0), (x_{441}, x_{442}, x_{443}) = (0,0,0), (x_{451}, x_{452}, x_{453}) = (0,0,0)$.

(iv) $(x_{111}, x_{112}, x_{113}) = (0,0,0), (x_{121}, x_{122}, x_{123}) = (0,0,0), (x_{131}, x_{132}, x_{133}) = (30,30,50.001), (x_{141}, x_{142}, x_{143}) = (0,0,10), (x_{151}, x_{152}, x_{153}) = (10,10,10), (x_{211}, x_{212}, x_{213}) = (10,20.13,20.13), (x_{221}, x_{222}, x_{223}) = (0,0,0), (x_{231}, x_{232}, x_{233}) = (0,0,0), (x_{241}, x_{242}, x_{243}) = (20,20,29.86), (x_{251}, x_{252}, x_{253}) = (20,20,20), (x_{311}, x_{312}, x_{313}) = (0,0,0), (x_{321}, x_{322}, x_{323}) = (10,10,29.86), (x_{331}, x_{332}, x_{333}) = (0,0,0), (x_{341}, x_{342}, x_{343}) = (0,0.136,0.136), (x_{351}, x_{352}, x_{353}) = (60,60,60), (x_{411}, x_{412}, x_{413}) = (0,0,19.86), (x_{421}, x_{422}, x_{423}) = (0,0,0.135), (x_{431}, x_{432}, x_{433}) = (0,0,0), (x_{441}, x_{442}, x_{443}) = (0,0,0), (x_{451}, x_{452}, x_{453}) = (0,0,0)$.

(v) $(x_{111}, x_{112}, x_{113}) = (0,0,30.01), (x_{121}, x_{122}, x_{123}) = (0,0,0), (x_{131}, x_{132}, x_{133}) = (30.01,30.01,30.01), (x_{141}, x_{142}, x_{143}) = (0,0,0), (x_{151}, x_{152}, x_{153}) = (10,10,10), (x_{211}, x_{212}, x_{213}) = (10,10,10), (x_{221}, x_{222}, x_{223}) = (0,0,0), (x_{231}, x_{232}, x_{233}) = (0,0,6.14), (x_{241}, x_{242}, x_{243}) = (20,33.88,33.88)$,

$(x_{251}, x_{252}, x_{253}) = (20,20,20), (x_{311}, x_{312}, x_{313}) = (0,0,0), (x_{321}, x_{322}, x_{323}) = (10,10,10), (x_{331}, x_{332}, x_{333}) = (0,0,13.87), (x_{341}, x_{342}, x_{343}) = (0,0,6.14), (x_{351}, x_{352}, x_{353}) = (60,60,60), (x_{411}, x_{412}, x_{413}) = (0,0,0), (x_{421}, x_{422}, x_{423}) = (0,0,20), (x_{431}, x_{432}, x_{433}) = (0,0,0), (x_{441}, x_{442}, x_{443}) = (0,0,0),$ $(x_{451}, x_{452}, x_{453}) = (0,0,0)$.

Step 6: Using the fuzzy efficient solutions, the fuzzy optimal values of the FlFuLFpP $(P2.7.1)$ are

(i) $(0.6447,0.6447,0.6447)$

(ii) $(0.1885,0.6447,2.982)$

(iii) $(0.2774,0.6724,2.1966)$

(iv) $(0.4139,0.6839,1.0572)$

(v) $(0.2773,0.6447,2.3334)$

2.8 Conclusions

In this chapter,

(i) It is shown that it is inappropriate to use the existing methods [9, 10, 45-50, 95, 110, 153, 154, 164] for solving unbalanced FlFuLFtPs.

(ii) An efficient method (named as Mehar method) is proposed for solving FlFuLFtPs.

Chapter 3

Efficient Method for Solving Fuzzy Linear Fractional Minimal Cost Flow Problems[1]

After reviewing the literature, it may be concluded that there does not exist any method except Mahmoodirad et al.'s method [117] for solving FuLFMiCfPs. In this chapter,

(i) It is pointed out that it is inappropriate to use the existing method [117].

(ii) An efficient method is proposed for solving FuLFMiCfPs.

3.1 LFpP corresponding to a LFMiCfP

An optimal solution of a linear minimal cost flow problem consisting of a non-empty set V of vertices and a set E disjoint from V as the set of arcs, can be obtained by solving its equivalent LpP $(P3.1.1)$ [4].

LpP $(\boldsymbol{P3.1.1})$

$Minimize\left(\sum_{(i,j)\in E} c_{ij}x_{ij}\right)$

Subject to

$\sum_{(i,j)\in E} x_{ij} - \sum_{(j,i)\in E} x_{ji} = b_i \quad \forall\, i \in \{1,2,\ldots,m\}$

$l_{ij} \le x_{ij} \le u_{ij} \quad \forall\, (i,j) \in E$

where,

(i) The non-negative real number x_{ij} represents the amount of flow on the arc (i,j).

[1] The contents of this chapter are published in "Journal of Intelligent & Fuzzy Systems 43 (2022) 1035-1051".

(ii) The non-negative real number b_i represents the difference between the amount sent from vertex i and the amount received by this vertex.

(iii) The positive real number c_{ij} represents the cost for transporting one unit quantity of the commodity across the arc (i, j).

(iv) The non-negative real number l_{ij} represents the lower bound capacity of the arc $(i, j) \in E$.

(v) The non-negative real number u_{ij} represents the upper bound capacity of the arc $(i, j) \in E$.

If it is assumed that the aim of a minimal cost flow problem is to minimize the efficiency of some activity, e.g., cost of production per unit of produced goods, instead of determining the least cost shipment of a commodity. Then, such a minimal cost flow problem is known as LFMiCfP.

An optimal solution of a LFMiCfP consisting of a non-empty set V of vertices and a set E disjoint from V as the set of arcs, can be obtained by solving its equivalent LFpP $(P3.1.2)$ [170].

LFpP $(\boldsymbol{P3.1.2})$

$$Minimize \left(\frac{\sum_{(i,j)\in E} c_{ij}x_{ij}+\gamma}{\sum_{(i,j)\in E} d_{ij}x_{ij}+\beta}\right)$$

Subject to

$$\sum_{(i,j)\in E} x_{ij} - \sum_{(j,i)\in E} x_{ji} = b_i \quad \forall\, i \in \{1,2, \ldots, m\}$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad \forall\, (i, j) \in E$$

where,

(i) The positive real number d_{ij} represents the profit for transporting one unit quantity of the commodity across the arc (i, j).

(ii) The non-negative real numbers γ and β are given constants.

3.2 Mahmoodirad et al.'s method for solving FuLFMiCfPs

Mahmoodirad et al. [117] claimed that to find an optimal solution of FuLFMiCfPs (LFMiCfP in which each known arc cost is either represented by a TFuN or a TrFuN) is equivalent to find an optimal solution of its equivalent FuLFpP $(P3.2.1)$. Hence, Mahmoodirad et al. [117] proposed the following method to find an optimal solution of the FuLFpP $(P3.2.1)$.

FuLFpP $(\boldsymbol{P3.2.1})$

$$Minimize \left(\frac{\sum_{(i,j)\in E}\tilde{c}_{ij}x_{ij}+\gamma}{\sum_{(i,j)\in E}\tilde{d}_{ij}x_{ij}+\beta}\right)$$

Subject to

$$\sum_{(i,j)\in E}x_{ij}-\sum_{(j,i)\in E}x_{ji}=b_i \quad \forall\, i\in\{1,2,\dots,m\}$$

$$l_{ij}\le x_{ij}\le u_{ij} \quad \forall\,(i,j)\in E$$

where,

(i) The positive TFuN $\tilde{c}_{ij}=(c_{ij1},c_{ij2},c_{ij3})$ or the positive TrFuN $\tilde{c}_{ij}=(c_{ij1},c_{ij2},c_{ij3},c_{ij4})$ represents the cost for transporting one unit quantity of the commodity across the arc (i,j).

(ii) The positive TFuN $\tilde{d}_{ij}=(d_{ij1},d_{ij2},d_{ij3})$ or the positive TrFuN $\tilde{d}_{ij}=(d_{ij1},d_{ij2},d_{ij3},d_{ij4})$ represents the profit for transporting one unit quantity of the commodity across the arc (i,j).

Step 1: Transform the FuLFpP $(P3.2.1)$ into its equivalent CrMpPs $(P3.2.2)$ and $(P3.2.3)$.

CrMpP $(\boldsymbol{P3.2.2})$

$$Z_\alpha^L=\min_{\substack{(C_{ij})_\alpha^L\le c_{ij}\le (C_{ij})_\alpha^U\\ (D_{ij})_\alpha^L\le d_{ij}\le (D_{ij})_\alpha^U\\ \forall\,(i,j)\in E}}\left[\min\left(\frac{\sum_{(i,j)\in E}c_{ij}x_{ij}+\gamma}{\sum_{(i,j)\in E}d_{ij}x_{ij}+\beta}\right)\right]$$

Subject to

Constraints of the FuLFpP $(P3.2.1)$.

where,

(i) $\left(C_{ij}\right)_{\alpha}^{L}$ is the lower bound of the α −cut for the fuzzy number $\tilde{c}_{ij}$.

(ii) $\left(C_{ij}\right)_{\alpha}^{U}$ is the upper bound of the α −cut for the fuzzy number $\tilde{c}_{ij}$.

(iii) $\left(D_{ij}\right)_{\alpha}^{L}$ is the lower bound of the α −cut for the fuzzy number $\tilde{d}_{ij}$.

(iv) $\left(D_{ij}\right)_{\alpha}^{U}$ is the upper bound of the α −cut for the fuzzy number $\tilde{d}_{ij}$.

CrMpP $(\boldsymbol{P3.2.3})$

$$Z_{\alpha}^{U} = \max_{\substack{\left(C_{ij}\right)_{\alpha}^{L} \leq c_{ij} \leq \left(C_{ij}\right)_{\alpha}^{U} \\ \left(D_{ij}\right)_{\alpha}^{L} \leq d_{ij} \leq \left(D_{ij}\right)_{\alpha}^{U} \\ \forall (i,j)\in E}} [\min\left(\frac{\sum_{(i,j)\in E} c_{ij}x_{ij}+\gamma}{\sum_{(i,j)\in E} d_{ij}x_{ij}+\beta}\right)]$$

Subject to

Constraints of the FuLFpP $(P3.2.1)$.

Step 2: Use the following steps to transform the CrMpP $(P3.2.2)$ into its equivalent CrLpP.

Step 2a: Transform the CrMpP $(P3.2.2)$ into its equivalent CrMpP $(P3.2.4)$.

CrMpP $(\boldsymbol{P3.2.4})$

$$Z_{\alpha}^{L} = \min\left(\frac{\sum_{(i,j)\in E} c_{ij}x_{ij}+\gamma}{\sum_{(i,j)\in E} d_{ij}x_{ij}+\beta}\right)$$

Subject to

$$\sum_{(i,j)\in E} x_{ij} - \sum_{(j,i)\in E} x_{ji} = b_i \quad \forall\, i \in \{1,2,\ldots,m\}$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \quad \forall\, (i,j) \in E$$

$$\left(C_{ij}\right)_{\alpha}^{L} \leq c_{ij} \leq \left(C_{ij}\right)_{\alpha}^{U} \quad \forall\, (i,j) \in E$$

$$\left(D_{ij}\right)_{\alpha}^{L} \leq d_{ij} \leq \left(D_{ij}\right)_{\alpha}^{U} \quad \forall\, (i,j) \in E$$

Step 2b: Transform the CrMpP $(P3.2.4)$ into its equivalent CrMpP $(P3.2.5)$.

CrMpP $(\boldsymbol{P3.2.5})$

$Z_\alpha^L = \min\left(\sum_{(i,j)\in E} c_{ij} y_{ij} + \gamma t\right)$

Subject to

$\sum_{(i,j)\in E} y_{ij} - \sum_{(j,i)\in E} y_{ji} = b_i t \quad \forall\, i \in \{1,2,\dots,m\}$

$\sum_{(i,j)\in E} d_{ij} y_{ij} + \beta t = 1$

$l_{ij} t \le y_{ij} \le u_{ij} t \qquad \forall\,(i,j) \in E$

$\left(C_{ij}\right)_\alpha^L \le c_{ij} \le \left(C_{ij}\right)_\alpha^U \qquad \forall\,(i,j) \in E$

$\left(D_{ij}\right)_\alpha^L \le d_{ij} \le \left(D_{ij}\right)_\alpha^U \qquad \forall\,(i,j) \in E$

$t > 0.$

Step 2c: Transform the CrMpP $(P3.2.5)$ into its equivalent CrMpP $(P3.2.6)$.

CrMpP $(\boldsymbol{P3.2.6})$

$Z_\alpha^L = \min\left(\sum_{(i,j)\in E} \left(C_{ij}\right)_\alpha^L y_{ij} + \gamma t\right)$

Subject to

$\sum_{(i,j)\in E} y_{ij} - \sum_{(j,i)\in E} y_{ji} = b_i t \quad \forall\, i \in \{1,2,\dots,m\}$

$\sum_{(i,j)\in E} d_{ij} y_{ij} + \beta t = 1$

$l_{ij} t \le y_{ij} \le u_{ij} t \qquad \forall\,(i,j) \in E$

$\left(D_{ij}\right)_\alpha^L \le d_{ij} \le \left(D_{ij}\right)_\alpha^U \qquad \forall\,(i,j) \in E$

$t > 0.$

Step 2d: Transform the CrMpP $(P3.2.6)$ into its equivalent CrLpP $(P3.2.7)$.

CrLpP $(\boldsymbol{P3.2.7})$

$Z_\alpha^L = \min\left(\sum_{(i,j)\in E} \left(C_{ij}\right)_\alpha^L y_{ij} + \gamma t\right)$

Subject to

$$\sum_{(i,j)\in E} y_{ij} - \sum_{(j,i)\in E} y_{ji} = b_i t \quad \forall\, i \in \{1,2,\dots,m\}$$

$$\sum_{(i,j)\in E} \rho_{ij} + \beta t = 1$$

$$l_{ij} t \le y_{ij} \le u_{ij} t \qquad \forall\, (i,j) \in E$$

$$\left(D_{ij}\right)_\alpha^L y_{ij} \le \rho_{ij} \le \left(D_{ij}\right)_\alpha^U y_{ij} \qquad \forall\, (i,j) \in E$$

$$t > 0.$$

Step 3: Use the following steps to transform the CrMpP $(P3.2.3)$ into its equivalent CrLpP.

Step 3a: Transform the CrMpP $(P3.2.3)$ into its equivalent CrMpP $(P3.2.8)$.

CrMpP $(\boldsymbol{P3.2.8})$

$$Z_\alpha^U = \max_{\substack{\left(C_{ij}\right)_\alpha^L \le c_{ij} \le \left(C_{ij}\right)_\alpha^U \\ \left(D_{ij}\right)_\alpha^L \le d_{ij} \le \left(D_{ij}\right)_\alpha^U \\ \forall\, (i,j)\in E}} \left[\min\ \left(\sum_{(i,j)\in E} c_{ij} y_{ij} + \gamma\, t\right)\right]$$

Subject to

$$\sum_{(i,j)\in E} y_{ij} - \sum_{(j,i)\in E} y_{ji} = b_i t \quad \forall\, i \in \{1,2,\dots,m\}$$

$$\sum_{(i,j)\in E} d_{ij} y_{ij} + \beta t = 1$$

$$l_{ij} t \le y_{ij} \le u_{ij} t \qquad \forall\, (i,j) \in E$$

$$t > 0.$$

Step 3b: Transform the CrMpP $(P3.2.8)$ into its equivalent CrMpP $(P3.2.9)$.

CrMpP $(\boldsymbol{P3.2.9})$

$$Z_\alpha^U = \max p$$

Subject to

$$\pi_i - \pi_j + d_{ij} p + s_{ij} - n_{ij} \le c_{ij} \quad \forall\, (i,j) \in E$$

$$\sum_{i=1}^{m} (-b_i \pi_i) + \beta p - \sum_{(i,j)\in E} l_{ij} s_{ij} + \sum_{(i,j)\in E} u_{ij} n_{ij} \le \gamma$$

$\left(C_{ij}\right)_{\alpha}^{L} \leq c_{ij} \leq \left(C_{ij}\right)_{\alpha}^{U} \qquad \forall\ (i,j) \in E$

$\left(D_{ij}\right)_{\alpha}^{L} \leq d_{ij} \leq \left(D_{ij}\right)_{\alpha}^{U} \qquad \forall\ (i,j) \in E$

$s_{ij}, n_{ij} \geq 0 \quad \forall\ (i,j) \in E$

π_i, p is unrestricted $\forall\ i \in \{1,2,\dots,m\}$

Step 3c: Transform the CrMpP $(P3.2.9)$ into its equivalent CrMpP $(P3.2.10)$.

CrMpP $(\boldsymbol{P3.2.10})$

$Z_{\alpha}^{U} = \max p$

Subject to

$\pi_i - \pi_j + d_{ij}p + s_{ij} - n_{ij} \leq \left(C_{ij}\right)_{\alpha}^{U} \quad \forall\ (i,j) \in E$

$\sum_{i=1}^{m}(-b_i\pi_i) + \beta p - \sum_{(i,j)\in E} l_{ij}s_{ij} + \sum_{(i,j)\in E} u_{ij}n_{ij} \leq \gamma$

$\left(D_{ij}\right)_{\alpha}^{L} \leq d_{ij} \leq \left(D_{ij}\right)_{\alpha}^{U} \qquad \forall\ (i,j) \in E$

$s_{ij}, n_{ij} \geq 0 \quad \forall\ (i,j) \in E$

$p > 0, \pi_i$ is unrestricted $\forall\ i \in \{1,2,\dots,m\}$

Step 3d: Transform the CrMpP $(P3.2.10)$ into its equivalent CrLpP $(P3.2.11)$.

CrLpP $(\boldsymbol{P3.2.11})$

$Z_{\alpha}^{U} = \max p$

Subject to

$\pi_i - \pi_j + w_{ij} + s_{ij} - n_{ij} \leq \left(C_{ij}\right)_{\alpha}^{U} \quad \forall\ (i,j) \in E$

$\sum_{i=1}^{m}(-b_i\pi_i) + \beta p - \sum_{(i,j)\in E} l_{ij}s_{ij} + \sum_{(i,j)\in E} u_{ij}n_{ij} \leq \gamma$

$\left(D_{ij}\right)_{\alpha}^{L} p \leq w_{ij} \leq \left(D_{ij}\right)_{\alpha}^{U} p \qquad \forall\ (i,j) \in E$

$s_{ij}, n_{ij} \geq 0 \quad \forall\ (i,j) \in E$

$p > 0, \pi_i$ is unrestricted $\forall\ i \in \{1,2,\ldots,m\}$

Step 4: Find an optimal solution $\left\{\left(x_{ij}\right)_{\alpha=0}\right\}$ and the corresponding optimal value ($Z^L_{\alpha=0}$) of the CrLpP ($P3.2.7$).

Step 5: Find an optimal solution $\left\{\left(x_{ij}\right)_{\alpha=1}\right\}$ and the corresponding optimal value ($Z^L_{\alpha=1}$) of the CrLpP ($P3.2.7$).

Step 6: Find an optimal solution $\left\{\left(x_{ij}\right)_{\alpha=0}\right\}$ and the corresponding optimal value ($Z^U_{\alpha=0}$) of the CrLpP ($P3.2.11$).

Step 7: Find an optimal solution $\left\{\left(x_{ij}\right)_{\alpha=1}\right\}$ and the corresponding optimal value ($Z^U_{\alpha=1}$) of the CrLpP ($P3.2.11$).

Step 8: Using the optimal values, obtained in Step 4 to Step 7, the obtained fuzzy optimal value is $(Z^L_{\alpha=0}, Z^L_{\alpha=1}, Z^U_{\alpha=1}, Z^U_{\alpha=0})$.

3.3 Inappropriateness of Mahmoodirad et al.'s method

It is pertinent to mention that as Liu's method [110], discussed in Section 2.4.1.1, and Mahmoodirad et al.'s method [117] are exactly same. Therefore, it can be easily concluded that the mathematical incorrect results considered in Liu's method [110], pointed out in Section 2.4.2.1, are also considered by Mahmoodirad et al. [117]. Hence, it is inappropriate to use Mahmoodirad et al.'s method [117].

3.4 Proposed Mehar method for solving FuLFMiCfPs

In this section, with the help of the proposed Mehar method for solving FlFuLFtPs, a new method (named as Mehar method) is proposed for solving FuLFMiCfPs.

The steps of the proposed Mehar method are as follows.

Step 1: Write the FuLFpP $(P3.4.1)$ corresponding to the FuLFMiCfP $(P3.2.1)$, in which each known arc cost is represented by a TrFuN.

FuLFpP $(\boldsymbol{P3.4.1})$

$$Minimize\left(\frac{\sum_{(i,j)\in E}(c_{ij1},c_{ij2},c_{ij3},c_{ij4})x_{ij}+\gamma}{\sum_{(i,j)\in E}(d_{ij1},d_{ij2},d_{ij3},d_{ij4})x_{ij}+\beta}\right)$$

Subject to

$$\sum_{(i,j)\in E} x_{ij} - \sum_{(j,i)\in E} x_{ji} = b_i \quad \forall\, i \in \{1,2,\dots,m\}$$

$$l_{ij} \le x_{ij} \le u_{ij} \quad \forall\, (i,j) \in E$$

Step 2: Using Definition 2.1.26 and Definition 2.1.19, transform the FuLFpP $(P3.4.1)$ into its equivalent FuLFpP $(P3.4.2)$.

FuLFpP $(\boldsymbol{P3.4.2})$

$$Minimize\left(\frac{(\sum_{(i,j)\in E} c_{ij1}x_{ij}+\gamma,\sum_{(i,j)\in E} c_{ij2}x_{ij}+\gamma,\sum_{(i,j)\in E} c_{ij3}x_{ij}+\gamma,\sum_{(i,j)\in E} c_{ij4}x_{ij}+\gamma)}{(\sum_{(i,j)\in E} d_{ij1}x_{ij}+\beta,\sum_{(i,j)\in E} d_{ij2}x_{ij}+\beta,\sum_{(i,j)\in E} d_{ij3}x_{ij}+\beta,\sum_{(i,j)\in E} d_{ij4}x_{ij}+\beta)}\right)$$

Subject to

Constraints of the FuLFpP $(P3.4.1)$.

Step 3: Using Definition 2.1.30, transform the FuLFpP $(P3.4.2)$ into its equivalent CrMoLFpP $(P3.4.3)$.

CrMoLFpP $(\boldsymbol{P3.4.3})$

$$Minimize\left(\frac{\sum_{(i,j)\in E} c_{ij1}x_{ij}+\gamma}{\sum_{(i,j)\in E} d_{ij4}x_{ij}+\beta}\right)$$

$$Minimize\left(\frac{\sum_{(i,j)\in E} c_{ij2}x_{ij}+\gamma}{\sum_{(i,j)\in E} d_{ij3}x_{ij}+\beta}\right)$$

$$Minimize\left(\frac{\sum_{(i,j)\in E} c_{ij3}x_{ij}+\gamma}{\sum_{(i,j)\in E} d_{ij2}x_{ij}+\beta}\right)$$

$$Minimize\left(\frac{\sum_{(i,j)\in E} c_{ij4}x_{ij}+\gamma}{\sum_{(i,j)\in E} d_{ij1}x_{ij}+\beta}\right)$$

Subject to

Constraints of the FuLFpP $(P3.4.1)$.

Step 4: Using the lexicographic approach [18], find all the efficient solutions $\{x_{ij}\}$ of the CrMoLFpP $(P3.4.3)$.

Step 5: Using the obtained efficient solutions $\{x_{ij}\}$, find the fuzzy optimal value of the FuLFpP $(P3.4.1)$ i.e., $\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij1},c_{ij2},c_{ij3},c_{ij4})x_{ij}+\gamma}{\sum_{i=1}^{m}\sum_{j=1}^{n}(d_{ij1},d_{ij2},d_{ij3},d_{ij4})x_{ij}+\beta}$.

3.5 Correct fuzzy optimal solutions of an existing FuLFMiCfP

Mahmoodirad et al. [117] solved the FuLFMiCfP (represented by Fig. 3.1, Table 3.1 and Table 3.2) to illustrate their proposed method. However, as discussed in Section 3.3, that Mahmoodirad et al.'s method [117] is not valid. Therefore, the optimal solutions of this network flow problem, obtained by Mahmoodirad et al. [117], are also not valid. In this section, correct fuzzy optimal solutions of the same problem are obtained by the proposed Mehar method.

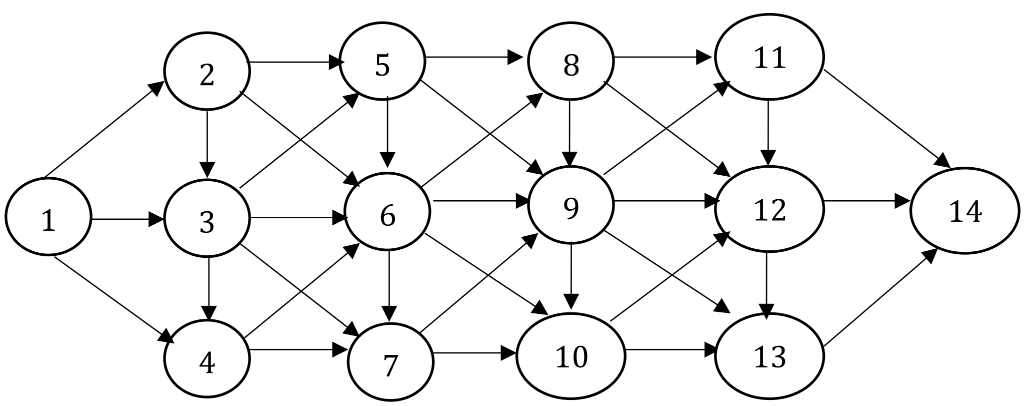

Fig. 3.1 [117] Graph of FuLFMiCfP

Table 3.1 [117] Fuzzy costs, fuzzy profit and crisp capacities of network arcs

Arc index	Tail node	Head node	Arc fuzzy cost (c_{ij})	Arc fuzzy profit (d_{ij})	Lower bound	Upper bound
1	1	2	(80,10,150,182)	(30,40,50,80)	155	450
2	1	3	(242,250,300,320)	(20,25,35,40)	160	600
3	1	4	(21,25,75,91)	(10,20,30,40)	20	55
4	2	3	(137,150,200,225)	(18,22,34,46)	45	470
5	2	5	(90,100,400,434)	(12,13,14,17)	105	125
6	2	6	(155,180,220,265)	(30,40,50,80)	70	800
7	3	4	(105,125,225,261)	(20,25,35,40)	0	450
8	3	5	(95,100,200,225)	(30,40,50,80)	0	250
9	3	6	(98,110,240,260)	(30,45,55,80)	0	125
10	3	7	(270,300,400,470)	(28,42,74,96)	0	725
11	4	6	(78,120,280,310)	(10,20,30,40)	105	625
12	4	7	(140,150,200,222)	(10,20,30,40)	20	100
13	5	6	(0,0,0,0)	(10,12,13,14)	0	100
14	5	8	(166,200,400,410)	(50,55,65,70)	0	135
15	5	9	(105,125,225,265)	(10,20,30,40)	0	625
16	6	7	(195,215,235,267)	(20,25,35,40)	15	175
17	6	8	(140,150,250,272)	(72,74,80,85)	70	160
18	6	9	(82,100,300,330)	(20,24,32,38)	80	190
19	6	10	(65,75,125,147)	(15,20,25,31)	30	750
20	7	9	(19,25,75,93)	(20,23,29,34)	0	230
21	7	10	(30,50,150,182)	(40,46,49,56)	0	220
22	8	9	(115,125,175,197)	(55,62,63,68)	30	110
23	8	11	(93,115,185,195)	(20,29,35,39)	25	120

24	8	12	(7,25,75,85)	(14,18,26,29)	0	215
25	9	10	(78,110,190,206)	(22,24,30,32)	5	200
26	9	11	(155,185,215,237)	(55,64,70,72)	40	140
27	9	12	(305,325,475,515)	(50,58,63,68)	80	190
28	9	13	(65,75,125,147)	(55,62,69,78)	5	340
29	10	12	(130,150,250,290)	(51,62,63,88)	0	165
30	10	13	(90,100,200,222)	(70,80,85,89)	0	220
31	11	12	(43,55,125,145)	(80,90,100,120)	15	340
32	11	14	(255,275,325,335)	(50,60,70,80)	0	110
33	12	13	(78,100,150,160)	(62,82,102,122)	25	215
34	12	14	(88,100,150,160)	(100,120,140,150)	20	135
35	13	14	(190,200,300,350)	(25,45,65,90)	0	265

Table 3.2 [117] Given capacities for the vertices of the graph

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}
350	100	-50	50	-50	100	-200	50	200	100	50	0	-100	-400

Step 1: According to Step 1 of the proposed Mehar method, the fuzzy optimal solution of the FuLFMiCfP can be obtained by solving its equivalent FuLFpP $(P3.5.1)$.

FuLFpP $(\boldsymbol{P3.5.1})$

$Minimize\left(\frac{\tilde{u}}{\tilde{v}}\right)$

Subject to

$x_{12}+x_{13}+x_{14}=350,$

$x_{23}+x_{25}+x_{26}-x_{12}=100,$

$-x_{13} - x_{23} + x_{34} + x_{35} + x_{36} + x_{37} = -50,$

$x_{46} + x_{47} - x_{14} - x_{34} = 50,$

$-x_{25} - x_{35} + x_{56} + x_{58} + x_{59} = -50,$

$x_{67} + x_{68} + x_{69} + x_{610} - x_{26} - x_{36} - x_{46} - x_{56} = 100,$

$-x_{37} - x_{47} - x_{67} + x_{79} + x_{710} = -200,$

$x_{89} + x_{811} + x_{812} - x_{58} - x_{68} = 50,$

$x_{910} + x_{911} + x_{912} + x_{913} - x_{59} - x_{69} - x_{79} - x_{89} = 200,$

$-x_{610} - x_{710} - x_{910} + x_{1012} + x_{1013} = -100,$

$x_{1112} + x_{1114} - x_{811} - x_{911} = 50,$

$x_{1213} + x_{1214} - x_{812} - x_{912} - x_{1012} - x_{1112} = 0,$

$-x_{913} - x_{1013} - x_{1213} + x_{1314} = -100,$

$-x_{1114} - x_{1214} - x_{1314} = -400,$

$155 \leq x_{12} \leq 450, \quad 160 \leq x_{13} \leq 600,$

$20 \leq x_{14} \leq 550, \quad 45 \leq x_{23} \leq 470,$

$105 \leq x_{25} \leq 125, \quad 70 \leq x_{26} \leq 800,$

$0 \leq x_{34} \leq 450, \quad 0 \leq x_{35} \leq 250,$

$0 \leq x_{36} \leq 125, \quad 0 \leq x_{37} \leq 725,$

$105 \leq x_{46} \leq 625, \quad 20 \leq x_{47} \leq 100,$

$0 \leq x_{56} \leq 100, \quad 0 \leq x_{58} \leq 135,$

$0 \leq x_{59} \leq 625, \quad 15 \leq x_{67} \leq 175,$

$70 \leq x_{68} \leq 160, \quad 80 \leq x_{69} \leq 190,$

$30 \leq x_{610} \leq 750, \quad 0 \leq x_{79} \leq 230,$

$0 \leq x_{710} \leq 220, \quad 30 \leq x_{89} \leq 110,$

$25 \leq x_{811} \leq 120, \qquad 0 \leq x_{812} \leq 215,$

$5 \leq x_{910} \leq 200, \qquad 40 \leq x_{911} \leq 140,$

$80 \leq x_{912} \leq 190, \qquad 5 \leq x_{913} \leq 340,$

$0 \leq x_{1012} \leq 165, \qquad 0 \leq x_{1013} \leq 220,$

$15 \leq x_{1112} \leq 340, \qquad 0 \leq x_{1114} \leq 110,$

$25 \leq x_{1213} \leq 215, \qquad 20 \leq x_{1214} \leq 135,$

$0 \leq x_{1314} \leq 265.$

where,

$\tilde{u} = (80,100,150,182)x_{12} + (242,250,300,320)x_{13} + (21,25,75,91)x_{14} +$

$(137,150,200,225)x_{23} + (90,100,400,434)x_{25} + (155,180,220,265)x_{26} +$

$(105,125,225,261)x_{34} + (95,100,200,225)x_{35} + (98,110,240,260)x_{36} +$

$(270,300,400,470)x_{37} + (78,120,280,310)x_{46} + (140,150,200,222)x_{47} +$

$(0,0,0,0)x_{56} + (166,200,400,410)x_{58} + (105,125,225,265)x_{59} +$

$(195,215,235,267)x_{67} + (140,150,250,272)x_{68} + (82,100,300,330)x_{69} +$

$(65,75,125,147)x_{610} + (19,25,75,93)x_{79} + (30,50,150,182)x_{710} +$

$(115,125,175,197)x_{89} + (93,115,185,195)x_{811} + (7,25,75,85)x_{812} +$

$(78,110,190,206)x_{910} + (155,185,215,237)x_{911} + (305,325,475,515)x_{912} +$

$(65,75,125,147)x_{913} + (130,150,250,290)x_{1012} + (90,100,200,222)x_{1013} +$

$(43,55,125,145)x_{1112} + (255,275,325,335)x_{1114} + (78,100,150,160)x_{1213} +$

$(88,100,200,224)x_{1214} + (190,200,300,350)x_{1314}$

$\tilde{v} = (30,40,50,80)x_{12} + (20,25,35,40)x_{13} + (10,20,30,40)x_{14} + (18,22,34,46)x_{23} +$

$(12,13,14,17)x_{25} + (30,40,50,80)x_{26} + (20,25,35,40)x_{34} + (30,40,50,80)x_{35} +$

$(30,45,55,80)x_{36} + (28,42,74,96)x_{37} + (10,20,30,40)x_{46} + (10,20,30,40)x_{47} +$

$(10,12,13,14)x_{56} + (50,55,65,70)x_{58} + (10,20,30,40)x_{59} + (20,25,35,40)x_{67} + (72,74,80,85)x_{68} + (20,24,32,38)x_{69} + (15,20,25,31)x_{610} + (20,23,29,34)x_{79} + (40,46,49,56)x_{710} + (55,62,63,68)x_{89} + (20,29,35,39)x_{811} + (14,18,26,29)x_{812} + (22,24,30,32)x_{910} + (55,64,70,72)x_{911} + (50,58,63,68)x_{912} + (55,62,69,78)x_{913} + (51,62,63,88)x_{1012} + (70,80,85,89)x_{1013} + (80,90,100,120)x_{1112} + (50,60,70,80)x_{1114} + (62,82,102,122)x_{1213} + (100,120,140,150)x_{1214} + (25,45,65,90)x_{1314}$

Step 2: According to Step 2 of the proposed Mehar method, the FuLFpP $(P3.5.1)$ can be transformed into its equivalent FuLFpP $(P3.5.2)$.

FuLFpP $(\boldsymbol{P3.5.2})$

$Minimize\left(\frac{(u_1,u_2,u_3,u_4)}{(v_1,v_2,v_3,v_4)}\right)$

Subject to

Constraints of the FuLFpP $(P3.5.1)$

where,

$u_1 = 80x_{12} + 242x_{13} + 21x_{14} + 137x_{23} + 90x_{25} + 155x_{26} + 105x_{34} + 95x_{35} + 98x_{36} + 270x_{37} + 78x_{46} + 140x_{47} + 166x_{58} + 105x_{59} + 195x_{67} + 140x_{68} + 82x_{69} + 65x_{610} + 19x_{79} + 30x_{710} + 115x_{89} + 93x_{811} + 7x_{812} + 78x_{910} + 155x_{911} + 305x_{912} + 65x_{913} + 130x_{1012} + 90x_{1013} + 43x_{1112} + 255x_{1114} + 78x_{1213} + 88x_{1214} + 190x_{1314}$,

$u_2 = 100x_{12} + 250x_{13} + 25x_{14} + 150x_{23} + 100x_{25} + 180x_{26} + 125x_{34} + 100x_{35} + 110x_{36} + 300x_{37} + 120x_{46} + 150x_{47} + 200x_{58} + 125x_{59} + 215x_{67} + 150x_{68} + 100x_{69} + 75x_{610} + 25x_{79} + 50x_{710} + 125x_{89} + 115x_{811} + 25x_{812} + 110x_{910} +$

$185x_{911} + 325x_{912} + 75x_{913} + 150x_{1012} + 100x_{1013} + 55x_{1112} + 275x_{1114} +$

$100x_{1213} + 100x_{1214} + 200x_{1314},$

$u_3 = 150x_{12} + 300x_{13} + 75x_{14} + 200x_{23} + 400x_{25} + 220x_{26} + 225x_{34} + 200x_{35} +$

$240x_{36} + 400x_{37} + 280x_{46} + 200x_{47} + 400x_{58} + 225x_{59} + 235x_{67} + 250x_{68} +$

$300x_{69} + 125x_{610} + 75x_{79} + 150x_{710} + 175x_{89} + 185x_{811} + 75x_{812} + 190x_{910} +$

$215x_{911} + 475x_{912} + 125x_{913} + 250x_{1012} + 200x_{1013} + 125x_{1112} + 325x_{1114} +$

$150x_{1213} + 200x_{1214} + 300x_{1314},$

$u_4 = 182x_{12} + 320x_{13} + 91x_{14} + 225x_{23} + 434x_{25} + 265x_{26} + 261x_{34} + 225x_{35} +$

$260x_{36} + 470x_{37} + 310x_{46} + 222x_{47} + 410x_{58} + 265x_{59} + 267x_{67} + 272x_{68} +$

$330x_{69} + 147x_{610} + 93x_{79} + 182x_{710} + 197x_{89} + 195x_{811} + 85x_{812} + 206x_{910} +$

$237x_{911} + 515x_{912} + 147x_{913} + 290x_{1012} + 222x_{1013} + 145x_{1112} + 335x_{1114} +$

$160x_{1213} + 224x_{1214} + 350x_{1314},$

$v_1 = 30x_{12} + 20x_{13} + 10x_{14} + 18x_{23} + 12x_{25} + 30x_{26} + 20x_{34} + 30x_{35} + 30x_{36} +$

$28x_{37} + 10x_{46} + 10x_{47} + 10x_{56} + 50x_{58} + 10x_{59} + 20x_{67} + 72x_{68} + 20x_{69} +$

$15x_{610} + 20x_{79} + 40x_{710} + 55x_{89} + 20x_{811} + 14x_{812} + 22x_{910} + 55x_{911} + 50x_{912} +$

$55x_{913} + 51x_{1012} + 70x_{1013} + 80x_{1112} + 50x_{1114} + 62x_{1213} + 100x_{1214} + 25x_{1314},$

$v_2 = 40x_{12} + 25x_{13} + 20x_{14} + 22x_{23} + 13x_{25} + 40x_{26} + 25x_{34} + 40x_{35} + 45x_{36} +$

$42x_{37} + 20x_{46} + 20x_{47} + 12x_{56} + 55x_{58} + 20x_{59} + 25x_{67} + 74x_{68} + 24x_{69} +$

$20x_{610} + 23x_{79} + 46x_{710} + 62x_{89} + 29x_{811} + 18x_{812} + 24x_{910} + 64x_{911} + 58x_{912} +$

$62x_{913} + 62x_{1012} + 80x_{1013} + 90x_{1112} + 60x_{1114} + 82x_{1213} + 120x_{1214} + 45x_{1314},$

$v_3 = 50x_{12} + 35x_{13} + 30x_{14} + 34x_{23} + 14x_{25} + 50x_{26} + 35x_{34} + 50x_{35} + 55x_{36} +$

$74x_{37} + 30x_{46} + 30x_{47} + 13x_{56} + 65x_{58} + 30x_{59} + 35x_{67} + 80x_{68} + 32x_{69} +$

$25x_{610} + 29x_{79} + 49x_{710} + 63x_{89} + 35x_{811} + 26x_{812} + 30x_{910} + 70x_{911} + 63x_{912} +$

$69x_{913} + 63x_{1012} + 85x_{1013} + 100x_{1112} + 70x_{1114} + 102x_{1213} + 140x_{1214} + 65x_{1314}$,

$v_4 = 80x_{12} + 40x_{13} + 40x_{14} + 46x_{23} + 17x_{25} + 80x_{26} + 40x_{34} + 80x_{35} + 80x_{36} + 96x_{37} + 40x_{46} + 40x_{47} + 14x_{56} + 70x_{58} + 40x_{59} + 40x_{67} + 85x_{68} + 38x_{69} + 31x_{610} + 34x_{79} + 56x_{710} + 68x_{89} + 39x_{811} + 29x_{812} + 32x_{910} + 72x_{911} + 68x_{912} + 78x_{913} + 88x_{1012} + 89x_{1013} + 120x_{1112} + 80x_{1114} + 122x_{1213} + 150x_{1214} + 90x_{1314}$.

Step 3: According to Step 3 of the proposed Mehar method, the FuLFpP $(P3.5.2)$ can be transformed into its equivalent CrMoLFpP $(P3.5.3)$.

CrMoLFpP $(\boldsymbol{P3.5.3})$

$Minimize\left(\frac{u_1}{v_4}\right)$

$Minimize\left(\frac{u_2}{v_3}\right)$

$Minimize\left(\frac{u_3}{v_2}\right)$

$Minimize\left(\frac{u_4}{v_1}\right)$

Subject to

Constraints of the FuLFpP $(P3.5.1)$.

Step 4: According to Step 4 of the proposed Mehar method, there is a need to apply lexicographic approach [18] to solve the CrMoLFpP $(P3.5.3)$.

Using the lexicographic approach [18], the obtained crisp efficient solutions are

(i) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 45, x_{35} = 45, x_{36} = 65, x_{37} = 0, x_{46} = 105, x_{47} = 25, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 140, x_{69} = 80, x_{610} = 80, x_{79} = 0, x_{710} = 0, x_{89} = 110, x_{811} =$

$80, x_{812} = 0, x_{910} = 20, x_{911} = 140, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(ii) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 71, x_{25} = 105, x_{26} = 79, x_{34} = 45, x_{35} = 45, x_{36} = 91, x_{37} = 0, x_{46} = 105, x_{47} = 25, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 140, x_{69} = 80, x_{610} = 80, x_{79} = 0, x_{710} = 0, x_{89} = 110, x_{811} = 80, x_{812} = 0, x_{910} = 20, x_{911} = 140, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(iii) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 40, x_{35} = 45, x_{36} = 65, x_{37} = 5, x_{46} = 105, x_{47} = 20, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 159, x_{69} = 80, x_{610} = 61, x_{79} = 0, x_{710} = 0, x_{89} = 110, x_{811} = 99, x_{812} = 0, x_{910} = 39, x_{911} = 140, x_{912} = 80, x_{913} = 131, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 19, x_{1213} = 215, x_{1214} = 135, x_{1314} = 246.$

(iv) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 40, x_{35} = 45, x_{36} = 65, x_{37} = 5, x_{46} = 105, x_{47} = 20, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 134, x_{69} = 80, x_{610} = 86, x_{79} = 0, x_{710} = 0, x_{89} = 104, x_{811} = 80, x_{812} = 0, x_{910} = 14, x_{911} = 140, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(v) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 80, x_{25} = 105, x_{26} = 70, x_{34} = 45, x_{35} = 45, x_{36} = 100, x_{37} = 0, x_{46} = 105, x_{47} = 25, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 160, x_{69} = 80, x_{610} = 60, x_{79} = 0, x_{710} = 0, x_{89} = 80, x_{811} = 120, x_{812} = 10, x_{910} = 40, x_{911} = 90, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 260, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(vi) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 67, x_{25} = 105, x_{26} = 83, x_{34} = 45, x_{35} = 45, x_{36} = 87, x_{37} = 0, x_{46} = 105, x_{47} = 25, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 160, x_{69} = 80, x_{610} = 60, x_{79} = 0, x_{710} = 0, x_{89} = 90, x_{811} = 120, x_{812} = 0, x_{910} = 40, x_{911} = 100, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(vii) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 73, x_{25} = 105, x_{26} = 77, x_{34} = 45, x_{35} = 45, x_{36} = 93, x_{37} = 0, x_{46} = 105, x_{47} = 25, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 160, x_{69} = 80, x_{610} = 60, x_{79} = 0, x_{710} = 0, x_{89} = 90, x_{811} = 120, x_{812} = 0, x_{910} = 40, x_{911} = 100, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(viii) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 40, x_{35} = 45, x_{36} = 65, x_{37} = 5, x_{46} = 105, x_{47} = 20, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 160, x_{69} = 80, x_{610} = 60, x_{79} = 0, x_{710} = 0, x_{89} = 110, x_{811} = 100, x_{812} = 0, x_{910} = 40, x_{911} = 128, x_{912} = 80, x_{913} = 142, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 8, x_{1213} = 215, x_{1214} = 135, x_{1314} = 257.$

(ix) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 40, x_{35} = 45, x_{36} = 65, x_{37} = 5, x_{46} = 105, x_{47} = 20, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 125, x_{69} = 80, x_{610} = 95, x_{79} = 0, x_{710} = 0, x_{89} = 83, x_{811} = 92, x_{812} = 0, x_{910} = 5, x_{911} = 128, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(x) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 40, x_{35} = 45, x_{36} = 65, x_{37} = 5, x_{46} = 105, x_{47} = 20, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 150, x_{69} = 80, x_{610} = 70, x_{79} = 0, x_{710} = 0, x_{89} = 110, x_{811} =$

$90, x_{812} = 0, x_{910} = 30, x_{911} = 140, x_{912} = 80, x_{913} = 140, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 10, x_{1213} = 215, x_{1214} = 135, x_{1314} = 255.$

(xi) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 73, x_{25} = 105, x_{26} = 76, x_{34} = 45, x_{35} = 45, x_{36} = 93, x_{37} = 0, x_{46} = 105, x_{47} = 25, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 160, x_{69} = 80, x_{610} = 60, x_{79} = 0, x_{710} = 0, x_{89} = 90, x_{811} = 120, x_{812} = 0, x_{910} = 40, x_{911} = 100, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(xii) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 80, x_{25} = 105, x_{26} = 70, x_{34} = 45, x_{35} = 45, x_{36} = 100, x_{37} = 0, x_{46} = 105, x_{47} = 25, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 160, x_{69} = 80, x_{610} = 60, x_{79} = 0, x_{710} = 0, x_{89} = 35, x_{811} = 120, x_{812} = 55, x_{910} = 40, x_{911} = 45, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 215, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(xiii) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 40, x_{35} = 45, x_{36} = 65, x_{37} = 5, x_{46} = 105, x_{47} = 20, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 160, x_{69} = 80, x_{610} = 60, x_{79} = 0, x_{710} = 0, x_{89} = 106, x_{811} = 104, x_{812} = 0, x_{910} = 40, x_{911} = 116, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(xiv) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 40, x_{35} = 45, x_{36} = 65, x_{37} = 5, x_{46} = 105, x_{47} = 20, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 125, x_{69} = 80, x_{610} = 95, x_{79} = 0, x_{710} = 0, x_{89} = 94, x_{811} = 81, x_{812} = 0, x_{910} = 5, x_{911} = 139, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(xv) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 40, x_{35} = 45, x_{36} = 65, x_{37} = 5, x_{46} = 105, x_{47} = 20, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 130, x_{69} = 80, x_{610} = 90, x_{79} = 0, x_{710} = 0, x_{89} = 100, x_{811} = 80, x_{812} = 0, x_{910} = 10, x_{911} = 140, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(xvi) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 45, x_{35} = 45, x_{36} = 65, x_{37} = 0, x_{46} = 105, x_{47} = 25, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 149, x_{69} = 80, x_{610} = 71, x_{79} = 0, x_{710} = 0, x_{89} = 79, x_{811} = 120, x_{812} = 0, x_{910} = 29, x_{911} = 100, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

(xvii) $x_{12} = 155, x_{13} = 160, x_{14} = 35, x_{23} = 45, x_{25} = 105, x_{26} = 105, x_{34} = 45, x_{35} = 45, x_{36} = 65, x_{37} = 0, x_{46} = 105, x_{47} = 25, x_{56} = 100, x_{58} = 0, x_{59} = 0, x_{67} = 175, x_{68} = 127, x_{69} = 80, x_{610} = 93, x_{79} = 0, x_{710} = 0, x_{89} = 57, x_{811} = 120, x_{812} = 0, x_{910} = 7, x_{911} = 100, x_{912} = 80, x_{913} = 150, x_{1012} = 0, x_{1013} = 0, x_{1112} = 270, x_{1114} = 0, x_{1213} = 215, x_{1214} = 135, x_{1314} = 265.$

Step 5: Using the crisp efficient solutions, the fuzzy optimal values of the FuLFpP $(P3.5.1)$ are

(i) $(1.59989, 2.19416, 4.22094, 5.86549)$

(ii) $(1.60066, 2.19335, 4.23949, 5.89031)$

(iii) $(1.62685, 2.21313, 3.78379, 5.85903)$

(iv) $(1.59927, 2.12159, 4.22464, 5.87849)$

(v) $(1.60305, 2.20242, 4.31577, 6)$

(vi) $(1.63354, 2.19633, 4.06151, 6.05462)$

(vii) $(1.59987, 2.19613, 4.28278, 5.94617)$

(viii) $(1.60713, 2.20259, 4.23675, 5.85609)$

(ix) $(1.59747, 2.19696, 4.24794, 5.90112)$

(x) $(1.60853, 2.14845, 4.23041, 5.86204)$

(xi) $(1.59974, 2.19574, 4.04527, 5.40142)$

(xii) $(1.60155, 2.23370, 4.45201, 6.23112)$

(xiii) $(1.60016, 2.19535, 4.38487, 5.20228)$

(xiv) $(1.59828, 2.19569, 4.37985, 5.20228)$

(xv) $(1.59886, 2.19515, 4.22741, 5.25513)$

(xvi) $(1.59918, 2.12588, 4.26662, 5.21362)$

(xvii) $(1.59690, 2.12647, 4.28766, 5.24240)$

3.6 Conclusions

In this chapter,

(i) It is shown that it is inappropriate to use the existing method [117] for solving FuLFMiCfPs.

(ii) An efficient method (named as Mehar method) is proposed for solving FuLFMiCfPs.

Chapter 4

Efficient Method for Solving Pythagorean Fuzzy Transportation Problems[1]

In this chapter, it is pointed out that some mathematical incorrect results are considered in all the existing methods [102, 127, 163] for solving PyFuTpS (transportation problems in which the transportation cost for supplying one unit quantity of the product from a source to a destination is represented by a PyFuN. While, all other parameters are represented by a non-negative real number). Therefore, it is inappropriate to use any of the existing methods [102, 127, 163] for solving PyFuTpS. Also, a new method (named as Mehar method) is proposed for solving PyFuTpS. Finally, the proposed Mehar method is illustrated with the help of a numerical example.

4.1 Preliminaries

In this section, some basic definitions are discussed.

Definition 4.1.1 [13] Let X be a universal set. Then, the set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)\rangle; x \in X\}$ is said to be an intuitionistic fuzzy number defined over the universal set X, where

(i) $\mu_{\tilde{A}}: X \to [0,1]$ and $v_{\tilde{A}}: X \to [0,1]$ are said to be the membership function and non-membership function respectively.

(ii) The values $\mu_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ are called the degree of membership and degree of non-membership for $x \in \tilde{A}$ respectively.

[1] The contents of this chapter are communicated in "International Journal of System Assurance Engineering and Management" for publication.

(iii) The values $\mu_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ satisfy the condition $0 \leq \mu_{\tilde{A}}(x) + v_{\tilde{A}}(x) \leq 1$.

An intuitionistic fuzzy number $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)\rangle; x \in X\}$ is also represented as $\tilde{A} = (\mu_{\tilde{A}}, v_{\tilde{A}})$.

Definition 4.1.2 [172] Let X be a universal set. Then, the set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)\rangle; x \in X\}$ is said to be a PyFuN defined over the universal set X, where

(i) $\mu_{\tilde{A}}: X \to [0,1]$ and $v_{\tilde{A}}: X \to [0,1]$ are said to be the membership function and non-membership function respectively.

(ii) The values $\mu_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ are called the degree of membership and degree of non-membership for $x \in \tilde{A}$ respectively.

(iii) The values $\mu_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ satisfy the condition $0 \leq \left(\mu_{\tilde{A}}(x)\right)^2 + \left(v_{\tilde{A}}(x)\right)^2 \leq 1$.

A PyFuN $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)\rangle; x \in X\}$ is also represented as $\tilde{A} = (\mu_{\tilde{A}}, v_{\tilde{A}})$.

4.2 Existing arithmetic operations of PyFuNs

In the existing methods [102, 127, 163], the following arithmetic operations of PyFuNs are used.

Let $\tilde{A}_k = \left(x_k; \mu_{\tilde{A}_k}, v_{\tilde{A}_k}\right); k = 1,2, \dots, m$, be PyFuNs where, $0 \leq \mu_{\tilde{A}_k}(x), v_{\tilde{A}_k}(x) \leq 1 \; \forall \; k, 0 \leq \left(\mu_{\tilde{A}_k}(x)\right)^2 + \left(v_{\tilde{A}_k}(x)\right)^2 \leq 1 \; \forall \; k$ and $p > 0$. Then,

(i) $$\tilde{A}_1 + \tilde{A}_2 + \dots + \tilde{A}_m = \left(\sum_{k=1}^{m} x_k \,; \sqrt{1 - \prod_{k=1}^{m}\left(1 - \left(\mu_{\tilde{A}_k}\right)^2\right)}, \prod_{k=1}^{m}\left(v_{\tilde{A}_k}\right)\right) \tag{4.2.1}$$

(ii) $$p\tilde{A}_k = \left(\sqrt{1 - \left(1 - \left(\mu_{\tilde{A}_k}\right)^2\right)^p}, \left(v_{\tilde{A}_k}\right)^p\right) \tag{4.2.2}$$

4.3 Existing methods for comparing PyFuNs

In this section, the methods for comparing PyFuNs, used in the existing methods [102, 127, 163], are discussed.

4.3.1 First existing comparing method

Kumar et al. [102] have used the following method for comparing two PyFuNs $\tilde{A}_1 = (\mu_{\tilde{A}_1}, v_{\tilde{A}_1})$ and $\tilde{A}_2 = (\mu_{\tilde{A}_2}, v_{\tilde{A}_2})$.

Step 1: Check that $S(\tilde{A}_1) > S(\tilde{A}_2)$ or $S(\tilde{A}_1) < S(\tilde{A}_2)$ or $S(\tilde{A}_1) = S(\tilde{A}_2)$, where

$$S(\tilde{A}_k) = \frac{1}{2}\left(1 - (\mu_{\tilde{A}_k})^2 - (v_{\tilde{A}_k})^2\right); k = 1,2 \tag{4.3.1.1}$$

Case (a) If $S(\tilde{A}_1) > S(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $S(\tilde{A}_1) < S(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $S(\tilde{A}_1) = S(\tilde{A}_2)$. Then, go to Step 2.

Step 2: Check that $H(\tilde{A}_1) > H(\tilde{A}_2)$ or $H(\tilde{A}_1) < H(\tilde{A}_2)$ or $H(\tilde{A}_1) = H(\tilde{A}_2)$, where

$$H(\tilde{A}_k) = (\mu_{\tilde{A}_k})^2 + (v_{\tilde{A}_k})^2; k = 1,2 \tag{4.3.1.2}$$

Case (a) If $H(\tilde{A}_1) > H(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $H(\tilde{A}_1) < H(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $H(\tilde{A}_1) = H(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

4.3.2 Second existing comparing method

Umamageswari and Uthra [163] have used the following method for comparing two PyFuNs $\tilde{A}_1 = (\mu_{\tilde{A}_1}, v_{\tilde{A}_1})$ and $\tilde{A}_2 = (\mu_{\tilde{A}_2}, v_{\tilde{A}_2})$.

Check that $S(\tilde{A}_1) > S(\tilde{A}_2)$ or $S(\tilde{A}_1) < S(\tilde{A}_2)$ or $S(\tilde{A}_1) = S(\tilde{A}_2)$, where

$$S(\tilde{A}_k) = \frac{1}{2}\left((\mu_{\tilde{A}_k})^2 + (v_{\tilde{A}_k})^2\right); k = 1,2 \tag{4.3.2.1}$$

Case (a) If $S(\tilde{A}_1) > S(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $S(\tilde{A}_1) < S(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $S(\tilde{A}_1) = S(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

4.3.3 Third existing comparing method

Nagar et al. [127] have used the following method for comparing two PyFuNs $\tilde{A}_1 = \left(\mu_{\tilde{A}_1}, v_{\tilde{A}_1}\right)$ and $\tilde{A}_2 = \left(\mu_{\tilde{A}_2}, v_{\tilde{A}_2}\right)$.

Step 1: Check that $S(\tilde{A}_1) > S(\tilde{A}_2)$ or $S(\tilde{A}_1) < S(\tilde{A}_2)$ or $S(\tilde{A}_1) = S(\tilde{A}_2)$, where

$$S(\tilde{A}_k) = \frac{2}{3}\left(\frac{2^{\left(\mu_{\tilde{A}_k}\right)^2-\left(v_{\tilde{A}_k}\right)^2}}{2-\left(\mu_{\tilde{A}_k}\right)^2-\left(v_{\tilde{A}_k}\right)^2} - \frac{1}{2}\right); k = 1,2 \tag{4.3.3.1}$$

Case (a) If $S(\tilde{A}_1) > S(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $S(\tilde{A}_1) < S(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $S(\tilde{A}_1) = S(\tilde{A}_2)$. Then, go to Step 2.

Step 2: Check that $\pi(\tilde{A}_1) > \pi(\tilde{A}_2)$ or $\pi(\tilde{A}_1) < \pi(\tilde{A}_2)$ or $\pi(\tilde{A}_1) = \pi(\tilde{A}_2)$, where

$$\pi(\tilde{A}_k) = \sqrt{1 - \left(\mu_{\tilde{A}_k}\right)^2 - \left(v_{\tilde{A}_k}\right)^2}; k = 1,2 \tag{4.3.3.2}$$

Case (a) If $\pi(\tilde{A}_1) > \pi(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $\pi(\tilde{A}_1) < \pi(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $\pi(\tilde{A}_1) = \pi(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

4.4 Existing methods for solving PyFuTpS

In this section, the existing methods [102, 127, 163] for solving the PyFuTp (represented by Table 4.1) are discussed.

Table 4.1 PyFuTp

	D_1	D_2	...	D_n	Availability
S_1	$(c;\mu_{11},v_{11})$	$(c;\mu_{12},v_{12})$	...	$(c;\mu_{1n},v_{1n})$	a_1
S_2	$(c;\mu_{21},v_{21})$	$(c;\mu_{22},v_{22})$	...	$(c;\mu_{2n},v_{2n})$	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_m	$(c;\mu_{m1},v_{m1})$	$(c;\mu_{m2},v_{m2})$	...	$(c;\mu_{mn},v_{mn})$	a_m
Demand	b_1	b_2	...	b_n	

where,

(i) m represents the number of sources.

(ii) n represents the number of destinations.

(iii) $(c;\mu_{ij},v_{ij})$ is a PyFuN.

(iv) c represents the cost for transporting one unit quantity of the product from the i^{th} source to the j^{th} destination.

(v) μ_{ij} represents the degree of membership associated with c.

(vi) v_{ij} represents the degree of non-membership associated with c.

(vii) a_i represents the availability of the product at the i^{th} source S_i.

(viii) b_j represents the demand of the product at the j^{th} destination D_j.

(ix) $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = A$ (say).

The steps of the existing methods [102, 127, 163] are as follows.

Step 1: Transform the PyFuTp (represented by Table 4.1) into the CrTp (represented by Table 4.2).

Table 4.2 CrTp

	D_1	D_2	...	D_n	Availability
S_1	$S(\mu_{11}, v_{11})$	$S(\mu_{12}, v_{12})$	...	$S(\mu_{1n}, v_{1n})$	a_1
S_2	$S(\mu_{21}, v_{21})$	$S(\mu_{22}, v_{22})$	...	$S(\mu_{2n}, v_{2n})$	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_m	$S(\mu_{m1}, v_{m1})$	$S(\mu_{m2}, v_{m2})$	...	$S(\mu_{mn}, v_{mn})$	a_m
Demand	b_1	b_2	...	b_n	

where,

(i) According to Kumar et al.'s method [102],

$$S(\mu, v) = \frac{1}{2}(1 - \mu^2 - v^2)$$

(ii) According to Umamageswari and Uthra's method [163],

$$S(\mu, v) = \frac{\mu^2 + v^2}{2}$$

(iii) According to Nagar et al.'s method [127],

$$S(\mu, v) = \frac{2}{3}\left(\frac{2^{\mu^2 - v^2}}{2 - \mu^2 - v^2} - \frac{1}{2}\right)$$

Step 2: Using any appropriate method, find an optimal solution $\{x_{ij}\}$ of the transformed CrTp (represented by Table 4.2). The obtained optimal solution also represents an optimal solution of PyFuTp (represented by Table 4.1).

Step 3: Using the optimal solution $\{x_{ij}\}$, obtained in Step 2, find the optimal transportation cost $\sum_{i=1}^{m} \sum_{j=1}^{n} \left(S(\mu_{ij}, v_{ij})\right) x_{ij}$.

4.5 Existing transformation method

In this section, the method, used in all the existing methods [102, 127, 163] to transform the PyFuTp (represented by Table 4.1) into the CrTp (represented by Table 4.2), is discussed.

The following method is used to transform the PyFuTp (represented by Table 4.1) into the CrTp (represented by Table 4.2).

Step 1: Write the FuLpP ($P4.5.1$) corresponding to the PyFuTp (represented by Table 4.1).

FuLpP ($\boldsymbol{P4.5.1}$)

$$Minimize\ \left(\tilde{Z} \approx \left(\sum_{i=1}^{m}\sum_{j=1}^{n} c\, x_{ij}\, ; \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\mu_{ij}, v_{ij}\right)x_{ij}\right)\right)\right)$$

Subject to

$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1,2,\dots,m,$

$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1,2,\dots,n,$

$x_{ij} \geq 0\ \forall\ i,j.$

Step 2: Since, c is constant. So, to find an optimal solution of the FuLpP ($P4.5.1$) is equivalent to find an optimal solution of the FuLpP ($P4.5.2$).

FuLpP ($\boldsymbol{P4.5.2}$)

$$Minimize\ \left(\tilde{Z} \approx \left(c\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\, ; \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\mu_{ij}, v_{ij}\right)x_{ij}\right)\right)\right)$$

Subject to

Constraints of the FuLpP ($P4.5.1$).

Step 3: Since, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = A$, $\sum_{j=1}^{n} x_{ij} = a_i; i = 1,2,\dots,m$, $\sum_{i=1}^{m} x_{ij} = b_j; j = 1,2,\dots,n \Rightarrow \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = A$. So, to find an optimal solution of the FuLpP ($P4.5.2$) is equivalent to find an optimal solution of the FuLpP ($P4.5.3$).

FuLpP ($\boldsymbol{P4.5.3}$)

$$Minimize\ \left(\tilde{Z} \approx \left(cA;\ \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\mu_{ij}, v_{ij}\right)x_{ij}\right)\right)\right)$$

Subject to

Constraints of the FuLpP ($P4.5.1$).

Step 4: Since cA is independent from x_{ij}. So, to find an optimal solution, there is a need to compare only $\sum_{i=1}^{m}\sum_{j=1}^{n}\left((\mu_{ij}, \nu_{ij})x_{ij}\right)$ corresponding to all the feasible solutions. Hence, using the relation $\tilde{A}_1 \succcurlyeq \tilde{A}_2$ if $S(\tilde{A}_1) \geq S(\tilde{A}_2)$, the FuLpP $(P4.5.3)$ can be transformed into its equivalent CrLpP $(P4.5.4)$.

CrLpP $(\boldsymbol{P4.5.4})$

$$Minimize\ \left(S(\tilde{Z}) = S\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left((\mu_{ij}, \nu_{ij})x_{ij}\right)\right)\right)$$

Subject to

Constraints of the FuLpP $(P4.5.1)$

where,

(i) According to Kumar et al.'s method [102],

$$S(\mu, \nu) = \frac{1}{2}(1 - \mu^2 - \nu^2)$$

(ii) According to Umamageswari and Uthra's method [163],

$$S(\mu, \nu) = \frac{\mu^2 + \nu^2}{2}$$

(iii) According to Nagar et al.'s method [127],

$$S(\mu, \nu) = \frac{2}{3}\left(\frac{2^{\mu^2 - \nu^2}}{2 - \mu^2 - \nu^2} - \frac{1}{2}\right)$$

Step 5: Using the relation $S(\tilde{A}_1 + \tilde{A}_2) = S(\tilde{A}_1) + S(\tilde{A}_2)$, the CrLpP $(P4.5.4)$ can be transformed into its equivalent CrLpP $(P4.5.5)$.

CrLpP $(\boldsymbol{P4.5.5})$

$$Minimize\ \left(S(\tilde{Z}) = \sum_{i=1}^{m}\sum_{j=1}^{n} S\left((\mu_{ij}, \nu_{ij})x_{ij}\right)\right)$$

Subject to

Constraints of the FuLpP $(P4.5.1)$.

Step 6: Using the relation $S(\tilde{A}p) = \left(S(\tilde{A})\right)p$, where $p \geq 0$, the CrLpP $(P4.5.5)$ can be transformed into its equivalent CrLpP $(P4.5.6)$.

CrLpP $(\boldsymbol{P4.5.6})$

$Minimize\ \left(S(\tilde{Z}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(S(\mu_{ij}, v_{ij})\right)x_{ij}\right)$

Subject to

Constraints of the FuLpP $(P4.5.1)$.

Step 7: The CrLpP $(P4.5.6)$ can be represented in a tabular form as shown by Table 4.2.

4.6 Limitations of existing methods

In this section, limitations of the existing methods [102, 127, 163] are discussed.

(i) It is a well-known fact that in every optimal solution $\{x_{ij}\}$, obtained in Step 2, at least $(m-1)(n-1)$ decision variables will be zero. Therefore, to find the optimal Pythagorean fuzzy transportation cost i.e., $\left(cA; \sum_{i=1}^{m}\sum_{j=1}^{n}\left((\mu_{ij}, v_{ij})x_{ij}\right)\right)$, there is a need to evaluate the value of $0(\mu_{ij}, v_{ij})$. However, as the existing expression (4.2.2) [102, 127, 163] i.e., $p\tilde{A} = \left(\sqrt{1-(1-(\mu_{\tilde{A}})^2)^p}, (v_{\tilde{A}})^p\right)$ is defined only for $p > 0$. Therefore, it is not possible to find the optimal transportation cost in terms of PyFuNs.

(ii) The existing methods [102, 127, 163] cannot be used to solve such PyFuTpS for which the condition $c_{ij} = c\ \forall\ i, j$ will not be satisfied. Therefore, the existing methods [102, 127, 163] cannot be used to solve PyFuTp (represented by Table 4.3).

Table 4.3 PyFuTp

	D_1	D_2	...	D_n	Availability
S_1	$(c_{11};\mu_{11},v_{11})$	$(c_{12};\mu_{12},v_{12})$	...	$(c_{1n};\mu_{1n},v_{1n})$	a_1
S_2	$(c_{21};\mu_{21},v_{21})$	$(c_{22};\mu_{22},v_{22})$	...	$(c_{2n};\mu_{2n},v_{2n})$	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_m	$(c_{m1};\mu_{m1},v_{m1})$	$(c_{m2};\mu_{m2},v_{m2})$	...	$(c_{mn};\mu_{mn},v_{mn})$	a_m
Demand	b_1	b_2	...	b_n	

where,

(i) $(c_{ij};\mu_{ij},v_{ij})$ is a PyFuN.

(ii) c_{ij} represents the cost for transporting one unit quantity of the product from the i^{th} source to the j^{th} destination.

(iii) μ_{ij} represents the degree of membership associated with c_{ij}.

(iv) v_{ij} represents the degree of non-membership associated with c_{ij}.

(v) $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = A$ (say).

4.7 Inappropriateness of existing methods

In this section, it is pointed out that some mathematical incorrect results are considered in the existing methods [102, 127, 163]. Hence, it is inappropriate to use any of the existing methods [102, 127, 163].

(i) It is obvious from Step 5 of the transformation method, discussed in Section 4.5, that to transform the CrLpP $(P4.5.4)$ into the CrLpP $(P4.5.5)$, it is assumed that the relation $S(\tilde{A}_1+\tilde{A}_2)=S(\tilde{A}_1)+S(\tilde{A}_2)$ will be satisfied for two arbitrary PyFuNs $\tilde{A}_1$ and $\tilde{A}_2$.

While, the following example clearly indicates that $S(\tilde{A}_1+\tilde{A}_2)\neq S(\tilde{A}_1)+S(\tilde{A}_2)$. Therefore, the CrLpP $(P4.5.4)$ cannot be transformed into the CrLpP $(P4.5.5)$. Hence, the

PyFuTp (represented by Table 4.1) is not equivalent to the CrTp (represented by Table 4.2).

Let $\tilde{A}_1 = (0.4, 0.8)$ and $\tilde{A}_2 = (0.6, 0.7)$ be two PyFuNs. Then, using the existing expression (4.2.1) [102, 127, 163],

$$\tilde{A}_1 + \tilde{A}_2 = \left(\sqrt{1-(1-(0.4)^2)(1-(0.6)^2)}, (0.8)(0.7)\right) = (0.68, 0.56).$$

Using

(a) The existing expression (4.3.1.1) [102],

$$S(\tilde{A}_1 + \tilde{A}_2) = S(0.68, 0.56) = \frac{1}{2}(1 - 0.68^2 - 0.56^2) = 0.112 \quad (4.7.1)$$

$$S(\tilde{A}_1) = S(0.4, 0.8) = \frac{1}{2}(1 - 0.4^2 - 0.8^2) = 0.1$$

$$S(\tilde{A}_2) = S(0.6, 0.7) = \frac{1}{2}(1 - 0.6^2 - 0.7^2) = 0.075$$

$$S(\tilde{A}_1) + S(\tilde{A}_2) = 0.1 + 0.075 = 0.175 \quad (4.7.2)$$

It is obvious from (4.7.1) and (4.7.2) that $S(\tilde{A}_1 + \tilde{A}_2) \neq S(\tilde{A}_1) + S(\tilde{A}_2)$.

(b) The existing expression (4.3.2.1) [163],

$$S(\tilde{A}_1 + \tilde{A}_2) = S(0.68, 0.56) = \frac{0.68^2 + 0.56^2}{2} = 0.388 \quad (4.7.3)$$

$$S(\tilde{A}_1) = S(0.4, 0.8) = \frac{0.4^2 + 0.8^2}{2} = 0.4$$

$$S(\tilde{A}_2) = S(0.6, 0.7) = \frac{0.6^2 + 0.7^2}{2} = 0.425$$

$$S(\tilde{A}_1) + S(\tilde{A}_2) = 0.4 + 0.425 = 0.825 \quad (4.7.4)$$

It is obvious from (4.7.3) and (4.7.4) that $S(\tilde{A}_1 + \tilde{A}_2) \neq S(\tilde{A}_1) + S(\tilde{A}_2)$.

(c) The existing expression (4.3.3.1) [127],

$$S(\tilde{A}_1 + \tilde{A}_2) = S(0.68, 0.56) = \frac{2}{3}\left(\frac{2^{0.68^2 - 0.56^2}}{2 - 0.68^2 - 0.56^2} - \frac{1}{2}\right) = 0.2705 \quad (4.7.5)$$

$$S(\tilde{A}_1) = S(0.4,0.8) = \frac{2}{3}\left(\frac{2^{0.4^2-0.8^2}}{2-0.4^2-0.8^2} - \frac{1}{2}\right) = 0.0650$$

$$S(\tilde{A}_2) = S(0.6,0.7) = \frac{2}{3}\left(\frac{2^{0.6^2-0.7^2}}{2-0.6^2-0.7^2} - \frac{1}{2}\right) = 0.1964$$

$$S(\tilde{A}_1) + S(\tilde{A}_2) = 0.0650 + 0.1964 = 0.2614 \quad (4.7.6)$$

It is obvious from (4.7.5) and (4.7.6) that $S(\tilde{A}_1 + \tilde{A}_2) \neq S(\tilde{A}_1) + S(\tilde{A}_2)$.

(ii) It is obvious from Step 6 of the transformation method, discussed in Section 4.5, that to transform the CrLpP ($P4.5.5$) into the CrLpP ($P4.5.6$), it is assumed that the relation $S(\tilde{A}p) = \left(S(\tilde{A})\right)p$ will be satisfied for any arbitrary PyFuN $\tilde{A}$ and $p \geq 0$.

While, the following example clearly indicates that $S(\tilde{A}p) \neq \left(S(\tilde{A})\right)p$. Therefore, the CrLpP ($P4.5.5$) cannot be transformed into the CrLpP ($P4.5.6$). Hence, the PyFuTp (represented by Table 4.1) is not equivalent to the CrTp (represented by Table 4.2).

Let $\tilde{A} = (0.6,0.5)$ be a PyFuN and $p = 4$ be a positive real number. Then, using the existing expression (4.2.2) [102, 127, 163],

$$\tilde{A}p = (0.6,0.5)4 = \left(\sqrt{1-(1-0.6^2)^4}, 0.5^4\right) = (0.9123,0.0625).$$

Using

(a) The existing expression (4.3.1.1) [102],

$$S(\tilde{A}p) = S(0.9123,0.0625) = \frac{1}{2}(1 - 0.9123^2 - 0.0625^2) = 0.4142 \quad (4.7.7)$$

$$S(\tilde{A}) = S(0.6,0.5) = \frac{1}{2}(1 - 0.6^2 - 0.5^2) = 0.195$$

$$\left(S(\tilde{A})\right)p = \left(S(0.6,0.5)\right)4 = (0.195)4 = 0.78 \quad (4.7.8)$$

It is obvious from (4.7.7) and (4.7.8) that $S(\tilde{A}p) \neq \left(S(\tilde{A})\right)p$.

(b) The existing expression (4.3.2.1) [163],

$$S(\tilde{A}p) = S(0.9123{,}0.0625) = \frac{0.9123^2+0.0625^2}{2} = 0.6115 \quad (4.7.9)$$

$$S(\tilde{A}) = S(0.6{,}0.5) = \frac{0.6^2+0.5^2}{2} = 0.305$$

$$\left(S(\tilde{A})\right)p = \left(S(0.6{,}0.5)\right)4 = (0.305)4 = 1.22 \quad (4.7.10)$$

It is obvious from (4.7.9) and (4.7.10) that $S(\tilde{A}p) \neq \left(S(\tilde{A})\right)p$.

(c) The existing expression (4.3.3.1) [127],

$$S(\tilde{A}p) = S(0.9123{,}0.0625) = \frac{2}{3}\left(\frac{2^{0.9123^2-0.0625^2}}{2-0.9123^2-0.0625^2} - \frac{1}{2}\right) = 0.6838 \quad (4.7.11)$$

$$S(\tilde{A}) = S(0.6{,}0.5) = \frac{2}{3}\left(\frac{2^{0.6^2-0.5^2}}{2-0.6^2-0.5^2} - \frac{1}{2}\right) = 0.1843$$

$$S(\tilde{A})p = S(0.6{,}0.5)4 = (0.1843)4 = 0.7372 \quad (4.7.12)$$

It is obvious from (4.7.11) and (4.7.12) that $S(\tilde{A}p) \neq \left(S(\tilde{A})\right)p$.

4.8 Inappropriateness of existing expression to evaluate sum of PyFuNs

In the existing methods [102, 127, 163], the existing expression (4.2.1) is used to evaluate sum of PyFuNs. However, in the literature [121], it is pointed out that it is inappropriate to use this expression due to the following reasons.

(i) It is obvious from expression (4.2.1) that if $\mu_{\tilde{A}_k} = 1$ for any k, then the degree of membership i.e., $\sqrt{1 - \prod_{k=1}^{m}\left(1 - \left(\mu_{\tilde{A}_k}\right)^2\right)}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 1. This indicates that the degree of membership i.e., $\sqrt{1 - \prod_{k=1}^{m}\left(1 - \left(\mu_{\tilde{A}_k}\right)^2\right)}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ is independent from the degree of membership of remaining PyFuNs.

(ii) It is obvious from expression (4.2.1) that if $v_{\tilde{A}_k} = 0$ for any k, then the degree of non-membership i.e., $\prod_{k=1}^{m}\left(v_{\tilde{A}_k}\right)$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 0. This indicates that the degree of non-membership i.e., $\prod_{k=1}^{m}\left(v_{\tilde{A}_k}\right)$ of $\tilde{A}_1+\tilde{A}_2+\cdots+\tilde{A}_m$ is independent from the degree of non-membership of remaining PyFuNs.

4.9 Limitation of existing expression to evaluate scalar multiplication

It is obvious from the existing expression (4.2.2) [102, 127, 163] that it can be used only to evaluate the multiplication of any arbitrary PyFuN with a positive real number. However, the existing expression (4.2.2) [102, 127, 163] cannot be used to evaluate the multiplication of any arbitrary PyFuN with a non-negative real number.

4.10 Inappropriateness of existing methods for comparing two PyFuNs

In this section, it is shown that the methods for comparing two PyFuNs, used in the existing methods [102, 163], are not appropriate.

(i) The following clearly indicates that Kumar et al.'s comparing method [102], discussed in Section 4.3.1, fails to distinguish two distinct PyFuNs. Hence, it is inappropriate to use Kumar et al.'s comparing method [102].

Let $\tilde{A}_1=(0.8,0.4)$ and $\tilde{A}_2=(0.4,0.8)$ be two PyFuNs. It is obvious that $\tilde{A}_1\neq\tilde{A}_2$. While, according to Kumar et al.'s comparing method [102], discussed in Section 4.3.1, $\tilde{A}_1=\tilde{A}_2$ as $S\left(\tilde{A}_1\right)=S\left(\tilde{A}_2\right)=\frac{1}{2}(1-(0.8)^2-(0.4)^2)=0.1$ and $H\left(\tilde{A}_1\right)=H\left(\tilde{A}_2\right)=(0.8)^2+(0.4)^2=0.8$.

(ii) The following clearly indicates that Umamageswari and Uthra's comparing method [163], discussed in Section 4.3.2, fails to distinguish two distinct PyFuNs. Hence, it is inappropriate to use Umamageswari and Uthra's comparing method [163].

Let $\tilde{A}_1=(0.8,0.4)$ and $\tilde{A}_2=(0.4,0.8)$ be two PyFuNs. It is obvious that $\tilde{A}_1\neq\tilde{A}_2$. While, according to Umamageswari and Uthra's comparing method [163], discussed in Section 4.3.2, $\tilde{A}_1=\tilde{A}_2$ as $S\left(\tilde{A}_1\right)=S\left(\tilde{A}_2\right)=\frac{1}{2}((0.8)^2+(0.4)^2)=0.4$.

4.11 Appropriate method for comparing two PyFuNs

The following clearly indicates that Nagar et al.'s comparing method [127], discussed in Section 4.3.3, will never fail to distinguish two distinct PyFuNs i.e., $S(\tilde{A}_1) = S(\tilde{A}_2)$ and $\pi(\tilde{A}_1) = \pi(\tilde{A}_2) \Rightarrow \tilde{A}_1 = \tilde{A}_2$ i.e., $\mu_{\tilde{A}_1} = \mu_{\tilde{A}_2}$ and $v_{\tilde{A}_1} = v_{\tilde{A}_2}$. Hence, it is appropriate to use Nagar et al.'s comparing method [127].

$$S(\tilde{A}_1) = S(\tilde{A}_2) \Rightarrow \frac{2}{3}\left(\frac{2^{(\mu_{\tilde{A}_1})^2-(v_{\tilde{A}_1})^2}}{2-(\mu_{\tilde{A}_1})^2-(v_{\tilde{A}_1})^2} - \frac{1}{2}\right) = \frac{2}{3}\left(\frac{2^{(\mu_{\tilde{A}_2})^2-(v_{\tilde{A}_2})^2}}{2-(\mu_{\tilde{A}_2})^2-(v_{\tilde{A}_2})^2} - \frac{1}{2}\right) \tag{4.11.1}$$

$$\pi(\tilde{A}_1) = \pi(\tilde{A}_2) \Rightarrow \sqrt{1-(\mu_{\tilde{A}_1})^2-(v_{\tilde{A}_1})^2} = \sqrt{1-(\mu_{\tilde{A}_2})^2-(v_{\tilde{A}_2})^2} \tag{4.11.2}$$

On simplifying the expression (4.11.1), we have

$$\frac{2^{(\mu_{\tilde{A}_1})^2-(v_{\tilde{A}_1})^2}}{2-(\mu_{\tilde{A}_1})^2-(v_{\tilde{A}_1})^2} = \frac{2^{(\mu_{\tilde{A}_2})^2-(v_{\tilde{A}_2})^2}}{2-(\mu_{\tilde{A}_2})^2-(v_{\tilde{A}_2})^2} \tag{4.11.3}$$

On simplifying the expression (4.11.2), we have

$$(\mu_{\tilde{A}_1})^2 = (\mu_{\tilde{A}_2})^2 + (v_{\tilde{A}_2})^2 - (v_{\tilde{A}_1})^2 \tag{4.11.4}$$

Putting (4.11.4) in (4.11.3), we have

$$\frac{2^{(\mu_{\tilde{A}_2})^2+(v_{\tilde{A}_2})^2-(v_{\tilde{A}_1})^2-(v_{\tilde{A}_1})^2}}{2-(\mu_{\tilde{A}_2})^2-(v_{\tilde{A}_2})^2+(v_{\tilde{A}_1})^2-(v_{\tilde{A}_1})^2} = \frac{2^{(\mu_{\tilde{A}_2})^2-(v_{\tilde{A}_2})^2}}{2-(\mu_{\tilde{A}_2})^2-(v_{\tilde{A}_2})^2}$$

$$\Rightarrow 2^{(\mu_{\tilde{A}_2})^2+(v_{\tilde{A}_2})^2-2(v_{\tilde{A}_1})^2} = 2^{(\mu_{\tilde{A}_2})^2-(v_{\tilde{A}_2})^2}$$

$$\Rightarrow (\mu_{\tilde{A}_2})^2 + (v_{\tilde{A}_2})^2 - 2(v_{\tilde{A}_1})^2 = (\mu_{\tilde{A}_2})^2 - (v_{\tilde{A}_2})^2$$

$$\Rightarrow (v_{\tilde{A}_1})^2 = (v_{\tilde{A}_2})^2$$

$$\Rightarrow \pm v_{\tilde{A}_1} = \pm v_{\tilde{A}_2}.$$

Since, $0 \le v_{\tilde{A}_1}, v_{\tilde{A}_2} \le 1$. So,

$$v_{\tilde{A}_1} = v_{\tilde{A}_2} \tag{4.11.5}$$

Putting (4.11.5) in (4.11.4), we have

$$\left(\mu_{\tilde{A}_1}\right)^2 = \left(\mu_{\tilde{A}_2}\right)^2 + \left(v_{\tilde{A}_1}\right)^2 - \left(v_{\tilde{A}_1}\right)^2$$

$$\Rightarrow \left(\mu_{\tilde{A}_1}\right)^2 = \left(\mu_{\tilde{A}_2}\right)^2$$

$$\Rightarrow \pm\mu_{\tilde{A}_1} = \pm\mu_{\tilde{A}_2}.$$

Since, $0 \leq \mu_{\tilde{A}_1}, \mu_{\tilde{A}_2} \leq 1$. So,

$$\mu_{\tilde{A}_1} = \mu_{\tilde{A}_2} \tag{4.11.6}$$

Remark 4.1: It is pertinent to mention that the existing method [134] will also never fail to distinguish two distinct PyFuNs. However, Nagar et al. [127] have pointed out that their method is better than the existing method [134] as the degree of hesitation is not considered in the existing method [134]. While, the degree of hesitation is considered in their method [127].

4.12 Proposed expression to evaluate sum of PyFuNs

In this section, an expression is proposed to evaluate sum of PyFuNs.

Let $\tilde{A}_k = \left(x_k; \mu_{\tilde{A}_k}, v_{\tilde{A}_k}\right); k = 1,2, \dots, m,$ be PyFuNs where, $0 \leq \mu_{\tilde{A}_k}(x), v_{\tilde{A}_k}(x) \leq 1\ \forall\ k, 0 \leq \left(\mu_{\tilde{A}_k}(x)\right)^2 + \left(v_{\tilde{A}_k}(x)\right)^2 \leq 1\ \forall\ k$. Also, let p_k be a non-negative real number. Then,

$$p_1\tilde{A}_1 + p_2\tilde{A}_2 + \dots + p_m\tilde{A}_m =$$

$$\left(\sum_{k=1}^m p_k x_k ; \left(\frac{\sum_{k=1}^m p_k\left(\mu_{\tilde{A}_k}\right)^2}{\sum_{k=1}^m p_k}\right)^{\frac{1}{2}}, \left(\frac{\sum_{k=1}^m p_k\left(v_{\tilde{A}_k}\right)^2}{\sum_{k=1}^m p_k}\right)^{\frac{1}{2}}\right); \qquad if \sum_{k=1}^m p_k \neq 0 \tag{4.12.1}$$

The following clearly indicates that $p_1\tilde{A}_1 + p_2\tilde{A}_2 + \dots + p_m\tilde{A}_m$ is a PyFuN.

(i) $0 \leq \mu_{\tilde{A}_k} \leq 1,\ p_k \geq 0$

$$\Longrightarrow 0 \leq \left(\mu_{\tilde{A}_k}\right)^2 \leq 1,\ p_k \geq 0$$

$$\Longrightarrow 0 \leq p_k\left(\mu_{\tilde{A}_k}\right)^2 \leq p_k$$

$$\Longrightarrow 0 \le \sum_{k=1}^{m} p_k\left(\mu_{\tilde{A}_k}\right)^2 \le \sum_{k=1}^{m} p_k$$

$$\Longrightarrow 0 \le \frac{\sum_{k=1}^{m} p_k\left(\mu_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k} \le 1$$

$$\Rightarrow 0 \le \left(\frac{\sum_{k=1}^{m} p_k\left(\mu_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{2}} \le 1.$$

(ii) $0 \le v_{\tilde{A}_k} \le 1,\ p_k \ge 0$

$$\Longrightarrow 0 \le \left(v_{\tilde{A}_k}\right)^2 \le 1,\ p_k \ge 0$$

$$\Longrightarrow 0 \le p_k\left(v_{\tilde{A}_k}\right)^2 \le p_k$$

$$\Longrightarrow 0 \le \sum_{k=1}^{m} p_k\left(v_{\tilde{A}_k}\right)^2 \le \sum_{k=1}^{m} p_k$$

$$\Longrightarrow 0 \le \frac{\sum_{k=1}^{m} p_k\left(v_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k} \le 1$$

$$\Rightarrow 0 \le \left(\frac{\sum_{k=1}^{m} p_k\left(v_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{2}} \le 1.$$

(iii) $p_k \ge 0,\ \mu_{\tilde{A}_k} \ge 0,\ v_{\tilde{A}_k} \ge 0,\ \sum_{k=1}^{m} p_k \ne 0$

$$\Rightarrow \left(\left(\frac{\sum_{k=1}^{m} p_k\left(\mu_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{2}}\right)^2 + \left(\left(\frac{\sum_{k=1}^{m} p_k\left(v_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{2}}\right)^2 \ge 0.$$

(iv) $$\left(\left(\frac{\sum_{k=1}^{m} p_k\left(\mu_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{2}}\right)^2 + \left(\left(\frac{\sum_{k=1}^{m} p_k\left(v_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{2}}\right)^2 = \frac{\sum_{k=1}^{m} p_k\left(\mu_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k} + \frac{\sum_{k=1}^{m} p_k\left(v_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k} = \frac{\sum_{k=1}^{m} p_k\left(\left(\mu_{\tilde{A}_k}\right)^2 + \left(v_{\tilde{A}_k}\right)^2\right)}{\sum_{k=1}^{m} p_k}.$$

Since, $\left(\mu_{\tilde{A}_k}\right)^2 + \left(v_{\tilde{A}_k}\right)^2 \le 1$. So,

$$\left(\left(\frac{\sum_{k=1}^{m} p_k\left(\mu_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{2}}\right)^2 + \left(\left(\frac{\sum_{k=1}^{m} p_k\left(v_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{2}}\right)^2 \leq \frac{\sum_{k=1}^{m} p_k}{\sum_{k=1}^{m} p_k}.$$

Since, $\frac{\sum_{k=1}^{m} p_k}{\sum_{k=1}^{m} p_k} = 1$. So,

$$\left(\left(\frac{\sum_{k=1}^{m} p_k\left(\mu_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{2}}\right)^2 + \left(\left(\frac{\sum_{k=1}^{m} p_k\left(v_{\tilde{A}_k}\right)^2}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{2}}\right)^2 \leq 1.$$

4.13 Proposed Mehar method for solving PyFuTpS

In this section, a new method (named as Mehar method) is proposed for solving PyFuTp (represented by Table 4.3).

The steps of the proposed Mehar method are as follows.

Step 1: Write the FuLpP $(P4.13.1)$ corresponding to the PyFuTp (represented by Table 4.3).

FuLpP $(\boldsymbol{P4.13.1})$

$$Minimize\left(\tilde{Z} \approx \left(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}\,;\, \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(\mu_{ij}, v_{ij}\right)x_{ij}\right)\right)\right)$$

Subject to

Constraints of the FuLpP $(P4.5.1)$.

Step 2: Using the proposed expression (4.12.1), the FuLpP $(P4.13.1)$ can be transformed into its equivalent FuMpP $(P4.13.2)$.

FuMpP $(\boldsymbol{P4.13.2})$

$$Minimize\left(\tilde{Z} \approx \left(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}\,;\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\mu_{ij}\right)^2 x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}}\right)^{\frac{1}{2}}, \left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(v_{ij}\right)^2 x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}}\right)^{\frac{1}{2}}\right)\right)$$

Subject to

Constraints of the FuLpP $(P4.5.1)$.

Step 3: Find the optimal value of the CrMpP $(P4.13.3)$.

CrMpP $(\boldsymbol{P4.13.3})$

$Minimize\left(\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\right)$

Subject to

Constraints of the FuLpP $(P4.5.1)$.

Step 4: Find the optimal value of the CrMpP $(P4.13.4)$.

CrMpP $(\boldsymbol{P4.13.4})$

$$Minimize\left(S\left(\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\,;\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(\mu_{ij})^{2}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)^{\frac{1}{2}},\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(v_{ij})^{2}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)^{\frac{1}{2}}\right)\right)$$

Subject to

Constraints of the FuLpP $(P4.5.1)$ with the additional constraint $\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}=\alpha$,

where,

(i) α is the optimal value of the CrMpP $(P4.13.3)$.

(ii) $S\left(\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\,;\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(\mu_{ij})^{2}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)^{\frac{1}{2}},\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(v_{ij})^{2}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)^{\frac{1}{2}}\right)=$

$$\frac{2}{3}\left(\frac{2^{\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(\mu_{ij})^{2}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)-\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(v_{ij})^{2}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)}}{2-\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(\mu_{ij})^{2}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)-\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(v_{ij})^{2}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)}-\frac{1}{2}\right).$$

Step 5: Find an optimal solution $\{x_{ij}\}$ of the CrMpP $(P4.13.5)$.

CrMpP $(\boldsymbol{P4.13.5})$

$$Minimize\left(\pi\left(\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\,;\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(\mu_{ij})^{2}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)^{\frac{1}{2}},\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(v_{ij})^{2}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)^{\frac{1}{2}}\right)\right)$$

Subject to

Constraints of the CrMpP $(P4.13.4)$ with the additional constraint

$$\frac{2}{3}\left(\frac{2^{\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(\mu_{ij})^2 x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)-\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(v_{ij})^2 x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)}}{2-\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(\mu_{ij})^2 x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)-\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(v_{ij})^2 x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)}-\frac{1}{2}\right)=\beta, \text{ where,}$$

(i) β is the optimal value of the CrMpP $(P4.13.4)$.

(ii) $$\pi\left(\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\,;\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(\mu_{ij})^2 x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)^{\frac{1}{2}},\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(v_{ij})^2 x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)^{\frac{1}{2}}\right)=$$

$$\sqrt{1-\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(\mu_{ij})^2 x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)-\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(v_{ij})^2 x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)}.$$

Step 6: Using the optimal solution $\{x_{ij}\}$, obtained in Step 5, find the optimal Pythagorean fuzzy

transportation cost $$\left(\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\,;\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}(\mu_{ij})^2}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)^{\frac{1}{2}},\left(\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}(v_{ij})^2}{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}}\right)^{\frac{1}{2}}\right).$$

4.14 Illustrative example

In this section, the existing PyFuTp [102, 127, 163] (represented by Table 4.4) is solved to illustrate the proposed Mehar method.

Table 4.4 [102, 127, 163] Existing PyFuTp

	D_1	D_2	D_3	D_4	Availability
S_1	$(2; 0.4,0.7)$	$(2; 0.5,0.4)$	$(2; 0.8,0.3)$	$(2; 0.6,0.3)$	26
S_2	$(2; 0.4,0.2)$	$(2; 0.7,0.3)$	$(2; 0.4,0.8)$	$(2; 0.7,0.3)$	24
S_3	$(2; 0.7,0.1)$	$(2; 0.8,0.1)$	$(2; 0.6,0.4)$	$(2; 0.9,0.1)$	30
Demand	17	23	28	12	

Step 1: According to Step 1 of the proposed Mehar method, the FuLpP $(P4.14.1)$ is obtained corresponding to the PyFuTp (represented by Table 4.4).

FuLpP $(\boldsymbol{P4.14.1})$

$Minimize\,(\tilde{Z} \approx (2;0.4,0.7)x_{11} + (2;0.5,0.4)x_{12} + (2;0.8,0.3)x_{13} + (2;0.6,0.3)x_{14} + (2;0.4,0.2)x_{21} + (2;0.7,0.3)x_{22} + (2;0.4,0.8)x_{23} + (2;0.7,0.3)x_{24} + (2;0.7,0.1)x_{31} + (2;0.8,0.1)x_{32} + (2;0.6,0.4)x_{33} + (2;0.9,0.1)x_{34})$

Subject to

$x_{11} + x_{12} + x_{13} + x_{14} = 26,$

$x_{21} + x_{22} + x_{23} + x_{24} = 24,$

$x_{31} + x_{32} + x_{33} + x_{34} = 30,$

$x_{11} + x_{21} + x_{31} = 17,$

$x_{12} + x_{22} + x_{32} = 23,$

$x_{13} + x_{23} + x_{33} = 28,$

$x_{14} + x_{24} + x_{34} = 12,$

$x_{ij} \geq 0; i = 1,2,3, j = 1,2,3,4.$

Step 2: According to Step 2 of the proposed Mehar method, the FuLpP $(P4.14.1)$ can be transformed into its equivalent FuMpP $(P4.14.2)$.

FuMpP $(\boldsymbol{P4.14.2})$

$$Minimize\left(\tilde{Z} \approx \left(2x_{11} + 2x_{12} + 2x_{13} + 2x_{14} + 2x_{21} + 2x_{22} + 2x_{23} + 2x_{24} + 2x_{31} + 2x_{32} + 2x_{33} + 2x_{34}; (r)^{\frac{1}{2}}, (s)^{\frac{1}{2}}\right)\right)$$

Subject to

Constraints of the FuLpP $(P4.14.1)$,

where,

(i) $r=\frac{1}{\sum_{i=1}^{3}\sum_{j=1}^{4}x_{ij}}((0.4)^2x_{11}+(0.5)^2x_{12}+(0.8)^2x_{13}+(0.6)^2x_{14}+(0.4)^2x_{21}+$

$(0.7)^2x_{22}+(0.4)^2x_{23}+(0.7)^2x_{24}+(0.7)^2x_{31}+(0.8)^2x_{32}+(0.6)^2x_{33}+$

$(0.9)^2x_{34})$.

(ii) $s=\frac{1}{\sum_{i=1}^{3}\sum_{j=1}^{4}x_{ij}}((0.7)^2x_{11}+(0.4)^2x_{12}+(0.3)^2x_{13}+(0.3)^2x_{14}+(0.2)^2x_{21}+$

$(0.3)^2x_{22}+(0.8)^2x_{23}+(0.3)^2x_{24}+(0.1)^2x_{31}+(0.1)^2x_{32}+(0.4)^2x_{33}+$

$(0.1)^2x_{34})$.

Step 3: According to Step 3 of the proposed Mehar method, there is a need to find the optimal value of the CrMpP $(P4.14.3)$.

CrMpP $(\boldsymbol{P4.14.3})$

$Minimize\ (2x_{11}+2x_{12}+2x_{13}+2x_{14}+2x_{21}+2x_{22}+2x_{23}+2x_{24}+2x_{31}+$

$2x_{32}+2x_{33}+2x_{34})$

Subject to

Constraints of the FuLpP $(P4.14.1)$.

It can be easily verified that the optimal value of the CrMpP $(P4.14.3)$ is 160.

Step 4: According to Step 4 of the proposed Mehar method, there is a need to find the optimal value of the CrMpP $(P4.14.4)$.

CrMpP $(\boldsymbol{P4.14.4})$

$Minimize\left(\frac{2}{3}\left(\frac{2^{r-s}}{2-r-s}-\frac{1}{2}\right)\right)$

Subject to

Constraints of the CrMpP $(P4.14.3)$ with the additional constraint $2x_{11}+2x_{12}+2x_{13}+$

$2x_{14}+2x_{21}+2x_{22}+2x_{23}+2x_{24}+2x_{31}+2x_{32}+2x_{33}+2x_{34}=160$.

It can be easily verified that the optimal value of the CrMpP $(P4.14.4)$ is $\frac{5285}{75527}$.

Step 5: According to Step 5 of the proposed Mehar method, there is a need to find an optimal solution of the CrMpP $(P4.14.5)$.

CrMpP $(\boldsymbol{P4.14.5})$

$Minimize\left(\sqrt{1-r-s}\right)$

Subject to

Constraints of the CrMpP $(P4.14.4)$ with the additional constraint $\frac{2}{3}\left(\frac{2^{r-s}}{2-r-s}-\frac{1}{2}\right)=\frac{5285}{75527}$.

It can be easily verified that an optimal solution of the CrMpP $(P4.14.5)$ is $x_{11}=0, x_{12}=0, x_{13}=26, x_{14}=0, x_{21}=0, x_{22}=23, x_{23}=1, x_{24}=0, x_{31}=17, x_{32}=0, x_{33}=1, x_{34}=12$.

Step 6: According to Step 6 of the proposed Mehar method, substituting $x_{11}=0, x_{12}=14, x_{13}=0, x_{14}=12, x_{21}=17, x_{22}=0, x_{23}=7, x_{24}=0, x_{31}=0, x_{32}=9, x_{33}=21, x_{34}=0$ in

$$\left(2x_{11}+2x_{12}+2x_{13}+2x_{14}+2x_{21}+2x_{22}+2x_{23}+2x_{24}+2x_{31}+2x_{32}+2x_{33}+2x_{34};\left(\frac{1}{\sum_{i=1}^{3}\sum_{j=1}^{4}x_{ij}}((0.4)^2x_{11}+(0.5)^2x_{12}+(0.8)^2x_{13}+(0.6)^2x_{14}+(0.4)^2x_{21}+(0.7)^2x_{22}+(0.4)^2x_{23}+(0.7)^2x_{24}+(0.7)^2x_{31}+(0.8)^2x_{32}+(0.6)^2x_{33}+(0.9)^2x_{34})\right)^{\frac{1}{2}},\left(\frac{1}{\sum_{i=1}^{3}\sum_{j=1}^{4}x_{ij}}((0.7)^2x_{11}+(0.4)^2x_{12}+(0.3)^2x_{13}+(0.3)^2x_{14}+(0.2)^2x_{21}+(0.3)^2x_{22}+(0.8)^2x_{23}+(0.3)^2x_{24}+(0.1)^2x_{31}+(0.1)^2x_{32}+(0.4)^2x_{33}+(0.1)^2x_{34})\right)^{\frac{1}{2}}\right)$$

, the optimal Pythagorean fuzzy transportation cost is $(160; 0.5588, 0.3313)$.

4.15 Conclusions

In this chapter,

(i) Some mathematical incorrect results, considered in the existing methods [102, 127, 163] for solving PyFuTpS, are pointed out.

(ii) It is pointed out that existing expression [102, 127, 163] to evaluate sum of PyFuNs is not appropriate.

(iii) An appropriate expression to evaluate sum of PyFuNs is proposed.

(iv) A new method (named as Mehar method) is proposed for solving PyFuTpS.

Chapter 5

Efficient Methods for Solving Shortest Path Problems Under Fuzzy Environment and Their Extensions[1]

In this chapter, limitations and/or shortcomings of some recently proposed methods for solving SpPs under fuzzy environment and their extensions [57, 63, 162, 175] are discussed. Also, to resolve the shortcomings and/or to overcome the limitations, efficient methods are proposed.

5.1 Efficient method for solving interval-valued fuzzy SpPs

After reviewing the literature, it may be concluded that there does not exist any method Ebrahimnejad et al.'s method [57] for solving IvTFuSpPs. In this section,

(i) It is pointed out that much computational efforts are required to apply Ebrahimnejad et al.'s method [57].

(ii) An efficient method is proposed for solving interval-valued fuzzy SpPs.

5.1.1 Preliminaries

In this section, some basic definitions are discussed.

Definition 5.1.1.1 [57] A fuzzy number $\tilde{A}$ is said to be an IvTFuN if its lower and upper membership functions are defined as

[1] Some of the contents of this chapter are published in "Sustainability 13 (2021) 4016". Some of the contents of this chapter are accepted for publication in "International Journal of Fuzzy System Applications". Some of the contents of this chapter are communicated in "International Journal of Fuzzy Systems" for publication.

$$\mu_{\tilde{A}^L}(x)=\begin{cases}0, & x\le a_1^L, x\ge a_3^L,\\ h_{\tilde{A}}^L\left(\frac{x-a_1^L}{a_2-a_1^L}\right), & a_1^L< x\le a_2,\\ h_{\tilde{A}}^L\left(\frac{a_3^L-x}{a_3^L-a_2}\right), & a_2\le x< a_3^L\end{cases}$$

$$\mu_{\tilde{A}^U}(x)=\begin{cases}0, & x\le a_1^U, x\ge a_3^U,\\ h_{\tilde{A}}^U\left(\frac{x-a_1^U}{a_2-a_1^U}\right), & a_1^U< x\le a_2,\\ h_{\tilde{A}}^U\left(\frac{a_3^U-x}{a_3^U-a_2}\right), & a_2\le x< a_3^U\end{cases}$$

where $a_1^U\le a_1^L\le a_2\le a_3^L\le a_3^U, 0<h_{\tilde{A}}^L\le h_{\tilde{A}}^U\le 1$.

It is represented as $\tilde{A}=\left\langle\left(a_1^L,a_2,a_3^L;h_{\tilde{A}}^L\right),\left(a_1^U,a_2,a_3^U;h_{\tilde{A}}^U\right)\right\rangle$.

Definition 5.1.1.2 [64] A fuzzy number $\tilde{A}$ is said to be an IvTrFuN if its lower and upper membership functions are defined as

$$\mu_{\tilde{A}^L}(x)=\begin{cases}0, & x\le a_1^L, x\ge a_4^L,\\ h_{\tilde{A}}^L\left(\frac{x-a_1^L}{a_2^L-a_1^L}\right), & a_1^L< x\le a_2^L,\\ h_{\tilde{A}}^L, & a_2^L\le x\le a_3^L,\\ h_{\tilde{A}}^L\left(\frac{a_4^L-x}{a_4^L-a_3^L}\right), & a_3^L< x\le a_4^L\end{cases}$$

$$\mu_{\tilde{A}^U}(x)=\begin{cases}0, & x\le a_1^U, x\ge a_4^U,\\ h_{\tilde{A}}^U\left(\frac{x-a_1^U}{a_2^U-a_1^U}\right), & a_1^U< x\le a_2^U,\\ h_{\tilde{A}}^U, & a_2^U\le x\le a_3^U,\\ h_{\tilde{A}}^U\left(\frac{a_4^U-x}{a_4^U-a_3^U}\right), & a_3^U< x\le a_4^U\end{cases}$$

where $a_1^U\le a_1^L\le a_2^U\le a_2^L\le a_3^L\le a_3^U\le a_4^L\le a_4^U, 0<h_{\tilde{A}}^L\le h_{\tilde{A}}^U\le 1$.

It is represented as $\tilde{A}=\left\langle\left(a_1^L,a_2^L,a_3^L,a_4^L;h_{\tilde{A}}^L\right),\left(a_1^U,a_2^U,a_3^U,a_4^U;h_{\tilde{A}}^U\right)\right\rangle$.

Definition 5.1.1.3 [57] Let $\tilde{A}_1=\langle(a_{11}^L,a_{12},a_{13}^L;h^L),(a_{11}^U,a_{12},a_{13}^U;h^U)\rangle$ and $\tilde{A}_2=\langle(a_{21}^L,b_{22},a_{23}^L;h^L),(a_{21}^U,a_{22},a_{23}^U;h^U)\rangle$ be two IvTFuNs, then $\tilde{A}_1+\tilde{A}_2=\langle(a_{11}^L+a_{21}^L,a_{12}+b_{22},a_{13}^L+a_{23}^L;h^L),(a_{11}^U+a_{21}^U,a_{12}+a_{22},a_{13}^U+a_{23}^U;h^U)\rangle$.

Definition 5.1.1.4 [64] Let $\tilde{A}_1 = \langle(a_{11}^L, a_{12}^L, a_{13}^L, a_{14}^L; h^L), (a_{11}^U, a_{12}^U, a_{13}^U, a_{14}^U; h^U)\rangle$ and $\tilde{A}_2 = \langle(a_{21}^L, a_{22}^L, a_{23}^L, a_{24}^L; h^L), (a_{21}^U, a_{22}^U, a_{23}^U, a_{24}^U; h^U)\rangle$ be two IvTrFuNs, then, $\tilde{A}_1 + \tilde{A}_2 = \langle(a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L; h^L),(a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U; h^U)\rangle$.

Definition 5.1.1.5 [57] Let $\tilde{A} = \left\langle\left(a_1^L, a_2, a_3^L; h_{\tilde{A}}^L\right), \left(a_1^U, a_2, a_3^U; h_{\tilde{A}}^U\right)\right\rangle$ be an IvTFuN and $k \geq 0$, then $k\tilde{A} = \left\langle\left(ka_1^L, ka_2, ka_3^L; h_{\tilde{A}}^L\right), \left(ka_1^U, ka_2, ka_3^U; h_{\tilde{A}}^U\right)\right\rangle$.

5.1.2 Ebrahimnejad et al.'s method

Ebrahimnejad et al. [57] claimed that for solving an IvTFuSpP is equivalent to solve the IvTFuLpP $(P5.1.2.1)$. Hence, Ebrahimnejad et al. [57] proposed the following method for solving the IvTFuLpP $(P5.1.2.1)$.

IvTFuLpP $(\boldsymbol{P5.1.2.1})$

$Minimize\left(\sum_{(i,j)\in E}\left(\left\langle\left(d_{ij1}^L, d_{ij2}, d_{ij3}^L; h^L\right), \left(d_{ij1}^U, d_{ij2}, d_{ij3}^U; h^U\right)\right\rangle x_{ij}\right)\right)$

Subject to

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} 1, & i = s, \\ 0, & i \neq s, r, \\ -1, & i = r, \end{cases}$$

$x_{ij} \geq 0 \; \forall \; (i,j) \in E$.

where,

(i) $E = \{(i,j): i,j \in V, i \neq j\}$ represents the set of arcs which directly connects the i^{th} node to the j^{th} node.

(ii) V represents the set of nodes.

(iii) $\left\langle\left(d_{ij1}^L, d_{ij2}, d_{ij3}^L; h^L\right), \left(d_{ij1}^U, d_{ij2}, d_{ij3}^U; h^U\right)\right\rangle$ represents the IvTFuD from the i^{th} node to the j^{th} node.

(iv) s and r represents the source node and the destination node respectively.

Step 1: Using Definition 5.1.1.5, transform the IvTFuLpP $(P5.1.2.1)$ into its equivalent IvTFuLpP $(P5.1.2.2)$.

IvTFuLpP $(\boldsymbol{P5.1.2.2})$

$$Minimize\left(\sum_{(i,j)\in E}\left(\left\langle\left(d^L_{ij1}x_{ij}, d_{ij2}x_{ij}, d^L_{ij3}x_{ij}; h^L\right), \left(d^U_{ij1}x_{ij}, d_{ij2}x_{ij}, d^U_{ij3}x_{ij}; h^U\right)\right\rangle\right)\right)$$

Subject to

Constraints of the IvTFuLpP $(P5.1.2.1)$.

Step 2: Using Definition 5.1.1.3, transform the IvTFuLpP $(P5.1.2.2)$ into its equivalent IvTFuLpP $(P5.1.2.3)$.

IvTFuLpP $(\boldsymbol{P5.1.2.3})$

$$Minimize\begin{pmatrix}\left\langle\left(\sum_{(i,j)\in E} d^L_{ij1}x_{ij}, \sum_{(i,j)\in E} d_{ij2}x_{ij}, \sum_{(i,j)\in E} d^L_{ij3}x_{ij}; h^L\right),\right.\\ \left.\left(\sum_{(i,j)\in E} d^U_{ij1}x_{ij}, \sum_{(i,j)\in E} d_{ij2}x_{ij}, \sum_{(i,j)\in E} d^U_{ij3}x_{ij}; h^U\right)\right\rangle\end{pmatrix}$$

Subject to

Constraints of the IvTFuLpP $(P5.1.2.1)$.

Step 3: Transform IvTFuLpP $(P5.1.2.3)$ into its equivalent CrMoLpP $(P5.1.2.4)$.

CrMoLpP $(\boldsymbol{P5.1.2.4})$

$$Minimize\left(\sum_{(i,j)\in E} d^U_{ij1}x_{ij}\right)$$

$$Minimize\left(\sum_{(i,j)\in E} d^L_{ij1}x_{ij}\right)$$

$$Minimize\left(\sum_{(i,j)\in E} d_{ij2}x_{ij}\right)$$

$$Minimize\left(\sum_{(i,j)\in E} d^L_{ij3}x_{ij}\right)$$

$$Minimize\left(\sum_{(i,j)\in E} d^U_{ij3}x_{ij}\right)$$

Subject to

Constraints of the IvTFuLpP $(P5.1.2.1)$.

Step 4: Solve the CrLpP $(P5.1.2.5)$. If a unique optimal solution exists, then go to Step 9, otherwise go to Step 5.

CrLpP $(\boldsymbol{P5.1.2.5})$

$Minimize\left(\sum_{(i,j)\in E} d_{ij1}^{U} x_{ij}\right)$

Subject to

Constraints of the IvTFuLpP $(P5.1.2.1)$.

Step 5: Solve the CrLpP $(P5.1.2.6)$. If a unique optimal solution exists, then go to Step 9, otherwise go to Step 6.

CrLpP $(\boldsymbol{P5.1.2.6})$

$Minimize\left(\sum_{(i,j)\in E} d_{ij1}^{L} x_{ij}\right)$

Subject to

Constraints of the IvTFuLpP $(P5.1.2.1)$ with additional constraint $\sum_{(i,j)\in E} d_{ij1}^{U} x_{ij} = \alpha_1^U$, where, α_1^U is the optimal value of the CrLpP $(P5.1.2.5)$.

Step 6: Solve the CrLpP $(P5.1.2.7)$. If a unique optimal solution exists, then go to Step 9, otherwise go to Step 7.

CrLpP $(\boldsymbol{P5.1.2.7})$

$Minimize\left(\sum_{(i,j)\in E} d_{ij2} x_{ij}\right)$

Subject to

Constraints of the CrLpP $(P5.1.2.6)$ with additional constraint $\sum_{(i,j)\in E} d_{ij1}^{L} x_{ij} = \alpha_1^L$, where, α_1^L is the optimal value of the CrLpP $(P5.1.2.6)$.

Step 7: Solve the CrLpP $(P5.1.2.8)$. If a unique optimal solution exists, then go to Step 9, otherwise go to Step 8.

CrLpP $(\boldsymbol{P5.1.2.8})$

$Minimize\left(\sum_{(i,j)\in E} d_{ij3}^{L} x_{ij}\right)$

Subject to

Constraints of the CrLpP $(P5.1.2.7)$ with additional constraint $\sum_{(i,j)\in E} d_{ij2} x_{ij} = \alpha_2$, where, α_2 is the optimal value of the CrLpP $(P5.1.2.7)$.

Step 8: Find an optimal solution of the CrLpP $(P5.1.2.9)$.

CrLpP $(\boldsymbol{P5.1.2.9})$

$Minimize\left(\sum_{(i,j)\in E} d_{ij3}^{U} x_{ij}\right)$

Subject to

Constraints of the CrLpP $(P5.1.2.8)$ with additional constraint $\sum_{(i,j)\in E} d_{ij3}^{L} x_{ij} = \alpha_3^{L}$, where, α_3^{L} is the optimal value of the CrLpP $(P5.1.2.8)$.

Step 9: Using the values of $x_{st_1} = x_{t_1 t_2} = x_{t_2 t_3} = \cdots = x_{t_l r} = 1$, obtained in the optimal solution, find the shortest path $s - t_1 - t_2 \ldots - r$.

Step 10: Using the optimal solution $\{x_{ij}\}$, find the shortest IvTFuD $\left(\sum_{(i,j)\in E}\left(\left\langle\left(d_{ij1}^{L}, d_{ij2}, d_{ij3}^{L}; h^{L}\right), \left(d_{ij1}^{U}, d_{ij2}, d_{ij3}^{U}; h^{U}\right)\right\rangle x_{ij}\right)\right)$.

5.1.3 Computational complexity of Ebrahimnejad et al.'s method

In this section, it is shown that to find the shortest path from the source node 1 to the destination node 6 in the existing IvTFuSpP (represented by Fig. 5.1 and Table 5.1) by Ebrahimnejad et al.'s method [57], there is a need to solve five LpPs. Hence, much computational efforts are required to apply Ebrahimnejad et al.'s method [57].

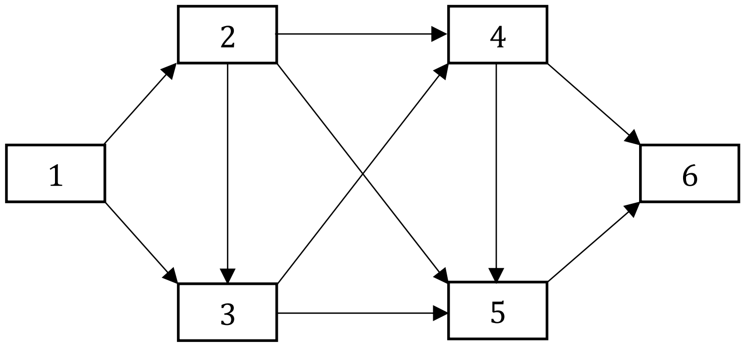

Fig. 5.1 [57] Existing IvTFuSpP

Table 5.1 [57] Existing IvTFuD between two nodes

Node i to node j	IvTFuD
Node 1 to node 2	$\langle(10,12,13;0.5),(9,12,15;1)\rangle$
Node 1 to node 3	$\langle(8,10,11;0.5),(7,10,14;1)\rangle$
Node 2 to node 3	$\langle(10,12,13;0.5),(9,12,16;1)\rangle$
Node 2 to node 4	$\langle(2,3,7;0.5),(1,3,8;1)\rangle$
Node 2 to node 5	$\langle(3,5,6;0.5),(2,5,7;1)\rangle$
Node 3 to node 4	$\langle(4,8,10;0.5),(3,8,12;1)\rangle$
Node 3 to node 5	$\langle(5,7,8;0.5),(4,7,9;1)\rangle$
Node 4 to node 5	$\langle(4,6,8;0.5),(1,6,10;1)\rangle$
Node 4 to node 6	$\langle(4,7,11;0.5),(3,7,12;1)\rangle$
Node 5 to node 6	$\langle(3,5,7;0.5),(2,5,8;1)\rangle$

According to Ebrahimnejad et al.'s method [57], to solve the considered IvTFuSpP is equivalent to solve the IvTFuLpP $(P5.1.3.1)$.

IvTFuLpP $(\boldsymbol{P5.1.3.1})$

$Minimize\ \big((\langle(10,12,13;0.5),(9,12,15;1)\rangle x_{12}) + (\langle(8,10,11;0.5),(7,10,14;1)\rangle x_{13}) +$

$(\langle(10,12,13;0.5),(9,12,16;1)\rangle x_{23}) + (\langle(2,3,7;0.5),(1,3,8;1)\rangle x_{24}) +$

$(\langle(3,5,6;0.5),(2,5,7;1)\rangle x_{25}) + (\langle(4,8,10;0.5),(3,8,12;1)\rangle x_{34}) +$

$(\langle(5,7,8;0.5),(4,7,9;1)\rangle x_{35}) + (\langle(4,6,8;0.5),(1,6,10;1)\rangle x_{45}) +$

$(\langle(4,7,11;0.5),(3,7,12;1)\rangle x_{46}) + (\langle(3,5,7;0.5),(2,5,8;1)\rangle x_{56})\big)$

Subject to

$x_{12} + x_{13} = 1,$

$x_{23} + x_{24} + x_{25} - x_{12} = 0,$

$x_{34} + x_{35} - x_{13} - x_{23} = 0,$

$x_{45} + x_{46} - x_{24} - x_{34} = 0,$

$x_{56} - x_{25} - x_{35} - x_{45} = 0,$

$-x_{46} - x_{56} = -1,$

$x_{12} \geq 0, x_{13} \geq 0, x_{23} \geq 0, x_{24} \geq 0, x_{25} \geq 0, x_{34} \geq 0, x_{35} \geq 0, x_{45} \geq 0, x_{46} \geq 0, x_{56} \geq 0.$

Using Ebrahimnejad et al.'s method [57], the IvTFuLpP $(P5.1.3.1)$ can be solved as follows.

Step 1: Using Step 1 of Ebrahimnejad et al.'s method [57], the IvTFuLpP $(P5.1.3.1)$ can be transformed into its equivalent IvTFuLpP $(P5.1.3.2)$.

IvTFuLpP $(\boldsymbol{P5.1.3.2})$

$Minimize\ (\langle(10x_{12}, 12x_{12}, 13x_{12}; 0.5), (9x_{12}, 12x_{12}, 15x_{12}; 1)\rangle +$

$\langle(8x_{13}, 10x_{13}, 11x_{13}; 0.5), (7x_{13}, 10x_{13}, 14x_{13}; 1)\rangle +$

$\langle(10x_{23}, 12x_{23}, 13x_{23}; 0.5), (9x_{23}, 12x_{23}, 16x_{23}; 1)\rangle +$

$\langle(2x_{24}, 3x_{24}, 7x_{24}; 0.5), (x_{24}, 3x_{24}, 8x_{24}; 1)\rangle +$

$\langle(3x_{25}, 5x_{25}, 6x_{25}; 0.5), (2x_{25}, 5x_{25}, 7x_{25}; 1)\rangle +$

$\langle(4x_{34}, 8x_{34}, 10x_{34}; 0.5), (3x_{34}, 8x_{34}, 12x_{34}; 1)\rangle +$

$\langle(5x_{35}, 7x_{35}, 8x_{35}; 0.5), (4x_{35}, 7x_{35}, 9x_{35}; 1)\rangle +$

$\langle(4x_{45}, 6x_{45}, 8x_{45}; 0.5), (x_{45}, 6x_{45}, 10x_{45}; 1)\rangle +$

$\langle(4x_{46}, 7x_{46}, 11x_{46}; 0.5), (3x_{46}, 7x_{46}, 12x_{46}; 1)\rangle +$

$\langle(3x_{56}, 5x_{56}, 7x_{56}; 0.5), (2x_{56}, 5x_{56}, 8x_{56}; 1)\rangle)$

Subject to

Constraints of the IvTFuLpP $(P5.1.3.1)$.

Step 2: Using Step 2 of Ebrahimnejad et al.'s method [57], the IvTFuLpP $(P5.1.3.2)$ can be transformed into its equivalent IvTFuLpP $(P5.1.3.3)$.

IvTFuLpP $(\boldsymbol{P5.1.3.3})$

$Minimize\ (\langle(10x_{12} + 8x_{13} + 10x_{23} + 2x_{24} + 3x_{25} + 4x_{34} + 5x_{35} + 4x_{45} + 4x_{46} + 3x_{56}, 12x_{12} + 10x_{13} + 12x_{23} + 3x_{24} + 5x_{25} + 8x_{34} + 7x_{35} + 6x_{45} + 7x_{46} + 5x_{56}, 13x_{12} + 11x_{13} + 13x_{23} + 7x_{24} + 6x_{25} + 10x_{34} + 8x_{35} + 8x_{45} + 11x_{46} + 7x_{56}; 0.5), (9x_{12} + 7x_{13} + 9x_{23} + x_{24} + 2x_{25} + 3x_{34} + 4x_{35} + x_{45} + 3x_{46} + 2x_{56}, 12x_{12} + 10x_{13} + 12x_{23} + 3x_{24} + 5x_{25} + 8x_{34} + 7x_{35} + 6x_{45} + 7x_{46} + 5x_{56}, 15x_{12} + 14x_{13} + 16x_{23} + 8x_{24} + 7x_{25} + 12x_{34} + 9x_{35} + 10x_{45} + 12x_{46} + 8x_{56}; 1)\rangle)$

Subject to

Constraints of the IvTFuLpP $(P5.1.3.1)$.

Step 3: Using Step 3 of Ebrahimnejad et al.'s method [57], the IvTFuLpP $(P5.1.3.3)$ can be transformed into its equivalent CrMoLpP $(P5.1.3.4)$.

CrMoLpP $(\boldsymbol{P5.1.3.4})$

$Minimize\ (9x_{12} + 7x_{13} + 9x_{23} + x_{24} + 2x_{25} + 3x_{34} + 4x_{35} + x_{45} + 3x_{46} + 2x_{56})$

$Minimize\ (10x_{12} + 8x_{13} + 10x_{23} + 2x_{24} + 3x_{25} + 4x_{34} + 5x_{35} + 4x_{45} + 4x_{46} + 3x_{56})$

$Minimize\ (12x_{12} + 10x_{13} + 12x_{23} + 3x_{24} + 5x_{25} + 8x_{34} + 7x_{35} + 6x_{45} + 7x_{46} + 5x_{56})$

$Minimize\ (13x_{12} + 11x_{13} + 13x_{23} + 7x_{24} + 6x_{25} + 10x_{34} + 8x_{35} + 8x_{45} + 11x_{46} + 7x_{56})$

$Minimize\ (15x_{12} + 14x_{13} + 16x_{23} + 8x_{24} + 7x_{25} + 12x_{34} + 9x_{35} + 10x_{45} + 12x_{46} + 8x_{56})$

Subject to

Constraints of the IvTFuLpP $(P5.1.3.1)$.

Step 4: According to Step 4 of Ebrahimnejad et al.'s method [57], there is a need to solve the CrLpP $(P5.1.3.5)$.

CrLpP $(\boldsymbol{P5.1.3.5})$

$Minimize\ (9x_{12} + 7x_{13} + 9x_{23} + x_{24} + 2x_{25} + 3x_{34} + 4x_{35} + x_{45} + 3x_{46} + 2x_{56})$

Subject to

Constraints of the IvTFuLpP $(P5.1.3.1)$.

Since, on solving the CrLpP $(P5.1.3.5)$, there does not exist a unique optimal solution. So, according to Step 4 of Ebrahimnejad et al.'s method [57], there is a need to go to Step 5.

Step 5: According to Step 5 of Ebrahimnejad et al.'s method [57], there is a need to solve the CrLpP $(P5.1.3.6)$.

CrLpP $(\boldsymbol{P5.1.3.6})$

$Minimize\ (10x_{12} + 8x_{13} + 10x_{23} + 2x_{24} + 3x_{25} + 4x_{34} + 5x_{35} + 4x_{45} + 4x_{46} + 3x_{56})$

Subject to

Constraints of the IvTFuLpP $(P5.1.3.1)$ and an additional constraint $9x_{12} + 7x_{13} + 9x_{23} + x_{24} + 2x_{25} + 3x_{34} + 4x_{35} + x_{45} + 3x_{46} + 2x_{56} = 13$, where 13 is the optimal value of the CrLpP $(P5.1.3.5)$.

Since, on solving the CrLpP $(P5.1.3.6)$, there does not exist a unique optimal solution. So, according to Step 5 of Ebrahimnejad et al.'s method [57], there is a need to go to Step 6.

Step 6: According to Step 6 of Ebrahimnejad et al.'s method [57], there is a need to solve the CrLpP $(P5.1.3.7)$.

CrLpP $(\boldsymbol{P5.1.3.7})$

$Minimize\ (12x_{12} + 10x_{13} + 12x_{23} + 3x_{24} + 5x_{25} + 8x_{34} + 7x_{35} + 6x_{45} + 7x_{46} + 5x_{56})$

Subject to

Constraints of the CrLpP $(P5.1.3.6)$ and an additional constraint $10x_{12} + 8x_{13} + 10x_{23} + 2x_{24} + 3x_{25} + 4x_{34} + 5x_{35} + 4x_{45} + 4x_{46} + 3x_{56} = 16$, where 16 is the optimal value of the CrLpP $(P5.1.3.6)$.

Since, on solving the CrLpP $(P5.1.3.7)$, there does not exist a unique optimal solution. So, according to Step 6 of Ebrahimnejad et al.'s method [57], there is a need to go to Step 7.

Step 7: According to Step 7 of Ebrahimnejad et al.'s method [57], there is a need to solve the CrLpP $(P5.1.3.8)$.

CrLpP $(\boldsymbol{P5.1.3.8})$

$Minimize\ (13x_{12} + 11x_{13} + 13x_{23} + 7x_{24} + 6x_{25} + 10x_{34} + 8x_{35} + 8x_{45} + 11x_{46} + 7x_{56})$

Subject to

Constraints of the CrLpP $(P5.1.3.7)$ and an additional constraint $12x_{12} + 10x_{13} + 12x_{23} + 3x_{24} + 5x_{25} + 8x_{34} + 7x_{35} + 6x_{45} + 7x_{46} + 5x_{56} = 22$, where 22 is the optimal value of the CrLpP $(P5.1.3.7)$.

Since, on solving the CrLpP $(P5.1.3.8)$, there does not exist a unique optimal solution. So, according to Step 7 of Ebrahimnejad et al.'s method [57], there is a need to go to Step 8.

Step 8: According to Step 8 of Ebrahimnejad et al.'s method [57], there is a need to find an optimal solution of the CrLpP $(P5.1.3.9)$.

CrLpP $(\boldsymbol{P5.1.3.9})$

$Minimize\ (15x_{12} + 14x_{13} + 16x_{23} + 8x_{24} + 7x_{25} + 12x_{34} + 9x_{35} + 10x_{45} + 12x_{46} + 8x_{56})$

Subject to

Constraints of the CrLpP $(P5.1.3.8)$ and an additional constraint $13x_{12} + 11x_{13} + 13x_{23} + 7x_{24} + 6x_{25} + 10x_{34} + 8x_{35} + 8x_{45} + 11x_{46} + 7x_{56} = 26$, where 26 is the optimal value of the CrLpP $(P5.1.3.8)$.

On solving the CrLpP $(P5.1.3.9)$, the optimal solution $x_{12} = 1, x_{25} = 1, x_{56} = 1, x_{13} = x_{23} = x_{24} = x_{34} = x_{35} = x_{45} = x_{46} = 0$ is obtained.

Step 9: Using the optimal solution, the shortest path from the source node 1 to the destination node 6 is $1 - 2 - 5 - 6$.

Step 10: Using the obtained optimal solution $x_{12} = 1, x_{25} = 1, x_{56} = 1, x_{13} = x_{23} = x_{24} = x_{34} = x_{35} = x_{45} = x_{46} = 0$, the shortest IvTFuD from the source node 1 to the destination node 6 is $\langle(16,22,26; 0.5), (13,22,30; 1)\rangle$.

5.1.4 Existing method for comparing IvTFuNs

Ebrahimnejad et al. [57] have used the following method for comparing IvTFuNs $\tilde{A}_1 = \langle(a_{11}^L, a_{12}, a_{13}^L; h^L), (a_{11}^U, a_{12}, a_{13}^U; h^U)\rangle$ and $\tilde{A}_2 = \langle(a_{21}^L, a_{22}, a_{23}^L; h^L), (a_{21}^U, a_{22}, a_{23}^U; h^U)\rangle$.

Step 1: Check whether $a_{11}^U < a_{21}^U$ or $a_{11}^U > a_{21}^U$ or $a_{11}^U = a_{21}^U$.

Case (a) If $a_{11}^U < a_{21}^U$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{11}^U > a_{21}^U$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{11}^U = a_{21}^U$, then go to Step 2.

Step 2: Check whether $a_{11}^L < a_{21}^L$ or $a_{11}^L > a_{21}^L$ or $a_{11}^L = a_{21}^L$.

Case (a) If $a_{11}^L < a_{21}^L$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{11}^L > a_{21}^L$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{11}^L = a_{21}^L$, then go to Step 3.

Step 3: Check whether $a_{12} < a_{22}$ or $a_{12} > a_{22}$ or $a_{12} = a_{22}$.

Case (a) If $a_{12} < a_{22}$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{12} > a_{22}$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{12} = a_{22}$, then go to Step 4.

Step 4: Check whether $a_{13}^L < a_{23}^L$ or $a_{13}^L > a_{23}^L$ or $a_{13}^L = a_{23}^L$.

Case (a) If $a_{13}^L < a_{23}^L$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{13}^L > a_{23}^L$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{13}^{L} = a_{23}^{L}$, then go to Step 5.

Step 5: Check whether $a_{13}^{U} < a_{23}^{U}$ or $a_{13}^{U} > a_{23}^{U}$ or $a_{13}^{U} = a_{23}^{U}$.

Case (a) If $a_{13}^{U} < a_{23}^{U}$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{13}^{U} > a_{23}^{U}$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{13}^{U} = a_{23}^{U}$, then $\tilde{A}_1 = \tilde{A}_2$.

5.1.5 Extended method for comparing IvTrFuNs

In this section, Ebrahimnejad et al.'s method [57] for comparing IvTFuNs is extended for comparing IvTrFuNs.

Let $\tilde{A}_1 = \langle (a_{11}^{L}, a_{12}^{L}, a_{13}^{L}, a_{14}^{L}; h^{L}), (a_{11}^{U}, a_{12}^{U}, a_{13}^{U}, a_{14}^{U}; h^{U}) \rangle$ and $\tilde{A}_2 = \langle (a_{21}^{L}, a_{22}^{L}, a_{23}^{L}, a_{24}^{L}; h^{L}), (a_{21}^{U}, a_{22}^{U}, a_{23}^{U}, a_{24}^{U}; h^{U}) \rangle$ be two IvTrFuNs. Then, $\tilde{A}_1$ and $\tilde{A}_2$ can be compared as follows.

Step 1: Check whether $a_{11}^{U} < a_{21}^{U}$ or $a_{11}^{U} > a_{21}^{U}$ or $a_{11}^{U} = a_{21}^{U}$.

Case (a) If $a_{11}^{U} < a_{21}^{U}$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{11}^{U} > a_{21}^{U}$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{11}^{U} = a_{21}^{U}$, then go to Step 2.

Step 2: Check whether $a_{11}^{L} < a_{21}^{L}$ or $a_{11}^{L} > a_{21}^{L}$ or $a_{11}^{L} = a_{21}^{L}$.

Case (a) If $a_{11}^{L} < a_{21}^{L}$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{11}^{L} > a_{21}^{L}$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{11}^{L} = a_{21}^{L}$, then go to Step 3.

Step 3: Check whether $a_{12}^{U} < a_{22}^{U}$ or $a_{12}^{U} > a_{22}^{U}$ or $a_{12}^{U} = a_{22}^{U}$.

Case (a) If $a_{12}^{U} < a_{22}^{U}$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{12}^{U} > a_{22}^{U}$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{12}^{U} = a_{22}^{U}$, then go to Step 4.

Step 4: Check whether $a_{12}^L < a_{22}^L$ or $a_{12}^L > a_{22}^L$ or $a_{12}^L = a_{22}^L$.

Case (a) If $a_{12}^L < a_{22}^L$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{12}^L > a_{22}^L$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{12}^L = a_{22}^L$, then go to Step 5.

Step 5: Check whether $a_{13}^L < a_{23}^L$ or $a_{13}^L > a_{23}^L$ or $a_{13}^L = a_{23}^L$.

Case (a) If $a_{13}^L < a_{23}^L$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{13}^L > a_{23}^L$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{13}^L = a_{23}^L$, then go to Step 6.

Step 6: Check whether $a_{13}^U < a_{23}^U$ or $a_{13}^U > a_{23}^U$ or $a_{13}^U = a_{23}^U$.

Case (a) If $a_{13}^U < a_{23}^U$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{13}^U > a_{23}^U$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{13}^U = a_{23}^U$, then go to Step 7.

Step 7: Check whether $a_{14}^L < a_{24}^L$ or $a_{14}^L > a_{24}^L$ or $a_{14}^L = a_{24}^L$.

Case (a) If $a_{14}^L < a_{24}^L$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{14}^L > a_{24}^L$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{14}^L = a_{24}^L$, then go to Step 8.

Step 8: Check whether $a_{14}^U < a_{24}^U$ or $a_{14}^U > a_{24}^U$ or $a_{14}^U = a_{24}^U$.

Case (a) If $a_{14}^U < a_{24}^U$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $a_{14}^U > a_{24}^U$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $a_{14}^U = a_{24}^U$, then $\tilde{A}_1 = \tilde{A}_2$.

5.1.6 Proposed Mehar method for solving interval-valued fuzzy SpPs

In this section, an efficient method (named as Mehar method) is proposed to solve an IvTrFuSpPs. The same method can also be used to solve IvTFuSpPs.

Using the proposed Mehar method, the shortest path and the corresponding shortest IvTrFuD from a node (called as source node) to any other node (called as destination node) can be obtained as follows.

Step 1: Transform the IvTrFuSpP into its equivalent CrSpP by replacing the IvTrFuD $\langle (d_{ij1}^L, d_{ij2}^L, d_{ij3}^L, d_{ij4}^L; h^L), (d_{ij1}^U, d_{ij2}^U, d_{ij3}^U, d_{ij4}^U; h^U) \rangle$ with the crisp distance d_{ij1}^U $\forall$ $(i, j) \in E$.

Step 2: Solve the transformed CrSpP by Dijkstra's algorithm [54].

Case (i): If a unique optimal path exists. Then, the total IvTrFuD corresponding to the obtained unique optimal path represents the shortest IvTrFuD from the node i to the node j.

Case (ii): If alternative optimal paths exist. Then, find all the IvTrFuNs, representing the total IvTrFuD from the node i to the node j, corresponding to all the obtained alternative optimal paths. Finally, use the extended comparing method, discussed in Section 5.1.5, to find the smallest IvTrFuN out of all the obtained IvTrFuNs. The obtained smallest IvTrFuN represents the shortest IvTrFuD from the node i to the node j as well as the corresponding path(s) represents the shortest path(s) from the node i to the node j.

5.1.7 Illustrative example

In this section, exact result of the existing IvTFuSpP (represented by Fig. 5.1 and Table 5.1) is obtained by the proposed Mehar method.

Using the proposed Mehar method, the shortest path and the corresponding shortest IvTFuD from the source node 1 to the destination node 6 for the network (represented by Fig. 5.1 and Table 5.1) can be obtained as follows.

Step 1: Using Step 1 of the proposed Mehar method, the IvTFuSpP (represented by Fig. 5.1 and Table 5.1) can be transformed into its equivalent CrSpP (represented by Fig. 5.2).

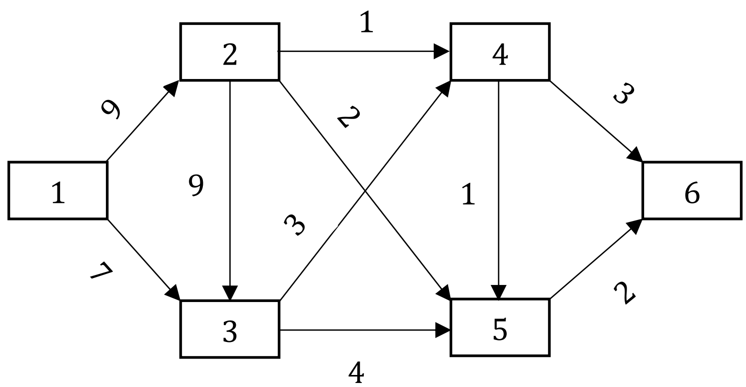

Fig. 5.2 Transformed CrSpP

Step 2: According to Step 2 of the proposed Mehar method, there is a need to apply Dijkstra algorithm [54] to find the shortest path from the source node 1 to the destination node 6 of the transformed CrSpP (represented by Fig. 5.2).

It can be easily verified that on applying Dijkstra algorithm [54] for the transformed CrSpP (represented by Fig. 5.2), the following four alternative optimal paths are obtained.

(i) $1-2-4-6$

(ii) $1-2-5-6$

(iii) $1-3-4-6$

(iv) $1-3-5-6$

Since, there are four alternative optimal paths. So, according to Case (ii) of Step 2 of the proposed Mehar method, there is a need to find the total IvTFuD corresponding to all the four alternative optimal paths.

It is obvious that

(i) The total IvTFuD corresponding to the path $1-2-4-6$ is $\langle(10,12,13;0.5),(9,12,15;1)\rangle+\langle(2,3,7;0.5),(1,3,8;1)\rangle+\langle(4,7,11;0.5),(3,7,12;1)\rangle=\langle(16,22,31;0.5),(13,22,35;1)\rangle$.

(ii) The total IvTFuD corresponding to the path $1-2-5-6$ is $\langle(10,12,13;0.5),(9,12,15;1)\rangle+\langle(3,5,6;0.5),(2,5,7;1)\rangle+\langle(3,5,7;0.5),(2,5,8;1)\rangle=\langle(16,22,26;0.5),(13,22,30;1)\rangle$.

(iii) The total IvTFuD corresponding to the path $1-3-4-6$ is $\langle(8,10,11;0.5),(7,10,14;1)\rangle+\langle(4,8,10;0.5),(3,8,12;1)\rangle+\langle(4,7,11;0.5),(3,7,12;1)\rangle=\langle(16,25,32;0.5),(13,25,38;1)\rangle$.

(iv) The total IvTFuD corresponding to the path $1-3-5-6$ is $\langle(8,10,11;0.5),(7,10,14;1)\rangle+\langle(5,7,8;0.5),(4,7,9;1)\rangle+\langle(3,5,7;0.5),(2,5,8;1)\rangle=\langle(16,22,26;0.5),(13,22,31;1)\rangle$.

Now, according to Case (ii) of Step 2 of the proposed Mehar method, there is a need to apply the existing comparing method, discussed in Section 5.1.4, to find minimum of the IvTFuNs $\tilde{A}_1=\langle(16,22,31;0.5),(13,22,35;1)\rangle$, $\tilde{A}_2=\langle(16,22,26;0.5),(13,22,30;1)\rangle$, $\tilde{A}_3=\langle(16,25,32;0.5),(13,25,38;1)\rangle$ and $\tilde{A}_4=\langle(16,22,26;0.5),(13,22,31;1)\rangle$.

Using the existing comparing method, discussed in Section 5.1.4, minimum of $\tilde{A}_1=\langle(16,22,31;0.5),(13,22,35;1)\rangle$, $\tilde{A}_2=\langle(16,22,26;0.5),(13,22,30;1)\rangle$, $\tilde{A}_3=\langle(16,25,32;0.5),(13,25,38;1)\rangle$ and $\tilde{A}_4=\langle(16,22,26;0.5),(13,22,31;1)\rangle$ can be obtained as follows.

Step 2a: Since $a_{11}^U=a_{21}^U=a_{31}^U=a_{41}^U=13$. So, according to Step 1 of the existing comparing method, discussed in Section 5.1.4, there is a need to go to Step 2.

Step 2b: Since $a_{11}^L = a_{21}^L = a_{31}^L = a_{41}^L = 16$. So, according to Step 2 of the existing comparing method, discussed in Section 5.1.4, there is a need to go to Step 3.

Step 2c: Since $a_{12} = a_{22} = a_{42} = 22$ and $a_{32} = 25$. So, according to Step 3 of the existing comparing method, discussed in Section 5.1.4, $\tilde{A}_3 = \langle(16,25,32; 0.5), (13,25,38; 1)\rangle$ does not represent the minimum fuzzy number out of the fuzzy numbers $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ and $\tilde{A}_4$. To find the minimum of the rest of the fuzzy numbers $\tilde{A}_1, \tilde{A}_2$ and $\tilde{A}_4$, there is a need to go to Step 4.

Step 2d: Since $a_{13}^L = 31$ is greater than $a_{23}^L = a_{43}^L = 26$. So, according to Step 4 of the existing comparing method, discussed in Section 5.1.4, $\tilde{A}_1 = \langle(16,22,31; 0.5), (13,22,35; 1)\rangle$ does not represent the minimum fuzzy number out of the fuzzy numbers $\tilde{A}_1, \tilde{A}_2$ and $\tilde{A}_4$. To find the minimum of the rest of the fuzzy numbers $\tilde{A}_2$ and $\tilde{A}_4$, there is a need to go to Step 5.

Step 2e: Since $a_{23}^U = 30$ is lesser than $a_{43}^L = 31$. So, according to Step 5 of the existing comparing method, discussed in Section 5.1.4, $\tilde{A}_2 = \langle(16,22,26; 0.5), (13,22,30; 1)\rangle$ represents the minimum of $\tilde{A}_2$ and $\tilde{A}_4$.

Hence, the total IvTFuD i.e., $\langle(16,22,26; 0.5), (13,22,30; 1)\rangle$ represents the shortest IvTFuD from the source node 1 to the destination node 6 as well as the corresponding path $1 - 2 - 5 - 6$ represents the shortest path from the source node 1 to the destination node 6.

5.1.8 Computational efficiency of the proposed Mehar method

As discussed in Section 5.1.2, that to apply Ebrahimnejad et al.'s method [57], at least one or at most five LpPs need to be solved to find the shortest IvTFuD from the source node to the destination node. While, to apply the proposed Mehar method, at most one LpP need to be solved (if one solve the transformed CrSpP by linear programming approach). Hence, less computational efforts are required to apply the proposed Mehar method as compared to Ebrahimnejad et al.'s method [57].

The following validates this claim.

It is obvious from Section 5.1.3 that for applying Ebrahimnejad et al.'s method [57], five LpPs need to be solved to find the shortest IvTFuD from the source node 1 to the destination node 6. While, it is obvious from Section 5.1.7 that for applying the proposed Mehar method, there is a need to solve only the LpP $(P5.1.8.1)$ (if one solve the transformed CrSpP using the linear programming approach) to find the shortest IvTFuD from the source node 1 to the destination node 6.

LpP $(\boldsymbol{P5.1.8.1})$

$Minimize\ (9x_{12} + 7x_{13} + 9x_{23} + x_{24} + 2x_{25} + 3x_{34} + 4x_{35} + x_{45} + 3x_{46} + 2x_{56})$

Subject to

Constraints of the problem IvTFuLpP $(P5.1.3.1)$.

5.2 Efficient method for solving IvPyFuSpPs

After reviewing the literature, it may be concluded that there does not exist any method except Enayattabar et al.'s method [63] for solving IvPyFuSpPs. In this section,

(i) It is shown that it is inappropriate to use Enayattabar et al.'s method [63].

(ii) It is pointed out that the reasons for the inappropriateness in Enayattabar et al.'s method [63] are the inappropriateness of existing expression [63] to evaluate sum of IvPyFuNs as well as the inappropriateness of the existing method [63] for comparing two IvPyFuNs.

(iii) Appropriate expression to evaluate sum of IvPyFuNs is proposed.

(iv) The existing method [78] for comparing two interval-valued intuitionistic fuzzy numbers is extended for comparing two IvPyFuNs.

(v) Using the proposed expression as well as the extended method, an efficient method is proposed for solving IvPyFuSpPs.

5.2.1 IvPyFuN

Let X be a universal set. Then, the set $\tilde{A} = \{\langle x, [\mu_{A^L}(x), \mu_{A^U}(x)], [v_{A^L}(x), v_{A^U}(x)]\rangle; x \in X\}$ is said to be an IvPyFuN [63] defined over the universal set X, where

(i) The values $[\mu_{A^L}(x), \mu_{A^U}(x)] \subseteq [0,1]$ and $[v_{A^L}(x), v_{A^U}(x)] \subseteq [0,1]$ are called the interval of degree of membership and the interval of degree of non-membership for $x \in \tilde{A}$ respectively.

(ii) The values $\mu_{A^U}(x)$ and $v_{A^U}(x)$ satisfy the condition $0 \leq \left(\mu_{A^U}(x)\right)^2 + \left(v_{A^U}(x)\right)^2 \leq 1$.

An IvPyFuN $\tilde{A} = \{\langle x, [\mu_{A^L}(x), \mu_{A^U}(x)], [v_{A^L}(x), v_{A^U}(x)]\rangle; x \in X\}$ is also represented as $\tilde{A} = \langle [\mu_{A^L}, \mu_{A^U}], [v_{A^L}, v_{A^U}]\rangle$.

5.2.2 Existing expression to evaluate sum of IvPyFuNs

In the existing method [63], the following expression is used to evaluate sum of IvPyFuNs.

Let $\tilde{A}_k = \left\langle x; \left[\mu_{A_k^L}, \mu_{A_k^U}\right], \left[v_{A_k^L}, v_{A_k^U}\right]\right\rangle; k = 1,2,\dots,m,$ be IvPyFuNs where $0 \leq \left(\mu_{A_k^U}\right)^2 + \left(v_{A_k^U}\right)^2 \leq 1 \ \forall\, k$. Then,

$$\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m =$$

$$\left\langle x; \left[\sqrt{1 - \prod_{k=1}^{m}\left(1 - \left(\mu_{A_k^L}\right)^2\right)}, \sqrt{1 - \prod_{k=1}^{m}\left(1 - \left(\mu_{A_k^U}\right)^2\right)}\right], \left[\prod_{k=1}^{m}\left(v_{A_k^L}\right), \prod_{k=1}^{m}\left(v_{A_k^U}\right)\right]\right\rangle \tag{5.2.2.1}$$

Remark 5.1: In the published papers [65, 74, 84, 98, 108, 109, 142, 173, 178], the expression (5.2.2.1) is used to evaluate sum of IvPyFuNs.

5.2.3 Existing method for comparing two IvPyFuNs

Enayattabar et al. [63] used the following method for comparing IvPyFuNs $\tilde{A}_1 = \left\langle x; \left[\mu_{A_1^L}, \mu_{A_1^U}\right], \left[v_{A_1^L}, v_{A_1^U}\right]\right\rangle$ and $\tilde{A}_2 = \left\langle x; \left[\mu_{A_2^L}, \mu_{A_2^U}\right], \left[v_{A_2^L}, v_{A_2^U}\right]\right\rangle$.

(i) If $M(\tilde{A}_1) < M(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

(ii) If $M(\tilde{A}_1) > M(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

(iii) If $M(\tilde{A}_1) = M(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

where,

$$M(\tilde{A}_k) = \frac{1}{2}\left(\left(\left(\mu_{A_k^L}\right)^2 - \left(v_{A_k^L}\right)^2\right)\left(1 + \sqrt{1 - \left(\mu_{A_k^L}\right)^2 - \left(v_{A_k^L}\right)^2}\right) + \left(\left(\mu_{A_k^U}\right)^2 - \left(v_{A_k^U}\right)^2\right)\left(1 + \sqrt{1 - \left(\mu_{A_k^U}\right)^2 - \left(v_{A_k^U}\right)^2}\right)\right); k = 1,2. \quad (5.2.3.1)$$

5.2.4 Enayattabar et al.'s method

Enayattabar et al. [63] extended the existing crisp Dijkstra algorithm [54] to interval-valued Pythagorean fuzzy Dijkstra algorithm by considering the following assumptions:

(i) $[\tilde{u}_j, i] = [\tilde{u}_i + \tilde{l}_{ij}, i]$ represent the label for the node j which is directly connected to the node i.

(ii) $[\tilde{u}_1, -] = [\langle 0; [0,0], [1,1]\rangle, -]$ represent the label for the source node (say, node 1), where $-$ indicates that the node 1 has no predecessor.

(iii) The IvPyFuN $\tilde{u}_i = \left\langle u; \left[\mu_{u_i^L}, \mu_{u_i^U}\right], \left[v_{u_i^L}, v_{u_i^U}\right]\right\rangle$ represent the shortest IvPyFuD from the source node (say, node 1) to the node i.

(iv) The IvPyFuN $\tilde{l}_{ij} = \left\langle l; \left[\mu_{l_{ij}^L}, \mu_{l_{ij}^U}\right], \left[v_{l_{ij}^L}, v_{l_{ij}^U}\right]\right\rangle$ represent the distance from the node i to the node j.

Step 1: Assign the permanent label $[\tilde{u}_1, -] = [0; \langle[0,0], [1,1]\rangle, -]$ for the source node (say, node 1). Set $i = 1$ and go to Step 2.

Step 2: Using the expression (5.2.2.1), compute the temporary label $\left[\tilde{u}_j, i\right] = \left[\tilde{u}_i + \tilde{l}_{ij}, i\right]$ for each node j which is directly connected to the node i, provided the node j is not permanently labelled.

If the node j is already labelled with $\left[\tilde{u}_j, k\right]$ through another node k and if $M\left(\tilde{u}_i + \tilde{l}_{ij}\right) < M\left(\tilde{u}_j\right)$, then replace the label $\left[\tilde{u}_j, k\right]$ with the label $\left[\tilde{u}_i + \tilde{l}_{ij}, i\right]$.

Step 3: If there is only one node (say, node r) having temporary label. Then, change the computed temporary label $[\tilde{u}_r, i]$ of the node r to the permanent label $[\tilde{u}_r, i]$. Otherwise, find $minimum\ \left(\tilde{u}_j\right)$ i.e., $minimum\ \left(\tilde{u}_i + \tilde{l}_{ij}\right)$, where j represents all those temporary nodes which are directly connected to the nodes having permanent labels.

Case (i): If $minimum\ \left(\tilde{u}_j\right)$ is corresponding to a unique value of j (say, r) then change the computed temporary label $[\tilde{u}_r, i]$ of the node r to the permanent label $[\tilde{u}_r, i]$ and go to Step 4.

Case (ii): If $minimum\ \left(\tilde{u}_j\right)$ is corresponding to more than one values of j then proceed further by considering any one from the obtained values of j (say, r). Also, change the computed temporary label $[\tilde{u}_r, i]$ of the node r to the permanent label $[\tilde{u}_r, i]$ and go to Step 4.

Step 4: Check that all the nodes have permanent labels or not.

Case (i): If yes then, the IvPyFuN $\tilde{u}_r$ represent the shortest IvPyFuD from the source node 1 to the destination node r. Use the existing backtracking process [54] to find the corresponding shortest path.

Case (ii): If no then, set $i = r$ and go to Step 2.

5.2.5 Inappropriateness of Enayattabar et al.'s method

The following clearly indicates that it is inappropriate to use Enayattabar et al.'s method [63] in its present form.

(i) In Step 2 of Enayattabar et al.'s method [63], the expression (5.2.2.1) is used to evaluate sum of IvPyFuNs $\tilde{u}_i$ and $\tilde{l}_{ij}$ i.e., to evaluate $\tilde{u}_i + \tilde{l}_{ij}$.

However, it is inappropriate to use the expression (5.2.2.1) due to the following reasons.

(a) If $v_{A_k^L} = 0$ for any k, then the lower value of the interval of degree of non-membership i.e., $\prod_{k=1}^{m}\left(v_{A_k^L}\right)$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 0. This indicates that the lower value of the interval of degree of non-membership i.e., $\prod_{k=1}^{m}\left(v_{A_k^L}\right)$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ is independent from the lower values of the interval of degree of non-membership of remaining IvPyFuNs.

(b) If $v_{A_k^U} = 0$ for any k, then the upper value of the interval of degree of non-membership i.e., $\prod_{k=1}^{m}\left(v_{A_k^U}\right)$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 0. This indicates that the upper value of the interval of degree of non-membership i.e., $\prod_{k=1}^{m}\left(v_{A_k^U}\right)$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ is independent from the upper values of the interval of degree of non-membership of remaining IvPyFuNs.

(ii) In Step 2 of Enayattabar et al.'s method [63], the expression (5.2.3.1) is used to check that $\tilde{u}_i + \tilde{l}_{ij}$ is less than $\tilde{u}_j$ i.e., to check that the condition $M(\tilde{u}_i + \tilde{l}_{ij}) < M(\tilde{u}_j)$ is satisfying or not.

However, the expression (5.2.3.1) is not valid as it fails to distinguish two distinct IvPyFuNs. Hence, on solving an IvPyFuSpP by Enayattabar et al.'s method [63], more than one IvPyFuNs, representing the shortest distance from the source node to the destination node, may be obtained, which is mathematically incorrect.

To validate this claim, it is shown that on solving the IvPyFuSpP (represented by Fig. 5.3 and Table 5.2) by Enayattabar et al.'s method [63], two different IvPyFuNs, representing the shortest distance from the source node 1 to the destination node 4, are obtained, which is mathematically incorrect.

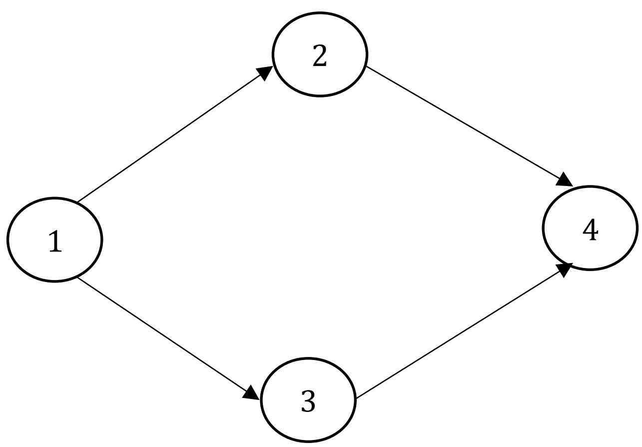

Fig. 5.3 Considered network

Table 5.2 IvPyFuD between two nodes

Node i to node j	IvPyFuD
Node 1 to node 2	$\left\langle l; \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{(17)^{\frac{1}{4}}}{3}, \frac{(7)^{\frac{1}{4}}}{2}\right]\right\rangle$
Node 1 to node 3	$\left\langle l; \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{(31)^{\frac{1}{4}}}{4}, \frac{(17)^{\frac{1}{4}}}{3}\right]\right\rangle$
Node 2 to node 4	$\left\langle l; \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{(17)^{\frac{1}{4}}}{3}, \frac{(7)^{\frac{1}{4}}}{2}\right]\right\rangle$
Node 3 to node 4	$\left\langle l; \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{(31)^{\frac{1}{4}}}{4}, \frac{(17)^{\frac{1}{4}}}{3}\right]\right\rangle$

Using Enayattabar et al.'s method [63], the shortest IvPyFuD and the corresponding shortest path from the source node 1 to the destination node 4 for the network (represented by Fig. 5.3 and Table 5.2) can be obtained as follows.

Iteration 1

Step 1: According to Step 1 of Enayattabar et al.'s method [63], there is a need to assign the permanent label $[\tilde{u}_1, -] = [\langle [0,0], [1,1] \rangle, -]$ for the source node 1.

Step 2: It is obvious from Fig. 5.3 that node 2 and node 3 are directly connected to the node 1. So, according to Step 2 of Enayattabar et al.'s method [63], there is a need to compute the temporary labels $[\tilde{u}_2, 1] = \left[\tilde{u}_1 + \tilde{l}_{12}, 1\right]$ and $[\tilde{u}_3, 1] = \left[\tilde{u}_1 + \tilde{l}_{13}, 1\right]$ for the node 2 and the node 3 respectively. All the labelled nodes (temporary and permanent) are shown in Table 5.3.

Table 5.3: All labelled nodes

Node	Label	Status
1	$[\tilde{u}_1, -] = [\langle [0,0], [1,1] \rangle, -]$	Permanent
2	$[\tilde{u}_2, 1] = [\tilde{u}_1 + \tilde{l}_{12}, 1] = \left[\langle [0,0], [1,1] \rangle + \left\langle \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{(17)^{\frac{1}{4}}}{3}, \frac{(7)^{\frac{1}{4}}}{2}\right] \right\rangle, 1\right] = \left[\left\langle \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{(17)^{\frac{1}{4}}}{3}, \frac{(7)^{\frac{1}{4}}}{2}\right] \right\rangle, 1\right]$	Temporary
3	$[\tilde{u}_3, 1] = [\tilde{u}_1 + \tilde{l}_{13}, 1] = \left[\langle [0,0], [1,1] \rangle + \left\langle \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{(31)^{\frac{1}{4}}}{4}, \frac{(17)^{\frac{1}{4}}}{3}\right] \right\rangle, 1\right] = \left[\left\langle \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{(31)^{\frac{1}{4}}}{4}, \frac{(17)^{\frac{1}{4}}}{3}\right] \right\rangle, 1\right]$	Temporary

Step 3: It is obvious from Table 5.3 that there exist two nodes having temporary label i.e., node 2 and node 3 which are directly connected to the node 1. Also, as $minimum\,\{M(\tilde{u}_2), M(\tilde{u}_3)\} =$
$minimum\left\{M\left\langle \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{(17)^{\frac{1}{4}}}{3}, \frac{(7)^{\frac{1}{4}}}{2}\right] \right\rangle, M\left\langle \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{(31)^{\frac{1}{4}}}{4}, \frac{(17)^{\frac{1}{4}}}{3}\right] \right\rangle\right\} =$
$minimum\,\{-0.55, -0.54\} = -0.55 = M(\tilde{u}_2)$. Therefore, according to Step 3 of Enayattabar et al.'s method [63], $r = 2$. Hence, the temporary label $[\tilde{u}_2, 1]$ of the node 2 will be changed to the permanent label $[\tilde{u}_2, 1]$, as shown in Table 5.4.

Table 5.4: Updated labelled nodes

Node	Label	Status
1	$[\tilde{u}_1, -] = [\langle[0,0],[1,1]\rangle, -]$	Permanent
2	$[\tilde{u}_2, 1] = \left[\left\langle\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{(17)^{\frac{1}{4}}}{3}, \frac{(7)^{\frac{1}{4}}}{2}\right]\right\rangle, 1\right]$	Permanent
3	$[\tilde{u}_3, 1] = \left[\left\langle\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{(31)^{\frac{1}{4}}}{4}, \frac{(17)^{\frac{1}{4}}}{3}\right]\right\rangle, 1\right]$	Temporary

Step 4: Since, all the nodes of Table 5.4 does not have permanent labels. So, according to Case (ii) of Step 4 of Enayattabar et al.'s method [63], there is a need to go to Step 2 of Enayattabar et al.'s method [63] by considering $i = 2$.

Iteration 2

Step 2: It is obvious from Fig. 5.3 that only node 4 is directly connected to the node 2. Therefore, according to Step 2 of Enayattabar et al.'s method [63], there is a need to compute the temporary label $[\tilde{u}_4, 2] = [\tilde{u}_2 + \tilde{l}_{24}, 2]$ for the node 4. All the labelled nodes are shown in Table 5.5.

Table 5.5: All labelled nodes

Node	Label	Status
1	$[\tilde{u}_1, -] = [\langle[0,0],[1,1]\rangle, -]$	Permanent
2	$[\tilde{u}_2, 1] = \left[\left\langle\left[\frac{1}{3},\frac{1}{2}\right],\left[\frac{(17)^{\frac{1}{4}}}{3},\frac{(7)^{\frac{1}{4}}}{2}\right]\right\rangle, 1\right]$	Permanent
3	$[\tilde{u}_3, 1] = \left[\left\langle\left[\frac{1}{4},\frac{1}{3}\right],\left[\frac{(31)^{\frac{1}{4}}}{4},\frac{(17)^{\frac{1}{4}}}{3}\right]\right\rangle, 1\right]$	Temporary
4	$[\tilde{u}_4, 2] = [\tilde{u}_2 + \tilde{l}_{24}, 2] = \left[\left\langle\left[\frac{1}{3},\frac{1}{2}\right],\left[\frac{(17)^{\frac{1}{4}}}{3},\frac{(7)^{\frac{1}{4}}}{2}\right]\right\rangle + \left\langle\left[\frac{1}{3},\frac{1}{2}\right],\left[\frac{(17)^{\frac{1}{4}}}{3},\frac{(7)^{\frac{1}{4}}}{2}\right]\right\rangle, 2\right] = \left[\left\langle\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right],\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right]\right\rangle, 2\right]$	Temporary

Step 3: It is obvious from Table 5.5 that there exist two nodes having temporary label i.e., node 3 and node 4. Also, as $minimum\,\{M(\tilde{u}_3), M(\tilde{u}_4)\} = minimum\left\{M\left\langle\left[\frac{1}{4},\frac{1}{3}\right],\left[\frac{(31)^{\frac{1}{4}}}{4},\frac{(17)^{\frac{1}{4}}}{3}\right]\right\rangle, M\left\langle\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right],\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right]\right\rangle\right\} = minimum\,\{-0.54, 0\} = -0.54 = M(\tilde{u}_3)$. Therefore, according to Step 3 of Enayattabar et al.'s method [63], $r = 3$. Hence, the temporary label $[\tilde{u}_3, 1]$ of the node 3 will be changed to the permanent label $[\tilde{u}_3, 1]$ as shown in Table 5.6.

Table 5.6: Updated labelled nodes

Node	Label	Status
1	$[\tilde{u}_1, -] = [\langle[0,0],[1,1]\rangle, -]$	Permanent
2	$[\tilde{u}_2, 1] = \left[\left\langle\left[\frac{1}{3},\frac{1}{2}\right],\left[\frac{(17)^{\frac{1}{4}}}{3},\frac{(7)^{\frac{1}{4}}}{2}\right]\right\rangle, 1\right]$	Permanent
3	$[\tilde{u}_3, 1] = \left[\left\langle\left[\frac{1}{4},\frac{1}{3}\right],\left[\frac{(31)^{\frac{1}{4}}}{4},\frac{(17)^{\frac{1}{4}}}{3}\right]\right\rangle, 1\right]$	Permanent
4	$[\tilde{u}_4, 2] = \left[\left\langle\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right],\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right]\right\rangle, 2\right]$	Temporary

Step 4: Since, all the nodes of Table 5.6 does not have permanent labels. So, according to Case (ii) of Step 4 of Enayattabar et al.'s method [63], there is a need to go to Step 2 of Enayattabar et al.'s method [63] by considering $i = 3$.

Iteration 3

Step 2: It is obvious from Fig. 5.3 that only node 4 is directly connected to the node 3. Since, node 4 is already labelled with $\left[\left\langle\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right],\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right]\right\rangle, 2\right]$ through another node 2 and $M(\tilde{u}_3 + \tilde{l}_{34}) = M(\tilde{u}_4)$ as $M\left(\left\langle\left[\frac{1}{4},\frac{1}{3}\right],\left[\frac{(31)^{\frac{1}{4}}}{4},\frac{(17)^{\frac{1}{4}}}{3}\right]\right\rangle + \left\langle\left[\frac{1}{4},\frac{1}{3}\right],\left[\frac{(31)^{\frac{1}{4}}}{4},\frac{(17)^{\frac{1}{4}}}{3}\right]\right\rangle = \left\langle\left[\frac{\sqrt{31}}{16},\frac{\sqrt{17}}{9}\right],\left[\frac{\sqrt{31}}{16},\frac{\sqrt{17}}{9}\right]\right\rangle\right) = M\left\langle\left[\frac{\sqrt{31}}{16},\frac{\sqrt{17}}{9}\right],\left[\frac{\sqrt{31}}{16},\frac{\sqrt{17}}{9}\right]\right\rangle = 0$ and $M\left\langle\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right],\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right]\right\rangle = 0$ i.e. $M\left\langle\left[\frac{\sqrt{31}}{16},\frac{\sqrt{17}}{9}\right],\left[\frac{\sqrt{31}}{16},\frac{\sqrt{17}}{9}\right]\right\rangle = M\left\langle\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right],\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right]\right\rangle$. Therefore, according to Step 2 of Enayattabar et al.'s method [63], the label of the node 4 is $\left[\left\langle\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right],\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right]\right\rangle, 2\right]$ as well as $\left[\left\langle\left[\frac{\sqrt{31}}{16},\frac{\sqrt{17}}{9}\right],\left[\frac{\sqrt{31}}{16},\frac{\sqrt{17}}{9}\right]\right\rangle,3\right]$. All the labelled nodes (temporary and permanent) are shown in Table 5.7.

Table 5.7: All labelled nodes

Node	Label	Status
1	$[\tilde{u}_1,-]=[\langle[0,0],[1,1]\rangle,-]$	Permanent
2	$[\tilde{u}_2,1]=\left[\left\langle\left[\frac{1}{3},\frac{1}{2}\right],\left[\frac{(17)^{\frac{1}{4}}}{3},\frac{(7)^{\frac{1}{4}}}{2}\right]\right\rangle,1\right]$	Permanent
3	$[\tilde{u}_3,1]=\left[\left\langle\left[\frac{1}{4},\frac{1}{3}\right],\left[\frac{(31)^{\frac{1}{4}}}{4},\frac{(17)^{\frac{1}{4}}}{3}\right]\right\rangle,1\right]$	Permanent
4	$[\tilde{u}_4,2]=\left[\left\langle\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right],\left[\frac{\sqrt{17}}{9},\frac{\sqrt{7}}{4}\right]\right\rangle,2\right]$ and $[\tilde{u}_4,3]=\left[\left\langle\left[\frac{\sqrt{31}}{16},\frac{\sqrt{17}}{9}\right],\left[\frac{\sqrt{31}}{16},\frac{\sqrt{17}}{9}\right]\right\rangle,3\right]$	Temporary

Step 3: It is obvious from Table 5.7 that there exists only one node having temporary label i.e., node 4. Therefore, according to Step 3 of Enayattabar et al.'s method [63], $r=4$. Hence, the temporary label $[\tilde{u}_4,3]$ of the node 4 will be changed to the permanent label $[\tilde{u}_4,3]$ as shown in Table 5.8.

Table 5.8: Updated labelled nodes

Node	Label	Status
1	$[\tilde{u}_1, -] = [\langle[0,0],[1,1]\rangle, -]$	Permanent
2	$[\tilde{u}_2, 1] = \left[\left\langle\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{(17)^{\frac{1}{4}}}{3}, \frac{(7)^{\frac{1}{4}}}{2}\right]\right\rangle, 1\right]$	Permanent
3	$[\tilde{u}_3, 1] = \left[\left\langle\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{(31)^{\frac{1}{4}}}{4}, \frac{(17)^{\frac{1}{4}}}{3}\right]\right\rangle, 1\right]$	Permanent
4	$[\tilde{u}_4, 2] = \left[\left\langle\left[\frac{\sqrt{17}}{9}, \frac{\sqrt{7}}{4}\right], \left[\frac{\sqrt{17}}{9}, \frac{\sqrt{7}}{4}\right]\right\rangle, 2\right]$ and $[\tilde{u}_4, 3] = \left[\left\langle\left[\frac{\sqrt{31}}{16}, \frac{\sqrt{17}}{9}\right], \left[\frac{\sqrt{31}}{16}, \frac{\sqrt{17}}{9}\right]\right\rangle, 3\right]$	Permanent

Step 4: Since, all the nodes of Table 5.8 have permanent labels. So, according to Case (i) of Step 4 of Enayattabar et al.'s method [63], both the IvPyFuNs $\left\langle\left[\frac{\sqrt{17}}{9}, \frac{\sqrt{7}}{4}\right], \left[\frac{\sqrt{17}}{9}, \frac{\sqrt{7}}{4}\right]\right\rangle$ and $\left\langle\left[\frac{\sqrt{31}}{16}, \frac{\sqrt{17}}{9}\right], \left[\frac{\sqrt{31}}{16}, \frac{\sqrt{17}}{9}\right]\right\rangle$ represents the shortest IvPyFuD from the source node 1 to the destination node 4. Also, using the existing backtracking process [54], the shortest paths from the source node 1 to the destination node 4 are $1-2-4$ and $1-3-4$ respectievly.

5.2.6 Proposed expression to evaluate sum of IvPyFuNs

In this section, a valid expression (5.2.6.1) is proposed to evaluate sum of IvPyFuNs.

Let $\tilde{A}_k = \left\langle x_k; \left[\mu_{A_k^L}, \mu_{A_k^U}\right], \left[\nu_{A_k^L}, \nu_{A_k^U}\right]\right\rangle; k = 1,2,\ldots,m,$ be IvPyFuNs where $0 \leq \left(\mu_{A_k^U}(x)\right)^2 + \left(\nu_{A_k^U}(x)\right)^2 \leq 1\ \forall\ k$ and $p_k \geq 0\ \forall\ k$. Then,

$p_1\tilde{A}_1 + p_2\tilde{A}_2 + \cdots + p_m\tilde{A}_m =$

$$\left\langle \sum_{k=1}^{m} p_k x_k\, ; \left[\sqrt{\frac{\sum_{k=1}^{m} p_k \left(\mu_{A_k^L}\right)^2}{\sum_{k=1}^{m} p_k}}, \sqrt{\frac{\sum_{k=1}^{m} p_k \left(\mu_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}}\right], \left[\sqrt{\frac{\sum_{k=1}^{m} p_k \left(v_{A_k^L}\right)^2}{\sum_{k=1}^{m} p_k}}, \sqrt{\frac{\sum_{k=1}^{m} p_k \left(v_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}}\right]\right\rangle;\ if \sum_{k=1}^{m} p_k \neq 0 \qquad (5.2.6.1)$$

The following clearly indicates that $p_1\tilde{A}_1 + p_2\tilde{A}_2 + \cdots + p_m\tilde{A}_m$ is an IvPyFuN.

(i) $0 \leq \mu_{A_k^L} \leq 1,\ p_k \geq 0$

$$\Longrightarrow 0 \leq \left(\mu_{A_k^L}\right)^2 \leq 1,\ p_k \geq 0$$

$$\Longrightarrow 0 \leq p_k \left(\mu_{A_k^L}\right)^2 \leq p_k$$

$$\Longrightarrow 0 \leq \sum_{k=1}^{m} p_k \left(\mu_{A_k^L}\right)^2 \leq \sum_{k=1}^{m} p_k$$

$$\Longrightarrow 0 \leq \frac{\sum_{k=1}^{m} p_k \left(\mu_{A_k^L}\right)^2}{\sum_{k=1}^{m} p_k} \leq 1$$

$$\Rightarrow 0 \leq \sqrt{\frac{\sum_{k=1}^{m} p_k \left(\mu_{A_k^L}\right)^2}{\sum_{k=1}^{m} p_k}} \leq 1.$$

(ii) $0 \leq \mu_{A_k^U} \leq 1,\ p_k \geq 0$

$$\Longrightarrow 0 \leq \left(\mu_{A_k^U}\right)^2 \leq 1,\ p_k \geq 0$$

$$\Longrightarrow 0 \leq p_k \left(\mu_{A_k^U}\right)^2 \leq p_k$$

$$\Longrightarrow 0 \leq \sum_{k=1}^{m} p_k \left(\mu_{A_k^U}\right)^2 \leq \sum_{k=1}^{m} p_k$$

$$\Longrightarrow 0 \leq \frac{\sum_{k=1}^{m} p_k \left(\mu_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k} \leq 1$$

$$\Rightarrow 0 \leq \sqrt{\frac{\sum_{k=1}^{m} p_k \left(\mu_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}} \leq 1.$$

(iii) $0 \leq v_{A_k^L} \leq 1,\ p_k \geq 0$

$$\Longrightarrow 0 \leq \left(v_{A_k^L}\right)^2 \leq 1,\ p_k \geq 0$$

$$\Longrightarrow 0 \leq p_k \left(v_{A_k^L}\right)^2 \leq p_k$$

$$\Longrightarrow 0 \leq \textstyle\sum_{k=1}^{m} p_k \left(v_{A_k^L}\right)^2 \leq \sum_{k=1}^{m} p_k$$

$$\Longrightarrow 0 \leq \frac{\sum_{k=1}^{m} p_k \left(v_{A_k^L}\right)^2}{\sum_{k=1}^{m} p_k} \leq 1$$

$$\Rightarrow 0 \leq \sqrt{\frac{\sum_{k=1}^{m} p_k \left(v_{A_k^L}\right)^2}{\sum_{k=1}^{m} p_k}} \leq 1.$$

(iv) $0 \leq v_{A_k^U} \leq 1,\ p_k \geq 0$

$$\Longrightarrow 0 \leq \left(v_{A_k^U}\right)^2 \leq 1,\ p_k \geq 0$$

$$\Longrightarrow 0 \leq p_k \left(v_{A_k^U}\right)^2 \leq p_k$$

$$\Longrightarrow 0 \leq \textstyle\sum_{k=1}^{m} p_k \left(v_{A_k^U}\right)^2 \leq \sum_{k=1}^{m} p_k$$

$$\Longrightarrow 0 \leq \frac{\sum_{k=1}^{m} p_k \left(v_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k} \leq 1$$

$$\Rightarrow 0 \leq \sqrt{\frac{\sum_{k=1}^{m} p_k \left(v_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}} \leq 1.$$

(v) $p_k \geq 0,\ \mu_{A_k^U} \geq 0,\ v_{A_k^U},\ \sum_{k=1}^{m} p_k \neq 0$

$$\Rightarrow \left(\sqrt{\frac{\sum_{k=1}^{m} p_k \left(\mu_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}}\right)^2 + \left(\sqrt{\frac{\sum_{k=1}^{m} p_k \left(v_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}}\right)^2 \geq 0.$$

(vi) $\left(\sqrt{\frac{\sum_{k=1}^{m} p_k\left(\mu_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}}\right)^2 + \left(\sqrt{\frac{\sum_{k=1}^{m} p_k\left(v_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}}\right)^2 = \frac{\sum_{k=1}^{m} p_k\left(\mu_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k} + \frac{\sum_{k=1}^{m} p_k\left(v_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k} =$

$\frac{\sum_{k=1}^{m} p_k\left(\left(\mu_{A_k^U}\right)^2 + \left(v_{A_k^U}\right)^2\right)}{\sum_{k=1}^{m} p_k}$.

Since, $\left(\mu_{A_k^U}\right)^2 + \left(v_{A_k^U}\right)^2 \leq 1$. So,

$$\left(\sqrt{\frac{\sum_{k=1}^{m} p_k\left(\mu_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}}\right)^2 + \left(\sqrt{\frac{\sum_{k=1}^{m} p_k\left(v_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}}\right)^2 \leq \frac{\sum_{k=1}^{m} p_k}{\sum_{k=1}^{m} p_k}.$$

Since, $\frac{\sum_{k=1}^{m} p_k}{\sum_{k=1}^{m} p_k} = 1$. So,

$$\left(\sqrt{\frac{\sum_{k=1}^{m} p_k\left(\mu_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}}\right)^2 + \left(\sqrt{\frac{\sum_{k=1}^{m} p_k\left(v_{A_k^U}\right)^2}{\sum_{k=1}^{m} p_k}}\right)^2 \leq 1.$$

5.2.7 Extended method for comparing IvPyFuNs

In this section, the existing method [78] for comparing interval-valued intuitionistic fuzzy numbers is extended for comparing IvPyFuNs.

Let $\tilde{A}_1 = \left\langle x_1; \left[\mu_{A_1^L}, \mu_{A_1^U}\right], \left[v_{A_1^L}, v_{A_1^U}\right]\right\rangle$ and $\tilde{A}_2 = \left\langle x_2; \left[\mu_{A_2^L}, \mu_{A_2^U}\right], \left[v_{A_2^L}, v_{A_2^U}\right]\right\rangle$ be two IvPyFuNs. Then, the IvPyFuNs $\tilde{A}_1$ and $\tilde{A}_2$ can be compared as follows.

Step 1: Check whether $x_1 < x_2, x_1 > x_2$ or $x_1 = x_2$.

Case (a) If $x_1 < x_2$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $x_1 > x_2$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $x_1 = x_2$, then go to Step 2.

Step 2: Check whether $S(\tilde{A}_1) < S(\tilde{A}_2)$ or $S(\tilde{A}_1) > S(\tilde{A}_2)$ or $S(\tilde{A}_1) = S(\tilde{A}_2)$, where

$$S(\tilde{A}_k) = \frac{1}{2}\left(\left(\mu_{A_k^L}\right)^2 + \left(\mu_{A_k^U}\right)^2 - \left(v_{A_k^L}\right)^2 - \left(v_{A_k^U}\right)^2\right); k = 1,2 \tag{5.2.7.1}$$

Case (a) If $S(\tilde{A}_1) < S(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $S(\tilde{A}_1) > S(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $S(\tilde{A}_1) = S(\tilde{A}_2)$, then go to Step 3.

Step 3: Check whether $E_1(\tilde{A}_1) < E_1(\tilde{A}_2)$ or $E_1(\tilde{A}_1) > E_1(\tilde{A}_2)$ or $E_1(\tilde{A}_1) = E_1(\tilde{A}_2)$, where

$$E_1(\tilde{A}_k) = \frac{1}{2}\left(\left(\mu_{A_k^L}\right)^2 + \left(\mu_{A_k^U}\right)^2 + \left(v_{A_k^L}\right)^2 + \left(v_{A_k^U}\right)^2\right); k = 1,2 \tag{5.2.7.2}$$

Case (a) If $E_1(\tilde{A}_1) < E_1(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $E_1(\tilde{A}_1) > E_1(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $E_1(\tilde{A}_1) = E_1(\tilde{A}_2)$, then go to Step 4.

Step 4: Check whether $E_2(\tilde{A}_1) < E_2(\tilde{A}_2)$ or $E_2(\tilde{A}_1) > E_2(\tilde{A}_2)$ or $E_2(\tilde{A}_1) = E_2(\tilde{A}_2)$, where

$$E_2(\tilde{A}_k) = \frac{1}{2}\left(\left(\mu_{A_k^U}\right)^2 - \left(\mu_{A_k^L}\right)^2 + \left(v_{A_k^U}\right)^2 - \left(v_{A_k^L}\right)^2\right); k = 1,2 \tag{5.2.7.3}$$

Case (a) If $E_2(\tilde{A}_1) < E_2(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $E_2(\tilde{A}_1) > E_2(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $E_2(\tilde{A}_1) = E_2(\tilde{A}_2)$, then go to Step 5.

Step 5: Check whether $E_3(\tilde{A}_1) < E_3(\tilde{A}_2)$ or $E_3(\tilde{A}_1) > E_3(\tilde{A}_2)$ or $E_3(\tilde{A}_1) = E_3(\tilde{A}_2)$, where

$$E_3(\tilde{A}_k) = \left(\mu_{A_k^U}\right)^2 - \left(\mu_{A_k^L}\right)^2; k = 1,2 \tag{5.2.7.4}$$

Case (a) If $E_3(\tilde{A}_1) < E_3(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $E_3(\tilde{A}_1) > E_3(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $E_3(\tilde{A}_1) = E_3(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

5.2.8 Proposed Mehar method for solving IvPyFuSpPs

In this section, using the proposed expression (5.2.6.1) to evaluate sum of IvPyFuNs and the extended method for comparing IvPyFuNs, discussed in Section 5.2.7, a new method (named as Mehar method) is proposed to find a shortest path from a node (called as source node) to any other node (called as destination node) as well as the corresponding shortest IvPyFuD of an IvPyFuSpP.

The steps of the proposed Mehar method are as follows.

Step 1: Transform the IvPyFuSpP into its equivalent CrSpP by replacing the IvPyFuD $\left\langle l_{ij}; \left[\mu_{l_{ij}^L}, \mu_{l_{ij}^U}\right], \left[\nu_{l_{ij}^L}, \nu_{l_{ij}^U}\right]\right\rangle$ with the crisp distance l_{ij} $\forall$ $(i,j) \in E$.

Step 2: Solve the transformed CrSpP by Dijkstra's algorithm [54].

Case (i): If a unique optimal path exists. Then, the total IvPyFuD corresponding to the obtained unique optimal path represents the shortest IvPyFuD from the node i to the node j.

Case (ii): If alternative optimal paths exist. Then, find all the IvPyFuNs, representing the total IvPyFuD from the node i to the node j, corresponding to all the obtained alternative optimal paths. Finally, use the extended comparing method, discussed in Section 5.2.7, to find the smallest IvPyFuN out of all the obtained IvPyFuNs. The obtained smallest IvPyFuN represents the shortest IvPyFuD from the node i to the node j as well as the corresponding path(s) represents the shortest path(s) from the node i to the node j.

5.2.9 Exact result of an existing IvPyFuSpP

Enayattabar et al. [63] applied their proposed method for the network (represented by Fig. 5.1 and Table 5.9) to find the shortest path and the corresponding shortest IvPyFuD from the source node 1 to the destination node 6. However, as discussed in Section 5.2.5, Enayattabar et al. [63] have considered some mathematical incorrect results in their proposed method. So, the obtained shortest path and the corresponding shortest IvPyFuD are not correct.

In this section, a correct shortest path from the source node 1 to the destination node 6 and the corresponding shortest IvPyFuD are obtained by the proposed Mehar method.

Table 5.9 [63] Existing IvPyFuD between two nodes

Node i to node j	IvPyFuD
Node 1 to node 2	$\langle 5; [0.4,0.5], [0.3,0.4] \rangle$
Node 1 to node 3	$\langle 5; [0.6,0.7], [0.2,0.3] \rangle$
Node 2 to node 3	$\langle 5; [0.3,0.6], [0.3,0.4] \rangle$
Node 2 to node 4	$\langle 5; [0.7,0.8], [0.1,0.2] \rangle$
Node 2 to node 5	$\langle 5; [0.6,0.7], [0.2,0.3] \rangle$
Node 3 to node 4	$\langle 5; [0.4,0.6], [0.2,0.4] \rangle$
Node 3 to node 5	$\langle 5; [0.7,0.8], [0.3,0.5] \rangle$
Node 4 to node 5	$\langle 5; [0.5,0.6], [0.1,0.3] \rangle$
Node 4 to node 6	$\langle 5; [0.4,0.7], [0.1,0.2] \rangle$
Node 5 to node 6	$\langle 5; [0.3,0.4], [0.1,0.2] \rangle$

Using the proposed Mehar method, the shortest path and the corresponding shortest IvPyFuD from the source node 1 to the destination node 6 for the network (represented by Fig. 5.1 and Table 5.9) can be obtained as follows.

Step 1: Using Step 1 of the proposed Mehar method, the IvPyFuSpP (represented by Fig. 5.1 and Table 5.9) can be transformed into its equivalent CrSpP (represented by Fig. 5.4).

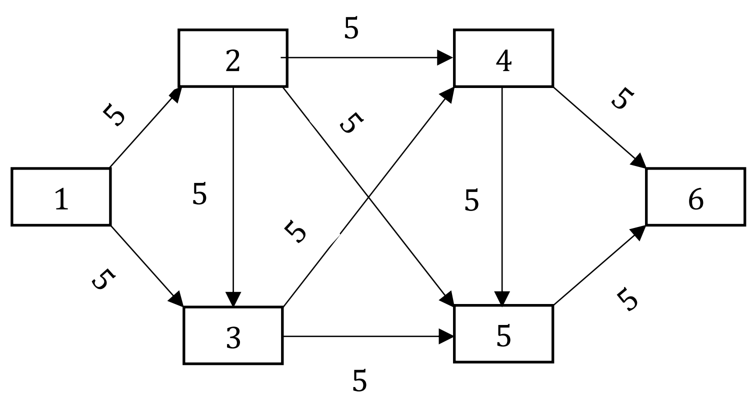

Fig. 5.4 Transformed CrSpP

Step 2: According to Step 2 of the proposed Mehar method, there is a need to apply Dijkstra algorithm [54] to find the shortest path from the source node 1 to the destination node 6 of the transformed CrSpP (represented by Fig. 5.4).

It can be easily verified that on applying Dijkstra algorithm [54] for the transformed CrSpP (represented by Fig. 5.4), the following four alternative optimal paths are obtained.

(i) $1-2-4-6$

(ii) $1-2-5-6$

(iii) $1-3-4-6$

(iv) $1-3-5-6$

Since, there are four alternative optimal paths. So, according to Case (ii) of Step 2 of the proposed Mehar method, there is a need to find the total IvPyFuD corresponding to all the four alternative optimal paths.

It is obvious that

(i) The total IvPyFuD corresponding to the path $1-2-4-6$ is $\langle 5;[0.4,0.5],[0.3,0.4]\rangle + \langle 5;[0.7,0.8],[0.1,0.2]\rangle + \langle 5;[0.4,0.7],[0.1,0.2]\rangle =$

$$\left\langle 15;\left[\sqrt{\frac{(0.4)^2+(0.7)^2+(0.4)^2}{3}},\sqrt{\frac{(0.5)^2+(0.8)^2+(0.7)^2}{3}}\right],\left[\sqrt{\frac{(0.3)^2+(0.1)^2+(0.1)^2}{3}},\sqrt{\frac{(0.4)^2+(0.2)^2+(0.2)^2}{3}}\right]\right\rangle =$$

$$\left\langle 15;\left[\sqrt{\frac{0.81}{3}},\sqrt{\frac{1.38}{3}}\right],\left[\sqrt{\frac{0.11}{3}},\sqrt{\frac{0.24}{3}}\right]\right\rangle.$$

(ii) The total IvPyFuD corresponding to the path $1-2-5-6$ is $\langle 5;[0.4,0.5],[0.3,0.4]\rangle+\langle 5;[0.6,0.7],[0.2,0.3]\rangle+\langle 5;[0.3,0.4],[0.1,0.2]\rangle=$

$$\left\langle 15;\left[\sqrt{\frac{(0.4)^2+(0.6)^2+(0.3)^2}{3}},\sqrt{\frac{(0.5)^2+(0.7)^2+(0.4)^2}{3}}\right],\left[\sqrt{\frac{(0.3)^2+(0.2)^2+(0.1)^2}{3}},\sqrt{\frac{(0.4)^2+(0.3)^2+(0.2)^2}{3}}\right]\right\rangle =$$

$$\left\langle 15;\left[\sqrt{\frac{0.61}{3}},\sqrt{\frac{0.9}{3}}\right],\left[\sqrt{\frac{0.14}{3}},\sqrt{\frac{0.29}{3}}\right]\right\rangle.$$

(iii) The total IvPyFuD corresponding to the path $1-3-4-6$ is $\langle 5;[0.6,0.7],[0.2,0.3]\rangle+\langle 5;[0.4,0.6],[0.2,0.4]\rangle+\langle 5;[0.4,0.7],[0.1,0.2]\rangle=$

$$\left\langle 15;\left[\sqrt{\frac{(0.6)^2+(0.4)^2+(0.4)^2}{3}},\sqrt{\frac{(0.7)^2+(0.6)^2+(0.7)^2}{3}}\right],\left[\sqrt{\frac{(0.2)^2+(0.2)^2+(0.1)^2}{3}},\sqrt{\frac{(0.3)^2+(0.4)^2+(0.2)^2}{3}}\right]\right\rangle =$$

$$\left\langle 15;\left[\sqrt{\frac{0.68}{3}},\sqrt{\frac{1.34}{3}}\right],\left[\sqrt{\frac{0.09}{3}},\sqrt{\frac{0.29}{3}}\right]\right\rangle.$$

(iv) The total IvPyFuD corresponding to the path $1-3-5-6$ is $\langle 5;[0.6,0.7],[0.2,0.3]\rangle+\langle 5;[0.7,0.8],[0.3,0.5]\rangle+\langle 5;[0.3,0.4],[0.1,0.2]\rangle=$

$$\left\langle 15;\left[\sqrt{\frac{(0.6)^2+(0.7)^2+(0.3)^2}{3}},\sqrt{\frac{(0.7)^2+(0.8)^2+(0.4)^2}{3}}\right],\left[\sqrt{\frac{(0.2)^2+(0.3)^2+(0.1)^2}{3}},\sqrt{\frac{(0.3)^2+(0.5)^2+(0.2)^2}{3}}\right]\right\rangle =$$

$$\left\langle 15;\left[\sqrt{\frac{0.94}{3}},\sqrt{\frac{1.29}{3}}\right],\left[\sqrt{\frac{0.14}{3}},\sqrt{\frac{0.38}{3}}\right]\right\rangle.$$

Now, according to Case (ii) of Step 2 of the proposed Mehar method, there is a need to apply the extended comparing method, discussed in Section 5.2.7, to find minimum of the IvPyFuNs $\tilde{A}_1 = \left\langle 15; \left[\sqrt{\frac{0.81}{3}}, \sqrt{\frac{1.38}{3}}\right], \left[\sqrt{\frac{0.11}{3}}, \sqrt{\frac{0.24}{3}}\right]\right\rangle$, $\tilde{A}_2 = \left\langle 15; \left[\sqrt{\frac{0.61}{3}}, \sqrt{\frac{0.9}{3}}\right], \left[\sqrt{\frac{0.14}{3}}, \sqrt{\frac{0.29}{3}}\right]\right\rangle$, $\tilde{A}_3 = \left\langle 15; \left[\sqrt{\frac{0.68}{3}}, \sqrt{\frac{1.34}{3}}\right], \left[\sqrt{\frac{0.09}{3}}, \sqrt{\frac{0.29}{3}}\right]\right\rangle$ and $\tilde{A}_4 = \left\langle 15; \left[\sqrt{\frac{0.94}{3}}, \sqrt{\frac{1.29}{3}}\right], \left[\sqrt{\frac{0.14}{3}}, \sqrt{\frac{0.38}{3}}\right]\right\rangle$.

Using the extended comparing method, discussed in Section 5.2.7, minimum of $\tilde{A}_1 = \left\langle 15; \left[\sqrt{\frac{0.81}{3}}, \sqrt{\frac{1.38}{3}}\right], \left[\sqrt{\frac{0.11}{3}}, \sqrt{\frac{0.24}{3}}\right]\right\rangle$, $\tilde{A}_2 = \left\langle 15; \left[\sqrt{\frac{0.61}{3}}, \sqrt{\frac{0.9}{3}}\right], \left[\sqrt{\frac{0.14}{3}}, \sqrt{\frac{0.29}{3}}\right]\right\rangle$, $\tilde{A}_3 = \left\langle 15; \left[\sqrt{\frac{0.68}{3}}, \sqrt{\frac{1.34}{3}}\right], \left[\sqrt{\frac{0.09}{3}}, \sqrt{\frac{0.29}{3}}\right]\right\rangle$ and $\tilde{A}_4 = \left\langle 15; \left[\sqrt{\frac{0.94}{3}}, \sqrt{\frac{1.29}{3}}\right], \left[\sqrt{\frac{0.14}{3}}, \sqrt{\frac{0.38}{3}}\right]\right\rangle$ can be obtained as follows.

Step 2a: Since $x_1 = x_2 = x_3 = x_4 = 15$. So, according to Step 1 of the extended comparing method, discussed in Section 5.2.7, there is a need to go to Step 2.

Step 2b: Since $S(\tilde{A}_k) = \frac{1}{2}\left(\left(\mu_{A_k^L}\right)^2 + \left(\mu_{A_k^U}\right)^2 - \left(v_{A_k^L}\right)^2 - \left(v_{A_k^U}\right)^2\right)$ i.e., $S(\tilde{A}_1) = 0.307, S(\tilde{A}_2) = 0.18, S(\tilde{A}_3) = 0.273, S(\tilde{A}_4) = 0.458$. So, according to Step 2 of the extended comparing method, discussed in Section 5.2.7, $\tilde{A}_2 = \left\langle 15; \left[\sqrt{\frac{0.61}{3}}, \sqrt{\frac{0.9}{3}}\right], \left[\sqrt{\frac{0.14}{3}}, \sqrt{\frac{0.29}{3}}\right]\right\rangle$ represents the minimum of $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ and $\tilde{A}_4$.

Hence, the total IvPyFuD, i.e., $\left\langle 15; \left[\sqrt{\frac{0.61}{3}}, \sqrt{\frac{0.9}{3}}\right], \left[\sqrt{\frac{0.14}{3}}, \sqrt{\frac{0.29}{3}}\right]\right\rangle$ or $\langle 15; [0.45,0.55], [0.22,0.31]\rangle$ represents the shortest IvPyFuD from the source node 1 to the destination node 6 as well as the corresponding path $1-2-5-6$ represents the shortest path from the source node 1 to the destination node 6.

5.3 Efficient method for solving TsFuSpPs

After reviewing the literature, it may be concluded that only the existing methods [162, 175] are proposed for solving TsFuSpPs. In this section,

(i) It is shown that it is inappropriate to use the existing methods [162, 175].

(ii) It is pointed out that the reasons for the inappropriateness in the existing methods [162, 175] are the inappropriateness of existing expression [162, 175] to evaluate sum of TsFuNs as well as the inappropriateness of the existing method [162, 175] for comparing two TsFuNs.

(iii) Appropriate expression to evaluate sum of TsFuNs is proposed.

(iv) By aggregating the existing methods [5, 115] for comparing two TsFuNs, a new method is proposed for comparing two TsFuNs.

(v) Using the proposed expression as well as the proposed method for comparing TsFuNs, an efficient method is proposed for solving TsFuSpPs.

5.3.1 Preliminaries

In this section, some basic definitions are discussed.

Definition 5.3.1.1 [171] Let X be a universal set. Then, the set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)\rangle; x \in X\}$ is said to be a q-rung ortho pair fuzzy number defined over the universal set X, where

(i) $\mu_{\tilde{A}}: X \to [0,1]$ and $v_{\tilde{A}}: X \to [0,1]$ are said to be the membership function and non-membership function respectively.

(ii) The values $\mu_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ are called the degree of membership and degree of non-membership for $x \in \tilde{A}$ respectively.

(iii) The values $\mu_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ satisfy the condition $0 \leq \left(\mu_{\tilde{A}}(x)\right)^q + \left(v_{\tilde{A}}(x)\right)^q \leq 1$ for some natural number q.

A q-rung ortho pair fuzzy number $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)\rangle; x \in X\}$ is also represented as $\tilde{A} = (\mu_{\tilde{A}}, v_{\tilde{A}})$.

Definition 5.3.1.2 [37] Let X be a universal set. Then, the set $\tilde{A} = \{\langle x, s_{\tilde{A}}(x), i_{\tilde{A}}(x), d_{\tilde{A}}(x)\rangle; x \in X\}$ is said to be a picture fuzzy number defined over the universal set X, where

(i) $s_{\tilde{A}}: X \to [0,1]$, $i_{\tilde{A}}: X \to [0,1]$ and $d_{\tilde{A}}: X \to [0,1]$ are said to be the membership function, hesitation function and non-membership function respectively.

(ii) The values $s_{\tilde{A}}(x)$, $i_{\tilde{A}}(x)$ and $d_{\tilde{A}}(x)$ are called the degree of membership, degree of hesitation and degree of non-membership for $x \in \tilde{A}$ respectively.

(iii) The values $s_{\tilde{A}}(x)$, $i_{\tilde{A}}(x)$ and $d_{\tilde{A}}(x)$ satisfy the condition $0 \leq s_{\tilde{A}}(x) + i_{\tilde{A}}(x) + d_{\tilde{A}}(x) \leq 1$.

A picture fuzzy number $\tilde{A} = \{\langle x, s_{\tilde{A}}(x), i_{\tilde{A}}(x), d_{\tilde{A}}(x)\rangle; x \in X\}$ is also represented as $\tilde{A} = (s_{\tilde{A}}, i_{\tilde{A}}, d_{\tilde{A}})$.

Definition 5.3.1.3 [115] Let X be a universal set. Then, the set $\tilde{A} = \{\langle x, s_{\tilde{A}}(x), i_{\tilde{A}}(x), d_{\tilde{A}}(x)\rangle; x \in X\}$ is said to be a spherical fuzzy number defined over the universal set X, where

(i) $s_{\tilde{A}}: X \to [0,1]$, $i_{\tilde{A}}: X \to [0,1]$ and $d_{\tilde{A}}: X \to [0,1]$ are said to be the membership function, hesitation function and non-membership function respectively.

(ii) The values $s_{\tilde{A}}(x)$, $i_{\tilde{A}}(x)$ and $d_{\tilde{A}}(x)$ are called the degree of membership, degree of hesitation and degree of non-membership for $x \in \tilde{A}$ respectively.

(iii) The values $s_{\tilde{A}}(x)$, $i_{\tilde{A}}(x)$ and $d_{\tilde{A}}(x)$ satisfy the condition $0 \leq \left(s_{\tilde{A}}(x)\right)^2 + \left(i_{\tilde{A}}(x)\right)^2 + \left(d_{\tilde{A}}(x)\right)^2 \leq 1$.

A spherical fuzzy number $\tilde{A} = \{\langle x, s_{\tilde{A}}(x), i_{\tilde{A}}(x), d_{\tilde{A}}(x)\rangle; x \in X\}$ is also represented as $\tilde{A} = (s_{\tilde{A}}, i_{\tilde{A}}, d_{\tilde{A}})$.

Definition 5.3.1.4 [115] Let X be a universal set. Then, the set $\tilde{A} = \{\langle x, s_{\tilde{A}}(x), i_{\tilde{A}}(x), d_{\tilde{A}}(x)\rangle; x \in X\}$ is said to be a TsFuN defined over the universal set X, where

(i) $s_{\tilde{A}}: X \to [0,1]$, $i_{\tilde{A}}: X \to [0,1]$ and $d_{\tilde{A}}: X \to [0,1]$ are said to be the membership function, hesitation function and non-membership function respectively.

(ii) The values $s_{\tilde{A}}(x)$, $i_{\tilde{A}}(x)$ and $d_{\tilde{A}}(x)$ are called the degree of membership, degree of hesitation and degree of non-membership for $x \in \tilde{A}$ respectively.

(iii) The values $s_{\tilde{A}}(x)$, $i_{\tilde{A}}(x)$ and $d_{\tilde{A}}(x)$ satisfy the condition $0 \leq \left(s_{\tilde{A}}(x)\right)^n + \left(i_{\tilde{A}}(x)\right)^n + \left(d_{\tilde{A}}(x)\right)^n \leq 1$ for some natural number n.

A TsFuN $\tilde{A} = \{\langle x, s_{\tilde{A}}(x), i_{\tilde{A}}(x), d_{\tilde{A}}(x)\rangle; x \in X\}$ is also represented as $\tilde{A} = (s_{\tilde{A}}, i_{\tilde{A}}, d_{\tilde{A}})$.

5.3.2 Existing expression to evaluate sum of TsFuNs

In the existing methods [162, 175], the following expression (5.3.2.1) is used to evaluate sum of TsFuNs.

Let $\tilde{A}_k = \left(x; s_{\tilde{A}_k}, i_{\tilde{A}_k}, d_{\tilde{A}_k}\right); k = 1,2,\ldots,m$, be TsFuNs and n be the smallest natural number where $0 \leq \left(s_{\tilde{A}_k}\right)^n + \left(i_{\tilde{A}_k}\right)^n + \left(d_{\tilde{A}_k}\right)^n \leq 1 \; \forall \, k$. Then,

$$\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m = \left(x; \sqrt[n]{1 - \prod_{k=1}^{m}\left(1 - s_{\tilde{A}_k}^n\right)}, \sqrt[n]{1 - \prod_{k=1}^{m}\left(1 - i_{\tilde{A}_k}^n\right)}, \prod_{k=1}^{m} d_{\tilde{A}_k}\right) \quad (5.3.2.1)$$

5.3.3 Existing method for comparing two TsFuNs

In the existing methods [162, 175], the following method is used for comparing two TsFuNs.

Let $\tilde{A}_1 = \left(x; s_{\tilde{A}_1}, i_{\tilde{A}_1}, d_{\tilde{A}_1}\right)$ and $\tilde{A}_2 = \left(x; s_{\tilde{A}_2}, i_{\tilde{A}_2}, d_{\tilde{A}_2}\right)$ be two TsFuNs and n be the smallest natural number where $0 \leq \left(s_{\tilde{A}_k}\right)^n + \left(i_{\tilde{A}_k}\right)^n + \left(d_{\tilde{A}_k}\right)^n \leq 1 \ \forall\, k$. Then,

(i) If $SC\left(\tilde{A}_1\right) < SC\left(\tilde{A}_2\right)$ then, $\tilde{A}_1 \prec \tilde{A}_2$.

(ii) If $SC\left(\tilde{A}_1\right) > SC\left(\tilde{A}_2\right)$ then, $\tilde{A}_1 \succ \tilde{A}_2$.

(iii) If $SC\left(\tilde{A}_1\right) = SC\left(\tilde{A}_2\right)$ then, $\tilde{A}_1 = \tilde{A}_2$.

where,

$$SC\left(\tilde{A}_k\right) = \frac{1}{3}\left(\left(s_{\tilde{A}_k}\right)^n\left(1 - \left(i_{\tilde{A}_k}\right)^n - \left(d_{\tilde{A}_k}\right)^n\right)\right); k = 1,2 \qquad (5.3.3.1)$$

5.3.4 Zedam et al.'s method for solving TsFuSpPs

Zedam et al. [175] proposed a method to find the shortest TsFuD and the corresponding shortest path from a node (called as source node) to any other node (called as destination node) of the considered network by considering the following assumptions:

(i) $\left[\tilde{u}_j, i\right] = \left[\tilde{u}_i + \tilde{l}_{ij}, i\right]$ represent the label for the node j which is directly connected to the node i.

(ii) $[\tilde{u}_1, -] = [(0; 0,0,1), -]$ represent the label for the source node (say, node 1), where $-$ indicates that the node 1 has no predecessor.

(iii) The TsFuN $\tilde{u}_i = \left(u_i; s_{\tilde{u}_i}, i_{\tilde{u}_i}, d_{\tilde{u}_i}\right)$ represent the shortest TsFuD from the source node (say, node 1) to the node i.

(iv) The TsFuN $\tilde{l}_{ij} = \left(l_{ij}; s_{\tilde{l}_{ij}}, i_{\tilde{l}_{ij}}, d_{\tilde{l}_{ij}}\right)$ represent the distance from the node i to the node j.

Step 1: Assign the permanent label $[\tilde{u}_1, -] = [(0; 0,0,1), -]$ for the source node (say, node 1). Set $i = 1$ and go to Step 2.

Step 2: Using the expression (5.3.2.1), compute the temporary label $\left[\tilde{u}_j, i\right] = \left[\tilde{u}_i + \tilde{l}_{ij}, i\right]$ for each node j that is directly connected to the node i, provided the node j is not permanently labelled.

If the node j is already labelled with $[\tilde{u}_j, k]$ through another node k and if $SC(\tilde{u}_i + \tilde{l}_{ij}) < SC(\tilde{u}_j)$, then replace the label $[\tilde{u}_j, k]$ with the label $[\tilde{u}_i + \tilde{l}_{ij}, i]$.

Step 3: If there is only one node (say, node r) for which temporary label is computed in Step 2. Then, change the computed temporary label $[\tilde{u}_r, i]$ of the node r to the permanent label $[\tilde{u}_r, i]$. Otherwise, find $minimum\,(\tilde{l}_{ij})$ i.e., $minimum\,\{SC(\tilde{l}_{ij})\}$, where j represents all those temporary nodes which are directly connected to the node i.

Case (i): If $minimum\,(\tilde{l}_{ij})$ is corresponding to a unique value of j (say, r) then change the computed temporary label $[\tilde{u}_r, i]$ of the node r to the permanent label $[\tilde{u}_r, i]$ and go to Step 4.

Case (ii): If $minimum\,(\tilde{l}_{ij})$ is corresponding to more than one values of j then proceed further by considering any one from the obtained values of j (say, r). Also, change the computed temporary label $[\tilde{u}_r, i]$ of the node r to the permanent label $[\tilde{u}_r, i]$ and go to Step 4.

Step 4: Check that r is the destination node or not.

Case (i): If yes then the TsFuN $\tilde{u}_r$ represent the shortest TsFuD from the source node 1 to the destination node r. Use the existing backtracking process [54] to find the corresponding shortest path.

Case (ii): If no then set $i = r$ and go to Step 2.

Remark 5.2: It is pertinent to mention that Ullah et al.'s method [162] is obtained by considering the following modification in Step 4 of Zedam et al.'s method [175].

Check that all the nodes have permanent labels or not.

Case (i): If yes then the TsFuN $\tilde{u}_r$ represent the shortest TsFuD from the source node 1 to the destination node r. Use the existing backtracking process [54] to find the corresponding shortest path.

Case (ii): If no then set $i = r$ and go to Step 2.

5.3.5 Inappropriateness of existing methods to solve TsFuSpPs

The following clearly indicates that it is inappropriate to use the existing methods [162, 175] in its present form.

(i) In Step 2 of the existing methods [162, 175], the expression (5.3.2.1) is used to evaluate sum of TsFuNs $\tilde{u}_i$ and $\tilde{l}_{ij}$ i.e., to evaluate $\tilde{u}_i + \tilde{l}_{ij}$. However, the expression (5.3.2.1) is not valid as the sum of TsFuNs, obtained by the expression (5.3.2.1), will not necessarily be a TsFuN. The following validates this claim.

Let $\tilde{A}_1 = \left(x; s_{\tilde{A}_1}, i_{\tilde{A}_1}, d_{\tilde{A}_1}\right) = (x; 1,0,0)$ and $\tilde{A}_2 = \left(x; s_{\tilde{A}_2}, i_{\tilde{A}_2}, d_{\tilde{A}_2}\right) = (x; 0,1,0)$ be two TsFuNs.

Then, using the expression (5.3.2.1), $\tilde{A}_1 + \tilde{A}_2 = \left(x; \sqrt[n]{1 - \prod_{k=1}^{2}\left(1 - s_{\tilde{A}_k}^{n}\right)}, \sqrt[n]{1 - \prod_{k=1}^{2}\left(1 - i_{\tilde{A}_k}^{n}\right)}, \prod_{k=1}^{2} d_{\tilde{A}_k}^{n}\right) = (x; 1 - (1 - 1)(1 - 0), 1 - (1 - 0)(1 - 1), 0) = (x; 1,1,0)$. It is obvious that $(x; 1,1,0)$ is not a TsFuN as for every natural number n, $(1)^n + (1)^n + (0)^n = 2$ which is greater than 1.

(ii) In Step 2 of the existing methods [162, 175], the expression (5.3.3.1) is used to check that $\tilde{u}_i + \tilde{l}_{ij}$ is less than $\tilde{u}_j$ i.e., to check that the condition $SC\left(\tilde{u}_i + \tilde{l}_{ij}\right) < SC\left(\tilde{u}_j\right)$ is satisfying or not.

However, the expression (5.3.3.1) is not valid as it fails to distinguish two distinct TsFuNs. The following validates this claim.

Let $\tilde{A}_1 = (x; 0.1,0.2,0.3)$ and $\tilde{A}_2 = (x; 0.1,0.3,0.2)$ be two TsFuNs. It is obvious that $\tilde{A}_1 \neq \tilde{A}_2$.

While, according to the expression (5.3.3.1), $\tilde{A}_1 = \tilde{A}_2$ as $SC(\tilde{A}_1) = SC(\tilde{A}_2) = \frac{1}{3}\left((0.1)(1 - 0.3 - 0.2)\right) = 0.0167$.

5.3.6 Invalidity of existing expressions to evaluate sum of TsFuNs

It is obvious from Section 5.3.5 that to resolve the first inappropriateness of the existing methods [162, 175], there is a need to use a valid expression to evaluate sum of TsFuNs. Although, in the literature [66, 112, 124], different expressions are proposed to evaluate sum of TsFuNs. But the following clearly indicates that none of the existing expressions [66, 112, 124] are valid. Hence, none of the existing expressions [66, 112, 124] can be used to resolve the first inappropriateness of the existing methods [162, 175].

(i) In the published papers [5, 35, 111, 112, 116, 118, 167, 176], the expression (5.3.6.1) is used to evaluate sum of TsFuNs $\tilde{A}_k = \left(x; s_{\tilde{A}_k}, i_{\tilde{A}_k}, d_{\tilde{A}_k}\right); k = 1,2, \dots, m$.

$$\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m = \left(x; \sqrt[n]{1 - \prod_{k=1}^{m}\left(1 - s_{\tilde{A}_k}^{n}\right)}, \prod_{k=1}^{m} i_{\tilde{A}_k}, \prod_{k=1}^{m} d_{\tilde{A}_k}\right) \quad (5.3.6.1)$$

However, the following clearly indicates that the expression (5.3.6.1) is not valid.

(a) If $s_{\tilde{A}_k} = 1$ for any k, then the degree of membership i.e., $\sqrt[n]{1 - \prod_{k=1}^{m}\left(1 - s_{\tilde{A}_k}^{n}\right)}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 1. This indicates that the degree of membership i.e., $\sqrt[n]{1 - \prod_{k=1}^{m}\left(1 - s_{\tilde{A}_k}^{n}\right)}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ is independent from the degree of membership of remaining TsFuNs.

(b) If $i_{\tilde{A}_k} = 0$ for any k, then the degree of hesitation i.e., $\prod_{k=1}^{m} i_{\tilde{A}_k}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 0. This indicates that the degree of hesitation i.e., $\prod_{k=1}^{m} i_{\tilde{A}_k}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ is independent from the degree of hesitation of remaining TsFuNs.

(c) If $d_{\tilde{A}_k} = 0$ for any k, then the degree of non-membership i.e., $\prod_{k=1}^{m} d_{\tilde{A}_k}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 0. This indicates that the degree of non-membership i.e., $\prod_{k=1}^{m} d_{\tilde{A}_k}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ is independent from the degree of non-membership of remaining TsFuNs.

(ii) In the published paper [124], the expression (5.3.6.2) is used to evaluate sum of TsFuNs $\tilde{A}_k = \left(x; s_{\tilde{A}_k}, i_{\tilde{A}_k}, d_{\tilde{A}_k}\right); k = 1,2,\dots,m.$

$$\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m = \left(x; \prod_{k=1}^{m} s_{\tilde{A}_k}, \prod_{k=1}^{m} i_{\tilde{A}_k}, \sqrt[n]{1 - \prod_{k=1}^{m}\left(1 - d_{\tilde{A}_k}^{n}\right)}\right) \tag{5.3.6.2}$$

However, the following clearly indicates that the expression (5.3.6.2) is not valid.

(a) If $s_{\tilde{A}_k} = 0$ for any k, then the degree of membership i.e., $\prod_{k=1}^{m} s_{\tilde{A}_k}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 0. This indicates that the degree of membership i.e., $\prod_{k=1}^{m} s_{\tilde{A}_k}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ is independent from the degree of membership of remaining TsFuNs.

(b) If $i_{\tilde{A}_k} = 0$ for any k, then the degree of hesitation i.e., $\prod_{k=1}^{m} i_{\tilde{A}_k}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 0. This indicates that the degree of hesitation i.e., $\prod_{k=1}^{m} i_{\tilde{A}_k}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ is independent from the degree of hesitation of remaining TsFuNs.

(c) If $d_{\tilde{A}_k} = 1$ for any k, then the degree of non-membership i.e., $\sqrt[n]{1-\prod_{k=1}^{m}\left(1-d_{\tilde{A}_k}^{n}\right)}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 1. This indicates that the degree of non-membership i.e., $\sqrt[n]{1-\prod_{k=1}^{m}\left(1-d_{\tilde{A}_k}^{n}\right)}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ is independent from the degree of non-membership of remaining TsFuNs.

(iii) In the published paper [66], the expression (5.3.6.3) is used to evaluate sum of TsFuNs $\tilde{A}_k = \left(x; s_{\tilde{A}_k}, i_{\tilde{A}_k}, d_{\tilde{A}_k}\right); k = 1,2,\dots,m$.

$\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m =$

$$\left(\begin{array}{c} x; \sqrt[n]{\prod_{k=1}^{m}\left(1-d_{\tilde{A}_k}^{n}\right)-\prod_{k=1}^{m}\left(1-s_{\tilde{A}_k}^{n}-i_{\tilde{A}_k}^{n}-d_{\tilde{A}_k}^{n}\right)-\prod_{k=1}^{m} i_{\tilde{A}_k}^{n}}, \\ \sqrt[n]{1-\prod_{k=1}^{m}\left(1-i_{\tilde{A}_k}^{n}\right)}, \sqrt[n]{1-\prod_{k=1}^{m}\left(1-d_{\tilde{A}_k}^{n}\right)} \end{array}\right) \tag{5.3.6.3}$$

However, the following clearly indicates that the expression (5.3.6.3) is not valid.

(a) If $i_{\tilde{A}_k} = 1$ for any k, then the degree of hesitation i.e., $\sqrt[n]{1-\prod_{k=1}^{m}\left(1-i_{\tilde{A}_k}^{n}\right)}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 1. This indicates that the degree of hesitation i.e., $\sqrt[n]{1-\prod_{k=1}^{m}\left(1-i_{\tilde{A}_k}^{n}\right)}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$is independent from the degree of hesitation of remaining TsFuNs.

(b) If $d_{\tilde{A}_k} = 1$ for any k, then the degree of non-membership i.e., $\sqrt[n]{1-\prod_{k=1}^{m}\left(1-d_{\tilde{A}_k}^{n}\right)}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ will be 1. This indicates that the degree of non-membership i.e., $\sqrt[n]{1-\prod_{k=1}^{m}\left(1-d_{\tilde{A}_k}^{n}\right)}$ of $\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_m$ is independent from the degree of non-membership of remaining TsFuNs.

5.3.7 Proposed expression to evaluate sum of TsFuNs

In this section, to resolve the first inappropriateness of the existing methods [162, 175], discussed in Section 5.3.5, a valid expression is proposed to evaluate sum of TsFuNs.

Let $\tilde{A}_k = \left(x_k; s_{\tilde{A}_k}, i_{\tilde{A}_k}, d_{\tilde{A}_k}\right); k = 1,2,\dots,m,$ be TsFuNs and n be the smallest natural number where $0 \le \left(s_{\tilde{A}_k}\right)^n + \left(i_{\tilde{A}_k}\right)^n + \left(d_{\tilde{A}_k}\right)^n \le 1\ \forall\ k$ and $p_k \ge 0\ \forall\ k$. Then,

$p_1\tilde{A}_1 + p_2\tilde{A}_2 + \dots + p_m\tilde{A}_m =$

$$\left(\sum_{k=1}^{m} p_k x_k\,; \left(\frac{\sum_{k=1}^{m} p_k\left(s_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}, \left(\frac{\sum_{k=1}^{m} p_k\left(i_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}, \left(\frac{\sum_{k=1}^{m} p_k\left(d_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right);\ if \sum_{k=1}^{m} p_k \ne 0 \quad (5.3.7.1)$$

The following clearly indicates that $p_1\tilde{A}_1 + p_2\tilde{A}_2 + \dots + p_m\tilde{A}_m$ is a TsFuN.

(i) $0 \le s_{\tilde{A}_k} \le 1,\ p_k \ge 0$

$\Rightarrow 0 \le \left(s_{\tilde{A}_k}\right)^n \le 1,\ p_k \ge 0$

$\Rightarrow 0 \le p_k\left(s_{\tilde{A}_k}\right)^n \le p_k$

$\Rightarrow 0 \le \sum_{k=1}^{m} p_k\left(s_{\tilde{A}_k}\right)^n \le \sum_{k=1}^{m} p_k$

$\Rightarrow 0 \le \frac{\sum_{k=1}^{m} p_k\left(s_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k} \le 1$

$\Rightarrow 0 \le \left(\frac{\sum_{k=1}^{m} p_k\left(s_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}} \le 1.$

(ii) $0 \le i_{\tilde{A}_k} \le 1,\ p_k \ge 0$

$\Rightarrow 0 \le \left(i_{\tilde{A}_k}\right)^n \le 1,\ p_k \ge 0$

$\Rightarrow 0 \le p_k\left(i_{\tilde{A}_k}\right)^n \le p_k$

$\Rightarrow 0 \le \sum_{k=1}^{m} p_k\left(i_{\tilde{A}_k}\right)^n \le \sum_{k=1}^{m} p_k$

$$\Longrightarrow 0 \le \frac{\sum_{k=1}^{m} p_k \left(i_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k} \le 1$$

$$\Rightarrow 0 \le \left(\frac{\sum_{k=1}^{m} p_k \left(i_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}} \le 1.$$

(iii) $0 \le d_{\tilde{A}_k} \le 1,\ p_k \ge 0$

$$\Longrightarrow 0 \le \left(d_{\tilde{A}_k}\right)^n \le 1,\ p_k \ge 0$$

$$\Longrightarrow 0 \le p_k\left(d_{\tilde{A}_k}\right)^n \le p_k$$

$$\Longrightarrow 0 \le \textstyle\sum_{k=1}^{m} p_k \left(d_{\tilde{A}_k}\right)^n \le \sum_{k=1}^{m} p_k$$

$$\Longrightarrow 0 \le \frac{\sum_{k=1}^{m} p_k \left(d_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k} \le 1$$

$$\Rightarrow 0 \le \left(\frac{\sum_{k=1}^{m} p_k \left(d_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}} \le 1.$$

(iv) $p_k \ge 0,\ s_{\tilde{A}_k} \ge 0,\ i_{\tilde{A}_k} \ge 0,\ d_{\tilde{A}_k} \ge 0,\ \sum_{k=1}^{m} p_k \ne 0$

$$\Rightarrow \left(\left(\frac{\sum_{k=1}^{m} p_k \left(s_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n + \left(\left(\frac{\sum_{k=1}^{m} p_k \left(i_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n + \left(\left(\frac{\sum_{k=1}^{m} p_k \left(d_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n \ge 0.$$

(v) $$\left(\left(\frac{\sum_{k=1}^{m} p_k \left(s_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n + \left(\left(\frac{\sum_{k=1}^{m} p_k \left(i_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n + \left(\left(\frac{\sum_{k=1}^{m} p_k \left(d_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n = \frac{\sum_{k=1}^{m} p_k \left(s_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k} +$$

$$\frac{\sum_{k=1}^{m} p_k \left(i_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k} + \frac{\sum_{k=1}^{m} p_k \left(d_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k} = \frac{\sum_{k=1}^{m} p_k \left(\left(s_{\tilde{A}_k}\right)^n + \left(i_{\tilde{A}_k}\right)^n + \left(d_{\tilde{A}_k}\right)^n\right)}{\sum_{k=1}^{m} p_k}.$$

Since, $\left(s_{\tilde{A}_k}\right)^n + \left(i_{\tilde{A}_k}\right)^n + \left(d_{\tilde{A}_k}\right)^n \le 1$. So,

$$\left(\left(\frac{\sum_{k=1}^{m} p_k \left(s_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n + \left(\left(\frac{\sum_{k=1}^{m} p_k \left(i_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n + \left(\left(\frac{\sum_{k=1}^{m} p_k \left(d_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n \le \frac{\sum_{k=1}^{m} p_k}{\sum_{k=1}^{m} p_k}.$$

Since, $\frac{\sum_{k=1}^{m} p_k}{\sum_{k=1}^{m} p_k} = 1$ So,

$$\left(\left(\frac{\sum_{k=1}^{m} p_k \left(s_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n + \left(\left(\frac{\sum_{k=1}^{m} p_k \left(i_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n + \left(\left(\frac{\sum_{k=1}^{m} p_k \left(d_{\tilde{A}_k}\right)^n}{\sum_{k=1}^{m} p_k}\right)^{\frac{1}{n}}\right)^n \leq 1.$$

5.3.8 Invalidity of existing methods for comparing two TsFuNs

It is obvious from Section 5.3.5 that to resolve the second inappropriateness of the existing methods [162, 175], there is a need to use a valid method for comparing two TsFuNs. Although, in the literature [5, 35, 85, 115, 118, 176], different methods are proposed to compare two TsFuNs. But the following clearly indicates that none of the existing methods [5, 35, 85, 115, 118, 176] are valid. Hence, none of the existing methods [5, 35, 85, 115, 118, 176] can be used to resolve the second inappropriateness of the existing methods [162, 175].

(i) In the published papers [66, 111, 112, 115], the following method is used to compare two TsFuNs $\tilde{A}_1 = \left(x; s_{\tilde{A}_1}, i_{\tilde{A}_1}, d_{\tilde{A}_1}\right)$ and $\tilde{A}_2 = \left(x; s_{\tilde{A}_2}, i_{\tilde{A}_2}, d_{\tilde{A}_2}\right)$.

Step 1: Check that $S(\tilde{A}_1) > S(\tilde{A}_2)$ or $S(\tilde{A}_1) < S(\tilde{A}_2)$ or $S(\tilde{A}_1) = S(\tilde{A}_2)$, where

$$S(\tilde{A}_k) = \left(s_{\tilde{A}_k}\right)^n - \left(d_{\tilde{A}_k}\right)^n; k = 1,2 \tag{5.3.8.1}$$

Case (a) If $S(\tilde{A}_1) > S(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $S(\tilde{A}_1) < S(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $S(\tilde{A}_1) = S(\tilde{A}_2)$. Then, go to Step 2.

Step 2: Check that $A(\tilde{A}_1) > A(\tilde{A}_2)$ or $A(\tilde{A}_1) < A(\tilde{A}_2)$ or $A(\tilde{A}_1) = A(\tilde{A}_2)$, where

$$A(\tilde{A}_k) = \left(s_{\tilde{A}_k}\right)^n + \left(i_{\tilde{A}_k}\right)^n + \left(d_{\tilde{A}_k}\right)^n; k = 1,2 \tag{5.3.8.2}$$

Case (a) If $A(\tilde{A}_1) > A(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $A(\tilde{A}_1) < A(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $A(\tilde{A}_1) = A(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

However, this method is not valid as it fails to distinguish two distinct TsFuNs. The following validates this claim.

Let $\tilde{A}_1 = (x; 0.1,0.3,0.1)$ and $\tilde{A}_2 = (x; 0,0.5,0)$ be two TsFuNs. It is obvious that $\tilde{A}_1 \neq \tilde{A}_2$. While, according to this method, $\tilde{A}_1 = \tilde{A}_2$ as $S(\tilde{A}_1) = S(\tilde{A}_2) = 0$ and $A(\tilde{A}_1) = A(\tilde{A}_2) = 0.5$.

(ii) In the published paper [176], the following method is used to compare two TsFuNs $\tilde{A}_1 = (x; s_{\tilde{A}_1}, i_{\tilde{A}_1}, d_{\tilde{A}_1})$ and $\tilde{A}_2 = (x; s_{\tilde{A}_2}, i_{\tilde{A}_2}, d_{\tilde{A}_2})$.

Step 1: Check that $SC(\tilde{A}_1) > SC(\tilde{A}_2)$ or $SC(\tilde{A}_1) < SC(\tilde{A}_2)$ or $SC(\tilde{A}_1) = SC(\tilde{A}_2)$,

where

$$SC(\tilde{A}_k) = (s_{\tilde{A}_k})^n - (i_{\tilde{A}_k})^n - (d_{\tilde{A}_k})^n + \left(\frac{e^{(s_{\tilde{A}_k})^n - (i_{\tilde{A}_k})^n - (d_{\tilde{A}_k})^n}}{e^{(s_{\tilde{A}_k})^n - (i_{\tilde{A}_k})^n - (d_{\tilde{A}_k})^n} + 1} - \frac{1}{2}\right)\left(1 - \left((s_{\tilde{A}_k})^n + (i_{\tilde{A}_k})^n + (d_{\tilde{A}_k})^n\right)\right); k = 1,2 \quad (5.3.8.3)$$

Case (a) If $SC(\tilde{A}_1) > SC(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $SC(\tilde{A}_1) < SC(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $SC(\tilde{A}_1) = SC(\tilde{A}_2)$. Then, go to Step 2.

Step 2: Check that $r(\tilde{A}_1) > r(\tilde{A}_2)$ or $r(\tilde{A}_1) < r(\tilde{A}_2)$ or $r(\tilde{A}_1) = r(\tilde{A}_2)$, where

$$r(\tilde{A}_k) = \sqrt[n]{1 - \left((s_{\tilde{A}_k})^n + (i_{\tilde{A}_k})^n + (d_{\tilde{A}_k})^n\right)}; k = 1,2 \quad (5.3.8.4)$$

Case (a) If $r(\tilde{A}_1) > r(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (b) If $r(\tilde{A}_1) < r(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (c) If $r(\tilde{A}_1) = r(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

However, this method is not valid as it fails to distinguish two distinct TsFuNs. The following validates this claim.

Let $\tilde{A}_1 = (x; 0.6,0.1,0.2)$ and $\tilde{A}_2 = (x; 0.6,0.2,0.1)$ be two TsFuNs. It is obvious that $\tilde{A}_1 \neq \tilde{A}_2$. While, according to this method, $\tilde{A}_1 = \tilde{A}_2$ as $SC(\tilde{A}_1) = SC(\tilde{A}_2) = 0.6 - 0.1 - 0.2 + \left(\frac{e^{0.6-0.1-0.2}}{e^{0.6-0.1-0.2}+1} - \frac{1}{2}\right)\left(1-(0.6+0.1+0.2)\right) = 0.3235$ and $r(\tilde{A}_1) = r(\tilde{A}_2) = \sqrt{1-(0.6+0.1+0.2)} = 0.3162$.

(iii) In the published papers [5, 35, 124], the following method is used to compare two TsFuNs $\tilde{A}_1 = \left(x; s_{\tilde{A}_1}, i_{\tilde{A}_1}, d_{\tilde{A}_1}\right)$ and $\tilde{A}_2 = \left(x; s_{\tilde{A}_2}, i_{\tilde{A}_2}, d_{\tilde{A}_2}\right)$.

Step 1: Check that $Sc(\tilde{A}_1) > Sc(\tilde{A}_2)$ or $Sc(\tilde{A}_1) < Sc(\tilde{A}_2)$ or $Sc(\tilde{A}_1) = Sc(\tilde{A}_2)$,

where

$$Sc(\tilde{A}_k) = \left(s_{\tilde{A}_k}\right)^n - \left(i_{\tilde{A}_k}\right)^n - \left(d_{\tilde{A}_k}\right)^n; k = 1,2 \tag{5.3.8.5}$$

Case (a) If $Sc(\tilde{A}_1) > Sc(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $Sc(\tilde{A}_1) < Sc(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $Sc(\tilde{A}_1) = Sc(\tilde{A}_2)$. Then, go to Step 2.

Step 2: Check that $Ac(\tilde{A}_1) > Ac(\tilde{A}_2)$ or $Ac(\tilde{A}_1) < Ac(\tilde{A}_2)$ or $Ac(\tilde{A}_1) = Ac(\tilde{A}_2)$,

where

$$Ac(\tilde{A}_k) = \left(s_{\tilde{A}_k}\right)^n + \left(i_{\tilde{A}_k}\right)^n + \left(d_{\tilde{A}_k}\right)^n; k = 1,2 \tag{5.3.8.6}$$

Case (a) If $Ac(\tilde{A}_1) > Ac(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $Ac(\tilde{A}_1) < Ac(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $Ac(\tilde{A}_1) = Ac(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

However, this method is not valid as it fails to distinguish two distinct TsFuNs. The following validates this claim.

Let $\tilde{A}_1 = (x; 0.8,0.2,0.1)$ and $\tilde{A}_2 = (x; 0.8,0.1,0.2)$ be two TsFuNs. It is obvious that $\tilde{A}_1 \neq \tilde{A}_2$. While, according to this method, $\tilde{A}_1 = \tilde{A}_2$ as $Sc(\tilde{A}_1) = Sc(\tilde{A}_2) = (0.8)^2 - (0.2)^2 - (0.1)^2 = 0.59$ and $Ac(\tilde{A}_1) = Ac(\tilde{A}_2) = (0.8)^2 + (0.2)^2 + (0.1)^2 = 0.69$.

(iv) In the published papers [85, 116, 167], the following method is used to compare two TsFuNs $\tilde{A}_1 = \left(x; s_{\tilde{A}_1}, i_{\tilde{A}_1}, d_{\tilde{A}_1}\right)$ and $\tilde{A}_2 = \left(x; s_{\tilde{A}_2}, i_{\tilde{A}_2}, d_{\tilde{A}_2}\right)$.

Step 1: Check that $S(\tilde{A}_1) > S(\tilde{A}_2)$ or $S(\tilde{A}_1) < S(\tilde{A}_2)$ or $S(\tilde{A}_1) = S(\tilde{A}_2)$, where

$$S(\tilde{A}_k) = \frac{1}{2}\left(1 + \left(s_{\tilde{A}_k}\right)^n - \left(i_{\tilde{A}_k}\right)^n - \left(d_{\tilde{A}_k}\right)^n\right); k = 1,2 \tag{5.3.8.7}$$

Case (a) If $S(\tilde{A}_1) > S(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $S(\tilde{A}_1) < S(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $S(\tilde{A}_1) = S(\tilde{A}_2)$. Then, go to Step 2.

Step 2: Check that $H(\tilde{A}_1) > H(\tilde{A}_2)$ or $H(\tilde{A}_1) < H(\tilde{A}_2)$ or $H(\tilde{A}_1) = H(\tilde{A}_2)$, where

$$H(\tilde{A}_k) = \left(s_{\tilde{A}_k}\right)^n + \left(i_{\tilde{A}_k}\right)^n + \left(d_{\tilde{A}_k}\right)^n; k = 1,2 \tag{5.3.8.8}$$

Case (a) If $H(\tilde{A}_1) > H(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $H(\tilde{A}_1) < H(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $H(\tilde{A}_1) = H(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

However, this method is not valid as it fails to distinguish two distinct TsFuNs. The following validates this claim.

Let $\tilde{A}_1 = (x; 0.8,0.4,0.5)$ and $\tilde{A}_2 = (x; 0.8,0.5,0.4)$ be two TsFuNs. It is obvious that $\tilde{A}_1 \neq \tilde{A}_2$. While, according to this method, $\tilde{A}_1 = \tilde{A}_2$ as $S(\tilde{A}_1) = S(\tilde{A}_2) =$

$\frac{1}{2}(1+(0.8)^3-(0.4)^3-(0.5)^3)=0.6615$ and $H(\tilde{A}_1)=H(\tilde{A}_2)=(0.8)^3+(0.4)^3+(0.5)^3=0.701$.

(v) In the published paper [118], the following method is used to compare two TsFuNs $\tilde{A}_1=(x;s_{\tilde{A}_1},i_{\tilde{A}_1},d_{\tilde{A}_1})$ and $\tilde{A}_2=(x;s_{\tilde{A}_2},i_{\tilde{A}_2},d_{\tilde{A}_2})$.

Step 1: Check that $S(\tilde{A}_1)>S(\tilde{A}_2)$ or $S(\tilde{A}_1)<S(\tilde{A}_2)$ or $S(\tilde{A}_1)=S(\tilde{A}_2)$, where

$$S(\tilde{A}_k)=(s_{\tilde{A}_k})^n-(i_{\tilde{A}_k})^n-(d_{\tilde{A}_k})^n+\left(\frac{e^{(s_{\tilde{A}_k})^n-(i_{\tilde{A}_k})^n-(d_{\tilde{A}_k})^n}}{e^{(s_{\tilde{A}_k})^n-(i_{\tilde{A}_k})^n-(d_{\tilde{A}_k})^n}+1}-\frac{1}{2}\right)\left(1-\left((s_{\tilde{A}_k})^n+(i_{\tilde{A}_k})^n+(d_{\tilde{A}_k})^n\right)\right);k=1,2 \tag{5.3.8.9}$$

Case (a) If $S(\tilde{A}_1)>S(\tilde{A}_2)$, then $\tilde{A}_1\succ\tilde{A}_2$.

Case (b) If $S(\tilde{A}_1)<S(\tilde{A}_2)$, then $\tilde{A}_1\prec\tilde{A}_2$.

Case (c) If $S(\tilde{A}_1)=S(\tilde{A}_2)$. Then, go to Step 2.

Step 2: Check that $A(\tilde{A}_1)>A(\tilde{A}_2)$ or $A(\tilde{A}_1)<A(\tilde{A}_2)$ or $A(\tilde{A}_1)=A(\tilde{A}_2)$, where

$$A(\tilde{A}_k)=(s_{\tilde{A}_k})^n+(i_{\tilde{A}_k})^n+(d_{\tilde{A}_k})^n;k=1,2 \tag{5.3.8.10}$$

Case (a) If $A(\tilde{A}_1)>A(\tilde{A}_2)$, then $\tilde{A}_1\succ\tilde{A}_2$.

Case (b) If $A(\tilde{A}_1)<A(\tilde{A}_2)$, then $\tilde{A}_1\prec\tilde{A}_2$.

Case (c) If $A(\tilde{A}_1)=A(\tilde{A}_2)$, then $\tilde{A}_1=\tilde{A}_2$.

However, this method is not valid as it fails to distinguish two distinct TsFuNs. The following validates this claim.

Let $\tilde{A}_1=(x;0.6,0.1,0.2)$ and $\tilde{A}_2=(x;0.6,0.2,0.1)$ be two TsFuNs. It is obvious that $\tilde{A}_1\neq\tilde{A}_2$. While, according to this method, $\tilde{A}_1=\tilde{A}_2$ as $S(\tilde{A}_1)=S(\tilde{A}_2)=0.6-$

$0.1-0.2+\left(\frac{e^{0.6-0.1-0.2}}{e^{0.6-0.1-0.2}+1}-\frac{1}{2}\right)\left(1-(0.6+0.1+0.2)\right)=0.3235$ and $A(\tilde{A}_1)=A(\tilde{A}_2)=0.6+0.1+0.2=0.9$.

5.3.9 Proposed method for comparing two TsFuNs

In this section, to resolve the second inappropriateness of the existing methods [162, 175], discussed in Section 5.3.5, by aggregating the existing comparing methods [5, 115], a new method is proposed for comparing two TsFuNs.

Let $\tilde{A}_1=\left(x_1; s_{\tilde{A}_1}, i_{\tilde{A}_1}, d_{\tilde{A}_1}\right)$ and $\tilde{A}_2=\left(x_2; s_{\tilde{A}_2}, i_{\tilde{A}_2}, d_{\tilde{A}_2}\right)$ be two TsFuNs and n be the smallest natural number where $0 \le \left(s_{\tilde{A}_k}\right)^n+\left(i_{\tilde{A}_k}\right)^n+\left(d_{\tilde{A}_k}\right)^n \le 1 \; \forall \, k$. Then,

Step 1: Check that $x_1 > x_2$ or $x_1 < x_2$ or $x_1 = x_2$.

Case (a) If $x_1 > x_2$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $x_1 < x_2$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $x_1 = x_2$. Then, go to Step 2.

Step 2: Check that $Sc(\tilde{A}_1) > Sc(\tilde{A}_2)$ or $Sc(\tilde{A}_1) < Sc(\tilde{A}_2)$ or $Sc(\tilde{A}_1) = Sc(\tilde{A}_2)$, where

$Sc(\tilde{A}_k)=\left(s_{\tilde{A}_k}\right)^n-\left(i_{\tilde{A}_k}\right)^n-\left(d_{\tilde{A}_k}\right)^n; k=1,2.$

Case (a) If $Sc(\tilde{A}_1) > Sc(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $Sc(\tilde{A}_1) < Sc(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $Sc(\tilde{A}_1) = Sc(\tilde{A}_2)$. Then, go to Step 3.

Step 3: Check that $S(\tilde{A}_1) > S(\tilde{A}_2)$ or $S(\tilde{A}_1) < S(\tilde{A}_2)$ or $S(\tilde{A}_1) = S(\tilde{A}_2)$, where

$S(\tilde{A}_k)=\left(s_{\tilde{A}_k}\right)^n-\left(d_{\tilde{A}_k}\right)^n; k=1,2.$

Case (a) If $S(\tilde{A}_1) > S(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $S(\tilde{A}_1) < S(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $S(\tilde{A}_1) = S(\tilde{A}_2)$. Then, go to Step 4.

Step 4: Check that $Ac(\tilde{A}_1) > Ac(\tilde{A}_2)$ or $Ac(\tilde{A}_1) < Ac(\tilde{A}_2)$ or $Ac(\tilde{A}_1) = Ac(\tilde{A}_2)$, where

$Ac(\tilde{A}_k) = (s_{\tilde{A}_k})^n + (i_{\tilde{A}_k})^n + (d_{\tilde{A}_k})^n; k = 1,2.$

Case (a) If $Ac(\tilde{A}_1) > Ac(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (b) If $Ac(\tilde{A}_1) < Ac(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (c) If $Ac(\tilde{A}_1) = Ac(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

It is obvious that $\tilde{A}_1 = \tilde{A}_2$ only if $x_1 = x_2, s^n_{\tilde{A}_1} = s^n_{\tilde{A}_2}, i^n_{\tilde{A}_1} = i^n_{\tilde{A}_2}$ and $d^n_{\tilde{A}_1} = d^n_{\tilde{A}_2}$ i.e., the proposed method will never fail to distinguish two distinct TsFuNs. Hence, the proposed method is valid.

5.3.10 Proposed Mehar method for solving TsFuSpPs

In this section, using the proposed expression (5.3.7.1) to evaluate sum of TsFuNs and the proposed method for comparing TsFuNs, discussed in Section 5.3.9, a new method (named as Mehar method) is proposed to find a shortest path from a node (called as source node) to any other node (called as destination node) as well as the corresponding shortest TsFuD of a TsFuSpP.

The steps of the proposed Mehar method are as follows.

Step 1: Find the smallest natural number n such that $\left(s_{\tilde{l}_{ij}}\right)^n + \left(i_{\tilde{l}_{ij}}\right)^n + \left(d_{\tilde{l}_{ij}}\right)^n \leq 1 \ \forall \, \tilde{l}_{ij} = \left(l_{ij}; s_{\tilde{l}_{ij}}, i_{\tilde{l}_{ij}}, d_{\tilde{l}_{ij}}\right)$.

Step 2: Transform the TsFuSpP into its equivalent CrSpP by replacing the TsFuD $\left(l_{ij}; s_{\tilde{l}_{ij}}, i_{\tilde{l}_{ij}}, d_{\tilde{l}_{ij}}\right)$ with the crisp distance $l_{ij} \ \forall \, (i,j) \in E$.

Step 3: Solve the transformed CrSpP by Dijkstra's algorithm [54].

Case (i): If a unique optimal path exists. Then, the total TsFuD corresponding to the obtained unique optimal path represents the shortest TsFuD from the node i to the node j.

Case (ii): If alternative optimal paths exist. Then, find all the TsFuNs, representing the total TsFuD from the node i to the node j, corresponding to all the obtained alternative optimal paths. Finally, use the proposed comparing method, discussed in Section 5.3.9, to find the smallest TsFuN out of all the obtained TsFuNs. The obtained smallest TsFuN represents the shortest TsFuD from the node i to the node j as well as the corresponding path(s) represents the shortest path(s) from the node i to the node j.

5.3.11 Exact result of an existing TsFuSpP

Zedam et al. [175] and Ullah et al. [162] applied their proposed method for the network (represented by Fig. 5.5 and Table 5.10) to find the shortest path and the corresponding shortest TsFuD from the source node 1 to the destination node 6. However, as discussed in Section 5.3.5 that in the existing methods [162, 175] some mathematical incorrect results are considered in their proposed methods. So, the obtained shortest path and the corresponding shortest TsFuD are not correct.

In this section, a correct shortest path from the source node 1 to the destination node 6 and the corresponding shortest TsFuD for the network (represented by Fig. 5.5 and Table 5.10) are obtained by the proposed Mehar method.

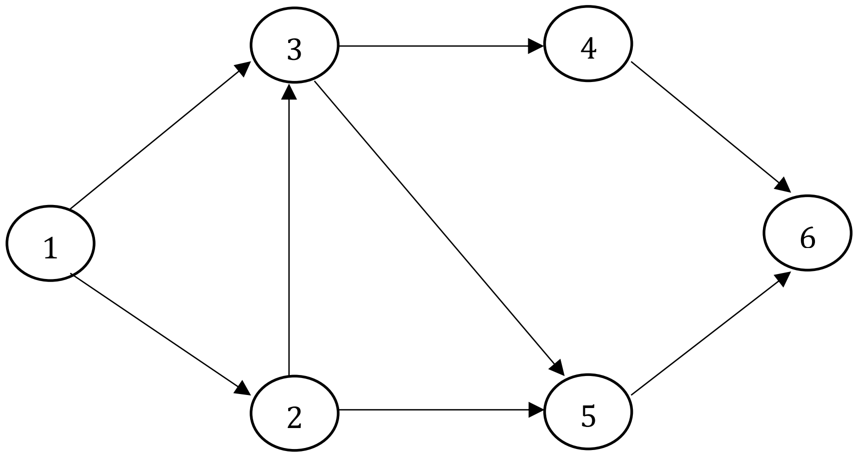

Fig. 5.5 [162, 175] TsFuSpP

Table 5.10 [162, 175] Existing TsFuD between two directed nodes

Node i to node j	TsFuD
Node 1 to node 2	(8; 0.5,0.5,0.7)
Node 1 to node 2	(8; 0.3,0.6,0.8)
Node 2 to node 3	(8; 0.8,0.4,0.8)
Node 2 to node 5	(8; 0.9,0.6,0.8)
Node 3 to node 4	(8; 0.7,0.4,0.3)
Node 3 to node 5	(8; 0.7,0.8,0.9)
Node 4 to node 6	(8; 0.5,0.4,0.8)
Node 5 to node 6	(8; 0.6,0.5,0.3)

Using the proposed Mehar method, the shortest path and the corresponding shortest TsFuD from the source node 1 to the destination node 6 for the network (represented by Fig. 5.5 and Table 5.10) can be obtained as follows.

Step 1: It can be easily verified that 6 is the smallest natural number such that $\left(s_{\tilde{l}_{ij}}\right)^6 + \left(i_{\tilde{l}_{ij}}\right)^6 + \left(d_{\tilde{l}_{ij}}\right)^6 \leq 1 \ \forall\ (i,j) \in E,$ where $E = \{(1,2),(1,3),(2,3),(2,5),(3,4),(3,5),(4,6),(5,6)\}$. Hence, according to Step 1 of the proposed Mehar method, $n = 6$.

Step 2: Using Step 2 of the proposed Mehar method, the TsFuSpP (represented by Fig. 5.5 and Table 5.10) can be transformed into its equivalent CrSpP (represented by Fig. 5.6).

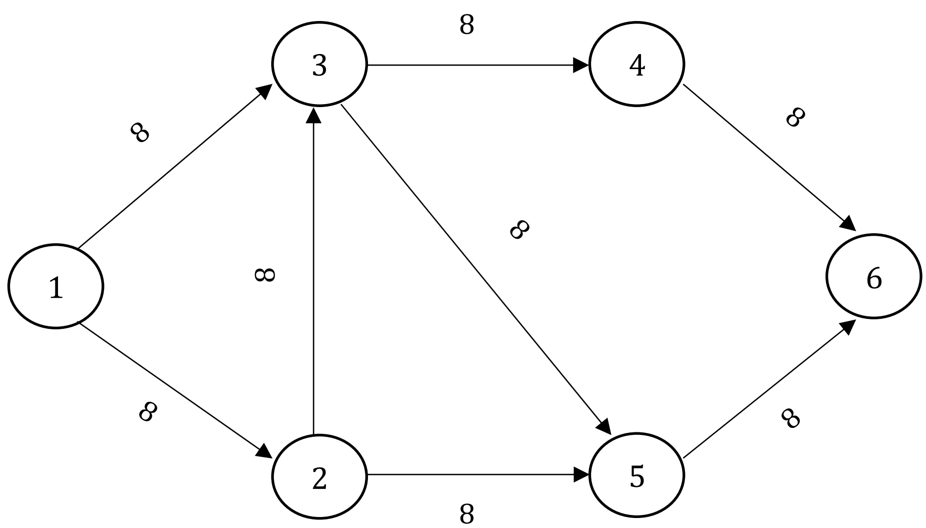

Fig. 5.6 Transformed CrSpP

Step 3: According to Step 3 of the proposed Mehar method, there is a need to apply Dijkstra algorithm [54] to find the shortest path from the source node 1 to the destination node 6 of the transformed CrSpP (represented by Fig. 5.6).

It can be easily verified that on applying Dijkstra algorithm [54] for the transformed CrSpP (represented by Fig. 5.6), the following three alternative optimal paths are obtained.

(i) $1-2-5-6$

(ii) $1-3-4-6$

(iii) $1-3-5-6$

Since, there are three alternative optimal paths. So, according to Case (ii) of Step 3 of the proposed Mehar method, there is a need to find the total TsFuD corresponding to all the three alternative optimal paths.

It is obvious that

(i) The total TsFuD corresponding to the path $1-2-5-6$ is $(8;0.5,0.5,0.7)+(8;0.9,0.6,0.8)+(8;0.6,0.5,0.3)=$

$$\left(24;\left(\frac{(0.5)^6+(0.9)^6+(0.6)^6}{3}\right)^{\frac{1}{6}},\left(\frac{(0.5)^6+(0.6)^6+(0.5)^6}{3}\right)^{\frac{1}{6}},\left(\frac{(0.7)^6+(0.8)^6+(0.3)^6}{3}\right)^{\frac{1}{6}}\right)=$$

$$\left(24;\left(\frac{0.594}{3}\right)^{\frac{1}{6}},\left(\frac{0.078}{3}\right)^{\frac{1}{6}},\left(\frac{0.38}{3}\right)^{\frac{1}{6}}\right).$$

(ii) The total TsFuD corresponding to the path $1-3-4-6$ is $(8;0.3,0.6,0.8)+(8;0.7,0.4,0.3)+(8;0.5,0.4,0.8)=$

$$\left(24;\left(\frac{(0.3)^6+(0.7)^6+(0.5)^6}{3}\right)^{\frac{1}{6}},\left(\frac{(0.6)^6+(0.4)^6+(0.4)^6}{3}\right)^{\frac{1}{6}},\left(\frac{(0.8)^6+(0.3)^6+(0.8)^6}{3}\right)^{\frac{1}{6}}\right)=$$

$$\left(24;\left(\frac{0.134}{3}\right)^{\frac{1}{6}},\left(\frac{0.055}{3}\right)^{\frac{1}{6}},\left(\frac{0.53}{3}\right)^{\frac{1}{6}}\right).$$

(iii) The total TsFuD corresponding to the path $1-3-5-6$ is $(8;0.3,0.6,0.8)+(8;0.7,0.8,0.9)+(8;0.6,0.5,0.3)=$

$$\left(24;\left(\frac{(0.3)^6+(0.7)^6+(0.6)^6}{3}\right)^{\frac{1}{6}},\left(\frac{(0.6)^6+(0.8)^6+(0.5)^6}{3}\right)^{\frac{1}{6}},\left(\frac{(0.8)^6+(0.9)^6+(0.3)^6}{3}\right)^{\frac{1}{6}}\right)=$$

$$\left(24;\left(\frac{0.165}{3}\right)^{\frac{1}{6}},\left(\frac{0.324}{3}\right)^{\frac{1}{6}},\left(\frac{0.79}{3}\right)^{\frac{1}{6}}\right).$$

Now, according to Case (ii) of Step 3 of the proposed Mehar method, there is a need to apply the proposed comparing method, discussed in Section 5.3.9, to find minimum of the TsFuNs $\tilde{A}_1=\left(24;\left(\frac{0.594}{3}\right)^{\frac{1}{6}},\left(\frac{0.078}{3}\right)^{\frac{1}{6}},\left(\frac{0.38}{3}\right)^{\frac{1}{6}}\right)$, $\tilde{A}_2=\left(24;\left(\frac{0.134}{3}\right)^{\frac{1}{6}},\left(\frac{0.055}{3}\right)^{\frac{1}{6}},\left(\frac{0.53}{3}\right)^{\frac{1}{6}}\right)$ and $\tilde{A}_3=\left(24;\left(\frac{0.165}{3}\right)^{\frac{1}{6}},\left(\frac{0.324}{3}\right)^{\frac{1}{6}},\left(\frac{0.79}{3}\right)^{\frac{1}{6}}\right)$.

Using the proposed comparing method, discussed in Section 5.3.9, minimum of $\tilde{A}_1=\left(24;\left(\frac{0.594}{3}\right)^{\frac{1}{6}},\left(\frac{0.078}{3}\right)^{\frac{1}{6}},\left(\frac{0.38}{3}\right)^{\frac{1}{6}}\right)$, $\tilde{A}_2=\left(24;\left(\frac{0.134}{3}\right)^{\frac{1}{6}},\left(\frac{0.055}{3}\right)^{\frac{1}{6}},\left(\frac{0.53}{3}\right)^{\frac{1}{6}}\right)$ and $\tilde{A}_3=\left(24;\left(\frac{0.165}{3}\right)^{\frac{1}{6}},\left(\frac{0.324}{3}\right)^{\frac{1}{6}},\left(\frac{0.79}{3}\right)^{\frac{1}{6}}\right)$ can be obtained as follows.

Step 2a: Since $x_1=x_2=x_3=24$. So, according to Step 1 of the proposed comparing method, discussed in Section 5.3.9, there is a need to go to Step 2.

Step 2b: Since $Sc(\tilde{A}_k)=\left(s_{\tilde{A}_k}\right)^n-\left(i_{\tilde{A}_k}\right)^n-\left(d_{\tilde{A}_k}\right)^n$ i.e., $Sc(\tilde{A}_1)=0.045, Sc(\tilde{A}_2)=-0.15, Sc(\tilde{A}_3)=-0.32$. So, according to Step 2 of the proposed comparing method, discussed in Section 5.3.9, $\tilde{A}_3=\left(24;\left(\frac{0.165}{3}\right)^{\frac{1}{6}},\left(\frac{0.324}{3}\right)^{\frac{1}{6}},\left(\frac{0.79}{3}\right)^{\frac{1}{6}}\right)$ represents the minimum of $\tilde{A}_1,\tilde{A}_2$ and $\tilde{A}_3$.

Hence, the total TsFuD, i.e., $\left(24;\left(\frac{0.165}{3}\right)^{\frac{1}{6}},\left(\frac{0.324}{3}\right)^{\frac{1}{6}},\left(\frac{0.79}{3}\right)^{\frac{1}{6}}\right)$ or $(24;0.62,0.69,0.8)$ represents the shortest TsFuD from the source node 1 to the destination node 6 as well as the corresponding path $1-3-5-6$ represents the shortest path from the source node 1 to the destination node 6.

5.4 Conclusions

In this chapter,

(i) It is shown that much computational efforts are required to apply the existing method [57].

(ii) It is shown that it is inappropriate to use the existing methods for solving SpPs under fuzzy environment and their extensions [63, 162, 175].

(iii) An efficient method (named as Mehar method) is proposed for solving SpPs under fuzzy environment and their extensions.

Chapter 6

Efficient Method for Solving Linear Programming Problems Under Neutrosophic Environment[1]

Khatter [100] pointed out that although several methods are proposed in the literature to solve SvNeLpPs (LpPs in which all the parameters except decision variables are represented by SvNeNs). However, all the methods for comparing SvNeNs, used in existing methods, are independent from the attitude of the decision maker towards the risk. To fill this gap, Khatter [100] proposed a method for comparing two SvNeNs by considering the attitude of the decision maker towards the risk. Then, using the proposed comparing method, Khatter [100] proposed a method for solving SvNeLpPs.

In this chapter,

(i) It is pointed out that a mathematical incorrect result is considered in Khatter's method [100].

(ii) It is pointed out that some mathematical incorrect results are considered in other existing methods for solving SvNeLpPs.

(iii) An efficient method (named as Mehar method) is proposed for solving SvNeLpPs.

(iv) Limitation of the proposed Mehar method is discussed.

6.1 Preliminaries

In this section, the basic definitions are discussed.

[1] The contents of this chapter are published in "Soft Computing 26 (2022) 8479-8495".

Definition 6.1.1 [51] Let X be a universal set. Then, the set $\tilde{A} = \{\langle x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\rangle : x \in X\}$ is said to be a SvNeN defined over the universal set X, where

(i) $T_{\tilde{A}}: X \to [0,1]$, $I_{\tilde{A}}: X \to [0,1]$ and $F_{\tilde{A}}: X \to [0,1]$ are said to be the truth membership function, indeterminacy membership function and falsity membership function respectively.

(ii) The values $T_{\tilde{A}}(x), I_{\tilde{A}}(x)$ and $F_{\tilde{A}}(x)$ are called the degree of truth membership, degree of indeterminacy membership and degree of falsity membership for $x \in \tilde{A}$ respectively.

(iii) The values $T_{\tilde{A}}(x), I_{\tilde{A}}(x)$ and $F_{\tilde{A}}(x)$ satisfy the condition $0 \leq T_{\tilde{A}}(x) + I_{\tilde{A}}(x) + F_{\tilde{A}}(x) \leq 3$.

A SvNeN $\tilde{A} = \{\langle x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\rangle : x \in X\}$ is also represented as $\tilde{A} = \left(T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\right)$.

Definition 6.1.2 [51] A single-valued neutrosophic set $\tilde{A} = (a_1, a_2, a_3; w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}})$, where $0 \leq w_{\tilde{A}} \leq 1, 0 \leq u_{\tilde{A}} \leq 1, , 0 \leq y_{\tilde{A}} \leq 1, 0 \leq w_{\tilde{A}} + u_{\tilde{A}} + y_{\tilde{A}} \leq 3$, is said to be SvTNeN if its membership functions are defined as

$$T_{\tilde{A}}(x) = \begin{cases} w_{\tilde{A}}\left(\frac{x-a_1}{a_2-a_1}\right), & a_1 \leq x < a_2, \\ w_{\tilde{A}}, & x = a_2, \\ w_{\tilde{A}}\left(\frac{a_3-x}{a_3-a_2}\right), & a_2 < x \leq a_3, \\ 0, & \text{otherwise} \end{cases}$$

$$I_{\tilde{A}}(x) = \begin{cases} \frac{a_2-x+u_{\tilde{A}}(x-a_1)}{a_2-a_1}, & a_1 \leq x < a_2, \\ u_{\tilde{A}}, & x = a_2, \\ \frac{x-a_2+u_{\tilde{A}}(a_3-x)}{a_3-a_2}, & a_2 < x \leq a_3, \\ 0, & \text{otherwise} \end{cases}$$

$$F_{\tilde{A}}(x) = \begin{cases} \frac{a_2-x+y_{\tilde{A}}(x-a_1)}{a_2-a_1}, & a_1 \leq x < a_2, \\ y_{\tilde{A}}, & x = a_2, \\ \frac{x-a_2+y_{\tilde{A}}(a_3-x)}{a_3-a_2}, & a_2 < x \leq a_3, \\ 0, & \text{otherwise} \end{cases}.$$

Definition 6.1.3 [51] A single-valued neutrosophic set $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}})$, where $0 \leq w_{\tilde{A}} \leq 1, 0 \leq u_{\tilde{A}} \leq 1, , 0 \leq y_{\tilde{A}} \leq 1, 0 \leq w_{\tilde{A}} + u_{\tilde{A}} + y_{\tilde{A}} \leq 3$, is said to be SvTrNeN if its membership functions are defined as

$$T_{\tilde{A}}(x) = \begin{cases} w_{\tilde{A}}\left(\frac{x-a_1}{a_2-a_1}\right), & a_1 \leq x < a_2, \\ w_{\tilde{A}}, & a_2 \leq x \leq a_3, \\ w_{\tilde{A}}\left(\frac{a_4-x}{a_4-a_3}\right), & a_3 < x \leq a_4, \\ 0, & \text{otherwise} \end{cases}$$

$$I_{\tilde{A}}(x) = \begin{cases} \frac{a_2-x+u_{\tilde{A}}(x-a_1)}{a_2-a_1}, & a_1 \leq x < a_2, \\ u_{\tilde{A}}, & a_2 \leq x \leq a_3, \\ \frac{x-a_3+u_{\tilde{A}}(a_4-x)}{a_4-a_3}, & a_3 < x \leq a_4, \\ 1, & \text{otherwise} \end{cases}$$

$$F_{\tilde{A}}(x) = \begin{cases} \frac{a_2-x+y_{\tilde{A}}(x-a_1)}{a_2-a_1}, & a_1 \leq x < a_2, \\ y_{\tilde{A}}, & a_2 \leq x \leq a_3, \\ \frac{x-a_3+y_{\tilde{A}}(a_4-x)}{a_4-a_3}, & a_3 < x \leq a_4, \\ 1, & \text{otherwise} \end{cases}$$

Definition 6.1.4 [51] Let $\tilde{A}_1 = \left(a_{11}, a_{12}, a_{13}; w_{\tilde{A}_1}, u_{\tilde{A}_1}, y_{\tilde{A}_1}\right)$ and $\tilde{A}_2 = \left(a_{21}, a_{22}, a_{23}; w_{\tilde{A}_2}, u_{\tilde{A}_2}, y_{\tilde{A}_2}\right)$ be two SvTNeNs, then $\tilde{A}_1 + \tilde{A}_2 = \left(a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}; minimum\{w_{\tilde{A}_1}, w_{\tilde{A}_2}\}, maximum\{u_{\tilde{A}_1}, u_{\tilde{A}_2}\}, maximum\{y_{\tilde{A}_1}, y_{\tilde{A}_2}\}\right)$.

Definition 6.1.5 [51] Let $\tilde{A}_1 = \left(a_{11}, a_{12}, a_{13}, a_{14}; w_{\tilde{A}_1}, u_{\tilde{A}_1}, y_{\tilde{A}_1}\right)$ and $\tilde{A}_2 = \left(a_{21}, a_{22}, a_{23}, a_{24}; w_{\tilde{A}_2}, u_{\tilde{A}_2}, y_{\tilde{A}_2}\right)$ be two SvTrNeNs, then $\tilde{A}_1 + \tilde{A}_2 = \left(a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}, a_{14} + a_{24}; minimum\{w_{\tilde{A}_1}, w_{\tilde{A}_2}\}, maximum\{u_{\tilde{A}_1}, u_{\tilde{A}_2}\}, maximum\{y_{\tilde{A}_1}, y_{\tilde{A}_2}\}\right)$.

Definition 6.1.6 [19] Let $\tilde{A} = (a_1, a_2, a_3; w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}})$ be a SvTNeN and k be a real number, then

$$k\tilde{A} = \begin{cases} (ka_1, ka_2, ka_3; w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}}), & \text{if } k \geq 0, \\ (ka_3, ka_2, ka_1; w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}}), & \text{if } k < 0. \end{cases}$$

Definition 6.1.7 [19] Let $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}})$ be a SvTrNeN and k be a real number, then $k\tilde{A} = \begin{cases} (ka_1, ka_2, ka_3, ka_4; w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}}), & \text{if} \quad k \geq 0, \\ (ka_4, ka_3, ka_2, ka_1; w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}}), & \text{if} \quad k < 0. \end{cases}$

Definition 6.1.8 [100] Let $\tilde{A}_1 = \left(a_{11}, a_{12}, a_{13}; w_{\tilde{A}_1}, u_{\tilde{A}_1}, y_{\tilde{A}_1}\right)$ and $\tilde{A}_2 = \left(a_{21}, a_{22}, a_{23}; w_{\tilde{A}_2}, u_{\tilde{A}_2}, y_{\tilde{A}_2}\right)$ be two SvTNeNs. Then, $\tilde{A}_1$ and $\tilde{A}_2$ can be compared as follows.

(i) If $V(\tilde{A}_1) < V(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

(ii) If $V(\tilde{A}_1) > V(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

(iii) If $V(\tilde{A}_1) = V(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

where,

(a) $$V(\tilde{A}_i) = \lambda\left(\frac{a_{i1}+4a_{i2}+a_{i3}}{6}\right)w_{\tilde{A}_i}^2 + (1-\lambda)\left(\frac{\left[2(a_{i1}+a_{i2}+a_{i3})-(a_{i1}-2a_{i2}+a_{i3})u_{\tilde{A}_i}-(a_{i1}+4a_{i2}+a_{i3})u_{\tilde{A}_i}^2\right]}{6} + \frac{\left[2(a_{i1}+a_{i2}+a_{i3})-(a_{i1}-2a_{i2}+a_{i3})y_{\tilde{A}_i}-(a_{i1}+4a_{i2}+a_{i3})y_{\tilde{A}_i}^2\right]}{6}\right), \lambda \in [0,1]; i = 1,2 \tag{6.1.1}$$

(b) λ reflects the attitude of the decision maker towards the risk.

(c) $\lambda \in [0, 0.5)$ indicates that the expert is risk taker and gives preference to uncertainty.

(d) $\lambda = 0.5$ indicates that the expert is neutral about deciding the parameters of SvTNeLpP.

(e) $\lambda \in (0.5, 1]$ indicates that the expert is risk aversive about deciding the parameters of SvTNeLpP and gives preference to certainty.

Definition 6.1.9 [100] Let $\tilde{A}_1 = \left(a_{11}, a_{12}, a_{13}, a_{14}; w_{\tilde{A}_1}, u_{\tilde{A}_1}, y_{\tilde{A}_1}\right)$ and $\tilde{A}_2 = \left(a_{21}, a_{22}, a_{23}, a_{24}; w_{\tilde{A}_2}, u_{\tilde{A}_2}, y_{\tilde{A}_2}\right)$ be two SvTrNeNs. Then, $\tilde{A}_1$ and $\tilde{A}_2$ can be compared as follows.

(i) If $V(\tilde{A}_1) < V(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

(ii) If $V(\tilde{A}_1) > V(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$.

(iii) If $V(\tilde{A}_1) = V(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

where,

$$V(\tilde{A}_i) = \lambda\left(\frac{a_{i1}+2a_{i2}+2a_{i3}+a_{i4}}{6}\right)w_{\tilde{A}_i}^2 + (1-\lambda)\left(\frac{\left[(2a_{i1}+a_{i2}+a_{i3}+2a_{i4})-(a_{i1}-a_{i2}-a_{i3}+a_{i4})u_{\tilde{A}_i}-(a_{i1}+2a_{i2}+2a_{i3}+a_{i4})u_{\tilde{A}_i}^2\right]}{6} + \frac{\left[(2a_{i1}+a_{i2}+a_{i3}+2a_{i4})-(a_{i1}-a_{i2}-a_{i3}+a_{i4})y_{\tilde{A}_i}-(a_{i1}+2a_{i2}+2a_{i3}+a_{i4})y_{\tilde{A}_i}^2\right]}{6}\right), \lambda \in [0,1]; i = 1,2 \quad (6.1.2)$$

6.2 Inappropriateness of existing methods

In this section,

(i) A mathematical incorrect result, considered in Singh et al.'s method [148] and Khatter's method [100], is pointed out. It can be easily verified that the same mathematical incorrect result is also considered in the existing methods [15, 16, 19, 22, 41, 42, 58, 60, 126, 137, 152, 156, 168].

(ii) A mathematical incorrect result, considered in Abdelfattah's method [3], is pointed out. It can be easily verified that the same mathematical incorrect result is also considered in the existing method [43].

(iii) A mathematical incorrect result, considered in Das et al.'s method [44], is pointed out. It can be easily verified that the same mathematical incorrect result is also considered in the existing methods [2, 59, 80].

(iv) A mathematical incorrect result, considered in Kar et al.'s method [89], is pointed out.

6.2.1 Inappropriateness of Singh et al.'s method

In Singh et al.'s method [148] firstly, the SvTrNeLpP $(P6.2.1.1)$ is transformed into the CrLpP $(P6.2.1.2)$. Then, the CrLpP $(P6.2.1.2)$ is transformed into the CrLpP $(P6.2.1.3)$. After that, the CrLpP $(P6.2.1.3)$ is transformed into the CrLpP $(P6.1.1.4)$. Finally, it is assumed that an optimal solution of the CrLpP $(P6.2.1.4)$ also represents an optimal solution of the SvTrNeLpP $(P6.2.1.1)$.

SvTrNeLpP $(\boldsymbol{P6.2.1.1})$

$Maximize/Minimize\left(\sum_{j=1}^{n}\left(c_{j1},c_{j2},c_{j3},c_{j4};w_{\tilde{c}_j},u_{\tilde{c}_j},y_{\tilde{c}_j}\right)x_j\right)$

Subject to

$\sum_{j=1}^{n}\left(a_{ij1},a_{ij2},a_{ij3},a_{ij4};w_{\tilde{a}_{ij}},u_{\tilde{a}_{ij}},y_{\tilde{a}_{ij}}\right)x_j\ (\preccurlyeq,\approx,\succcurlyeq)\left(b_{i1},b_{i2},b_{i3},b_{i4};w_{\tilde{b}_i},u_{\tilde{b}_i},y_{\tilde{b}_i}\right),i=1,2,\dots,m,$

$x_j\geq 0,\ j=1,2,\dots,n.$

where,

(i) m represents the number of constraints.

(ii) n represents the number of variables.

(iii) $\left(c_{j1},c_{j2},c_{j3},c_{j4};w_{\tilde{c}_j},u_{\tilde{c}_j},y_{\tilde{c}_j}\right)$ is a SvTrNeN for each $j=1,2,\dots,n$.

(iv) $\left(b_{i1},b_{i2},b_{i3},b_{i4};w_{\tilde{b}_i},u_{\tilde{b}_i},y_{\tilde{b}_i}\right)$ is a SvTrNeN for each $i=1,2,\dots,m$.

(v) $\left(a_{ij1},a_{ij2},a_{ij3},a_{ij4};w_{\tilde{a}_{ij}},u_{\tilde{a}_{ij}},y_{\tilde{a}_{ij}}\right)$ is a SvTrNeN for each $i=1,2,\dots,m;\ j=1,2,\dots,n.$

CrLpP $(\boldsymbol{P6.2.1.2})$

$Maximize/Minimize\ \left(R\left(\sum_{j=1}^{n}\left(c_{j1},c_{j2},c_{j3},c_{j4};w_{\tilde{c}_j},u_{\tilde{c}_j},y_{\tilde{c}_j}\right)x_j\right)\right)$

Subject to

$R\left(\sum_{j=1}^{n}\left(a_{ij1},a_{ij2},a_{ij3},a_{ij4};w_{\tilde{a}_{ij}},u_{\tilde{a}_{ij}},y_{\tilde{a}_{ij}}\right)x_j\right)(\leq,=,\geq)\ R\left(b_{i1},b_{i2},b_{i3},b_{i4};w_{\tilde{b}_i},u_{\tilde{b}_i},y_{\tilde{b}_i}\right),i=1,2,\dots,m,$

$x_j\geq 0,\ j=1,2,\dots,n,$

where,

(a) If the problem is of maximization [1],

$$R(\tilde{A}_i) = \left(\frac{a_{i1}+2(a_{i2}+a_{i3})+a_{i4}}{2}\right) + \left(w_{\tilde{A}_i} - u_{\tilde{A}_i} - y_{\tilde{A}_i}\right) \tag{6.2.1.1}$$

(b) If the problem is of minimization [1],

$$R(\tilde{A}_i) = \left(\frac{a_{i1}-3(a_{i2}+a_{i3})+a_{i4}}{2}\right) + \left(w_{\tilde{A}_i} - u_{\tilde{A}_i} - y_{\tilde{A}_i}\right) \tag{6.2.1.2}$$

CrLpP $(\boldsymbol{P6.2.1.3})$

$$Maximize/Minimize\left(\sum_{j=1}^{n} R\left(c_{j1}, c_{j2}, c_{j3}, c_{j4}; w_{\tilde{c}_j}, u_{\tilde{c}_j}, y_{\tilde{c}_j}\right) x_j - \sum_{j=1}^{n} w_{\tilde{c}_j x_j} + \sum_{j=1}^{n} u_{\tilde{c}_j x_j} + \sum_{j=1}^{n} y_{\tilde{c}_j x_j} + \underset{1\le j\le n}{minimum}\left\{w_{\tilde{c}_j x_j}\right\} - \underset{1\le j\le n}{maximum}\left\{u_{\tilde{c}_j x_j}\right\} - \underset{1\le j\le n}{maximum}\left\{y_{\tilde{c}_j x_j}\right\}\right)$$

Subject to

Constraints of the CrLpP $(P6.2.1.2)$.

CrLpP $(\boldsymbol{P6.2.1.4})$

$$Maximize/Minimize\left(\sum_{j=1}^{n} R\left(c_{j1}, c_{j2}, c_{j3}, c_{j4}; w_{\tilde{c}_j}, u_{\tilde{c}_j}, y_{\tilde{c}_j}\right) x_j - \sum_{j=1}^{n} w_{\tilde{c}_j x_j} + \sum_{j=1}^{n} u_{\tilde{c}_j x_j} + \sum_{j=1}^{n} y_{\tilde{c}_j x_j} + \underset{1\le j\le n}{minimum}\left\{w_{\tilde{c}_j x_j}\right\} - \underset{1\le j\le n}{maximum}\left\{u_{\tilde{c}_j x_j}\right\} - \underset{1\le j\le n}{maximum}\left\{y_{\tilde{c}_j x_j}\right\}\right)$$

Subject to

$$\sum_{j=1}^{n}\left(R\left(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}; w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}}, y_{\tilde{a}_{ij}}\right) x_j\right) (\le, =, \ge)\ R\left(b_{i1}, b_{i2}, b_{i3}, b_{i4}; w_{\tilde{b}_i}, u_{\tilde{b}_i}, y_{\tilde{b}_i}\right), i = 1,2,\dots,m,$$

$$x_j \ge 0,\ j = 1,2,\dots,n.$$

It is pertinent to mention that Singh et al. [148] have used the relation $R(\tilde{A}_1 + \tilde{A}_2) = R(\tilde{A}_1) + R(\tilde{A}_2)$ to transform the CrLpP $(P6.2.1.3)$ into the CrLpP $(P6.2.1.4)$. While, the following example clearly indicates that $R(\tilde{A}_1 + \tilde{A}_2) \ne R(\tilde{A}_1) + R(\tilde{A}_2)$ i.e., the CrLpP $(P6.2.1.3)$ is not equivalent to the CrLpP $(P6.2.1.4)$. Hence, it is inappropriate to use Singh et al.'s method [148].

Let $\tilde{A}_1 = (10,20,30,40; 0.8,0.5,0.3)$ and $\tilde{A}_2 = (30,50,70,90; 0.7,0.3,0.2)$ be two SvTrNeNs. Then, using Definition 6.1.5,

$\tilde{A}_1 + \tilde{A}_2 = (10 + 30,20 + 50,30 + 70,40 + 90; minimum\{0.8,0.7\}, maximum\{0.5,0.3\}, maximum\{0.3,0.2\}) = (40,70,100,130; 0.7,0.5,0.3)$

Using the expression (6.2.1.1) [1],

$$R(\tilde{A}_1 + \tilde{A}_2) = R(40,70,100,130; 0.7,0.5,0.3) = 254.9. \quad (6.2.1.3)$$

$R(\tilde{A}_1) = R(10,20,30,40; 0.8,0.5,0.3) = 75.$

$R(\tilde{A}_2) = R(30,50,70,90; 0.7,0.3,0.2) = 180.2.$

$$R(\tilde{A}_1) + R(\tilde{A}_2) = 255.2. \quad (6.2.1.4)$$

It is obvious from (6.2.1.3) and (6.2.1.4) that $R(\tilde{A}_1 + \tilde{A}_2) \neq R(\tilde{A}_1) + R(\tilde{A}_2)$.

6.2.2 Inappropriateness of Khatter's method

In Khatter's method [100] firstly, the SvTrNeLpP $(P6.2.1.1)$ is transformed into the CrLpP $(P6.2.2.1)$. Then, the CrLpP $(P6.2.2.1)$ is transformed into the CrLpP $(P6.2.2.2)$. Finally, it is assumed that an optimal solution of the CrLpP $(P6.2.2.2)$ also represent an optimal solution of the SvTrNeLpP $(P6.2.1.1)$.

CrLpP $(\boldsymbol{P6.2.2.1})$

$$Maximize/Minimize \left(V\left(\sum_{j=1}^{n}\left(c_{j1}, c_{j2}, c_{j3}, c_{j4}; w_{\tilde{c}_j}, u_{\tilde{c}_j}, y_{\tilde{c}_j}\right)x_j\right)\right)$$

Subject to

$$V\left(\sum_{j=1}^{n}\left(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}; w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}}, y_{\tilde{a}_{ij}}\right)x_j\right) (\leq, =, \geq)\, V\left(b_{i1}, b_{i2}, b_{i3}, b_{i4}; w_{\tilde{b}_i}, u_{\tilde{b}_i}, y_{\tilde{b}_i}\right), i = 1,2, \dots, m,$$

$x_j \geq 0,\ j = 1,2, \dots, n.$

where,

$V(\tilde{A}_i) = \lambda\left(\frac{a_{i1}+2a_{i2}+2a_{i3}+a_{i4}}{6}\right)w_{\tilde{A}_i}^2 + (1-$

$\lambda)\left(\frac{\left[(2a_{i1}+a_{i2}+a_{i3}+2a_{i4})-(a_{i1}-a_{i2}-a_{i3}+a_{i4})u_{\tilde{A}_i}-(a_{i1}+2a_{i2}+2a_{i3}+a_{i4})u_{\tilde{A}_i}^2\right]}{6}+\right.$

$\left.\frac{\left[(2a_{i1}+a_{i2}+a_{i3}+2a_{i4})-(a_{i1}-a_{i2}-a_{i3}+a_{i4})y_{\tilde{A}_i}-(a_{i1}+2a_{i2}+2a_{i3}+a_{i4})y_{\tilde{A}_i}^2\right]}{6}\right), \lambda \in [0,1]; i = 1,2.$

CrLpP $(P6.2.2.2)$

$Maximize/Minimize\left(\sum_{j=1}^{n} V\left(c_{j1}, c_{j2}, c_{j3}, c_{j4}; w_{\tilde{c}_j}, u_{\tilde{c}_j}, y_{\tilde{c}_j}\right)x_j\right)$

Subject to

$\sum_{j=1}^{n}\left(V\left(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}; w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}}, y_{\tilde{a}_{ij}}\right)x_j\right) (\leq, =, \geq)\ V\left(b_{i1}, b_{i2}, b_{i3}, b_{i4}; w_{\tilde{b}_i}, u_{\tilde{b}_i}, y_{\tilde{b}_i}\right), i =$

$1,2, \ldots, m,$

$x_j \geq 0,\ j = 1,2, \ldots, n.$

It is pertinent to mention that Khatter [100] has used the relation $V(\tilde{A}_1 + \tilde{A}_2) = V(\tilde{A}_1) + V(\tilde{A}_2)$ to transform the CrLpP $(P6.2.2.1)$ into the CrLpP $(P6.2.2.2)$. While, the following example clearly indicates that $V(\tilde{A}_1 + \tilde{A}_2) \neq V(\tilde{A}_1) + V(\tilde{A}_2)$ i.e., the CrLpP $(P6.2.2.1)$ is not equivalent to the CrLpP $(P6.2.2.2)$. Hence, it is inappropriate to use Khatter's method [100].

Let $\tilde{A}_1 = (30,40,50,70; 0.7,0.4,0.3)$ and $\tilde{A}_2 = (40,50,60,70; 0.6,0.5,0.2)$ be two SvTrNeNs. Then, using Definition 6.1.5,

$\tilde{A} + \tilde{B} = (30 + 40,40 + 50,50 + 60,70 +$

$70; minimum\{0.7,0.6\}, maximum\{0.4,0.5\}, maximum\{0.3,0.2\}) =$

$(70,90,110,140; 0.6,0.5,0.3).$

Using the expression (6.1.2) [100],

$V(\tilde{A}_1 + \tilde{A}_2) = V(70,90,110,140; 0.6,0.5,0.3) = 36.6\lambda + (1-\lambda)(77.08 + 93.68)$

$= 170.76 - 134.16\lambda.$ (6.2.2.1)

$V(\tilde{A}_1) = V(30,40,50,70; 0.7,0.4,0.3) = 22.87\lambda + (1-\lambda)(40.2 + 43.63) = 83.83 - 60.96\lambda$

$V(\tilde{A}_2) = V(40,50,60,70; 0.6,0.5,0.2) = 19.8\lambda + (1-\lambda)(41.25 + 52.8) = 94.05 - 74.25\lambda$

$$V(\tilde{A}_1) + V(\tilde{A}_2) = 177.88 - 135.21\lambda. \tag{6.2.2.2}$$

It is obvious from (6.2.2.1) and (6.2.2.2) that $V(\tilde{A}_1 + \tilde{A}_2) \neq V(\tilde{A}_1) + V(\tilde{A}_2)$.

6.2.3 Inappropriateness of Abdelfattah's method

Abdelfattah [3] claimed that on solving the SvTNeLpP $(P6.2.3.1)$, the results presented in Table 6.1, are obtained.

SvTNeLpP $(\boldsymbol{P6.2.3.1})$

$Maximize\ \left((30,40,50; 0.7,0.4,0.3)x_1 + (40,50,60; 0.6,0.5,0.2)x_2\right)$

Subject to

$(0.5,1,3; 0.6,0.4,0.1)x_1 + (0,2,6; 0.6,0.4,0.1)x_2 \preccurlyeq (20,40,60; 0.4,0.3,0.5),$

$(1,4,12; 0.4,0.3,0.2)x_1 + (1,3,10; 0.7,0.4,0.3)x_2 \preccurlyeq (100,120,140; 0.7,0.4,0.3),$

$x_1, x_2 \geq 0.$

Table 6.1 [3] Optimal solutions and optimal values

(α,β,γ)	$x^{B^*}_{1(\alpha,\beta,\gamma)}$	$x^{B^*}_{2(\alpha,\beta,\gamma)}$	$x^{W^*}_{1(\alpha,\beta,\gamma)}$	$x^{W^*}_{2(\alpha,\beta,\gamma)}$	$Z^{B^*}_{(\alpha,\beta,\gamma)}$	$Z^{W^*}_{(\alpha,\beta,\gamma)}$
(0,0.5,0.5)	33.64	19.48	12.53	2.24	2523	554.62
(0,1,1)	0	140	6.67	0	8400	200
(0.4,0.5,0.5)	28.43	12.79	16.46	3.68	1874	793.07
(0.4,1,1)	0	106.31	9.36	0	6201	294.29
(0.2,0.8,0.7)	0	77.07	11.42	0.37	4334	398.46

It is pertinent to mention that as in the SvTNeLpP $(P6.2.3.1)$, x_1 and x_2 are considered as non-negative real numbers. So, the obtained optimal values of x_1 and x_2 should be same for all values of α, β, γ. While, it is obvious from Table 6.1 that the values of x_1 and x_2 are different for different values of α, β, γ. This clearly indicates that x_1 and x_2, obtained by Abdelfattah's method [3], are not non-negative real numbers. Hence, it is inappropriate to use Abdelfattah's method [3].

6.2.4 Inappropriateness of Das et al.'s method

It is pertinent to mention that in one of the steps of Das et al.'s method [44], the scalar multiplication $\lambda\tilde{A} = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4; \lambda w_{\tilde{A}}, \lambda u_{\tilde{A}}, \lambda y_{\tilde{A}}), \lambda > 0$, is used to transform the SvTrNeLpP $(P6.2.1.1)$ into the SvTrNeLpP $(P6.2.4.1)$.

SvTrNeLpP $(\boldsymbol{P6.2.4.1})$

$$Maximize/Minimize\left(\sum_{j=1}^{n}\left(c_{j1}x_j, c_{j2}x_j, c_{j3}x_j, c_{j4}x_j; w_{\tilde{c}_j}, u_{\tilde{c}_j}, y_{\tilde{c}_j}x_j\right)\right)$$

Subject to

$$\sum_{j=1}^{n}\left(a_{ij1}x_j, a_{ij2}x_j, a_{ij3}x_j, a_{ij4}x_j; w_{\tilde{a}_{ij}}x_j, u_{\tilde{a}_{ij}}x_j, y_{\tilde{a}_{ij}}x_j\right)(\preccurlyeq, \approx$$

$$, \succcurlyeq)\left(b_{i1}, b_{i2}, b_{i3}, b_{i4}; w_{\tilde{b}_i}, u_{\tilde{b}_i}, y_{\tilde{b}_i}\right), i = 1,2, \dots, m,$$

$$x_j \geq 0, \; j = 1,2, \dots, n.$$

However, this scalar multiplication is not valid as the following clearly indicates that the number $(\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4; \lambda w_{\tilde{A}}, \lambda u_{\tilde{A}}, \lambda y_{\tilde{A}})$ is not a SvTrNeN. Hence, it is inappropriate to use Das et al.'s method [44].

According to Definition 6.1.3, discussed in Section 6.1, the number $(\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4; \lambda w_{\tilde{A}}, \lambda u_{\tilde{A}}, \lambda y_{\tilde{A}})$ will be a SvTrNeN if

(i) $\lambda a_1 \leq \lambda a_2 \leq \lambda a_3 \leq \lambda a_4$

(ii) $0 \leq \lambda w_{\tilde{A}} \leq 1, 0 \leq \lambda u_{\tilde{A}} \leq 1, 0 \leq \lambda y_{\tilde{A}} \leq 1$

(iii) $0 \leq \lambda w_{\tilde{A}} + \lambda u_{\tilde{A}} + \lambda y_{\tilde{A}} \leq 3$

While,

(i) $0 \leq w_{\tilde{A}} \leq 1, 0 \leq u_{\tilde{A}} \leq 1, 0 \leq y_{\tilde{A}} \leq 1 \Rightarrow 0 \leq \lambda w_{\tilde{A}} \leq \lambda, 0 \leq \lambda u_{\tilde{A}} \leq \lambda, 0 \leq \lambda y_{\tilde{A}} \leq \lambda$ i.e., the necessary condition $0 \leq \lambda w_{\tilde{A}} \leq 1, 0 \leq \lambda u_{\tilde{A}} \leq 1, 0 \leq \lambda y_{\tilde{A}} \leq 1$ is not satisfying.

(ii) $0 \leq w_{\tilde{A}} + u_{\tilde{A}} + y_{\tilde{A}} \leq 3 \Rightarrow 0 \leq \lambda w_{\tilde{A}} + \lambda u_{\tilde{A}} + \lambda y_{\tilde{A}} \leq 3\lambda$ i.e., the necessary condition $0 \leq \lambda w_{\tilde{A}} + \lambda u_{\tilde{A}} + \lambda y_{\tilde{A}} \leq 3$ is not satisfying.

6.2.5 Inappropriateness of Kar et al.'s method

It pertinent to mention that in one of the steps of Kar et al.'s method [89], it is assumed that if $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}; a_{14}, a_{15}, a_{16}; a_{17}, a_{18}, a_{19})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}; a_{24}, a_{25}, a_{26}; a_{27}, a_{28}, a_{29})$ are two SvTNeNs. Then, $\frac{\tilde{A}_1}{\tilde{A}_2} = \left(\frac{a_{11}}{a_{21}}, \frac{a_{12}}{a_{22}}, \frac{a_{13}}{a_{23}}; \frac{a_{14}}{a_{24}}, \frac{a_{15}}{a_{25}}, \frac{a_{16}}{a_{26}}; \frac{a_{17}}{a_{27}}, \frac{a_{18}}{a_{28}}, \frac{a_{19}}{a_{29}}\right)$ will also be a SvTNeN. While, the following clearly indicates that $\left(\frac{a_{11}}{a_{21}}, \frac{a_{12}}{a_{22}}, \frac{a_{13}}{a_{23}}; \frac{a_{14}}{a_{24}}, \frac{a_{15}}{a_{25}}, \frac{a_{16}}{a_{26}}; \frac{a_{17}}{a_{27}}, \frac{a_{18}}{a_{28}}, \frac{a_{19}}{a_{29}}\right)$ will not necessarily be a SvTNeN. Hence, it is inappropriate to use Kar et al.'s method [89].

Let $\tilde{A}_1 = (1,2,5; 6,7,8; 9,10,11)$ and $\tilde{A}_2 = (2,3,4; 8,9,10; 11,12,13)$ be two SvTNeNs. Then, $\frac{\tilde{A}_1}{\tilde{A}_2} = \left(\frac{1}{2}, \frac{2}{3}, \frac{5}{4}; \frac{6}{8}, \frac{7}{9}, \frac{8}{10}; \frac{9}{11}, \frac{10}{12}, \frac{11}{13}\right) = (0.5,0.67,1.25; 0.75,0.78,0.8; 0.81,0.83,0.85)$ is not a SvTNeN as the necessary condition $\frac{a_{11}}{a_{21}} \leq \frac{a_{12}}{a_{22}} \leq \frac{a_{13}}{a_{23}} \leq \frac{a_{14}}{a_{24}} \leq \frac{a_{15}}{a_{25}} \leq \frac{a_{16}}{a_{26}} \leq \frac{a_{17}}{a_{27}} \leq \frac{a_{18}}{a_{28}} \leq \frac{a_{19}}{a_{29}}$ is not satisfying.

Remark 6.1: It can be easily verified that the shortcoming, pointed out by Singh et al. [148] in Abdel-Basset et al.'s method [1], also occurs in the existing methods [61, 106]. Hence, it is inappropriate to use the existing methods [61, 106].

6.3 Proposed Mehar method for solving SvTrNeLpPs

In this section, a new method (named as Mehar method) is proposed for solving the SvTrNeLpP ($P6.2.1.1$). The proposed Mehar method can also be used to solve SvTNeLpPs.

Step 1: Using Definition 6.1.7, transform the SvTrNeLpP $(P6.2.1.1)$ into its equivalent SvTrNeLpP $(P6.3.1)$.

SvTrNeLpP $(\boldsymbol{P6.3.1})$

$$Maximize/Minimize\left(\sum_{j=1}^{n}\left(c_{j1}x_j, c_{j2}x_j, c_{j3}x_j, c_{j4}x_j; w_{\tilde{c}_j}, u_{\tilde{c}_j}, y_{\tilde{c}_j}\right)\right)$$

Subject to

$$\sum_{j=1}^{n}\left(a_{ij1}x_j, a_{ij2}x_j, a_{ij3}x_j, a_{ij4}x_j; w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}}, y_{\tilde{a}_{ij}}\right)(\preccurlyeq, \approx, \succcurlyeq)\left(b_{i1}, b_{i2}, b_{i3}, b_{i4}; w_{\tilde{b}_i}, u_{\tilde{b}_i}, y_{\tilde{b}_i}\right), i = 1,2, \dots, m,$$

$$x_j \geq 0,\ j = 1,2, \dots, n.$$

Step 2: Using Definition 6.1.5, transform the SvTrNeLpP $(P6.3.1)$ into its equivalent SvTrNeLpP $(P6.3.2)$.

SvTrNeLpP $(\boldsymbol{P6.3.2})$

$$Maximize/Minimize\begin{pmatrix}\sum_{j=1}^{n} c_{j1}x_j, \sum_{j=1}^{n} c_{j2}x_j, \sum_{j=1}^{n} c_{j3}x_j, \sum_{j=1}^{n} c_{j4}x_j; \\ \underset{1\leq j\leq n}{minimum}\left\{w_{\tilde{c}_j}\right\}, \underset{1\leq j\leq n}{maximum}\left\{u_{\tilde{c}_j}\right\}, \underset{1\leq j\leq n}{maximum}\left\{y_{\tilde{c}_j}\right\}\end{pmatrix}$$

Subject to

$$\begin{pmatrix}\sum_{j=1}^{n} a_{ij1}x_j, \sum_{j=1}^{n} a_{ij2}x_j, \sum_{j=1}^{n} a_{ij3}x_j, \sum_{j=1}^{n} a_{ij4}x_j; \\ \underset{1\leq j\leq n}{minimum}\left\{w_{\tilde{a}_{ij}}\right\}, \underset{1\leq j\leq n}{maximum}\left\{u_{\tilde{a}_{ij}}\right\}, \underset{1\leq j\leq n}{maximum}\left\{y_{\tilde{a}_{ij}}\right\}\end{pmatrix}(\preccurlyeq, \approx, \succcurlyeq)\left(b_{\tilde{b}_i}^{1}, b_{\tilde{b}_i}^{2}, b_{\tilde{b}_i}^{3}, b_{\tilde{b}_i}^{4}; w_{\tilde{b}_i}, u_{\tilde{b}_i}, y_{\tilde{b}_i}\right), i = 1,2, \dots, m,$$

$$x_j \geq 0,\ j = 1,2, \dots, n.$$

Step 3: Using Definition 6.1.9, transform the SvTrNeLpP $(P6.3.2)$ into its equivalent CrLpP $(P6.3.3)$.

CrLpP $(\boldsymbol{P6.3.3})$

$$Maximize/Minimize\left(V\left(\begin{matrix}\sum_{j=1}^{n} c_{j1}x_j\,,\sum_{j=1}^{n} c_{j2}x_j\,,\sum_{j=1}^{n} c_{j3}x_j\,,\sum_{j=1}^{n} c_{j4}x_j\,;\\ \underset{1\le j\le n}{minimum}\left\{w_{\tilde{c}_j}\right\},\underset{1\le j\le n}{maximum}\left\{u_{\tilde{c}_j}\right\},\underset{1\le j\le n}{maximum}\left\{y_{\tilde{c}_j}\right\}\end{matrix}\right)\right)$$

Subject to

$$V\left(\begin{matrix}\sum_{j=1}^{n} a_{ij1}x_j\,,\sum_{j=1}^{n} a_{ij2}x_j\,,\sum_{j=1}^{n} a_{ij3}x_j\,,\sum_{j=1}^{n} a_{ij4}x_j\,;\\ \underset{1\le j\le n}{minimum}\left\{w_{\tilde{a}_{ij}}\right\},\underset{1\le j\le n}{maximum}\left\{u_{\tilde{a}_{ij}}\right\},\underset{1\le j\le n}{maximum}\left\{y_{\tilde{a}_{ij}}\right\}\end{matrix}\right)(\preccurlyeq,\approx$$

$$,\succcurlyeq)\ V\left(b_{\tilde{b}_i}^1,b_{\tilde{b}_i}^2,b_{\tilde{b}_i}^3,b_{\tilde{b}_i}^4;w_{\tilde{b}_i},u_{\tilde{b}_i},y_{\tilde{b}_i}\right),i=1,2,\dots,m,$$

$$x_j\ge 0,\ \ j=1,2,\dots,n$$

where,

$$V(\tilde{A})=\lambda\left(\frac{a_1+2a_2+2a_3+a_4}{6}\right)w_{\tilde{A}}^2+(1-$$

$$\lambda)\left(\frac{\left[(2a_1+a_2+a_3+2a_4)-(a_1-a_2-a_3+a_4)u_{\tilde{A}}-(a_1+2a_2+2a_3+a_4)u_{\tilde{A}}^2\right]}{6}+\right.$$

$$\left.\frac{\left[(2a_1+a_2+a_3+2a_4)-(a_1-a_2-a_3+a_4)y_{\tilde{A}}-(a_1+2a_2+2a_3+a_4)y_{\tilde{A}}^2\right]}{6}\right),\lambda\in[0,1].$$

Step 4: Find an optimal solution of the CrLpP $(P6.3.3)$ for some values of $\lambda\in[0,1]$. The obtained optimal solution also represents an optimal solution of the SvTrNeLpP $(P6.2.1.1)$.

6.4 Correct optimal solution of some existing SvTNeLpPs

In this section, the correct optimal solution of some existing SvTNeLpPs is obtained by the proposed Mehar method.

Hussian et al. [79] as well as Khatter [100] have considered the following real-life problem to illustrate their proposed method.

A Pottery Company, run by a Native American tribal council, desires to find the number of bowls and mugs to be produced each day in order to maximize the profit by considering

(i) The data presented in Table 6.2.

(ii) The data presented in Table 6.3.

(iii) The data presented in Table 6.4.

However, as some mathematical incorrect results are considered in Hussian et al.'s method [79] as well as in Khatter's method [100]. So, the existing optimal solution [79, 100] is not correct. In this section, a correct optimal solution of this real-life problem is obtained by the proposed Mehar method.

Table 6.2 [79, 100] Resource requirements of two products

Product	Resource requirements		
	Labour (Hr./unit)	Clay (Lb./unit)	Profit($/unit)
Bowl	$(0.5,1,3;0.6,0.4,0.1)$	$(1,4,12;0.4,0.3,0.2)$	$(30,40,50;0.7,0.4,0.3)$
Mug	$(0,2,6;0.6,0.4,0.1)$	$(1,3,10;0.7,0.4,0.3)$	$(40,50,60;0.6,0.5,0.2)$
	Total available hr of labour $=(20,40,60;0.4,0.3,0.5)$	Total available pounds of clay $=(100,120,140;0.7,0.4,0.3)$	

Table 6.3 [79, 100] Resource requirements of two products

Product	Resource requirements		
	Labour (Hr./unit)	Clay (Lb./unit)	Profit($/unit)
Bowl	$(3.5,4,4.1;0.75,0.5,0.25)$	$(0,1,2;0.15,0.5,0)$	$(4,5,6;0.5,0.8,0.3)$
Mug	$(2.5,3,3.2;0.2,0.8,0.4)$	$(2.8,3,3.2;0.75,0.5,0.25)$	$(2.5,3,3.2;0.6,0.4,0)$
	Total available hr of labour $=(11,12,13;0.2,0.6,0.5)$	Total available pounds of clay $=(5.5,6,7.5;0.8,0.6,0.4)$	

Table 6.4 [79, 100] Resource requirements of two products

Product	Resource requirements			
	Skilled Labour (Hr./unit)	Unskilled Labour (Hr./unit)	Clay (Lb./unit)	Profit($/unit)
Bowl	15	24	21	$(19,25,33;0.8,0.1,0.4)$
Mug	30	6	14	$(44,48,54;0.75,0.25,0)$
	Total available hr of skilled labour $=45000$	Total available hr of unskilled labour $=24000$	Total available pounds of clay $=28000$	

6.4.1 First illustrative example

If the data, presented in Table 6.2, is considered. Then, to find an optimal solution of the real-life problem is equivalent to find an optimal solution of the SvTNeLpP $(P6.4.1.1)$.

SvTNeLpP $(\boldsymbol{P6.4.1.1})$

$Maximize\left((30,40,50;0.7,0.4,0.3)x_1+(40,50,60;0.6,0.5,0.2)x_2\right)$

Subject to

$(0.5,1,3;0.6,0.4,0.1)x_1+(0,2,6;0.6,0.4,0.1)x_2\preccurlyeq(20,40,60;0.4,0.3,0.5)$,

$(1,4,12;0.4,0.3,0.2)x_1+(1,3,10;0.7,0.4,0.3)x_2\preccurlyeq(100,120,140;0.7,0.4,0.3)$,

$x_1,x_2\geq 0$.

Using the proposed Mehar method, an optimal solution of the SvTNeLpP $(P6.4.1.1)$ can be obtained as follows:

Step 1: Using Step 1 of the proposed Mehar method, the SvTNeLpP $(P6.4.1.1)$ can be transformed into its equivalent SvTNeLpP $(P6.4.1.2)$.

SvTNeLpP $(\boldsymbol{P6.4.1.2})$

$Maximize\left((30x_1,40x_1,50x_1;0.7,0.4,0.3)+(40x_2,50x_2,60x_2;0.6,0.5,0.2)\right)$

Subject to

$(0.5x_1, 1x_1, 3x_1; 0.6,0.4,0.1) + (0x_2, 2x_2, 6x_2; 0.6,0.4,0.1) \preccurlyeq (20,40,60; 0.4,0.3,0.5)$,

$(1x_1, 4x_1, 12x_1; 0.4,0.3,0.2) + (1x_2, 3x_2, 10x_2; 0.7,0.4,0.3) \preccurlyeq$

$(100,120,140; 0.7,0.4,0.3)$,

$x_1, x_2 \geq 0$.

Step 2: Using Step 2 of the proposed Mehar method, the SvTNeLpP $(P6.4.1.2)$ can be transformed into its equivalent SvTNeLpP $(P6.4.1.3)$.

SvTNeLpP $(\boldsymbol{P6.4.1.3})$

$Maximize\ (30x_1 + 40x_2, 40x_1 + 50x_2, 50x_1 +$

$60x_2; minimum\{0.7,0.6\}, maximum\{0.4,0.5\}, maximum\{0.3,0.2\})$

Subject to

$(0.5x_1 + 0x_2, 1x_1 + 2x_2, 3x_1 +$

$6x_2; minimum\{0.6,0.6\}, maximum\{0.4,0.4\}, maximum\{0.1,0.1\}) \preccurlyeq$

$(20,40,60; 0.4,0.3,0.5)$,

$(1x_1 + 1x_2, 4x_1 + 3x_2, 12x_1 +$

$10x_2; minimum\{0.4,0.7\}, maximum\{0.3,0.4\}, maximum\{0.2,0.3\}) \preccurlyeq$

$(100,120,140; 0.7,0.4,0.3)$,

$x_1, x_2 \geq 0$.

Step 3: Using Step 3 of the proposed Mehar method, the SvTNeLpP $(P6.4.1.3)$ can be transformed into its equivalent CrLpP $(P6.4.1.4)$.

CrLpP $(\boldsymbol{P6.4.1.4})$

$Maximize\ \left(V(30x_1 + 40x_2, 40x_1 + 50x_2, 50x_1 + 60x_2; 0.6,0.5,0.3)\right)$

Subject to

$V(0.5x_1 + 0x_2, 1x_1 + 2x_2, 3x_1 + 6x_2; 0.6,0.4,0.1) \leq V(20,40,60; 0.4,0.3,0.5)$,

$V(1x_1 + 1x_2, 4x_1 + 3x_2, 12x_1 + 10x_2; 0.4,0.4,0.3) \leq V(100,120,140; 0.7,0.4,0.3),$

$x_1, x_2 \geq 0,$

where,

$$V(\tilde{A}) = \lambda \left(\frac{a_1+4a_2+a_3}{6}\right) w_{\tilde{A}}^2 + (1-\lambda)\left(\frac{[2(a_1+a_2+a_3)-(a_1-2a_2+a_3)u_{\tilde{A}}-(a_1+4a_2+a_3)u_{\tilde{A}}^2]}{6} + \frac{[2(a_1+a_2+a_3)-(a_1-2a_2+a_3)y_{\tilde{A}}-(a_1+4a_2+a_3)y_{\tilde{A}}^2]}{6}\right), \lambda \in [0,1].$$

Step 4: The obtained optimal solution of the CrLpP $(P6.4.1.4)$ for some values of $\lambda \in [0,1]$ are shown in Table 6.5. It is pertinent to mention that according to Step 4 of the proposed Mehar method, the obtained optimal solution also represents an optimal solution of the SvTNeLpP $(P6.4.1.1)$.

Table 6.5 Correct optimal solution for different values of λ

λ	Optimal solution	
	x_1	x_2
0	19.55	3.02
0.1	20.60	2.32
0.2	21.89	1.47
0.3	23.49	0.41
0.4	23.87	0
0.5	23.41	0
0.6	22.79	0
0.7	21.92	0
0.8	20.63	0
0.9	18.48	0
1	14.22	0

6.4.2 Second illustrative example

If the data, presented in Table 6.3, is considered. Then, to find an optimal solution of the real-life problem is equivalent to find an optimal solution of the SvTNeLpP $(P6.4.2.1)$.

SvTNeLpP $(\boldsymbol{P6.4.2.1})$

$Maximize\ \big((4,5,6; 0.5,0.8,0.3)x_1 + (2.5,3,3.2; 0.6,0.4,0)x_2\big)$

Subject to

$(3.5,4,4.1; 0.75,0.5,0.25)x_1 + (2.5,3,3.2; 0.2,0.8,0.4)x_2 \preccurlyeq (11,12,13; 0.2,0.6,0.5),$

$(0,1,2; 0.15,0.5,0)x_1 + (2.8,3,3.2; 0.75,0.5,0.25)x_2 \preccurlyeq (5.5,6,7.5; 0.8,0.6,0.4),$

$x_1, x_2 \geq 0.$

Using the proposed Mehar method, an optimal solution of the SvTNeLpP $(P6.4.2.1)$ can be obtained as follows:

Step 1: Using Step 1 of the proposed Mehar method, the SvTNeLpP $(P6.4.2.1)$ can be transformed into its equivalent SvTNeLpP $(P6.4.2.2)$.

SvTNeLpP $(\boldsymbol{P6.4.2.2})$

$Maximize\ \big((4x_1, 5x_1, 6x_1; 0.5,0.8,0.3) + (2.5x_2, 3x_2, 3.2x_2; 0.6,0.4,0)\big)$

Subject to

$(3.5x_1, 4x_1, 4.1x_1; 0.75,0.5,0.25) + (2.5x_2, 3x_2, 3.2x_2; 0.2,0.8,0.4) \preccurlyeq$

$(11,12,13; 0.2,0.6,0.5),$

$(0x_1, 1x_1, 2x_1; 0.15,0.5,0) + (2.8x_2, 3x_2, 3.2x_2; 0.75,0.5,0.25) \preccurlyeq$

$(5.5,6,7.5; 0.8,0.6,0.4),$

$x_1, x_2 \geq 0.$

Step 2: Using Step 2 of the proposed Mehar method, the SvTNeLpP $(P6.4.2.2)$ can be transformed into its equivalent SvTNeLpP $(P6.4.2.3)$.

SvTNeLpP $(\boldsymbol{P6.4.2.3})$

$Maximize\ (4x_1 + 2.5x_2, 5x_1 + 3x_2, 6x_1 +$

$3.2x_2; minimum\{0.5,0.6\}, maximum\{0.8,0.4\}, maximum\{0.3,0\})$

Subject to

$(3.5x_1 + 2.5x_2, 4x_1 + 3x_2, 4.1x_1 +$

$3.2x_2; minimum\{0.75,0.2\}, maximum\{0.5,0.8\}, maximum\{0.25,0.4\}) \preccurlyeq$

$(11,12,13; 0.2,0.6,0.5)$,

$(2.8x_2, x_1 + 3x_2, 2x_1 +$

$3.2x_2; minimum\{0.15,0.75\}, maximum\{0.5,0.5\}, maximum\{0,0.25\}) \preccurlyeq$

$(5.5,6,7.5; 0.8,0.6,0.4)$,

$x_1, x_2 \geq 0$.

Step 3: Using Step 3 of the proposed Mehar method, the SvTNeLpP $(P6.4.2.3)$ can be transformed into its equivalent CrLpP $(P6.4.2.4)$.

CrLpP $(\boldsymbol{P6.4.2.4})$

$Maximize\left(V(4x_1 + 2.5x_2, 5x_1 + 3x_2, 6x_1 + 3.2x_2; 0.5,0.8,0.3)\right)$

Subject to

$V(3.5x_1 + 2.5x_2, 4x_1 + 3x_2, 4.1x_1 + 3.2x_2; 0.2,0.8,0.4) \leq V(11,12,13; 0.2,0.6,0.5)$,

$V(2.8x_2, x_1 + 3x_2, 2x_1 + 3.2x_2; 0.15,0.5,0.25) \leq V(5.5,6,7.5; 0.8,0.6,0.4)$,

$x_1, x_2 \geq 0$,

where,

$V(\tilde{A}) = \lambda\left(\frac{a_1+4a_2+a_3}{6}\right)w_{\tilde{A}}^2 + (1-\lambda)\left(\frac{[2(a_1+a_2+a_3)-(a_1-2a_2+a_3)u_{\tilde{A}}-(a_1+4a_2+a_3)u_{\tilde{A}}^2]}{6} +\right.$

$\left.\frac{[2(a_1+a_2+a_3)-(a_1-2a_2+a_3)y_{\tilde{A}}-(a_1+4a_2+a_3)y_{\tilde{A}}^2]}{6}\right), \lambda \in [0,1]$.

Step 4: The obtained optimal solution of the CrLpP $(P6.4.2.4)$ for some values of $\lambda \in [0,1]$ are shown in Table 6.6. It is pertinent to mention that according to Step 4 of the proposed

Mehar method, the obtained optimal solution also represents an optimal solution of the SvTNeLpP $(P6.4.2.1)$.

Table 6.6 Correct optimal solution for different values of $\boldsymbol{\lambda}$

λ	Optimal solution	
	x_1	x_2
0	3.574	0
0.1	3.572	0
0.2	3.570	0
0.3	3.567	0
0.4	3.563	0
0.5	3.557	0
0.6	3.549	0
0.7	3.536	0
0.8	3.512	0
0.9	3.452	0
1	3.051	0

6.4.3 Third illustrative example

If the data, presented in Table 6.4, is considered. Then, to find an optimal solution of the real-life problem is equivalent to find an optimal solution of the SvTNeLpP $(P6.4.3.1)$.

SvTNeLpP $(\boldsymbol{P6.4.3.1})$

$Maximize\left((19,25,33;0.8,0.1,0.4)x_1 + (44,48,54;0.75,0.25,0)x_2\right)$

Subject to

$15x_1 + 30x_2 \leq 45000,$

$24x_1 + 6x_2 \leq 24000,$

$21x_1 + 14x_2 \leq 28000,$

$x_1, x_2 \geq 0.$

Using the proposed Mehar method, an optimal solution of the SvTNeLpP $(P6.4.3.1)$ can be obtained as follows:

Step 1: Using Step 1 of the proposed Mehar method, the SvTNeLpP $(P6.4.3.1)$ can be transformed into its equivalent SvTNeLpP $(P6.4.3.2)$.

SvTNeLpP $(\boldsymbol{P6.4.3.2})$

$Maximize\ \big((19x_1, 25x_1, 33x_1; 0.8,0.1,0.4) + (44x_2, 48x_2, 54x_2; 0.75,0.25,0)\big)$

Subject to

Constraints of the SvTNeLpP $(P6.4.3.1)$.

Step 2: Using Step 2 of the proposed Mehar method, the SvTNeLpP $(P6.4.3.2)$ can be transformed into its equivalent SvTNeLpP $(P6.4.3.3)$.

SvTNeLpP $(\boldsymbol{P6.4.3.3})$

$Maximize\ (19x_1 + 44x_2, 25x_1 + 48x_2, 33x_1 +$

$54x_2; minimum\{0.8,0.75\}, maximum\{0.1,0.25\}, maximum\{0.4,0\})$

Subject to

Constraints of the SvTNeLpP $(P6.4.3.1)$.

Step 3: Using Step 3 of the proposed Mehar method, the SvTNeLpP $(P6.4.3.3)$ can be transformed into its equivalent CrLpP $(P6.4.3.4)$.

CrLpP $(\boldsymbol{P6.4.3.4})$

$Maximize\ \big(V(19x_1 + 44x_2, 25x_1 + 48x_2, 33x_1 + 54x_2; 0.75,0.25,0.4)\big)$

Subject to

Constraints of the SvTNeLpP $(P6.4.3.1)$.

where,

$$V(\tilde{A}) = \lambda\left(\frac{a_1+4a_2+a_3}{6}\right)w_{\tilde{A}}^2 + (1-\lambda)\left(\frac{[2(a_1+a_2+a_3)-(a_1-2a_2+a_3)u_{\tilde{A}}-(a_1+4a_2+a_3)u_{\tilde{A}}^2]}{6} + \frac{[2(a_1+a_2+a_3)-(a_1-2a_2+a_3)y_{\tilde{A}}-(a_1+4a_2+a_3)y_{\tilde{A}}^2]}{6}\right), \lambda \in [0,1].$$

Step 4: The obtained optimal solution of the CrLpP $(P6.4.3.4)$ for some values of $\lambda \in [0,1]$ are shown in Table 6.7. It is pertinent to mention that according to Step 4 of the proposed Mehar method, the obtained optimal solution also represents an optimal solution of the SvTNeLpP $(P6.4.3.1)$.

Table 6.7 Correct optimal solution for different values of λ

λ	Optimal solution	
	x_1	x_2
0	500	1250
0.1	500	1250
0.2	500	1250
0.3	500	1250
0.4	500	1250
0.5	500	1250
0.6	500	1250
0.7	500	1250
0.8	500	1250
0.9	500	1250
1	500	1250

6.5 Correct optimal solution of an existing SvTrNeLpP

In this section, the correct optimal solution of an existing SvTrNeLpP is obtained by the proposed Mehar method.

Das et al. [44] have considered the following real-life problem to illustrate their proposed method.

An electric cable maker desires to find the number of cable 1 and cable 2 to be produced each day in order to maximize the profit by considering the data presented in Table 6.8.

Table 6.8 [44] Resource requirements of two cables

	Resource requirements		
	Metal (meter)	Plastic (meter)	Profit
Cable 1	$(2,4,6,8; 0.6,0.1,0.3)$	$(4,7,10,13; 0.7,0.4,0.2)$	$(1,3,4,7; 0.8,0.2,0.4)$
Cable 2	$(3,5,9,12; 0.7,0.2,0.1)$	$(3,6,9,14; 0.8,0.5,0.3)$	$(4,6,8,10; 0.9,0.3,0.5)$
	Total available meters of metal $= (10,15,20,25; 0.6,0,0.5)$	Total available meters of plastic $= (10,20,25,30; 0.9,0.45,0.3)$	

However, as some mathematical incorrect results are considered in Das et al.'s method [44]. So, the existing optimal solution [44] is not correct. In this section, a correct optimal solution of this real-life problem is obtained by the proposed Mehar method.

If the data, presented in Table 6.8, is considered. Then, to find an optimal solution of the real-life problem is equivalent to find an optimal solution of the SvTrNeLpP $(P6.5.1)$.

SvTrNeLpP $(\boldsymbol{P6.5.1})$

$Maximize\ \big((1,3,4,7; 0.8,0.2,0.4)x_1 + (4,6,8,10; 0.9,0.3,0.5)x_2\big)$

Subject to

$(2,4,6,8; 0.6,0.1,0.3)x_1 + (3,5,9,12; 0.7,0.2,0.1)x_2 \preccurlyeq (10,15,20,25; 0.6,0,0.5)$,

$(4,7,10,13; 0.7,0.4,0.2)x_1 + (3,6,9,14; 0.8,0.5,0.3)x_2 \preccurlyeq (10,20,25,30; 0.9,0.45,0.3)$,

$x_1, x_2 \geq 0$.

Using the proposed Mehar method, an optimal solution of the SvTrNeLpP $(P6.5.1)$ can be obtained as follows:

Step 1: Using Step 1 of the proposed Mehar method, the SvTrNeLpP $(P6.5.1)$ can be transformed into its equivalent SvTrNeLpP $(P6.5.2)$.

SvTrNeLpP $(\boldsymbol{P6.5.2})$

$Maximize\left((x_1, 3x_1, 4x_1, 7x_1; 0.8,0.2,0.4) + (4x_2, 6x_2, 8x_2, 10x_2; 0.9,0.3,0.5)\right)$

Subject to

$(2x_1, 4x_1, 6x_1, 8x_1; 0.6,0.1,0.3) + (3x_2, 5x_2, 9x_2, 12x_2; 0.7,0.2,0.1) \preccurlyeq$

$(10,15,20,25; 0.6,0,0.5)$,

$(4x_1, 7x_1, 10x_1, 13x_1; 0.7,0.4,0.2) + (3x_2, 6x_2, 9x_2, 14x_2; 0.8,0.5,0.3) \preccurlyeq$

$(10,20,25,30; 0.9,0.45,0.3)$,

$x_1, x_2 \geq 0$.

Step 2: Using Step 2 of the proposed Mehar method, the SvTrNeLpP $(P6.5.2)$ can be transformed into its equivalent SvTrNeLpP $(P6.5.3)$.

SvTrNeLpP $(\boldsymbol{P6.5.3})$

$Maximize\left((x_1 + 4x_2, 3x_1 + 6x_2, 4x_1 + 8x_2, 7x_1 +\right.$

$\left.10x_2; minimum\{0.8,0.9\}, maximum\{0.2,0.3\}, maximum\{0.4,0.5\})\right)$

Subject to

$(2x_1 + 3x_2, 4x_1 + 5x_2, 6x_1 + 9x_2, 8x_1 +$

$12x_2; minimum\{0.6,0.7\}, maximum\{0.1,0.2\}, maximum\{0.3,0.1\}) \preccurlyeq$

$(10,15,20,25; 0.6,0,0.5)$,

$(4x_1 + 3x_2, 7x_1 + 6x_2, 10x_1 + 9x_2, 13x_1 +$

$14x_2; minimum\{0.7,0.8\}, maximum\{0.4,0.5\}, maximum\{0.2,0.3\}) \preccurlyeq$

$(10,20,25,30; 0.9,0.45,0.3)$,

$x_1, x_2 \geq 0$.

Step 3: Using Step 3 of the proposed Mehar method, the SvTrNeLpP $(P6.5.3)$ can be transformed into its equivalent CrLpP $(P6.5.4)$.

CrLpP $(P6.5.4)$

$Maximize\left(V(x_1+4x_2,3x_1+6x_2,4x_1+8x_2,7x_1+10x_2;0.8,0.3,0.5)\right)$

Subject to

$V(2x_1+3x_2,4x_1+5x_2,6x_1+9x_2,8x_1+12x_2;0.6,0.2,0.3)\leq$

$V(10,15,20,25;0.6,0,0.5),$

$V(4x_1+3x_2,7x_1+6x_2,10x_1+9x_2,13x_1+14x_2;0.7,0.5,0.3)\leq$

$V(10,20,25,30;0.9,0.45,0.3),$

$x_1,x_2\geq 0,$

where,

$V(\tilde{A})=\lambda\left(\frac{a_1+2a_2+2a_3+a_4}{6}\right)w_{\tilde{A}}^2+(1-$

$\lambda)\left(\frac{[(2a_1+a_2+a_3+2a_4)-(a_1-a_2-a_3+a_4)u_{\tilde{A}}-(a_1+2a_2+2a_3+a_4)u_{\tilde{A}}^2]}{6}+\right.$

$\left.\frac{[(2a_1+a_2+a_3+2a_4)-(a_1-a_2-a_3+a_4)y_{\tilde{A}}-(a_1+2a_2+2a_3+a_4)y_{\tilde{A}}^2]}{6}\right),\lambda\in[0,1].$

Step 4: The obtained optimal solution of the CrLpP $(P6.5.4)$ for some values of $\lambda\in[0,1]$ are shown in Table 6.9. It is pertinent to mention that according to Step 4 of the proposed Mehar method, the obtained optimal solution also represents an optimal solution of the SvTrNeLpP $(P6.5.1)$.

Table 6.9 Correct optimal solution for different values of λ

λ	Optimal solution	
	x_1	x_2
0	0	2.243
0.1	0	2.248
0.2	0	2.252
0.3	0	2.258
0.4	0	2.266
0.5	0	2.275
0.6	0	2.287
0.7	0	2.304
0.8	0	2.329
0.9	0	2.368
1	0	2.442

6.6 Limitation of the proposed Mehar method

The proposed Mehar method is based upon the existing method [100], discussed in Definition 6.1.9, for comparing two SvTrNeNs. However, it can be easily verified that the existing comparing method [100] fails to distinguish two distinct SvTrNeNs. Hence, the proposed Mehar method can be used only if a unique optimal solution exists for the CrLpP $(P6.3.3)$.

6.7 Conclusions

It is shown that some mathematical incorrect results are considered in all existing methods for solving mathematical programming problems under neutrosophic environment. Hence, it is inappropriate to use any existing method to solve mathematical programming problems under neutrosophic environment. Also, a new method (named as Mehar method) is proposed to solve SvNeLpPs. Furthermore, correct optimal solutions of some existing real-life problems under neutrosophic environment [44, 79, 100] are obtained by the proposed Mehar method.

Chapter 7

Future Scope

The following open research problems may be considered as future research directions.

(i) To analyse the efficiency of the proposed methods for really large problems.

(ii) In the existing method [11], a multi-objective interval-valued FuLFpP with crisp decision variables is solved to find a non-dominated fuzzy optimal solution of a balanced multi-objective interval-valued FlFuLFtP. However, as discussed in Section 2.3 of Chapter 2, the decision variables should be considered as interval-valued fuzzy numbers. Therefore, it is inappropriate to use the existing method [11] to solve multi-objective interval-valued FlFuLFtPs. To propose an appropriate method to solve multi-objective interval-valued FlFuLFtPs is an open research problem.

(iii) In the existing method [62] for solving intuitionistic fully fuzzy multi-objective LFtPs as well as in the existing method [12] for solving fully fuzzy multi-objective LFpPs, the existing method [30] is used. However, in the literature [155], it is pointed out that it is inappropriate to use the existing method [30]. Hence, it is inappropriate to use the existing method [62] for solving intuitionistic fully fuzzy multi-objective LFtPs as well as it is inappropriate to use the existing method [12] for solving fully fuzzy multi-objective LFpPs. To propose an appropriate method to solve intuitionistic fully fuzzy multi-objective LFtPs as well as to propose an appropriate method to solve fully fuzzy multi-objective LFpPs are open research problems.

(iv) In the existing method [25], an intuitionistic FuLFpP with crisp decision variables is solved to find a fuzzy optimal solution of balanced intuitionistic FlFuLFtPs. However, as discussed in Section 2.3 of Chapter 2, the decision variables should be considered as intuitionistic fuzzy numbers. Therefore, it is inappropriate to use the existing method [25] to solve intuitionistic FlFuLFtPs. To propose an appropriate method to solve intuitionistic FlFuLFtPs is an open research problem.

(v) In the existing methods [7, 113], Charnes and Cooper transformation method under fuzzy environment is used to transform a fully fuzzy multi-objective LFpP into a fully fuzzy multi-objective LpP. However, as discussed in Section 2.4.2.3 of Chapter 2, the fully fuzzy multi-objective LpP, obtained by Charnes and Cooper transformation method under fuzzy environment, will not be equivalent to the fully fuzzy multi-objective LFpP. Hence, it is inappropriate to use the existing methods [7, 113] to solve fully fuzzy multi-objective LFpPs. To propose an appropriate method to solve fully fuzzy multi-objective LFpPs is an open research problem.

(vi) In Chapter 6, it is pointed out that it is inappropriate to use the existing methods for solving SvTNeLFpPs [2, 43, 80] and SvTrNeLFpPs [59]. To propose an appropriate method to solve SvTNeLFpPs as well as to propose an appropriate method to solve SvTrNeLFpPs are open research problems.

(vii) It can be easily verified that the relation $S(\tilde{A}_1\tilde{A}_2) = \left(S(\tilde{A}_1)\right)\left(S(\tilde{A}_2)\right)$ is considered in Khalifa and Kumar's method [97] for solving single-valued trapezoidal fully neutrosophic LpPs (LpPs in which all the parameters including decision variables are represented by SvTrNeNs),

where,

(a) $\tilde{A}_i = \left(a_{i1}, a_{i2}, a_{i3}; w_{\tilde{A}_i}, u_{\tilde{A}_i}, y_{\tilde{A}_i}\right); i = 1,2$ be non-negative SvTrNeNs i.e., $a_{i1} \geq 0$.

(b) $\tilde{A}_1\tilde{A}_2 = \left(\begin{matrix} a_{11}a_{21}, a_{12}a_{22}, a_{13}a_{23}, a_{14}a_{24}; \\ minimum\{w_{\tilde{A}_1}, w_{\tilde{A}_2}\}, maximum\{u_{\tilde{A}_1}, u_{\tilde{A}_2}\}, maximum\{y_{\tilde{A}_1}, y_{\tilde{A}_2}\}\end{matrix}\right)$.

(c) $S(\tilde{A}_i) = \frac{1}{16}(a_{i1} + a_{i2} + a_{i3} + a_{i4})\left(w_{\tilde{A}_i} + \left(1 - u_{\tilde{A}_i}\right) + \left(1 - y_{\tilde{A}_i}\right)\right); i = 1,2.$

While, the following example clearly indicates that $S(\tilde{A}_1\tilde{A}_2) \neq \left(S(\tilde{A}_1)\right)\left(S(\tilde{A}_2)\right)$.

Let $\tilde{A}_1 = (1,3,4,5; 0.1,0.8,0.1)$ and $\tilde{A}_2 = (3,4,6,7; 0.1,0.8,0.9)$ be two SvTrNeNs. Then,

$\tilde{A}_1\tilde{A}_2 = \left(\begin{matrix} 3,12,24,35; \\ minimum\{0.1,0.1\}, maximum\{0.8,0.8\}, maximum\{0.1,0.9\}\end{matrix}\right) =$

$(3,12,24,35; 0.1,0.8,0.9)$.

Using the existing expression (c) [97],

$S(\tilde{A}_1\tilde{A}_2) = S(3,12,24,35; 0.1,0.8,0.9)$

$$= \frac{1}{16}(3 + 12 + 24 + 35)\big(0.1 + (1 - 0.8) + (1 - 0.9)\big) = 1.85 \tag{7.1}$$

$S(\tilde{A}_1) = S(1,3,4,5; 0.1,0.8,0.1)$

$= \frac{1}{16}(1 + 3 + 4 + 5)\big(0.1 + (1 - 0.8) + (1 - 0.1)\big) = 0.975$

$S(\tilde{A}_2) = S(3,4,6,7; 0.1,0.8,0.9)$

$= \frac{1}{16}(3 + 4 + 6 + 7)\big(0.1 + (1 - 0.8) + (1 - 0.9)\big) = 0.5$

$$\left(S(\tilde{A}_1)\right)\left(S(\tilde{A}_2)\right) = (0.975)(0.5) = 0.4875 \tag{7.2}$$

It is obvious from (7.1) and (7.2) that $S(\tilde{A}_1\tilde{A}_2) \neq \left(S(\tilde{A}_1)\right)\left(S(\tilde{A}_2)\right)$.

Hence, it is inappropriate to use Khalifa and Kumar's method [97]. To propose an appropriate method to solve single-valued trapezoidal fully neutrosophic LpPs is an open research problem.

(viii) It is pertinent to mention that the shortcoming pointed out in Khalifa and Kumar's method [97] also occurs in the existing methods [23, 24] for solving SvTrNeLpPs with single-valued trapezoidal neutrosophic decision variables. Hence, it is inappropriate to use the existing methods [23, 24]. To propose an appropriate method to solve SvTrNeLpPs with single-valued trapezoidal neutrosophic decision variables is an open research problem.

(ix) Nishad and Abhishekh [130] proposed a method to solve NoTrFlItFuTpS/NoTFlItFuTpS (transportation problems in which each parameter is represented by a NoTrItFuN/NoTItFuN). Nishad and Abhishekh [130] claimed that their proposed method is superior than the existing methods [8, 103, 149]. In Appendix A, it is shown that Nishad and Abhishekh [130] have considered a mathematical incorrect result in their proposed method. Therefore, it is inappropriate to use Nishad and Abhishekh's method [130] to solve NoTrFlItFuTpS/NoTFlItFuTpS. To propose an appropriate method to resolve the inappropriateness of Nishad and Abhishekh's method [130] is an open research problem.

Appendix A

A note on "A new ranking approach for solving fully fuzzy transportation problem in intuitionistic fuzzy environment"[1]

Nishad and Abhishekh [130] proposed a method to solve NoTrFlItFuTpS/NoTFlItFuTpS (transportation problems in which each parameter is represented by a NoTrItFuN/NoTItFuN). Nishad and Abhishekh [130] claimed that their proposed method is superior than the existing methods [8, 103, 149]. In this appendix, it is shown that Nishad and Abhishekh [130] have considered a mathematical incorrect result in their proposed method. Therefore, it is inappropriate to use Nishad and Abhishekh's method [130] to solve NoTrFlItFuTpS/NoTFlItFuTpS.

A.1. Introduction

Nishad and Abhishekh [130] claimed that to find an optimal solution of a NoTrFlItFuTp/NoTFlItFuTp is equivalent to find an optimal solution of the FlItFuLpP (P_1). Hence, Nishad and Abhishekh [130] proposed a method to find an optimal solution of the FlItFuLpP (P_1).

Problem (P_1)

$Minimize\left(\tilde{Z} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij}\tilde{x}_{ij}\right)$

Subject to

[1] The contents of this appendix are accepted for publication in Journal of Control, Automation and Electrical Systems.

$\sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_i,$ $\quad i = 1,2, \dots, m,$

$\sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_j,$ $\quad j = 1,2, \dots, n,$

$\tilde{x}_{ij} \succcurlyeq \tilde{0} \; \forall \; i, j.$

where,

(i) m represents the number of sources.

(ii) n represents the number of destinations.

(iii) $\tilde{c}_{ij} = \left(c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4\right), \left(C_{ij}^1, C_{ij}^2, C_{ij}^3, C_{ij}^4\right)$ or $\tilde{c}_{ij} = \left(c_{ij}^1, c_{ij}^2, c_{ij}^4\right), \left(C_{ij}^1, C_{ij}^2, C_{ij}^4\right)$ represents the NoTrItFuN/NoTItFuN.

(iv) $\tilde{x}_{ij} = \left(x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4\right), \left(X_{ij}^1, X_{ij}^2, X_{ij}^3, X_{ij}^4\right)$ or $\tilde{x}_{ij} = \left(x_{ij}^1, x_{ij}^2, x_{ij}^4\right), \left(X_{ij}^1, X_{ij}^2, X_{ij}^4\right)$ represents the NoTrItFuN/NoTItFuN.

(v) $\tilde{a}_i = (a_i^1, a_i^2, a_i^3, a_i^4), (A_i^1, A_i^2, A_i^3, A_i^4)$ or $\tilde{a}_i = (a_i^1, a_i^2, a_i^4), (A_i^1, A_i^2 A_i^4)$ represents the NoTrItFuN/NoTItFuN.

(vi) $\tilde{b}_j = \left(b_j^1, b_j^2, b_j^3, b_j^4\right), \left(B_j^1, B_j^2, B_j^3, B_j^4\right)$ or $\tilde{b}_j = \left(b_j^1, b_j^2, b_j^4\right), \left(B_j^1, B_j^2, B_j^4\right)$ represents the NoTrItFuN/NoTItFuN.

(vii) Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}, a_{14}), (A_{11}, A_{12}, A_{13}, A_{14})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}, a_{24}), (A_{21}, A_{22}, A_{23}, A_{24})$ be two NoTrItFuNs. Then, $\tilde{A}_1$ and $\tilde{A}_2$ can be compared as follows.

Check whether $\Re\left(\tilde{A}_1, \delta\right) > \Re\left(\tilde{A}_2, \delta\right)$ or $\Re\left(\tilde{A}_1, \delta\right) < \Re\left(\tilde{A}_2, \delta\right)$ or $\Re\left(\tilde{A}_1, \delta\right) = \Re\left(\tilde{A}_2, \delta\right)$.

(a) If $\Re\left(\tilde{A}_1, \delta\right) > \Re\left(\tilde{A}_2, \delta\right)$ then, $\tilde{A}_1 \succ \tilde{A}_2$.

(b) If $\Re\left(\tilde{A}_1, \delta\right) \geq \Re\left(\tilde{A}_2, \delta\right)$ then, $\tilde{A}_1 \succcurlyeq \tilde{A}_2$.

(c) If $\Re\left(\tilde{A}_1, \delta\right) = \Re\left(\tilde{A}_2, \delta\right)$ then, $\tilde{A}_1 = \tilde{A}_2$.

where,

$\Re\left(\tilde{A}_k, \delta\right) = I_\alpha\left(\tilde{A}_k\right) + \delta I'_\alpha\left(\tilde{A}_k\right); k = 1,2,$

$$\delta = \begin{cases} 0 & \text{if } I_\alpha(\tilde{A}_1) \neq I_\alpha(\tilde{A}_2) \\ 1 & \text{if } I_\alpha(\tilde{A}_1) = I_\alpha(\tilde{A}_2) \end{cases}; k = 1,2.$$

$$I_\alpha(\tilde{A}_k) = \frac{1}{12}\big(a_{k1} + a_{k4} + 2(a_{k2} + a_{k3}) + A_{k1} + A_{k4} + 2(A_{k2} + A_{k3})\big); k = 1,2,$$

$$I'_\alpha(\tilde{A}_k) = \frac{1}{12}\big(a_{k4} - a_{k1} + 2(a_{k3} - a_{k2}) + A_{k4} - A_{k1} + 2(A_{k3} - A_{k2})\big); k = 1,2.$$

In Nishad and Abhishekh's method [130] firstly, an initial basic feasible solution is obtained by using any appropriate method. Then, the constraints of the dual problem (P_2) are used to check that the obtained initial basic feasible solution is optimal or not.

Problem (P_2)

$Maximize\ \left(\tilde{Z} \approx \left(\sum_{i=1}^{m} \tilde{u}_i\, \tilde{a}_i + \sum_{j=1}^{n} \tilde{v}_j\, \tilde{b}_j\right)\right)$

Subject to

$\tilde{u}_i + \tilde{v}_j \preccurlyeq \tilde{c}_{ij}, \qquad i = 1,2,\dots,m, j = 1,2,\dots,n,$

$\tilde{u}_i, \tilde{v}_j$ are unrestricted in sign.

where, $\tilde{u}_i, \tilde{v}_j$ represents the fuzzy dual variables.

In Section A.3., it is shown that the FlItFuLpP (P_2) does not represents the dual of the FlItFuLpP (P_1). Hence, it is inappropriate to use Nishad and Abhishekh's method [130] to solve the FlItFuLpP (P_1).

A.2. Origin of the dual problem

Nishad and Abhishekh [130] have used the following method to obtain the dual problem (P_2) corresponding to the FlItFuLpP (P_1).

Step 1: Using the relation $\tilde{A}_1 \succcurlyeq \tilde{A}_2$ if $\Re(\tilde{A}_1, \delta) \geq \Re(\tilde{A}_2, \delta)$, the FlItFuLpP ($P_1$) can be transformed into its equivalent CrLpP (P_3).

Problem (P_3)

$Minimize\ \left(\Re(\tilde{Z}, \delta) = \Re\left(\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij}\tilde{x}_{ij}, \delta\right)\right)$

Subject to

$\Re\left(\sum_{j=1}^{n} \tilde{x}_{ij}, \delta\right) = \Re(\tilde{a}_i, \delta), \qquad i = 1,2, \dots, m,$

$\Re\left(\sum_{i=1}^{m} \tilde{x}_{ij}, \delta\right) = \Re\left(\tilde{b}_j, \delta\right), \qquad j = 1,2, \dots, n,$

$\Re\left(\tilde{x}_{ij}, \delta\right) \geq 0 \;\forall\; i, j.$

Step 2: Using the relation $\Re\left(\tilde{A}_1 + \tilde{A}_2, \delta\right) = \Re\left(\tilde{A}_1, \delta\right) + \Re\left(\tilde{A}_2, \delta\right)$, the CrLpP (P_3) can be transformed into its equivalent CrLpP (P_4).

Problem (P_4)

$Minimize\ \left(\Re\left(\tilde{Z}, \delta\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \Re\left(\tilde{c}_{ij}\tilde{x}_{ij}\right), \delta\right)$

Subject to

$\sum_{j=1}^{n} \Re\left(\tilde{x}_{ij}, \delta\right) = \Re(\tilde{a}_i, \delta), \qquad i = 1,2, \dots, m,$

$\sum_{i=1}^{m} \Re\left(\tilde{x}_{ij}, \delta\right) = \Re\left(\tilde{b}_j, \delta\right), \qquad j = 1,2, \dots, n,$

$x_{ij} \geq 0 \;\forall i, j.$

Step 3: Using the relation $\Re\left(\tilde{A}_1\tilde{A}_2, \delta\right) = \Re\left(\tilde{A}_1, \delta\right)\Re\left(\tilde{A}_2, \delta\right)$, the CrLpP (P_4) can be transformed into its equivalent CrLpP (P_5).

Problem (P_5)

$Minimize\ \left(\Re\left(\tilde{Z}, \delta\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \Re\left(\tilde{c}_{ij}, \delta\right)\Re\left(\tilde{x}_{ij}, \delta\right)\right)$

Subject to

Constraints of the problem (P_4).

Step 4: Assuming $\Re\left(\tilde{c}_{ij}, \delta\right) = c_{ij}, \Re\left(\tilde{x}_{ij}, \delta\right) = x_{ij}, \Re(\tilde{a}_i, \delta) = a_i, \Re\left(\tilde{b}_j, \delta\right) = b_j$ and $\Re\left(\tilde{Z}, \delta\right) = Z$, the CrLpP (P_5) can be transformed into its equivalent CrLpP (P_6).

Problem (P_6)

$Minimize\ \left(Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}\right)$

Subject to

$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1,2, \dots, m,$

$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1,2, \dots, n,$

$x_{ij} \geq 0 \;\forall\; i, j.$

Step 5: Using duality theory, the problem (P_7) represents the dual of the CrLpP (P_6).

Problem (P_7)

$Maximize\left(Z = \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j\right)$

Subject to

$u_i + v_j \leq c_{ij}, \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$

u_i, v_j are unrestricted in sign.

where, u_i, v_j represents the dual variables.

Step 6: Assuming $c_{ij} = \Re(\tilde{c}_{ij}, \delta), x_{ij} = \Re(\tilde{x}_{ij}, \delta), a_i = \Re(\tilde{a}_i, \delta), b_j = \Re(\tilde{b}_j, \delta), u_i = \Re(\tilde{u}_i, \delta),$ $v_j = \Re(\tilde{v}_j, \delta)$ and $Z = \Re(\tilde{Z}, \delta)$, the CrLpP (P_7) can be transformed into its equivalent CrLpP (P_8).

Problem (P_8)

$Maximize\left(\Re(\tilde{Z}, \delta) = \sum_{i=1}^{m} \Re(\tilde{u}_i, \delta)\Re(\tilde{a}_i, \delta) + \sum_{j=1}^{n} \Re(\tilde{v}_j, \delta)\Re(\tilde{b}_j, \delta)\right)$

Subject to

$\Re(\tilde{u}_i, \delta) + \Re(\tilde{v}_j, \delta) \leq \Re(\tilde{c}_{ij}, \delta), \qquad i = 1,2, \dots, m, j = 1,2, \dots, n,$

u_i, v_j are unrestricted in sign.

Step 7: Using the relation $\Re(\tilde{A}_1, \delta)\Re(\tilde{A}_2, \delta) = \Re(\tilde{A}_1\tilde{A}_2, \delta)$, the CrLpP ($P_8$) can be transformed into its equivalent CrLpP (P_9).

Problem (P_9)

$Maximize\left(\Re(\tilde{Z}, \delta) = \sum_{i=1}^{m} \Re(\tilde{u}_i\tilde{a}_i, \delta) + \sum_{j=1}^{n} \Re(\tilde{v}_j\tilde{b}_j, \delta)\right)$

Subject to

Constraints of the problem (P_8).

Step 8: Using the relation $\Re(\tilde{A}_1, \delta) + \Re(\tilde{A}_2, \delta) = \Re(\tilde{A}_1 + \tilde{A}_2, \delta)$, the CrLpP ($P_9$) can be transformed into its equivalent CrLpP (P_{10}).

Problem (P_{10})

$Maximize\left(\Re(\tilde{Z}, \delta) = \Re\left(\sum_{i=1}^{m} \tilde{u}_i\, \tilde{a}_i + \sum_{j=1}^{n} \tilde{v}_j\, \tilde{b}_j, \delta\right)\right)$

Subject to

Constraints of the problem (P_8).

Step 9: The CrLpP (P_{10}) can be transformed into its equivalent FlItFuLpP (P_2).

A.3. Inappropriateness of Nishad and Abhishekh's method

Nishad and Abhishekh's method [130] to solve NoTrFlItFuTpS/NoTFlItFuTpS is not appropriate as this method is based upon the dual problem (P_2). While, the dual problem (P_2) is not valid as the following mathematical incorrect result is considered to obtain it from the FlItFuLpP (P_1).

It is obvious from Step 3 and Step 7 of Nishad and Abhishekh's method [130], discussed in Section A.2., that to transform the CrLpP (P_4) into its equivalent CrLpP (P_5) and to transform the CrLpP (P_8) into its equivalent CrLpP (P_9) respectively, Nishad and Abhishekh [130] have assumed that the relation $\Re(\tilde{A}_1\tilde{A}_2,,\delta) = \Re(\tilde{A}_1,\delta)\Re(\tilde{A}_2,\delta)$ will be satisfied for two NoTrItFuNs/NoTItFuNs $\tilde{A}_1$ and $\tilde{A}_2$. While, the following clearly indicates that $\Re(\tilde{A}_1\tilde{A}_2,\delta) \neq \Re(\tilde{A}_1,\delta)\Re(\tilde{A}_2,\delta)$. Therefore, the problem (P_2) does not represent the dual problem of the FlItFuLpP (P_1). Hence, it is inappropriate to use Nishad and Abhishekh's method [130] to solve NoTrFlItFuTpS/NoTFlItFuTpS.

Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}, a_{14}), (A_{11}, A_{12}, A_{13}, A_{14})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}, a_{24}), (A_{21}, A_{22}, A_{23}, A_{24})$ be two non-negative NoTrItFuNs. Then, using the existing multiplication of two non-negative NoTrItFuNs [130],

$\tilde{A}_1\tilde{A}_2 = (a_{11}a_{21}, a_{12}a_{22}, a_{13}a_{23}, a_{14}a_{24}), (A_{11}A_{21}, A_{12}A_{22}, A_{13}A_{23}, A_{14}A_{24})$.

Therefore, using the relation $\Re(\tilde{A},\delta) = \Re\left(\left((a_1, a_2, a_3, a_4), (A_1, A_2, A_3, A_4)\right),\delta\right) = I_\alpha(\tilde{A}) + \delta I'_\alpha(\tilde{A}) = I_\alpha(\tilde{A}) = \frac{1}{12}\left(a_1 + a_4 + 2(a_2 + a_3) + A_1 + A_4 + 2(A_2 + A_3)\right)$

$$\Re(\tilde{A}_1\tilde{A}_2,\delta) = \Re\left(\left((a_{11}a_{21}, a_{12}a_{22}, a_{13}a_{23}, a_{14}a_{24}), (A_{11}A_{21}, A_{12}A_{22}, A_{13}A_{23}, A_{14}A_{24})\right),\delta\right) =$$

$$\frac{1}{12}\left(a_{11}a_{21} + a_{14}a_{24} + 2(a_{12}a_{22} + a_{13}a_{23}) + A_{11}A_{21} + A_{14}A_{24} + 2(A_{12}A_{22} + A_{13}A_{23})\right) \quad \text{(A1)}$$

$\Re(\tilde{A}_1,\delta) = \Re\left(((a_{11},a_{12},a_{13},a_{14}),(A_{11},A_{12},A_{13},A_{14})),\delta\right) = \frac{1}{12}\big(a_{11}+a_{14}+2(a_{12}+a_{13})+ A_{11}+A_{14}+2(A_{12}+A_{13})\big)$

$\Re(\tilde{A}_2,\delta) = \Re\left(((a_{21},a_{22},a_{23},a_{24}),(A_{21},A_{22},A_{23},A_{24})),\delta\right) = \frac{1}{12}\big(a_{21}+a_{24}+2(a_{22}+a_{23})+ A_{21}+A_{24}+2(A_{22}+A_{23})\big)$

$$\Re(\tilde{A}_1,\delta)\Re(\tilde{A}_2,\delta) = \left(\frac{1}{12}\big(a_{11}+a_{14}+2(a_{12}+a_{13})+A_{11}+A_{14}+2(A_{12}+A_{13})\big)\right)\left(\frac{1}{12}\big(a_{21}+ a_{24}+2(a_{22}+a_{23})+A_{21}+A_{24}+2(A_{22}+A_{23})\big)\right) = \frac{1}{144}\big(a_{11}+a_{14}+2(a_{12}+a_{13})+A_{11}+ A_{14}+2(A_{12}+A_{13})\big)\big(a_{21}+a_{24}+2(a_{22}+a_{23})+A_{21}+A_{24}+2(A_{22}+A_{23})\big) \qquad \text{(A2)}$$

It is obvious from (A1) and (A2) that $\Re(\tilde{A}_1\tilde{A}_2,\delta) \neq \Re(\tilde{A}_1,\delta)\Re(\tilde{A}_2,\delta)$.

A.4. Conclusions

It is shown that the FlItFuLpP (P_2) does not represent the dual problem of the FlItFuLpP (P_1). Hence, it is inappropriate to use Nishad and Abhishekh's method [130] to solve NoTrFlItFuTpS/NoTFlItFuTpS.

Bibliography

1. M. Abdel-Basset, M. Gunasekaran, M. Mohamed and F. Smarandache, A novel method for solving the fully neutrosophic linear programming problems, Neural Computing and Applications 31 (2019) 1595-1605.
2. M. Abdel-Basset, M. Mohamed and F. Smarandache, Linear fractional programming based on triangular neutrosophic numbers, International Journal of Applied Management Science 11 (2019) 1-20.
3. W. Abdelfattah, A parametric approach to solve neutrosophic linear programming models, Journal of Information and Optimization Sciences 42 (2021) 631-654.
4. R. K. Ahuja, T. L. Magnanti and J. B. Orlin, Network flows: Theory, algorithms and applications, Prentice-Hall, Englewood Cliffs, NJ, (1993).
5. Z. Ali, T. Mahmood and M. S. Yang, Complex T-spherical fuzzy aggregation operators with application to multi-attribute decision making, Symmetry 12 (2020) 2020.
6. D. Almaatani, S. G. Diagne, Y. Gningue and P. M. Takouda, Solving the linear transportation problem by modified vogel method. Interdisciplinary Topics in Applied Mathematics, Modeling and Computational Science (2015) (pp. 13-19) Springer, Cham.
7. E. E. Ammar and H. A. Khalifa, Solving fully fuzzy multi-objective linear fractional programming problems based on fuzzy programming approach, The Journal of Fuzzy Mathematics 27 (2019) 301-312.

8. R. J. P. Antony, S. J. Savarimuthu and T. Pathinathan, Method for solving the transportation problem using triangular intuitionistic fuzzy number, International Journal of Computing Algorithm 3 (2014) 590-605.
9. P. Anukokila, A. Anju and B. Radhakrishnan, Lexicographic approach for solving fully fuzzy fractional transportation problem, International Journal of Pure and Applied Mathematics 117 (2017) 117-127.
10. P. Anukokila and B. Radhakrishnan, Goal programming approach to fully fuzzy fractional transportation problem, Journal of Taibah University for Science 13 (2019) 864-874.
11. P. Anukokila, B. Radhakrishnan and A. Anju, Goal programming approach for solving multi-objective fractional transportation problem with fuzzy parameters, RAIRO-Operations Research 53 (2019) 157-178.
12. R. Arya, P. Singh, S. Kumari and M. S. Obaidat, An approach for solving fully fuzzy multi-objective linear fractional optimization problems, Soft Computing 24 (2020) 9105-90119.
13. K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87-96.
14. K. Atanassov, P. Vassilev, E. Szmidt and J. Kacprzyk, On interval valued intuitionistic fuzzy pairs, Journnal of Universal Mathematics 1 (2018) 261-268.
15. E. Badr, S. Nada, S. Ali and A. Elrokh, Solving neutrosophic linear programming problems using exterior point simplex algorithm, Neutrosophic Sets and Systems 45 (2021) 320-339.
16. E. M. Badr, M. A. Salam and F. Smarandache, Solving neutrosophic linear programming problems with two-phase approach, In Neutrosophic Sets in Decision Analysis and Operations Research (2020) (pp. 391-412). IGI Global.

17. L. Bai, C. Shi, Y. Guo, Q. Du and Y. Huang, Quality risk evaluation of the food supply chain using a fuzzy comprehensive evaluation model and failure mode, effects, and criticality analysis, Journal of Food Quality 2018 (2018) 2637075.
18. E. B. Bajalinov, Linear-fractional programming: Theory, methods, applications and software, Kluwer Academic Publishers, Boston, (2003).
19. B. Basumatary and B. Said, Interval-valued triangular neutrosophic linear programming problem, International Journal of Neutrosophic Science 10 (2020) 105-115.
20. M. S. Bazaraa, J. J. Jarvis and H. D. Sherali, Linear Programming and Network Flows. Third edition, John Wiley & Sons, New-York, (2005).
21. C. R. Bector and S. Chandra, Fuzzy mathematical programming and fuzzy matrix games, Springer, Berlin, (2005).
22. T. Bera and N. K. Mahapatra, Generalised single valued neutrosophic number and its application to neutrosophic linear programming, Neutrosophic Sets and Systems 25 (2019) 85-103.
23. T. Bera and N. K. Mahapatra, Neutrosophic linear programming problem and its application to real life, Afrika Matematika 31 (2020) 709-726.
24. T. Bera and N. K. Mahapatra, An approach to solve the linear programming problem using single valued trapezoidal neutrosophic number, International Journal of Neutrosophic Science 3 (2020) 54-66.
25. S. K. Bharati, Trapezoidal intuitionistic fuzzy fractional transportation problem, Soft Computing for Problem Solving. Advances in Intelligent Systems and Computing, 817 (2019) 833-842, Springer, Singapore.

26. H. Bordbar, S. Z. Song, M. R. Bordbar and Y. B. Jun, Fuzzy soft set theory with applications in hyper BCK-algebras, Journal of Intelligent & Fuzzy Systems 38 (2020) 1789-1797.
27. S. Broumi, M. Talea, A. Bakali, F. Smarandache, D. Nagarajan, M. Lathamaheswari and M. Parimala, Shortest path problem in fuzzy, intuitionistic fuzzy and neutrosophic environment: an overview, Complex & Intelligent Systems 5 (2019) 371-378.
28. H. Bustince, E. Barrenechea, M. Pagola, J. Fernandez, Z. Xu, B. Bedregal, J. Montero, H. Hagras, F. Herrera and B. D. Baets, A historical account of types of fuzzy sets and their relationships, IEEE Transactions on Fuzzy Systems 24 (2016) 179-194.
29. O. Castillo, P. K. Muhuri, H. Mo, C. Li, R. Li and H. R. Karimi, Editorial on Special Issue: "Trends and Developments on Type-2 Fuzzy Sets and Systems", International Journal of Fuzzy Systems 23 (2021) 1055-1056.
30. M. Chakraborty and S. Gupta, Fuzzy mathematical programming for multi-objective linear fractional programming problem, Fuzzy Sets and Systems 125 (2002) 335-342.
31. D. Chakraborty, D. K. Jana and T. K. Roy, A new approach to solve fully fuzzy transportation problem using triangular fuzzy number, International Journal of Operational Research 26 (2016) 153-179.
32. S. Chanas, W. Kolodziejczyk and A. Machaj, A fuzzy approach to the transportation problem, Fuzzy Sets and Systems 13 (1984) 211-221.
33. S. Chandra, Jayadeva and A. Mehra, Numerical Optimization with Applications, Narosa Publishing House, New Delhi, (2009).
34. A. Charnes and W. W. Cooper, Programming with linear fractional functionals, Naval Research Logistics Quarterly 9 (1962) 181-186.

35. Y. Chen, M. Munir, T. Mahmood, A. Hussain and S. Zeng, Some generalized T-spherical and group-generalized fuzzy geometric aggregation operators with application in MADM problems, Journal of Mathematics 2021 (2021) 5578797.
36. D. Chhibber, P. K. Srivastava and D. C. Bisht, From fuzzy transportation problem to non-linear intuitionistic fuzzy multi-objective transportation problem: A literature review, International Journal of Modelling and Simulation 41 (2021) 335-350.
37. B. C. Cuong, Picture fuzzy sets, Journal of Computer Science and Cybernetics 30 (2014) 409-420.
38. H. Dalman and M. Sivri, A fuzzy logic based approach to solve interval multiobjective nonlinear transportation problem, Proceedings of the National Academy of Sciences, India, Section A: Physical Sciences 89 (2019) 279-289.
39. G. B. Dantzig, Linear programming and extensions, Princeton University Press, (1963).
40. B. A. Dar, A. Jayswal and D. Singh, Optimality, duality and saddle point analysis for interval-valued nondifferentiable multiobjective fractional programming problems, Optimization 70 (2021) 1275-1305.
41. S. Das and J. Dash, Modified solution for neutrosophic linear programming problems with mixed constraints, International Journal of Research in Industrial Engineering 9 (2020) 13-24.
42. S. K. Das and S. A. Edalatpanah, A new ranking function of triangular neutrosophic number and its application in integer programming, International Journal of Neutrosophic Science 4 (2020) 82-92.

43. S. K. Das, S. A. Edalatpanah and J. K. Dash, An intelligent dual simplex method to solve triangular neutrosophic linear fractional programming problem, Neutrosophic Sets and Systems 36 (2020) 50-69.
44. S. K. Das, S. A. Edalatpanah and J. K. Dash, A novel lexicographical-based method for trapezoidal neutrosophic linear programming problem, Neutrosophic Sets and Systems 46 (2021) 151-179.
45. S. K. Das, S. A. Edalatpanah and T. Mandal, Development of unrestricted fuzzy linear fractional programming problems applied in real case, Fuzzy Information and Engineering 13 (2021) 184-195.
46. S. K. Das and T. Mandal, A MOLFP method for solving linear fractional programming under fuzzy environment, International Journal of Research in Industrial Engineering 6 (2017) 202-213.
47. S. K. Das, T. Mandal and S. A. Edalatpanah, A note on "A new method for solving fully fuzzy linear fractional programming with a triangular fuzzy numbers", Applied Mathematics and Computational Intelligence 4 (2015) 361-367.
48. S. K. Das, T. Mandal and S. A. Edalatpanah, A new approach for solving fully fuzzy linear fractional programming problems using the multi-objective linear programming, RAIRO-Operations Research 51 (2017) 285-297.
49. M. Deb, A study of fully fuzzy linear fractional programming problems by signed distance ranking technique, Optimization techniques for problem solving in uncertainty (2018) 73-115, IGI Global.
50. M. Deb and P. K. De, Optimal solution of a fully fuzzy linear fractional programming problem by using graded mean integration representation method, Applications and Applied Mathematics 10 (2015) 571-587.

51. I. Deli and Y. Şubas, A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems, International Journal of Machine Learning and Cybernetics 8 (2017) 1309-1322.

52. K. Devi and S. P. Yadav, A multicriteria intuitionistic fuzzy group decision making for plant location selection with ELECTRE method, The International Journal of Advanced Manufacturing Technology 66 (2013) 1219-1229.

53. S. Dhaneskar, S. Hariharan and P. Sekar, Fuzzy Hugarian MODI algorithm to solve fully fuzzy transportation problems, International Journal of Fuzzy Systems 19 (2017) 1479-1491.

54. E. W. Dijkstra, A note on two problems in connexion with graphs, Numerische Mathematik 1 (1959) 269-271.

55. D. Dubey, S. Chandra and A. Mehra, Fuzzy linear programming under interval uncertainty based on IFS representation, Fuzzy Sets and Systems 188 (2012) 68-87.

56. D. Dubois and H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, (1980).

57. A. Ebrahimnejad, S. Tabatabaei and F. J. Santos-Arteaga, A novel lexicographic optimization method for solving shortest path problems with interval-valued triangular fuzzy arc weights, Journal of Intelligent & Fuzzy Systems 39 (2020) 1277-1287.

58. A. M. ElHadidi, O. E. Emam and A. M. Abdelfadel, New method to optimize the fully neutrosophic linear programming problems, Applied Mathematics & Information Sciences 15 (2021) 211-216.

59. A. M. ElHadidi, O. E. Emam, A. M. Abdelfadel and M. Lotayif, Linear fractional programming based on trapezoidal neutrosophic numbers, Applied Mathematics & Information Sciences 15 (2021) 571-576.

60. O. E. Emam, M. A. Abdel-Fattah and S. M. Azzam, Interactive approach for solving three-level linear programming problem with neutrosophic parameters in the objective functions, Journal of Statistics Applications & Probability 8 (2019) 219-227.

61. O. E. Emam, M. A. Abdel-Fattah and S. M. Azzam, A decomposition algorithm for solving multi-level large-scale linear programming problems with neutrosophic parameters in the constrains, Applied Mathematics & Information Sciences 14 (2020) 913-919.

62. M. A. El Sayed and M. A. Abo-Sinna, A novel approach for fully intuitionistic fuzzy multi-objective fractional transportation problem, Alexandria Engineering Journal 60 (2021) 1447-1463.

63. M. Enayattabar, A. Ebrahimnejad and H. Motameni, Dijkstra algorithm for shortest path problem under interval-valued Pythagorean fuzzy environment, Complex & Intelligent Systems 5 (2019) 93-100.

64. M. Enayattabr, A. Ebrahimnejad, H. Motameni and H. Garg, A novel approach for solving all-pairs shortest path problem in an interval-valued fuzzy network, Journal of Intelligent & Fuzzy Systems 37 (2019) 6865-6877.

65. X. Fu, T. Ouyang, Z. Yang and S. Liu, A product ranking method combining the features–opinion pairs mining and interval-valued Pythagorean fuzzy sets, Applied Soft Computing 97 (2020) 106803.

66. H. Garg, M. Munir, K. Ullah, T. Mahmood and N. Jan, Algorithm for T-spherical fuzzy multi-attribute decision making based on improved interactive aggregation operators, Symmetry 10 (2018) 670.
67. I. Georgescu, Inventory problems with fuzzy numbers as demands, Soft Computing 26 (2022) 3947-3955.
68. R. Ghanbari, K. Ghorbani-Moghadam, N. Mahdavi-Amiri and B. De Baets, Fuzzy linear programming problems: models and solutions, Soft Computing 24 (2020) 10043-10073.
69. S. K. Gupta, D. Dangar, I. Ahmad and S. Al-Homidan, Duality in non-linear programming problems under fuzzy environment with exponential membership functions, Journal of Inequalities and Applications 218 (2018) 1-10.
70. P. Gupta and M. K. Mehlawat, An algorithm for a fuzzy transportation problem to select a new type of coal for a steel manufacturing unit, TOP 15 (2007) 114-137.
71. A. Gupta, A. Mehra and S. S. Appadoo, Mixed solution strategy for MCGDM problems using entropy/cross entropy in interval-valued intuitionistic fuzzy environment, International Game Theory Review 17 (2015) 1540007.
72. M. Gupta and B. K. Mohanty, An algorithmic approach to group decision making problems under fuzzy and dynamic environment, Expert Systems with Applications 55 (2016) 118-132.
73. G. Hadley, Linear Programming, Berkley, Addison-Wesley, Reading, MA (1962).
74. E. Haktanır and C. Kahraman, A novel interval-valued Pythagorean fuzzy QFD method and its application to solar photovoltaic technology development, Computers & Industrial Engineering 132 (2019) 361-372.

75. K. B. Haley, New methods in mathematical programming-the solid transportation problem, Operations Research 10 (1962) 448-463.

76. F. L. Hitchcock, The distribution of a product from several sources to numerous localities, Journal of Mathematics and Physics 20 (1941) 224-230.

77. M. Hojati and A. S. Patil, An integer linear programming-based heuristic for scheduling heterogeneous, part-time service employees, European Journal of Operational Research 209 (2011) 37-50.

78. W. Huang, F. Zhang and S. Xu, A complete ranking method for interval-valued intuitionistic fuzzy numbers and its applications to multicriteria decision making, Soft Computing 25 (2021) 2513-2520.

79. A. Hussian, M. Mohamed, M. Abdel-Baset and F. Smarandache, Neutrosophic linear programming problems, Neutrosophic Operational Research I (2017) 15-27.

80. A. Hussian, M. Mohamed, A. Mohamed, Z. Yongquan and F. Smarandache, Neutrosophic linear fractional programming problems, Neutrosophic Operational Research I (2018) 29-47.

81. M. Inuiguchi, Z. Gao and C. O. Henriques, Robust optimality analysis of non-degenerate basic feasible solutions in linear programming problems with fuzzy objective coefficients, Fuzzy Optimization and Decision Making (2022) https://doi.org/10.1007/s10700-022-09383-2

82. V. A. Jadhav and D. M. Doke, Solution procedure to solve fractional transportation problem with fuzzy cost and profit coefficients, International Journal of Mathematics and Computer Research 4 (2016) 1554-1562.

83. N. Jan, M. Aslam, K. Ullah, T. Mahamood and J. Wang, An approach towards decision making and shortest path problems using the concepts of interval-valued

Pythagorean fuzzy information, International Journal of Intelligent Systems 34 (2019) 2403-2428.

84. Y. Ji, Y. Xu, S. Qu, Z. Xu, Z. Wu and M. Nabe, A novel two-stage multi-criteria decision-making method based on interval-valued Pythagorean fuzzy aggregation operators with self-confidence levels, Arabian Journal for Science and Engineering 46 (2021) 1561-1584.

85. Y. Ju, Y. Liang, C. Luo, P. Dong, E. D. Gonzalez and A. Wang, T-spherical fuzzy TODIM method for multi-criteria group decision-making problem with incomplete weight information, Soft Computing 25 (2021) 2981-3001.

86. Y. Kacher and P. Singh, A comprehensive literature review on transportation problems, International Journal of Applied and Computational Mathematics 7 (2021) 206.

87. J. Kacprzyk and M. Krawczak, On an inexact transportation problem, Lecture Notes in Control and Information Sciences 23 (2005) 373-379.

88. S. Kalyani, L. Maragatham and S. Nagarani, An algorithm for linear fuzzy fractional transportation problem, International Journal of Engineering Technology, Management and Applied Sciences 4 (2016) 61-68.

89. R. Kar, A. K. Shaw and B. Das, Neutrosophic fuzzy number and its application in solving linear programming problem by simplex algorithm: An alternative approach, Annals of Optimization Theory and Practice 4 (2021) 29-39.

90. H. S. Kasana and K. D. Kumar, Introductory Operations Research: Theory and Applications, Springer Science & Business Media, Springer-Verlag, Berlin Heidelberg, (2013).

91. A. Kaufmann and M. M. Gupta, Introduction to Fuzzy Arithmetic Theory and Applications, Van Nostrand Reinhold, New York, (1985).

92. A. Kaufmann and M. M. Gupta, Fuzzy Mathematical Models in Engineering and Management science, Amsterdam Netherland, Elsevier, (1988).

93. A. Kaur, J. Kacprzyk and A. Kumar, Fuzzy transportation and transshipment problems, Springer Nature Switzerland AG, Switzerland, (2020).

94. J. Kaur and A. Kumar, An introduction to fuzzy linear programming problems, Springer International Publishing, Switzerland, (2016).

95. J. Kaur and A. Kumar, A novel method for solving fully fuzzy linear fractional programming problems, Journal of Intelligent & Fuzzy Systems 33 (2017) 1983-1990.

96. M. Kaur and R. Sadiq, A new method for solving single- and multi-objective capacitated solid minimum cost flow problems under uncertainty, Journal of Intelligent Systems 25 (2016) 159-183.

97. H. A. E. W. Khalifa and P. Kumar, Solving fully neutrosophic linear programming problem with application to stock portfolio selection, Croatian Operational Research Review 11 (2020) 165-176.

98. M. S. A. Khan and S. Abdullah, Interval-valued Pythagorean fuzzy GRA method for multiple-attribute decision making with incomplete weight information, International Journal of Intelligent Systems 33 (2018) 1689-1716.

99. M. A. Khan, I. Ahmad and A. Aljohani, Criterion for generalized weakly fuzzy invex monotonocities, Advances in Fuzzy Systems 2018 (2018) 3749650.

100. K. Khatter, Neutrosophic linear programming using possibilistic mean, Soft Computing 24 (2020) 16847-16867.

101. G. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall, New Jersey, (1995).
102. R. Kumar, S. A. Edalatpanah, S. Jha and R. Singh, A Pythagorean fuzzy approach to the transportation problem, Complex & Intelligent Systems 5 (2019) 255-263.
103. P. S. Kumar and R. J. Hussain, Computationally simple approach for solving fully intuitionistic fuzzy real life transportation problems, International Journal of System Assurance Engineering and Management 7 (2016) 90-101.
104. A. Kumar and J. Kaur, Fuzzy optimal solution of fully fuzzy linear programming problems using ranking function, Journal of Intelligent & Fuzzy Systems 26 (2014) 337-344.
105. A. Kumar, J. Kaur and P. Singh, A new method for solving fully fuzzy linear programming problems, Applied Mathematical Modelling 35 (2011) 817-823.
106. K. Lachhwani, Solving the general fully neutrosophic multi-level multiobjective linear programming problems, OPSEARCH 58 (2021) 1192-1216.
107. S. M. Lee and L. J. Moore, Optimizing transportation problems with multiple objectives, AIIE transactions 5 (1973) 333-338.
108. H. Li, L. Lv, F. Li, L. Wang and Q. Xia, A novel approach to emergency risk assessment using FMEA with extended MULTIMOORA method under interval-valued Pythagorean fuzzy environment, International Journal of Intelligent Computing and Cybernetics 13 (2020) 41-65.
109. D. Liang, A. P. Darko and J. Zeng, Interval-valued Pythagorean fuzzy power average-based MULTIMOORA method for multi-criteria decision-making, Journal of Experimental & Theoretical Artificial Intelligence 32 (2020) 845-874.

110. S. T. Liu, Fractional transportation problem with fuzzy parameters, Soft Computing 20 (2016) 3629-3636.
111. P. Liu and D. Wang, An extended taxonomy method based on normal T-spherical fuzzy numbers for multiple-attribute decision-making, International Journal of Fuzzy Systems 24 (2022) 73-90.
112. P. Liu, B. Zhu and P. Wang, A multi-attribute decision-making approach based on spherical fuzzy sets for Yunnan Baiyao's R&D project selection problem, International Journal of Fuzzy Systems 21 (2019) 2168-2191.
113. T. Loganathan and K. Ganesan, Fuzzy solution of fully fuzzy multi-objective linear fractional programming problems, Pakistan Journal of Statistics and Operation Research 17 (2021) 817-825.
114. Z. M. Ma and Z. S. Xu, Symmetric Pythagorean fuzzy weighted geometric/averaging operators and their application in multicriteria decision-making problems, International Journal of Intelligent Systems 31 (2016) 1198-1219.
115. T. Mahmood, K. Ullah, Q. Khan and N. Jan, An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets, Neural Computing and Applications 31 (2019) 7041-7053.
116. T. Mahmood, M. S. Warraich, Z. Ali and D. Pamucar, Generalized MULTIMOORA method and Dombi prioritized weighted aggregation operators based on T-spherical fuzzy sets and their applications, International Journal of Intelligent Systems 36 (2021) 4659-4692.
117. A. Mahmoodirad, S. Niroomand, N. Mirzaei and A. Shoja, Fuzzy fractional minimal cost flow problem, Intrenational Journal of Fuzzy Systems 20 (2018) 174-186.

118. S. Mahnaz, J. Ali, M. A. Malik and Z. Bashir, T-spherical fuzzy frank aggregation operators and their application to decision making with unknown weight information, IEEE Access 10 (2021) 7408-7438.
119. S. S. Meghwani and M. Thakur, Multi-objective heuristic algorithms for practical portfolio optimization and rebalancing with transaction cost, Applied Soft Computing 67 (2018) 865-894.
120. R. Mesiar and A. Kolesarova, On the fuzzy set theory and aggregation functions: History and some recent advances, Iranian Journal of Fuzzy Systems 15 (2018) 1-12.
121. A. Mishra and A. Kumar, Aggregation operators for various extensions of fuzzy set and its applications in transportation problems, Springer Nature, Singapore, (2020).
122. S. Mohanaselvia and K. Ganesan, A new approach for solving linear fuzzy fractional transportation problem, International Journal of Civil Engineering and Technology 8 (2017) 1123-1129.
123. B. K. Mohanty and T. A. S. Vijayaraghavan, A multi-objective programming problem and its equivalent goal programming problem with appropriate priorities and aspiration levels: A fuzzy approach, Computers & Operations Research 22 (1995) 771-778.
124. M. Munir, T. Mahmood and A. Hussain, Algorithm for T-spherical fuzzy MADM based on associated immediate probability interactive geometric aggregation operators, Artificial Intelligence Review 54 (2021) 6033-6061.
125. A. H. Nafei and S. H. Nasseri, A new approach for solving neutrosophic integer programming problems, International Journal of Applied Operational Research 9 (2019) 1-9.

126. A. Nafei, Y. U. A. N. Wenjun and H. Nasseri, A new method for solving interval neutrosophic linear programming problems, Gazi University Journal of Science 33 (2020) 796-808.

127. P. Nagar, P. K. Srivastava and A. Srivastava, A new dynamic score function approach to optimize a special class of Pythagorean fuzzy transportation problem, International Journal of System Assurance Engineering and Management (2021) https://doi.org/10.1007/s13198-021-01339-w

128. S. Narayanamoorthy and S. Kalyani, The intelligence of dual simplex method to solve linear fractional fuzzy transportation problem, Computational Intelligence and Neuroscience 2015 (2015) 103618.

129. S. H. Nasseri, A. Ebrahimnejad and B. Y. Cao, Fuzzy linear programming: Solution techniques and applications, Springer Nature Switzerland AG, Switzerland, (2019).

130. A. K. Nishad and Abhishekh, A new ranking approach for solving fully fuzzy transportation problem in intuitionistic fuzzy environment, Journal of Control, Automation and Electrical Systems 31 (2020) 900-911.

131. M. Ojha, K. P. Singh, P. Chakraborty and S. Verma, A review of multi-objective optimisation and decision making using evolutionary algorithms, International Journal of Bio-Inspired Computation 14 (2019) 69-84.

132. V. K. Ojha, V. Snášel and A. Abraham, Multiobjective programming for type-2 hierarchical fuzzy inference trees, IEEE Transactions on Fuzzy Systems 26 (2017) 915-936.

133. R. Pelissari, M. C. Oliveira, A. J. Abackerli, S. Ben-Amor and M. R. P. Assumpção, Techniques to model uncertain input data of multi-criteria decision-making problems:

a literature review, International Transactions in Operational Research 28 (2021) 523-559.

134. X. Peng and Y. Yang, Some results for Pythagorean fuzzy sets, International Journal of Intelligent Systems 30 (2015) 1133-1160.

135. X. Peng and Y. Yang, Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators, International Journal of Intelligent Systems 31 (2016) 444-487.

136. B. Pop and I.M. Stancu-Minasian, A method of solving fully fuzzified linear fractional programming problems, Journal of Applied Mathematics and Computing 27 (2008) 227-242.

137. A. Rabie, E. El Seidy, A. Elrayes and E. Badr, Dual artificial variable-free simplex algorithm for solving neutrosophic linear programming problems, Neutrosophic Sets and Systems 46 (2021) 36-49.

138. B. Radhakrishnan and P. Anukokila, A compensatory approach to fuzzy fractional transportation problem, International Journal of Mathematics in Operational Research 6 (2014) 176-192.

139. D. Rani, T. R. Gulati and A. Kumar, A method for unbalanced transportation problems in fuzzy environment, Sadhana: Academy Proceedings in Engineering Sciences 39 (2014) 573-581.

140. L. V. Reddy and R. N. Mukherjee, Composite nonsmooth multiobjective programs with $V-\rho$ invexity, Journal of Mathematical Analysis and Applications 235 (1999) 567-577.

141. R. Sadiq and S. Tesfamariam, Environmental decision-making under uncertainty using intuitionistic fuzzy analytic hierarchy process (IF-AHP), Stochastic Environmental Research and Risk Assessment 23 (2009) 75-91.

142. M. Sajjad Ali Khan, S. Abdullah, M. Yousaf Ali, I. Hussain and M. Farooq, Extension of TOPSIS method base on Choquet integral under interval-valued Pythagorean fuzzy environment, Journal of Intelligent & Fuzzy Systems 34 (2018) 267-282.

143. B. Sharma, V. K. Katiyar and A. K. Gupta, Fuzzy logic model for the prediction of traffic volume in week days, International Journal of Computer Applications 107 (2014) 1-6.

144. A. Sharma, V. Verma, P. Kaur and K. Dhahiya, An iterative algorithm for two level hierarchical time minimization transportation problem, European Journal of Operational Research 246 (2015) 700-707.

145. A. Sheikhi, S. M. Karbassi and N. Bidabadi, A novel algorithm for solving bi-objective fractional transportation problems with fuzzy numbers, Journal of Mathematical Extension 14 (2019) 29-47.

146. H. D. Sherali, On a fractional minimal cost flow problem on networks, Optimization Letters 6 (2012) 1945-1949.

147. A. H. Siddiqi, P. Manchanda and R. Bhardwaj, Mathematical models, methods and applications, Springer Science, Business Media Singapore (2015).

148. A. Singh, A. Kumar and S. S. Appadoo, A novel method for solving the fully neutrosophic linear programming problems: Suggested modifications, Journal of Intelligent & Fuzzy Systems 37 (2019) 885-895.

149. S. K. Singh, and S. P. Yadav, Efficient approach for solving type-1 intuitionistic fuzzy transportation problem, International Journal of System Assurance Engineering and Management 6 (2015) 259-267.

150. S. K. Singh and S. P. Yadav, A new approach for solving intuitionistic fuzzy transportation problem of type-2, Annals of Operations Research 243 (2016) 349-363.

151. F. Smarandache, A unifying field in logics. Neutrosophy: neutrsophic probability, set and logic, Rehoboth: American Research Press, (1998).

152. M. A. SN and D. Ulaganathan, A study on a new method for solving interval neutrosophic linear programming problems, Journal of Computational Mathematica 5 (2021) 48-54.

153. B. Stanojevic and I. M. Stancu-Minasian, On solving fuzzified linear fractional programs, Advanced Modeling and Optimization 11 (2009) 503-523.

154. B. Stanojević and I. M. Stancu-Minasian, Evaluating fuzzy inequalities and solving fully fuzzified linear fractional programs, Yugoslav Journal of Operations Research 22 (2012) 41-50.

155. B. Stanojević and M Stanojević, Comment on "Fuzzy mathematical programming for multi objective linear fractional programming problem", Fuzzy Sets and Systems 246 (2014) 156-159.

156. S. Stephen and M. Helen, Interval valued neutrosophic linear programming with trapezoidal numbers, The International Journal Of Analytical And Experimental Modal Analysis 12 (2020) 533-542.

157. B. Sun, D. Zhu and S. X. Yang, An optimized fuzzy control algorithm for three-dimensional AUV path planning, International Journal of Fuzzy Systems 20 (2018) 597-610.

158. P. Suryawanshi and P. Dutta, Optimization models for supply chains under risk, uncertainty, and resilience: A state-of-the-art review and future research directions, Transportation Research Part E: Logistics and Transportation Review 157 (2022) 102553.

159. K. Swarup, Transportation technique in linear fractional functional programming, Journal of Royal Naval Scientific Service 21 (1966) 256-260.

160. E. Szmidt and J. Kacprzyk, Distances between intuitionistic fuzzy sets, Fuzzy Sets and Systems 114 (2000) 505-518.

161. H. A. Taha, Operations Research: An Introduction, 8th edition, Pearson Education India, (2008).

162. Z. Ullah, H. Bashir, R. Anjum, S. A. AlQahtani, S. Al-Hadhrami and A. Ghaffar, Analysis of the shortest path in spherical fuzzy networks using the novel Dijkstra algorithm, Mathematical Problems in Engineering 2021 (2021) 7946936.

163. R. M. Umamageswari and G. Uthra, A Pythagorean fuzzy approach to solve transportation problem, Adalya Journal 9 (2020) 1301-1308.

164. C. Veeramani and M. Sumathi, Fuzzy mathematical programming approach for solving fuzzy linear fractional programming problem, RAIRO-Operations research 48 (2014) 109-122.

165. C. Veeramani and M. Sumathi, Solving the linear fractional programming problem in a fuzzy environment: Numerical approach, Applied Mathematical Modelling, 40 (2016) 6148-6164.

166. S. Vij, A. Jain, D. Tayal and O. Castillo, An analytical insight to investigate the research patterns in the realm of type-2 fuzzy logic, Journal of Automation Mobile Robotics and Intelligent Systems 12 (2018) 3-32.

167. H. Wang, T-spherical fuzzy rough interactive power Heronian mean aggregation operators for multiple attribute group decision-making, Symmetry 13 (2021) 2422.
168. Q. Wang, Y. Huang, S. Kong, X. Ma, Y. Liu, S. K. Das and S. A. Edalatpanah, A novel method for solving multiobjective linear programming problems with triangular neutrosophic numbers, Journal of Mathematics 2021 (2021) 6631762.
169. X. Wang and E. E. Kerre, Reasonable properties for ordering fuzzy quantities (I), Fuzzy Sets and Systems 18 (2001) 375-385.
170. C. Xu, X. M. Xu and H. F. Wang, The fractional minimal cost flow problem on network, Optimization Letters 5 (2011) 307-317.
171. R.R. Yager, Generalized orthopair fuzzy sets, IEEE Transactions on Fuzzy Systems 25 (2017) 1222-1230.
172. R. R. Yager and A. M. Abbasov, Pythagorean membership grades, complex numbers, and decision making, International Journal of Intelligent Systems 28 (2013) 436-452.
173. O. Yanmaz, Y. Turgut, E. N. Can and C. Kahraman, Interval-valued Pythagorean fuzzy EDAS method: an application to car selection problem, Journal of Intelligent & Fuzzy Systems 38 (2020) 4061-4077.
174. L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.
175. L. Zedam, N. Jan, E. Rak and T. Mahmood, An approach towards decision-making and shortest path problems based on T-spherical fuzzy information, International Journal of Fuzzy Systems 22 (2020) 1521-1534.
176. S. Zeng, H. Garg, M. Munir, T. Mahmood and A. Hussain, A multi-attribute decision making process with immediate probabilistic interactive averaging aggregation operators of T-spherical fuzzy sets and its application in the selection of solar cells, Energies 12 (2019) 4436.

177. X. L. Zhang and Z. S. Xu, Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, International Journal of Intelligent Systems 29 (2014) 1061-1078.

178. M. Zhao, G. Wei, C. Wei and J. Wu, TODIM method for interval-valued Pythagorean fuzzy MAGDM based on cumulative prospect theory and its application to green supplier selection, Arabian Journal for Science and Engineering 46 (2021) 1899-1910.

179. H. J. Zimmermann, Fuzzy programming and linear programming with several objective functions, Fuzzy Sets and Systems 1 (1978) 45-55.

180. H. J. Zimmermann, Fuzzy Set Theory and Its Applications, Kluwer Academic Publisher, Dordrecht, (2001).