EFFICIENT METHODS FOR SOLVING SOME MATHEMATICAL PROGRAMMING PROBLEMS WITH FUZZY PARAMETERS

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CERTIFICATE

This is to certify that the thesis entitled, **"Efficient Methods for Solving Some Mathematical Programming Problems with Fuzzy Parameters"**, submitted by Sukhpreet Kaur Sidhu in the fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar University, Patiala, is a record of candidate's own work carried out by her under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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DECLARATION

It is certified that the thesis is entirely my own work and the ideas and references cited herein have been duly acknowledged.

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TO

THE ALMIGHTY

&

MY HUSBAND

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Abstract

In this thesis, new methods are proposed to solve the linear programming problems with fuzzy parameters and the linear programming problems with intuitionistic fuzzy parameters. Also, the advantages of the proposed methods, over the existing methods, are presented.

The chapter wise summary of the thesis is as follows:

Chapter 1

In this chapter, a brief review of the work done in the area of linear programming problems with fuzzy parameters as well as linear programming problems with intuitionistic fuzzy parameters, is presented.

Chapter 2

Ganesan and Veeramani [52] proposed the product ($\otimes_G$) of symmetric trapezoidal fuzzy numbers and proposed a method to find the fuzzy optimal solution of fuzzy linear programming problem with symmetric trapezoidal fuzzy numbers (P_1).

$$\begin{aligned}
&\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right] \\
&\text{Subject to} \qquad\qquad\qquad\qquad\qquad\qquad (P_1) \\
&\sum_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, ..., m, \\
&\tilde{x}_j \succeq \tilde{0}, \qquad\qquad j = 1, 2, ..., n,
\end{aligned}$$

where $\tilde{c}_j, \tilde{x}_j$ and $\tilde{b}_i$ are symmetric trapezoidal fuzzy numbers and a_{ij} are real numbers.

Since, then the different methods [45,82,84] have been proposed for the same. In this chapter, a new method (named as Mehar method) is proposed for the same. It is shown that all the fuzzy linear programming problems which can be solved by the existing methods [45, 52, 82, 84] can also be solved by the proposed Mehar method. However, it is much easy to apply the proposed Mehar method as compared to the existing methods [45,52,82,84].

Chapter 3

Ebrahimnejad et al. [42] pointed out that if the existing method [52] will be used for solving bounded fuzzy linear programming problem with symmetric trapezoidal fuzzy numbers (P_2), then the problem size and computational effort would increase significantly. Therefore, Ebrahimnejad et al. [42] proposed an alternative method to find the fuzzy optimal solution of problem (P_2).

$$\begin{aligned}
&\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right] \\
&\text{Subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_2) \\
&\sum_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, ..., m, \\
&\tilde{l}_j \preceq \tilde{x}_j \preceq \tilde{u}_j, \qquad\quad j = 1, 2, ..., n,
\end{aligned}$$

where $\tilde{c}_j, \tilde{x}_j, \tilde{b}_i, \tilde{l}_j$ and $\tilde{u}_j$ are symmetric trapezoidal fuzzy numbers and a_{ij} are real numbers.

Hatami and Kazemipoor [60] pointed out that there is no method in literature for solving fully fuzzy linear programming problems with symmetric trapezoidal fuzzy numbers (P_3) and extended the existing method [84] for solving fully fuzzy linear programming problems with symmetric trapezoidal fuzzy numbers (P_3).

$$\begin{aligned}
&\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right] \\
&\text{Subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_3) \\
&\sum_{j=1}^{n} \tilde{a}_{ij} \otimes_G \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m, \\
&\tilde{x}_j \succeq \tilde{0}, \qquad\qquad j = 1, 2, ..., n,
\end{aligned}$$

where $\tilde{c}_j, \tilde{x}_j, \tilde{a}_{ij}$ and $\tilde{b}_i$ are symmetric trapezoidal fuzzy numbers.

However, the method, proposed by Ebrahimnejad et al. [42], can not be used to find the fuzzy optimal solution of problem (P_3) and the method, proposed by Hatami and Kazemipoor [60], can not be used to find the fuzzy optimal solution of problem (P_2). Therefore, to overcome the limitations of both the existing methods [42, 60], in this chapter, the Mehar method, proposed in Chapter 2, is extended for solving such bounded fully fuzzy linear programming problems (P_4) in which all the parameters and variables are represented by symmetric trapezoidal fuzzy numbers.

$$\begin{aligned}
&\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right] \\
&\text{Subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_4) \\
&\sum_{j=1}^{n} \tilde{a}_{ij} \otimes_G \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \qquad i = 1, 2, ..., m, \\
&\tilde{l}_j \preceq \tilde{x}_j \preceq \tilde{u}_j, \qquad\qquad j = 1, 2, ..., n,
\end{aligned}$$

where $\tilde{c}_j, \tilde{x}_j, \tilde{a}_{ij}, \tilde{b}_i, \tilde{l}_j$ and $\tilde{u}_j$ are symmetric trapezoidal fuzzy numbers.

Chapter 4

Ebrahimnejad [38] proposed a method for solving bounded fuzzy linear programming problem (P_5) in which coefficients of variables in objective function as well as in constraints are represented by real numbers while all other parameters and variables are represented by trapezoidal fuzzy numbers.

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} c_j \tilde{x}_j\right]$$

Subject to $\qquad (P_5)$

$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, ..., m,$$

$$\tilde{l}_j \preceq \tilde{x}_j \preceq \tilde{u}_j, \qquad j = 1, 2, ..., n,$$

where $\tilde{x}_j, \tilde{b}_i, \tilde{l}_j$ and $\tilde{u}_j$ are trapezoidal fuzzy numbers and $c_j,\ a_{ij}$ are real numbers.

However, the existing method [38] can be used only for solving bounded fuzzy linear programming problem (P_5) if the initial fuzzy basic solution is optimal but not feasible.

To overcome this limitation of the existing method [38], Ebrahimnejad and Verdegay [46] proposed a method for solving the same problem (P_5). In this chapter, a new method (named as Mehar method) is proposed to find the fuzzy optimal solution of problem (P_5) and it is shown that all the problems which can be solved by the existing methods [38, 46] can also be solved by the proposed Mehar method and there are several advantages of applying proposed Mehar method over applying other existing methods [38, 46].

Chapter 5

Parvathi and Malathi [129] proposed intuitionistic fuzzy simplex method to solve such intuitionistic fuzzy linear programming problems (P_6) in which the coefficients of the variables, in the objective function and in all the constraints, are represented by real numbers whereas the variables and other remaining parameters are represented by symmetric trapezoidal intuitionistic fuzzy numbers.

$$\text{Maximize/Minimize } \left[\tilde{z}^I \approx \sum_{j=1}^{n} c_j \tilde{x}_j^I\right]$$

Subject to $\qquad (P_6)$

$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j^I \preceq, \approx, \succeq \tilde{b}_i^I, \quad i = 1, 2, ..., m,$$
$$\tilde{x}_j^I \succeq 0, \qquad\qquad j = 1, 2, ..., n,$$

where $\tilde{x}_j^I$ and $\tilde{b}_i^I$ are symmetric trapezoidal intuitionistic fuzzy numbers and $c_j,\ a_{ij}$ are real numbers.

In this chapter, it is shown that, the existing method [129] can be used to solve (P_6) only if all the coefficients c_j and a_{ij} in problem (P_6) are non-negative real numbers. However, if any of the coefficients c_j or a_{ij} is negative real number. Then, existing method [129] cannot be used for solving problem (P_6). So, to overcome these limitations of the existing method [129], new methods (named as Mehar methods), are proposed.

Chapter 6

The methods, proposed in Chapter 5, cannot be used to find the intuitionistic fuzzy optimal solution of intuitionistic fully fuzzy linear programming problems (intuitionistic fuzzy linear programming problems in which all the variables and parameters are represented by intuitionistic fuzzy numbers).

In this chapter, flaws in the existing method [111] for finding the intuitionistic fuzzy optimal solution of intuitionistic fully fuzzy linear programming problems, are pointed out. Also, the product of two unrestricted trapezoidal intuitionistic fuzzy numbers is proposed as well as a new method (named as Mehar method) is proposed to find the intuitionistic fuzzy optimal solution of such intuitionistic fully fuzzy linear programming problems in which all the variables and parameters are represented by trapezoidal intuitionistic fuzzy numbers.

Chapter 7

Suresh et al. [143] proposed the ranking function for comparing triangular intuitionistic fuzzy numbers and applied this ranking function to solve different types of intuitionistic fuzzy linear programming problems.

In this chapter, it is pointed out that the ranking function, proposed by Suresh et al. [143], is not valid. Hence, the results of intuitionistic fuzzy linear programming problems, obtained by using this ranking function, are also not valid. Further, the exact ranking function is obtained by modifying existing ranking function and using the exact ranking function, the exact results of intuitionistic fuzzy linear programming problems, considered by Suresh et al. [143], are obtained.

Chapter 8

Finally, in this chapter, future work has been suggested.

List of Research Papers

1. **S.K. Sidhu**, A. Kumar, S.S. Appadoo, Mehar methods for fuzzy optimal solution and sensitivity analysis of fuzzy linear programming with symmetric trapezoidal fuzzy numbers, Mathematical Problems in Engineering, 2014 (2014) 1-8, **(SCI)** (**Impact factor 0.762**).

2. **S.K. Sidhu**, A. Kumar, A. Kaur, A note on "A fuzzy approach to transport optimization problem", Optimization and Engineering, DOI: 10.1007/s 11081-015-9279-9, **(SCI)** (**Impact factor 1.233**).

3. **S.K. Sidhu**, A. Kumar, A note on "Solving intuitionistic fuzzy linear programming problems by ranking function", Journal of Intelligent and Fuzzy Systems, 30 (2016) 2787-2790, **(SCI)** (**Impact factor 1.812**).

4. **S.K. Sidhu**, A new approach to solve intuitionistic fuzzy linear programming problems with symmetric trapezoidal intuitionistic fuzzy numbers, Mathematical Theory and Modeling, 5 (2015), 66-74, **(Non-SCI)**.

5. **S.K. Sidhu**, A. Kumar, Mehar method to find the fuzzy optimal solution of bounded fully fuzzy linear programs with symmetric trapezoidal fuzzy numbers, Proceedings of National Academy of Sciences: Physical Sciences. (Communicated)

6. **S.K. Sidhu**, A. Kumar, Mehar method for sensitivity analysis in fully fuzzy linear programs, Journal of Applied Research and Technology. (Communicated)
7. **S.K. Sidhu**, A. Kumar, Mehar method for solving bounded linear programming problems with fuzzy variables, Neural Computing and Applications. (Communicated)
8. **S.K. Sidhu**, A. Kumar, Mehar methods to solve intuitionistic fuzzy linear programming problems, Journal of Intelligent and Fuzzy Systems. (Communicated)
9. **S.K. Sidhu**, A. Kumar, Mehar method to solve intuitionistic fully fuzzy linear programming problems, Applied Mathematical Modelling. (Communicated)

Papers presented in International Conferences

1. **S.K. Sidhu**, A. Kumar, S.S. Appadoo, "Mehar methods for fuzzy optimal solution and sensitivity analysis of fuzzy linear programming with symmetric trapezoidal fuzzy numbers", International Conference on Special Functions and Their Applications, held on October 16-18, 2014, organized by School of Mathematics and Computer Applications, Thapar University, Patiala.
2. **S.K. Sidhu**, "Modified approach for solving intuitionistic fuzzy linear programming problem with triangular intuitionistic fuzzy numbers", International Conference on Emerging Areas of Mathematics for Science and Technology,

held on January 30-February 1, 2015, organized by Department of Mathematics, Punjabi University, Patiala.

3. **S.K. Sidhu**, "A new approach to solve intuitionistic fuzzy linear programming problems with symmetric trapezoidal intuitionistic fuzzy numbers", International Conference on Modeling, Simulation and Optimizing Techniques, held on February 12-14, 2015, organized by Post-Graduate Department of Mathematics, DAV College, Jalandhar.

4. **S.K. Sidhu**, "A new method for solving intuitionistic fuzzy linear programming problems by ranking function", International Conference on Advances in Mathematical Sciences, on March 19-21, 2015, organized by GSSDGS Khalsa College, Patiala and International Multidisciplinary Research Foundation, India.

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Chapter 1

INTRODUCTION

To find the solution of most of the real life problems, there is a need to develop a mathematical model for it and then this mathematical model is solved to obtain the solution of these real life problems. In mathematical model of several real life problems, there is a need to optimize (maximize/minimize) one or more functions subject to certain constraints. The techniques, used for obtaining the optimal (maximum/minimum) value of such problems, are called optimization techniques and such problems are called mathematical programming problems.

If there is only one function which have to be optimized then mathematical programming problem is known as single-objective mathematical programming problem. Further, mathematical programming problems, in which the objective function (function which have to be optimized) as well as the constraints appear as linear functions, are called linear programming problems.

A single-objective linear programming problem can be generally stated as "Given a set of m linear inequalities in n variables, finding non-negative values of these variables which will satisfy all the linear inequalities and will optimize some linear function of the variables".

Optimize $\left[\sum_{j=1}^{n} c_j x_j\right]$

Subject to $(P_{1.1})$

$\sum_{j=1}^{n} a_{ij}x_j \leq, =, \geq b_i; \qquad i = 1, 2, ..., m,$

$x_j \geq 0; \qquad j = 1, 2, ..., n.$

Linear programming problem $(P_{1.1})$ involves a lot of parameters (c_j, a_{ij} and b_i) whose values are assigned by experts. However, both experts and decision maker frequently do not precisely know the values of those parameters. Rather than the particular value, only the vague, imprecise and incomplete information about the parameter is present, which is called the uncertain information.

In the literature, several researchers have used fuzzy numbers [163] to represent such parameters. Linear programming problems in which c_j and/or x_j and/or a_{ij} and/or b_i are represented by fuzzy numbers are called linear programming problems with fuzzy parameters or fuzzy linear programming problems. Several methods have been proposed in the literature for solving different types of fuzzy linear programming problems.

However, if there exist some hesitation about the value of a parameter then that parameter cannot be represented by fuzzy number and in such a case intuitionistic fuzzy number [6], generalization of fuzzy number, may be used to represent that parameter. Linear programming problems in which c_j and/or x_j and/or a_{ij} and/or b_i are represented by intuitionistic fuzzy numbers are called linear programming problems with intuitionistic fuzzy parameters or intuitionistic fuzzy linear programming problems. Very few methods have been proposed in the literature for solving different types of intuitionistic fuzzy linear programming problems.

1.1 Literature Review

The existing methods for solving fuzzy/intuitionistic fuzzy linear programming problems can be broadly divided into two classes as follows:

1. Solving fuzzy/intuitionistic fuzzy linear programming problems without transforming into equivalent crisp linear programming problems.

2. Solving fuzzy/intuitionistic fuzzy linear programming problems by transforming into equivalent crisp linear programming problems.

In this section, a brief review of the work done by some of the researchers in the last few years is presented.

(1) Solving fuzzy/intuitionistic fuzzy linear programming problems without transforming into equivalent crisp linear programming problems

Maleki et al. [108] proposed a method to find the crisp optimal solution of fuzzy linear programming problem ($P_{1.2}$).

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j x_j\right]$$

$$\text{Subject to} \qquad (P_{1.2})$$

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$$

$$x_j \geq 0,\ j = 1, 2, ..., n,$$

where $\tilde{c}_j, \tilde{a}_{ij}$ and $\tilde{b}_i$ are trapezoidal fuzzy numbers.

Maleki et al. [108] also proved that the fuzzy optimal solution of fuzzy linear programming problem ($P_{1.3}$), can be obtained with the help of the crisp optimal solution of fuzzy linear programming problem ($P_{1.2}$).

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} c_j \tilde{x}_j\right]$$

$$\text{Subject to} \qquad (P_{1.3})$$

$\sum\limits_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$

$\tilde{x}_j \succeq 0,\ j = 1, 2, ..., n,$

where c_j, a_{ij} are real numbers and $\tilde{x}_j$, $\tilde{b}_i$ are trapezoidal fuzzy numbers.

Nasseri and Ardil [115] proposed simplex method to find the fuzzy optimal solution of fuzzy linear programming problem $(P_{1.4})$.

Maximize/Minimize $\left[\tilde{z} \approx \sum\limits_{j=1}^{n} c_j\tilde{x}_j\right]$

Subject to $\qquad (P_{1.4})$

$\sum\limits_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$

$\tilde{x}_j \succeq 0,\ j = 1, 2, ..., n,$

where c_j, a_{ij} are real numbers and $\tilde{x}_j$, $\tilde{b}_i$ are trapezoidal fuzzy numbers.

Ganesan and Veeramani [52] introduced a new type of fuzzy multiplication $(\otimes_G)$ for symmetric trapezoidal fuzzy numbers and proposed a method to find fuzzy optimal solution of fuzzy linear programming problem $(P_{1.5})$.

Maximize/Minimize $\left[\tilde{z} \approx \sum\limits_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right]$

Subject to $\qquad (P_{1.5})$

$\sum\limits_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$

$\tilde{x}_j \succeq \tilde{0},\ j = 1, 2, ..., n,$

where $\tilde{c}_j, \tilde{x}_j$, $\tilde{b}_i$ are symmetric trapezoidal fuzzy numbers and a_{ij} is a real number.

Mahdavi-Amiri and Nasseri [105] proposed a dual simplex method to find fuzzy optimal solution of fuzzy linear programming problem $(P_{1.6})$.

Maximize/Minimize $\left[\tilde{z} \approx \sum\limits_{j=1}^{n} c_j\tilde{x}_j\right]$

Subject to $\qquad (P_{1.6})$

$\sum\limits_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$

$\tilde{x}_j \succeq 0,\ j = 1, 2, ..., n,$

where c_j, a_{ij} are real numbers and $\tilde{x}_j$, $\tilde{b}_i$ are trapezoidal fuzzy numbers.

Mahdavi-Amiri et al. [106] proposed fuzzy primal simplex algorithm to find crisp optimal solution of fuzzy linear programming problem $(P_{1.7})$ as well as fuzzy optimal solution of fuzzy linear programming problem $(P_{1.8})$.

$$\begin{aligned}
&\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j x_j\right] \\
&\text{Subject to} \qquad\qquad\qquad\qquad (P_{1.7}) \\
&\sum_{j=1}^{n} a_{ij} x_j \preceq, \approx, \succeq b_i,\ \ i = 1, 2, ..., m, \\
&\tilde{x}_j \succeq 0,\ \ j = 1, 2, ..., n,
\end{aligned}$$

where $\tilde{c}_j$ is a trapezoidal fuzzy number and x_j, a_{ij}, b_i are real numbers.

$$\begin{aligned}
&\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} c_j \tilde{x}_j\right] \\
&\text{Subject to} \qquad\qquad\qquad\qquad (P_{1.8}) \\
&\sum_{j=1}^{n} a_{ij} \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ \ i = 1, 2, ..., m, \\
&\tilde{x}_j \succeq 0,\ \ j = 1, 2, ..., n,
\end{aligned}$$

where $\tilde{b}_i$, $\tilde{x}_j$ are trapezoidal fuzzy numbers and a_{ij}, c_j are real numbers.

Nasseri and Mahdavi-Amiri [123] defined the dual of fuzzy linear programming problem $(P_{1.5})$, as well as proposed some duality results for these problems. Nasseri and Khabiri [121] proposed a revised fuzzy simplex algorithm to find fuzzy optimal solution of fuzzy linear programming problem $(P_{1.9})$.

$$\begin{aligned}
&\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} c_j \tilde{x}_j\right] \\
&\text{Subject to} \qquad\qquad\qquad\qquad (P_{1.9}) \\
&\sum_{j=1}^{n} a_{ij} \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ \ i = 1, 2, ..., m, \\
&\tilde{x}_j \succeq 0,\ \ j = 1, 2, ..., n,
\end{aligned}$$

where $\tilde{b}_i$, $\tilde{x}_j$ are trapezoidal fuzzy numbers and a_{ij}, c_j are real numbers.

Dubey and Mehra [34] proposed a more general definition of triangular intuitionistic fuzzy numbers. Dubey and Mehra [34] also defined a ranking function based on value and ambiguity indexes and applied this ranking function to solve the intuitionistic fuzzy linear programming problem $(P_{1.10})$.

Maximize/Minimize $\left[\tilde{z}^I \approx \sum_{j=1}^{n} \tilde{c}_j^I x_j\right]$

Subject to $\qquad (P_{1.10})$

$\sum_{j=1}^{n} \tilde{a}_{ij}^I x_j \preceq, \approx, \succeq \tilde{b}_i^I; \quad i = 1, 2, ..., m;$

$x_j \geq 0; \qquad j = 1, 2, ..., n,$

where $\tilde{c}_j^I$, $\tilde{a}_{ij}^I$, $\tilde{b}_i^I$ are triangular intuitionistic fuzzy numbers and x_j is a real number.

Ebrahimnejad et al. [42] pointed out that the existing method [52] is not efficient for the situations in which some or all variables are restricted to lie within fuzzy lower bounds and fuzzy upper bounds. So, to overcome this limitation, Ebrahimnejad et al. [42] proposed a method to find the fuzzy optimal solution of bounded fuzzy linear programming problem $(P_{1.11})$.

Maximize/Minimize $\left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right]$

Subject to $\qquad (P_{1.11})$

$\sum_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \ i = 1, 2, ..., m,$

$\tilde{l}_j \preceq \tilde{x}_j \preceq \tilde{u}_j, \ j = 1, 2, ..., n,$

where $\tilde{c}_j, \tilde{x}_j, \tilde{b}_i, \tilde{l}_j$ and $\tilde{u}_j$ are symmetric trapezoidal fuzzy numbers and a_{ij} is a real number.

Ebrahimnejad [36] pointed out that if there exist a feasible solution for the existing fuzzy linear programming problem $(P_{1.5})$, then there will also exist a fuzzy basic feasible solution for this problem. Also, if a fuzzy optimal solution exist for fuzzy linear programming problem $(P_{1.5})$ then there will also exist a fuzzy optimal basic solution for this problem. Ebrahimnejad [36] also proposed the fuzzy revised simplex method to find the fuzzy optimal solution of fuzzy linear programming problem $(P_{1.5})$.

Ebrahimnejad [37] pointed out that the existing method [52] can be used only if it is easily possible to find a basic feasible solution of primal problem. However, if this condition is not satisfied then the existing method [52] cannot be used. To overcome this limitation, Ebrahimnejad [37] proposed a primal dual simplex algorithm for solving existing fuzzy linear programming problem $(P_{1.5})$.

Nagoorgani and Ponnalagu [111] defined the division operation for triangular intuitionistic fuzzy number using $\alpha-$ cut and also defined scoring and accuracy function to rank triangular intuitionistic fuzzy number. Based on this approach, Nagoorgani and Ponnalagu [111] proposed a method to find the solution of intuitionistic fuzzy linear programming problem $(P_{1.12})$.

Maximize/Minimize $\left[\tilde{z}^I \approx \sum_{j=1}^{n} \tilde{c}_j^I \tilde{x}_j^I\right]$

Subject to $\qquad (P_{1.12})$

$\sum_{j=1}^{n} \tilde{a}_{ij}^I \tilde{x}_j^I \preceq, \approx, \succeq \tilde{b}_i^I, i = 1, 2, ..., m,$

$\tilde{x}_j^I \succeq 0,\ j = 1, 2, ..., n,$

where $\tilde{c}_j^I, \tilde{x}_j^I, \tilde{a}_{ij}^I$ and $\tilde{b}_i^I$ are triangular intuitionistic fuzzy numbers.

Parvathi and Malathi [129] proposed intuitionistic fuzzy simplex method to solve intuitionistic fuzzy linear programming problem $(P_{1.13})$.

Maximize/Minimize $\left[\tilde{z}^I \approx \sum_{j=1}^{n} c_j \tilde{x}_j^I\right]$

Subject to $\qquad (P_{1.13})$

$\sum_{j=1}^{n} a_{ij} \tilde{x}_j^I \preceq, \approx, \succeq \tilde{b}_i^I,\ i = 1, 2, ..., m,$

$\tilde{x}_j^I \succeq 0,\ j = 1, 2, ..., n,$

where $c_j,\ a_{ij}$ are real numbers and $\tilde{x}_j^I,\ \tilde{b}_i^I$ are symmetric trapezoidal intuitionistic fuzzy numbers.

Ebrahimnejad [38] proposed a method to find the fuzzy optimal solution of bounded fuzzy linear programming problem $(P_{1.14})$.

Maximize/Minimize $\left[\tilde{z} \approx \sum_{j=1}^{n} c_j \tilde{x}_j\right]$

Subject to $\qquad (P_{1.14})$

$\sum_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$

$\tilde{l}_j \preceq \tilde{x}_j \preceq \tilde{u}_j,\ j = 1, 2, ..., n,$

where $\tilde{x}_j, \tilde{b}_i, \tilde{l}_j$ and $\tilde{u}_j$ are trapezoidal fuzzy numbers and $c_j,\ a_{ij}$ are real numbers.

Kheirfam and Verdegay [82] proposed an approach, based on dual simplex method, to find the fuzzy optimal solution of existing fuzzy linear programming problem $(P_{1.5})$ as well as proposed a method to deal with sensitivity analysis of fuzzy linear programming problem $(P_{1.5})$.

Suresh et al. [143] proposed the ranking of triangular intuitionistic fuzzy numbers by means of magnitude and using this ranking, Suresh et al. [143] solved the following type of intuitionistic fuzzy linear programming problems $(P_{1.15})$, $(P_{1.16})$ and $(P_{1.17})$.

Maximize/Minimize $\left[\tilde{z}^I \approx \sum_{j=1}^{n} \tilde{c}_j^I x_j\right]$

Subject to $\qquad (P_{1.15})$

$\sum_{j=1}^{n} a_{ij}x_j \leq, =, \geq b_i; \quad i = 1, 2, ..., m,$

$x_j \geq 0; \quad j = 1, 2, ..., n,$

where $\tilde{c}_j^I$ is a triangular intuitionistic fuzzy number and $x_j,\ a_{ij},\ b_i$ are real numbers.

Maximize/Minimize $\left[z = \sum_{j=1}^{n} c_j x_j\right]$

Subject to $\qquad (P_{1.16})$

$\sum\limits_{j=1}^{n} \tilde{a}_{ij}^{I} x_j \preceq, \approx, \succeq \tilde{b}_i^{I}; \quad i = 1, 2, ..., m,$

$x_j \geq 0; \quad j = 1, 2, ..., n,$

where $\tilde{a}_{ij}^{I}$, $\tilde{b}_i^{I}$ are triangular intuitionistic fuzzy numbers and c_j, x_j are real numbers.

Maximize/Minimize $\left[\tilde{z}^{I} \approx \sum\limits_{j=1}^{n} \tilde{c}_j^{I} x_j\right]$

Subject to $\qquad (P_{1.17})$

$\sum\limits_{j=1}^{n} \tilde{a}_{ij}^{I} x_j \preceq, \approx, \succeq \tilde{b}_i^{I}; \quad i = 1, 2, ..., m,$

$x_j \geq 0; \quad j = 1, 2, ..., n,$

where $\tilde{c}_j^{I}$, $\tilde{a}_{ij}^{I}$, $\tilde{b}_i^{I}$ are triangular intuitionistic fuzzy numbers and x_j is real number.

Ezzati et al. [48] extended the existing fuzzy linear programming problem $(P_{1.5})$, into fuzzy lexicographic multi-objective linear programming problems and proposed an algorithm to find preemptive fuzzy optimal solution of problem $(P_{1.18})$.

lex Maximize/Minimize $\left[\sum\limits_{j=1}^{n} \tilde{c}_j^{1} \otimes_G \tilde{x}_j, \sum\limits_{j=1}^{n} \tilde{c}_j^{2} \otimes_G \tilde{x}_j, ..., \sum\limits_{j=1}^{n} \tilde{c}_j^{p} \otimes_G \tilde{x}_j\right]$

Subject to $\qquad (P_{1.18})$

$\sum\limits_{j=1}^{n} a_{ij} \tilde{x}_j \approx \tilde{b}_i; \quad i = 1, 2, ..., m,$

$\tilde{x}_j \succeq \tilde{0}; \quad j = 1, 2, ..., n,$

where $\tilde{c}_j^{k} (k = 1, 2, ..., p), \tilde{x}_j$, $\tilde{b}_i$ are symmetric trapezoidal fuzzy numbers and a_{ij} is real number.

Khan et al. [80] proposed a simplex method to find fuzzy optimal solution of fuzzy linear programming problem $(P_{1.19})$.

Maximize/Minimize $\left[\tilde{z} \approx \sum\limits_{j=1}^{n} \tilde{c}_j \tilde{x}_j\right]$

Subject to $\qquad (P_{1.19})$

$\sum\limits_{j=1}^{n} \tilde{a}_{ij} \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \; i = 1, 2, ..., m,$

$\tilde{x}_j \succeq \tilde{0}, \; j = 1, 2, ..., n,$

where $\tilde{c}_j, \tilde{x}_j, \tilde{b}_i$ and $\tilde{a}_{ij}$ are triangular fuzzy numbers.

Bhardwaj and Kumar [13] pointed out the error in existing method [80].

(2) Solving fuzzy/intuitionistic fuzzy linear programming problems by transforming into equivalent crisp linear programming problems

Hashemi et al. [59] proposed a method to find the fuzzy optimal solution of fuzzy linear programming problem ($P_{1.20}$).

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j\right]$$
$$\text{Subject to} \qquad (P_{1.20})$$
$$\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$$
$$\tilde{x}_j \succeq 0,\ j = 1, 2, ..., n,$$

where $\tilde{c}_j,\ \tilde{x}_j,\ \tilde{a}_{ij}$ and $\tilde{b}_i$ are symmetric LL fuzzy numbers.

Hashemi et al. [59] also generalized the concept of duality and proposed weak duality theory for the fuzzy linear programming problem ($P_{1.20}$).

Mahdavi-Amiri and Nasseri [104] proposed a method to find crisp optimal solution of fuzzy linear programming problem ($P_{1.21}$).

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j x_j\right]$$
$$\text{Subject to} \qquad (P_{1.21})$$
$$\sum_{j=1}^{n} \tilde{a}_{ij}x_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$$
$$x_j \geq 0,\ j = 1, 2, ..., n,$$

where $\tilde{c}_j,\ \tilde{a}_{ij}$ and $\tilde{b}_i$ are trapezoidal fuzzy numbers and x_j is a real number.

Mahdavi-Amiri and Nasseri [104] also introduced the dual of fuzzy linear programming problem ($P_{1.21}$) and deduced some duality results.

Allahviranloo et al. [2] proposed a method to find the fuzzy optimal solution of fuzzy linear programming problem ($P_{1.22}$).

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j\right]$$

Subject to $(P_{1.22})$

$\sum\limits_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$

$\tilde{x}_j \succeq 0,\ j = 1, 2, ..., n,$

where $\tilde{c}_j,\ \tilde{x}_j,\ \tilde{a}_{ij}$ and $\tilde{b}_i$ are represented by either non-negative or non-positive LR fuzzy numbers.

Lotfi et al. [97] pointed out that there is no method in literature for solving such fuzzy linear programming problems with equality constraints in which all parameters as well as variables are represented by fuzzy numbers and proposed a method to find the fuzzy optimal solution of fuzzy linear programming problem $(P_{1.23})$.

Maximize/Minimize $\left[\tilde{z} \approx \sum\limits_{j=1}^{n} \tilde{c}_j\tilde{x}_j\right]$

Subject to $(P_{1.23})$

$\sum\limits_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j = \tilde{b}_i,\ i = 1, 2, ..., m,$

$\tilde{x}_j \succeq 0,\ j = 1, 2, ..., n,$

where $\tilde{c}_j,\ \tilde{x}_j,\ \tilde{a}_{ij}$ and $\tilde{b}_i$ are non-negative triangular fuzzy numbers.

Kumar et al. [89] pointed out that the results, obtained by using the existing method [97], are approximate and proposed a method to find exact non-negative fuzzy optimal solution of fuzzy linear programming problem $(P_{1.24})$.

Maximize/Minimize $\left[\tilde{z} \approx \sum\limits_{j=1}^{n} \tilde{c}_j\tilde{x}_j\right]$

Subject to $(P_{1.24})$

$\sum\limits_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j = \tilde{b}_i,\ i = 1, 2, ..., m,$

$\tilde{x}_j \succeq 0,\ j = 1, 2, ..., n,$

where $\tilde{c}_j,\ \tilde{a}_{ij}$ and $\tilde{b}_i$ are unrestricted triangular fuzzy numbers and $\tilde{x}_j$ is non-negative triangular fuzzy number.

Kumar and Kaur [84] proposed an alternative method for solving existing symmetric fuzzy linear programming problems ($P_{1.5}$) and shown that it is much easy to apply their proposed method as compared to the existing method [52].

Kaur and Kumar [77] proposed a method to find the exact fuzzy optimal solution of fuzzy linear programming problem ($P_{1.25}$).

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j\right]$$

Subject to $\quad (P_{1.25})$

$$\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j = \tilde{b}_i,\ i = 1, 2, ..., m,$$

$$\tilde{x}_j \succeq 0,\ j = 1, 2, ..., n,$$

where $\tilde{c}_j,\ \tilde{x}_j\ \tilde{a}_{ij}$ and $\tilde{b}_i$ are unrestricted triangular/trapezoidal fuzzy numbers.

Kaur and Kumar [76] pointed out that the fuzzy optimal value, obtained by using the existing method [89], is not necessarily a unique fuzzy number and proposed a method to obtain the non-negative unique fuzzy optimal value of such fuzzy linear programming problems with equality constraints in which all the parameters are represented by trapezoidal fuzzy numbers.

Kaur and Kumar [79] proposed a method for solving fuzzy linear programming problem ($P_{1.26}$).

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j\right]$$

Subject to $\quad (P_{1.26})$

$$\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$$

$$\tilde{x}_j \succeq 0,\ j = 1, 2, ..., n,$$

where $\tilde{c}_j,\ \tilde{x}_j,\ \tilde{a}_{ij}$ and $\tilde{b}_i$ are LR flat fuzzy numbers.

Kumar and Kaur [86] proposed a method to find the non-negative fuzzy optimal solution of fuzzy linear programming problem ($P_{1.27}$).

$$\begin{aligned}&\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j\right]\\&\text{Subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{1.27})\\&\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,\\&\tilde{x}_j \succeq 0,\ j = 1, 2, ..., n,\end{aligned}$$

where $\tilde{c}_j,\ \tilde{a}_{ij},\ \tilde{b}_i$ are unrestricted triangular fuzzy numbers and $\tilde{x}_j$ is a non-negative triangular fuzzy number.

Ebrahimnejad and Tavana [45] pointed out the drawbacks of the existing method [84] and proposed a simplified method for solving fuzzy linear programming problem ($P_{1.5}$).

After reviewing the literature, it is found that there are some limitations and shortcomings in the existing methods for solving fuzzy/intuitionistic fuzzy linear programming problems. In this thesis, these limitations and shortcomings are pointed out and to overcome the limitations as well as to resolve the shortcomings of the existing methods, new methods are proposed.

Chapter 2

Mehar Method For Fuzzy Optimal Solution Of Fuzzy Linear Programming Problems With Symmetric Trapezoidal Fuzzy Numbers

Ganesan and Veeramani [52] proposed the product ($\otimes_G$) of symmetric trapezoidal fuzzy numbers and proposed a method to find the fuzzy optimal solution of fuzzy linear programming problem with symmetric trapezoidal fuzzy numbers ($P_{2.1}$).

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right]$$

$$\text{Subject to} \qquad (P_{2.1})$$

$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$$

$$\tilde{x}_j \succeq \tilde{0},\ j = 1, 2, ..., n,$$

where $\tilde{c}_j, \tilde{x}_j$ and $\tilde{b}_i$ are symmetric trapezoidal fuzzy numbers and a_{ij} are real numbers.

Since, then the different methods [45, 82, 84] have been proposed for the same. In this chapter, a new method (named as Mehar method) is proposed for the same and the advantages of proposed Mehar method over the existing methods

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[45, 52, 82, 84], are discussed.

2.1 Preliminaries

In this section, some basic definitions, arithmetic operations and comparison of symmetric trapezoidal fuzzy numbers are presented [38].

2.1.1 Some basic definitions

In this section, some basic definitions are presented.

Definition 2.1 Let X be the universal set. $\tilde{A}$ is called a fuzzy set in X if $\tilde{A}$ is a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$, where $\mu_{\tilde{A}}(.)$ is a membership function of $\tilde{A}$ and assigns to each element $x \in X$ a real number $\mu_{\tilde{A}}(x)$ in the interval [0,1].

Definition 2.2 Given a fuzzy set $\tilde{A}$ defined on X and any number $\alpha \in [0, 1]$, the $\alpha-$ cut is the crisp set $[\tilde{A}]_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}$.

Definition 2.3 The support of a fuzzy set $\tilde{A}$ within a universal set X is a crisp set that contains all the elements of X that have a non-zero membership grade in $\tilde{A}$, i.e. Supp($\tilde{A}$) $= \{x \in X | \mu_{\tilde{A}}(x) > 0\}$.

Definition 2.4 The height $h(\tilde{A})$ of a fuzzy set $\tilde{A}$ is the largest membership grade obtained by any element in that set, i.e., $h(\tilde{A}) = sup\{\mu_{\tilde{A}}(x) | x \in X\}$. Also, a fuzzy set $\tilde{A}$ is called normal when $h(\tilde{A}) = 1$.

Definition 2.5 A fuzzy set $\tilde{A}$ is called convex if and only if for each $x, y \in R$ (set of real numbers) and each $\lambda \in [0, 1], \mu_{\tilde{A}}(\lambda x + (1 - \lambda)y) \geq min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$.

Definition 2.6 A fuzzy set $\tilde{A}$ on R is said to be a fuzzy number, if

(i) $\tilde{A}$ is normal.

(ii) $[\tilde{A}]_\alpha$ is a closed interval for every $\alpha \in (0, 1]$.

(iii) The support of $\tilde{A}$ is bounded.

Definition 2.7 A fuzzy number on R is said to be a symmetric trapezoidal fuzzy number, if there exist real numbers a_1 and a_2, $a_1 \leq a_2$ and $\alpha > 0$, such that

$$\mu_{\tilde{A}(x)} = \begin{cases} \frac{x}{\alpha} + \frac{\alpha - a_1}{\alpha}, & x \in [a_1 - \alpha, a_1); \\ 1, & x \in [a_1, a_2]; \\ \frac{-x}{\alpha} + \frac{a_2 + \alpha}{\alpha}, & x \in (a_2, a_2 + \alpha]; \\ 0, & \text{otherwise.} \end{cases}$$

A symmetric trapezoidal fuzzy number $\tilde{A}$ is denoted by $\tilde{A} = (a_1, a_2, \alpha, \alpha)$.

2.1.2 Arithmetic operations on symmetric trapezoidal fuzzy numbers

In this section, arithmetic operations on symmetric trapezoidal fuzzy numbers, are presented.

Let $\tilde{A} = (a_1, a_2, \alpha, \alpha)$ and $\tilde{B} = (b_1, b_2, \beta, \beta)$ be two symmetric trapezoidal fuzzy numbers. Then the arithmetic operations on $\tilde{A}$ and $\tilde{B}$ are as follows:

(i) $x \geq 0, x \in R; x\tilde{A} = (xa_1, xa_2, x\alpha, x\alpha)$.

(ii) $x < 0, x \in R; x\tilde{A} = (xa_2, xa_1, -x\alpha, -x\alpha)$.

(iii) $\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, \alpha + \beta, \alpha + \beta)$.

(iv) $\tilde{A} - \tilde{B} = (a_1 - b_2, a_2 - b_1, \alpha + \beta, \alpha + \beta)$.

(v) $\tilde{A} \otimes_G \tilde{B} = \big((\frac{a_1 + a_2}{2})(\frac{b_1 + b_2}{2}) - \omega, (\frac{a_1 + a_2}{2})(\frac{b_1 + b_2}{2}) + \omega, |a_2\beta + b_2\alpha|, |a_2\beta + b_2\alpha| \big)$

where, $\omega = \frac{h-k}{2}, k = min(a_1b_1, a_1b_2, a_2b_1, a_2b_2)$ and $h = max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)$.

2.1.3 Comparison of symmetric trapezoidal fuzzy numbers

To find the fuzzy optimal solution of a fuzzy linear programming problem, there is need to compare fuzzy numbers. Several methods have been proposed in the literature for comparing fuzzy numbers. In this section, the method, used in the existing methods [45, 52, 82, 84], for comparing fuzzy numbers, is presented.

If $\tilde{A} = (a_1, a_2, \alpha, \alpha)$ and $\tilde{B} = (b_1, b_2, \beta, \beta)$ are two symmetric trapezoidal fuzzy numbers, then

(i) $\tilde{A} \succeq \tilde{B}$ if and only if $\Re(\tilde{A}) \geq \Re(\tilde{B})$.

(ii) $\tilde{A} \succ \tilde{B}$ if and only if $\Re(\tilde{A}) > \Re(\tilde{B})$.

(iii) $\tilde{A} \approx \tilde{B}$ if and only if $\Re(\tilde{A}) = \Re(\tilde{B})$.

where $\Re(\tilde{A}) = \frac{1}{2}[a_1 + a_2]$ and $\Re(\tilde{B}) = \frac{1}{2}[b_1 + b_2]$.

2.2 Proposed Mehar method

In this section, a new method (named as Mehar method) is proposed to find the fuzzy optimal solution of problem $(P_{2.1})$.

The steps of the proposed Mehar method are as follows:

Step 1 Using Section 2.1.3, the problem $(P_{2.1})$ can be transformed into problem $(P_{2.2})$.

$$\begin{aligned}
&\text{Maximize/Minimize } \left[\Re(\tilde{z}) = \Re\left(\sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right)\right] \\
&\text{Subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{2.2}) \\
&\Re\left(\sum_{j=1}^{n} a_{ij}\tilde{x}_j\right) \leq, =, \geq \Re(\tilde{b}_i),\ i = 1, 2, ..., m, \\
&\Re(\tilde{x}_j) \geq \Re(\tilde{0}),\ j = 1, 2, ..., n.
\end{aligned}$$

Step 2 Using the properties $\Re\left(\sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right) = \sum_{j=1}^{n} \Re\left(\tilde{c}_j \otimes_G \tilde{x}_j\right) = \sum_{j=1}^{n} \Re(\tilde{c}_j)\Re(\tilde{x}_j)$ and $\Re\left(\sum_{j=1}^{n} a_{ij}\tilde{x}_j\right) = \sum_{j=1}^{n} \Re(a_{ij}\tilde{x}_j) = \sum_{j=1}^{n} a_{ij}\Re(\tilde{x}_j)$, the problem $(P_{2.2})$ can be transformed into problem $(P_{2.3})$.

$$\begin{aligned}
&\text{Maximize/Minimize } \left[\Re(\tilde{z}) = \sum_{j=1}^{n} \Re(\tilde{c}_j)\Re(\tilde{x}_j)\right] \\
&\text{Subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{2.3}) \\
&\sum_{j=1}^{n} a_{ij}\Re(\tilde{x}_j) \leq, =, \geq \Re(\tilde{b}_i),\ i = 1, 2, ..., m, \\
&\Re(\tilde{x}_j) \geq \Re(\tilde{0}),\ j = 1, 2, ..., n.
\end{aligned}$$

Step 3 Since $\Re(\tilde{A})$ is a real number, so assuming $\Re(\tilde{c}_j) = c_j,\ \Re(\tilde{b}_i) = b_i, \Re(\tilde{z}) = z$ and $\Re(\tilde{x}_j) = x_j$ and putting $\Re(\tilde{0}) = 0$, the problem $(P_{2.3})$ can be transformed into problem $(P_{2.4})$.

Maximize/Minimize $\left[z = \sum\limits_{j=1}^{n} c_j x_j\right]$

Subject to $\hfill (P_{2.4})$

$\sum\limits_{j=1}^{n} a_{ij}x_j \leq, =, \geq b_i,\ i = 1, 2, ..., m,$

$x_j \geq 0, \quad j = 1, 2, ..., n.$

Step 4 Using an appropriate existing method [144], find the optimal solution of the problem $(P_{2.4})$.

Step 5 Since, there exist infinite symmetric trapezoidal fuzzy numbers having the same rank. So, if $x_1 = a_1,\ x_2 = a_2, ..., x_n = a_n$ is an optimal solution of the problem $(P_{2.4})$, then all the symmetric trapezoidal fuzzy numbers $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n$ such that $\Re(\tilde{x}_1) = a_1$, $\Re(\tilde{x}_2) = a_2$,...,$\Re(\tilde{x}_n) = a_n$ will also be the fuzzy optimal solution of the problem $(P_{2.1})$.

2.3 Fuzzy optimal solution of an existing problem by proposed Mehar method

In this section, the existing problem [82, Example 1, pp. 178] is solved by proposed Mehar method.

Example 2.1 [82, Example 1, pp. 178]

Maximize $[\tilde{z} \approx -(13, 15, 2, 2) \otimes_G \tilde{x}_1 - (12, 14, 3, 3) \otimes_G \tilde{x}_2 - (15, 17, 2, 2) \otimes_G \tilde{x}_3]$

Subject to $\hfill (P_{2.5})$

$2\tilde{x}_1 + 3\tilde{x}_2 + 2\tilde{x}_3 \succeq (45, 55, 6, 6),$

$4\tilde{x}_1 + 3\tilde{x}_3 \succeq (60, 80, 8, 8),$

$2\tilde{x}_1 + 5\tilde{x}_2 \succeq (65, 95, 5, 5),$

$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \succeq \tilde{0}.$

Using the Mehar method, proposed in Section 2.2, the fuzzy optimal solution of the problem $(P_{2.5})$ can be obtained as follows:

Step 1 Using Step 1 of the proposed Mehar method, the problem $(P_{2.5})$ can be transformed into its equivalent problem $(P_{2.6})$.

Maximize $[\Re(\tilde{z}) = \Re(-(13, 15, 2, 2) \otimes_G \tilde{x}_1 - (12, 14, 3, 3) \otimes_G \tilde{x}_2 - (15, 17, 2, 2) \otimes_G \tilde{x}_3)]$

Subject to $(P_{2.6})$

$\Re(2\tilde{x}_1 + 3\tilde{x}_2 + 2\tilde{x}_3) \geq \Re(45, 55, 6, 6),$

$\Re(4\tilde{x}_1 + 3\tilde{x}_3) \geq \Re(60, 80, 8, 8),$

$\Re(2\tilde{x}_1 + 5\tilde{x}_2) \geq \Re(65, 95, 5, 5),$

$\Re(\tilde{x}_1), \Re(\tilde{x}_2), \Re(\tilde{x}_3) \geq \Re(\tilde{0}).$

Step 2 Using Step 2 of the proposed Mehar method, the problem $(P_{2.6})$ can be transformed into its equivalent problem $(P_{2.7})$.

Maximize $[\Re(\tilde{z}) = -\Re(13, 15, 2, 2)\Re(\tilde{x}_1) - \Re(12, 14, 3, 3)\Re(\tilde{x}_2) - \Re(15, 17, 2, 2)\Re(\tilde{x}_3)]$

Subject to $(P_{2.7})$

$2\Re(\tilde{x}_1) + 3\Re(\tilde{x}_2) + 2\Re(\tilde{x}_3) \geq \Re(45, 55, 6, 6),$

$4\Re(\tilde{x}_1) + 3\Re(\tilde{x}_3) \geq \Re(60, 80, 8, 8),$

$2\Re(\tilde{x}_1) + 5\Re(\tilde{x}_2) \geq \Re(65, 95, 5, 5),$

$\Re(\tilde{x}_1), \Re(\tilde{x}_2), \Re(\tilde{x}_3) \geq \Re(\tilde{0}).$

Step 3 Using Step 3 of the proposed Mehar method, the problem $(P_{2.7})$ can be transformed into its equivalent problem $(P_{2.8})$.

Maximize $[z = -14x_1 - 13x_2 - 16x_3]$

Subject to $(P_{2.8})$

$2x_1 + 3x_2 + 2x_3 \geq 50,$

$4x_1 + 3x_3 \geq 70,$

$2x_1 + 5x_2 \geq 80,$

$x_1, x_2, x_3 \geq 0.$

Step 4 On solving the problem $(P_{2.8})$, the obtained optimal solution is $x_1 = \frac{35}{2}, x_2 = 9, x_3 = 0$.

Step 5 Using Step 5 of the proposed Mehar method, all the symmetric trapezoidal fuzzy numbers $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ such that $\Re(\tilde{x}_1) = \frac{35}{2}, \Re(\tilde{x}_2) = 9$ and $\Re(\tilde{x}_3) = 0$ will be the fuzzy optimal solutions of the problem $(P_{2.5})$; for example, the following three are fuzzy optimal solutions of the problem $(P_{2.5})$.

(i) $\tilde{x}_1 = (10, 25, 3, 3)$, $\tilde{x}_2 = (8, 10, 2, 2)$, $\tilde{x}_3 = (-1, 1, 3, 3)$.

(ii) $\tilde{x}_1 = (17, 18, 4, 4)$, $\tilde{x}_2 = (9, 9, 3, 3)$, $\tilde{x}_3 = (-3, 3, 4, 4)$.

(iii) $\tilde{x}_1 = (12, 23, 3, 3)$, $\tilde{x}_2 = (6, 12, 4, 4)$, $\tilde{x}_3 = (-2, 2, 3, 3)$.

2.4 Fuzzy optimal solution of existing problem by existing methods

To show the advantages of proposed Mehar method over other existing methods [45, 52, 82, 84], there is need to solve the same existing problem [82, Example 1, pp. 178] by the existing methods [45,52,82,84]. Therefore, in this section, the fuzzy optimal solution of the existing problem [82, Example 1, pp. 178] is obtained by the existing methods [45, 52, 82, 84].

2.4.1 Fuzzy optimal solution by Ganesan and Veeramani method

Using the method, proposed by Ganesan and Veeramani [52], the fuzzy optimal solution of existing problem [82, Example 1, pp. 178], can be obtained as follows:

The problem $(P_{2.5})$ can be written as:

Maximize $\left[\tilde{z} \approx -(13,15,2,2) \otimes_G \tilde{x}_1 - (12,14,3,3) \otimes_G \tilde{x}_2 - (15,17,2,2) \otimes_G \tilde{x}_3 - M\tilde{A}_1 - M\tilde{A}_2 - M\tilde{A}_3\right]$

Subject to $(P_{2.9})$

$2\tilde{x}_1 + 3\tilde{x}_2 + 2\tilde{x}_3 - \tilde{s}_1 + \tilde{A}_1 \approx (45,55,6,6)$,

$4\tilde{x}_1 + 3\tilde{x}_3 - \tilde{s}_2 + \tilde{A}_2 \approx (60,80,8,8)$,

$2\tilde{x}_1 + 5\tilde{x}_2 - \tilde{s}_3 + \tilde{A}_3 \approx (65,95,5,5)$,

$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \succeq \tilde{0}$.

Initial iteration The initial table for the problem $(P_{2.9})$ is given as follows:

Table 2.1

The initial table for problem $(P_{2.9})$

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	Solution
$\tilde{A}_1$	2	3	2	-1	0	0	1	0	0	(45,55,6,6)
$\tilde{A}_2$	4	0	3	0	-1	0	0	1	0	(60,80,8,8)
$\tilde{A}_3$	2	5	0	0	0	-1	0	0	1	(65,95,5,5)
$\tilde{z}_j - \tilde{c}_j$	$-8M + (13,15,2,2)$	$-8M + (12,14,3,3)$	$-5M + (15,17,2,2)$	M	M	M	0	0	0	$(-230M, -170M, 19M, 19M)$

Since, minimum $\{-8M+(13,15,2,2), -8M+(12,14,3,3), -5M+(15,17,2,2)\}$ = minimum $\{\Re(-8M+(13,15,2,2)), \Re(-8M+(12,14,3,3)), \Re(-5M+(15,17,2,2))\}$ is corresponding to $-8M+(12,14,3,3)$ i.e., $\tilde{x}_2$, so $\tilde{x}_2$ will be the entering variable and minimum $\{\frac{(45,55,6,6)}{3}, \frac{(65,95,5,5)}{5}\}$ = minimum $\{\frac{\Re(45,55,6,6)}{3}, \frac{\Re(65,95,5,5)}{5}\}$ is corresponding to $\frac{(65,95,5,5)}{5}$ i.e., $\tilde{A}_3$, so $\tilde{A}_3$ will be the leaving variable. The next updated table is given as follows:

Table 2.2

The first iteration table

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$	$\tilde{A}_1$	$\tilde{A}_2$	Solution
$\tilde{A}_1$	$\frac{4}{5}$	0	2	-1	0	$\frac{3}{5}$	1	0	$(-12, 16, 9, 9)$
$\tilde{A}_2$	4	0	3	0	-1	0	0	1	(60,80,8,8)
$\tilde{x}_2$	$\frac{2}{5}$	1	0	0	0	$-\frac{1}{5}$	0	0	(13,19,1,1)
$\tilde{z}_j - \tilde{c}_j$	$-\frac{24}{5}M + (\frac{37}{5}, \frac{51}{5}, \frac{16}{5}, \frac{16}{5})$	0	$-5M + (15, 17, 2, 2)$	M	M	$-\frac{3}{5}M + (\frac{12}{5}, \frac{14}{5}, \frac{3}{5}, \frac{3}{5})$	0	0	$(-96M - 263, -48M - 153, 17M + 71, 17M + 71)$

Since, minimum $\{-\frac{24}{5}M+(\frac{37}{5}, \frac{51}{5}, \frac{16}{5}, \frac{16}{5}), -5M+(15, 17, 2, 2), -\frac{3}{5}M+(\frac{12}{5}, \frac{14}{5}, \frac{3}{5}, \frac{3}{5})\} =$ minimum $\{\Re(-\frac{24}{5}M+(\frac{37}{5}, \frac{51}{5}, \frac{16}{5}, \frac{16}{5})), \Re(-5M+(15, 17, 2, 2)), \Re(-\frac{3}{5}M+(\frac{12}{5}, \frac{14}{5}, \frac{3}{5}, \frac{3}{5}))\}$ is corresponding to $-5M + (15, 17, 2, 2)$ i.e., $\tilde{x}_3$, so $\tilde{x}_3$ will be the entering variable and minimum $\{\frac{(-12,16,9,9)}{2}, \frac{(60,80,8,8)}{3}\}$ = minimum $\{\frac{\Re(-12,16,9,9)}{2}, \frac{\Re(60,80,8,8)}{3}\}$ is corresponding to $\frac{(-12,16,9,9)}{2}$ i.e., $\tilde{A}_1$ so $\tilde{A}_1$ will be the leaving variable. The next updated table is given as follows:

Table 2.3

The second iteration table

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$	$\tilde{A}_2$	Solution
$\tilde{x}_3$	$\frac{2}{5}$	0	1	$-\frac{1}{2}$	0	$\frac{3}{10}$	0	$(-6, 8, \frac{9}{2}, \frac{9}{2})$
$\tilde{A}_2$	$\frac{14}{5}$	0	0	$\frac{3}{2}$	-1	$-\frac{9}{10}$	1	$(36, 98, \frac{43}{2}, \frac{43}{2})$
$\tilde{x}_2$	$\frac{2}{5}$	1	0	0	0	$-\frac{1}{5}$	0	(13,19,1,1)
$\tilde{z}_j - \tilde{c}_j$	$-\frac{14}{5}M + (\frac{3}{5}, \frac{21}{5}, 4, 4)$	0	0	$-\frac{3}{2}M + (\frac{15}{2}, \frac{17}{2}, 1, 1)$	M	$\frac{9}{10}M + (-\frac{27}{10}, -\frac{17}{10}, \frac{6}{5}, \frac{6}{5})$	0	$(-98M, -36M, \frac{43}{2}M, \frac{43}{2}M) + (-398, -50, \frac{327}{2}, \frac{327}{2})$

Since, minimum $\{-\frac{14}{5}M+(\frac{3}{5}, \frac{21}{5}, 4, 4), -\frac{3}{2}M+(\frac{15}{2}, \frac{17}{2}, 1, 1)\} =$ minimum $\{\Re(-\frac{14}{5}M+(\frac{3}{5}, \frac{21}{5}, 4, 4)), \Re(-\frac{3}{2}M+(\frac{15}{2}, \frac{17}{2}, 1, 1))\}$ is corresponding to $-\frac{14}{5}M+(\frac{3}{5}, \frac{21}{5}, 4, 4)$ i.e., $\tilde{x}_1$, so $\tilde{x}_1$ will be the entering variable and minimum $\left\{\frac{(-6,8,\frac{9}{2},\frac{9}{2})}{\frac{2}{5}}, \frac{(36,98,\frac{43}{2},\frac{43}{2})}{\frac{14}{5}}, \frac{(13,19,1,1)}{\frac{2}{5}}\right\} =$ minimum $\left\{\frac{\Re(-6,8,\frac{9}{2},\frac{9}{2})}{\frac{2}{5}}, \frac{\Re(36,98,\frac{43}{2},\frac{43}{2})}{\frac{14}{5}}, \frac{\Re(13,19,1,1)}{\frac{2}{5}}\right\}$ is corresponding to $\frac{(-6,8,\frac{9}{2},\frac{9}{2})}{\frac{2}{5}}$, i.e., $\tilde{x}_3$, so $\tilde{x}_3$ will be the leaving variable. The next updated table is given as follows:

Table 2.4

The third iteration table

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$	$\tilde{A}_2$	Solution
$\tilde{x}_1$	1	0	$\frac{5}{2}$	$-\frac{5}{4}$	0	$\frac{3}{4}$	0	$(-15, 20, \frac{45}{4}, \frac{45}{4})$
$\tilde{A}_2$	0	0	-7	5	-1	-3	1	$(-20, 140, \frac{53}{2}, \frac{53}{2})$
$\tilde{x}_2$	0	1	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$(5, 25, \frac{11}{2}, \frac{11}{2})$
$\tilde{z}_j - \tilde{c}_j$	0	0	$7M + (-\frac{51}{2}, -\frac{37}{2}, 8, 8)$	$-5M + (\frac{37}{4}, \frac{51}{4}, 4, 4)$	M	$3M + (-\frac{21}{4}, -\frac{11}{4}, 3, 3)$	0	$(-140M, 20M, \frac{53}{2}, \frac{53}{2}) + (-\frac{1275}{2}, \frac{355}{2}, \frac{1751}{4}, \frac{1751}{4})$

Since, $-5M + (\frac{37}{4}, \frac{51}{4}, 4, 4)$ is the only negative value of $\tilde{z}_j - \tilde{c}_j$ which is corresponding to $\tilde{s}_1$, so $\tilde{s}_1$ will be the entering variable and minimum $\{\frac{(-20,140,\frac{53}{2},\frac{53}{2})}{5}, \frac{(5,25,\frac{11}{2},\frac{11}{2})}{\frac{1}{2}}\} =$ minimum $\{\frac{\Re(-20,140,\frac{53}{2},\frac{53}{2})}{5}, \frac{\Re(5,25,\frac{11}{2},\frac{11}{2})}{\frac{1}{2}}\}$ is corresponding to $\frac{(-20,140,\frac{53}{2},\frac{53}{2})}{5}$ i.e., $\tilde{A}_2$, so $\tilde{A}_2$ will be the leaving variable. The next updated table is given as follows:

Table 2.5

The optimal table

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$	Solution
$\tilde{x}_1$	1	0	$\frac{3}{4}$	0	$-\frac{1}{4}$	0	$(-20, 55, \frac{49}{2}, \frac{49}{2})$
$\tilde{s}_1$	0	0	$-\frac{7}{5}$	1	$-\frac{1}{5}$	$-\frac{3}{5}$	$(-4, 28, \frac{53}{10}, \frac{53}{10})$
$\tilde{x}_2$	0	1	$-\frac{3}{10}$	0	$\frac{1}{10}$	$-\frac{1}{5}$	$(-9, 27, \frac{163}{20}, \frac{163}{20})$
$\tilde{z}_j - \tilde{c}_j$	0	0	$(-\frac{113}{20}, \frac{153}{20}, \frac{4}{5}, \frac{4}{5})$	0	$(\frac{37}{20}, \frac{51}{20}, \frac{4}{5}, \frac{4}{5})$	0	$(-\frac{2353}{2}, \frac{905}{2}, \frac{3363}{5}, \frac{3363}{5})$

Since $\Re(-\frac{113}{20}, \frac{153}{20}, \frac{4}{5}, \frac{4}{5}) \geq 0$ and $\Re(\frac{37}{20}, \frac{51}{20}, \frac{4}{5}, \frac{4}{5}) \geq 0$. So, the obtained solution is fuzzy optimal solution.

Hence, the fuzzy optimal solution is $\tilde{x}_1 = (-20, 55, \frac{49}{2}, \frac{49}{2})$, $\tilde{s}_1 = (-4, 28, \frac{53}{10}, \frac{53}{10})$ and $\tilde{x}_2 = (-9, 27, \frac{163}{20}, \frac{163}{20})$ and fuzzy optimal value is $\tilde{z} = (-\frac{2353}{2}, \frac{905}{2}, \frac{3363}{5}, \frac{3363}{5})$.

2.4.2 Fuzzy optimal solution by Kumar and Kaur method

Using the method, proposed by Kumar and Kaur [84], the fuzzy optimal solution of existing problem [82, Example 1, pp. 178], can be obtained as follows:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, \alpha_1, \alpha_1), \tilde{x}_2 = (x_2, y_2, \alpha_2, \alpha_2)$ and $\tilde{x}_3 = (x_3, y_3, \alpha_3, \alpha_3)$, the problem $(P_{2.5})$ can be transformed into problem $(P_{2.10})$.

Maximize $[\tilde{z} \approx -(13, 15, 2, 2) \otimes_G (x_1, y_1, \alpha_1, \alpha_1) - (12, 14, 3, 3) \otimes_G (x_2, y_2, \alpha_2, \alpha_2) - (15, 17, 2, 2) \otimes_G (x_3, y_3, \alpha_3, \alpha_3)]$

Subject to $(P_{2.10})$

$2(x_1, y_1, \alpha_1, \alpha_1) + 3(x_2, y_2, \alpha_2, \alpha_2) + 2(x_3, y_3, \alpha_3, \alpha_3) \succeq (45, 55, 6, 6),$

$4(x_1, y_1, \alpha_1, \alpha_1) + 3(x_3, y_3, \alpha_3, \alpha_3) \succeq (60, 80, 8, 8),$

$2(x_1, y_1, \alpha_1, \alpha_1) + 5(x_2, y_2, \alpha_2, \alpha_2) \succeq (65, 95, 5, 5),$

$(x_1, y_1, \alpha_1, \alpha_1), (x_2, y_2, \alpha_2, \alpha_2), (x_3, y_3, \alpha_3, \alpha_3) \succeq \tilde{0}.$

Step 2 The problem $(P_{2.10})$ can be transformed into problem $(P_{2.11})$.

Maximize $[\tilde{z} \approx -(13, 15, 2, 2) \otimes_G (x_1, y_1, \alpha_1, \alpha_1) - (12, 14, 3, 3) \otimes_G (x_2, y_2, \alpha_2, \alpha_2) - (15, 17, 2, 2) \otimes_G (x_3, y_3, \alpha_3, \alpha_3)]$

Subject to $(P_{2.11})$

$(2x_1, 2y_1, 2\alpha_1, 2\alpha_1) + (3x_2, 3y_2, 3\alpha_2, 3\alpha_2) + (2x_3, 2y_3, 2\alpha_3, 2\alpha_3) \succeq (45, 55, 6, 6),$

$(4x_1, 4y_1, 4\alpha_1, 4\alpha_1) + (3x_3, 3y_3, 3\alpha_3, 3\alpha_3) \succeq (60, 80, 8, 8),$

$(2x_1, 2y_1, 2\alpha_1, 2\alpha_1) + (5x_2, 5y_2, 5\alpha_2, 5\alpha_2) \succeq (65, 95, 5, 5),$

$(x_1, y_1, \alpha_1, \alpha_1), (x_2, y_2, \alpha_2, \alpha_2), (x_3, y_3, \alpha_3, \alpha_3) \succeq \tilde{0}.$

Step 3 Using the property of ranking function $\Re((13, 15, 2, 2) \otimes_G (x_1, y_1, \alpha_1, \alpha_1)) = \Re(13, 15, 2, 2)\Re(x_1, y_1, \alpha_1, \alpha_1)$, $\Re((12, 14, 3, 3) \otimes_G (x_2, y_2, \alpha_2, \alpha_2)) = \Re(12, 14, 3, 3)\Re(x_2, y_2, \alpha_2, \alpha_2)$, $\Re((15, 17, 2, 2) \otimes_G (x_3, y_3, \alpha_3, \alpha_3)) = \Re(15, 17, 2, 2)\Re(x_3, y_3, \alpha_3, \alpha_3)$, the problem $(P_{2.11})$ can be transformed into problem $(P_{2.12})$.

Maximize $[-7(x_1 + y_1) - \frac{13}{2}(x_2 + y_2) - 8(x_3 + y_3)]$

Subject to $(P_{2.12})$

$x_1 + y_1 + \frac{3}{2}x_2 + \frac{3}{2}y_2 + x_3 + y_3 \geq 50,$

$2x_1 + 2y_1 + \frac{3}{2}x_3 + \frac{3}{2}y_3 \geq 70,$

$x_1 + y_1 + \frac{5}{2}x_2 + \frac{5}{2}y_2 \geq 80,$

$x_1 + y_1 \geq 0,$

$x_2 + y_2 \geq 0,$

$x_3 + y_3 \geq 0,$

$x_1 \leq y_1, x_2 \leq y_2, x_3 \leq y_3,$

$\alpha_1, \alpha_2, \alpha_3 \geq 0.$

Step 4 Solving the crisp linear programming problem $(P_{2.12})$, the obtained solution is $x_1 = \frac{35}{2}, y_1 = \frac{35}{2}, x_2 = 9, y_2 = 9, x_3 = 0, y_3 = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0.$

Putting these values in $\tilde{x}_1 = (x_1, y_1, \alpha_1, \alpha_1), \tilde{x}_2 = (x_2, y_2, \alpha_2, \alpha_2), \tilde{x}_3 = (x_3, y_3, \alpha_3, \alpha_3)$, the obtained fuzzy optimal solution is $\tilde{x}_1 = (\frac{35}{2}, \frac{35}{2}, 0, 0)$, $\tilde{x}_2 = (9, 9, 0, 0), \tilde{x}_3 = (0, 0, 0, 0)$.

Step 5 Putting the values of $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ in $\tilde{z} \approx -(13, 15, 2, 2) \otimes_G (x_1, y_1, \alpha_1, \alpha_1) - (12, 14, 3, 3) \otimes_G (x_2, y_2, \alpha_2, \alpha_2) - (15, 17, 2, 2) \otimes_G (x_3, y_3, \alpha_3, \alpha_3)$, the fuzzy optimal value is $\tilde{z} \approx (-\frac{777}{2}, -\frac{671}{2}, 0, 0)$.

2.4.3 Fuzzy optimal solution by Kheirfam and Verdegay method

Using the method, proposed by Kheirfam and Verdegay [82], the fuzzy optimal solution of existing problem [82, Example 1, pp. 178], can be obtained as follows:

Step 1 Assuming $\tilde{x}_4, \tilde{x}_5, \tilde{x}_6$ as fuzzy surplus variables, problem $(P_{2.5})$ can be transformed into problem $(P_{2.13})$.

Maximize $[\tilde{z} \approx -(13, 15, 2, 2) \otimes_G \tilde{x}_1 - (12, 14, 3, 3) \otimes_G \tilde{x}_2 - (15, 17, 2, 2) \otimes_G \tilde{x}_3]$

Subject to $(P_{2.13})$

$2\tilde{x}_1 + 3\tilde{x}_2 + 2\tilde{x}_3 - \tilde{x}_4 \approx (45, 55, 6, 6),$

$4\tilde{x}_1 + 3\tilde{x}_3 - \tilde{x}_5 \approx (60, 80, 8, 8),$

$2\tilde{x}_1 + 5\tilde{x}_2 - \tilde{x}_6 \approx (65, 95, 5, 5),$

$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6 \succeq \tilde{0}$.

Step 2 The problem $(P_{2.13})$ can be transformed into problem $(P_{2.14})$.

Maximize $[\tilde{z} \approx -(13, 15, 2, 2) \otimes_G \tilde{x}_1 - (12, 14, 3, 3) \otimes_G \tilde{x}_2 - (15, 17, 2, 2) \otimes_G \tilde{x}_3]$

Subject to $(P_{2.14})$

$-2\tilde{x}_1 - 3\tilde{x}_2 - 2\tilde{x}_3 + \tilde{x}_4 \approx (-55, -45, 6, 6)$,

$-4\tilde{x}_1 - 3\tilde{x}_3 + \tilde{x}_5 \approx (-80, -60, 8, 8)$,

$-2\tilde{x}_1 - 5\tilde{x}_2 + \tilde{x}_6 \approx (-95, -65, 5, 5)$,

$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6 \succeq \tilde{0}$.

Step 3 The initial table for problem $(P_{2.14})$ is given as follows:

Table 2.6

The initial table for problem $(P_{2.14})$

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	Solution
$\tilde{x}_4$	-2	-3	-2	1	0	0	$(-55, -45, 6, 6)$
$\tilde{x}_5$	-4	0	-3	0	1	0	$(-80, -60, 8, 8)$
$\tilde{x}_6$	-2	-5	0	0	0	1	$(-95, -65, 5, 5)$
$\tilde{z}_j - \tilde{c}_j$	(13,15,2,2)	(12,14,3,3)	(15,17,2,2)	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$

Since, minimum $\{(-55, -45, 6, 6), (-80, -60, 8, 8), (-95, -65, 5, 5)\}$ = minimum $\{\Re(-55, -45, 6, 6), \Re(-80, -60, 8, 8), \Re(-95, -65, 5, 5)\}$ is corresponding to $(-95, -65, 5, 5)$ i.e., $\tilde{x}_6$, so $\tilde{x}_6$ is a leaving variable and maximum $\left\{(-\frac{15}{2}, -\frac{13}{2}, 1, 1), (-\frac{14}{5}, -\frac{12}{5}, \frac{3}{5}, \frac{3}{5})\right\}$ = maximum $\left\{\Re(-\frac{15}{2}, -\frac{13}{2}, 1, 1), \Re(-\frac{14}{5}, -\frac{12}{5}, \frac{3}{5}, \frac{3}{5})\right\}$ is corresponding to $(-\frac{14}{5}, -\frac{12}{5}, \frac{3}{5}, \frac{3}{5})$, i.e., $\tilde{x}_2$, so $\tilde{x}_2$ is an entering variable. Now, after applying required row-operations, the next updated table is given as follows:

Table 2.7

The first iteration table

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	Solution
$\tilde{x}_4$	$-\frac{4}{5}$	0	-2	1	0	$-\frac{3}{5}$	$(-16, 12, 9, 9)$
$\tilde{x}_5$	-4	0	-3	0	1	0	$(-80, -60, 8, 8)$
$\tilde{x}_2$	$\frac{2}{5}$	1	0	0	0	$-\frac{1}{5}$	$(13, 19, 1, 1)$
$\tilde{z}_j - \tilde{c}_j$	$(\frac{37}{5}, \frac{51}{5}, \frac{16}{5}, \frac{16}{5})$	$\tilde{0}$	(15,17,2,2)	$\tilde{0}$	$\tilde{0}$	$(\frac{12}{5}, \frac{14}{5}, \frac{3}{5}, \frac{3}{5})$	$(-263, -153, 71, 71)$

Since, minimum $\{(-16,12,9,9),(-80,-60,8,8)\}$ = minimum $\{\Re(-16,12,9,9),$ $\Re(-80,-60,8,8)\}$ is corresponding to $(-80,-60,8,8)$ i.e., $\tilde{x}_5$, so $\tilde{x}_5$ is a leaving variable and maximum $\left\{(-\frac{51}{20},-\frac{37}{20},\frac{16}{20},\frac{16}{20}),(-\frac{17}{3},-\frac{15}{3},\frac{2}{3},\frac{2}{3})\right\}$ = maximum $\left\{\Re(-\frac{51}{20},-\frac{37}{20},\frac{16}{20},\frac{16}{20}),\right.$ $\left.\Re(-\frac{17}{3},-\frac{15}{3},\frac{2}{3},\frac{2}{3})\right\}$ is corresponding to $(-\frac{51}{20},-\frac{37}{20},\frac{16}{20},\frac{16}{20})$, i.e., $\tilde{x}_1$, so $\tilde{x}_1$ is an entering variable. The next updated table is given as follows:

Table 2.8

The optimal table

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	Solution
$\tilde{x}_4$	0	0	$-\frac{7}{5}$	1	$-\frac{1}{5}$	$-\frac{3}{5}$	$(-4,28,\frac{48}{5},\frac{48}{5})$
$\tilde{x}_1$	1	0	$\frac{3}{4}$	0	$-\frac{1}{4}$	0	$(15,20,2,2)$
$\tilde{x}_2$	0	1	$-\frac{3}{10}$	0	$\frac{1}{10}$	$-\frac{1}{5}$	$(5,13,\frac{9}{5},\frac{9}{5})$
$\tilde{z}_j-\tilde{c}_j$	$\tilde{0}$	$\tilde{0}$	$(\frac{147}{20},\frac{229}{20},\frac{22}{5},\frac{22}{5})$	$\tilde{0}$	$(\frac{37}{20},\frac{51}{20},\frac{4}{5},\frac{4}{5})$	$(\frac{12}{5},\frac{14}{5},\frac{3}{5},\frac{3}{5})$	$(-\frac{951}{2},-\frac{499}{2},\frac{157}{5},\frac{157}{5})$

Since, $\Re(-4,28,\frac{48}{5},\frac{48}{5}) \geq 0, \Re(15,20,2,2) \geq 0, \Re(5,13,\frac{9}{5},\frac{9}{5}) \geq 0$. Therefore, the obtained solution is fuzzy optimal solution. Hence, the fuzzy optimal solution is $\tilde{x}_1 = (15,20,2,2), \tilde{x}_2 = (5,13,\frac{9}{5},\frac{9}{5}), \tilde{x}_3 = \tilde{0}$ and fuzzy optimal value is $\tilde{z} = (-\frac{951}{2},-\frac{499}{2},\frac{157}{5},\frac{157}{5})$.

2.4.4 Fuzzy optimal solution by Ebrahimnejad and Tavana method

Using the method, proposed by Ebrahimnejad and Tavana [45], the fuzzy optimal solution of existing problem [82, Example 1, pp. 178], can be obtained as follows:

Following Step 1 to Step 3 of the Mehar method, proposed in Section 2.3, the fuzzy linear programming problem $(P_{2.5})$ can be transformed into the crisp linear programming problem $(P_{2.15})$:

Maximize $[z = -14x_1 - 13x_2 - 16x_3]$

Subject to $(P_{2.15})$

$2x_1 + 3x_2 + 2x_3 \geq 50,$

$4x_1 + 3x_3 \geq 70,$

$2x_1 + 5x_2 \geq 80,$

$x_1, x_2, x_3 \geq 0.$

After solving the problem $(P_{2.15})$ by using the Big-M method or two phase method, the obtained optimal table is given as follows:

Table 2.9

The optimal table of problem $(P_{2.15})$

Basis	x_1	x_2	x_3	s_1	s_2	s_3	Solution
x_1	1	0	$\frac{3}{4}$	0	$-\frac{1}{4}$	0	$\frac{35}{2}$
s_1	0	0	$-\frac{7}{5}$	1	$-\frac{1}{5}$	$-\frac{3}{5}$	12
x_2	0	1	$-\frac{3}{10}$	0	$\frac{1}{10}$	$-\frac{1}{5}$	9
$z_j - c_j$	0	0	$\frac{47}{5}$	0	$\frac{11}{5}$	0	-362

Then, the fuzzy optimal solution is obtained as follows:

$$\tilde{x}_B = \begin{pmatrix} \tilde{x}_1 \\ \tilde{s}_1 \\ \tilde{x}_2 \end{pmatrix} \approx B^{-1}\tilde{b} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ -1 & \frac{1}{5} & \frac{3}{5} \\ 0 & -\frac{1}{10} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} (45, 55, 6, 6) \\ (60, 80, 8, 8) \\ (65, 95, 5, 5) \end{pmatrix} \approx \begin{pmatrix} (15, 20, 2, 2) \\ (-4, 28, \frac{53}{5}, \frac{53}{5}) \\ (5, 13, \frac{9}{5}, \frac{9}{5}) \end{pmatrix}.$$

Putting the values of $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ in $[\tilde{z} \approx -(13, 15, 2, 2) \otimes_G \tilde{x}_1 - (12, 14, 3, 3) \otimes_G \tilde{x}_2 - (15, 17, 2, 2) \otimes_G \tilde{x}_3] = -(13, 15, 2, 2) \otimes_G (15, 20, 2, 2) - (12, 14, 3, 3) \otimes_G (-4, 28, \frac{53}{5}, \frac{53}{5}) - (15, 17, 2, 2) \otimes_G (5, 13, \frac{9}{5}, \frac{9}{5})$, the fuzzy optimal value is $\tilde{z} \approx (-\frac{951}{2}, -\frac{497}{2}, \frac{671}{5}, \frac{671}{5})$.

2.5 Advantages of the proposed Mehar method over other existing methods

In this section, the advantages of the proposed Mehar method over the other existing methods [45, 52, 82, 84], are discussed.

2.5.1 Advantages of the proposed Mehar method over Ganesan and Veeramani method as well as Kheirfam and Verdegay method

In this section, the advantages of the proposed Mehar method over Ganesan and Veeramani method [52] as well as Kheirfam and Verdegay method [82], are discussed.

1. In the crisp environment, an alternative optimal solution of a crisp linear programming problem will not exist if the relative profit corresponding to none of the basic variable is zero. On the same direction, in the existing methods [52, 82], it is assumed that if $\Re$(relative fuzzy profit) corresponding to none of the fuzzy basic variable is zero in the optimal table then alternative fuzzy optimal solution will not exist.

   For example, Table 2.8 is the optimal table of problem $(P_{2.5})$. Since, in Table 2.8, the $\Re$(fuzzy relative profit) corresponding to none of the fuzzy basic variables is zero, so, according to the existing methods [52, 82], there will not exist any alternative fuzzy optimal solution for this fuzzy linear programming problem.

   However, it can be easily verified that all the symmetric trapezoidal fuzzy numbers $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ such that $\Re(\tilde{x}_1) = \frac{35}{2}$, $\Re(\tilde{x}_2) = 9$ and $\Re(\tilde{x}_3) = 0$ will be the fuzzy optimal solutions of the problem $(P_{2.5})$.

   For example, the following are also fuzzy optimal solutions of the problem $(P_{2.5})$.

   (i) $\tilde{x}_1 = (10, 25, 3, 3)$, $\tilde{x}_2 = (8, 10, 2, 2)$, $\tilde{x}_3 = (-1, 1, 3, 3)$.

   (ii) $\tilde{x}_1 = (17, 18, 4, 4)$, $\tilde{x}_2 = (9, 9, 3, 3)$, $\tilde{x}_3 = (-3, 3, 4, 4)$.

(iii) $\tilde{x}_1 = (12, 23, 3, 3)$, $\tilde{x}_2 = (6, 12, 4, 4)$, $\tilde{x}_3 = (-2, 2, 3, 3)$.

While, all these fuzzy optimal solutions can be obtained by using Step 5 of the proposed Mehar method.

2. Since, for applying the proposed Mehar method, there is need to solve crisp linear programming problems. So, the existing and easily available software such as TORA, LINDO etc. can be used for the same. However, for applying the existing methods [52, 82] there is need to solve fuzzy linear programming problems. So, the existing and easily available software's such as TORA, LINDO etc. cannot be used and need to develop new softwares.
3. To find the fuzzy optimal solution by using the existing methods [52,82], there is need to use arithmetic operations of fuzzy numbers. While, if the proposed Mehar method is used for the same then there is need to use arithmetic operations on real numbers. Since, it is much complicated to apply the arithmetic operations on fuzzy numbers as compared to the arithmetic operations of real numbers. So, it is much easy to apply the proposed Mehar method as compared to the existing methods [52, 82].

2.5.2 Advantages of the proposed Mehar method over Kumar and Kaur method

In this section, the advantages of the proposed Mehar method over the method, proposed by Kumar and Kaur [84], are discussed.

1. To find the fuzzy optimal solution of problem ($P_{2.1}$) by using the method, proposed by Kumar and Kaur [84], firstly it is transformed into ($P_{2.16}$) then using the optimal solution of ($P_{2.16}$), the fuzzy optimal solution of problem ($P_{2.1}$) is obtained.

Maximize/Minimize $\left[\sum\limits_{j=1}^{n}(\frac{c_{j1}+c_{j2}}{2})(\frac{x_{j1}+x_{j2}}{2})\right]$

Subject to $(P_{2.16})$

$\sum\limits_{j=1}^{n} a_{ij}(\frac{x_{j1}+x_{j2}}{2}) \leq, =, \geq \frac{b_{i1}+b_{i2}}{2}, i = 1, 2, ..., m,$

$\frac{x_{j1}+x_{j2}}{2} \geq 0,\ j = 1, 2, ..., n$

$x_{j1} \leq x_{j2}.$

While, using the proposed Mehar method, problem $(P_{2.1})$ is firstly transformed into problem $(P_{2.17})$ then using the optimal solution of problem $(P_{2.17})$, the fuzzy optimal solution of problem $(P_{2.1})$ is obtained.

Maximize/Minimize $\left[\sum\limits_{j=1}^{n}(\frac{c_{j1}+c_{j2}}{2})x_j\right]$

Subject to $(P_{2.17})$

$\sum\limits_{j=1}^{n} a_{ij}x_j \leq, =, \geq \frac{b_{i1}+b_{i2}}{2}, i = 1, 2, ..., m,$

$x_j \geq 0.$

It is obvious that number of variables and number of constraints in problem $(P_{2.16})$, obtained on transforming problem $(P_{2.1})$ by Kumar and Kaur method [84], are more than number of variables and number of constraints in problem $(P_{2.17})$, obtained on transforming problem $(P_{2.1})$ by proposed Mehar method. Therefore, it is better to use proposed Mehar method as compared to Kumar and Kaur method [84].

2. It is obvious from Step 5 of the proposed Mehar method, there exist infinite symmetric trapezoidal fuzzy numbers having the same rank. So, if the solution of the problem $(P_{2.1})$ will exist then infinite alternative fuzzy optimal solution of problem $(P_{2.1})$ will also exist. Kumar and Kaur [84] have not discussed any condition about the existence of alternative fuzzy optimal solutions of problem $(P_{2.1})$.

2.5.3 Advantages of the proposed Mehar method over Ebrahimnejad and Tavana method

In this section, the advantages of the proposed Mehar method over the method, proposed by Ebrahimnejad and Tavana [45], are discussed.

1. To find the fuzzy optimal solution of problem ($P_{2.1}$) with the help of optimal solution of the problem ($P_{2.4}$) by using the method, proposed by Ebrahimnejad and Tavana [45], there is need to use arithmetic operations on fuzzy numbers. While, in the proposed Mehar method with the help of optimal solution of the problem ($P_{2.4}$), the fuzzy optimal solution of problem ($P_{2.1}$) can be directly obtained without arithmetic operations of fuzzy numbers.
2. It is obvious from Step 5 of the proposed Mehar method, there exist infinite symmetric trapezoidal fuzzy numbers having the same rank. So, if the solution of the problem ($P_{2.1}$) will exist then infinite alternative fuzzy optimal solution of problem ($P_{2.1}$) will also exist. Ebrahimnejad and Tavana [45] have not discussed any condition about the existence of alternative fuzzy optimal solutions of problem ($P_{2.1}$).

2.6 Conclusions

On the basis of the present study, it can be concluded that all the fuzzy linear programming problems which can be solved by the existing methods [45, 52, 82, 84] can also be solved by the proposed Mehar method. However, it is much easy to apply the proposed Mehar method as compared to the existing methods [45, 52, 82, 84]. Also, it is shown that if the solution of the problem ($P_{2.1}$) will exist then infinite alternative fuzzy optimal solutions of problem ($P_{2.1}$) will also exist. These alternative fuzzy optimal solutions cannot be obtained by any of existing methods [45,52,82,84] but can be obtained by proposed Mehar method. Hence, it is better to use the proposed Mehar method as compared to the existing methods [45, 52, 82, 84].

Chapter 3

MEHAR METHOD FOR SOLVING BOUNDED FULLY FUZZY LINER PROGRAMMING PROBLEMS WITH SYMMETRIC TRAPEZOIDAL FUZZY NUMBERS

Ebrahimnejad et al. [42] pointed out that if the existing method [52] will be used for solving bounded fuzzy linear programming problem ($P_{3.1}$) with symmetric trapezoidal fuzzy numbers, then the problem size and computational effort would increase significantly. Therefore, Ebrahimnejad et al. [42] proposed an alternative method to find the fuzzy optimal solution of problem ($P_{3.1}$).

Maximize/Minimize $\left[\tilde{z} \approx \sum\limits_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right]$

Subject to $\qquad (P_{3.1})$

$\sum\limits_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$

$\tilde{l}_j \preceq \tilde{x}_j \preceq \tilde{u}_j,\ j = 1, 2, ..., n,$

where $\tilde{c}_j, \tilde{x}_j, \tilde{b}_i, \tilde{l}_j$ and $\tilde{u}_j$ are symmetric trapezoidal fuzzy numbers and a_{ij} are real numbers.

Hatami and Kazemipoor [60] pointed out that there is no method in literature

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for solving fully fuzzy linear programming problems $(P_{3.2})$ with symmetric trapezoidal fuzzy numbers and extended the existing method [84] for solving fully fuzzy linear programming problems $(P_{3.2})$ with symmetric trapezoidal fuzzy numbers.

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right]$$

$$\text{Subject to} \qquad (P_{3.2})$$

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes_G \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$$

$$\tilde{x}_j \succeq \tilde{0},\ j = 1, 2, ..., n,$$

where $\tilde{c}_j, \tilde{x}_j, \tilde{a}_{ij}$ and $\tilde{b}_i$ are symmetric trapezoidal fuzzy numbers.

However, the method, proposed by Ebrahimnejad et al. [42], can not be used to find the fuzzy optimal solution of problem $(P_{3.2})$ and the method, proposed by Hatami and Kazemipoor [60], can not be used to find the fuzzy optimal solution of problem $(P_{3.1})$. Therefore, to overcome the limitations of both the existing methods [42, 60], in this chapter, the Mehar method, proposed in Chapter 2, is extended for solving such bounded fully fuzzy linear programming problems $(P_{3.3})$ in which all the parameters and variables are represented by symmetric trapezoidal fuzzy numbers.

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right]$$

$$\text{Subject to} \qquad (P_{3.3})$$

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes_G \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, ..., m,$$

$$\tilde{l}_j \preceq \tilde{x}_j \preceq \tilde{u}_j, \quad j = 1, 2, ..., n,$$

where $\tilde{c}_j, \tilde{x}_j, \tilde{a}_{ij}, \tilde{b}_i, \tilde{l}_j$ and $\tilde{u}_j$ are symmetric trapezoidal fuzzy numbers.

3.1 Proposed Mehar method

In this section, to overcome the limitations of the existing methods [42, 60], a new method (named as Mehar method) is proposed to find the fuzzy optimal solution of problem ($P_{3.3}$).

The steps of the proposed Mehar method are as follows:

Step 1 Using Section 2.1.3, the problem ($P_{3.3}$) can be transformed into problem ($P_{3.4}$).

$$\text{Maximize/Minimize } \left[\Re(\tilde{z}) = \Re\left(\sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right)\right]$$

Subject to $\qquad (P_{3.4})$

$$\Re\left(\sum_{j=1}^{n} \tilde{a}_{ij} \otimes_G \tilde{x}_j\right) \leq, =, \geq \Re(\tilde{b}_i), \qquad i = 1, 2, ..., m,$$

$$\Re(\tilde{l}_j) \leq \Re(\tilde{x}_j) \leq \Re(\tilde{u}_j), \qquad j = 1, 2, ..., n.$$

Step 2 Using the properties $\Re\left(\sum_{j=1}^{n} \tilde{c}_j \otimes_G \tilde{x}_j\right) = \sum_{j=1}^{n} \Re(\tilde{c}_j \otimes_G \tilde{x}_j) = \sum_{j=1}^{n} \Re(\tilde{c}_j)\Re(\tilde{x}_j)$ and $\Re\left(\sum_{j=1}^{n} \tilde{a}_{ij} \otimes_G \tilde{x}_j\right) = \sum_{j=1}^{n} \Re(\tilde{a}_{ij})\Re(\tilde{x}_j)$, the problem ($P_{3.4}$) can be transformed into problem ($P_{3.5}$).

$$\text{Maximize/Minimize } \left[\Re(\tilde{z}) = \sum_{j=1}^{n} \Re(\tilde{c}_j)\Re(\tilde{x}_j)\right]$$

Subject to $\qquad (P_{3.5})$

$$\sum_{j=1}^{n} \Re(\tilde{a}_{ij})\Re(\tilde{x}_j) \leq, =, \geq \Re(\tilde{b}_i), \qquad i = 1, 2, ..., m,$$

$$\Re(\tilde{l}_j) \leq \Re(\tilde{x}_j) \leq \Re(\tilde{u}_j), \qquad j = 1, 2, ..., n.$$

Step 3 Since, $\Re(\tilde{A})$ is a real number. So, assuming $\Re(\tilde{z}) = z, \Re(\tilde{c}_j) = c_j, \Re(\tilde{x}_j) = x_j, \Re(\tilde{a}_{ij}) = a_{ij}$, $\Re(\tilde{b}_i) = b_i, \Re(\tilde{l}_j) = l_j$, and $\Re(\tilde{u}_j) = u_j$, the problem ($P_{3.5}$) can be transformed into problem ($P_{3.6}$).

$$\text{Maximize/Minimize } \left[z = \sum_{j=1}^{n} c_j x_j\right]$$

Subject to $\qquad (P_{3.6})$

$\sum\limits_{j=1}^{n} a_{ij}x_j \leq, =, \geq b_i, \qquad i = 1, 2, ..., m,$

$l_j \leq x_j \leq u_j, \qquad j = 1, 2, ..., n.$

Step 4 Find the optimal solution $\{x_j; j = 1, 2, ..., n\}$ of problem $(P_{3.6})$ by using an appropriate existing method [144].

Step 5 Since, there exist infinite trapezoidal fuzzy numbers having same rank. So, if $x_1 = a_1, x_2 = a_2, ..., x_n = a_n$ is the optimal solution of problem $(P_{3.6})$ then all the symmetric trapezoidal fuzzy numbers $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n$ such that $\Re(\tilde{x}_1) = a_1, \Re(\tilde{x}_2) = a_2, ..., \Re(\tilde{x}_n) = a_n$ will also be the fuzzy optimal solution of the problem $(P_{3.3})$.

3.2 Fuzzy optimal solution of some existing problems by proposed Mehar method

In this section, the existing problems [42, Section 4, pp. 281] and [60, Section 4, pp. 468] are solved by proposed Mehar method.

3.2.1 Fuzzy optimal solution of existing problem considered by Ebrahimnejad et al.

In this section, the existing problem [42, Section 4, pp. 281] is solved by proposed Mehar method.

Example 3.1 [42, Section 4, pp. 281]

Minimize $[\tilde{z} \approx (-4, -2, 1, 1) \otimes_G \tilde{x}_1 + (-6, -4, 1, 1) \otimes_G \tilde{x}_2 + (-3, -1, 1, 1) \otimes_G \tilde{x}_3]$

Subject to $\qquad (P_{3.7})$

$\tilde{x}_1 + 2\tilde{x}_2 + 2\tilde{x}_3 \preceq (12, 16, 4, 4),$

$2\tilde{x}_1 + 4\tilde{x}_2 + 3\tilde{x}_3 \preceq (20, 26, 6, 6),$

$\tilde{0} \preceq \tilde{x}_1 \preceq (3, 5, 2, 2),\ (1, 3, 1, 1) \preceq \tilde{x}_2 \preceq (4, 6, 1, 1),\ \tilde{0} \preceq \tilde{x}_3 \preceq (2, 4, 1, 1).$

Using the proposed Mehar method, presented in Section 3.1, the fuzzy optimal solution of the problem ($P_{3.7}$) can be obtained as follows:

Step 1 Using Step 1 of proposed Mehar method, presented in Section 3.1, the problem ($P_{3.7}$) can be transformed into problem ($P_{3.8}$).

Minimize $[\Re(\tilde{z}) = \Re\left((-4,-2,1,1) \otimes_G \tilde{x}_1 + (-6,-4,1,1) \otimes_G \tilde{x}_2 + (-3,-1,1,1) \otimes_G \tilde{x}_3\right)]$

Subject to ($P_{3.8}$)

$\Re\left(\tilde{x}_1 + 2\tilde{x}_2 + 2\tilde{x}_3\right) \leq \Re(12,16,4,4)$,

$\Re\left(2\tilde{x}_1 + 4\tilde{x}_2 + 3\tilde{x}_3\right) \leq \Re(20,26,6,6)$,

$\Re(\tilde{0}) \leq \Re(\tilde{x}_1) \leq \Re(3,5,2,2)$, $\Re(1,3,1,1) \leq \Re(\tilde{x}_2) \leq \Re(4,6,1,1)$, $\Re(\tilde{0}) \leq \Re(\tilde{x}_3) \leq \Re(2,4,1,1)$.

Step 2 Using Step 2 of proposed Mehar method, presented in Section 3.1, the problem ($P_{3.8}$) can be transformed into problem ($P_{3.9}$).

Minimize $[\Re(\tilde{z}) = \Re(-4,-2,1,1)\Re(\tilde{x}_1) + \Re(-6,-4,1,1)\Re(\tilde{x}_2) + \Re(-3,-1,1,1)\Re(\tilde{x}_3)]$

Subject to ($P_{3.9}$)

$\Re(\tilde{x}_1) + 2\Re(\tilde{x}_2) + 2\Re(\tilde{x}_3) \leq \Re(12,16,4,4)$,

$2\Re(\tilde{x}_1) + 4\Re(\tilde{x}_2) + 3\Re(\tilde{x}_3) \leq \Re(20,26,6,6)$,

$\Re(\tilde{0}) \leq \Re(\tilde{x}_1) \leq \Re(3,5,2,2)$, $\Re(1,3,1,1) \leq \Re(\tilde{x}_2) \leq \Re(4,6,1,1)$, $\Re(\tilde{0}) \leq \Re(\tilde{x}_3) \leq \Re(2,4,1,1)$.

Step 3 Using Step 3 of proposed Mehar method, presented in Section 3.1, the problem ($P_{3.9}$) can be transformed into problem ($P_{3.10}$).

Minimize $[z = -3x_1 - 5x_2 - 2x_3]$

Subject to ($P_{3.10}$)

$x_1 + 2x_2 + 2x_3 \leq 14$,

$2x_1 + 4x_2 + 3x_3 \leq 23$,

$0 \leq x_1 \leq 4, 2 \leq x_2 \leq 5, 0 \leq x_3 \leq 3$.

Step 4 On solving the problem $(P_{3.10})$, the obtained optimal solution and optimal value are $x_1 = 4, x_2 = \frac{15}{4}, x_3 = 0$ and $-\frac{123}{4}$ respectively.

Step 5 Using Step 5 of proposed Mehar method, presented in Section 3.1, all the symmetric trapezoidal fuzzy numbers $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ such that $\Re(\tilde{x}_1) = 4, \Re(\tilde{x}_2) = \frac{15}{4}, \Re(\tilde{x}_3) = 0$ will be the fuzzy optimal solutions of the problem $(P_{3.7})$; for example, the following are the fuzzy optimal solutions of the problem $(P_{3.7})$.

(i) $\tilde{x}_1 = (3, 5, 2, 2), \tilde{x}_2 = (\frac{-3}{2}, 9, 5, 5), \tilde{x}_3 = (0, 0, 0, 0)$.

(ii) $\tilde{x}_1 = (2, 6, 3, 3), \tilde{x}_2 = (\frac{3}{2}, 6, 1, 1), \tilde{x}_3 = (-2, 2, 1, 1)$.

3.2.2 Fuzzy optimal solution of existing problem considered by Hatami and Kazemipoor

In this section, the fuzzy optimal solution of existing problem [60, Section 4, pp. 468] is obtained by proposed Mehar method.

Example 3.2 [60, Section 4, pp. 468]

Maximize $[\tilde{z} \approx (-2, 2, 2, 2) \otimes_G \tilde{x}_1 + (-2, 0, 1, 1) \otimes_G \tilde{x}_2]$

Subject to $\qquad (P_{3.11})$

$(2, 4, 1, 1) \otimes_G \tilde{x}_1 + (1, 3, 2, 2) \otimes_G \tilde{x}_2 \preceq (-4, 4, 2, 2)$,

$(-2, 0, 1, 1) \otimes_G \tilde{x}_1 + (-2, 4, 2, 2) \otimes_G \tilde{x}_2 \preceq (-1, 5, 3, 3)$,

$\tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}$.

Solution Using the proposed Mehar method, presented in Section 3.1, the fuzzy optimal solution of the problem $(P_{3.11})$ can be obtained as follows:

Step 1 Using Step 1 of proposed Mehar method, the problem $(P_{3.11})$ can be transformed into problem $(P_{3.12})$.

Maximize $[\Re(\tilde{z}) = \Re((-2, 2, 2, 2) \otimes_G \tilde{x}_1 + (-2, 0, 1, 1) \otimes_G \tilde{x}_2)]$

Subject to $\qquad (P_{3.12})$

$\Re((2,4,1,1) \otimes_G \tilde{x}_1 + (1,3,2,2) \otimes_G \tilde{x}_2) \leq \Re(-4,4,2,2)$,

$\Re((-2,0,1,1) \otimes_G \tilde{x}_1 + (-2,4,2,2) \otimes_G \tilde{x}_2) \leq \Re(-1,5,3,3)$,

$\Re(\tilde{x}_1), \Re(\tilde{x}_2) \geq \Re(\tilde{0})$.

Step 2 Using Step 2 of proposed Mehar method, the problem $(P_{3.12})$ can be transformed into problem $(P_{3.13})$.

Maximize $[\Re(\tilde{z}) = \Re(-2,2,2,2)\Re(\tilde{x}_1) + \Re(-2,0,1,1)\Re(\tilde{x}_2)]$

Subject to $\qquad (P_{3.13})$

$\Re(2,4,1,1)\Re(\tilde{x}_1) + \Re(1,3,2,2)\Re(\tilde{x}_2) \leq \Re(-4,4,2,2)$,

$\Re(-2,0,1,1)\Re(\tilde{x}_1) + \Re(-2,4,2,2)\Re(\tilde{x}_2) \leq \Re(-1,5,3,3)$,

$\Re(\tilde{x}_1), \Re(\tilde{x}_2) \geq \Re(\tilde{0})$.

Step 3 Using Step 3 of the proposed Mehar method, the problem $(P_{3.13})$ can be transformed into problem $(P_{3.14})$.

Maximize $[0x_1 - 1x_2]$

Subject to $\qquad (P_{3.14})$

$3x_1 + 2x_2 \leq 0$,

$-1x_1 + 1x_2 \leq 2$,

$x_1, x_2 \geq 0$.

Step 4 On solving problem $(P_{3.14})$, the obtained optimal solution is $x_1 = 0, x_2 = 0$.

Step 5 Using Step 5 of the proposed Mehar method, all the symmetric trapezoidal fuzzy numbers $\tilde{x}_1$ and $\tilde{x}_2$ such that $\Re(\tilde{x}_1) = 0$ and $\Re(\tilde{x}_2) = 0$ will be the fuzzy optimal solutions of the problem $(P_{3.11})$; for example, the following are fuzzy optimal solutions of the problem $(P_{3.11})$.

(i) $\tilde{x}_1 = (-1,1,3,3)$, $\tilde{x}_2 = (-2,2,2,2)$.

(ii) $\tilde{x}_1 = (0,0,4,4)$, $\tilde{x}_2 = (-3,3,3,3)$.

(iii) $\tilde{x}_1 = (-2, 2, 1, 1)$, $\tilde{x}_2 = (-4, 4, 3, 3)$.

3.2.3 Fuzzy optimal solution of chosen problem by proposed Mehar method

The bounded fully fuzzy linear programming problem, chosen in Example 3.3, in which all the parameters and variables are represented by symmetric trapezoidal fuzzy numbers can neither be solved by the existing method [42] nor by the existing method [60]. The fuzzy optimal solution of this problem is obtained by proposed Mehar method.

Example 3.3

Maximize $[\tilde{z} \approx (3, 5, 4, 4) \otimes_G \tilde{x}_1 + (1, 3, 2, 2) \otimes_G \tilde{x}_2 + (4, 8, 1, 1) \otimes_G \tilde{x}_3]$

Subject to $(P_{3.15})$

$(2, 6, 3, 3) \otimes_G \tilde{x}_1 - (1, 1, 4, 4) \otimes_G \tilde{x}_2 - (2, 4, 2, 2) \otimes_G \tilde{x}_3 \preceq (7, 11, 4, 4)$,

$(-4, 2, 2, 2) \otimes_G \tilde{x}_1 + (-1, 3, 1, 1) \otimes_G \tilde{x}_2 + (2, 2, 2, 2) \otimes_G \tilde{x}_3 \succeq (7, 9, 6, 6)$,

$(-4, -2, 2, 2) \otimes_G \tilde{x}_1 + (1, 1, 4, 4) \otimes_G \tilde{x}_2 + (3, 5, 3, 3) \otimes_G \tilde{x}_3 \preceq (10, 12, 2, 2)$,

$(0, 2, 1, 1) \preceq \tilde{x}_1 \preceq (2, 4, 2, 2)$, $(-1, 1, 2, 2) \preceq \tilde{x}_2 \preceq (5, 5, 2, 2)$, $(-2, 2, 3, 3) \preceq \tilde{x}_3 \preceq (-6, 10, 5, 5)$.

Solution Using the Mehar method, proposed in Section 3.1, the fuzzy optimal solution of the problem $(P_{3.15})$ can be obtained as follows:

Step 1 Using Step 1 of proposed Mehar method, the problem $(P_{3.15})$ can be transformed into problem $(P_{3.16})$.

Maximize $[\Re(\tilde{z}) = \Re\left((3, 5, 4, 4) \otimes_G \tilde{x}_1 + (1, 3, 2, 2) \otimes_G \tilde{x}_2 + (4, 8, 1, 1) \otimes_G \tilde{x}_3\right)]$

Subject to $(P_{3.16})$

$\Re\left((2, 6, 3, 3) \otimes_G \tilde{x}_1 - (1, 1, 4, 4) \otimes_G \tilde{x}_2 - (2, 4, 2, 2) \otimes_G \tilde{x}_3\right) \leq \Re(7, 11, 4, 4)$,

$\Re\left((-4, 2, 2, 2) \otimes_G \tilde{x}_1 + (-1, 3, 1, 1) \otimes_G \tilde{x}_2 + (2, 2, 2, 2) \otimes_G \tilde{x}_3\right) \geq \Re(7, 9, 6, 6)$,

$\Re\left((-4,-2,2,2)\otimes_G \tilde{x}_1 + (1,1,4,4)\otimes_G \tilde{x}_2 + (3,5,3,3)\otimes_G \tilde{x}_3\right) \leq \Re(10,12,2,2)$,

$\Re(0,2,1,1) \leq \Re(\tilde{x}_1) \leq \Re(2,4,2,2)$, $\Re(-1,1,2,2) \leq \Re(\tilde{x}_2) \leq \Re(5,5,2,2)$, $\Re(-2,2,3,3) \leq \Re(\tilde{x}_3) \leq \Re(-6,10,5,5)$.

Step 2 Using Step 2 of proposed Mehar method, the problem $(P_{3.16})$ can be transformed into problem $(P_{3.17})$.

Maximize $[\Re(\tilde{z}) = \Re(3,5,4,4)\Re(\tilde{x}_1) + \Re(1,3,2,2)\Re(\tilde{x}_2) + \Re(4,8,1,1)\Re(\tilde{x}_3)]$

Subject to $(P_{3.17})$

$\Re(2,6,3,3)\Re(\tilde{x}_1) - \Re(1,1,4,4)\Re(\tilde{x}_2) - \Re(2,4,2,2)\Re(\tilde{x}_3) \leq \Re(7,11,4,4)$,

$\Re(-4,2,2,2)\Re(\tilde{x}_1) + \Re(-1,3,1,1)\Re(\tilde{x}_2) + \Re(2,2,2,2)\Re(\tilde{x}_3) \geq \Re(7,9,6,6)$,

$\Re(-4,-2,2,2)\Re(\tilde{x}_1) + \Re(1,1,4,4)\Re(\tilde{x}_2) + \Re(3,5,3,3)\Re(\tilde{x}_3) \leq \Re(10,12,2,2)$,

$\Re(0,2,1,1) \leq \Re(\tilde{x}_1) \leq \Re(2,4,2,2)$, $\Re(-1,1,2,2) \leq \Re(\tilde{x}_2) \leq \Re(5,5,2,2)$, $\Re(-2,2,3,3) \leq \Re(\tilde{x}_3) \leq \Re(-6,10,5,5)$.

Step 3 Using Step 3 of proposed Mehar method, the problem $(P_{3.17})$ can be transformed into problem $(P_{3.18})$.

Maximize $[4x_1 + 2x_2 + 6x_3]$

Subject to $(P_{3.18})$

$4x_1 - x_2 - 3x_3 \leq 9$,

$-x_1 + x_2 + 2x_3 \geq 8$,

$-3x_1 + x_2 + 4x_3 \leq 11$,

$1 \leq x_1 \leq 3, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 2$.

Step 4 On solving the problem $(P_{3.18})$, the obtained optimal solution and optimal value are $x_1 = 1, x_2 = 5, x_3 = 2$ and 26 respectively.

Step 5 Using Step 5 of proposed Mehar method, all the symmetric trapezoidal fuzzy numbers $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ such that $\Re(\tilde{x}_1) = 1, \Re(\tilde{x}_2) = 5, \Re(\tilde{x}_3) = 2$ will be the fuzzy optimal solutions of the problem ($P_{3.15}$); for example, the following are the fuzzy optimal solutions of the problem ($P_{3.15}$).

(i) $\tilde{x}_1 = (1,1,2,2), \tilde{x}_2 = (4,6,3,3), \tilde{x}_3 = (1,3,2,2)$.

(ii) $\tilde{x}_1 = (0,2,3,3), \tilde{x}_2 = (3,7,2,2), \tilde{x}_3 = (2,2,1,1)$.

(iii) $\tilde{x}_1 = (1,1,4,4), \tilde{x}_2 = (5,5,2,2), \tilde{x}_3 = (0,4,1,1)$.

3.3 Fuzzy optimal solution of existing problems by existing methods

To show the advantages of proposed Mehar method over existing methods [42, 60], there is need to solve the same existing problems Example 3.1 [42, Section 4, pp. 281] and Example 3.2 [60, Section 4, pp. 468] by the existing methods [42,60]. Therefore, in this section, the fuzzy optimal solution of the existing problem Example 3.1 [42, Section 4, pp. 281] and Example 3.2 [60, Section 4, pp. 468] is obtained by the existing methods [42, 60].

Further, to understand the numerical example, solved by Ebrahimnejad et al. [42], there is need to explain the algorithm proposed by Ebrahimnejad et al. [42]. Therefore, the algorithm, proposed by Ebrahimnejad et al. [42], is also presented in a brief manner.

3.3.1 Existing algorithm to solve bounded fuzzy linear programming problems

In this section, the existing algorithm, proposed by Ebrahimnejad et al. [42], to solve bounded fuzzy linear programming problem ($P_{3.1}$), is presented.

The steps of the existing algorithm [42] are as follows:

Step 1 Convert all inequalities into equalities by using fuzzy slack or fuzzy surplus variables.

Step 2 Obtain an initial fuzzy basic feasible solution. Let $B = [a_1, a_2, ..., a_m]$ be initial basis vector. Further, non-basic vectors can be decomposed into N_1 and N_2. So, $\tilde{x}$ can be partitioned as $(\tilde{x}_B, \tilde{x}_{N_1}, \tilde{x}_{N_2})$, where $\tilde{x}_B$ are fuzzy basic variables and $\tilde{x}_{N_1}$ and $\tilde{x}_{N_2}$ are fuzzy non-basic variables at their lower and upper bounds respectively. Then the current fuzzy basic solution of problem $(P_{3.1})$ is given as:

$$\begin{pmatrix} \tilde{x}_B \\ \tilde{x}_{N_1} \\ \tilde{x}_{N_2} \end{pmatrix} \approx \begin{pmatrix} \tilde{\tilde{b}} \approx B^{-1}\tilde{b} - B^{-1}N_1\tilde{l}_{N_1} - B^{-1}N_2\tilde{u}_{N_2} \\ \tilde{l}_{N_1} \\ \tilde{u}_{N_2} \end{pmatrix}.$$

The fuzzy value of the objective function for this solution is

$$\tilde{\tilde{z}} \approx \tilde{c}\tilde{x} \approx \tilde{c}_B B^{-1}\tilde{b} + (\tilde{c}_{N_1} - \tilde{c}_B B^{-1}N_1)\tilde{l}_{N_1} + (\tilde{c}_{N_2} - \tilde{c}_B B^{-1}N_2)\tilde{u}_{N_2}.$$

In the initial basic feasible solution, it is assumed that all the non-basic variables will be at their lower bound.

Step 3 Calculate $\tilde{z} \approx \tilde{c}_B B^{-1}\tilde{b} - \sum\limits_{j\in R_1} (\tilde{z}_j - \tilde{c}_j)\tilde{l}_j + \sum\limits_{j\in R_2} (\tilde{z}_j - \tilde{c}_j)\tilde{u}_j$ where R_1 and R_2 are the set of indices of non-basic variables at their lower and upper bounds respectively. If for all $j \in R_1, \tilde{z}_j - \tilde{c}_j \preceq \tilde{0}$ and for all $j \in R_2, \tilde{z}_j - \tilde{c}_j \succeq \tilde{0}$, then optimality conditions are obtained. Stop the algorithm, the current fuzzy basic feasible solution is optimal. Otherwise, go to the next step.

Step 4 Choose the pivot column by applying the following test:

maximum(maximum$\{\tilde{z}_j - \tilde{c}_j; j \in R_1\}$,maximum$\{\tilde{c}_j - \tilde{z}_j, j \in R_2\}$)

Without loss of generality, let $\tilde{x}_k$ be pivot column. Then x_k will be tentative entering variable.

Step 5 Check whether $k \in R_1$ or $k \in R_2$.

Case 1 If $k \in R_1$.

Calculate $\tilde{\Delta}_k$ to decide leaving variable, $\tilde{\Delta}_k \approx$ minimum$\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3\}$, where $\tilde{\theta}_1 \approx$

$\tilde{u}_k - \tilde{l}_k$, but if there is no upper bound restriction on $\tilde{x}_k$, then $\tilde{\theta}_1 \approx \tilde{\infty}$.

$\tilde{\theta}_2 = \underset{1 \le i \le m}{\text{minimum}} \left\{ \frac{\tilde{\hat{b}}_i - \tilde{l}_{B_i}}{y_{ik}} | y_{ik} > 0, 1 \le i \le m \right\}$, but if $y_{ik} \le 0$ then $\tilde{\theta}_2 \approx \tilde{\infty}$.

$\tilde{\theta}_3 = \underset{1 \le i \le m}{\text{minimum}} \left\{ \frac{\tilde{u}_{B_i} - \tilde{\hat{b}}_i}{-y_{ik}} | y_{ik} < 0, 1 \le i \le m \right\}$.

Again if $y_{ik} \ge 0$ then $\tilde{\theta}_3 \approx \tilde{\infty}$.

So, $\tilde{\Delta}_k$ can be taken as minimum of $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3$. If $\tilde{\Delta}_k \approx \tilde{\theta}_1$ then $\tilde{x}_k$ is still non-basic variable but at its upper bound and there is no change in basis matrix B. The variable corresponding to which $\tilde{\Delta}_k$ is minimum, will be the leaving variable.

New fuzzy solution is given by:

$$\begin{cases} \tilde{x}_{B_i} \approx \tilde{\hat{b}}_i - y_{ik}\tilde{\Delta}_k, & i = 1, 2, ..., m; \\ \tilde{x}_k \approx \tilde{l}_k + \tilde{\Delta}_k, & \\ \tilde{x}_j \approx \tilde{l}_j, & j \neq k, j \in R_1; \\ \tilde{x}_j \approx \tilde{u}_j, & j \in R_2. \end{cases}$$

and new fuzzy objective value is $\tilde{\tilde{z}} \approx \tilde{\hat{z}} - (\tilde{z}_k - \tilde{c}_k)\tilde{\Delta}_k$.

Case 2 If $k \in R_2$.

Calculate $\tilde{\Delta}_k$ to decide leaving variable, $\tilde{\Delta}_k$ = minimum $\{\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3\}$, where $\tilde{\delta}_1 = \tilde{u}_k - \tilde{l}_k$, but if there is no upper bound restriction on $\tilde{x}_k$, then $\tilde{\delta}_1 \approx \tilde{\infty}$.

$\tilde{\delta}_2 = \underset{1 \le i \le m}{\text{minimum}} \left\{ \frac{\tilde{\hat{b}}_i - \tilde{l}_{B_i}}{-y_{ik}} | y_{ik} < 0 \right\}$, but if $y_{ik} \ge 0$ then $\tilde{\delta}_2 \approx \tilde{\infty}$.

$\tilde{\delta}_3 = \underset{1 \le i \le m}{\text{minimum}} \left\{ \frac{\tilde{u}_{B_i} - \tilde{\hat{b}}_i}{y_{ik}} | y_{ik} > 0 \right\}$.

Again if $y_{ik} \le 0$ then $\tilde{\delta}_3 \approx \tilde{\infty}$.

So, $\tilde{\Delta}_k$ can be taken as minimum of $\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3$. If $\tilde{\Delta}_k \approx \tilde{\delta}_1$ then $\tilde{x}_k$ is still non-basic variable but at its lower bound and there is no change in basis matrix B. The variable corresponding to which $\tilde{\Delta}_k$ is minimum, will be the leaving variable.

New fuzzy solution is given by:

$$\begin{cases} \tilde{x}_{B_i} \approx \tilde{\hat{b}}_i + y_{ik}\tilde{\Delta}_k, & i = 1, 2, ..., m; \\ \tilde{x}_j \approx \tilde{l}_j, & j \in R_1 \\ \tilde{x}_k \approx \tilde{u}_k - \tilde{\Delta}_k, & \\ \tilde{x}_j \approx \tilde{u}_j, & j \neq k, j \in R_2. \end{cases}$$

and new fuzzy objective value is $\tilde{\tilde{z}} \approx \tilde{\tilde{z}} + (\tilde{z}_k - \tilde{c}_k)\tilde{\Delta}_k$.

Step 6 Again check the optimality conditions, if these are satisfied then stop the algorithm. The current solution is optimal solution otherwise repeat the Step 4 and Step 5.

3.3.2 Fuzzy optimal solution by Ebrahimnejad et al. method

In this section, the fuzzy optimal solution of existing problem [42, Section 4, pp. 281], presented in Example 3.1, is obtained by the method proposed by Ebrahimnejad et al. [42].

The solution of the problem $(P_{3.7})$, by using the existing method [42], can be obtained as follows:

Step 1 Assuming $\tilde{x}_4, \tilde{x}_5$ as slack variables, problem $(P_{3.7})$ can be transformed into problem $(P_{3.19})$.

Minimize $[\tilde{z} \approx (-4,-2,1,1) \otimes_G \tilde{x}_1 + (-6,-4,1,1) \otimes_G \tilde{x}_2 + (-3,-1,1,1) \otimes_G \tilde{x}_3]$

Subject to $\qquad (P_{3.19})$

$\tilde{x}_1 + 2\tilde{x}_2 + 2\tilde{x}_3 + \tilde{x}_4 \approx (12,16,4,4)$,

$2\tilde{x}_1 + 4\tilde{x}_2 + 3\tilde{x}_3 + \tilde{x}_5 \approx (20,26,6,6)$,

$\tilde{0} \preceq \tilde{x}_1 \preceq (3,5,2,2),\ (1,3,1,1) \preceq \tilde{x}_2 \preceq (4,6,1,1),\ \tilde{0} \preceq \tilde{x}_3 \preceq (2,4,1,1), \tilde{x}_4, \tilde{x}_5 \succeq \tilde{0}$.

Step 2 Let initially $\tilde{x}_4$ and $\tilde{x}_5$ be the fuzzy basic variables and all fuzzy non-basic variables are at their lower bound i.e. $\tilde{x}_1 \approx \tilde{0}, \tilde{x}_2 \approx (1,3,1,1)$ and $\tilde{x}_3 \approx \tilde{0}$. Thus

$$B = [a_4, a_5] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, N_1 = [a_1, a_2, a_3], \tilde{x}_{N_1} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} \approx \begin{pmatrix} (0,0,0,0) \\ (1,3,1,1) \\ (0,0,0,0) \end{pmatrix}.$$

The initial fuzzy basic solution and its fuzzy objective value is given as:

$$\tilde{x}_B = \begin{pmatrix} \tilde{x}_4 \\ \tilde{x}_5 \end{pmatrix} \approx \begin{pmatrix} (12,16,4,4) - 2(1,3,1,1) = (6,14,6,6) \\ (20,26,6,6) - 4(1,3,1,1) = (8,22,10,10) \end{pmatrix},$$

Step 3 Calculating $\tilde{z}_j - \tilde{c}_j, j = 1,2,3,$ we have $\tilde{z}_1 - \tilde{c}_1 \approx (2,4,1,1), \tilde{z}_2 - \tilde{c}_2 \approx$

$(4,6,1,1)$, $\tilde{z}_3-\tilde{c}_3 \approx (1,3,1,1)$. So, $\tilde{z} \approx (0,0,0,0)-[(4,6,1,1)(1,3,1,1)] \approx (-17,-3,7,7)$.

Step 4 Iteration 1 Now, maximum value of $\tilde{z}_j - \tilde{c}_j, j = 1,2,3$ is (4,6,1,1) corresponding to $\tilde{x}_2$. Therefore, $\tilde{x}_k = \tilde{x}_2$. Then $y_2 = B^{-1}a_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\tilde{\Delta}_2$ is given by minimum $\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3\}$, where $\tilde{\theta}_1 = \tilde{u}_2 - \tilde{l}_2 = (4,6,1,1) - (1,3,1,1) \approx (1,5,2,2)$,

$\tilde{\theta}_2 =$ minimum $\{(\frac{6}{2}, \frac{14}{2}, \frac{6}{2}, \frac{6}{2}), (\frac{8}{4}, \frac{22}{4}, \frac{10}{4}, \frac{10}{4})\} \approx (\frac{8}{4}, \frac{22}{4}, \frac{10}{4}, \frac{10}{4})$, corresponding to $\tilde{x}_{B_2} = \tilde{x}_5$. Also, $\tilde{\theta}_3 \approx \tilde{\infty}$ as $y_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Therefore, $\tilde{\Delta}_2 =$ minimum $\{(1,5,2,2), (\frac{8}{4}, \frac{22}{4}, \frac{10}{4}, \frac{10}{4}), \tilde{\infty}\} \approx (1,5,2,2)$ which means that $\tilde{x}_2$ is still non-basic variable but at its upper bound. Thus

$B = [a_4, a_5], N_1 = [a_1, a_3], \tilde{x}_{N_1} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_3 \end{pmatrix} \approx \begin{pmatrix} (0,0,0,0) \\ (0,0,0,0) \end{pmatrix}, \tilde{x}_{N_2} = \tilde{x}_2 \approx (4,6,1,1), \begin{pmatrix} \tilde{x}_4 \\ \tilde{x}_5 \end{pmatrix} \approx \begin{pmatrix} (6,14,6,6) \\ (8,22,10,10) \end{pmatrix} - y_2\tilde{\Delta}_2 \approx \begin{pmatrix} (6,14,6,6) \\ (8,22,10,10) \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} (1,5,2,2) \approx \begin{pmatrix} (-4,12,10,10) \\ (-12,18,18,18) \end{pmatrix}$

and $\tilde{z} \approx (-17,-3,7,7) - (\tilde{z}_2 - \tilde{c}_2)\tilde{\Delta}_2 = (-17,-3,7,7) - (4,6,1,1)(1,5,2,2) \approx (-45,-5,24,24)$.

Also, $\tilde{z}_1 - \tilde{c}_1 \approx (2,4,1,1) \succ \tilde{0}, \tilde{z}_2 - \tilde{c}_2 \approx (4,6,1,1) \succ \tilde{0}, \tilde{z}_3 - \tilde{c}_3 \approx (1,3,1,1) \succ \tilde{0}$.

Iteration 2 The maximum value of $\tilde{z}_j - \tilde{c}_j$ for $j \in R_1 = \{j = 1,3\}$ is (2,4,1,1) corresponding to $\tilde{x}_1$. Therefore $\tilde{x}_k = \tilde{x}_1$. Then $y_1 = B^{-1}a_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\tilde{\Delta}_1$ is given by minimum $\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3\}$, where $\tilde{\theta}_1 \approx \tilde{u}_1 - \tilde{l}_1 \approx (3,5,2,2)$.

$\tilde{\theta}_2 =$ minimum $\{(\frac{-4}{1}, \frac{12}{1}, \frac{10}{1}, \frac{10}{1}), (\frac{-12}{2}, \frac{18}{2}, \frac{18}{2}, \frac{18}{2})\} \approx (\frac{-12}{2}, \frac{18}{2}, \frac{18}{2}, \frac{2}{2})$, corresponding to $\tilde{x}_5$.

Also, $\tilde{\theta}_3 \approx \tilde{\infty}$ as $y_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Therefore $\tilde{\Delta}_1 =$ minimum $\{(3,5,2,2), (\frac{-12}{2}, \frac{18}{2}, \frac{18}{2}, \frac{18}{2}), \tilde{\infty}\} \approx (\frac{-12}{2}, \frac{18}{2}, \frac{18}{2}, \frac{18}{2})$, which means that $\tilde{x}_1$ enters the basis and $\tilde{x}_{B_2} = \tilde{x}_5$ drops to its lower bound and leaves the basis. Thus

$B = [a_4, a_1], N_1 = [a_5, a_3], \tilde{x}_{N_1} = \begin{pmatrix} \tilde{x}_5 \\ \tilde{x}_3 \end{pmatrix} \approx \begin{pmatrix} (0,0,0,0) \\ (0,0,0,0) \end{pmatrix}, \tilde{x}_{N_2} = \tilde{x}_2 \approx (4,6,1,1),$

$\begin{pmatrix} \tilde{x}_4 \\ \tilde{x}_1 \end{pmatrix} \approx \begin{pmatrix} (-4,12,10,10)-(-6,9,9,9) \approx (-13,18,19,19) \\ (0,0,0,0)+(-6,9,9,9) \approx (-6,9,9,9) \end{pmatrix}, \tilde{z} \approx (-45,-5,24,24)-$ $(\tilde{z}_1-\tilde{c}_1)\tilde{\Delta}_k \approx (\frac{-159}{2}, \frac{41}{2}, 69, 69)$.

Again, $\tilde{z}_2-\tilde{c}_2 \approx (-4,2,3,3), \tilde{z}_3-\tilde{c}_3 \approx (-5,-2,\frac{5}{2},\frac{5}{2}), \tilde{z}_5-\tilde{c}_5 \approx (-2,-1,\frac{1}{2},\frac{1}{2})$.

Iteration 3 Now, the fuzzy non-basic variable $\tilde{x}_2$ is at its upper bound such that $\tilde{z}_2-\tilde{c}_2 \approx (-4,2,3,3) \prec \tilde{0}$. So, $\tilde{x}_k=\tilde{x}_2$. Then $y_2 = B^{-1}a_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\tilde{\Delta}_2$ is given by minimum $\{\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3\}$, where $\tilde{\delta}_1 \approx \tilde{u}_2 - \tilde{l}_2 \approx (1,5,2,2)$.

Since, $y_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, So, $\tilde{\delta}_2 \approx \tilde{\infty}, \tilde{\delta}_3 = (\frac{(3,5,2,2)-(-6,9,9,9)}{2}) \approx (-3,\frac{11}{2},\frac{11}{2},\frac{11}{2})$, corresponding to $\tilde{x}_{B_2}=\tilde{x}_1$.

Therefore $\tilde{\Delta}_2 = \text{minimum } \{(1,5,2,2), \tilde{\infty}, (-3,\frac{11}{2},\frac{11}{2},\frac{11}{2})\} = (-3,\frac{11}{2},\frac{11}{2},\frac{11}{2})$, which means that $\tilde{x}_2$ enters the basis and $\tilde{x}_{B_2}=\tilde{x}_1$ reaches to its upper bound and leaves the basis. Finally, $B=[a_4,a_2], N_1=[a_5,a_3], \tilde{x}_{N_1} = \begin{pmatrix} \tilde{x}_5 \\ \tilde{x}_3 \end{pmatrix} \approx \begin{pmatrix} (0,0,0,0) \\ (0,0,0,0) \end{pmatrix}, \tilde{x}_{N_2} =$ $\tilde{x}_1 \approx (3,5,2,2), \begin{pmatrix} \tilde{x}_4 \\ \tilde{x}_2 \end{pmatrix} \approx \begin{pmatrix} (-13,18,19,19) \\ (\frac{-3}{2},9,\frac{13}{2},\frac{13}{2}) \end{pmatrix}, \tilde{z} \approx (-99,35,\frac{77}{2},\frac{77}{2})$.

Also, $\tilde{z}_1-\tilde{c}_1 \approx (2,5,\frac{3}{2},\frac{3}{2}) \succeq \tilde{0}, \tilde{z}_3-\tilde{c}_3 \approx (-3,-\frac{3}{2},\frac{3}{4},\frac{3}{4}) \preceq \tilde{0}$ and $\tilde{z}_5-\tilde{c}_5 \approx (-1,-\frac{1}{2},\frac{1}{4},\frac{1}{4}) \preceq \tilde{0}$, which means that for all fuzzy non-basic variables at lower bound $\tilde{z}_j-\tilde{c}_j \preceq \tilde{0}$ and for all fuzzy non-basic variables at upper bound $\tilde{z}_j-\tilde{c}_j \succeq \tilde{0}$.

Hence, the current fuzzy basic feasible solution is a fuzzy optimal solution.

3.3.3 Fuzzy optimal solution by Hatami and Kazemipoor method

In this section, the fuzzy optimal solution of existing problem [60, Section 4, pp. 468], presented in Example 3.2, is obtained by Hatami and Kazemipoor method [60].

The solution of the problem ($P_{3.11}$), by using the existing method [60], can be obtained as follows:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, \alpha_1, \alpha_1)$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \alpha_2)$, the problem $(P_{3.11})$ is transformed into problem $(P_{3.20})$.

Maximize $\tilde{z} \approx (-2, 2, 2, 2) \otimes_G (x_1, y_1, \alpha_1, \alpha_1) + (-2, 0, 1, 1) \otimes_G (x_2, y_2, \alpha_2, \alpha_2)$

Subject to $(P_{3.20})$

$(2, 4, 1, 1) \otimes_G (x_1, y_1, \alpha_1, \alpha_1) + (1, 3, 2, 2) \otimes_G (x_2, y_2, \alpha_2, \alpha_2) \preceq (-4, 4, 2, 2)$,

$(-2, 0, 1, 1) \otimes_G (x_1, y_1, \alpha_1, \alpha_1) + (-2, 4, 2, 2) \otimes_G (x_2, y_2, \alpha_2, \alpha_2) \preceq (-1, 5, 3, 3)$,

$(x_1, y_1, \alpha_1, \alpha_1), (x_2, y_2, \alpha_2, \alpha_2) \succeq \tilde{0}$.

Step 2 Using the property of ranking function $\Re((p_j, q_j, \beta_j, \beta_j) \otimes_G (x_j, y_j, \alpha_j, \alpha_j)) = \Re(p_j, q_j, \beta_j, \beta_j)\Re(x_j, y_j, \alpha_j, \alpha_j)$, the problem $(P_{3.20})$ is transformed into problem $(P_{3.21})$.

Maximize $\Re(\tilde{z}) = \Re(-2, 2, 2, 2)\Re(x_1, y_1, \alpha_1, \alpha_1) + \Re(-2, 0, 1, 1)\Re(x_2, y_2, \alpha_2, \alpha_2)$

Subject to $(P_{3.21})$

$\Re(2, 4, 1, 1)\Re(x_1, y_1, \alpha_1, \alpha_1) + \Re(1, 3, 2, 2)\Re(x_2, y_2, \alpha_2, \alpha_2) \leq \Re(-4, 4, 2, 2)$,

$\Re(-2, 0, 1, 1)\Re(x_1, y_1, \alpha_1, \alpha_1) + \Re(-2, 4, 2, 2)\Re(x_2, y_2, \alpha_2, \alpha_2) \leq \Re(-1, 5, 3, 3)$,

$\Re(x_1, y_1, \alpha_1, \alpha_1), \Re(x_2, y_2, \alpha_2, \alpha_2) \geq \Re(\tilde{0})$.

Step 3 Using Section 2.1.3, the problem $(P_{3.21})$ is transformed into problem $(P_{3.22})$.

Maximize $z = 0 \times \frac{x_1+y_1}{2} + (-1) \times \frac{x_2+y_2}{2}$

Subject to $(P_{3.22})$

$3 \times \frac{x_1+y_1}{2} + 2 \times \frac{x_2+y_2}{2} \leq 0$,

$(-1) \times \frac{x_1+y_1}{2} + 1 \times \frac{x_2+y_2}{2} \leq 2$,

$x_1 + y_1 \geq 0,\ x_2 + y_2 \geq 0$,

$x_1 \leq y_1, x_2 \leq y_2$,

$\alpha_1 \geq 0, \alpha_2 \geq 0$.

Step 4 On solving the problem $(P_{3.22})$, the obtained solution is $x_1 = 0, y_1 = 0, \alpha_1 = 0$ and $x_2 = 2, y_2 = 2, \alpha_2 = 0$. Hence $\tilde{x}_1 = (0, 0, 0, 0)$ and $\tilde{x}_2 = (2, 2, 0, 0)$.

Step 5 The obtained fuzzy optimal value is $\tilde{z} = (-4, 0, 2, 2)$.

3.4 Advantages of the proposed Mehar method over existing methods

In this section, advantages of the proposed Mehar method over existing methods [42, 60], are discussed.

1. Since, Ebrahimnejad et al. [42] proposed a method for solving bounded symmetric fuzzy linear programming ($P_{3.1}$) without transforming into crisp linear programming problems. So, the method proposed by Ebrahimnejad et al. [42] is similar to the methods, proposed by Ganesan and Veermani [52] as well as Kheirfam and Verdegay [82]. Therefore, the advantages of proposed Mehar method over Ebrahimnejad et al. [42] are same as advantages of proposed Mehar method over Ganesan and Veermani [52] method as well as Kheirfam and Verdegay [82] method discussed in Section 2.5.1 of Chapter 2. Further, the proposed Mehar method can be used to find the fuzzy optimal solution of problem ($P_{3.1}$) and problem ($P_{3.2}$) but the method proposed by Ebrahimnejad et al. [42] can not be used to find the fuzzy optimal solution of problem ($P_{3.2}$).
2. Since, Hatami and Kazemipoor [60] method is an extension of the method, proposed by Kumar and Kaur [84]. Therefore, the advantages of proposed Mehar method over the method, proposed by Hatami and Kazemipoor [60], is same as the advantages of the proposed Mehar method over Kumar and Kaur [84] discussed in Section 2.5.2 of Chapter 2. Further, the proposed Mehar method can be used to find the fuzzy optimal solution of problem ($P_{3.2}$) and problem ($P_{3.3}$) but the method proposed by Hatami and Kazemipoor [60] can

not be used to find the fuzzy optimal solution of problem $(P_{3.3})$.

3. None of the existing methods [42, 60] as well as the Mehar method, proposed in Chapter 2, can be used to find the fuzzy optimal solution of problem $(P_{3.3})$. While, all the problems which can be solved by any of the existing methods [42, 60] as well as Mehar method, proposed in Chapter 2, can be solved by proposed Mehar method.

3.5 Conclusions

On the basis of present study, it can be concluded that all the problems which can be solved by the existing methods [42, 60] can also be solved by the proposed Mehar method and there are several advantages of applying proposed Mehar method over applying other existing methods [42, 60]. Hence, it is better to use proposed Mehar method as compared to existing methods [42, 60].

Chapter 4

MEHAR METHOD FOR SOLVING BOUNDED FUZZY LINER PROGRAMMING PROBLEMS WITH TRAPEZOIDAL FUZZY NUMBERS

Ebrahimnejad [38] proposed a method for solving bounded fuzzy linear programming problem ($P_{4.1}$) in which coefficients of variables in objective function as well as in constraints are represented by real numbers while all other parameters and variables are represented by trapezoidal fuzzy numbers.

$$\text{Maximize/Minimize } \left[\tilde{z} \approx \sum_{j=1}^{n} c_j \tilde{x}_j\right]$$

$$\text{Subject to} \qquad (P_{4.1})$$

$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i,\ i = 1, 2, ..., m,$$

$$\tilde{l}_j \preceq \tilde{x}_j \preceq \tilde{u}_j,\ j = 1, 2, ..., n,$$

where $\tilde{x}_j, \tilde{b}_i, \tilde{l}_j$ and $\tilde{u}_j$ are trapezoidal fuzzy numbers and $c_j,\ a_{ij}$ are real numbers.

However, the existing method [38] can be used only for solving bounded fuzzy linear programming problem ($P_{4.1}$) if the initial fuzzy basic solution is optimal but not feasible.

To overcome this limitation of the existing method [38], Ebrahimnejad and

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Verdegay [46] proposed a method for solving the same problem ($P_{4.1}$). In this chapter, a new method (named as Mehar method) is proposed to find the fuzzy optimal solution of problem ($P_{4.1}$) and the advantages of proposed Mehar method over existing methods [38, 46], are discussed.

4.1 Preliminaries

In this section, arithmetic operations and comparison of trapezoidal fuzzy numbers are presented [38].

4.1.1 Arithmetic operations on trapezoidal fuzzy numbers

In this section, arithmetic operations on trapezoidal fuzzy numbers are presented.

A fuzzy number $\tilde{A} = (a_1, a_2, \alpha_1, \alpha_2)$ is said to be trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}(x)} = \begin{cases} \frac{x-(a_1-\alpha_1)}{\alpha_1}, & x \in [a_1-\alpha_1, a_1); \\ 1, & x \in [a_1, a_2]; \\ \frac{(a_2+\alpha_2)-x}{\alpha_2}, & x \in (a_2, a_2+\alpha_2]; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\tilde{A} = (a_1, a_2, \alpha_1, \alpha_2)$ and $\tilde{B} = (b_1, b_2, \beta_1, \beta_2)$ be two trapezoidal fuzzy numbers. Then the arithmetic operations on $\tilde{A}$ and $\tilde{B}$ are as follows.

(i) $x \geq 0, x \in R; x\tilde{A} = (xa_1, xa_2, x\alpha_1, x\alpha_2)$.

(ii) $x < 0, x \in R; x\tilde{A} = (xa_2, xa_1, -x\alpha_2, -x\alpha_1)$.

(iii) $\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$.

(iv) $\tilde{A} - \tilde{B} = (a_1 - b_2, a_2 - b_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1)$.

4.1.2 Comparison of trapezoidal fuzzy numbers

In this section, the method for comparing fuzzy numbers, used in the existing methods [38, 46], is presented.

If $\tilde{A} = (a_1, a_2, \alpha_1, \alpha_2)$ and $\tilde{B} = (b_1, b_2, \beta_1, \beta_2)$ are two trapezoidal fuzzy numbers, then

(i) $\tilde{A} \succeq \tilde{B}$ if and only if $\Re(\tilde{A}) \geq \Re(\tilde{B})$.

(ii) $\tilde{A} \succ \tilde{B}$ if and only if $\Re(\tilde{A}) > \Re(\tilde{B})$.

(iii) $\tilde{A} \approx \tilde{B}$ if and only if $\Re(\tilde{A}) = \Re(\tilde{B})$.

where $\Re(\tilde{A}) = \frac{1}{2}\left[a_1 + a_2 + \frac{\alpha_2 - \alpha_1}{2}\right]$ and $\Re(\tilde{B}) = \frac{1}{2}\left[b_1 + b_2 + \frac{\beta_2 - \beta_1}{2}\right]$.

4.2 Proposed Mehar method

In this section, a new method (named as Mehar method) is proposed to find the fuzzy optimal solution of problem $(P_{4.1})$.

The steps of the proposed Mehar method are as follows:

Step 1 Using Section 4.1.2, the problem $(P_{4.1})$ can be transformed into problem $(P_{4.2})$.

Maximize/Minimize $\left[\Re(\tilde{z}) = \Re\left(\sum_{j=1}^{n} c_j \tilde{x}_j\right)\right]$

Subject to $\qquad (P_{4.2})$

$\Re\left(\sum_{j=1}^{n} a_{ij} \tilde{x}_j\right) \leq, =, \geq \Re(\tilde{b}_i), \qquad i = 1, 2, ..., m,$

$\Re(\tilde{l}_j) \leq \Re(\tilde{x}_j) \leq \Re(\tilde{u}_j), \qquad j = 1, 2, ..., n.$

Step 2 Using the properties $\Re\left(\sum_{j=1}^{n} c_j \tilde{x}_j\right) = \sum_{j=1}^{n} c_j \Re(\tilde{x}_j)$ and $\Re\left(\sum_{j=1}^{n} a_{ij} \tilde{x}_j\right) = \sum_{j=1}^{n} a_{ij} \Re(\tilde{x}_j)$, the problem $(P_{4.2})$ can be transformed into problem $(P_{4.3})$.

Maximize/Minimize $\left[\Re(\tilde{z}) = \sum_{j=1}^{n} c_j \Re(\tilde{x}_j)\right]$

Subject to $\qquad (P_{4.3})$

$\sum_{j=1}^{n} a_{ij}\Re(\tilde{x}_j) \leq, =, \geq \Re(\tilde{b}_i), \qquad i = 1, 2, ..., m,$

$\Re(\tilde{l}_j) \leq \Re(\tilde{x}_j) \leq \Re(\tilde{u}_j), \qquad j = 1, 2, ..., n.$

Step 3 Since, $\Re(\tilde{A})$ is a real number. So, assuming $\Re(\tilde{z}) = z, \Re(\tilde{x}_j) = x_j, \Re(\tilde{b}_i) = b_i, \Re(\tilde{l}_j) = l_j$, and $\Re(\tilde{u}_j) = u_j$, the problem $(P_{4.3})$ can be transformed into problem $(P_{4.4})$.

Maximize/Minimize $\left[z = \sum_{j=1}^{n} c_j x_j\right]$

Subject to $\hfill (P_{4.4})$

$\sum_{j=1}^{n} a_{ij} x_j \leq, =, \geq b_i, \qquad i = 1, 2, ..., m,$

$l_j \leq x_j \leq u_j, \qquad j = 1, 2, ..., n.$

Step 4 Find the optimal solution $\{x_j; j = 1, 2, ..., n\}$ of problem $(P_{4.4})$ by using an appropriate existing method [144].

Step 5 Since, there exist infinite trapezoidal fuzzy numbers having same rank. So, if $x_1 = a_1, x_2 = a_2, ..., x_n = a_n$ is the optimal solution of problem $(P_{4.4})$ then all the trapezoidal fuzzy numbers $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n$ such that $\Re(\tilde{x}_1) = a_1, \Re(\tilde{x}_2) = a_2, ..., \Re(\tilde{x}_n) = a_n$ will also be the fuzzy optimal solution of the problem $(P_{4.1})$.

4.3 Fuzzy optimal solution of existing problems by proposed Mehar method

In this section, the fuzzy optimal solution of existing problem [38, Example 2, pp. 2058] and the fuzzy optimal solution of existing problem [46, Example 1, pp. 2273] is obtained by proposed Mehar method.

4.3.1 Fuzzy optimal solution of bounded fuzzy linear programming problem considered by Ebrahimnejad

Ebrahimnejad [38] formulated the existing problem [38, Example 2, pp. 2058], presented in Example 4.1, into bounded fuzzy linear programming problem $(P_{4.5})$ [38, equation 22, pp. 2058] and claimed that on solving this problem, the obtained fuzzy optimal solution and fuzzy optimal value is $\tilde{x}_{11} = (96, 102, 1, 5), \tilde{x}_{12} = (96, 102, 1, 5), \tilde{x}_{13} = (192, 204, 2, 10), \tilde{x}_{21} = (192, 204, 2, 10), \tilde{x}_{22} = (384, 408, 4, 20), \tilde{x}_{23} = (288, 306, 3, 15), \tilde{x}_{31} = (192, 204, 2, 10), \tilde{x}_{32} = (192, 204, 2, 10), \tilde{x}_{33} = (288, 306, 3, 15)$ and $(8736, 9282, 91, 350)$ respectively. However, it can be easily verified that it is not possible to find any initial fuzzy basic solution for problem $(P_{4.5})$ [38, equation 22, pp. 2058] which is optimal but not feasible. Therefore, the existing method [38] cannot be used to find the fuzzy optimal solution of the bounded fuzzy linear programming problem $(P_{4.5})$ [38, equation 22, pp. 2058] and hence the claim of Ebrahimnejad [38] that on solving the bounded fuzzy linear programming problem $(P_{4.5})$ [38, equation 22, pp. 2058] by existing method [38], the obtained fuzzy optimal solution and fuzzy optimal value is $\tilde{x}_{11} = (96, 102, 1, 5), \tilde{x}_{12} = (96, 102, 1, 5), \tilde{x}_{13} = (192, 204, 2, 10), \tilde{x}_{21} = (192, 204, 2, 10), \tilde{x}_{22} = (384, 408, 4, 20), \tilde{x}_{23} = (288, 306, 3, 15), \tilde{x}_{31} = (192, 204, 2, 10), \tilde{x}_{32} = (192, 204, 2, 10), \tilde{x}_{33} = (288, 306, 3, 15)$ and $(8736, 9282, 91, 350)$ respectively, is not valid.

In this section, the exact fuzzy optimal solution of bounded fuzzy linear programming problem $(P_{4.5})$ [38, equation 22, pp. 2058], presented in Example 4.1, is obtained by the proposed Mehar method.

Example 4.1 [38, Example 2, pp. 2058] A furniture manufacturer has three plants that need approximately 500, 700 and 800 of lumber weekly. The manufacturer may purchase the lumber from three companies. Because of other commitments, these companies can ship about 400, 900 and 700 tonnes weekly. The second lumber manufacturer uses rail for transportation and the minimum and maximum tonnage that can be shipped to any of the furniture companies is about 200 and 500 tonnes, respectively. On the other hand, the first and third lumber companies use trucks and the minimum and maximum tonnage that can be shipped to any of the furniture companies is approximately 100 and 300 tonnes respectively. The transportation cost from the lumber companies to the furniture manufacturer is given in the Table 4.1.

Table 4.1

The data for Example 4.1

Lumber company	Furniture facility 1	Furniture facility 2	Furniture facility 3
1	2	4	3
2	9	5	7
3	3	4	2

Since, the quantities of supplies or demands are approximate values, the tonnage that can be shipped to any of the furniture companies from plants will be uncertain.

The quantities of supplies which are approximately 400, 900 and 700 tonnes can be modeled as (384, 408, 4, 20), (864, 918, 9, 45) and (672, 714, 7, 35) respectively. Also, the quantities of demands which are approximately 500, 700 and 800 tonnes can be modeled as (480, 510, 5, 25), (672, 714, 7, 35) and (768, 816, 8, 40) respectively. In the similar way, other parameters are also modelled as trapezoidal fuzzy numbers. So, the given data is modelled into following bounded fuzzy linear programming problem $(P_{4.5})$ [38, equation 22, pp. 2058].

Minimize $[\tilde{z} \approx 6\tilde{x}_{11} + 4\tilde{x}_{12} + 3\tilde{x}_{13} + 5\tilde{x}_{21} + 3\tilde{x}_{22} + 4\tilde{x}_{23} + 3\tilde{x}_{31} + 2\tilde{x}_{32} + 5\tilde{x}_{33}]$

Subject to $(P_{4.5})$

$\tilde{x}_{11} + \tilde{x}_{12} + \tilde{x}_{13} \approx (384, 408, 4, 20)$,

$\tilde{x}_{21} + \tilde{x}_{22} + \tilde{x}_{23} \approx (864, 918, 9, 45)$,

$\tilde{x}_{31} + \tilde{x}_{32} + \tilde{x}_{33} \approx (672, 714, 7, 35)$,

$\tilde{x}_{11} + \tilde{x}_{21} + \tilde{x}_{31} \approx (480, 510, 5, 25)$,

$\tilde{x}_{12} + \tilde{x}_{22} + \tilde{x}_{32} \approx (672, 714, 7, 35)$,

$\tilde{x}_{13} + \tilde{x}_{23} + \tilde{x}_{33} \approx (768, 816, 8, 40)$,

$(96, 102, 1, 5) \preceq \tilde{x}_{1j} \preceq (288, 306, 3, 15),\ j = 1, 2, 3$,

$(192, 204, 2, 10) \preceq \tilde{x}_{2j} \preceq (480, 510, 5, 25),\ j = 1, 2, 3$,

$(96, 102, 1, 5) \preceq \tilde{x}_{3j} \preceq (288, 306, 3, 15),\ j = 1, 2, 3$.

Solution Using the proposed Mehar method, the exact fuzzy optimal solution of the bounded fuzzy linear programming problem $(P_{4.5})$ can be obtained as follows:

Step 1 Using Step 1 of the proposed Mehar method, the problem $(P_{4.5})$ can be transformed into problem $(P_{4.6})$.

Minimize $[\Re(\tilde{z}) = \Re(6\tilde{x}_{11} + 4\tilde{x}_{12} + 3\tilde{x}_{13} + 5\tilde{x}_{21} + 3\tilde{x}_{22} + 4\tilde{x}_{23} + 3\tilde{x}_{31} + 2\tilde{x}_{32} + 5\tilde{x}_{33})]$

Subject to $(P_{4.6})$

$\Re(\tilde{x}_{11} + \tilde{x}_{12} + \tilde{x}_{13}) = \Re(384, 408, 4, 20)$,

$\Re(\tilde{x}_{21} + \tilde{x}_{22} + \tilde{x}_{23}) = \Re(864, 918, 9, 45)$,

$\Re(\tilde{x}_{31} + \tilde{x}_{32} + \tilde{x}_{33}) = \Re(672, 714, 7, 35)$,

$\Re(\tilde{x}_{11} + \tilde{x}_{21} + \tilde{x}_{31}) = \Re(480, 510, 5, 25)$,

$\Re(\tilde{x}_{12} + \tilde{x}_{22} + \tilde{x}_{32}) = \Re(672, 714, 7, 35)$,

$\Re(\tilde{x}_{13}+\tilde{x}_{23}+\tilde{x}_{33})=\Re(768,816,8,40)$,

$\Re(96,102,1,5)\leq\Re(\tilde{x}_{1j})\leq\Re(288,306,3,15),\ j=1,2,3$,

$\Re(192,204,2,10)\leq\Re(\tilde{x}_{2j})\leq\Re(480,510,5,25),\ j=1,2,3$,

$\Re(96,102,1,5)\leq\Re(\tilde{x}_{3j})\leq\Re(288,306,3,15),\ j=1,2,3$.

Step 2 Using Step 2 of the proposed Mehar method, the problem $(P_{4.6})$ can be transformed into problem $(P_{4.7})$.

Minimize $[\Re(\tilde{z})=6\Re(\tilde{x}_{11})+4\Re(\tilde{x}_{12})+3\Re(\tilde{x}_{13})+5\Re(\tilde{x}_{21})+3\Re(\tilde{x}_{22})+4\Re(\tilde{x}_{23})+3\Re(\tilde{x}_{31})+2\Re(\tilde{x}_{32})+5\Re(\tilde{x}_{33})]$

Subject to $(P_{4.7})$

$\Re(\tilde{x}_{11})+\Re(\tilde{x}_{12})+\Re(\tilde{x}_{13})=\Re(384,408,4,20)$,

$\Re(\tilde{x}_{21})+\Re(\tilde{x}_{22})+\Re(\tilde{x}_{23})=\Re(864,918,9,45)$,

$\Re(\tilde{x}_{31})+\Re(\tilde{x}_{32})+\Re(\tilde{x}_{33})=\Re(672,714,7,35)$,

$\Re(\tilde{x}_{11})+\Re(\tilde{x}_{21})+\Re(\tilde{x}_{31})=\Re(480,510,5,25)$,

$\Re(\tilde{x}_{12})+\Re(\tilde{x}_{22})+\Re(\tilde{x}_{32})=\Re(672,714,7,35)$,

$\Re(\tilde{x}_{13})+\Re(\tilde{x}_{23})+\Re(\tilde{x}_{33})=\Re(768,816,8,40)$,

$\Re(96,102,1,5)\leq\Re(\tilde{x}_{1j})\leq\Re(288,306,3,15),\ j=1,2,3$,

$\Re(192,204,2,10)\leq\Re(\tilde{x}_{2j})\leq\Re(480,510,5,25),\ j=1,2,3$,

$\Re(96,102,1,5)\leq\Re(\tilde{x}_{3j})\leq\Re(288,306,3,15),\ j=1,2,3$.

Step 3 Using Step 3 of the proposed Mehar method, the problem $(P_{4.7})$ can be transformed into problem $(P_{4.8})$.

Minimize $[z=6x_{11}+4x_{12}+3x_{13}+5x_{21}+3x_{22}+4x_{23}+3x_{31}+2x_{32}+5x_{33}]$

Subject to $(P_{4.8})$

$x_{11}+x_{12}+x_{13}=400$,

$x_{21}+x_{22}+x_{23}=900$,

$x_{31} + x_{32} + x_{33} = 700,$

$x_{11} + x_{21} + x_{31} = 500,$

$x_{12} + x_{22} + x_{32} = 700,$

$x_{13} + x_{23} + x_{33} = 800,$

$100 \leq x_{1j} \leq 300, j = 1, 2, 3,$

$200 \leq x_{2j} \leq 500, j = 1, 2, 3,$

$100 \leq x_{3j} \leq 300, j = 1, 2, 3.$

Step 4 On solving the problem ($P_{4.8}$), the obtained optimal solution and optimal value are $x_{11} = 100, x_{12} = 100, x_{13} = 200, x_{21} = 200, x_{22} = 300, x_{23} = 400, x_{31} = 200, x_{32} = 300, x_{33} = 200$ and 7300 respectively.

Step 5 Using Step 5 of the proposed Mehar method, all the trapezoidal fuzzy numbers $\tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{13}, \tilde{x}_{21}, \tilde{x}_{22}, \tilde{x}_{23}, \tilde{x}_{31}, \tilde{x}_{32}, \tilde{x}_{33}$ such that $\Re(\tilde{x}_{11}) = 100, \Re(\tilde{x}_{12}) = 100, \Re(\tilde{x}_{13}) = 200, \Re(\tilde{x}_{21}) = 200, \Re(\tilde{x}_{22}) = 300, \Re(\tilde{x}_{23}) = 400, \Re(\tilde{x}_{31}) = 200, \Re(\tilde{x}_{32}) = 300, \Re(\tilde{x}_{33}) = 200$ will be the fuzzy optimal solutions of the problem ($P_{4.5}$) e.g., following are the three different fuzzy optimal solutions of the problem ($P_{4.5}$):

(i) $\tilde{x}_{11} = (96, 102, 1, 5), \tilde{x}_{12} = (96, 102, 1, 5), \tilde{x}_{13} = (192, 204, 2, 10), \tilde{x}_{21} = (192, 204, 2, 10),$
$\tilde{x}_{22} = (288, 306, 3, 15), \tilde{x}_{23} = (384, 408, 4, 20), \tilde{x}_{31} = (192, 204, 2, 10), \tilde{x}_{32} = (288, 306, 3, 15),$
$\tilde{x}_{33} = (192, 204, 2, 10).$

(ii)$\tilde{x}_{11} = (97, 99, 4, 12), \tilde{x}_{12} = (97, 99, 4, 12), \tilde{x}_{13} = (190, 202, 10, 26), \tilde{x}_{21} = (190, 202, 10, 26),$
$\tilde{x}_{22} = (276, 314, 12, 32), \tilde{x}_{23} = (390, 404, 6, 18), \tilde{x}_{31} = (190, 202, 10, 26), \tilde{x}_{32} = (276, 314, 12, 32),$
$\tilde{x}_{33} = (190, 202, 10, 26).$

(iii) $\tilde{x}_{11} = (93, 101, 3, 15), \tilde{x}_{12} = (93, 101, 3, 15), \tilde{x}_{13} = (196, 200, 4, 12), \tilde{x}_{21} = (196, 200, 4, 12),$
$\tilde{x}_{22} = (290, 304, 5, 17), \tilde{x}_{23} = (378, 412, 8, 28), \tilde{x}_{31} = (196, 200, 4, 12), \tilde{x}_{32} = (290, 304, 5, 17),$
$\tilde{x}_{33} = (196, 200, 4, 12).$

4.3.2 Fuzzy optimal solution of bounded fuzzy linear programming problem considered by Ebrahimnejad and Verdegay

In this section, fuzzy optimal solution of bounded fuzzy linear programming problem, considered by Ebrahimnejad and Verdegay [46, Example 1, pp. 2273], is obtained by proposed Mehar method.

Example 4.2 [46, Example 1, pp. 2273] A company has discontinued the production of a certain unprofitable product line. This act created considerable excess production capacity. Management is considering devoting this excess capacity to one or more of three products; call them products 1,2,3. The available approximate capacity on the machines that might limit output is summarized in Table 4.2.

Table 4.2

The data of Example 4.2

	Product 1	Product 2	Product 3	Available Time (Machine hours per week)
Milling machine	9	3	5	360
Lathe	2	4	3	250
Grinder	5	0	2	150

The number of machine hours required for each unit of the respective products are given in Table 4.2. The sales department indicates that the sales potential for products 1, 2 and 3 are close to 25, 30 and 35 units per week respectively. The unit profit would be 50, 20 and 25 dollars on products 1,2 and 3 respectively. The objective is to determine how much of each product the company should produce to maximize profit.

This problem is evidently an uncertain optimization problem due to variations in available capacity of machines and in sales potential for products. So, the amount of each unit of product will be uncertain. Hence, the problem can be formulated as a bounded fuzzy variable linear programming problem. The available capacity on the milling machine, lathe and grinder which are approximately 360, 250 and 150 units is modeled as (340,370,5,25), (245,265,25,5) and (140,160,5,5) respectively. In a similar way, the other parameters also modeled as trapezoidal fuzzy numbers taking into account the nature of the problem and other requirements. So, the problem is formulated as problem $(P_{4.9})$ [46, equation 31, pp. 2273].

Maximize $[\tilde{z} \approx 50\tilde{x}_1 + 20\tilde{x}_2 + 25\tilde{x}_3]$

Subject to $\qquad (P_{4.9})$

$9\tilde{x}_1 + 3\tilde{x}_2 + 5\tilde{x}_3 \preceq (340, 370, 5, 25),$

$2\tilde{x}_1 + 4\tilde{x}_2 + 3\tilde{x}_3 \preceq (245, 265, 25, 5),$

$5\tilde{x}_1 + 2\tilde{x}_3 \preceq (140, 160, 5, 5),$

$\tilde{0} \preceq \tilde{x}_1 \preceq (24, 26, 2, 2),$

$\tilde{0} \preceq \tilde{x}_2 \preceq (26, 32, 2, 10),$

$\tilde{0} \preceq \tilde{x}_3 \preceq (34, 38, 5, 1).$

Solution Using the proposed Mehar method, the fuzzy optimal solution of the bounded fuzzy linear programming problem $(P_{4.9})$ can be obtained as follows:

Step 1 Using Step 1 of the proposed Mehar method, the problem $(P_{4.9})$ can be transformed into problem $(P_{4.10})$.

Maximize $[\Re(\tilde{z}) = \Re(50\tilde{x}_1 + 20\tilde{x}_2 + 25\tilde{x}_3)]$

Subject to $\qquad (P_{4.10})$

$\Re(9\tilde{x}_1 + 3\tilde{x}_2 + 5\tilde{x}_3) \leq \Re(340, 370, 5, 25),$

$\Re(2\tilde{x}_1 + 4\tilde{x}_2 + 3\tilde{x}_3) \leq \Re(245, 265, 25, 5),$

$\Re(5\tilde{x}_1 + 2\tilde{x}_3) \leq \Re(140, 160, 5, 5),$

$\Re(\tilde{0}) \leq \Re(\tilde{x}_1) \leq \Re(24, 26, 2, 2),$

$\Re(\tilde{0}) \leq \Re(\tilde{x}_2) \leq \Re(26, 32, 2, 10),$

$\Re(\tilde{0}) \leq \Re(\tilde{x}_3) \leq \Re(34, 38, 5, 1)$.

Step 2 Using Step 2 of the proposed Mehar method, the problem ($P_{4.10}$) can be transformed into problem ($P_{4.11}$).

Maximize $[\Re(\tilde{z}) = 50\Re(\tilde{x}_1) + 20\Re(\tilde{x}_2) + 25\Re(\tilde{x}_3)]$

Subject to ($P_{4.11}$)

$9\Re(\tilde{x}_1) + 3\Re(\tilde{x}_2) + 5\Re(\tilde{x}_3) \leq \Re(340, 370, 5, 25)$,

$2\Re(\tilde{x}_1) + 4\Re(\tilde{x}_2) + 3\Re(\tilde{x}_3) \leq \Re(245, 265, 25, 5)$,

$5\Re(\tilde{x}_1) + 2\Re(\tilde{x}_3) \leq \Re(140, 160, 5, 5)$,

$\Re(\tilde{0}) \leq \Re(\tilde{x}_1) \leq \Re(24, 26, 2, 2)$,

$\Re(\tilde{0}) \leq \Re(\tilde{x}_2) \leq \Re(26, 32, 2, 10)$,

$\Re(\tilde{0}) \leq \Re(\tilde{x}_3) \leq \Re(34, 38, 5, 1)$.

Step 3 Using Step 3 of the proposed Mehar method, the problem ($P_{4.11}$) can be transformed into problem ($P_{4.12}$).

Maximize $[z = 50x_1 + 20x_2 + 25x_3]$

Subject to ($P_{4.12}$)

$9x_1 + 3x_2 + 5x_3 \leq 360$,

$2x_1 + 4x_2 + 3x_3 \leq 250$,

$5x_1 + 2x_3 \leq 150$,

$0 \leq x_1 \leq 25$, $0 \leq x_2 \leq 31$, $0 \leq x_3 \leq 35$.

Step 4 On solving the problem ($P_{4.12}$), the obtained optimal solution and optimal value are $x_1 = 25, x_2 = 31, x_3 = \frac{42}{5}$ and 2080 respectively.

Step 5 Using Step 5 of the proposed Mehar method, all the trapezoidal fuzzy numbers $\tilde{x}_1, \tilde{x}_2$, and $\tilde{x}_3$ such that $\Re(\tilde{x}_1) = 25, \Re(\tilde{x}_2) = 31$ and $\Re(\tilde{x}_3) = \frac{42}{5}$ will be the fuzzy optimal solutions of the problem ($P_{4.9}$); for example, the following are fuzzy optimal solutions of the problem $(P_{4.9})$.

(i) $\tilde{x}_1 = (24, 26, 2, 2)$, $\tilde{x}_2 = (26, 34, 1, 5)$, $\tilde{x}_3 = (\frac{32}{5}, \frac{42}{5}, 2, 6)$.

(ii) $\tilde{x}_1 = (24, 25, 2, 4)$, $\tilde{x}_2 = (24, 36, 2, 6)$, $\tilde{x}_3 = (\frac{34}{5}, 8, 3, 7)$.

(iii) $\tilde{x}_1 = (23, 26, 3, 5)$, $\tilde{x}_2 = (28, 32, 3, 7)$, $\tilde{x}_3 = (7, \frac{39}{5}, 1, 5)$.

4.4 Fuzzy optimal solution of existing problem by existing method

To show the advantages of proposed Mehar method over existing method [46], there is need to solve the same existing problem [46, Example 1, pp. 2273] by the existing method [46]. Therefore, in this section, the fuzzy optimal solution of the existing problem [46, Example 1, pp. 2273] is obtained by the existing method [46].

Further, to understand the numerical example, solved by Ebrahimnejad and Verdegay [46], there is need to explain the algorithm proposed by Ebrahimnejad and Verdegay [46]. Therefore, the algorithm, proposed by Ebrahimnejad and Verdegay [46], is also presented in a brief manner.

4.4.1 Existing algorithm to solve bounded fuzzy linear programming problems

In this section, the algorithm, proposed by Ebrahimnejad and Verdegay [46] to solve a bounded fuzzy variable linear programming problem $(P_{4.1})$, is presented.

Step 1 (i) Check whether the problem $(P_{4.1})$ is of maximization or minimization type. If problem is of minimization type then convert the problem $(P_{4.1})$ into maximization problem using the following relationship:

Minimize $\tilde{z}=-$ Maximize $(-\tilde{z})$.

(ii) Check whether all $\tilde{b}_i (i = 1, 2, ..., m)$ are positive. If any one of $\tilde{b}_i$ is negative, then multiply the corresponding constraint by -1 to make it positive.

(iii) Convert all inequalities into equalities by using fuzzy slack or fuzzy surplus variables.

Step 2 Obtain an initial fuzzy basic feasible solution. Let $B = [a_1, a_2, ..., a_m]$ be initial basis vector. Further, non-basic vectors can be decomposed into N_1 and N_2. So, $\tilde{x}$ can be partitioned as $(\tilde{x}_B, \tilde{x}_{N_1}, \tilde{x}_{N_2})$, where $\tilde{x}_B$ are fuzzy basic variables and $\tilde{x}_{N_1}$ and $\tilde{x}_{N_2}$ are fuzzy non-basic variables at their lower and upper bounds respectively. Then the current fuzzy basic solution of problem $(P_{4.1})$ is given by:

$$\begin{pmatrix} \tilde{x}_B \\ \tilde{x}_{N_1} \\ \tilde{x}_{N_2} \end{pmatrix} \approx \begin{pmatrix} \tilde{\tilde{b}} \approx B^{-1}\tilde{b} - B^{-1}N_1\tilde{l}_{N_1} - B^{-1}N_2\tilde{u}_{N_2} \\ \tilde{l}_{N_1} \\ \tilde{u}_{N_2} \end{pmatrix}.$$

The fuzzy value of the objective function for this solution is

$$\tilde{\tilde{z}} \approx \sum_{i=1}^{m} c_{B_i}\tilde{\tilde{b}}_i + \sum_{j\in R_1} c_j\tilde{l}_j + \sum_{j\in R_2} c_j\tilde{u}_j$$

where R_1 and R_2 are the set of indices of non-basic variables at their lower and upper bounds respectively. In the initial basic feasible solution, it is assumed that all the non-basic variables will be at their lower bound.

Step 3 Calculate $\bar{c}_j = c_j - c_B B^{-1} a_j \ \forall j \in \{R_1 \cup R_2\}$, If $\bar{c}_j \leq 0 \ \forall$ non-basic variables $\tilde{x}_j$ at their lower bounds and $\bar{c}_j \geq 0 \ \forall$ non-basic variables $\tilde{x}_j$ at their upper bounds, then optimality conditions are obtained. Stop the algorithm, the current fuzzy basic feasible solution is optimal. Otherwise, go to the next step.

Step 4 Choose the pivot column by applying the following test:

maximum(maximum$\{\bar{c}_j; j \in R_1\}$,maximum$\{-\bar{c}_j, j \in R_2\}$)

Without lose of generality, let $\tilde{x}_k$ be pivot column. Then x_k will be tentative entering variable.

Step 5 Check whether $k \in R_1$ or $k \in R_2$.

Case 1 If $k \in R_1$.

Calculate $\tilde{\Delta}_k$ to decide leaving variable, $\tilde{\Delta}_k$ =minimum$\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3\}$, where $\tilde{\theta}_1 = \tilde{u}_k - \tilde{l}_k$, but if there is no upper bound restriction on $\tilde{x}_k$, then $\tilde{\theta}_1 \approx \tilde{\infty}$.

$\tilde{\theta}_2 = \underset{1 \leq i \leq m}{\text{minimum}} \left\{ \frac{\tilde{\tilde{b}}_i - \tilde{l}_{B_i}}{y_{ik}} | y_{ik} > 0, 1 \leq i \leq m \right\}$, but if $y_{ik} \leq 0$ then $\tilde{\theta}_2 \approx \tilde{\infty}$.

$\tilde{\theta}_3 = \underset{1 \leq i \leq m}{\text{minimum}} \left\{ \frac{\tilde{u}_{B_i} - \tilde{\tilde{b}}_i}{-y_{ik}} | y_{ik} < 0, 1 \leq i \leq m \right\}$.

Again if $y_{ik} \geq 0$ then $\tilde{\theta}_3 \approx \tilde{\infty}$.

So, $\tilde{\Delta}_k$ can be taken as minimum of $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3$. If $\tilde{\Delta}_k \approx \tilde{\theta}_1$ then $\tilde{x}_k$ is still non-basic variable but at its upper bound and there is no change in basis matrix B. The variable corresponding to which $\tilde{\Delta}_k$ is minimum, will be the leaving variable.

New solution is given by:

$$\begin{cases} \tilde{x}_{B_i} \approx \tilde{\tilde{b}}_i - y_{ik}\tilde{\Delta}_k, & i = 1, 2, ..., m; \\ \tilde{x}_k \approx \tilde{l}_k + \tilde{\Delta}_k, & \\ \tilde{x}_j \approx \tilde{l}_j, & j \neq k, j \in R_1; \\ \tilde{x}_j \approx \tilde{u}_j, & j \in R_2. \end{cases}$$

Case 2 If $k \in R_2$.

Calculate $\tilde{\Delta}_k$ to decide leaving variable, $\tilde{\Delta}_k$ = minimum $\{\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3\}$, where $\tilde{\delta}_1 = \tilde{u}_k - \tilde{l}_k$, but if there is no upper bound restriction on $\tilde{x}_k$, then $\tilde{\delta}_1 \approx \tilde{\infty}$.

$\tilde{\delta}_2 = \underset{1 \leq i \leq m}{\text{minimum}} \left\{ \frac{\tilde{\tilde{b}}_i - \tilde{l}_{B_i}}{-y_{ik}} | y_{ik} < 0 \right\}$, but if $y_{ik} \geq 0$ then $\tilde{\delta}_2 \approx \tilde{\infty}$.

$\tilde{\delta}_3 = \underset{1 \leq i \leq m}{\text{minimum}} \left\{ \frac{\tilde{u}_{B_i} - \tilde{\tilde{b}}_i}{y_{ik}} | y_{ik} > 0 \right\}$.

Again if $y_{ik} \leq 0$ then $\tilde{\delta}_3 \approx \tilde{\infty}$.

So, $\tilde{\Delta}_k$ can be taken as minimum of $\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3$. If $\tilde{\Delta}_k \approx \tilde{\delta}_1$ then $\tilde{x}_k$ is still non-basic variable but at its lower bound and there is no change in basis matrix B. The variable corresponding to which $\tilde{\Delta}_k$ is minimum, will be the leaving variable.

New fuzzy solution is given by:

$$\begin{cases} \tilde{x}_{B_i} \approx \tilde{\tilde{b}}_i + y_{ik}\tilde{\Delta}_k, & i = 1, 2, ..., m; \\ \tilde{x}_j \approx \tilde{l}_j, & j \in R_1 \\ \tilde{x}_k \approx \tilde{u}_k - \tilde{\Delta}_k, & \\ \tilde{x}_j \approx \tilde{u}_j, & j \neq k, j \in R_2. \end{cases}$$

and new fuzzy objective value is $\tilde{\tilde{z}} \approx \tilde{\hat{z}} + (\tilde{z}_k - \tilde{c}_k)\tilde{\Delta}_k$.

Step 6 Update the table entries by making pivot element unity and other entries of pivot column zero.

Step 7 Again check the optimality conditions, if these are satisfied then stop the algorithm. The current solution is optimal solution otherwise repeat the Step 4 and Step 5.

Remark 4.1 If $\tilde{\Delta}_k = \tilde{\infty}$, then the problem $(P_{4.1})$ has unbounded solution.

4.4.2 Fuzzy optimal solution of existing problem by Ebrahimnejad and Verdegay method

In this section, the fuzzy optimal solution of existing problem [46, Example 1, pp. 2273] is obtained by the method, proposed by Ebrahimnejad and Verdegay [46].

Converting the problem $(P_{4.9})$ into standard form by adding fuzzy slack variables $\tilde{x}_4, \tilde{x}_5$ and $\tilde{x}_6$, the problem $(P_{4.9})$ is transformed into problem $(P_{4.13})$.

Maximize $[\tilde{z} \approx 50\tilde{x}_1 + 20\tilde{x}_2 + 25\tilde{x}_3]$

Subject to $\quad (P_{4.13})$

$9\tilde{x}_1 + 3\tilde{x}_2 + 5\tilde{x}_3 + \tilde{x}_4 \approx (340, 370, 5, 25)$,

$2\tilde{x}_1 + 4\tilde{x}_2 + 3\tilde{x}_3 + \tilde{x}_5 \approx (245, 265, 25, 5)$,

$5\tilde{x}_1 + 2\tilde{x}_3 + \tilde{x}_6 \approx (140, 160, 5, 5)$,

$\tilde{0} \preceq \tilde{x}_1 \preceq (24, 26, 2, 2)$,

$\tilde{0} \preceq \tilde{x}_2 \preceq (26, 32, 2, 10)$,

$\tilde{0} \preceq \tilde{x}_3 \preceq (34, 38, 5, 1)$,

$\tilde{x}_4, \tilde{x}_5, \tilde{x}_6 \succeq \tilde{0}$.

Initialization Step Consider $B = [a_4, a_5, a_6] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. as initial working basis and $N_1 = [a_1, a_2, a_3]$, $\tilde{x}_{N_1} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} \approx \begin{pmatrix} \tilde{0} \\ \tilde{0} \\ \tilde{0} \end{pmatrix}$

The initial fuzzy basic solution and fuzzy objective value is given by:

$\tilde{x}_B = \begin{pmatrix} \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \end{pmatrix} \approx \begin{pmatrix} (340, 370, 5, 25) \\ (245, 265, 25, 5) \\ (140, 160, 5, 5) \end{pmatrix}, \tilde{z} \approx (0, 0, 0, 0)$.

The fuzzy values of $\bar{c}_j = c_j - c_B B^{-1} a_j = c_j - c_B y_j$ for all $j \in N = \{1, 2, 3\}$ is obtained as $\bar{c}_1 = 50, \bar{c}_2 = 20, \bar{c}_3 = 25$. The initial table is given as Table 4.3.

Table 4.3

The initial table for problem $(P_{4.13})$

	l	l	l					
Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	Solution	$\Re$(Solution)
$\tilde{x}_4$	9	3	5	1	0	0	(340,370,5,25)	360
$\tilde{x}_5$	2	4	3	0	1	0	(245,265,25,5)	250
$\tilde{x}_6$	5	0	2	0	0	1	(140,160,5,5)	150
$\bar{c}_j$	50	20	25	0	0	0	(0,0,0,0)	0

Iteration 1 The maximum value of $\bar{c}_j$ for $j \in R_1 = \{1, 2, 3\}$ is 50, corresponding to $\tilde{x}_1$.Therfore $\tilde{x}_k = \tilde{x}_1$. Then $y_1 = B^{-1} a_1 = \begin{pmatrix} 9 \\ 2 \\ 5 \end{pmatrix}$ and $\tilde{\Delta}_1$ is given by minimum$\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3\}$, where $\tilde{\theta}_1, \tilde{\theta}_2$ and $\tilde{\theta}_3$ are obtained as:

$\tilde{\theta}_1 \approx \tilde{u}_1 - \tilde{l}_1 \approx (24, 26, 2, 2)$,

$\tilde{\theta}_2$ =minimum $\{(\frac{340}{9}, \frac{370}{9}, \frac{5}{9}, \frac{25}{9}), (\frac{245}{2}, \frac{265}{2}, \frac{25}{2}, \frac{5}{2}), (\frac{140}{5}, \frac{160}{5}, \frac{5}{5}, \frac{5}{5})\} \approx (\frac{140}{5}, \frac{160}{5}, \frac{5}{5}, \frac{5}{5})$, corresponding to $\tilde{x}_{B_3} = \tilde{x}_6$. Since, $y_1 = \begin{pmatrix} 9 \\ 2 \\ 5 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. So, $\tilde{\theta}_3 \approx \tilde{\infty}$.

Hence, $\tilde{\Delta}_1 = $ minimum $\{(24, 26, 2, 2), (\frac{140}{5}, \frac{160}{5}, \frac{5}{5}, \frac{5}{5}), \tilde{\infty}\} \approx (24, 26, 2, 2)$. So $\tilde{x}_1$ is still non-basic variable at its upper bound. Therefore $B = [a_4, a_5, a_6], N_1 = [a_2, a_3], \tilde{x}_{N_1} = \begin{pmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} \approx \begin{pmatrix} (0,0,0,0) \\ (0,0,0,0) \end{pmatrix}$, $\tilde{x}_{N_2} = \tilde{x}_1 \approx (24, 26, 2, 2)$, $\begin{pmatrix} \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \end{pmatrix} \approx \begin{pmatrix} (340,370,5,25) \\ (245,265,25,5) \\ (140,160,5,5) \end{pmatrix} - y_1\tilde{\Delta}_1 \approx \begin{pmatrix} (340,370,5,25) \\ (245,265,25,5) \\ (140,160,5,5) \end{pmatrix} - \begin{pmatrix} 9 \\ 2 \\ 5 \end{pmatrix} (24,26,2,2) \approx \begin{pmatrix} (106,154,23,43) \\ (193,217,29,9) \\ (10,40,15,15) \end{pmatrix}$ and $\tilde{z} \approx (0,0,0,0) + \bar{c}_1\tilde{\Delta}_1 \approx (0,0,0,0) + 50(24,26,2,2) = (1200,1300,100,100)$.

The next updated table is given as follows:

Table 4.4

The first iteration

	u	l	l					
Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	Solution	$\Re$(Solution)
$\tilde{x}_4$	9	3	5	1	0	0	(106,154,23,43)	135
$\tilde{x}_5$	2	4	3	0	1	0	(193,217,29,9)	200
$\tilde{x}_6$	5	0	2	0	0	1	(10,40,15,15)	25
$\bar{c}_j$	50	20	25	0	0	0	(1200,1300,100,100)	1250

Iteration 2 The maximum value of $\bar{c}_j$ for $j \in R_1 = \{2, 3\}$ is 25 corresponding to $\tilde{x}_3$. So, $\tilde{x}_k = \tilde{x}_3$. Also, $y_3 = B^{-1}a_3 = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ and $\tilde{\Delta}_3$ is given by minimum $\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3\}$, where $\tilde{\theta}_1, \tilde{\theta}_2$ and $\tilde{\theta}_3$ are calculated as follows:

$\tilde{\theta}_1 \approx \tilde{u}_3 - \tilde{l}_3 \approx (34, 38, 5, 1)$,

$\tilde{\theta}_2 = \text{minimum } \{(\frac{106}{5}, \frac{154}{5}, \frac{23}{5}, \frac{43}{5}), (\frac{193}{3}, \frac{217}{3}, \frac{29}{3}, \frac{9}{3}), (\frac{10}{2}, \frac{40}{2}, \frac{15}{2}, \frac{15}{2})\} \approx (\frac{10}{2}, \frac{40}{2}, \frac{15}{2}, \frac{15}{2})$ corresponding to $\tilde{x}_{B_3} = \tilde{x}_6$.

Since, $y_3 = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. So, $\tilde{\theta}_3 \approx \tilde{\infty}$.

Therefore, $\tilde{\Delta}_3$ =minimum $\{(34, 38, 5, 1), (\frac{10}{2}, \frac{40}{2}, \frac{15}{2}, \frac{15}{2}), \tilde{\infty}\} \approx (\frac{10}{2}, \frac{40}{2}, \frac{15}{2}, \frac{15}{2})$. So,

$\tilde{x}_3$ enters the basis and $\tilde{x}_6$ leaves the basis and drops to its lower bound. Therefore $B = [a_4, a_5, a_3]$, $N_1 = [a_2, a_6], \tilde{x}_{N_1} = \begin{pmatrix} \tilde{x}_2 \\ \tilde{x}_6 \end{pmatrix} \approx \begin{pmatrix} (0,0,0,0) \\ (0,0,0,0) \end{pmatrix}, \tilde{x}_{N_2} = \tilde{x}_1 \approx$ $(24,26,2,2), \begin{pmatrix} \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \end{pmatrix} \approx \begin{pmatrix} (106,154,23,43) \\ (193,217,29,9) \\ (10,40,15,15) \end{pmatrix} - y_3\tilde{\Delta}_3 \approx \begin{pmatrix} (106,154,23,43) \\ (193,217,29,9) \\ (10,40,15,15) \end{pmatrix} -$ $\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} (\frac{10}{2}, \frac{40}{2}, \frac{15}{2}, \frac{15}{2}) \approx \begin{pmatrix} (6,129,\frac{121}{2},\frac{161}{2}) \\ (133,202,\frac{103}{2},\frac{63}{2}) \\ (0,0,0,0) \end{pmatrix}$, $\tilde{x}_3 \approx (\frac{10}{2}, \frac{40}{2}, \frac{15}{2}, \frac{15}{2})$ and $\tilde{z} \approx (1200, 1300, 100, 100)$ $+\bar{c}_3\tilde{\Delta}_3 \approx (1200,1300,100,100) + 25(\frac{10}{2}, \frac{40}{2}, \frac{15}{2}, \frac{15}{2}) = (1325, 1800, \frac{575}{2}, \frac{575}{2})$.

The updated table is given as Table 4.5.

Table 4.5

The second iteration

	u	l				l		
Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	Solution	$\Re$(Solution)
$\tilde{x}_4$	$\frac{-7}{2}$	3	0	1	0	$\frac{-5}{2}$	$(6,129,\frac{121}{2},\frac{161}{2})$	$\frac{145}{2}$
$\tilde{x}_5$	$\frac{-11}{2}$	4	0	0	1	$\frac{-3}{2}$	$(133,202,\frac{103}{2},\frac{63}{2})$	$\frac{325}{2}$
$\tilde{x}_3$	$\frac{5}{2}$	0	1	0	0	$\frac{1}{2}$	$(5,20,\frac{15}{2},\frac{15}{2})$	$\frac{25}{2}$
$\bar{c}_j$	$\frac{-25}{2}$	20	0	0	0	$\frac{-25}{2}$	$(1325,1800,\frac{575}{2},\frac{575}{2})$	$\frac{3125}{2}$

Iteration 3 The next maximum value of $\bar{c}_j, j \in R_1$ is 20 corresponds to $\tilde{x}_2$. So $\tilde{x}_k = \tilde{x}_2$ and $y_2 = B^{-1}a_2 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ and $\tilde{\Delta}_2$ is given by minimum $\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3\}$, where $\tilde{\theta}_1, \tilde{\theta}_2$ and $\tilde{\theta}_3$ are obtained as follows:

$\tilde{\theta}_1 \approx \tilde{u}_2 - \tilde{l}_2 \approx (26, 32, 2, 10)$,

$\tilde{\theta}_2 =$minimum $\{(\frac{6}{3}, \frac{129}{3}, \frac{121}{6}, \frac{161}{6}), (\frac{133}{4}, \frac{202}{4}, \frac{103}{8}, \frac{103}{8})\} \approx (\frac{6}{3}, \frac{129}{3}, \frac{121}{6}, \frac{161}{6})$ corresponding to $\tilde{x}_4$.

Since, $y_2 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. So, $\tilde{\theta}_3 \approx \tilde{\infty}$.

Therefore $\tilde{\Delta}_2 =$ minimum $\{(26, 32, 2, 10), (\frac{6}{3}, \frac{129}{3}, \frac{121}{6}, \frac{161}{6}), \tilde{\infty}\} \approx (\frac{6}{3}, \frac{129}{3}, \frac{121}{6}, \frac{161}{6})$.

So, $\tilde{x}_2$ enters the basis and $\tilde{x}_4$ drops to its lower bound and leaves the basis. Thus we have

$B = [a_4, a_5, a_3], N_1 = [a_2, a_6], \tilde{x}_{N_1} = \begin{pmatrix} \tilde{x}_2 \\ \tilde{x}_6 \end{pmatrix} \approx \begin{pmatrix} (0,0,0,0) \\ (0,0,0,0) \end{pmatrix}, \tilde{x}_{N_2} = \tilde{x}_1 \approx (24, 26, 2, 2), \begin{pmatrix} \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_3 \end{pmatrix} \approx \begin{pmatrix} (6, 129, \frac{121}{2}, \frac{161}{2}) \\ (133, 202, \frac{103}{2}, \frac{63}{2}) \\ (5, 20, \frac{15}{2}, \frac{15}{2}) \end{pmatrix} - y_2\tilde{\Delta}_2 \approx \begin{pmatrix} (6, 129, \frac{121}{2}, \frac{161}{2}) \\ (133, 202, \frac{103}{2}, \frac{63}{2}) \\ (5, 20, \frac{15}{2}, \frac{15}{2}) \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} (\frac{6}{3}, \frac{129}{3}, \frac{121}{6}, \frac{161}{6}) \approx \begin{pmatrix} (0,0,0,0) \\ (-39, 194, \frac{953}{6}, \frac{673}{6}) \\ (5, 20, \frac{15}{2}, \frac{15}{2}) \end{pmatrix}$ and $\tilde{x}_2 \approx \tilde{l}_2 + \tilde{\Delta}_2 \approx (\frac{6}{3}, \frac{129}{3}, \frac{121}{6}, \frac{161}{6}) = (2, 43, \frac{121}{6}, \frac{161}{6}), \tilde{z} = (1325, 1800, \frac{575}{2}, \frac{575}{2}) + \bar{c}_2\tilde{\Delta}_2 = (1325, 1800, \frac{575}{2}, \frac{575}{2}) + 20(\frac{6}{3}, \frac{129}{3}, \frac{121}{6}, \frac{161}{6}) \approx (1365, 2660, \frac{4145}{6}, \frac{4945}{6})$.

The updated table is given as Table 4.6.

Table 4.6
The third iteration

	u			l		l		
Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	Solution	$\Re$(Solution)
$\tilde{x}_2$	$\frac{-7}{6}$	1	0	$\frac{1}{3}$	0	$\frac{-5}{6}$	(2,43,$\frac{121}{6}$, $\frac{161}{6}$)	$\frac{145}{6}$
$\tilde{x}_5$	$\frac{-5}{6}$	0	0	$\frac{-4}{3}$	1	$\frac{11}{6}$	(-39,194,$\frac{953}{6}$, $\frac{673}{6}$)	$\frac{395}{6}$
$\tilde{x}_3$	$\frac{5}{2}$	0	1	0	0	$\frac{1}{2}$	(5,20,$\frac{15}{2}$, $\frac{15}{2}$)	$\frac{25}{2}$
$\bar{c}_j$	$\frac{65}{6}$	0	0	$\frac{-20}{3}$	0	$\frac{25}{6}$	(1365,2660,$\frac{4145}{6}$, $\frac{4945}{6}$)	$\frac{12275}{6}$

Iteration 4 The next maximum value of $\bar{c}_j, j \in R_1$ is $\frac{25}{6}$ corresponding to $\tilde{x}_6$. So, $\tilde{x}_k = \tilde{x}_6$. Then $y_6 = \begin{pmatrix} -\frac{5}{6} \\ \frac{11}{6} \\ \frac{1}{2} \end{pmatrix}$. Since, $\tilde{x}_6$ has no upper bound, so, $\tilde{\theta}_1 \approx \tilde{\infty}$.

$\tilde{\theta}_2$ =minimum $\left\{(-\frac{39}{\frac{11}{6}}, \frac{194}{\frac{11}{6}}, \frac{953}{11}, \frac{673}{11}), (\frac{5}{\frac{1}{2}}, \frac{20}{\frac{1}{2}}, 15, 15)\right\} \approx (\frac{5}{\frac{1}{2}}, \frac{20}{\frac{1}{2}}, 15, 15)$,

$\tilde{\theta}_3 = \frac{(26,32,2,10)-(2,43,\frac{121}{6},\frac{161}{6})}{-(\frac{-5}{6})} = (-\frac{102}{5}, 36, \frac{173}{5}, \frac{181}{5})$.

Thus $\tilde{\Delta}_6$ =minimum $\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3\}$ =minimum $\{\infty, (\frac{5}{\frac{1}{2}}, \frac{20}{\frac{1}{2}}, 15, 15), (-\frac{102}{5}, 36, \frac{173}{5}, \frac{181}{5})\} = (-\frac{102}{5}, 36, \frac{173}{5}, \frac{181}{5})$ corresponding to $\tilde{x}_{B_1} = \tilde{x}_2$. So, $\tilde{x}_6$ enters the basis and $\tilde{x}_2$ drops to its upper bound and leaves the basis. Therefore $B = [a_6, a_5, a_3], \tilde{x}_{N_1} = \tilde{x}_4 = (0,0,0,0), \tilde{x}_{N_2} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \approx \begin{pmatrix} (24, 26, 2, 2) \\ (26, 32, 2, 10) \end{pmatrix}, \begin{pmatrix} \tilde{x}_6 \\ \tilde{x}_5 \\ \tilde{x}_3 \end{pmatrix} \approx \begin{pmatrix} (2, 43, \frac{121}{6}, \frac{161}{6}) \\ (-39, 194, \frac{953}{6}, \frac{673}{6}) \\ (5, 20, \frac{15}{2}, \frac{15}{2}) \end{pmatrix} -$

$$y_6\tilde{\Delta}_6 \approx \begin{pmatrix} (2, 43, \frac{121}{6}, \frac{161}{6}) \\ (-39, 194, \frac{953}{6}, \frac{673}{6}) \\ (5, 20, \frac{15}{2}, \frac{15}{2}) \end{pmatrix} - \begin{pmatrix} -\frac{5}{6} \\ \frac{11}{6} \\ \frac{1}{2} \end{pmatrix} (-\tfrac{102}{5}, 36, \tfrac{173}{5}, \tfrac{181}{5}) \approx \begin{pmatrix} (-15, 73, 49, 57) \\ (-105, \frac{1157}{5}, \frac{3378}{15}, \frac{2634}{15}) \\ (-13, \frac{151}{5}, \frac{128}{5}, \frac{124}{5}) \end{pmatrix}$$

and $\tilde{z} \approx (1365, 2660, \frac{4145}{6}, \frac{4945}{6}) + \bar{c}_6\tilde{\Delta}_6 \approx (1365, 2660, \frac{4145}{6}, \frac{4945}{6}) + (\frac{25}{6})(-\frac{102}{5}, 36, \frac{173}{5}, \frac{181}{5}) =$ $(1280, 2810, 835, 975)$.

The optimal table is given as Table 4.7.

Table 4.7
The optimal table

	u	u				l		
Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	Solution	$\Re$(Solution)
$\tilde{x}_6$	$\frac{7}{5}$	$\frac{-6}{5}$	0	$\frac{-2}{5}$	0	1	(−15,73,49,57)	31
$\tilde{x}_5$	$\frac{-17}{5}$	$\frac{11}{5}$	0	$\frac{-3}{5}$	1	0	$(-105,\frac{1157}{5},\frac{3378}{15},\frac{2634}{15})$	$\frac{254}{5}$
$\tilde{x}_3$	$\frac{9}{5}$	$\frac{3}{5}$	1	$\frac{1}{5}$	0	0	$(-13,\frac{151}{5},\frac{128}{5},\frac{124}{5})$	$\frac{42}{5}$
$\bar{c}_j$	5	5	0	−5	0	0	(1280,2810,835,975)	2080

Now $\bar{c}_1 = 5, \bar{c}_2 = 5$ and $\bar{c}_4 = -5$, which means that for all fuzzy non-basic variables at lower bound $\bar{c}_j \leq 0$ and for all fuzzy non-basic variables at upper bound $\bar{c}_j \geq 0$. So, the current fuzzy basic feasible solution is a fuzzy optimal solution. Therefore, the values of fuzzy optimal solution and fuzzy optimal value are $\tilde{x}_1 = (24, 26, 2, 2), \tilde{x}_2 = (26, 32, 1, 5), \tilde{x}_3 = (-13, \frac{151}{5}, \frac{113}{5}, \frac{121}{5})$ and $\tilde{z} = (1280, 2810, 830, 950)$ respectively.

4.5 Advantages of proposed Mehar method over the existing methods

In this section, the advantages of the proposed Mehar method over existing methods [38, 46] are discussed.

1. The existing method [38] can be used only for solving bounded fuzzy linear programming problem $(P_{4.1})$ if for the initial fuzzy basic solution, the optimality criterion [38, Theorem 1, pp. 2051] is satisfied but the feasibility

criterion [38, Theorem 2, pp. 2052] is not satisfied. However, for applying the proposed Mehar method for solving bounded fuzzy linear programming problem ($P_{4.1}$), there is no need to check any such criterion for initial fuzzy basic solution. Hence, all the bounded fuzzy linear programming problems which can be solved by the existing method [38] can also be solved by both the existing method [46] and the proposed Mehar method. But, such bounded fuzzy linear programming problems in which for initial fuzzy basic solution, the feasibility criterion is satisfied but optimality criterion is not satisfied, cannot be solved by the existing method [38] e.g., it can be easily verified that for the initial fuzzy basic solution of the existing bounded fuzzy linear programming problem [38, Example 1, pp. 2054], the optimality criterion is satisfying but feasibility criterion is not satisfying. So, this problem can be solved by the existing methods [38,46] as well as proposed Mehar method. However, for the initial fuzzy basic solution of the existing bounded fuzzy linear programming problems [46, Example 1, pp. 2273], the feasibility criterion is satisfying but optimality criterion is not satisfying. So, this problem cannot be solved by the existing method [38] but can be solved by the existing method [46] and the proposed Mehar method.

2. For solving a bounded fuzzy linear programming problem by the existing methods [38,46], there is need to use arithmetic operations of fuzzy numbers. So, the existing and easily available softwares like TORA, LINDO for solving crisp linear programming problems cannot be used and there is need to develop new softwares to solve large scale problems with the help of the existing methods [38,46]. While, for solving a bounded fuzzy linear programming problem

by the proposed Mehar method, firstly it is transformed into an equivalent crisp linear programming problem and then the crisp optimal solution of the transformed crisp linear programming is used to obtain the fuzzy optimal solution of bounded fuzzy linear programming problem. So, there is no need to develop any new softwares and the easily available softwares like TORA, LINDO etc. can be used for solving large scale problems.

3. It is obvious from Step 5 of the proposed Mehar method that if there will exist a fuzzy optimal solution of bounded fuzzy linear programming problem ($P_{4.1}$) then there will exist infinite alternative fuzzy optimal solutions of bounded fuzzy linear programming problem ($P_{4.1}$). However, according to existing methods [38,46], the alternative fuzzy optimal solutions of bounded fuzzy linear programming problem ($P_{4.1}$) may exist only if there will exist a non-basic variable in the optimal table corresponding to which value of $z_j - c_j$ is 0.

4.6 Conclusions

On the basis of present study, it can be concluded that all the problems which can be solved by the existing methods [38,46] can also be solved by the proposed Mehar method and there are several advantages of applying proposed Mehar method over applying other existing methods [38,46]. Hence, it is better to use proposed Mehar method as compared to existing methods [38,46].

Chapter 5

MEHAR METHODS TO SOLVE INTUITIONISTIC FUZZY LINEAR PROGRAMMING PROBLEMS WITH TRAPEZOIDAL INTUITIONISTIC FUZZY NUMBERS

Parvathi and Malathi [129] proposed intuitionistic fuzzy simplex method to solve such intuitionistic fuzzy linear programming problems in which the coefficients of the variables, in the objective function and in all the constraints, are represented by real numbers. While, the variables and other remaining parameters are represented by symmetric trapezoidal intuitionistic fuzzy numbers. In this chapter, limitations of the existing method [129] are pointed out. Also, new methods (named as Mehar methods) are proposed to overcome these limitations of the existing method [129].

5.1 Preliminaries

In this section, some basic definitions, arithmetic operations and comparison of trapezoidal intuitionistic fuzzy numbers are presented.

The contents of this chapter are communicated for publication in *Journal of Intelligent and Fuzzy Systems*.

5.1.1 Some basic definitions

In this section, some basic definitions related to intuitionistic fuzzy numbers are presented.

Definition 5.1 [6] An intuitionistic fuzzy set $\tilde{A}$ in X is defined as an object of the form $\tilde{A} = \{< x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) >: x \in X\}$ where the functions $\mu_{\tilde{A}} : X \to [0,1]$ and $\nu_{\tilde{A}} : X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ respectively and for every $x \in X$ in $\tilde{A}$, $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$ holds.

Definition 5.2 [6] The intuitionistic fuzzy index of x in $\tilde{A}$ is defined as $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$. It is also known as degree of hesitancy or degree of uncertainty of the element x in $\tilde{A}$. Obviously, for every $x \in X$, $0 \leq \pi_{\tilde{A}}(x) \leq 1$.

Definition 5.3 [6] An intuitionistic fuzzy set $\tilde{A} = \{< x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) >: x \in X\}$ is said to be intuitionistic fuzzy normal if there exist at least two points $x_0, x_1 \in X$ such that $\mu_{\tilde{A}}(x_0)$=1, $\nu_{\tilde{A}}(x_1)$=1.

Definition 5.4 [102] An intuitionistic fuzzy set $\tilde{A}$ is said to be intuitionistic fuzzy number $\tilde{A}^I$ if it is

(a) Intuitionistic fuzzy normal.

(b) Convex for the membership function $\mu_{\tilde{A}^I}(x)$ i.e., $\mu_{\tilde{A}^I}(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_{\tilde{A}^I}(x_1), \mu_{\tilde{A}^I}(x_2))$ for every $x_1, x_2 \in R, \lambda \in [0,1]$.

(c) Concave for the non-membership function $\nu_{\tilde{A}^I}(x)$ i.e., $\nu_{\tilde{A}^I}(\lambda x_1 + (1-\lambda)x_2) \leq max(\nu_{\tilde{A}^I}(x_1), \nu_{\tilde{A}^I}(x_2))$ for every $x_1, x_2 \in R, \lambda \in [0,1]$.

Definition 5.5 [101] An intuitionistic fuzzy number is said to be a trapezoidal intuitionistic fuzzy number if it has the following membership function $\mu_{\tilde{A}^I}(x)$ and non-membership function $\nu_{\tilde{A}^I}(x)$:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \frac{x-(a_1-\alpha)}{\alpha}, & x \in [a_1-\alpha, a_1); \\ 1, & x \in [a_1, a_2]; \\ \frac{a_2+\beta-x}{\beta}, & x \in (a_2, a_2+\beta]; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\nu_{\tilde{A}^I}(x) = \begin{cases} \frac{a_1-x}{\alpha'}, & x \in [a_1-\alpha', a_1); \\ 0, & x \in [a_1, a_2]; \\ \frac{x-a_2}{\beta'}, & x \in (a_2, a_2+\beta']; \\ 1, & \text{otherwise.} \end{cases}$$

where $\alpha, \beta, \alpha^{'}, \beta^{'} > 0$.

The trapezoidal intuitionistic fuzzy number is denoted by $\tilde{A}^I = [a_1, a_2, \alpha, \beta; a_1, a_2, \alpha^{'}, \beta^{'}]$.

Definition 5.6 [128] A trapezoidal intuitionistic fuzzy number is said to be symmetric trapezoidal intuitionistic fuzzy number if $\alpha = \beta$ (say h) and $\alpha^{'} = \beta^{'}$ (say $h^{'}$), i.e., if there exist real numbers $a_1, a_2, h, h^{'}$ where $a_1 \leq a_2, h \leq h^{'}$ and $h, h^{'} > 0$ such that the membership and non-membership functions are as follows:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \frac{x-(a_1-h)}{h}, & x \in [a_1-h, a_1); \\ 1, & x \in [a_1, a_2]; \\ \frac{a_2+h-x}{h}, & x \in (a_2, a_2+h]; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\nu_{\tilde{A}^I}(x) = \begin{cases} \frac{a_1-x}{h'}, & x \in [a_1 - h', a_1); \\ 0, & x \in [a_1, a_2]; \\ \frac{x-a_2}{h'}, & x \in (a_2, a_2 + h']; \\ 1, & \text{otherwise.} \end{cases}$$

The symmetric trapezoidal intuitionistic fuzzy number is denoted by $\tilde{A}^I = [a_1, a_2, h, h; a_1, a_2, h', h']$.

5.1.2 Arithmetic operations on trapezoidal intuitionistic fuzzy numbers

In this section, the arithmetic operations on trapezoidal intuitionistic fuzzy numbers are presented [128].

If $\tilde{A}^I = [a_1, a_2, h_1, h_2; a_1, a_2, h_1', h_2']$ and $\tilde{B}^I = [b_1, b_2, k_1, k_2; b_1, b_2, k_1', k_2']$ are two trapezoidal intuitionistic fuzzy numbers. Then

(i) $\tilde{A}^I + \tilde{B}^I = [a_1 + b_1, a_2 + b_2, h_1 + k_1, h_2 + k_2; a_1 + b_1, a_2 + b_2, h_1' + k_1', h_2' + k_2']$.

(ii) $\tilde{A}^I - \tilde{B}^I = [a_1 - b_2, a_2 - b_1, h_1 + k_2, h_2 + k_1; a_1 - b_2, a_2 - b_1, h_1' + k_2', h_2' + k_1']$.

(iii)$\lambda\tilde{A}^I = \begin{cases} ([\lambda a_1, \lambda a_2, \lambda h_1, \lambda h_2; \lambda a_1, \lambda a_2, \lambda h_1', \lambda h_2']), & \text{if } \lambda \geq 0; \\ ([\lambda a_2, \lambda a_1, -\lambda h_2, -\lambda h_1; \lambda a_2, \lambda a_1, -\lambda h_2', -\lambda h_1']), & \text{if } \lambda < 0. \end{cases}$

5.1.3 Comparison of symmetric trapezoidal intuitionistic fuzzy numbers

To find the intuitionistic fuzzy optimal solution of the intuitionistic fuzzy linear programming problem, there is need to compare intuitionistic fuzzy numbers. In this section, the method, used by the Parvathi and Malathi [129], for comparing intuitionistic fuzzy numbers, is presented.

If $\tilde{A}^I = [a_1, a_2, h, h; a_1, a_2, h', h']$ and $\tilde{B}^I = [b_1, b_2, k, k; b_1, b_2, k', k']$ are two symmetric trapezoidal intuitionistic fuzzy numbers, then

$\tilde{A}^I \succeq \tilde{B}^I$ if and only if $\Re(\tilde{A}^I) \geq \Re(\tilde{B}^I)$,

$\tilde{A}^I \succ \tilde{B}^I$ if and only if $\Re(\tilde{A}^I) > \Re(\tilde{B}^I)$,

$\tilde{A}^I \approx \tilde{B}^I$ if and only if $\Re(\tilde{A}^I) = \Re(\tilde{B}^I)$,

where $\Re(\tilde{A}^I) = a_1 + a_2 + \frac{1}{2}(h' - h)$ and $\Re(\tilde{B}^I) = b_1 + b_2 + \frac{1}{2}(k' - k)$.

Remark 5.1 If $\tilde{A}^I = [a_1, a_2, h_1, h_2; a_1, a_2, h'_1, h'_2]$ and $\tilde{B}^I = [b_1, b_2, k_1, k_2; b_1, b_2, k'_1, k'_2]$ are non-symmetric trapezoidal intuitionistic fuzzy numbers then $\Re(\tilde{A}^I) = a_1 + a_2 + \frac{1}{4}(h'_1 + h'_2 - h_1 - h_2)$ and $\Re(\tilde{B}^I) = b_1 + b_2 + \frac{1}{4}(k'_1 + k'_2 - k_1 - k_2)$.

5.2 Existing method to find solution of intuitionistic fuzzy linear programming problem

Parvathi and Malathi [129] proposed intuitionistic fuzzy simplex method to solve the intuitionistic fuzzy linear programming problems with symmetric trapezoidal intuitionistic fuzzy numbers $(P_{5.1})$.

Maximize/Minimize $\left[\tilde{z}^I \approx \sum_{j=1}^{n} c_j \tilde{x}_j^I\right]$

Subject to $\qquad (P_{5.1})$

$\sum_{j=1}^{n} a_{ij}\tilde{x}_j^I \preceq, \approx, \succeq \tilde{b}_i^I,\ i = 1, 2, ..., m,$

$\tilde{x}_j^I \succeq 0,\ j = 1, 2, ..., n.$

The steps of the existing method [129] are as follows:

Step 1 Using Section 5.1.3, the problem $(P_{5.1})$ can be transformed into problem $(P_{5.2})$.

Maximize/Minimize $\left[\Re(\tilde{z}^I) = \Re\left(\sum_{j=1}^{n} c_j \tilde{x}_j^I\right)\right]$

Subject to $\qquad (P_{5.2})$

$\Re\left[\sum_{j=1}^{n} a_{ij}\tilde{x}_j^I\right] \leq, =, \geq \Re(\tilde{b}_i^I),\ i = 1, 2, ..., m,$

$\Re(\tilde{x}_j^I) \geq 0,\ j = 1, 2, ..., n.$

Step 2 Using the linearity property $\Re\left(\sum_{i=1}^{n} \lambda_i \tilde{A}_i^I\right) = \sum_{i=1}^{n} \lambda_i \Re(\tilde{A}_i^I)$, where λ is a real number, the problem $(P_{5.2})$ can be transformed into problem $(P_{5.3})$.

$$\text{Maximize/Minimize } \left[\Re(\tilde{z}^I) = \sum_{j=1}^{n} c_j \Re(\tilde{x}_j^I)\right]$$

$$\text{Subject to} \qquad (P_{5.3})$$

$$\left[\sum_{j=1}^{n} a_{ij}\Re(\tilde{x}_j^I)\right] \leq, =, \geq \Re(\tilde{b}_i^I),\ i = 1, 2, ..., m,$$

$$\Re(\tilde{x}_j^I) \geq 0,\ j = 1, 2, ..., n.$$

Step 3 Use any appropriate existing method to find the fuzzy optimal solution $\{\Re(\tilde{x}_j^I)\}$ of the problem $(P_{5.3})$.

5.3 Intuitionistic fuzzy optimal solution of existing problem by existing method

In this section, the intuitionistic fuzzy linear programming problem $(P_{5.4})$, considered by Parvathi and Malathi [129, Section 5.5, pp. 45], is solved by the existing method [129].

Example 5.1 [129, Section 5.5, pp. 45]

Maximize $\left[\tilde{z}^I \approx 5\tilde{x}_1^I + 4\tilde{x}_2^I\right]$

Subject to

$6\tilde{x}_1^I + 4\tilde{x}_2^I \preceq [23, 25, 1, 1; 23, 25, 3, 3]$,

$\tilde{x}_1^I + 2\tilde{x}_2^I \preceq [5, 7, 2, 2; 5, 7, 4, 4]$, $\qquad (P_{5.4})$

$-\tilde{x}_1^I + \tilde{x}_2^I \preceq [3, 5, 4, 4; 3, 5, 6, 6]$,

$\tilde{x}_2^I \preceq [1, 3, 2, 2; 1, 3, 4, 4]$,

$\tilde{x}_1^I, \tilde{x}_2^I \succeq 0$,

where, $\tilde{x}_1^I$ and $\tilde{x}_2^I$ are symmetric trapezoidal intuitionistic fuzzy numbers.

Using the existing method [129], the intuitionistic fuzzy optimal solution of problem $(P_{5.4})$ can be obtained as follows:

Step 1 Using Step 1 of the existing method [129], presented in Section 5.2, the problem ($P_{5.4}$) can be transformed into problem ($P_{5.5}$).

$$\begin{aligned}
&\text{Maximize } \left[\Re(\tilde{z}^I) = \Re(5\tilde{x}_1^I + 4\tilde{x}_2^I)\right]\\
&\text{Subject to}\\
&\Re(6\tilde{x}_1^I + 4\tilde{x}_2^I) \leq \Re[23, 25, 1, 1; 23, 25, 3, 3],\\
&\Re(\tilde{x}_1^I + 2\tilde{x}_2^I) \leq \Re[5, 7, 2, 2; 5, 7, 4, 4], \qquad (P_{5.5})\\
&\Re(-\tilde{x}_1^I + \tilde{x}_2^I) \leq \Re[3, 5, 4, 4; 3, 5, 6, 6],\\
&\Re(\tilde{x}_2^I) \leq \Re[1, 3, 2, 2; 1, 3, 4, 4],\\
&\Re(\tilde{x}_1^I), \Re(\tilde{x}_2^I) \geq 0.
\end{aligned}$$

Step 2 Using Step 2 of the existing method [129], presented in Section 5.2, the problem ($P_{5.5}$) can be transformed into problem ($P_{5.6}$).

$$\begin{aligned}
&\text{Maximize } \left[\Re(\tilde{z}^I) = 5\Re(\tilde{x}_1^I) + 4\Re(\tilde{x}_2^I)\right]\\
&\text{Subject to}\\
&6\Re(\tilde{x}_1^I) + 4\Re(\tilde{x}_2^I) \leq \Re[23, 25, 1, 1; 23, 25, 3, 3],\\
&\Re(\tilde{x}_1^I) + 2\Re(\tilde{x}_2^I) \leq \Re[5, 7, 2, 2; 5, 7, 4, 4], \qquad (P_{5.6})\\
&-\Re(\tilde{x}_1^I) + \Re(\tilde{x}_2^I) \leq \Re[3, 5, 4, 4; 3, 5, 6, 6],\\
&\Re(\tilde{x}_2^I) \leq \Re[1, 3, 2, 2; 1, 3, 4, 4],\\
&\Re(\tilde{x}_1^I), \Re(\tilde{x}_2^I) \geq 0.
\end{aligned}$$

Step 3 Adding slack variables $\Re(\tilde{x}_3^I), \Re(\tilde{x}_4^I), \Re(\tilde{x}_5^I)$ and $\Re(\tilde{x}_6^I)$ into first, second, third and fourth constraints of problem ($P_{5.6}$) respectively, the problem ($P_{5.6}$) can be transformed into problem ($P_{5.7}$).

$$\begin{aligned}
&\text{Maximize } \left[\Re(\tilde{z}^I) = 5\Re(\tilde{x}_1^I) + 4\Re(\tilde{x}_2^I)\right]\\
&\text{Subject to}\\
&6\Re(\tilde{x}_1^I) + 4\Re(\tilde{x}_2^I) + \Re(\tilde{x}_3^I) = \Re[23, 25, 1, 1; 23, 25, 3, 3],\\
&\Re(\tilde{x}_1^I) + 2\Re(\tilde{x}_2^I) + \Re(\tilde{x}_4^I) = \Re[5, 7, 2, 2; 5, 7, 4, 4], \qquad (P_{5.7})
\end{aligned}$$

$-\Re(\tilde{x}_1^I) + \Re(\tilde{x}_2^I) + \Re(\tilde{x}_5^I) = \Re[3,5,4,4;3,5,6,6]$,

$\Re(\tilde{x}_2^I) + \Re(\tilde{x}_6^I) = \Re[1,3,2,2;1,3,4,4]$,

$\Re(\tilde{x}_1^I), \Re(\tilde{x}_2^I), \Re(\tilde{x}_3^I), \Re(\tilde{x}_4^I), \Re(\tilde{x}_5^I), \Re(\tilde{x}_6^I) \geq 0.$

Table 5.1 is the initial simplex table of the problem $(P_{5.7})$.

Table 5.1

The initial simplex table

Basis $(\tilde{x}_B^I)$	$\Re(\tilde{x}_1^I)$	$\Re(\tilde{x}_2^I)$	$\Re(\tilde{x}_3^I)$	$\Re(\tilde{x}_4^I)$	$\Re(\tilde{x}_5^I)$	$\Re(\tilde{x}_6^I)$	Solution
$\Re(\tilde{x}_3^I)$	6	4	1	0	0	0	$\Re[23,25,1,1;23,25,3,3] = 49$
$\Re(\tilde{x}_4^I)$	1	2	0	1	0	0	$\Re[5,7,2,2;5,7,4,4] = 13$
$\Re(\tilde{x}_5^I)$	-1	1	0	0	1	0	$\Re[3,5,4,4;3,5,6,6] = 9$
$\Re(\tilde{x}_6^I)$	0	1	0	0	0	1	$\Re[1,3,2,2;1,3,4,4] = 5$
$z_j - c_j$	-5	-4	0	0	0	0	$\Re(\tilde{0}^I) = 0$

Since, minimum $\{-5,-4\} = -5$, So $\Re(\tilde{x}_1^I)$ is entering variable. Also, minimum $\{\frac{49}{6}, \frac{13}{1}\} = \frac{49}{6}$ corresponding to $\Re(\tilde{x}_3^I)$. So, $\Re(\tilde{x}_3^I)$ is a leaving variable. Now, after applying the required row-operations, Table 5.2 is obtained.

Table 5.2

The first iteration table

Basis $(\tilde{x}_B^I)$	$\Re(\tilde{x}_1^I)$	$\Re(\tilde{x}_2^I)$	$\Re(\tilde{x}_3^I)$	$\Re(\tilde{x}_4^I)$	$\Re(\tilde{x}_5^I)$	$\Re(\tilde{x}_6^I)$	Solution
$\Re(\tilde{x}_1^I)$	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	$\Re[\frac{23}{6},\frac{25}{6},\frac{1}{6},\frac{1}{6};\frac{23}{6},\frac{25}{6},\frac{3}{6},\frac{3}{6}] = \frac{49}{6}$
$\Re(\tilde{x}_4^I)$	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	$\Re[\frac{5}{6},\frac{19}{6},\frac{13}{6},\frac{13}{6};\frac{5}{6},\frac{19}{6},\frac{27}{6},\frac{27}{6}] = \frac{31}{6}$
$\Re(\tilde{x}_5^I)$	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	$\Re[\frac{41}{6},\frac{55}{6},\frac{25}{6},\frac{25}{6};\frac{41}{6},\frac{55}{6},\frac{39}{6},\frac{39}{6}] = \frac{103}{6}$
$\Re(\tilde{x}_6^I)$	0	1	0	0	0	1	$\Re[1,3,2,2;1,3,4,4] = 5$
$z_j - c_j$	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	$\Re[\frac{115}{6},\frac{125}{6},\frac{5}{6},\frac{5}{6};\frac{115}{6},\frac{125}{6},\frac{15}{6},\frac{15}{6}] = \frac{245}{6}$

Now, $-\frac{2}{3}$ is only negative entry in $z_j - c_j$. So, $\Re(\tilde{x}_2^I)$ is an entering variable and minimum $\left\{\frac{\frac{49}{6}}{\frac{2}{3}}, \frac{\frac{31}{6}}{\frac{4}{3}}, \frac{\frac{103}{6}}{\frac{5}{3}}, \frac{5}{1}\right\} = \frac{\frac{31}{6}}{\frac{4}{3}} = \frac{31}{8}$ corresponding to $\Re(\tilde{x}_4^I)$. So, $\Re(\tilde{x}_4^I)$ is a leaving variable. The next updated table is Table 5.3.

Table 5.3

The final iteration table

Basis $(\tilde{x}_B^I)$	$\Re(\tilde{x}_1^I)$	$\Re(\tilde{x}_2^I)$	$\Re(\tilde{x}_3^I)$	$\Re(\tilde{x}_4^I)$	$\Re(\tilde{x}_5^I)$	$\Re(\tilde{x}_6^I)$	Solution
$\Re(\tilde{x}_1^I)$	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	$\Re[\frac{9}{4},\frac{15}{4},\frac{5}{4},\frac{5}{4};\frac{9}{4},\frac{15}{4},\frac{11}{4},\frac{11}{4}]=\frac{27}{4}$
$\Re(\tilde{x}_2^I)$	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\Re[\frac{5}{8},\frac{19}{8},\frac{13}{8},\frac{13}{8};\frac{5}{8},\frac{19}{8},\frac{27}{8},\frac{27}{8}]=\frac{31}{8}$
$\Re(\tilde{x}_5^I)$	0	0	$\frac{9}{24}$	$-\frac{5}{4}$	1	0	$\Re[\frac{23}{8},\frac{65}{8},\frac{55}{8},\frac{55}{8};\frac{23}{8},\frac{65}{8},\frac{97}{8},\frac{97}{8}]=\frac{109}{8}$
$\Re(\tilde{x}_6^I)$	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\Re[-\frac{11}{8},\frac{19}{8},\frac{29}{8},\frac{29}{8};\frac{11}{8},\frac{19}{8},\frac{59}{8},\frac{59}{8}]=\frac{23}{8}$
z_j-c_j	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	$\Re[\frac{55}{4},\frac{113}{4},\frac{51}{4},\frac{51}{4};\frac{55}{4},\frac{113}{4},\frac{109}{4},\frac{109}{4}]=\frac{197}{4}$

Since, all the values of $z_j-c_j \geq 0$. So, obtained fuzzy solution is a fuzzy optimal solution. The obtained fuzzy optimal solution is $\tilde{x}_1^I \approx [\frac{9}{4},\frac{15}{4},\frac{5}{4},\frac{5}{4};\frac{9}{4},\frac{15}{4},\frac{11}{4},\frac{11}{4}]$ with $\Re(\tilde{x}_1^I)=\Re[\frac{9}{4},\frac{15}{4},\frac{5}{4},\frac{5}{4};\frac{9}{4},\frac{15}{4},\frac{11}{4},\frac{11}{4}]=\frac{27}{4}=6.75$; $\tilde{x}_2^I \approx [\frac{5}{8},\frac{19}{8},\frac{13}{8},\frac{13}{8};\frac{5}{8},\frac{19}{8},\frac{27}{8},\frac{27}{8}]$ with $\Re(\tilde{x}_2^I)=\Re[\frac{5}{8},\frac{19}{8},\frac{13}{8},\frac{13}{8};\frac{5}{8},\frac{19}{8},\frac{27}{8},\frac{27}{8}]=\frac{31}{8}=3.875$ and the obtained fuzzy optimal value is $\tilde{z}^I \approx [\frac{55}{4},\frac{113}{4},\frac{51}{4},\frac{51}{4};\frac{55}{4},\frac{113}{4},\frac{109}{4},\frac{109}{4}]$ with $\Re(\tilde{z}^I)=\Re[\frac{55}{4},\frac{113}{4},\frac{51}{4},\frac{51}{4};\frac{55}{4},\frac{113}{4},\frac{109}{4},\frac{109}{4}]=\frac{197}{4}=49.25$.

5.4 Linearity property of existing ranking function

In this section, it is shown that for ranking function $\Re$, used by Parvathi and Malathi [129], the property $\Re\left(\sum\limits_{i=1}^{m}\lambda_i\tilde{A}_i^I\right)=\sum\limits_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I)$ will be satisfied only if $\lambda_i \geq 0$. But, if $\lambda_i < 0$, then this property is not satisfied.

Let $\tilde{A}_i^I=[x_i,y_i,h,h;x_i,y_i,h',h']$, $i=1,2,...,m$ be a symmetric trapezoidal intuitionistic fuzzy number.

Case 1 If $\lambda_i \geq 0$.

Then

$$\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right)=\Re\left(\sum_{i=1}^{m}\lambda_i[x_i,y_i,h,h;x_i,y_i,h',h']\right)$$
$$=\Re\left(\sum_{i=1}^{m}[\lambda_i x_i,\lambda_i y_i,\lambda_i h,\lambda_i h;\lambda_i x_i,\lambda_i y_i,\lambda_i h',\lambda_i h']\right)$$

$$= \Re\left[\sum_{i=1}^{m}\lambda_i x_i, \sum_{i=1}^{m}\lambda_i y_i, \sum_{i=1}^{m}\lambda_i h, \sum_{i=1}^{m}\lambda_i h; \sum_{i=1}^{m}\lambda_i x_i, \sum_{i=1}^{m}\lambda_i y_i, \sum_{i=1}^{m}\lambda_i h', \sum_{i=1}^{m}\lambda_i h'\right]$$
$$= \sum_{i=1}^{m}\lambda_i x_i + \sum_{i=1}^{m}\lambda_i y_i + \tfrac{1}{2}\sum_{i=1}^{m}\lambda_i(h'-h) = \sum_{i=1}^{m}\lambda_i(x_i + y_i + \tfrac{1}{2}(h'-h)) = \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I).$$

Hence, $\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right) = \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I)$.

Case 2 If $\lambda_i < 0$ say $\lambda_i = -k_i, k_i > 0$.

Then $\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right) = \Re\left(\sum_{i=1}^{m}(-k_i)[x_i, y_i, h, h; x_i, y_i, h^{'}, h^{'}]\right)$

$$= \Re\left(\sum_{i=1}^{m}[(-k_i)y_i, (-k_i)x_i, k_i h, k_i h; (-k_i)y_i, (-k_i)x_i, k_i h^{'}, k_i h^{'}]\right)$$
$$= \Re\left[\sum_{i=1}^{m}(-k_i)y_i, \sum_{i=1}^{m}(-k_i)x_i, \sum_{i=1}^{m}k_i h, \sum_{i=1}^{m}k_i h; \sum_{i=1}^{m}(-k_i)y_i, \sum_{i=1}^{m}(-k_i)x_i, \sum_{i=1}^{m}k_i h', \sum_{i=1}^{m}k_i h'\right]$$
$$= \sum_{i=1}^{m}(-k_i)y_i + \sum_{i=1}^{m}(-k_i)x_i + \tfrac{1}{2}\sum_{i=1}^{m}k_i(h'-h) = \sum_{i=1}^{m}(-k_i)(x_i + y_i - \tfrac{1}{2}(h'-h)) \neq \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I).$$

Hence, $\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right) \neq \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I)$.

5.5 Limitations of the existing method

In this section, the limitations of the existing method [129], are presented.

1. Parvathi and Malathi [129] have used the linearity property $\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right) = \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I)$ in Step 2 of their proposed method. However, as proved in Section 5.4, this property is valid only if all the coefficients c_j and a_{ij} in problem $(P_{5.1})$ are non-negative real numbers. Hence, if any of the coefficients c_j or a_{ij} is negative real number. Then, existing method [129] cannot be used for solving problem $(P_{5.1})$.

2. The method, proposed by Parvathi and Malathi [129], is applicable only if variables and right hand side vector are represented by symmetric trapezoidal intuitionistic fuzzy numbers. But, this method [129] is not applicable to solve intuitionistic fuzzy linear programming problems $(P_{5.8})$ in which variables and right hand side vector are represented by non-symmetric trapezoidal intuitionistic fuzzy numbers.

$$\text{Maximize/Minimize } \left[\tilde{z}^I \approx \sum_{j=1}^{n} c_j \tilde{x}_j^I\right]$$

$$\text{Subject to} \qquad (P_{5.8})$$

$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j^I \preceq, \approx, \succeq \tilde{b}_i^I,\ i = 1, 2, ..., m;$$

$$\tilde{x}_j^I \succeq 0,\ j = 1, 2, ..., n.$$

5.6 Error in the existing results

In Section 5.4, it is proved that if λ_i is a negative real number then $\Re\left(\sum_{i=1}^{m} \lambda_i \tilde{A}_i^I\right) \neq \sum_{i=1}^{m} \lambda_i \Re(\tilde{A}_i^I)$. It is obvious that the coefficients of $\tilde{x}_1^I$ in 3^{rd} constraint of the problem $(P_{5.4})$ is the negative real number. So, $\Re(-\tilde{x}_1^I + \tilde{x}_2^I) \neq -\Re(\tilde{x}_1^I) + \Re(\tilde{x}_2^I)$.

However, it is obvious from Step 2 of Section 5.3 that Parvathi and Malathi [129] have used the property $\Re(-\tilde{x}_1^I + \tilde{x}_2^I) = -\Re(\tilde{x}_1^I) + \Re(\tilde{x}_2^I)$ to transform the problem $(P_{5.5})$ into problem $(P_{5.6})$. Due to the same reason, the optimal solution of the problem $(P_{5.4})$, obtained by Parvathi and Malathi [129], is not satisfying the first and second constraints of problem $(P_{5.4})$. This is shown as given below:

(i) Putting the values of $\tilde{x}_1^I$ and $\tilde{x}_2^I$ in left hand side of first constraint, we have

$6[\frac{9}{4}, \frac{15}{4}, \frac{5}{4}, \frac{5}{4}; \frac{9}{4}, \frac{15}{4}, \frac{11}{4}, \frac{11}{4}] + 4[\frac{5}{8}, \frac{19}{8}, \frac{13}{8}, \frac{13}{8}; \frac{5}{8}, \frac{19}{8}, \frac{27}{8}, \frac{27}{8}] = [\frac{27}{2}, \frac{45}{2}, \frac{15}{2}, \frac{15}{2}; \frac{27}{2}, \frac{45}{2}, \frac{33}{2}, \frac{33}{2}] + [\frac{5}{2}, \frac{19}{2}, \frac{13}{2}, \frac{13}{2}; \frac{5}{2}, \frac{19}{2}, \frac{27}{2}, \frac{27}{2}] = [16, 32, 14, 14; 16, 32, 30, 30]$

and $\Re[16, 32, 14, 14; 16, 32, 30, 30] = 56$.

Right hand side of first constraint is $[23, 25, 1, 1; 23, 25, 3, 3]$ and $\Re[23, 25, 1, 1; 23, 25, 3, 3] = 49$. It is obvious that $[16, 32, 14, 14; 16, 32, 30, 30] \not\preceq [23, 25, 1, 1; 23, 25, 3, 3]$

(ii) Putting the values of $\tilde{x}_1^I$ and $\tilde{x}_2^I$ in left hand side of second constraint, we have

$[\frac{9}{4}, \frac{15}{4}, \frac{5}{4}, \frac{5}{4}; \frac{9}{4}, \frac{15}{4}, \frac{11}{4}, \frac{11}{4}] + 2[\frac{5}{8}, \frac{19}{8}, \frac{13}{8}, \frac{13}{8}; \frac{5}{8}, \frac{19}{8}, \frac{27}{8}, \frac{27}{8}] = [\frac{9}{4}, \frac{15}{4}, \frac{5}{4}, \frac{5}{4}; \frac{9}{4}, \frac{15}{4}, \frac{11}{4}, \frac{11}{4}] + [\frac{5}{4}, \frac{19}{4}, \frac{13}{4}, \frac{13}{4}; \frac{5}{4}, \frac{19}{4}, \frac{27}{4}, \frac{27}{4}] = [\frac{7}{2}, \frac{17}{2}, \frac{9}{2}, \frac{9}{2}; \frac{7}{2}, \frac{17}{2}, \frac{19}{2}, \frac{19}{2}]$ and $\Re[\frac{7}{2}, \frac{17}{2}, \frac{9}{2}, \frac{9}{2}; \frac{7}{2}, \frac{17}{2}, \frac{19}{2}, \frac{19}{2}] = \frac{29}{2}$.

Right hand side of second constraint is $[5, 7, 2, 2; 5, 7, 4, 4]$ and $\Re[5, 7, 2, 2; 5, 7, 4, 4] =$

13. It is obvious that $[\frac{7}{2},\frac{17}{2},\frac{9}{2},\frac{9}{2};\frac{7}{2},\frac{17}{2},\frac{19}{2},\frac{19}{2}]\not\preceq[5,7,2,2;5,7,4,4]$.

Hence, the fuzzy optimal solution, obtained by Parvathi and Malathi [129], is not valid.

5.7 Proposed Mehar methods to find the solution of intuitionistic fuzzy linear programming problems with trapezoidal intuitionistic fuzzy numbers

To overcome the limitations of the existing method [129], discussed in Section 5.5, new methods (named as Mehar methods) are proposed to find the intuitionistic fuzzy optimal solution of $(P_{5.8})$.

5.7.1 Proposed Mehar method to find the solution of intuitionistic fuzzy linear programming problems with trapezoidal intuitionistic fuzzy numbers with non-negative coefficients.

In this section, a new method is proposed to solve intuitionistic fuzzy linear programming problem $(P_{5.8})$ with trapezoidal intuitionistic fuzzy numbers in which all the coefficients of the variables are non-negative real numbers. The same method can be applied to solve intuitionistic fuzzy linear programming problems with symmetric trapezoidal intuitionistic fuzzy numbers in which all the coefficients of the variables are non-negative real numbers.

The steps of the proposed method are as follows:

Step 1 Using Section 5.1.3, the problem $(P_{5.8})$ can be transformed into problem $(P_{5.9})$.

Maximize/Minimize $\left[\Re(\tilde{z}^I)=\Re\left(\sum\limits_{j=1}^{n}c_j\tilde{x}_j^I\right)\right]$

Subject to $\qquad (P_{5.9})$

$\Re\left[\sum_{j=1}^{n} a_{ij}\tilde{x}_j^I\right] \leq, =, \geq \Re(\tilde{b}_i^I),\ i = 1, 2, ..., m;$

$\Re(\tilde{x}_j^I) \geq 0,\ j = 1, 2, ..., n.$

Step 2 Using the linearity property $\Re\left(\sum_{i=1}^{n} \lambda_i \tilde{A}_i^I\right) = \sum_{i=1}^{n} \lambda_i \Re(\tilde{A}_i^I)$, where λ is a non-negative real number, the problem $(P_{5.9})$ can be transformed into problem $(P_{5.10})$.

Maximize/Minimize $\left[\Re(\tilde{z}^I) = \sum_{j=1}^{n} c_j \Re(\tilde{x}_j^I)\right]$

Subject to $(P_{5.10})$

$\left[\sum_{j=1}^{n} a_{ij}\Re(\tilde{x}_j^I)\right] \leq, =, \geq \Re(\tilde{b}_i^I),\ i = 1, 2, ..., m;$

$\Re(\tilde{x}_j^I) \geq 0,\ j = 1, 2, ..., n.$

Step 3 Since, the rank of an intuitionistic fuzzy number is a real number. So, assuming $\Re(\tilde{z}^I) = z, \Re(\tilde{x}_j^I) = x_j$ and $\Re(\tilde{b}_i^I) = b_i$, the problem $(P_{5.10})$ can be transformed into $(P_{5.11})$.

Maximize/Minimize $\left[z = \sum_{j=1}^{n}(c_j x_j)\right]$

Subject to $(P_{5.11})$

$\left[\sum_{j=1}^{n} a_{ij}x_j\right] \leq, =, \geq b_i,\ i = 1, 2, ..., m;$

$x_j \geq 0,\ j = 1, 2, ..., n.$

Step 4 Using an appropriate existing method, find the optimal solution of the problem $(P_{5.11})$.

Step 5 Since, there exist infinite intuitionistic fuzzy numbers having the same rank. So, if $x_1 = a_1$, $x_2 = a_2, ..., x_n = a_n$ is an optimal solution of the problem $(P_{5.11})$, then all the trapezoidal intuitionistic fuzzy numbers $\tilde{x}_1^I, \tilde{x}_2^I, ..., \tilde{x}_n^I$ such that $\Re(\tilde{x}_1^I) = a_1$, $\Re(\tilde{x}_2^I) = a_2$,...,$\Re(\tilde{x}_n^I) = a_n$ will be the fuzzy optimal solution of the problem $(P_{5.8})$.

5.7.2 Proposed Mehar method to find the solution of intuitionistic fuzzy linear programming problems with trapezoidal intuitionistic fuzzy numbers with unrestricted coefficients

In this section, a new method (named as Mehar method) is proposed to solve intuitionistic fuzzy linear programming problems ($P_{5.8}$). The same method can be applied to solve intuitionistic fuzzy linear programming problems ($P_{5.1}$), by replacing $h_{1j} = h_{2j} = h_j$ and $h'_{1j} = h'_{2j} = h'_j$.

The steps of proposed Mehar method are as follows:

Step 1 Substituting $\tilde{x}^I_j = [x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}]$ and $\tilde{b}^I_i = [b_i, g_i, k_{1i}, k_{2i}; b_i, g_i, k'_{1i}, k'_{2i}]$ into the problem ($P_{5.8}$), the problem ($P_{5.8}$) can be transformed into problem ($P_{5.12}$).

$$\text{Maximize/Minimize} \left[\tilde{z}^I \approx \sum_{j=1}^{n} c_j [x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \right]$$

Subject to $\qquad (P_{5.12})$

$$\sum_{j=1}^{n} a_{ij} [x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \preceq, \approx, \succeq [b_i, g_i, k_{1i}, k_{2i}; b_i, g_i, k'_{1i}, k'_{2i}],$$

$$[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \succeq 0.$$

Step 2 Assuming $c_j [x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] = [p_j, q_j, r_{1j}, r_{2j}; p_j, q_j, r'_{1j}, r'_{2j}]$ and $a_{ij} [x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] = [d_j, e_j, f_{1j}, f_{2j}; d_j, e_j, f'_{1j}, f'_{2j}]$, the problem ($P_{5.12}$) can be transformed into problem ($P_{5.13}$).

$$\text{Maximize/Minimize} \left[\tilde{z}^I \approx \sum_{j=1}^{n} [p_j, q_j, r_{1j}, r_{2j}; p_j, q_j, r'_{1j}, r'_{2j}] \right]$$

Subject to $\qquad (P_{5.13})$

$$\sum_{j=1}^{n} [d_j, e_j, f_{1j}, f_{2j}; d_j, e_j, f'_{1j}, f'_{2j}] \preceq, \approx, \succeq [b_i, g_i, k_{1i}, k_{2i}; b_i, g_i, k'_{1i}, k'_{2i}],$$

$$[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \succeq 0.$$

Step 3 The problem ($P_{5.13}$) can be transformed into problem ($P_{5.14}$).

$$\text{Maximize/Minimize} \left[\tilde{z}^I \approx [\sum_{j=1}^{n} p_j, \sum_{j=1}^{n} q_j, \sum_{j=1}^{n} r_{1j}, \sum_{j=1}^{n} r_{2j}; \sum_{j=1}^{n} p_j, \sum_{j=1}^{n} q_j, \sum_{j=1}^{n} r'_{1j}, \sum_{j=1}^{n} r'_{2j}] \right]$$

Subject to $\qquad (P_{5.14})$

$$\left[\sum_{j=1}^{n} d_j, \sum_{j=1}^{n} e_j, \sum_{j=1}^{n} f_{1j}, \sum_{j=1}^{n} f_{2j}; \sum_{j=1}^{n} d_j, \sum_{j=1}^{n} e_j, \sum_{j=1}^{n} f'_{1j}, \sum_{j=1}^{n} f'_{2j}\right] \preceq, \approx, \succeq [b_i, g_i, k_{1i}, k_{2i}; b_i, g_i, k'_{1i}, k'_{2i}],$$

$[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \succeq 0.$

Step 4 The problem $(P_{5.14})$ can be transformed into problem $(P_{5.15})$.

$$\text{Maximize/Minimize } \left[\tilde{z}^I = \Re\left[\sum_{j=1}^{n} p_j, \sum_{j=1}^{n} q_j, \sum_{j=1}^{n} r_{1j}, \sum_{j=1}^{n} r_{2j}; \sum_{j=1}^{n} p_j, \sum_{j=1}^{n} q_j, \sum_{j=1}^{n} r'_{1j}, \sum_{j=1}^{n} r'_{2j}\right]\right]$$

Subject to $(P_{5.15})$

$$\Re\left[\sum_{j=1}^{n} d_j, \sum_{j=1}^{n} e_j, \sum_{j=1}^{n} f_{1j}, \sum_{j=1}^{n} f_{2j}; \sum_{j=1}^{n} d_j, \sum_{j=1}^{n} e_j, \sum_{j=1}^{n} f'_{1j}, \sum_{j=1}^{n} f'_{2j}\right] \le, =, \ge$$

$\Re[b_i, g_i, k_{1i}, k_{2i}; b_i, g_i, k'_{1i}, k'_{2i}],$

$\Re[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \ge 0.$

Step 5 Using Section 5.1.3, the problem $(P_{5.15})$ can be transformed into problem $(P_{5.16})$.

$$\text{Maximize/Minimize } \left[\tilde{z}^I = \sum_{j=1}^{n} p_j + \sum_{j=1}^{n} q_j + \frac{1}{4}\left(\sum_{j=1}^{n} r'_{1j} + \sum_{j=1}^{n} r'_{2j} - \sum_{j=1}^{n} r_{1j} - \sum_{j=1}^{n} r_{2j}\right)\right]$$

Subject to $(P_{5.16})$

$$\left(\sum_{j=1}^{n} d_j + \sum_{j=1}^{n} e_j + \frac{1}{4}\left(\sum_{j=1}^{n} f'_{1j} + \sum_{j=1}^{n} f'_{2j} - \sum_{j=1}^{n} f_{1j} - \sum_{j=1}^{n} f_{2j}\right)\right) \le, =, \ge$$

$\left(b_i + g_i + \frac{1}{4}(k'_{1i} + k'_{2i} - k_{1i} - k_{2i})\right),$

$x_j + y_j + \frac{1}{4}(h'_{1j} + h'_{2j} - h_{1j} - h_{2j}) \ge 0,$

$x_j \le y_j, h_{1j} \le h'_{1j}, h_{2j} \le h'_{2j}$

x_j, y_j are unrestricted and $h_{1j}, h_{2j}, h'_{1j}, h'_{2j} \ge 0.$

Step 6 Solve the problem $(P_{5.16})$ by using an appropriate existing method to find the values of $x_j, y_j, h_{1j}, h_{2j}, h'_{1j}, h'_{2j}$ and put these values in $\tilde{x}^I_j = [x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}]$ to find the intuitionistic fuzzy optimal solution.

Step 7 Find the intuitionistic fuzzy optimal value $\tilde{z}^I$ by putting the values of $\tilde{x}^I_j$ in $\sum_{j=1}^{n}(c_j\tilde{x}^I_j)$.

5.8 Exact solution of existing problem

In this section, the exact solution of the problem $(P_{5.4})$ is obtained by using proposed Mehar method.

Step 1 Substituting the values of $\tilde{x}_1^I = [x_1, y_1, h_1, h_1; x_1, y_1, h_1', h_1']$ and $\tilde{x}_2^I = [x_2, y_2, h_2, h_2; x_2, y_2, h_2', h_2']$ in the problem $(P_{5.4})$, it can be transformed into problem $(P_{5.17})$.

Maximize $\left[\tilde{z}^I \approx 5\,[x_1, y_1, h_1, h_1; x_1, y_1, h_1', h_1'] + 4\,[x_2, y_2, h_2, h_2; x_2, y_2, h_2', h_2']\right]$

Subject to

$$
\begin{aligned}
&6\,[x_1, y_1, h_1, h_1; x_1, y_1, h_1', h_1'] + 4\,[x_2, y_2, h_2, h_2; x_2, y_2, h_2', h_2'] \preceq [23, 25, 1, 1; 23, 25, 3, 3],\\
&[x_1, y_1, h_1, h_1; x_1, y_1, h_1', h_1'] + 2\,[x_2, y_2, h_2, h_2; x_2, y_2, h_2', h_2'] \preceq [5, 7, 2, 2; 5, 7, 4, 4], \qquad (P_{5.17})\\
&-[x_1, y_1, h_1, h_1; x_1, y_1, h_1', h_1'] + [x_2, y_2, h_2, h_2; x_2, y_2, h_2', h_2'] \preceq [3, 5, 4, 4; 3, 5, 6, 6],\\
&[x_2, y_2, h_2, h_2; x_2, y_2, h_2', h_2'] \preceq [1, 3, 2, 2; 1, 3, 4, 4],\\
&[x_1, y_1, h_1, h_1; x_1, y_1, h_1', h_1'] \succeq 0,\\
&[x_2, y_2, h_2, h_2; x_2, y_2, h_2', h_2'] \succeq 0.
\end{aligned}
$$

Step 2 Using Step 2 of proposed Mehar method, the problem $(P_{5.17})$ can be transformed into problem $(P_{5.18})$.

Maximize $\left[\tilde{z}^I = [5x_1, 5y_1, 5h_1, 5h_1; 5x_1, 5y_1, 5h_1', 5h_1'] + [4x_2, 4y_2, 4h_2, 4h_2; 4x_2, 4y_2, 4h_2', 4h_2']\right]$

Subject to

$$
\begin{aligned}
&[6x_1, 6y_1, 6h_1, 6h_1; 6x_1, 6y_1, 6h_1', 6h_1'] + [4x_2, 4y_2, 4h_2, 4h_2; 4x_2, 4y_2, 4h_2', 4h_2'] \preceq [23, 25, 1, 1; 23, 25, 3, 3],\\
&[x_1, y_1, h_1, h_1; x_1, y_1, h_1', h_1'] + [2x_2, 2y_2, 2h_2, 2h_2; 2x_2, 2y_2, 2h_2', 2h_2'] \preceq [5, 7, 2, 2; 5, 7, 4, 4],\\
&[-y_1, -x_1, h_1, h_1; -y_1, -x_1, h_1', h_1'] + [x_2, y_2, h_2, h_2; x_2, y_2, h_2', h_2'] \preceq [3, 5, 4, 4; 3, 5, 6, 6],\\
&[x_2, y_2, h_2, h_2; x_2, y_2, h_2', h_2'] \preceq [1, 3, 2, 2; 1, 3, 4, 4], \qquad (P_{5.18})\\
&[x_1, y_1, h_1, h_1; x_1, y_1, h_1', h_1'] \succeq 0,\\
&[x_2, y_2, h_2, h_2; x_2, y_2, h_2', h_2'] \succeq 0.
\end{aligned}
$$

Step 3 Using Step 3 of proposed Mehar method, the problem ($P_{5.18}$) can be transformed into problem ($P_{5.19}$).

Maximize

$\left[\tilde{z}^I = [5x_1 + 4x_2, 5y_1 + 4y_2, 5h_1 + 4h_2, 5h_1 + 4h_2; 5x_1 + 4x_2, 5y_1 + 4y_2, 5h'_1 + 4h'_2, 5h'_1 + 4h'_2]\right]$

Subject to

$[6x_1 + 4x_2, 6y_1 + 4y_2, 6h_1 + 4h_2, 6h_1 + 4h_2; 6x_1 + 4x_2, 6y_1 + 4y_2, 6h'_1 + 4h'_2, 6h'_1 + 4h'_2] \preceq$

$[23, 25, 1, 1; 23, 25, 3, 3]$,

$[x_1 + 2x_2, y_1 + 2y_2, h_1 + 2h_2, h_1 + 2h_2; x_1 + 2x_2, y_1 + 2y_2, h'_1 + 2h'_2, h'_1 + 2h'_2] \preceq$

$[5, 7, 2, 2; 5, 7, 4, 4]$,

$[-y_1 + x_2, -x_1 + y_2, h_1 + h_2, h_1 + h_2; -y_1 + x_2, -x_1 + y_2, h'_1 + h'_2, h'_1 + h'_2] \preceq [3, 5, 4, 4; 3, 5, 6, 6]$,

$[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2] \preceq [1, 3, 2, 2; 1, 3, 4, 4]$, ($P_{5.19}$)

$[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1] \succeq 0$,

$[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2] \succeq 0$.

Step 4 Using Step 4 of proposed Mehar method, the problem ($P_{5.19}$) can be transformed into problem ($P_{5.20}$).

Maximize

$\left[\tilde{z}^I = \Re\left[5x_1 + 4x_2, 5y_1 + 4y_2, 5h_1 + 4h_2, 5h_1 + 4h_2; 5x_1 + 4x_2, 5y_1 + 4y_2, 5h'_1 + 4h'_2, 5h'_1 + 4h'_2\right]\right]$

Subject to

$\Re\left[6x_1 + 4x_2, 6y_1 + 4y_2, 6h_1 + 4h_2, 6h_1 + 4h_2; 6x_1 + 4x_2, 6y_1 + 4y_2, 6h'_1 + 4h'_2, 6h'_1 + 4h'_2\right] \leq$

$\Re\left[23, 25, 1, 1; 23, 25, 3, 3\right]$,

$\Re\left[x_1 + 2x_2, y_1 + 2y_2, h_1 + 2h_2, h_1 + 2h_2; x_1 + 2x_2, y_1 + 2y_2, h'_1 + 2h'_2, h'_1 + 2h'_2\right] \leq$

$\Re[5, 7, 2, 2; 5, 7, 4, 4]$,

$\Re\left[-y_1 + x_2, -x_1 + y_2, h_1 + h_2, h_1 + h_2; -y_1 + x_2, -x_1 + y_2, h'_1 + h'_2, h'_1 + h'_2\right] \leq$

$\Re[3, 5, 4, 4; 3, 5, 6, 6]$,

$\Re\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \leq \Re[1,3,2,2;1,3,4,4],$ $(P_{5.20})$

$\Re\left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right] \geq 0,$

$\Re\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \geq 0.$

Step 5 Using Step 5 of proposed Mehar method, the problem $(P_{5.20})$ can be transformed into problem $(P_{5.21})$.

Maximize $\left[\tilde{z}^I = \left[5x_1 + 4x_2 + 5y_1 + 4y_2 + \frac{1}{2}[5h'_1 + 4h'_2 - 5h_1 - 4h_2]\right]\right]$

Subject to

$6x_1 + 4x_2 + 6y_1 + 4y_2 + \frac{1}{2}[6h'_1 + 4h'_2 - 6h_1 - 4h_2] \leq 49,$

$x_1 + 2x_2 + y_1 + 2y_2 + \frac{1}{2}[h'_1 + 2h'_2 - h_1 - 2h_2] \leq 13,$

$-y_1 + x_2 - x_1 + y_2 + \frac{1}{2}[h'_1 + h'_2 - h_1 - h_2] \leq 9,$

$x_2 + y_2 + \frac{1}{2}[h'_2 - h_2] \leq 5,$ $(P_{5.21})$

$x_1 + y_1 + \frac{1}{2}[h'_1 - h_1] \geq 0,$

$x_2 + y_2 + \frac{1}{2}[h'_2 - h_2] \geq 0,$

$x_1 \leq y_1, x_2 \leq y_2, h_1 \leq h'_1, h_2 \leq h'_2,$

where x_1, x_2, y_1, y_2 are unrestricted and $h_1, h'_1, h_2, h'_2 \geq 0$.

Step 6 Solving the problem $(P_{5.21})$, the obtained values of x_1, y_1, h_1, h'_1 are $\frac{3}{32}, \frac{3}{32}, 0, \frac{89}{8}$ and x_2, y_2, h_2, h'_2 are $0, 0, 0, \frac{29}{4}$ respectively. Hence, the intuitionistic fuzzy optimal solution is $\tilde{x}^I_1 = \left[\frac{3}{32}, \frac{3}{32}, 0, 0; \frac{3}{32}, \frac{3}{32}, \frac{89}{8}, \frac{89}{8}\right]$ and $\tilde{x}^I_2 = \left[0, 0, 0, 0; 0, 0, \frac{29}{4}, \frac{29}{4}\right]$.

Step 7 Putting the values of $\tilde{x}^I_1$ and $\tilde{x}^I_2$ in $5\tilde{x}^I_1 + 4\tilde{x}^I_2$, the intuitionistic fuzzy optimal value is $\left[\frac{15}{32}, \frac{15}{32}, 0, 0; \frac{15}{32}, \frac{15}{32}, \frac{677}{8}, \frac{677}{8}\right]$.

5.9 Conclusions

In this chapter, limitations of the existing method [129] are pointed out and to overcome these limitations, new methods (named as Mehar methods) are proposed to solve intuitionistic fuzzy linear programming problems with trapezoidal intuitionistic fuzzy numbers.

Chapter 6

MEHAR METHOD TO SOLVE INTUITIONISTIC FULLY FUZZY LINEAR PROGRAMMING PROBLEMS WITH TRAPEZOIDAL INTUITIONISTIC FUZZY NUMBERS

The methods, proposed in Chapter 5, cannot be used to find the intuitionistic fuzzy optimal solution of intuitionistic fully fuzzy linear programming problems (intuitionistic fuzzy linear programming problems in which all the variables and parameters are represented by intuitionistic fuzzy numbers).

In this chapter, flaws in the existing method [111] for finding the intuitionistic fuzzy optimal solution of intuitionistic fully fuzzy linear programming problems, is pointed out. Also, the product of two unrestricted trapezoidal intuitionistic fuzzy numbers is proposed as well as a new method (named as Mehar method) is proposed to find the intuitionistic fuzzy optimal solution of such intuitionistic fully fuzzy linear programming problems in which all the variables and parameters are represented by trapezoidal intuitionistic fuzzy numbers.

The contents of this chapter are communicated for publication in *Applied Mathematical Modelling*.

6.1 Preliminaries

In this section, arithmetic operations and comparison of trapezoidal intuitionistic fuzzy numbers are presented [111].

6.1.1 Arithmetic operations on trapezoidal intuitionistic fuzzy numbers

If $\tilde{A}^I = \{(a_1, a_2^L, a_2^M, a_3); (a_1', a_2^L, a_2^M, a_3')\}$ and $\tilde{B}^I = \{(b_1, b_2^L, b_2^M, b_3); (b_1', b_2^L, b_2^M, b_3')\}$ are two trapezoidal intuitionistic fuzzy numbers. Then

(i) $\tilde{A}^I + \tilde{B}^I = \big\{(a_1 + b_1, a_2^L + b_2^L, a_2^M + b_2^M, a_3 + b_3); (a_1' + b_1', a_2^L + b_2^L, a_2^M + b_2^M, a_3' + b_3')\big\}$.

(ii) $\tilde{A}^I - \tilde{B}^I = \big\{(a_1 - b_3, a_2^L - b_2^M, a_2^M - b_2^L, a_3 - b_1); (a_1' - b_3', a_2^L - b_2^M, a_2^M - b_2^L, a_3' - b_1')\big\}$.

(iii) $\tilde{A}^I \times \tilde{B}^I = \big\{(a_1 b_1, a_2^L b_2^L, a_2^M b_2^M, a_3 b_3); (a_1' b_1', a_2^L b_2^L, a_2^M b_2^M, a_3' b_3')\big\}$

(iv) $k\tilde{A}^I = \begin{cases} \{(ka_1, ka_2^L, ka_2^M, ka_3); (ka_1', ka_2^L, ka_2^M, ka_3')\}, & \text{if } k \geq 0; \\ \{(ka_3, ka_2^M, ka_2^L, ka_1); (ka_3', ka_2^M, ka_2^L, ka_1')\}, & \text{if } k < 0. \end{cases}$

Remark 6.1 The trapezoidal intuitionistic fuzzy number $[a_1, a_2, h_1, h_2; a_1, a_2, h_1^{'}, h_2^{'}]$ can also be represented in the form $\{(a_1 - h_1, a_1, a_2, a_2 + h_2); (a_1 - h_1', a_1, a_2, a_2 + h_2')\}$ where $a_1 - h_1' \leq a_1 - h_1 \leq a_1 \leq a_2 \leq a_2 + h_2 \leq a_2 + h_2'$.

In the present chapter $\{(a_1, a_2^L, a_2^M, a_3); (a_1', a_2^L, a_2^M, a_3')\}$ representation of trapezoidal intuitionistic fuzzy number is used everywhere.

6.1.2 Comparison of trapezoidal intuitionistic fuzzy numbers

To find the intuitionistic fuzzy optimal solution of the intuitionistic fuzzy linear programming problem, there is need to compare intuitionistic fuzzy numbers. In this section, the method, used by Nagoorgani and Ponnalagu [111], for comparing intuitionistic fuzzy numbers, is presented.

Let $\tilde{A}^I = \{(a_1, a_2^L, a_2^M, a_3); (a_1', a_2^L, a_2^M, a_3')\}$ and $\tilde{B}^I = \{(b_1, b_2^L, b_2^M, b_3); (b_1', b_2^L, b_2^M, b_3')\}$

be two trapezoidal intuitionistic fuzzy numbers. Then

$\tilde{A}^I \succeq \tilde{B}^I$ if and only if $\Re(\tilde{A}^I) \geq \Re(\tilde{B}^I)$,

$\tilde{A}^I \succ \tilde{B}^I$ if and only if $\Re(\tilde{A}^I) > \Re(\tilde{B}^I)$,

$\tilde{A}^I \approx \tilde{B}^I$ if and only if $\Re(\tilde{A}^I) = \Re(\tilde{B}^I)$,

where $\Re(\tilde{A}^I) = \frac{(a_1+a_2^L+a_2^M+a_3)+(a_1'+a_2^L+a_2^M+a_3')}{8}$ and $\Re(\tilde{B}^I) = \frac{(b_1+b_2^L+b_2^M+b_3)+(b_1'+b_2^L+b_2^M+b_3')}{8}$.

6.2 Existing method to find solution of intuitionistic fully fuzzy linear programming problems

Nagoorgani and Ponnalagu [111] proposed a method to find the intuitionistic fuzzy optimal solution of intuitionistic fully fuzzy linear programming problem with triangular intuitionistic fuzzy numbers ($P_{6.1}$).

$$\text{Maximize/Minimize } \left[\tilde{z}^I \approx \sum_{j=1}^{n} \tilde{c}_j^I \tilde{x}_j^I\right]$$

Subject to $\qquad (P_{6.1})$

$$\sum_{j=1}^{n} \tilde{a}_{ij}^I \tilde{x}_j^I \preceq, \approx, \succeq \tilde{b}_i^I, i = 1, 2, ..., m;$$

$$\tilde{x}_j^I \succeq 0, \ j = 1, 2, ..., n.$$

The steps of the existing method [111] are as follows:

Step 1 Using Section 6.1.2, the problem ($P_{6.1}$) can be transformed into problem ($P_{6.2}$).

$$\text{Maximize/Minimize } \left[\Re(\tilde{z}^I) = \Re\left(\sum_{j=1}^{n} \tilde{c}_j^I \tilde{x}_j^I\right)\right]$$

Subject to $\qquad (P_{6.2})$

$$\Re\left[\sum_{j=1}^{n} \tilde{a}_{ij}^I \tilde{x}_j^I\right] \leq, =, \geq \Re(\tilde{b}_i^I), i = 1, 2, ..., m;$$

$$\Re(\tilde{x}_j^I) \geq 0, \ j = 1, 2, ..., n.$$

Step 2 Using $\Re\left(\sum_{j=1}^{n}(\tilde{a}_{ij}^I \tilde{x}_j^I)\right) = \sum_{j=1}^{n}(\tilde{a}_{ij}^I)\Re(\tilde{x}_j^I)$, the problem ($P_{6.2}$) can be transformed into problem ($P_{6.3}$).

$$\text{Maximize/Minimize } \left[\Re(\tilde{z}^I)=\sum_{j=1}^{n}(\tilde{c}_j^I)\Re(\tilde{x}_j^I)\right]$$

Subject to $\qquad (P_{6.3})$

$$\left[\sum_{j=1}^{n}(\tilde{a}_{ij}^I)\Re(\tilde{x}_j^I)\right] \leq,=,\geq \Re(\tilde{b}_i^I), i=1,2,...,m;$$

$$\Re(\tilde{x}_j^I)\geq 0,\ \ j=1,2,...,n.$$

Step 3 Use an appropriate existing method to find the fuzzy optimal solution of problem $(P_{6.3})$.

6.3 Intuitionistic fuzzy optimal solution of existing problem

In this section, the intuitionistic fully fuzzy linear programming problem $(P_{6.4})$, considered by Nagoorgani and Ponnalagu [111], is solved by existing method [111].

Example 6.1 [111, Section 6, pp. 3472]

Maximize $[\tilde{z}^I \approx \{(4,5,6);(4,5,6.1)\}\tilde{x}_1^I+\{(2.5,3,3.2);(2,3,3.5)\}\tilde{x}_2^I]$

Subject to $\qquad (P_{6.4})$

$\{(3.5,4,4.1);(3,4,5)\}\tilde{x}_1^I+\{(2.5,3,3.5);(2.4,3,3.6)\}\tilde{x}_2^I\preceq\{(11,12,13);(11,12,14)\},$

$\{(0.8,1,2);(0.5,1,2.1)\}\tilde{x}_1^I+\{(2.8,3,3.2);(2.5,3,3.2)\}\tilde{x}_2^I\preceq\{(5.5,6,7.5);(5,6,8.1)\},$

$\tilde{x}_1^I,\tilde{x}_2^I\succeq 0.$

Using the existing method [111], the intuitionistic fully fuzzy linear programming problem $(P_{6.4})$ can be solved as follows:

Step 1 Using Step 1 of the existing method [111], presented in Section 6.2, the problem $(P_{6.4})$ can be transformed into problem $(P_{6.5})$.

Maximize $[\Re(\tilde{z}^I)=\Re(\{(4,5,6);(4,5,6.1)\}\tilde{x}_1^I+\{(2.5,3,3.2);(2,3,3.5)\}\tilde{x}_2^I)]$

Subject to $\qquad (P_{6.5})$

$\Re(\{(3.5,4,4.1);(3,4,5)\}\tilde{x}_1^I + \{(2.5,3,3.5);(2.4,3,3.6)\}\tilde{x}_2^I) \leq \Re(\{(11,12,13);(11,12,14)\})$,

$\Re(\{(0.8,1,2);(0.5,1,2.1)\}\tilde{x}_1^I + \{(2.8,3,3.2);(2.5,3,3.2)\}\tilde{x}_2^I) \leq \Re(\{(5.5,6,7.5);(5,6,8.1)\})$,

$\Re(\tilde{x}_1^I), \Re(\tilde{x}_2^I) \geq 0$.

Step 2 Using Step 2 of the existing method [111], presented in Section 6.2, the problem $(P_{6.5})$ can be transformed into problem $(P_{6.6})$.

Maximize $[\Re(\tilde{z}^I) = \{(4,5,6);(4,5,6.1)\}\Re(\tilde{x}_1^I) + \{(2.5,3,3.2);(2,3,3.5)\}\Re(\tilde{x}_2^I)]$

Subject to $(P_{6.6})$

$\{(3.5,4,4.1);(3,4,5)\}\Re(\tilde{x}_1^I) + \{(2.5,3,3.5);(2.4,3,3.6)\}\Re(\tilde{x}_2^I) \leq \Re(\{(11,12,13);(11,12,14)\})$,

$\{(0.8,1,2);(0.5,1,2.1)\}\Re(\tilde{x}_1^I) + \{(2.8,3,3.2);(2.5,3,3.2)\}\Re(\tilde{x}_2^I) \leq \Re(\{(5.5,6,7.5);(5,6,8.1)\})$,

$\Re(\tilde{x}_1^I), \Re(\tilde{x}_2^I) \geq 0$.

Step 3 Adding slack variables $\Re(\tilde{s}_1^I)$ and $\Re(\tilde{s}_2^I)$ into first and second constraint of problem $(P_{6.6})$ respectively, the problem $(P_{6.6})$ can be transformed into problem $(P_{6.7})$.

Maximize $[\Re(\tilde{z}^I) = \{(4,5,6);(4,5,6.1)\}\Re(\tilde{x}_1^I) + \{(2.5,3,3.2);(2,3,3.5)\}\Re(\tilde{x}_2^I)]$

Subject to $(P_{6.7})$

$\{(3.5,4,4.1);(3,4,5)\}\Re(\tilde{x}_1^I) + \{(2.5,3,3.5);(2.4,3,3.6)\}\Re(\tilde{x}_2^I) + \Re(\tilde{s}_1^I) = \Re(\{(11,12,13);(11,12,14)\})$

$\{(0.8,1,2);(0.5,1,2.1)\}\Re(\tilde{x}_1^I) + \{(2.8,3,3.2);(2.5,3,3.2)\}\Re(\tilde{x}_2^I) + \Re(\tilde{s}_2^I) = \Re(\{(5.5,6,7.5);(5,6,8.1)\})$

$\Re(\tilde{x}_1^I), \Re(\tilde{x}_2^I), \Re(\tilde{s}_1^I), \Re(\tilde{s}_1^I) \geq 0$.

Table 6.1 is the initial simplex table of problem $(P_{6.7})$.

Table 6.1

The initial simplex table

Basis $(\tilde{x}_B^I)$	$\Re(\tilde{x}_1^I)$	$\Re(\tilde{x}_2^I)$	$\Re(\tilde{s}_1^I)$	$\Re(\tilde{s}_2^I)$	Solution	Minimum Ratio
$\Re(\tilde{s}_1^I)$	$\{(3.5,4,4.1);(3,4,5)\}$	$\{(2.5,3,3.5);(2.4,3,3.6)\}$	$\{(1,1,1);(1,1,1)\}$	$\{(0,0,0);(0,0,0)\}$	$\Re\left(\{(11,12,13);(11,12,14)\}\right)$	$\Re\left(\frac{\{(11,12,13);(11,12,14)\}}{\{(3.5,4,4.1);(3,4,5)\}}\right)=\frac{\Re(\{(11,12,13);(11,12,14)\})}{\Re(\{(3.5,4,4.1);(3,4,5)\})}=\frac{97}{31.6}=3.07$
$\Re(\tilde{s}_2^I)$	$\{(0.8,1,2);(0.5,1,2.1)\}$	$\{(2.8,3,3.2);(2.5,3,3.2)\}$	$\{(0,0,0);(0,0,0)\}$	$\{(1,1,1);(1,1,1)\}$	$\Re((\{(5.5,6,7.5);(5,6,8.1)\})$	$\Re\left(\frac{\{(5.5,6,7.5);(5,6,8.1)\}}{\{(0.8,1,2);(0.5,1,2.1)\}}\right)=\frac{\Re(\{(5.5,6,7.5);(5,6,8.1)\})}{\Re(\{(0.8,1,2);(0.5,1,2.1)\})}=\frac{50.1}{9.4}=5.33$
$\Re(\tilde{z}_j^I-\tilde{c}_j^I)$	$-\{(4,5,6);(4,5,6.1)\}$	$-\{(2.5,3,3.2);(2,3,3.5)\}$	$\{(0,0,0);(0,0,0)\}$	$\{(0,0,0);(0,0,0)\}$	-	-

Table 6.2

The optimal table

Basis $(\tilde{x}_B^I)$	$\Re(\tilde{x}_1^I)$	$\Re(\tilde{x}_2^I)$	$\Re(\tilde{s}_1^I)$	$\Re(\tilde{s}_2^I)$	Solution
$\Re(\tilde{x}_1^I)$	$\{(0.85,1,1.17);(0.6,1,1.67)\}$	$\{(0.61,0.75,1);(0.46,0.75,1.2)\}$	$\{(0.24,0.25,0.29);(0.2,0.25,0.67)\}$	$\{(0,0,0);(0,0,0)\}$	$\Re\left(\{(2.68,3,3.71);(2.2,3,4.67)\}\right)=3.1575$
$\Re(\tilde{s}_2^I)$	$\{(-0.37,0,1.15);(-1.17,0,1.5)\}$	$\{(1.8,2.25,2.59);(1.3,2.25,2.74)\}$	$-\{(0.24,0.25,0.29);(0.2,0.25,0.67)\}$	$\{(1,1,1);(1,1,1)\}$	$\Re\left(\{(1.79,3,4.82);(0.33,3,5.9)\}\right)3.105$
$\Re(\tilde{z}_j^I-\tilde{c}_j^I)$	$\{(0,0,0);(0,0,0)\}$	$\{(-0.76,0.75,0.76);(-1.66,0.75,1.66)\}$	$\{(0.96,1.25,1.74);(0.8,1.25,4.08)\}$	$\{(0,0,0);(0,0,0)\}$	$\Re\left(\{(10.72,15,22.26);(8.8,15,28.48)\}\right)=16.2825$

Since, minimum $(\Re(-\{(4,5,6);(4,5,6.1)\}),\Re(-\{(2.5,3,3.2);(2,3,3.5)\}))=$ $\Re(-\{(4,5,6);(4,5,6.1)\})$ corresponding to $\Re(\tilde{x}_1^I)$. So, $\Re(\tilde{x}_1^I)$ is the entering variable. Also, minimum $\left(\frac{\Re(\{(11,12,13);(11,12,14)\})}{\Re(\{(3.5,4,4.1);(3,4,5)\})},\frac{\Re(\{(5.5,6,7.5);(5,6,8.1)\})}{\Re(\{(0.8,1,2);(0.5,1,2.1)\})}\right)=\frac{\Re(\{(11,12,13);(11,12,14)\})}{\Re(\{(3.5,4,4.1);(3,4,5)\})}$ corresponding to $\Re(\tilde{s}_1^I)$. So, $\Re(\tilde{s}_1^I)$ is the leaving variable. Now, after applying the required row-operations, Table 6.2 is obtained.

Since, $\Re(\tilde{z}_j^I-\tilde{c}_j^I)\geq 0$. So, the obtained intuitionistic fuzzy solution is intuitionistic fuzzy optimal solution. The obtained intuitionistic fuzzy optimal solution is $\tilde{x}_1^I=\{(2.68,3,3.71);(2.2,3,4.67)\}$ with $\Re(\tilde{x}_1^I)=\Re(\{(2.68,3,3.71);(2.2,3,4.67)\})=$ $3.1575, \tilde{x}_2^I=\{(0,0,0);(0,0,0)\}$ with $\Re(\{(0,0,0);(0,0,0)\})=0, \tilde{s}_1^I=\{(0,0,0);(0,0,0)\}$ with $\Re(\{(0,0,0);(0,0,0)\})=0, \tilde{s}_2^I=\{(1.79,3,4.82);(0.33,3,5.9)\}$ with $\Re(\tilde{s}_2^I)=$ $\Re(\{(1.79,3,4.82);(0.33,3,5.9)\})=3.105$ and the obtained intuitionistic fuzzy optimal value is $\tilde{z}_j^I=\{(10.72,15,22.26);(8.8,15,28.48)\}$ with $\Re(\{(10.72,15,22.26);$ $(8.8,15,28.48)\})=16.2825$.

6.4 Flaws in the existing method

It is obvious from Step 2 of the existing method [111], described in Section 6.2, that Nagoorgani and Ponnalagu [111] has assumed $\Re(\tilde{A}^I\times\tilde{B}^I)=\tilde{A}^I\times\Re(\tilde{B}^I)$ to transform the constraints and objective function of problem $(P_{6.2})$ into constraints and objective function of problem $(P_{6.3})$. Since, $\Re(\tilde{A}^I\times\tilde{B}^I)$ is a real number and $\tilde{A}^I\times\Re(\tilde{B}^I)$ is an intuitionistic fuzzy number. So, $\Re(\tilde{A}^I\times\tilde{B}^I)$ will not be equal to $\tilde{A}^I\times\Re(\tilde{B}^I)$ and hence, the problem $(P_{6.2})$ cannot be transformed into problem $(P_{6.3})$. Therefore, the existing method [111], in which the intuitionistic fuzzy optimal solution of problem $(P_{6.2})$ is obtained with the help of intuitionistic fuzzy optimal solution of problem $(P_{6.3})$, is not valid.

6.5 Proposed product of unrestricted trapezoidal intuitionistic fuzzy numbers

It is obvious from problem $(P_{6.1})$ that to find the fuzzy optimal solution of problem $(P_{6.1})$, there is need to multiply $\tilde{x}_j^I$ with $\tilde{c}_j^I$ in the objective function and with $\tilde{a}_{ij}^I$ in constraints. In problem $(P_{6.1})$, it is assumed that $\tilde{x}_j^I$ is an intuitionistic fuzzy number whose rank is non-negative. However, $\Re(\tilde{x}_j^I) \geq 0$ does not imply that $\tilde{x}_j^I$ is a non-negative intuitionistic fuzzy number. If $\Re(\tilde{x}_j^I) \geq 0$, there exist several negative and unrestricted (neither negative nor positive) intuitionistic fuzzy numbers, whose rank will be non-negative. For example, $\tilde{x}_j^I = \{(-4, 6, 8, 37); (-10, 6, 8, 56)\}$ is an unrestricted intuitionistic fuzzy number but $\Re(\tilde{x}_j^I)$ is non-negative.

Therefore, to find fuzzy optimal solution of problem $(P_{6.1})$, there is need to use product of unrestricted intuitionistic fuzzy number $(\tilde{x}_j^I)$ with other unrestricted intuitionistic fuzzy number $\tilde{c}_j^I$ and $\tilde{a}_{ij}^I$. However, product of unrestricted intuitionistic fuzzy number is not defined by anyone in literature. Therefore, in this section, with the help of existing product of unrestricted trapezoidal fuzzy number [79], the product of unrestricted trapezoidal intuitionistic fuzzy number is proposed.

If $\tilde{A}^I = \{(a_1, a_2^L, a_2^M, a_3); (a_1', a_2^L, a_2^M, a_3')\}$ and $\tilde{B}^I = \{(b_1, b_2^L, b_2^M, b_3); (b_1', b_2^L, b_2^M, b_3')\}$ are two unrestricted trapezoidal intuitionistic fuzzy numbers. Then

$$\tilde{A}^I \times \tilde{B}^I =$$

$$\begin{cases}
\begin{array}{l}\{(min(a_1b_1, a_3b_1), min(a_2^Lb_2^L, a_2^Mb_2^L), max(a_2^Lb_2^M, a_2^Mb_2^M), max(a_1b_3, a_3b_3));\\ (min(a_1'b_1', a_3'b_1'), min(a_2^Lb_2^L, a_2^Mb_2^L), max(a_2^Lb_2^M, a_2^Mb_2^M), max(a_1'b_3', a_3'b_3'))\},\end{array} & \text{if } a_1' \geq 0;\\[2ex]
\begin{array}{l}\{(min(a_1b_3, a_3b_3), min(a_2^Lb_2^M, a_2^Mb_2^M), max(a_2^Lb_2^L, a_2^Mb_2^L), max(a_1b_1, a_3b_1));\\ (min(a_1'b_3', a_3'b_3'), min(a_2^Lb_2^M, a_2^Mb_2^M), max(a_2^Lb_2^L, a_2^Mb_2^L), max(a_1'b_1', a_3'b_1'))\},\end{array} & \text{if } a_3' < 0;\\[2ex]
\begin{array}{l}\{(min(a_1b_3, a_3b_3), min(a_2^Lb_2^M, a_2^Mb_2^M), max(a_2^Lb_2^L, a_2^Mb_2^L), max(a_1b_1, a_3b_1));\\ (min(a_1'b_1', a_3'b_1'), min(a_2^Lb_2^L, a_2^Mb_2^L), max(a_2^Lb_2^M, a_2^Mb_2^M), max(a_1'b_3', a_3'b_3'))\},\end{array} & \text{if } a_1' \leq a_1 \leq a_2^L \leq a_2^M \leq a_3 < 0 < a_3';\\[2ex]
\begin{array}{l}\{(min(a_1b_3, a_3b_1), min(a_2^Lb_2^M, a_2^Mb_2^M), max(a_2^Lb_2^L, a_2^Mb_2^L), max(a_1b_1, a_3b_3));\\ (min(a_1'b_3', a_3'b_1'), min(a_2^Lb_2^M, a_2^Mb_2^M), max(a_2^Lb_2^M, a_2^Mb_2^L), max(a_1'b_1', a_3'b_3'))\},\end{array} & \text{if } a_1' \leq a_1 \leq a_2^L < a_2^M < 0 < a_3 \leq a_3';\\[2ex]
\begin{array}{l}\{(min(a_1b_3, a_3b_1), min(a_2^Lb_2^L, a_2^Mb_2^M), max(a_2^Lb_2^M, a_2^Mb_2^L), max(a_1b_1, a_3b_3));\\ (min(a_1'b_3', a_3'b_1'), min(a_2^Lb_2^L, a_2^Mb_2^M), max(a_2^Lb_2^M, a_2^Mb_2^L), max(a_1'b_1', a_3'b_3'))\},\end{array} & \text{if } a_1' \leq a_1 \leq a_2^L < 0 < a_2^M \leq a_3 \leq a_3';\\[2ex]
\begin{array}{l}\{(min(a_1b_3, a_3b_1), min(a_2^Lb_2^L, a_2^Mb_2^L), max(a_2^Lb_2^M, a_2^Mb_2^M), max(a_1b_1, a_3b_3));\\ (min(a_1'b_3', a_3'b_1'), min(a_2^Lb_2^L, a_2^Mb_2^L), max(a_2^Lb_2^M, a_2^Mb_2^M), max(a_1'b_1', a_3'b_3'))\},\end{array} & \text{if } a_1' \leq a_1 < 0 < a_2^L \leq a_2^M \leq a_3 \leq a_3';\\[2ex]
\begin{array}{l}\{(min(a_1b_1, a_3b_1), min(a_2^Lb_2^L, a_2^Mb_2^L), max(a_2^Lb_2^M, a_2^Mb_2^M), max(a_1b_3, a_3b_3));\\ (min(a_1'b_3', a_3'b_1'), min(a_2^Lb_2^L, a_2^Mb_2^L), max(a_2^Lb_2^M, a_2^Mb_2^M), max(a_1'b_1', a_3'b_3'))\},\end{array} & \text{if } a_1' < 0 < a_1 \leq a_2^L \leq a_2^M \leq a_3 \leq a_3'.
\end{cases}$$

Remark 6.2 $min(a, b)$ represents minimum $(a, b) = \frac{a+b}{2} - |\frac{a-b}{2}|$ and $max(a, b)$ represents maximum $(a, b) = \frac{a+b}{2} + |\frac{a-b}{2}|$.

Remark 6.3 A trapezoidal intuitionistic fuzzy number $\tilde{A}^I = \{(a_1, a_2^L, a_2^M, a_3); (a_1', a_2^L, a_2^M, a_3')\}$ is said to be unrestricted trapezoidal intuitionistic fuzzy number if a_1' is a real number.

Remark 6.4 A trapezoidal intuitionistic fuzzy number $\tilde{A}^I = \{(a_1, a_2^L, a_2^M, a_3); (a_1', a_2^L, a_2^M, a_3')\}$ is said to be non-negative trapezoidal intuitionistic fuzzy number if $a_1' \geq 0$.

Remark 6.5 A trapezoidal intuitionistic fuzzy number $\tilde{A}^I = \{(a_1, a_2^L, a_2^M, a_3); (a_1', a_2^L, a_2^M, a_3')\}$ is said to be non-positive trapezoidal intuitionistic fuzzy number if $a_1' \leq 0$.

6.6 Proposed Mehar method to find the intuitionistic fuzzy optimal solution of intuitionistic fuzzy linear programming problems with trapezoidal intuitionistic fuzzy numbers

In this section, to resolve the flaws of the existing method [111], a new method (named as Mehar method) is proposed to find the intuitionistic fuzzy optimal solution of problem ($P_{6.1}$).

The steps of the proposed Mehar method are as follows:

Step 1 Substituting $\tilde{c}_j^I = \{(c_j, d_j^L, d_j^M, e_j); (c'_j, d_j^L, d_j^M, e'_j)\}$, $\tilde{x}_j^I = \{(x_j, y_j^L, y_j^M, z_j); (x'_j, y_j^L, y_j^M, z'_j)\}$, $\tilde{a}_{ij}^I = \{(f_{ij}, g_{ij}^L, g_{ij}^M, h_{ij}); (f'_{ij}, g_{ij}^L, g_{ij}^M, h'_{ij})\}$ and $\tilde{b}_i^I = \{(k_i, l_i^L, l_i^M, o_i); (k'_i, l_i^L, l_i^M, o'_i)\}$ in problem ($P_{6.1}$), it can be transformed into problem ($P_{6.8}$).

Maximize/Minimize $\left[\sum_{j=1}^{n} \{(c_j, d_j^L, d_j^M, e_j); (c'_j, d_j^L, d_j^M, e'_j)\}\{(x_j, y_j^L, y_j^M, z_j); (x'_j, y_j^L, y_j^M, z'_j)\}\right]$

Subject to ($P_{6.8}$)

$\sum_{j=1}^{n} \{(f_{ij}, g_{ij}^L, g_{ij}^M, h_{ij}); (f'_{ij}, g_{ij}^L, g_{ij}^M, h'_{ij})\}\{(x_j, y_j^L, y_j^M, z_j); (x'_j, y_j^L, y_j^M, z'_j)\} \preceq, \approx, \succeq$ $\{(k_i, l_i^L, l_i^M, o_i); (k'_i, l_i^L, l_i^M, o'_i)\}$,

$\{(x_j, y_j^L, y_j^M, z_j); (x'_j, y_j^L, y_j^M, z'_j)\} \succeq 0$.

Step 2 Assuming $\{(c_j, d_j^L, d_j^M, e_j); (c'_j, d_j^L, d_j^M, e'_j)\}\{(x_j, y_j^L, y_j^M, z_j); (x'_j, y_j^L, y_j^M, z'_j)\} = \{(p_j, q_j^L, q_j^M, r_j); (p'_j, q_j^L, q_j^M, r'_j)\}$, $\{(f_{ij}, g_{ij}^L, g_{ij}^M, h_{ij}); (f'_{ij}, g_{ij}^L, g_{ij}^M, h'_{ij})\}\{(x_j, y_j^L, y_j^M, z_j); (x'_j, y_j^L, y_j^M, z'_j)\} = \{(s_j, t_j^L, t_j^M, u_j); (s'_j, t_j^L, t_j^M, u'_j)\}$, the problem ($P_{6.8}$) can be transformed into problem ($P_{6.9}$).

Maximize/Minimize $\left[\sum_{j=1}^{n} \{(p_j, q_j^L, q_j^M, r_j); (p'_j, q_j^L, q_j^M, r'_j)\}\right]$

Subject to ($P_{6.9}$)

$\sum_{j=1}^{n} \{(s_j, t_j^L, t_j^M, u_j); (s'_j, t_j^L, t_j^M, u'_j)\} \preceq, \approx, \succeq \{(k_i, l_i^L, l_i^M, o_i); (k'_i, l_i^L, l_i^M, o'_i)\}$,

$\{(x_j, y_j^L, y_j^M, z_j); (x'_j, y_j^L, y_j^M, z'_j)\} \succeq 0$.

Step 3 Using Section 6.1.1, the problem $(P_{6.9})$ can be transformed into problem $(P_{6.10})$.

$$\text{Maximize/Minimize } \left[\{(\sum_{j=1}^{n} p_j, \sum_{j=1}^{n} q_j^L, \sum_{j=1}^{n} q_j^M, \sum_{j=1}^{n} r_j); (\sum_{j=1}^{n} p'_j, \sum_{j=1}^{n} q_j^L, \sum_{j=1}^{n} q_j^M, \sum_{j=1}^{n} r'_j)\}\right]$$

Subject to $(P_{6.10})$

$$\{(\sum_{j=1}^{n} s_j, \sum_{j=1}^{n} t_j^L, \sum_{j=1}^{n} t_j^M, \sum_{j=1}^{n} u_j); (\sum_{j=1}^{n} s'_j, \sum_{j=1}^{n} t_j^L, \sum_{j=1}^{n} t_j^M \sum_{j=1}^{n} u'_j)\} \preceq, \approx, \succeq \{(k_i, l_i^L, l_i^M, o_i); (k'_i, l_i^L, l_i^M, o'_i)\},$$

$$\{(x_j, y_j^L, y_j^M, z_j); (x'_j, y_j^L, y_j^M, z'_j)\} \succeq 0.$$

Step 4 Using Section 6.1.2, the problem $(P_{6.10})$ can be transformed into problem $(P_{6.11})$.

$$\text{Maximize/Minimize } \Re\left[\{(\sum_{j=1}^{n} p_j, \sum_{j=1}^{n} q_j^L, \sum_{j=1}^{n} q_j^M, \sum_{j=1}^{n} r_j); (\sum_{j=1}^{n} p'_j, \sum_{j=1}^{n} q_j^L, \sum_{j=1}^{n} q_j^M, \sum_{j=1}^{n} r'_j)\}\right]$$

Subject to $(P_{6.11})$

$$\Re\left[\{(\sum_{j=1}^{n} s_j, \sum_{j=1}^{n} t_j^L, \sum_{j=1}^{n} t_j^M, \sum_{j=1}^{n} u_j); (\sum_{j=1}^{n} s'_j, \sum_{j=1}^{n} t_j^L, \sum_{j=1}^{n} t_j^M, \sum_{j=1}^{n} u'_j)\}\right] \leq, =, \geq$$

$$\Re\left\{(k_i, l_i^L, l_i^M o_i); (k'_i, l_i^L, l_i^M, o'_i)\right\},$$

$$\Re\{(x_j, y_j^L, y_j^M, z_j); (x'_j, y_j^L, y_j^M, z'_j)\} \geq 0.$$

Step 5 Using Section 6.1.2, the problem $(P_{6.11})$ can be transformed into problem $(P_{6.12})$.

$$\text{Maximize/Minimize } \left[\frac{\sum_{j=1}^{n} p_j + 2\sum_{j=1}^{n} q_j^L + \sum_{j=1}^{n} r_j + \sum_{j=1}^{n} p'_j + 2\sum_{j=1}^{n} q_j^M + \sum_{j=1}^{n} r'_j}{8}\right]$$

Subject to $(P_{6.12})$

$$\left(\frac{\sum_{j=1}^{n} s_j + 2\sum_{j=1}^{n} t_j^L + \sum_{j=1}^{n} u_j + \sum_{j=1}^{n} s'_j + 2\sum_{j=1}^{n} t_j^M + \sum_{j=1}^{n} u'_j}{8}\right) \leq, =, \geq \frac{(k_i + 2l_i^L + o_i) + (k'_i + 2l_i^M + o'_i)}{8},$$

$$\frac{(x_j + 2y_j^L + z_j) + (x'_j + 2y_j^M + z'_j)}{8} \geq 0,$$

$$x_j - x'_j \geq 0, y_j^M - y_j^L \geq 0, y_j^L - x_j \geq 0, z_j - y_j^M \geq 0, z'_j - z_j \geq 0,$$

$x_j, x'_j, y_j^L, y_j^M, z_j, z'_j$ are unrestricted variables.

Step 6 Solve the problem $(P_{6.12})$ by using an appropriate existing method to find the values of $x_j, x'_j, y_j^L, y_j^M, z_j, z'_j$ and put these values in $\tilde{x}_j^I = \{(x_j, y_j^L, y_j^M, z_j); (x'_j, y_j^L, y_j^M, z'_j)\}$ to find the intuitionistic fuzzy optimal solution.

Step 7 Find the intuitionistic fuzzy optimal value by putting the values of $\tilde{x}_j^I$ in $\sum\limits_{j=1}^{n}(\tilde{c}_j^I\tilde{x}_j^I)$.

6.7 Intuitionistic fuzzy optimal solution by proposed Mehar method

In this section, to illustrate the proposed Mehar method, the following examples are solved by proposed Mehar method.

Example 6.1 [111, Section 6, pp. 3472]

Using the Mehar method, proposed in Section 6.6, the intuitionistic fuzzy optimal solution of the problem $(P_{6.4})$ can be obtained as follows:

Step 1 Substituting the values of $\tilde{x}_1^I = \{(x_1, y_1, z_1); (x_1', y_1, z_1')\}$ and $\tilde{x}_2^I = \{(x_2, y_2, z_2); (x_2', y_2, z_2')\}$ in the problem $(P_{6.4})$, it can be transformed into problem $(P_{6.13})$.

Maximize

$[\{(4, 5, 6); (4, 5, 6.1)\}\{(x_1, y_1, z_1); (x_1', y_1, z_1')\}+\{(2.5, 3, 3.2); (2, 3, 3.5)\}\{(x_2, y_2, z_2); (x_2', y_2, z_2')\}]$

Subject to $\qquad (P_{6.13})$

$\{(3.5, 4, 4.1); (3, 4, 5)\}\{(x_1, y_1, z_1); (x_1', y_1, z_1')\}+\{(2.5, 3, 3.5); (2.4, 3, 3.6)\}\{(x_2, y_2, z_2); (x_2', y_2, z_2')\}$

$\preceq \{(11, 12, 13); (11, 12, 14)\}$,

$\{(0.8, 1, 2); (0.5, 1, 2.1)\}\{(x_1, y_1, z_1); (x_1', y_1, z_1')\}+\{(2.8, 3, 3.2); (2.5, 3, 3.2)\}\{(x_2, y_2, z_2); (x_2', y_2, z_2')\}$

$\preceq \{(5.5, 6, 7.5); (5, 6, 8.1)\}$,

$\{(x_1, y_1, z_1); (x_1', y_1, z_1')\}, \{(x_2, y_2, z_2); (x_2', y_2, z_2')\} \succeq 0$.

Step 2 Using Step 2 of the proposed Mehar method, the problem $(P_{6.13})$ can be transformed into problem $(P_{6.14})$.

Maximize $[\{(min(4x_1, 6x_1), 5y_1, max(4z_1, 6z_1)); (min(4x_1', 6.1x_1'), 5y_1, max(4z_1', 6.1z_1'))\}+$

$\{(min(2.5x_2, 3.2x_2), 3y_2, max(2.5z_2, 3.2z_2)); (min(2x_2', 3.5x_2'), 3y_2, max(2z_2', 3.5z_2'))\}]$

Subject to $(P_{6.14})$

$\{(min(3.5x_1, 4.1x_1), 4y_1, max(3.5z_1, 4.1z_1)); (min(3x'_1, 5x'_1), 4y_1, max(3z'_1, 5z'_1))\}+\{(min(2.5x_2,$

$3.5x_2), 3y_2, max(2.5z_2, 3.5z_2)); (min(2.4x'_2, 3.6x'_2), 3y_2, max(2.4z'_2, 3.6z'_2))\} \preceq$

$\{(11, 12, 13); (11, 12, 14)\},$

$\{(min(0.8x_1, 2x_1), y_1, max(0.8z_1, 2z_1)); (min(0.5x'_1, 2.1x'_1), y_1, max(0.5z'_1, 2.1z'_1))\}+\{(min(2.8x_2,$

$3.2x_2), 3y_2, max(2.8z_2, 3.2z_2)); (min(2.5x'_2, 3.2x'_2), 3y_2, max(2.5z'_2, 3.2z'_2))\} \preceq$

$\{(5.5, 6, 7.5); (5, 6, 8.1)\},$

$\{(x_1, y_1, z_1); (x'_1, y_1, z'_1)\}, \{(x_2, y_2, z_2); (x'_2, y_2, z'_2)\} \succeq 0.$

Step 3 The problem $(P_{6.14})$ can be transformed into problem $(P_{6.15})$.

Maximize

$\Big[\{(\frac{4x_1+6x_1}{2} - |\frac{4x_1-6x_1}{2}|, 5y_1, \frac{4z_1+6z_1}{2} + |\frac{4z_1-6z_1}{2}|); (\frac{4x'_1+6.1x'_1}{2} - |\frac{4x'_1-6.1x'_1}{2}|, 5y_1, \frac{4z'_1+6.1z'_1}{2} + |\frac{4z'_1-6.1z'_1}{2}|)\}$

$+\{(\frac{2.5x_2+3.2x_2}{2} - |\frac{2.5x_2-3.2x_2}{2}|, 3y_2, \frac{2.5z_2+3.2z_2}{2} + |\frac{2.5z_2-3.2z_2}{2}|); (\frac{2x'_2+3.5x'_2}{2} - |\frac{2x'_2-3.5x'_2}{2}|, 3y_2, \frac{2z'_2+3.5z'_2}{2}+$

$|\frac{2z'_2-3.5z'_2}{2}|)\}\Big]$

Subject to $(P_{6.15})$

$\{(\frac{3.5x_1+4.1x_1}{2}-|\frac{3.5x_1-4.1x_1}{2}|, 4y_1, \frac{3.5z_1+4.1z_1}{2}+|\frac{3.5z_1-4.1z_1}{2}|); (\frac{3x'_1+5x'_1}{2}-|\frac{3x'_1-5x'_1}{2}|, 4y_1, \frac{3z'_1+5z'_1}{2}+$

$|\frac{3z'_1-5z'_1}{2}|)\}+\{(\frac{2.5x_2+3.5x_2}{2}-|\frac{2.5x_2-3.5x_2}{2}|, 3y_2, \frac{2.5z_2+3.5z_2}{2}+|\frac{2.5z_2-3.5z_2}{2}|); (\frac{2.4x'_2+3.6x'_2}{2}-|\frac{2.4x'_2-3.6x'_2}{2}|,$

$3y_2, \frac{2.4z'_2+3.6z'_2}{2} + |\frac{2.4z'_2-3.6z'_2}{2}|)\} \preceq \{(11, 12, 13); (11, 12, 14)\},$

$\{(\frac{0.8x_1+2x_1}{2}-|\frac{0.8x_1-2x_1}{2}|, y_1, \frac{0.8z_1+2z_1}{2}+|\frac{0.8z_1-2z_1}{2}|); (\frac{0.5x'_1+2.1x'_1}{2}-|\frac{0.5x'_1-2.1x'_1}{2}|, y_1, \frac{0.5z'_1+2.1z'_1}{2}+$

$|\frac{0.5z'_1-2.1z'_1}{2}|)\} + \{(\frac{2.8x_2+3.2x_2}{2} - |\frac{2.8x_2-3.2x_2}{2}|, 3y_2, \frac{2.8z_2+3.2z_2}{2} + |\frac{2.8z_2-3.2z_2}{2}|); (\frac{2.5x'_2+3.2x'_2}{2} -$

$|\frac{2.5x'_2-3.2x'_2}{2}|, 3y_2, \frac{2.5z'_2+3.2z'_2}{2} + |\frac{2.5z'_2-3.2z'_2}{2}|)\} \preceq \{(5.5, 6, 7.5); (5, 6, 8.1)\},$

$\{(x_1, y_1, z_1); (x'_1, y_1, z'_1)\}, \{(x_2, y_2, z_2); (x'_2, y_2, z'_2)\} \succeq 0.$

Step 4 The problem $(P_{6.15})$ can be transformed into problem $(P_{6.16})$.

Maximize $\big[\{(5x_1 - \frac{1}{2}|x_1|, 5y_1, 5z_1 + \frac{1}{2}|z_1|); (\frac{10.1}{2}x'_1 - \frac{2.1}{2}|x'_1|, 5y_1, \frac{10.1}{2}z'_1 + \frac{2.1}{2}|z'_1|)\}+$

$\{(\frac{5.7}{2}x_2 - \frac{0.7}{2}|x_2|, 3y_2, \frac{5.7}{2}z_2 + \frac{0.7}{2}|z_2|); (\frac{5.5}{2}x'_2 - \frac{1.5}{2}|x'_2|, 3y_2, \frac{5.5}{2}z'_2 + \frac{1.5}{2}|z'_2|)\}\big]$

Subject to $(P_{6.16})$

$\{(3.8x_1-0.3|x_1|, 4y_1, 3.8z_1+0.3|z_1|); (4x'_1-|x'_1|, 4y_1, 4z'_1+|z'_1|)\}+\{(3x_2-\frac{1}{2}|x_2|, 3y_2, 3z_2+\frac{1}{2}|z_2|); (3x'_2-0.6|x'_2|, 3y_2, 3z'_2+0.6|z'_2|)\} \preceq \{(11,12,13); (11,12,14)\}$,

$\{(1.4x_1-0.6|x_1|, y_1, 1.4z_1+0.6|z_1|); (1.3x'_1-0.8|x'_1|, y_1, 1.3z'_1+0.8|z'_1|)\}+\{(3x_2-0.2|x_2|, 3y_2, 3z_2+0.2|z_2|)); (\frac{5.7}{2}x'_2-\frac{0.7}{2}|x'_2|, 3y_2, \frac{5.7}{2}z'_2+\frac{0.7}{2}|z'_2|)\} \preceq \{(5.5,6,7.5); (5,6,8.1)\}$,

$\{(x_1, y_1, z_1); (x'_1, y_1, z'_1)\}, \{(x_2, y_2, z_2); (x'_2, y_2, z'_2)\} \succeq 0.$

Step 5 The problem $(P_{6.16})$ can be transformed into problem $(P_{6.17})$.

Maximize

$\big[\{(5x_1-\frac{1}{2}|x_1|+\frac{5.7}{2}x_2-\frac{0.7}{2}|x_2|, 5y_1+3y_2, 5z_1+\frac{1}{2}|z_1|+\frac{5.7}{2}z_2+\frac{0.7}{2}|z_2|); (\frac{10.1}{2}x'_1-\frac{2.1}{2}|x'_1|+\frac{5.5}{2}x'_2-\frac{1.5}{2}|x'_2|, 5y_1+3y_2, \frac{10.1}{2}z'_1+\frac{2.1}{2}|z'_1|+\frac{5.5}{2}z'_2+\frac{1.5}{2}|z'_2|)\}\big]$

Subject to $(P_{6.17})$

$\{(3.8x_1-0.3|x_1|+3x_2-\frac{1}{2}|x_2|, 4y_1+3y_2, 3.8z_1+0.3|z_1|+3z_2+\frac{1}{2}|z_2|); (4x'_1-|x'_1|+3x'_2-0.6|x'_2|, 4y_1+3y_2, 4z'_1+|z'_1|+3z'_2+0.6|z'_2|)\} \preceq \{(11,12,13); (11,12,14)\}$,

$\{(1.4x_1-0.6|x_1|+3x_2-0.2|x_2|, y_1+3y_2, 1.4z_1+0.6|z_1|+3z_2+0.2|z_2|); (1.3x'_1-0.8|x'_1|+\frac{5.7}{2}x'_2-\frac{0.7}{2}|x'_2|, y_1+3y_2, 1.3z'_1+0.8|z'_1|+\frac{5.7}{2}z'_2+\frac{0.7}{2}|z'_2|)\} \preceq \{(5.5,6,7.5); (5,6,8.1)\}$,

$\{(x_1, y_1, z_1); (x'_1, y_1, z'_1)\}, \{(x_2, y_2, z_2); (x'_2, y_2, z'_2)\} \succeq 0.$

Step 6 The problem $(P_{6.17})$ can be transformed into problem $(P_{6.18})$.

Maximize

$\frac{1}{8}[5x_1-\frac{1}{2}|x_1|+\frac{5.7}{2}x_2-\frac{0.7}{2}|x_2|+10y_1+6y_2+5z_1+\frac{1}{2}|z_1|+\frac{5.7}{2}z_2+\frac{0.7}{2}|z_2|+\frac{10.1}{2}x'_1-\frac{2.1}{2}|x'_1|+\frac{5.5}{2}x'_2-\frac{1.5}{2}|x'_2|+10y_1+6y_2+\frac{10.1}{2}z'_1+\frac{2.1}{2}|z'_1|+\frac{5.5}{2}z'_2+\frac{1.5}{2}|z'_2|]$

Subject to $(P_{6.18})$

$\frac{1}{8}[3.8x_1-0.3|x_1|+3x_2-\frac{1}{2}|x_2|+8y_1+6y_2+3.8z_1+0.3|z_1|+3z_2+\frac{1}{2}|z_2|+4x'_1-|x'_1|+3x'_2-0.6|x'_2|+8y_1+6y_2+4z'_1+|z'_1|+3z'_2+0.6|z'_2|] \leq \frac{97}{8}$,

$\frac{1}{8}[1.4x_1-0.6|x_1|+3x_2-0.2|x_2|+2y_1+6y_2+1.4z_1+0.6|z_1|+3z_2+0.2|z_2|+1.3x'_1-$

$0.8|x'_1| + \frac{5.7}{2}x'_2 - \frac{0.7}{2}|x'_2| + 2y_1 + 6y_2 + 1.3z'_1 + 0.8|z'_1| + \frac{5.7}{2}z'_2 + \frac{0.7}{2}|z'_2|] \leq \frac{50.1}{8}$,

$\frac{(x_1+2y_1+z_1)+(x'_1+2y_1+z'_1)}{8} \geq 0$,

$\frac{(x_2+2y_2+z_2)+(x'_2+2y_2+z'_2)}{8} \geq 0$,

$x_1 - x'_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, z'_1 - z_1 \geq 0$,

$x_2 - x'_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, z'_2 - z_2 \geq 0$,

$x_1, x'_1, y_1, z_1, z'_1, x_2, x'_2, y_2, z_2, z'_2$ are unrestricted variables.

Step 7 Since, on solving the problem $(P_{6.18})$, no feasible solution is obtained. So, there does not exist any intuitionistic feasible solution and hence there does not exist intuitionistic fuzzy optimal solution for problem $(P_{6.4})$.

Example 6.2

Maximize $[\{(2,3,4,5);(1,3,4,6)\}\tilde{x}^I_1 + \{(3,6,7,9);(2,6,7,10)\}\tilde{x}^I_2]$

Subject to $(P_{6.19})$

$\{(1,2,3,4);(0,2,3,5)\}\tilde{x}^I_1 + \{(0,2,3,5);(0,2,3,6)\}\tilde{x}^I_2 \preceq \{(-4,6,8,37);(-10,6,8,56)\}$,

$\{(5,6,7,8);(3,6,7,10)\}\tilde{x}^I_1 + \{(3,6,7,9);(1,6,7,10)\}\tilde{x}^I_2 \preceq \{(1,18,42,69);(-18,18,42,100)\}$,

$\tilde{x}^I_1, \tilde{x}^I_2 \succeq 0$.

Solution The intuitionistic fuzzy optimal solution of the problem $(P_{6.19})$ can be obtained as follows.

Step 1 Substituting the values of $\tilde{x}^I_1 = \{(x_1, y^L_1, y^M_1, z_1);(x'_1, y^L_1, y^M_1, z'_1)\}$ and $\tilde{x}^I_2 = \{(x_2, y^L_2, y^M_2, z_2);(x'_2, y^L_2, y^M_2, z'_2)\}$ in the problem $(P_{6.19})$, it can be transformed into problem $(P_{6.20})$.

Maximize $[\{(2,3,4,5);(1,3,4,6)\}\{(x_1, y^L_1, y^M_1, z_1);(x'_1, y^L_1, y^M_1, z'_1)\} + \{(3,6,7,9);(2,6,7,10)\}$
$\{(x_2, y^L_2, y^M_2, z_2);(x'_2, y^L_2, y^M_2, z'_2)\}]$

Subject to $(P_{6.20})$

$\{(1,2,3,4);(0,2,3,5)\}\{(x_1, y^L_1, y^M_1, z_1);(x'_1, y^L_1, y^M_1, z'_1)\} + \{(0,2,3,5);(0,2,3,6)\}\{(x_2, y^L_2, y^M_2, z_2);$

$(x'_2, y^L_2, y^M_2, z'_2)\} \preceq \{(-4, 6, 8, 37); (-10, 6, 8, 56)\},$

$\{(5, 6, 7, 8); (3, 6, 7, 10)\}\{(x_1, y^L_1, y^M_1, z_1); (x'_1, y^L_1, y^M_1, z'_1)\}+\{(3, 6, 7, 9); (1, 6, 7, 10)\}\{(x_2, y^L_2, y^M_2, z_2);$

$(x'_2, y^L_2, y^M_2, z'_2)\} \preceq \{(1, 18, 42, 69); (-18, 18, 42, 100)\},$

$\{(x_1, y^L_1, y^M_1, z_1); (x'_1, y^L_1, y^M_1, z'_1)\}, \{(x_2, y^L_2, y^M_2, z_2); (x'_2, y^L_2, y^M_2, z'_2)\} \succeq 0.$

Step 2 Using Section 6.1.1 and Step 2 of the proposed method, the problem $(P_{6.20})$ can be transformed into problem $(P_{6.21})$.

Maximize $\big[\{(min(2x_1, 5x_1), min(3y^L_1, 4y^L_1), max(3y^M_1, 4y^M_1), max(2z_1, 5z_1)); (min(x'_1, 6x'_1), min(3y^L_1, 4y^L_1),$

$max(3y^M_1, 4y^M_1), max(z'_1, 6z'_1))\} + \{(min(3x_2, 9x_2), min(6y^L_2, 7y^L_2), max(6y^M_2, 7y^M_2), max(3z_2, 9z_2));$

$(min(2x'_2, 10x'_2), min(6y^L_2, 7y^L_2), max(6y^M_2, 7y^M_2), max(2z'_2, 10z'_2))\}\big]$

Subject to $(P_{6.21})$

$\{(min(x_1, 4x_1), min(2y^L_1, 3y^L_1), max(2y^M_1, 3y^M_1), max(z_1, 4z_1)); (min(0x'_1, 5x'_1), min(2y^L_1, 3y^L_1),$

$max(2y^M_1, 3y^M_1), max(0z'_1, 5z'_1))\}+\{(min(0x_2, 5x_2), min(2y^L_2, 3y^L_2), max(2y^M_2, 3y^M_2), max(0z_2, 5z_2));$

$(min(0x'_2, 6x'_2), min(2y^L_2, 3y^L_2), max(2y^M_2, 3y^M_2), max(0z'_2, 6z'_2))\} \preceq \{(-4, 6, 8, 37); (-10, 6, 8, 56)\},$

$\{(min(5x_1, 8x_1), min(6y^L_1, 7y^L_1), max(6y^M_1, 7y^M_1), max(5z_1, 8z_1)); (min(3x'_1, 10x'_1), min(6y^L_1, 7y^L_1),$

$max(6y^M_1, 7y^M_1), max(3z'_1, 10z'_1))\}+\{(min(3x_2, 9x_2), min(6y^L_2, 7y^L_2), max(6y^M_2, 7y^M_2), max(3z_2, 9z_2));$

$(min(x'_2, 10x'_2), min(6y^L_2, 7y^L_2), max(6y^M_2, 7y^M_2), max(z'_2, 10z'_2))\} \preceq \{(1, 18, 42, 69); (-18, 18, 42, 100)\},$

$\{(x_1, y^L_1, y^M_1, z_1); (x'_1, y^L_1, y^M_1, z'_1)\}, \{(x_2, y^L_2, y^M_2, z_2); (x'_2, y^L_2, y^M_2, z'_2)\} \succeq 0.$

Step 3 The problem $(P_{6.21})$ can be transformed into problem $(P_{6.22})$.

Maximize

$\big[\{(\frac{7}{2}x_1 - \frac{3}{2}|x_1|, \frac{7}{2}y^L_1 - \frac{1}{2}|y^L_1|, \frac{7}{2}y^M_1 + \frac{1}{2}|y^M_1|, \frac{7}{2}z_1 + \frac{3}{2}|z_1|); (\frac{7}{2}x'_1 - \frac{5}{2}|x'_1|, \frac{7}{2}y^L_1 - \frac{1}{2}|y^L_1|, \frac{7}{2}y^M_1 + \frac{1}{2}|y^M_1|, \frac{7}{2}z'_1 + \frac{5}{2}$

$|z'_1|)\} + \{(6x_2 - 3|x_2|, \frac{13}{2}y^L_2 - \frac{1}{2}|y^L_2|, \frac{13}{2}y^M_2 + \frac{1}{2}|y^M_2|, 6z_2 + 3|z_2|); (6x'_2 - 4|x'_2|, \frac{13}{2}y^L_2 - \frac{1}{2}|y^L_2|, \frac{13}{2}y^M_2 + \frac{1}{2}|y^M_2|,$

$6z'_2 + 4|z'_2|)\}\big]$

Subject to $(P_{6.22})$

$\{(\frac{5}{2}x_1 - \frac{3}{2}|x_1|, \frac{5}{2}y^L_1 - \frac{1}{2}|y^L_1|, \frac{5}{2}y^M_1 + \frac{1}{2}|y^M_1|, \frac{5}{2}z_1 + \frac{3}{2}|z_1|); (\frac{5}{2}x'_1 - \frac{5}{2}|x'_1|, \frac{5}{2}y^L_1 - \frac{1}{2}|y^L_1|, \frac{5}{2}y^M_1 +$

$\frac{1}{2}|y_1^M|, \frac{5}{2}z_1' + \frac{5}{2}|z_1'|)\} + \{(\frac{5}{2}x_2 - \frac{5}{2}|x_2|, \frac{5}{2}y_2^L - \frac{1}{2}|y_2^L|, \frac{5}{2}y_2^M + \frac{1}{2}|y_2^M|, \frac{5}{2}z_2 + \frac{5}{2}|z_2|); (3x_2' - 3|x_2'|, \frac{5}{2}y_2^L - \frac{1}{2}|y_2^L|, \frac{5}{2}y_2^M + \frac{1}{2}|y_2^M|, 3z_2' + 3|z_2'|)\} \preceq \{(-4, 6, 8, 37); (-10, 6, 8, 56)\},$

$\{(\frac{13}{2}x_1 - \frac{3}{2}|x_1|, \frac{13}{2}y_1^L - \frac{1}{2}|y_1^L|, \frac{13}{2}y_1^M + \frac{1}{2}|y_1^M|, \frac{13}{2}z_1 + \frac{3}{2}|z_1|); (\frac{13}{2}x_1' - \frac{7}{2}|x_1'|, \frac{13}{2}y_1^L - \frac{1}{2}|y_1^L|, \frac{13}{2}y_1^M + \frac{1}{2}|y_1^M|, \frac{13}{2}z_1' + \frac{7}{2}|z_1'|)\} + \{(6x_2 - 3|x_2|, \frac{13}{2}y_2^L - \frac{1}{2}|y_2^L|, \frac{13}{2}y_2^M + \frac{1}{2}|y_2^M|, 6z_2 + 3|z_2|); (\frac{11}{2}x_2' - \frac{9}{2}|x_2'|, \frac{13}{2}y_2^L - \frac{1}{2}|y_2^L|, \frac{13}{2}y_2^M + \frac{1}{2}|y_2^M|, \frac{11}{2}z_2' + \frac{9}{2}|z_2'|)\} \preceq \{(1, 18, 42, 69); (-18, 18, 42, 100)\},$

$\{(x_1, y_1^L, y_1^M, z_1); (x_1', y_1^L, y_1^M, z_1')\}, \{(x_2, y_2^L, y_2^M, z_2); (x_2', y_2^L, y_2^M, z_2')\} \succeq 0.$

Step 4 The problem $(P_{6.22})$ can be transformed into problem $(P_{6.23})$.

Maximize $[\{(\frac{7}{2}x_1 - \frac{3}{2}|x_1| + 6x_2 - 3|x_2|, \frac{7}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{13}{2}y_2^L - \frac{1}{2}|y_2^L|, \frac{7}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{13}{2}y_2^M + \frac{1}{2}|y_2^M|, \frac{7}{2}z_1 + \frac{3}{2}|z_1| + 6z_2 + 3|z_2|); (\frac{7}{2}x_1' - \frac{5}{2}|x_1'| + 6x_2' - 4|x_2'|, \frac{7}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{13}{2}y_2^L - \frac{1}{2}|y_2^L|, \frac{7}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{13}{2}y_2^M + \frac{1}{2}|y_2^M|, \frac{7}{2}z_1' + \frac{5}{2}|z_1'| + 6z_2' + 4|z_2'|)\}]$

Subject to $(P_{6.23})$

$\{(\frac{5}{2}x_1 - \frac{3}{2}|x_1| + \frac{5}{2}x_2 - \frac{5}{2}|x_2|, \frac{5}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{5}{2}y_2^L - \frac{1}{2}|y_2^L|, \frac{5}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{5}{2}y_2^M + \frac{1}{2}|y_2^M|, \frac{5}{2}z_1 + \frac{3}{2}|z_1| + \frac{5}{2}z_2 + \frac{5}{2}|z_2|); (\frac{5}{2}x_1' - \frac{5}{2}|x_1'| + 3x_2' - 3|x_2'|, \frac{5}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{5}{2}y_2^L - \frac{1}{2}|y_2^L|, \frac{5}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{5}{2}y_2^M + \frac{1}{2}|y_2^M|, \frac{5}{2}z_1' + \frac{5}{2}|z_1'| + 3z_2' + 3|z_2'|)\} \preceq \{(-4, 6, 8, 37); (-10, 6, 8, 56)\},$

$\{(\frac{13}{2}x_1 - \frac{3}{2}|x_1| + 6x_2 - 3|x_2|, \frac{13}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{13}{2}y_2^L - \frac{1}{2}|y_2^L|, \frac{13}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{13}{2}y_2^M + \frac{1}{2}|y_2^M|, \frac{13}{2}z_1 + \frac{3}{2}|z_1| + 6z_2 + 3|z_2|); (\frac{13}{2}x_1' - \frac{7}{2}|x_1'| + \frac{11}{2}x_2' - \frac{9}{2}|x_2'|, \frac{13}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{13}{2}y_2^L - \frac{1}{2}|y_2^L|, \frac{13}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{13}{2}y_2^M + \frac{1}{2}|y_2^M|, \frac{13}{2}z_1' + \frac{7}{2}|z_1'| + \frac{11}{2}z_2' + \frac{9}{2}|z_2'|)\} \preceq \{(1, 18, 42, 69); (-18, 18, 42, 100)\},$

$\{(x_1, y_1^L, y_1^M, z_1); (x_1', y_1^L, y_1^M, z_1')\}, \{(x_2, y_2^L, y_2^M, z_2); (x_2', y_2^L, y_2^M, z_2')\} \succeq 0.$

Step 5 The problem $(P_{6.23})$ can be transformed into problem $(P_{6.24})$.

Maximize

$\frac{1}{8}\big(\frac{7}{2}x_1 - \frac{3}{2}|x_1| + 6x_2 - 3|x_2| + \frac{7}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{13}{2}y_2^L - \frac{1}{2}|y_2^L| + \frac{7}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{13}{2}y_2^M + \frac{1}{2}|y_2^M| + \frac{7}{2}z_1 + \frac{3}{2}|z_1| + 6z_2 + 3|z_2| + \frac{7}{2}x_1' - \frac{5}{2}|x_1'| + 6x_2' - 4|x_2'| + \frac{7}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{13}{2}y_2^L - \frac{1}{2}|y_2^L| + \frac{7}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{13}{2}y_2^M + \frac{1}{2}|y_2^M| + \frac{7}{2}z_1' + \frac{5}{2}|z_1'| + 6z_2' + 4|z_2'|\big)$

Subject to $(P_{6.24})$

$\frac{1}{8}\left(\frac{5}{2}x_1 - \frac{3}{2}|x_1| + \frac{5}{2}x_2 - \frac{5}{2}|x_2| + \frac{5}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{5}{2}y_2^L - \frac{1}{2}|y_2^L| + \frac{5}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{5}{2}y_2^M + \frac{1}{2}|y_2^M| + \frac{5}{2}z_1 + \right.$

$\frac{3}{2}|z_1| + \frac{5}{2}z_2 + \frac{5}{2}|z_2| + \frac{5}{2}x_1' - \frac{5}{2}|x_1'| + 3x_2' - 3|x_2'| + \frac{5}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{5}{2}y_2^L - \frac{1}{2}|y_2^L| + \frac{5}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{5}{2}y_2^M$

$\left. + \frac{1}{2}|y_2^M| + \frac{5}{2}z_1' + \frac{5}{2}|z_1'| + 3z_2' + 3|z_2'|\right) \leq \frac{107}{8}$,

$\frac{1}{8}\left(\frac{13}{2}x_1 - \frac{3}{2}|x_1| + 6x_2 - 3|x_2| + \frac{13}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{13}{2}y_2^L - \frac{1}{2}|y_2^L| + \frac{13}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{13}{2}y_2^M + \frac{1}{2}|y_2^M| + \frac{13}{2}z_1 + \right.$

$\frac{3}{2}|z_1| + 6z_2 + 3|z_2| + \frac{13}{2}x_1' - \frac{7}{2}|x_1'| + \frac{11}{2}x_2' - \frac{9}{2}|x_2'| + \frac{13}{2}y_1^L - \frac{1}{2}|y_1^L| + \frac{13}{2}y_2^L - \frac{1}{2}|y_2^L| + \frac{13}{2}y_1^M + \frac{1}{2}|y_1^M| + \frac{13}{2}y_2^M +$

$\left. \frac{1}{2}|y_2^M| + \frac{13}{2}z_1' + \frac{7}{2}|z_1'| + \frac{11}{2}z_2' + \frac{9}{2}|z_2'|\right) \leq 34$,

$\frac{1}{8}(x_1 + 2(y_1^L + y_1^M) + z_1 + x_1' + z_1') \geq 0$,

$\frac{1}{8}(x_2 + 2(y_2^L + y_2^M) + z_2 + x_2' + z_2') \geq 0$,

$x_1 - x_1' \geq 0, y_1^L - x_1 \geq 0, y_1^M - y_1^L \geq 0, z_1 - y_1^M \geq 0, z_1' - z_1 \geq 0$,

$x_2 - x_2' \geq 0, y_2^L - x_2 \geq 0, y_2^M - y_2^L \geq 0, z_2 - y_2^M \geq 0, z_2' - z_2 \geq 0$,

$x_1, x_1', y_1^L, y_1^M, z_1, z_1', x_2, x_2', y_2^L, y_2^M, z_2, z_2'$ are unrestricted variables.

Step 7 On solving the problem $(P_{6.24})$, the obtained values of $x_1, x_1', y_1^L, y_1^M, z_1, z_1'$ are $-1, -2, 0, 2, 3, 4$ and $x_2, x_2', y_2^L, y_2^M, z_2, z_2'$ are $3, 2, 3, 4, 5, 6$ respectively. Hence, the obtained intuitionistic fuzzy optimal solution of problem $(P_{6.19})$ is $\tilde{x}_1^I = \{(-1, 0, 2, 3); (-2, 0, 2, 4)\}$ and $\tilde{x}_2^I = \{(3, 3, 4, 5); (2, 3, 4, 6)\}$.

Step 8 Putting the values of $\tilde{x}_1^I$ and $\tilde{x}_2^I$ in $[\{(2, 3, 4, 5); (1, 3, 4, 6)\}\tilde{x}_1^I + \{(3, 6, 7, 9); (2, 6, 7, 10)\}\tilde{x}_2^I]$, the obtained intuitionistic fuzzy optimal value of problem $(P_{6.19})$ is $\{(4, 18, 36, 60); (-8, 18, 36, 84)\}$.

6.8 Conclusions

In this chapter, flaws in existing method [111], are pointed out and a new method is proposed to solve intuitionistic fully fuzzy linear programming problems with trapezoidal intuitionistic fuzzy numbers.

Chapter 7

A NOTE ON "SOLVING INTUITIONISTIC FUZZY LINEAR PROGRAMMING PROBLEMS BY RANKING FUNCTION"

Suresh et al. [143] proposed the ranking function for comparing triangular intuitionistic fuzzy numbers and applied this ranking function to solve different types of intuitionistic fuzzy linear programming problems.

In this chapter, it is pointed out that the ranking function, proposed by authors, is not valid. Hence, the results of intuitionistic fuzzy linear programming problems, obtained by using this ranking function, are also not valid. Further, the exact ranking function is obtained by modifying existing ranking function and using the exact ranking function, the exact results of intuitionistic fuzzy linear programming problems, considered by Suresh et al. [143], are obtained.

7.1 Applicability of existing method

Suresh et al. [143, Definition 4.2, pp. 3084] proposed a ranking function (named as Mag) for comparing triangular intuitionistic fuzzy numbers and used it to find

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the solution of the following types of intuitionistic fuzzy linear programming problems.

(i) Intuitionistic fuzzy linear programming problems ($P_{7.1}$) in which the coefficients of the variables in objective function are represented by triangular intuitionistic fuzzy numbers whereas all other variables and parameters are represented by real numbers.

Maximize/Minimize $\left[\tilde{z} \approx \sum\limits_{j=1}^{n} \tilde{c}_j^I x_j\right]$

Subject to $\qquad\qquad (P_{7.1})$

$\sum\limits_{j=1}^{n} a_{ij} x_j \leq, =, \geq b_i; \quad i = 1, 2, ..., m,$

$x_j \geq 0; \quad j = 1, 2, ..., n,$

where $\tilde{c}_j^I = \{(\underline{c}_j^{\mu}, c_j, \overline{c}_j^{\mu}; w_{\tilde{c}_j}), (\underline{c}_j^{\nu}, c_j, \overline{c}_j^{\nu}; u_{\tilde{c}_j})\}$ is a triangular intuitionistic fuzzy number.

(ii) Intuitionistic fuzzy linear programming problems ($P_{7.2}$) in which the variables and coefficients of the variables in objective function are represented by real numbers whereas coefficients of the variables in constraints and right hand side vector is represented by triangular intuitionistic fuzzy numbers.

Maximize/Minimize $\left[z = \sum\limits_{j=1}^{n} c_j x_j\right]$

Subject to $\qquad\qquad (P_{7.2})$

$\sum\limits_{j=1}^{n} \tilde{a}_{ij}^I x_j \preceq, \approx, \succeq \widetilde{b_i^I}; \quad i = 1, 2, ..., m,$

$x_j \geq 0; \quad j = 1, 2, ..., n,$

where $\tilde{a}_{ij}^I = \{(\underline{a}_{ij}^{\mu}, a_{ij}, \overline{a}_{ij}^{\mu}; w_{\tilde{a}_{ij}}), (\underline{a}_{ij}^{\nu}, a_{ij}, \overline{a}_{ij}^{\nu}; u_{\tilde{a}_{ij}})\}, \widetilde{b_i^I} = \{(\underline{b}_i^{\mu}, b_i, \overline{b}_i^{\mu}; w_{\tilde{b}_i}), (\underline{b}_i^{\nu}, b_i, \overline{b}_i^{\nu}; u_{\tilde{b}_i})\}$ are triangular intuitionistic fuzzy numbers.

(iii) Intuitionistic fuzzy linear programming problems ($P_{7.3}$) in which only variables are represented by real numbers whereas all other parameters are represented by triangular intuitionistic fuzzy numbers.

Maximize/Minimize $\left[z \approx \sum\limits_{j=1}^{n} \tilde{c}_j^I x_j\right]$

Subject to $\qquad (P_{7.3})$

$\sum\limits_{j=1}^{n} \tilde{a}_{ij}^I x_j \preceq, \approx, \succeq \tilde{b}_i^I; \quad i = 1, 2, ..., m,$

$x_j \geq 0; \quad j = 1, 2, ..., n,$

where $\tilde{c}_j^I = \{(\underline{c}_j^\mu, c_j, \overline{c}_j^\mu; w_{\tilde{c}_j}), (\underline{c}_j^\nu, c_j, \overline{c}_j^\nu; u_{\tilde{c}_j})\}$, $\tilde{a}_{ij}^I = \{(\underline{a}_{ij}^\mu, a_{ij}, \overline{a}_{ij}^\mu; w_{\tilde{a}_{ij}}), (\underline{a}_{ij}^\nu, a_{ij}, \overline{a}_{ij}^\nu; u_{\tilde{a}_{ij}})\}$ and $\tilde{b}_i^I = \{(\underline{b}_i^\mu, b_i, \overline{b}_i^\mu; w_{\tilde{b}_i}), (\underline{b}_i^\nu, b_i, \overline{b}_i^\nu; u_{\tilde{b}_i})\}$ are triangular intuitionistic fuzzy numbers.

Remark 7.1 The membership function $\mu_{\tilde{A}^I}(x)$ and non-membership function $\nu_{\tilde{A}^I}(x)$ for a triangular intuitionistic fuzzy number $\tilde{A}^I = \{(\underline{a}^\mu, a, \overline{a}^\mu; w_{\tilde{a}}), (\underline{a}^\nu, a, \overline{a}^\nu; u_{\tilde{a}})\}$ is defined as

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \frac{(x-\underline{a}^\mu)w_{\tilde{a}}}{a-\underline{a}^\mu}, & x \in [\underline{a}^\mu, a); \\ w_{\tilde{a}}, & x = a; \\ \frac{(\overline{a}^\mu - x)w_{\tilde{a}}}{\overline{a}^\mu - a}, & x \in (a, \overline{a}^\mu]; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\nu_{\tilde{A}^I}(x) = \begin{cases} \frac{a-x+u_{\tilde{a}}(x-\underline{a}^\nu)}{a-\underline{a}^\nu}, & x \in [\underline{a}^\nu, a); \\ u_{\tilde{a}}, & x = a; \\ \frac{x-a+u_{\tilde{a}}(\overline{a}^\nu - x)}{\overline{a}^\nu - a}, & x \in (a, \overline{a}^\nu]; \\ 1, & \text{otherwise.} \end{cases}$$

The values $w_{\tilde{a}}$ and $u_{\tilde{a}}$ represent the maximum degree of membership and minimum degree of non-membership respectively such that $0 \leq w_{\tilde{a}} \leq 1$, $0 \leq u_{\tilde{a}} \leq 1$ and $0 \leq w_{\tilde{a}} + u_{\tilde{a}} \leq 1$.

Remark 7.2 If $\tilde{A}^I = \{(\underline{a}^\mu, a, \overline{a}^\mu; w_{\tilde{a}}), (\underline{a}^\nu, a, \overline{a}^\nu; u_{\tilde{a}})\}$ and $\tilde{B}^I = \{(\underline{b}^\mu, b, \overline{b}^\mu; w_{\tilde{b}}), (\underline{b}^\nu, b, \overline{b}^\nu; u_{\tilde{b}})\}$ are two triangular intuitionistic fuzzy numbers. Then

(i) $\tilde{A}^I+\tilde{B}^I = \{(\underline{a}^\mu+\underline{b}^\mu, a+b, \overline{a}^\mu+\overline{b}^\nu; min\{w_{\tilde{a}}, w_{\tilde{b}}\}), (\underline{a}^\nu+\underline{b}^\nu, a+b, \overline{a}^\nu+\overline{b}^\nu; max\{u_{\tilde{a}}, u_{\tilde{b}}\})\}$.

(ii) $\tilde{A}^I-\tilde{B}^I = \{(\underline{a}^\mu-\overline{b}^\mu, a-b, \overline{a}^\mu+\underline{b}^\nu; min\{w_{\tilde{a}}, w_{\tilde{b}}\}), (\underline{a}^\nu-\overline{b}^\nu, a-b, \overline{a}^\nu-\underline{b}^\nu; max\{u_{\tilde{a}}, u_{\tilde{b}}\})\}$.

(iii) $k\tilde{A}^I = \{(k\underline{a}^\mu, ka, k\overline{a}^\mu; w_{\tilde{a}}), (k\underline{a}^\nu, ka, k\overline{a}^\nu; u_{\tilde{a}})\}$ if $k \geq 0$,

$k\tilde{A}^I = \{(k\overline{a}^\mu, ka, k\underline{a}^\mu; w_{\tilde{a}}), (k\overline{a}^\nu, ka, k\underline{a}^\nu; u_{\tilde{a}})\}$ if $k < 0$.

7.2 Existing ranking function

Suresh et al. [143] proposed a ranking function $\left(Mag(\tilde{A}^I) = \frac{1}{12}\left\{\frac{4a-2(\overline{a}^\mu+\underline{a}^\mu)+3w_{\tilde{a}}(\overline{a}^\mu+\underline{a}^\mu)}{w_{\tilde{a}}} + \frac{2(a+\overline{a}^\nu+\underline{a}^\nu)-3u_{\tilde{a}}(\overline{a}^\nu+\underline{a}^\nu)}{1-u_{\tilde{a}}}\right\}\right)$ to obtain a real number corresponding to a triangular intuitionistic fuzzy number $\tilde{A}^I = \{(\underline{a}^\mu, a, \overline{a}^\mu; w_{\tilde{a}}), (\underline{a}^\nu, a, \overline{a}^\nu; u_{\tilde{a}})\}$.

Suresh et al. [143] have used the following method to obtain the ranking function $\left(Mag(\tilde{A}^I) = \frac{1}{12}\left\{\frac{4a-2(\overline{a}^\mu+\underline{a}^\mu)+3w_{\tilde{a}}(\overline{a}^\mu+\underline{a}^\mu)}{w_{\tilde{a}}} + \frac{2(a+\overline{a}^\nu+\underline{a}^\nu)-3u_{\tilde{a}}(\overline{a}^\nu+\underline{a}^\nu)}{1-u_{\tilde{a}}}\right\}\right)$.

Step 1 Find the $\alpha-$ cut $\left[\underline{a}^\mu + \frac{\alpha(a-\underline{a}^\mu)}{w_{\tilde{a}}},\ \overline{a}^\mu - \frac{\alpha(\overline{a}^\mu-a)}{w_{\tilde{a}}}\right]$ corresponding to membership function of triangular intuitionistic fuzzy number $\tilde{A}^I = \{(\underline{a}^\mu, a, \overline{a}^\mu; w_{\tilde{a}}), (\underline{a}^\nu, a, \overline{a}^\nu; u_{\tilde{a}})\}$.

Step 2 Find $\alpha-$ cut $\left[\frac{a-u_{\tilde{a}}\underline{a}^\nu-\alpha(a-\underline{a}^\nu)}{1-u_{\tilde{a}}}, \frac{a-u_{\tilde{a}}\overline{a}^\nu+\alpha(\overline{a}^\nu-a)}{1-u_{\tilde{a}}}\right]$ corresponding to non-membership function of triangular intuitionistic fuzzy number $\tilde{A}^I = \{(\underline{a}^\mu, a, \overline{a}^\mu; w_{\tilde{a}}), (\underline{a}^\nu, a, \overline{a}^\nu; u_{\tilde{a}})\}$.

Step 3 Find $Mag(\tilde{A}^I) = \frac{1}{2}\left\{\int\limits_0^1(\underline{a}^\mu + \frac{\alpha(a-\underline{a}^\mu)}{w_{\tilde{a}}} + \overline{a}^\mu - \frac{\alpha(\overline{a}^\mu-a)}{w_{\tilde{a}}})f(\alpha)d\alpha\right\}+\frac{1}{2}\left\{\int\limits_0^1(\frac{a-u_{\tilde{a}}\underline{a}^\nu-\alpha(a-\underline{a}^\nu)}{1-u_{\tilde{a}}}+\frac{a-u_{\tilde{a}}\overline{a}^\nu+\alpha(\overline{a}^\nu-a)}{1-u_{\tilde{a}}})f(\alpha)d\alpha\right\}$ where $f(\alpha)$ is non-negative and increasing weighting function on $[0,1]$ with $f(0) = 0, f(1) = 1$ and $\int\limits_0^1 f(\alpha)d\alpha = \frac{1}{2}$.

7.3 Invalidity of the existing ranking function

The ranking function $\left(Mag(\tilde{A}^I) = \frac{1}{12}\left\{\frac{4a-2(\overline{a}^\mu+\underline{a}^\mu)+3w_{\tilde{a}}(\overline{a}^\mu+\underline{a}^\mu)}{w_{\tilde{a}}} + \frac{2(a+\overline{a}^\nu+\underline{a}^\nu)-3u_{\tilde{a}}(\overline{a}^\nu+\underline{a}^\nu)}{1-u_{\tilde{a}}}\right\}\right)$, proposed by Suresh et al. [143], is not valid due to the following reasons.

1. It is obvious from Step 3 of Section 7.2 that Suresh et al. [143] have assumed that the upper limit of integration as 1. While, as in Step 1, the membership function

is considered for finding the $\alpha-$ cut and the maximum membership value is $w_{\tilde{a}}$. So, in Step 3 of the method, discussed in Section 7.2, the upper limit of integration $\frac{1}{2}\left\{\int\left(\underline{a}^{\mu}+\frac{\alpha(a-\underline{a}^{\mu})}{w_{\tilde{a}}}+\overline{a}^{\mu}-\frac{\alpha(\overline{a}^{\mu}-a)}{w_{\tilde{a}}}\right)f(\alpha)d\alpha\right\}$ should be $w_{\tilde{a}}$. Similarly, in Step 2, non-membership is considered for finding $\alpha-$ cut and maximum value of non-membership from $X-$ axis is $1-u_{\tilde{a}}$. So, in Step 3 of method, discussed in Section 7.2, the upper limit of integration $\frac{1}{2}\left\{\int\left(\frac{a-u_{\tilde{a}}\underline{a}^{\nu}-\alpha(a-\underline{a}^{\nu})}{1-u_{\tilde{a}}}+\frac{a-u_{\tilde{a}}\overline{a}^{\nu}+\alpha(\overline{a}^{\nu}-a)}{1-u_{\tilde{a}}}\right)f(\alpha)d\alpha\right\}$ should be $1-u_{\tilde{a}}$.

2. The values of $\alpha-$ cut, mentioned in Step 2 of Section 7.2, is obtained by putting $\frac{a-x+u_{\tilde{a}}(x-\underline{a}^{\nu})}{a-\underline{a}^{\nu}}=\alpha$, $\frac{x-a+u_{\tilde{a}}(\overline{a}^{\nu}-x)}{\overline{a}^{\nu}-a}=\alpha$, with the assumption that α will be distance from $X-$ axis. While in case of non-membership function distance from $X-$ axis will be $1-\alpha$ instead of α i.e., the exact value of $\alpha-$ cuts, obtained by putting $\frac{a-x+u_{\tilde{a}}(x-\underline{a}^{\nu})}{a-\underline{a}^{\nu}}=1-\alpha$, $\frac{x-a+u_{\tilde{a}}(\overline{a}^{\nu}-x)}{\overline{a}^{\nu}-a}=1-\alpha$ is $\left[\underline{a}^{\nu}+\frac{\alpha(a-\underline{a}^{\nu})}{1-u_{\tilde{a}}},\overline{a}^{\nu}-\frac{\alpha(\overline{a}^{\nu}-a)}{1-u_{\tilde{a}}}\right]$.

7.4 Exact ranking function

It is obvious from Section 7.3 that the exact ranking function $\left(Mag(\tilde{A}^{I})\right)$ for triangular intuitionistic fuzzy number $\tilde{A}^{I}$ will be

$$Mag(\tilde{A}^{I})=\frac{1}{2}\left\{\int_{0}^{w_{\tilde{a}}}\left(\underline{a}^{\mu}+\frac{\alpha(a-\underline{a}^{\mu})}{w_{\tilde{a}}}+\overline{a}^{\mu}-\frac{\alpha(\overline{a}^{\mu}-a)}{w_{\tilde{a}}}\right)f(\alpha)d\alpha+\right.$$

$$\left.\int_{0}^{1-u_{\tilde{a}}}\left(\underline{a}^{\nu}+\frac{\alpha(a-\underline{a}^{\nu})}{1-u_{\tilde{a}}}+\overline{a}^{\nu}-\frac{\alpha(\overline{a}^{\nu}-a)}{1-u_{\tilde{a}}}\right)f(\alpha)d\alpha\right\}.$$

Assuming $f(\alpha)=\alpha$, we have

$$Mag(\tilde{A}^{I})=\frac{1}{2}\left\{\int_{0}^{w_{\tilde{a}}}\left(\underline{a}^{\mu}+\frac{\alpha(a-\underline{a}^{\mu})}{w_{\tilde{a}}}+\overline{a}^{\mu}-\frac{\alpha(\overline{a}^{\mu}-a)}{w_{\tilde{a}}}\right)\alpha d\alpha+\int_{0}^{1-u_{\tilde{a}}}\left(\underline{a}^{\nu}+\frac{\alpha(a-\underline{a}^{\nu})}{1-u_{\tilde{a}}}+\overline{a}^{\nu}-\frac{\alpha(\overline{a}^{\nu}-a)}{1-u_{\tilde{a}}}\right)\alpha d\alpha\right\}.$$

After simplification, $Mag(\tilde{A}^{I})=\frac{1}{12}\left\{w_{\tilde{a}}^{2}(4a+\overline{a}^{\mu}+\underline{a}^{\mu})+(1-u_{\tilde{a}})^{2}(4a+\overline{a}^{\nu}+\underline{a}^{\nu})\right\}$.

7.5 Exact solution of existing problems

Suresh et al. [143] used the ranking function $Mag(\tilde{A}^I) = \left(\frac{1}{12}\left\{\frac{4a-2(\overline{a}^{\mu}+\underline{a}^{\mu})+3w_{\tilde{a}}(\overline{a}^{\mu}+\underline{a}^{\mu})}{w_{\tilde{a}}} + \frac{2(a+\overline{a}^{\nu}+\underline{a}^{\nu})-3u_{\tilde{a}}(\overline{a}^{\nu}+\underline{a}^{\nu})}{1-u_{\tilde{a}}}\right\}\right)$ for finding the optimal solution of intuitionistic fuzzy linear programming problems $(P_{7.4})$ [143, Example 1, pp. 3085], $(P_{7.5})$ [143, Example 2, pp. 3085], $(P_{7.6})$ [143, Example 3, pp. 3086] and $(P_{7.7})$ [143, Example 4, pp. 3086].

$$\left.\begin{array}{l}
\text{Maximize} \quad \tilde{z}^I = \{(19,25,33;0.9),(18,25,34;0)\}x_1 + \{(44,48,54;0.9),(43,48,56;0)\}x_2 \\
\text{Subject to} \\
15x_1 + 30x_2 \leq 45000, \\
24x_1 + 6x_2 \leq 24000, \\
21x_1 + 14x_2 \leq 28000, \\
x_1, x_2 \geq 0.
\end{array}\right\} (P_{7.4})$$

$$\left.\begin{array}{l}
\text{Maximize} \quad z = 25x_1 + 48x_2 \\
\text{Subject to} \\
\{(14,15,17;0.9),(10,15,18;0)\}x_1 + \{(25,30,34;0.9),(23,30,38;0)\}x_2 \preceq \\
\{(44980,45000,45030;0.9),(44970,45000,45070;0)\}, \\
\{(21,24,26;0.9),(20,24,33;0)\}x_1 + \{(4,6,8;0.9),(2,6,11;0)\}x_2 \preceq \\
\{(23980,24000,24050;0.9),(23940,24000,24060;0)\}, \\
\{(17,21,22;0.9),(16,21,26;0)\}x_1 + \{(12,14,19;0.9),(8,14,22;0)\}x_2 \leq \\
\{(27990,28000,28030;0.9),(27950,28000,28040;0)\}, \\
x_1, x_2 \geq 0.
\end{array}\right\} (P_{7.5})$$

$$\left.\begin{array}{l}
\text{Maximize} \quad \tilde{z} = \{(19,25,33;0.9),(18,25,34;0)\}x_1 + \{(44,48,54;0.9),(43,48,56;0)\}x_2 \\
\text{Subject to} \\
\{(14,15,17;0.9),(10,15,18;0)\}x_1 + \{(25,30,34;0.9),(23,30,38;0)\}x_2 \preceq \\
\{(44980,45000,45030;0.9),(44970,45000,45070;0)\}, \\
\{(21,24,26;0.9),(20,24,33;0)\}x_1 + \{(4,6,8;0.9),(2,6,11;0)\}x_2 \preceq \\
\{(23980,24000,24050;0.9),(23940,24000,24060;0)\}, \\
\{(17,21,22;0.9),(16,21,26;0)\}x_1 + \{(12,14,19;0.9),(8,14,22;0)\}x_2 \leq \\
\{(27990,28000,28030;0.9),(27950,28000,28040;0)\}, \\
x_1, x_2 \geq 0.
\end{array}\right\} (P_{7.6})$$

$$\left.\begin{array}{l}\text{Maximize} \quad z = 3x_1 + 4x_2 \\ \text{Subject to} \\ \{(0,1,3;0.9),(0,1,4;0)\}x_1 + \{(1,2,6;0.9),(0,2,8;0)\}x_2 \preceq \{(18,20,32;0.9),(16,20,34;0)\}, \\ \{(2,3,8;0.9),(1,3,10;0)\}x_1 + \{(3,5,6;0.9),(1,5,9;0)\}x_2 \preceq \{(56,60,66;0.9),(55,60,68;0)\}, \\ x_1, x_2 \geq 0. \end{array}\right\} (P_{7.7})$$

However, as discussed in Section 7.3 that the ranking function, proposed by Suresh et al. [143], is not valid. Hence, the optimal solutions of problems $(P_{7.4}), (P_{7.5}), (P_{7.6})$ and $(P_{7.7})$, obtained by Suresh et al. [143], are also not exact optimal solutions of these problems.

The exact optimal solutions of these problems, obtained by using the existing algorithm [143, Section 4, pp. 3084], with exact ranking function $Mag(\tilde{A}^I) = \frac{1}{12}\left\{w_{\tilde{a}}^2(4a + \overline{a}^\mu + \underline{a}^\mu) + (1 - u_{\tilde{a}})^2(4a + \overline{a}^\nu + \underline{a}^\nu)\right\}$ instead of using the invalid ranking function $Mag(\tilde{A}^I) = \frac{1}{12}\left\{\frac{4a - 2(\overline{a}^\mu + \underline{a}^\mu) + 3w_{\tilde{a}}(\overline{a}^\mu + \underline{a}^\mu)}{w_{\tilde{a}}} + \frac{2(a + \overline{a}^\nu + \underline{a}^\nu) - 3u_{\tilde{a}}(\overline{a}^\nu + \underline{a}^\nu)}{1 - u_{\tilde{a}}}\right\}$, are shown in Table 7.1.

Table 7.1
Exact optimal solutions of problems $(P_{7.4})$, $(P_{7.5})$, $(P_{7.6})$ and $(P_{7.7})$

Problem	Optimal solution	Optimal value
$(P_{7.4})$	$x_1 = 500, x_2 = 1250$	$\frac{794935}{12}$
$(P_{7.5})$	$x_1 = 0, x_2 = \frac{30408620}{15647}$	$\frac{1459613760}{15647}$
$(P_{7.6})$	$x_1 = 0, x_2 = \frac{30408620}{15647}$	$\frac{2665315543}{31294}$
$(P_{7.7})$	$x_1 = \frac{58863960}{4178747}, x_2 = \frac{6344186}{4178747}$	$\frac{201968624}{4178747}$

7.6 Conclusion

It is shown that the ranking function, proposed by Suresh et al. [143], is not valid and the exact ranking function is proposed. Also, the exact optimal solutions of the problems, solved by Suresh et al. [143], are obtained.

Chapter 8

FUTURE SCOPE

The following work may be considered as future research work:

1. To the best of my knowledge, till now there is no method in the literature to solve bounded intuitionistic fully fuzzy linear programming problems. In future, it may be tried to propose a method for the same.
2. It is well known that the optimal value of a crisp linear programming problem, corresponding to all alternative optimal solutions, will be a unique real number. However, the fuzzy optimal value of a fuzzy linear programming problem corresponding to different alternative optimal solutions, obtained by the existing methods as well as proposed Mehar methods, is not a unique fuzzy/intuitionistic fuzzy number. This flaw is occurring due to flaw in the method used in the existing methods/proposed Mehar methods for comparing fuzzy numbers/intuitionistic fuzzy numbers. In future, a new method for comparing fuzzy numbers/intuitionistic fuzzy numbers may be proposed and used to resolve this flaw of existing methods/proposed Mehar methods.

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