A Thesis

On

FLAVOR OSCILLATIONS IN NEUTRINO

Submitted in partial fulfillment of the requirement for the award of the
degree of

Master of Science (Physics)
Submitted by
Prabhjot Kaur
Roll No: 300904010

Supervisor
Dr. Alka Upadhyay

Under
School of Physics and Materials Science
Thapar University, Patiala-147004
July 2011

CERTIFICATE

This is to certify that the thesis entitled "Flavor Oscillations in Neutrino" submitted by Miss Prabhjot Kaur , Roll No. 300904010 in the partial fulfillment of the requirement for award of the degree of MASTERS OF PHYSICS from the School of Physics and Material Science, Thapar University, Patiala. It is to certify that the matter embodied in this report is of the candidates own record and has not been submitted in part or full to any other university or institute for the award of any degree.

15/07/11

Dr. Alka Upadhyay
(Assistant Professor)
Supervisor
School of Physics and Material Science
Thapar University, Patiala

Countersigned by:

Dr.O.P.Pandey
(Prof. & Head)
School of Physics and Material Science
Thapar University, Patiala.

Dr. S.K. Mohapatra
Dean of academic affairs
School of Physics and Material Science
Thapar University, Patiala.

ACKNOWLEDGEMENT

I express my deep sense of gratitude and respect to my guide **Dr. Alka Upadhyay, Assistant Professor, School of Physics and Material Science, Thapar University, Patiala** for her keen interest and valuable guidance, strong motivation and constant encouragement during the course of the work. I thank her from the bottom of my heart for introducing me to Particle Physics. I thank her for her great patience, constructive criticism and useful suggestions apart from invaluable guidance to me. I am sure that the knowledge gained through my association with my supervisor shall go a long way in helping me to realize my goals in life.

I would also like to give special thanks to Research Scholar **Mrs. Meenakshi Batra**, for her keen interest and valuable guidance, strong motivation and constant encouragement during the course of the work.

I am also thankful to **My friends** for their help and support at every stage during the project report work.

Finally, I would like to express my deepest gratitude to **GOD** and **My Parents**, without whom I am nothing, to provide me great opportunities, everlasting support, big encouragement and lots of love.

Prabhjot Kaur

Date: 15/07/11

Prabhjot Kaur

ABSTRACT

Neutrino oscillations are sign for small masses of neutrinos. In standard model neutrinos are considered to be massless, hence this observation points to the possibility of physics beyond standard model. Thus the precise determination of neutrino oscillations are important for formulating theories beyond standard model. In this work, we study the phenomenology of two and three-flavor neutrino oscillations, focusing in particular on the prospects of recent experiments. We analyzed various experiments, which are currently going on, for their potential to determine these parameters precisely. After discussing the details about the detectors and reactors working on neutrino oscillations, we have given a theoretical formalism on solar and atmospheric neutrino oscillation probabilities for three generation case. We also discuss the three-flavor effects appearing in $\nu_e \rightarrow \nu_e$, $\nu_\mu \rightarrow \nu_e$, and $\nu_\mu \rightarrow \nu_\tau$ oscillation probabilities through their plots. The probability curves have been compared with the theoretical and experimental paper's curves and found to be consistent.

LIST OF TABLES, FIGURES & PLOTS

CONTENTS

Chapter 1

Introduction: **Page No.**

Chapter 2

Two Flavor Neutrino Oscillations:

Chapter 3

Three Flavor Neutrino Oscillations:

CHAPTER:1 INTRODUCTION

1.1 HIGH ENERGY PHYSICS

High Energy Physics is the study of the smallest components of matter and how they interact. Our world is made up of very tiny particles known as fundamental particles[1]. These tiny particles are invisible to the naked eyes even most powerful electron microscope cannot see them. Through the experiments, high-energy physicists have pieced together a theory of the basic structure of matter and its forces.

At one time, scientists thought the smallest possible unit of matter was the atom. But experiments showed that the atom itself was made of electrons plus a nucleus containing protons and neutrons-and that protons and neutrons themselves break down into fundamental particles called quarks. According to the Standard Model, all matter is composed of two categories of elementary particles: quarks and leptons, six kinds of each. These particles of matter interact through four fundamental forces of nature: gravity, electromagnetism, the strong force (which holds together the nucleus), and the weak force (which governs certain kinds of radioactive decay). These forces are also composed of particles, called gauge bosons. One of these, the photon, transmits electromagnetism. The W and Z particles carry the weak force. (The Standard Model actually considers electromagnetism and the weak force to be different aspects of one underlying “electroweak” force.) The gluon is responsible for the strong force. The graviton is presumed to carry gravity, but this particle has not yet been found and is not incorporated into the current theory.

1.2 STANDARD MODEL

The Standard Model of Particle Physics (abbreviated as SM) Summarizes all of our knowledge of the particle world. It gives us a picture of how our universe looks on the fundamental level. Through a combination of theories and experiments, a mathematical model that describes all particles observed so far by physicists has been worked out. This model is called the Standard model.

SM Proposes that all of the Nature can be explained through a list of fundamental particle and a set of equations that can explain the interaction between and within these particles. The Standard Model of particle physics is a theory concerning the electromagnetic, weak and strong nuclear interactions which mediate the dynamics of the known subatomic particles. It is a Single framework that describes the behavior of all known subatomic particle incorporating quarks and leptons through weak and electromagnetic interactions.

The Standard Model consists of elementary particles divided into two classes:

Bosons: Particles that transmit forces

Bosons are particles which obey Bose–Einstein statistics and when we interchange the two bosons the wave function of the system is unchanged. Several Bosons can occupy the same quantum state. So Bosons are having spin either 0 or 1 or 2.

Fermions: Particles that make up matter

Fermions are the particles which obeys Fermi–Dirac statistics and the Pauli Exclusion Principle and when we interchange the two fermions the wave function of the system is changed. Two fermions cannot occupy the same quantum state. So fermions are having spin 1/2. Neutrinos are also having existence in the well known standard model but they are mass less fermions.

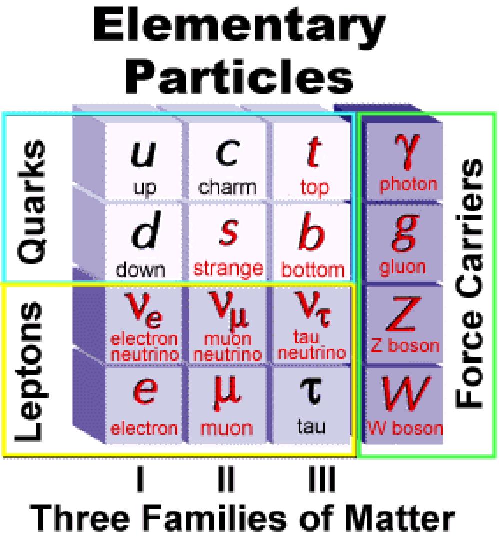


Fig 1.1: Six leptons, six quarks and force carriers

1.3 Fundamental Particles & Their Interactions

Quarks and leptons are the fundamental particles. Quarks are the building blocks of protons and neutrons but not electrons.

There are six types of quarks. Named as Q

$$Q = \begin{pmatrix} u \\ d \\ s \\ c \\ b \\ t \end{pmatrix}$$

i.e. up, down, strange, charm, bottom, and top. All quarks have their antiparticles know to be antiquarks. Quarks have positive charges, while antiquarks have negative charges. Notice that the quarks with charge +2/3 come in a group of three, as do the quarks with charge $-1/3$, as do the electron, muon and tau, and the electron, muon and tau neutrinos respectively.

NAME of the Quark	**SPIN**	**ELECTRIC CHARGE**
Up	½	+2/3
Charm	½	+2/3
Top	½	+2/3
Down	½	-1/3
Strange	½	-1/3
Bottom	½	-1/3

Table 1.1: Quarks with their quantum numbers.

There is also another type of matter that makes up atoms. They are known as leptons.

$$L = \begin{pmatrix} e \\ \mu \\ \tau \\ \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

Leptons are similar to quarks. Although some leptons are larger than quarks and some are smaller. The most well known lepton is the electron. The other types of leptons are tau, muon and neutrinos. Tau and muon leptons have some electrical charge while Neutrinos are having very little charge.

Fundamental Interactions:

There are four basic fundamental interaction or forces between them.

1) Electromagnetic Interaction: One of the four fundamental forces, the electromagnetic force itself through the forces between charges (Coulomb's Law) and the magnetic force, both of which are summarized in the Lorentz force law. Fundamentally, both magnetic and electric forces are manifestations of an exchange force involving the exchange of photon. The quantum approach to the electromagnetic force is called quantum electrodynamics or QED. The electromagnetic force is a force of infinite range which obeys the inverse square law. The electromagnetic force holds atoms and molecules together.

2) Strong Interaction: A force which can hold a nucleus together against the enormous forces of repulsion of the protons is strong indeed. However, it is not an inverse square force like the electromagnetic force and it has a very short range. It is the strongest of the four fundamental forces. Since the protons and neutrons which make up the nucleus are themselves considered to be made up of quarks, and the quarks are considered to be held together by the color force, the strong force between nucleons may be considered to be a residual color force. In the standard model, therefore, the basic exchange particle is the gluon which mediates the forces between quarks.

3) Weak Interactions: One of the four fundamental forces, the weak interaction involves the exchange of the intermediate vector bosons, the W and the Z. Since the mass of these particles is on the order of 80 GeV. The role of the weak force in the transmutation of quarks makes it the interaction involved in many decays of nuclear particles which require a change of a quark from one flavor to another. It was in radioactive decay such as beta decay that the existence of the weak interaction was first revealed. The weak interaction is the only process in which a quark

can change to another quark, or a lepton to another lepton - the so-called "flavor changes". The weak interaction acts between both quarks and leptons.

4) Gravitational Interaction: Gravity is the weakest of the four fundamental forces, yet it is the dominant force in the universe for shaping the large scale structure of galaxies, stars, etc. The gravitational force between two masses m_1 and m_2 is given by the relationship:

$$F_{gravity} = \frac{Gm_1m_2}{r^2} \text{ where } G = 6.67 \times 10^{-11} Nm^2/kg^2$$

This is often called the "universal law of gravitation" and G the universal gravitation constant. It is an example of an inverse square law force. The force is always attractive and acts along the line joining the centers of mass of the two masses.

1.4 LEPTONS:

A lepton is an elementary particle with no fundamental constituent into it. leptons are produced in radioactive decays of many atomic nuclei. In these processes they appear along with their sibling, the electron. For example, so long as it is not trapped in a nucleus, a neutron turns into a proton by emitting an electron and neutrino in the process. This is called beta decay, where the instability of the neutron is due to it having a slightly greater mass than does a proton. Nature seeks the state of lowest energy, which translates in this case to the state of lowest mass. It is the small excess mass of a neutron that makes it (slightly) unstable when left in isolation. If you had a large sample of neutrons, each of them free of the others, then after about ten minutes, half will have decayed by beta radioactivity. If we denote the neutron and proton by the symbols n, p, and the electron and neutrino by e^-, ν, then beta decay of the neutron is summarized by the expression:

$$n \rightarrow p + e^- + \nu$$

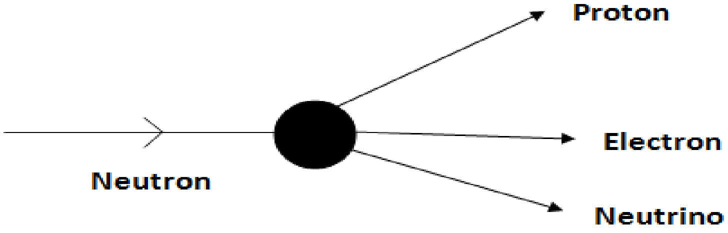


Fig 1.2: Beta decay

Thus the neutron has no electrical charge overall; this is preserved in the beta decay as the proton has one unit positive, counterbalancing the negative electron. The proton, being the lightest state made of three quarks, is stable (or, at least, if protons are unstable, their mean lifetime is greater than 1032 years!)

It was first verified in the Cowan–Reines neutrino experiment[2] conducted by Clyde Cowan and Frederick Reines in 1956 by the inverse beta decay. Basically inverse beta decay is just that, the reverse of regular beta decay or positron emission. It involves a proton absorbing an antineutrino, transforming into a neutron and a positron.

$$\bar{\nu}_e + p \rightarrow n + e^+$$

To be more precise, what happens in the first case is the antineutrino comes close enough to one of the up quarks inside the proton, the antineutrino emits a W^- particle, turning into a positron in the process. The up quark absorbs the W particle and becomes a down quark as a result. In the second case the roles are reversed, neutrino emits a W^+ particle, turning a down quark into an up quark as a result. This is essentially the reverse process involved in normal beta decays.

1.41 GENERATION OF LEPTONS

Two main classes of leptons exist, charged leptons (also known as the electron-like leptons), or neutral leptons (better known as neutrinos). Charged leptons are very common, and can combine with other particles to form various composite Particles such as atoms and positronium, while neutrinos rarely interact with anything, and are consequently rarely observed.

Flavor	**Generation**	**Electric charge**	**Spin**	**Lepton Number**
Electron (e^-)	First	-1	1/2	1
Electron neutrino (ν_e)	First	0	1/2	1
Muon (μ^-)	Second	-1	1/2	1
Muon neutrino (ν_μ)	Second	0	1/2	1
Tau (τ^-)	Third	-1	1/2	1
Tau neutrino (ν_τ)	Third	0	1/2	1

Table : 1.2 Generation of leptons

There are six types of leptons, known as flavors, forming three generations. The first generation is the electronic leptons, comprising the electrons (e^-) and electron neutrinos (ν_e); the second is the muonic leptons, comprising muons (μ^-) and muon neutrinos (ν_μ); and the third is the tauonic leptons, comprising taus (τ^-) and tau neutrinos (ν_τ). Electrons have the least mass of all the charged-leptons. The heavier muons and taus will rapidly change into electrons through a process of particle decay: the transformation from a higher mass state to a lower mass state.

As we are discussing about presence of neutrinos, Thus we can say that neutrinos are:

- Particles that accompany radioactive beta decay.
- The products of nuclear reactions.
- The most abundant particles.
- The lightest massive particles.
- The most weakly interacting particles.
- Particles that may be their own antiparticles.

Neutrinos are most often created or detected with a well define manner. Neutrinos are coming from sun, atmosphere, earth and stars.

Current excitement in neutrinos

Solar neutrinos problem[3].

Atmospheric neutrinos problem[4].

First let us talk about solar neutrinos.

The Sun is a natural nuclear fusion reactor, powered by proton-proton chain reaction which converts four Hydrogen nuclei (protons) into Helium, neutrinos and energy. That means when neutron is changing into proton or proton is changing into proton there is a emission of neutrino and the excess energy is released as gamma rays and as kinetic energy of particles including the neutrinos which travel from Sun's core to earth without any appreciable loss by Sun's outer layers. Therefore the Sun sends enormous numbers of neutrinos in all directions. Every second, about 65 billion (6.5×10^{10}) solar neutrinos pass through every square centimeter on the part of the Earth that faces the Sun. As sun send enormous numbers of neutrinos but on earth we are getting nearly half of the neutrinos than that of original value. Therefore this arises a very big problem that where these half neutrinos are going??? This is generally know as the *''SOLAR NEUTRINO PROBLEM''* than further research shows that neutrinos are having flavor.

1.42 Types of Neutrinos

There are three known types of neutrinos:

1) Electron neutrino (ν_e)

2) Muon neutrino (ν_μ)

3) Tau neutrino (ν_τ)

Similarly the antineutrinos of these are ($\overline{\nu}_e$, $\overline{\nu_\mu}$ and $\overline{\nu}_\tau$). Here electron neutrino means it consist a some proportion of mass of electron, mass of muon and mass of tauon but mass of electron is more than that of muon and tauon so for that it was given a name electron neutrino similarly for muon and tauon neutrino. The distinction between neutrinos and antineutrinos lies in spin of these particles that is spin of neutrinos is opposite in the direction to the direction of its motion.

The possibility of sterile neutrinos-relatively light neutrinos which do not participate in the weak interaction but which could be created through flavor oscillation is unaffected by these Z-boson-based measurements, and the existence of such particles is in fact hinted by experimental data from the LSND[5] experiment. A sterile neutrino is hypothetical neutrinos which do not interact via any of the fundamental interactions. It is a right handed neutrino or left handed antineutrino.

1.5 Neutrino & their Sources

Atmospheric Neutrinos: Atmospheric neutrinos result from interaction of cosmic rays with atomic nuclei in earth's atmosphere creating shower of particles many of which are unstable produce neutrinos when they decay.

In the year 1965, collaboration of particle physicists from Tata Institute of fundamental and Research (TIFR), India, Osaka City University, Japan and Dusham University, UK recorded the first cosmic ray neutrino interaction in an underground laboratory in KGF Goldmines[6] in India.

Akhmedov in 1993[7] showed that atmospheric neutrinos are decay particles or products of hadrons produced by cosmic ray interactions in atmosphere.

F. Ronga in 2000[8] gave the result on atmospheric neutrinos and the main emphasis was given to Soudan-2 and Macro experiments. Both experiments observed atmospheric neutrinos anomalies in agreement with ν_μ to ν_τ oscillations with maximum mixing.

Kajita in 2003[9] discovered the atmospheric neutrino oscillations. The atmospheric neutrino anomaly is resolved due to neutrino oscillations by the high statistic measurement of atmospheric neutrinos in Super-KamioKande.

Solar Neutrinos: Solar neutrinos originate from the nuclear fusion powering the Sun and other stars. The details of the operation of the Sun are explained by the Standard Solar Model. In short: when four protons fuse to become one helium nucleus, two of them have to convert into neutrons, and each such conversion releases one electron neutrino.

The Sun sends enormous numbers of neutrinos in all directions. Every second, about 65 billion (6.5×10^{10}) solar neutrinos pass through every square centimeter on the part of the Earth that faces the Sun[10].Since neutrinos are insignificantly absorbed by the mass of the Earth, the surface area on the side of the Earth opposite the Sun receives about the same number of neutrinos as the side facing the Sun.

Cosmological Neutrinos: It is thought that, just like the cosmic microwave background radiation left over from the Big Bang, there is a background of low energy neutrinos in our Universe. In the 1980s it was proposed that these may be the explanation for the dark matter thought to exist in the universe. From particle experiments, it is known that neutrinos are very light. This means that they move at speeds close to the speed of light. Thus, dark matter made from neutrinos is termed "hot dark matter". The problem is that being fast moving, the neutrinos would tend to have spread out evenly in the universe before cosmological expansion made them cold enough to congregate in clumps.

The energy of supernova neutrinos ranges from a few to several tens of MeV. However, the sites where cosmic rays are accelerated are expected to produce neutrinos that are at least one million times more energetic. The origin of the cosmic rays was attributed to supernovas by Walter Baade and Fritz Zwicky this hypothesis was refined by Vitaly L. Ginzburg and Sergei I. Syrovatsky who attributed the origin to supernova remnants, if the efficiency of acceleration in supernova remnants is about 10 percent. Ginzburg and Syrovatskii's hypothesis is supported by the specific mechanism of "shock wave acceleration" happening in supernova remnants, which is consistent with the original theoretical picture drawn by of Enrico Fermi, and it is receiving support from observational data.

1.6 NEUTRINO OSCILLATIONS

In five distinct measurements, Super-Kamiokande finds neutrinos apparently "disappearing". Since it is unlikely that momentum and energy are actually vanishing from the universe, a more plausible explanation is that the types of neutrinos we can detect are changing into types we cannot detect. This phenomenon is known as neutrino oscillation. Neutrino oscillation is not black magic - there are very specific predictions for the behavior of our data if neutrinos oscillate, and we have uniformly found the data in good agreement with these predictions. Unfortunately, a non-mathematical explanation of why neutrino oscillation and neutrino mass are inseparable is difficult. A more detailed, yet still non-mathematical explanation, and a full calculation, using basic quantum mechanics, are provided for those inquiring minds who want to know.

It is much easier to explain why our data appear to lead inexorably to the conclusion that neutrinos are oscillating. The secret lies in the fact that the neutrinos observed by Super-

Kamiokande are without exception produced at great distances from the detector. Neutrinos produced in the atmosphere arrive at the detector from distances of about 40 km (if produced above it) to 12,000 km (if produced on the other side of the Earth). These distances are significantly greater than any measurements made to date with neutrinos from accelerators or nuclear reactors on Earth. The Sun, clearly, is much, much further still. The great distances not only allow us to detect effects which would be invisible with a closer neutrino source, they also allow us to measure the behavior of neutrinos produced over a great range of distances. This ability in fact leads to some of the most dramatic evidence that oscillations are occurring.

The probability of a neutrino changing type is related to the distance travelled by the neutrino from its point of production to its point of detection. As a general rule, neutrinos travelling greater distances will exhibit greater depletion from oscillation. Moreover, the oscillation probability varies smoothly over increasing distance. Hence, tests of the angular variation of the muon rate are the best to determine whether the overall deficit of muon neutrinos fits the hypothesis of oscillation as opposed to other some unaccounted-for systematic effect. In all cases, the angular (neutrino path length) variation of the muon rate agrees with the oscillation hypothesis.

The oscillation probability is also a function of the neutrino energy, and here again we sample a large range of values, roughly a factor of 1000 from the Sub-GeV sample to the through-going upward muons. Although estimation of the neutrino energy giving rise to a given interaction is necessary approximate, there are no inconsistencies in the behavior of the different data sets, and all the atmospheric data agree well with the oscillation hypothesis regardless of energy. It was observed that neutrinos are changing their flavor from one into another. This is possible only if neutrinos are having some mass. If neutrinos are massless than how they change flavor. Neutrino oscillations are a quantum mechanical phenomenon predicted by Bruno Pontecorvo where by a neutrino created with a specific lepton flavor (electron, muon or tau) can later be measured to have a different flavor. The probability of measuring a particular flavor for a neutrino varies periodically as it propagates. Neutrino oscillation is of theoretical and experimental interest since observation of the phenomenon implies that the neutrino has a non-zero mass, which is not part of the original Standard Model of particle physics.

Why the neutrino mass must be non-zero???

The reason neutrino oscillation is relevant to the question of neutrino mass is that massless neutrinos cannot oscillate. Put another way, observation of oscillation implies that the masses of the neutrinos involved cannot be equal to one another. Since they cannot be equal to one another, they cannot both be zero. In fact it is quite likely that if any neutrinos have non-zero mass, all of them do.

1.7 Experiments: Detections, Type of Detectors, Location

EXPERIMENTS STUDYING SOLAR NEUTRINOS

Abbreviation	**Full Name**	**Location**	**Operation**	**Type**	**Type of detector**
BOREXINO	Boron Experiment	Gran Sasso,Italy	May 2007	ν_e	scintillation
Double Chooz	Double Chooz Reactor Neutrino Experiment	Chooz, France	2011-	$\bar{\nu}_e$	scintillation
SNO	Sudbury Neutrino Observatory	Creighton Mine, Ontario	1999-2006	ν_e, ν_μ, ν_τ	cherenkov
HOMESTAKE–CHLORINE[11]	Homestake chlorine Experiment	Homestake Mine, South Dakota	1967–1998	ν_e	Radiochemical
GALLEX	GALLium Experiment	Gran Sasso, Italy	1991-1997	ν_e	Radiochemical

Table 1.3: Experiments for Solar Neutrinos

EXPERIMENTS STUDYING ATMOSPHERIC NEUTRINOS

Abbreviation	Full Name	Location	Operation	Type	Type of detector
MINOS	Main Injector Neutrino Oscillation Search	Illinois and Minnesota, United States	2005–	ν_e, ν_μ	scintillation
Kamiokande	Kamioka Nucleon Decay Experiment	Kamioka, Japan	1986–1995	ν_e	cherenkov
Super-K	Super-Kamiokande	Kamioka, Japan	1996–	ν_e, ν_μ, ν_τ	cherenkov
IceCube	IceCube Neutrino Detector	South Pole, Antarctica	2006–	ν_e, ν_μ, ν_τ	cherenkov
NEVOD	Cherenkov water detector NEVOD	Moscow, Russia	1993–	ν_μ	cherenkov

Table 1.4: Experiments for Atmospheric Neutrinos

India-based Neutrino Observatory

INO(India-based Neutrino Observatory) collaboration is progressing well in India.It has presently about 55 members from about fifteen Institutions and Universities. INO is a proposed particle physics research project to primarily study atmospheric neutrinos. This is one of the biggest experimental particle physics projects undertaken in India.The project is expected to be completed in 2015 at an estimated cost of $250 million. The location of the site was supposed to be southwest of Masinagudi in the Nilgiri Hills of South India. INO is looking for the study of the problems via stimulations. The geometry of the detector is stimulated on the computer and interactions are fed in. Different processes like interaction of the neutrinos with the detector materials are studied and this is used to determine which are more promising processes to study.

References:

[1] High Energy Physics: ”An Introduction by Sharon Butler”(2007)

[2] The Reines-Cowan Experiments,Los Alamos Science No.**25**: (1997) Retrieved (2010)

[3] Herbert H. Chen,Phys.Rev.Lett 55,1534-1536(1985)

[4] A. J. Davies, Xiao-Gang He,Phys.Rev.D 46,3208-3210(1992)

[5] Sergio Palomares-Ruiz, J.Phys.Conf.Ser. 39,307-309, arXiv:0602083v1(2006)

[6] V.S.Smirinov, Science Academy, **70, A,** No.1, (2004) 11–25,

[7] E. Kh . Akhmedov , W. Fratt et. al. Phys Rev D-48 1140 (1993)

[8] F. Ronga, vol-87, Issue 1-3, (2000) 135-144

[9] K. Takaak, N. Butsuri : Gakkaishi (Fo221.A), Vol -58, No-5,(2003) 326-331

[10] K. Kodama et al. Phys Lett B **504**: 218. doi:10.1016/S0370-2693(01)00307-0

[11] M. Fukugita, T. Yanagida, Phys. Rev. Lett. 65,(1990)

Chapter 2 Two Flavor Neutrino Oscillations

2.1 TWO GENERATION CASE

Let ν_e, ν_μ be the flavor eigen states and ν_1 , ν_2 the mass eigen states with masses m_1 and m_2, respectively and both are having momentum p. States in the flavor and mass bases are related by a mixing matrix U[1]. where U is an orthogonal transformation for two dimensions. Then a matrix Equation can be written that relates the eigen states to the mass eigen states.

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \qquad (2.1)$$

where

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (2.2)$$

where θ is the mixing angle, therefore by putting the value of eqn (2.2) in eqn (2.1), we get

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \qquad (2.3)$$

Now let us see the time evolution of ν_e state

At time t = 0

$$|\nu_e(t=0)> = |\nu_e > = -\sin\theta|\nu_1 > + \cos\theta|\nu_2 > \qquad (2.4)$$

After time t = t

$$|\nu_e(t=t)> = |\nu_e(t) > = -\sin\theta|\nu_1 > e^{\frac{-iE_1t}{h}} + \cos\theta|\nu_2 > e^{\frac{-iE_2t}{h}} \qquad (2.5)$$

Where E_1 and E_2 represents the energy of mass eigen state and is given by:

$$E_1 = (p^2c^2 + m_1^2c^4)^{1/2} \quad \text{and} \quad E_2 = (p^2c^2 + m_2^2c^4)^{1/2} \qquad (2.6)$$

and $p_1 = p_2$

According to standard natural units

$$\hbar = c = 1$$

Therefore eqn (2.6) becomes:

$$E_1 = (p^2 + m_1^2)^{1/2} \quad \text{and} \quad E_2 = (p^2 + m_2^2)^{1/2} \qquad (2.7)$$

Also, neutrinos are assumed to be relativistic:

$$\gamma = \frac{E}{m_0 c^2} = \frac{(p^2c^2 + m_0^2 c^4)^{1/2}}{m_0 c^2} \gg 1$$

Then $p \gg m_0$

$$E = (p^2 + m_0^2)^{1/2} = p(1 + {m_0^2}/{p^2})^{1/2} \qquad (2.8)$$

Now using binomial expansion therefore the equation (2.8) becomes:

$$E = p + \frac{1}{2}\frac{m_0^2}{p}$$

Therefore energy of mass eigen states can be written as:

$$E_1 = p + \frac{1}{2}\frac{m_1^2}{p} \quad \text{and} \quad E_2 = p + \frac{1}{2}\frac{m_2^2}{p} \qquad (2.9)$$

Now let us discuss how does the ν_e propagates in time by putting the value of eqn (2.9) in eqn (2.5)

$$|\nu_e(t)> = -\sin\theta|\nu_1> e^{-i\left(p+\frac{1}{2}\frac{m_1^2}{p}\right)t} + \cos\theta|\nu_2> e^{-i\left(p+\frac{1}{2}\frac{m_2^2}{p}\right)t}$$

$$|\nu_e(t)> = e^{-i\left(p+\frac{1}{2}\frac{m_1^2}{p}\right)t}\left(-\sin\theta|\nu_1> + \cos\theta|\nu_2> e^{-i(p+\frac{1}{2}\frac{m_1^2-m_2^2}{p})t}\right) \qquad (2.10)$$

Now using some definitions and substitutions

$$\Delta m^2 = m_1^2 - m_2^2$$

As

$$\text{Speed} = \frac{\text{dist}}{\text{time}}, \quad \text{i.e.} \quad t = \frac{x}{c} = x$$

And

$$e^{-iz} = e^{-i\left(p+\frac{1}{2}\frac{m_1^2}{p}\right)t}$$

By substituting these value in equation (2.10), we get

$$|\nu_e(t)> \; = \; e^{-iz} \, (-\sin\theta|\nu_1> + \cos\theta|\nu_2> e^{-i(p+\frac{1}{2}\frac{\Delta m^2}{p})x}) \qquad (2.11)$$

Therefore we can say that when it propagates it contains both the mass eigen states i.e. ν_1&ν_2[2]

2.2 <u>Oscillation probabilities in Vacuum</u>

To calculate the probability for a “pure" ν_e state to oscillate into a ν_μ state, we must square the quantum mechanical amplitude[3] that describes this transition.

$$P\,(\nu_e \longrightarrow \nu_\mu) = |<\nu_\mu|\nu_e(t)>|^2$$

Where $\quad < \nu_\mu| = \cos\theta < \nu_1| + \sin\theta < \nu_2|$

Therefore amplitude can be written as:

$$< \nu_\mu|\nu_e(t)> = \; e^{-iz}(-\sin\theta\cos\theta + \sin\theta\cos\theta \; e^{\frac{i}{2}\frac{\Delta m^2}{p}x})$$

As probability is square of the amplitude, thus

$$P\,(\nu_e \longrightarrow \nu_\mu) = |<\nu_\mu|\nu_e(t)>|^2$$

$$= e^{iz-iz} \, \sin^2\theta\cos^2\theta \, (-1 + e^{\frac{i}{2}\frac{\Delta m^2}{p}x}) \, (-1 + e^{\frac{-i}{2}\frac{\Delta m^2}{p}x})$$

Since the neutrino is relativistic, we can also make the substitution: p = E_v, and likewise, we will make the substitution x = L.

$$=\sin^2\theta\cos^2\theta \, (-1 + e^{\frac{i}{2}\frac{\Delta m^2}{E_v}L}) \, (-1 + e^{\frac{-i}{2}\frac{\Delta m^2}{E_v}L})$$

for making our expression simpler put $A = \frac{1}{2}\frac{\Delta m^2}{E_v} L$

$$=\sin^2\theta\cos^2\theta \, (-1 + e^{iA}) \, (-1 + e^{-iA})$$

$$= \frac{4 \, \mathrm{Sin}^2\theta \, \mathrm{Cos}^2\theta}{4} \, (e^{iA} - 1) \, (e^{-iA} - 1)$$

$$= \sin^2 2\theta/4 \, (1 - e^{iA} - e^{-iA} + 1)$$

$$= \sin^2 2\theta/4 \, (2 - (e^{iA} + e^{-iA})$$

As $\quad \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$$= \sin^2 2\theta/4 \, (2 - 2\cos A)$$

$$= \sin^2 2\theta/2\ (1-\cos A)$$

By putting the value of A, Therefore we write probability as

$$P\ (\nu_e \longrightarrow \nu_\mu) = \frac{1}{2}\sin^2 2\theta\ (\ 1-\cos(\frac{1}{2}\frac{\Delta m^2}{E_v} L\ )\ )$$

Also $\quad 1 - \cos 2\theta = 2\sin^2\theta$

$$= \frac{1}{2}\sin^2 2\theta\ (\ 1-\cos 2(\frac{1}{4}\frac{\Delta m^2}{E_v} L\ )\ )$$

Thus we can write it as:-

$$P\ (\nu_e \longrightarrow \nu_\mu) = \sin^2 2\theta\ \sin^2(\ \frac{1}{4}\frac{\Delta m^2}{E_v} L\ ) \qquad (2.12)$$

Now, we can write the argument of the second sin term above so it's dimensionless by introducing the appropriate number of h's and c's.

$$(\ \frac{1}{4}\frac{\Delta m^2}{E_v} L\ ) \quad \rightarrow \quad (\ \frac{1}{4}\frac{\Delta m^2}{E_v}\frac{c^4}{\hbar c} L\ )$$

Let's write the above quantity in units that are convenient for an experimental physicist[1].

We would like the variables in the above equation to have the following units:

$\Delta m^2 c^4 (eV^2)$,

L (meters),

and E_v (MeV).

Now we have to solve the value of ħc:

$$\hbar c = \frac{h}{2\Pi} \times c$$

$$= \frac{6.62\times10^{-34}}{2\times3.14} \times 3\times10^8$$

$$=3.15\times10^{-26}$$

$$=\frac{3.15\times10^{-26}}{1.6\times10^{-19}}$$

$$=1.968\times10^{-7}$$

$$=196.8\times10^{-9} \cong 197$$

Now, If we substitute ħc with 197 nm eV, then we can write the quantity as:

$$(\ \frac{1}{4}\frac{\Delta m^2}{E_v}\frac{c^4}{\hbar c} L\ ) \quad \rightarrow \quad (\ \frac{\Delta m^2 c^4}{4\times197\ \text{nm eV}}\ \frac{L}{E_v}\ )\ (\ \frac{10^{-6}}{10^{-9}}\ \frac{\text{MeV}/\text{eV}}{\text{m}/\text{nm}}\ )$$

$$= (\ \frac{1}{788}\frac{\Delta m^2 c^4}{E_v} L\ )\ (\frac{10^{-6}}{10^{-9}}\ \frac{\text{MeV}/\text{eV}}{\text{m}/\text{nm}}\ )$$

$$= (1.269\times10^{-3}\ \Delta m^2 c^4\ \frac{L}{E_v})\ (\frac{10^{-6}}{10^{-9}}\ \frac{MeV/_{eV}}{m/_{nm}})$$

$$= (1.27\ \Delta m^2\ \frac{L}{E_v}\ )$$

Substitute this value in equation (2.12) , we get

$$\boxed{\mathbf{P(\nu_e \longrightarrow \nu_\mu)(L,E) = \ sin^2 2\theta\ sin^2(\ 1.27\ \Delta m^2\ \frac{L}{E_v})}} \quad (2.13)$$

If neutrino oscillations occur, the mixing probability ($\sin^2 2\theta$) and the mass difference (Δm^2) are determined by nature. Physicists can probe different regions of Δm^2 by adjusting the distance between the neutrino source and the detector (L) as well as the neutrino energy E_v .

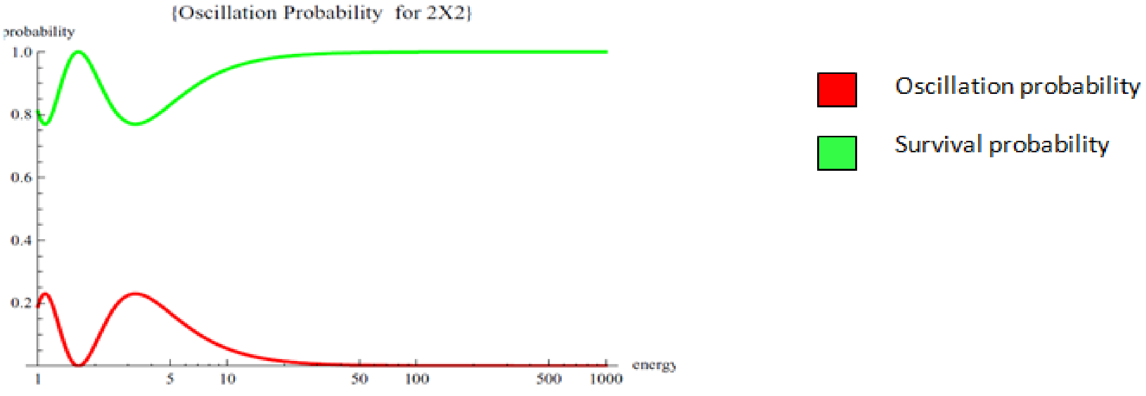
Maximal mixing occurs if $\sin^2 2\theta$ is ~ 1 (i.e.,$\theta=45^o$). In the case of atmospheric neutrinos, it is suspected that maximal mixing occurs.

The L/E term is the quantity of interest when exploring different mass regions[5]. In the LSND[8] experiment, L was about 30 meters and E was about 30 MeV giving an L/ E[9] of ~ 1.

If MiniBooE[4] is going to explore the same Δm^2 and $\sin^2\theta$ region, then its L/E must be similar to the LSND[6] value. In the case of MiniBooNE, the neutrinos travel about 500 m and have energies on the order of 500 MeV. So, roughly speaking, the MiniBooNE experiment is designed to explore the same Δm^2 region as LSND, but with higher sensitivity (i.e., down to lower values of $\sin^2 2\theta$).

2.3 Probability oscillations for Solar Neutrino

For studying the probability for solar neutrinos, we are taking parameters from CHOOZ Experiment[7] which is a long-baseline Experiment[10]. In this experiment length is taken as L=5000m and mass square difference as $\Delta m_{12}{}^2 = 8.1\times10^{-4} ev^2$ and the mixing angle is $\sin^2 2\theta_{12}=0.23$ and the energy is in MeV's. We are plotting a graph between probability and energy. In This red is showing Oscillation Probability for one neutrino changing into another and green colour shows survival probability.



Plot 2.1: Probability Vs Energy(MeV)for solar neutrinos.

The two flavor oscillation probability expression has dependence on following parameters:

Mixing Angle θ where mixing angle dependence is contained in term $Sin^2 2\theta$. Second is mass squared difference denoted as Δm^2_{ij}. Both values exist in nature. The physicist probe the mass eigen values and mixing angle by manipulating energy of neutrinos coming from source and distance between source and detector.

Here changing θ to Π/2-θ, mixing angle dependence does not change. Second factor in neutrino oscillation probability expression is $Sin^2 1.27\ \Delta m^2_{ij} L/E$. The oscillation cannot occur if $\Delta m^2_{ij} L/E << 1$That why the experimentalist work for L and E in such a way so as to make $\Delta m^2_{ij} L/E \sim 1$.

The graph here shows that two flavor oscillations do not completely describe the phenomenology there should be participation of third flavor as well this led to oscillation among three flavors.

References:

[1] Darrel Smith,Embry-Riddle University July 18 (2001), Lecture Series for MiniBooNE.

[2] Introduction to the physics of massive and mixed neutrinos,S.M. Bilenky, 99-107 (2010)

[3] C. Giunti, P. Giuria, Nuclear Phys B,Vol 521(1998) 3-36

[4] S. J. Brice, L. Bugel, et.al, arXiv:1005.3525v1(2006)

[5] W.Grimus, Phys.Polon.B30:3067-3088,arXiv:9910340v1(1999)

[6] Sergio Palomares-Ruiz,Phys.Conf.Ser.39,307-309, arXiv:0602083v1(2006)

[7] Phys.: Conf. Ser. 120 052023,(http://iopscience.iop.org/1742-6596/120/5/052023)(2003)

[8] Sergio Palomares-Ruiz,J.Phys.Conf.Ser. 39,307-309, arXiv:0602083v1(2006)

[9] Stephen J. Parke "Neutrino Oscillation Phenomenology" 60510,parke@fnal.gov

[10] V. Barger, M. Bishai,et.al, arXiv:0705.4396v1

Chapter 3 Three Flavor Neutrino Oscillations

3.1 Three Generation Case

According to standard model of particle physics neutrinos are massless. The phenomenon of neutrino oscillation comes from the fact that neutrino has non-zero mass. The neutrinos produced are $\nu_e, \overline{\nu_e}$ or others, which are linear combinations of the mass Eigen states that is:

$$\nu_e = a\,\nu_1 + b\,\nu_2 + c\,\nu_3$$

While this shows that neutrinos have mass, the absolute neutrino mass scale is still not known. This is because neutrino oscillations are sensitive only to the difference in the squares of the masses.

Neutrino mixing is expressed mathematically by the mixing matrix U as shown where the mixing angles θ_{ij} (ij = 12, 13, 23) and the phase angle δ

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

Where U is a mixing matrix.

3.11 Standard Parameterization of Mixing Matrix[1]

Consider the unitary 3×3 mixing matrix for Dirac neutrinos and introduce the standard parameters (three mixing angles and one phase) which characterizes it. Let us consider three orthogonal and normalized vectors

$$|i > \qquad (i = 1, 2, 3)$$

$$\langle i|k\rangle = \delta_{ik} \tag{3.1}$$

In order to obtain three general "mixed" vectors we will perform three Euler rotations. The first rotation will be performed at the angle θ_{12} around the vector |3>. The new orthogonal and normalized vectors are

$$\begin{aligned} &|1>^{(1)} = c_{12}\,|1> + s_{12}\,|2> \\ &|2>^{(1)} = -s_{12}\,|1> + c_{12}\,|2> \\ &|3>^{(1)} = |3> \end{aligned} \tag{3.2}$$

Where $c_{12} = \cos\theta_{12}$ and $s_{12} = \sin\theta_{12}$

Eqn (3.2) can be written as

$$|\nu>^{(1)} = U^{(1)}\,|\nu> \tag{3.3}$$

Here

$$|\nu>^{(1)} = \begin{pmatrix} |1^{(1)}> \\ |2^{(1)}> \\ |3^{(1)}> \end{pmatrix} \quad \text{and} \quad |\nu> = \begin{pmatrix} |1> \\ |2> \\ |3> \end{pmatrix} \tag{3.4}$$

And mixing matrix:

$$U^{(1)} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.5}$$

Let us perform second rotation at the angle θ_{13} around vector the $|2>^{(1)}$ At this we introduce the CP phase δ,

$$\begin{aligned} &|1>^{(2)} = c_{13}|1>^{(1)}| + s_{13}e^{-i\delta}|3>^{(1)} \\ &|2>^{(2)} = |2>^{(1)} \\ &|3>^{(2)} = -s_{13}e^{-i\delta}|1>^{(1)} + c_{13}|3>^{(1)} \end{aligned} \tag{3.6}$$

In the matrix

$$|\nu>^{(2)} = U^{(2)}|\nu>^{(1)} \tag{3.7}$$

Here

$$U^{(2)} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \qquad (3.8)$$

Now peform rotation around vector $|1>^{(2)}$ at the angle θ_{23}

$$|1>^{\text{mix}} = |1>^{(2)}$$

$$|2>^{\text{mix}} = c_{23}|2>^{(2)} + s_{23}|3>^{(2)}$$

$$|3>^{\text{mix}} = -s_{23}|2>^{(2)} + c_{23}|3>^{(2)} \qquad (3.9)$$

Therefore

$$\left|v^{mix}\ >\right. = U^{(3)}|v>^{(2)} \qquad (3.10)$$

And

$$U(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \qquad (3.11)$$

From equation (3.3),(3.7) and (3.10)

$$\left|v^{mix}\ >\right. = U|v>$$

Where

$$U = U^{(3)}U^{(2)}U^{(1)}$$

From equation (3.5),(3.8) and (3.11)

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ -s_{13}s_{23}e^{-i\delta} & c_{23} & s_{23}c_{13} \\ -c_{23}s_{13}e^{-i\delta} & -s_{23} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - s_{13}c_{12}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{13}c_{23}s_{12}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \qquad (3.12)$$

Where U is the mixing matrix. This matrix is characterized by three mixing angles θ_{12}, θ_{23} and θ_{13} and the phase δ. In the case of CP conservation in the lepton sector $U^* = U$. Thus, the phase δ is responsible for effects of the CP violation: if CP is conserved, $\delta = 0$.The mixing angles are parameters which can take values in the ranges $0 \leq \theta_{12} \leq \pi$, $0 \leq \theta_{13} \leq \pi$, $0 \leq \theta_{23} \leq \pi$ and the phase δ can take values in the range $0 \leq \delta \leq 2\pi$. The origin of mixing is not explained by the Standard Model (SM). Indeed, the massive neutrinos are the first experimental evidence for physics beyond SM. Neutrino mixing is then considered as a correction within SM.

3.2 Normal And Inverted Hierarchy

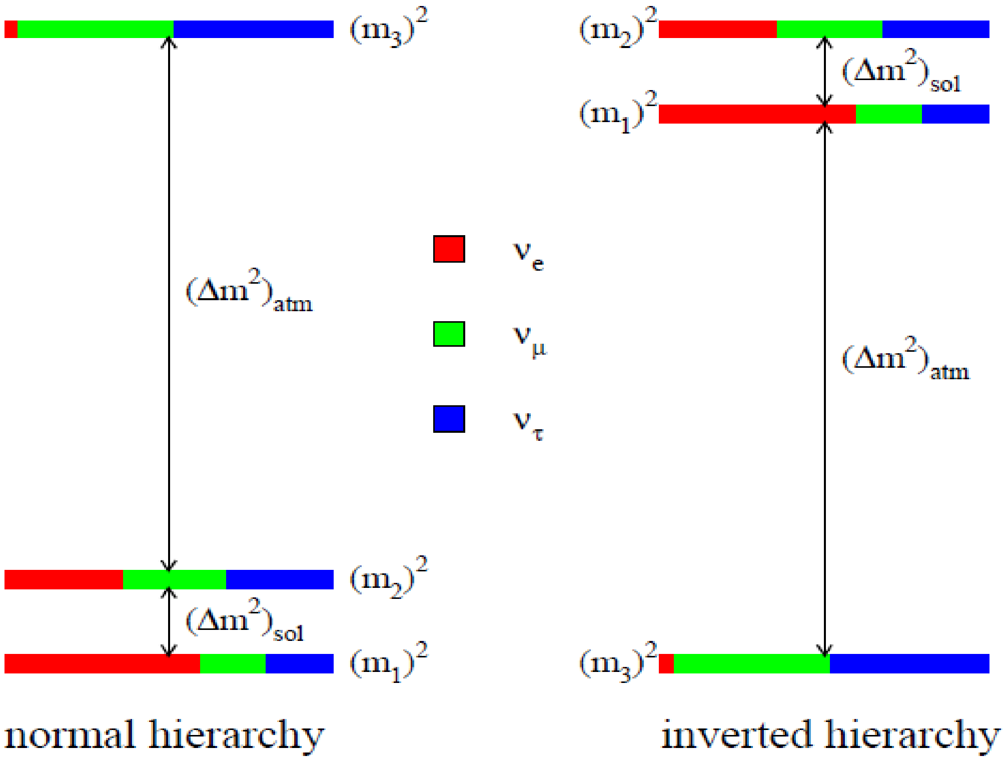


Fig 3.1: The two possible hierarchies of neutrino mass eigenstates, normal hierarchy NH (left) and inverted hierarchy IH(right)

The given fig.3.1 shows the normal and inverted hierarchy, As we earlier told that neutrinos[2] are sensitive to the mass square difference therefore m_1^2 shows the mass of electron neutrino and m_2^2 shows the mass of muon neutrino and m_3^2 shows the mass of tau neutrino. The two independent mass-squared differences: $\Delta m_{21}{}^2 = m_2^2 - m_1^2$ and $\Delta m_{31}^2 = m_3^2 - m_1^2$. Without loss of generality, we take Δm_{21}^2 to be the smaller mass-squared difference (known as the solar mass-squared difference, since it governs the oscillations of solar neutrinos) and Δm_{31}^2 to be the larger mass-squared difference (known as the atmospheric mass-squared difference, since it governs the oscillations of atmospheric neutrinos). Solar neutrino data require Δm_{21}^2 to be positive. However, data from atmospheric neutrinos as well as accelerator neutrino experiments (K2K and MINOS) constrain only the magnitude of Δm_{31}^2 but not its sign. Determination of sign (Δm_{31}^2) is also called mass hierarchy determination, in the limit where the lightest mass Eigen state is essentially mass less.

If sign of (Δm_{13}^2) is positive,

$$\Delta m_{13}^2 > 0 \text{ -------- Normal Mass Hierarchy}$$

Then we have the following mass pattern $m_3 \gg m_2 \gg m_1$, This is referred to as normal hierarchy(NH).

If sign (Δm_{31}^2) is negative,

$$\Delta m_{13}^2 < 0 \text{ ------Inverted Mass Hierarchy}$$

Then the mass pattern is $m_2 \geq m_1 \gg m_3$. This is referred to as inverted hierarchy (IH).

Therefore for solar neutrino electron neutrino changing into muon neutrino and vice-versa for both the hierarchies but for the case of atmospheric neutrino in normal hierarchy muon neutrino changing into tau neutrino and vice-versa and for inverted hierarchy tau neutrino changing into electron neutrino and vice-versa. Thus we can say that for atmospheric neutrino all the three neutrino changing into one another. These three active neutrinos mix to form three mass eigen states. The mixing is parameterized by three mixing angles θ_{12}, θ_{13}, θ_{23} and one CP violating phase δ_{CP}.

3.3 Oscillations of Flavor Neutrino

Neutrino oscillations are periodic transitions[3] between different flavor neutrinos in neutrino beams. We will consider a beam of neutrinos which was produced in CC weak decays

$$(\Pi^+ \rightarrow \mu^+ + \nu_\mu \,, \mu^+ \rightarrow e^+ + \nu_e + \overline{\nu_\mu}\,)$$

at an accelerator or at a reactor or at a Neutrino Factory or in decays of radioactive nuclei (β-beam), etc.

In the quantum field theory the dependence of states on the time is given by them Schrodinger equation.

$$i\frac{\partial|\Psi(t)>}{\partial t} = H|\Psi(t)> \qquad (3.13)$$

Where H is the total Hamiltonian and the general solution of equation (3.13) is:

$$|\Psi(t)> = e^{-iHt}|\Psi(0)> \qquad (3.14)$$

Where $|\Psi(0)>$ is the state at the initial time (t=0)

Let the initial state of flavor neutrino is ν_l (l= e,μ,τ) and is given by:

$$|\nu_l = \sum_{i=0}^{3} U_{li}^{*}|\nu_i> \qquad (3.15)$$

Where U_{li}^{*} is the unitary matrix and $|\nu_i>$ is the mass eigen state.

Thus we have at time (t=0)

$$|\Psi(0)> = |\nu_i> \qquad (3.16)$$

As when we apply Hamiltonian on some wave function then we get the energy operator therefore taking into account that

$$H|\nu_i> = E_i|\nu_i> \qquad (3.17)$$

Where $E_i = \sqrt{p_i^2 + m_i^2}$ (3.18)

From eqn (3.14) & (3.15), we find the state of the left-handed neutrino at the time t ≥ 0 ,we have

$$|\nu_l>_t = e^{-iHt}|\nu_l> = \sum_{i=1}^{3} e^{-iE_i t}U_{li}^{*}|\nu_i> \qquad (3.19)$$

Similarly, for the state of the right-handed antineutrino

$$|\overline{\nu_l}>_t = e^{-iHt}|\overline{\nu_l}> = \sum_{i=1}^{3} e^{-iE_i t}U_{li}|\overline{\nu_i}> \qquad (3.20)$$

Generally, neutrino energies E_i (i=1,2,3) are different

As neutrinos are having flavor and they oscillate into one another therefore we can write the amplitude of the transition $\nu_l \rightarrow \nu_{l'}$ during the time t

$$A(\nu_l \rightarrow \nu_{l'}) = \sum_{i=1}^{3} U_{il'}\ e^{-iE_i t}\ U_{li}^{*} \qquad (3.21)$$

Where

$U_{il'}$ is the amplitude of transition from the state $|\nu_i >$ into $|\nu_{l'} >$

$e^{-iE_i t}$ is the propagation in the state with definite mass.

U_{li}^{*} is the amplitude of transition from initial flavor state $|\nu_l >$ into the state of neutrino with definite mass $|\nu_i >$

Analogously, for the amplitude of the transition $\overline{\nu_l} \rightarrow \overline{\nu_{l'}}$ during the time t is given by:

$$A(\overline{\nu_l} \rightarrow \overline{\nu_{l'}}) = \sum_{i=1}^{3} U_{il'}^{*}\, e^{-iE_i t}\ U_{li} \qquad (3.22)$$

Probability is the square of the amplitude, thus we can write it as:

$$P(\nu_l \rightarrow \nu_{l'}) = \left| \sum_{i=1}^{3} U_{il'}\, e^{-iE_i t}\ U_{li}^{*} \right|^2 \qquad (3.23)$$

And

$$P(\overline{\nu_l} \rightarrow \overline{\nu_{l'}}) = \left| \sum_{i=1}^{3} U_{il'}^{*}\, e^{-iE_i t}\ U_{li} \right|^2 \qquad (3.24)$$

From eqn (3.23) & (3.24) we can find possible relations between probabilities:

$$\sum_{l} P(\nu_l \rightarrow \nu_{l'}) = 1\ ,\ \ \sum_{l'} P(\overline{\nu_l} \rightarrow \overline{\nu_{l'}}) = 1$$
$$\sum_{l'} P(\nu_l \rightarrow \nu_{l'}) = 1\ ,\ \ \sum_{l} P(\overline{\nu_l} \rightarrow \overline{\nu_{l'}}) = 1 \qquad (3.25)$$

These relations are a consequence of the unitary of the mixing matrix. In fact we have

$$\sum_{l'} P(\nu_l \rightarrow \nu_{l'}) = \left| \sum_{i=1}^{3} U_{il'}\ e^{-iE_i t}\ U_{li}^{*} \right|^2$$
$$= \left(\sum_{i=1}^{3} U_{il'}\ e^{-iE_i t} U_{li}^{*} \right) \left(\sum_{k=1}^{3} U_{kl'}^{*}\ \ e^{iE_k t} U_{lk} \right)$$
$$= \sum_{i,k,l'} \sum U_{il'}\ U_{kl'}^{*}\ e^{-i(E_i - E_k)t}\ U_{li}^{*} U_{lk}$$
$$= \sum_i \delta_{ik}\, e^{-i(E_i - E_k)t} U_{li}^{*} U_{lk}$$

When k = i

$$= \sum_i \delta_{ii}\, e^{-i(E_i - E_i)t} U_{li}^{*} U_{li}$$

Using the property $\delta_{ii} = 1$

$$= \sum_{i} U_{li}^{*} U_{li} = 1 \tag{3.26}$$

The following equation can easily obtained by direct comparison of eqn (3.23) &(3.24) i.e.

$$P(\nu_l \rightarrow \nu_{l'}) = P(\overline{\nu_{l'}} \rightarrow \overline{\nu_l}) \tag{3.27}$$

Eqn (3.27) follows the probabilities of the transitions $(\nu_l \rightarrow \nu_l)$ and $(\overline{\nu_l} \rightarrow \overline{\nu_l})$ are equal.

$$P(\nu_l \rightarrow \nu_l) = P(\overline{\nu_l} \rightarrow \overline{\nu_l}) \tag{3.28}$$

$$P(\nu_{l'} \rightarrow \nu_l) = P(\overline{\nu_l} \rightarrow \overline{\nu_{l'}}) \tag{3.29}$$

This eqn is valid in the case of CP invariance in the lepton sector, therefore we have

$$U_{li} = U_{li}^{*} \quad \text{(Dirac neutrino)}$$
$$\rho_i U_{li} = U_{li}^{*} \text{ (Majorana neutrino)} \tag{3.30}$$

Where $\rho_i = \pm 1$

Using Eqn (3.30) in eqn (3.23) & (3.24) we may get the relation for the eqn (3.29)

The probabilities of the transitions $(\nu_l \rightarrow \nu_l)$ and $(\overline{\nu_l} \rightarrow \overline{\nu_l})$ are equal due to CPT invariance.

Let us notice that in case of CP invariance eqn (3.27) & (3.29) follows that

$$P(\nu_l \rightarrow \nu_{l'}) = P(\nu_{l'} \rightarrow \nu_l)$$
$$P(\overline{\nu_l} \rightarrow \overline{\nu_{l'}}) = P(\overline{\nu_{l'}} \rightarrow \overline{\nu_l})$$

These relations are consequence on the invariance under time reversal (T) which holds for both CP invariance and CPT invariance.

$$P(\nu_\alpha \rightarrow \nu_{\alpha'}) = \left| \sum_{i=1}^{3} U_{\alpha' i}\, e^{-i(E_i - E_j)t}\, U_{\alpha i}^{*} \right| \tag{3.31}$$

Where E_i and E_j are the two different energy levels.

In quantum field theory, states of particles are characterized by their momenta, helicities, mass etc. let us assume that a mixed neutrino states is characterized by their momentum p, masses m_i

Let us assume

$$p_i = p \tag{3.32}$$

As neutrinos are having less mass but high energy, if the energies of neutrinos are more therefore their momentum is also more thus we can take into account that

$$\frac{m_i^2}{p^2} \ll 1$$

Therefore we have,

$$\mathrm{E_i} \cong \mathrm{p} + \frac{\mathrm{m_i^2}}{\mathrm{2p}}$$

$$\mathrm{E_j} \cong \mathrm{P} + \frac{\mathrm{m_j^2}}{\mathrm{2p}}$$

Thus, $$\mathrm{E_i} - \mathrm{E_j} = \mathrm{p} + \frac{\mathrm{m_i^2}}{\mathrm{2p}} - \mathrm{p} - \frac{\mathrm{m_j^2}}{\mathrm{2p}}$$

$$= \frac{\mathrm{m_i^2} - \mathrm{m_j^2}}{\mathrm{2p}}$$

$$= \frac{\Delta \mathrm{m_{ji}^2}}{\mathrm{2p}}$$

Where $\Delta \mathrm{m_{ji}^2} = \mathrm{m_i^2} - \mathrm{m_j^2}$

As E ≈ p , therefore

$$\mathrm{E_i} - \mathrm{E_j} = \frac{\Delta \mathrm{m_{ji}^2}}{\mathrm{2E}} \tag{3.33}$$

Let us suppose t is the difference of neutrino production time and detection time for the ultrarelavistic neutrino.

$$\mathrm{t} \cong \mathrm{L} \tag{3.34}$$

where L is the distance between source of neutrino and the detector therefore Eqn (3.33) becomes

$$(\mathrm{E_i} - \mathrm{E_j})\mathrm{t} = \frac{\Delta \mathrm{m_{ji}^2}}{\mathrm{2E}}\,\mathrm{L} \tag{3.35}$$

Substitute this value in Eqn (3.31), we get

$$\mathrm{P}(\nu_\alpha \rightarrow \nu_{\alpha'}) = \left| \sum_{i=1}^{3} U_{\alpha' i}\, e^{-i\frac{\Delta m_{ji}^2 L}{2E}}\, U_{\alpha i}^* \right|^2 \tag{3.36}$$

Now let us take the unitary condition

$$\sum_i U_{\alpha' i}\, U_{\alpha i}^* = \delta_{\alpha\alpha'} \tag{3.37}$$

From Eqn (3.36) & (3.37), we find the convenient expression for the neutrino probabaility

$$\mathrm{P}(\nu_\alpha \rightarrow \nu_{\alpha'}) = \left| \delta_{\alpha\alpha'} + \sum_{i \neq j} U_{i\alpha'}\, (e^{-i\frac{\Delta m_{ji}^2 L}{2E}} - 1) U_{\alpha i}^* \right|^2 \tag{3.38}$$

Analogously, for the case of antineutrino we can write it as:

$$P(\overline{\nu_\alpha} \to \overline{\nu_{\alpha'}}) = \left| \delta_{\alpha\alpha'} + \sum_{i\neq j} U^*_{i\alpha'} \left(e^{-i\frac{\Delta m^2_{ji} L}{2E}} - 1\right) U_{\alpha i} \right|^2 \tag{3.39}$$

From Eqn (3.31) the transition probability $\nu_\alpha \to \nu_{\alpha'}$ can also be presented as:

$$P(\nu_\alpha \to \nu_{\alpha'}) = \sum_{i,k} U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k} e^{-i\frac{\Delta m^2_{ji}}{2E}L}$$

$$= \sum_i \left|U_{\alpha' i}\right|^2 |U_{\alpha i}|^2 + 2\mathrm{Re} \sum_{i>k} (U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k} e^{-i\frac{\Delta m^2_{ji}}{2E}L}) \tag{3.40}$$

Further, from the unitary relation we can easily obtain the following relation

$$\sum_i \left|U_{\alpha' i}\right|^2 |U_{\alpha i}|^2 = \delta_{\alpha\alpha'} - 2\,\mathrm{Re}\sum_{i>k} (U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k}) \tag{3.41}$$

Substitute eqn (3.41) in eqn (3.40), we get:

$$= \delta_{\alpha\alpha'} - 2\,\mathrm{Re}\sum_{i>k} (U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k} + 2\mathrm{Re} \sum_{i>k} (U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k} e^{-i\frac{\Delta m^2_{ji}}{2E}L}$$

$$= \delta_{\alpha\alpha'} - 2\,\mathrm{Re}\sum_{i>k} (U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k} \left(1 - e^{-i\,\frac{\Delta m^2_{ki} L}{2E}}\right) \tag{3.42}$$

Finally, taking into account that for any complex a and b

$$\mathrm{Re}(ab) = \mathrm{Re}(a)\mathrm{Re}(b) - \mathrm{Im}(a)\mathrm{Im}(b)$$

$$= \delta_{\alpha\alpha'} - 2\,\mathrm{Re}\sum_{i>k} (U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k} \left(1 - \mathrm{Cos}\frac{\Delta m^2_{ki} L}{2E} + i\,\mathrm{Sin}\frac{\Delta m^2_{ki} L}{2E}\right)$$

$$\boxed{\begin{aligned} P(\nu_\alpha \to \nu_{\alpha'}) = \; & \delta_{\alpha\alpha'} - 2\sum_{i>k} Re(U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k}) \left(1 - \mathrm{Cos}\frac{\Delta m^2_{ki} L}{2E}\right) \\ & + 2\sum_{i>k} Im(U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k})\, \mathrm{Sin}\frac{\Delta m^2_{ki} L}{2E} \end{aligned}} \tag{3.43}$$

Analogously, for the probability of the transition $\overline{\nu_\alpha} \to \overline{\nu_{\alpha'}}$,we get

$$\boxed{\begin{aligned} P(\overline{\nu_\alpha} \to \overline{\nu_{\alpha'}}) = \; & \delta_{\alpha\alpha'} - 2\sum_{i>k} Re(U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k}) \left(1 - \mathrm{Cos}\frac{\Delta m^2_{ki} L}{2E}\right) \\ & - 2\sum_{i>k} Im(U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k})\, \mathrm{Sin}\frac{\Delta m^2_{ki} L}{2E} \end{aligned}} \tag{3.44}$$

From these expressions we conclude that neutrino transition probabilities depend on the parameter $\frac{L}{E}$. They are determined by the elements of the neutrino mixing $U_{\alpha i}$ and independent mass-squared differences Δm^2_{ki} .

3.4 Neutrino oscillations in vacuum: Atmospheric Neutrino Case

In the case of three-neutrinos mixing, there are three flavor transition channels[4] for neutrinos:

$$\nu_e \leftrightarrow \nu_\mu, \nu_e \leftrightarrow \nu_\tau , \nu_\mu \leftrightarrow \nu_\tau$$

and the corresponding three channels for antineutrinos. The probabilities of neutrinos oscillations depend on six independent parameters: two independent Δm^2(often $\Delta m_{21}{}^2$ and $\Delta m_{31}{}^2$ are used), three mixing angles and one Dirac phase (ϑ_{12},ϑ_{13}, ϑ_{23}, and δ_{13} in the parameterization given in eqn (3.12)

Real parts of the quartic products of elements of the mixing matrix entering in the three-neutrino oscillation probabilities is given by the eqn (3.43)

$$= Re(U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k})$$

Now let us take first case when electron neutrino changing into muon neutrino i.e. $\mathbf{P(\nu_e \rightarrow \nu_\mu)}$, for our convenience let us label for electron as index 1 and for muon as index 2 that means $\boldsymbol{\alpha}$ =1 and α' =2. Now further when electron neutrino changing into muon we have to see from which mass eigen state that change will occur it can change from three mass eigen states i.e. 2→ 1, 3→1, 3→2 , therefore we have to discuss all the three cases one by one.

Case 1

When $\boldsymbol{\alpha}$ =1 and α' =2 and i=2 and k=1

We have to calculate only the real part i.e.

$$= Re(U_{\alpha' i} U^*_{\alpha' k} U^*_{\alpha i} U_{\alpha k})$$

$$= (U_{22} U^*_{21} U^*_{12} U_{11})$$

Now we will find these coefficient from the mixing matrix i.e. from eqn (3.12)

$$= (c_{23} c_{12} - s_{13} s_{23} s_{12} e^{i\delta}) \left(-c_{23} s_{12} - s_{13} s_{23} c_{12} e^{-i\delta}\right) (c_{13} s_{12})(c_{13} c_{12})$$

$$= c_{13}^2 c_{12} s_{12} \,(-c_{23}^2 c_{12} s_{12} - c_{23} c_{12} s_{13} s_{23} c_{12} e^{-i\delta} + c_{23} s_{23} s_{13} s_{12}^2 e^{-i\delta} + s_{12} c_{12} s_{13}^2 s_{23}^2)$$

$$= c_{13}^2 \,(-c_{23}^2 c_{12}^2 s_{12}^2 - c_{23} c_{12}^3 s_{13} s_{12} s_{23} e^{-i\delta} + c_{23} s_{23} s_{13} c_{12} s_{12}^3 e^{-i\delta} + s_{12}^2 c_{12}^2 s_{13}^2 s_{23}^2)$$

$$= c_{13}^2 \,(-\frac{1}{4} c_{23}^2 4\cos_{\theta_{12}}^2 \sin_{\theta_{12}}^2 - \frac{1}{4} 2\cos_{\theta_{12}} \sin_{\theta_{12}} 2\cos_{\theta_{23}} \sin_{\theta_{23}} s_{13} c_{12}^2 e^{-i\delta}$$
$$+ \frac{1}{4} 2\cos_{\theta_{12}} \sin_{\theta_{12}} 2\cos_{\theta_{23}} \sin_{\theta_{23}} s_{13} s_{12}^2 e^{-i\delta} + \frac{1}{4} 4\cos_{\theta_{12}}^2 \sin_{\theta_{12}}^2 s_{13}^2 s_{23}^2)$$

$$= c_{13}^2(-\frac{1}{4}c_{23}^2\sin^2 2\nu_{12} - \frac{1}{4}s_{13}c_{12}^2\sin 2\nu_{23}\sin 2\nu_{12}e^{-i\delta} + \frac{1}{4}s_{13}s_{12}^2\sin 2\nu_{23}\sin 2\nu_{12}e^{-i\delta} +$$

$$\frac{1}{4}s_{23}^2 s_{13}^2\sin^2 2\nu_{12})$$

$$= -\frac{1}{4}c_{13}^2\sin 2\nu_{12}(c_{23}^2\sin 2\nu_{12} + s_{13}c_{12}^2\sin 2\nu_{23}e^{-i\delta} - s_{13}s_{12}^2\sin 2\nu_{23}e^{-i\delta} - s_{23}^2 s_{13}^2\sin 2\nu_{12})$$

$$= -\frac{1}{4}c_{13}^2\sin 2\nu_{12}(c_{23}^2\sin 2\nu_{12} + s_{13}c_{12}^2\sin 2\nu_{23}\cos\delta_{13} - s_{13}s_{12}^2\sin 2\nu_{23}\cos\delta_{13} - s_{23}^2 s_{13}^2\sin 2\nu_{12})$$

$$= -\frac{1}{4}c_{13}^2\sin 2\nu_{12}[\sin 2\nu_{12}(c_{23}^2 - s_{23}^2 s_{13}^2) + \sin 2\nu_{23}s_{13}\cos\delta_{13}(\cos^2\theta_{12} - \sin^2\theta_{12})]$$

Where

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$= -\frac{1}{4}c_{13}^2\sin 2\nu_{12}[\sin 2\nu_{12}(c_{23}^2 - s_{23}^2 s_{13}^2) + \cos 2\nu_{12}\sin 2\nu_{23}s_{13}\cos\delta_{13}]$$

<u>Case 2</u>

When $\boldsymbol{\alpha}$ =1 and α' =2 and i=3 and k=2

We have to calculate only the real part i.e.

$$= Re(U_{\alpha' i}U^*_{\alpha' k}U^*_{\alpha i}U_{\alpha k})$$

$$= (U_{23}U^*_{22}U^*_{13}U_{12})$$

By putting these coefficients, we get

$$= (c_{13}s_{23})\left(c_{23}c_{12} - s_{13}s_{23}s_{12}e^{-i\delta}\right)\left(s_{13}e^{i\delta}\right)(c_{13}s_{12})$$

$$= c_{13}^2 s_{13}s_{12}s_{23}e^{i\delta}(c_{23}c_{12} - s_{13}s_{23}s_{12}e^{-i\delta})$$

$$= c_{13}^2 s_{13}s_{12}s_{23}(c_{23}c_{12}e^{i\delta} - s_{13}s_{23}s_{12})$$

$$= -c_{13}^2 s_{13}s_{12}s_{23}(s_{13}s_{23}s_{12} - c_{23}c_{12}\cos\delta_{13})$$

<u>Case 3</u>

When $\boldsymbol{\alpha}$ =1 and α' =2 and i=3 and k=1

We have to calculate only the real part i.e.

$$= Re(U_{\alpha' i}U^*_{\alpha' k}U^*_{\alpha i}U_{\alpha k})$$

$$= (U_{23}U^*_{21}U^*_{13}U_{11})$$

By putting these coefficients, we get

$$= (c_{13}s_{23})\left(-c_{23}s_{12} - s_{13}s_{23}c_{12}e^{-i\delta}\right)\left(s_{13}e^{i\delta}\right)(c_{13}c_{12})$$

$$= c_{13}^2 s_{13}c_{12}s_{23}e^{i\delta}(-c_{23}s_{12} - s_{13}s_{23}c_{12}e^{-i\delta})$$

$$= c_{13}^2 s_{13}c_{12}s_{23}\left(-c_{23}s_{12}e^{i\delta} - s_{13}s_{23}c_{12}\right)$$

$$= -c_{13}^2 s_{13}c_{12}s_{23}(c_{23}s_{12}e^{i\delta} + s_{13}s_{23}c_{12})$$

Similarly solving real coefficients for electron neutrino changing into tau neutrino **P($\boldsymbol{\nu_e \rightarrow \nu_\tau}$)** as per the three possible cases we get:

i=2,k=1	$\frac{1}{4} c_{13}^2 \sin 2\nu_{12} [\sin 2\nu_{12} (c_{23}^2 s_{13}^2 - s_{23}^2) + \cos 2\nu_{12} \sin 2\nu_{23} s_{13} \cos\delta_{13}]$
i=3,k=2	$-c_{13}^2 s_{13} s_{12} c_{23} (s_{13} c_{23} s_{12} + s_{23} c_{12} \cos\delta_{13})$
i=3,k=1	$-c_{13}^2 s_{13} c_{12} c_{23} (s_{13} c_{23} c_{12} - s_{23} s_{12} \cos\delta_{13})$

Similarly let us take the case of electron neutrino when it is not changing into another neutrino i.e. **P($\boldsymbol{\nu_e \rightarrow \nu_e}$)** as per the three possible cases we get:

i=2,k=1	$\frac{1}{4} c_{13}^4 \sin^2 2\nu_{12}$
i=3,k=2	$\frac{1}{4} s_{12}^2 \sin^2 2\nu_{13}$
i=3,k=1	$\frac{1}{4} c_{12}^2 \sin^2 2\nu_{13}$

Similarly solving real coefficients for muon neutrino changing into tau neutrino **P($\boldsymbol{\nu_\mu \rightarrow \nu_\tau}$) as** per the three possible cases we get:

i=2,k=1	$\frac{1}{16}\sin^2 2\vartheta_{12} \sin^2 2\vartheta_{23}(1 + s^2{}_{13})^2 - \frac{1}{4}(\sin^2 2\vartheta_{12} + \sin^2 2\vartheta_{23}) s^2{}_{13} - \frac{1}{16} \sin 4\vartheta_{12} \sin 4\vartheta_{23} (1 + s^2{}_{13}) s_{13} \cos\delta_{13} + \frac{1}{4} \sin^2 2\vartheta_{12} \sin^2 2\vartheta_{23} s_{13}^2 \cos^2 \delta_{13}$
i=3,k=2	$\frac{-1}{4} \sin 2\vartheta_{23} c_{13}^2 [\sin 2\vartheta_{23} (c^2{}_{12} - s^2{}_{12} s^2{}_{13}) + \sin 2\vartheta_{12} \cos 2\vartheta_{23} s_{13} \cos\delta_{13}]$
i=3,k=1	$\frac{1}{4}\sin 2\vartheta_{23} c_{13}^2 [\sin 2\vartheta_{23} (c^2{}_{12} s^2{}_{13} - s^2{}_{12}) + \sin 2\vartheta_{12} \cos 2\vartheta_{23} s_{13} \cos\delta_{13}]$

Similarly solving real coefficients for muon neutrino changing into the same neutrino **P($\boldsymbol{\nu_\mu \rightarrow \nu_\mu}$) as** per the three possible cases we get

i=2,k=1	$\frac{1}{4} \sin^2 2\vartheta_{12} (c^4{}_{23} + s^4{}_{23} s^2{}_{13}) + \frac{1}{4} (1 - \frac{1}{2}\sin^2 2\vartheta_{12}) \sin^2 2\vartheta_{23} s^2{}_{13} + \frac{1}{4} \sin 4\vartheta_{12} \sin 2\vartheta_{23} (c^2{}_{23} - s^2{}_{23} s^2{}_{13}) s_{13} \cos\delta_{13} - \frac{1}{4} \sin^2 2\vartheta_{12} \sin^2 2\vartheta_{23} s^2{}_{13} \cos^2\delta_{13}$
i=3,k=2	$s^2{}_{23} c^2{}_{13} (c^2{}_{12} c^2{}_{23} + s^2{}_{12} s^2{}_{23} s^2{}_{13} - \frac{1}{2} \sin 2\vartheta_{12} \sin 2\vartheta_{23} s_{13} \cos\delta_{13})$
i=3,k=1	$s^2{}_{23} c^2{}_{13} (s^2{}_{12} c^2{}_{23} + c^2{}_{12} s^2{}_{23} s^2{}_{13} + \frac{1}{2} \sin 2\vartheta_{12} \sin 2\vartheta_{23} s_{13} \cos\delta_{13})$

Similarly solving real coefficients for muon neutrino changing into the same neutrino **$P(\nu_\tau \rightarrow \nu_\tau)$ as** per the three possible cases we get

i=2,k=1	$\frac{1}{4}\sin^2 2\vartheta_{12}(s^4{}_{23} + c^4{}_{23}s^2{}_{13}) + \frac{1}{4}(1 - \frac{1}{2}\sin^2 2\vartheta_{12})\sin^2 2\vartheta_{23}s^2{}_{13} + \frac{1}{4}\sin 4\vartheta_{12}\sin 2\vartheta_{23}$ $(s^2{}_{23} - c^2{}_{23}s^2{}_{13})s_{13}\cos\delta_{13} - \frac{1}{4}\sin^2 2\vartheta_{12}\sin^2 2\vartheta_{23}s^2{}_{13}\cos^2\delta_{13}$
i=3,k=2	$c_{23}^2 c_{13}^2 \ (c^2{}_{12}s^2{}_{23} + s^2{}_{12}c^2{}_{23}s^2{}_{13} + \frac{1}{2}\sin 2\vartheta_{12}\sin 2\vartheta_{23}s_{13}\cos\delta_{13})$
i=3,k=1	$c_{23}^2 c_{13}^2 (s_{12}^2 s_{23}^2 + c_{12}^2 c_{23}^2 s_{13}^2 - \frac{1}{2}\sin 2\nu_{12}\sin 2\nu_{23}s_{13}\cos\delta_{13})$

Now we have to calculate the probability for one neutrino changing into another neutrino by substituting these real coefficients into the eqn (3.43) i.e.

$$P(\nu_\alpha \rightarrow \nu_{\alpha'}) = \delta_{\alpha\alpha'} - 2\sum_{i>k} Re(U_{\alpha' i}U^*_{\alpha' k}U^*_{\alpha i}U_{\alpha k})\,(1 - \mathrm{Cos}\frac{\Delta m^2_{ki}L}{2E})$$

$$+2\sum_{i>k} Im(U_{\alpha' i}U^*_{\alpha' k}U^*_{\alpha i}U_{\alpha k})\,\mathrm{Sin}\frac{\Delta m^2_{ki}L}{2E}$$

As we are calculating probabilities for vacuum so we can neglect the imaginary part of above eqn therefore we have:

$$P(\nu_\alpha \rightarrow \nu_{\alpha'}) = \delta_{\alpha\alpha'} - 2\sum_{i>k} Re(U_{\alpha' i}U^*_{\alpha' k}U^*_{\alpha i}U_{\alpha k})\,(1 - \mathrm{Cos}\frac{\Delta m^2_{ki}L}{2E})$$

$$= \delta_{\alpha\alpha'} - 4\sum_{i>k} Re(U_{\alpha' i}U^*_{\alpha' k}U^*_{\alpha i}U_{\alpha k})\,\frac{(1 - \mathrm{os}\frac{\Delta m^2_{ki}L}{2E})}{2}$$

$$= \delta_{\alpha\alpha'} - 4\sum_{i>k} Re(U_{\alpha' i}U^*_{\alpha' k}U^*_{\alpha i}U_{\alpha k})\,\mathrm{Sin}\frac{\Delta m^2_{ki}L}{4E}$$

Let us calculate probability for electron neutrino changing into muon neutrino, as we already discussed that there are three possible cases for electron neutrino changing into muon neutrino i.e. from one mass eigen state to another.

$$P(\nu_e \rightarrow \nu_\mu) = \delta_{12} - 4[-\frac{1}{4}c_{13}^2\sin 2\nu_{12}[\sin 2\nu_{12}(c_{23}^2 - s_{23}^2 s_{13}^2) + \cos 2\nu_{12}\sin 2\nu_{23}s_{13}\cos\delta_{13}]$$

$$\mathrm{Sin}\frac{\Delta m^2_{21}L}{2E} + \delta_{32} - 4[-c_{13}^2 s_{13}s_{12}s_{23}(s_{13}s_{23}s_{12} - c_{23}c_{12}\cos\delta_{13})]\mathrm{Sin}\frac{\Delta m^2_{32}L}{4E} + \delta_{31} -$$

$$4[-c_{13}^2 s_{13}c_{12}s_{23}(c_{23}s_{12}e^{i\delta} + s_{13}s_{23}c_{12})]\mathrm{Sin}\frac{\Delta m^2_{31}L}{4E}$$

For atmospheric neutrinos we take

$$\Delta m^2_{32} \cong \Delta m^2_{31} \cong \Delta m^2_{atm}$$

And we can neglect $\Delta m^2_{21} \cong \Delta m^2_{sol} \cong 0$

Also $$\delta_{\alpha\alpha'} = 1 \; if \; \alpha = \alpha'$$

$$\delta_{\alpha\alpha'} = 0 \; if \; \alpha \neq \alpha'$$

Therefore above eqn becomes:

$$P(\nu_e \to \nu_\mu) = -4(-s_{12}^2 s_{23}^2 s_{13}^2 c_{13}^2 + s_{12}s_{23}s_{13}c_{23}c_{12}c_{13}^2 cos\delta_{13} - c_{12}^2 c_{23}^2 s_{13}^2 c_{13}^2 - s_{12}s_{23}s_{13}c_{23}c_{12}c_{13}^2 cos\delta_{13}) \text{ Sin}\frac{\Delta m_{atm}^2 L}{4E}$$

$$= 4[s_{23}^2 s_{13}^2 c_{13}^2 (s_{12}^2 + c_{12}^2)] \text{ Sin}\frac{\Delta m_{atm}^2 L}{4E}$$

As $(s_{12}^2 + c_{12}^2) = 1$

$$= 4[s_{23}^2 s_{13}^2 c_{13}^2] \text{ Sin}\frac{\Delta m_{atm}^2 L}{4E}$$

$$=4\sin^2\theta_{23}\cos^2\theta_{13}\sin^2\theta_{13}\text{Sin}\frac{\Delta m_{atm}^2 L}{4E}$$

$$\boxed{\mathbf{P(\nu_e \to \nu_\mu) = \sin^2(2\theta_{13})\sin^2(\theta_{23})Sin\frac{\Delta m_{atm}^2 L}{4E}}} \qquad (3.45)$$

Similarly we can calculate the probability for electron neutrinos changing into tauon neutrinos.

$$\boxed{\mathbf{P(\nu_e \to \nu_\tau) = \sin^2(2\theta_{13}) \cos^2(\theta_{23}) \; Sin\frac{\Delta m_{atm}^2 L}{4E}}} \qquad (3.46)$$

Now we will calculate the survival probability of electron neutrinos i.e.

$$P(\nu_e \to \nu_e) = 1 - [P(\nu_e \to \nu_\mu) + P(\nu_e \to \nu_\tau)]$$

$$=1- [\sin^2(2\theta_{13})\sin^2(\theta_{23})\text{Sin}\frac{\Delta m_{atm}^2 L}{4E} + \sin^2(2\theta_{13}) \cos^2(\theta_{23}) \text{ Sin}\frac{\Delta m_{atm}^2 L}{4E}]$$

$$= 1- [\sin^2(2\theta_{13}) \text{ Sin}\frac{\Delta m_{atm}^2 L}{4E}\{\sin^2(\theta_{23})+ \cos^2(\theta_{23})\}]$$

$$\boxed{\mathbf{P(\nu_e \to \nu_e) = 1- \sin^2(2\theta_{13}) \; Sin\frac{\Delta m_{atm}^2 L}{4E}}} \qquad (3.47)$$

Similarly we can calculate the probability for muon neutrinos changing into another one. In the case of oscillation in vacuum probability for changing electron neutrinos into muon neutrinos and vice-versa probability remains the same.

Therefore in case of vacuum

$$P(\nu_e \rightarrow \nu_\mu) = P(\nu_\mu \rightarrow \nu_e)$$

Thus we can write it as:

$$\boxed{\mathbf{P(\nu_\mu \rightarrow \nu_e) = \sin^2(2\theta_{13})\sin^2(\theta_{23})Sin\frac{\Delta m^2_{atm}L}{4E}}} \qquad (3.48)$$

Similarly we can calculate the probability for muon neutrinos changing into tauon neutrinos

$$\boxed{\mathbf{P(\nu_\mu \rightarrow \nu_\tau) = \sin^2(2\theta_{23})\cos^4(\theta_{13})\ Sin\frac{\Delta m^2_{atm}L}{4E}}} \qquad (3.49)$$

Now we will calculate the survival probability of muon neutrinos i.e.

$$P(\nu_\mu \rightarrow \nu_\mu) = 1 - [P(\nu_\mu \rightarrow \nu_e) + P(\nu_\mu \rightarrow \nu_\tau)]$$

$$= 1 - [\ \sin^2(2\theta_{13})\sin^2(\theta_{23})Sin\frac{\Delta m^2_{atm}\ L}{4E} + \sin^2(2\theta_{23})\cos^4(\theta_{13})\ Sin\frac{\Delta m^2_{atm}\ L}{4E}]$$

$$= 1 - [\ \sin^2(2\theta_{13})\sin^2(\theta_{23}) + \sin^2(2\theta_{23})\cos^4(\theta_{13})]\ Sin\frac{\Delta m^2_{atm}\ L}{4E}$$

$$\boxed{\mathbf{P(\nu_\mu \rightarrow \nu_\mu) = 1 - [\ \sin^2(2\theta_{13})\sin^2(\theta_{23}) + \sin^2(2\theta_{23})\cos^4(\theta_{13})]\ Sin\frac{\Delta m^2_{atm}L}{4E}}} \qquad 50)$$

These are the six probability terms from where we can find the probability for one neutrinos changing into another depending upon their flavor.

3.5 Neutrino Oscillation in Vacuum: Solar Neutrino Case

Let us calculate probability for solar neutrinos i.e. electron neutrinos changing into muon neutrinos, electron neutrinos into tauon neutrinos and survival probability for electron neutrinos into electron neutrinos as we already discussed that there are three possible cases for electron neutrinos changing into muon neutrinos i.e. from one mass eigen state to another. For solar neutrino we are taking only mass squared difference 21

$$P(\nu_e \rightarrow \nu_\mu) = \delta_{12} - 4[-\frac{1}{4}c^2_{13}\sin 2\nu_{12}[\sin 2\nu_{12}(c^2_{23} - s^2_{23}s^2_{13}) + \cos 2\nu_{12}\ \sin 2\nu_{23}s_{13}\ \cos\delta_{13}]$$
$$Sin\frac{\Delta m^2_{21}L}{2E}$$

As $\delta_{ij} = 1$ for i = j

$$\delta_{ij} = 0 \quad \text{for } i \neq j$$

Therefore $\delta_{12} = 0$

$P(\nu_e \rightarrow \nu_\mu) = -4[-\frac{1}{4} c_{13}^2 \sin 2\nu_{12}[\sin 2\nu_{12}(c_{23}^2 - s_{23}^2 s_{13}^2) + \cos 2\nu_{12} \sin 2\nu_{23} s_{13} \cos\delta_{13}] \operatorname{Sin}\frac{\Delta m_{21}^2 L}{2E}$

$= [\sin^2 2\theta_{12} \cos^2\theta_{13}(\cos^2\theta_{23} - \sin^2\theta_{23} \sin^2\theta_{13}) + \sin 2\theta_{12} \cos 2\theta_{12} \cos^2\theta_{13} \sin\theta_{13} \sin 2\theta_{23} \cos\delta_{13}] \operatorname{Sin}\frac{\Delta m_{21}^2 L}{2E}$

$= [\sin^2 2\theta_{12} \cos^2\theta_{13}(\cos^2\theta_{23} - \sin^2\theta_{23} \sin^2\theta_{13}) + \frac{1}{4} \sin 4\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \cos\theta_{13} \cos\delta_{13}] \operatorname{Sin}\frac{\Delta m_{21}^2 L}{2E}$

For oscillation in vacuum $\delta = 0$, therefore $\cos\delta_{13} = 1$

$$\boxed{\begin{aligned} P(\nu_e \rightarrow \nu_\mu) = {} & [\sin^2 2\theta_{12} \cos^2\theta_{13}(\cos^2\theta_{23} - \sin^2\theta_{23} \sin^2\theta_{13}) \\ & + \frac{1}{4} \sin 4\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \cos\theta_{13}] \operatorname{Sin}\frac{\Delta m_{21}^2 L}{2E} \end{aligned}} \qquad (3.51)$$

This is gernalized probability term for electron neutrinos changing into muon neutrinos.

Now calculating the probability term for electron neutrino changing into tauon neutrino in case of solar neutrinos

$P(\nu_e \rightarrow \nu_\tau)$
$= \delta_{12} - 4[\frac{1}{4} c_{13}^2 \sin 2\nu_{12}\{\sin 2\nu_{12}(c_{23}^2 s_{13}^2 - s_{23}^2) + \cos 2\nu_{12} \sin 2\nu_{23} s_{13} \cos\delta_{13}\}] \operatorname{Sin}\frac{\Delta m_{21}^2 L}{4E}$

As $\delta_{12} = 0$

$= \sin^2 2\theta_{12} \cos^2\theta_{13}(\cos^2\theta_{23} \sin^2\theta_{13} - \sin^2\theta_{23}) + \sin 2\theta_{12} \cos 2\theta_{12} \cos^2\theta_{13} \sin\theta_{13} \sin 2\theta_{23} \cos\delta_{13}] \operatorname{Sin}\frac{\Delta m_{21}^2 L}{2E}$

$= [-\sin^2 2\theta_{12} \cos^2\theta_{13}(\cos^2\theta_{23} \sin^2\theta_{13} - \sin^2\theta_{23}) - \frac{1}{4} \sin 4\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \cos\theta_{13} \cos\delta_{13}] \operatorname{Sin}\frac{\Delta m_{21}^2 L}{2E}$

Neglecting $\cos\delta_{13}$, we have probability term for electron neutrinos changing into tauon neutrinos

$$\boxed{\begin{aligned} P(\nu_e \rightarrow \nu_\tau) = {} & [-\sin^2 2\theta_{12} \cos^2\theta_{13}(\cos^2\theta_{23} \sin^2\theta_{13} - \sin^2\theta_{23}) \\ & - \frac{1}{4} \sin 4\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \cos\theta_{13}] \operatorname{Sin}\frac{\Delta m_{21}^2 L}{2E} \end{aligned}} \qquad (3.52)$$

Now we calculate the survival probability i.e.

$P(\nu_e \rightarrow \nu_e)=1-[\sin^2 2\theta_{12}\cos^2\theta_{13}(\cos^2\theta_{23}-\sin^2\theta_{23}\sin^2\theta_{13})+\frac{1}{4}\sin 4\theta_{12}$
$\sin 2\theta_{13}\sin 2\theta_{23}\cos\theta_{13}-\sin^2 2\theta_{12}\cos^2\theta_{13}(\cos^2\theta_{23}\sin^2\theta_{13}-\sin^2\theta_{23})-\frac{1}{4}\sin 4\theta_{12}\sin 2\theta_{13}$
$\sin 2\theta_{23}\cos\theta_{13}\cos\delta_{13}]\ \mathrm{Sin}\frac{\Delta m_{21}^2 L}{2E}$

$=1-[\ \sin^2 2\theta_{12}\cos^2\theta_{13}(\cos^2\theta_{23}-\sin^2\theta_{23}\ \sin^2\theta_{13})-\sin^2 2\theta_{12}\cos^2\theta_{13}(\cos^2\theta_{23}\sin^2\theta_{13}-\sin^2\theta_{23})]$
$\mathrm{Sin}\frac{\Delta m_{21}^2 L}{2E}$

$=1-[\ \sin^2 2\theta_{12}\cos^2\theta_{13}(\cos^2\theta_{23}-\sin^2\theta_{23}\ \sin^2\theta_{13}-\cos^2\theta_{23}\sin^2\theta_{13}+\sin^2\theta_{23})]\ \mathrm{Sin}\frac{\Delta m_{21}^2 L}{2E}$

$=1-[\ \sin^2 2\theta_{12}\cos^2\theta_{13}\{\ \cos^2\theta_{23}(1-\sin^2\theta_{13})+\sin^2\theta_{23}(1-\sin^2\theta_{13})\}]\ \mathrm{Sin}\frac{\Delta m_{21}^2 L}{2E}$

$=1-[\ \sin^2 2\theta_{12}\cos^2\theta_{13}\{\ \cos^2\theta_{23}\ \cos^2\theta_{13}+\sin^2\theta_{23}\ \cos^2\theta_{13}\}]\ \mathrm{Sin}\frac{\Delta m_{21}^2 L}{2E}$

$=1-[\ \sin^2 2\theta_{12}\cos^2\theta_{13}\ \cos^2\theta_{13}(\cos^2\theta_{23}+\sin^2\theta_{23})]\ \mathrm{Sin}\frac{\Delta m_{21}^2 L}{2E}$

Using $\cos^2\theta_{23}+\sin^2\theta_{23}=1$

$=1-[\ \sin^2 2\theta_{12}\cos^4\theta_{13}]\ \mathrm{Sin}\frac{\Delta m_{21}^2 L}{2E}$

therefore survival probability term is

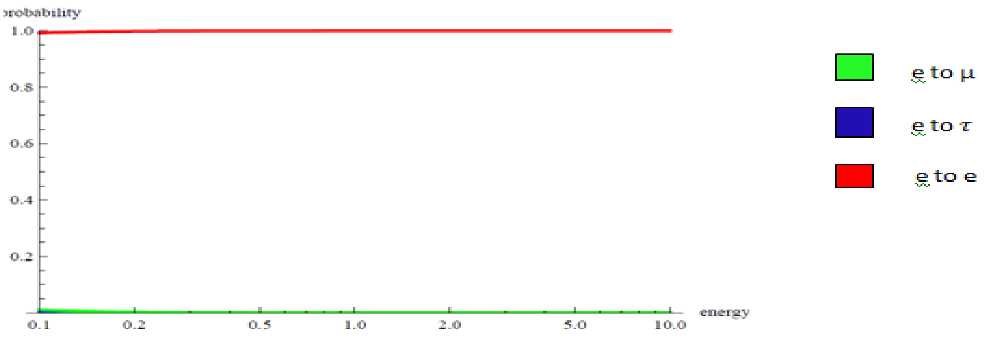
$$\boxed{\mathbf{P(\nu_e \rightarrow \nu_e)=1-[\ \sin^2 2\theta_{12}\cos^4\theta_{13}]\ Sin\frac{\Delta m_{21}^2 L}{2E}}} \qquad (3.53)$$

3.6 Oscillation Plots & Analysis

3.61 Probability Plots for Three Neutrino Case:

For studying the probability for three neutrinos oscillations[6], we are taking parameters from KamLAND[5] Experiment. In this experiment length is taken as L=100 m and mass square difference as $\Delta m_{12}{}^2 = 7.9\times10^{-5}\mathrm{eV}^2$ and the mixing angle is θ_{12}=32.2, $\theta_{13}=12.7, \theta_{23}=35.9$ and the energy is in MeV's. We are plotting a graph between probability and energy by using the probability expression no. (3.51). Probability is along Y-axis and energy is along

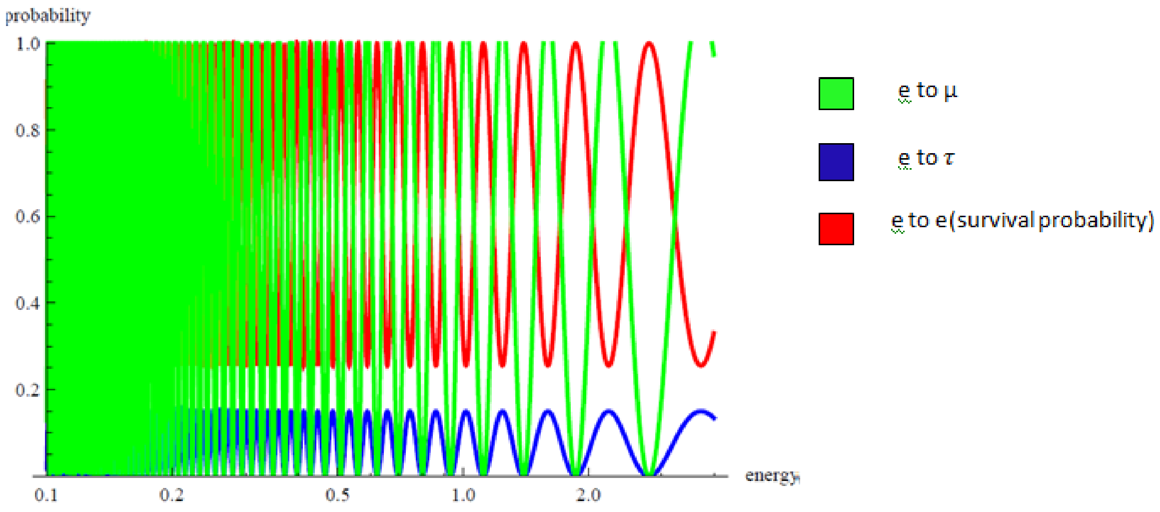
X-axis. In this plot all the three probabilities are shown i.e. electron to muon, electron to tauon and electron to electron survival probability. This is a short baseline experiment.



Plot 3.1: Probability vs Energy (MeV) for solar vacuum oscillation short baseline.

For short baseline vacuum oscillations in the case of electron neutrino, the graph here comes to be a straight line giving oscillation probability value approximately zero for electron to muon transitions whereas electron to electron, again there is a straight line having probability value as one. This concludes for SBL the neutrino does not change its flavor as it propagates leading to existence of only survival probability. This was predicted by CHOOZ[7] experiment for the case of antineutrino, no change of flavor was observed upto this length and for this particular energy range.

Next we plot the same graph for Long baseline[8] i.e. we have to increase the length for long baseline here we are taking a length L= 17500m and other parameters remains the same. Then we plot a graph between probability and energy. Probability is along Y-axis and energy is along X-axis. The length here is chosen from KAMLAND experiment data[9] which first observed the change of flavor for antineutrinos at this energy range.

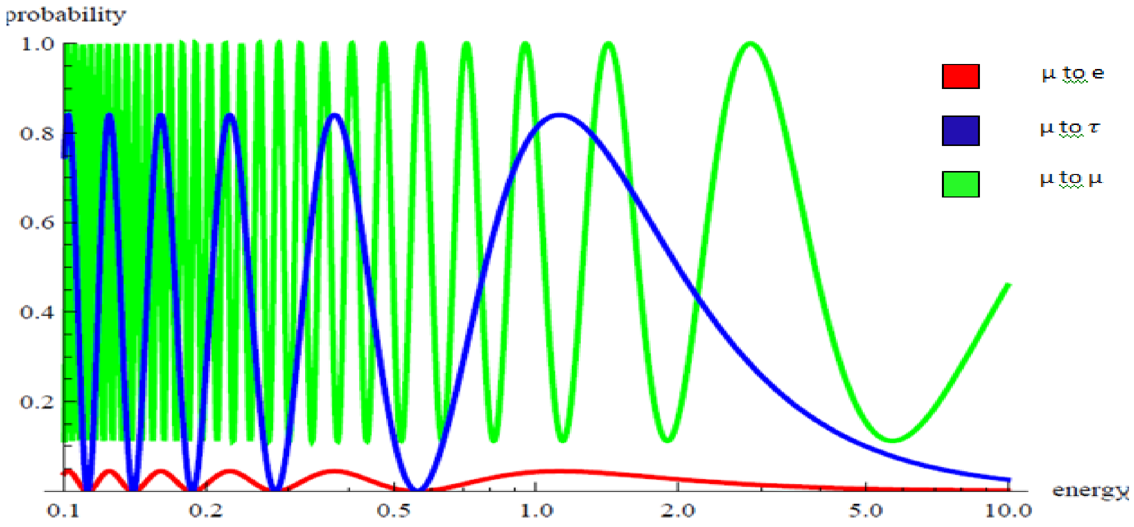


Plot 3.2: Probability Vs Energy (MeV) for solar vacuum oscillation long baseline.

The graph here shows that by increasing length from few km to hundreds of km the oscillation came into existence for this energy range. KamLAND was the first experiment to provide direct evidence for this oscillation. Still major part of flavor transitions is from electron to muon, not from electron to tauon.

3.62 Probability Plot for muon neutrinos changing into another neutrinos

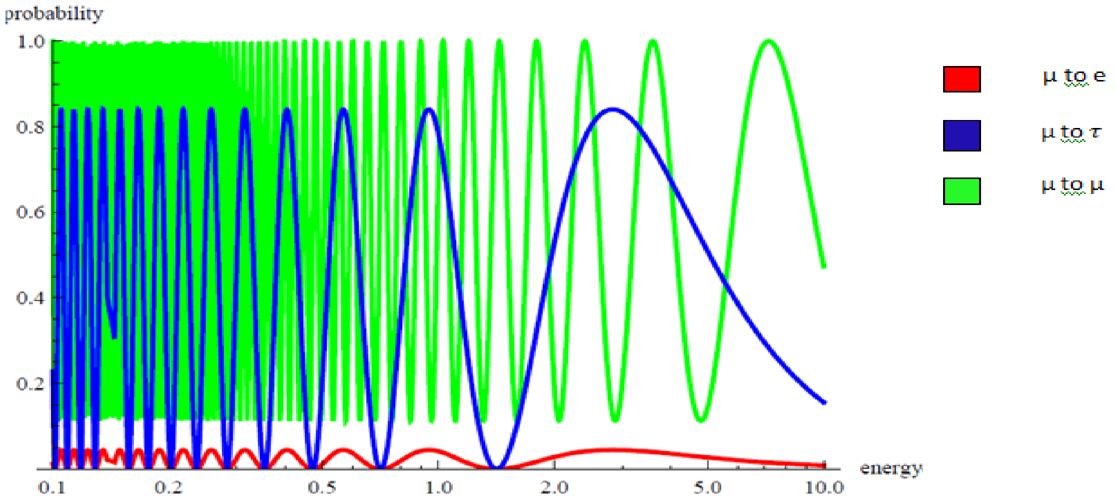
In this we are taking parameters from Particle data group[10]. In this length is taken as L=2900m and mass square difference as $\Delta m_{12}{}^{2}$ =2.43×10^{-3}ev^2 and the mixing angle is $\theta_{13} = 10.55, \theta_{23} = 35.9$ and the energy is in GeV's. We are plotting a graph between probability and energy by using probability expression no. (3.48). Probability is along Y-axis and energy is along X-axis. In this plot all the three probabilities are shown i.e. muon to electron, muon to tauon and muon to muon survival probability. This is a short baseline experiment.



Plot 3.3: Probability vs Energy(GeV) for atmospheric vacuum oscillation short baseline.

The atmospheric neutrino consists initially of muon neutrinos and as they propagate muon changes into either electron or muon but change from muon to tauon is more frequent rather than electron. The energy range for this probability lies in GeV's. At short baseline the change of flavor transition is not very rapid.

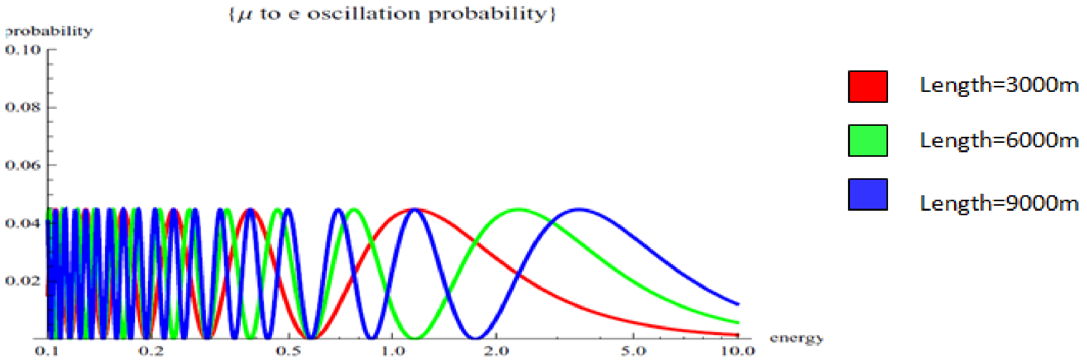
But if we plot the same graph for long baseline i.e. we have to increase the length for long baseline here we are taking a length L= 7330 m and other parameters remains the same the oscillations are more rapid and change of flavor is more fast as they propagate in vacuum.



Plot 3.4: Probability vs Energy(GeV) for atmospheric vacuum oscillation long baseline.

3.63 Probability Plot for muon neutrinos changing into electron neutrinos at different lengths

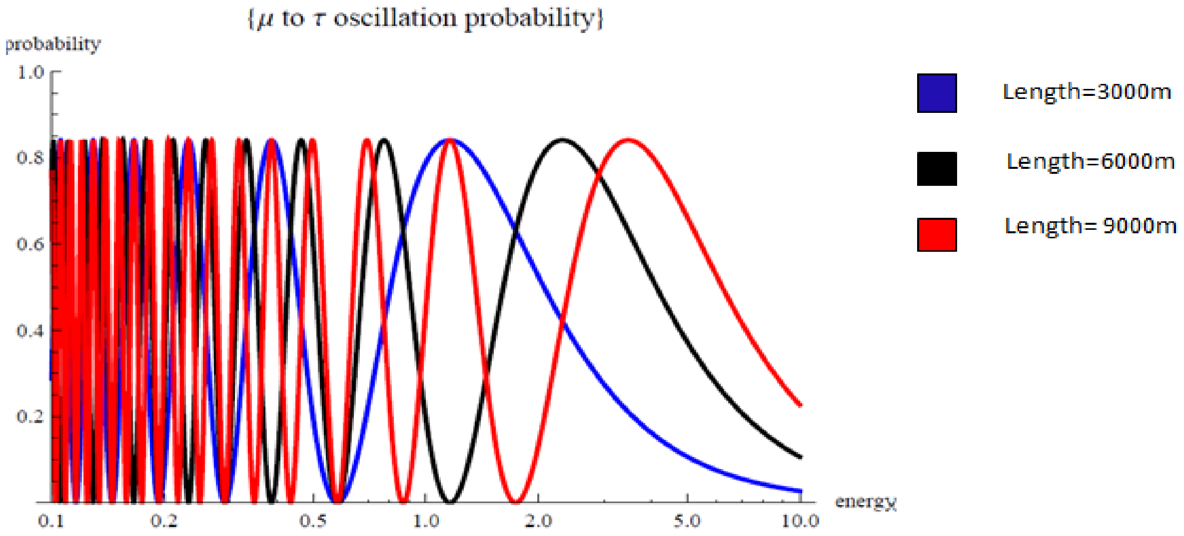
All the parameters remains same as in the case of muon neutrinos changing into electron neutrinos. we can change only length. In this plot Probability is along Y-axis and energy is along X-axis. when length is taken as 3000m it is shown by red line when we further increase the length from 3000 to 6000m it is shown by green line then further increase in length from 6000 to 9000m it is shown by the blue line.



Plot 3.5: Probability vs Energy for atmospheric vacuum oscillation for different lengths for muon neutrino changing into electron neutrino.

3.64 Probability Plot for muon neutrinos changing into tauon neutrinos at different lengths

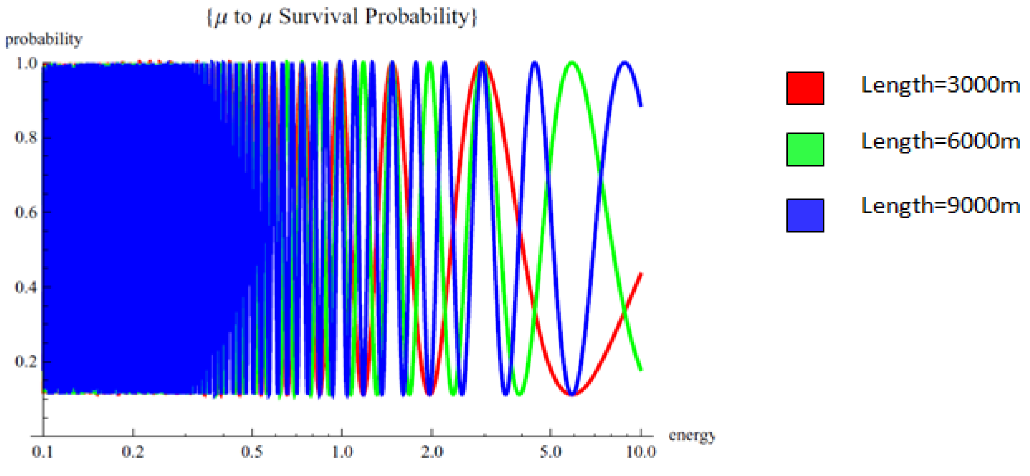
All the parameters remains same as in the case of muon neutrinos changing into tauon neutrinos. we can change only length. In this plot Probability is along Y-axis and energy is along X-axis. We plot a probability graph by using eqn (3.49). when length is taken as 3000m it is shown by blue line when we further increase the length from 3000 to 6000m it is shown by black line then further increase in length from 6000 to 9000m it is shown by the red line.



Plot 3.6: Probability vs Energy for atmospheric vacuum oscillation for different lengths for muon neutrino changing into tauon neutrino.

3.65 Probability Plot for muon neutrinos changing into muon neutrinos at different lengths

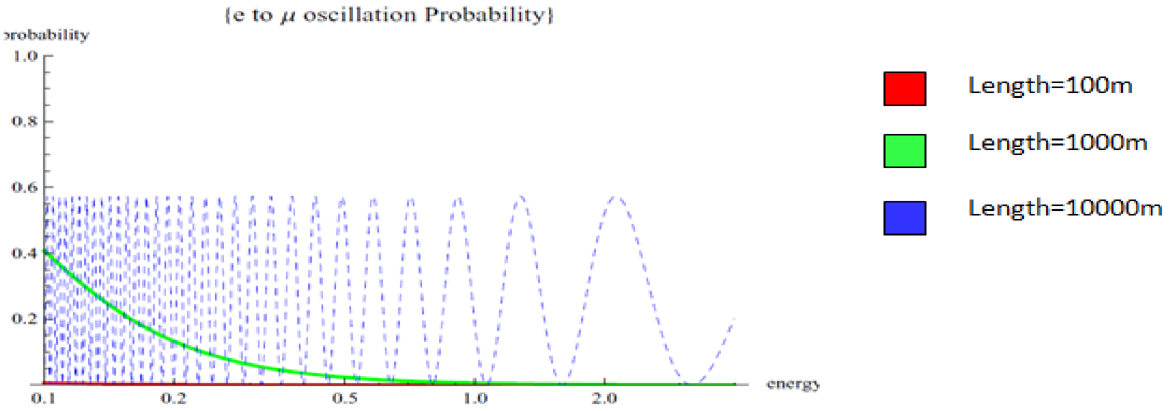
All the parameters remains same as in the case of muon neutrinos changing into muon neutrinos. we can change only length. In this plot Probability is along Y-axis and energy is along X-axis. For this Probability curve we are using eqn no(3.50)



Plot 3.7: Probability vs Energy for atmospheric vacuum oscillation for different lengths for survival probability for muon neutrino.

3.66 Probability Plot for electron neutrinos changing into muon neutrinos at different lengths

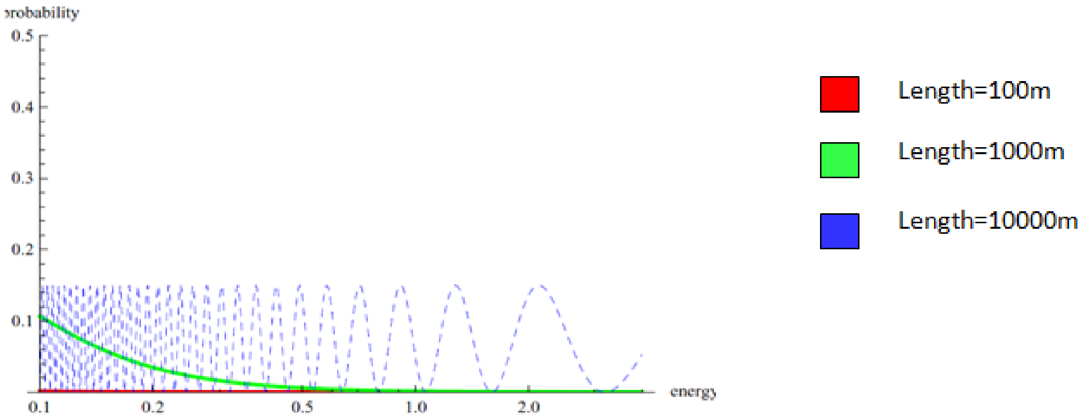
All the parameters remains same as in the case of electron neutrinos changing into muon neutrinos. we can change only length. In this plot Probability is along Y-axis and energy is along X-axis. when length is taken as 100m it is shown by red line when we further increase the length from 100 to 1000m it is shown by green line then further increase in length from 1000 to 10000m it is shown by the blue line.



Plot 3.8: Probability vs Energy plot for different lengths for electron neutrino changing into muon neutrino.

3.67 Probability Plot for electron neutrinos changing into tauon neutrinos at different lengths

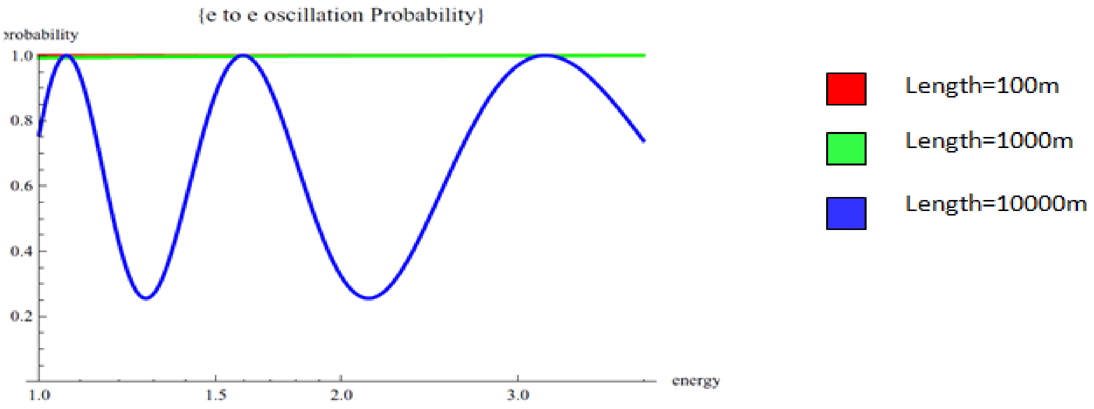
All the parameters remains same as in the case of electron neutrinos changing into muon neutrinos. we can change only length. In this plot Probability is along Y-axis and energy is along X-axis and we plot this curve by using eqn no (3.52). when length is taken as 100m it is shown by red line when we further increase the length from 100 to 1000m it is shown by green line then further increase in length from 1000 to 10000m it is shown by the blue line but the probability range for this plot is from 0 to 0.5



Plot 3.9: Probability vs Energy plot for different lengths for electron neutrino changing into tauon neutrino.

3.68 Probability Plot for electron neutrinos changing into electron neutrinos at different lengths

All the parameters remains same as in the case of electron neutrinos changing into muon neutrinos. we can change only length. In this plot Probability is along Y-axis and energy is along X-axis and in this plot we are using probability expression by eqn (3.53)



Plot 3.10: Probability vs Energy plot for different lengths for survival probability for electron neutrino.

Plot 3.8,3.9,3.10 confirms that oscillation came into picture only for solar neutrinos when the length between source and detector exceeds up to hundreds of Km.

Recent Update

According to the new updates from recent papers of A.B. Balantekin[11] and Takeshi Araki[12] the third neutrino mixing angle θ_{13} is likely to be the very high as compared to the early value of θ_{13}. Two of the mixing angles are well measured i.e. θ_{12} from solar and reactor experiment and θ_{23} from the atmospheric neutrino experiment. To have the complete information on neutrino mixing, we still need to know three more pieces of information:

i) The value of the remaining mixing angle θ_{13}

ii) The mass hierarchy.

iii) The value of the CP-violating phase.

References:

[1] Introduction to the physics of massive and mixed neutrinos,S.M. Bilenky, 99-107 (2010)

[2] Pomita Ghoshal, et.al, Phys.Rev. D.73.053001(2006)

[3] V.Barger,K.Whisnant, Phys.Rev.D.59.093007 (1999)

[4] Fundamentals of Neutrino Physics and Astrophysics

Carlo Giunti: Istituto Nazionale di Fisica Nucleare, Sezione di Torino and Department di Fisica Teorica, University di Torino, Italy

Chung W. Kim: Korea Institute for Advanced Study, Seoul, Korea and The Johns Hopkins University, Baltimore, MD, USA, pp 452-455

[5] Junpei Shirai,Phys.Rev.Lett.94.081801

[6] K. Nakamura et.at, Phys.Rev.Lett.107:041801 (2010)

[7] M. Apollonio,et.al, Phys.Rev.B.420: 390-404 (1998)

[8] Irina Mocioiu and Robert Shrock, Phys.Rev.D.Vol 62 (2000)

[9] Gary J.Feldman,Robert D.Cousins,Phys.Rev.D.Vol 57, no.7 (1998)

[10] C. Amsler,et.al, (Particle Data Group), Physics Letters B.667, 1 (2008)

[11] A.B. Balantekin, hep/ph:1106.5021v1(2011)

[12] Takeshi Araki, Phys.Rev.D.Vol 84 (2011)