Symmetry Analysis and Conservation Laws for Some Systems of Nonlinear Partial Differential Equations

A Thesis

submitted in partial fulfillment of the requirements for
the award of the degree of

Doctor of Philosophy
in
Mathematics

by
Bikramjeet Kaur
(Registration No.: 901211004)

to the

School of Mathematics
Thapar Institute of Engineering and Technology,
Patiala-147004, INDIA

July-2019

DEDICATED

To

THE ALMIGHTY

&

MY FAMILY

Certificate

I hereby certify that the work, which is being presented in the thesis, entitled **Symmetry Analysis and Conservation Laws for Some Systems of Nonlinear Partial Differential Equations**, in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** and submitted to the institution is an authentic record of my own work carried out during the period **January 11, 2013** to **July 22, 2019** under the supervision of **Dr. Rajesh Kumar Gupta**, Associate Professor, Department of Mathematics, School of Physical and Mathematical Sciences, Central University of Haryana, Mahendergarh-123031, Haryana, INDIA.

Date: June 3, 2020

Bikramjeet Kaur.

(**Bikramjeet Kaur**)
Candidate

It is certified that the above statement made by the candidate is correct to the best of my knowledge.

Date: June 3, 2020

Rajesh

(**Dr. Rajesh Kumar Gupta**)
Supervisor

Acknowledgments

There are several people in my life who were there for me when I need them and those great persons I wish to mention here. An acknowledgment in my thesis is just a small gesture and doesn't truly represent how much I appreciate the influence of these people in my life.

Firstly, I wish to express my genuine thanks and acknowledgement to my esteemed supervisor **Dr. Rajesh Kumar Gupta** (Associate Professor, Department of Mathematics, School of Physical and Mathematical Sciences, Central University of Haryana, Mahendergarh-123031, Haryana, India), who was always ready to answer any questions I had, no matter how trivial is and you replied to all my inquiries and put a tremendous amount of efforts into every aspect of my thesis.

I would like to thank the authorities of the Thapar Institute of Engineering and Technology, Patiala for giving prominent research facilities. I also express my sincere thanks to Dr. Satish Kumar Sharma (Head of School of Mathematics), Doctoral committee members, faculty members, and Dr. Rafat Siddique (Dean of Research and Sponsored Project) for encouraging me to carry out the research work.

I express my most profound appreciation to my mom and dad, my husband and my sweet daughter (Aaradhya) for their blessings, unconditional love, support, and encouragement. I will forever be grateful for everything you have done for me.

Also, I would like to thank the University Grants Commission, New Delhi, India for financial assistance under the grant No. (F1-17.1/2013-14/MANF-2013-14-SIK-PUN-21763).

Finally and most importantly, I would like to thank Almighty, whose blessing gave me the motivation, patience, confidence, and power to complete this work.

Bikramjeet Kaur.

Bikramjeet Kaur

List of Abbreviations

EJEFE	Extended Jacobi elliptic function expansion method
IG	Infinitesimal generator
KdV	Korteweg and de Vries
PDEs	Partial differential equations
RLF	Riemann-Liouville fractional
YTSF	Yu-Toda-Sasa-Fukuyama
ZK	Zakharov-Kuznetsov
1LGTs	One-parameter Lie group of transformations

Abstract

The work complied in this thesis includes the investigation of nonlinear partial differential equations (PDEs) of integer and fractional order representing some physical phenomena for exact solutions, symmetries, and conservation laws. Linear dispersion analysis of fractional order PDEs is carried out to identify the normal/anomalous dispersion of waves. The techniques to retrieve solutions have thoroughly described and successfully implemented. The thesis consists of five chapters compiled for the investigation of seven nonlinear PDEs which are (2+1)-dimensional new coupled Zakharov-Kuznetsov (ZK) system, generalised 7^{th} order Korteweg and de Vries (KdV) equation, new coupled ZK system as well as Wu-Zhang system in (2+1)-dimensions having time derivatives of fractional order, time fractional 5^{th} order equation from Burgers hierarchy, space-time fractional potential Yu-Toda-Sasa-Fukuyama (YTSF) equation and space-time fractional Maccari model in (2+1)-dimensions.

Thesis is organised into five chapters as follows.

Chapter 1

The chapter introduces some important nonlinear PDEs of integer and fractional orders. The physical phenomena inherited by different types of PDEs are tabulated. Brief literature reviews on Lie group of transformations, methods for finding exact solutions, and conservation laws are presented. Introduction of linear dispersion analysis is briefed out. The frame work of the thesis is also presented systematically in this chapter.

Chapter 2

This chapter consists the preliminaries including some definitions, theorems related to Lie group theory, conservation laws, exact solutions, and dispersion analysis. Lie infinitesimal criterion to examine integer and time fractional PDEs is presented in an algorithmic way. The various methodologies used for finding solutions in terms of solitary waves, exact travelling waves, and doubly periodic waves have thoroughly described. Also, method known as improved F-expansion is proposed for examining the space-time fractional PDEs and subsequently, applied to space-time fractional potential YTSF equation in chapter 5. The methods to derive conservation laws for nonlinear PDEs have been discussed in details and algorithms are constructed for the same. For fractional PDEs, the linear dispersion analysis is also suggested.

Chapter 3

This chapter is devoted to study the integer order nonlinear PDEs with variable coefficients such as (2+1)-dimensional new coupled ZK system and generalised 7^{th} order KdV equation. The infinitesimal symmetries, symmetry groups, optimal system, invariants and reductions are systematically determined for new coupled ZK system. The variety of solutions in terms of Jacobi, trigonometric and hyperbolic functions are obtained, and analysed graphically to discuss the effect of arbitrary function on the wave profile. The generalised 7^{th} order KdV equation is also examined for Lie symmetries. Vector fields of the optimal system give solutions in an explicit form appeared as power series and involved Jacobi elliptic functions. The conservation laws are constructed for these equations by applying direct method and new conservation theorem with nonlinear self-adjointness.

Chapter 4

This chapter presents the comprehensive investigation of nonlinear PDEs having time derivatives of fractional order. It includes new coupled ZK system in (2+1)-dimensions, Wu-Zhang system in (2+1)-dimensions and 5^{th} order equation from Burgers hierarchy. The Lie classical technique is adopted to examine Lie symmetries with the use of Riemann-Liouville fractional (RLF) order derivative and corresponding invariants for these equations. The dimensions of fractional PDEs are reduced from (2+1) to (1+1) using invariants. The solutions show bright, dark and singular solitary wave like character for new coupled ZK system. The methodology of exponential rational function method has been utilized to seek solutions of Wu-Zhang system. Solutions in form of power series have obtained for 5^{th}order equation from Burgers hierarchy. The solutions of these equations are discussed graphically. The conservation laws for the equations are obtained by new conservation theorem. These equations are also studied for deriving dispersion relations, group and phase velocities.

Chapter 5

This chapter deals with important nonlinear PDEs from mathematical physics having space-time variations of fractional form such as potential YTSF equation and (2+1)-dimensional Maccari system. The improved F-expansion method suggested in chapter 2 for space-time fractional PDEs is applied to potential YTSF equation in this chapter and exact travelling waves are obtained as solutions. The Maccari system in (2+1)-dimensions is investigated using an extended Jacobi elliptic function expansion (EJEFE) method for deriving solutions having doubly periodic waves. The solutions of these equations are discussed graphically to show the influence of fractional parameters onto wave profile. The dispersion relations for space-time fractional PDEs are systematically derived and the anomalous/normal dispersion of waves is shown graphically.

At last, the summary of the thesis and some concluding remarks are given of the work conducted in different chapters.

List of Research Papers

1. B. Kaur and R. K. Gupta. Invariance properties, conservation laws, and soliton solutions of the time-fractional (2+1)-dimensional new coupled ZK system in magnetized dusty plasmas. *Computational and Applied Mathematics*, 37(5), 5981-6004 (2018). (**Impact factor-0.863, DOI: 10.1007/s40314-018-0674-7**).
2. B. Kaur and R. K. Gupta. Dispersion analysis and improved F-expansion method for space-time fractional differential equations. *Nonlinear Dynamics*, 96(2), 837-852 (2019). (**Impact factor-4.339, DOI: 10.1007/s11071-019-04825-w**).
3. B. Kaur and R. K. Gupta. Multiple types of exact solutions and conservation laws of new coupled (2+1)-dimensional ZK system with time dependent coefficients. *Pramana −Journal of Physics*, 93(4), 59 (2019). (**Impact factor-0.699, DOI: 10.1007/s12043-019-1806-3**).
4. B. Kaur and R. K. Gupta. Time fractional (2+1)-dimensional Wu-Zhang system: Dispersion analysis, similarity reductions, conservation laws, and exact solutions. *Computers & Mathematics with Applications*, 79(4), 1031-1048 (2019). (**Impact factor-2.811, DOI: 10.1016/j.camwa.2019.08.014**).
5. B. Kaur and R. K. Gupta. Dispersion and fractional Lie group analysis of time fractional equation from Burgers hierarchy. *Journal of Applied Analysis and Computation*. (**Communicated**).

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Chapter 1

Introduction

1.1 Nonlinear partial differential equations (PDEs)

Intensive research is going on to investigate physical systems modelled through nonlinear PDEs in the numerous branches of applied sciences particularly in fluid dynamics, condensed matter physics, physics of plasma, physical chemistry, mechanical engineering, aerospace engineering, and biomedical and health sciences [67, 104]. PDEs can be classified as linear or nonlinear and further, according to their order, these are categorized as integer or fractional. The physical systems have been modelled mathematically in terms of differential equations involving integer order derivatives as defined by Leibniz and Newton. Associated to this concept, the numerous examples have been encountered in literature, some of which are given in Table 1.1. However, the modern research on PDEs has pointed out that the concept of fractional order derivatives in mathematical modelling map the complex physical and engineering systems more relevantly [73, 116, 161, 193, 196, 216, 225, 254, 266, 289]. This fact boosts the study of fractional differential equations. For the basic introduction to fractional calculus number of books/publications are available in the literature [156, 157, 216, 226] and study reveals that fractional calculus is an advanced version of classical calculus. The concept of half derivative was first advised in Leibniźs letter to ĺHospital. Various prominent mathematicians such as Liouville, Grünwald, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel, etc. have been influenced by fractional calculus and contributed in its advancement. There are various definitions to understand the key idea of fractional orders. The most significant definition adopted

Table 1.1: Physical phenomena modelled and solved through integer order PDEs

Physical phenomena	References
Phase transitions	[77, 294]
Hysteresis phenomena	[199, 292, 295]
Homogenization phenomena	[53, 293]
Fluid flow through Porous media	[15, 208]
Boundary layer flow	[15, 102, 223]
Large deflections, stress distribution in plastic and non-coherent masses	[296]
Electrostatic plasma oscillations	[15, 52]
Diffusion with second order reaction	[15, 149]
Heat conduction in an isotropic medium	[15]
Heat conduction with radioactive decay	[15]
Heat flow and wave propagation phenomena	[81, 171, 308]

Table 1.2: Physical phenomena modelled and solved through fractional PDEs

Physical phenomena	References
Continuous-time random walk process	[253]
Anomalous diffusion	[325, 328]
Memory term effect	[50, 70, 226, 324]
Modelling mechanical and electrical properties of real materials	[66, 226]
Description of rheological properties of rocks	[226]
Power laws	[324]
Fractals	[72, 329]

for fractional derivatives is presented by Riemann and Liouville, and other definitions given by Gr$\ddot{u}$nwald–Letnikov [226], Caputo [49, 226], etc. are the variations of Riemann-Liouville definition. Most of the problems expressed in terms of fractional PDEs involve space/time or both derivatives of fractional order. In Table 1.2 and the references therein tabulate some of the fields where fractional PDEs have been successfully presented the various physical phenomena. It is necessary to draft the various methods to find well-defined exact solutions of PDEs. It is an important task and performs a crucial role in exploring nonlinear science. In recent years, researchers have shown more interest to modify the existing methods for solving PDEs. However, the development of simple and effective methodologies for solving PDEs always remains a challenge for researchers. Therefore, our main focus in the present study is to solve various types of nonlinear PDEs.

Table 1.3: Geometrical interpretation of fractional derivative of polynomial functions

Fractional order derivatives	Fractional derivative values at $x = 2$	Area of triangle (Δ)
$D^{0.1}[f(x)]$	17.34	7.38
$D^{0.25}[f(x)]$	19.47	6.57
$D^{0.5}[f(x)]$	23.34	5.48
$D^{1.0}[f(x)]$	32	4

1.1.1 Geometric and physical interpretation of fractional derivatives

As we know that, the classical derivatives have geometrical meanings but for fractional derivatives the geometrical meaning is different because the fractional calculus deals with memory and it gives the knowledge about present as well as the future. To express the geometrical significance of fractional derivatives, a polynomial functions is considered and the fractional derivative of polynomial functions is given as $D^\alpha[x^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}$, where α is the order of the derivative and $0 < \alpha < 1$. By using this formula, the fractional derivatives of the function $f(x) = x^4$, at point $(2, 16)$ are computed and given in table 1.3. The figure 1.1 shows the function $f(x) = x^4$ has first order derivative $D^{1.0}[f(x)]$ is computed as 32 and it is the tangent line at $P(2, 16)$ which passes through X-axis at A. Draw the perpendicular line from $(2, 16)$ meets the X-axis at $B(2, 0)$. The area of the triangle (ΔPAB) is 4. Similarly, for other fractional derivatives the tangent lines have drawn and the area enclosed by corresponding triangles are given in table 1.3. From the table, it has been concluded that as the order of fractional derivative increases, the area of the corresponding triangle decreases and vice-versa. The fractional derivatives and area of triangle are found to possess a inversely proportional relation to each other i.e.,

$$D^\alpha[f(x)] \propto \frac{1}{\Delta}, \tag{1.1}$$

or $D^\alpha[f(x)].\Delta$=constant. As the change in area is physical property, so the fractional derivatives can be used to measure the changes in temperature, pressure, gradient, divergence and curl, etc.

1.2 Reported literature on exact solutions of PDEs

Success to find exact solutions measures the extent to which physical phenomena are completely described using nonlinear PDEs. The task becomes more complex in the

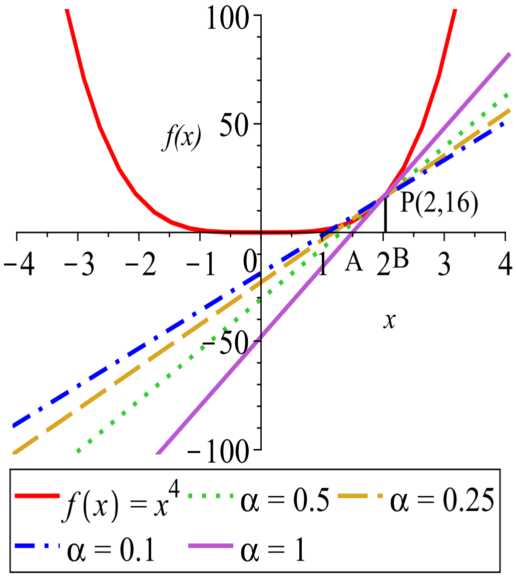

Figure 1.1: Geometrical interpretation of fractional derivatives of the function $f(x) = x^4$ and triangles formed with the fractional order derivatives.

case when various parameters of the system are considered as variables rather than constants. Thus, finding solutions of such nonlinear PDEs is more challenging mathematically and from related application perspective.

The exploration of literature related to PDEs shows that Lie symmetry group method is an efficient procedure for getting analytical solutions. The concept of symmetry was introduced centuries ago by scientists in physics and chemistry by Greeks, Kepler, Newton (laws of mechanics), Galileo (principle of relativity), Einstein (modified the theory of relativity), crystal symmetries in solid state physics. Presently, symmetries are used to grasp the various concepts in mathematical physics, chemistry of materials, engineering, electrodynamics, classical/quantum mechanics, condensed matter and particle physics. Basically, symmetry is a transformation whose application results into a transformed system in an invariant form and is one of the most appealing practical application of the Lie group theory. This concept was explored by Sophus Lie in the nineteenth century 1872-1899 [177, 179] as Lie groups. He further formulates theory of symmetries for differential equations that provides solutions in systematic manner. The work of G. Birkhoff [37] and Sedov [259] on dimensional analysis, and the pioneering efforts of Ovsiannikov [222] have extended this concept to an advanced state. New developments of the group theory were addressed by Bluman and Cole [41], Bluman and Kumei [42], Stephani [281], Bluman and Anco [40, 43], Olver [219], Ibragimov [120], Bhutani et al. [33–35], Clarkson and Mansfield

[61], etc.

The group invariance is a fundamental concept behind symmetry analysis related to PDEs. The system under the action of such transformations remains in an invariant form. By applying this technique, it is possible to reduce independent variables or system dimensionality. The reduced system of equations can be easily solved rather than the original system.

There are number of methods found in literature which provide exact solutions of system of PDEs. The symmetry methods include methods based upon Lie group theoretic ideas such as Lie's classical [151, 165, 167, 267–269], nonclassical [41, 108, 214], Steinberg's symmetry reduction method [107, 279] and so on. Some of other methods which are not based upon Lie group theory such as Direct method [54, 60, 352], B*ä*cklund transformation [239], Painlev*é* analysis [7, 314, 315], Inverse scattering transformation [6] and so on.

With the current progresses, the numerous applications have been found in the area of Lie classical theory. Lie classical theory is the root of the various generalisations such as nonclassical method [41, 214], equivalence transformations [180], generalised symmetries [219], nonlocal symmetries [42, 173, 219], approximate symmetries [25, 128], general method of differential constraints [220, 232] and so on. Some recent contributions are reported in the direction of Lie group theory by Gandarias et al. [48, 88–91], Nucci [211–214], Y. K. Gupta et al. [110, 111], Wazwaz [303, 309–311], Biswas et al. [4, 32, 38, 39], Sharma et al. [224, 262–264], Lakshmanan et al. [170, 230, 231, 249, 290], Gupta et al. [106, 107, 109, 150–153, 163, 165, 166, 267, 269–273], Kara [146–148], Ibragimov et al. [120, 125, 129], Olver [219, 220], Oliveri [217, 218], Singh et al. [267, 268], Amaranath et al. [14, 260], Ramaswamy et al. [237, 238].

There are number of integer order PDEs having variable coefficients which were investigated successfully by Lie symmetry method [107, 108, 151, 268, 273]. In the last decades, this approach was further modified to solve time or space-time fractional PDEs [94, 95, 113, 118, 139, 140, 151, 229, 234, 241, 244, 245, 247, 274–276, 297, 299, 331], some of them are as follows

1. Time-fractional model of Cahn–Allen as well as Klein–Gordon equations [134].
2. Generalized time-fractional modified KdV equation [235].
3. Time fractional form of generalized equation coming from Burgers–Huxley [133], so on.

Besides of Lie symmetry method, there exist some other methodologies or ansätz based methods in literature which can be used in solving nonlinear PDEs of integer or fractional as follows

- Tanh-sech procedure [302] and the extended tanh method [303, 304, 351].
- Sine-cosine method [320].
- Hirota method [307].
- Homogeneous balance method [79].
- Jacobi elliptic function expansion technique [64].
- F-expansion [136–138] as well as improved F-expansion methods [10, 135–138, 150].
- Homotopy perturbation method [93].
- Variational iteration method [114].
- G'/G-expansion method [36, 200, 301].
- Fractional sub-equation method [348].
- Exponential function method [8, 36] and exponential rational function method [11, 100, 204].
- Factorization methodology using travelling wave transformations [330].
- Travelling wave solutions focused on the development of local fractional derivatives [327, 329].
- The first integral method [30, 287], and so on.

The various methodologies provide multiple types solutions for PDEs such as kink, singular kink, solitary and periodic. The research carried out in last few years has proved that nonlinear PDEs possess not only the soliton type exact solutions but there is probability to derive lump solutions for linear and nonlinear PDEs. Lump solutions are termed as a special type of exact solutions with hidden vital physical information. Recently, Hirota bilinear method [57–59] is proposed to find the lump solutions of nonlinear PDEs. There are reports [188, 189] devoted to the interaction of lump solutions with kinks and solitons for nonlinear PDEs.

Apart from the importance of exact solutions, the numerical solutions can also be obtained for nonlinear PDEs. There are numerous numerical techniques, e.g. the finite difference method [103, 206], finite element [98, 99], finite volume [103], spectral and pseudo spectral methods, and differential quadrature method [31, 144, 202] which results in the approximate solutions of PDEs. Exact solutions assist in designing and testing validity of numerical algorithms.

In this thesis, our work is primarily focused on solving three different types of nonlinear PDEs of integer, time fractional and space-time fractional for exact solutions as given by

1. Integer order (2+1)-dimensional new coupled ZK system.
2. Integer order generalised 7^{th} order KdV equation.
3. New coupled ZK system in (2+1)-dimensions with time fractional derivatives.
4. Wu-Zhang system in (2+1)-dimensions with time fractional derivatives.
5. 5^{th} order equation from Burgers hierarchy with time fractional derivatives.
6. Potential YTSF equation with space-time fractional derivatives.
7. Maccari system in (2+1)-dimensions with space-time fractional derivatives.

1.3 Conservation laws

Conservation law for a family of PDEs has been expressed as divergence equation that approaches zero for all solutions and for physical systems it appears as conservation of mass, charge, momentum, energy and other constants of motion. However, special nonlinear PDEs may also have a plenitude of conservation laws without physical meaning. The laws are helpful in revealing existence as well as uniqueness of its various solutions [172], such equations can be linearised or explicitly solved and also helps in establishing the numerical tools/programs, e.g. finite element method [62, 175]. An infinite number of conservation laws reveals integrability [16]. Many methods developed in literature to find conservation laws such as

- **Noether's method**: There is a strong association among the symmetries and corresponding conservation laws of PDEs as evidenced by Emmy Noether (1918) in her theorem [210]. The Noether's theorem is applicable to variational

PDEs (the PDEs possess classical Lagrangian). Noether suggested explicit formula for generating conservation laws. Bessel-Hagen (1921) and Boyer (1967) extended the Noether's theorem and obtained results strongly rely on Lie group of transformations [43].

- **Partial Lagrangian technique**: This approach is similar to Noether's method and applicable to even order PDEs. This technique uses a partial Lagrangian [148] instead of standard Lagrangian. Partial Lagrangian is easier to handle mathematically than the standard Lagrangian which is difficult or not possible to found for some PDEs.

- **Direct method of multipliers**: To overcome the limitation of Noether's method, the direct method for conservation laws is developed by Anco and Bluman [18–20] and presented by Olver [219]. This method is applicable to any PDE without the existence of Lagrangian or action functional.

- **New conservation theorem**: As the Noether's theorem is applicable to variational PDEs, and Lagrangian can exist for some special type of PDEs. Thus to overcome this restriction, the new conservation theorem which is based on formal Lagrangian and adjoint equations of PDEs given by Ibragimov [121]. Any symmetry (Lie point, Lie B*ä*cklund or nonlocal) yields conservation laws by the general theorem on conservation laws. Moreover, this method of self-adjointness is extended to quasi/weak/nonlinear self-adjointness [88, 122, 123, 288].

- S. C. Anco shows that Ibragimov's new conservation theorem gives trivial conservation laws [17]. Recently, an approach by utilizing both symmetries and adjoint symmetries has been reported, and it is applicable to all PDEs inspite of the existence of a Lagrangian [186, 187] and used to derive solutions of PDEs [23, 124, 126].

Some reports describing a systematic way to find conservation laws are given in [24, 69, 89, 127, 146, 147, 280, 291]. The method used in deriving the conserved density and fluxes for integer order PDEs has been extended to fractional order PDEs. The generalisation of Noether's theorem and its fractional generalisation using Noether's operator with new conservation theorem for time fractional PDEs are given in [22, 45, 83, 121, 151, 185, 197, 215] and for the space-time fractional PDEs are reported in [26, 84, 131, 275]. Some of the examples of well-known PDEs which were investigated in literature for conservation laws are described as follows

1. Klein–Gordon–Fock equation [5].
2. Krichever–Novikov equation [87].
3. Burgers-Fisher equation in two dimensions [240].
4. Time fractional (2+1)-dimensional Davey-Stewartson equations [337].
5. Kudryashov-Sinelshchikov equation [48].
6. Nizhink-Novikov-Veselov equation [339].
7. Nonlinear Schr$\ddot{o}$dinger's equation for compressional dispersive Alv$\grave{e}$n waves [130], etc.

In present work, the following PDEs are to be investigated for conservation laws

1. Integer order new coupled ZK system with time dependent coefficients in (2+1)-dimensions.
2. Integer order generalised 7^{th} order KdV equation with variable coefficients.
3. New coupled ZK system in (2+1)-dimensions having time derivatives of fractional order.
4. (2+1)-dimensional form of Wu-Zhang system with time derivatives of fractional order.
5. 5^{th} order equation from Burgers hierarchy involving fractional order time derivative.

1.4 Linear dispersion analysis

Linear dispersive wave analysis of the PDEs helps to derive the relation connecting wave number (s) and angular frequency (ω), known as dispersion relation, and it takes the form as follows

$$\mathcal{D}(\omega, s) = 0, \tag{1.2}$$

where $\mathcal{D}$ represents a suitable real function involving ω and s. This relation is satisfied for ω, $s \in \mathbb{C}$. The expression for wave velocity is given by

$$\frac{\omega}{s} = \pm c, \tag{1.3}$$

which is independent of ω and s, and c is arbitrary constant. More precisely, this is termed as phase velocity. The speed of a sinusoidal wave $\sin(sx-\omega\, t)$ represents how fast a point with constant phase, $sx-\omega\, t$, moves. So, the phase velocity is given by

$$sx-\omega\, t = constant. \tag{1.4}$$

$$\frac{d(sx-\omega\, t)}{dt} = 0. \tag{1.5}$$

$$s\frac{dx}{dt} - \omega = 0. \tag{1.6}$$

$$\frac{dx}{dt} = \frac{\omega}{s}. \tag{1.7}$$

However, the systems in which the phase velocity is not constant are called dispersive systems. In particular, a new feature that arises is group velocity [205] defined as the speed with which wave packet (basically, a bump in the wave) moves and is given by

$$v_g = \frac{d\omega}{ds}. \tag{1.8}$$

If group velocity differs from phase velocity, then in such dispersive systems, the waves having non-identical frequencies move with different speeds. If the signs of phase and group velocities are same then the linear waves are termed as the forward waves and in case, when the signs of the phase and group velocities are opposite then the linear waves are known as the backward waves. Similar type of discussion on the forward and backward moving solitons is discussed in [194].

Dispersion leads to different behavior of phase velocity v_p and group velocity v_g with the variation in wave number. The theory of linear dispersive waves is a very well established and developed field of mathematical physics. Effects of fractional extensions of such linear systems on the dispersion of waves can still represent an interesting, and utterly non-trivial, research topic. Analysis of dispersion of waves performs a vital role in hydrodynamics including water waves, nonlinear optics, and Bose-Einstein condensates [278]. However, the various types of nonlinear PDEs used in modelling different physical phenomena for real-life applications are successfully solved but there is always a need of drafting a simple algorithm and discussion of the solutions in terms of various system parameters requires a serious attention. For dispersion analysis, some of PDEs such as fractional KdV equation [63] and time-fractional Cattaneo-Maxwell heat equation [101] are investigated in literature.

Under this work, the following PDEs are to be studied for linear dispersion analysis

1. New coupled ZK system in (2+1)-dimensions (time fractional) .
2. Wu-Zhang system in (2+1)-dimensions (time fractional).
3. 5^{th} order equation from Burgers hierarchy (time fractional).
4. Space-time fractional potential YTSF equation.
5. Maccari system in (2+1)-dimensions with space-time fractional derivatives.

1.5 Framework of the thesis

Keeping in view the above facts, work complied in this thesis helps to understand the symmetry analysis, procedures to find conservation laws, exact travelling wave solutions, and the linear dispersion analysis for some of the important nonlinear PDEs. The work is thoroughly discussed with concrete examples. The solutions achieved in different cases are analysed graphically for better understanding. The dispersion relation obtained for various nonlinear fractional PDEs helps to formulate conditions for normal or anomalous dispersion of waves.

Thesis contains five different chapters, details of which are as under:

In **chapter 1**, introduction of nonlinear PDEs is given and important nonlinear phenomena described by PDEs are mentioned. The literature of PDEs for Lie group analysis, conservation laws, and dispersion relation are addressed.

Chapter 2 presents some vital preliminaries related to Lie group of transformations, infinitesimals, invariant solutions, new conservation theorem, direct method and dispersion analysis. The various methods used in exploring Lie symmetries, exact solutions and conservation laws are described algorithmically. Linear analysis of fractional PDEs is also proposed and discussed in an algorithmic way.

Chapter 3 investigates two integer order nonlinear PDEs. Firstly, (2+1)-dimensional new coupled ZK system is considered in the following form

$$\begin{aligned} &u_t + T_1(t)(uv)_x + T_2(t)(u_{xx} + u_{yy})_x + T_3(t)(vw)_x = 0, \\ &v_t + T_4(t)(uw)_x + T_2(t)(v_{xx} + v_{yy})_x = 0, \\ &w_t + T_4(t)(uv)_x + T_2(t)(w_{xx} + w_{yy})_x = 0, \end{aligned} \tag{1.9}$$

where $T_1(t)$, $T_2(t)$, $T_3(t)$ and $T_4(t)$ represent functions of t in an arbitrary form. Secondly, the following generalised 7^{th} order KdV equation having coefficients as function of t is selected

$$u_t + T_5(t)u_x u^3 + T_6(t)u_x^3 + T_7(t)uu_x u_{xx} + T_8(t)u^2 u_{xxx} + T_9(t)u_{xx}u_{xxx} + T_{10}(t)u_x u_{xxxx} + T_{11}(t)uu_{xxxxx} + T_{12}(t)u_{xxxxxxx} = 0, \tag{1.10}$$

where $T_5(t)$, $T_6(t)$, $T_7(t)$, $T_8(t)$, $T_9(t)$, $T_{10}(t)$, $T_{11}(t)$ and $T_{12}(t)$ are time dependent coefficients. In this chapter, exact solutions are constructed via Lie symmetry transformation technique. Conservation laws are derived by direct method of multipliers and new conservation theorem with nonlinear self-adjointness.

Chapter 4 is devoted to the study three nonlinear time fractional PDEs. First is new coupled ZK system in (2+1)-dimensions having time derivatives of fractional order is defined by

$$\begin{aligned}
&\frac{\partial^\alpha u}{\partial t^\alpha} - a\left(uv_x + u_x v\right) - g\left(vw_x + v_x w\right) - b\left(u_{xxx} + u_{xyy}\right) = 0,\\
&\frac{\partial^\alpha v}{\partial t^\alpha} - l\left(wu_x + w_x u\right) - b\left(v_{xxx} + v_{xyy}\right) = 0,\\
&\frac{\partial^\alpha w}{\partial t^\alpha} - l\left(uv_x + u_x v\right) - b\left(w_{xxx} + w_{xyy}\right) = 0,
\end{aligned} \tag{1.11}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes RLF derivative [226, 251] with respect to time and $(0 < \alpha < 1)$. Second equation is (2+1)-dimensional Wu-Zhang system given by

$$\begin{aligned}
&\frac{\partial^\alpha u}{\partial t^\alpha} + uu_x + vu_y + w_x = 0,\\
&\frac{\partial^\alpha v}{\partial t^\alpha} + uv_x + vv_y + w_y = 0,\\
&\frac{\partial^\alpha w}{\partial t^\alpha} + u_x w + uw_x + v_y w + vw_y + \frac{1}{3}\left(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}\right) = 0,\ 0 < \alpha < 1.
\end{aligned} \tag{1.12}$$

Third equation is time fractional 5^{th} order equation from Burgers hierarchy described by

$$\begin{aligned}
&\frac{\partial^\alpha u}{\partial t^\alpha} + \mu\Big(u_{xxxxx} + 10\,{u_{xx}}^2 + 15\,u_x u_{xxx} + 5\,uu_{xxxx} + 15\,{u_x}^3 + 50\,uu_x u_{xx}\\
&+ 10\,u^2 u_{xxx} + 30\,u^2 {u_x}^2 + 10\,u^3 u_{xx} + 5\,u^4 u_x\Big) = 0,\ 0 < \alpha < 1.
\end{aligned} \tag{1.13}$$

The Lie symmetries corresponding conserved densities/fluxes (conservation laws) and exact solutions for these equations are determined. Also, the dispersion relation, phase velocity, and group velocities expressions are obtained. Physical representation of exact solutions, dispersion, phase, and group velocities are shown by 2D and 3D plots.

Chapter 5 is devoted to investigate two space-time fractional PDEs. In this chapter, an improved F-expansion method is applied to potential YTSF equation having space-time derivatives of fractional order in the following form

$$\begin{aligned}&-4\,\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\right)+\frac{\partial^{3\beta}}{\partial x^{3\beta}}\left(\frac{\partial^{\delta}u}{\partial z^{\delta}}\right)+3\,\frac{\partial^{2\gamma}u}{\partial y^{2\gamma}}+4\left(\frac{\partial^{\beta}u}{\partial x^{\beta}}\right)\left(\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{\partial^{\delta}u}{\partial z^{\delta}}\right)\right)\\&+2\left(\frac{\partial^{2\beta}u}{\partial x^{2\beta}}\right)\left(\frac{\partial^{\delta}u}{\partial z^{\delta}}\right)=0,\end{aligned}\tag{1.14}$$

where α, β, γ and δ $(0<\alpha,\beta,\gamma,\delta<1)$ denote the fractional parameters representing the orders of derivatives with respect to independent variables t, x, y and z, respectively. The YTSF equation has been explored for analysing dispersion of waves and exact solutions. Also, Maccari system in (2+1)-dimensions having space-time derivatives of fractional order has been considered as follows

$$\begin{aligned}&i\,\frac{\partial^{\alpha}P}{\partial t^{\alpha}}+\frac{\partial^{2\beta}P}{\partial x^{2\beta}}+PW=0,\\&\frac{\partial^{\alpha}W}{\partial t^{\alpha}}+\frac{\partial^{\gamma}W}{\partial y^{\gamma}}+\frac{\partial^{\beta}|P|^{2}}{\partial x^{\beta}}=0,\ 0<\alpha,\beta,\gamma<1,\end{aligned}\tag{1.15}$$

where $i=\sqrt{-1}$, P and W are the complex and real functions, respectively, of the variables t, x and y. The Maccari system is investigated by utilizing EJEFE algorithm for solutions and dispersion of waves.

Chapter 2

Preliminaries

In this chapter, the fundamentals of the Lie group of transformations are introduced which describe group, infinitesimal transformations, invariant function, Lie algebra and classification of subalgebras. It also includes the extension of one parameter Lie groups, their prolongation formulas and description of invariant solutions of PDEs. The algorithmic descriptions of various methods for the investigation of nonlinear PDEs are described with relevant details. For more information on the symmetry groups and the proof of various theorems, readers can refer to Bluman and Anco [40], Bluman and Cole [41], Bluman and Kumei [42], and Olver [219]. The concepts related to conservation laws can be found in [19, 20, 43, 121, 123, 210]. Also, more details related to all of these concepts can be found in [27, 63, 87, 96, 101, 164, 168, 209] and references therein.

2.1 Symmetry

A symmetry [40] is some kind of transformation under which an object state or its neighborhood remains unchanged (invariant). More precisely, the symmetry of a system supports the following properties

(i) it preserves the structure of the object and maps the object into itself.

(ii) the symmetry transformation and corresponding inverse transformation are continuous, hence, infinitely differentiable, e.g. rotational symmetry of an equilateral triangle through an angle of 120° or rotation of sphere around any one of its diameter.

2.2 Group

A non-empty set G with a law of composition φ between elements is said to be a group [40] if it satisfies following axioms

(i) Closure: If θ_1, $\theta_2 \in G$ then $\varphi(\theta_1, \theta_2) \in G$.

(ii) Associative: If θ_1, θ_2, $\theta_3 \in G$ then $\varphi(\theta_1, \varphi(\theta_2, \theta_3)) = \varphi(\varphi(\theta_1, \theta_2), \theta_3)$.

(iii) Existence of identity: For all $\theta_1 \in G$ $\exists$ $e \in G$ such that $\varphi(\theta_1, e) = \varphi(e, \theta_1) = \theta_1$, where e is called identity element of G.

(iv) Existence of inverse: For all $\theta_1 \in G$ $\exists$ θ_1^{-1} in G such that $\varphi(\theta_1, \theta_1^{-1}) = \varphi(\theta_1^{-1}, \theta_1) = e$, where θ_1^{-1} is called inverse of θ_1.

2.3 Commutative group

A group G is called commutative group or an abelian group [40] if $\varphi(\theta_1, \theta_2) = \varphi(\theta_2, \theta_1)$, $\forall$ θ_1, $\theta_2 \in G$.

2.4 One-parameter Lie group of transformations (1LGTs)

Consider a set $\bar{\phi}$ with the elements $\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3, ..., \bar{\phi}_n \in D \subset \mathbb{R}^n$ and the transformations

$$\bar{\phi}^* = \Omega(\bar{\phi}; \epsilon) \tag{2.1}$$

are defined $\forall$ $\bar{\phi}$ and these transformations constitute 1LGTs if for parameters ϵ, $\sigma \in S \subset \mathbb{R}$, with the law of composition $\varphi(\epsilon, \sigma)$ [40], the following conditions should satisfy:

1. the transformations (2.1) are one to-one and onto $\forall$ ϵ in S .
2. S forms a group G under law of composition φ.
3. $\forall$ $\bar{\phi} \in D$, $\bar{\phi}^* = \bar{\phi}$ holds if parameter $\epsilon = \epsilon_0$ represents identity, i.e., $\bar{\phi} = \Omega(\bar{\phi}; \epsilon_0)$.
4. If $\bar{\phi}^* = \Omega(\bar{\phi}; \epsilon)$, $\bar{\phi}^{**} = \Omega(\bar{\phi}^*; \sigma)$, then $\bar{\phi}^{**} = \Omega(\bar{\phi}; \varphi(\epsilon, \sigma))$.
5. ϵ must be a continuous parameter and termed as identity for $\epsilon = 0$.

6. Ω is an analytic function of ϵ in set S and infinitely differentiable w.r.t. $\bar{\phi}$ in D.

7. $\varphi(\epsilon, \sigma)$ is an analytic function of ϵ and σ.

2.5 Infinitesimal transformations

The infinitesimal transformations [40] can be achieved by expanding 1LGTs (2.1) around parameter $\epsilon = 0$ as follows

$$\begin{aligned}\bar{\phi}^* &= \bar{\phi} + \epsilon \left(\frac{\partial \Omega(\bar{\phi}; \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0} \right) + \frac{\epsilon^2}{2} \left(\frac{\partial^2 \Omega(\bar{\phi}; \epsilon)}{\partial \epsilon^2}\Big|_{\epsilon=0} \right) + ... \\ &= \bar{\phi} + \epsilon \mathcal{X}(\bar{\phi}) + O(\epsilon^2),\end{aligned} \tag{2.2}$$

where

$$\mathcal{X}(\bar{\phi}) = \frac{\partial \Omega(\bar{\phi}; \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{2.3}$$

The $\bar{\phi} + \epsilon \mathcal{X}(\bar{\phi})$ is known as infinitesimal transformation of 1LGTs (2.1), where $\mathcal{X}(\bar{\phi})$ termed as the infinitesimals of (2.1).

2.6 Invariant function

A function $\mathcal{M}(\bar{\phi})$ is termed as an invariant function [27, 40, 164, 168] for 1LGTs (2.1) if and only if

$$\mathcal{M}(\bar{\phi}^*) = \mathcal{M}(\bar{\phi}), \tag{2.4}$$

or $\mathcal{M}(\bar{\phi})$ justifies the following

$$\sum_{i=1}^{n} \mathcal{X}_i(\bar{\phi}) \frac{\partial \mathcal{M}(\bar{\phi})}{\partial \bar{\phi}^i} = 0. \tag{2.5}$$

The operator

$$X = X(\bar{\phi}) = \sum_{i=1}^{n} \mathcal{X}_i(\bar{\phi}) \frac{\partial}{\partial \bar{\phi}^i} \tag{2.6}$$

is known as an infinitesimal generator (IG) [40] and $X\mathcal{M}$ is defined as Lie derivative of $\mathcal{M}$. Expand $\mathcal{M}(\bar{\phi})$ as a Taylor series around $\epsilon = 0$ then Eq. (2.4) results into following

$$\mathcal{M}(\bar{\phi}^*) = \mathcal{M}(\bar{\phi}) + \epsilon X\,\mathcal{M}(\bar{\phi}) + \frac{\epsilon^2}{2}\,X^2\,\mathcal{M}(\bar{\phi}) + ... + \frac{\epsilon^n}{n!}\,X^n\,\mathcal{M}(\bar{\phi}) + O(\epsilon^{n+1}). \tag{2.7}$$

If the series given in Eq. (2.7) converges then it called as Lie series and Eq. (2.7) becomes

$$\mathcal{M}(\bar{\phi}^*) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\,X^n\,\mathcal{M}(\bar{\phi}), \tag{2.8}$$

which can also be described as follows

$$\mathcal{M}(\bar{\phi}^*) = \exp(\epsilon\,X)\,\mathcal{M}(\bar{\phi}). \tag{2.9}$$

2.7 Lie algebra

Lie algebra constitute a vector space (X) over the field of invariant functions. If X_l and X_m are the IGs then the commutator or Lie bracket is established as follows

$$[X_l, X_m] = X_l\,X_m - X_m\,X_l, \;\; \forall\, l, m. \tag{2.10}$$

The commutator exhibits the following properties

1. **Closure:**

   $$[X_l, X_m] \in X. \tag{2.11}$$

   It implies that if X_l and X_m are two IGs, then commutation $[X_l, X_m]$ of both the operators will also become a generator of a symmetry group [119, 281]. Thus, the set of all IGs is closed under commutation relation.

2. **Bilinearity:**

   $$\begin{aligned} [a\,X_l + b\,X_m, X_n] &= a\,[X_l, X_n] + b\,[X_m, X_n], \\ [X_l, a\,X_m + b\,X_n] &= a\,[X_l, X_m] + b\,[X_l, X_n], \end{aligned} \tag{2.12}$$

   where a, $b \in \mathbb{R}$.

3. **Skew-symmetry:**

   $$[X_l, X_m] = -[X_m, X_l]. \tag{2.13}$$

Table 2.1: Commutator table

	X_1	X_2	$\cdots$	X_k
X_1	0	X_{12}	$\cdots$	X_{1k}
X_2	-X_{12}	0	$\cdots$	X_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_k	-X_{1k}	-X_{2k}	$\cdots$	0

4. **Jacobi identity:**

$$[X_l,[X_m,X_n]]+[X_n,[X_l,X_m]]+[X_m,[X_n,X_l]]=0. \tag{2.14}$$

If X_1, X_2,..., X_k are the infinitesimal symmetries of a k-parameter Lie symmetry group then new IGs can be found by using the commutators of the known IGs. The formation of Lie algebra can be examined by listing all the IGs called as commutator table [219]. The (l,m)-th element in commutator table can be displayed as $X_{lm} = [X_l,X_m] = X_lX_m - X_mX_l$. The commutator of two IGs is antisymmetric in nature ($X_{lm} = [X_l,X_m]$=-$[X_m,X_l] = -X_{ml}$) and all diagonal entities are zero ($[X_l,X_l]=0$), so there is only need to determine the entities above the diagonal as given in Table 2.1.

2.8 Classification of subalgebras

Classification of subgroups for a given Lie symmetry group performs a crucial role in the study of differential equations. Classification assists to measure group invariant solutions without the chance of an appearance of equivalent solutions. For each subgroup of a symmetry group, family of group invariant solutions will be obtained and if there exist subgroups (infinite number) then it is not feasible to enlist all group invariant solutions. In such situations, optimal system [40, 219, 222, 312] for a given system is obtained. To construct an optimal system, the Lie series is used to find the adjoint representation as follows

$$Ad(exp(\epsilon\, X_i))\, X_j = X_j - \epsilon\,[X_i,X_j] + \frac{\epsilon^2}{2}\,[X_i,\,[X_i,X_j]] + \cdots\,. \tag{2.15}$$

The procedure for classification of one-dimensional subalgebras is carried out by the technique given in [219]. If X_1, X_2,..., X_k are the elements of the basis of Lie algebra, then the most general IG can be represented as follows

$$X = g_1\, X_1 + g_2\, X_2 + \ldots + g_k\, X_k. \tag{2.16}$$

Simplification of the most general IG can be done by use of adjoint actions and coefficients g_i, $i = 1, \ldots, k$, appears in the list of inequivalent one-dimensional subalgebras. On utilizing the inequivalent one-dimensional subalgebras, the group invariant reductions and corresponding invariant solutions can be determined.

2.9 Extension of 1LGTs and prolongation formulas

Consider a $\tilde{s}^{th}$ order PDE as given by

$$\Xi(\bar{\phi}, \bar{\varrho}, \partial\bar{\varrho}, ..., \partial^{\tilde{s}}\bar{\varrho}) = 0, \tag{2.17}$$

where $\bar{\phi}$ corresponds to independent variables $\bar{\phi}_1, \bar{\phi}_2, ..., \bar{\phi}_n$ and $\bar{\varrho}$ corresponds to dependent variable. The term $\partial^j\bar{\varrho}$ denotes j^{th} order partial derivative w.r.t. $\bar{\phi}_i$. The 1LGTs (point) are represented by

$$\begin{aligned} \bar{\phi}_i^* &= \Omega_i(\bar{\phi}, \bar{\varrho}; \epsilon) = \bar{\phi}_i + \epsilon\, \mathcal{X}_i(\bar{\phi}, \bar{\varrho}) + O(\epsilon^2), i = 1, 2, ..., n \\ \bar{\varrho}^* &= \Phi(\bar{\phi}, \bar{\varrho}; \epsilon) = \bar{\varrho} + \epsilon\, \mathcal{Z}(\bar{\phi}, \bar{\varrho}) + O(\epsilon^2), \end{aligned} \tag{2.18}$$

where $\mathcal{X}_i$, $\mathcal{Z}$ are termed as infinitesimals corresponding to ϕ_i's and $\bar{\varrho}$, respectively, in $(\bar{\phi}, \bar{\varrho})$-space. The IG for 1LGTs (2.18) is given by

$$X = \mathcal{X}_i(\bar{\phi}, \bar{\varrho})\, \frac{\partial}{\partial\bar{\phi}_i} + \mathcal{Z}(\bar{\phi}, \bar{\varrho})\, \frac{\partial}{\partial\bar{\varrho}}. \tag{2.19}$$

The $\tilde{s}^{th}$ extension [40] of point transformations given in Eqs. (2.18) is represented as follows

$$\begin{aligned} &\bar{\phi}_i^* = \Omega_i(\bar{\phi}, \bar{\varrho}; \epsilon) = \bar{\phi}_i + \epsilon\, \mathcal{X}_i(\bar{\phi}, \bar{\varrho}) + O(\epsilon^2), \\ &\bar{\varrho}^* = \Phi(\bar{\phi}, \bar{\varrho}; \epsilon) = \bar{\varrho} + \epsilon\, \mathcal{Z}(\bar{\phi}, \bar{\varrho}) + O(\epsilon^2), \\ &\bar{\varrho}_i^* = \Phi_i(\bar{\phi}, \bar{\varrho}, \partial\bar{\varrho}; \epsilon) = \bar{\varrho}_i + \epsilon\, \mathcal{Z}_i^{(1)}(\bar{\phi}, \bar{\varrho}, \partial\bar{\varrho}) + O(\epsilon^2), \\ &\vdots \\ &\bar{\varrho}^*_{i_1 i_2 \ldots i_{\tilde{s}}} = \Phi_{i_1 i_2 \ldots i_{\tilde{s}}}(\bar{\phi}, \bar{\varrho}, \partial\bar{\varrho}, ..., \partial^{\tilde{s}}\bar{\varrho}; \epsilon) = \bar{\varrho}_{i_1 i_2 \ldots i_{\tilde{s}}} + \epsilon\, \mathcal{Z}^{(\tilde{s})}_{i_1 i_2 \ldots i_{\tilde{s}}}(\bar{\phi}, \bar{\varrho}, \partial\bar{\varrho}, ..., \partial^{\tilde{s}}\bar{\varrho}) + O(\epsilon^2), \end{aligned} \tag{2.20}$$

where i varies from 1 to n and i_l for one particular l varies from 1 to $\tilde{s}$ with $\tilde{s} \geq 1$. The corresponding $\tilde{s}^{th}$-extended IG is represented as follows

$$\begin{aligned} X^{(\tilde{s})} = {} & \mathcal{X}_i(\bar{\phi}, \bar{\varrho}) \frac{\partial}{\partial \bar{\phi}_i} + \mathcal{Z}(\bar{\phi}, \bar{\varrho}) \frac{\partial}{\partial \bar{\varrho}} + \mathcal{Z}_i^{(1)}(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}) \frac{\partial}{\partial \bar{\varrho}_i} + \cdots \\ & + \mathcal{Z}^{(\tilde{s})}_{i_1 i_2 \ldots i_{\tilde{s}}}(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}, ..., \partial^{\tilde{s}} \bar{\varrho}) \frac{\partial}{\partial \bar{\varrho}_{i_1 i_2 \ldots i_{\tilde{s}}}}, \ \ \tilde{s} \geq 1 \end{aligned} \tag{2.21}$$

The extended infinitesimals $\mathcal{Z}^{(\tilde{s})}$ can be found by following recurrence relations

$$\begin{aligned} \mathcal{Z}_i^{(1)} &= D_i \mathcal{Z} - (D_i \mathcal{X}_j) \bar{\varrho}_j, \ \ i = 1, 2, ..., n, \\ \mathcal{Z}^{(\tilde{s})}_{i_1 i_2 \ldots i_{\tilde{s}}} &= D_{i_s} \mathcal{Z}^{(\tilde{s}-1)}_{i_1 i_2 \ldots i_{\tilde{s}-1}} - (D_{i_{\tilde{s}}} \mathcal{X}_j) \bar{\varrho}_{i_1 i_2 \ldots i_{\tilde{s}} j}, \end{aligned} \tag{2.22}$$

i_l for one particular l varies from 1 to $\tilde{s}$ with $\tilde{s} \geq 2$.
Herein, D_i corresponds to the total derivative operator and is defined as follows

$$D_i = \frac{\partial}{\partial \bar{\phi}_i} + \bar{\varrho}_i \frac{\partial}{\partial \bar{\varrho}} + \bar{\varrho}_{ij} \frac{\partial}{\partial \bar{\varrho}_j} + ... + \bar{\varrho}_{i i_1 i_2 \ldots i_n} \frac{\partial}{\partial \bar{\varrho}_{i i_1 i_2 \ldots i_n}} + ..., \ \ i = 1, 2, ..., n. \tag{2.23}$$

In case of a system PDEs having $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2, ..., \bar{\phi}_n)$ and $\bar{\varrho} = (\bar{\varrho}^1, \bar{\varrho}^2, ..., \bar{\varrho}^\kappa)$ as independent and dependent variables, respectively, with $\kappa \geq 2$ then extended transformation from $(\bar{\phi}, \bar{\varrho})$ space to $(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}, \partial^2 \bar{\varrho}, ..., \partial^{\tilde{s}} \bar{\varrho})$-space are given by

$$\begin{aligned} & \bar{\phi}_i^* = \Omega_i(\bar{\phi}, \bar{\varrho}; \epsilon) = \bar{\phi}_i + \epsilon \, \mathcal{X}_i(\bar{\phi}, \bar{\varrho}) + O(\epsilon^2), \\ & (\bar{\varrho}^\varsigma)^* = \Phi^\varsigma(\bar{\phi}, \bar{\varrho}; \epsilon) = \bar{\varrho}^\varsigma + \epsilon \, \mathcal{Z}^\varsigma(\bar{\phi}, \bar{\varrho}) + O(\epsilon^2), \\ & (\bar{\varrho}_i^\varsigma)^* = \Phi_i^\varsigma(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}; \epsilon) = \bar{\varrho}_i^\varsigma + \epsilon \, \mathcal{Z}_i^{(1)\varsigma}(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}) + O(\epsilon^2), \\ & \vdots \\ & (\bar{\varrho}^\varsigma_{i_1 i_2 \ldots i_{\tilde{s}}})^* = \Phi^\varsigma_{i_1 i_2 \ldots i_{\tilde{s}}}(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}, ..., \partial^{\tilde{s}} \bar{\varrho}; \epsilon) = \bar{\varrho}_{i_1 i_2 \ldots i_{\tilde{s}}} + \epsilon \, \mathcal{Z}^{(\tilde{s})\varsigma}_{i_1 i_2 \ldots i_{\tilde{s}}}(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}, ..., \partial^{\tilde{s}} \bar{\varrho}) + O(\epsilon^2), \end{aligned} \tag{2.24}$$

with the extended infinitesimals are given by

$$\begin{aligned} \mathcal{Z}_i^{(1)\varsigma} &= D_i \mathcal{Z}^\varsigma - (D_i \mathcal{X}_j) \bar{\varrho}_j^\varsigma, \\ \mathcal{Z}^{(\tilde{s})\varsigma}_{i_1 i_2 \ldots i_{\tilde{s}}} &= D_{i_{\tilde{s}}} \mathcal{Z}^{(\tilde{s}-1)\varsigma}_{i_1 i_2 \ldots i_{\tilde{s}-1}} - (D_{i_{\tilde{s}}} \mathcal{X}_j) \bar{\varrho}^\varsigma_{i_1 i_2 \ldots i_{\tilde{s}} j}, \end{aligned} \tag{2.25}$$

where $\varsigma = 1, 2, ..., \kappa$, i_l for one particular l varies from 1 to $\tilde{s}$ with $\tilde{s} \geq 2$. Here, the $\tilde{s}^{th}$-extended IG is represented as follows

$$\begin{aligned} X^{(\tilde{s})} = {} & \mathcal{X}_i(\bar{\phi}, \bar{\varrho}) \frac{\partial}{\partial \bar{\phi}_i} + \mathcal{Z}^{\varsigma}(\bar{\phi}, \bar{\varrho}) \frac{\partial}{\partial \bar{\varrho}^{\varsigma}} + \mathcal{Z}^{(1)\varsigma}_i(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}) \frac{\partial}{\partial \bar{\varrho}^{\varsigma}_i} + \cdots \\ & + \mathcal{Z}^{(\tilde{s})\varsigma}_{i_1 i_2 \ldots i_{\tilde{s}}}(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}, ..., \partial^{\tilde{s}} \bar{\varrho}) \frac{\partial}{\partial \bar{\varrho}^{\varsigma}_{i_1 i_2 \ldots i_{\tilde{s}}}}, \ \tilde{s} \geq 1 \end{aligned} \tag{2.26}$$

2.10 Invariant solutions

For invariant solutions, consider N number of PDEs having $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2, ..., \bar{\phi}_n)$ and $\bar{\varrho} = (\bar{\varrho}^1, \bar{\varrho}^2, ..., \bar{\varrho}^{\kappa})$ as independent and dependent variables, respectively, represented as follows

$$\Xi^{\varsigma}(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}, ..., \partial^{\tilde{s}} \bar{\varrho}) = 0, \ \varsigma = 1, 2, ..., N, \tag{2.27}$$

where ς denotes one particular PDE out of N PDEs, the term $\partial^j \bar{\varrho} = \frac{\partial^j \bar{\varrho}}{\partial \bar{\phi}_{i_1} \partial \bar{\phi}_{i_2} \ldots \partial \bar{\phi}_{i_j}}$, $i_j = 1, 2, ..., n$ for every j varies from 1 to $\tilde{s}$. In this case, let us write a 1LGTs in the subsequent form

$$\begin{aligned} \bar{\phi}^* &= \Omega(\bar{\phi}, \bar{\varrho}; \epsilon), \\ \bar{\varrho}^* &= \Phi(\bar{\phi}, \bar{\varrho}; \epsilon). \end{aligned} \tag{2.28}$$

Assume, the IG of 1LGTs (2.28) as follows

$$X = \mathcal{X}_i(\bar{\phi}, \bar{\varrho}) \frac{\partial}{\partial \bar{\phi}_i} + \mathcal{Z}^{\bar{\nu}}(\bar{\phi}, \bar{\varrho}) \frac{\partial}{\partial \bar{\varrho}^{\bar{\nu}}} \tag{2.29}$$

and the $\tilde{s}^{th}$-extended IG of Eq. (2.29) is represented by following form

$$\begin{aligned} X^{(\tilde{s})} = {} & \mathcal{X}_i(\bar{\phi}, \bar{\varrho}) \frac{\partial}{\partial \bar{\phi}_i} + \mathcal{Z}^{\bar{\nu}}(\bar{\phi}, \bar{\varrho}) \frac{\partial}{\partial \bar{\varrho}^{\bar{\nu}}} + \mathcal{Z}^{(1)\bar{\nu}}_i(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}) \frac{\partial}{\partial \bar{\varrho}^{\bar{\nu}}_i} + \cdots \\ & + \mathcal{Z}^{(\tilde{s})\bar{\nu}}_{i_1 i_2 \ldots . i_{\tilde{s}}}(\bar{\phi}, \bar{\varrho}, \partial \bar{\varrho}, ..., \partial^{\tilde{s}} \bar{\varrho}) \frac{\partial}{\partial \bar{\varrho}^{\bar{\nu}}_{i_1 i_2 \ldots . i_{\tilde{s}}}}, \end{aligned} \tag{2.30}$$

where $\mathcal{Z}^{(1)\bar{\nu}}_i$ and $\mathcal{Z}^{(\tilde{s})\bar{\nu}}_{i_1 i_2 \ldots . i_{\tilde{s}}}$ are given by (2.25), $\bar{\nu} = 1, 2, ..., \kappa$ and $i_j = 1, 2, ..., n$ for $j = 1, 2, ..., \tilde{s}$.

The function $\bar{\varrho} = \Theta(\bar{\phi})$ is said to be invariant solution of PDEs (2.27) and it remains invariant under 1LGTs (2.28), iff $\bar{\varrho} = \Theta(\bar{\phi})$ satisfies the following conditions

(i)

$$\mathcal{X}_i(\bar{\phi}, \Theta(\bar{\phi})) \frac{\partial \Theta(\bar{\phi})}{\partial \bar{\phi}_i} = \mathcal{Z}^{\bar{\nu}}(\bar{\phi}, \Theta(\bar{\phi})), \ \bar{\nu} = 1, 2, ..., \kappa. \tag{2.31}$$

(ii)

$$\Xi^{\varsigma}(\bar{\phi}, \Theta(\bar{\phi}), \partial \Theta(\phi), ..., \partial^{\tilde{s}} \Theta(\phi)) = 0, \ \varsigma = 1, 2, ..., N. \tag{2.32}$$

Eqs. (2.31) represent invariant surface conditions for invariant solutions of PDEs (2.32). These conditions result from the invariance under the point symmetry (2.29). The invariant form method [40] can be used to find invariant solutions. The set procedure [40] for applying invariant form method using following characteristic equations

$$\frac{d\bar{\phi}_1}{\mathcal{X}_1(\bar{\phi},\bar{\varrho})}=\frac{d\bar{\phi}_2}{\mathcal{X}_2(\bar{\phi},\bar{\varrho})}=...=\frac{d\bar{\phi}_n}{\mathcal{X}_n(\bar{\phi},\bar{\varrho})}=\frac{d\bar{\varrho}^1}{\mathcal{Z}^1(\bar{\phi},\bar{\varrho})}=\frac{d\bar{\varrho}^2}{\mathcal{Z}^2(\bar{\phi},\bar{\varrho})}=...=\frac{d\bar{\varrho}^\kappa}{\mathcal{Z}^\kappa(\bar{\phi},\bar{\varrho})}. \tag{2.33}$$

Theorem 2.10.1. *A connected group of transformations G is a symmetry group of a non-degenerate system of PDEs* (2.27) *if and only if for every IG X of G*

$$X^{(\bar{s})}(\Xi^\varsigma(\bar{\phi},\bar{\varrho},\partial\bar{\varrho},...,\partial^{\tilde{s}}\bar{\varrho}))=0,\ \varsigma=1,2,...,N, \tag{2.34}$$

for all the solutions $\bar{\varrho}=f(\bar{\phi})$ of the system (2.27)*.*
The Eqs. (2.34) *are the determining equations of the symmetry group to $\Xi^\varsigma=0$. For non-degenerate systems, this is equivalent to*

$$X^{(\bar{s})}(\Xi^\varsigma)=A\cdot\Xi^\varsigma=\sum_{\nu=1}^{N}A^\nu\Xi^\nu. \tag{2.35}$$

Maximal rank: Let the system of PDEs (2.27) is said to be of maximal rank if $N\times(l+\kappa l^{(\bar{s})})$ Jacobin matrix

$$J_\Xi(\bar{\phi},\bar{\varrho},\partial\bar{\varrho},...,\partial^{\tilde{s}}\bar{\varrho})=(\frac{\partial\Xi^\varsigma}{\partial\bar{\phi}^i},\frac{\partial\Xi^\varsigma}{\partial\bar{\varrho}^\varsigma_j}) \tag{2.36}$$

of Ξ with respect to all variables $(\bar{\phi},\bar{\varrho},\partial\bar{\varrho},...,\partial^{\tilde{s}}\bar{\varrho})$ is of rank N, whenever $\Xi=0$. Often one says somewhat loosely, that the infinitesimal invariance conditions (2.34) should hold "whenever $(\bar{\phi},\bar{\varrho}^{(\bar{s})})$ satisfy (2.27)." It is not true in general that if $\bar{\phi}_0$ is point in $\bar{\phi}=\mathbf{R^n}$ and $\bar{\varrho}_0^{(\bar{s})}$ is collection of derivatives of $\bar{\varrho}$ at $\bar{\phi}_0$ that satisfy the algebraic conditions imposed by the system (2.27), then there exists a smooth solution $\bar{\varrho}=f(\bar{\phi})$ to the system whose derivatives at the point $\bar{\phi}_0$ agree with the values of $\bar{\varrho}_0^{(\bar{s})}$. A point $(\bar{\phi}_0,\bar{\varrho}_0^{(\bar{s})})$, which does satisfy this condition, and so pertains to an actual solution $\bar{\varrho}=f(\bar{\phi})$ of the system, is said to be a point of local solvability of the system. Thus the local solvability condition can be stated as follows.
Local Solvability: Any each point $(\bar{\phi}_0,\bar{\varrho}_0^{(\bar{s})})$ such that

$$\Xi(\bar{\phi}_0,\bar{\varrho}_0^{(\bar{s})})=0, \tag{2.37}$$

there exists a solution $\bar{\varrho} = f(\bar{\phi})$ with

$$\bar{\varrho}_0^{(\tilde{s})} = X^{(\tilde{s})} f(\bar{\phi}_0). \tag{2.38}$$

Non-degenerate = maximal rank+locally solvable.

2.11 Lie infinitesimal criterion for system of integer order PDEs

This part of the chapter is briefed the various important steps of Lie infinitesimal criterion for integer order PDEs [40, 219] as follows

1. Consider 1LGTs (2.28) for invariance of PDEs (2.27) of maximal rank.
2. Write down the corresponding IG (2.29) and its $\tilde{s}^{th}$ order extension (2.30)
3. Apply the extended operator $X^{(\tilde{s})}$ (2.34) [219] for a symmetry group of the system (2.27) along with the conditions (2.36) and (2.37).
4. Using the extended infinitesimals, obtain over-determining system of linear PDEs in terms of $\mathcal{X}$ and $\mathcal{Z}$.
5. And the solution of over-determining equations yields the expressions for $\mathcal{X}$ and $\mathcal{Z}$.
6. By the use of characteristic Eqs. (2.33), $\bar{\varrho}$ can be expressed as a function of new independent variables.
7. Rewrite the given system (2.27) in terms of new coordinates to get the reduced system of PDEs.

2.12 Lie infinitesimal criterion for system of time fractional PDEs

This subsection describes the important steps to calculate infinitesimal symmetries of time fractional PDEs [151, 174, 276, 300] having l independent variables $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2, ..., \bar{\phi}_l)$ and κ dependent variables $\bar{\varrho} = (\bar{\varrho}^1, \bar{\varrho}^2, ..., \bar{\varrho}^\kappa)$ as follows

$$\Xi^\varsigma(\bar{\phi}, \bar{\varrho}, D^\alpha_{\bar{\phi}_1}\bar{\varrho}, \bar{\varrho}_{\bar{\phi}_i}, ...) = 0, \ \varsigma = 1, 2, ..., \kappa, \ i = 2, 3, ..., l, \tag{2.39}$$

where $\alpha > 0$ and $D^{\alpha}_{\bar{\phi}_1}$ denotes RLF derivative [226] with respect to $\bar{\phi}_1$ of order α and other subscripts denote the partial derivatives of integer orders, i.e., $\bar{\varrho}_{\bar{\phi}_i} = \dfrac{\partial \bar{\varrho}}{\partial \bar{\phi}_i}$. Here, $\bar{\phi}_1$ is considered as the time variable and other, $\bar{\phi}_i$ $(i = 2, 3, ...)$ are considered as the space variables. The RLF derivative [226] of order α (> 0) is given by following

$$D^{\alpha}_{\bar{\phi}_1} f(\bar{\phi}_1) = \begin{cases} \frac{d^n}{d\bar{\phi}_1^n} f(\bar{\phi}_1), & \alpha = n, \text{ where } n \in \mathbb{N}; \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\bar{\phi}_1^n} \int_0^{\bar{\phi}_1} \frac{f(\theta)}{(\bar{\phi}_1 - \theta)^{\alpha+1-n}} d\theta, & n-1 < \alpha < n,\ n \in \mathbb{N}. \end{cases} \tag{2.40}$$

RLF derivative exhibits the following property [226]

$$D^{\alpha} \bar{\phi}_1^{\bar{\zeta}} = \frac{\Gamma(\bar{\zeta}+1) \bar{\phi}_1^{\bar{\zeta}-\alpha}}{\Gamma(\bar{\zeta}-\alpha+1)},\ \bar{\zeta} > \alpha - 1, \tag{2.41}$$

where the symbol $\Gamma(\cdot)$ represents gamma function.
Let us assume 1LGTs (point) [151, 244, 247, 276] which leaves the system (2.39) in an invariant form and can be described as follows

$$\begin{aligned} \tilde{\bar{\phi}}_i &\to \bar{\phi}_i + \varepsilon \mathcal{X}_i(\bar{\phi}, \bar{\varrho}) + O(\varepsilon^2),\ i = 1, 2, ..., l \\ \tilde{\bar{\varrho}}^{\varsigma} &\to \bar{\varrho}^{\varsigma} + \varepsilon \mathcal{Z}^{\varsigma}(\bar{\phi}, \bar{\varrho}) + O(\varepsilon^2), \varsigma = 1, 2, ..., \kappa, \end{aligned} \tag{2.42}$$

where $\mathcal{X}_i$, $\mathcal{Z}^{\varsigma}$ are infinitesimals corresponding to various ϕ_i's and $\bar{\varrho}^i$'s, respectively, and group parameter $\varepsilon << 1$. The IG X for 1LGTs is defined by the following expression

$$X = \mathcal{X}_i(\bar{\phi}, \bar{\varrho}) \frac{\partial}{\partial_{\bar{\phi}_i}} + \mathcal{Z}^{\varsigma}(\bar{\phi}, \bar{\varrho}) \frac{\partial}{\partial_{\bar{\varrho}^{\varsigma}}}, \tag{2.43}$$

where $\mathcal{X}_i = \dfrac{d\tilde{\bar{\phi}}_i}{d\epsilon}|_{\epsilon=0}$ and $\mathcal{Z}^{\varsigma} = \dfrac{d\tilde{\bar{\varrho}}^{\varsigma}}{d\epsilon}|_{\epsilon=0}$.

Theorem 2.12.1. *A connected group of transformations G is a symmetry group of a non-degenerate system of time fractional PDEs* (2.39) *if and only if*

$$Pr^{(n,\alpha)} X(\Xi^{\varsigma}(\bar{\phi}, \bar{\varrho}, D^{\alpha}_{\bar{\phi}_1} \bar{\varrho}, \bar{\varrho}_{\bar{\phi}_i}, ...)) = 0,\ n = 1, 2, ..., \tag{2.44}$$

whenever $\bar{\varrho}$ is a solution to the system (2.39) *for every IG X of G. The Eqs.* (2.44) *are the determining equations of the symmetry group to $\Xi^{\varsigma} = 0$. For non-degenerate systems, this is equivalent to*

$$Pr^{(n,\alpha)} X(\Xi^{\varsigma}) = A \cdot \Xi^{\varsigma} = \sum_{\nu=1}^{N} A^{\nu} \Xi^{\nu}. \tag{2.45}$$

Maximal rank: Let system (2.39) is said to be of maximal rank if $N \times (l + \kappa l^{(\bar{s})})$ Jacobin matrix

$$J_{\Xi}(\bar{\phi}, \bar{\varrho}, D^{\alpha}_{\bar{\phi}_1}\bar{\varrho}, \bar{\varrho}_{\bar{\phi}_i}, ...) = (\frac{\partial^{\alpha}\Xi^{\varsigma}}{\partial\bar{\phi}_1^{\alpha}}, \frac{\partial\Xi^{\varsigma}}{\partial\bar{\phi}_2}, \frac{\partial\Xi^{\varsigma}}{\partial\bar{\phi}_3}, ..., \frac{\partial\Xi^{\varsigma}}{\partial\bar{\phi}_l}, \frac{\partial\Xi^{\varsigma}}{\partial\bar{\varrho}_j^{\varsigma}}) \tag{2.46}$$

of Ξ with respect to all variables $(\bar{\phi}, \bar{\varrho}, D^{\alpha}_{\bar{\phi}_1}\bar{\varrho}, \bar{\varrho}_{\bar{\phi}_i}, ...)$ is of rank N, whenever $\Xi = 0$.
Local Solvability: Any each point $(\bar{\phi}_0, \bar{\varrho}_0^{(\bar{s})})$ such that

$$\Xi(\bar{\phi}_0, \bar{\varrho}_0^{(\bar{s})}) = 0, \tag{2.47}$$

there exists a solution $\bar{\varrho} = f(\bar{\phi})$ with

$$\bar{\varrho}_0^{(\bar{s})} = X^{(\bar{s})} f(\bar{\phi}_0). \tag{2.48}$$

On the basis of group of transformations (2.42), the structure of RLF derivative is framed. It should be noticed that, lower limit of integral in RLF derivative is fixed, so it must be invariant under the application of 1LGTs given in Eq. (2.42).
Thus, invariance condition is appeared finally as follows

$$\mathcal{X}_1(\bar{\phi}, \bar{\varrho})|_{\bar{\phi}_1 = 0} = 0. \tag{2.49}$$

The extended infinitesimals can be found by applying theorem 2.12.2 given below.

Theorem 2.12.2. *[276] The* α^{th} *extended infinitesimals* $(\mathcal{Z}^{\varsigma})^{\alpha,\bar{\phi}_1} = \frac{d}{d\epsilon}\left(D^{\alpha}_{\bar{\phi}_1}\tilde{\bar{\varrho}}^{\varsigma}(\tilde{\bar{\phi}})\right)|_{\epsilon=0}$*, for* $\alpha > 0$ $(\varsigma = 1, 2, ..., \kappa)$ *with general IG* (2.43) *are defined as follows*

$$\begin{aligned}
(\mathcal{Z}^{\varsigma})^{\alpha,\bar{\phi}_1} = {} & \frac{\partial^{\alpha}\mathcal{Z}^{\varsigma}}{\partial\bar{\phi}_1^{\alpha}} + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^{n}\mathcal{Z}^{\varsigma}_{\bar{\varrho}^{\varsigma}}}{\partial\bar{\phi}_1^{n}} - \binom{\alpha}{n+1}D^{n+1}_{\bar{\phi}_1}(\mathcal{X}_1)\right]\partial^{\alpha-n}_{\bar{\phi}_1}(\bar{\varrho}^{\varsigma}) \\
& + \sum_{s\neq\varsigma,s=1}^{m}\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^{n}\mathcal{Z}^{\varsigma}_{\bar{\varrho}^{s}}}{\partial\bar{\phi}_1^{n}}\partial^{\alpha-n}_{\bar{\phi}_1}(\bar{\varrho}^{s}) - \sum_{j\neq 1}\sum_{n=1}^{\infty}\binom{\alpha}{n}D^{n}_{\bar{\phi}_1}\mathcal{X}_j D^{\alpha-n}_{\bar{\phi}_1}\bar{\varrho}^{\varsigma}_{\bar{\phi}_j} \\
& + \left(\mathcal{Z}^{\varsigma}_{\bar{\varrho}^{\varsigma}} - \alpha D_{\bar{\phi}_1}(\mathcal{X}_{\bar{\phi}_1})\right)\partial^{\alpha}_{\bar{\phi}_1}\bar{\varrho}^{\varsigma} - \bar{\varrho}^{\varsigma}\frac{\partial^{\alpha}\mathcal{Z}^{\varsigma}_{\bar{\varrho}^{\varsigma}}}{\partial\bar{\phi}_1^{\alpha}} + \sum_{s\neq\varsigma,s=1}^{\kappa}\left(\mathcal{Z}^{\varsigma}_{\bar{\varrho}^{s}}\frac{\partial^{\alpha}\bar{\varrho}^{s}}{\partial\bar{\phi}_1^{\alpha}} - \bar{\varrho}^{s}\frac{\partial^{\alpha}\mathcal{Z}^{\varsigma}_{\bar{\varrho}^{s}}}{\partial\bar{\phi}_1^{\alpha}}\right) \\
& + \sum_{j=1}^{\kappa}\mu_{\varsigma,s},
\end{aligned} \tag{2.50}$$

where $\mathcal{Z}^{\varsigma}_{\bar{\varrho}^{s}} = \frac{\partial\mathcal{Z}^{\varsigma}}{\partial\bar{\varrho}^{s}}$, $D_{\bar{\phi}_1}$ *denotes total derivative operator and* $\mu_{\varsigma,s}$ *are defined as follows*

$$\mu_{\varsigma,s} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{\bar{\phi}_1^{n-\alpha}}{\Gamma(n+1-\alpha)}(-\bar{\varrho}^s)^r\frac{\partial^m}{\partial\bar{\phi}_1^m}((\bar{\varrho}^s)^{k-r})\frac{\partial^{n-m+k}\mathcal{Z}^\varsigma}{\partial\bar{\phi}_1^{n-m}(\partial\bar{\varrho}^s)^k}. \tag{2.51}$$

Proof. The prolongations for IG (2.43) are used as follows

$$(\mathcal{Z}^\varsigma)^{\alpha,\bar{\phi}_1} = D^\alpha_{\bar{\phi}_1}(\mathcal{Z}^\varsigma) + \sum_{j=1}^{l}\left(\mathcal{X}_j D^\alpha_{\bar{\phi}_1}\left(\bar{\varrho}^\varsigma_{\bar{\phi}_j}\right) - D^\alpha_{\bar{\phi}_1}\left(\mathcal{X}_j\bar{\varrho}^\varsigma_{\bar{\phi}_j}\right)\right), \tag{2.52}$$

where $\varsigma = 1, 2, ...\kappa$, and $\bar{\varrho}^\varsigma_{\bar{\phi}_j} = \dfrac{\partial\bar{\varrho}^\varsigma}{\partial\bar{\phi}_j}$.

The generalised Leibnitz rule [201, 226] for $\alpha \in \mathbb{R}$ can be written as follows

$$D^\alpha_\varpi(f(\varpi)g(\varpi)) = \sum_{n=0}^{\infty}\binom{\alpha}{n}D^n_\varpi f(\varpi)D^{\alpha-n}_\varpi g(\varpi), \tag{2.53}$$

where $D^n_\varpi f = \dfrac{d^n f}{d\varpi^n}$, $\dbinom{\alpha}{n} = \dfrac{\alpha\Gamma(\alpha)}{n(\alpha-n)\Gamma(n)\Gamma(\alpha-n)}$, for all $n \in \mathbb{N}$. Using (2.53), the following result is obtained

$$\mathcal{X}_1 D^\alpha_{\bar{\phi}_1}(\bar{\varrho}^\varsigma_{\bar{\phi}_1}) - D^\alpha_{\bar{\phi}_1}(\mathcal{X}_1\bar{\varrho}^\varsigma_{\bar{\phi}_1}) = -\alpha D_{\bar{\phi}_1}\mathcal{X}_1\partial^\alpha_{\bar{\phi}_1}\bar{\varrho}^\varsigma - \sum_{n=1}^{\infty}\binom{\alpha}{n+1}D^{n+1}_{\bar{\phi}_1}\mathcal{X}_1 D^{\alpha-n}_{\bar{\phi}_1}\bar{\varrho}^\varsigma, \tag{2.54}$$

where $D_{\bar{\phi}_1}$ denotes total differential operator with respect to $\bar{\phi}$ given by

$$D_{\bar{\phi}_1} = \partial_{\bar{\phi}_1} + \bar{\varrho}^\varsigma_{\bar{\phi}_1}\partial_{\bar{\varrho}^\varsigma} + \bar{\varrho}^\varsigma_{\bar{\phi}_1\bar{\phi}_1}\partial\bar{\varrho}^\varsigma_{\bar{\phi}_1} + \sum_{j\neq 1}\bar{\varrho}^\varsigma_{\bar{\phi}_j\bar{\phi}_1}\partial\bar{\varrho}^\varsigma_{\bar{\phi}_j} + ..., \tag{2.55}$$

Equivalently, for $j \neq 1$,

$$\mathcal{X}_j D^\alpha_{\bar{\phi}_1}(\bar{\varrho}^\varsigma_{\bar{\phi}_j}) - D^\alpha_{\bar{\phi}_1}(\mathcal{X}_j\bar{\varrho}^\varsigma_{\bar{\phi}_j}) = -\sum_{n=1}^{\infty}\binom{\alpha}{n}(D^n_{\bar{\phi}_1}\mathcal{X}_j)D^{\alpha-n}_{\bar{\phi}_1}\bar{\varrho}^\varsigma_{\bar{\phi}_j}. \tag{2.56}$$

The Eq. (2.52) can be expressed as follows

$$
\begin{aligned}
(\mathcal{Z}^{\varsigma})^{\alpha,\bar{\phi}_1} = {} & D^{\alpha}_{\bar{\phi}_1}(\mathcal{Z}^{\varsigma}) - \alpha(D_{\bar{\phi}_1}\mathcal{X}_1)\partial^{\alpha}_{\bar{\phi}_1}\bar{\varrho}^{\varsigma} - \sum_{j\neq 1, j=2}^{l}\sum_{n=1}^{\infty}\binom{\alpha}{n}(D^{n}_{\bar{\phi}_1}\mathcal{X}_j)D^{\alpha-n}_{\bar{\phi}_1}\bar{\varrho}^{\varsigma}_{\bar{\phi}_j} \\
& -\sum_{n=1}^{\infty}\binom{\alpha}{n+1}(D^{n+1}_{x_1}\mathcal{X}_1)D^{\alpha-n}_{x_1}\bar{\varrho}^{\varsigma}.
\end{aligned}
\tag{2.57}
$$

The extended form of the formula form time fractional derivative is given by Osler [221] for $\alpha > 0$ and is quoted as follows

$$
\begin{aligned}
D^{\alpha}_{\bar{\phi}_1} f(\bar{\phi}_1, g_1(\bar{\phi}_1), g_2(\bar{\phi}_1), ..., g_l(\bar{\phi}_1)) = {} & \sum_{j=1}^{l}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{r=0}^{k}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r} \\
& \frac{\bar{\phi}_1^{n-\alpha}}{k!\Gamma(n+1-\alpha)}(-g_j)^r\frac{d^m}{d\bar{\phi}_1^m}(g_j^{k-r})\frac{\partial^{n-m+k}f(\bar{\phi}_1, g_1, ..., g_l)}{(\partial\bar{\phi}_1)^{n-m}(\partial g_j)^k}.
\end{aligned}
\tag{2.58}
$$

Since $\mathcal{Z}^{\varsigma} = \mathcal{Z}^{\varsigma}(\bar{\phi}_1, ..., \bar{\phi}_l, \bar{\varrho}^1, ..., \bar{\varrho}^{\kappa})$, using (2.58) and the generalised multi-variable chain rule [216, 300], the term $D^{\alpha}_{\bar{\phi}_1}(\mathcal{Z}^{\varsigma})$ in (2.57) can be obtained as follows

$$
D^{\alpha}_{\bar{\phi}_1}(\mathcal{Z}^{\varsigma}) = \frac{\partial^{\alpha}\mathcal{Z}^{\varsigma}}{\partial\bar{\phi}_1^{\alpha}} + \sum_{s=1}^{\kappa}\left[\left(\mathcal{Z}^{\varsigma}_{\bar{\varrho}^s}\frac{\partial^{\alpha}\bar{\varrho}^s}{\partial\bar{\phi}_1^{\alpha}} - \bar{\varrho}^s\frac{\partial^{\alpha}\mathcal{Z}^{\varsigma}_{\bar{\varrho}^s}}{\partial\bar{\phi}_1^{\alpha}}\right) - \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\mathcal{Z}^{\varsigma}_{\bar{\varrho}^s}}{\partial\bar{\phi}_1^n}D^{\alpha-n}_{\bar{\phi}_1}(\bar{\varrho}^s) + \mu_{\varsigma,s}\right],
\tag{2.59}
$$

where $\mathcal{Z}^{\varsigma}_{\bar{\varrho}^s} = \frac{\partial\mathcal{Z}^{\varsigma}}{\partial\bar{\varrho}^s}$ and the exact form of $\mu_{\varsigma,s}$ is given as follows

$$
\mu_{\varsigma,s} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{\bar{\phi}_1^{n-\alpha}}{\Gamma(n+1-\alpha)}(-\bar{\varrho}^s)^r\frac{\partial^m}{\partial\bar{\phi}_1^m}((\bar{\varrho}^s)^{k-r})\frac{\partial^{n-m+k}\mathcal{Z}^{\varsigma}}{\partial\bar{\phi}_1^{n-m}(\partial\bar{\varrho}^s)^k}.
\tag{2.60}
$$

Therefore, the expression for $(\mathcal{Z}^{\varsigma})^{\alpha,\bar{\phi}_1}$ is expressed as follows

$$
\begin{aligned}
(\mathcal{Z}^{\varsigma})^{\alpha,\bar{\phi}_1} = {} & \frac{\partial^{\alpha}\mathcal{Z}^{\varsigma}}{\partial\bar{\phi}_1^{\alpha}} + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n\mathcal{Z}^{\varsigma}_{\bar{\varrho}^{\varsigma}}}{\partial\bar{\phi}_1^n} - \binom{\alpha}{n+1}D^{n+1}_{\bar{\phi}_1}(\mathcal{X}_1)\right]\partial^{\alpha-n}_{\bar{\phi}_1}(\bar{\varrho}^{\varsigma}) \\
& + \sum_{s\neq\varsigma, s=1}^{\kappa}\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\mathcal{Z}^{\varsigma}_{\bar{\varrho}^s}}{\partial\bar{\phi}_1^n}\partial^{\alpha-n}_{\bar{\phi}_1}(\bar{\varrho}^s) - \sum_{j\neq 1}\sum_{n=1}^{\infty}\binom{\alpha}{n}D^{n}_{\bar{\phi}_1}\mathcal{X}_j D^{\alpha-n}_{\bar{\phi}_1}\bar{\varrho}^{\varsigma}_{\bar{\phi}_j} \\
& + \left(\mathcal{Z}^{\varsigma}_{\bar{\varrho}^{\varsigma}} - \alpha D_{\bar{\phi}_1}(\mathcal{X}_1)\right)\partial^{\alpha}_{\bar{\phi}_1}\bar{\varrho}^{\varsigma} - \bar{\varrho}^{\varsigma}\frac{\partial^{\alpha}\mathcal{Z}^{\varsigma}_{\bar{\varrho}^{\varsigma}}}{\partial\bar{\phi}_1^{\alpha}} + \sum_{s\neq\varsigma, s=1}^{\kappa}\left(\mathcal{Z}^{\varsigma}_{\bar{\varrho}^s}\frac{\partial^{\alpha}\bar{\varrho}^s}{\partial\bar{\phi}_1^{\alpha}} - \bar{\varrho}^s\frac{\partial^{\alpha}\mathcal{Z}^{\varsigma}_{\bar{\varrho}^s}}{\partial\bar{\phi}_1^{\alpha}}\right) \\
& + \sum_{s=1}^{\kappa}\mu_{\varsigma,s}.
\end{aligned}
\tag{2.61}
$$

Hence, completes the proof. □

Back substitution of the fractional extended infinitesimals and integer extended infinitesimals into the invariance condition (2.44) gives the over-determining equations whose solution with additional invariance condition gives the required infinitesimals. The invariants retrieved from characteristic equations provide reduced system of fractional PDEs.

In the next section of the chapter, algorithms for various methods to find solutions of fractional PDEs are described in a systematic way. In the other chapters of the thesis, these methods are successfully applied and discussed with suitable examples.

2.13 Solutions for fractional PDEs

This section involves the methods which are being used for exact solutions of fractional PDEs. The bright, dark and singular solitary, and exact travelling waves by exponential rational function method for time fractional PDEs are presented. Also, an improved F-expansion method is proposed to analyse space-time fractional PDEs for exact solutions. The EJEFE method gives doubly periodic wave type solutions and described systematically for space-time fractional PDEs.

2.13.1 Solitary wave solutions of time fractional PDEs

This subsection describes the solitary wave solutions of time fractional PDEs of order α having three independent (ϕ_1, ϕ_2, ϕ_3) and three dependent variables $(\varrho_1, \varrho_2, \varrho_3)$ as follows

$$\begin{aligned} &\Xi^1(\phi_1, D^\alpha_{\phi_1}\varrho_1, \phi_2, \varrho_{1\phi_2}, \phi_3, \varrho_{2\phi_2}, \varrho_1, \varrho_{3\phi_2}, \varrho_2, \varrho_3, ...) = 0,\\ &\Xi^2(\phi_1, D^\alpha_{\phi_1}\varrho_1, \phi_2, \varrho_{1\phi_2}, \phi_3, \varrho_{2\phi_2}, \varrho_1, \varrho_{3\phi_2}, \varrho_2, \varrho_3, ...) = 0,\\ &\Xi^3(\phi_1, D^\alpha_{\phi_1}\varrho_1, \phi_2, \varrho_{1\phi_2}, \phi_3, \varrho_{2\phi_2}, \varrho_1, \varrho_{3\phi_2}, \varrho_2, \varrho_3, ...) = 0. \end{aligned} \tag{2.62}$$

The fractional complex transformation [115, 176, 243, 282, 326] connecting the variables is defined as folllows

$$\varrho_1 = f(\zeta),\ \varrho_2 = g(\zeta),\ \varrho_3 = h(\zeta),\ \zeta = k_1\phi_2 + k_2\phi_3 + \frac{k_3\,{\phi_1}^\alpha}{\Gamma(1+\alpha)}, \tag{2.63}$$

where k_1, k_2 and k_3 are taken as arbitrary constants and ζ is complex variable. By using the chain rule

$$\begin{aligned} D^{\alpha}_{\phi_1}\varrho_1 &= (\sigma_1)_{\phi_1} D_{\zeta} f \, D^{\alpha}_{\phi_1}\zeta = (\sigma_1)_{\phi_1} f' k_3, \\ D^{\alpha}_{\phi_1}\varrho_2 &= (\sigma_2)_{\phi_1} D_{\zeta} g \, D^{\alpha}_{\phi_1}\zeta = (\sigma_2)_{\phi_1} g' k_3, \\ D^{\alpha}_{\phi_1}\varrho_3 &= (\sigma_3)_{\phi_1} D_{\zeta} h \, D^{\alpha}_{\phi_1}\zeta = (\sigma_3)_{\phi_1} h' k_3, \end{aligned} \tag{2.64}$$

where $(\sigma_1)_{\phi_1}$, $(\sigma_2)_{\phi_1}$ and $(\sigma_3)_{\phi_1}$ are the fractional indexes [115, 243] and without the loss of generality, consider $(\sigma_1)_{\phi_1} = (\sigma_2)_{\phi_1} = (\sigma_3)_{\phi_1} = \rho$, where ρ is the arbitrary constant, and prime $'$ denotes derivative of $f(\zeta)$, $g(\zeta)$, $h(\zeta)$ with respect to ζ. Thus, the transformation (2.63) leads to the ODEs for the given system (2.62) in following form

$$\begin{aligned} R^1(f, g, h, (\sigma_1)_{\phi_1} f' k_3, (\sigma_2)_{\phi_1} g' k_3, (\sigma_3)_{\phi_1} h' k_3, ..) &= 0, \\ R^2(f, g, h, (\sigma_1)_{\phi_1} f' k_3, (\sigma_2)_{\phi_1} g' k_3, (\sigma_3)_{\phi_1} h' k_3, ..) &= 0, \\ R^3(f, g, h, (\sigma_1)_{\phi_1} f' k_3, (\sigma_2)_{\phi_1} g' k_3, (\sigma_3)_{\phi_1} h' k_3, ..) &= 0. \end{aligned} \tag{2.65}$$

The obtained system of ODEs (2.65) has been tested to possess bright, dark and singular solitary wave solutions.

1. For bright solitary wave solution, let us consider

   $$f(\zeta) = A_1 \operatorname{sech}^{p_1}(\zeta), \ g(\zeta) = A_2 \operatorname{sech}^{p_2}(\zeta), \ h(\zeta) = A_3 \operatorname{sech}^{p_3}(\zeta), \tag{2.66}$$

   where A_i and p_i $(i = 1, 2, 3)$ termed as wave amplitudes and arbitrary constants, respectively. In Eq. (2.66), ζ is considered same as in Eq. (2.63). Back substitution of (2.66) into ODEs (2.65) gives some set of equations and then using balancing principle, the values for p_1, p_2 and p_3 are obtained. The substitution of assumed solution (2.66) with the values of p_1, p_2 and p_3, gives determining equations for A_i, $i = 1, 2, 3$. Solution of these equations gives corresponding values of A_1, A_2 and A_3. Thus, solution of given system of time fractional PDEs are obtained.

2. Similarly, for dark solitary wave solution, assume

   $$f(\zeta) = A_1 \tanh^{p_1}(\zeta), \ g(\zeta) = A_2 \tanh^{p_2}(\zeta), \ h(\zeta) = A_3 \tanh^{p_3}(\zeta). \tag{2.67}$$

   All other steps are same as mentioned for bright solitary wave solution.

3. For singular solitary wave solution, consider

$$f(\zeta) = A_1 \operatorname{csch}^{p_1}(\zeta), \; g(\zeta) = A_2 \operatorname{csch}^{p_2}(\zeta), \; h(\zeta) = A_3 \operatorname{csch}^{p_3}(\zeta). \tag{2.68}$$

2.13.2 Exponential rational function method for exact solutions of time fractional PDEs

This subsection deals with the procedure to obtain exact solutions for time fractional PDEs (2.62) with three independent and three dependent variables using the exponential rational function method [11, 28, 204, 286, 338]. To find the exact solutions, consider, the fractional complex transformation (2.63) that converts given fractional PDEs into ODEs (2.65).
The exact solution of ODEs (2.65) has been written as follows

$$f(\zeta) = \sum_{r=0}^{N_1} \frac{\mathcal{A}_r}{(1+e^{\zeta})^r}, \; g(\zeta) = \sum_{r=0}^{N_2} \frac{\mathcal{B}_r}{(1+e^{\zeta})^r}, \; h(\zeta) = \sum_{r=0}^{N_3} \frac{\mathcal{C}_r}{(1+e^{\zeta})^r}, \tag{2.69}$$

where $\mathcal{A}_r$, $\mathcal{B}_r$ and $\mathcal{C}_r$ $(\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r \neq 0)$ are the arbitrary constants.

- The upper limit in the summations N_1, N_2 and N_3 can be computed by balancing principle from obtained system of ODEs (2.65). This gives the values for N_1, N_2 and N_3.

- Then, substitution of system (2.69) into ODEs (2.65) and set of polynomial equations in $e^{j\zeta}$ is obtained. By equating the coefficients of $e^{j\zeta}$ to zero, algebraic equations are obtained in terms of unknown constants $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{C}_n$.

- Using symbolic computational software Maple, solutions of algebraic equations can be found. The solutions of reduced ODEs (2.65) can be computed.

- Using the solutions of reduced ODEs into Eq. (2.63), one can get the solutions of given time fractional PDEs (2.62).

2.13.3 Exact travelling wave solutions of space-time fractional PDEs by an improved F-expansion method

In case of fractional PDEs having space as well as time derivatives of fractional order, the solutions of such systems can be obtained using the improved F-expansion method

[10, 137, 138, 150]. In this subsection, the improved F-expansion method for space-time fractional PDEs is proposed and uses the Riccati equation as follows

$$\frac{d\phi}{d\zeta} = l + \phi^2(\zeta), \tag{2.70}$$

where l is the real parameter, $\phi(\zeta)$ is real function of ζ, and the general solutions of Riccati Eq. (2.70) are classified in the following cases.
Case 1 For $l < 0$, two solutions ϕ_1 and ϕ_2 in terms of hyperbolic functions are given as follows

$$\begin{aligned} \phi_1 &= -\sqrt{-l}\tanh(\sqrt{-l}\,\zeta), \\ \phi_2 &= -\sqrt{-l}\coth(\sqrt{-l}\,\zeta). \end{aligned} \tag{2.71}$$

Case 2 When $l > 0$, two solutions ϕ_3 and ϕ_4 can be represented in terms of the trigonometric functions as follows

$$\begin{aligned} \phi_3 &= \sqrt{l}\tan(\sqrt{l}\,\zeta), \\ \phi_4 &= -\sqrt{l}\cot(\sqrt{l}\,\zeta). \end{aligned} \tag{2.72}$$

Case 3 For $l = 0$, the solution of the Riccati Eq. (2.70) is represented in the following rational form

$$\phi_5 = -\frac{1}{\zeta}. \tag{2.73}$$

To find the exact solutions by an improved F-expansion method, the space-time fractional PDE is considered in the following form

$$\begin{aligned} \Xi(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^\beta u}{\partial x^\beta}, \frac{\partial^\gamma u}{\partial y^\gamma}, \frac{\partial^\delta u}{\partial z^\delta}, \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right), \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\delta u}{\partial z^\delta}\right), \frac{\partial^{2\beta} u}{\partial x^{2\beta}}, ...) = 0, \\ 0 < \alpha, \beta, \gamma, \delta < 1, \end{aligned} \tag{2.74}$$

where $u = u(t, x, y, z)$ and α, β, γ and δ are the fractional orders of the derivatives with respect to t, x, y and z, respectively. Some of the solution of fractional PDE (2.74) can be obtained by applying fractional complex transformation [115, 176, 243, 282, 326] given as follows

$$u(t, x, y, z) = U(\zeta), \tag{2.75}$$

where ζ is the complex variable given as follows

$$\zeta = \frac{K_1 x^\beta}{\Gamma(\beta+1)} + \frac{K_2 y^\gamma}{\Gamma(\gamma+1)} + \frac{K_3 z^\delta}{\Gamma(\delta+1)} + \frac{K_4 t^\alpha}{\Gamma(\alpha+1)}, \tag{2.76}$$

where K_1, K_2, K_3 and K_4 are the arbitrary constants. Using the chain rule, the fractional derivatives are obtained as follows

$$\begin{aligned}
&\frac{\partial^\alpha u}{\partial t^\alpha} = \sigma_t D_\zeta U D_t^\alpha \zeta = \sigma_t K_4 (D_\zeta U) = \sigma_t K_4 U', \\
&\frac{\partial^\beta u}{\partial x^\beta} = \sigma_x D_\zeta U D_x^\beta \zeta = \sigma_x K_1 (D_\zeta U) = \sigma_x K_1 U', \\
&\frac{\partial^\gamma u}{\partial y^\gamma} = \sigma_y D_\zeta U D_y^\gamma \zeta = \sigma_y K_2 (D_\zeta U) = \sigma_y K_2 U', \\
&\frac{\partial^\delta u}{\partial z^\delta} = \sigma_z D_\zeta U D_z^\delta \zeta = \sigma_z K_3 (D_\zeta U) = \sigma_z K_3 U', \\
&\vdots
\end{aligned} \tag{2.77}$$

where σ_t, σ_x, σ_y and σ_z are the fractional indexes and without the loss of generality, consider $\sigma_t = \sigma_x = \sigma_y = \sigma_z = \rho$, where ρ is the arbitrary constant [115, 145, 243]. Thus, the fractional transformation (2.75) converts the space-time fractional PDE (2.74) into an ODE in the form as follows

$$\Phi(U, \rho K_4 U', \rho K_1 U', \rho K_2 U', \rho K_3 U', \rho^2 K_4 K_1 U'', \rho^2 K_1 K_3 U'', \rho^2 K_1^2 U'', ...) = 0. \tag{2.78}$$

The obtained ODE (2.78) can also be integrated once or more number of times and the constants of integration can be equated to zero for getting the solutions.
The solution in traveling wave form for obtained ODE (2.78) has been considered as given by

$$U(\zeta) = \sum_{i=0}^{P} \mu_i (m + \phi(\zeta))^i + \sum_{i=1}^{P} \nu_i (m + \phi(\zeta))^{-i}, \tag{2.79}$$

where $\phi(\zeta)$ satisfies the Riccati Eq. (2.70). The μ_i, ν_i and m are the arbitrary constants and μ_i, $\nu_i \neq 0$, simultaneously.

- The parameter P can be determined by using the balancing principle in which the highest order derivative term is balanced with the highest order nonlinear term from ODE (2.78).

- Using the value of P, the travelling wave solution (2.79) is substituted back into ODE (2.78) and it gives the polynomial in terms of function $\phi(\zeta)$.

- By equating the coefficients of similar powers of $\phi(\zeta)$ to zero, it gives a system of over determining equations whose solution provides μ_i, ν_i and m values.

- The solutions of ODE (2.78) can be drafted by inserting the values of μ_i, ν_i and

m into the Eq. (2.79) with known solutions of the Riccati Eq. (2.70).

- The back substitution of solutions of ODE (2.78) into Eq. (2.75) and finally, it gives exact travelling wave solutions of the space-time fractional PDE (2.74).

2.13.4 Doubly periodic waves for space-time fractional PDEs by EJEFE method

This subsection describes EJEFE method [4, 39, 71, 277, 346] for getting the doubly periodic wave solutions. For this, the space-time fractional PDE of the form (2.74) is considered and using the fractional complex transformation (2.75), the reduced ODE (2.78) is obtained. For the solutions of the reduced ODE, the function $U(\zeta)$ is expressed as a series given by

$$U(\zeta) = \sum_{i=0}^{n} Q_i \operatorname{sn}^i(\zeta) + \sum_{j=1}^{n} R_j \operatorname{sn}^{-j}(\zeta), \tag{2.80}$$

where $\operatorname{sn}(\zeta)$ denotes the Jacobi elliptic function, Q_i and R_j are arbitrary constants. The value of upper limit (n) in the summation can be found by balancing principle from reduced ODE. Relations between the Jacobi functions are described as follows

$$\begin{gathered}\operatorname{cn}^2(\zeta) = 1 - \operatorname{sn}^2(\zeta),\ \operatorname{dn}^2(\zeta) = 1 - m^2 \operatorname{sn}^2(\zeta),\\ \frac{d}{d\zeta}\operatorname{sn}(\zeta) = \operatorname{cn}(\zeta)\operatorname{dn}(\zeta),\ \frac{d}{d\zeta}\operatorname{cn}(\zeta) = -\operatorname{sn}(\zeta)\operatorname{dn}(\zeta),\ \frac{d}{d\zeta}\operatorname{dn}(\zeta) = -m^2\operatorname{sn}(\zeta)\operatorname{cn}(\zeta),\end{gathered} \tag{2.81}$$

where $\operatorname{cn}(\zeta)$, $\operatorname{dn}(\zeta)$ denote Jacobi elliptic functions of cosine and third kind, respectively for the modulus m $(0 < m < 1)$. If m value approaches to 1, then Jacobi functions become hyperbolic functions [4, 39, 71, 277, 346] and if the m value approaches to 0, then Jacobi functions result into triangular functions [4, 39, 71, 277, 346]. Back substitution of the solution (2.80) into the ODE (2.78), results into a polynomial in terms of Jacobi function $\operatorname{sn}(\zeta)$. By equating the coefficients of different powers of $\operatorname{sn}(\zeta)$, the determining equations are obtained and their solution gives the values of various constants A_i and B_j. Back substitution of these variables into the complex transformation, the general solution of space-time fractional PDE is obtained.

2.14 Conservation laws

This section deals with the algorithmic view to derive conservation laws for PDEs.

2.14.1 Conservation laws of integer order PDEs

This subsection gives detail to find conservation laws of integer order PDEs by the application of two methods, one is direct method given by Anco and Bluman [19, 43] and second is Ibragimov's method [121, 123].
For conservation laws, let us assume $\tilde{s}^{th}$-order PDEs with $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2, ..., \bar{\phi}_n)$ and $\bar{\varrho} = (\bar{\varrho}^1, \bar{\varrho}^2, ..., \bar{\varrho}^\kappa)$ as independent and dependent variables, respectively, as follows

$$\Xi_\varsigma(\bar{\phi}, \bar{\varrho}, \partial\bar{\varrho}, ..., \partial^{\tilde{s}}\bar{\varrho}) = 0, \ \varsigma = 1, 2, ..., \kappa. \tag{2.82}$$

A vector $C = (C^1, C^2, ...C^n)$ with components C^1, C^2,...C^n is said to be a conserved vector if it justifies following continuity equation

$$D_i C^i\big|_{(2.82)} = 0, \tag{2.83}$$

for all the solutions of system (2.82).

2.14.1.1 Direct method

In direct method, the local conservation laws can be obtained from the linear combinations of system of PDEs with the set of multipliers [43]. For conservation laws, multiplier $\Lambda_\varsigma(\bar{\phi}, \bar{\varrho}, \partial\bar{\varrho}, ...)$ depends upon $\bar{\phi}$, $\bar{\varrho}$ as well as derivatives of $\bar{\varrho}$ upto some finite order and it has the following property

$$\Lambda_\varsigma(\bar{\phi}, \bar{\varrho}, \partial\bar{\varrho}, ...)\Xi_\varsigma(\bar{\phi}, \bar{\varrho}, \partial\bar{\varrho}, ..., \partial^s\bar{\varrho}) = D_i C^i \tag{2.84}$$

and holds identically for all the solutions of system of PDEs. The local conservation law of PDEs (2.82) can be found from non-singular multipliers $\Lambda_\varsigma(\bar{\phi}, \bar{\varrho}, \partial\bar{\varrho}, ...)$ [43] if and only if

$$\frac{\delta}{\delta\bar{\varrho}^\varsigma}(\Lambda_\varsigma\Xi_\varsigma) = 0, \tag{2.85}$$

where $\frac{\delta}{\delta\bar{\varrho}^\varsigma}$ represents Euler operator, also known as variational derivative defined by

$$\frac{\delta}{\delta\bar{\varrho}^\varsigma} = \frac{\partial}{\partial\bar{\varrho}^\varsigma} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}...D_{i_s}\frac{\partial}{\partial\bar{\varrho}^\varsigma_{i_1...i_s}}. \tag{2.86}$$

The solution of determining equations given by Eq. (2.85) provides multipliers Λ_ς. Once the multipliers are found, the conserved vectors can be constructed by integrating the divergence expression (2.84) [43]. In some cases, the inversion of divergence expression is difficult or complicated then in these cases the inversion of divergence equation is done by the use of homotopy operator [228] and multipliers can be found.

2.14.1.2 Ibragimov's method for conservation laws

Ibragimov proposes the new conservation theorem [121] and theory of nonlinear self-adjointness [123] for system (2.82) associated with the symmetries.
The adjoint equations to the Eqs. (2.82) are defined by

$$\Xi^*_\varsigma(\bar{\phi},\bar{\varrho},\partial\bar{\varrho},\partial\Upsilon,...,\partial^s\bar{\varrho},\partial^s\Upsilon)=0,\ \varsigma=1,2,...,\kappa, \tag{2.87}$$

with the adjoint operator Ξ^*_ς as given by

$$(\Xi_\varsigma)^*(\bar{\phi},\bar{\varrho},\partial\bar{\varrho},\partial\Upsilon,...,\partial^s\bar{\varrho},\partial^s\Upsilon)=\frac{\delta\mathcal{L}}{\delta\bar{\varrho}^\varsigma}, \tag{2.88}$$

where the formal Lagrangian ($\mathcal{L}$) for Eqs. (2.82) is expressed below

$$\mathcal{L}=\sum_{\varpi=1}^{m}\Upsilon^\varpi\Xi_\varsigma(\bar{\phi},\bar{\varrho},\partial\bar{\varrho},...,\partial^s\bar{\varrho}) \tag{2.89}$$

and $\dfrac{\delta}{\delta\bar{\varrho}^\varsigma}$ is the variational derivative defined in (2.86).
A system (2.82) is called nonlinearly self-adjoint [123] if Eqs. (2.87) justifies ∀ solutions of system (2.82) after some substitution of Υ^ς given by

$$\Upsilon^\varsigma=\phi^\varsigma(\bar{\phi},\bar{\varrho}),\ \varsigma=1,2,...\kappa, \tag{2.90}$$

provided that not all ϕ^ς's vanish simultaneously [123]. Or in other words, the system (2.82) is defined as nonlinearly self-adjoint if it satisfies the following condition

$$\Xi^*_\varsigma(\bar{\phi},\bar{\varrho},\partial\bar{\varrho},\partial\phi(\bar{\phi},\bar{\varrho}),...,\partial^s\bar{\varrho},\partial^s\phi(\bar{\phi},\bar{\varrho}))=\lambda^\varpi_\varsigma(\Xi_\varsigma(\bar{\phi},\bar{\varrho},\partial\bar{\varrho},...,\partial^s\bar{\varrho})),\ \varsigma=1,2,...,\kappa, \tag{2.91}$$

where $\lambda^\varrho_\varsigma$ are the undetermined coefficients.

Theorem 2.14.1. *(Ibragimov's new conservation theorem) [121] Infinitesimal symmetry given by*

$$X=\mathcal{X}_i\frac{\partial}{\partial\bar{\phi}_i}+\mathcal{Z}^\varsigma\frac{\partial}{\partial\bar{\varrho}^\varsigma} \tag{2.92}$$

of nonlinear self-adjoint system (2.82) *generates a conservation law* $D_iC^i=0$ *by* *using formula given below*

$$\begin{aligned} C^i = \mathcal{X}_i\mathcal{L} + W^\varsigma &\left(\left(\frac{\partial \mathcal{L}}{\partial \bar{\varrho}^\varsigma_i}\right) - D_j \left(\frac{\partial\mathcal{L}}{\partial\bar{\varrho}^\varsigma_{ij}}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial\bar{\varrho}^\varsigma_{ijk}}\right) - \cdots \right) \\ + D_j(W^\varsigma)&\left(\left(\frac{\partial\mathcal{L}}{\partial\bar{\varrho}^\varsigma_{ij}}\right) - D_k\left(\frac{\partial\mathcal{L}}{\partial\bar{\varrho}^\varsigma_{ijk}}\right)+\cdots\right) + D_jD_k(W^\varsigma)\left(\frac{\partial\mathcal{L}}{\partial\bar{\varrho}^\varsigma_{ijk}} - \cdots\right), \end{aligned} \tag{2.93}$$

where $W^\varsigma = \mathcal{Z}^\varsigma - \mathcal{X}_j\bar{\varrho}^\varsigma_j$. $\mathcal{L}$ *must be taken in symmetric form with respect to mixed derivative terms* $\bar{\varrho}^\varsigma_{ij}$, $\bar{\varrho}^\varsigma_{ijk}$,... .

2.14.2 Conservation laws for PDEs of time fractional derivatives

To obtain conservation laws of time fractional PDEs (2.39), consider the formal Lagrangian as expressed by

$$\mathcal{L} = \sum_{\varpi=1}^{\kappa} \Upsilon^{\varpi}\,\Xi^\varsigma(\bar{\phi}, \bar{\varrho}^\varsigma, D^\alpha_{\bar{\phi}_1}\bar{\varrho}^\varsigma, \bar{\varrho}^\varsigma_{\bar{\phi}_i}, ...) \tag{2.94}$$

Then adjoint equations to the system (2.39) are given as follows

$$\frac{\delta\mathcal{L}}{\delta\bar{\varrho}^\varsigma} = 0, \tag{2.95}$$

where $\dfrac{\delta}{\delta\bar{\varrho}^\varsigma}$ is the variational derivative defined by

$$\frac{\delta}{\delta\bar{\varrho}^\varsigma} = \frac{\partial}{\partial\bar{\varrho}^\varsigma} - (D^\alpha_{\bar{\phi}_1})^*\frac{\partial}{\partial\left(D^\alpha_{\bar{\phi}_1}\bar{\varrho}^\varsigma\right)} - D_{\bar{\phi}_i}\frac{\partial}{\partial\bar{\varrho}^\varsigma_{\bar{\phi}_i}} + D_{\bar{\phi}_i}D_{\bar{\phi}_j}\frac{\partial}{\partial\bar{\varrho}^\varsigma_{\bar{\phi}_i\bar{\phi}_j}} + \cdots,\ i = 2, 3, ..., \tag{2.96}$$

and $(D^\alpha_{\bar{\phi}_1})^*$ is the adjoint operator of $(D^\alpha_{\bar{\phi}_1})$. By using the RLF derivative [96, 185, 244, 247, 297], the $\bar{\phi}_1$ component '$C^{\bar{\phi}_1}$' of conserved vector is given by

$$C^{\bar{\phi}_1} = \sum_{k=0}^{n-1}(-1)^k\,{}_0D^{\alpha-1-k}_{\bar{\phi}_1}(W^\varsigma)D^k_{\bar{\phi}_1}\frac{\partial\mathcal{L}}{\partial_0D^\alpha_{\bar{\phi}_1}\bar{\varrho}^\varsigma_i} - (-1)^nJ\left(W^\varsigma, D^n_{\bar{\phi}_1}\frac{\partial\mathcal{L}}{\partial_0D^\alpha_{\bar{\phi}_1}\bar{\varrho}^\varsigma_i}\right), \tag{2.97}$$

where $n = [\alpha] + 1$ and J is the integral defined by

$$J(F, G) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^\vartheta\frac{F(\bar{\phi}_2, \bar{\phi}_3, ...\bar{\phi}_l, \phi)G(\bar{\phi}_2, \bar{\phi}_3, ...\bar{\phi}_l, q)}{(q-\phi)^{\alpha+1-n}}dqd\phi. \tag{2.98}$$

The conserved vectors C^i corresponding to other independent variables for $\bar{\phi}_i$ $i = 2, 3, ...$ can be found by similar formula given for integer order PDEs in Eq. (2.93).

2.15 Linear dispersion analysis

This section gives linear dispersion analysis for fractional PDEs using the definition of RLF derivative (2.40) with lower terminal a at $-\infty$ [226]. The RLF derivative with lower terminal a at $-\infty$ has the following property

$$ {}_{-\infty}D_t^{\alpha} e^{\lambda t} = \lambda^{\alpha} e^{\lambda t},\ \lambda > 0. \tag{2.99}$$

The analysis describes the dispersion of waves in the nonlinear dynamical systems. It gives the dispersion relation which is used to find the phase velocity v_p and the group velocity v_g. For linear analysis, consider dispersive waves in the form of the wave function $\psi(t, x, y, z)$ having sinusoidal form with the periodic spatial and time dependence as follows [63, 101, 154, 195, 196, 316]

$$\psi(t, x, y, z) = \texttt{Re}\{A e^{i(\omega t - \vec{S}\cdot\vec{r})}\}, \tag{2.100}$$

where A is the complex amplitude, $\vec{S} = s_x\,\hat{i} + s_y\,\hat{j} + s_z\,\hat{k}$ is the three dimensional wave vector, and $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ is the three dimensional displacement vector. The relation between ω and $\vec{S}$ is called dispersion relation defined by

$$\mathcal{D}(\omega, \vec{S}) = 0, \tag{2.101}$$

where $\mathcal{D}$ is the real function of ω and $\vec{S}$. This relation is satisfied with certain ω, $\vec{S}$ $\in \mathbb{C}$. The Eq. (2.101) can be solved for a real parameter ω by utilizing the following condition

$$\bar{\omega}(\vec{S}) = \bar{\omega}(s_x, s_y, s_z) \in \mathbb{C},\ \ s_x,\ s_y,\ s_z \in \mathbb{R}, \tag{2.102}$$

Then, the normal mode solution of (3+1)-dimensional dynamical system can be written as follows

$$\psi(t, x, y, z; \vec{S}) = \texttt{Re}\{A(\vec{S}) e^{i\left(\bar{\omega} t - \vec{S}\cdot\vec{r}\right)}\}. \tag{2.103}$$

The phase and group velocity for (2.103) from dispersion can be defined as follows

$$v_p(\vec{S}) = \frac{\vec{S}\,\texttt{Re}\bar{\omega}(\vec{S})}{|\vec{S}|^2},\ \ v_g(\vec{S}) = \vec{\nabla}_{\vec{S}}\,\texttt{Re}\bar{\omega}(\vec{S}), \tag{2.104}$$

where $\vec{\nabla}_{\vec{S}}$ denotes the gradient and $|\vec{S}| = \sqrt{s_x^2 + s_y^2 + s_z^2}$.
Eq. (2.103) shows that $\psi(t, x, y, z; \vec{S})$ is oscillating sinusoidally in space with a wavelength $\lambda = \frac{2\pi}{|\vec{S}|}$, but with time, the sinusoidal variation depends upon whether $\bar{\omega}$ is real or imaginary. Thus,

$$\bar{\omega}(\vec{S}) = \omega_r(\vec{S}) + i\,\omega_i(\vec{S}), \tag{2.105}$$

where $\omega_r(\vec{S}) = \mathtt{Re}\,\bar{\omega}$ and $\omega_i(\vec{S}) = \mathtt{Im}\,\bar{\omega}$. If $\omega_i \geq 0$, $\gamma(S) = \omega_i(S)$ is the time damping factor.

Chapter 3

Integer Order PDEs[1]

This chapter examines two nonlinear integer order PDEs with variable coefficients which are (2+1)-dimensional new coupled Zakharov-Kuznetsov (ZK) system and generalised 7^{th} order Korteweg and de Vries (KdV) equation. These equations are studied for Lie group analysis which includes the Lie infinitesimal symmetries, group invariant forms, coefficient functions, and corresponding reductions. The exact explicit solutions are obtained corresponding to reduced ODEs and are reframed for given equations. The conserved density and fluxes are derived using general theorem on conservation laws with nonlinear self-adjointness technique and direct method.

3.1 New coupled ZK system in (2+1)-dimensions

The nonlinear physical phenomena can be disseminated by solving nonlinear PDEs. The evolution profile of solutions for such equations plays a key role in exploring various models taken from mathematical, physical and engineering sciences. Thus, it is required to develop an algorithm to find a variety of solutions. The present work addresses to obtain the multiple type solutions by symmetry transformations [107, 108, 219, 268]. Further, the Lie symmetries are used to construct conservation laws [19, 121, 123, 148, 210, 312] associated with PDEs. The one such nonlinearly evolved new coupled ZK system in (2+1)-dimensions with time dependent coefficients

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[1] The contents of section (3.1) are published in *Pramana −Journal of Physics*, 93(4), 59 (2019).

is considered as follows

$$\begin{aligned}
&\Xi_1 \equiv u_t + T_1(t)(uv)_x + T_3(t)(vw)_x + T_2(t)(u_{xx} + u_{yy})_x = 0,\\
&\Xi_2 \equiv v_t + T_4(t)(uw)_x + T_2(t)(v_{xx} + v_{yy})_x = 0,\\
&\Xi_3 \equiv w_t + T_4(t)(uv)_x + T_2(t)(w_{xx} + w_{yy})_x = 0,
\end{aligned} \tag{3.1}$$

where $T_1(t)$, $T_2(t)$, $T_3(t)$ and $T_4(t)$ are the random functions of t.
The ZK system spanned dust acoustic solitary waves, nonlinear ion-acoustic waves and hot isothermal electrons [236, 256, 257, 317, 340, 347]. The ZK system with constant coefficients (3.1) is solved successfully by Elboree [75], Wei and Tang [313], Khalique [155] and Fahmy [78] for different types of exact solutions. The system (3.1) has not been investigated with time dependent coefficients by the symmetry transformation method for exact solutions as well as for conservation laws.

3.1.1 Maximal rank and local solvability

The maximal rank of the system is computed by using the formula given in Eq. (2.36) is as follows

$$\begin{pmatrix} \frac{\partial \Xi_1}{\partial x^i} & \frac{\partial \Xi_1}{\partial u^\varsigma_j} \\ \frac{\partial \Xi_2}{\partial x^i} & \frac{\partial \Xi_2}{\partial u^\varsigma_j} \\ \frac{\partial \Xi_3}{\partial x^i} & \frac{\partial \Xi_3}{\partial u^\varsigma_j} \end{pmatrix}, \tag{3.2}$$

where $i, \varsigma = 1, 2, 3$ and $x^1 = t$, $x^2 = x$, $x^3 = y$, $u^1 = u$, $u^2 = v$, $u^3 = w$, and u_j are various derivatives with respect to independent variables. According to the maximal rank condition, the resulting jacobian matrix rank becomes 3 which is maximum i.e., maximal rank condition is satisfied. Also, the given Eq. (3.1) is locally solvable. Because, any assignation of initial values
$(t^0, x^0, y^0; u^0, v^0, w^0; u^0_t, u^0_x, u^0_y, v^0_t, v^0_x, v^0_y; u^0_{tx}, ...; u^0_{xxx}, u^0_{xyy}, v^0_{xxx}, v^0_{xyy}, w^0_{xxx}, w^0_{xyy})$, which satisfies given system and will give smooth solution. Any solution of the system is smooth if it satisfies to every prolongation of the system.

3.1.2 Lie symmetry transformations

Lie infinitesimal symmetries have been extracted using an IG considered in the following form [219]

$$X = \mathcal{X}_1 \frac{\partial}{\partial t} + \mathcal{X}_2 \frac{\partial}{\partial x} + \mathcal{X}_3 \frac{\partial}{\partial y} + \mathcal{Z}_1 \frac{\partial}{\partial u} + \mathcal{Z}_2 \frac{\partial}{\partial v} + \mathcal{Z}_3 \frac{\partial}{\partial w}, \tag{3.3}$$

where $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, $\mathcal{Z}_1$, $\mathcal{Z}_2$ and $\mathcal{Z}_3$ are infinitesimals with respect to t, x, y, u, v and w, respectively. The third order prolongation Pr^3X of X for the system (3.1) is given as follows

$$\begin{aligned} Pr^3X = X &+ {\mathcal{Z}_1}^t\frac{\partial}{\partial u_t} + {\mathcal{Z}_1}^x\frac{\partial}{\partial u_x} + {\mathcal{Z}_1}^{xxx}\frac{\partial}{\partial u_{xxx}} + {\mathcal{Z}_1}^{xyy}\frac{\partial}{\partial u_{xyy}} + {\mathcal{Z}_2}^t\frac{\partial}{\partial u_t} + {\mathcal{Z}_2}^x\frac{\partial}{\partial v_x} \\ &+ {\mathcal{Z}_2}^{xxx}\frac{\partial}{\partial v_{xxx}} + {\mathcal{Z}_2}^{xyy}\frac{\partial}{\partial v_{xyy}} + {\mathcal{Z}_3}^t\frac{\partial}{\partial u_t} + {\mathcal{Z}_3}^x\frac{\partial}{\partial w_x} + {\mathcal{Z}_3}^{xxx}\frac{\partial}{\partial w_{xxx}} + {\mathcal{Z}_3}^{xyy}\frac{\partial}{\partial w_{xyy}}, \end{aligned} \tag{3.4}$$

where ${\mathcal{Z}_1}^t$, ${\mathcal{Z}_2}^t$, ${\mathcal{Z}_3}^t$, ${\mathcal{Z}_1}^x$, ${\mathcal{Z}_2}^x$, ${\mathcal{Z}_3}^x$, ${\mathcal{Z}_1}^{xxx}$, ${\mathcal{Z}_2}^{xxx}$, ${\mathcal{Z}_3}^{xxx}$, ${\mathcal{Z}_1}^{xyy}$, ${\mathcal{Z}_2}^{xyy}$ and ${\mathcal{Z}_3}^{xyy}$ are the extended infinitesimals [219] (detail is given in Appendix A.1). The system (3.3) represents Lie point symmetry of ZK system if the following conditions hold

$$\begin{aligned} &Pr^3X(\Xi_1)|_{\Xi_1=0,\Xi_2=0,\Xi_3=0} = 0,\ Pr^3X(\Xi_2)|_{\Xi_1=0,\Xi_2=0,\Xi_3=0} = 0, \\ &Pr^3X(\Xi_3)|_{\Xi_1=0,\Xi_2=0,\Xi_3=0} = 0. \end{aligned} \tag{3.5}$$

The analysis suggests the following symmetry equations

$$\begin{aligned} &{\mathcal{Z}_1}^t + {T_1}'(t)\mathcal{X}_1(uv)_x + T_1(t)(u{\mathcal{Z}_2}^x + \mathcal{Z}_1 v_x + {\mathcal{Z}_1}^x v + u_x\mathcal{Z}_2) + {T_3}'(t)\mathcal{X}_1(vw)_x \\ &+ T_3(t)(v{\mathcal{Z}_3}^x + \mathcal{Z}_2 w_x + {\mathcal{Z}_2}^x w + v_x\mathcal{Z}_3) + {T_2}'(t)\mathcal{X}_1 u_{xxx} + T_2(t){\mathcal{Z}_1}^{xxx} + {T_2}'(t)\mathcal{X}_1 u_{xyy} \\ &+ T_2(t){\mathcal{Z}_1}^{xyy} = 0, \\ &{\mathcal{Z}_2}^t + {T_4}'(t)\mathcal{X}_1(uw)_x + T_4(t)(u{\mathcal{Z}_3}^x + \mathcal{Z}_1 w_x + {\mathcal{Z}_1}^x w + u_x\mathcal{Z}_3) + {T_2}'(t)\mathcal{X}_1 v_{xxx} + T_2(t){\mathcal{Z}_2}^{xxx} \\ &+ {T_2}'(t)\mathcal{X}_1 v_{xyy} + T_2(t){\mathcal{Z}_2}^{xyy} = 0, \\ &{\mathcal{Z}_3}^t + {T_4}'(t)\mathcal{X}_1(uv)_x + T_4(t)(u{\mathcal{Z}_2}^x + \mathcal{Z}_1 v_x + {\mathcal{Z}_1}^x v + u_x\mathcal{Z}_2) + {T_2}'(t)\mathcal{X}_1 w_{xxx} + T_2(t){\mathcal{Z}_3}^{xxx} \\ &+ {T_2}'(t)\mathcal{X}_1 w_{xyy} + T_2(t){\mathcal{Z}_3}^{xyy} = 0. \end{aligned} \tag{3.6}$$

Using the extended infinitesimals in Eqs. (3.6) and by matching similar terms, an over-determined set of linear PDEs are retrieved and whose solution can be expressed as follows

$$\begin{aligned} &\mathcal{X}_1 = \frac{1}{T_2(t)}(3\,f_1\int T_2(t)\,dt + f_4),\ \mathcal{X}_2 = f_1 x + f_3,\ \mathcal{X}_3 = f_1 y + f_2,\ \mathcal{Z}_1 = f_5 u, \\ &\mathcal{Z}_2 = f_5 v,\ \mathcal{Z}_3 = f_5 w, \end{aligned} \tag{3.7}$$

where f_1, f_2, f_3, f_4, f_5 are arbitrary constants and the various coefficients $T_1(t)$, $T_2(t)$, $T_3(t)$, $T_4(t)$ justify following conditions

$$\begin{aligned}
&T_2(t)(T_3(t)\,(2\,f_1+f_5)+T_3{}'(t))\,\mathcal{X}_1-T_{2t}T_3(t)=0,\\
&T_2(t)(T_1(t)\,(2\,f_1+f_5)+T_1{}'(t))\,\mathcal{X}_1-T_{2t}T_1(t)=0,\\
&T_2(t)(T_4(t)\,(2\,f_1+f_5)+T_4{}'(t))\,\mathcal{X}_1-T_{2t}T_4(t)=0.
\end{aligned} \tag{3.8}$$

From Eq. (3.7), the one-dimensional Lie algebra [219, 312] is generated by following vector fields

$$\begin{aligned}
X_1 &= \frac{1}{T_2(t)}\partial t,\\
X_2 &= \partial x,\\
X_3 &= \partial y,\\
X_4 &= u\partial u+v\partial v+w\partial w,\\
X_5 &= x\partial x+\frac{3\int T_2(t)\,dt}{T_2(t)}\partial t+y\partial y.
\end{aligned} \tag{3.9}$$

The above IGs satisfied the condition given in Eq. (2.34), which shows that $Pr^3X_i(\Xi_i)=0$, $i=1,2,3$, whenever $\Xi_i=0$. Thus above above IGs become symmetry groups and infinitesimal criterion of invariance (2.34) is verified.

Further, vector fields X_i generate a set of one-parameter groups G_i given in following table.

The transformed point is given by the entries $\exp(\epsilon X_i)(t,x,y,u,v,w)=(\hat{t},\hat{x},\hat{y},\hat{u},\hat{v},\hat{w})$:

$$\begin{aligned}
&G_1:\left(t+\frac{\epsilon}{T_2(t)},x,y,u,v,w\right),\\
&G_2:(t,x+\epsilon,y,u,v,w),\\
&G_3:(t,x,y+\epsilon,u,v,w),\\
&G_4:(t,x,y,u\,e^{\epsilon},v\,e^{\epsilon},w\,e^{\epsilon}),\\
&G_5:\left(t+\epsilon\,\frac{3\int T_2(t)\,dt}{T_2(t)},x\,e^{\epsilon},y\,e^{\epsilon},u,v,w\right),
\end{aligned} \tag{3.10}$$

where various G_i; $(i=1,2,...,5)$ represent the symmetry groups and the following theorem can be stated which ultimately leads to the optimal system [219, 312].

Theorem 3.1.1. *If* $\left(u=\bar{\mathcal{D}}(t,x,y), v=\bar{\mathcal{E}}(t,x,y), w=\bar{\mathcal{F}}(t,x,y)\right)$ *represents solution of the system* (3.1)*, so are the functions*

$$\begin{aligned}
(u^{(1)},v^{(1)},w^{(1)}) &= \left(\bar{\mathcal{D}}\left(t-\frac{\epsilon}{T_2(t)},x,y\right),\bar{\mathcal{E}}\left(t-\frac{\epsilon}{T_2(t)},x,y\right),\bar{\mathcal{F}}\left(t-\frac{\epsilon}{T_2(t)},x,y\right)\right),\\
(u^{(2)},v^{(2)},w^{(2)}) &= \left(\bar{\mathcal{D}}(t,x-\epsilon,y),\bar{\mathcal{E}}(t,x-\epsilon,y),\bar{\mathcal{F}}(t,x-\epsilon,y)\right),\\
(u^{(3)},v^{(3)},w^{(3)}) &= \left(\bar{\mathcal{D}}\left(t,x,y-\epsilon\right),\bar{\mathcal{E}}\left(t,x,y-\epsilon\right),\bar{\mathcal{F}}\left(t,x,y-\epsilon\right)\right),\\
(u^{(4)},v^{(4)},w^{(4)}) &= \left(e^{\epsilon}\,\bar{\mathcal{D}}(t,x,y),e^{\epsilon}\,\bar{\mathcal{E}}(t,x,y),e^{\epsilon}\,\bar{\mathcal{F}}(t,x,y)\right),\\
(u^{(5)},v^{(5)},w^{(5)}) &= \left(\bar{\mathcal{D}}\left(t-\epsilon\,\frac{3\int T_2(t)\,dt}{T_2(t)},x\,e^{-\epsilon},y\,e^{-\epsilon}\right),\bar{\mathcal{E}}\left(t-\epsilon\,\frac{3\int T_2(t)\,dt}{T_2(t)},x\,e^{-\epsilon},y\,e^{-\epsilon}\right),\right.\\
&\qquad \left.\bar{\mathcal{F}}\left(t-\epsilon\,\frac{3\int T_2(t)\,dt}{T_2(t)},x\,e^{-\epsilon},y\,e^{-\epsilon}\right)\right).
\end{aligned} \tag{3.11}$$

A family of group invariant solutions will be obtained for each subgroup of symmetry groups G_i $(i=1,2,...,5)$, in present case, an infinitely many such subgroups exists, and difficult to enlist acceptable group invariant solutions for ZK system. The group invariant solution introduces concept of optimal system [219] using the adjoint representation and Lie brackets. The non zero Lie brackets can be obtained from the Lie algebra (3.9) as follows

$$\begin{aligned}
&[X_1,X_5]=-[X_5,X_1]=3\,X_1,\ [X_2,X_5]=-[X_5,X_2]=X_2,\\
&[X_3,X_5]=-[X_5,X_3]=X_3.
\end{aligned} \tag{3.12}$$

The Lie series (2.15) and Lie commutation relation (3.12) further help to write down the adjoint representation of ZK system

$$Ad(\exp(\epsilon X_i))X_i = X_i, i=1,2,3,4,5, \tag{3.13}$$

and

$$\begin{aligned}
&Ad(\exp(\epsilon X_1))X_2 = X_2,\ ,Ad(\exp(\epsilon X_1))X_3 = X_3,\\
&Ad(\exp(\epsilon X_1))X_4 = X_4,\ Ad(\exp(\epsilon X_1)X_5 = X_5 - 3\,\epsilon X_1,\\
&Ad(\exp(\epsilon X_2))X_1 = X_1,\ Ad(\exp(\epsilon X_2))X_3 = X_3,\\
&Ad(\exp(\epsilon X_2))X_4 = X_4,\ Ad(\exp(\epsilon X_2))X_5 = X_5 - \epsilon X_2,\\
&Ad(\exp(\epsilon X_3))X_1 = X_1,\ Ad(\exp(\epsilon X_3))X_2 = X_2,\\
&Ad(\exp(\epsilon X_3))X_4 = X_4,\ Ad(\exp(\epsilon X_3))X_5 = X_2 - \epsilon X_3,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&Ad(\exp(\epsilon X_4))X_1 = X_1,\ Ad(\exp(\epsilon X_4))X_2 = X_2,\\
&Ad(\exp(\epsilon X_4))X_3 = X_3,\ Ad(\exp(\epsilon X_4))X_5 = X_5,\\
&Ad(\exp(\epsilon X_5))X_1 = X_1\, e^{3\,\epsilon},\ Ad(\exp(\epsilon X_5))X_2 = X_2\, e^{\epsilon},\\
&Ad(\exp(\epsilon X_5))X_3 = X_3\, e^{\epsilon},\ Ad(\exp(\epsilon X_5))X_4 = X_4.
\end{aligned} \tag{3.15}$$

Then, adjoint representation is used to construct optimal system [219] generated by following vector fields

(i) $X_5 + \rho\, X_4$,

(ii) $X_4 + \mu\, X_3 + \theta\, X_2 + \nu\, X_1$,

(iii) $X_3 + r\, X_2 + s\, X_1$,

(iv) $X_2 + p\, X_1$,

(v) X_1,

where ρ, μ, θ, ν, r, s and p are arbitrary constants. These vectors fields are used in next section for similarity reductions.

3.1.3 Similarity reductions and exact solutions

For similarity reductions with respect to the vector fields described in the optimal system, the following characteristic equations are used

$$\frac{dt}{\mathcal{X}_1} = \frac{dx}{\mathcal{X}_2} = \frac{dy}{\mathcal{X}_3} = \frac{du}{\mathcal{Z}_1} = \frac{dv}{\mathcal{Z}_2} = \frac{dw}{\mathcal{Z}_3}. \tag{3.16}$$

3.1.3.1 Vector field $X_5 + \rho\, X_4$

The characteristic equations (3.16) for this vector field give the following invariants

$$\begin{aligned}
&\zeta_1 = \frac{x}{(\int T_2(t)\, dt)^{1/3}},\ \zeta_2 = \frac{y}{(\int T_2(t)\, dt)^{1/3}},\ u(t,x,y) = F_1(\zeta_1,\zeta_2)\left(\int T_2\,(t)\, dt\right)^{\rho/3},\\
&v(t,x,y) = G_1(\zeta_1,\zeta_2)\left(\int T_2\,(t)\, dt\right)^{\rho/3},\ w(t,x,y) = H_1(\zeta_1,\zeta_2)\left(\int T_2\,(t)\, dt\right)^{\rho/3}.
\end{aligned} \tag{3.17}$$

Using Eq. (3.8), the time dependent coefficients can be written as follows

$$
\begin{aligned}
&T_1(t) = p_4\, T_2(t) \left(\int T_2(t)\, dt \right)^{-(2+\rho)/3}, \; T_3(t) = p_5\, T_2(t) \left(\int T_2(t)\, dt \right)^{-(2+\rho)/3}, \\
&T_4(t) = p_6\, T_2(t) \left(\int T_2(t)\, dt \right)^{-(2+\rho)/3},
\end{aligned}
\tag{3.18}
$$

where p_4, p_5 and p_6 are arbitrary constants.

Using the invariants and coefficient functions given in Eqs (3.17) and (3.18), the ZK system is transformed into following reduced PDEs

$$
\begin{aligned}
&F_{1\zeta_1}\zeta_1 + F_{1\zeta_2}\zeta_2 - F_1\rho - 3p_4\, F_1 G_{1\zeta_1} - 3p_4\, F_{1\zeta_1} G_1 - 3p_5\, G_1 H_{1\zeta_1} - 3p_5\, G_{1\zeta_1} H_1 \\
&- 3F_{1\zeta_1\zeta_1\zeta_1} - 3F_{1\zeta_1\zeta_2\zeta_2} = 0, \\
&G_{1\zeta_1}\zeta_1 + G_{1\zeta_2}\zeta_2 - G_1\rho - 3p_6\, H_1 F_{1\zeta_1} - 3p_6\, H_{1\zeta_1} F_1 - 3G_{1\zeta_1\zeta_1\zeta_1} - 3G_{1\zeta_1\zeta_2\zeta_2} = 0, \\
&H_{1\zeta_1}\zeta_1 + H_{1\zeta_2}\zeta_2 - H_1\rho - 3p_6\, F_1 G_{1\zeta_1} - 3p_6\, F_{1\zeta_1} G_1 - 3H_{1\zeta_1\zeta_1\zeta_1} - 3H_{1\zeta_1\zeta_2\zeta_2} = 0.
\end{aligned}
\tag{3.19}
$$

These Eqs. (3.19) possessed only trivial solutions, hence are not physically important.

3.1.3.2 Vector field $X_4 + \mu\, X_3 + \theta\, X_2 + \nu\, X_1$

In this case, the invariants and the corresponding variable coefficients are obtained as follows

$$
\begin{aligned}
&\zeta_1 = -\frac{\theta}{\nu} \int T_2(t)\, dt + x, \; \zeta_2 = -\frac{\mu}{\nu} \int T_2(t)\, dt + y, \; u(t,x,y) = F_1(\zeta_1, \zeta_2)\, e^{\frac{1}{\nu} \int T_2(t)\, dt}, \\
&v(t,x,y) = G_1(\zeta_1, \zeta_2)\, e^{\frac{1}{\nu} \int T_2(t)\, dt}, \; w(t,x,y) = H_1(\zeta_1, \zeta_2)\, e^{\frac{1}{\nu} \int T_2(t)\, dt}, \\
&T_1(t) = q_1\, T_2(t)\, e^{-\frac{1}{\nu} \int T_2(t)\, dt}, \; T_3(t) = q_2\, T_2(t)\, e^{-\frac{1}{\nu} \int T_2(t)\, dt}, \; T_4(t) = q_3\, T_2(t)\, e^{-\frac{1}{\nu} \int T_2(t)\, dt},
\end{aligned}
\tag{3.20}
$$

where q_1, q_2 and q_3 are arbitrary constants. The reduced PDEs for ZK system reads as follows

$$
\begin{aligned}
&F_{1\zeta_1}\theta - F_{1\zeta_2}\mu + F_1 + q_1\, \nu\, F_1 G_{1\zeta_1} + q_1\, \nu\, F_{1\zeta_1} G_1 + q_2\, \nu\, G_1 H_{1\zeta_1} + q_2\, \nu\, G_{1\zeta_1} H_1 + \nu\, F_{1\zeta_1\zeta_1\zeta_1} \\
&+ \nu\, F_{1\zeta_1\zeta_2\zeta_2} = 0, \\
&G_{1\zeta_1}\theta + G_{1\zeta_2}\mu - G_1 - q_3\, \nu\, H_1 F_{1\zeta_1} - q_3\, \nu\, H_{1\zeta_1} F_1 - \nu\, G_{1\zeta_1\zeta_1\zeta_1} - \nu\, G_{1\zeta_1\zeta_2\zeta_2} = 0, \\
&- H_{1\zeta_1}\theta - H_{1\zeta_2}\mu + H_1 + q_3\, \nu\, F_1 G_{1\zeta_1} + q_3\, \nu\, F_{1\zeta_1} G_1 + \nu\, H_{1\zeta_1\zeta_1\zeta_1} + \nu\, H_{1\zeta_1\zeta_2\zeta_2} = 0.
\end{aligned}
\tag{3.21}
$$

By applying the Lie symmetry transformation algorithm on the reduced PDEs, the following infinitesimals are obtained

$$\xi^1 = q_5,\, \xi^2 = q_4,\, \eta^1 = \eta^2 = \eta^3 = 0, \tag{3.22}$$

where q_4 and q_5 are arbitrary constants. The new invariants are obtained using the characteristic equations $\dfrac{d\zeta_1}{\xi^1} = \dfrac{d\zeta_2}{\xi^2} = \dfrac{dF_1}{\eta^1} = \dfrac{dG_1}{\eta^2} = \dfrac{dH_1}{\eta^3}$ as follows

$$\zeta = q_4\zeta_1 - q_5\zeta_2,\, F_1(\zeta_1,\zeta_2) = f(\zeta),\, G_1(\zeta_1,\zeta_2) = g(\zeta),\, H_1(\zeta_1,\zeta_2) = h(\zeta). \tag{3.23}$$

Finally, the reduced ODEs for the ZK system are given as follows

$$\begin{aligned}
&f_\zeta q_4\,\theta + f_\zeta q_5\,\mu + f + q_1\,\nu\, f g_\zeta q_4 + q_1\,\nu\, f_\zeta q_4\, g + q_2\,\nu\, g h_\zeta q_4 + q_2\,\nu\, g_\zeta q_4\, h + \nu\, f_{\zeta\zeta\zeta} {q_4}^3\\
&+ \nu\, f_{\zeta\zeta\zeta} {q_5}^2 q_4 = 0,\\
&g_\zeta q_4\,\theta - g_\zeta q_5\,\mu - g - q_3\,\nu\, h f_\zeta q_4 - q_3\,\nu\, h_\zeta c_4\, f - \nu\, g_{\zeta\zeta\zeta} {q_4}^3 - \nu\, g_{\zeta\zeta\zeta} {q_5}^2 q_4 = 0,\\
&- h_\zeta q_4\,\theta + h_\zeta q_5\,\mu + h + q_3\,\nu\, f g_\zeta q_4 + q_3\,\nu\, f_\zeta c_4\, g + \nu\, h_{\zeta\zeta\zeta} {q_4}^3 + \nu\, h_{\zeta\zeta\zeta} {q_5}^2 q_4 = 0.
\end{aligned} \tag{3.24}$$

The power series solution of system (3.24) is considered as follows

$$f(\zeta) = \sum_{\sigma_1=0}^{\infty} D_{\sigma_1}\zeta^{\sigma_1},\; g(\zeta) = \sum_{\sigma_1=0}^{\infty} E_{\sigma_1}\zeta^{\sigma_1},\; h(\zeta) = \sum_{\sigma_1=0}^{\infty} K_{\sigma_1}\zeta^{\sigma_1}, \tag{3.25}$$

where D_{σ_1}, E_{σ_1} and K_{σ_1} are unknown coefficients and are to be determined later. Substitution of (3.25) into reduced ODEs (3.24), gives the following recurrence relations

$$
\begin{aligned}
D_{\sigma_1+3} = {} & \frac{-1}{(\nu\, q_4^3 + \sigma_1 u\, q_5^2 q_4)(\sigma_1+1)(\sigma_1+2)(\sigma_1+3)} \, [\{(q_4\theta + q_5\mu)\,(\sigma_1+1) D_{\sigma_1+1} + D_{\sigma_1} \\
& + \nu\, q_1 q_4 \{ \left(\sum_{\sigma_2=0}^{\sigma_1} (\sigma_1 - \sigma_2 + 1)(D_{\sigma_2} E_{\sigma_1-\sigma_2+1} + E_{\sigma_2} D_{\sigma_1-\sigma_2+1}) \right) \\
& + \nu\, q_2 q_4 \left(\sum_{\sigma_2=0}^{\sigma_1} (\sigma_1 - \sigma_2 + 1)(E_{\sigma_2} K_{\sigma_1-\sigma_2+1} + K_{\sigma_2} E_{\sigma_1-\sigma_2+1}) \right) \Bigg) \Bigg] , \\
E_{\sigma_1+3} = {} & \frac{1}{(\nu\, q_4^3 + \sigma_1 u\, q_5^2 q_4)(\sigma_1+1)(\sigma_1+2)(\sigma_1+3)} \, [(q_4\theta - q_5\mu)(\sigma_1+1) E_{\sigma_1+1} - E_{\sigma_1} \\
& - \nu\, q_3 q_4 \left(\sum_{\sigma_2=0}^{\sigma_1} (\sigma_1 - \sigma_2 + 1)(K_{\sigma_2} D_{\sigma_1-\sigma_2+1} + D_{\sigma_2} K_{\sigma_1-\sigma_2+1}) \right) \Bigg] , \\
K_{\sigma_1+3} = {} & \frac{1}{(\nu\, q_4^3 + \sigma_1 u\, q_5^2 q_4)(\sigma_1+1)(\sigma_1+2)(\sigma_1+3)} \, [(q_4\theta - q_5\mu)(\sigma_1+1) K_{\sigma_1+1} - K_{\sigma_1} \\
& - \nu\, q_3 q_4 \left(\sum_{\sigma_2=0}^{\sigma_1} (\sigma_1 - \sigma_2 + 1)(D_{\sigma_2} E_{\sigma_1-\sigma_2+1} + E_{\sigma_2} D_{\sigma_1-\sigma_2+1}) \right) \Bigg] ,
\end{aligned}
\tag{3.26}
$$

here D_0, D_1, D_2, E_0, E_1, E_2, K_0, K_1, K_2 are arbitrary constants, and
$D_3 = \frac{-1}{6\,(\nu\, q_4^3 + \nu\, q_5^2 q_4)} \, [(q_4\theta + q_5\mu) D_1 + D_0 + \nu\, q_1 q_4\, (D_0 E_1 + E_0 D_1)) \, + \nu\, q_2 q_4\, ((E_0 K_1 + K_0 E_1))]$,
$E_3 = \frac{1}{6\,(\nu\, q_4^3 + \nu\, q_5^2 q_4)} \, [(q_4\theta - q_5\mu) E_1 - E_0 + \nu\, q_3 q_4\, ((K_0 D_1 + D_0 K_1))]$,
$K_3 = \frac{1}{6\,(\nu\, q_4^3 + \nu\, q_5^2 q_4)} \, [(q_4\theta - q_5\mu) K_1 - K_0 + \nu\, q_3 q_4\, ((D_0 E_1 + E_0 D_1))]$.
Then, the explicit solutions for the given system appeared in form as given below

$$
\begin{aligned}
u(t,x,y) = {} & e^{\frac{1}{\nu}\int T_2(t)\,dt} \Bigg(D_0 + D_1 \zeta + D_2 \zeta^2 - \frac{1}{6\,(\nu\, q_4^3 + \nu\, q_5^2 q_4)} \, [(q_4\theta + q_5\mu) D_1 + D_0 \\
& + \nu\, q_1 q_4\, (D_0 E_1 + E_0 D_1)) + \nu\, q_2 q_4\, ((E_0 K_1 + K_0 E_1)] \, \zeta^3 \\
& - \sum_{\sigma_1=1}^{\infty} \frac{1}{(\nu\, q_4^3 + \nu\, q_5^2 q_4)(\sigma_1+1)(\sigma_1+2)(\sigma_1+3)} \, [(q_4\theta + q_5\mu)(\sigma_1+1) D_{\sigma_1+1} + D_{\sigma_1} \\
& + \nu\, q_1 q_4 \left(\sum_{\sigma_2=0}^{\sigma_1} (\sigma_1 - \sigma_2 + 1)(D_{\sigma_2} E_{\sigma_1-\sigma_2+1} + E_{\sigma_2} D_{\sigma_1-\sigma_2+1}) \right) \\
& + \nu\, q_2 q_4 \left(\sum_{\sigma_2=0}^{\sigma_1} (\sigma_1 - \sigma_2 + 1)(E_{\sigma_2} K_{\sigma_1-\sigma_2+1} + K_{\sigma_2} E_{\sigma_1-\sigma_2+1}) \right) \Bigg] \, \zeta^{\sigma_1+3} \Bigg)
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
v(t,x,y) &= e^{\frac{1}{\nu}\int T_2(t)\,dt}\Bigg(E_0+E_1\zeta+E_2\zeta^2+\frac{1}{6\left(\nu\, q_4^3+\nu\, q_5^2 q_4\right)}\left[(q_4\theta-q_5\mu)E_1-E_0\right.\\
&\quad\left.+\nu\, q_3q_4\left((K_0D_1+D_0K_1)\right)\right]\zeta^3\\
&\quad+\sum_{\sigma_1=1}^{\infty}\frac{1}{(\nu\, q_4^3+\nu\, q_5^2q_4)(\sigma_1+1)(\sigma_1+2)(\sigma_1+3)}\left[(q_4\theta-q_5\mu)(\sigma_1+1)E_{\sigma_1+1}-E_{\sigma_1}\right.\\
&\quad\left.+\nu\, q_3q_4\left(\sum_{\sigma_2=0}^{\sigma_1}(\sigma_1-\sigma_2+1)(K_{\sigma_2}D_{\sigma_1-\sigma_2+1}+D_{\sigma_2}K_{\sigma_1-\sigma_2+1})\right)\right]\zeta^{\sigma_1+3}\Bigg),\\
w(t,x,y) &= e^{\frac{1}{\nu}\int T_2(t)\,dt}\Bigg(K_0+K_1\zeta+K_2\zeta^2+\frac{1}{6\left(\nu\, q_4^3+\nu\, q_5^2 q_4\right)}\left[(q_4\theta-q_5\mu)K_1-K_0\right.\\
&\quad\left.+\nu\, q_3q_4\left((D_0E_1+E_0D_1)\right)\right]\zeta^3\\
&\quad+\sum_{\sigma_1=1}^{\infty}\frac{1}{(\nu\, q_4^3+\nu\, q_5^2q_4)(\sigma_1+1)(\sigma_1+2)(\sigma_1+3)}\left[(q_4\theta-q_5\mu)(\sigma_1+1)K_{\sigma_1+1}-K_{\sigma_1}\right.\\
&\quad\left.+\nu\, q_3q_4\left(\sum_{\sigma_2=0}^{\sigma_1}(\sigma_1-\sigma_2+1)(D_{\sigma_2}E_{\sigma_1-\sigma_2+1}+E_{\sigma_2}D_{\sigma_1-\sigma_2+1})\right)\right]\zeta^{\sigma_1+3}\Bigg),
\end{aligned}
\tag{3.28}
$$

where $\zeta = q_4\left(-\frac{\theta}{\nu}\int T_2(t)\,dt+x\right)-q_5\left(-\frac{\mu}{\nu}\int T_2(t)\,dt+y\right)$.

3.1.3.3 Vector field $X_3 + r\,X_2 + s\,X_1$

The invariants, coefficient functions and reduced PDEs for the present vector field are obtained as follows

$$
\begin{aligned}
&\zeta_1=-\frac{r}{s}\int T_2(t)\,dt+x,\ \zeta_2=-\frac{1}{s}\int T_2(t)\,dt+y,\\
&u=F_1(\zeta_1,\zeta_2),\ v=G_1(\zeta_1,\zeta_2),\ w=H_1(\zeta_1,\zeta_2),\\
&T_1(t)=c_1\,T_2(t),\ T_3(t)=c_2\,T_2(t),\ T_4(t)=c_3\,T_2(t),
\end{aligned}
\tag{3.29}
$$

and

$$
\begin{aligned}
&-F_{1\zeta_1}r-F_{1\zeta_2}+c_1\,sF_1G_{1\zeta_1}+c_1\,sF_{1\zeta_1}G_1+c_2\,sG_1H_{1\zeta_1}+c_2\,sG_{1\zeta_1}H_1+sF_{1\zeta_1\zeta_1\zeta_1}\\
&+sF_{1\zeta_1\zeta_2\zeta_2}=0,\\
&G_{1\zeta_1}r+G_{1\zeta_2}-c_3\,sH_1F_{1\zeta_1}-c_3\,sH_{1\zeta_1}F_1-sG_{1\zeta_1\zeta_1\zeta_1}-sG_{1\zeta_1\zeta_2\zeta_2}=0,\\
&-H_{1\zeta_1}r-H_{1\zeta_2}+c_3\,sF_1G_{1\zeta_1}+c_3\,sF_{1\zeta_1}G_1+sH_{1\zeta_1\zeta_1\zeta_1}+sH_{1\zeta_1\zeta_2\zeta_2}=0,
\end{aligned}
\tag{3.30}
$$

where c_j, $j=1,2,3$ are considered as random constants. The again applying similarity transformations, the following infinitesimals are obtained

$$\xi^1 = c_5,\ \xi^2 = c_4,\ \eta^1 = \eta^2 = \eta^3 = 0, \tag{3.31}$$

where c_4 and c_5 are arbitrary constants. The corresponding invariants and ODEs are obtained as follows

$$\zeta = c_4\,\zeta_1 - c_5\,\zeta_2,\ F_1(\zeta_1,\zeta_2) = f(\zeta),\ G_1(\zeta_1,\zeta_2) = g(\zeta),\ H_1(\zeta_1,\zeta_2) = h(\zeta), \tag{3.32}$$

and

$$\begin{aligned}
&-f_{\zeta}c_4\,r + f_{\zeta}c_5 + c_1\,sfg_{\zeta}c_4 + c_1\,sf_{\zeta}c_4\,g + c_2\,sgh_{\zeta}c_4 + c_2\,sg_{\zeta}c_4\,h + sf_{\zeta\zeta\zeta}{c_4}^3\\
&+ sf_{\zeta\zeta\zeta}{c_5}^2c_4 = 0,\\
&g_{\zeta}c_4\,r - g_{\zeta}c_5 - c_3\,shf_{\zeta}c_4 - c_3\,sh_{\zeta}c_4\,f - sg_{\zeta\zeta\zeta}{c_4}^3 - sg_{\zeta\zeta\zeta}{c_5}^2c_4 = 0,\\
&-h_{\zeta}c_4\,r + h_{\zeta}c_5 + c_3\,sfg_{\zeta}c_4 + c_3\,sf_{\zeta}c_4\,g + sh_{\zeta\zeta\zeta}{c_4}^3 + sh_{\zeta\zeta\zeta}{c_5}^2c_4 = 0.
\end{aligned} \tag{3.33}$$

These ODEs are solved by using computational software Maple and the exact solutions for ZK system are obtained as follows

(i)

$$\begin{aligned}
c_2 &= \frac{6\,{c_7}^2\left(12\,{c_7}^2{c_5}^2{c_4}^2 + {c_4}^2c_1\,c_8 + 6\,{c_7}^2{c_4}^4 + 6\,{c_7}^2{c_5}^4 + {c_5}^2c_1\,c_8\right)}{{c_8}^2c_3},\\
u\left(t,x,y\right) &= \frac{8\,A - c_5 + rc_4}{2\,c_3\,c_4\,s} - \frac{6\,B}{c_3}\tanh^2\left(c_6 + c_7\,\zeta\right),\\
v\left(t,x,y\right) &= -\frac{c_8\,\left(8\,A - c_5 + rc_4\right)}{12\,c_4\,s\,B} + c_8\,\tanh^2\left(c_6 + c_7\,\zeta\right),\\
w\left(t,x,y\right) &= -\frac{c_8\,\left(8\,A - c_5 + rc_4\right)}{12\,c_4\,s\,B} + c_8\,\tanh^2\left(c_6 + c_7\,\zeta\right).
\end{aligned} \tag{3.34}$$

(ii)

$$\begin{aligned}
c_2 &= -\frac{3\,{c_1}^2}{16\,c_3},\\
u\left(t,x,y\right) &= \frac{-4\,A - c_5 + rc_4}{2\,c_3\,c_4\,s} + \frac{6\,B}{c_3}\operatorname{sech}^2\left(c_6 + c_7\,\zeta\right),\\
v\left(t,x,y\right) &= \frac{2\left(-4\,A - c_5 + rc_4\right)}{3\,c_1\,sc_4} + \frac{8\,B}{c_1}\operatorname{sech}^2\left(c_6 + c_7\,\zeta\right),\\
w\left(t,x,y\right) &= \frac{2\left(-4\,A - c_5 + rc_4\right)}{3\,c_1\,sc_4} + \frac{8\,B}{c_1}\operatorname{sech}^2\left(c_6 + c_7\,\zeta\right).
\end{aligned} \tag{3.35}$$

(iii)

$$\begin{aligned}
c_2 &= \frac{6\,{c_7}^2\left(12\,{c_7}^2{c_5}^2{c_4}^2+{c_4}^2c_1\,c_8+6\,{c_7}^2{c_4}^4+6\,{c_7}^2{c_5}^4+{c_5}^2c_1\,c_8\right)}{{c_8}^2c_3},\\
u\left(t,x,y\right) &= -\frac{-4\,A-c_5+rc_4}{2\,c_3\,c_4\,s}+\frac{6\,B}{c_3}\operatorname{csch}^2\left(c_6+c_7\,\zeta\right),\\
v\left(t,x,y\right) &= -\frac{c_8\,\left(-4\,A-c_5+rc_4\right)}{12\,c_4\,s\,B}+c_8\operatorname{csch}^2\left(c_6+c_7\,\zeta\right),\\
w\left(t,x,y\right) &= \frac{c_8\,\left(-4\,A-c_5+rc_4\right)}{12\,c_4\,s}-c_8\operatorname{csch}^2\left(c_6+c_7\,\zeta\right).
\end{aligned}\tag{3.36}$$

(iv)

$$\begin{aligned}
c_2 &= \frac{6\,{c_7}^2\left(12\,{c_7}^2{c_5}^2{c_4}^2+{c_4}^2c_1\,c_8+6\,{c_7}^2{c_4}^4+6\,{c_7}^2{c_5}^4+{c_5}^2c_1\,c_8\right)}{{c_8}^2c_3},\\
u\left(t,x,y\right) &= -\frac{-8\,A-c_5+rc_4}{2\,c_4\,sc_3}+\frac{6\,B}{c_3}\,\cot^2\left(c_6+c_7\,\zeta\right),\\
v\left(t,x,y\right) &= -\frac{c_8\,\left(-8\,A-c_5+rc_4\right)}{12\,c_4\,s\,B}+c_8\,\cot^2\left(c_6+c_7\,\zeta\right),\\
w\left(t,x,y\right) &= \frac{c_8\,\left(-8\,A-c_5+rc_4\right)}{12\,c_4\,s\,B}-c_8\,\cot^2\left(c_6+c_7\,\zeta\right).
\end{aligned}\tag{3.37}$$

(v)

$$\begin{aligned}
c_2 &= -\frac{3\,{c_1}^2}{16\,c_3},\\
u\left(t,x,y\right) &= -\frac{C}{2\,c_3}+\frac{6\,B\,{c_8}^2}{c_3c_7^2}\operatorname{ns}^2\left(c_7+c_8\,\zeta,c_6\right),\\
v\left(t,x,y\right) &= \frac{2\,C}{3\,c_1}-\frac{8\,B\,{c_8}^2}{c_1\,c_7^2}\operatorname{ns}^2\left(c_7+c_8\,\zeta,c_6\right),\\
w\left(t,x,y\right) &= -\frac{2\,C}{3\,c_1}+\frac{8\,B\,{c_8}^2}{c_1\,c_7^2}\operatorname{ns}^2\left(c_7+c_8\,\zeta,c_6\right).
\end{aligned}\tag{3.38}$$

(vi)

$$\begin{aligned}
c_2 &= -\frac{3\,{c_1}^2}{16\,c_3},\\
u\left(t,x,y\right) &= -\frac{C}{2\,c_3}+\frac{6\,{c_8}^2{c_6}^2\,B}{c_3\,c_7^2}\operatorname{sn}^2\left(c_7+c_8\,\zeta,c_6\right),\\
v\left(t,x,y\right) &= \frac{2\,C}{3\,c_1}-\frac{8\,{c_8}^2{c_6}^2\,B}{c_1\,c_7^2}\operatorname{sn}^2\left(c_7+c_8\,\zeta,c_6\right),\\
w\left(t,x,y\right) &= -\frac{2\,C}{3\,c_1}+\frac{8\,{c_8}^2{c_6}^2\,B}{c_1\,c_7^2}\operatorname{sn}^2\left(c_7+c_8\,\zeta,c_6\right).
\end{aligned}\tag{3.39}$$

Herein,

$$\begin{aligned}
&\zeta = \left(-\frac{(rc_4 - c_5)\int T_2(t)\,dt + s(-c_4x + c_5y)}{s}\right),\\
&A = {c_7}^2{c_4}^3s + {c_7}^2c_4\,s{c_5}^2\\
&B = {c_7}^2\left({c_5}^2 + {c_4}^2\right)\\
&C = \frac{4\,{c_8}^2{c_4}^3s{c_6}^2 + 4\,{c_8}^2{c_4}^3s + 4\,{c_8}^2c_4\,s{c_5}^2{c_6}^2 + 4\,{c_8}^2c_4\,s{c_5}^2 - c_5 + rc_4}{c_4\,s},
\end{aligned} \tag{3.40}$$

where c_i are arbitrary constants.

Figures 3.1-3.6 show graphical representation of dark, bright and periodic waves for solutions (3.34), (3.35) and (3.39), respectively by considering suitable parametric values. The Figure 3.1 shows dark solitary wave solution (3.34) for u, v and w when $T_2(t) = \sin(t)$. Figure 3.2 describes the effect of coefficient function $T_2(t)$ on the wave profile of the solution when it is considered as linear, exponential and trigonometric function of t. Figure 3.3 depicts bright solitary wave solution (3.35) at particular $T_2(t) = e^t$ and Figure 3.4 reveals the influence of various $T_2(t)$ on solution profile. Figure 3.5 and Figure 3.6 show periodic wave profile of the solution (3.39) in 3D plots for $T_2(t) = t$ and 2D plots for different $T_2(t)$, respectively.

3.1.3.4 Vector field $X_2 + p\,X_1$

By following similar procedure as described in previous section, the reduced ODEs are obtained for vector field $X_2 + p\,X_1$ as follows

$$\begin{aligned}
&-pf_{\zeta\zeta\zeta}{r_4}^2 + f_\zeta - r_1\,pfg_\zeta - r_1\,pf_\zeta g - r_2\,pgh_\zeta - r_2\,pg_\zeta h - pf_{\zeta\zeta\zeta}{r_5}^2 = 0,\\
&pg_{\zeta\zeta\zeta}{r_4}^2 - g_\zeta + r_3\,phf_\zeta + r_3\,ph_\zeta f + pg_{\zeta\zeta\zeta}{r_5}^2 = 0,\\
&-ph_{\zeta\zeta\zeta}{r_4}^2 + h_\zeta - r_3\,pfg_\zeta - r_3\,pf_\zeta g - ph_{\zeta\zeta\zeta}{r_5}^2 = 0,
\end{aligned} \tag{3.41}$$

where $\zeta = r_5\zeta_1 - r_4\zeta_2$, $\zeta_1 = y$, $\zeta_2 = -\frac{1}{p}\int T_2(t)\,dt + x$, $f(\zeta) = u$, $g(\zeta) = v$, $h(\zeta) = w$, and $T_1(t) = r_1T_2(t)$, $T_3(t) = r_2T_2(t)$ and $T_4(t) = r_3T_2(t)$. Herein, r_i, $i = 1, 2, ..., 5$ are arbitrary constants. Using Maple software, the exact solutions for given system are obtained as follows

(i)

$$
\begin{aligned}
&r_2 = \frac{2\,r_1\,r_{10}\,p+1}{4\,r_3\,p^2 {r_{10}}^2},\\
&u\left(t,x,y\right) = -\frac{1}{2\,p\,r_3} + \frac{6\,{r_9}^2\,D}{r_3}\,\mathrm{WeierstrassP}\left(r_8+r_9\,\zeta, r_7, r_6\right),\\
&v\left(t,x,y\right) = -r_{10} + E\,\mathrm{WeierstrassP}\left(r_8+r_9\,\zeta, r_7, r_6\right),\\
&w\left(t,x,y\right) = r_{10} - E\,\mathrm{WeierstrassP}\left(r_8+r_9\,\zeta, r_7, r_6\right).
\end{aligned}
\tag{3.42}
$$

(ii)

$$
\begin{aligned}
&r_2 = \frac{6\,{r_7}^2\left(r_8\,{r_5}^2 r_1 + r_8\,{r_4}^2 r_1 + 6\,{r_7}^2 {r_4}^4 + 12\,{r_7}^2 {r_4}^2 {r_5}^2 + 6\,{r_7}^2 {r_5}^4\right)}{{r_8}^2 r_3},\\
&u\left(t,x,y\right) = \frac{8\,J+1}{2\,p\,r_3} - \frac{6\,{r_7}^2 D}{r_3}\,\coth^2\left(r_6+r_7\,\zeta\right),\\
&v\left(t,x,y\right) = -\frac{r_8\,\left(8\,J+1\right)}{12\,{r_7}^2 p\,D} + c_8\,\coth^2\left(r_6+r_7\,\zeta\right),\\
&w\left(t,x,y\right) = -\frac{r_8\,\left(8\,J+1\right)}{12\,{r_7}^2 p\,D} + c_8\,\coth^2\left(r_6+r_7\,\zeta\right).
\end{aligned}
\tag{3.43}
$$

(iii)

$$
\begin{aligned}
&r_2 = \frac{6\,{r_7}^2\left(r_8\,{r_5}^2 r_1 + r_8\,{r_4}^2 r_1 + 6\,{r_7}^2 {r_4}^4 + 12\,{r_7}^2 {r_4}^2 {r_5}^2 + 6\,{r_7}^2 {r_5}^4\right)}{{r_8}^2 r_3},\\
&u\left(t,x,y\right) = -\frac{4\,J+1}{2\,p\,r_3} + \frac{6\,{r_7}^2\,D}{r_3}\,\sec^2\left(r_6+r_7\,\zeta\right),\\
&v\left(t,x,y\right) = -\frac{r_8\,\left(4\,J+1\right)}{12\,{r_7}^2 p\,D} + r_8\,\sec^2\left(r_6+r_7\,\zeta\right),\\
&w\left(t,x,y\right) = \frac{r_8\,\left(4\,J+1\right)}{12\,{r_7}^2 p\,D} - r_8\,\sec^2\left(r_6+r_7\,\zeta\right).
\end{aligned}
\tag{3.44}
$$

(iv)

$$
\begin{aligned}
&r_2 = \frac{6\,{r_7}^2\left(r_8\,{r_5}^2 r_1 + r_8\,{r_4}^2 r_1 + 6\,{r_7}^2 {r_4}^4 + 12\,{r_7}^2 {r_4}^2 {r_5}^2 + 6\,{r_7}^2 {r_5}^4\right)}{{r_8}^2 r_3},\\
&u\left(t,x,y\right) = \frac{8\,J-1}{2\,p r_3} + \frac{6\,{r_7}^2\,D}{r_3}\,\tan^2\left(r_6+r_7\,\zeta\right),\\
&v\left(t,x,y\right) = \frac{r_8\,\left(8\,J-1\right)}{12\,{r_7}^2 p\,D} + r_8\,\tan^2\left(r_6+r_7\,\zeta\right),\\
&w\left(t,x,y\right) = -\frac{r_8\,\left(8\,J-1\right)}{12\,{r_7}^2 p\,D} - r_8\,\tan^2\left(r_6+r_7\,\zeta\right).
\end{aligned}
\tag{3.45}
$$

Herein,

$$\begin{aligned}
&\zeta = \left(\frac{r_5 yp + r_4 \int T_2\left(t\right) dt - r_4 xp}{p}\right), \\
&D = \left({r_4}^2 + {r_5}^2\right) \\
&E = \left(12\, r_{10}\, p{r_9}^2{r_5}^2 + 12\, r_{10}\, p{r_9}^2{r_4}^2\right) \\
&J = {r_7}^2{r_4}^2p + {r_7}^2{r_5}^2p,
\end{aligned} \tag{3.46}$$

where r_i are arbitrary constants.

The solutions (3.42) and (3.45) are presented graphically for various parametric values. Figure 3.7 and Figure 3.9 show 3D plots representation when $T_2(t) = t$ and $T_2(t) = e^t$, respectively and reveal singular solitary wave profile of solutions. The effect of function $T_2(t)$ is shown in Figure 3.8 and Figure 3.10.

The graphical representation of other solutions given in Eqs. (3.43)-(3.44) reflects singular nature which is similar to as discussed for solutions (3.42) and (3.45). Similarly, the other solutions in Eqs. (3.36)-(3.38) also follow singular solitary wave structure.

3.1.3.5 Vector field X_1

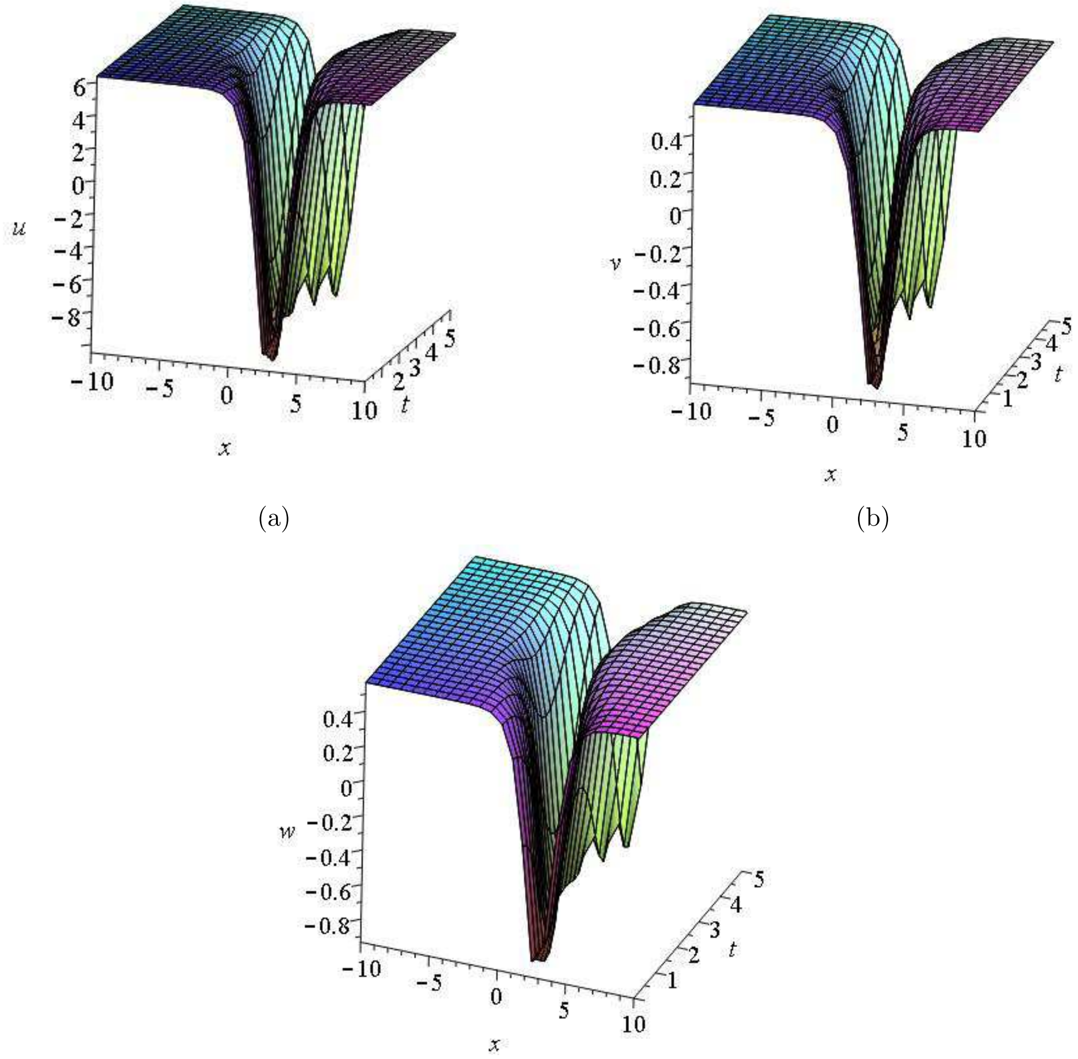
In this case, the solutions are obtained by using Maple software and are given as follows

$$\begin{aligned}
&s_2 = \frac{6\, {s_7}^2 s_1\, s_9 + 6\, {s_7}^4 + 12\, {s_7}^2{s_8}^2 + {s_8}^2 c_1\, s_9 + 6\, {s_8}^4}{{s_9}^2 s_3}, \\
&u\left(t,x,y\right) = \frac{6\, \left({s_8}^2 + {s_7}^2\right)\, \mathrm{WeierstrassP}\left(s_6 + s_7\, x + s_8\, y, s_5, s_4\right)}{s_3}, \\
&v\left(t,x,y\right) = s_9\, \mathrm{WeierstrassP}\left(c_6 + s_7\, x + s_8\, y, s_5, s_4\right), \\
&w\left(t,x,y\right) = -s_9\, \mathrm{WeierstrassP}\left(c_6 + s_7\, x + s_8\, y, s_5, s_4\right).
\end{aligned} \tag{3.47}$$

Remark: In case, when T_1, T_2, T_3 and T_4 are considered as constant functions with respect to variable t then, the solutions (3.34)-(3.39) and (3.42)-(3.45) transform to solutions obtained for constant coefficients ZK system [75, 155, 313].

3.1.4 Nonlinear self-adjointness and conservation laws

This subsection describes the stepwise procedure to prove nonlinear self-adjointness for the system (3.1). The formal Lagrangian [121] for underlying equations in terms

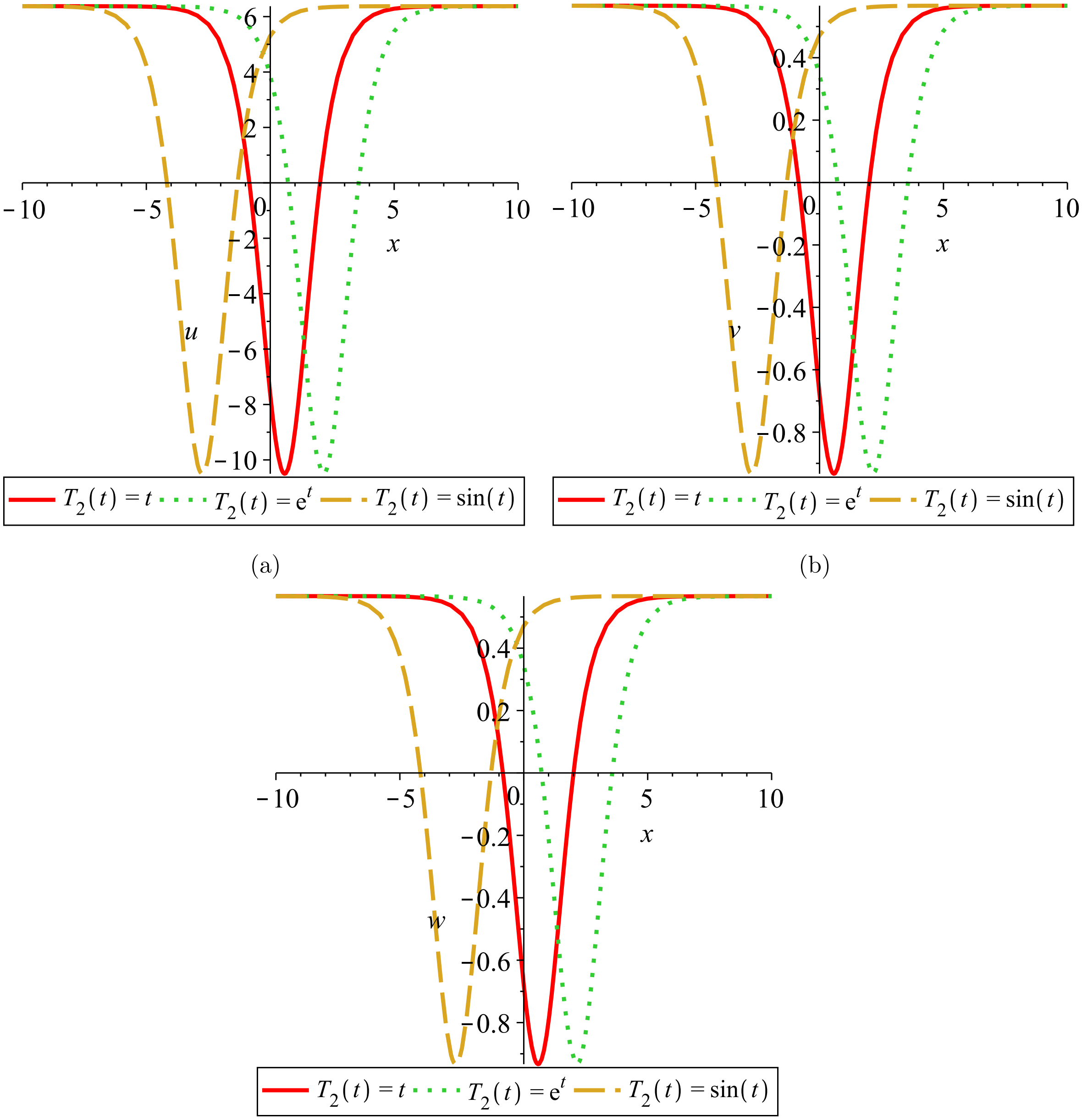


(a)

(b)

(c)

Figure 3.1: Dark solitary wave solution (3.34) profile with c_1, c_3, c_4, c_5, c_6, c_7 and c_8 are taken as 0.25, -1, 1, 2, 0.5, 0.75 and 1.5, respectively for $r = 0.5$, $s = 1$, $y = 1$, $T_2(t) = \sin(t)$.

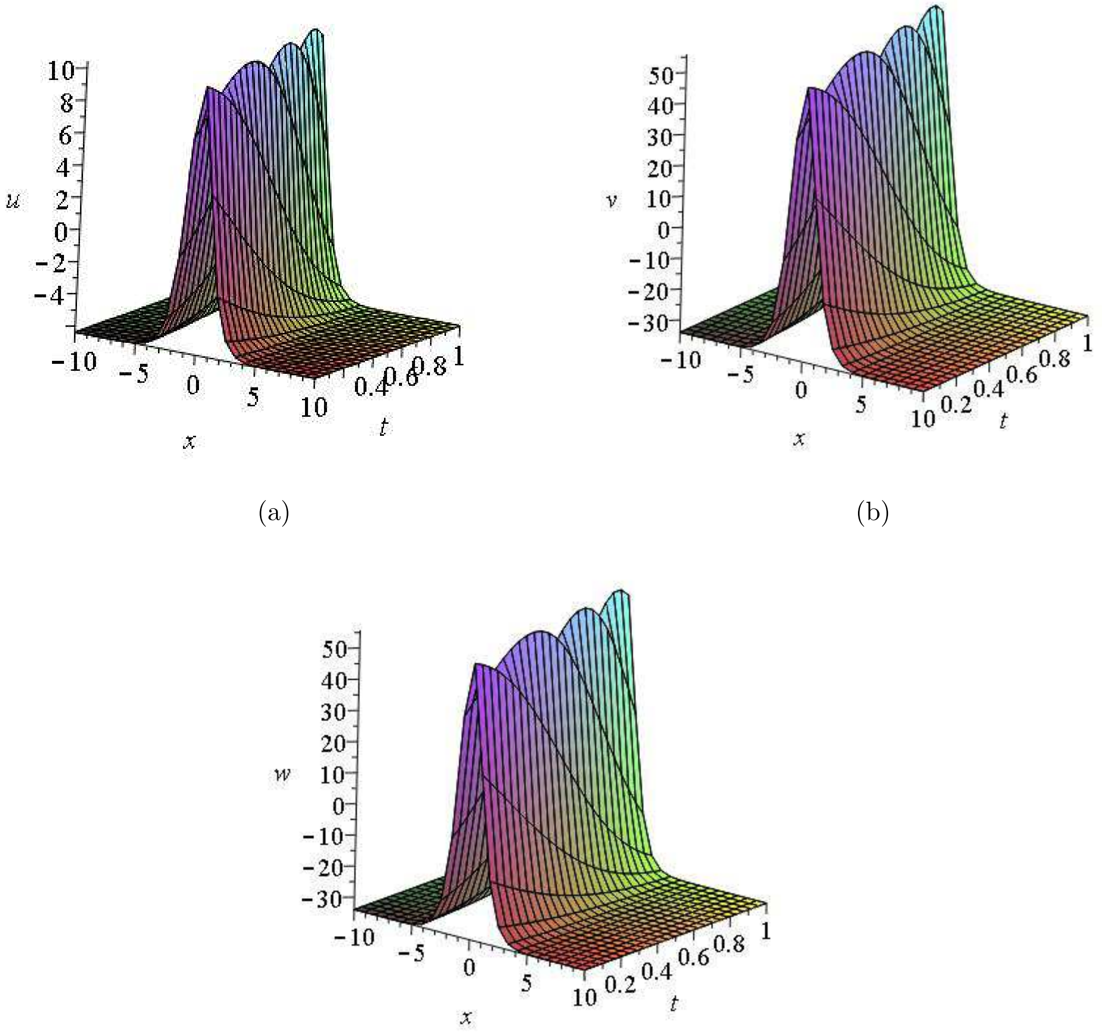


(a)

(b)

(c)

Figure 3.2: Influence of $T_2(t)$ on the profile of the solution (3.34) with c_1, c_3, c_4, c_5, c_6, c_7 and c_8 are taken as 0.25, −1, 1, 2, 0.5, 0.75 and 1.5, respectively for $r = 0.5$, $s = 1$, $y = 1$, $t = 1$.

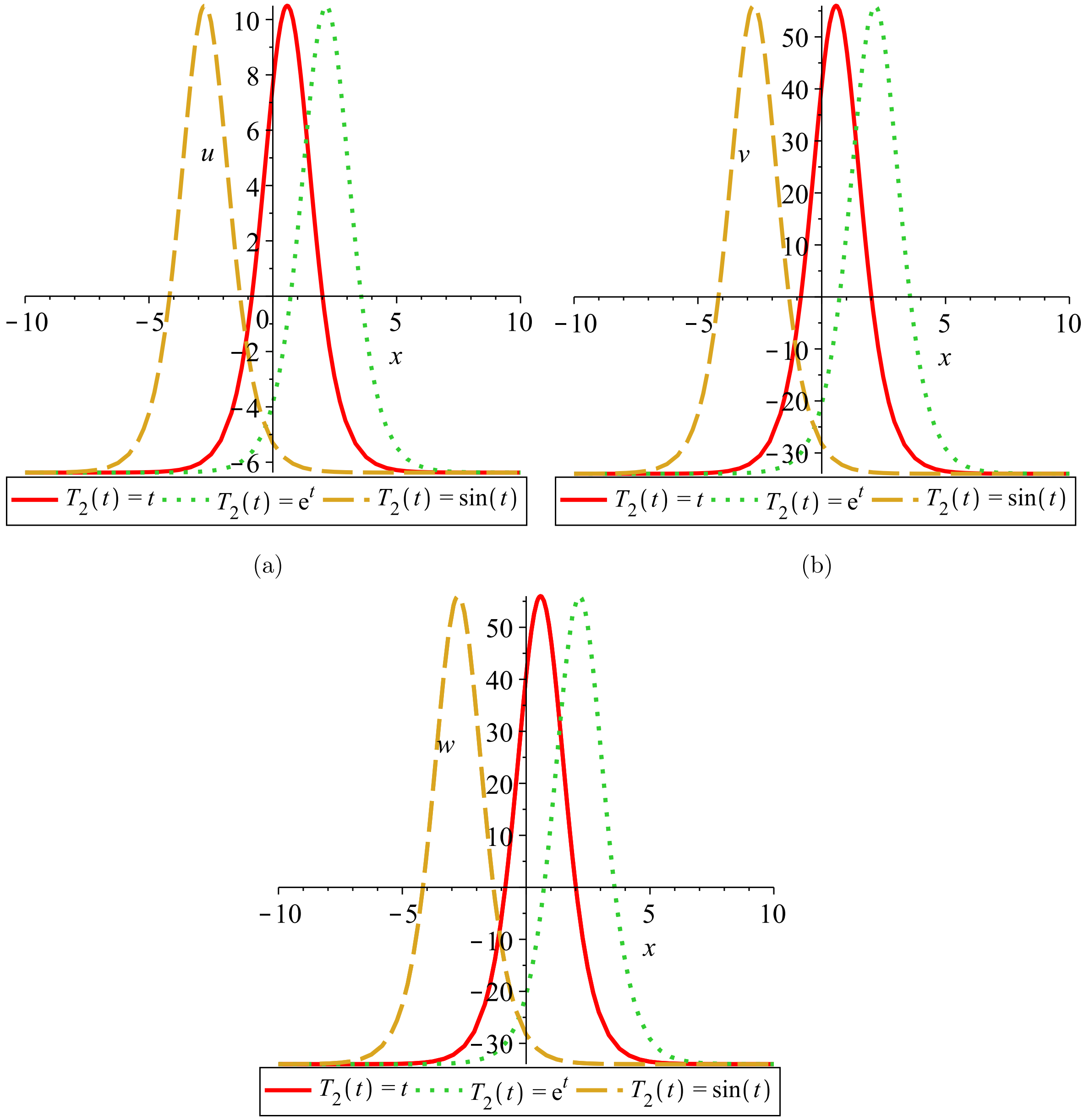


(a)

(b)

(c)

Figure 3.3: Bright solitary wave profile of solution (3.35) with c_1, c_3, c_4, c_5, c_6, c_7 and c_8 are taken as 0.25, 1, 1, 2, 0.5, 0.75 and 1.5, respectively for $r = 0.5$, $s = 1$, $y = 1$, $T_2(t) = e^t$.

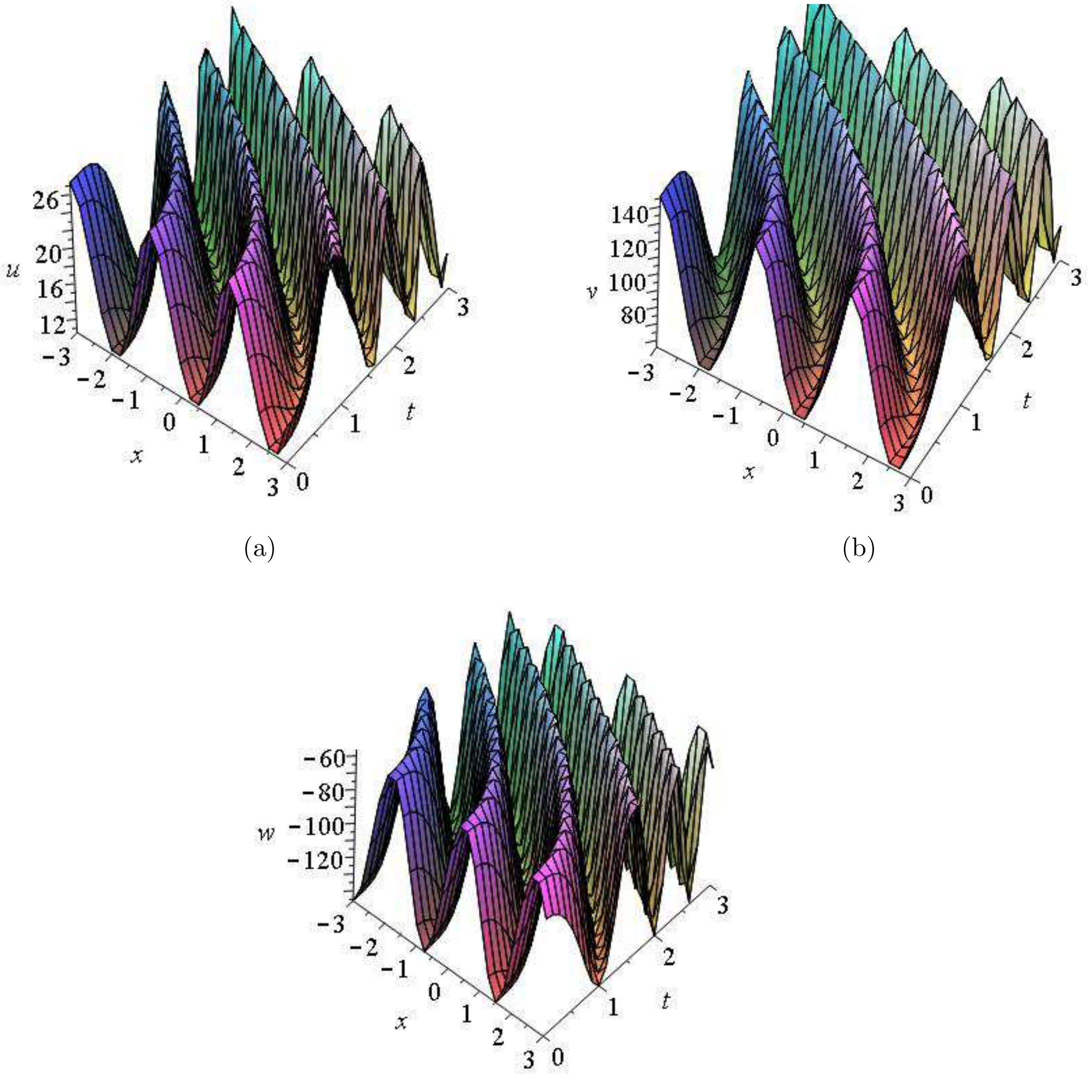


(a)

(b)

(c)

Figure 3.4: Influence of time dependent coefficient $T_2(t)$ on solution (3.35) with c_1, c_3, c_4, c_5, c_6, c_7 and c_8 are taken as 0.25, 1, 1, 2, 0.5, 0.75 and 1.5, respectively for $r = 0.5$, $s = 1$, $y = 1$, $t = 1$.

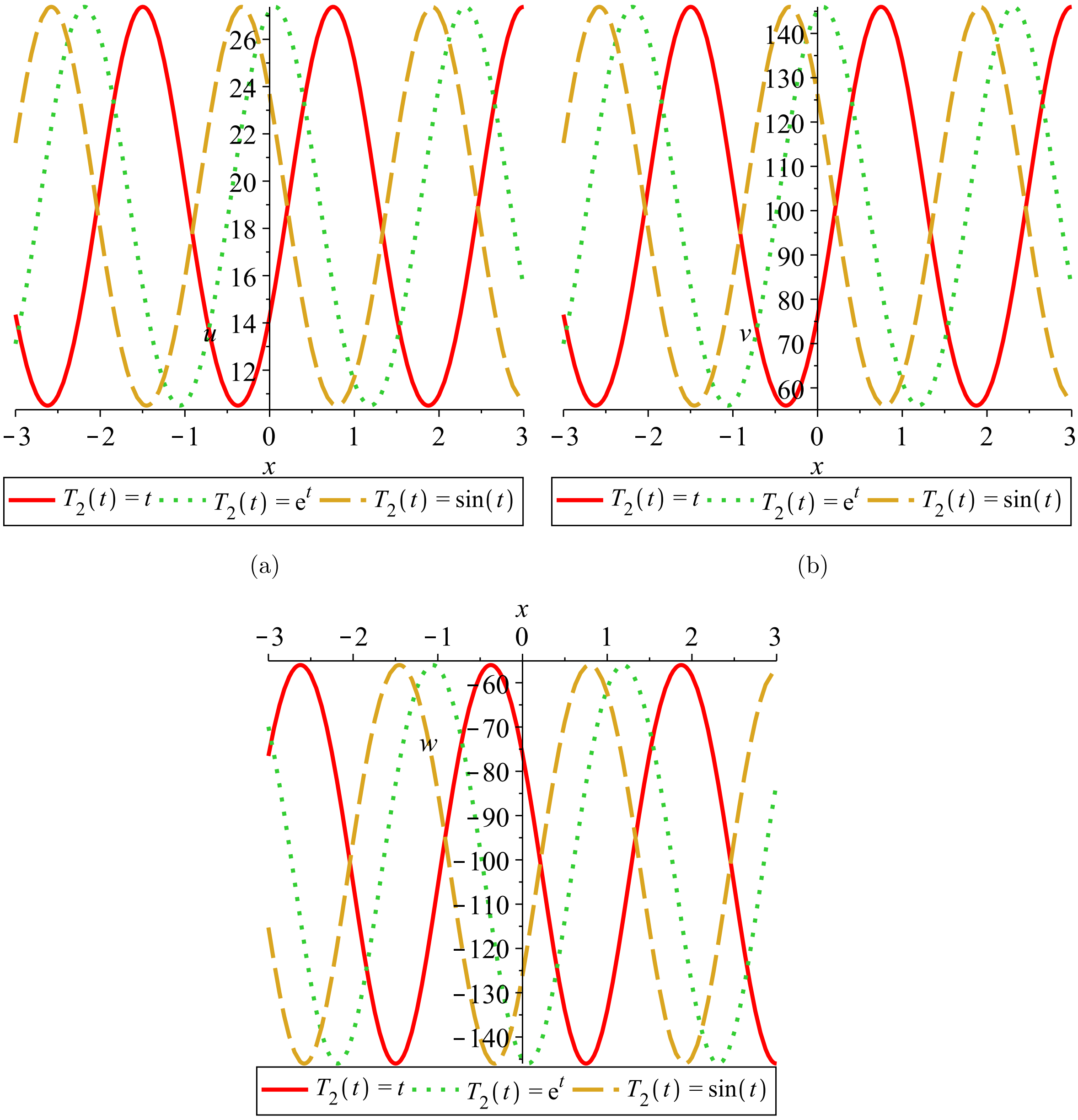


(a)

(b)

(c)

Figure 3.5: Periodic wave profile of the solution (3.39) with c_1, c_3, c_4, c_5, c_6, c_7 and c_8 are taken as 0.25, -1, 1, 2, 0.5, 0.75 and 1.5, respectively for $r = 0.5$, $s = 1$, $y = 1$, $T_2(t) = t$.

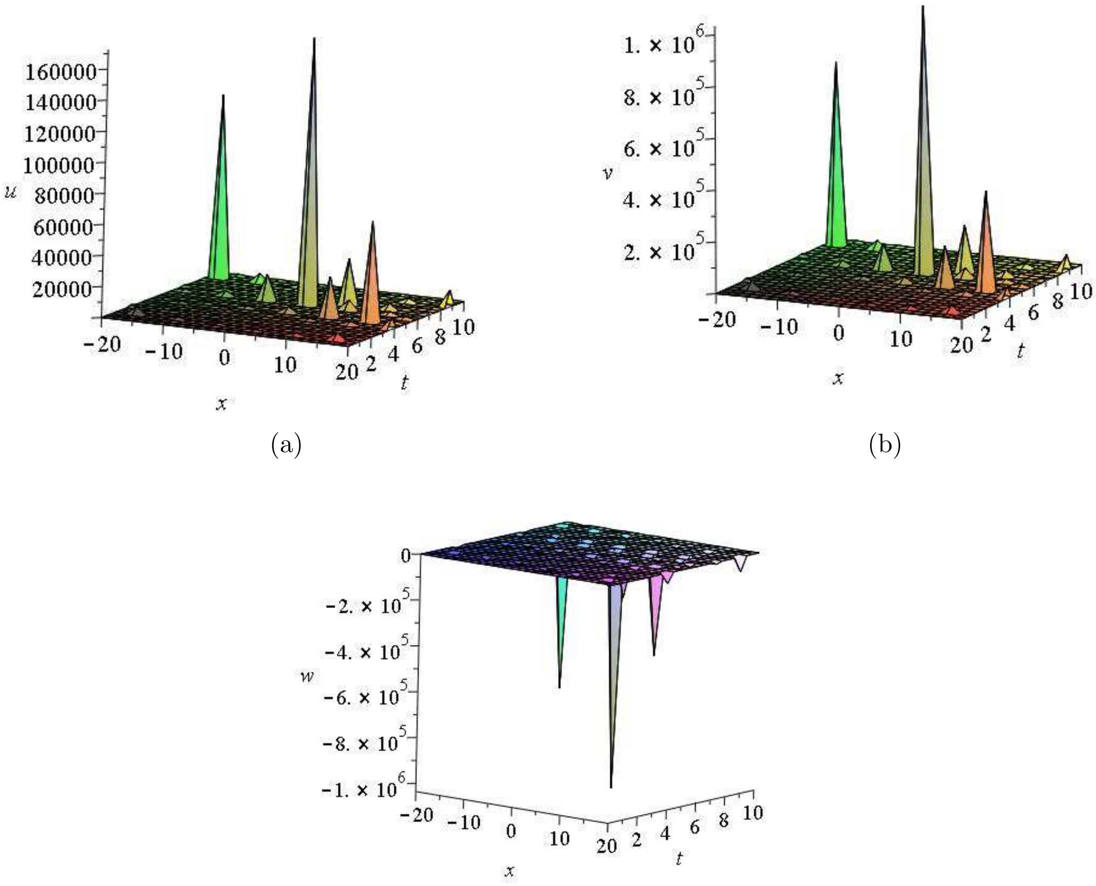


(a)

(b)

(c)

Figure 3.6: 2D representations of solution (3.39) with c_1, c_3, c_4, c_5, c_6, c_7 and c_8 are taken as 0.25, -1, 1, 2, 0.5, 0.75 and 1.5, respectively for $r = 0.5$, $s = 1$, $y = 1$, $t = 1$.

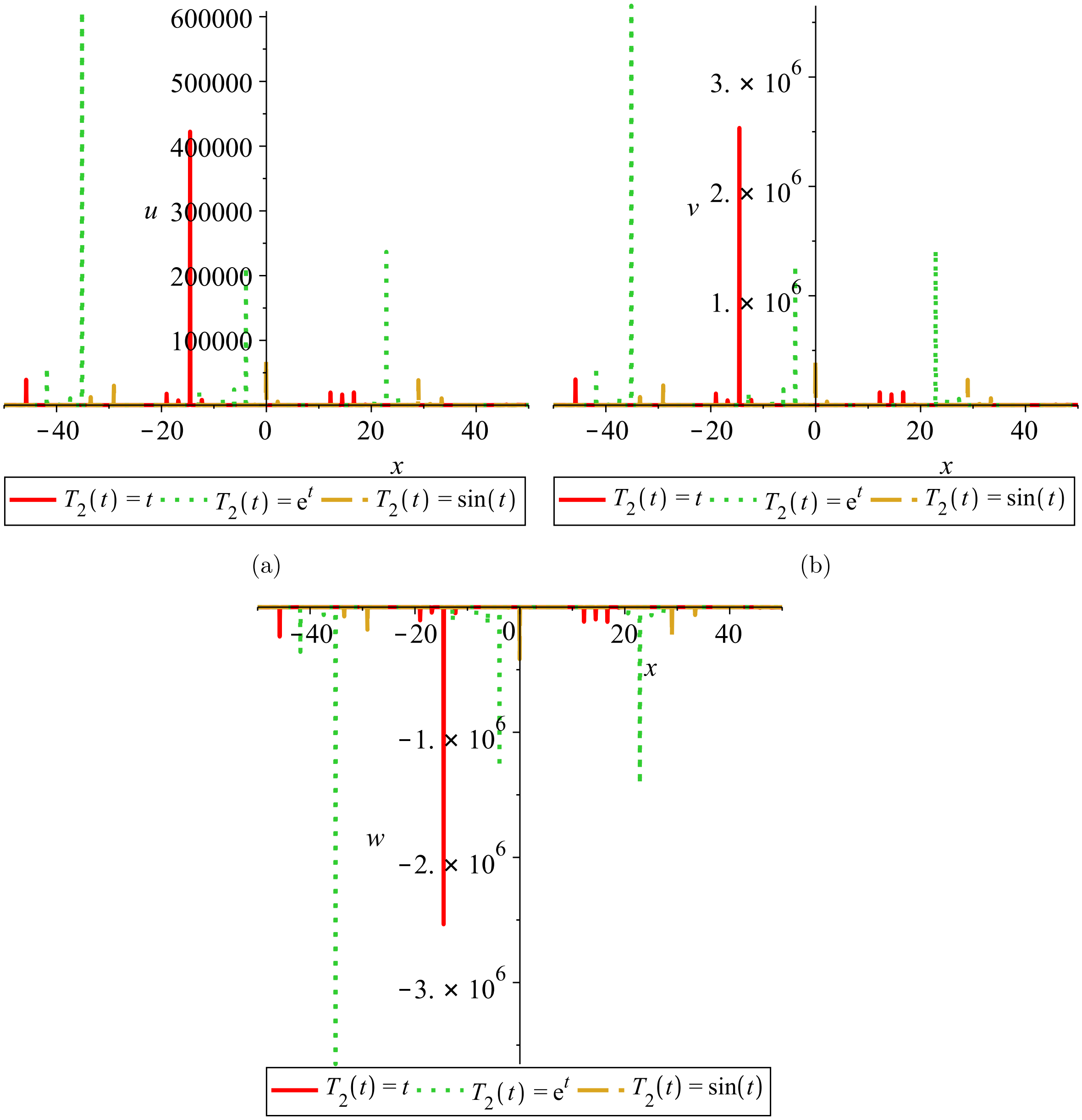


(a)

(b)

(c)

Figure 3.7: Singular solitary wave solution (3.42) with $r_1 = 0.25$, $r_3 = 1$, $r_4 = 2$, $r_5 = 1$, $r_6 = 0.5$, $r_7 = 0.25$, $r_8 = 0.5$, $r_9 = 0.75$, $r_{10} = 1.5$, $p = 2$, $y = 1$, $T_2(t) = t$.

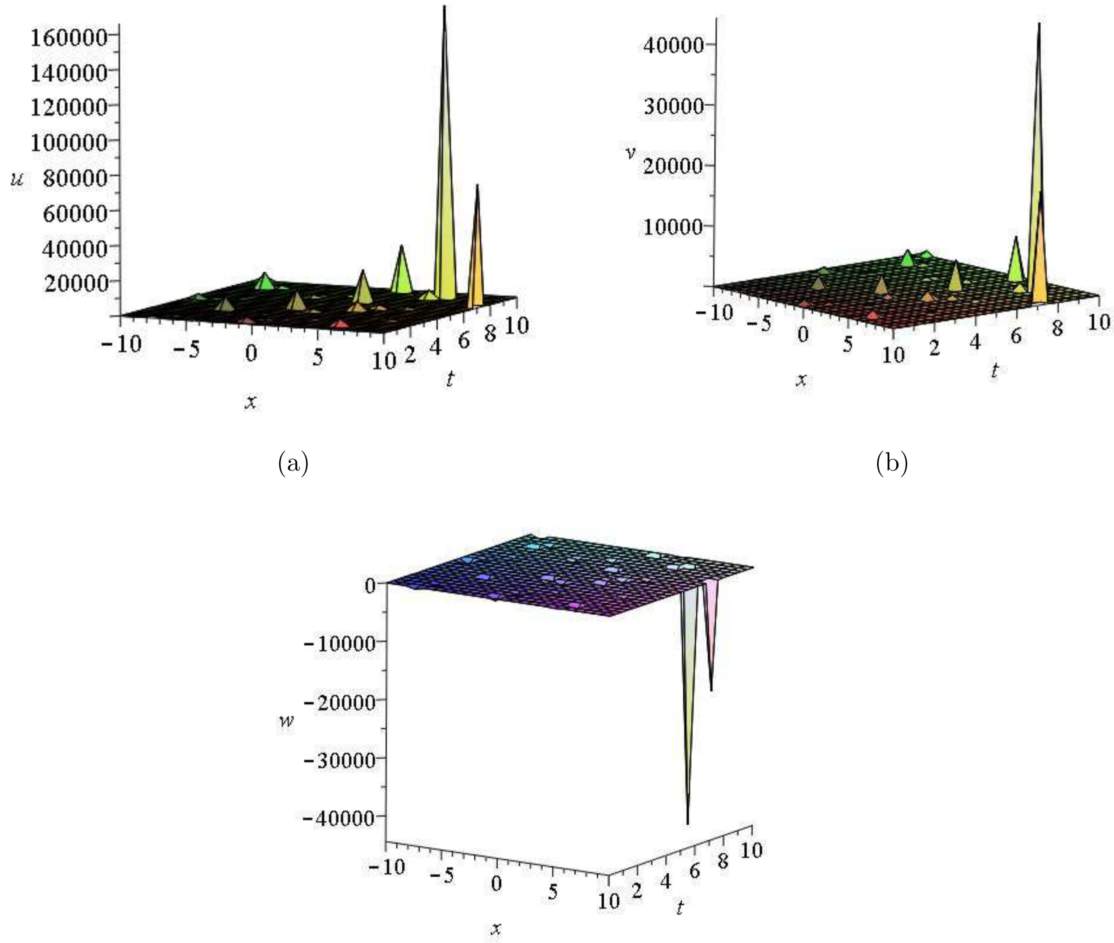


(a)

(b)

(c)

Figure 3.8: Influence of parameter $T_2(t)$ on the profile of solution (3.42) with $r_1 = 0.25$, $r_3 = 1$, $r_4 = 2$, $r_5 = 1$, $r_6 = 0.5$, $r_7 = 0.25$, $r_8 = 0.5$, $r_9 = 0.75$, $r_{10} = 1.5$, $p = 2$, $y = 1$, $t = 1$.

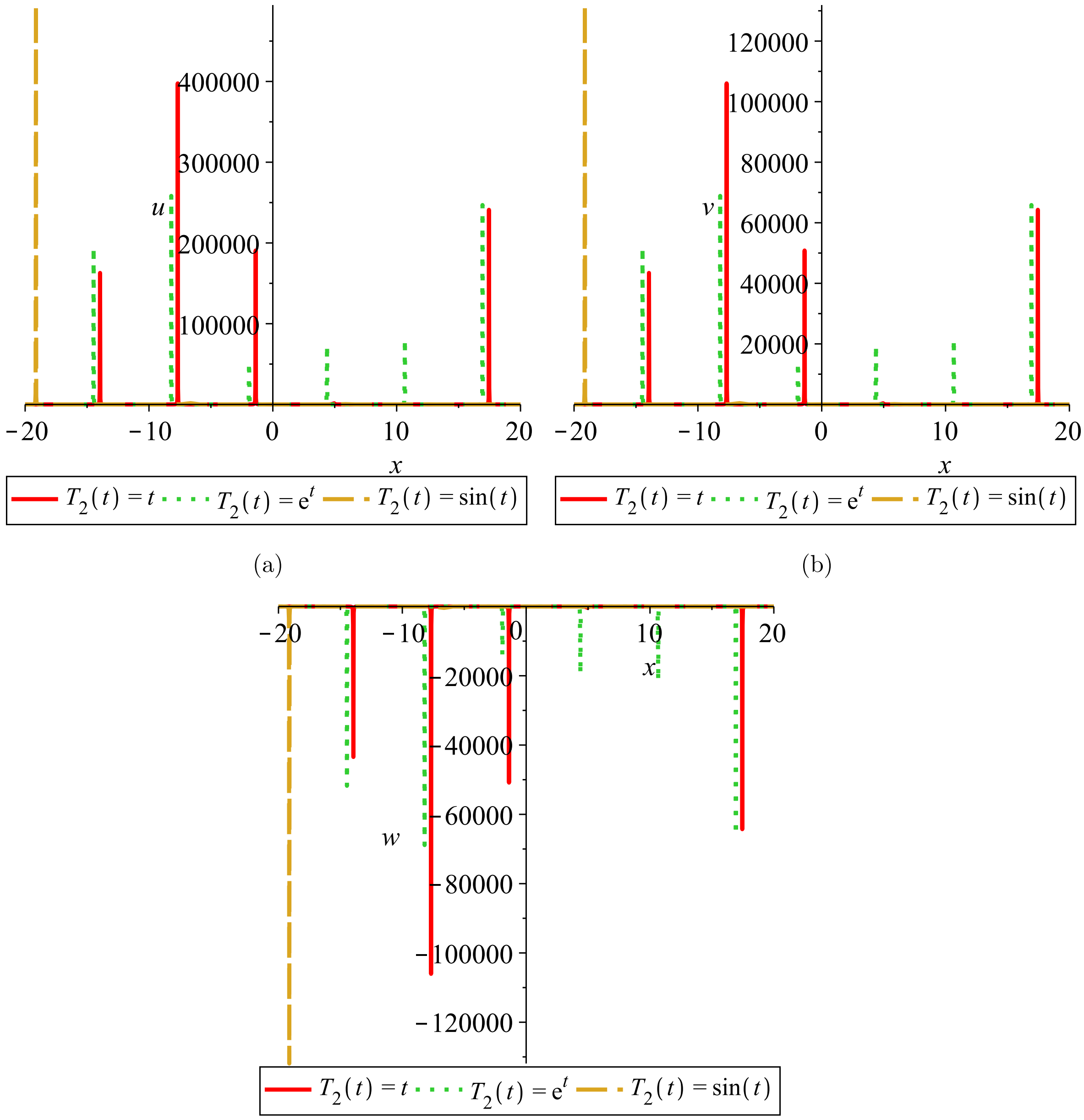


(a)

(b)

(c)

Figure 3.9: Singular solitary wave solution (3.45) with $r_1 = 0.25$, $r_3 = 1$, $r_4 = 2$, $r_5 = 1$, $r_6 = 0.5$, $r_7 = 0.25$, $r_8 = 0.5$, $r_9 = 0.75$, $r_{10} = 1.5$, $p = 2$, $y = 1$, $T_2(t) = e^t$.



(a)

(b)

(c)

Figure 3.10: 2D profile of solitary wave solution (3.45) with $r_1 = 0.25$, $r_3 = 1$, $r_4 = 2$, $r_5 = 1$, $r_6 = 0.5$, $r_7 = 0.25$, $r_8 = 0.5$, $r_9 = 0.75$, $r_{10} = 1.5$, $p = 2$, $y = 1$, $t = 1$.

of new dependent variables $(a(t,x,y), b(t,x,y), c(t,x,y))$ is expressed as follows

$$\begin{aligned}\mathcal{L} &= a(t,x,y)\left(u_t + T_1(t)\left(uv_x + u_x v\right) + T_3(t)\left(vw_x + v_x w\right) + T_2(t)\left(u_{xxx} + u_{xyy}\right)\right)\\ &+ b(t,x,y)\left(v_t + T_4(t)\left(wu_x + w_x u\right) + T_2(t)\left(v_{xxx} + v_{xyy}\right)\right)\\ &+ c(t,x,y)\left(w_t + T_4(t)\left(uv_x + u_x v\right) + T_2(t)\left(w_{xxx} + w_{xyy}\right)\right).\end{aligned} \tag{3.48}$$

Adjoint equations for system of Eqs. (3.1) can be determined as follows

$$\begin{aligned}\Xi_1^* &\equiv -a_x T_1(t)\,v - b_x T_4(t)\,w - c_x T_4(t)\,v - a_t - a_{xxx}T_2(t) - a_{xyy}T_2(t) = 0,\\ \Xi_2^* &\equiv -a_x T_1(t)\,u - a_x T_3(t) w - c_x T_4(t)\,u - b_t - b_{xxx}T_2(t) - b_{xyy}T_2(t) = 0,\\ \Xi_3^* &\equiv -a_x T_3(t) v - b_x T_4(t)\,u - c_t - c_{xxx}T_2(t) - c_{xyy}T_2(t) = 0.\end{aligned} \tag{3.49}$$

The ZK system (3.1) is termed as nonlinear self-adjoint, if following substitution in adjoint equations (3.49)

$$a(t,x,y) = \Phi_1,\ b(t,x,y) = \Phi_2,\ c(t,x,y) = \Phi_3, \tag{3.50}$$

where Φ_i $i = 1,2,3$ are functions of (t,x,y,u,v,w), satisfies equations

$$\begin{aligned}\Xi_1^*\big|_{(3.50)} &= \lambda_1\Xi_1 + \lambda_2\Xi_2 + \lambda_3\Xi_3,\\ \Xi_2^*\big|_{(3.50)} &= \lambda_4\Xi_1 + \lambda_5\Xi_2 + \lambda_6\Xi_3,\\ \Xi_3^*\big|_{(3.50)} &= \lambda_7\Xi_1 + \lambda_8\Xi_2 + \lambda_9\Xi_3,\end{aligned} \tag{3.51}$$

for all the solutions of u, v and w. Here λ_i, $i = 1,2,...,9$ are the undetermined coefficients and the functions $\Phi_i(t,x,y,u,v,w) \neq 0$, simultaneously, $i = 1,2,...,3$.
By matching the coefficients of all possible derivatives of u, v as well as w in Eq. (3.51), the determining equations are obtained and solved with the aid of Maple software. The various $\Phi_i's$ are given as follows

$$\Phi_1 = \mathcal{F}_1(y),\ \Phi_2 = \mathcal{F}_2(y),\ \Phi_3 = \mathcal{F}_3(y), \tag{3.52}$$

where $\mathcal{F}_i(y)$, $i = 1,2,3$ represent arbitrary functions of y. The nonlinear self-adjointness substitutions (3.52) and theorem 2.14.1 provide conservation laws.
Symmetry $X_1 = \frac{\partial}{\partial x}$ with Lie characteristic functions $W_1 = -u_x$, $W_2 = -v_x$ and

$W_3 = -w_x$ gives following conserved vector components

$$\begin{aligned}
C^1 &= -u_x\mathcal{F}_1 - v_x\mathcal{F}_2 - w_x\mathcal{F}_3,\\
C^2 &= \frac{1}{3}\,u_t\mathcal{F}_1 + \frac{1}{3}\,w_t\mathcal{F}_3 + \frac{1}{3}\,v_t\mathcal{F}_2 - \frac{1}{3}\,u_x\mathcal{F}_{1yy}T_2(t) + \frac{1}{3}\,u_{xy}\mathcal{F}_{1y}T_2(t) - \frac{2}{3}\,\mathcal{F}_1T_1(t)\,uv_x\\
&\quad - \frac{2}{3}\,\mathcal{F}_1T_1(t)\,u_xv - \frac{2}{3}\,\mathcal{F}_1T_3(t)vw_x - \frac{2}{3}\,\mathcal{F}_1T_3(t)v_xw - \frac{2}{3}\,\mathcal{F}_1T_2(t)\,u_{xxx} - \frac{1}{3}\,v_x\mathcal{F}_{2yy}T_2(t)\\
&\quad + \frac{1}{3}\,v_{xy}\mathcal{F}_{2y}T_2(t) - \frac{2}{3}\,\mathcal{F}_2T_4(t)\,wu_x - \frac{2}{3}\,\mathcal{F}_2T_4(t)\,w_xu - \frac{2}{3}\,\mathcal{F}_2T_2(t)\,v_{xxx} - \frac{1}{3}\,w_x\mathcal{F}_{3yy}T_2(t)\\
&\quad + \frac{1}{3}\,w_{xy}\mathcal{F}_{3y}T_2(t) - \frac{2}{3}\,\mathcal{F}_3T_4(t)\,uv_x - \frac{2}{3}\,\mathcal{F}_3T_4(t)\,u_xv - \frac{2}{3}\,\mathcal{F}_3T_2(t)\,w_{xxx},\\
C^3 &= -\frac{1}{3}\,T_2(t)\,\left(-u_{xx}\mathcal{F}_{1y} + 2\,u_{xxy}\mathcal{F}_1 - v_{xx}\mathcal{F}_{2y} + 2\,v_{xxy}\mathcal{F}_2 - w_{xx}\mathcal{F}_{3y} + 2\,w_{xxy}\mathcal{F}_3\right).
\end{aligned}\tag{3.53}$$

Here, the divergence condition $D_tC^1 + D_xC^2 + D_yC^3 = 0$ is satisfied using Maple softwares for all the solutions of said system (3.1).

Symmetry $X_2 = \frac{1}{T_2(t)}\frac{\partial}{\partial t}$ with the Lie characteristic functions $W_1 = -\frac{1}{T_2(t)}u_t$, $W_2 = -\frac{1}{T_2(t)}v_t$ and $W_3 = -\frac{1}{T_2(t)}w_t$ gives the following nontrivial conservation laws

$$\begin{aligned}
C^1 &= -\frac{u_t\mathcal{F}_1 + v_t\mathcal{F}_2 + w_t\mathcal{F}_3}{T_2(t)},\\
C^2 &= \left(-\frac{1}{3}\,w_{tyy} - w_{txx}\right)\mathcal{F}_3 + \left(-v_{txx} - \frac{1}{3}\,v_{tyy}\right)\mathcal{F}_2 + \left(-u_{txx} - \frac{1}{3}\,u_{tyy}\right)\mathcal{F}_1\\
&\quad + \frac{1}{3}\,u_{ty}\mathcal{F}_{1y} + \frac{1}{3}\,w_{ty}\mathcal{F}_{3y} - \frac{1}{3}\,v_t\mathcal{F}_{2yy} - \frac{1}{3}\,u_t\mathcal{F}_{1yy} - \frac{1}{3}\,w_t\mathcal{F}_{3yy} + \frac{1}{3}\,v_{ty}\mathcal{F}_{2y}\\
&\quad + \frac{1}{T_2(t)}\left(\left(-u_tT_4(t)\,v - v_tT_4(t)\,u\right)\mathcal{F}_3 + \left(-u_tT_4(t)\,w - w_tT_4(t)\,u\right)\mathcal{F}_2\right)\\
&\quad + \frac{1}{T_2(t)}\left(-u_tT_1(t)\,v - v_tT_3(t)w - w_tT_3(t)v - v_tT_1(t)\,u\right)\mathcal{F}_1,\\
C^3 &= \frac{1}{3}\,u_{tx}\mathcal{F}_{1y} - \frac{2}{3}\,u_{txy}\mathcal{F}_1 + \frac{1}{3}\,v_{tx}\mathcal{F}_{2y} - \frac{2}{3}\,v_{txy}\mathcal{F}_2 + \frac{1}{3}\,w_{tx}\mathcal{F}_{3y}\frac{1}{T_2(t)} - \frac{2}{3}\,w_{txy}\mathcal{F}_3.
\end{aligned}\tag{3.54}$$

In this case, divergence condition becomes

$$\begin{aligned}
D_tC^1 + D_xC^2 + D_yC^3 = D_x\Bigg(&\frac{\left(-\mathcal{F}_3T_4(t)\,uv - \mathcal{F}_2T_4(t)\,wu + \left(-T_1(t)\,uv - T_3(t)vw\right)\mathcal{F}_1\right)T_2{}'(t)}{T_2(t)^2}\\
&+\frac{T_4{}'(t)uv\mathcal{F}_3 + T_4{}'(t)wu\mathcal{F}_2 + \left(T_1{}'(t)uv + T_3{}'(t)vw\right)\mathcal{F}_1}{T_2(t)}\Bigg).
\end{aligned}\tag{3.55}$$

A simple rearrangement of terms gives the following relation

$$\begin{aligned} &D_tC^1 + D_x\left(C^2 - \frac{(-\mathcal{F}_3T_4(t)\,uv - \mathcal{F}_2T_4(t)\,wu + (-T_1(t)\,uv - T_3(t)vw)\,\mathcal{F}_1)\,T_2{}'(t)}{T_2(t)^2}\right. \\ &\left.-\frac{T_4{}'(t)uv\mathcal{F}_3 + T_4{}'(t)wu\mathcal{F}_2 + (T_1{}'(t)uv + T_3{}'(t)vw)\,\mathcal{F}_1}{T_2(t)}\right) + D_yC^3 = 0. \end{aligned} \tag{3.56}$$

Therefore, components of conserved vectors are modified as follows

$$\begin{aligned} \tilde{C}^1 &= C^1, \\ \tilde{C}^2 &= \frac{1}{3}\,u_{ty}\mathcal{F}_{1y} - \frac{1}{3}\,w_{tyy}\mathcal{F}_3 - \frac{1}{3}\,w_t\mathcal{F}_{3yy} - \frac{1}{3}\,v_{tyy}\mathcal{F}_2 - \frac{1}{3}\,u_{tyy}\mathcal{F}_1 \\ &+ \frac{1}{3}\,w_{ty}\mathcal{F}_{3y} - u_{txx}\mathcal{F}_1 - v_{txx}\mathcal{F}_2 - w_{txx}\mathcal{F}_3 + \frac{1}{3}\,v_{ty}\mathcal{F}_{2y} - \frac{1}{3}\,v_t\mathcal{F}_{2yy} \\ &+ \frac{1}{T_2(t)}\,((-\mathcal{F}_1T_3{}'(t)v - \mathcal{F}_2T_4{}'(t)u - v_t\mathcal{F}_1T_3(t) - u_t\mathcal{F}_2T_4(t))\,w \\ &+ \left(-\frac{1}{3}\,\left(3\,\mathcal{F}_3T_4{}'(t) + 3\,\mathcal{F}_1T_1{}'(t)\right)u - u_t\mathcal{F}_1T_1(t) - u_t\mathcal{F}_3T_4(t) - w_t\mathcal{F}_1T_3(t)\right)v \\ &\left.-\frac{1}{3}\,\left(3\,v_t\mathcal{F}_1T_1(t) + 3\,v_t\mathcal{F}_3T_4(t) + 3\,w_t\mathcal{F}_2T_4(t)\right)u\right) - \frac{1}{3}\,u_t\mathcal{F}_{1yy} \\ &+ \frac{1}{T_2(t)^2}\,\left((\mathcal{F}_1T_2{}'(t)T_3(t)v + \mathcal{F}_2T_2{}'(t)T_4(t)\,u)\,w\right. \\ &\left.-\frac{1}{3}\,\left(-3\,\mathcal{F}_3T_2{}'(t)T_4(t) - 3\,\mathcal{F}_1T_2{}'(t)T_1(t)\right)uv\right), \\ \tilde{C}^3 &= C^3. \end{aligned} \tag{3.57}$$

Symmetry $X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{3\int T_2(t)dt}{T_2(t)}\frac{\partial}{\partial t}$ with $W_1 = -xu_x - yu_y - \frac{3\int T_2(t)dt}{T_2(t)}u_t$, $W_2 = -xv_x - yv_y - \frac{3\int T_2(t)dt}{T_2(t)}v_t$ and $W_3 = -xw_x - yw_y - \frac{3\int T_2(t)dt}{T_2(t)}w_t$ yields the following conservation laws

$$\begin{aligned} C^1 = &-\frac{3\int T_2(t)dt\,(u_t\mathcal{F}_1 + v_t\mathcal{F}_2 + w_t\mathcal{F}_3)}{T_2(t)} - y\,(\mathcal{F}_2v_y + \mathcal{F}_3w_y + \mathcal{F}_1u_y) \\ &- x\,(w_x\mathcal{F}_3 + v_x\mathcal{F}_2 + u_x\mathcal{F}_1)\,, \end{aligned}$$

$$
\begin{aligned}
C^2 = {} & \Bigg(-\frac{1}{3}\,\left(9\,w_{txx}+3\,w_{tyy}\right)\mathcal{F}_3-\frac{1}{3}\,\left(9\,v_{txx}+3\,v_{tyy}\right)\mathcal{F}_2+v_{ty}\mathcal{F}_{2y}\\
& -u_t\mathcal{F}_{1yy}-v_t\mathcal{F}_{2yy}-w_t\mathcal{F}_{3yy}+u_{ty}\mathcal{F}_{1y}+w_{ty}\mathcal{F}_{3y}-\frac{1}{3}\,\left(9\,u_{txx}+3\,u_{tyy}\right)\mathcal{F}_1\\
& +\frac{1}{T_2(t)}\Bigg(-\frac{1}{3}\,\left(9\,u_tT_4(t)\,v+9\,v_tT_4(t)\,u\right)\mathcal{F}_3\\
& -\frac{1}{3}\,\left(9\,v_tT_1(t)\,u+9\,u_tT_1(t)\,v+9\,w_tT_3(t)v+9\,v_tT_3(t)w\right)\mathcal{F}_1\\
& -\frac{1}{3}\,\left(9\,w_tT_4(t)\,u+9\,u_tT_4(t)\,w\right)\mathcal{F}_2\Bigg)\Bigg)\left(\int T_2(t)dt\right)\\
& +\Bigg(-\frac{1}{3}\,\left(2\,xw_{xxx}+2\,w_{yy}+3\,yw_{xxy}+6\,w_{xx}+yw_{yyy}\right)\mathcal{F}_3\\
& -\frac{1}{3}\,\left(yu_{yyy}+2\,xu_{xxx}+3\,yu_{xxy}+2\,u_{yy}+6\,u_{xx}\right)\mathcal{F}_1\\
& -\frac{1}{3}\,\left(3\,yv_{xxy}+yv_{yyy}+6\,v_{xx}+2\,v_{yy}+2\,xv_{xxx}\right)\mathcal{F}_2+\frac{1}{3}\,\mathcal{F}_{1y}yu_{yy}\\
& +\frac{1}{3}\,\mathcal{F}_{2y}xv_{xy}+\frac{1}{3}\,\mathcal{F}_{2y}yv_{yy}-\frac{1}{3}\,xv_x\mathcal{F}_{2yy}-\frac{1}{3}\,yu_y\mathcal{F}_{1yy}-\frac{1}{3}\,xw_x\mathcal{F}_{3yy}\\
& +\frac{1}{3}\,\mathcal{F}_{1y}xu_{x,y}+\frac{1}{3}\,\mathcal{F}_{3y}yw_{yy}+\frac{1}{3}\,\mathcal{F}_{2y}v_y-\frac{1}{3}\,xu_x\mathcal{F}_{1yy}+\frac{1}{3}\,\mathcal{F}_{3y}xw_{xy}\\
& +\frac{1}{3}\,\mathcal{F}_{1y}u_y-\frac{1}{3}\,yv_y\mathcal{F}_{2yy}+\frac{1}{3}\,\mathcal{F}_{3y}w_y-\frac{1}{3}\,yw_y\mathcal{F}_{3yy}\Bigg)T_2(t)\\
& -\frac{1}{3}\,\left(2\,xT_4(t)\,uv_x+3\,yu_yT_4(t)\,v+2\,xT_4(t)\,u_xv+3\,yv_yT_4(t)\,u-xw_t\right)\mathcal{F}_3\\
& -\frac{1}{3}\,\left(3\,yu_yT_4(t)\,w+3\,yw_yT_4(t)\,u+2\,xT_4(t)\,w_xu-xv_t+2\,xT_4(t)\,wu_x\right)\mathcal{F}_2\\
& -\frac{1}{3}\,\big(3\,yv_yT_1(t)\,u+2\,xT_1(t)\,u_xv+3\,yv_yT_3(t)w+2\,xT_1(t)\,uv_x+3\,yu_yT_1(t)\,v+3\,yw_yT_3(t)v\\
& +2\,xT_3(t)v_xw+2\,xT_3(t)vw_x-xu_t\big)\mathcal{F}_1,
\end{aligned}
$$

$$
\begin{aligned}
C^3 = {} & \left(\frac{2}{3}\,y\mathcal{F}_1vw_x+\frac{2}{3}\,y\mathcal{F}_1v_xw\right)T_3(t)+\left(\frac{2}{3}\,y\mathcal{F}_1uv_x+\frac{2}{3}\,y\mathcal{F}_1u_xv\right)T_1(t)\\
& +\left(\frac{2}{3}\,y\mathcal{F}_2w_xu+\frac{2}{3}\,y\mathcal{F}_3u_xv+\frac{2}{3}\,y\mathcal{F}_2wu_x+\frac{2}{3}\,y\mathcal{F}_3uv_x\right)T_4(t)\\
& +\Bigg(\frac{1}{3}\,\mathcal{F}_{3y}w_x+\frac{1}{3}\,\mathcal{F}_{1y}yu_{xy}+\frac{1}{3}\,\mathcal{F}_{3y}yw_{xy}-\frac{2}{3}\,\mathcal{F}_1xu_{xxy}+\frac{1}{3}\,\mathcal{F}_{3y}xw_{xx}\\
& -\frac{2}{3}\,\mathcal{F}_3xw_{xxy}-\frac{4}{3}\,\mathcal{F}_3w_{xy}+\frac{1}{3}\,\mathcal{F}_{1y}xu_{xx}+\frac{2}{3}\,y\mathcal{F}_1u_{xxx}+\frac{1}{3}\,\mathcal{F}_{1y}u_x\\
& +\frac{2}{3}\,y\mathcal{F}_2v_{xxx}-\frac{4}{3}\,\mathcal{F}_1u_{xy}+\frac{2}{3}\,y\mathcal{F}_3w_{xxx}-\frac{4}{3}\,\mathcal{F}_2v_{xy}+\frac{1}{3}\,\mathcal{F}_{2y}xv_{xx}\\
& -\frac{2}{3}\,\mathcal{F}_2xv_{xxy}+\frac{1}{3}\,\mathcal{F}_{2y}yv_{xy}+\frac{1}{3}\,\mathcal{F}_{2y}v_x\Bigg)T_2(t)+\frac{2}{3}\,y\mathcal{F}_2v_t+\frac{2}{3}\,y\mathcal{F}_3w_t\\
& +\left(\int T_2(t)dt\right)\left(v_{tx}\mathcal{F}_{2y}-2\,w_{txy}\mathcal{F}_3+u_{tx}\mathcal{F}_{1y}-2\,u_{txy}\mathcal{F}_1-2\,v_{txy}\mathcal{F}_2+w_{tx}\mathcal{F}_{3y}\right)\\
& +\frac{2}{3}\,y\mathcal{F}_1u_t.
\end{aligned}
\tag{3.58}
$$

Total divergence condition gives the following result

$$
\begin{aligned}
D_tC^1 + D_xC^2 + D_yC^3 = & \\
& D_x \Bigg(\frac{\left(\left(-3\,T_1(t)\,uv - 3\,T_3(t)vw\right)\mathcal{F}_1 - 3\,\mathcal{F}_2T_4(t)\,wu\right)\left(\int T_2(t)dt\right)T_2{}'(t)}{T_2(t)^2} \\
& + \frac{\left(3\,T_4{}'(t)uv\mathcal{F}_3 + 3\,T_4{}'(t)wu\mathcal{F}_2 + \left(3\,T_1{}'(t)uv + 3\,T_3{}'(t)vw\right)\mathcal{F}_1\right)\left(\int T_2(t)dt\right)}{T_2(t)} \\
& + 2\,\mathcal{F}_2T_4(t)\,wu + 2\,\mathcal{F}_3T_4(t)\,uv + \left(2\,T_3(t)vw + 2\,T_1(t)\,uv\right)\mathcal{F}_1 \\
& \frac{-3\,\mathcal{F}_3T_4(t)\,uv}{T_2(t)^2} \Bigg).
\end{aligned}
\tag{3.59}
$$

Then, modified components of conserved vectors have been represented as follows

$$
\begin{aligned}
\tilde{C}^1 &= C^1, \\
\tilde{C}^2 &= x\mathcal{F}_3w_t + x\mathcal{F}_1u_t + x\mathcal{F}_2v_t + \left(-w_{tyy}\mathcal{F}_3 - 3\,w_{txx}\mathcal{F}_3 - w_t\mathcal{F}_{3yy} - u_t\mathcal{F}_{1yy}\right. \\
&\quad + w_{ty}\mathcal{F}_{3y} + v_{ty}\mathcal{F}_{2y} + u_{ty}\mathcal{F}_{1y} - u_{tyy}\mathcal{F}_1 - 3\,u_{txx}\mathcal{F}_1 - v_t\mathcal{F}_{2yy} - 3\,v_{txx}\mathcal{F}_2 \\
&\quad \left. - v_{tyy}\mathcal{F}_2\right)\left(\int T_2(t)dt\right) - yv_y\mathcal{F}_1T_1(t)\,u - yv_y\mathcal{F}_1T_3(t)w - yv_y\mathcal{F}_3T_4(t)\,u - yw_y\mathcal{F}_1T_3(t)v \\
&\quad - yw_y\mathcal{F}_2T_4(t)\,u - yu_y\mathcal{F}_1T_1(t)\,v - yu_y\mathcal{F}_3T_4(t)\,v - yu_y\mathcal{F}_2T_4(t)\,w \\
&\quad - 2\,\mathcal{F}_2T_4(t)\,wu - 2\,\mathcal{F}_3T_4(t)\,uv - 2\,\mathcal{F}_1T_3(t)vw - 2\,\mathcal{F}_1T_1(t)\,uv + \Bigg(\frac{1}{3}\,\mathcal{F}_{1y}u_y - \frac{2}{3}\,\mathcal{F}_2v_{yy} \\
&\quad - \frac{2}{3}\,\mathcal{F}_3w_{yy} - 2\,\mathcal{F}_1u_{xx} - 2\,\mathcal{F}_3w_{xx} + \frac{1}{3}\,\mathcal{F}_{2y}v_y - \frac{2}{3}\,\mathcal{F}_1u_{yy} - 2\,\mathcal{F}_2v_{xx} + \frac{1}{3}\,\mathcal{F}_{3y}w_y \\
&\quad + \frac{1}{3}\,\mathcal{F}_{2y}yv_{yy} + \frac{2}{3}\,\mathcal{F}_3xw_{xyy} + \frac{1}{3}\,\mathcal{F}_{1y}yu_{yy} - \frac{1}{3}\,xw_x\mathcal{F}_{3yy} - \frac{1}{3}\,y\mathcal{F}_1u_{yyy} \\
&\quad + \frac{1}{3}\,\mathcal{F}_{2y}xv_{xy} - \frac{1}{3}\,yw_y\mathcal{F}_{3yy} + \frac{2}{3}\,\mathcal{F}_1xu_{xyy} + \frac{1}{3}\,\mathcal{F}_{3y}yw_{yy} - y\mathcal{F}_2v_{xxy} - \frac{1}{3}\,xv_x\mathcal{F}_{2yy} \\
&\quad - \frac{1}{3}\,xu_x\mathcal{F}_{1yy} + \frac{1}{3}\,\mathcal{F}_{1y}xu_{xy} + \frac{2}{3}\,\mathcal{F}_2xv_{xyy} - y\mathcal{F}_3w_{xxy} - \frac{1}{3}\,yu_y\mathcal{F}_{1yy} \\
&\quad - \frac{1}{3}\,y\mathcal{F}_2v_{yyy} - \frac{1}{3}\,yv_y\mathcal{F}_{2yy} - \frac{1}{3}\,y\mathcal{F}_3w_{yyy} - y\mathcal{F}_1u_{xxy} + \frac{1}{3}\,\mathcal{F}_{3y}xw_{xy}\Bigg)T_2(t) \\
&\quad + \left(\frac{\int T_2(t)dt}{T_2(t)}\right)\left(-3\,\mathcal{F}_1T_1{}'(t)uv - 3\,\mathcal{F}_3T_4{}'(t)uv - 3\,\mathcal{F}_2T_4{}'(t)wu - 3\,\mathcal{F}_1T_3{}'(t)vw\right. \\
&\quad - 3\,v_t\mathcal{F}_1T_1(t)\,u - 3\,u_t\mathcal{F}_3T_4(t)\,v - 3\,v_t\mathcal{F}_1T_3(t)w - 3\,u_t\mathcal{F}_2T_4(t)\,w - 3\,w_t\mathcal{F}_1T_3(t)v \\
&\quad \left. - 3\,w_t\mathcal{F}_2T_4(t)\,u - 3\,u_t\mathcal{F}_1T_1(t)\,v - 3\,v_t\mathcal{F}_3T_4(t)\,u\right) + \frac{\left(\int T_2(t)dt\right)\left(T_2{}'(t)\right)}{T_2(t)^2}\left(3\,\mathcal{F}_3T_4(t)\,uv\right. \\
&\quad \left. + 3\,\mathcal{F}_1T_3(t)vw + 3\,\mathcal{F}_1T_1(t)\,uv + 3\,\mathcal{F}_2T_4(t)\,wu\right), \\
\tilde{C}^3 &= C^3.
\end{aligned}
\tag{3.60}
$$

The symmetry $X_4 = \frac{\partial}{\partial y}$ yields following conserved vectors

$$\begin{aligned}
C^1 &= -u_y\mathcal{F}_1 - v_y\mathcal{F}_2 - w_y\mathcal{F}_3,\\
C^2 &= \left(\left(-w_{xxy} - \frac{1}{3}\,w_{yyy}\right)\mathcal{F}_3 + \left(-v_{xxy} - \frac{1}{3}\,v_{yyy}\right)\mathcal{F}_2\right.\\
&\quad + \left(-u_{xxy} - \frac{1}{3}\,u_{yyy}\right)\mathcal{F}_1 + \frac{1}{3}\,w_{yy}\mathcal{F}_{3y} - \frac{1}{3}\,v_y\mathcal{F}_{2yy} - \frac{1}{3}\,u_y\mathcal{F}_{1yy}\\
&\quad \left. + \frac{1}{3}\,u_{yy}\mathcal{F}_{1y} + \frac{1}{3}\,v_{yy}\mathcal{F}_{2y} - \frac{1}{3}\,w_y\mathcal{F}_{3yy}\right) T_2(t) + \left(-u_yT_4(t)\,v - v_yT_4(t)\,u\right)\mathcal{F}_3\\
&\quad + \left(-u_yT_4(t)\,w - w_yT_4(t)\,u\right)\mathcal{F}_2 + \left(-w_yT_3(t)v - v_yT_1(t)\,u - v_yT_3(t)w - u_yT_1(t)\,v\right)\mathcal{F}_1,\\
C^3 &= \left(\frac{2}{3}\,\mathcal{F}_1u_xv + \frac{2}{3}\,\mathcal{F}_1uv_x\right)T_1(t) + \left(\frac{2}{3}\,\mathcal{F}_1vw_x + \frac{2}{3}\,\mathcal{F}_1v_xw\right)T_3(t)\\
&\quad + \left(\frac{2}{3}\,\mathcal{F}_2wu_x + \frac{2}{3}\,\mathcal{F}_2w_xu + \frac{2}{3}\,\mathcal{F}_3u_xv + \frac{2}{3}\,\mathcal{F}_3uv_x\right)T_4(t)\\
&\quad + \left(\frac{1}{3}\,u_{xy}\mathcal{F}_{1y} + \frac{1}{3}\,v_{xy}\mathcal{F}_{2y} + \frac{2}{3}\,\mathcal{F}_3w_{xxx} + \frac{2}{3}\,\mathcal{F}_2v_{xxx} + \frac{2}{3}\,\mathcal{F}_1u_{xxx}\right.\\
&\quad \left. + \frac{1}{3}\,w_{xy}\mathcal{F}_{3y}\right)T_2(t) + \frac{2}{3}\,\mathcal{F}_1u_t + \frac{2}{3}\,\mathcal{F}_2v_t + \frac{2}{3}\,\mathcal{F}_3w_t.
\end{aligned} \tag{3.61}$$

Divergence condition yields the following expression

$$\begin{aligned}
D_tC^1 + D_xC^2 + D_yC^3 &= D_t\left(\mathcal{F}_{1y}u + \mathcal{F}_{2y}v + \mathcal{F}_{3y}w\right)\\
&\quad + D_x\left(\left(\mathcal{F}_{1y}T_3(t)v + \mathcal{F}_{2y}T_4(t)\,u\right)w + \left(\mathcal{F}_{3y}T_4(t) + \mathcal{F}_{1y}T_1(t)\right)uv\right.\\
&\quad + \left(\left(w_{xx} + w_{yy}\right)\mathcal{F}_{3y} + \left(v_{xx} + v_{yy}\right)\mathcal{F}_{2y}\right.\\
&\quad \left.\left. + \left(u_{yy} + u_{xx}\right)\mathcal{F}_{1y}\right)T_2(t)\right).
\end{aligned} \tag{3.62}$$

Then, components of conserved vectors are changed as follows

$$\begin{aligned}
\tilde{C}^1 &= -\mathcal{F}_1u_y - \mathcal{F}_2v_y - \mathcal{F}_3w_y - \mathcal{F}_{1y}u - \mathcal{F}_{2y}v - \mathcal{F}_{3y}w,\\
\tilde{C}^2 &= \left(-v_y\mathcal{F}_3u - \mathcal{F}_{2y}wu - \mathcal{F}_{3y}uv - u_y\mathcal{F}_3v - u_y\mathcal{F}_2w - w_y\mathcal{F}_2u\right)T_4(t)\\
&\quad + \left(-v_y\mathcal{F}_1u - u_y\mathcal{F}_1v - \mathcal{F}_{1y}uv\right)T_1(t) + \left(-\mathcal{F}_{1y}vw - w_y\mathcal{F}_1v - v_y\mathcal{F}_1w\right)T_3(t)\\
&\quad + \left(-\frac{2}{3}\,v_{yy}\mathcal{F}_{2y} - \frac{1}{3}\,v_y\mathcal{F}_{2yy} - \mathcal{F}_{2y}v_{xx} - \frac{2}{3}\,w_{yy}\mathcal{F}_{3y} - \frac{2}{3}\,u_{yy}\mathcal{F}_{1y}\right.\\
&\quad - \frac{1}{3}\,w_y\mathcal{F}_{3yy} - u_{xxy}\mathcal{F}_1 - \frac{1}{3}\,u_{yyy}\mathcal{F}_1 - \frac{1}{3}\,w_{yyy}\mathcal{F}_3 - \mathcal{F}_{3y}w_{xx} - v_{xxy}\mathcal{F}_2\\
&\quad \left. - \frac{1}{3}\,u_y\mathcal{F}_{1yy} - w_{xxy}\mathcal{F}_3 - \mathcal{F}_{1y}u_{xx} - \frac{1}{3}\,v_{yyy}\mathcal{F}_2\right)T_2(t),\\
\tilde{C}^3 &= C^3.
\end{aligned} \tag{3.63}$$

For symmetry $X_5 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}$, conservation laws are determined as follows

$$\begin{aligned}
C^1 &= \mathcal{F}_1 u + v\mathcal{F}_2 + w\mathcal{F}_3,\\
C^2 &= \left(2\,\mathcal{F}_3 uv + 2\,\mathcal{F}_2 wu\right) T_4(t) + \left(\left(w_{xx} + \frac{1}{3}\,w_{yy}\right)\mathcal{F}_3 + \left(v_{xx} + \frac{1}{3}\,v_{yy}\right)\mathcal{F}_2\right.\\
&\quad + \left(u_{xx} + \frac{1}{3}\,u_{yy}\right)\mathcal{F}_1 - \frac{1}{3}\,\mathcal{F}_{3y} w_y - \frac{1}{3}\,\mathcal{F}_{2y} v_y + \frac{1}{3}\,u\mathcal{F}_{1yy} - \frac{1}{3}\,\mathcal{F}_{1y} u_y \\
&\quad \left. + \frac{1}{3}\,w\mathcal{F}_{3yy} + \frac{1}{3}\,v\mathcal{F}_{2yy}\right) T_2(t) + \left(2\,T_3(t) vw + 2\,T_1(t)\,uv\right)\mathcal{F}_1,\\
C^3 &= -\frac{1}{3}\,T_2(t)\,\left(u_x\mathcal{F}_{1y} - 2\,u_{xy}\mathcal{F}_1 + v_x\mathcal{F}_{2y} - 2\,v_{xy}\mathcal{F}_2 + w_x\mathcal{F}_{3y} - 2\,w_{xy}\mathcal{F}_3\right).
\end{aligned} \tag{3.64}$$

Total divergence condition has the following form

$$D_t C^1 + D_x C^2 + D_y C^3 = D_x\left(\left(\left(T_3(t) w + T_1(t)\,u\right)\mathcal{F}_1 + u\mathcal{F}_3 T_4(t)\right) v + \mathcal{F}_2 T_4(t)\,wu\right). \tag{3.65}$$

Then, conserved vectors are modified as follows

$$\begin{aligned}
\tilde{C}^1 &= C^1,\\
\tilde{C}^2 &= \left(u\mathcal{F}_2 w + u\mathcal{F}_3 v\right) T_4(t) + \left(-\frac{1}{3}\,u_y\mathcal{F}_{1y} + u_{xx}\mathcal{F}_1 + \frac{1}{3}\,u_{yy}\mathcal{F}_1 + \frac{1}{3}\,u\mathcal{F}_{1yy}\right.\\
&\quad + \frac{1}{3}\,v\mathcal{F}_{2yy} - \frac{1}{3}\,v_y\mathcal{F}_{2y} + v_{xx}\mathcal{F}_2 + \frac{1}{3}\,v_{yy}\mathcal{F}_2 + \frac{1}{3}\,w\mathcal{F}_{3yy} - \frac{1}{3}\,w_y\mathcal{F}_{3y}\\
&\quad \left. + w_{xx}\mathcal{F}_3 + \frac{1}{3}\,w_{yy}\mathcal{F}_3\right) T_2(t) + u\mathcal{F}_1 T_1(t)\,v + v\mathcal{F}_1 T_3(t) w,\\
\tilde{C}^3 &= C^3.
\end{aligned} \tag{3.66}$$

3.1.5 Direct method and conservation laws

This section presents conservation laws by direct method. In direct method [19], the multipliers of the form $\Lambda_i(t,x,y,u,v,w)$, $i = 1,2,3$ are to be determined. The multipliers should satisfy the following divergence expression

$$\Lambda_1\Xi_1 + \Lambda_2\Xi_2 + \Lambda_3\Xi_3 = D_t C^1 + D_x C^2 + D_y C^3, \tag{3.67}$$

for arbitrary functions u, v and w. Multipliers yield determining equations using variational derivatives of the following expressions

$$\begin{aligned}
&\frac{\delta}{\delta u}\left(\Lambda_1\Xi_1+\Lambda_2\Xi_2+\Lambda_3\Xi_3\right)=0,\\
&\frac{\delta}{\delta v}\left(\Lambda_1\Xi_1+\Lambda_2\Xi_2+\Lambda_3\Xi_3\right)=0,\\
&\frac{\delta}{\delta w}\left(\Lambda_1\Xi_1+\Lambda_2\Xi_2+\Lambda_3\Xi_3\right)=0,
\end{aligned} \tag{3.68}$$

where $\frac{\delta}{\delta u}$, $\frac{\delta}{\delta v}$ and $\frac{\delta}{\delta w}$ are variational derivatives defined in (2.86). Expanding the above system by variational derivatives and then coefficients of derivatives of dependent variables u, v and w give some determining equations. After solving determining equations, we get following multipliers

$$\Lambda_1=\mathcal{G}_1(y),\ \Lambda_2=\mathcal{G}_2(y),\ \Lambda_3=\mathcal{G}_3(y), \tag{3.69}$$

where $\mathcal{G}_1(y)$, $\mathcal{G}_2(y)$ and $\mathcal{G}_3(y)$ are arbitrary functions of y.
Conservation laws corresponding to multipliers Λ_1, Λ_2 and Λ_3 are given as follows

$$\begin{aligned}
C^1 &= u\mathcal{G}_1(y),\\
C^2 &= u\mathcal{G}_1(y)T_1v+w\mathcal{G}_1(y)T_3v+\frac{1}{3}u\mathcal{G}_{1yy}T_2-\frac{1}{3}\mathcal{G}_{1y}T_2u_y+\frac{1}{3}u_{yy}\mathcal{G}_1(y)T_2\\
&\quad+u_{xx}\mathcal{G}_1(y)T_2,\\
C^3 &= -\frac{1}{3}T_2(\mathcal{G}_{1y}u_x-2\mathcal{G}_1(y)u_{xy}).
\end{aligned} \tag{3.70}$$

$$\begin{aligned}
C^1 &= v\mathcal{G}_2(y),\\
C^2 &= u\mathcal{G}_2(y)T_4w+\frac{1}{3}v\mathcal{G}_{2yy}T_2-\frac{1}{3}\mathcal{G}_{2y}T_2v_y+\frac{1}{3}v_{yy}\mathcal{G}_2(y)T_2\\
&\quad+v_{xx}\mathcal{G}_2(y)T_2,\\
C^3 &= -\frac{1}{3}T_2(\mathcal{G}_{2y}v_x-2\mathcal{G}_2(y)v_{xy})
\end{aligned} \tag{3.71}$$

$$\begin{aligned}
C^1 &= w\mathcal{G}_3(y),\\
C^2 &= u\mathcal{G}_3(y)T_4v+\frac{1}{3}w\mathcal{G}_{3yy}T_2-\frac{1}{3}\mathcal{G}_{3y}T_2wy+\frac{1}{3}w_{yy}\mathcal{G}_3(y)T_2\\
&\quad+w_{xx}\mathcal{G}_3(y)T_2,\\
C^3 &= -\frac{1}{3}T_2(\mathcal{G}_{3y}w_x-2\mathcal{G}_3(y)w_{xy}).
\end{aligned} \tag{3.72}$$

The infinitely many conservation laws are obtained by direct method (3.70)-(3.72) because of arbitrary functions $\mathcal{G}_1(y)$, $\mathcal{G}_2(y)$ and $\mathcal{G}_3(y)$.

3.1.6 Conclusion

The Lie symmetries, similarity solutions and conservation laws are successfully derived for (2+1)-dimensional ZK system with time dependent coefficients. The solutions appeared in terms of power series, trigonometric, hyperbolic, Jacobi and Weierstrass functions. The graphical analysis of the solutions reveals bright, dark, singular solitary wave and periodic wave profiles. The effect of arbitrary coefficient function $T_2(t)$ on solutions wave profiles is successfully represented in 2D/3D plots. The system is found to possess nontrivial infinitely many conservation laws because of arbitrary functions $\mathcal{F}_1(y)$, $\mathcal{F}_2(y)$, $\mathcal{F}_3(y)$, $\mathcal{G}_1(y)$, $\mathcal{G}_2(y)$ and $\mathcal{G}_3(y)$. The authenticity of solutions and conservation laws is verified by Maple software.

3.2 Generalised 7^{th} order KdV equation

The generalised 7^{th} order KdV equation has been explored in literature for representing waves in shallow water, ion-acoustics and stratified internal waves and so on [46, 141, 158, 203, 319]. To discuss the impact of higher-order dispersion terms onto the wave profile, a model in the form of KdV equation has been successfully analysed in literature [51, 92, 169, 227, 242, 250, 252, 265, 284, 322, 336]. The prominent forms of KdV equation, the nature of various constants and the different techniques used to solve as well as the type of solutions obtained, all are tabulated in Table 3.1.

Table 3.1: Various forms of KdV equation

Form of the equation	Coefficient values	Technique used for obtaining exact solutions	Type of exact solutions	References
Generalized 7^{th} order KdV equation $u_t + T_5 u_x u^3 + T_6 u_x^3 + T_7 u u_x u_{xx} + T_8 u^2 u_{xxx} + T_9 u_{xx} u_{xxx} + T_{10} u_x u_{xxxx} + T_{11} u u_{xxxxx} + u_{xxxxxxx} = 0$	T_5, T_6,...,T_{11} are arbitrary constants	Cole Hopf transformation, Nonlinear transformations and symbolic computation in Maple for conservation laws, Painlevé analysis, Lax pairs and conservation laws.	Travelling wave, solitary wave and soliton solutions	[242, 250, 322]
7^{th} order Sawada-Kotera-Ito equation	$T_5 = 252$, $T_6 = 63$, $T_7 = 378$, $T_8 = 126$, $T_9 = 63$, $T_{10} = 48$ and $T_{11} = 21$	Bell polynomial approach	Lax pair and infinitely many conservation laws	[265]
7^{th} order Lax's equation	$T_5 = 140$, $T_6 = 70$, $T_7 = 280$, $T_8 = 70$, $T_9 = 70$, $T_{10} = 42$ and $T_{11} = 14$	Sech and rational exp function method	Travelling wave solutions	[92]
7^{th} order Kaup Kupershmidt equation	$T_5 = 2016$, $T_6 = 630$, $T_7 = 2268$, $T_8 = 504$, $T_9 = 252$, $T_{10} = 147$ and $T_{11} = 42$	Cole Hopf transformation	Travelling wave solutions	[250]

In present analysis, the generalised 7^{th} order KdV equation in form as given below

$$\begin{aligned}&u_t + T_5(t)u_x u^3 + T_6(t)u_x^3 + T_7(t)uu_x u_{xx} + T_8(t)u^2 u_{xxx} + T_9(t)u_{xx}u_{xxx} + T_{10}(t)u_x u_{xxxx}\\ &+ T_{11}(t)uu_{xxxxx} + T_{12}(t)u_{xxxxxxx} = 0.\end{aligned} \tag{3.73}$$

The equation involves u (real function) in terms of independent variables x, t and $T_i(t)$, $i = 5, 6, ..., 12$, are assumed to be dependent on time t. Symmetry transformations are employed to get symmetries and further transformed the PDEs into a system of ODEs. The generated ODEs can be solved by applying number of efficient techniques from the literature. Also, the conservation laws of the 7^{th} order KdV equation have been derived with the help of nonlinear self-adjointness technique and the direct method.

3.2.1 Symmetry reductions and exact solutions

To find the symmetries associated with the Eq. (3.73), consider the IG as a function of independent and dependent variables in following form

$$X = \mathcal{X}_4\frac{\partial}{\partial t} + \mathcal{X}_5\frac{\partial}{\partial x} + \mathcal{Z}_4\frac{\partial}{\partial u}. \tag{3.74}$$

This gives symmetry determining equation in terms of infinitesimals $\mathcal{X}_4$, $\mathcal{X}_5$ and $\mathcal{Z}_4$ for Eq. (3.73) as follows

$$\begin{aligned}&{\mathcal{Z}_4}^t + T_5'(t)\mathcal{X}_4 u_x u^3 + T_5(t){\mathcal{Z}_4}^x u^3 + 3T_5(t)u_x u^2\mathcal{Z}_4 + T_6'(t)\mathcal{X}_4 u_x^3 + 3T_6(t)u_x^2{\mathcal{Z}_4}^x\\ &+ T_7'(t)\mathcal{X}_4 uu_x u_{xx} + T_7(t)\mathcal{Z}_4 u_x u_{xx} + T_7(t)u{\mathcal{Z}_4}^x u_{xx} + T_7(t)uu_x{\mathcal{Z}_4}^{xx} + T_8'(t)\mathcal{X}_4 u^2 u_{xxx}\\ &+ 2T_8(t)u\mathcal{Z}_4 u_{xxx} + T_8(t)u^2{\mathcal{Z}_4}^{xxx} + T_9'(t)\mathcal{X}_4 u_{xx}u_{xxx} + T_9(t){\mathcal{Z}_4}^{xx}u_{xxx} + T_9(t)u_{xx}{\mathcal{Z}_4}^{xxx}\\ &+ T_{10}'(t)\mathcal{X}_4 u_x u_{xxxx} + T_{10}(t){\mathcal{Z}_4}^x u_{xxxx} + T_{10}(t)u_x{\mathcal{Z}_4}^{xxxx} + T_{11}'(t)\mathcal{X}_4 uu_{xxxxx} + T_{11}(t)\mathcal{Z}_4 u_{xxxxx}\\ &+ T_{11}(t)u{\mathcal{Z}_4}^{xxxxx} + T_{12}'(t)\mathcal{X}_4 u_{xxxxxxx} + T_{12}(t){\mathcal{Z}_4}^{xxxxxxx} = 0,\end{aligned} \tag{3.75}$$

where ${\mathcal{Z}_4}^x$, ${\mathcal{Z}_4}^{xx}$,...,${\mathcal{Z}_4}^{xxxxxxx}$ are the extended infinitesimals. The solution of symmetry determining equations has been expressed in the form as follows

$$\mathcal{X}_4 = \frac{\int 7l_1 T_{12}(t)dt + l_3}{T_{12}(t)},\ \mathcal{X}_5 = l_1 x + l_2,\ \mathcal{Z}_4 = l_4 u, \tag{3.76}$$

where l_1, l_2, l_3, l_4 are taken as arbitrary constants and $T_{12}(t)$ is considered as a nonzero arbitrary function of variable t.

The other coefficients $T_5(t)$, $T_6(t)$,...,$T_{11}(t)$ can be extracted from following equations

$$\begin{gathered}
T_5'(t)\mathcal{X}_4 + T_5(t)\mathcal{X}_{4t} + (3l_4 - l_1)T_5(t) = 0,\ T_6'(t)\mathcal{X}_4 + T_6(t)\mathcal{X}_{4t} + (2l_4 - 3l_1)T_6(t) = 0,\\
T_7'(t)\mathcal{X}_4 + T_7(t)\mathcal{X}_{4t} + (2l_4 - 3l_1)T_7(t) = 0,\ T_8'(t)\mathcal{X}_4 + T_8(t)\mathcal{X}_{4t} + (2l_4 - 3l_1)T_8(t) = 0,\\
T_9'(t)\mathcal{X}_4 + T_9(t)\mathcal{X}_{4t} + (l_4 - 5l_1)T_9(t) = 0,\ T_{10}'(t)\mathcal{X}_4 + T_{10}(t)\mathcal{X}_{4t} + (l_4 - 5l_1)T_{10}(t) = 0,\\
T_{11}'(t)\mathcal{X}_4 + T_{11}(t)\mathcal{X}_{4t} + (l_4 - 5l_1)T_{11}(t) = 0.
\end{gathered} \tag{3.77}$$

The IG X as given by equation (3.74) admits following group of symmetries

$$X_6 = \frac{\partial}{\partial x},\ X_7 = \frac{1}{T_{12}(t)}\frac{\partial}{\partial t},\ X_8 = u\frac{\partial}{\partial u},\ X_9 = \frac{7\int T_{12}(t)dt}{T_{12}(t)}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{3.78}$$

These group of symmetries generates an optimal system [107, 219] spanned by following vector fields

$$\begin{gathered}
(i)X_9 + pX_8,\ \ (ii)X_8 + qX_7 + X_6,\\
(iii)X_8 + qX_7,\ \ (iv)X_7 + X_6,\\
(v)X_7,\ \ (vi)X_6
\end{gathered} \tag{3.79}$$

where p and q are randomly selected as constants. The invariant solutions and reduced ODEs corresponding to these vector fields are given in the following sections.

Vector field (i) $X_9 + pX_8$

For current vector field, the $u(x,t)$ of following form is obtained

$$u(t,x) = H(\zeta)\left(\int T_{12}(t)dt\right)^{\frac{1}{7}p}, \tag{3.80}$$

where $\zeta = \frac{x}{\sqrt[7]{\int T_{12}(t)dt}}$ and the time dependent coefficients $T_i(t)$, $i = 5, 6, ..., 11$ are assembled in the following form

$$\begin{gathered}
T_5(t) = l_5\left(\int T_{12}(t)dt\right)^{-\frac{6}{7}-\frac{3}{7}p}T_{12}(t),\ T_6(t) = l_6\left(\int T_{12}(t)dt\right)^{-\frac{4}{7}-\frac{2}{7}p}T_{12}(t),\\
T_7(t) = l_7\left(\int T_{12}(t)dt\right)^{-\frac{4}{7}-\frac{2}{7}p}T_{12}(t),\ T_8(t) = l_8\left(\int T_{12}(t)dt\right)^{-\frac{4}{7}-\frac{2}{7}p}T_{12}(t),\\
T_9(t) = l_9\left(\int T_{12}(t)dt\right)^{-\frac{2}{7}-\frac{1}{7}p}T_{12}(t),\ T_{10}(t) = l_{10}\left(\int T_{12}(t)dt\right)^{-\frac{2}{7}-\frac{1}{7}p}T_{12}(t),\\
T_{11}(t) = l_{11}\left(\int T_{12}(t)dt\right)^{-\frac{2}{7}-\frac{1}{7}p}T_{12}(t),
\end{gathered} \tag{3.81}$$

where l_i, $i = 5, 6, ..., 11$ are arbitrary constants.
Using these, the reduced ODE of the following form has been obtained

$$\begin{aligned} &- H_\zeta\zeta + p\,H + 7\,l_5 H^3 H_\zeta + 7\,l_6 {H_\zeta}^3 + 7\,l_7 H H_\zeta H_{\zeta\zeta} + 7\,l_8 H^2 H_{\zeta\zeta\zeta} + 7\,l_9 H_{\zeta\zeta} H_{\zeta\zeta\zeta} \\ &+ 7\,l_{10} H_\zeta H_{\zeta\zeta\zeta\zeta} + 7\,l_{11} H H_{\zeta\zeta\zeta\zeta\zeta} + 7 H_{\zeta\zeta\zeta\zeta\zeta\zeta\zeta} = 0. \end{aligned} \tag{3.82}$$

To solve the above reduced ODE, the solution in terms of power series with variable ζ is suggested as follows

$$H(\zeta) = \sum_{\sigma_3=0}^{\infty} a_{\sigma_3} \zeta^{\sigma_3}, \tag{3.83}$$

where $\{a_{\sigma_3}\}_{\sigma_3=0}^{\infty}$ are expansion coefficients. Using above power series, the solution has been obtained as follows

$$\begin{aligned} u(t,x) = \left(\int T_{12}(t)dt\right)^{\frac{1}{7}p} \Big[& a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + a_5\zeta^5 + a_6\zeta^6 + a_7\zeta^7 \\ & + \sum_{\sigma_3=1}^{\infty} a_{\sigma_3+7} \zeta^{\sigma_3+7} \Big], \end{aligned} \tag{3.84}$$

where a_{σ_3}, $\sigma_3 = 0, 1, ..., 6$ are arbitrary constants. The coefficients $\{a_{\sigma_3}\}_{\sigma_3=7}^{\infty}$ can be evaluated from the following recurrence relation in terms of arbitrary constants

$$\begin{aligned} a_{\sigma_3+7} = & \frac{1}{7(\sigma_3+7)(\sigma_3+6)(\sigma_3+5)(\sigma_3+4)(\sigma_3+3)(\sigma_3+2)(\sigma_3+1)} \Big(\sigma_3 a_{\sigma_3} - p\,a_{\sigma_3} \\ & - 7k_{10} \sum_{\sigma_6=0}^{\sigma_3} \sum_{\sigma_5=0}^{\sigma_6} \sum_{\sigma_4=0}^{\sigma_5} a_{\sigma_4} a_{\sigma_5-\sigma_4} a_{\sigma_3-\sigma_5} (\sigma_3 - \sigma_6 + 1) a_{\sigma_3-\sigma_6+1} \\ & - 7l_6 \sum_{\sigma_5=0}^{\sigma_3} \left(\sum_{\sigma_4=0}^{\sigma_5} (\sigma_4+1)\, a_{\sigma_4+1}\, (\sigma_5 - \sigma_4 + 1)\, a_{\sigma_5-\sigma_4+1}\, (\sigma_3 - \sigma_5 + 1)\, a_{\sigma_3-\sigma_5+1} \right) \\ & - 7l_7 \sum_{\sigma_5=0}^{\sigma_3} \left(\sum_{\sigma_4=0}^{\sigma_5} a_{\sigma_4}\, (\sigma_5 - \sigma_4 + 1)\, a_{\sigma_5-\sigma_4+1}\, (\sigma_3 - \sigma_5 + 1)\, (\sigma_3 - \sigma_5 + 2)\, a_{\sigma_3-\sigma_5+2} \right) \\ & - 7k_7 \sum_{\sigma_5=0}^{\sigma_3} \left(\sum_{\sigma_4=0}^{\sigma_5} a_{\sigma_4} a_{\sigma_5-\sigma_4}\, (\sigma_3 - \sigma_5 + 3)\, (\sigma_3 - \sigma_5 + 2)\, (\sigma_3 - \sigma_5 + 1)\, a_{\sigma_3-\sigma_5+3} \right) \\ & - 7l_9 \sum_{\sigma_4=0}^{\sigma_3} (\sigma_4+1)\, (\sigma_4+2)\, a_{\sigma_4+2}\, (\sigma_3 - \sigma_4 + 3)\, (\sigma_3 - \sigma_4 + 2)\, (\sigma_3 - \sigma_4 + 1)\, a_{\sigma_3-\sigma_4+3} \\ & - 7l_{10} \sum_{\sigma_4=0}^{\sigma_3} (\sigma_4+1)\, a_{\sigma_4+1}\, (\sigma_3 - \sigma_4 + 4)\, (\sigma_3 - \sigma_4 + 3)\, (\sigma_3 - \sigma_4 + 2)\, (\sigma_3 - \sigma_4 + 1)\, a_{\sigma_3-\sigma_4+4} \\ & - 7l_{11} \sum_{\sigma_4=0}^{\sigma_3} a_{\sigma_4}\, (\sigma_3 - \sigma_4 + 5)\, (\sigma_3 - \sigma_4 + 4)\, (\sigma_3 - \sigma_4 + 3)\, (\sigma_3 - \sigma_4 + 2)\, (\sigma_3 - \sigma_4 + 1)\, a_{\sigma_3-\sigma_4+5} \Big). \end{aligned} \tag{3.85}$$

Vector field (ii) $X_8+qX_7+X_6$
The expression for $u(t,x)$ is obtained as follows

$$u(t,x)=\mathrm{e}^{\int\frac{T_{12}(t)}{q}dt}H\left(-\int\frac{T_{12}(t)}{q}dt+x\right), \tag{3.86}$$

where $\zeta=-\int\frac{T_{12}(t)}{q}dt+x$ and coefficients $T_5(t)$, $T_6(t)$,..., $T_{11}(t)$ are given in following form

$$\begin{aligned}&T_5(t)=l_5T_{12}(t)\mathrm{e}^{-3\frac{\int h(t)dt}{q}},\ T_6(t)=l_6T_{12}(t)\mathrm{e}^{-2\frac{\int h(t)dt}{q}},\ T_7(t)=l_7T_{12}(t)\mathrm{e}^{-2\frac{\int h(t)dt}{q}},\\&T_8(t)=l_8T_{12}(t)\mathrm{e}^{-2\frac{\int h(t)dt}{q}},\ T_9(t)=l_9T_{12}(t)\mathrm{e}^{-\frac{\int h(t)dt}{q}},\ T_{10}(t)=l_{10}T_{12}(t)\mathrm{e}^{-\frac{\int h(t)dt}{q}},\\&T_{11}(t)=l_{11}T_{12}(t)\mathrm{e}^{-\frac{\int h(t)dt}{q}},\end{aligned} \tag{3.87}$$

where $l_i,\ i=5,6,...,11$ are arbitrary constants. The reduced ODE of the following form has been obtained

$$\begin{aligned}&-H_\zeta+H+ql_5H^3H_\zeta+ql_6{H_\zeta}^3+ql_7HH_\zeta H_{\zeta\zeta}+ql_8H^2H_{\zeta\zeta\zeta}+ql_9H_{\zeta\zeta}H_{\zeta\zeta\zeta}\\&+ql_{10}H_\zeta H_{\zeta\zeta\zeta\zeta}+ql_{11}HH_{\zeta\zeta\zeta\zeta\zeta}+qH_{\zeta\zeta\zeta\zeta\zeta\zeta\zeta}=0.\end{aligned} \tag{3.88}$$

The power series solution of ODE (3.88) is taken in the following form

$$H(\zeta)=\sum_{\sigma_3=0}^{\infty}a_{\sigma_3}\zeta^{\sigma_3}, \tag{3.89}$$

where a_{σ_3}'s are expansion coefficients. The power series solution (3.89) of Eq. (3.73) is expressed as follows

$$\begin{aligned}u(t,x)=\mathrm{e}^{\int\frac{T_{12}(t)}{q}dt}\Big[&a_0+a_1\zeta+a_2\zeta^2+a_3\zeta^3+a_4\zeta^4+a_5\zeta^5+a_6\zeta^6+a_7\zeta^7\\&+\sum_{\sigma_3=1}^{\infty}a_{\sigma_3+7}\zeta^{\sigma_3+7}\Big],\end{aligned} \tag{3.90}$$

where a_{σ_3}, $\sigma_3 = 0, 1, ..., 6$ are arbitrary constants. The coefficients $\{a_{\sigma_3}\}_{\sigma_3=7}^{\infty}$ can be obtained from following recurrence relation as function of arbitrary constants.

$$\begin{aligned}
a_{\sigma_3+7} = {} & \frac{1}{7(\sigma_3+7)(\sigma_3+6)(\sigma_3+5)(\sigma_3+4)(\sigma_3+3)(\sigma_3+2)(\sigma_3+1)}\Big((\sigma_3+1)a_{\sigma_3+1} - a_{\sigma_3} \\
& - qk_{10}\sum_{\sigma_6=0}^{\sigma_3}\sum_{\sigma_5=0}^{\sigma_6}\sum_{\sigma_4=0}^{\sigma_5} a_{\sigma_4}a_{\sigma_5-\sigma_4}a_{\sigma_3-\sigma_5}(\sigma_3-\sigma_6+1)a_{\sigma_3-\sigma_6+1} \\
& - ql_6\sum_{\sigma_5=0}^{\sigma_3}\left(\sum_{\sigma_4=0}^{\sigma_5}(\sigma_4+1)\,a_{\sigma_4+1}\,(\sigma_5-\sigma_4+1)\,a_{\sigma_5-\sigma_4+1}\,(\sigma_3-\sigma_5+1)\,a_{\sigma_3-\sigma_5+1}\right) \\
& - ql_7\sum_{\sigma_5=0}^{\sigma_3}\left(\sum_{\sigma_4=0}^{\sigma_5}a_{\sigma_4}\,(\sigma_5-\sigma_4+1)\,a_{\sigma_5-\sigma_4+1}\,(\sigma_3-\sigma_5+1)\,(\sigma_3-\sigma_5+2)\,a_{\sigma_3-\sigma_5+2}\right) \\
& - qk_7\sum_{\sigma_5=0}^{\sigma_3}\left(\sum_{\sigma_4=0}^{\sigma_5}a_{\sigma_4}a_{\sigma_5-\sigma_4}\,(\sigma_3-\sigma_5+3)\,(\sigma_3-\sigma_5+2)\,(\sigma_3-\sigma_5+1)\,a_{\sigma_3-\sigma_5+3}\right) \\
& - ql_9\sum_{\sigma_4=0}^{\sigma_3}(\sigma_4+1)\,(\sigma_4+2)\,a_{\sigma_4+2}\,(\sigma_3-\sigma_4+3)\,(\sigma_3-\sigma_4+2)\,(\sigma_3-\sigma_4+1)\,a_{\sigma_3-\sigma_4+3} \\
& - ql_{10}\sum_{\sigma_4=0}^{\sigma_3}(\sigma_4+1)\,a_{\sigma_4+1}\,(\sigma_3-\sigma_4+4)\,(\sigma_3-\sigma_4+3)\,(\sigma_3-\sigma_4+2)\,(\sigma_3-\sigma_4+1)\,a_{\sigma_3-\sigma_4+4} \\
& - ql_{11}\sum_{\sigma_4=0}^{\sigma_3}a_{\sigma_4}\,(\sigma_3-\sigma_4+5)\,(\sigma_3-\sigma_4+4)\,(\sigma_3-\sigma_4+3)\,(\sigma_3-\sigma_4+2)\,(\sigma_3-\sigma_4+1)\,a_{\sigma_3-\sigma_4+5}\Big).
\end{aligned} \tag{3.91}$$

Vector field (iii) $X_8 + qX_7$

For this vector filed, $u(t,x)$ is presented as follows

$$u(t,x) = \mathrm{e}^{\int\frac{T_{12}(t)}{q}dt}H\left(x\right), \tag{3.92}$$

where $\zeta = x$ and coefficients $T_i(t)$, $i = 5, 6, ..., 11$ can be obtained form following

$$\begin{aligned}
& T_5(t) = l_5T_{12}(t)\mathrm{e}^{-3\frac{\int T_{12}(t)dt}{q}},\ T_6(t) = l_6T_{12}(t)\mathrm{e}^{-2\frac{\int T_{12}(t)dt}{q}},\ T_7(t) = l_7T_{12}(t)\mathrm{e}^{-2\frac{\int T_{12}(t)dt}{q}}, \\
& T_8(t) = l_8T_{12}(t)\mathrm{e}^{-2\frac{\int T_{12}(t)dt}{q}},\ T_9(t) = l_9T_{12}(t)\mathrm{e}^{-\frac{\int T_{12}(t)dt}{q}},\ T_{10}(t) = l_{10}T_{12}(t)\mathrm{e}^{-\frac{\int T_{12}(t)dt}{q}}, \\
& T_{11}(t) = l_{11}T_{12}(t)\mathrm{e}^{-\frac{\int T_{12}(t)dt}{q}},
\end{aligned} \tag{3.93}$$

where l_i, $i = 5, 6, ..., 11$ are arbitrary constants.

Using these, the reduced ODE in the following form has been obtained

$$\begin{aligned}
& H + ql_5H^3H_\zeta + ql_6{H_\zeta}^3 + ql_7HH_\zeta H_{\zeta\zeta} + ql_8H^2H_{\zeta\zeta\zeta} + ql_9H_{\zeta\zeta}H_{\zeta\zeta\zeta} \\
& + ql_{10}H_\zeta H_{\zeta\zeta\zeta\zeta} + ql_{11}HH_{\zeta\zeta\zeta\zeta\zeta} + qH_{\zeta\zeta\zeta\zeta\zeta\zeta\zeta} = 0.
\end{aligned} \tag{3.94}$$

The solution of reduced ODE in terms of power series is described as follows

$$H(\zeta)=\sum_{\sigma_3=0}^{\infty}a_{\sigma_3}\zeta^{\sigma_3}, \tag{3.95}$$

The following expression presents solution of given equation

$$\begin{aligned} u(t,x)=\mathrm{e}^{\int\frac{T_{12}(t)}{q}dt}\Big[&a_0+a_1\zeta+a_2\zeta^2+a_3\zeta^3+a_4\zeta^4+a_5\zeta^5+a_6\zeta^6+a_7\zeta^7\\ &+\sum_{\sigma_3=1}^{\infty}a_{\sigma_3+7}\zeta^{\sigma_3+7}\Big], \end{aligned} \tag{3.96}$$

where a_{σ_3}, $\sigma_3=0,1,...,6$ are arbitrary constants. The other coefficients a_7, a_8,... in terms of arbitrary constants, can be determined from the following recurrence relation

$$\begin{aligned} a_{\sigma_3+7}=&\frac{1}{7(\sigma_3+7)(\sigma_3+6)(\sigma_3+5)(\sigma_3+4)(\sigma_3+3)(\sigma_3+2)(\sigma_3+1)}\Big(-a_{\sigma_3}\\ &-qk_{10}\sum_{\sigma_6=0}^{\sigma_3}\sum_{\sigma_5=0}^{\sigma_6}\sum_{\sigma_4=0}^{\sigma_5}a_{\sigma_4}a_{\sigma_5-\sigma_4}a_{\sigma_3-\sigma_5}(\sigma_3-\sigma_6+1)a_{\sigma_3-\sigma_6+1}\\ &-ql_6\sum_{\sigma_5=0}^{\sigma_3}\left(\sum_{\sigma_4=0}^{\sigma_5}(\sigma_4+1)\,a_{\sigma_4+1}\,(\sigma_5-\sigma_4+1)\,a_{\sigma_5-\sigma_4+1}\,(\sigma_3-\sigma_5+1)\,a_{\sigma_3-\sigma_5+1}\right)\\ &-ql_7\sum_{\sigma_5=0}^{\sigma_3}\left(\sum_{\sigma_4=0}^{\sigma_5}a_{\sigma_4}\,(\sigma_5-\sigma_4+1)\,a_{\sigma_5-\sigma_4+1}\,(\sigma_3-\sigma_5+1)\,(\sigma_3-\sigma_5+2)\,a_{\sigma_3-\sigma_5+2}\right)\\ &-qk_7\sum_{\sigma_5=0}^{\sigma_3}\left(\sum_{\sigma_4=0}^{\sigma_5}a_{\sigma_4}a_{\sigma_5-\sigma_4}\,(\sigma_3-\sigma_5+3)\,(\sigma_3-\sigma_5+2)\,(\sigma_3-\sigma_5+1)\,a_{\sigma_3-\sigma_5+3}\right)\\ &-ql_9\sum_{\sigma_4=0}^{\sigma_3}(\sigma_4+1)\,(\sigma_4+2)\,a_{\sigma_4+2}\,(\sigma_3-\sigma_4+3)\,(\sigma_3-\sigma_4+2)\,(\sigma_3-\sigma_4+1)\,a_{\sigma_3-\sigma_4+3}\\ &-ql_{10}\sum_{\sigma_4=0}^{\sigma_3}(\sigma_4+1)\,a_{\sigma_4+1}\,(\sigma_3-\sigma_4+4)\,(\sigma_3-\sigma_4+3)\,(\sigma_3-\sigma_4+2)\,(\sigma_3-\sigma_4+1)\,a_{\sigma_3-\sigma_4+4}\\ &-ql_{11}\sum_{\sigma_4=0}^{\sigma_3}a_{\sigma_4}\,(\sigma_3-\sigma_4+5)\,(\sigma_3-\sigma_4+4)\,(\sigma_3-\sigma_4+3)\,(\sigma_3-\sigma_4+2)\,(\sigma_3-\sigma_4+1)\,a_{\sigma_3-\sigma_4+5}\Big). \end{aligned} \tag{3.97}$$

Vector field *(iv)* X_7+X_6

Corresponding to this vector filed, the $u(t,x)$ is appeared in following form

$$u(t,x)=H(\zeta), \tag{3.98}$$

where $\zeta = -\int T_{12}(t)dt + x$ and coefficients $T_5(t)$, $T_6(t)$,..., $T_{11}(t)$ of 7^{th} order KdV equation are given as follows

$$\begin{gathered} T_5(t) = l_5 T_{12}(t),\ T_6(t) = l_6 T_{12}(t),\ T_7(t) = l_7 T_{12}(t),\ T_8(t) = l_8 T_{12}(t),\ T_9(t) = l_9 T_{12}(t), \\ T_{10}(t) = l_{10} T_{12}(t),\ T_{11}(t) = l_{11} T_{12}(t), \end{gathered} \tag{3.99}$$

where l_i, $i = 5, 6, ..., 11$ are arbitrary constants.
As a result of simple calculations, the reduced ODE has been obtained as follows

$$\begin{aligned} &- H_\zeta + l_5 H^3 H_\zeta + l_6 {H_\zeta}^3 + l_7 H H_\zeta H_{2\zeta} + l_8 H^2 H_{\zeta\zeta\zeta} + l_9 H_{\zeta\zeta} H_{\zeta\zeta\zeta} + l_{10} H_\zeta H_{\zeta\zeta\zeta\zeta} \\ &+ l_{11} H H_{\zeta\zeta\zeta\zeta\zeta} + H_{\zeta\zeta\zeta\zeta\zeta\zeta\zeta} = 0. \end{aligned} \tag{3.100}$$

The ODE solutions is assumed in form as below

$$H(\zeta) = A_{10} + A_{11} sn\left(\zeta, m\right) + A_{12} sn^2\left(\zeta, m\right), \tag{3.101}$$

where $sn(\zeta, m)$ represents Jacobi elliptic sine function having modulus parameter m $(0 < m < 1)$ and A_{10}, A_{11}, A_{12} are arbitrary constants. The various possible cases for the solution have been elaborated as follows.
The values of A_{10}, A_{11} and other arbitrary constants have been found as follows

$$\begin{aligned} &A_{10} = -\frac{A_{12}\left(1+m^2\right)}{3m^2},\ A_{11} = 0, \\ &l_5 = -\frac{6}{\left(2\,m^4 - 5\,m^2 + 2\right)\left(1+m^2\right){A_{12}}^3}\left(m^2\left(576\,m^{10} - 864\,m^8 - 6\,l_8 {A_{12}}^2 m^2\right.\right. \\ &-72\,l_{11} A_{12} m^4 + 48\,l_{11} A_{12} m^2 + 585\,m^4 - 864\,m^6 + 4\,m^6 {A_{12}}^2 l_8 + 48\,m^8 l_{11} A_{12} \\ &\left.\left.-6\,l_8 {A_{12}}^2 m^4 - 72\,l_{11} A_{12} m^6 + 4\,l_8 {A_{12}}^2\right)\right), \\ &l_6 = -\frac{3}{4{A_{12}}^2\left(2\,m^4 - 5\,m^2 + 2\right)\left(1+m^2\right)}\left(\left(1920\,m^8 + 32\,l_{10} A_{12} m^6 - 2880\,m^6\right.\right. \\ &\left.\left.-2880\,m^4 - 48\,l_{10} A_{12} m^4 - 48\,l_{10} A_{12} m^2 + 1929\,m^2 + 32\,l_{10} A_{12}\right) m^2\right), \\ &l_7 = -\frac{3}{2{A_{12}}^2\left(2\,m^4 - 5\,m^2 + 2\right)\left(1+m^2\right)}\left(\left(3456\,m^8 - 5184\,m^6 + 48\,l_{11} A_{12} m^6\right.\right. \\ &+16\,l_9 A_{12} m^6 + 16\,l_{10} A_{12} m^6 - 72\,l_{11} A_{12} m^4 - 24\,l_9 A_{12} m^4 - 24\,l_{10} A_{12} m^4 - 5184\,m^4 \\ &\left.\left.+3447\,m^2 - 72\,l_{11} A_{12} m^2 - 24\,l_9 A_{12} m^2 - 24\,l_{10} A_{12} m^2 + 16\,l_9 A_{12} + 16\,l_{10} A_{12} + 48\,l_{11} A_{12}\right) m^2\right), \end{aligned} \tag{3.102}$$

where l_8, l_9, l_{10}, l_{11}, A_{12} and m are arbitrary constants.
Using these values, solution of KdV equation has been appeared as follows

$$u(t,x) = -\frac{A_{12}\left(1+m^2\right)}{3m^2} + A_{12}\left(\mathrm{sn}\left(-\int T_{12}(t)dt + x, m\right)\right)^2. \tag{3.103}$$

In limiting case, when $m \to 1$, the following dark-singular solitary wave solution is obtained

$$-\frac{2A_{12}}{3} + A_{12}\left(\tanh\left(-\int T_{12}(t)dt + x\right)\right)^2. \tag{3.104}$$

Vector field (v) X_7
This IG imparts the solution $u(x,t)$ of KdV equation only as function of x variable. Hence, it is not physically important solution.

Vector field (vi) X_6
For this IG, the solution $u(x,t)$ is only function of t. Hence, it is not physically important.

3.2.2 Conservation laws by Ibragimov's method

Conservation laws for the Eq. (3.73) has been obtained by following the procedure as given in chapter 2. The Lagrangian has been written by following expression

$$\begin{aligned}\mathcal{L} = \Upsilon(t,x)\big(&u_t + T_5(t)u_x u^3 + T_6(t)u_x^3 + T_7(t)uu_xu_{xx} + T_8(t)u^2u_{xxx} + T_9(t)u_{xx}u_{xxx}\\ &+T_{10}(t)u_xu_{xxxx} + T_{11}(t)uu_{xxxxx} + T_{12}(t)u_{xxxxxxx}\big),\end{aligned} \tag{3.105}$$

where $\Upsilon(t,x)$ is new dependent variable.
Based upon Lagrangian, the adjoint equation can be written as $\dfrac{\delta L}{\delta u} = 0$ and is given by

$$\begin{aligned}&-\Upsilon_t + (-\Upsilon_{xxx}u_{xx} - \Upsilon_x u_{xxxx} - 2\,\Upsilon_{xx}u_{xxx})\,T_9(t)\\ &+\left(\Upsilon_x uu_{xx} + \Upsilon_{xx}uu_x + 2\,\Upsilon_x {u_x}^2 + 3\,\Upsilon u_xu_{xx}\right)T_7\,(t)\\ &+\left(-6\,\Upsilon_x {u_x}^2 - \Upsilon_{xxx}u^2 - 6\,\Upsilon_x uu_{xx} - 6\,\Upsilon_{xx}uu_x - 6\,\Upsilon u_xu_{xx}\right)T_8(t) - \Upsilon_x T_5(t)u^3\\ &+(-5\,\Upsilon_{xxxx}u_x - \Upsilon_{xxxxx}u - 5\,\Upsilon_x u_{xxxx} - 10\,\Upsilon_{xxx}u_{xx} - 10\,\Upsilon_{xx}u_{xxx})\,T_{11}(t)\\ &+(3\,\Upsilon_x u_{xxxx} + \Upsilon_{xxxx}u_x + 4\,\Upsilon_{xxx}u_{xx} + 6\,\Upsilon_{xx}u_{xxx})\,T_{10}(t)\\ &+\left(-6\,\Upsilon u_xu_{xx} - 3\,\Upsilon_x {u_x}^2\right)T_6\,(t) - \Upsilon_{xxxxxxx}T_{12}(t) = 0.\end{aligned} \tag{3.106}$$

The given Eq. (3.73) termed as nonlinear self-adjoint, if substituting $\Upsilon = \Phi_4$, where Φ_4 is taken as a function of t, x and u, satisfies the following equation

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta u}|_{\Upsilon(t,x)=\Phi_4(t,x,u)} = \lambda\big(&u_t + T_5(t)u_xu^3 + T_6(t)u_x^3 + T_7(t)uu_xu_{xx} + T_8(t)u^2u_{xxx}\\ &+T_9(t)u_{xx}u_{xxx} + T_{10}(t)u_xu_{xxxx} + T_{11}(t)uu_{xxxxx} + T_{12}(t)u_{xxxxxxx}\big),\end{aligned} \tag{3.107}$$

where λ is undetermined constant. Simplification of Eq. (3.107) results in determining equations for Φ_4 and random functions $T_5(t)$, $T_6(t)$,..., $T_{12}(t)$ as functions of t are obtained. The solution of equations can be written in following cases.

Case I

$$\begin{aligned}&\lambda = 0,\ T_5(t) = 0,\ T_7(t) = 6T_8(t),\ T_6(t) = 2T_8(t),\ T_9(t) = -5T_{11}(t) + 3T_{10}(t),\\ &\Phi_4 = b_0 + b_1 x + b_2 x^2,\end{aligned} \tag{3.108}$$

where b_0, b_1, b_2 are randomly taken constants, and $T_8(t)$, $T_{10}(t)$, $T_{11}(t)$, $T_{12}(t)$ are considered as random functions of t.

Case II

$$\begin{aligned}&\lambda = -\tilde{a}_0,\ T_5(t) = 0,\ T_6(t) = 0\ T_7(t) = 3T_8(t),\ T_9(t) = -10T_{11}(t) + 5T_{10}(t),\\ &\Phi_4 = \tilde{a}_0 u + \bar{M}_0(t) + \bar{M}_1(t)x + \bar{M}_2(t)x^2,\end{aligned} \tag{3.109}$$

where $\tilde{a}_0$ is arbitrary constant, and $\bar{M}_0(t)$, $\bar{M}_1(t)$, $\bar{M}_2(t)$ are arbitrary functions of t and satisfy following equation

$$-\bar{M}_0'(t) + \bar{M}_1'(t)x - \bar{M}_2'(t)x^2 = 0. \tag{3.110}$$

Case III

$$\lambda = 0,\ T_7(t) = 2T_6(t) + 2T_8(t),\ \Phi_4(t,x,u) = L_1, \tag{3.111}$$

where L_1 is arbitrary constant, and $T_5(t)$, $T_6(t)$, $T_8(t)$, $T_9(t)$, $T_{10}(t)$, $T_{11}(t)$ and $T_{12}(t)$ are functions of t.

Case IV

$$\begin{aligned}&\lambda = -\tilde{A}_1,\ T_6(t) = T_8(t), T_7(t) = 4T_8(t),\ T_9(t) = 5T_{10}(t) - 10T_{11}(t),\\ &\Phi_4(t,x,u) = \tilde{A}_0 + \tilde{A}_1 u,\end{aligned} \tag{3.112}$$

where $\tilde{A}_0$, $\tilde{A}_1$ are taken as constants, and $T_5(t)$, $T_{10}(t)$, $T_{11}(t)$, $T_{12}(t)$ are taken as arbitrary functions of t.

Conserved vector components associated with IGs using above nonlinear self-adjoint substitutions of Case I are given as follows.

The symmetry $X_6 = \frac{\partial}{\partial x}$ with Lie characteristic function $W = -u_x$ gives the following conserved vector components corresponding to arbitrary constants b_0, b_1 and b_2

$$C_0^t = -u_x,\ C_0^x = u_t. \tag{3.113}$$

$$
\begin{aligned}
C_1^t &= -xu_x, \\
C_1^x &= \big(2\,T_8(t)u{u_x}^2 - u_x u_{xxx} T_{11}(t) + u_{xxx} T_{10}(t) u_x + u_{xx} T_8(t) u^2 + {u_{xx}}^2 T_{10}(t) \\
&\qquad + u_{xxxx} T_{11}(t) u - 2\,{u_{xx}}^2 T_{11}(t) + x u_t + u_{xxxxxx} T_{12}(t)\big)\,.
\end{aligned}
\tag{3.114}
$$

$$
\begin{aligned}
C_2^t &= -x^2 u_x, \\
C_2^x &= \big(x^2 u_t + \big(-2\,u_x u_{xxx} T_{11}(t) + 2\,u_{xxxx} T_{11}(t) u - 4\,{u_{xx}}^2 T_{11}(t) + 2\,u_{xxxxxx} T_{12}(t) \\
&\quad + 2\,{u_{xx}}^2 T_{10}\,(t) + 2\,u_{xx} T_8(t) u^2 + 2\,u_{xxx} f\,(t)\,u_x + 4\,T_8(t) u {u_x}^2\big)\,x - 2\,u_{xxxxx} T_{12}(t) \\
&\quad - 2\,u_x T_8\,(t)\,u^2 + 4\,u_x T_{11}(t) u_{xx} - 2\,u_{xx} f\,(t)\,u_x - 2\,u_{xxx} T_{11}(t) u\big)\,.
\end{aligned}
\tag{3.115}
$$

For symmetry $X_7 = \frac{1}{T_{12}(t)} \frac{\partial}{\partial t}$, the conserved vector components corresponding to arbitrary constants b_0, b_1 and b_2 are found as follows

$$
\begin{aligned}
C_0^t &= -\frac{u_t}{T_{12}(t)}, \\
C_0^x &= -\frac{1}{{T_{12}}^2\,(t)}\,(-2\,T_{11t} T_{12}(t) {u_{xx}}^2 + T_{10t} T_{12}\,(t)\,{u_{xx}}^2 + 2\,T_{12t} T_{11}(t) {u_{xx}}^2 \\
&\quad - T_{12t} T_{10}(t) {u_{xx}}^2 - 2\,T_{12t} T_8\,(t)\,u{u_x}^2 + 2\,T_{8t} u T_{12}(t) {u_x}^2 + 2\,T_{12}(t) u_t T_8(t) {u_x}^2 \\
&\quad + T_{12}\,(t)\,u_t T_{11}(t) u_{xxxx} + T_{12}(t) u_{tx} u_{xxx} T_{10}(t) + T_{12}(t) u_{txx} T_8\,(t)\,u^2 \\
&\quad + 2\,T_{12}(t) u_{txx} u_{xx} T_{10}\,(t) + T_{12t} T_{11}(t) u_x u_{xxx} + T_{11t} T_{12}(t) u u_{xxxx} - T_{11t} T_{12}(t) u_x u_{xxx} \\
&\quad + T_{12}(t) u_{txxx} T_{10}(t) u_x - T_{12t} T_8(t) u^2 u_{xx} + T_{12}(t) u_{txxxx} T_{11}(t) u - T_{12}(t) u_{txxx} T_{11}\,(t)\,u_x \\
&\quad - T_{12}(t) u_{tx} u_{xxx} T_{11}\,(t) - 4\,T_{12}(t) u_{txx} u_{xx} T_{11}\,(t) + T_{8t} u^2 T_{12}(t) u_{xx} + T_{10t} T_{12}\,(t)\,u_x u_{xxx} \\
&\quad - T_{12t} T_{10}(t) u_x u_{xxx} - T_{12t} T_{11}(t) u u_{xxxx} + u_{txxxxxx} {T_{12}}^2\,(t) + 2\,T_{12}(t) u_t T_8(t) u u_{xx} \\
&\quad + 4\,T_{12}(t) u_{tx} T_8\,(t)\,u u_x)\,.
\end{aligned}
\tag{3.116}
$$

$$
\begin{aligned}
C_1^t &= -\frac{x u_t}{T_{12}(t)}, \\
C_1^x &= -\frac{1}{{T_{12}}^2\,(t)}\,(\big(-2\,T_{11t} T_{12}(t) {u_{xx}}^2 + T_{10t} T_{12}(t) {u_{xx}}^2 + 2\,T_{12t} T_{11}(t) {u_{xx}}^2 - T_{12t} T_{10}(t) {u_{xx}}^2 \\
&\quad - 2\,T_{12t} T_8\,(t)\,u{u_x}^2 + 2\,T_{8t} u T_{12}(t) {u_x}^2 + 2\,T_{12}(t) u_t T_8(t) {u_x}^2 + T_{12}\,(t)\,u_t T_{11}(t) u_{xxxx} \\
&\quad + T_{12}\,(t)\,u_{tx} u_{xxx} T_{10}(t) + T_{12}(t) u_{txx} T_8(t) u^2 + 2\,T_{12}(t) u_{txx} u_{xx} T_{10}(t) + T_{12t} T_{11}(t) u_x u_{xxx} \\
&\quad + T_{11t} T_{12}(t) u u_{xxxx} - T_{11t} T_{12}(t) u_x u_{xxx} + T_{12}(t) u_{txxx} T_{10}(t) u_x - T_{12t} T_8(t) u^2 u_{xx} \\
&\quad + T_{12}(t) u_{txxxx} T_{11}(t) u - T_{12}(t) u_{txxx} T_{11}\,(t)\,u_x - T_{12}(t) u_{tx} u_{xxx} T_{11}\,(t) \\
&\quad - 4\,T_{12}(t) u_{txx} u_{xx} T_{11}\,(t) + T_{8t} u^2 T_{12}(t) u_{xx} + T_{10t} T_{12}\,(t)\,u_x u_{xxx} - T_{12t} T_{10}(t) u_x u_{xxx} \\
&\quad - T_{12t} T_{11}(t) u u_{xxxx} + u_{txxxxxx} {T_{12}}^2\,(t) + 2\,T_{12}(t) u_t T_8(t) u u_{xx} + 4\,T_{12}(t) u_{tx} T_8\,(t)\,u u_x\big)\,x \\
&\quad - T_{12}(t) u_t u_{xxx} T_{11}\,(t) + T_{12t} T_8(t) u^2 u_x + T_{12t} T_{11}\,(t)\,u u_{xxx} + T_{12t} T_{10}(t) u_x u_{xx} \\
&\quad - T_{12}(t) u_{txxx} T_{11}(t) u - T_{12}\,(t)\,u_{tx} T_8(t) u^2 + 2\,T_{12}(t) u_{tx} u_{xx} T_{11}(t) - T_{12}(t) u_{txx} T_{10}\,(t)\,u_x \\
&\quad - 2\,T_{12t} T_{11}(t) u_x u_{xx} - T_{12}(t) u_{tx} u_{xx} T_{10}(t) + 2\,T_{12}\,(t)\,u_{txx} T_{11}(t) u_x - T_{10t} T_{12}\,(t)\,u_x u_{xx} \\
&\quad - T_{11t} T_{12}(t) u u_{xxx} - u_{txxxxx} {T_{12}}^2\,(t) - T_{8t} u^2 T_{12}(t) u_x - 2\,T_{12}(t) u_t T_8\,(t)\,u u_x \\
&\quad + 2\,T_{11t} T_{12}(t) u_x u_{xx})\,.
\end{aligned}
\tag{3.117}
$$

$$
\begin{aligned}
C_2^t &= -\frac{x^2 u_t}{T_{12}(t)},\\
C_2^x &= -\frac{1}{3{T_{12}}^2(t)}\big(\big(-6\,T_{11t}T_{12}(t){u_{xx}}^2 + 3\,T_{10t}T_{12}(t){u_{xx}}^2 + 6\,T_{12t}T_{11}(t)\,{u_{xx}}^2 - 3\,T_{12t}T_{10}(t){u_{xx}}^2\\
&\quad -6\,T_{12t}T_8(t)u{u_x}^2 + 6\,T_{8t}uT_{12}(t)\,{u_x}^2 + 6\,T_{12}(t)u_tT_8(t)\,{u_x}^2 + 3\,T_{12}(t)u_tT_{11}(t)\,u_{xxxx}\\
&\quad +3\,T_{12}(t)u_{tx}u_{xxx}T_{10}(t) + 3\,T_{12}(t)u_{txx}T_8(t)\,u^2 + 6\,T_{12}(t)u_{txx}u_{xx}T_{10}(t) + 3\,T_{12t}T_{11}(t)u_xu_{xxx}\\
&\quad +3\,T_{11t}T_{12}(t)\,uu_{xxxx} - 3\,T_{11t}T_{12}(t)u_xu_{xxx} + 3\,T_{12}(t)u_{txxx}T_{10}(t)u_x - 3\,T_{12t}T_8(t)u^2u_{xx}\\
&\quad +3\,T_{12}(t)\,u_{txxxx}T_{11}(t)u - 3\,T_{12}(t)u_{txxx}T_{11}(t)u_x - 3\,T_{12}(t)u_{tx}u_{xxx}T_{11}(t)\\
&\quad -12\,T_{12}(t)u_{txx}u_{xx}T_{11}(t) + 3\,T_{8t}u^2T_{12}(t)u_{xx} + 3\,T_{10t}T_{12}(t)u_xu_{xxx} - 3\,T_{12t}T_{10}(t)\,u_xu_{xxx}\\
&\quad -3\,T_{12t}T_{11}(t)uu_{xxxx} + 3\,u_{txxxxxx}{T_{12}}^2(t) + 6\,T_{12}(t)\,u_tT_8(t)uu_{xx} + 12\,T_{12}(t)\,u_{tx}T_8(t)uu_x\big)\,x^2\\
&\quad +\big(-6\,T_{12}(t)u_tu_{xxx}T_{11}(t) - 12\,T_{12t}T_{11}(t)u_xu_{xx} + 6\,T_{12t}T_8(t)u^2u_x - 6\,T_{10t}T_{12}(t)u_xu_{xx}\\
&\quad -6\,u_{txxxxx}{T_{12}}^2(t) - 6\,T_{12}(t)\,u_{tx}T_8(t)u^2 + 12\,T_{11t}T_{12}(t)\,u_xu_{xx} - 6\,T_{12}(t)u_{tx}u_{xx}T_{10}(t)\\
&\quad +6\,T_{12t}T_{10}(t)u_xu_{xx} + 12\,T_{10}(t)\,u_{txx}T_{11}(t)u_x - 6\,T_{11t}T_{12}(t)\,uu_{xxx} - 6\,T_{12}(t)u_{txx}T_{10}(t)\,u_x\\
&\quad -6\,T_{8t}u^2T_{12}(t)u_x - 12\,T_{12}(t)u_tT_8(t)uu_x + 6\,T_{12t}T_{11}(t)\,uu_{xxx} + 12\,T_{12}(t)u_{tx}u_{xx}T_{11}(t)\\
&\quad -6\,T_{12}(t)u_{txxx}T_{11}(t)\,u\big)\,x + 2\,T_{8t}u^3T_{12}(t) + 6\,T_{12}(t)\,u_tT_8(t)u^2 + 3\,T_{10t}T_{12}(t)\,{u_x}^2\\
&\quad +6\,T_{12}(t)u_tT_{11}(t)\,u_{xx} - 6\,T_{12t}T_{11}(t)uu_{xx} - 2\,T_{12t}T_8(t)\,u^3 + 9\,T_{12t}T_{11}(t){u_x}^2 + 6\,T_{11t}T_{12}(t)uu_{xx}\\
&\quad -3\,T_{12t}T_{10}(t){u_x}^2 - 18\,T_{12}(t)u_{tx}T_{11}(t)u_x - 9\,T_{11t}T_{12}(t){u_x}^2 + 6\,u_{txxxx}\,(T_{12}(t))^2\\
&\quad +6\,T_{12}(t)u_{tx}T_{10}(t)\,u_x + 6\,T_{12}(t)u_{txx}T_{11}(t)\,u\big).
\end{aligned}
\tag{3.118}
$$

For symmetry $X_8 = u\frac{\partial}{\partial u}$, the components of conserved vectors corresponding to arbitrary constants b_0, b_1 and b_2 are given as follows

$$
\begin{aligned}
C_0^t &= u,\\
C_0^x &= \left(u_{xxxx}u - u_xu_{xxx} - 2\,{u_{xx}}^2\right)T_{11}(t) + \left(u_xu_{xxx} + {u_{xx}}^2\right)T_{10}(t) + u_{xxxxxx}T_{12}(t)\\
&\quad + \left(u_{x,x}u^2 + 2\,u{u_x}^2\right)T_8(t).
\end{aligned}
\tag{3.119}
$$

$$
\begin{aligned}
C_1^t &= xu,\\
C_1^x &= \left(-xu_xu_{xxx} + xuu_{xxxx} - uu_{xxx} + 2\,u_xu_{x,x} - 2\,x{u_{xx}}^2\right)T_{11}(t)\\
&\quad + \left(-u_xu_{xx} + xu_xu_{xxx} + x{u_{xx}}^2\right)T_{10}(t) + \left(-u_{xxxxx} + u_{xxxxxx}x\right)T_{12}(t)\\
&\quad + \left(-u^2u_x + 2\,xu{u_x}^2 + xu^2u_{xx}\right)T_8(t).
\end{aligned}
\tag{3.120}
$$

$$
\begin{aligned}
C_2^t &= x^2u,\\
C_2^x &= \left(u_xx^2u_{xxx} + {u_x}^2 - 2\,u_xxu_{xx} + x^2{u_{xx}}^2\right)T_{10}(t)\\
&\quad + \left(4\,u_xxu_{x,x} + 2\,uu_{xx} - 2\,uxu_{xxx} - 2\,x^2{u_{xx}}^2 - 3\,{u_x}^2 + uu_{xxxx}x^2 - u_xx^2u_{xxx}\right)T_{11}(t)\\
&\quad + \left(2\,ux^2{u_x}^2 - 2\,xu^2u_x + x^2u^2u_{xx} + \frac{2}{3}\,u^3\right)T_8(t)\\
&\quad + \left(u_{xxxxxx}x^2 + 2\,u_{xxxx} - 2\,u_{xxxxx}x\right)T_{12}(t).
\end{aligned}
\tag{3.121}
$$

For symmetry $X_9 = \frac{7\int T_{12}(t)dt}{T_{12}(t)}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$, the conserved vectors are accomplished as follows

$$
\begin{aligned}
C_0^t &= -7\,\frac{\tilde{H}u_t}{T_{12}(t)} - xu_x,\\
C_0^x &= \frac{\left(-7\,u^2u_{xx} - 14\,uu_x{}^2\right)\tilde{H}T_{8t}}{T_{12}(t)} + \frac{\left(-7\,u_{xx}{}^2 - 7\,u_xu_{xxx}\right)\tilde{H}T_{10t}}{T_{12}(t)} - 7\,u_{txxxxxx}\tilde{H}\\
&\quad + \frac{\left(7\,u_{xx}{}^2 + 7\,u_xu_{xxx}\right)T_{10}(t)T_{12t}\tilde{H}}{\left(T_{12}(t)\right)^2} + \frac{\left(7\,u^2u_{xx} + 14\,uu_x{}^2\right)T_8(t)T_{12t}\tilde{H}}{\left(T_{12}(t)\right)^2}\\
&\quad + \frac{\left(-14\,u_{xx}{}^2 + 7\,uu_{xxxx} - 7\,u_xu_{xxx}\right)T_{11}(t)T_{12t}\tilde{H}}{\left(T_{12}\left(t\right)\right)^2}\\
&\quad + \frac{\left(-7\,uu_{xxxx} + 14\,u_{xx}{}^2 + 7\,u_xu_{xxx}\right)T_{11t}\tilde{H}}{T_{12}\left(t\right)}\\
&\quad + \frac{\left(-14\,u_tuu_{xx} - 28\,u_{tx}uu_x - 7\,u_{txx}u^2 - 14\,u_tu_x{}^2\right)\tilde{H}T_8\left(t\right)}{T_{12}(t)}\\
&\quad + \frac{\left(-7\,u_{tx}u_{xxx} - 7\,u_{txxx}u_x - 14\,u_{txx}u_{xx}\right)\tilde{H}T_{10}\left(t\right)}{T_{12}(t)}\\
&\quad + \frac{\left(-7\,u_{txxxx}u + 7\,u_{tx}u_{xxx} + 28\,u_{txx}u_{xx} - 7\,u_tu_{xxxx} + 7\,u_{txxx}u_x\right)\tilde{H}T_{11}(t)}{T_{12}\left(t\right)}\\
&\quad + xu_t + \left(12\,u_{xx}{}^2 + 6\,u_xu_{xxx} - 6\,uu_{xxxx}\right)T_{11}(t) - 6\,u_{xxxxxx}T_{12}(t)\\
&\quad + \left(-12\,uu_x{}^2 - 6\,u^2u_{xx}\right)T_8(t) + \left(-6\,u_xu_{xxx} - 6\,u_{xx}{}^2\right)T_{10}(t).
\end{aligned}
\tag{3.122}
$$

$$
\begin{aligned}
C_1^t &= \left(-7\,\frac{\tilde{H} u_t}{T_{12}\left(t\right)} - x u_x \right) x, \\
C_1^x &= \frac{\left(7\,u^2 u_x - 7\,x u^2 u_{xx} - 14\,x u {u_x}^2\right) \tilde{H} T_{8t}}{T_{12}(t)} + \frac{\left(-7\,x u_x u_{xxx} - 7\,x {u_{xx}}^2 + 7\,u_x u_{xx}\right) \tilde{H} T_{10t}}{T_{12}(t)} \\
&\quad + \left(7\,u_{txxxxx} - 7\,u_{txxxxxx} x\right) \tilde{H} + \left(-5\,x u_x u_{xxx} + 5\,u_x u_{xx} - 5\,x {u_{xx}}^2\right) T_{10}(t) \\
&\quad + \frac{\left(-28\,u_{tx} x u u_x - 14\,u_t x u u_{xx} + 14\,u_t u u_x - 14\,u_t x {u_x}^2 - 7\,u_{txx} x u^2 + 7\,u_{tx} u^2\right) \tilde{H} T_8(t)}{T_{12}\left(t\right)} \\
&\quad + \frac{\left(7\,u_{txx} u_x + 7\,u_{tx} u_{xx} - 7\,u_{txxx} u_x x - 14\,u_{txx} x u_{xx} - 7\,u_{tx} x u_{xxx}\right) \tilde{H} T_{10}(t)}{T_{12}(t)} \\
&\quad + \frac{\left(7\,u_{txxx} u_x x + 7\,u_{tx} x u_{xxx} + 28\,u_{txx} x u_{xx} + 7\,u_{txxx} u - 7\,u_t u_{xxxx} x\right) \tilde{H} T_{11}(t)}{T_{12}(t)} \\
&\quad \frac{\left(-7\,u_{txxxx} u x - 14\,u_{txx} u_x + 7\,u_t u_{xxx} - 14\,u_{tx} u_{xx}\right) \tilde{H} T_{11}(t)}{T_{12}(t)} \\
&\quad + \frac{\left(7\,x u_x u_{xxx} - 7\,x u u_{xxxx} + 14\,x {u_{xx}}^2 + 7\,u u_{xxx} - 14\,u_x u_{xx}\right) T_{11t} \tilde{H}}{T_{12}\left(t\right)} \\
&\quad + \frac{\left(-14\,x {u_{xx}}^2 + 7\,x u u_{xxxx} - 7\,x u_x u_{xxx} + 14\,u_x u_{xx} - 7\,u u_{xxx}\right) T_{11}\left(t\right) T_{12t} \tilde{H}}{{T_{12}}^2\left(t\right)} \\
&\quad + \frac{\left(-7\,u_x u_{xx} + 7\,x {u_{xx}}^2 + 7\,x u_x u_{xxx}\right) T_{10}(t) T_{12t} \tilde{H}}{\left(T_{12}\left(t\right)\right)^2} \\
&\quad + \frac{\left(14\,x u {u_x}^2 - 7\,u^2 u_x + 7\,x u^2 u_{xx}\right) T_8(t) T_{12t} \tilde{H}}{{T_{12}}^2\left(t\right)} \\
&\quad + x^2 u_t + \left(5\,u_{xxxxx} - 5\,x u_{xxxxxx}\right) T_{12}(t) + \left(5\,u^2 u_x - 5\,x u^2 u_{xx} - 10\,x u {u_x}^2\right) T_8(t) \\
&\quad + \left(5\,x u_x u_{xxx} + 5\,u u_{xxx} + 10\,x {u_{xx}}^2 - 5\,x u u_{xxxx} - 10\,u_x u_{xx}\right) T_{11}(t).
\end{aligned}
\tag{3.123}
$$

$$
\begin{aligned}
C_2^t &= \left(-7\,\frac{\tilde{H}u_t}{h\left(t\right)} - xu_x\right)x^2,\\
C_2^x &= \frac{\left(14\,u^2u_xx - 14\,x^2u{u_x}^2 - 7\,x^2u^2u_{xx} - 14/3\,u^3\right)\tilde{H}T_{8t}}{T_{12}(t)}\\
&+ \frac{\left(14\,u_xu_{xx}x - 7\,x^2{u_{xx}}^2 - 7\,{u_x}^2 - 7\,x^2u_xu_{xxx}\right)\tilde{H}T_{10t}}{T_{12}\left(t\right)}\\
&+ \left(14\,u_{txxxxx}x - 7\,u_{txxxxxx}x^2 - 14\,u_{txxxx}\right)\tilde{H}\\
&+ \frac{\left(21\,{u_x}^2 + 14\,xuu_{xxx} - 14\,uu_{xx} - 28\,u_xu_{xx}x + 14\,x^2{u_{xx}}^2 + 7\,x^2u_xu_{xxx}\right)T_{11t}\tilde{H}}{T_{12}(t)}\\
&- \frac{7\,x^2uu_{xxxx}T_{11t}\tilde{H}}{T_{12}(t)} + \frac{(28\,u_xu_{xx}x + 7\,x^2uu_{xxxx})T_{11}(t)T_{12t}\tilde{H}}{\left(T_{12}(t)\right)^2}\\
&+ \frac{\left(-14\,xuu_{xxx} - 7\,x^2u_xu_{xxx} + 14\,uu_{xx} - 14\,x^2{u_{xx}}^2 - 21\,{u_x}^2\right)T_{11}(t)T_{12t}\tilde{H}}{\left(T_{12}(t)\right)^2}\\
&+ \frac{\left(7\,x^2u_xu_{xxx} + 7\,x^2{u_{xx}}^2 - 14\,u_xu_{xx}x + 7\,{u_x}^2\right)T_{10}\left(t\right)T_{12t}\tilde{H}}{{T_{12}}^2\left(t\right)}\\
&+ \frac{\left(14\,x^2u{u_x}^2 - 14\,u^2u_xx + 7\,x^2u^2u_{xx} + 14/3\,u^3\right)T_8(t)T_{12t}\tilde{H}}{{T_{12}}^2\left(t\right)}\\
&+ \frac{\left(28\,u_txuu_x + 14\,u_{tx}xu^2 - 7\,u_{txx}x^2u^2 - 14\,u_tx^2{u_x}^2 - 14\,u_tx^2uu_{xx}\right)\tilde{H}T_8(t)}{T_{12}\left(t\right)}\\
&+ \frac{(-28\,u_{tx}x^2uu_x - 14\,u_tu^2)\tilde{H}T_8(t)}{T_{12}\left(t\right)} + \frac{(14\,u_{tx}xu_{xx} + 14\,u_{txx}u_xx)\tilde{H}T_{10}\left(t\right)}{T_{12}(t)}\\
&+ \frac{\left(-7\,u_{txxx}u_xx^2 - 14\,u_{tx}u_x - 7\,u_{tx}x^2u_{xxx} - 14\,u_{txx}x^2u_{xx}\right)\tilde{H}T_{10}\left(t\right)}{T_{12}(t)}\\
&+ \frac{\left(7\,u_{tx}x^2u_{xxx} + 28\,u_{txx}x^2u_{xx} - 14\,u_tu_{xx} - 28\,u_{txx}u_xx\right)\tilde{H}T_{11}(t)}{T_{12}(t)}\\
&+ \frac{(-28\,u_{tx}xu_{xx} + 14\,u_{txxx}ux - 7\,u_tu_{xxxx}x^2 + 7\,u_{txxx}u_xx^2)\tilde{H}T_{11}(t)}{T_{12}(t)}\\
&+ \frac{(42\,u_{tx}u_x + 14\,u_txu_{xxx} - 7\,u_{txxxx}ux^2 - 14\,u_{txx}u)\tilde{H}T_{11}(t)}{T_{12}(t)}\\
&+ x^3u_t + \left(-8\,u_{xxxx} + 8\,xu_{xxxxx} - 4\,x^2u_{xxxxxx}\right)T_{12}(t)\\
&+ \left(4\,x^2u_xu_{xxx} - 8\,uu_{xx} + 12\,{u_x}^2 - 16\,u_xu_{xx}x - 4\,x^2uu_{xxxx}\right)T_{11}\left(t\right)\\
&+ (8\,x^2{u_{xx}}^2 + 8\,xuu_{xxx})T_{11}\left(t\right)\\
&+ \left(-8/3\,u^3 - 4\,x^2u^2u_{xx} + 8\,u^2u_xx - 8\,x^2u{u_x}^2\right)T_8(t)\\
&+ \left(-4\,x^2u_xu_{xxx} - 4\,x^2{u_{xx}}^2 - 4\,{u_x}^2 + 8\,u_xu_{xx}x\right)T_{10}(t).
\end{aligned}
\tag{3.124}
$$

Herein, $\tilde{H} = \int T_{12}(t)dt$. Similarly, we can obtain conservation laws for other cases. To avoid repetition, we have omitted these conservation laws.

3.2.3 Direct method of conservation laws

The conservation laws are derived by using direct method of multipliers, the multiplier for the Eq. (3.73) is considered in the form $\Lambda(t, x, u)$ and satisfies the following equation

$$\begin{aligned}&\frac{\delta}{\delta u}\left(\Lambda\left(u_t + T_5(t)u_x u^3 + T_6(t)u_x^3 + T_7(t)uu_x u_{xx} + T_8(t)u^2 u_{xxx} + T_9(t)u_{xx}u_{xxx}\right.\right.\\&\left.\left.\quad +T_{10}(t)u_x u_{xxxx} + T_{11}(t)uu_{xxxxx} + T_{12}(t)u_{xxxxxxx}\right)\right) = 0,\end{aligned} \tag{3.125}$$

where $\frac{\delta}{\delta u}$ is Euler Lagrangian operator defined in Eq. (2.86). Then

$$\begin{aligned}&\Lambda\left(u_t + T_5(t)u_x u^3 + T_6(t)u_x^3 + T_7(t)uu_x u_{xx} + T_8(t)u^2 u_{xxx} + T_9(t)u_{xx}u_{xxx}\right.\\&\left.+T_{10}(t)u_x u_{xxxx} + T_{11}(t)uu_{xxxxx} + T_{12}(t)u_{xxxxxxx}\right) = D_t C^t + D_x C^x,\end{aligned} \tag{3.126}$$

where C^t and C^x are the conserved vector components. After solving Eq. (3.125), multiplier Λ and arbitrary functions have been obtained in following cases.
Case V

$$\begin{aligned}&\Lambda = \tilde{p}_0 + \tilde{p}_1 x + \tilde{p}_2 x^2,\\&T_5(t) = 0,\ T_6(t) = 6T_8(t),\ c = 6T_8(t),\ T_9(t) = 3T_{10}(t) - 5T_{11}(t),\end{aligned} \tag{3.127}$$

where $\tilde{p}_0$, $\tilde{p}_1$, $\tilde{p}_2$ are taken as random constants, and $T_8(t)$, $T_{10}(t)$, $T_{11}(t)$, $T_{12}(t)$ are arbitrary functions of t.
Case VI

$$\begin{aligned}&\Lambda = \tilde{K},\\&T_7(t) = 2T_6(t) + 6T_8(t),\ T_9(t) = 3T_{10}(t) - 5T_{11}(t),\end{aligned} \tag{3.128}$$

where $\tilde{K}$ is taken as random constants, and $T_5(t)$, $T_6(t)$, $T_8(t)$, $T_{10}(t)$, $T_{11}(t)$, $T_{12}(t)$ are functions of t. Then, using these two case, the conserved vector components are obtained as follows
For case V, the following conservation laws are constructed

$$\begin{aligned}&C_0^t = u,\\&C_0^x = u_t + T_8(t)\left(6\,u_x u u_{xx} + 2\,{u_x}^3 + u^2 u_{xxx}\right) + T_{10}(t)\left(3\,u_{xx}u_{xxx} + u_x u_{xxxx}\right)\\&\quad + T_{11}(t)\left(-5\,u_{xx}u_{xxx} + uu_{xxxxx}\right) + T_{12}(t)u_{xxxxxxx}.\end{aligned} \tag{3.129}$$

$$\begin{aligned}
&C_1^t = xu,\\
&C_1^x = \big(u_t + T_8(t)\left(6\,u_x u u_{xx} + 2\,{u_x}^3 + u^2 u_{xxx}\right) + T_{10}(t)\,(3\,u_{x,x} u_{xxx} + u_x u_{xxxx})\\
&\;+T_{11}(t)\,(-5\,u_{xx} u_{xxx} + u u_{xxxxx}) + T_{12}\,(t)\,u_{xxxxxxx}\big)\,x.
\end{aligned} \tag{3.130}$$

$$\begin{aligned}
&C_2^t = x^2 u,\\
&C_2^x = \big(u_t + T_8(t)\left(6\,u_x u u_{xx} + 2\,{u_x}^3 + u^2 u_{xxx}\right) + T_{10}(t)\,(3\,u_{x,x} u_{xxx} + u_x u_{xxxx})\\
&\;+T_{11}(t)\,(-5\,u_{xx} u_{xxx} + u u_{xxxxx}) + T_{12}\,(t)\,u_{xxxxxxx}\big)\,x^2.
\end{aligned} \tag{3.131}$$

For case VI, the following conserved vector components are obtained corresponding to randomly selected constant $\tilde{K}$ as follows

$$\begin{aligned}
&C^t = u,\\
&C^x = 1/4\,T_5(t)u^4 + T_6(t){u_x}^2 u + T_8\,(t)\,u^2 u_{xx} + T_{10}(t)\left(u_x u_{xxx} + {u_{xx}}^2\right)\\
&\;+ T_{11}(t)\left(-2\,{u_{xx}}^2 + u u_{xxxx} - u_x u_{xxx}\right) + T_{12}(t) u_{xxxxxx}.
\end{aligned} \tag{3.132}$$

3.2.4 Conclusion

The generalised 7^{th} order KdV equation in present analysis found to possess different type of solutions. The symmetry group analysis gives the solutions in the term of power series and Jacobi elliptic functions. The optimal system has been generated for inequivalent group invariant solutions. Similarity reductions associated with every symmetries are carried out. The nonlinear self-adjointness technique with new conservation theorem and the direct method have been applied to construct conserved vectors.

Chapter 4

Time Fractional PDEs[2]

In this chapter, three different types of time fractional PDEs have been investigated for the Lie symmetries, conserved densities/fluxes (conservation laws), and solutions. The symmetry analysis of the equations yields invariants which further help to reduce the dimensionality of the fractional PDEs. Conservation laws associated with the symmetries have been retrieved by applying new conservation theorem. Also, factional complex transformation converts the given fractional PDEs into the ODEs and further these ODEs are analysed for getting the exact solutions. A physical interpretation of the solutions is expressed by their graphical representations. The linear dispersion relations for the equations have been derived and the condition for the normal or anomalous dispersion of waves is also tested. In this chapter, section 4.1 presents the examination of new coupled ZK system in (2+1)-dimensions having time derivatives of fractional order for Lie symmetries and associated conserved densities/fluxes. The bright, dark and singular solitary wave solutions are obtained. In section 4.2, the details of steps used in studying the Wu-Zhang system in (2+1)-dimensions having time derivatives of fractional order are given. In section 4.3, 5^{th} order equation from Burgers hierarchy having time derivative of fractional order is

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[2] The contents of this chapter have been published and communicated in following journals:

(*i*) The contents of section (4.1) are published in *Computational and Applied Mathematics*, 37(5), 5981-6004 (2018).

(*ii*) The contents of section (4.2) are published in *Computers & Mathematics with Applications*, 79(4), 1031-1048 (2019).

(*iii*) The contents of section (4.3) are communicated in *Journal of Applied Analysis and Computation.*

being investigated for dispersion relation, Lie symmetries, solutions, and conserved densities/fluxes.

4.1 New coupled ZK system in (2+1)-dimensions having time derivatives of fractional order

The new coupled ZK system in (2+1)-dimensions in terms of equations Ξ_1, Ξ_2 and Ξ_3 is given in the following form

$$\begin{aligned}
\Xi_1 &\equiv \frac{\partial^\alpha u}{\partial t^\alpha} - a\left(uv_x + u_x v\right) - g\left(vw_x + v_x w\right) - b\left(u_{xxx} + u_{xyy}\right) = 0, \\
\Xi_2 &\equiv \frac{\partial^\alpha v}{\partial t^\alpha} - l\left(wu_x + w_x u\right) - b\left(v_{xxx} + v_{xyy}\right) = 0, \\
\Xi_3 &\equiv \frac{\partial^\alpha w}{\partial t^\alpha} - l\left(uv_x + u_x v\right) - b\left(w_{xxx} + w_{xyy}\right) = 0,
\end{aligned} \tag{4.1}$$

α denotes order of time derivative and $(0 < \alpha < 1)$. In system (4.1), a, b, g and l are the arbitrary constants. The ZK system successfully describes nonlinear ion-acoustic waves in a dusty plasma composed of hot isothermal electrons and cold ions [207, 236, 256, 257, 309, 317, 340, 347].

When $\alpha = 1$, the ZK system (4.1) has been investigated by many authors, and results were reported for exact solutions and conservation laws. The explicit solutions in terms of travelling waves of the ZK system have been reported by Wei and Tang [313] whereas Khalique [155] has discussed the way to find both the exact explicit solutions and conservation laws. The approximate analytic solutions of the ZK system have been obtained using homotopy perturbation method by Fahmy [78]. Elboree [75] has used a variational approach to find soliton and singular solitary wave solutions for the similar system. The stability analysis of new exact travelling-wave solutions has been examined by Seadawy et al. [258].

The system (4.1) considered under investigation has not been reported for the fractional Lie symmetries, conservation laws and solitary wave solutions. Therefore, the main thrust behind this investigation is to find the similarity solutions using the fractional Lie symmetry analysis and to derive the non-local conservation laws. The application of fractional type complex transformations results into ODEs, and further these ODEs are solved for the solitary wave solutions and analysed graphically in various conditions.

4.1.1 Lie symmetries and reductions

This subsection presents the Lie symmetries and corresponding similarity reductions of system (4.1). To obtain the Lie symmetries [244, 247, 276], 1LGTs (point) of the ZK system are considered as follows

$$\begin{aligned}
&\tilde{t} \to t + \varepsilon\, \mathcal{X}_6(t,x,y,u,v,w) + O(\varepsilon^2),\ \tilde{x} \to x + \varepsilon\, \mathcal{X}_7(t,x,y,u,v,w) + O(\varepsilon^2),\\
&\tilde{y} \to y + \varepsilon\, \mathcal{X}_8(t,x,y,u,v,w) + O(\varepsilon^2),\ \tilde{u} \to u + \varepsilon\, \mathcal{Z}_5(t,x,y,u,v,w) + O(\varepsilon^2),\\
&\tilde{v} \to v + \varepsilon\, \mathcal{Z}_6(t,x,y,u,v,w) + O(\varepsilon^2),\ \tilde{w} \to w + \varepsilon\, \mathcal{Z}_7(t,x,y,u,v,w) + O(\varepsilon^2),
\end{aligned} \tag{4.2}$$

where $\varepsilon << 1$ is a group parameter. The IG for a 1LGTs (4.2) is represented by following expression

$$X = \mathcal{X}_6\partial_t + \mathcal{X}_7\partial_x + \mathcal{X}_8\partial_y + \mathcal{Z}_5\partial_u + \mathcal{Z}_6\partial_v + \mathcal{Z}_7\partial_w. \tag{4.3}$$

The 3^{rd} order prolonged generator for IG (4.3) is given by

$$\begin{aligned}
Pr^{\alpha,3}X = {} & X + {\mathcal{Z}_5}^{x}\partial_{u_x} + {\mathcal{Z}_6}^{x}\partial_{v_x} + {\mathcal{Z}_7}^{x}\partial_{w_x} + {\mathcal{Z}_5}^{xxx}\partial_{u_{xxx}} + {\mathcal{Z}_6}^{xxx}\partial_{v_{xxx}} + {\mathcal{Z}_7}^{xxx}\partial_{w_{xxx}}\\
& + {\mathcal{Z}_5}^{xyy}\partial_{u_{xyy}} + {\mathcal{Z}_6}^{xyy}\partial_{v_{xyy}} + {\mathcal{Z}_7}^{xyy}\partial_{w_{xyy}} + {\mathcal{Z}_5}^{\alpha,t}\partial_{\partial_t^\alpha u} + {\mathcal{Z}_6}^{\alpha,t}\partial_{\partial_t^\alpha v} + {\mathcal{Z}_7}^{\alpha,t}\partial_{\partial_t^\alpha w},
\end{aligned} \tag{4.4}$$

where ${\mathcal{Z}_5}^{x}$, ${\mathcal{Z}_6}^{x}$, ${\mathcal{Z}_7}^{x}$, ${\mathcal{Z}_5}^{xxx}$,... are taken as integer order extended infinitesimals and ${\mathcal{Z}_5}^{\alpha,t}$, ${\mathcal{Z}_6}^{\alpha,t}$, ${\mathcal{Z}_7}^{\alpha,t}$ are corresponding extended infinitesimals having fractional order related to RLF derivative (detail has been given in the Appendix A.1).

The IG (4.3) will become a Lie point symmetry of the given system (4.1) if it satisfies the following conditions

$$\begin{aligned}
&Pr^{\alpha,3}X(\Xi_1)|_{\Xi_1=0,\Xi_2=0,\Xi_3=0} = 0,\ Pr^{\alpha,3}X(\Xi_2)|_{\Xi_1=0,\Xi_2=0,\Xi_3=0} = 0,\\
&Pr^{\alpha,3}X(\Xi_3)|_{\Xi_1=0,\Xi_2=0,\Xi_3=0} = 0.
\end{aligned} \tag{4.5}$$

The Eqs. (4.5) lead to the following symmetry determining equations

$$\begin{aligned}
&{\mathcal{Z}_5}^{\alpha,t} - a\left(u{\mathcal{Z}_6}^{x} + v_x\mathcal{Z}_5 + {\mathcal{Z}_5}^{x}v + u_x\mathcal{Z}_6\right) - g\left(v{\mathcal{Z}_7}^{x} + \mathcal{Z}_6 w_x + {\mathcal{Z}_6}^{x}w + v_x\mathcal{Z}_7\right)\\
&- b\left({\mathcal{Z}_5}^{xxx} + {\mathcal{Z}_5}^{xyy}\right) = 0,\\
&{\mathcal{Z}_6}^{\alpha,t} - l\left(u{\mathcal{Z}_7}^{x} + \mathcal{Z}_5 w_x + {\mathcal{Z}_5}^{x}w + u_x\mathcal{Z}_7\right) - b\left({\mathcal{Z}_6}^{xxx} + {\mathcal{Z}_6}^{xyy}\right) = 0,\\
&{\mathcal{Z}_7}^{\alpha,t} - l\left(u{\mathcal{Z}_6}^{x} + \mathcal{Z}_5 v_x + {\mathcal{Z}_5}^{x}v + u_x\mathcal{Z}_6\right) - b\left({\mathcal{Z}_7}^{xxx} + {\mathcal{Z}_7}^{xyy}\right) = 0.
\end{aligned} \tag{4.6}$$

Back substituting the integer and α-order extended infinitesimals into the Eq. (4.6), and equating coefficients of derivatives of the dependent variables to zero, we get over-determining system of equations in terms of infinitesimals. The solution of over-determining equations is given as follows

$$\begin{aligned}&\mathcal{X}_6 = 3e_3t + e_4,\ \mathcal{X}_7 = e_1 + e_3\alpha x,\ \mathcal{X}_8 = e_3\alpha y + e_2,\ \mathcal{Z}_5 = -2e_3\alpha u,\ \mathcal{Z}_6 = -2e_3\alpha v,\\ &\mathcal{Z}_7 = -2e_3\alpha w,\end{aligned} \tag{4.7}$$

where e_i, $i = 1, 2, 3, 4$ are arbitrary constants.

The lower limit in RLF derivative (2.40) is fixed, it remains invariant under the transformations given in Eq. (4.2) and it must holds the following condition

$$\mathcal{X}_6(t, x, y, u, v, w)|_{t=0} = 0. \tag{4.8}$$

This equality gives value of constant $e_4 = 0$ and the infinitesimals given in Eq. (4.7) reduce into following expressions

$$\begin{aligned}&\mathcal{X}_6 = 3e_3t,\ \mathcal{X}_7 = e_1 + e_3\alpha x,\ \mathcal{X}_8 = e_3\alpha y + e_2,\ \mathcal{Z}_5 = -2e_3\alpha u,\ \mathcal{Z}_6 = -2e_3\alpha v,\\ &\mathcal{Z}_7 = -2e_3\alpha w.\end{aligned} \tag{4.9}$$

Thus, the symmetry group for the system (4.1) is spanned by the following set of IGs

$$X_{10} = \frac{\partial}{\partial x},\ X_{11} = \frac{\partial}{\partial y}, X_{12} = \alpha\, x\frac{\partial}{\partial x} - 2\alpha\, u\frac{\partial}{\partial u} + 3t\frac{\partial}{\partial t} - 2\alpha\, v\frac{\partial}{\partial v} + \alpha\, y\frac{\partial}{\partial y} - 2\alpha\, w\frac{\partial}{\partial w}. \tag{4.10}$$

Similarity variables and related similarity solutions with corresponding IGs can be accessed with their associated characteristic equations. The solution of the characteristic equations will suggest the form of invariants for the system (4.1).

For the similarity solutions, consider the IG $X_{12} = \alpha\, x\frac{\partial}{\partial x} - 2\alpha\, u\frac{\partial}{\partial u} + 3t\frac{\partial}{\partial t} - 2\alpha\, v\frac{\partial}{\partial v} + \alpha\, y\frac{\partial}{\partial y} - 2\alpha\, w\frac{\partial}{\partial w}$.

The characteristic equations associated with the IG X_{12} are defined as follows

$$\frac{dx}{\alpha x} = \frac{dt}{3t} = \frac{dy}{\alpha y} = \frac{du}{-2\alpha u} = \frac{dv}{-2\alpha v} = \frac{dw}{-2\alpha w}. \tag{4.11}$$

Solution of characteristic equations gives similarity variables and the similarity solution in the following form

$$\zeta_3 = \frac{x}{t^{\alpha/3}}, \zeta_4 = \frac{y}{t^{\alpha/3}},\ u = F_2(\zeta_3, \zeta_4)t^{-2\alpha/3},\ v = G_2(\zeta_3, \zeta_4)t^{-2\alpha/3},\ w = H_2(\zeta_3, \zeta_4)t^{-2\alpha/3}. \tag{4.12}$$

The back substitution of similarity transformation (4.12) into the system (4.1) results in the reduced fractional order (1+1)-dimensional PDEs and the detailed procedure can be summarized theorem 4.1.1.

Theorem 4.1.1. *The similarity transformation* (4.12) *reduces the given (2+1)-dimensional system of fractional PDEs* (4.1) *into the following (1+1)-dimensional fractional PDEs*

$$\begin{aligned}
&(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F_2)(\zeta_3,\zeta_4) - aF_2G_{2\zeta_3} - aF_{2\zeta_3}G_2 - gG_2H_{2\zeta_3} - gG_{2\zeta_3}H_2 - bF_{2\zeta_3\zeta_3\zeta_3}\\
&- bF_{2\zeta_3\zeta_4\zeta_4} = 0,\\
&(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}G_2)(\zeta_3,\zeta_4) - lH_2F_{2\zeta_3} - lH_{2\zeta_3}F_2 - bG_{2\zeta_3\zeta_3\zeta_3} - bG_{2\zeta_3\zeta_4\zeta_4} = 0,\\
&(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}H_2)(\zeta_3,\zeta_4) - lF_2G_{2\zeta_3} - lF_{2\zeta_3}G_2 - bH_{2\zeta_3\zeta_3\zeta_3} - bH_{2\zeta_3\zeta_4\zeta_4} = 0,
\end{aligned} \tag{4.13}$$

where $(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}})$ *represents the extended left-hand sided Erdélyi-Kober fractional differential operator [157, 276] defined by*

$$\left(P^{\tau,\alpha}_{\beta_1,\beta_2}F_2\right)(\zeta_3,\zeta_4) = \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\beta_1}\zeta_3\frac{\partial}{\partial\zeta_3} - \frac{1}{\beta_2}\zeta_4\frac{\partial}{\partial\zeta_4}\right)\left(K^{\tau+\alpha,n-\alpha}_{\beta_1,\beta_2}F_2\right)(\zeta_3,\zeta_4), \tag{4.14}$$

$$\left(P^{\tau,\alpha}_{\beta_1,\beta_2}G_2\right)(\zeta_3,\zeta_4) = \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\beta_1}\zeta_3\frac{\partial}{\partial\zeta_3} - \frac{1}{\beta_2}\zeta_4\frac{\partial}{\partial\zeta_4}\right)\left(K^{\tau+\alpha,n-\alpha}_{\beta_1,\beta_2}G_2\right)(\zeta_3,\zeta_4), \tag{4.15}$$

and

$$\left(P^{\tau,\alpha}_{\beta_1,\beta_2}H_2\right)(\zeta_3,\zeta_4) = \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\beta_1}\zeta_3\frac{\partial}{\partial\zeta_3} - \frac{1}{\beta_2}\zeta_4\frac{\partial}{\partial\zeta_4}\right)\left(K^{\tau+\alpha,n-\alpha}_{\beta_1,\beta_2}H_2\right)(\zeta_3,\zeta_4), \tag{4.16}$$

where

$$n = \begin{cases} [\alpha]+1, & \alpha \notin N; \\ \alpha & \alpha \in N \end{cases}, \tag{4.17}$$

and $(K^{\tau+\alpha,n-\alpha}_{\beta_1,\beta_2})$ *is the extended left-hand sided Erdélyi-Kober fractional integral operator [157, 276] given by*

$$\begin{aligned}
&\left(K^{\tau,\alpha}_{\beta_1,\beta_2}F_2\right)(\zeta_3,\zeta_4)\\
&= \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty(\theta-1)^{\alpha-1}\theta^{-(\tau+\alpha)}F(\zeta_3\theta^{1/\beta_1},\zeta_4\theta^{1/\beta_2})d\theta, & \alpha > 0; \\ F_2(\zeta_3,\zeta_4) & \alpha = 0 \end{cases},
\end{aligned} \tag{4.18}$$

$$\begin{aligned}&\Big(K^{\tau,\alpha}_{\beta_1,\beta_2}G_2\Big)(\zeta_3,\zeta_4)\\ &=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_1^\infty(\theta-1)^{\alpha-1}\theta^{-(\tau+\alpha)}G(\zeta_3\theta^{1/\beta_1},\zeta_4\theta^{1/\beta_2})d\theta, & \alpha>0;\\ G_2(\zeta_3,\zeta_4) & \alpha=0\end{cases},\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}&\Big(K^{\tau,\alpha}_{\beta_1,\beta_2}H_2\Big)(\zeta_3,\zeta_4)\\ &=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_1^\infty(\theta-1)^{\alpha-1}\theta^{-(\tau+\alpha)}H(\zeta_3\theta^{1/\beta_1},\zeta_4\theta^{1/\beta_2})d\theta, & \alpha>0;\\ H_2(\zeta_3,\zeta_4) & \alpha=0.\end{cases}\end{aligned} \tag{4.20}$$

Proof. The fractional derivative of $u(t,x,y)=t^{-2\alpha/3}F_2(\zeta_3,\zeta_4)$ with respect to t of order α using (2.40) for $n-1<\alpha<n$, $n=1,2,3,...$ is given by

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\Big[\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\theta)^{n-\alpha-1}\theta^{\frac{-2\alpha}{3}}F_2(\theta^{\frac{-\alpha}{3}}x,\theta^{\frac{-\alpha}{3}}y)d\theta\Big]. \tag{4.21}$$

Let us consider $\vartheta=t/\theta$ and using it, the above equation can be drafted as follows

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{\partial^n}{\partial t^n}\Big[\frac{1}{\Gamma(n-\alpha)}t^{n-\frac{5\alpha}{3}}\int_1^\infty(\vartheta-1)^{n-\alpha-1}\vartheta^{-(n+1-\frac{5\alpha}{3})}F_2(\zeta_3\vartheta^{\frac{\alpha}{3}},\zeta_4\vartheta^{\frac{\alpha}{3}})\Big]d\vartheta\\ &=\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{5\alpha}{3}}\Big(K^{1-\frac{2\alpha}{3},n-\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F_2\Big)(\zeta_3,\zeta_4)\Big].\end{aligned} \tag{4.22}$$

Using the relation $\zeta_3(x,t)=x\,t^{-\frac{\alpha}{3}}$, $\zeta_4(y,t)=y\,t^{-\frac{\alpha}{3}}$, the chain rule of differentiation on the function $\varphi(\zeta_3,\zeta_4)$ yields

$$t\frac{\partial}{\partial t}\varphi(\zeta_3,\zeta_4)=t(\varphi_{\zeta_3}\zeta_{3t}+\varphi_{\zeta_4}\zeta_{4t})=-\frac{\alpha}{3}\zeta_3\varphi_{\zeta_3}-\frac{\alpha}{3}\zeta_4\varphi_{\zeta_4}. \tag{4.23}$$

The above relation simplifies the left-hand side of the Eq. (4.22) as follows

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{5\alpha}{3}}\Big(K^{1-\frac{2\alpha}{3},n-\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F_2\Big)(\zeta_3,\zeta_4)\Big]&=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[\frac{\partial}{\partial t}\Big(t^{n-\frac{5\alpha}{3}}\Big(K^{1-\frac{2\alpha}{3},n-\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F_2\Big)(\zeta_3,\zeta_4)\Big)\Big]\\ &=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[t^{n-\frac{5\alpha}{3}-1}\Big(n-\frac{5\alpha}{3}-\frac{\alpha}{3}\zeta_3\varphi_{\zeta_3}-\frac{\alpha}{3}\zeta_4\varphi_{\zeta_4}\Big)\\ &\qquad\Big(K^{1-\frac{2\alpha}{3},n-\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F_2\Big)(\zeta_3,\zeta_4)\Big].\end{aligned} \tag{4.24}$$

By reapplying the stated procedure $n-1$ times, the following result is obtained

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{5\alpha}{3}}\Big(K^{1-\frac{2\alpha}{3},n-\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F_2\Big)(\zeta_3,\zeta_4)\Big]&=t^{-\frac{5\alpha}{3}}\prod_{j=0}^{n-1}\Big(1-\frac{5\alpha}{3}+j-\frac{\alpha}{3}\zeta_3\varphi_{\zeta_3}-\frac{\alpha}{3}\zeta_4\varphi_{\zeta_4}\Big)\\ &\qquad\Big(K^{1-\frac{2\alpha}{3},n-\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F_2\Big)(\zeta_3,\zeta_4)\\ &=t^{-\frac{5\alpha}{3}}\Big(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}}F_2\Big)(\zeta_3,\zeta_4).\end{aligned} \tag{4.25}$$

Thus, the Eq. (4.22) reduces to

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = t^{-\frac{5\alpha}{3}} \Big(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}} F_2\Big)(\zeta_3,\zeta_4). \tag{4.26}$$

Similarly, the expressions of fractional derivative of other dependent variables v and w with respect to t are given as follows

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} = t^{-\frac{5\alpha}{3}} \Big(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}} G_2\Big)(\zeta_3,\zeta_4). \tag{4.27}$$

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = t^{-\frac{5\alpha}{3}} \Big(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}} H_2\Big)(\zeta_3,\zeta_4). \tag{4.28}$$

Hence, the given (2+1)-dimensional new coupled ZK system has been reduced to the (1+1)-dimensional system in the following form

$$\begin{aligned}
&(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}} F_2)(\zeta_3,\zeta_4) - aF_2G_{2\zeta_3} - aF_{2\zeta_3}G_2 - gG_2H_{2\zeta_3} - gG_{2\zeta_3}H_2 - bF_{2\zeta_3\zeta_3\zeta_3} \\
&- bF_{2\zeta_3\zeta_4\zeta_4} = 0, \\
&(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}} G_2)(\zeta_3,\zeta_4) - lH_2F_{2\zeta_3} - lH_{2\zeta_3}F_2 - bG_{2\zeta_3\zeta_3\zeta_3} - bG_{2\zeta_3\zeta_4\zeta_4} = 0, \\
&(P^{1-\frac{5\alpha}{3},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha}} H_2)(\zeta_3,\zeta_4) - lF_2G_{2\zeta_3} - lF_{2\zeta_3}G_2 - bH_{2\zeta_3\zeta_3\zeta_3} - bH_{2\zeta_3\zeta_4\zeta_4} = 0.
\end{aligned} \tag{4.29}$$

It completes the proof of theorem. □

The IGs $X_{10} = \frac{\partial}{\partial x}$ and $X_{11} = \frac{\partial}{\partial y}$ provide the trivial solutions, hence are not physically important.

4.1.2 Conservation laws of time fractional new coupled ZK system in (2+1)-dimensions

In this subsection, derivation of conservation laws for the system (4.1) by applying new conservation theorem [121] is elaborated.
Formulation of Lagrangian for Eqs. (4.1) is framed as follows

$$\begin{aligned}
\mathcal{L} = {} & \mathcal{A}(t,x,y)\Big({u_t}^{\alpha} - a\left(uv_x + u_xv\right) - g\left(vw_x + v_xw\right) - b\left(u_{xxx} + u_{xyy}\right)\Big) \\
& + \mathcal{B}(t,x,y)\Big({v_t}^{\alpha} - l\left(wu_x + w_xu\right) - b\left(v_{x,x,x} + v_{xyy}\right)\Big) \\
& + \mathcal{C}(t,x,y)\Big({w_t}^{\alpha} - l\left(uv_x + u_xv\right) - b\left(w_{x,x,x} + w_{xyy}\right)\Big),
\end{aligned} \tag{4.30}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are taken as new dependent variables.
The Euler-Lagrange operators $\dfrac{\delta}{\delta u}$, $\dfrac{\delta}{\delta v}$ and $\dfrac{\delta}{\delta w}$ [96, 185] from the Eq. (2.96) are defined in terms of the following expressions

$$\begin{aligned}
\frac{\delta}{\delta u} &= \partial_u + (D_t^\alpha)^* \partial_{\partial_t^\alpha u} - D_x \partial_{u_x} - D_x^3 \partial_{u_{xxx}} - D_x D_y^2 \partial_{u_{xyy}}, \\
\frac{\delta}{\delta v} &= \partial_v + (D_t^\alpha)^* \partial_{\partial_t^\alpha v} - D_x \partial_{v_x} - D_x^3 \partial_{v_{xxx}} - D_x D_y^2 \partial_{v_{xyy}}, \\
\frac{\delta}{\delta w} &= \partial_w + (D_t^\alpha)^* \partial_{\partial_t^\alpha w} - D_x \partial_{w_x} - D_x^3 \partial_{w_{xxx}} - D_x D_y^2 \partial_{w_{xyy}},
\end{aligned} \tag{4.31}$$

where $(D_t^\alpha)^*$ describes adjoint operator of (D_t^α).
Thus, the adjoint equations for the system (4.1) can be formulated as follows

$$\frac{\delta \mathcal{L}}{\delta u} = 0,\ \frac{\delta \mathcal{L}}{\delta v} = 0,\ \frac{\delta \mathcal{L}}{\delta w} = 0. \tag{4.32}$$

The relation [121, 244, 247] in case of three independent variables and three dependent variables can be presented as follows

$$\bar{X} + D_t \mathcal{X}_6\, e + D_x \mathcal{X}_7\, e + D_y \mathcal{X}_8\, e = W_4 \frac{\delta}{\delta u} + W_5 \frac{\delta}{\delta v} + W_6 \frac{\delta}{\delta w} + D_t C^t + D_x C^x + D_y C^y, \tag{4.33}$$

where e is the identity operator, and C^t, C^x, C^y are conserved vector components, and W_4, W_5 and W_6 are the Lie characteristic functions defined as follows

$$\begin{aligned}
W_4 &= \mathcal{Z}_5 - \mathcal{X}_6 u_t - \mathcal{X}_7 u_x - \mathcal{X}_8 u_y, \\
W_5 &= \mathcal{Z}_6 - \mathcal{X}_6 v_t - \mathcal{X}_7 v_x - \mathcal{X}_8 v_y, \\
W_6 &= \mathcal{Z}_7 - \mathcal{X}_6 w_t - \mathcal{X}_7 w_x - \mathcal{X}_8 w_y.
\end{aligned} \tag{4.34}$$

$\bar{X}$ is given by

$$\begin{aligned}
\bar{X} = {} & \mathcal{X}_6 \partial_t + \mathcal{X}_7 \partial_x + \mathcal{X}_8 \partial_y + \mathcal{Z}_5 \partial_u + \mathcal{Z}_6 \partial_v + \mathcal{Z}_7 \partial_w + {\mathcal{Z}_5}^{\alpha,t} \partial_{\partial_t^\alpha u} + {\mathcal{Z}_6}^{\alpha,t} \partial_{\partial_t^\alpha v} + {\mathcal{Z}_7}^{\alpha,t} \partial_{\partial_t^\alpha w} \\
& + {\mathcal{Z}_5}^{x} \partial_{u_x} + {\mathcal{Z}_6}^{x} \partial_{v_x} + {\mathcal{Z}_7}^{x} \partial_{w_x} + {\mathcal{Z}_5}^{xxx} \partial_{u_{xxx}} + {\mathcal{Z}_6}^{xxx} \partial_{v_{xxx}} + {\mathcal{Z}_7}^{xxx} \partial_{w_{xxx}} \\
& + {\mathcal{Z}_5}^{xyy} \partial_{u_{xyy}} + {\mathcal{Z}_6}^{xyy} \partial_{v_{xyy}} + {\mathcal{Z}_7}^{xyy} \partial_{w_{xyy}}.
\end{aligned} \tag{4.35}$$

By using the RLF derivative [96, 185, 244, 247, 297], and Eqs. (2.97) and (2.93), the t, x and y components C^t, C^x and C^y of the conserved vector are obtained corresponding to the IGs X_{10}, X_{11} and X_{12} as follows.
For IG $X_{10} = \frac{\partial}{\partial x}$, Lie characteristic functions are expressed as $W_4 = -u_x$, $W_5 = -v_x$, $W_6 = -w_x$, and corresponding conserved vector components can be determined as

follows

$$\begin{aligned} C^t &= {}_0D_t^{\alpha-1}(-u_x)\mathcal{A} + J(-u_x,\mathcal{A}_t) + {}_0D_t^{\alpha-1}(-v_x)\mathcal{B} + J(-v_x,\mathcal{B}_t) + {}_0D_t^{\alpha-1}(-w_x)\mathcal{C} \\ &\quad + J(-w_x,\mathcal{C}_t). \end{aligned} \tag{4.36}$$

$$\begin{aligned} C^x &= -u_x\left(-\mathcal{A}av - \mathcal{B}lw - \mathcal{C}lv - \frac{1}{3}\,\mathcal{A}_{yy}b - \mathcal{A}_{xx}b\right) - u_{xx}\mathcal{A}_x b - \frac{1}{3}\,u_{xy}\mathcal{A}_y b \\ &\quad + u_{xxx}\mathcal{A}b + \frac{1}{3}\,u_{xyy}\mathcal{A}b - v_x\left(\mathcal{A}\left(-au + gw\right) - \mathcal{C}lu - \frac{1}{3}\,\mathcal{B}_{yy}b - \mathcal{B}_{xx}b\right) \\ &\quad - v_{xx}\mathcal{B}_x b - \frac{1}{3}\,v_{xy}\mathcal{B}_y b + v_{xxx}\mathcal{B}b + \frac{1}{3}\,v_{xyy}\mathcal{B}b \\ &\quad - w_x\left(\mathcal{A}gv - \mathcal{B}lu - \frac{1}{3}\,\mathcal{C}_{yy}b - \mathcal{C}_{xx}b\right) - w_{xx}\mathcal{C}_x b - \frac{1}{3}\,w_{xy}\mathcal{C}_y b + w_{xxx}\mathcal{C}b \\ &\quad + \frac{1}{3}\,w_{xyy}\mathcal{C}b. \end{aligned} \tag{4.37}$$

$$\begin{aligned} C^y &= \frac{2}{3}\,u_x\mathcal{A}_{xy}b - \frac{1}{3}\,u_{xy}\mathcal{A}_x b - \frac{1}{3}\,u_{xx}\mathcal{A}_y b + \frac{2}{3}\,u_{xxy}\mathcal{A}b + \frac{2}{3}\,v_x\mathcal{B}_{xy}b \\ &\quad - \frac{1}{3}\,v_{xy}\mathcal{B}_x b - \frac{1}{3}\,v_{xx}\mathcal{B}_y b + \frac{2}{3}\,v_{xxy}\mathcal{B}b + \frac{2}{3}\,w_x\mathcal{C}_{xy}b - \frac{1}{3}\,w_{xy}\mathcal{C}_x b \\ &\quad - \frac{1}{3}\,w_{xx}\mathcal{C}_y b + \frac{2}{3}\,w_{xxy}\mathcal{C}b. \end{aligned} \tag{4.38}$$

The IG $X_{11} = \frac{\partial}{\partial y}$ with $W_4 = -u_y$, $W_5 = -v_y$ and $W_6 = -w_y$ gives the following conserved density and fluxes

$$\begin{aligned} C^t &= {}_0D_t^{\alpha-1}(-u_y)\mathcal{A} + J(-u_y,\mathcal{A}_t) + {}_0D_t^{\alpha-1}(-v_y)\mathcal{B} + J(-v_y,\mathcal{B}_t) + {}_0D_t^{\alpha-1}(-w_y)\mathcal{C} \\ &\quad + J(-w_y,\mathcal{C}_t). \end{aligned} \tag{4.39}$$

$$\begin{aligned} C^x &= -u_y\left(-\mathcal{A}av - \mathcal{B}lw - \mathcal{C}lv - \frac{1}{3}\,\mathcal{A}_{yy}b - \mathcal{A}_{xx}b\right) - u_{x,y}\mathcal{A}_x b - \frac{1}{3}\,u_{yy}\mathcal{A}_y b \\ &\quad + u_{xxy}\mathcal{A}b + \frac{1}{3}\,u_{yyy}\mathcal{A}b - v_y\left(\mathcal{A}\left(-au + gw\right) - \mathcal{C}lu - \frac{1}{3}\,\mathcal{B}_{yy}b - \mathcal{B}_{xx}b\right) \\ &\quad - v_{xy}\mathcal{B}_x b - \frac{1}{3}\,v_{yy}\mathcal{B}_y b + v_{xxy}\mathcal{B}b + \frac{1}{3}\,v_{yyy}\mathcal{B}b \\ &\quad - w_y\left(\mathcal{A}gv - \mathcal{B}lu - \frac{1}{3}\,\mathcal{C}_{yy}b - \mathcal{C}_{xx}b\right) - w_{xy}\mathcal{C}_x b - \frac{1}{3}\,w_{yy}\mathcal{C}_y b + w_{xxy}\mathcal{C}b \\ &\quad + \frac{1}{3}\,w_{yyy}\mathcal{C}b. \end{aligned} \tag{4.40}$$

$$\begin{aligned}C^y &= \frac{2}{3}\, u_y\mathcal{A}_{xy}b - \frac{1}{3}\, u_{yy}\mathcal{A}_x b - \frac{1}{3}\, u_{xy}\mathcal{A}_y b + 2/3\, u_{xyy}\mathcal{A}b + \frac{2}{3}\, v_y\mathcal{B}_{xy}b \\ &- \frac{1}{3}\, v_{yy}\mathcal{B}_x b - \frac{1}{3}\, v_{xy}\mathcal{B}_y b + \frac{2}{3}\, v_{xyy}\mathcal{B}b + \frac{2}{3}\, w_y\mathcal{C}_{xy}b - \frac{1}{3}\, w_{y,y}\mathcal{C}_x b \\ &- \frac{1}{3}\, w_{xy}\mathcal{C}_y b + \frac{2}{3}\, w_{xyy}\mathcal{C}b.\end{aligned} \tag{4.41}$$

For $X_{12} = \alpha\, x\frac{\partial}{\partial x} - 2\alpha\, u\frac{\partial}{\partial u} + 3t\frac{\partial}{\partial t} - 2\alpha\, v\frac{\partial}{\partial v} + \alpha\, y\frac{\partial}{\partial y} - 2\alpha\, w\frac{\partial}{\partial w}$ with $W_4 = -2\alpha\, u - 3tu_t - \alpha\, xu_x - \alpha\, yu_y$, $W_5 = -2\alpha\, v - 3tv_t - \alpha\, xv_x - \alpha\, yv_y$ and $W_6 = -2\alpha\, w - 3tw_t - \alpha\, xw_x - \alpha\, yw_y$, the following conservation laws are constructed

$$\begin{aligned}C^t &= {}_0D_t^{\alpha-1}(-2\alpha\, u - 3tu_t - \alpha\, xu_x - \alpha\, yu_y)\mathcal{A} + J(-2\alpha\, u - 3tu_t - \alpha\, xu_x - \alpha\, yu_y, \mathcal{A}_t) \\ &+ {}_0D_t^{\alpha-1}(-2\alpha\, v - 3tv_t - \alpha\, xv_x - \alpha\, yv_y)\mathcal{B} + J(-2\alpha\, v - 3tv_t - \alpha\, xv_x - \alpha\, yv_y, \mathcal{B}_t) \\ &+ {}_0D_t^{\alpha-1}(-2\alpha\, w - 3tw_t - \alpha\, xw_x - \alpha\, yw_y)\mathcal{C} + J(-2\alpha\, w - 3tw_t - \alpha\, xw_x - \alpha\, yw_y, \mathcal{C}_t).\end{aligned} \tag{4.42}$$

$$\begin{aligned}C^x &= (-2\,\alpha\, u - 3\, tu_t - \alpha\, xu_x - \alpha\, yu_y)\left(-\mathcal{A}av - \mathcal{B}lw - \mathcal{C}lv - \frac{1}{3}\,\mathcal{A}_{yy}b - \mathcal{A}_{xx}b\right) \\ &+ (-3\,\alpha\, u_x - \alpha\, xu_{xx} - 3\, tu_{tx} - \alpha\, yu_{xy})\,\mathcal{A}_x b \\ &+ \frac{1}{3}\,(-3\,\alpha\, u_y - \alpha\, xu_{xy} - 3\, tu_{ty} - \alpha\, yu_{yy})\,\mathcal{A}_y b \\ &- (-4\,\alpha\, u_{xx} - \alpha\, xu_{xxx} - 3\, tu_{txx} - \alpha\, yu_{xxy})\,\mathcal{A}b \\ &- \frac{1}{3}\,(-4\,\alpha\, u_{yy} - \alpha\, xu_{xyy} - 3\, tu_{tyy} - \alpha\, yu_{yyy})\,\mathcal{A}b \\ &+ (-2\,\alpha\, v - 3\, tv_t - \alpha\, xv_x - \alpha\, yv_y)\left(\mathcal{A}\,(-au + gw) - \mathcal{C}lu - \frac{1}{3}\,\mathcal{B}_{yy}b - \mathcal{B}_{xx}b\right) \\ &+ (-3\,\alpha\, v_x - \alpha\, xv_{xx} - 3\, tv_{tx} - \alpha\, yv_{xy})\,\mathcal{B}_x b \\ &+ \frac{1}{3}\,(-3\,\alpha\, v_y - \alpha\, xv_{xy} - 3\, tv_{t,y} - \alpha\, yv_{yy})\,\mathcal{B}_y b \\ &- (-4\,\alpha\, v_{xx} - \alpha\, xv_{xxx} - 3\, tv_{txx} - \alpha\, yv_{xxy})\,\mathcal{B}b \\ &- \frac{1}{3}\,(-4\,\alpha\, v_{yy} - \alpha\, xv_{xyy} - 3\, tv_{tyy} - \alpha\, yv_{yyy})\,\mathcal{B}b \\ &+ (-2\,\alpha\, w - 3\, tw_t - \alpha\, xw_x - \alpha\, yw_y)\left(\mathcal{A}gv - \mathcal{B}lu - \frac{1}{3}\,\mathcal{C}_{yy}b - \mathcal{C}_{xx}b\right) \\ &+ (-3\,\alpha\, w_x - \alpha\, xw_{xx} - 3\, tw_{tx} - \alpha\, yw_{xy})\,\mathcal{C}_x b \\ &+ \frac{1}{3}\,(-3\,\alpha\, w_y - \alpha\, xw_{xy} - 3\, tw_{ty} - \alpha\, yw_{yy})\,\mathcal{C}_y b \\ &- (-4\,\alpha\, w_{xx} - \alpha\, xw_{xxx} - 3\, tw_{txx} - \alpha\, yw_{xxy})\,\mathcal{C}b \\ &- \frac{1}{3}\,(-4\,\alpha\, w_{yy} - \alpha\, xw_{xyy} - 3\, tw_{tyy} - \alpha\, yw_{yyy})\,\mathcal{C}b.\end{aligned} \tag{4.43}$$

$$\begin{aligned}
C^y = &-\frac{2}{3}\left(-2\,\alpha\, u-3\,tu_t-\alpha\, xu_x-\alpha\, yu_y\right)\mathcal{A}_{xy}b\\
&+\frac{1}{3}\left(-3\,\alpha\, u_y-\alpha\, xu_{xy}-3\,tu_{ty}-\alpha\, yu_{yy}\right)\mathcal{A}_{x}b\\
&+\frac{1}{3}\left(-3\,\alpha\, u_x-\alpha\, xu_{xx}-3\,tu_{tx}-\alpha\, yu_{xy}\right)\mathcal{A}_{y}b\\
&-\frac{2}{3}\left(-4\,\alpha\, u_{xy}-\alpha\, xu_{xxy}-3\,tu_{txy}-\alpha\, yu_{xyy}\right)\mathcal{A}b\\
&-\frac{2}{3}\left(-2\,\alpha\, v-3\,tv_t-\alpha\, xv_x-\alpha\, yv_y\right)\mathcal{B}_{xy}b\\
&+\frac{1}{3}\left(-3\,\alpha\, v_y-\alpha\, xv_{xy}-3\,tv_{ty}-\alpha\, yv_{yy}\right)\mathcal{B}_{x}b\\
&+\frac{1}{3}\left(-3\,\alpha\, v_x-\alpha\, xv_{xx}-3\,tv_{tx}-\alpha\, yv_{xy}\right)\mathcal{B}_{y}b\\
&-\frac{2}{3}\left(-4\,\alpha\, v_{xy}-\alpha\, xv_{xxy}-3\,tv_{txy}-\alpha\, yv_{xyy}\right)\mathcal{B}b\\
&-\frac{2}{3}\left(-2\,\alpha\, w-3\,tw_t-\alpha\, xw_x-\alpha\, yw_y\right)\mathcal{C}_{xy}b\\
&+\frac{1}{3}\left(-3\,\alpha\, w_y-\alpha\, xw_{xy}-3\,tw_{ty}-\alpha\, yw_{yy}\right)\mathcal{C}_{x}b\\
&+\frac{1}{3}\left(-3\,\alpha\, w_x-\alpha\, xw_{xx}-3\,tw_{tx}-\alpha\, yw_{xy}\right)\mathcal{C}_{y}b\\
&-\frac{2}{3}\left(-4\,\alpha\, w_{xy}-\alpha\, xw_{xxy}-3\,tw_{txy}-\alpha\, yw_{xyy}\right)\mathcal{C}b.
\end{aligned} \tag{4.44}$$

4.1.3 Solutions in-terms of solitary waves

In this subsection, the solitary wave solutions for time fractional new coupled ZK system (4.1) are to be determined. The fractional complex transformation (2.63) and the chain rule (2.64) lead to the ODEs for given system (4.1) in the following form

$$\begin{aligned}
&k_3\rho\, f_\zeta-afg_\zeta k_1-af_\zeta k_1 g-ggh_\zeta k_1-gg_\zeta k_1 h-bf_{\zeta\zeta\zeta}{k_1}^3-bf_{\zeta\zeta\zeta}k_1{k_2}^2=0,\\
&k_3\rho\, g_\zeta-lhf_\zeta k_1-lh_\zeta k_1 f-bg_{\zeta\zeta\zeta}{k_1}^3-bg_{\zeta\zeta\zeta}k_1{k_2}^2=0,\\
&k_3\rho\, h_\zeta-lfg_\zeta k_1-lf_\zeta k_1 g-bh_{\zeta\zeta\zeta}{k_1}^3-bh_{\zeta\zeta\zeta}k_1{k_2}^2=0.
\end{aligned} \tag{4.45}$$

The ODEs (4.45) have been further examined to possess bright, dark and singular solitary wave solutions.

- For getting the bright solitary wave solution, the following form of solution is considered

$$f(\zeta)=A_1\,\mathrm{sech}^{p_1}(\zeta),\ \ g(\zeta)=A_2\,\mathrm{sech}^{p_2}(\zeta),\ \ h(\zeta)=A_3\,\mathrm{sech}^{p_3}(\zeta), \tag{4.46}$$

where A_i and p^i $(i = 1, 2, 3)$ are the wave amplitudes and the arbitrary constants, respectively. The back substitution of Eq. (4.46) into the ODEs results in the following set of equations

$$\begin{aligned}
&\left(-k_3\rho\, A_2p_2 + bk_1{}^3A_2p_2{}^3 + bk_1k_2{}^2A_2p_2{}^3\right)\operatorname{sech}^{p_2}(\zeta) + \left(-3\, bk_1{}^3A_2p_2{}^2\right.\\
&-2\, bk_1{}^3A_2p_2 - bk_1{}^3A_2p_2{}^3 - 3\, bk_1k_2{}^2A_2p_2{}^2 - 2\, bk_1k_2{}^2A_2p_2\\
&\left.-bk_1k_2{}^2A_2p_2{}^3\right)\operatorname{sech}^{p_2+2}(\zeta) + \left(lA_3A_1p_1k_1 + lA_3p_3k_1A_1\right)\operatorname{sech}^{p_3+p_1}(\zeta) = 0,\\
&\left(-k_3\rho\, A_3p_3 + bk_1{}^3A_3p_3{}^3 + bk_1k_2{}^2A_3p_3{}^3\right)\operatorname{sech}^{p_3}(\zeta)\\
&+\left(-3\, bk_1{}^3A_3p_3{}^2 - 2\, bk_1{}^3A_3p_3 - bk_1{}^3A_3p_3{}^3 - 3\, bk_1k_2{}^2A_3p_3{}^2 - 2\, bk_1k_2{}^2A_3p_3\right.\\
&\left.-bk_1k_2{}^2A_3p_3{}^3\right)\operatorname{sech}^{p_3+2}(\zeta) + \left(lA_1A_2p_2k_1 + lA_1p_1k_1A_2\right)\operatorname{sech}^{p_1+p_2}(\zeta) = 0,\\
&\left(gA_2A_3p_3k_1 + gA_2p_2k_1A_3\right)\operatorname{sech}^{p_2+p_3}(\zeta)\\
&+\left(-k_3\rho\, A_1p_1 + bk_1{}^3A_1p_1{}^3 + bk_1k_2{}^2A_1p_1{}^3\right)\operatorname{sech}^{p_1}(\zeta)\\
&+\left(-bk_1{}^3A_1p_1{}^3 - bk_1k_2{}^2A_1p_1{}^3 - 3\, bk_1{}^3A_1p_1{}^2 - 2\, bk_1{}^3A_1p_1 - 3\, bk_1k_2{}^2A_1p_1{}^2\right.\\
&\left.-2\, bk_1k_2{}^2A_1p_1\right)\operatorname{sech}^{p_1+2}(\zeta) + \left(aA_1A_2p_2k_1 + aA_1p_1k_1A_2\right)\operatorname{sech}^{p_1+p_2}(\zeta) = 0.
\end{aligned} \tag{4.47}$$

By applying the balancing principle on the system (4.47), one can find the values of $p_1 = p_2 = p_3 = 2$. Thus coefficients of linearly independent functions have been equated to zero, the solution of determining equations for the variables A_1, A_2, A_3 and ρ, generates the following constraints.

Case 1

$$\begin{aligned}
&A_1 = \frac{6\, b\left(k_1{}^2 + k_2{}^2\right)}{l},\ A_2 = \frac{3\left(-a + \sqrt{a^2 + 4\, lg}\right)\left(k_1{}^2 + k_2{}^2\right) b}{lg},\\
&A_3 = \frac{3\left(-a + \sqrt{a^2 + 4\, lg}\right)\left(k_1{}^2 + k_2{}^2\right) b}{lg},\ \rho = \frac{4bk_1\left(k_1{}^2 + k_2{}^2\right)}{k_3}.
\end{aligned} \tag{4.48}$$

Case 2

$$\begin{aligned}
&A_1 = -\frac{6\, b\left(k_1{}^2 + k_2{}^2\right)}{l},\ A_2 = \frac{3\left(-a + \sqrt{a^2 + 4\, lg}\right)\left(k_1{}^2 + k_2{}^2\right) b}{lg},\\
&A_3 = -\frac{3\left(-a + \sqrt{a^2 + 4\, lg}\right)\left(k_1{}^2 + k_2{}^2\right) b}{lg},\ \rho = \frac{4bk_1\left(k_1{}^2 + k_2{}^2\right)}{k_3}.
\end{aligned} \tag{4.49}$$

Thus, the bright solitary wave solutions for the system (4.1) using above constraints are retrieved as follows

$$
\begin{aligned}
u\left(t,x,y\right) &= \left(\frac{6\,b\left({k_1}^2+{k_2}^2\right)}{l}\right)\operatorname{sech}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma\left(1+\alpha\right)}\right),\\
v\left(t,x,y\right) &= \left(\frac{3\,\left(-a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg}\right)\\
&\qquad \operatorname{sech}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma\left(1+\alpha\right)}\right),\\
w\left(t,x,y\right) &= \left(\frac{3\,\left(-a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg}\right)\\
&\qquad \operatorname{sech}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma\left(1+\alpha\right)}\right),\\
a^2&+4\,lg\geq 0.
\end{aligned}
\tag{4.50}
$$

$$
\begin{aligned}
u\left(t,x,y\right) &= -\left(\frac{6\,b\left({k_1}^2+{k_2}^2\right)}{l}\right)\operatorname{sech}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma\left(1+\alpha\right)}\right),\\
v\left(t,x,y\right) &= \left(\frac{3\,\left(-a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg}\right)\\
&\qquad \operatorname{sech}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma\left(1+\alpha\right)}\right),\\
w\left(t,x,y\right) &= -\left(\frac{3\,\left(-a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg}\right)\\
&\qquad \operatorname{sech}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma\left(1+\alpha\right)}\right),\\
a^2&+4\,lg\geq 0.
\end{aligned}
\tag{4.51}
$$

- For getting the dark solitary wave solution, let us consider

$$f(\zeta)=A_1\tanh^{p_1}(\zeta),\;\; g(\zeta)=A_2\tanh^{p_2}(\zeta),\;\; h(\zeta)=A_3\tanh^{p_3}(\zeta). \tag{4.52}$$

By adopting exactly same procedure as discussed in previous subsection for the bright solitary wave solution gives the parameters values p_1, p_2 and p_3 as $p_1=p_2=p_3=2$. The determining equations are obtained when coefficients of linearly independent terms are equated to zero and their solution is discussed in the following cases.

Case 3

$$\begin{aligned}
A_1 &= \frac{6b\left({k_1}^2+{k_2}^2\right)}{l},\ A_2 = \frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg},\\
A_3 &= \frac{-3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg},\ \rho = -\frac{8bk_1\left({k_1}^2+{k_2}^2\right)}{k_3}.
\end{aligned} \tag{4.53}$$

Case 4

$$\begin{aligned}
A_1 &= -\frac{6b\left({k_1}^2+{k_2}^2\right)}{l},\ A_2 = \frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg},\\
A_3 &= \frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg},\ \rho = -\frac{8bk_1\left({k_1}^2+{k_2}^2\right)}{k_3}.
\end{aligned} \tag{4.54}$$

Thus, dark solitary wave solutions for the system (4.1) are given by

$$\begin{aligned}
u\left(t,x,y\right) &= \left(\frac{6\,b\left({k_2}^2+{k_1}^2\right)}{l}\right)\tanh^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma\left(1+\alpha\right)}\right),\\
v\left(t,x,y\right) &= \left(\frac{\left(3\,a+3\,\sqrt{a^2+4\,lg}\right)\left({k_2}^2+{k_1}^2\right)b}{lg}\right)\\
&\qquad \tanh^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma\left(1+\alpha\right)}\right),\\
w\left(t,x,y\right) &= -\left(\frac{3\,\left(a+\sqrt{a^2+4\,lg}\right)\left({k_2}^2+{k_1}^2\right)b}{lg}\right)\\
&\qquad \tanh^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma\left(1+\alpha\right)}\right),\\
&a^2+4\,lg \geq 0.
\end{aligned} \tag{4.55}$$

$$\begin{aligned}
u\left(t,x,y\right) &= -\left(\frac{6\,b\left({k_2}^2+{k_1}^2\right)}{l}\right)\tanh^2\left(k_1x+k_2y+\frac{k_3t^\alpha}{\Gamma\left(1+\alpha\right)}\right),\\
v\left(t,x,y\right) &= \left(\frac{\left(3\,a+3\,\sqrt{a^2+4\,lg}\right)\left({k_2}^2+{k_1}^2\right)b}{lg}\right)\\
&\qquad \tanh^2\left(k_1x+k_2y+\frac{k_3t^\alpha}{\Gamma\left(1+\alpha\right)}\right),\\
w\left(t,x,y\right) &= \left(\frac{3\,\left(a+\sqrt{a^2+4\,lg}\right)\left({k_2}^2+{k_1}^2\right)b}{lg}\right)\\
&\qquad \tanh^2\left(k_1x+k_2y+\frac{k_3t^\alpha}{\Gamma\left(1+\alpha\right)}\right),\\
a^2&+4\,lg\geq 0.
\end{aligned} \tag{4.56}$$

- To obtain the singular solitary wave solution, let us consider

$$f(\zeta)=A_1\,\mathrm{csch}^{p_1}(\zeta),\;\; g(\zeta)=A_2\,\mathrm{csch}^{p_2}(\zeta),\;\; h(\zeta)=A_3\,\mathrm{csch}^{p_3}(\zeta). \tag{4.57}$$

By using the balancing principle one can find the values $p_1=p_2=p_3=2$. A little simplification results in the expressions for the wave amplitudes and ρ are discussed in the following cases.

Case 5

$$\begin{aligned}
A_1 &= -\frac{6b\left({k_1}^2+{k_2}^2\right)}{l},\; A_2=\frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg},\\
A_3 &= \frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg},\; \rho=\frac{4bk_1\left({k_1}^2+{k_2}^2\right)}{k_3}.
\end{aligned} \tag{4.58}$$

Case 6

$$\begin{aligned}
A_1 &= \frac{6b\left({k_1}^2+{k_2}^2\right)}{l},\; A_2=\frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg},\\
A_3 &= -\frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_1}^2+{k_2}^2\right)b}{lg},\; \rho=\frac{4bk_1\left({k_1}^2+{k_2}^2\right)}{k_3}.
\end{aligned} \tag{4.59}$$

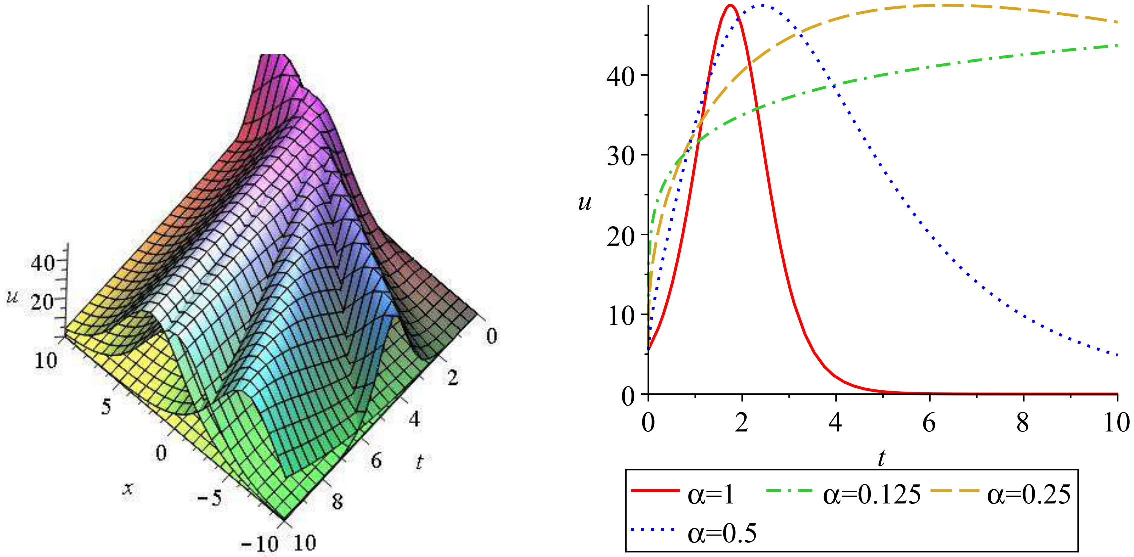
Thus, the singular solitary wave solutions for the system (4.1) can be formulated as follows

$$
\begin{aligned}
u(t,x,y) &= -\left(\frac{6\,b\left({k_2}^2+{k_1}^2\right)}{l}\right)\operatorname{csch}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma(1+\alpha)}\right),\\
v(t,x,y) &= \left(\frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_2}^2+{k_1}^2\right)b}{lg}\right)\operatorname{csch}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma(1+\alpha)}\right),\\
w(t,x,y) &= \left(\frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_2}^2+{k_1}^2\right)b}{lg}\right)\operatorname{csch}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma(1+\alpha)}\right),\\
&a^2+4\,lg\geq 0.
\end{aligned}
\tag{4.60}
$$

$$
\begin{aligned}
u(t,x,y) &= \frac{6\,b\left({k_2}^2+{k_1}^2\right)}{l}\operatorname{csch}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma(1+\alpha)}\right),\\
v(t,x,y) &= \frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_2}^2+{k_1}^2\right)b}{lg}\operatorname{csch}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma(1+\alpha)}\right),\\
w(t,x,y) &= -\frac{3\left(a+\sqrt{a^2+4\,lg}\right)\left({k_2}^2+{k_1}^2\right)b}{lg}\operatorname{csch}^2\left(k_1x+k_2y+\frac{k_3t^{\alpha}}{\Gamma(1+\alpha)}\right),\\
&a^2+4\,lg\geq 0.
\end{aligned}
\tag{4.61}
$$

4.1.3.1 Analysis of solutions

The bright (4.50), dark (4.55) and singular solitary wave solutions (4.60) are graphically presented in Figures 4.1-4.8 by choosing suitable values of the various parameters. Figure 4.1 reveals the 3D and 2D wave profiles for bright solitary wave solution (4.50) in terms of variable u for $y = 1$ and for $x = 1, y = 1$, respectively. The 3D wave profile as shown in Figure 4.1(a) represents the solitary wave obtained for different fractional order parameter α values. The Figure 4.1(b) describes the corresponding wave profile on the 2D graph. It has been noticed from the variation in wave profile as shown in the Figure 4.1(b) that the value of α has marked a strong influence on the curvature of the solitary wave. The shape of the wave is found to be almost symmetric for value of $\alpha = 1$ and but as the value of α decreases the corresponding width of the wave found to increase. Figure 4.2 describes 3D and 2D wave profiles of the bright solitary wave solution corresponding to the variable v. The solitary waves associated with variable v shows the similar behavior for variation in α values as noticed for the variable u. The solitary waves for the variable w resemble the wave shape as observed

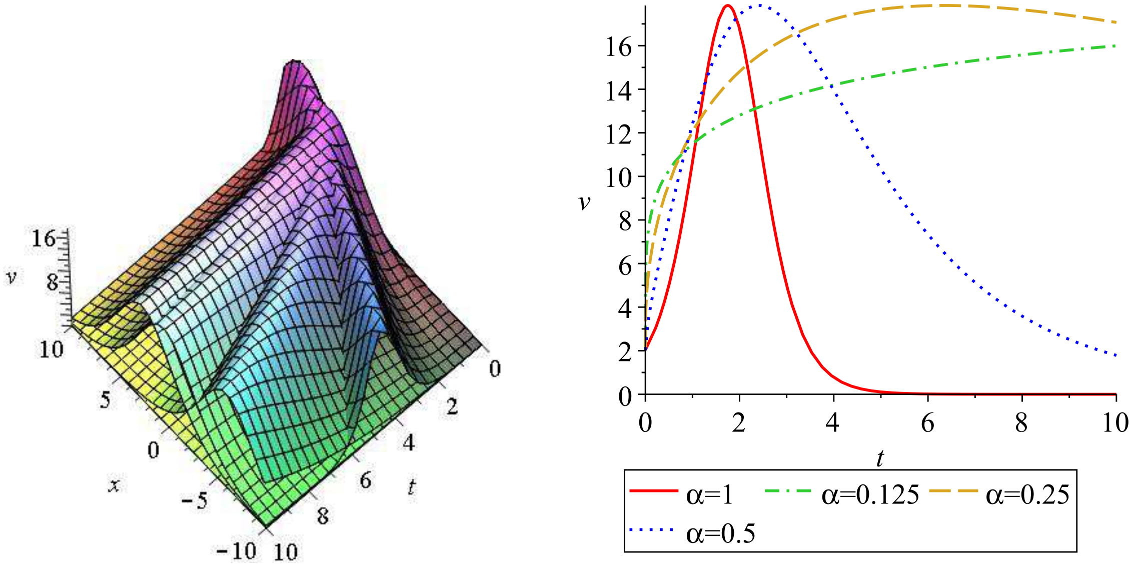


(a) (b)

Figure 4.1: Bright solitary wave solution (4.50) presented for variable $u(t, x, y)$ (a) by 3D plot for $y = 1$ and (b) by 2D plot for $x = 1$, $y = 1$ having $l = 0.5$, $b = 1$, $k_1 = 0.25$, $k_2 = -2$, $k_3 = 1$.

for the variable v. Figure 4.3(a-c) represents the solitary waves obtained at different time steps for the various values of fractional order α corresponding to the variables u, v and w, respectively. It has been found that for larger values of α, the waves are displaced widely, thus it suggests that α here controls the velocity of the solitary waves. A large value of α causes the solitary waves to move with higher velocity as depicted by Figure 4.3(a). As the value of α decreases, the waves are supposed to move with slightly different velocity (Figure 4.3(c)) and can be regarded as quasi-stationary waves as they are negligibly displaced. Figure 4.4 shows the solitary waves for the variables u, v and w for particular $y = 1$ and $y = 2$ for different α values. Figures 4.5-4.6 show the wave profiles of dark solitary wave solution (4.55) in terms of 3D and 2D plots for the variables u, v and w, respectively. The dark solitary wave solution can also be explored for the wave shape analysis of associated solitary waves for different α values. Figures 4.7-4.8 show the wave profiles of singular solitary wave solution (4.55) for the variables u and v, respectively. The variable w has a profile similar with v. Therefore, we have shown only profiles for u and v.

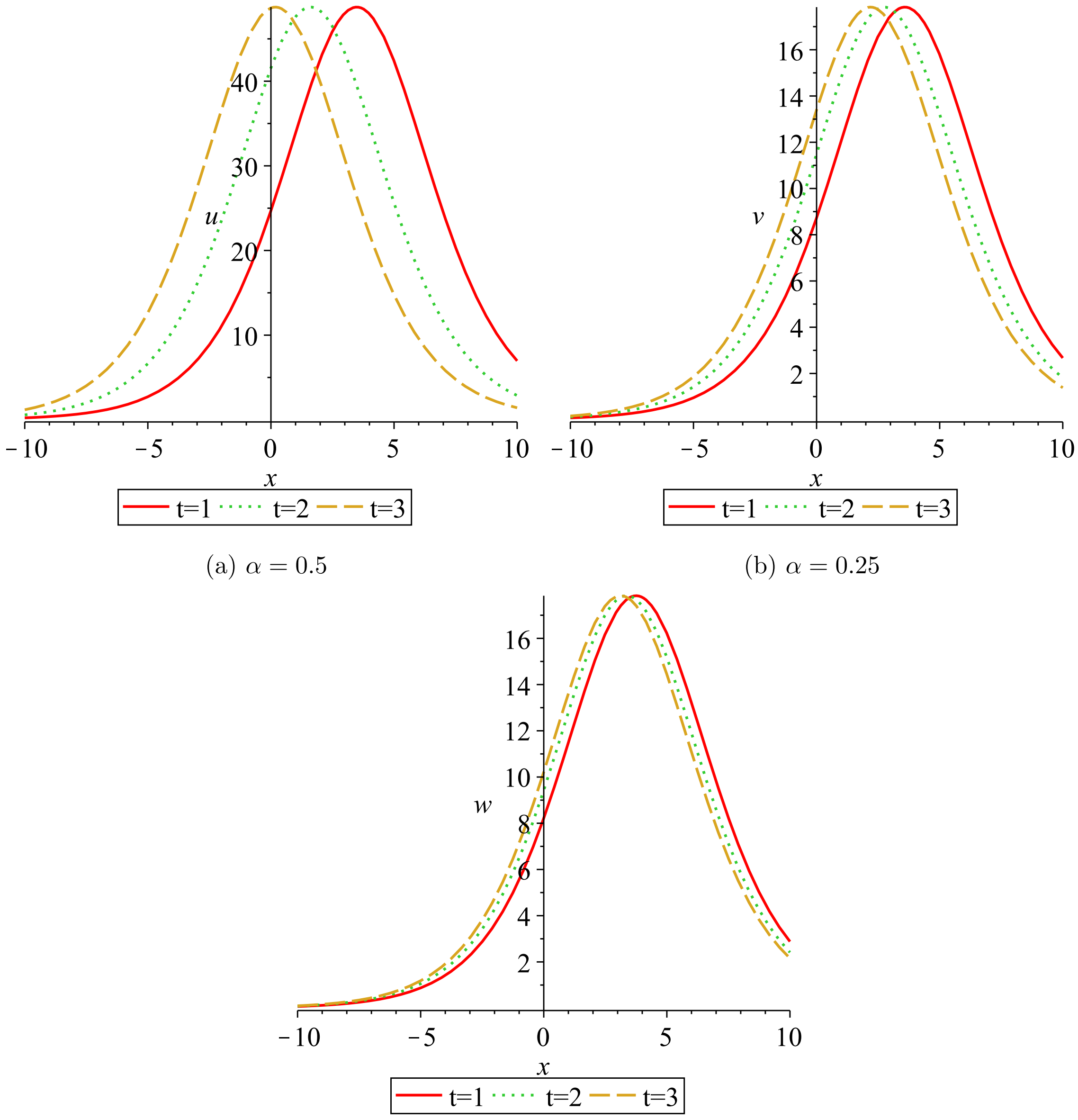
As a real world application, the new coupled ZK system having fractional order time derivative can be used to study the interaction of solitary waves between two layer fluids. The exact solutions obtained in the present analysis representing solitary waves



(a) (b)

Figure 4.2: Bright solitary wave solution (4.50) presented for variable $v(t,x,y)$ (a) by 3D plot for $y=1$ and (b) by 2D plot for $x=1$, $y=1$ having $g=1$, $l=0.5$, $a=1$, $b=1$, $k_1=0.25$, $k_2=-2$, $k_3=1$.

possessing a fluctuating relationship two waves having wave number k_1 and k_2. The solution shows that the resulting solitary wave is a function of wave number of each wave as well as amplitude and it signifies the typical characteristic of nonlinear wave interaction. Such solitary waves has a significant impact on the propagation stability of waves. The unstable waves after interaction are possessing unstability while the interaction of a unstable and a stable waves may lead to stability or unstability. It strongly depends upon the coupling terms in the system.

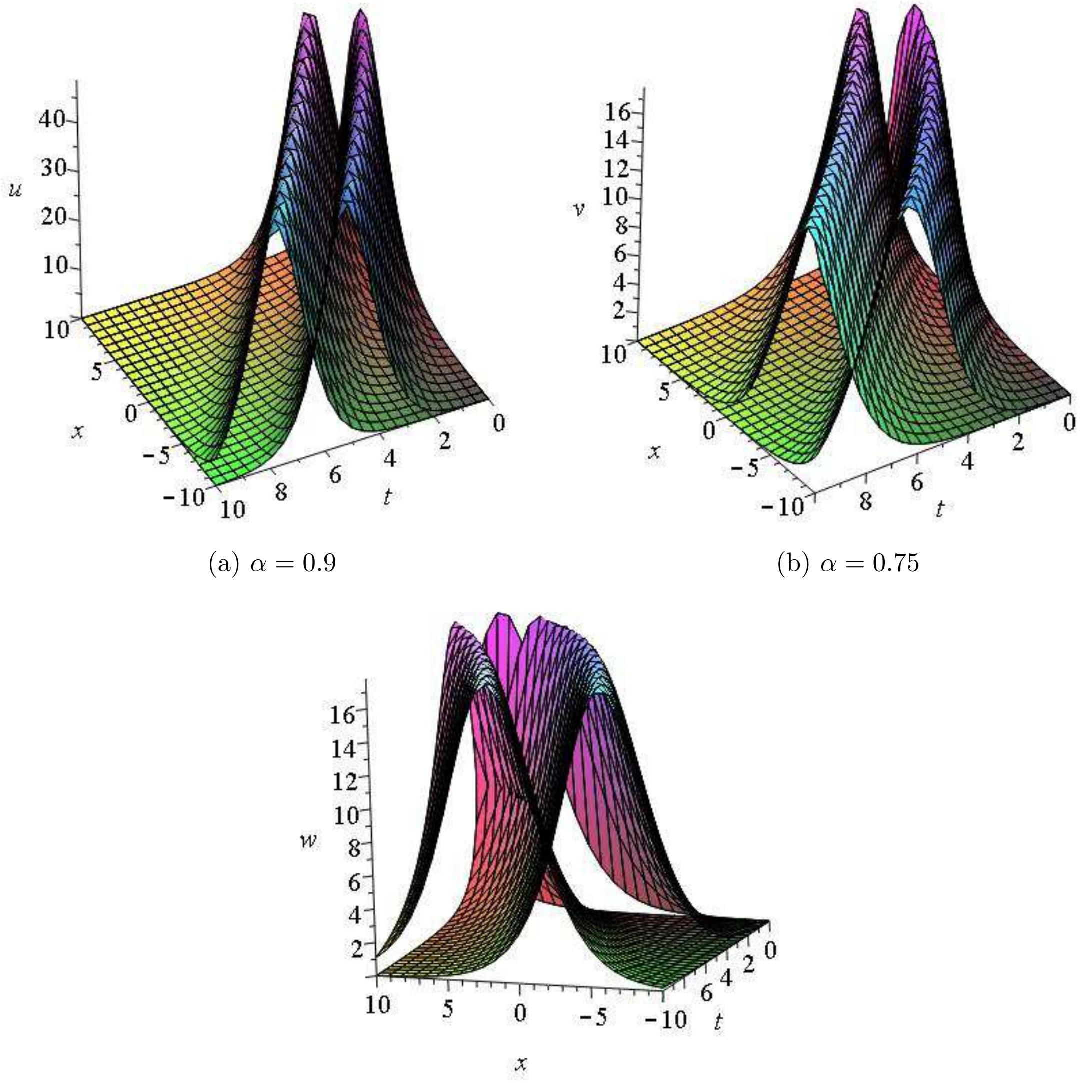


(a) $\alpha = 0.5$

(b) $\alpha = 0.25$

(c) $\alpha = 0.125$

Figure 4.3: Bright solitary wave profile of the Eq. (4.50) for α values (a) 0.5 (b) 0.25 and (c) 0.125.

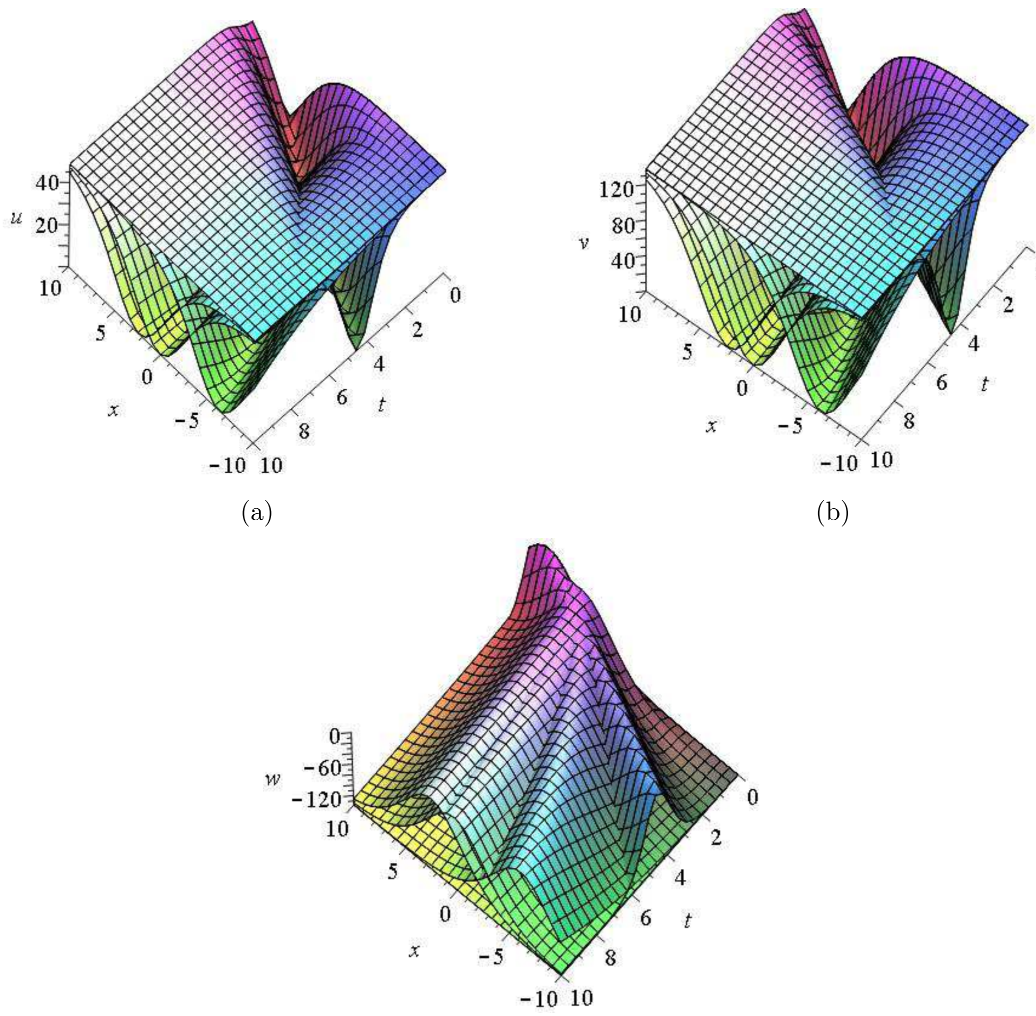


(a) $\alpha = 0.9$

(b) $\alpha = 0.75$

(c) $\alpha = 0.5$

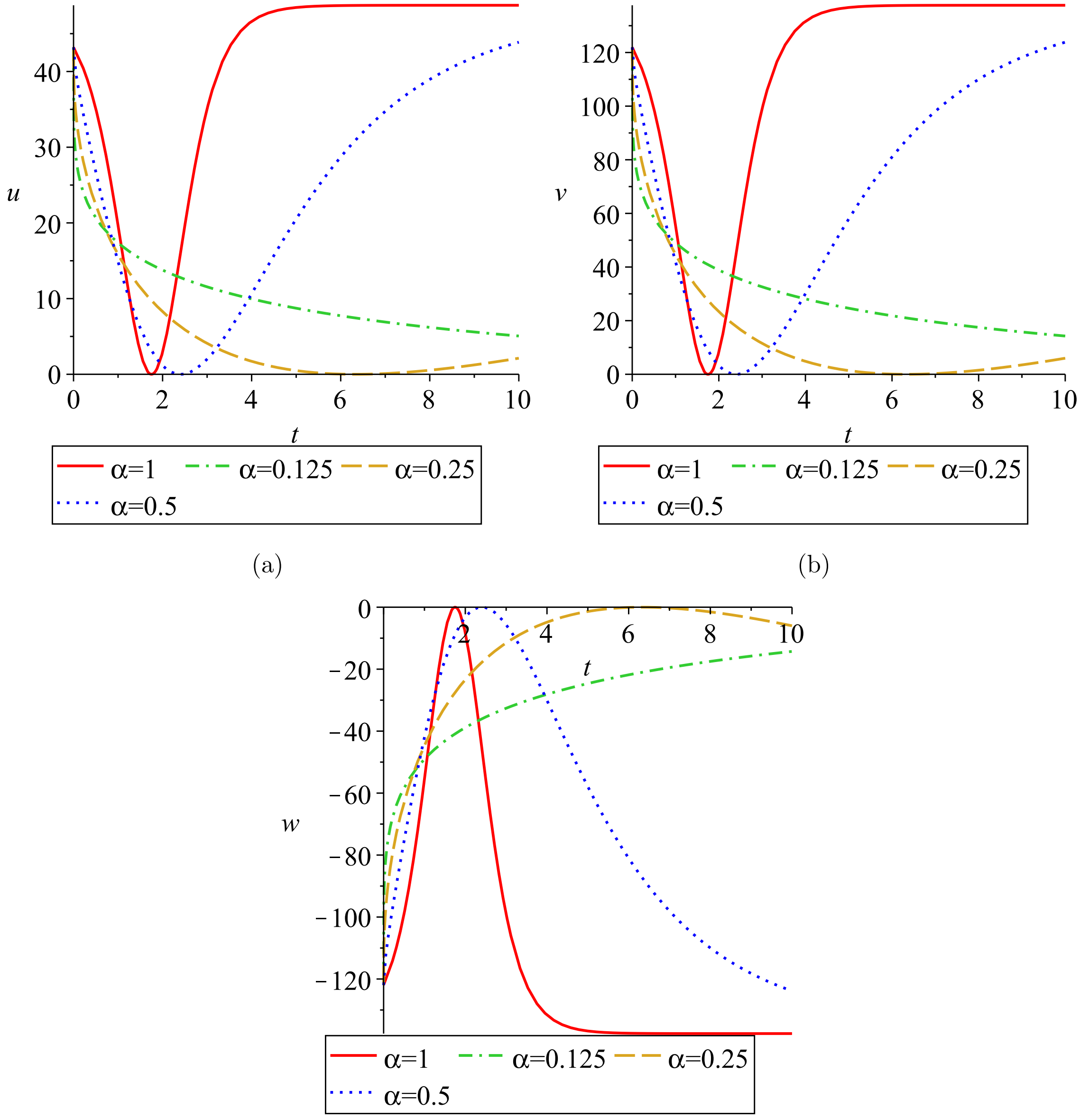
Figure 4.4: Solitary wave profile corresponding to bright solitary wave solution (4.50) for $y = 1$ and $y = 2$ having α values (a) 0.9 (b) 0.75 and (c) 0.5.



(a) (b)

(c)

Figure 4.5: 3D profile of dark solitary wave solution (4.55) for $y = 1$ with $l = -0.5$, $b = -1$, $k_1 = 0.25$, $k_2 = -2$, $k_3 = 1$ having $\alpha = 1, 0.5, 0.25, 0.125$ for (a) u (b) v and (c) w.

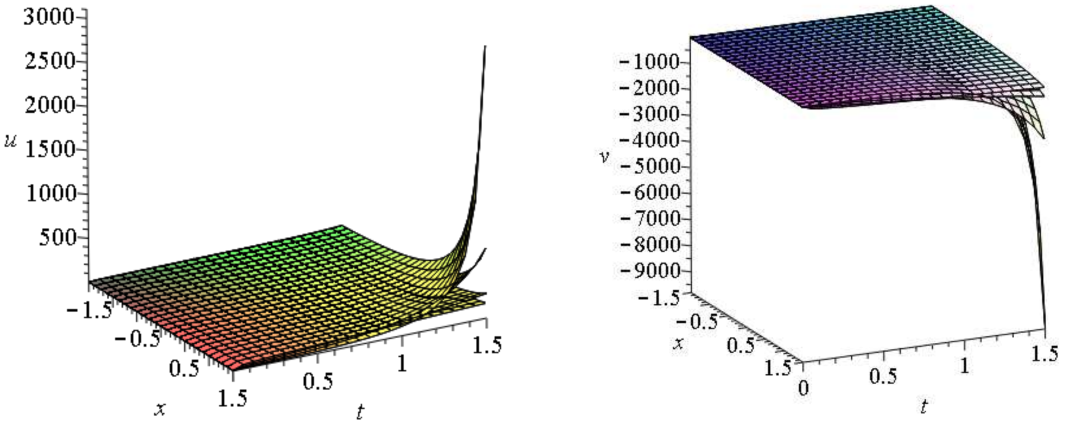


(a)

(b)

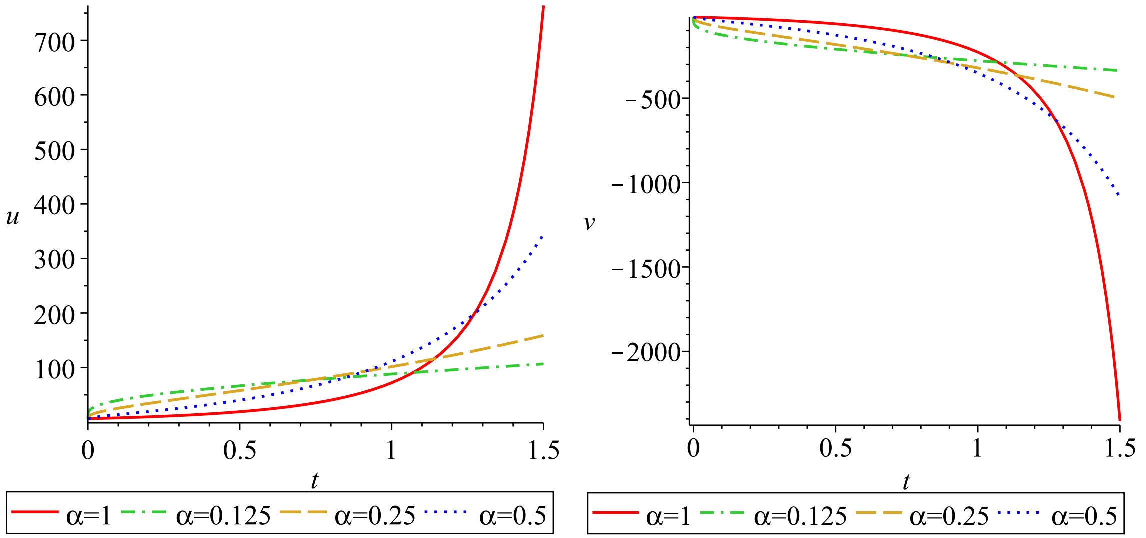
(c)

Figure 4.6: 2D profile of dark solitary wave solution (4.55) for $x = 1$, $y = 1$, with $g = 1$, $l = -0.5$, $a = 3$, $b = -1$, $k_1 = 0.25$, $k_2 = -2$, $k_3 = 1$ for (a) u (b) v and (c) w.



(a) (b)

Figure 4.7: 3D profile of singular solitary wave solution (4.60) for $y = 1$ with $l = 0.5$, $b = -1$, $k_1 = 0.25$, $k_2 = -2$, $k_3 = 1$ on values of $\alpha = 1, 0.5, 0.25, 0.125$ for (a) u and (b) v.



(a) (b)

Figure 4.8: 2D profile of singular solitary wave solution (4.60) for $x = 1$, $y = 1$, with $g = 1$, $l = 0.5$, $a = 3$, $b = -1$, $k_1 = 0.25$, $k_2 = -2$, $k_3 = 1$ for (a) u and (b) v.

4.1.4 Dispersion relation

This subsection gives the linear dispersion analysis of the system using the definition of RLF derivative with lower terminal at $-\infty$. The detailed procedure for deriving the dispersion relation has been outlined in chapter 2.

To study the dispersion relation for the system (4.1), the linearised form of system reads

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} - b\left(u_{xxx} + u_{xyy}\right) = 0,\\ &\frac{\partial^\alpha v}{\partial t^\alpha} - b\left(v_{xxx} + v_{xyy}\right) = 0,\\ &\frac{\partial^\alpha w}{\partial t^\alpha} - b\left(v_{xxx} + v_{xyy}\right) = 0. \end{aligned} \tag{4.62}$$

Let us consider, the sinusoidal solution of the linear system (4.62) for the dependent variables u, v and w as follows

$$\begin{aligned} u &= A_4 e^{i\left(\bar{\omega}t - \vec{S}\cdot\vec{r}\right)},\\ v &= A_5 e^{i\left(\bar{\omega}t - \vec{S}\cdot\vec{r}\right)},\\ w &= A_6 e^{i\left(\bar{\omega}t - \vec{S}\cdot\vec{r}\right)}. \end{aligned} \tag{4.63}$$

Here, $\bar{\omega} = \omega_r + i\,\omega_i$, and A_4, A_5 and A_6 are the amplitudes of u, v and w, respectively. Now, applying the Fourier transform for term ${}_{-\infty}D_t^\alpha f(t)$ [226], gives the dispersion relation for the linearised system (4.62) by solving following set of equations

$$\begin{aligned} &\left((i\bar{\omega})^\alpha A_4 - b\,A_4(-is_x)^3 - b\,A_4(-is_x)(-is_y)^2\right) e^{i\left(\bar{\omega}t - \vec{S}\cdot\vec{r}\right)} = 0,\\ &\left((i\bar{\omega})^\alpha A_5 - b\,A_5(-is_x)^3 - b\,A_5(-is_x)(-is_y)^2\right) e^{i\left(\bar{\omega}t - \vec{S}\cdot\vec{r}\right)} = 0,\\ &\left((i\bar{\omega})^\alpha A_6 - b\,A_6(-is_x)^3 - b\,A_6(-is_x)(-is_y)^2\right) e^{i\left(\bar{\omega}t - \vec{S}\cdot\vec{r}\right)} = 0. \end{aligned} \tag{4.64}$$

The solution of above system gives the expression for $\bar{\omega}$ as follows

$$\bar{\omega} = i^{1/\alpha - 1}\left(bs_x\left({s_x}^2 + {s_y}^2\right)\right)^{1/\alpha}. \tag{4.65}$$

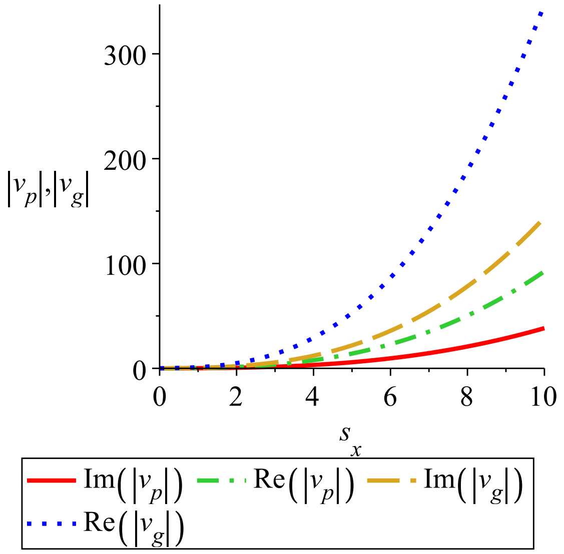


Figure 4.9: Comparison between phase and group velocities at $\alpha = 0.8$, $s_y = 1$, $b = 0.25$.

The mathematical form of phase and group velocities by using dispersion relation according to Eq. (2.104) is determined as follows

$$\begin{aligned}
v_p(\vec{S}) &= \frac{(s_x i + s_y j)\, i^{1/\alpha-1} \left(b s_x \left({s_x}^2+{s_y}^2\right)\right)^{1/\alpha}}{{s_x}^2+{s_y}^2},\\
v_g(\vec{S}) &= \frac{\partial}{\partial s_x}\omega_i(\vec{S})\hat{\imath} + \frac{\partial}{\partial s_y}\omega_i(\vec{S})\hat{\jmath}\\
&= \frac{i^{1/\alpha-1}\left(b s_x \left({s_x}^2+{s_y}^2\right)\right)^{1/\alpha}}{\alpha\,\left({s_x}^2+{s_y}^2\right)}\left(\frac{3\,{s_x}^2+{s_y}^2}{s_x}\,\hat{\imath} + 2\,s_y\,\hat{\jmath}\right).
\end{aligned} \tag{4.66}$$

The magnitude of the phase velocity and the group velocity (4.66) components can be obtained as follows

$$\begin{aligned}
|v_p| &= \frac{i^{1/\alpha-1}\left(b s_x \left({s_x}^2+{s_y}^2\right)\right)^{1/\alpha}}{\sqrt{{s_x}^2+{s_y}^2}},\\
|v_g| &= {s_x}^{1/\alpha}\sqrt{\frac{9\,{s_x}^4+10\,{s_y}^2{s_x}^2+{s_y}^4}{{s_x}^2}}\,i^{1/\alpha-1}b^{1/\alpha}\left({s_x}^2+{s_y}^2\right)^{1/\alpha-1}\alpha^{-1}.
\end{aligned} \tag{4.67}$$

A graph is plotted to investigate the variation of $|v_g|$ and $|v_p|$ with s_x and the obtained variation is shown in Figure 4.9. It shows that the real and imaginary parts of group velocity is greater than the real and imaginary parts of phase velocity. It indicates the anomalous dispersion of waves occurs for the ZK system.

4.1.5 Conclusion

The (2+1)-dimensional time fractional new coupled ZK system is successfully solved for Lie point symmetries, conservation laws and solitary wave solutions. Lie similarity reductions lead to reduced (1+1)-dimensional system of fractional PDEs from given (2+1)-dimensional system of fractional PDEs. The non-local conservation laws associated with Lie point symmetries using conservation theorem based on formal Lagrangian are determined. Bright, dark and singular solitary wave solutions are obtained using factional complex transformation. Effect of parameter $\alpha = 1, 0.5, 0.25, 0.125$ is shown graphically on the wave profile of solitary wave solutions. Also, solitary waves are shown graphically at $t = 1, 2, 3$ and different α's. The waves found to preserve their shape with the passage of time, so fundamental solitons exist [75, 76]. The solitary waves for $y = 1$, $y = 2$ are also presented which show the effect of α on the wave profile of the solutions. Also, the dispersion analysis gives relation between phase and group velocities which suggests anomalous dispersion of waves.

4.2 Time fractional Wu-Zhang system in (2+1)-dimensions

Herein, time fractional (2+1)-dimensional system suggested by Wu and Zhang [318] describing dispersive long waves in the shallow water moving along two horizontal directions is investigated. The system is given in terms of the following set of time fractional PDEs

$$\begin{aligned}
&\frac{\partial^\alpha u}{\partial t^\alpha} + uu_x + vu_y + w_x = 0,\\
&\frac{\partial^\alpha v}{\partial t^\alpha} + uv_x + vv_y + w_y = 0,\\
&\frac{\partial^\alpha w}{\partial t^\alpha} + u_x w + uw_x + v_y w + vw_y + \frac{1}{3}\left(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}\right) = 0,\ 0 < \alpha < 1,
\end{aligned} \tag{4.68}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the RLF derivative with respect to time [226, 251]. Here (u, v) represents the horizontal projection of surface velocity of a water particle and w denotes the water elevation. Individually, u and v corresponds to velocity of water particle along two perpendicular directions i.e. x and y, respectively. Dissipation terms u_{xxx}, u_{xyy}, v_{xxy}, v_{yyy} provide the damping and the other nonlinear terms in the system (4.68) acting as stabilizing agent in the transmission of dispersive long waves in area having shallow water. The coastal/civil engineers have employed the solutions of system (4.68) in applying nonlinear water wave model in harbors and coastal designing.

For $\alpha = 1$, the system (4.68) becomes classical or integer order Wu-Zhang system and the analysis of such system has been reported earlier by many authors such as Asgari et al. [21] have used extended tanh and exp-function techniques for getting exact solutions, Ji et al. [142] have found some new exact solutions by utilizing Lie symmetry analysis, Ma [191] has introduced Homotopy perturbation method for finding soliton solutions arising in fluid dynamics, the Adomian decomposition method for getting approximate solutions is carried out by Qasim and Ali [233], Taghizadeh et al. [285] have employed reduced differential transform and found solutions in terms of convergent power series, the one-dimensional optimal system and similarity reductions are performed by Xiong et al. [321], the approximate solutions are calculated by Zayed and Rahman [344] employing modified variational iteration method and Zheng et al. [351] have adopted generalised extended tanh method for exact travelling wave solutions.

The system (4.68) considered under investigation has not been explored yet in literature, so the main thrust in this section of the chapter is to study the fractional system (4.68) for dispersion analysis, Lie symmetries, derivation of conserved densities/fluxes and solutions.

The linear analysis of time fractional Wu-Zhang system is carried out to obtain the dispersion relation, the phase and group velocity of waves. The Lie symmetries and similarity reductions are obtained which further helps to derive the conservation laws by applying new conservation theorem with nonlinear self-adjointness. The exponential rational function method is applied to find the velocity profile of water particle as an exact solution of Wu-Zhang system (4.68). Thereafter, the water particle velocity is analysed graphically under different conditions.

4.2.1 Dispersion analysis of Wu-Zhang system

For deriving the dispersion relation of the time fractional Wu-Zhang system, let us consider the system (4.68) into its linearised form as follows

$$\begin{aligned}
&\frac{\partial^\alpha u}{\partial t^\alpha}+w_x=0,\\
&\frac{\partial^\alpha v}{\partial t^\alpha}+w_y=0,\\
&\frac{\partial^\alpha w}{\partial t^\alpha}+\frac{1}{3}\left(u_{xxx}+u_{xyy}+v_{xxy}+v_{yyy}\right)=0.
\end{aligned}\tag{4.69}$$

Consider the sinusoidal wave form for the variables u, v and w of Wu-Zhang system (4.69) in following form

$$u=A_7e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)},\ v=A_8e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)},\ w=A_9e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)}.\tag{4.70}$$

Here, A_7, A_8 and A_9 are the amplitudes of waves associated with the variables u, v and w, respectively. Here $\bar{\omega}$ and $\vec{S}$ represent the frequency and the wave vector, respectively. The application of the Fourier transform for the function $f(t)$ as ${}_{-\infty}D_t^\alpha f(t)$ [226] helps to find the dispersion relation for the linearised system (4.69) as follows

$$\begin{aligned}
&\left((i\bar{\omega})^\alpha A_7-is_xA_9\right)e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)}=0,\\
&\left((i\bar{\omega})^\alpha A_8-is_yA_9\right)e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)}=0,\\
&\left((i\bar{\omega})^\alpha A_9+\frac{1}{3}iA_7\left(s_x^3+s_xs_y^2\right)+\frac{1}{3}iA_8\left(s_x^2s_y+s_y^3\right)\right)e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)}=0.
\end{aligned}\tag{4.71}$$

The matrix form of the system (4.71) can be written as follows

$$\begin{pmatrix}(i\bar{\omega})^\alpha & 0 & -is_x\\ 0 & (i\bar{\omega})^\alpha & -is_y\\ \frac{1}{3}i\left(s_x^3+s_xs_y^2\right) & \frac{1}{3}i\left(s_x^2s_y+s_y^3\right) & (i\bar{\omega})^\alpha\end{pmatrix}\begin{pmatrix}A_7\\A_8\\A_9\end{pmatrix}=0.\tag{4.72}$$

The system (4.71) will generate non trivial solution if determinant of coefficients matrix from A_7, A_8 and A_9 vanishes identically; that is, if

$$(i\bar{\omega})^\alpha\left((i\bar{\omega})^{2\alpha}-\frac{2}{3}s_x^2s_y^2-\frac{1}{3}s_x^4-\frac{1}{3}s_y^4\right)=0.\tag{4.73}$$

It gives $(i\bar{\omega})^{2\alpha} - \frac{2}{3}s_x^2 s_y^2 - \frac{1}{3}s_x^4 - \frac{1}{3}s_y^4 = 0$; as $(i\bar{\omega})^{\alpha} \neq 0$. A little simplification for $\bar{\omega}$ gives

$$\bar{\omega} = \pm i \frac{\left(s_x^2 + s_y^2\right)^{1/\alpha}}{3^{1/2\alpha}}. \tag{4.74}$$

The above equation can also be written as

$$\bar{\omega} = \pm i \frac{(S)^{2/\alpha}}{3^{1/2\alpha}}, \tag{4.75}$$

where $S = |\vec{S}| = \sqrt{s_x^2 + s_y^2}$. The negative value of the frequency $\bar{\omega}$ is discarded as it gives a diverging wavefront for t approaches infinity. For the positive value of $\bar{\omega}$, the dispersion relation becomes

$$\bar{\omega} = i \frac{(S)^{2/\alpha}}{3^{1/2\alpha}}. \tag{4.76}$$

It gives the following values for the amplitudes A_8 and A_9

$$\begin{aligned} A_8 &= \frac{s_y A_7}{s_x}, \\ A_9 &= i \frac{\left(\frac{2}{3}s_x^2 s_y^2 + \frac{1}{3}s_x^4 + \frac{1}{3}s_y^4\right)^{1/2} A_7}{s_x}. \end{aligned} \tag{4.77}$$

Thus, the dispersion relation is found to be purely imaginary on comparison with the following equation

$$\bar{\omega} = \omega_r(\vec{S}) + i\,\omega_i(\vec{S}). \tag{4.78}$$

It gives $\omega_r(\vec{S}) = 0$ and $\omega_i(\vec{S}) = \frac{(S)^{2/\alpha}}{3^{1/2\alpha}}$. The damping factor $\gamma(\vec{S}) = \omega_i(\vec{S})$ is found to be $\vec{S}$ dependent. It reveals that as the waves travel in the horizontal plane, they exponentially damped with time. The expressions for the phase and group velocities using the dispersion relation are obtained as follows

$$\begin{aligned} v_p(\vec{S}) &= \frac{(s_x\hat{i} + s_y\hat{j})}{s_x^2 + s_y^2} \frac{\left(s_x^2 + s_y^2\right)^{1/\alpha}}{3^{1/2\alpha}}, \\ v_g(\vec{S}) &= \frac{\partial}{\partial s_x}\omega_i(\vec{S})\hat{i} + \frac{\partial}{\partial s_y}\omega_i(\vec{S})\hat{j} = \frac{2\,s_x(s_x^2 + s_y^2)^{1/\alpha - 1}}{\alpha\, 3^{1/2\alpha}}\,\hat{i} + \frac{2\,s_y(s_x^2 + s_y^2)^{1/\alpha - 1}}{\alpha\, 3^{1/2\alpha}}\,\hat{j}. \end{aligned} \tag{4.79}$$

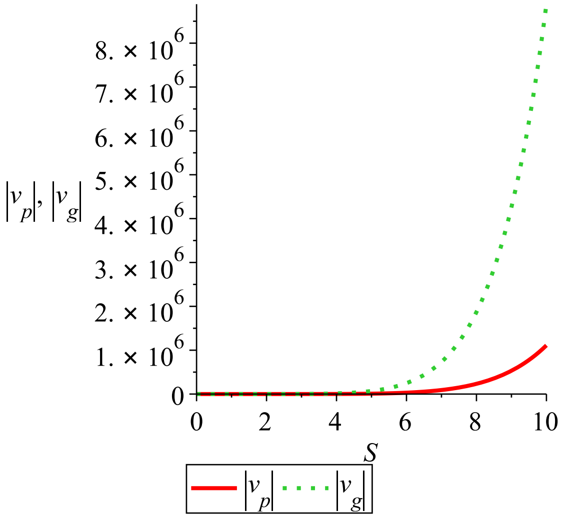


Figure 4.10: Comparison between phase and group velocities at $\alpha = 0.25$.

Then, magnitude of the phase velocity and the group velocity (4.79) components are obtained as follows

$$\begin{aligned} |v_p| &= \frac{S^{\frac{2}{\alpha}-1}}{3^{1/2\alpha}}, \\ |v_g| &= 2\,\frac{S^{\frac{2}{\alpha}-1}}{3^{1/2\alpha}\alpha}. \end{aligned} \tag{4.80}$$

The system shows an anomalous dispersion of waves since $|v_g| > |v_p|$ and it is proved as follows

$$\begin{aligned} |v_g| - |v_p| &= 2\,i\frac{S^{\frac{2}{\alpha}-1}}{3^{1/2\alpha}\alpha} - i\frac{S^{\frac{2}{\alpha}-1}}{3^{1/2\alpha}}, \\ &= i\frac{S^{\frac{2}{\alpha}-1}}{3^{1/2\alpha}}(\frac{2}{\alpha} - 1), \\ &= i\frac{S^{\frac{2}{\alpha}-1}}{3^{1/2\alpha}}(\frac{2-\alpha}{\alpha}). \end{aligned} \tag{4.81}$$

For the positive values of S and $0 < \alpha < 1$, the $|v_g| - |v_p| > 0$, or $|v_g| > |v_p|$. Thus, the system shows an anomalous dispersion of waves. A graph is plotted to investigate the variation of $|v_g|$ and $|v_p|$ with S and is shown in Figure 4.10.

4.2.2 Lie symmetry analysis

In this subsection, the Lie symmetries of Wu-Zhang system in (2+1)-dimensions with time fractional derivatives (4.68) are obtained using RLF derivative (2.40) when the lower terminal is lying at 0.

The analysis is similar to as discussed in previous section 4.1. For the system (4.68), the following infinitesimals $\mathcal{X}_9$, $\mathcal{X}_{10}$, $\mathcal{X}_{11}$, $\mathcal{Z}_8$, $\mathcal{Z}_9$ and $\mathcal{Z}_{10}$ corresponding to independent and dependent variables, respectively, are obtained

$$\begin{aligned}&\mathcal{X}_9 = 2d_3t,\ \mathcal{X}_{10} = d_1 + d_3\alpha x,\ \mathcal{X}_{11} = d_3\alpha y + d_2,\ \mathcal{Z}_8 = -d_3\alpha u,\ \mathcal{Z}_9 = -d_3\alpha v,\\ &\mathcal{Z}_{10} = -2d_3\alpha w,\end{aligned} \tag{4.82}$$

where d_1, d_2 and d_3 are arbitrary constants.
From the Eq. (4.82), Lie algebra for the given system (4.68) is obtained as follows

$$X_{13} = \alpha\, x\frac{\partial}{\partial x} - \alpha\, u\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial t} - \alpha\, v\frac{\partial}{\partial v} + \alpha\, y\frac{\partial}{\partial y} - 2\alpha\, w\frac{\partial}{\partial w},\ X_{14} = \frac{\partial}{\partial x},\ X_{15} = \frac{\partial}{\partial y}. \tag{4.83}$$

4.2.2.1 Similarity reductions

The similarity reductions for the system (4.68) corresponding to the infinitesimal symmetry $X_{13} = 2t\frac{\partial}{\partial t} - \alpha\, u\frac{\partial}{\partial u} - \alpha\, v\frac{\partial}{\partial v} - 2\alpha\, w\frac{\partial}{\partial w} + \alpha\, x\frac{\partial}{\partial x} + \alpha\, y\frac{\partial}{\partial y}$ can be carried out by solving the characteristic equations as follows

$$\frac{dx}{\alpha x} = \frac{dt}{2t} = \frac{dy}{\alpha y} = \frac{du}{-\alpha u} = \frac{dv}{-\alpha v} = \frac{dw}{-2\alpha w}. \tag{4.84}$$

The corresponding invariants mathematically expressed in the form as follows

$$\begin{aligned}&\zeta_5 = \frac{x}{t^{\alpha/2}},\ \zeta_6 = \frac{y}{t^{\alpha/2}},\ u = F_3(\zeta_5, \zeta_6)t^{-\alpha/2},\ v = G_3(\zeta_5, \zeta_6)t^{-\alpha/2},\\ &w = H_3(\zeta_5, \zeta_6)t^{-\alpha}.\end{aligned} \tag{4.85}$$

Using the invariants (4.85) into the system (4.68), the time-fractional Wu-Zhang system is transformed into system of reduced (1+1)-dimensional fractional PDEs as proved in theorem stated below.

Theorem 4.2.1. *Using the invariants* (4.85)*, the system* (4.68) *is reduced from (2+1)-dimensional time fractional PDEs to (1+1)-dimensional PDEs in fractional* *form as follows*

$$\begin{aligned}
&(P^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}F_3)(\zeta_5,\zeta_6)+F_3F_{3\zeta_5}+G_3F_{3\zeta_6}+F_{3\zeta_5}=0,\\
&(P^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}G_3)(\zeta_5,\zeta_6)+F_3G_{3\zeta_5}+G_3G_{3\zeta_6}+F_{3\zeta_6}=0,\\
&(P^{1-2\alpha,\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}H_3)(\zeta_5,\zeta_6)+2\,F_3F_{3\zeta_5}+G_3F_{3\zeta_6}+F_3G_{3\zeta_6}\\
&+\frac{1}{3}\left(F_{3\zeta_5\zeta_5\zeta_5}+F_{3\zeta_5\zeta_6\zeta_6}+G_{3\zeta_5\zeta_5\zeta_6}+G_{3\zeta_6\zeta_6\zeta_6}\right)=0,
\end{aligned} \tag{4.86}$$

where $P^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}$ *and* $P^{1-2\alpha,\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}$ *represent the extended left-hand sided Erdélyi-Kober fractional differential operators defined in the Eqs.* (4.14)-(4.16).

Proof. Theorem can be proved by following steps given in Theorem 4.1.1. □

The IGs $X_{14}=\frac{\partial}{\partial x}$ and $X_{15}=\frac{\partial}{\partial y}$ give only the trivial solutions. Hence, they are not physically important.

4.2.3 Conservation laws

Lagrangian and adjoint equations for deriving conservation laws of Wu-Zhang system (4.68) by using a new conservation theorem [121] and the nonlinear self-adjointness technique [123] is given as follows.

Let $\mathcal{L}$ denotes formal Lagrangian for given system (4.68) by following expression

$$\begin{aligned}
\mathcal{L}=&\,P(t,x,y)\left(\frac{\partial^\alpha u}{\partial t^\alpha}+uu_x+vu_y+w_x\right)+Q(t,x,y)\left(\frac{\partial^\alpha v}{\partial t^\alpha}+uv_x+vv_y+w_y\right)\\
&+R(t,x,y)\left(\frac{\partial^\alpha w}{\partial t^\alpha}+u_xw+uw_x+v_yw+vw_y+\frac{1}{3}\left(u_{xxx}+u_{xyy}+v_{xxy}+v_{yyy}\right)\right),
\end{aligned} \tag{4.87}$$

where new dependent variables (P,Q,R) are functions t,x,y. Thus the adjoint equations for the system (4.68) are defined as follows

$$\frac{\delta\mathcal{L}}{\delta u}=0,\ \frac{\delta\mathcal{L}}{\delta v}=0,\ \frac{\delta\mathcal{L}}{\delta w}=0, \tag{4.88}$$

where $\dfrac{\delta}{\delta u}$, $\dfrac{\delta}{\delta v}$ and $\dfrac{\delta}{\delta w}$ represent the Euler-Lagrange operators and defined according to the Eq. (2.96) as follows

$$\begin{aligned}
\frac{\delta}{\delta u} &= \partial_u + (D_t^\alpha)^* \partial_{\partial_t^\alpha u} - D_x \partial_{u_x} - D_x^3 \partial_{u_{xxx}} - D_x D_y^2 \partial_{u_{xyy}},\\
\frac{\delta}{\delta v} &= \partial_v + (D_t^\alpha)^* \partial_{\partial_t^\alpha v} - D_x \partial_{v_x} - D_x^2 D_y \partial_{v_{xxy}} - D_y^3 \partial_{v_{yyy}},\\
\frac{\delta}{\delta w} &= \partial_w + (D_t^\alpha)^* \partial_{\partial_t^\alpha w} - D_x \partial_{w_x},
\end{aligned} \tag{4.89}$$

where $(D_t^\alpha)^*$ represent the adjoint operator of (D_t^α).
The adjoint equations are obtained from the Eq. (4.88) as follows

$$\begin{aligned}
&Qv_x - P_x u - R_x w - (D_t^\alpha)^* P - P_y v - P v_y - \frac{1}{3} R_{xxx} - \frac{1}{3} R_{xyy} = 0,\\
&Pu_y - Q_x u - Q u_x - (D_t^\alpha)^* Q - Q_y v - R_y w - \frac{1}{3} R_{xxy} - \frac{1}{3} R_{yyy} = 0,\\
&- P_x - R_x u - (D_t^\alpha)^* R - Q_y - R_y v = 0.
\end{aligned} \tag{4.90}$$

The nonlinear self-adjointness principle [96, 123] is proved for all solution (u, v, w) upon substitution $P = \Phi_5(t, x, y, u, v, w)$, $Q = \Phi_6(t, x, y, u, v, w)$ and $R = \Phi_7(t, x, y, u, v, w)$, the following conditions should be satisfied

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta u}\Big|_{\{P=\Phi_5, Q=\Phi_6, R=\Phi_7\}} &= \lambda_{10}(\frac{\partial^\alpha u}{\partial t^\alpha} + uu_x + vu_y + w_x) + \lambda_{11}(\frac{\partial^\alpha v}{\partial t^\alpha} + uv_x + vv_y + w_y)\\
&\quad + \lambda_{12}(\frac{\partial^\alpha w}{\partial t^\alpha} + u_x w + uw_x + v_y w + vw_y + \frac{1}{3}(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy})),\\
\frac{\delta \mathcal{L}}{\delta v}\Big|_{\{P=\Phi_5, Q=\Phi_6, R=\Phi_7\}} &= \lambda_{13}(\frac{\partial^\alpha u}{\partial t^\alpha} + uu_x + vu_y + w_x) + \lambda_{14}(\frac{\partial^\alpha v}{\partial t^\alpha} + uv_x + vv_y + w_y)\\
&\quad + \lambda_{15}(\frac{\partial^\alpha w}{\partial t^\alpha} + u_x w + uw_x + v_y w + vw_y + \frac{1}{3}(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy})),\\
\frac{\delta \mathcal{L}}{\delta w}\Big|_{\{P=\Phi_5, Q=\Phi_6, R=\Phi_7\}} &= \lambda_{16}(\frac{\partial^\alpha u}{\partial t^\alpha} + uu_x + vu_y + w_x) + \lambda_{17}(\frac{\partial^\alpha v}{\partial t^\alpha} + uv_x + vv_y + w_y)\\
&\quad + \lambda_{18}(\frac{\partial^\alpha w}{\partial t^\alpha} + u_x w + uw_x + v_y w + vw_y + \frac{1}{3}(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy})),
\end{aligned} \tag{4.91}$$

with $\Phi_i(t, x, y, u, v, w)$, $(i = 5, 6, 7)$, not are all zero, simultaneously and λ_i $(i = 10, 11, ..., 18)$ are the regular undetermined coefficients. On comparing the coefficients of u, v, w and their partial derivatives, the over-determining system of equations for Φ_5, Φ_6 and Φ_7 is obtained. After solving the determining equations, the following results are obtained

$$\begin{aligned}&\lambda_i = 0\,(i = 10, 11, ..., 18),\ \ \Phi_5 = 0,\ \ \Phi_6 = 0,\\ &\Phi_7 = B,\end{aligned} \tag{4.92}$$

where B is the arbitrary constant, for simplicity, let us consider $B = 1$.

Thus every Lie point symmetry possesed by Wu-Zhang system (4.68), shows the existence of a conservation law $D_i(C^i) = 0$, whose components C^i are constructed by using formula as given in Chapter 2.

For the construction of conservation laws, formal form of Lagrangian $\mathcal{L}$ expressed in a symmetric form as follows

$$\begin{aligned}\mathcal{L} &= P\Big(\frac{\partial^\alpha u}{\partial t^\alpha} + uu_x + vu_y + w_x\Big) + Q\Big(\frac{\partial^\alpha v}{\partial t^\alpha} + uv_x + vv_y + w_y\Big)\\ &+ R\left(\frac{\partial^\alpha w}{\partial t^\alpha} + u_x w + uw_x + v_y w + vw_y + \frac{1}{3}u_{xxx} + \frac{1}{9}\left(u_{xyy} + u_{yxy} + u_{yyx}\right)\right.\\ &\left.+\frac{1}{9}\left(v_{xxy} + v_{xyx} + v_{yxx}\right) + \frac{1}{3}v_{yyy}\right).\end{aligned} \tag{4.93}$$

For the Lie infinitesimal symmetry generator $X_{13} = \alpha\, x\frac{\partial}{\partial x} - \alpha\, u\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial t} - \alpha\, v\frac{\partial}{\partial v} + \alpha\, y\frac{\partial}{\partial y} - 2\alpha\, w\frac{\partial}{\partial w}$, the Lie characteristic functions attain the following forms

$$\begin{aligned}W_7 &= -\alpha\, u - 2tu_t - \alpha\, xu_x - \alpha\, yu_y,\\ W_8 &= -\alpha\, v - 2tv_t - \alpha\, xv_x - \alpha\, yv_y,\\ W_9 &= -2\alpha\, w - 2tw_t - \alpha\, xw_x - \alpha\, yw_y.\end{aligned} \tag{4.94}$$

In this case, the conservation laws, using Eqs. (4.92), (2.93) and (2.97), are obtained as follows

$$C^t = -2\alpha I_t^{1-\alpha}(w) - 2I_t^{1-\alpha}(tw_t) - \alpha\, xI_t^{1-\alpha}(w_x) - \alpha\, yI_t^{1-\alpha}(w_y). \tag{4.95}$$

$$\begin{aligned}C^x &= (-u\alpha - 2\,tu_t - \alpha\, xu_x - y\alpha\, u_y)\, w - \alpha\, u_{xx} - \frac{1}{3}\,\alpha\, xu_{xxx} - \frac{2}{3}\, tu_{txx}\\ &- \frac{1}{3}\, y\alpha\, u_{xxy} - \frac{1}{3}\,\alpha\, u_{yy} - \frac{1}{9}\,\alpha\, xu_{xyy} - \frac{2}{9}\, tu_{tyy} - \frac{1}{9}\, y\alpha\, u_{yyy} - \frac{2}{3}\,\alpha\, v_{xy}\\ &- \frac{2}{9}\,\alpha\, xv_{xxy} - \frac{4}{9}\, tv_{txy} - \frac{2}{9}\, y\alpha\, v_{xyy} + (-2\, w\alpha - 2\, tw_t - \alpha\, xw_x - y\alpha\, w_y)\, u.\end{aligned} \tag{4.96}$$

$$\begin{aligned} C^y = & -\frac{2}{3}\,\alpha\, u_{xy} - \frac{2}{9}\,\alpha\, x u_{xxy} - \frac{4}{9}\, t u_{txy} - \frac{2}{9}\, y\alpha\, u_{xyy} \\ & + \left(-v\alpha - 2\, t v_t - \alpha\, x v_x - y\alpha\, v_y\right) w - \frac{1}{3}\,\alpha\, v_{xx} - \frac{1}{9}\,\alpha\, x v_{xxx} - \frac{2}{9}\, t v_{txx} \\ & - \frac{1}{9}\, y\alpha\, v_{xxy} - \alpha\, v_{yy} - \frac{1}{3}\,\alpha\, x v_{xyy} - \frac{2}{3}\, t v_{tyy} - \frac{1}{3}\, y\alpha\, v_{yyy} \\ & + \left(-2\, w\alpha - 2\, t w_t - \alpha\, x w_x - y\alpha\, w_y\right) v. \end{aligned} \tag{4.97}$$

For Lie infinitesimal symmetry generator $X_{14} = \frac{\partial}{\partial x}$, the Lie characteristic functions are $W_7 = -u_x$, $W_8 = -v_x$, $W_9 = -w_x$ and the corresponding conservation laws are given by following

$$C^t = -I_t^{1-\alpha}(w_x). \tag{4.98}$$

$$C^x = -u_x w - \frac{1}{3}\, u_{xxx} - \frac{1}{9}\, u_{xyy} - \frac{2}{9}\, v_{xxy} - w_x u. \tag{4.99}$$

$$C^y = -\frac{2}{9}\, u_{xxy} - v_x w - \frac{1}{9}\, v_{x,x,x} - \frac{1}{3}\, v_{xyy} - w_x v. \tag{4.100}$$

Similarly, the infinitesimal symmetry generator $X_{15} = \frac{\partial}{\partial y}$ with $W_7 = -u_y$, $W_8 = -v_y$ and $W_9 = -w_y$, the following conservation laws are obtained

$$C^t = -I_t^{1-\alpha}(w_y). \tag{4.101}$$

$$C^x = -u_y w - \frac{1}{3}\, u_{xxy} - \frac{1}{9}\, u_{yyy} - \frac{2}{9}\, v_{xyy} - w_y u. \tag{4.102}$$

$$C^y = -\frac{2}{9}\, u_{xyy} - v_y w - \frac{1}{9}\, v_{xxy} - \frac{1}{3}\, v_{yyy} - w_y v. \tag{4.103}$$

4.2.4 Exact solutions via exponential rational function method

This subsection describes the way to find water particle velocity (u, v) and elevation w as solutions of time fractional form of Wu-Zhang system (4.68) by applying exponential rational function technique [11, 28, 204, 286, 338]. This method gives the analytical solutions for various types of fractional PDEs as reported in [11, 204]. For solutions, the fractional complex transformation (2.63) converts the given system (4.68) into a following system of ODEs

$$\begin{aligned} & k_3\rho\, f_\zeta + k_1\, f f_\zeta + k_2\, g f_\zeta + k_1\, h_\zeta = 0, \\ & k_3\rho\, g_\zeta + k_1\, f g_\zeta + k_2\, g g_\zeta + k_2\, h_\zeta = 0, \\ & k_3\rho\, h_\zeta + k_1\, f h_\zeta + k_1\, h f_\zeta + k_2\, g h_\zeta + k_2 h g_\zeta + \frac{1}{3}\left(\left(k_1^3 + k_1 k_2^2\right) f_{\zeta\zeta\zeta} + \left(k_1^2 k_2 + k_2^3\right) g_{\zeta\zeta\zeta}\right) = 0. \end{aligned} \tag{4.104}$$

The solution of ODEs (4.104) according to Eq. (2.69) can be written as follows

$$f(\zeta)=\mathcal{A}_0+\frac{\mathcal{A}_1}{(1+e^{\zeta})},\ g(\zeta)=\mathcal{B}_0+\frac{\mathcal{B}_1}{(1+e^{\zeta})},\ h(\zeta)=\mathcal{C}_0+\frac{\mathcal{C}_1}{(1+e^{\zeta})}+\frac{\mathcal{C}_2}{(1+e^{\zeta})^2}. \tag{4.105}$$

The back substitution of the solutions (4.105) into ODEs (4.104) and produces a set of polynomial equations in $e^{j\zeta}$. The coefficients of different powers of e^{ζ} have been equated to zero for getting algebraic equations in terms of unknown parameters $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$. By using symbolic computational software Maple, the different solutions of algebraic equations are obtained and discussed in following cases.

Case VII

$$\begin{aligned}&\mathcal{A}_0=-\left(\frac{k_1^2+k_2^2}{\sqrt{3}k_1}+\frac{k_3\rho+k_2\mathcal{B}_0}{k_1}\right),\ \mathcal{A}_1=\frac{2k_1}{\sqrt{3}},\ \mathcal{B}_1=\frac{2k_2}{\sqrt{3}},\ \mathcal{C}_0=0,\\&\mathcal{C}_1=\frac{2}{3}\left({k_1}^2+{k_2}^2\right),\ \mathcal{C}_2=-\frac{2}{3}\left({k_1}^2+{k_2}^2\right).\end{aligned} \tag{4.106}$$

Case VIII

$$\mathcal{A}_0=-\frac{k_3\rho+k_2\mathcal{B}_0}{k_1},\ \mathcal{A}_1=-\frac{k_2\mathcal{B}_1}{k_1},\ \mathcal{C}_1=0,\ \mathcal{C}_2=0. \tag{4.107}$$

Using the above results for relations given in Eq. (4.105), solutions of ODEs are apreared as follows

$$\begin{aligned}&f(\zeta)=-\left(\frac{k_1^2+k_2^2}{\sqrt{3}k_1}+\frac{k_3\rho+k_2\mathcal{B}_0}{k_1}\right)+\frac{2}{\sqrt{3}}\frac{k_1}{1+\mathrm{e}^{\zeta}},\ g(\zeta)=\mathcal{B}_0+\frac{2}{\sqrt{3}}\frac{k_2}{1+\mathrm{e}^{\zeta}},\\&h(\zeta)=\frac{2({k_1}^2+{k_2}^2)}{3\left(1+\mathrm{e}^{\zeta}\right)}-\frac{2({k_1}^2+{k_2}^2)}{3\left(1+\mathrm{e}^{\zeta}\right)^2},\end{aligned} \tag{4.108}$$

and

$$f(\zeta)=-\frac{k_3\rho+k_2\mathcal{B}_0}{k_1}-\frac{k_2\mathcal{B}_1}{k_1\left(1+\mathrm{e}^{\zeta}\right)},\ g(\zeta)=\mathcal{B}_0+\frac{\mathcal{B}_1}{1+\mathrm{e}^{\zeta}},\ h(\zeta)=\mathcal{C}_0. \tag{4.109}$$

The back substitution of solutions of ODEs into the Eq. (2.63), the following horizontal projection of water particle velocity u, v and the elevation w as exact solutions of fractional Wu-Zhang system (4.68) are obtained

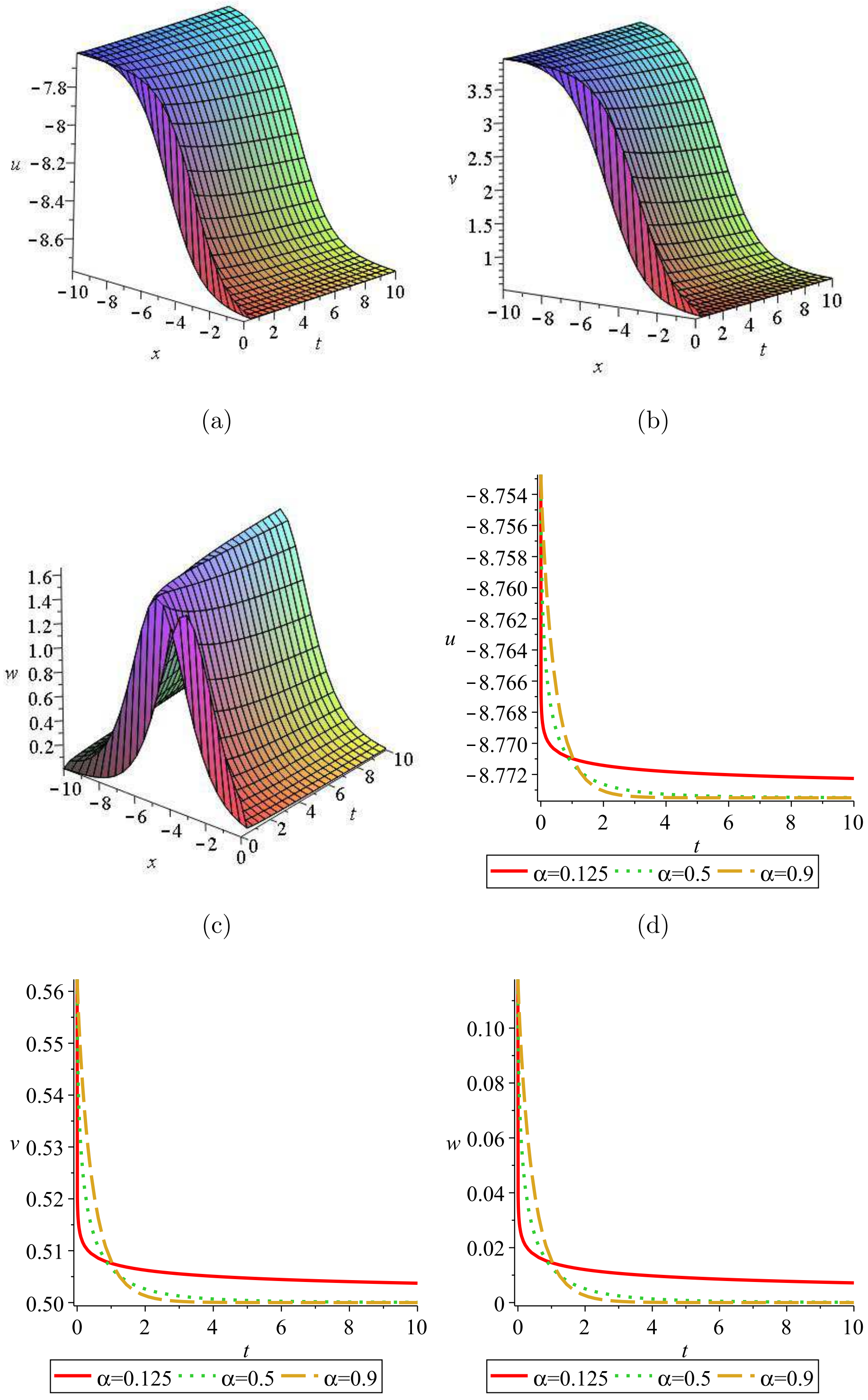
$$
\begin{aligned}
u(t,x,y) &= -\left(\frac{k_1^2+k_2^2}{\sqrt{3}k_1}+\frac{k_3\rho+k_2\mathcal{B}_0}{k_1}\right)+\frac{2k_1}{\sqrt{3}\left(1+\mathrm{e}^{k_1x+k_2y+\frac{k_3t^\alpha}{\Gamma(\alpha+1)}}\right)},\\
v(t,x,y) &= \mathcal{B}_0+\frac{2k_2}{\sqrt{3}\left(1+\mathrm{e}^{k_1x+k_2y+\frac{k_3t^\alpha}{\Gamma(\alpha+1)}}\right)},\\
w(t,x,y) &= \frac{2(k_1^2+k_2^2)}{3\left(1+\mathrm{e}^{k_1x+k_2y+\frac{k_3t^\alpha}{\Gamma(\alpha+1)}}\right)}-\frac{2(k_1^2+k_2^2)}{3\left(1+\mathrm{e}^{k_1x+k_2y+\frac{k_3t^\alpha}{\Gamma(\alpha+1)}}\right)^2},
\end{aligned}
\tag{4.110}
$$

and

$$
\begin{aligned}
u(t,x,y) &= -\frac{k_3\rho+k_2\mathcal{B}_0}{k_1}-\frac{k_2\mathcal{B}_1}{k_1\left(1+\mathrm{e}^{k_1x+k_2y+\frac{k_3t^\alpha}{\Gamma(\alpha+1)}}\right)},\\
v(t,x,y) &= \mathcal{B}_0+\frac{\mathcal{B}_1}{\left(1+\mathrm{e}^{k_1x+k_2y+\frac{k_3t^\alpha}{\Gamma(\alpha+1)}}\right)},\\
w(t,x,y) &= \mathcal{C}_0.
\end{aligned}
\tag{4.111}
$$

4.2.4.1 Physical interpretation of the solutions of Wu-Zhang system

The graphical representation of the solutions (4.110) is discussed in this subsection. The values of various parameters selected for plotting the graphs of the solutions are respectively given in the corresponding figure captions. Figure 4.11 shows 3D and 2D plots for the solution given in Eq.(4.110) for dependent variables u, v and w. Figure 4.11(a,b,c) shows 3D plots for the water particle velocity u, v and the elevation w, respectively at $\alpha = 0.125$. Figure 4.11(a,b) depicts the kink wave profile for the velocity represented through u and v variables. Figure 4.11(c) shows the bell type profile for the solution obtained for the variable w. Figure 4.11(d,e,f) shows the variation of water particle velocity (u, v) and elevation (w) of Wu-Zhang system for different values of fractional order α as 0.125, 0.5 and 0.9 by 2D plots. The Figure 4.11(d,e,f) shows that water particle velocity and elevation decrease with an increase in the variation of fractional order α values. Figure 4.12 represents the variation in wave profile of the solution (4.110) for u, v and w variables as a function of α having other parameters (t,x,y) fixed as $(1,1,1)$ and (3,1,1).

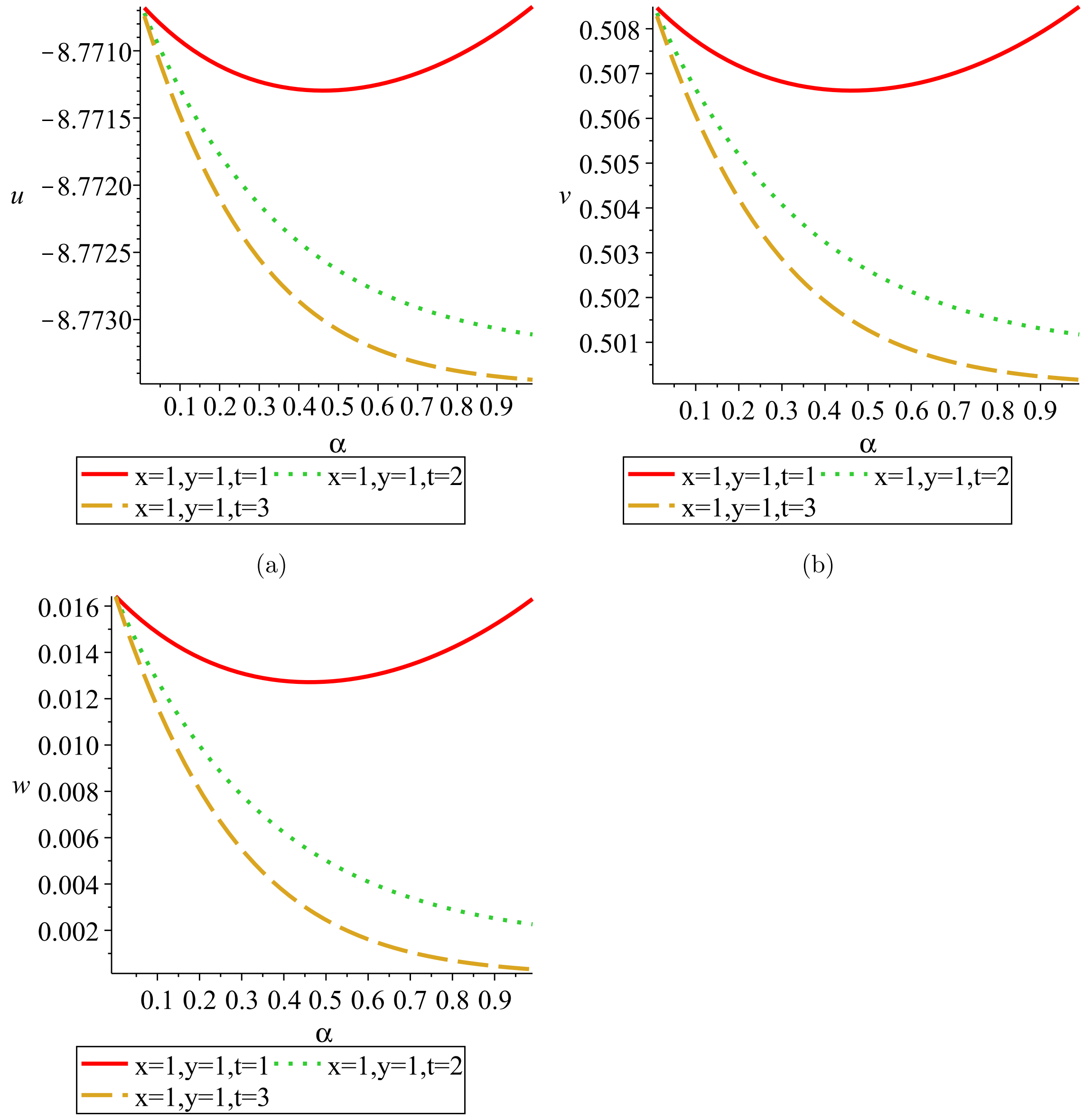


(a) (b)

(c) (d)

(e) (f)

Figure 4.11: Shows 3D representations of solution (4.110) (a) kink wave profile for u at $y = 1$, $\alpha = 0.125$ (b) kink wave profile for v at $y = 1$, $\alpha = 0.125$ and (c) bell shape profile for w at $y = 1$, $\alpha = 0.125$. 2D representations describe effect of fractional order α in (d), (e) and (f) for $k_1 = 1$, $k_2 = 3$, $k_3 = 2$, $\rho = 0.75$, $\mathcal{B}_0 = 0.5$, $x = 1$, $y = 1$.



(a) (b)

(c)

Figure 4.12: Effect of α on the solution (4.110) with $k_1 = 1$, $k_2 = 3$, $k_3 = 2$, $\rho = 0.75$, $\mathcal{B}_0 = 0.5$.

4.2.5 Conclusion

The (2+1)-dimensional time fractional Wu-Zhang system derived for dispersive long waves in shallow water model has been investigated for its dispersion analysis, Lie point symmetries and conservation laws. The dispersion relation signifies the anomalous dispersion of waves for the system. Similarity reductions reduce the system to (1+1)-dimensional fractional PDEs. The new conserved density and fluxes for Wu-Zhang system are obtained using nonlinear self-adjointness and new conservation theorem. Exact solutions are obtained by using exponential rational function method and analysed graphically to investigate the effect of fractional order α. The wave profile for the velocity of water particle and elevation in Wu-Zhang system is of kink and bell type, respectively. The 2D and 3D plots of solutions show that water particle velocity decreases with an increase in fractional order α.

4.3 5^{th} order equation from Burgers hierarchy having fractional order derivative with respect to time

In this section, the time fractional form of 5^{th} order equation from Burgers hierarchy [2, 38, 97, 310, 333] is studied for explicit solutions, symmetries, linear analysis and conservation laws. Burgers hierarchy with time derivative of fractional order can be determined from the following equation

$$D_t^{\alpha} u + \mu D_x (D_x + u)^{\upsilon} u = 0, \ \upsilon = 0, 1, 2, ..., \tag{4.112}$$

where μ is the arbitrary constant, D_t^{α} denotes α^{th} order RLF derivative, and α lies between 0 and 1. Fractional Burgers hierarchy (4.112) of order α is the generalisation of Burgers hierarchy of integer order [82, 159, 160, 310] given by

$$D_t u + \mu D_x (D_x + u)^{\upsilon} u = 0, \ \upsilon = 0, 1, 2, ..., . \tag{4.113}$$

The Burgers equation balances the dissipation and the nonlinear convection processes [308]. It efficiently describes the models of fluid mechanics, traffic flow, nonlinear acoustic transmission and gas dynamics [308]. The 2^{nd} and 3^{rd} order time fractional equations obtained from the Burgers hierarchy have investigated for the invariant solutions by Lie group method [245, 299]. The multiwave solutions were obtained in the investigation of 4^{th} and 5^{th} order space-time PDEs from the Burgers hierarchy [2].

The time fractional 5^{th} order equation from Burgers hierarchy found by substituting $s = 4$ in the Eq. (4.112) is given by

$$\begin{aligned} &D_t^{\alpha} u + \mu \Big(u_{xxxxx} + 10\, {u_{xx}}^2 + 15\, u_x u_{xxx} + 5\, u u_{xxxx} + 15\, {u_x}^3 + 50\, u u_x u_{xx} \\ &+ 10\, u^2 u_{xxx} + 30\, u^2 {u_x}^2 + 10\, u^3 u_{xx} + 5\, u^4 u_x \Big) = 0,\ 0 < \alpha < 1. \end{aligned} \tag{4.114}$$

The Eq. (4.114) derived using Burger's hierarchy is not investigated yet for the linear dispersion properties, symmetries, explicit solutions, and conservation laws. So, the main thrust in this section is to study the Eq. (4.114) for all above mentioned properties.

4.3.1 Linear dispersion analysis

The dispersion relation has been obtained in the following form using the detailed procedure described in section 2.15

$$\bar{\omega}(s) = \mu^{\frac{1}{\alpha}} s^{\frac{5}{\alpha}} i^{-1+\frac{1}{\alpha}},\ \alpha \neq 1. \tag{4.115}$$

It is to be noticed that the dispersion relation is complex in nature and its real as well as the imaginary parts for $s > 0$ are obtained as follows

$$\begin{aligned} \mathtt{Re}(\bar{\omega}(s)) &= \cos((\frac{1}{\alpha} - 1)\frac{\pi}{2})\mu^{\frac{1}{\alpha}} s^{\frac{5}{\alpha}}, \\ \mathtt{Im}(\bar{\omega}(s)) &= \sin((\frac{1}{\alpha} - 1)\frac{\pi}{2})\mu^{\frac{1}{\alpha}} s^{\frac{5}{\alpha}}. \end{aligned} \tag{4.116}$$

The complex forms of the phase and the group velocities for Eq. (4.114) have been obtained as follows

$$\begin{aligned} \bar{v_p}(s) &= \frac{\bar{\omega}(s)}{s}, \\ &= \mu^{\frac{1}{\alpha}} s^{\frac{5}{\alpha}-1} i^{\frac{1}{\alpha}-1}, \end{aligned} \tag{4.117}$$

and

$$\begin{aligned} \bar{v_g}(s) &= \frac{\partial}{\partial s}\bar{\omega}(s), \\ &= \frac{5}{\alpha}\mu^{\frac{1}{\alpha}} s^{\frac{5}{\alpha}-1} i^{\frac{1}{\alpha}-1}. \end{aligned} \tag{4.118}$$

The real as well as imaginary parts of phase and group velocities are obtained as follows

$$\begin{aligned} v_p(s) = \mathtt{Re}(\bar{v_p}(s)) &= \cos((\frac{1}{\alpha}-1)\frac{\pi}{2})\mu^{\frac{1}{\alpha}}s^{\frac{5}{\alpha}-1}, \\ = \mathtt{Im}(\bar{v_p}(s)) &= \sin((\frac{1}{\alpha}-1)\frac{\pi}{2})\mu^{\frac{1}{\alpha}}s^{\frac{5}{\alpha}-1}, \end{aligned} \tag{4.119}$$

and

$$\begin{aligned} v_g(s) = \mathtt{Re}(\bar{v_g}(s)) &= \frac{5}{\alpha}\cos((\frac{1}{\alpha}-1)\frac{\pi}{2})\mu^{\frac{1}{\alpha}}s^{\frac{5}{\alpha}-1}, \\ = \mathtt{Im}(\bar{v_g}(s)) &= \frac{5}{\alpha}\sin((\frac{1}{\alpha}-1)\frac{\pi}{2})\mu^{\frac{1}{\alpha}}s^{\frac{5}{\alpha}-1}. \end{aligned} \tag{4.120}$$

The variation of phase and group velocities with respect to wave vector s for given equation is shown in Figure 4.13 and Figure 4.14 for $\alpha = 0.75$ and $\alpha = 0.5$, respectively. The phase and group velocities are related to each other by relation $v_g = v_p + s\,\frac{dv_p}{ds}$. It has been found that the group velocity magnitude is higher than magnitude of phase velocity for all s values. Thus, the waves follow an anomalous dispersion and the long wavelengths propagate comparatively slower than short wavelengths. The variation of v_p and v_g with α for $s = 1$, $\mu = 1$ is shown in Figure 4.15 and it reveals that there are number of α values lie between 0 and 1 for which real parts of v_p and v_g approache zero and it forces the waves to travel in opposite directions. The real parts of the phase and group velocity vanishe at $\alpha = \frac{1}{2(m+1)}$, where m is an integer having values 0, 1, 2,... and the corresponding α values are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\cdots$. Similarly, the α values for which the imaginary parts of the v_p and v_g become zero are given by $\frac{1}{2n+1}$, where n is an integer. It is to be noted that imaginary part of the $\bar{\omega}(s)$ corresponds to the damping of the waves as they propagates in space with time. The damping in the propagation of waves will take place except at the points where $\alpha = \frac{1}{2n+1}$, $n = 1, 2, 3, \ldots.$

4.3.2 Lie point symmetries

This subsection provides the Lie point symmetries and reductions [94, 95, 118, 245, 299] for the Eq. (4.114) with the help of RLF derivative [157, 226] of order α.

The admitted Lie algebra for the Eq. (4.114) under 1LGTs is generated by following IG

$$X = \mathcal{X}_{12}\partial_t + \mathcal{X}_{13}\partial_x + \mathcal{Z}_{11}\partial_u, \tag{4.121}$$

where $\mathcal{X}_{12}$, $\mathcal{X}_{13}$ and $\mathcal{Z}_{11}$ are infinitesimals corresponding to t, x and u, respectively. If the IG (4.121) is assigned as the Lie point symmetry of the Eq. (4.114) then it must satisfies the invariance condition. A little simplification of the invariance condition,

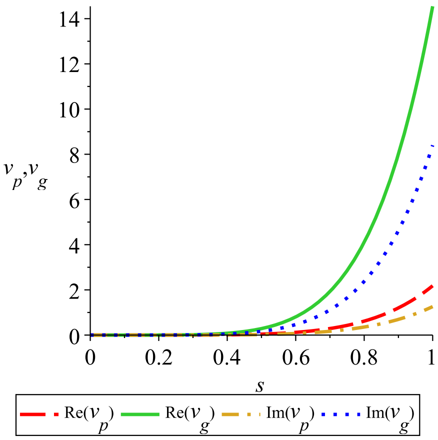


Figure 4.13: Variation of phase and group velocities with s at $\alpha = 0.75$.

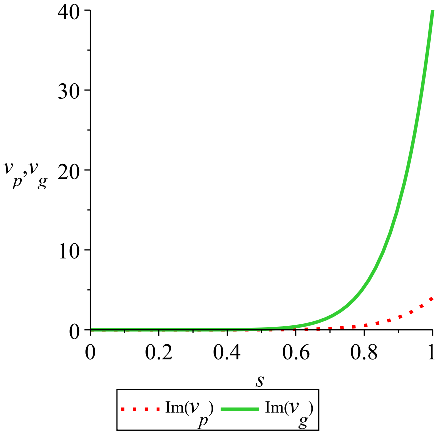


Figure 4.14: Variation of phase and group velocities with s at $\alpha = 0.5$.

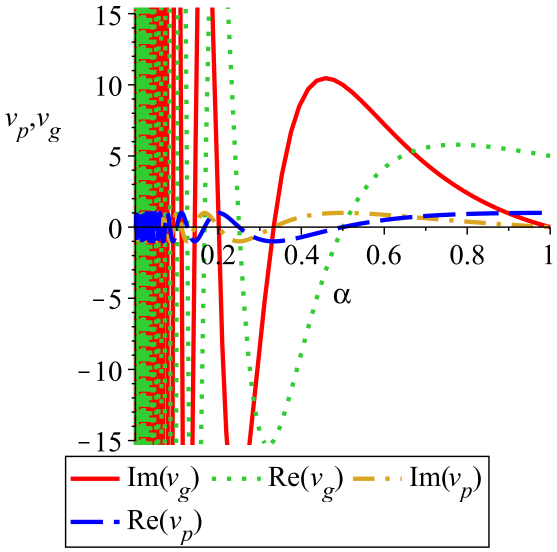


Figure 4.15: Variation of phase and group velocities with α having $\mu = 1$, $s = 1$.

further by equating the coefficients of dependent variable and their derivatives, results into the determining equations for $\mathcal{X}_{12}$, $\mathcal{X}_{13}$ and $\mathcal{Z}_{11}$. The solution of determining equations will provide the following infinitesimals

$$\mathcal{X}_{12} = \frac{5t}{\alpha}e_1,\ \mathcal{X}_{13} = e_1 x + e_2,\ \mathcal{Z}_{11} = -u e_1, \tag{4.122}$$

where e_1 and e_2 are randomly selected as constants. The associated Lie algebra obtained from above infinitesimals for Eq. (4.114) constitutes following two vector fields

$$X_{16} = \partial_x,\ X_{17} = -u\partial_u + \frac{5t}{\alpha}\partial_t + x\partial_x. \tag{4.123}$$

Thus the invariant solutions and the similarity reductions are compiled in the following cases.

Case IX: Vector field $X_{16} = \partial_x$ gives the invariant solution for the Eq. (4.114) as given by

$$u(x,t) = F(t). \tag{4.124}$$

The reduced fractional ODE obtained from the invariant solution is retrieved as follows

$$\frac{\partial^\alpha F(t)}{\partial t^\alpha} = 0. \tag{4.125}$$

The solution for reduced fractional ODE is obtained in the form as given by

$$u(x,t) = g_1\, t^{\alpha-1}, \tag{4.126}$$

where g_1 is an arbitrary constant.

Case X: For case $X_{17} = -u\partial_u + \frac{5t}{\alpha}\partial_t + x\partial_x$, associated characteristic equations are given as follows

$$\frac{dx}{x} = \frac{dt}{\frac{5t}{\alpha}} = \frac{du}{-u}. \tag{4.127}$$

Using solution of above characteristic equations, the following forms of invariants are determined

$$\zeta = \frac{x}{t^{\frac{\alpha}{5}}},\ u = F(\zeta)t^{-\frac{\alpha}{5}}. \tag{4.128}$$

The substitution of above invariants into Eq. (4.114) provides nonlinear ODE having fractional order by adopting procedure as given in previous section 4.1. Hence, the Eq. (4.114) is reduces to nonlinear fractional ODE in following form

$$\begin{aligned}(P_{\frac{5}{\alpha}}^{1-6\frac{\alpha}{5},\alpha}F)(\zeta) + \Big(&10\,F^3F_{\zeta\zeta} + 5\,F^4F_\zeta + 10\,F^2F_{\zeta\zeta\zeta} + 30\,F^2{F_\zeta}^2 + F_{\zeta\zeta\zeta\zeta\zeta} + 10\,{F_{\zeta\zeta}}^2\\ &+ 15\,F_\zeta F_{\zeta\zeta\zeta} + 5\,FF_{\zeta\zeta\zeta\zeta} + 15\,{F_\zeta}^3 + 50\,FF_\zeta F_{\zeta\zeta}\Big)\mu = 0,\end{aligned} \tag{4.129}$$

where $\left(P_{\frac{5}{\alpha}}^{1-\frac{6\alpha}{5},\alpha}F\right)$ is the Erdélyi-Kober fractional differential operator [157].

4.3.3 Explicit power series solution of the reduced nonlinear fractional ODE (4.129)

This subsection describes the procedure to find the explicit power series solutions to the Eq. (4.129), as this method is an excellent way to obtain the solution for fractional ODEs [86, 131–133, 234]. The Eq. (4.129) is considered to solve by power series as follows

$$F(\zeta)=\sum_{\sigma_7=0}^{\infty} a_{\sigma_7}\zeta^{\sigma_7}. \tag{4.130}$$

On substitution of Eq. (4.130) into the Eq. (4.129) and for $\sigma_7=0$, it gives

$$\begin{aligned}a_5 = &-\frac{1}{120}\,\frac{\Gamma\left(1-\frac{1}{5}\,\alpha\right)a_0}{\mu\,\Gamma\left(1-\frac{6}{5}\,\alpha\right)}-\frac{1}{120}\,\left(20\,{a_0}^3a_2+5\,{a_0}^4a_1+60\,{a_0}^2a_3\right.\\ &\left.+30\,{a_0}^2{a_1}^2+40\,{a_2}^2+90\,a_1a_3+120\,a_0a_4+15\,{a_1}^3+100\,a_0a_1a_2\right),\end{aligned} \tag{4.131}$$

where a_0, a_1, a_2, a_3 and a_4 are arbitrary constants. For $n\geq 1$, following recurrence relation is obtained

$$\begin{aligned}a_{\sigma_7+5} = &-\frac{1}{(\sigma_7+5)(\sigma_7+4)(\sigma_7+3)(\sigma_7+2)(\sigma_7+1)}\left(\frac{\Gamma\left(1-\frac{1}{5}\,\alpha-\frac{1}{5}\,\sigma_7\alpha\right)a_{\sigma_7}}{\mu\,\Gamma\left(1-\frac{6}{5}\,\alpha-\frac{1}{5}\,\sigma_7\alpha\right)}\right.\\ &+10\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9}\sum_{\sigma_{10}=0}^{\sigma_8} a_{\sigma_{10}}a_{\sigma_8-\sigma_{10}}a_{\sigma_9-\sigma_8}(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+2}\\ &+5\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9}\sum_{\sigma_{10}=0}^{\sigma_8}\sum_{m=0}^{\sigma_{10}} a_m a_{\sigma_{10}-m}a_{\sigma_8-\sigma_{10}}a_{\sigma_9-\sigma_8}(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+1}\\ &+10\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9} a_{\sigma_8}a_{\sigma_9-\sigma_8}(\sigma_7-\sigma_9+3)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+3}\\ &+30\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9}\sum_{\sigma_{10}=0}^{\sigma_8} a_{\sigma_{10}}a_{\sigma_8-\sigma_{10}}(\sigma_9-\sigma_8+1)\,a_{\sigma_9-\sigma_8+1}(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+1}\\ &+10\sum_{\sigma_9=0}^{\sigma_7}(\sigma_9+2)(\sigma_9+1)\,a_{\sigma_9+2}(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+2}\\ &+15\sum_{\sigma_9=0}^{\sigma_7}(\sigma_9+1)\,a_{\sigma_9+1}(\sigma_7-\sigma_9+3)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+3}\\ &+5\sum_{\sigma_9=0}^{\sigma_7}a_{\sigma_9}(\sigma_7-\sigma_9+4)(\sigma_7-\sigma_9+3)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+4}\\ &+15\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9}(\sigma_8+1)\,a_{\sigma_8+1}(\sigma_9-\sigma_8+1)\,a_{\sigma_9-\sigma_8+1}(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+1}\\ &\left.+50\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9}a_{\sigma_8}a_{\sigma_9-\sigma_8+1}(\sigma_9-\sigma_8+1)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+2}\right),\end{aligned} \tag{4.132}$$

$\sigma_7 = 0, 1, 2, ...$ Thus, other terms in sequence $\{a_{\sigma_7}\}_{\sigma_7=6}^{\infty}$ can be determined successively using Eqs. (4.131)-(4.132) in terms of arbitrary constants. Thus, solution for the Eq. (4.129) with the coefficients as given in Eqs (4.131) and (4.132) can be expressed as follows

$$
\begin{aligned}
F(\zeta) &= a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + a_5\zeta^5 + \sum_{\sigma_7=1}^{\infty} a_{\sigma_7+5}\zeta^{\sigma_7+5} \\
&= a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 - \frac{1}{120}\left(\frac{\Gamma\left(1-\frac{1}{5}\,\alpha\right)a_0}{\mu\,\Gamma\left(1-\frac{6}{5}\,\alpha\right)}\right. \\
&+20\,{a_0}^3 a_2 + 5\,{a_0}^4 a_1 + 60\,{a_0}^2 a_3 + 30\,{a_0}^2{a_1}^2 + 40\,{a_2}^2 + 90\,a_1a_3 \\
&\left.+120\,a_0a_4 + 15\,{a_1}^3 + 100\,a_0a_1a_2\right)\zeta^5 \\
&-\left(\sum_{\sigma_7=1}^{\infty}\frac{1}{(\sigma_7+5)(\sigma_7+4)(\sigma_7+3)(\sigma_7+2)(\sigma_7+1)}\left(\frac{\Gamma\left(1-\frac{1}{5}\,\alpha-\frac{1}{5}\,\sigma_7\alpha\right)a_{\sigma_7}}{\mu\,\Gamma\left(1-\frac{6}{5}\,\alpha-\frac{1}{5}\,\sigma_7\alpha\right)}\right.\right. \\
&+10\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9}\sum_{\sigma_{10}=0}^{\sigma_8} a_{\sigma_{10}}a_{\sigma_8-\sigma_{10}}a_{\sigma_9-\sigma_8}(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+2} \\
&+5\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9}\sum_{\sigma_{10}=0}^{\sigma_8}\sum_{m=0}^{\sigma_{10}} a_m a_{\sigma_{10}-m}a_{\sigma_8-\sigma_{10}}a_{\sigma_9-\sigma_8}(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+1} \\
&+10\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9} a_{\sigma_8}a_{\sigma_9-\sigma_8}(\sigma_7-\sigma_9+3)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+3} \\
&+30\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9}\sum_{\sigma_{10}=0}^{\sigma_8} a_{\sigma_{10}}a_{\sigma_8-\sigma_{10}}(\sigma_9-\sigma_8+1)\,a_{\sigma_9-\sigma_8+1}(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+1} \\
&+10\sum_{\sigma_9=0}^{\sigma_7}(\sigma_9+2)(\sigma_9+1)\,a_{\sigma_9+2}(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+2} \\
&+15\sum_{\sigma_9=0}^{\sigma_7}(\sigma_9+1)\,a_{\sigma_9+1}(\sigma_7-\sigma_9+3)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+3} \\
&+5\sum_{\sigma_9=0}^{\sigma_7} a_{\sigma_9}(\sigma_7-\sigma_9+4)(\sigma_7-\sigma_9+3)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+4} \\
&+15\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9}(\sigma_8+1)\,a_{\sigma_8+1}(\sigma_9-\sigma_8+1)\,a_{\sigma_9-\sigma_8+1}(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+1} \\
&\left.\left.+50\sum_{\sigma_9=0}^{\sigma_7}\sum_{\sigma_8=0}^{\sigma_9} a_{\sigma_8}a_{\sigma_9-\sigma_8+1}(\sigma_9-\sigma_8+1)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\,a_{\sigma_7-\sigma_9+2}\zeta^{\sigma_7+5}\right)\right).
\end{aligned}
\tag{4.133}
$$

Hence, the explicit power series solution for given Eq. (4.114) can be expressed as follows

$$\begin{aligned}
u &= \sum_{\sigma_7=0}^{5} a_{\sigma_7} x^{\sigma_7} t^{-(\sigma_7+1)\alpha/5} + \sum_{\sigma_7=1}^{\imath\sigma_7 fty} a_{\sigma_7+5} x^{\sigma_7+5} t^{\frac{-(\sigma_7+5)\alpha}{5}} \\
&= a_0 t^{\frac{-\alpha}{5}} + a_1 x t^{\frac{-2\alpha}{5}} + a_2 x^2 t^{\frac{-3\alpha}{5}} + a_3 x^3 t^{\frac{-4\alpha}{5}} + a_4 x^4 t^{-\alpha} \\
&- \frac{1}{120} \left(\frac{\Gamma\left(1 - \frac{1}{5}\,\alpha\right) a_0}{\mu\,\Gamma\left(1 - \frac{6}{5}\,\alpha\right)} + 20\,{a_0}^3 a_2 + 5\,{a_0}^4 a_1 + 60\,{a_0}^2 a_3 + 30\,{a_0}^2 {a_1}^2 + 40\,{a_2}^2 \right. \\
&\left. +90\,a_1 a_3 + 120\,a_0 a_4 + 15\,{a_1}^3 + 100\,a_0 a_1 a_2 \right) x^5 t^{\frac{-6\alpha}{5}} \\
&- \left(\sum_{\sigma_7=1}^{\imath\sigma_7 fty} \frac{1}{(\sigma_7+5)(\sigma_7+4)(\sigma_7+3)(\sigma_7+2)(\sigma_7+1)} \left(\frac{\Gamma\left(1 - \frac{1}{5}\,\alpha - \frac{1}{5}\,\sigma_7\alpha\right) a_{\sigma_7}}{\mu\,\Gamma\left(1 - \frac{6}{5}\,\alpha - \frac{1}{5}\,\sigma_7\alpha\right)} \right. \right. \\
&+10 \sum_{\sigma_9=0}^{\sigma_7} \sum_{\sigma_8=0}^{\sigma_9} \sum_{\sigma_{10}=0}^{\sigma_8} a_{\sigma_{10}} a_{\sigma_8-\sigma_{10}} a_{\sigma_9-\sigma_8} (\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\, a_{\sigma_7-\sigma_9+2} \\
&+5 \sum_{\sigma_9=0}^{\sigma_7} \sum_{\sigma_8=0}^{\sigma_9} \sum_{\sigma_{10}=0}^{\sigma_8} \sum_{m=0}^{\sigma_{10}} a_m a_{\sigma_{10}-m} a_{\sigma_8-\sigma_{10}} a_{\sigma_9-\sigma_8} (\sigma_7-\sigma_9+1)\, a_{\sigma_7-\sigma_9+1} \\
&+10 \sum_{\sigma_9=0}^{\sigma_7} \sum_{\sigma_8=0}^{\sigma_9} a_{\sigma_8} a_{\sigma_9-\sigma_8} (\sigma_7-\sigma_9+3)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\, a_{\sigma_7-\sigma_9+3} \\
&+30 \sum_{\sigma_9=0}^{\sigma_7} \sum_{\sigma_8=0}^{\sigma_9} \sum_{\sigma_{10}=0}^{\sigma_8} a_{\sigma_{10}} a_{\sigma_8-\sigma_{10}} (\sigma_9-\sigma_8+1)\, a_{\sigma_9-\sigma_8+1} (\sigma_7-\sigma_9+1)\, a_{\sigma_7-\sigma_9+1} \\
&+10 \sum_{\sigma_9=0}^{\sigma_7} (\sigma_9+2)(\sigma_9+1)\, a_{\sigma_9+2} (\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\, a_{\sigma_7-\sigma_9+2} \\
&+15 \sum_{\sigma_9=0}^{\sigma_7} (\sigma_9+1)\, a_{\sigma_9+1} (\sigma_7-\sigma_9+3)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\, a_{\sigma_7-\sigma_9+3} \\
&+5 \sum_{\sigma_9=0}^{\sigma_7} a_{\sigma_9} (\sigma_7-\sigma_9+4)(\sigma_7-\sigma_9+3)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\, a_{\sigma_7-\sigma_9+4} \\
&+15 \sum_{\sigma_9=0}^{\sigma_7} \sum_{\sigma_8=0}^{\sigma_9} (\sigma_8+1)\, a_{\sigma_8+1} (\sigma_9-\sigma_8+1)\, a_{\sigma_9-\sigma_8+1} (\sigma_7-\sigma_9+1)\, a_{\sigma_7-\sigma_9+1} \\
&\left. +50 \sum_{\sigma_9=0}^{\sigma_7} \sum_{\sigma_8=0}^{\sigma_9} a_{\sigma_8} a_{\sigma_9-\sigma_8+1} (\sigma_9-\sigma_8+1)(\sigma_7-\sigma_9+2)(\sigma_7-\sigma_9+1)\, a_{\sigma_7-\sigma_9+2} \right) \\
&\left. x^{\sigma_7+5} t^{\frac{-(\sigma_7+5)\alpha}{5}} \right).
\end{aligned} \tag{4.134}$$

The obtained power series solution given in Eq. (4.134) has been analysed graphically in terms of 2D and 3D wave profiles. The figure caption provides the details of various

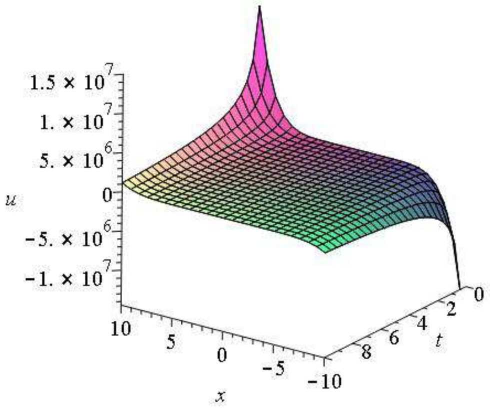


Figure 4.16: 3D plot of the solution (4.134) with $a_i = 1$, $i = 1, 2, 3, 4$, $a_5 = -4$, $a_6 = 0.17$, $a_7 = 0.76$, $\mu = 2$, $\alpha = 0.5$, $n = 0$ to 7.

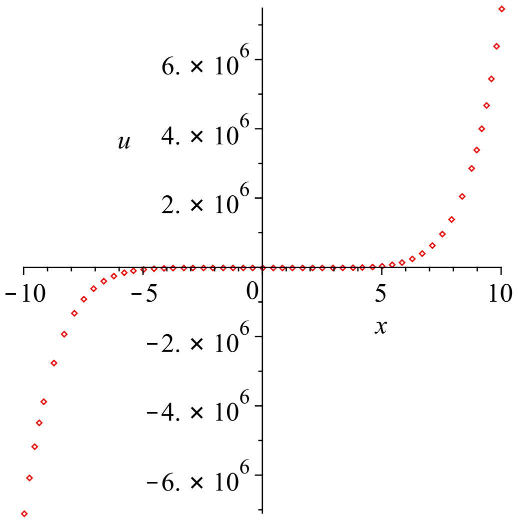


Figure 4.17: 2D plot of the solution (4.134) with $a_i = 1$, $i = 1, 2, 3, 4$, $a_5 = -4$, $a_6 = 0.17$, $a_7 = 0.76$, $\mu = 2$, $\alpha = 0.5$, $t = 1$, $n = 0$ to 7.

parameters selected for plotting the curves. Figure 4.16 and 4.17 correspond to the 3D and 2D plots, respectively for the solution when $\alpha = 0.5$. The Figures 4.18-4.19 and Figures 4.20-4.21 represent wave pattern for $\alpha = 0.75$ and $\alpha = 0.9$, respectively in 3D and 2D plots.

4.3.4 Conservation laws

For conservation laws, formation of Lagrangian for Eq. (4.114) is expressed as follows

$$\begin{aligned}\mathcal{L} = \Upsilon(x,t)\Big(D_t^{\alpha}u + \mu\Big(&u_{xxxxx} + 10\,{u_{xx}}^{2} + 15\,u_x u_{3x} + 5\,u u_{xxxx} + 15\,{u_x}^{3} + 50\,u u_x u_{xx} \\ &+ 10\,u^2 u_{xxx} + 30\,u^2 {u_x}^{2} + 10\,u^3 u_{xx} + 5\,u^4 u_x\Big)\Big),\end{aligned} \tag{4.135}$$

where $\Upsilon(x,t)$ is taken as new dependent variable.
The $\frac{\delta\mathcal{L}}{\delta u} = 0$ gives adjoint equation [121] to Eq. (4.114), where $\dfrac{\delta}{\delta u}$ represents Euler-Lagrange operator [96, 185] and given in Eq. (2.96). Using Eq. (4.135), the adjoint equation for Eq. (4.114) can be written as follows

$$\begin{aligned}&\big(-10\,\Upsilon_{xx} u u_x - 10\,\Upsilon_x u u_{xx} + 5\,\Upsilon_{xxx} u_x + 5\,\Upsilon_{xxxx} u + 5\,\Upsilon_{xx} u_{xx} - \Upsilon_{xxxxx} + 10\,\Upsilon_{xx} u^3 \\ &-5\,\Upsilon_x u^4 - 10\,\Upsilon_{xxx} u^2 - 5\,\Upsilon_x {u_x}^{2}\big)\,\mu - (D_t^{\alpha})^{*}\Upsilon = 0.\end{aligned} \tag{4.136}$$

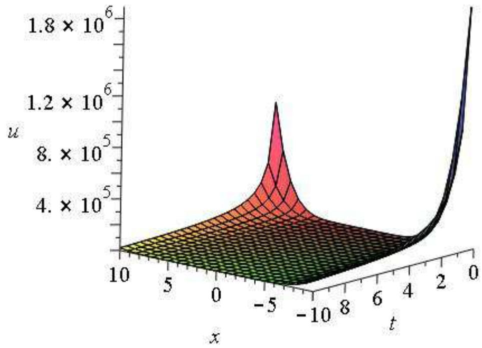


Figure 4.18: 3D plot of the solution (4.134) with a_i, $i = 1, 2, ..., 6$ are taken as $1, 0.5, 0.25, 0.125, 2, -2.37, 0.49$, and $\mu = 2$, $\alpha = 0.75$, $n = 0$ to 6.

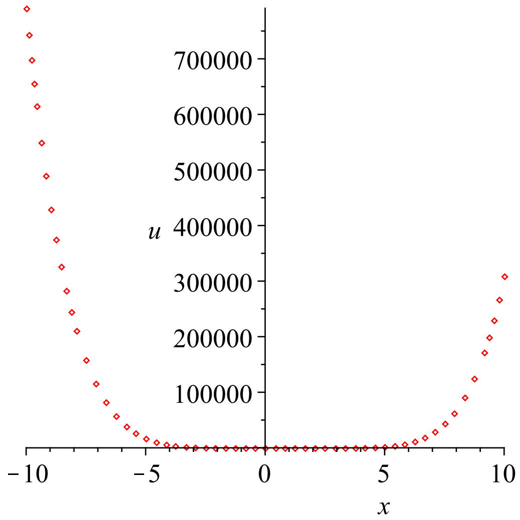


Figure 4.19: 2D plot of the solution (4.134) with a_i, $i = 1, 2, ..., 6$ are taken as $1, 0.5, 0.25, 0.125, 2, -2.37, 0.49$, and $\mu = 2$, $\alpha = 0.75$, $t = 1$, $n = 0$ to 6.

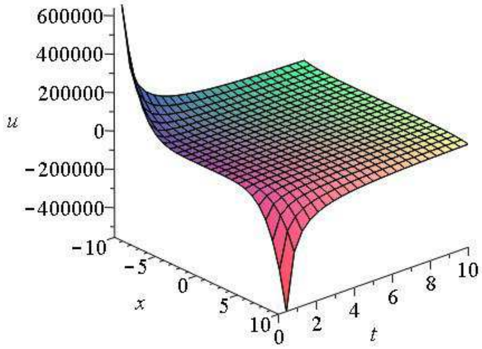


Figure 4.20: 3D plot of the solution (4.134) with a_i, $i = 1, 2, ..., 6$ are taken as $1, 0.5, 0.25, 0.125, 2, -2.37, 0.49$, and $\mu = 2$, $\alpha = 0.9$, $n = 0$ to 5.

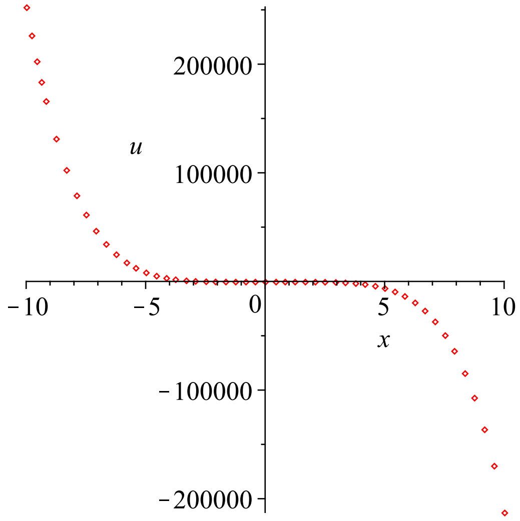


Figure 4.21: 2D plot of the solution (4.134) with a_i, $i = 1, 2, ..., 6$ are taken as $1, 0.5, 0.25, 0.125, 2, -2.37, 0.49$, and $\mu = 2$, $\alpha = 0.9$, $t = 1$, $n = 0$ to 5.

Thus t and x components of conserved vectors corresponding to X_{16} and X_{17} for given Eq. (4.114) are calculated as follows

$$\begin{aligned}
C_7^t &= TL + (-1)^0\,{}_0D_t^{\alpha-1}(W_{10})D_t^0\frac{\partial L}{\partial_0 D_t^\alpha u} - (-1)^1 J\Big(W_{10}, D_t^1\frac{\partial L}{\partial_0 D_t^\alpha u}\Big)\\
&= {}_0D_t^{\alpha-1}(W_{10})\Upsilon + J(W_{10}, \Upsilon_t),\\
&= {}_0D_t^{\alpha-1}(-u_x)\Upsilon + J(-u_x, \Upsilon_t),
\end{aligned} \tag{4.137}$$

$$\begin{aligned}
C_7^x = \big(&-u_{xxx}\Upsilon_{xx} - 50\,u\Upsilon u_x u_{xx} - 15\,u_x u_{xxx}\Upsilon - 30\,u^2\Upsilon {u_x}^2 - 5\,u^4\Upsilon u_x\\
&+10\,u_x u_{xx}\Upsilon_x + 10\,u\Upsilon_x {u_x}^2 + 10\,u^3\Upsilon_x u_x - 10\,u_x\Upsilon_{xx}u^2 + 5\,uu_x\Upsilon_{xxx}\\
&-10\,u^3\Upsilon u_{xx} + 10\,u^2\Upsilon_x u_{xx} - 5\,uu_{xx}\Upsilon_{xx} - 10\,u^2\Upsilon u_{xxx} + 5\,uu_{xxx}\Upsilon_x\\
&-5\,uu_{xxxx}\Upsilon - u_{xxxxx}\Upsilon - u_x\Upsilon_{xxxx} - 10\,{u_{xx}}^2\Upsilon - 15\,{u_x}^3\Upsilon + u_{xx}\Upsilon_{xxx}\\
&+u_{xxxx}\Upsilon_x\big)\,\mu.
\end{aligned} \tag{4.138}$$

$$\begin{aligned}
C_8^t &= TL + (-1)^0\,{}_0D_t^{\alpha-1}(W_{11})D_t^0\frac{\partial L}{\partial_0 D_t^\alpha u} - (-1)^1 J\Big(W_{11}, D_t^1\frac{\partial L}{\partial_0 D_t^\alpha u}\Big)\\
&= {}_0D_t^{\alpha-1}(W_{11})\Upsilon + J(W_{11}, \Upsilon_t),\\
&= {}_0D_t^{\alpha-1}(-u - xu_x - \frac{5t}{\alpha}u_t)\Upsilon + J(-u - xu_x - \frac{5t}{\alpha}u_t, \Upsilon_t).
\end{aligned} \tag{4.139}$$

$$\begin{aligned}
C_8^x = -\mu\;\big(&5\,\Upsilon u_{xxxx} - 4\,u_{xxx}\Upsilon_x + 3\,u_{xx}\Upsilon_{xx} - 2\,u_x\Upsilon_{xxx} - 10\,{u_x}^2\Upsilon_x\\
&+u\Upsilon_{xxxx} - 5\,u^2\Upsilon_{xxx} + 10\,u^3\Upsilon_{xx} - 10\,u^4\Upsilon_x + 5\,u^5\Upsilon + 15\,xu_x\Upsilon u_{xxx}\\
&+30\,x{u_x}^2\Upsilon u^2 + 5\,xu_x\Upsilon u^4 - 10\,xu_x\Upsilon_x u_{xx} - 10\,x{u_x}^2\Upsilon_x u - 10\,xu_x\Upsilon_x u^3\\
&+10\,xu_x\Upsilon_{xx}u^2 - 5\,xu_x\Upsilon_{xxx}u + 10\,xu_{xx}\Upsilon u^3 - 10\,xu_{xx}\Upsilon_x u^2 + 5\,xu_{xx}\Upsilon_{xx}u\\
&+10\,xu_{xxx}\Upsilon u^2 - 5\,xu_{xxx}\Upsilon_x u + 5\,xu_{xxxx}\Upsilon u + 50\,xu_x\Upsilon uu_{xx} - 20\,u\Upsilon_x u_{xx}\\
&+25\,u\Upsilon u_{xxx} + xu_x\Upsilon_{xxxx} + 50\,u_x\Upsilon u_{xx} + 10\,u_x\Upsilon_{xx}u + 50\,u^3\Upsilon u_x\\
&+50\,u^2\Upsilon u_{xx} - xu_{xxxx}\Upsilon_x + 15\,x{u_x}^3\Upsilon + 75\,u\Upsilon {u_x}^2 - 30\,u^2\Upsilon_x u_x\\
&-xu_{xx}\Upsilon_{xxx} + xu_{xxx}\Upsilon_{xx} + 10\,x{u_{xx}}^2\Upsilon + \Upsilon xu_{xxxxx}\big)\\
&-\frac{\mu}{\alpha}\,\big(25\,tu_{3xt}\Upsilon u + 25\,tu_t\Upsilon u_{xxx} + 75\,tu_t\Upsilon {u_x}^2 + 25\,tu_t\Upsilon u^4 - 25\,tu_t\Upsilon_x u_{xx}\\
&-50\,tu_t\Upsilon_x u^3 + 50\,tu_t\Upsilon_{xx}u^2 - 25\,tu_t\Upsilon_{xxx}u + 50\,tu_{tx}\Upsilon u_{xx} + 50\,tu_{tx}\Upsilon u^3\\
&-25\,tu_{tx}\Upsilon_x u_x - 50\,tu_{tx}\Upsilon_x u^2 + 25\,tu_{tx}\Upsilon_{xx}u + 50\,tu_{xx}\Upsilon u_x + 50\,tu_{2xt}\Upsilon u^2\\
&-25\,tu_{2xt}\Upsilon_x u + 5\,\Upsilon tu_{4xt} - 5\,tu_{3xt}\Upsilon_x + 5\,tu_{2xt}\Upsilon_{xx}\\
&-5\,tu_{tx}\Upsilon_{xxx} + 5\,tu_t\Upsilon_{4x} + 100\,tu_t\Upsilon uu_{xx} + 150\,tu_t\Upsilon u^2u_x - 50\,tu_t\Upsilon_x uu_x\\
&+150\,tu_{tx}\Upsilon uu_x\big)\,.
\end{aligned} \tag{4.140}$$

4.3.5 Conclusion

We present an algorithm to analyse systematically the 5^{th} order fractional equation from Burgers hierarchy. The linear analysis of the equation gives the dispersion relation whose real part reveals dispersion and imaginary part corresponds to wave damping. Relation between phase and group velocity signifies anomalous dispersion of waves and velocities are found to be a function of time fractional derivative order. The power series solution is obtained for the reduced fractional ODE by Lie symmetry analysis. The graphical analysis of the solution for different α values are presented. The new conservation theorem has been applied to derive conservation laws corresponding to infinitesimal symmetries.

Chapter 5

Space-Time Fractional PDEs[3]

In this chapter, the two physically important PDEs having space-time derivatives of fractional order are examined for solutions and dispersion relation. The potential Yu-Toda-Sasa-Fukuyama (YTSF) equation and (2+1)-dimensional Maccari system in fractional forms are chosen for investigation. For solutions, an improved F-expansion method is proposed for YTSF equation and EJEFE principle for (2+1)-dimensional Maccari system. A discussion on linear forms of the equations is carried out for dispersion relation, phase and group velocities.

5.1 Space-time fractional Potential YTSF equation

In this section, an improved F-expansion method is applied to space-time fractional potential YTSF equation [335] for deriving solutions. The integer order YTSF equation is given as follows

$$(-4w_t + \Phi(w)w_z)_x + 3w_{yy} = 0,\ \Phi(w) = \partial_x^2 + 4w + 2w_x\partial_x^{-1}, \tag{5.1}$$

where $w : \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z \times \mathbb{R}_t \to \mathbb{R}$ and different subscripts used in equations denote partial derivatives with respect to variables taken as x, y, z or t and $\partial_x^{-1}(\cdot) = \int_{-\infty}^{\infty}(\cdot)dx$. The Eq. (5.1) represents (3+1)-dimensional generalisation of the Calogero-

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[3] The contents of section (5.1) are published *Nonlinear Dynamics*, 96(2), 837-852 (2019)

Bogoyavlenskii-Schiff equation [255] which is given by

$$w_t + \Phi(w) w_z = 0,\ \Phi(w) = \partial_x^2 + 4w + 2w_x \partial_x^{-1}. \tag{5.2}$$

For $w = u_x$, the Eq. (5.1) transforms into (3+1)-dimensional potential YTSF equation [335] and is given by

$$-4\, u_{xt} + u_{xxxz} + 3\, u_{yy} + 4\, u_x u_{xz} + 2\, u_{xx} u_z = 0. \tag{5.3}$$

In the Eq. (5.3), the function $u(t, x, y, z)$ represents the amplitude of wave and is used to describe mobility of solitons. It also represents nonlinear waves in the fields of plasma physics, fluid dynamics, weakly dispersive media, etc. [181]. The Eq. (5.3) was solved for soliton, non-travelling wave, exact travelling wave, rogue wave, soliton solutions in Gramian, rational, lump and solitary waves type solutions [13, 44, 47, 56, 65, 117, 136, 183, 184, 190, 246, 283, 305, 306, 323, 332, 341–343, 345, 349]. In literature, no reports on space-time fractional potential YTSF Eq. (5.3) has been found for exact solutions and dispersion analysis. The real life applications of the equation have found in oceanography, waves in the two layer liquid medium and elastic medium [1, 56, 183, 283, 311, 332]. The fractional order version of YTSF Eq. (5.3) can be obtained by using the variational principle in fractional calculus [9, 198] and this principle is successfully implemented for deriving fractional order differential equations for other systems reported in [9, 84, 105, 198, 247]. In the present analysis, space-time fractional potential YTSF equation in normalised form is considered as follows

$$\begin{aligned} &-4\, \frac{\partial^{\beta}}{\partial x^{\beta}} \left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}} \right) + \frac{\partial^{3\beta}}{\partial x^{3\beta}} \left(\frac{\partial^{\delta} u}{\partial z^{\delta}} \right) + 3\, \frac{\partial^{2\gamma} u}{\partial y^{2\gamma}} + 4 \left(\frac{\partial^{\beta} u}{\partial x^{\beta}} \right) \left(\frac{\partial^{\beta}}{\partial x^{\beta}} \left(\frac{\partial^{\delta} u}{\partial z^{\delta}} \right) \right) \\ &+ 2 \left(\frac{\partial^{2\beta} u}{\partial x^{2\beta}} \right) \left(\frac{\partial^{\delta} u}{\partial z^{\delta}} \right) = 0, \end{aligned} \tag{5.4}$$

where α, β, γ and δ ($0 < \alpha, \beta, \gamma, \delta < 1$) denote the fractional orders of the derivatives with respect to independent variables t, x, y and z, respectively.

5.1.1 Improved F-expansion method to solve potential YTSF Eq. (5.4)

The fractional complex transformation (2.75) converts the given Eq. (5.4) after integration into the following ODE

$$\left(-4\,K_1K_4+3\,{K_2}^2\right)U'+{K_1}^3K_3\rho^2U'''+3\,\rho\,{K_1}^2K_3{U'}^2=0, \tag{5.5}$$

where, the constant of integration is considered as zero and $'$ denotes derivative of $U(\zeta)$ with respect to ζ. The homogeneous balancing between highest order derivative term U''' and highest order nonlinear term ${U'}^2$ in the Eq. (5.5) gives the solution of Eq. (5.5) in the following form

$$U(\zeta)=\mu_0+\mu_1(m+\phi(\zeta))+\nu_1(m+\phi(\zeta))^{-1}. \tag{5.6}$$

The use of Eq. (5.6) into Eq. (5.5) gives a polynomial in $\phi(\zeta)$ as follows

$$\begin{aligned}
&\left(3\,\rho\,{K_1}^2K_3{\mu_1}^2+6\,{K_1}^3K_3\rho^2\mu_1\right)\phi^8(\zeta)+\left(12\,\rho\,{K_1}^2K_3{\mu_1}^2m+24\,{K_1}^3K_3\rho^2\mu_1m\right)\phi^7(\zeta)\\
&+\big(36\,{K_1}^3K_3\rho^2\mu_1m^2+8\,{K_1}^3K_3\rho^2\mu_1l+18\,\rho\,{K_1}^2K_3{\mu_1}^2m^2+3\,{K_2}^2\mu_1-4\,K_1K_4\,\mu_1\\
&-6\,\rho\,{K_1}^2K_3\mu_1\nu_1+6\,\rho\,{K_1}^2K_3{\mu_1}^2l\big)\phi^6(\zeta)+\big(24\,\rho\,{K_1}^2K_3{\mu_1}^2ml+12\,{K_2}^2\mu_1m\\
&-16\,K_1K_4\,\mu_1m+12\,\rho\,{K_1}^2K_3{\mu_1}^2m^3+24\,{K_1}^3K_3\rho^2\mu_1m^3+32\,{K_1}^3K_3\rho^2\mu_1ml\\
&-12\,\rho\,{K_1}^2K_3\mu_1m\nu_1\big)\phi^5(\zeta)+\big(4\,K_1K_4\,\nu_1-12\,\rho\,{K_1}^2K_3\mu_1\nu_1l-6\,\rho\,{K_1}^2K_3\mu_1m^2\nu_1\\
&-24\,K_1K_4\,\mu_1m^2+3\,\rho\,{K_1}^2K_3{\nu_1}^2+3\,\rho\,{K_1}^2K_3{\mu_1}^2m^4+3\,\rho\,{K_1}^2K_3{\mu_1}^2l^2+2\,{K_1}^3K_3\rho^2\mu_1l^2\\
&+36\,\rho\,{K_1}^2K_3{\mu_1}^2m^2l+18\,{K_2}^2\mu_1m^2-2\,{K_1}^3K_3\rho^2\nu_1l-4\,K_1K_4\,\mu_1l-6\,{K_1}^3K_3\rho^2\nu_1m^2\\
&-3\,{K_2}^2\nu_1+6\,{K_1}^3K_3\rho^2\mu_1m^4+48\,{K_1}^3K_3\rho^2\mu_1m^2l+3\,{K_2}^2\mu_1l\big)\phi^4(\zeta)\\
&+\big(12\,\rho\,{K_1}^2K_3{\mu_1}^2ml^2-24\,\rho\,{K_1}^2K_3\mu_1m\nu_1l-16\,K_1K_4\,\mu_1m^3+8\,{K_1}^3K_3\rho^2\mu_1ml^2\\
&+12\,{K_2}^2\mu_1ml+32\,{K_1}^3K_3\rho^2\mu_1m^3l+8\,K_1K_4\,\nu_1m+24\,\rho\,{K_1}^2K_3{\mu_1}^2m^3l-6\,{K_2}^2\nu_1m\\
&+12\,{K_2}^2\mu_1m^3-16\,K_1K_4\,\mu_1ml+8\,{K_1}^3K_3\rho^2\nu_1ml\big)\phi^3(\zeta)\\
&+\big(-8\,{K_1}^3k_3\rho^2\nu_1l^2+6\,\rho\,{K_1}^2K_3{\nu_1}^2l+3\,{K_2}^2\mu_1m^4+18\,\rho\,{K_1}^2K_3{\mu_1}^2m^2l^2\\
&+12\,{K_1}^3K_3\rho^2\mu_1m^2l^2+18\,{K_2}^2\mu_1m^2l-12\,\rho\,{K_1}^2K_3\mu_1m^2\nu_1l+6\,\rho\,{K_1}^2K_3{\mu_1}^2m^4l\\
&+8\,{K_1}^3K_3\rho^2\mu_1m^4l-6\,\rho\,{K_1}^2K_3\mu_1\nu_1l^2+4\,K_1K_4\,\nu_1m^2-3\,{K_2}^2\nu_1m^2-4\,K_1K_4\,\mu_1m^4\\
&-24\,K_1K_4\,\mu_1m^2l-8\,{K_1}^3K_3\rho^2\nu_1m^2l-3\,{K_2}^2\nu_1l+4\,K_1K_4\,\nu_1l\big)\phi^2(\zeta)\\
&+\big(8\,{K_1}^3K_3\rho^2\mu_1m^3l^2-12\,\rho\,{K_1}^2K_3\mu_1m\nu_1l^2+8\,K_1K_4\,\nu_1ml-16\,K_1K_4\,\mu_1m^3l\\
&+8\,{K_1}^3K_3\rho^2\nu_1ml^2+12\,{K_2}^2\mu_1m^3l+12\,\rho\,{K_1}^2K_3{\mu_1}^2m^3l^2-6\,{K_2}^2\nu_1ml\big)\phi(\zeta)
\end{aligned}$$

$$\begin{aligned}&-3\,{K_2}^2\nu_1 m^2 l-2\,{K_1}^3K_3\rho^2\nu_1 m^2l^2-4\,K_1K_4\,\mu_1m^4l-6\,\rho\,{K_1}^2K_3\mu_1m^2\nu_1l^2\\&+3\,\rho\,{K_1}^2K_3{\mu_1}^2m^4l^2-6\,{K_1}^3K_3\rho^2\nu_1l^3+2\,{K_1}^3K_3\rho^2\mu_1m^4l^2+3\,{K_2}^2\mu_1m^4l\\&+4\,K_1K_4\,\nu_1m^2l+3\,\rho\,{K_1}^2K_3{\nu_1}^2l^2=0.\end{aligned}\tag{5.7}$$

Now, by equating the coefficients of various powers of $\phi^i(\zeta)$, $i = 1, 2, ...$ to zero, the over-determining system of equations is obtained. The system of equations is solved by using symbolic computational software Maple and solutions are given by the following expressions

$$K_4=-\frac{4\,{K_1}^3K_3\rho^2\,l-3\,{K_2}^2}{4\,K_1},\ \mu_1=0,\ \nu_1=2\,K_1\rho\,l+2\,K_1\rho\,m^2,\tag{5.8}$$

and

$$K_4=-\frac{4\,{K_1}^3K_3\rho^2\,l-3\,{K_2}^2}{4\,K_1},\ \mu_1=-2K_1\rho,\ \nu_1=0.\tag{5.9}$$

Using the above set of values for μ_i, ν_i, and K_4 into Eqs. (5.6) and (2.75), the exact solutions for the Eq. (5.4) are obtained in terms of hyperbolic, trigonometric and rational functions and discussed graphically in the following cases.

Case 1: For $l < 0$, the solutions involve hyperbolic functions as follows

Family 1

$$u(t,x,y,z)=\mu_0+\frac{(2\,\rho\,K_1l+2\,K_1\rho\,m^2)}{\left(m-\sqrt{-l}\tanh\left(\sqrt{-l}\zeta\right)\right)}.\tag{5.10}$$

$$u(t,x,y,z)=\mu_0+\frac{(2\,\rho\,K_1l+2\,K_1\rho\,m^2)}{\left(m-\sqrt{-l}\coth\left(\sqrt{-l}\zeta\right)\right)}.\tag{5.11}$$

The solutions derived in family 1 are plotted in Figure 5.1. Figure 5.1(a-c) represents the 3D and 2D kink wave profiles for the solution (5.10) whereas Figure 5.1(d-f) describes the 3D and 2D singular wave profiles of the solution (5.11). The 3D plots help to compare the wave profiles for solutions in family 1 for fractional order versus an integer order equation. The effect of variation in fractional parameters on the wave profile of solutions is depicted by 2D plots. Note that, there is no well-defined procedure to select a set of fractional order parameters, hence the solutions of fractional order systems can be simulated by selecting the various combinations of parameters.

Family 2

$$u(t,x,y,z)=\mu_0+(2\,\rho\,K_1)\left(m-\sqrt{-l}\tanh\left(\sqrt{-l}\zeta\right)\right).\tag{5.12}$$

$$u(t,x,y,z)=\mu_0+(2\,\rho\,K_1)\left(m-\sqrt{-l}\coth\left(\sqrt{-l}\zeta\right)\right).\tag{5.13}$$

The solutions obtained in family 2 are presented graphically in Figure 5.2. It describes the effect of variation in fractional order parameters on the kink wave profile of solution (5.12) in Figure 5.2(a-c) and singular wave profiles for the solution (5.13) in Figure 5.2(d-f).

Case 2: For $l > 0$, solutions are obtained in trigonometric functions as follows

Family 3

$$u(t,x,y,z) = \mu_0 + \frac{\left(2\,\rho\,K_1 l + 2\,K_1 \rho\,m^2\right)}{\left(m + \sqrt{l}\tan\left(\sqrt{l}\zeta\right)\right)}. \tag{5.14}$$

$$u(t,x,y,z) = \mu_0 + \frac{\left(2\,\rho\,K_1 l + 2\,K_1 \rho\,m^2\right)}{\left(m - \sqrt{l}\cot\left(\sqrt{l}\zeta\right)\right)}. \tag{5.15}$$

Figure 5.3 shows singular periodic wave profile for the solutions (5.14) and (5.15) obtained in the family 3. The wave profile of solutions with fractional order parameters is compared with the integer order in Figure 5.3(a,b,d,e). The result of variation in fractional parameters on the wave profile of solutions is revealed by 2D plots in Figure 5.3(c,f).

Family 4

$$u(t,x,y,z) = \mu_0 + \left(2\,\rho\,K_1\right)\left(m + \sqrt{l}\tan\left(\sqrt{l}\zeta\right)\right). \tag{5.16}$$

$$u(t,x,y,z) = \mu_0 + \left(2\,\rho\,K_1\right)\left(m - \sqrt{l}\cot\left(\sqrt{l}\zeta\right)\right). \tag{5.17}$$

Figure 5.4 also shows singular periodic wave profiles for the solutions (5.16) and (5.17) for the family 4 in terms of 3D and 2D representations.

Case 3: When $l = 0$, solutions are appeared in the form of following rational functions

Family 5

$$u(t,x,y,z) = \mu_0 + \frac{\left(2\,\rho\,K_1 l + 2\,K_1 \rho\,m^2\right)\zeta}{\left(m\,\zeta - 1\right)}. \tag{5.18}$$

Family 6

$$u(t,x,y,z) = \mu_0 + \left(2\,\rho\,K_1\right)\left(m - \frac{1}{\zeta}\right). \tag{5.19}$$

Herein $\zeta = \frac{K_1 x^\beta}{\Gamma(\beta+1)} + \frac{K_2 y^\gamma}{\Gamma(\gamma+1)} + \frac{K_3 z^\delta}{\Gamma(\delta+1)} - \frac{\left(4\,{K_1}^3 K_3 \rho^2 l - 3\,K_2\right)t^\alpha}{4\,K_1 \Gamma(\alpha+1)}$.

The graphical analysis of solutions obtained for Eq. (5.4) in families 5 and 6 is shown in Figure 5.5. It shows singular kink wave profiles for the solutions (5.18) and (5.19). Figure 5.5(a,d) represents 3D wave profiles at $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$ and $\delta = 0.15$ and Figure 5.5(b,e) depicts the 3D wave profiles at $\alpha = \beta = \gamma = \delta = 1$. The effect of fractional orders on the wave profile of the solutions is also shown by 2D representations in Figure 5.5(c,f).

The real life application of fractional order Eq. (5.4) helps to understand the rogue waves which were observed in deep sea and this is a field of intensive research in oceanography [56, 183].
The solutions (5.10)-(5.19) for the Eq. (5.4) are cross checked and verified by Maple software. It has been found that for parameters $\alpha = \beta = \gamma = \delta = 1$ and $\rho = K_1 = K_2 = K_3 = 1$, the solutions were found to be similar as reported in [13, 47, 136, 341, 342] for integer order YTSF equation. Hence, solutions obtained in the present work are generalised as the fractional order terms α, β, γ and δ can be set to any value lying between 0 to 1.

5.1.2 Dispersion analysis of potential YTSF equation

In this section, the linear dispersion analysis of space-time fractional potential YTSF Eq. (5.4) is discussed using the definition of RLF derivative [226] with lower integral limit is at $-\infty$ and is given by

$$ {}_{-\infty}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{-\infty}^{t} (t-\theta)^{n-\alpha-1} f(\theta)\, d\theta,\ n-1 < \alpha < n,\, n \in \mathbb{N}. \tag{5.20}$$

Consider linearised from of Eq. (5.4) as follows

$$ -4\, \frac{\partial^{\beta}}{\partial x^{\beta}} \left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}} \right) + \frac{\partial^{3\beta}}{\partial x^{3\beta}} \left(\frac{\partial^{\delta} u}{\partial z^{\delta}} \right) + 3\, \frac{\partial^{2\gamma} u}{\partial y^{2\gamma}} = 0. \tag{5.21}$$

The sinusoidal solution for the linear system of Eq. (5.21) is given by

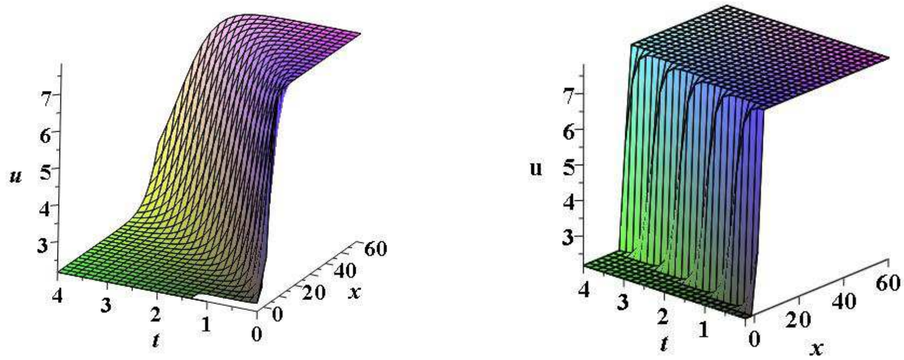
$$ u(t,x,y,z) = A\, e^{i\left(\bar{\omega} t - \vec{S}\cdot\vec{r}\right)}, \tag{5.22}$$

where A is the amplitude of u. Substituting Eq. (5.22) into Eq. (5.21) and using Fourier transform for fractional order derivative terms [226] with RLF derivative (5.20) gives

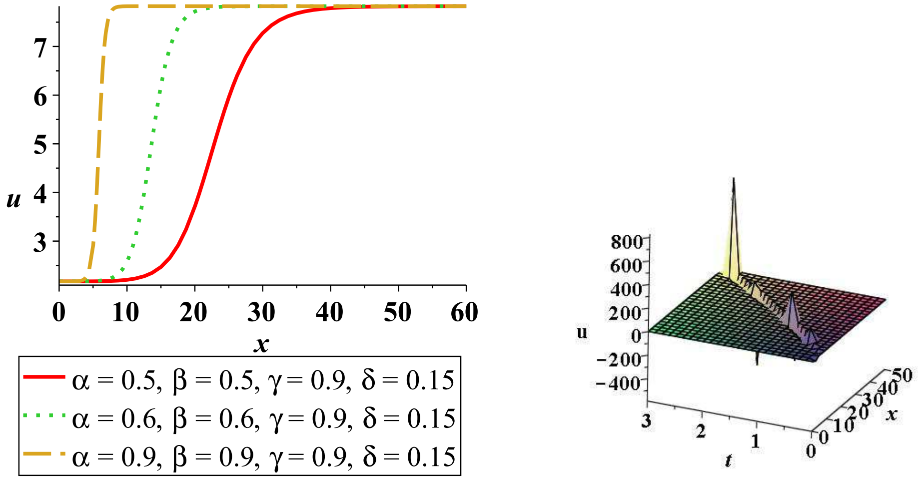
$$ \left(-4(i\bar{\omega})^{\alpha}(-is_x)^{\beta} + (-is_x)^{3\beta}(-is_z)^{\delta} + 3(-is_y)^{2\gamma}\right) A\, e^{i(\bar{\omega} t - S\cdot r)} = 0. \tag{5.23}$$

Solving the Eq. (5.23) for $\bar{\omega}$, we get,

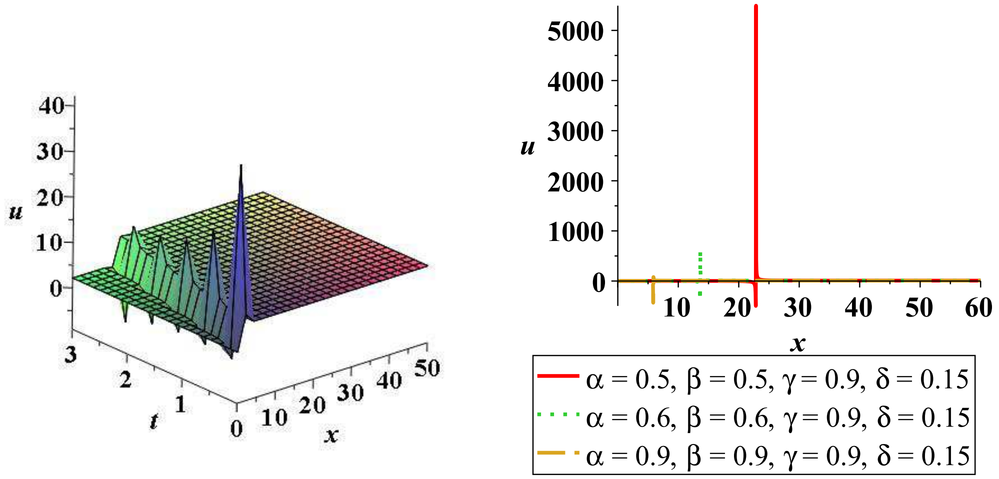
$$ \bar{\omega} = (\omega_0)^{1/\alpha}, \tag{5.24}$$



(a) $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.9,\ \delta = 0.15$ (b) $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$

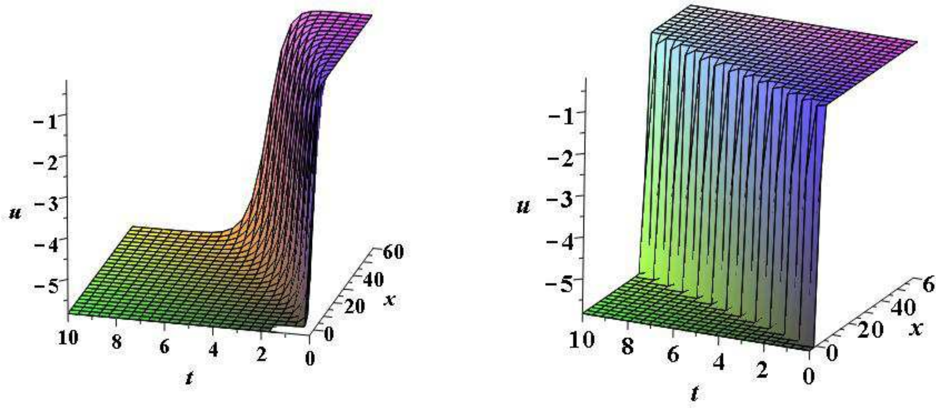


(c) Effect of fractional orders α, β, γ and δ (d) $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.9,\ \delta = 0.15$



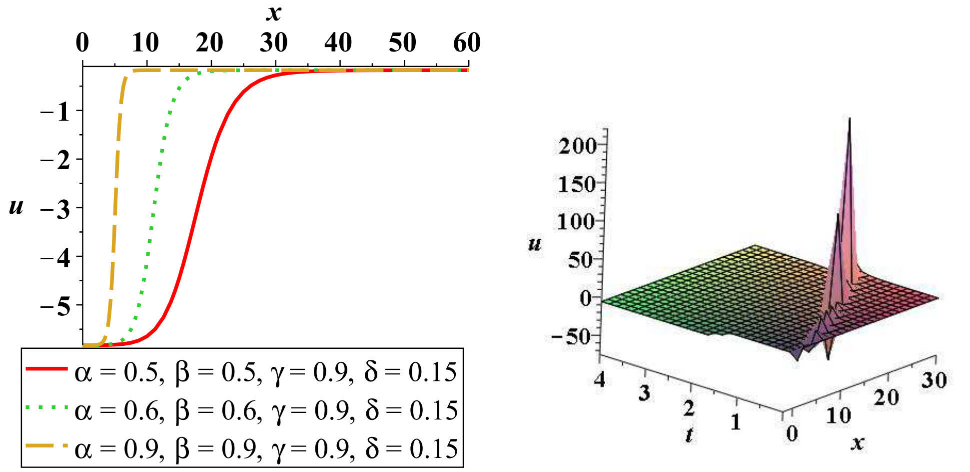
(e) $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$ (f) Effect of fractional orders α, β, γ and δ

Figure 5.1: Wave profiles of family 1 representing (a), (b), (c) for the solution (5.10) and (d), (e), (f) for the solution (5.11) with $\mu_0 = 1$, $l = -2$, $K_1 = 1$, $K_2 = 1$, $K_3 = -2$, $\rho = 1$, $m = 2$, $y = 1$, $z = 1$ for 3D plots and $y = 1$, $z = 1$, $t = 1$ for 2D plots.



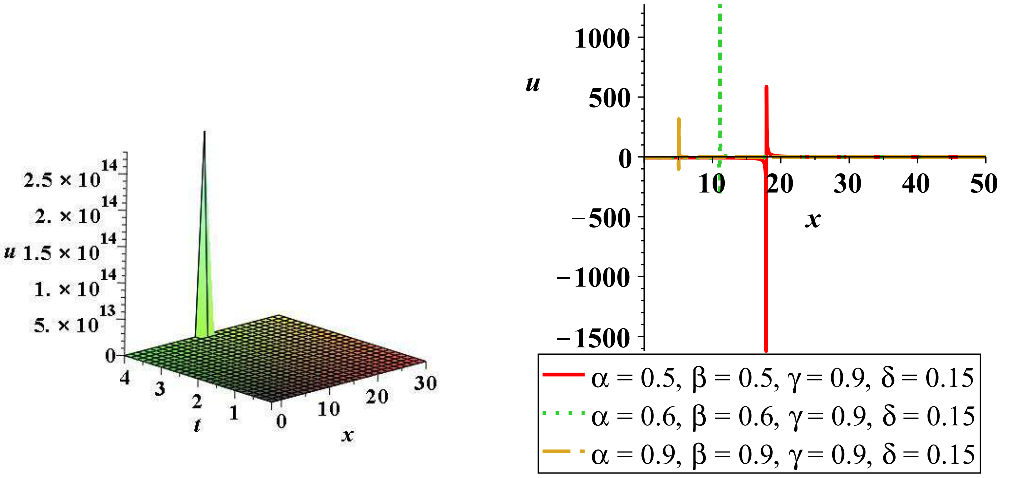
(a) $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.9,\ \delta = 0.15$

(b) $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$



(c) Effect of fractional orders α, β, γ and δ

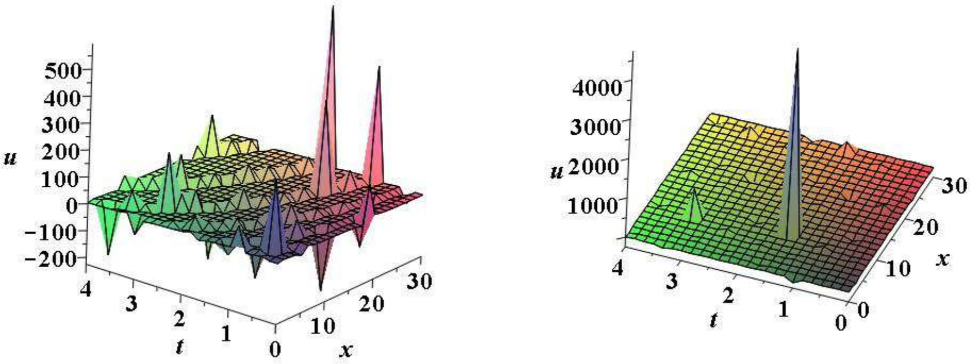
(d) $\alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.9,\ \delta = 0.15$



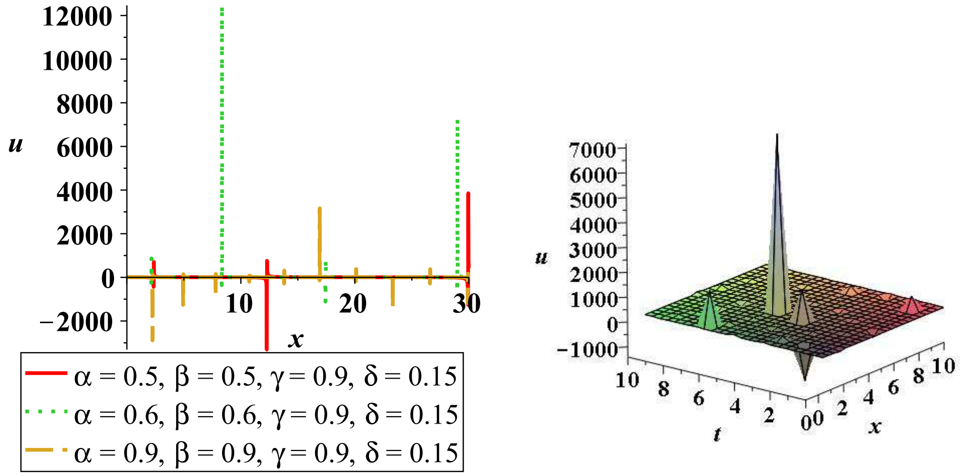
(e) $\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1$

(f) Effect of fractional orders α, β, γ and δ

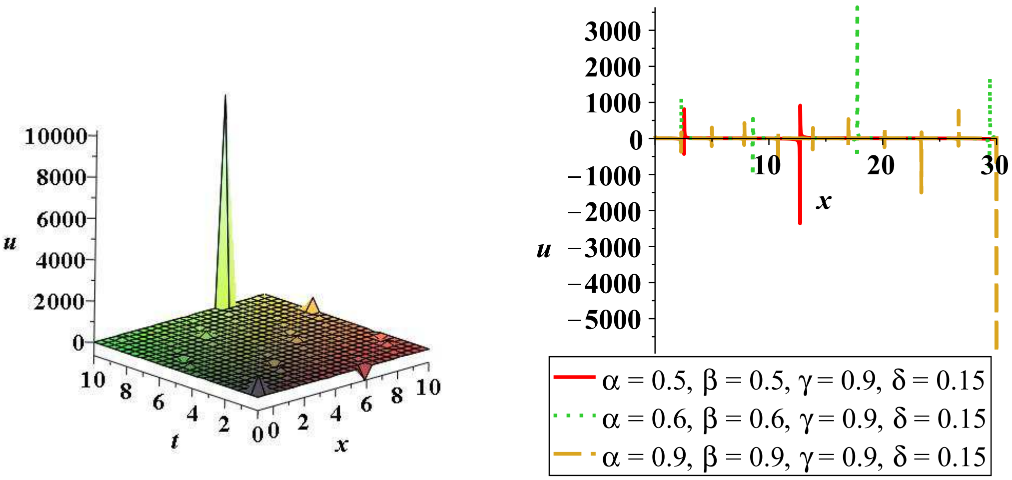
Figure 5.2: Wave profiles of family 2 representing (a), (b), (c) for the solution (5.12) and (d), (e), (f) for the solution (5.13) with $\mu_0 = 1$, $l = -2$, $K_1 = 1$, $K_2 = 1$, $K_3 = -2$, $\rho = 1$, $m = 2$, $y = 1$, $z = 1$ for 3D plots and $y = 1$, $z = 1$, $t = 1$ for 2D plots.



(a) $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.15$ (b) $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$

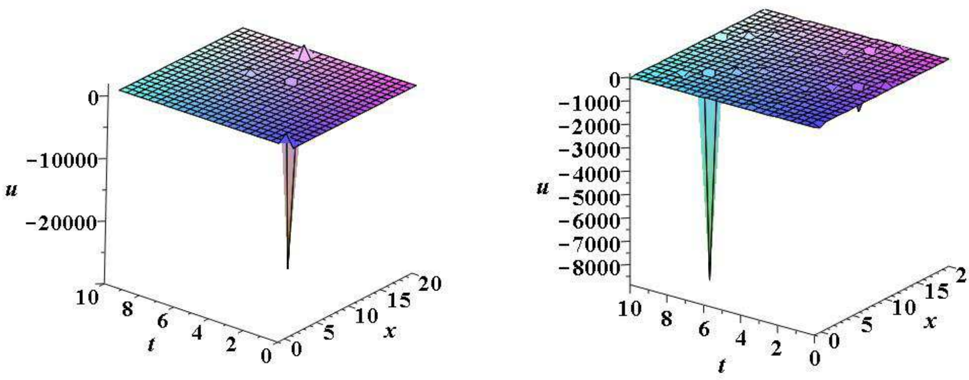


(c) Effect of fractional orders α, β, γ and δ (d) $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.15$

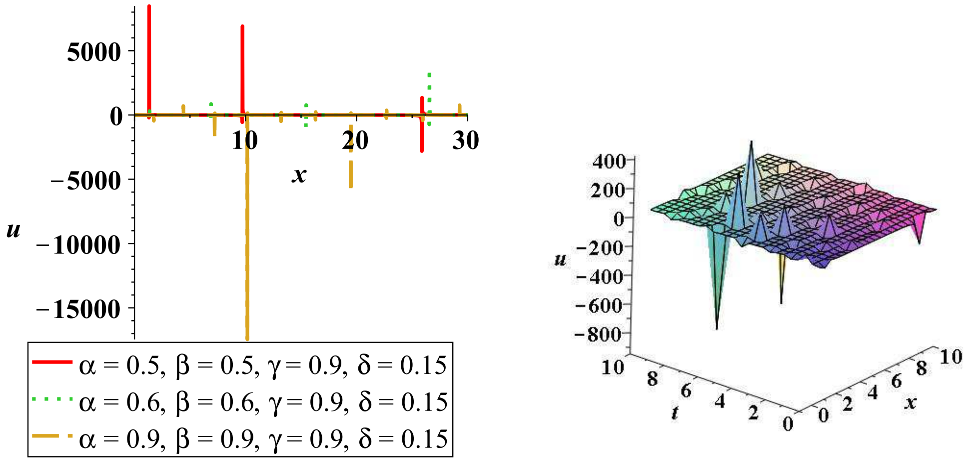


(e) $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$ (f) Effect of fractional orders α, β, γ and δ

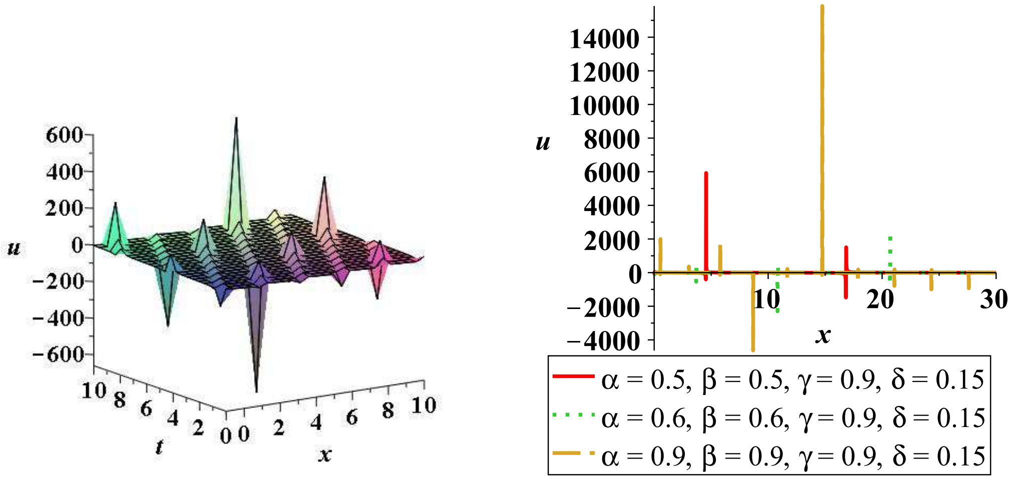
Figure 5.3: Wave profiles of family 3 representing (a), (b), (c) for the solution (5.14) and (d), (e), (f) for the solution (5.15) with $\mu_0 = 1$, $l = 2$, $K_1 = 1$, $K_2 = 1$, $K_3 = -2$, $\rho = 1$, $m = 2$, $y = 1$, $z = 1$ for 3D plots and $y = 1$, $z = 1$, $t = 1$ for 2D plots.



(a) $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.15$ (b) $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$

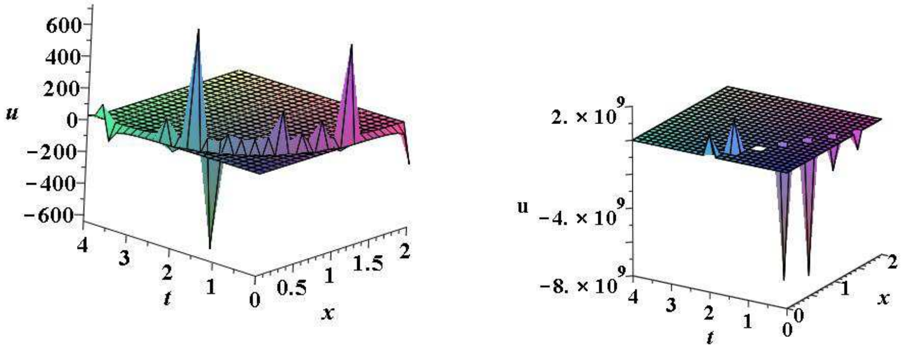


(c) Effect of fractional orders α, β, γ and δ (d) $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.15$

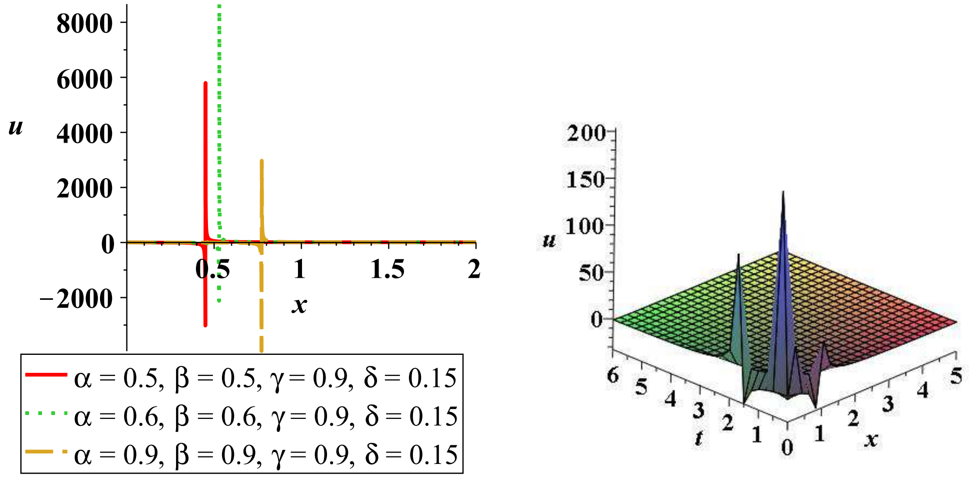


(e) $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$ (f) Effect of fractional orders α, β, γ and δ

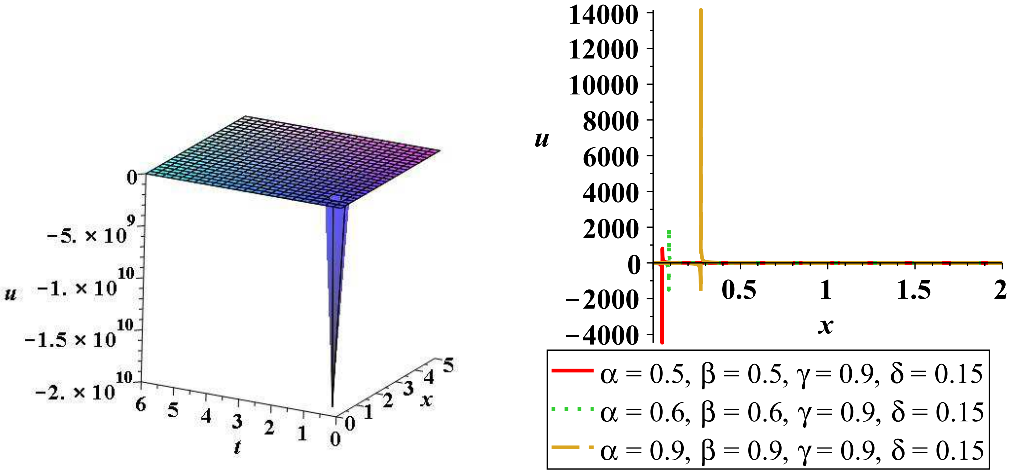
Figure 5.4: Wave profiles of family 4 representing (a), (b), (c) for the solution (5.16) and (d), (e), (f) for the solution (5.17) with $\mu_0 = 1$, $l = 2$, $K_1 = 1$, $K_2 = 1$, $K_3 = -2$, $\rho = 1$, $m = 2$, $y = 1$, $z = 1$ for 3D plots and $y = 1$, $z = 1$, $t = 1$ for 2D plots.



(a) $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.15$ (b) $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$



(c) Effect of fractional orders α, β, γ and δ (d) $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.15$



(e) $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$ (f) Effect of fractional orders α, β, γ and δ

Figure 5.5: Wave profiles of family 5 and 6 representing (a), (b), (c) for the solution (5.18) and (d), (e), (f) for the solution (5.19) with $\mu_0 = 1$, $l = 0$, $K_1 = 1$, $K_2 = 1$, $K_3 = -2$, $\rho = 1$, $m = 2$, $y = 1$, $z = 1$ for 3D plots and $y = 1$, $z = 1$, $t = 1$ for 2D plots.

where $\omega_0 = \frac{1}{4}\Big(\left(-1\right)^{2\,\beta+\delta}\,{s_x}^{2\,\beta}{s_z}^{\delta}\left(\cos\left(\theta_1\right)+i\sin\left(\theta_1\right)\right)$
$+3\,\left(-1\right)^{2\,\gamma-\beta}\,{s_y}^{2\,\gamma}{s_x}^{-\beta}\left(\cos\left(\theta_2\right)+i\sin\left(\theta_2\right)\right)\Big)$,
$\theta_1 = \frac{\pi}{2}\,\left(2\,\beta+\delta-\alpha\right)$, $\theta_2 = \frac{\pi}{2}\,\left(2\,\gamma-\beta-\alpha\right)$.

The expressions for phase and group velocities using dispersion relation are given by

$$\begin{aligned} v_p(\vec{S}) &= \frac{s_x\,\hat{i}+s_y\,\hat{j}+s_z\,\hat{k}}{s_x^2+s_y^2+s_z^2}\left(\omega_0\right)^{1/\alpha}, \\ v_g(\vec{S}) &= \frac{\partial}{\partial s_x}{\omega_0}^{1/\alpha}\,\hat{i}+\frac{\partial}{\partial s_y}{\omega_0}^{1/\alpha}\,\hat{j}+\frac{\partial}{\partial s_z}{\omega_0}^{1/\alpha}\,\hat{k}. \end{aligned} \tag{5.25}$$

Thus, magnitudes for phase and group velocity components are given as follows

$$\begin{aligned} |v_p| &= \frac{\left(4\right)^{-1/\alpha}{\omega_0}^{1/\alpha}}{\sqrt{{s_x}^2+{s_y}^2+{s_z}^2}}, \\ |v_g| &= \frac{\left(4\right)^{-1/\alpha}{\omega_0}^{(1/\alpha)-1}}{\alpha}\left({\omega_1}^2+36\,\frac{\left(-1\right)^{4\,\gamma-2\,\beta}\,{s_y}^{4\,\gamma}\gamma^2{s_x}^{-2\,\beta}\left(\cos(2\,\theta_2)+i\sin(2\,\theta_2)\right)}{{s_y}^2}\right. \\ &\left.+\frac{\left(-1\right)^{4\,\beta+2\,\delta}\,{s_x}^{4\,\beta}{s_z}^{2\,\delta}\delta^2\left(\cos\left(2\,\theta_1\right)+i\sin\left(2\,\theta_1\right)\right)}{{s_z}^2}\right)^{1/2}, \end{aligned} \tag{5.26}$$

where

$$\begin{aligned} \omega_1 = \frac{1}{s_x}\Big(&2\left(-1\right)^{2\,\beta+\delta}\,{s_x}^{2\,\beta}\beta\,{s_z}^{\delta}\left(\cos\left(\theta_1\right)+i\sin\left(\theta_1\right)\right) \\ &-3\left(-1\right)^{2\,\gamma-\beta}\,{s_y}^{2\,\gamma}{s_x}^{-\beta}\beta\,\left(\cos\left(\theta_2\right)+i\sin\left(\theta_2\right)\right)\Big). \end{aligned} \tag{5.27}$$

The Eq. (5.26) shows that $|v_p|$ is not constant, thus medium is found to be dispersive in nature. The magnitude of group velocity and phase velocity is plotted in Fig. 5.6 for different values of α, β, γ and δ. It has been noticed that imaginary part of phase velocity is greater than imaginary part of group velocity for $\alpha = 0.75$, $\beta = 0.4$, $\gamma = 0.9$ and $\delta = 0.6$ values. It implies that the sinusoidal waves with large wavelength travel faster as compared to waves having smaller wavelength. Thus, it suggests the normal dispersion of waves. Whereas, the real part of group velocity is greater than real part of phase velocity for the same set of parameters thus, it indicates the anomalous dispersion of waves. The relation given in Eq. (5.26) is not linear in s_x, s_y and s_z, thus it suggests that wave packet gets distorted in shape.

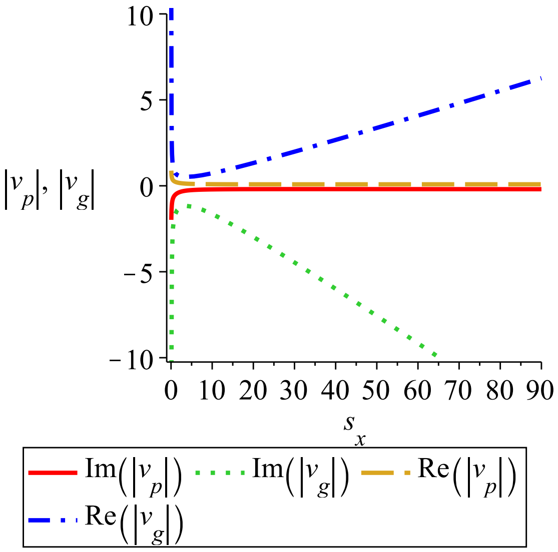


Figure 5.6: Comparison between phase and group velocities with $\alpha = 0.75$, $\beta = 0.4$, $\gamma = 0.9$, $\delta = 0.6$, $s_y = 1$, $s_z = 1$.

5.1.3 Conclusion

The improved F-expansion method successfully gives the solutions of potential YTSF equation in terms of hyperbolic, trigonometric and rational functions. The effect of variation in fractional parameters on the wave profile of solutions is described in terms of 2D and 3D plots. The solutions possess kink, periodic and singular wave profiles. The conditions for normal/anamlous dispersion of waves in terms of fractional parameters are formulated.

5.2 Maccari system in (2+1)-dimensions with fractional order space and time derivatives

The present section is focused on following Maccari system having space-time fractional form in (2+1)-dimensions

$$\begin{aligned} &i\,\frac{\partial^{\alpha} P}{\partial t^{\alpha}} + \frac{\partial^{2\beta} P}{\partial x^{2\beta}} + PW = 0, \\ &\frac{\partial^{\alpha} W}{\partial t^{\alpha}} + \frac{\partial^{\gamma} W}{\partial y^{\gamma}} + \frac{\partial^{\beta} |P|^2}{\partial x^{\beta}} = 0, \end{aligned} \tag{5.28}$$

where $i = \sqrt{-1}$, P and W represent complex and real functions, respectively, of the variables t, x and y. $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ denotes the RLF derivative [226, 251] corresponding time variable of order α. $\frac{\partial^{2\beta}}{\partial x^{2\beta}}$, $\frac{\partial^{\gamma}}{\partial y^{\gamma}}$ represent the fractional order derivatives w.r.t. x and y of order 2β and γ, respectively and $0 < \alpha, \beta, \gamma < 1$.

Clearly, the Maccari system (5.28) is complex in nature and expressed as $P = U + i\,V$, where U and V are real functions of t, x and y. Thus, the Maccari system (5.28) can be rewritten as follows

$$\begin{aligned} &\frac{\partial^{\alpha} U}{\partial t^{\alpha}} + \frac{\partial^{2\beta} V}{\partial x^{2\beta}} + VW = 0, \\ &-\frac{\partial^{\alpha} V}{\partial t^{\alpha}} + \frac{\partial^{2\beta} U}{\partial x^{2\beta}} + UW = 0, \\ &\frac{\partial^{\alpha} W}{\partial t^{\alpha}} + \frac{\partial^{\gamma} W}{\partial y^{\gamma}} + \frac{\partial^{\beta}(U^2 + V^2)}{\partial x^{\beta}} = 0. \end{aligned} \tag{5.29}$$

In literature, the derivation of Maccari system can be found in [192]. For $\alpha{=}\beta{=}\gamma{=}1$, the system (5.29) has been successfully solved by various methods reported in [55, 68, 112, 143, 162, 182, 261, 298, 334, 350]. The time fractional form Maccari system has been already reported for solutions using rational exponential method [12].

It is desirable to solve the space-time fractional order Maccari system as the fractional PDEs reveal the complex nonlinear phenomena judiciously in various scientific fields [201, 216, 226, 251]. In literature, various methods have been reported for getting analytical solutions of fractional PDEs such as Lie symmetry analysis [118, 140, 151, 229, 234, 244, 245, 276, 297, 299], fractional sub-equation method [348], (G′/G)-expansion method [36], exp function method [36], exponential rational function method [11, 204], fractional complex transformation method [29], new computational approach [330], travelling wave solutions method [327, 329], the first integral method [30] and so on. The Jacobi elliptic function expansion method provides doubly periodic waves and this method is applied on various types of nonlinear PDEs [3, 32, 74, 80, 85, 248]. In the present discussion, extended Jacobi elliptic function expansion (JEFE) technique is used to analyse fractional Maccari system of space-time form for exact solutions. Main thrust is to investigate the Maccari system (5.28) for dispersion relation and exact solutions.

5.2.1 Dispersion relation for Maccari system

For deriving dispersion relation of Maccari system (5.28), consider its linearised form as follows

$$\begin{aligned}
&\frac{\partial^{\alpha}U}{\partial t^{\alpha}}+\frac{\partial^{2\beta}V}{\partial x^{2\beta}}=0,\\
&-\frac{\partial^{\alpha}V}{\partial t^{\alpha}}+\frac{\partial^{2\beta}U}{\partial x^{2\beta}}=0,\\
&\frac{\partial^{\alpha}W}{\partial t^{\alpha}}+\frac{\partial^{\gamma}W}{\partial y^{\gamma}}=0.
\end{aligned} \tag{5.30}$$

The corresponding sinusoidal solutions of linear system (5.30) in terms of U, V and W are considered in the form as follows

$$\begin{aligned}
U&=A_{13}e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)},\\
V&=A_{14}e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)},\\
W&=A_{15}e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)},
\end{aligned} \tag{5.31}$$

where A_{13}, A_{14} and A_{15} are the amplitudes of real functions u, v and w, respectively. On applying Fourier transform to fractional derivative terms [226] using RLF derivative with lower terminal as $a=-\infty$, thus the dispersion relation for linearised system (5.30) obtained by solving following set of equations

$$\begin{aligned}
&\left((i\bar{\omega})^{\alpha}A_{13}+(-is_x)^{2\beta}A_{14}\right)e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)}=0,\\
&-\left((i\bar{\omega})^{\alpha}A_{14}+(-is_x)^{2\beta}A_{13}\right)e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)}=0,\\
&\left((i\bar{\omega})^{\alpha}A_{15}+(-i\,s_y)^{\gamma}A_{15}\right)e^{i\left(\bar{\omega}t-\vec{S}\cdot\vec{r}\right)}=0.
\end{aligned} \tag{5.32}$$

The Eqs. (5.32) expressed in matrix form as follows

$$\begin{pmatrix}
(i\bar{\omega})^{\alpha} & (-is_x)^{2\beta} & 0\\
(-is_x)^{2\beta} & -(i\bar{\omega})^{\alpha} & 0\\
0 & 0 & (i\bar{\omega})^{\alpha}+(-i\,s_y)^{\gamma}
\end{pmatrix}
\begin{pmatrix} A_{13}\\ A_{14}\\ A_{15}\end{pmatrix}=0. \tag{5.33}$$

The Eqs. (5.32) will possess solution in an non trivial form if determinant constituted out of coefficient matrix vanishes identically as follows

$$\left((i\bar{\omega})^{\alpha}+(-i\,s_y)^{\gamma}\right)\left(-(i\bar{\omega})^{2\alpha}+(is_x)^{4\beta}\right)=0. \tag{5.34}$$

On solving Eq. (5.34) for $\bar{\omega}$, the three possible roots (ω_0, ω_1, ω_2) of $\bar{\omega}$ are obtained as follows

$$
\begin{aligned}
\omega_0 &= (-1)^{\frac{1+\gamma}{\alpha}} i^{\frac{\gamma}{\alpha}-1} s_y^{\gamma/\alpha}, \\
\omega_1 &= i^{\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}} s_x^{\frac{2\beta}{\alpha}}, \\
\omega_2 &= (-1)^{\frac{1}{\alpha}} i^{\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}} s_x^{\frac{2\beta}{\alpha}}.
\end{aligned}
\tag{5.35}
$$

Using the complex algebra, the following relations can be written in the form of a complex number as follows

$$
\begin{aligned}
i^{\frac{\gamma}{\alpha}-1} &= \cos\left(\frac{\pi}{2}\left(\frac{\gamma}{\alpha}-1\right)\right) + i\,\sin\left(\frac{\pi}{2}\left(\frac{\gamma}{\alpha}-1\right)\right), \\
i^{\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}} &= \cos\left(\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right) + i\,\sin\left(\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right), \\
(-1)^{\frac{(1+\gamma)}{\alpha}} &= \cos\left(\frac{(1+\gamma)}{\alpha}\pi\right) + i\sin\left(\frac{(1+\gamma)}{\alpha}\pi\right), \\
(-1)^{1/\alpha} &= \cos\left(\frac{\pi}{\alpha}\right) + i\sin\left(\frac{\pi}{\alpha}\right).
\end{aligned}
\tag{5.36}
$$

Thus, from Eqs. (5.35) and (5.36), the following set of relations are obtained by equating real and imaginary parts of ω_0, ω_1 and ω_2,

$$
\begin{aligned}
\mathtt{R}e\,\omega_0 &= \cos\left(\frac{\pi}{\alpha}\,(1+\gamma) + \frac{\pi}{2\,\alpha}\,(\gamma-\alpha)\right)\, s_y^{\gamma/\alpha}, \\
\mathtt{I}m\,\omega_0 &= \sin\left(\frac{\pi}{\alpha}\,(1+\gamma) + \frac{\pi}{2\,\alpha}\,(\gamma-\alpha)\right)\, s_y^{\gamma/\alpha}, \\
\mathtt{R}e\,\omega_1 &= \cos\left(\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)\, s_x^{2\beta/\alpha}, \\
\mathtt{I}m\,\omega_1 &= \sin\left(\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)\, s_x^{2\beta/\alpha}, \\
\mathtt{R}e\,\omega_2 &= \cos\left(\frac{\pi}{\alpha}+\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)\, s_x^{2\beta/\alpha}, \\
\mathtt{I}m\,\omega_2 &= \sin\left(\frac{\pi}{\alpha}+\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)\, s_x^{2\beta/\alpha}.
\end{aligned}
\tag{5.37}
$$

For ω_0, the phase and group velocity in the component form of a two dimensional vector can be written as follows

$$
\begin{aligned}
v_p(\vec{S}) &= \frac{(s_x\hat{i} + s_y\hat{j})\omega_0}{s_x^2 + s_y^2}, \\
v_g(\vec{S}) &= \frac{\partial}{\partial s_x}\omega_0\,\hat{i} + \frac{\partial}{\partial s_y}\omega_0\,\hat{j}.
\end{aligned}
\tag{5.38}
$$

Table 5.1: Phase and group velocity expressions

Velocity	$\bar{\omega}$	
	ω_1	ω_2
$\mathtt{Re}\ \lvert v_p(\vec{S})\rvert$	$\frac{\cos\left(\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)s_x^{2\beta/\alpha}}{\sqrt{s_x^2+s_y^2}}$	$\frac{\cos\left(\frac{\pi}{\alpha}+\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)(s_x)^{2\beta/\alpha}}{\sqrt{s_x^2+s_y^2}}$
$\mathtt{Im}\ \lvert v_p(\vec{S})\rvert$	$\frac{\sin\left(\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)s_x^{2\beta/\alpha}}{\sqrt{s_x^2+s_y^2}}$	$\frac{\sin\left(\frac{\pi}{\alpha}+\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)(s_x)^{2\beta/\alpha}}{\sqrt{s_x^2+s_y^2}}$
$\mathtt{Re}\ \lvert v_g(\vec{S})\rvert$	$\frac{2\beta}{\alpha}\cos\left(\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)s_x^{2\beta/\alpha-1}$	$\frac{2\beta}{\alpha}\cos\left(\frac{\pi}{\alpha}+\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)s_x^{2\beta/\alpha-1}$
$\mathtt{Im}\ \lvert v_g(\vec{S})\rvert$	$\frac{2\beta}{\alpha}\sin\left(\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)s_x^{2\beta/\alpha-1}$	$\frac{2\beta}{\alpha}\sin\left(\frac{\pi}{\alpha}+\frac{\pi}{2}\left(\frac{1}{\alpha}-1+\frac{2\beta}{\alpha}\right)\right)s_x^{2\beta/\alpha-1}$

Substitution of $\omega_0 = (-1)^{\frac{1+\gamma}{\alpha}} i^{\frac{\gamma}{\alpha}-1} s_y^{\gamma/\alpha}$ into Eq. (5.38) gives

$$\begin{aligned} v_p(\vec{S}) &= \frac{(s_x\hat{i}+s_y\hat{j})((-1)^{\frac{1+\gamma}{\alpha}} i^{\frac{\gamma}{\alpha}-1} s_y^{\gamma/\alpha})}{s_x^2+s_y^2}, \\ v_g(\vec{S}) &= \frac{\partial}{\partial s_x}((-1)^{\frac{1+\gamma}{\alpha}} i^{\frac{\gamma}{\alpha}-1} s_y^{\gamma/\alpha})\,\hat{i} + \frac{\partial}{\partial s_y}((-1)^{\frac{1+\gamma}{\alpha}} i^{\frac{\gamma}{\alpha}-1} s_y^{\gamma/\alpha})\,\hat{j}. \end{aligned} \tag{5.39}$$

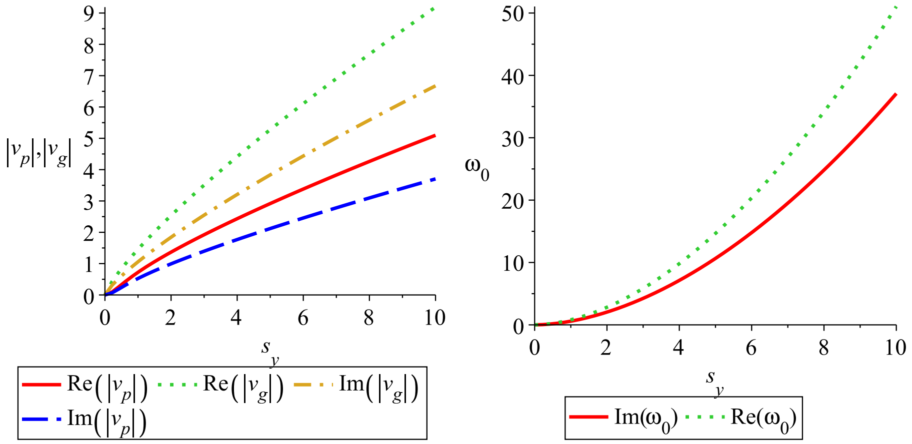
The magnitude of phase and group velocity vectors can be written as follows

$$\begin{aligned} |v_p(\vec{S})| &= \frac{((-1)^{\frac{1+\gamma}{\alpha}} i^{\frac{\gamma}{\alpha}-1} s_y^{\gamma/\alpha})}{\sqrt{s_x^2+s_y^2}}, \\ |v_g(\vec{S})| &= \frac{\gamma}{\alpha}(-1)^{\frac{1+\gamma}{\alpha}} i^{\frac{\gamma}{\alpha}-1} s_y^{\gamma/\alpha-1}. \end{aligned} \tag{5.40}$$

The real and imaginary parts of the phase and group velocities can be separated out from the Eq. (5.40) as given by

$$\begin{aligned} \mathtt{R}e\,|v_p(\vec{S})| &= \frac{\cos\left(\pi\ \left(\frac{1+\gamma}{\alpha}\right)+\frac{\pi}{2}\left(\frac{\gamma-\alpha}{\alpha}\right)\right)s_y^{\gamma/\alpha}}{\sqrt{s_x^2+s_y^2}}, \\ \mathtt{I}m\,|v_p(\vec{S})| &= \frac{\sin\left(\pi\ \left(\frac{1+\gamma}{\alpha}\right)+\frac{\pi}{2}\left(\frac{\gamma-\alpha}{\alpha}\right)\right)s_y^{\gamma/\alpha}}{\sqrt{s_x^2+s_y^2}}, \\ \mathtt{R}e\,|v_g(\vec{S})| &= \frac{\gamma}{\alpha}\cos\left(\frac{\pi}{\alpha}\ (1+\gamma)+\frac{\pi}{2\alpha}(\gamma-\alpha)\right)s_y^{\frac{\gamma}{\alpha}-1}, \\ \mathtt{I}m\,|v_g(\vec{S})| &= \frac{\gamma}{\alpha}\sin\left(\frac{\pi}{\alpha}\ (1+\gamma)+\frac{\pi}{2\alpha}(\gamma-\alpha)\right)s_y^{\frac{\gamma}{\alpha}-1}. \end{aligned} \tag{5.41}$$

On the similar way for ω_1 and ω_2, the real and imaginary parts of phase and group velocities are obtained and tabulated in Table 5.1. The variation of real and imaginary parts of phase and group velocities for ω_0, ω_1 and ω_2 is shown graphically in Figures 5.7(a), 5.8(a) and 5.9(a), respectively. It has been noticed that $\mathtt{R}e|v_g(\vec{S})|$ and $\mathtt{I}m|v_g(\vec{S})|$ values are found to be greater than the corresponding $\mathtt{R}e|v_p(\vec{S})|$ and



(a) $\alpha = 0.5$, $\gamma = 0.9$, $s_x = 0.5$ (b) $\alpha = 0.5$, $\gamma = 0.9$, $s_x = 0.5$

Figure 5.7: (a) Comparison between phase and group velocity for ω_0 and (b) shows dispersion relation ω_0.

$\mathtt{Im}|v_p(\vec{S})|$, respectively. It indicates the anomalous dispersion of waves obtained for ω_0, ω_1 and ω_2. The variation of real and imaginary parts of ω_0, ω_1 and ω_2 with s_x and s_y are shown in Figures 5.7(b), 5.8(b) and 5.9(b), respectively. In Figure 5.7(b), real part of ω_0 is greater than the imaginary part of ω_0, whereas in Figures 5.8(b) and 5.9(b), imaginary part of ω_1 and ω_2 are found to be greater than real part of ω_1 and ω_2.

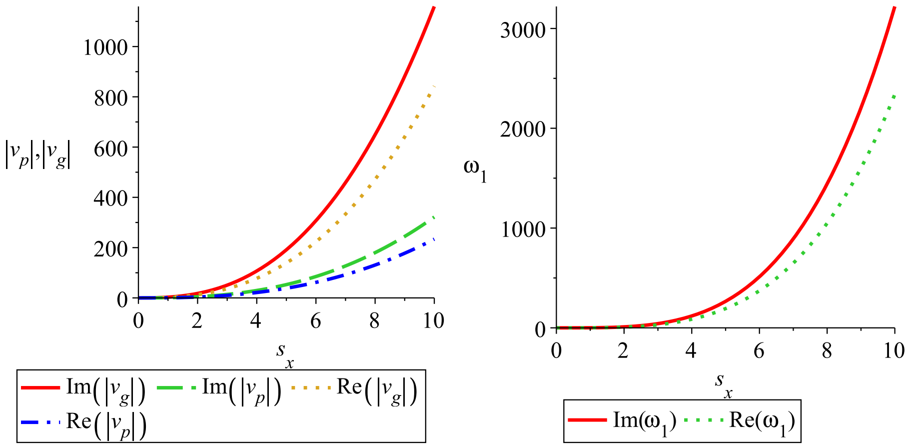
5.2.2 Exact solutions of Maccari system by applying EJEFE algorithm

In order to find the exact solutions of the Maccari system (5.28), the Fractional complex transformation for the complex function P and real function W are considered in a form to find solutions of said system as given by

$$\begin{aligned} P &= e^{i\mathcal{K}}p(\phi), \\ W &= w(\phi), \end{aligned} \tag{5.42}$$

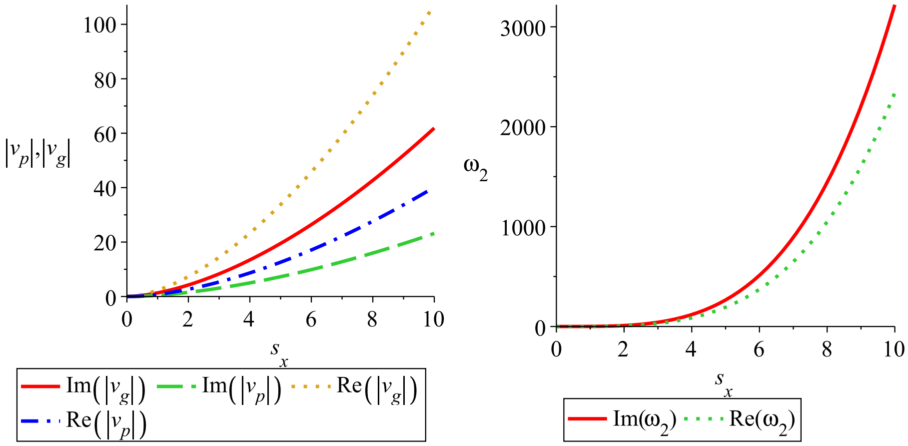
where

$$\begin{aligned} \mathcal{K} &= \frac{a_{11}x^{\beta}}{\Gamma(\beta+1)} + \frac{a_{12}y^{\gamma}}{\Gamma(\gamma+1)} + \frac{a_{13}t^{\alpha}}{\Gamma(\alpha+1)}, \\ \phi &= \frac{b_{11}x^{\beta}}{\Gamma(\beta+1)} + \frac{b_{12}y^{\gamma}}{\Gamma(\gamma+1)} + \frac{b_{13}t^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned} \tag{5.43}$$



(a) $\alpha = 0.5$, $\beta = 0.9$, $s_y = 0.5$ (b) $\alpha = 0.5$, $\gamma = 0.9$, $s_y = 0.5$

Figure 5.8: (a) Comparison between phase and group velocity for ω_1 and (b) shows dispersion relation ω_1.



(a) $\alpha = 0.45$, $\gamma = 0.6$, $s_y = 0.5$ (b) $\alpha = 0.5$, $\gamma = 0.9$, $s_y = 0.5$

Figure 5.9: (a) Comparison between phase and group velocity for ω_2 and (b) shows dispersion relation ω_2.

where a_{11}, a_{12}, a_{13}, b_{11}, b_{12} and b_{13} are arbitrary constants.
The various fractional derivative terms of the Maccari system (5.28) using complex transformation (5.43) read as follows

$$\begin{aligned}
D_t^{\alpha} P &= (\sigma_{11})_t (e^{i\mathcal{K}} i a_{13} p(\phi) + e^{i\mathcal{K}} p'(\phi) b_{13}),\\
D_x^{\beta} P &= (\sigma)_x (e^{i\mathcal{K}} i a_{11} p(\phi) + e^{i\mathcal{K}} p'(\phi) b_{11}),\\
D_t^{\alpha} W &= (\sigma_{12})_t w'(\phi) b_{13},\\
D_y^{\gamma} W &= (\sigma)_y w'(\phi) b_{12},
\end{aligned} \tag{5.44}$$

where $(\sigma_{11})_t$, $(\sigma_{12})_t$, σ_x and σ_y are the fractional indexes corresponding to time and space variables as indicated by their subscripts [115, 243]. The $p'(\phi)$ denotes derivative of $p(\phi)$ with respect to ϕ. The Eq. (5.44) converts the Maccari system (5.28) into ODE in terms of p and w, as given by

$$i(\sigma_{11})_t (i a_{13} p + p' b_{13}) + \sigma_x^2 (-a_{11}^2 p + 2 i a_{11} b_{11} p' + b_{11}^2 p'') + p w = 0, \tag{5.45}$$

and

$$(\sigma_{12})_t w' b_{13} + \sigma_y w' b_{12} + 2 \sigma_x b_{11} p p' = 0. \tag{5.46}$$

On integrating Eq. (5.46) with respect to ϕ, results in

$$w = -\frac{\sigma_x b_{11} p^2}{(\sigma_{12})_t b_{13} + \sigma_y b_{12}}. \tag{5.47}$$

The use of Eq. (5.47) into Eq. (5.45) provides following equations after equating real and imaginary parts

$${\sigma_x}^2 {b_{11}}^2 \left((\sigma_{12})_t b_{13} + \sigma_y b_{12}\right) p'' - \left((\sigma_{11})_t a_{13} + {a_{11}}^2 {\sigma_x}^2\right) \left((\sigma_{12})_t b_{13} + \sigma_y b_{12}\right) p - \sigma_x b_{11} p^3 = 0, \tag{5.48}$$

$$b_{13} = -2\, \frac{{\sigma_x}^2 a_{11} b_{11}}{(\sigma_{11})_t}. \tag{5.49}$$

The Eq. (5.48) possessed a solution in the form of Jacobi functions as follows

$$p(\phi) = A_0 + A_1 \operatorname{sn}(\phi) + B_1 \operatorname{sn}^{-1}(\phi). \tag{5.50}$$

Back substitution of the solution (5.50) into the Eq. (5.48), results into a polynomial in terms of Jacobi function sn(ϕ). By equating the coefficients of powers of sn(ϕ) to zero, the determining equations are obtained and their solution gives the values of various constants A_0, A_1, B_1, a_{13} as follows

$$\begin{aligned}
&A_0 = 0,\ A_1 = \sqrt{2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t}}m,\\
&B_1 = \sqrt{2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t}},\\
&a_{13} = -\frac{\left({a_{11}}^2 + 6\,{b_{11}}^2 m + {b_{11}}^2 + {b_{11}}^2 m^2\right){\sigma_x}^2}{(\sigma_{11})_t}
\end{aligned} \tag{5.51}$$

and

$$\begin{aligned}
&A_0 = 0,\ A_1 = -\sqrt{2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t}}m,\\
&B_1 = -\sqrt{2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t}},\\
&a_{13} = -\frac{\left({a_{11}}^2 + 6\,{b_{11}}^2 m + {b_{11}}^2 + {b_{11}}^2 m^2\right){\sigma_x}^2}{(\sigma_{11})_t}.
\end{aligned} \tag{5.52}$$

Using these values of constants, solutions for Maccari system can be written as follows

$$\begin{aligned}
&P(t,x,y) = e^{i\mathcal{K}}\sqrt{2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t}}\left(m\ \mathrm{sn}\left(\phi\right) + sn^{-1}\left(\phi\right)\right),\\
&W(t,x,y) = -2\,{b_{11}}^2{\sigma_x}^2\left(m\,\mathrm{sn}\left(\phi\right) + \mathrm{sn}^{-1}\left(\phi\right)\right)^2
\end{aligned} \tag{5.53}$$

and

$$\begin{aligned}
&P(t,x,y) = -e^{i\mathcal{K}}\sqrt{2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t}}\left(m\ \mathrm{sn}\left(\phi\right) + sn^{-1}\left(\phi\right)\right),\\
&W(t,x,y) = -2\,{b_{11}}^2{\sigma_x}^2\left(m\,\mathrm{sn}\left(\phi\right) + \mathrm{sn}^{-1}\left(\phi\right)\right)^2,
\end{aligned} \tag{5.54}$$

where $\mathcal{K} = \left(\frac{a_{11}x^\beta}{\Gamma(\beta+1)} + \frac{a_{12}y^\gamma}{\Gamma(\gamma+1)} - \frac{\left({a_{11}}^2+6\,{b_{11}}^2m+{b_{11}}^2+{b_{11}}^2m^2\right){\sigma_x}^2x^\alpha}{(\sigma_{11})_t\Gamma(\alpha+1)}\right)$,
$\phi = \frac{b_{11}x^\beta}{\Gamma(\beta+1)} + \frac{b_{12}y^\gamma}{\Gamma(\gamma+1)} - \frac{2{\sigma_x}^2a_{11}b_{11}x^\alpha}{(\sigma_{11})_t\Gamma(\alpha+1)}$, whenever $2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t} > 0$.
In limiting case, when the modulus $m \to 1$, the following dark-singular soliton solutions are evolved

$$\begin{aligned}
&P(t,x,y) = e^{i\mathcal{K}}\sqrt{2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t}}\left(\tanh\left(\phi\right) + \coth\left(\phi\right)\right),\\
&W(t,x,y) = -2\,{b_{11}}^2{\sigma_x}^2\left(\tanh\left(\phi\right) + \coth\left(\phi\right)\right)^2
\end{aligned} \tag{5.55}$$

and

$$\begin{aligned} P(t,x,y) &= -e^{i\mathcal{K}}\sqrt{2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t}}\left(\tanh\left(\phi\right)+\coth\left(\phi\right)\right), \\ W(t,x,y) &= -2\,{b_{11}}^2{\sigma_x}^2\left(\tanh\left(\phi\right)+\coth\left(\phi\right)\right)^2, \end{aligned} \tag{5.56}$$

$\mathcal{K} = \left(\frac{a_{11}x^\beta}{\Gamma(\beta+1)} + \frac{a_{12}y^\gamma}{\Gamma(\gamma+1)} - \frac{\left({a_{11}}^2+6\,{b_{11}}^2+{b_{11}}^2+{b_{11}}^2\right){\sigma_x}^2x^\alpha}{(\sigma_{11})_t\Gamma(\alpha+1)}\right)$, $\phi = \frac{b_{11}x^\beta}{\Gamma(\beta+1)} + \frac{b_{12}y^\gamma}{\Gamma(\gamma+1)} - \frac{2{\sigma_x}^2a_{11}b_{11}x^\alpha}{(\sigma_{11})_t\Gamma(\alpha+1)}$, whenever $2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t} > 0$.
When the modulus $m \to 0$, the following periodic-singular solutions are found

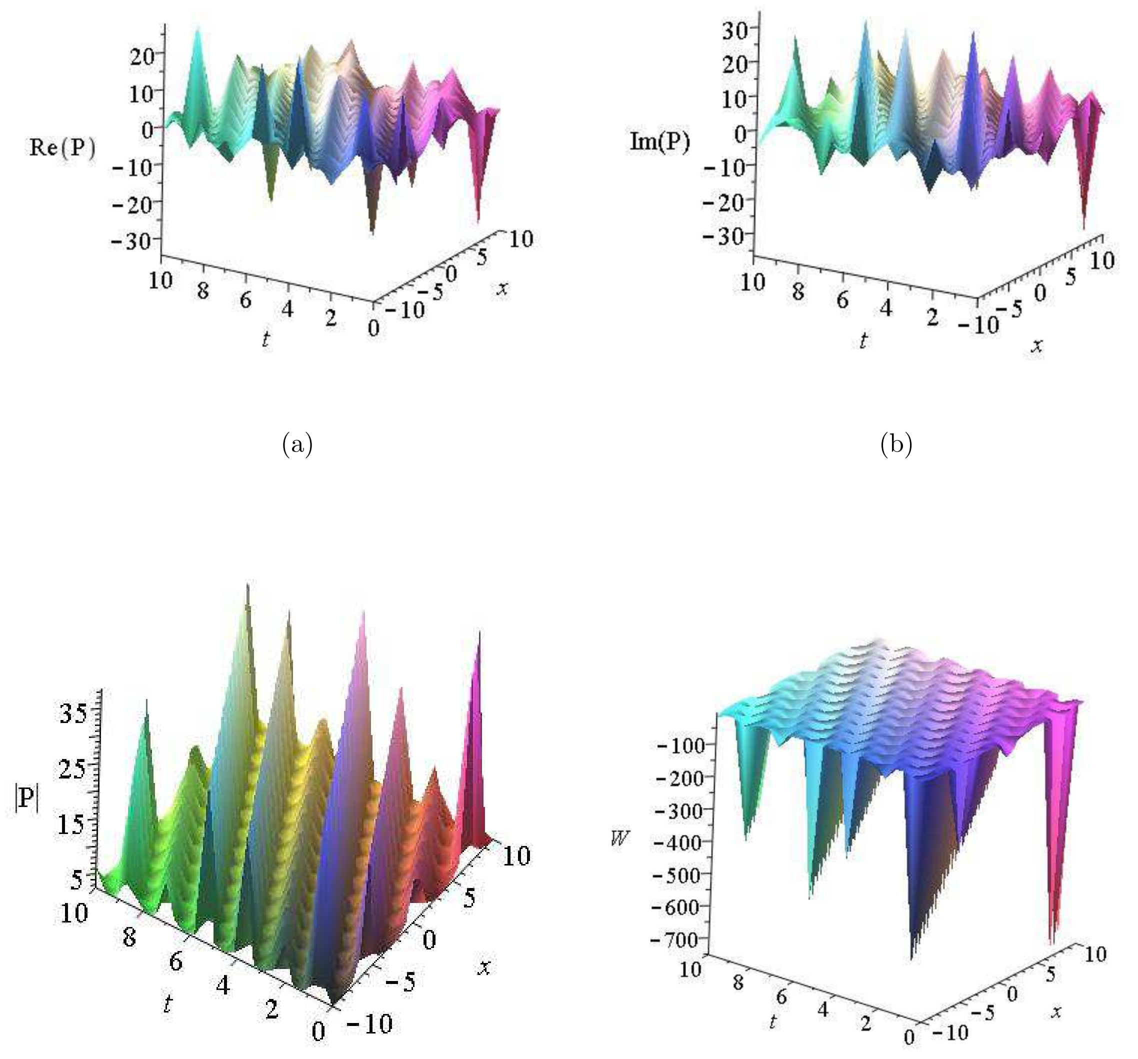
$$\begin{aligned} P &= e^{i\mathcal{K}}\sqrt{2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t}}\left(\csc\left(\phi\right)\right), \\ W &= -2\,{b_{11}}^2{\sigma_x}^2\left(\csc\left(\phi\right)\right)^2 \end{aligned} \tag{5.57}$$

and

$$\begin{aligned} P &= -e^{i\mathcal{K}}\sqrt{2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t}}\left(\csc\left(\phi\right)\right), \\ W &= -2\,{b_{11}}^2{\sigma_x}^2\left(\csc\left(\phi\right),\right)^2 \end{aligned} \tag{5.58}$$

$\mathcal{K} = \left(\frac{a_{11}x^\beta}{\Gamma(\beta+1)} + \frac{a_{12}y^\gamma}{\Gamma(\gamma+1)} - \frac{\left({a_{11}}^2+{b_{11}}^2\right){\sigma_x}^2x^\alpha}{(\sigma_{11})_t\Gamma(\alpha+1)}\right)$, $\phi = \frac{b_{11}x^\beta}{\Gamma(\beta+1)} + \frac{b_{12}y^\gamma}{\Gamma(\gamma+1)} - \frac{2{\sigma_x}^2a_{11}b_{11}x^\alpha}{(\sigma_{11})_t\Gamma(\alpha+1)}$, whenever $2\,\sigma_x b_{11}\sigma_y b_{12} - 4\,\frac{{\sigma_x}^3{b_{11}}^2(\sigma_{12})_t a_{11}}{(\sigma_{11})_t} > 0$.

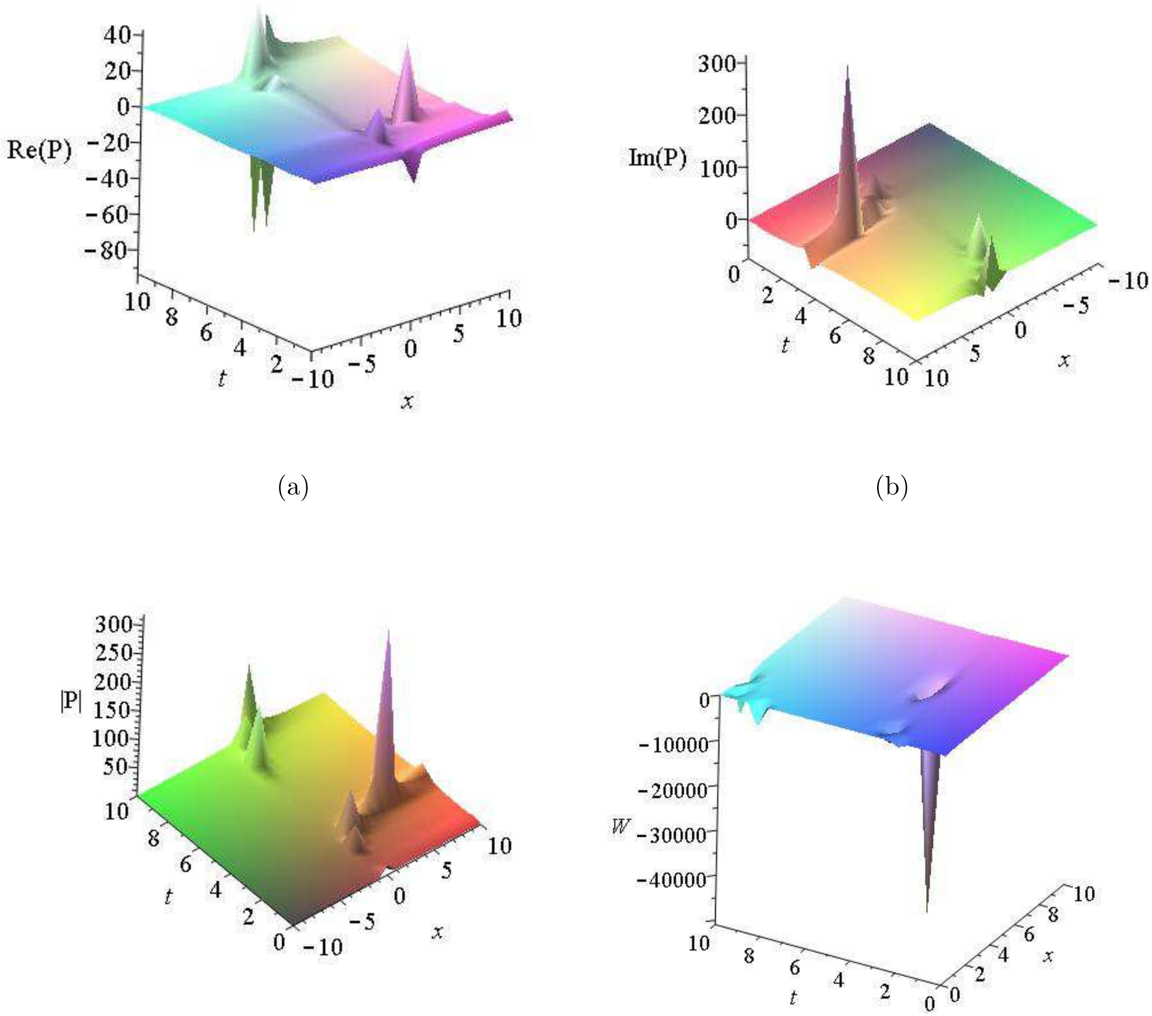
The wave profile of various solutions of Maccari system are further analysed graphically in terms of 2D and 3D plots in Figures 5.10-5.14. It shows the effect of various fractional parameters, arbitrary constants and modulus parameter on the wave profile. Figure 5.10 shows wave profiles for the solution given in Eq. (5.53) and it represents the complex function P and real function W of Maccari system in the form of a doubly periodic waves for $\alpha = \beta = \gamma = 1$ and $m = 0.25$. If the values of the fractional parameters are different from 1 then it causes the wave evolved as a singular doubly periodic as shown in Figure 5.11 for $\alpha = 0.5$, $\beta = 0.25$ and $\gamma = 0.7$ with $m = 0.25$. Figure 5.12 describes the effect of increase in the m value ($m = 1$) keeping the other fractional parameters unaltered ($\alpha = 0.5$, $\beta = 0.25$ and $\gamma = 0.7$), solution of Maccari system transforms to the singular wave. Singular periodic wave solutions are given in Figure 5.13 when $m = 0$. Figure 5.14 shows the effect of fractional parameters α, β and γ on wave profile of solution (5.53) for P and W. It represents singular periodic wave solutions.
To construct other double periodic wave solutions having Jacobi functions (cn(ϕ) and



(a) (b)

(c) (d)

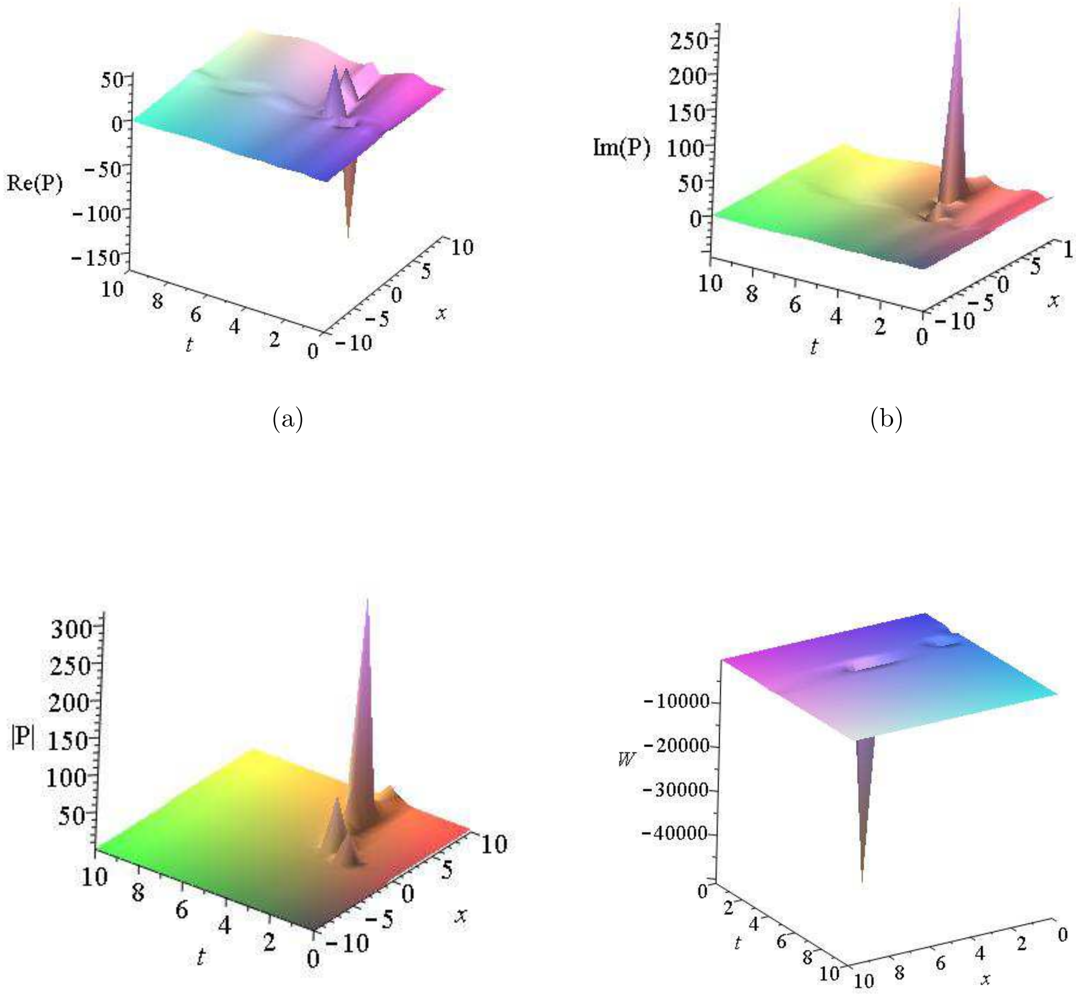
Figure 5.10: 3D profiles of Eq. (5.53) with $m = 0.25$, $\alpha = \beta = \gamma = 1$, $\sigma_x = 2$, $\sigma_y = 2$, $(\sigma_{11})_t = 2$, $(\sigma_{12})_t = 2$, $a_{11} = 1$, $a_{12} = 0.25$, $b_{11} = 0.5$, $b_{12} = 3$, $y = 1$.



(a) (b)

(c) (d)

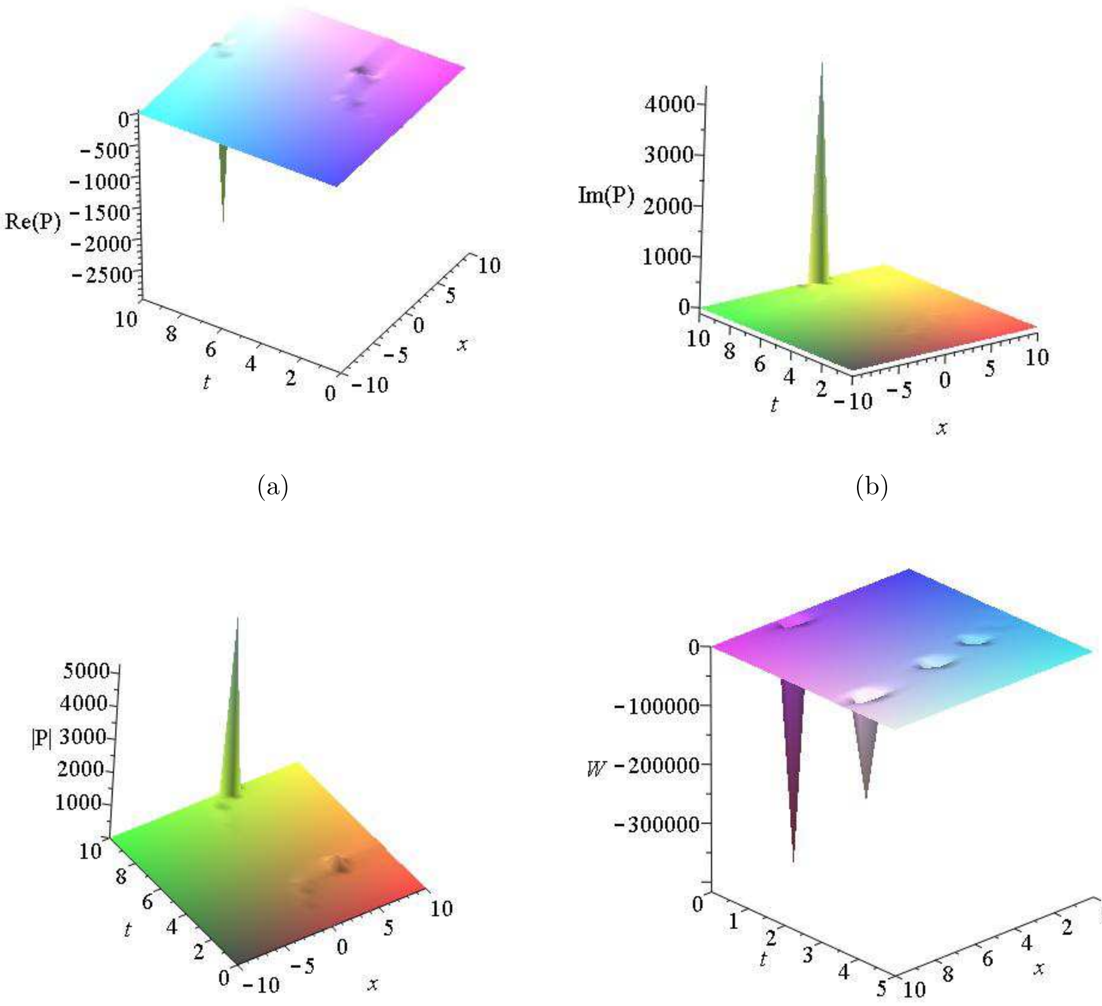
Figure 5.11: 3D profiles of Eq. (5.53) with $m = 0.25$, $\alpha = 0.5$, $\beta = 0.25$, $\gamma = 0.7$, $\sigma_x = 2$, $\sigma_y = 2$, $(\sigma_{11})_t = 2$, $(\sigma_{12})_t = 2$, $a_{11} = 1$, $a_{12} = 0.25$, $b_{11} = 0.5$, $b_{12} = 3$, $y = 1$.



(a) (b)

(c) (d)

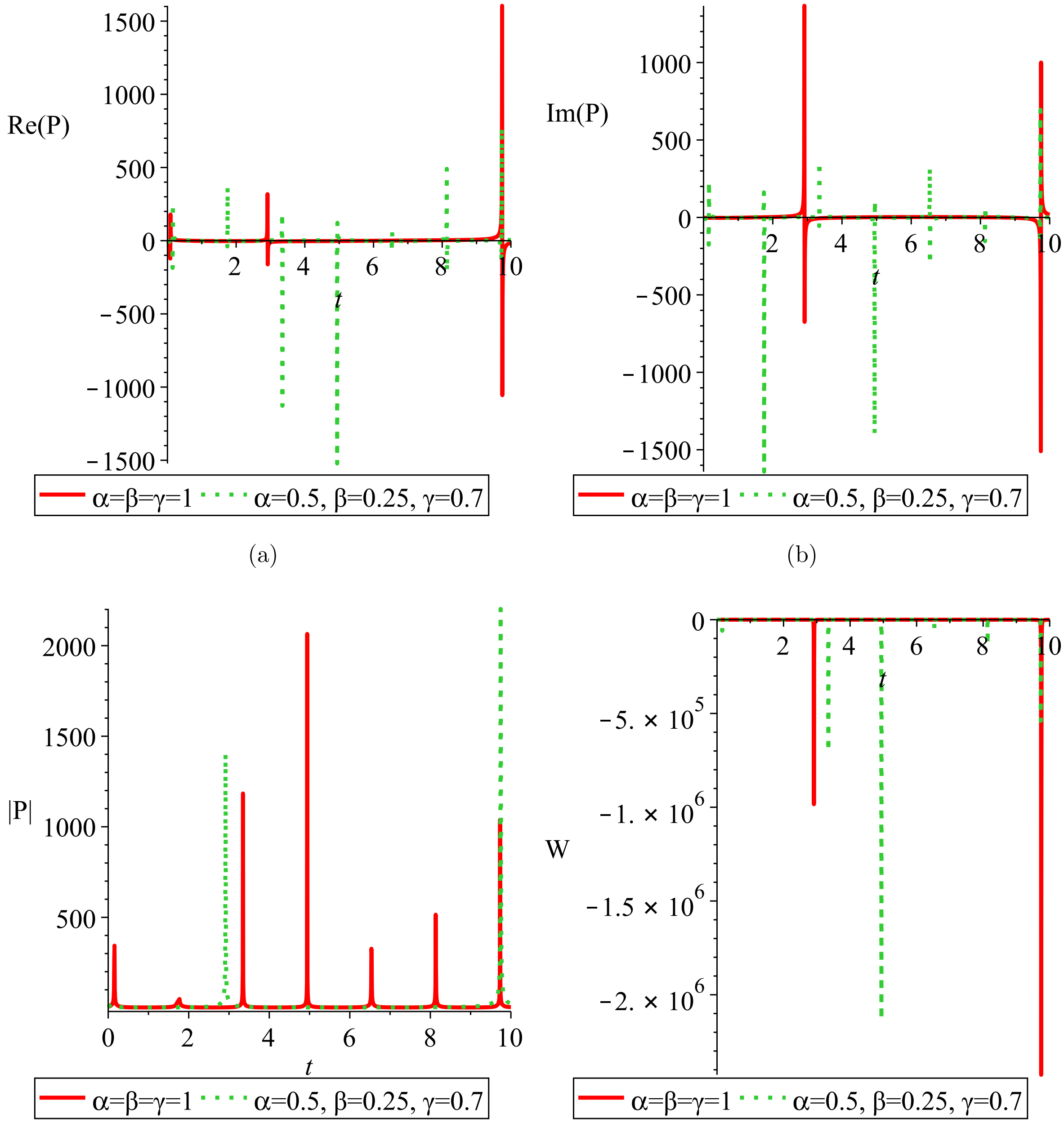
Figure 5.12: 3D profiles of Eq. (5.55) with $m = 1$, $\alpha = 0.5$, $\beta = 0.25$, $\gamma = 0.7$, $\sigma_x = 2$, $\sigma_y = 2$, $(\sigma_{11})_t = 2$, $(\sigma_{12})_t = 2$, $a_{11} = 1$, $a_{12} = 0.25$, $b_{11} = 0.5$, $b_{12} = 3$, $y = 1$.



(a) (b)

(c) (d)

Figure 5.13: 3D profiles of Eq. (5.57) with $m = 0$, $\alpha = 0.5$, $\beta = 0.25$, $\gamma = 0.7$, $\sigma_x = 2$, $\sigma_y = 2$, $(\sigma_{11})_t = 2$, $(\sigma_{12})_t = 2$, $a_{11} = 1$, $a_{12} = 0.25$, $b_{11} = 0.5$, $b_{12} = 3$, $y = 1$.



(a) (b)

(c) (d)

Figure 5.14: Effect of fractional parameters on solution (5.53) with $m = 0.25$, $\alpha = 0.5$, $\beta = 0.25$, $\gamma = 0.7$, $\sigma_x = 2$, $\sigma_y = 2$, $(\sigma_{11})_t = 2$, $(\sigma_{12})_t = 2$, $a_{11} = 1$, $a_{12} = 0.25$, $b_{11} = 0.5$, $b_{12} = 3$, $y = 1$.

dn(ϕ)) for the given system, the following expansion can be used

$$p(\phi) = \sum_{i=0}^{n} A_i \operatorname{cn}^i(\phi) + \sum_{j=1}^{n} B_j \operatorname{cn}^{-j}(\phi), \tag{5.59}$$

and

$$p(\phi) = \sum_{i=0}^{n} A_i \operatorname{dn}^i(\phi) + \sum_{j=1}^{n} B_j \operatorname{dn}^{-j}(\phi). \tag{5.60}$$

By utilizing Eqs. (5.59) and (5.60), one can find cnoidal waves and dnoidal waves solutions. Additional soliton and triangular periodic wave solutions for the system (5.28) can also be found for limiting case $m \to 0$ and $m \to 1$. We have omitted these expansions to avoid repetition.

5.2.3 Conclusion

The travelling wave solutions have been obtained for space-time fractional form of Maccari system in terms of doubly periodic waves using EJEFE method involving sn functions. In limiting case, as modulus parameter m approaches to 1 or 0, the singular or singular periodic wave solutions are obtained, respectively. The dispersion relation of the system is formulated which indicates the anomalous dispersion of waves.

Summary

Lie symmetries, conservation laws and exact solutions perform a crucial role in understanding modelling and physical importance of the systems of nonlinear PDEs. Linear dispersion analysis on the other hand helps to identify the normal/anomalous dispersion of waves. In present work, the nonlinear PDEs of integer and fractional order have been studied for symmetries, conservation laws, exact solutions and dispersion relation. Whole work is accordingly divided into five different chapters.

In chapter 1, the introduction of nonlinear PDEs has been given and some nonlinear phenomena are mentioned. In chapter 2, the relevant literature on the theory of Lie groups, various methodologies and linear analysis of PDEs is presented in an algorithmic way. In chapter 3, the integer order nonlinear PDEs with variable coefficients have been explored for solutions involving power series, trigonometric, hyperbolic, Jacobi and Weierstrass functions using Lie symmetry analysis, and conserved densities/fluxes (conservation laws) are successfully constructed for the equations by new conservation theorem and direct method. Chapter 4 has been concerned with (2+1)-dimensional time fractional new coupled ZK system which is successfully solved for Lie point symmetries, nonlocal conservation laws and solitary wave solutions including bright, dark and singular. The results are discussed graphically under the variation of system parameters. The anomalous dispersion of waves is noticed in the linear analysis. Also in this chapter, the other two examples of time fractional systems namely time fractional form of Wu-Zhang system in (2+1)-dimensions and time fractional 5^{th} order equation from Burgers hierarchy are successfully investigated for dispersion analysis, Lie point symmetries and conservation laws. Dispersion relation signifies the anomalous dispersion of waves for Wu-Zhang system whereas the wave profile of solutions resembles kink and bell shaped. The 5^{th} order Burgers equation found to possess power series solutions. In chapter 5, the solutions of potential YTSF equation with space and time fractional derivatives in (3+1)-dimensions have been obtained using improved F-expansion method and found to follow kink, periodic and singular

waves. The Maccari system in (2+1)-dimensions having space-time derivatives of fractional order have been solved for deriving anomalous/normal dispersion of waves and the equation possesses doubly periodic wave solutions. The software Maple is used to check the authenticity of obtained solutions and conservation laws, and for graphical representations of the solutions.

The exact solutions obtained in the present work may be used to interpret the various natural phenomenon arises in mathematical physics, chemistry and biological sciences. Further these solutions are helpful in analysing the stability analysis and can be compared with corresponding numerical solvers. The analysis of fission and fusion phenomena for solitons, electromagnetic interactions, quantum relativistic atom theory, phase isolation in several components bass system, and the relativistic energy-momentum relation can be explored. Numerous practical applications such as, the fluid-dynamics traffic model with fractional derivatives to eliminate the deficiency arising from the assumption of continuum traffic flow, and the nonlinear oscillation of earthquakes can be modeled by fractional derivatives. Fractional differentiation and integration operators can also be used for extending the diffusion and wave equations.

Future scope

In this work, the Lie classical symmetries are examined for integer and fractional order PDEs. The nonclassical symmetries, nonlocal symmetries and generalized symmetries can also be determined for these equations in future.

Also, for fractional PDEs, the conservation laws are calculated by new conservation theorem. For nontrivial conservation laws, direct method of multipliers can also be utilized for fractional PDEs, so we have a plan to construct conservation laws of fractional PDEs by direct method in future.

Appendix A

A.1 Extended infinitesimals of nonlinear PDEs

Extended infinitesimals of (2+1)-dimensional PDEs with dependent variables u, v, w as functions of t, x and y, and $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, $\mathcal{Z}_5$, $\mathcal{Z}_6$, $\mathcal{Z}_7$ are infinitesimals corresponding to t, x, y, u, v, w, respectively:

$$\begin{aligned}
{\mathcal{Z}_1}^{x} &= D_x\mathcal{Z}_1 - u_x(D_x\mathcal{X}_2) - u_t(D_x\mathcal{X}_1) - u_y(D_x\mathcal{X}_3),\\
{\mathcal{Z}_2}^{x} &= D_x\mathcal{Z}_2 - v_x(D_x\mathcal{X}_2) - v_t(D_x\mathcal{X}_1) - v_y(D_x\mathcal{X}_3),\\
{\mathcal{Z}_3}^{x} &= D_x\mathcal{Z}_3 - w_x(D_x\mathcal{X}_2) - w_t(D_x\mathcal{X}_1) - w_y(D_x\mathcal{X}_3),\\
{\mathcal{Z}_1}^{xxx} &= D_x{\mathcal{Z}_1}^{xx} - u_{xxx}(D_x\mathcal{X}_2) - u_{xxt}(D_x\mathcal{X}_1) - u_{xxy}(D_x\mathcal{X}_3),\\
{\mathcal{Z}_2}^{xxx} &= D_x{\mathcal{Z}_2}^{xx} - v_{xxx}(D_x\mathcal{X}_2) - v_{xxt}(D_x\mathcal{X}_1) - v_{xxy}(D_x\mathcal{X}_3),\\
{\mathcal{Z}_3}^{xxx} &= D_x{\mathcal{Z}_3}^{xx} - w_{xxx}(D_x\mathcal{X}_2) - w_{xxt}(D_x\mathcal{X}_1) - w_{xxy}(D_x\mathcal{X}_3),\\
{\mathcal{Z}_1}^{xyy} &= D_x{\mathcal{Z}_1}^{yy} - u_{xyy}(D_x\mathcal{X}_2) - u_{tyy}(D_x\mathcal{X}_1) - u_{yyy}(D_x\mathcal{X}_3),\\
{\mathcal{Z}_2}^{xyy} &= D_x{\mathcal{Z}_2}^{yy} - v_{xyy}(D_x\mathcal{X}_2) - v_{tyy}(D_x\mathcal{X}_1) - v_{yyy}(D_x\mathcal{X}_3),\\
{\mathcal{Z}_3}^{xyy} &= D_x{\mathcal{Z}_3}^{yy} - w_{xyy}(D_x\mathcal{X}_2) - w_{tyy}(D_x\mathcal{X}_1) - w_{yyy}(D_x\mathcal{X}_3),
\end{aligned} \tag{A.1.1}$$

where D_x, D_t and D_y denote total differential operators with respect to x, t and y, respectively and these are defined as follows

$$D_{x^j} = \partial_{x^j} + u_j\partial_u + v_j\partial_v + w_j\partial_w + u_{ji}\partial_{u_i} + v_{ji}\partial_{v_i} + w_{ji}\partial_{w_i} + \cdots\,,\ i,j = 1,...,3,$$

where, $x^1 = x$, $x^2 = t$, $x^3 = y$.

Extended infinitesimals for (1+1)-dimensional PDEs with one dependent variable $u(t, x)$, and $\mathcal{X}_4$, $\mathcal{X}_5$ and $\mathcal{Z}_4$ are infinitesimals corresponding to t, x and u, respectively:

$$\begin{aligned}
&{\mathcal{Z}_4}^{x} = D_x\mathcal{Z}_4 - u_x(D_x\mathcal{X}_5) - u_t(D_x\mathcal{X}_4),\\
&{\mathcal{Z}_4}^{xx} = D_x{\mathcal{Z}_4}^{x} - u_{xx}(D_x\mathcal{X}_5) - u_{xt}(D_x\mathcal{X}_4),\\
&{\mathcal{Z}_4}^{xxx} = D_x{\mathcal{Z}_4}^{xx} - u_{xxx}(D_x\mathcal{X}_5) - u_{xxt}(D_x\mathcal{X}_4),\\
&{\mathcal{Z}_4}^{xxxx} = D_x{\mathcal{Z}_4}^{xxx} - u_{xxxx}(D_x\mathcal{X}_5) - u_{xxxt}(D_x\mathcal{X}_4),\\
&{\mathcal{Z}_4}^{xxxxx} = D_x{\mathcal{Z}_4}^{xxxx} - u_{xxxxx}(D_x\mathcal{X}_5) - u_{xxxxt}(D_x\mathcal{X}_4),\\
&{\mathcal{Z}_4}^{xxxxxx} = D_x{\mathcal{Z}_4}^{xxxxx} - u_{xxxxxx}(D_x\mathcal{X}_5) - u_{xxxxxt}(D_x\mathcal{X}_4),\\
&{\mathcal{Z}_4}^{xxxxxxx} = D_x{\mathcal{Z}_4}^{xxxxxx} - u_{xxxxxxx}(D_x\mathcal{X}_5) - u_{xxxxxxt}(D_x\mathcal{X}_4).
\end{aligned} \tag{A.1.2}$$

α^{th} extended infinitesimals for (2+1)-dimensional fractional PDEs

The following α^{th} extended infinitesimals (${\mathcal{Z}_5}^{\alpha,t}$, ${\mathcal{Z}_6}^{\alpha,t}$, ${\mathcal{Z}_7}^{\alpha,t}$) related to RLF derivative using Eq. (2.61) are obtained for dependent variables u, v and w, and $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, $\mathcal{Z}_5$, $\mathcal{Z}_6$ and $\mathcal{Z}_7$ are infinitesimals corresponding to t, x, y, u, v and w, respectively:

$$\begin{aligned}
{\mathcal{Z}_5}^{\alpha,t} = {} & \frac{\partial^\alpha \mathcal{Z}_1}{\partial t^\alpha} + (\mathcal{Z}_{1u} - \alpha D_t(\mathcal{X}_6))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \mathcal{Z}_{1u}}{\partial t^\alpha} - \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix} D_t^n(\mathcal{X}_7)D_t^{\alpha-n}(u_x)\\
& + \mu_{11} + \mu_{12} + \mu_{13} - \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix} D_t^n(\mathcal{X}_8)D_t^{\alpha-n}(u_y)\\
& + \sum_{n=1}^{\infty}\Big[\begin{pmatrix}\alpha\\ n\end{pmatrix}\frac{\partial^n \mathcal{Z}_{1u}}{\partial t^n} - \begin{pmatrix}\alpha\\ n+1\end{pmatrix} D_t^{n+1}(\mathcal{X}_6)\Big] D_t^{\alpha-n}(u)\\
& + \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix}\frac{\partial^n \mathcal{Z}_{1v}}{\partial t^n}D_t^{\alpha-n}(v) + \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix}\frac{\partial^n \mathcal{Z}_{1w}}{\partial t^n}D_t^{\alpha-n}(w)\\
& + \left(\mathcal{Z}_{1v}\frac{\partial^\alpha v}{\partial t^\alpha} - v\frac{\partial^\alpha \mathcal{Z}_{1v}}{\partial t^\alpha}\right) + \left(\mathcal{Z}_{1w}\frac{\partial^\alpha w}{\partial t^\alpha} - w\frac{\partial^\alpha \mathcal{Z}_{1w}}{\partial t^\alpha}\right),
\end{aligned} \tag{A.1.3}$$

$$\begin{aligned}
\mathcal{Z}_6{}^{\alpha,t} ={}& \frac{\partial^\alpha \mathcal{Z}_2}{\partial t^\alpha} + (\mathcal{Z}_{2v} - \alpha D_t(\mathcal{X}_6))\frac{\partial^\alpha v}{\partial t^\alpha} - v\frac{\partial^\alpha \mathcal{Z}_{2v}}{\partial t^\alpha} - \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix} D_t^n(\mathcal{X}_7)D_t^{\alpha-n}(v_x)\\
&+ \mu_{21} + \mu_{22} + \mu_{23} - \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix} D_t^n(\mathcal{X}_8)D_t^{\alpha-n}(v_y)\\
&+ \sum_{n=1}^{\infty}\Big[\begin{pmatrix}\alpha\\ n\end{pmatrix}\frac{\partial^\alpha \mathcal{Z}_{2v}}{\partial t^\alpha} - \begin{pmatrix}\alpha\\ n+1\end{pmatrix} D_t^{n+1}(\mathcal{X}_6)\Big] D_t^{\alpha-n}(v)\\
&+ \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix}\frac{\partial^n \mathcal{Z}_{2u}}{\partial t^n}D_t^{\alpha-n}(u) + \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix}\frac{\partial^n \mathcal{Z}_{2w}}{\partial t^n}D_t^{\alpha-n}(w)\\
&+ \left(\mathcal{Z}_{2u}\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \mathcal{Z}_{2u}}{\partial t^\alpha}\right) + \left(\mathcal{Z}_{2w}\frac{\partial^\alpha w}{\partial t^\alpha} - w\frac{\partial^\alpha \mathcal{Z}_{2w}}{\partial t^\alpha}\right),
\end{aligned} \tag{A.1.4}$$

$$\begin{aligned}
\mathcal{Z}_7{}^{\alpha,t} ={}& \frac{\partial^\alpha \mathcal{Z}_3}{\partial t^\alpha} + (\mathcal{Z}_{3w} - \alpha D_t(\mathcal{X}_6))\frac{\partial^\alpha w}{\partial t^\alpha} - w\frac{\partial^\alpha \mathcal{Z}_{3w}}{\partial t^\alpha} - \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix} D_t^n(\mathcal{X}_7)D_t^{\alpha-n}(w_x)\\
&+ \mu_{31} + \mu_{32} + \mu_{33} - \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix} D_t^n(\mathcal{X}_8)D_t^{\alpha-n}(w_y)\\
&+ \sum_{n=1}^{\infty}\Big[\begin{pmatrix}\alpha\\ n\end{pmatrix}\frac{\partial^\alpha \mathcal{Z}_{3w}}{\partial t^\alpha} - \begin{pmatrix}\alpha\\ n+1\end{pmatrix} D_t^{n+1}(\mathcal{X}_6)\Big] D_t^{\alpha-n}(w)\\
&+ \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix}\frac{\partial^n \mathcal{Z}_{3u}}{\partial t^n}D_t^{\alpha-n}(u) + \sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix}\frac{\partial^n \mathcal{Z}_{3v}}{\partial t^n}D_t^{\alpha-n}(v)\\
&+ \left(\mathcal{Z}_{3u}\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \mathcal{Z}_{3u}}{\partial t^\alpha}\right) + \left(\mathcal{Z}_{3v}\frac{\partial^\alpha v}{\partial t^\alpha} - v\frac{\partial^\alpha \mathcal{Z}_{3v}}{\partial t^\alpha}\right),
\end{aligned} \tag{A.1.5}$$

where

$$\mu_{ij} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\begin{pmatrix}\alpha\\ n\end{pmatrix}\begin{pmatrix}n\\ m\end{pmatrix}\begin{pmatrix}k\\ r\end{pmatrix}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u_j)^r\frac{\partial^m}{\partial t^m}(u_j{}^{k-r})\frac{\partial^{n-m+k}\mathcal{Z}_j}{\partial t^{n-m}\partial u_j{}^k}, \tag{5.61}$$

where $i, j = 1, 2, 3$, and $u_1 = u$, $u_2 = v$, $u_3 = w$.

α^{th} extended infinitesimals for (1+1)-dimensional fractional PDEs

On using the Eq. (2.61), the following α^{th} extended infinitesimal has been obtained for the dependent variable $u(x,t)$ and $\mathcal{X}_{12}$, $\mathcal{X}_{13}$, $\mathcal{Z}_{11}$ are infinitesimals corresponding to t, x, u, respectively:

$$\begin{aligned}{\mathcal{Z}_{11}}^{\alpha,t} &= \frac{\partial^\alpha \mathcal{Z}_{11}}{\partial t^\alpha} + (\mathcal{Z}_{11u} - \alpha D_t(\mathcal{X}_{12}))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \mathcal{Z}_{11u}}{\partial t^\alpha} - \sum_{n=1}^{\infty} \begin{pmatrix} \alpha \\ n \end{pmatrix} D_t^n(\mathcal{X}_{13}) D_t^{\alpha-n}(u_x) \\ &\quad + \mu + \sum_{n=1}^{\infty} \Big[\begin{pmatrix} \alpha \\ n \end{pmatrix} \frac{\partial^\alpha \mathcal{Z}_{11u}}{\partial t^\alpha} - \begin{pmatrix} \alpha \\ n+1 \end{pmatrix} D_t^{n+1}(\mathcal{X}_{12}) \Big] D_t^{\alpha-n}(u),\end{aligned} \tag{A.1.6}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1} \begin{pmatrix} \alpha \\ n \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} k \\ r \end{pmatrix} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} (-u)^r \frac{\partial^m}{\partial t^m}(u^{k-r}) \frac{\partial^{n-m+k} \mathcal{Z}_{11}}{\partial t^{n-m} \partial u^k}.$$

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