AGGREGATION OPERATORS FOR VARIOUS EXTENSIONS OF FUZZY SET AND ITS APPLICATIONS IN TRANSPORTATION PROBLEMS

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CERTIFICATE

This is to certify that the thesis entitled, "**Aggregation operators for various extensions of fuzzy set and its applications in transportation problems**" submitted by **Akansha Mishra** in the fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar Institute of Engineering & Technology, Patiala, is a record of candidate's own work carried out by her under our supervision and guidance.

The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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DECLARATION

It is certified that the thesis is entirely my own. The ideas and references cited herein have been duly acknowledged.

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TO

MY PARENTS

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List of abbreviations

BFFTP	Balanced fully fuzzy transportation problem
BFFTPs	Balanced fully fuzzy transportation problems
BFIFTP	Balanced fully intuitionistic fuzzy transportation problem
BFIFTPs	Balanced fully intuitionistic fuzzy transportation problems
FFN	flat fuzzy number
FFNs	flat fuzzy numbers
FFTP	Fully fuzzy transportation problem
FFTPs	Fully fuzzy transportation problems
FIFTP	Fully intuitionistic fuzzy transportation problem
GIVTrF	Generalized interval-valued trapezoidal fuzzy
GIVTrFN	Generalized interval-valued trapezoidal fuzzy number
GIVTrFNTP	Generalized interval-valued trapezoidal fuzzy number transportation problem
GTrFN	Generalized trapezoidal fuzzy number
GTrFNs	Generalized trapezoidal fuzzy numbers
IF	Intuitionistic fuzzy
IFFLPP	Intuitionistic fully fuzzy linear programming problem
IFLPP	Intuitionistic fuzzy linear programming problem
IFMN	Intuitionistic fuzzy multiplicative number
IFMNs	Intuitionistic fuzzy multiplicative numbers
IFMW	Intuitionistic fuzzy multiplicative weighted
IFMWAO	Intuitionistic fuzzy multiplicative weighted averaging operator
IFMWGO	Intuitionistic fuzzy multiplicative weighted geometric operator

IFN	Intuitionistic fuzzy number
IFNs	Intuitionistic fuzzy numbers
IFTP	Intuitionistic fuzzy transportation problem
IFW	Intuitionistic fuzzy weighted
IFWAO	Intuitionistic fuzzy weighted averaging operator
IFWAOs	Intuitionistic fuzzy weighted averaging operators
IFWGO	Intuitionistic fuzzy weighted geometric operator
IFWGOs	Intuitionistic fuzzy weighted geometric operators
IVIF	Interval-valued intuitionistic fuzzy
IVIFN	Interval-valued intuitionistic fuzzy number
IVIFNs	Interval-valued intuitionistic fuzzy numbers
IVIFW	Interval-valued intuitionistic fuzzy weighted
IVIFWAO	Interval-valued intuitionistic fuzzy weighted averaging operator
IVIFWAOs	Interval-valued intuitionistic fuzzy weighted averaging operators
IVIFWGO	Interval-valued intuitionistic fuzzy weighted geometric operator
IVIFWGOs	Interval-valued intuitionistic fuzzy weighted geometric operators
JMD	Jai Mata (Mehar) Di
LPP	Linear programming problem
PF	Pythagorean fuzzy
PFN	Pythagorean fuzzy number
PFNs	Pythagorean fuzzy numbers
PFW	Pythagorean fuzzy weighted
PFWAO	Pythagorean fuzzy weighted averaging operator

PFWAOs	Pythagorean fuzzy weighted averaging operators
PFWGO	Pythagorean fuzzy weighted geometric operator
PFWGOs	Pythagorean fuzzy weighted geometric operators
SVN	Single-valued neutrosophic
SVNHFN	Single-valued neutrosophic hesitant fuzzy number
SVNHFNs	Single-valued neutrosophic hesitant fuzzy numbers
SVNHFOW	Single-valued neutrosophic hesitant fuzzy ordered weighted
SVNHFOWAO	Single-valued neutrosophic hesitant fuzzy ordered weighted averaging operator
SVNHFOWAOs	Single-valued neutrosophic hesitant fuzzy ordered weighted averaging operators
SVNHFWGO	Single-valued neutrosophic hesitant fuzzy weighted geometric operator
SVNHFWGOs	Single-valued neutrosophic hesitant fuzzy weighted geometric operators
SVNN	Single-valued neutrosophic number
SVNNs	Single-valued neutrosophic numbers
SVNW	Single-valued neutrosophic weighted
SVNWAO	Single-valued neutrosophic weighted averaging operator
SVNWAOs	Single-valued neutrosophic weighted averaging operators
SVNWGO	Single-valued neutrosophic weighted geometric operator
TIFN	Triangular intuitionistic fuzzy number
TIFNs	Triangular intuitionistic fuzzy numbers
TrFN	Trapezoidal fuzzy number
TrFNs	Trapezoidal fuzzy numbers
TrIFN	Trapezoidal intuitionistic fuzzy number

TrIFNs	Trapezoidal intuitionistic fuzzy numbers
TrIFWAO	Trapezoidal intuitionistic fuzzy weighted averaging operator

ABSTARCT

There exist several methods in the literature to solve fuzzy transportation problems under its various extensions. But, as all these methods have been proposed by considering the assumption that the aggregated value of all the parameters are known. Therefore, if instead of the aggregated values of the parameters, the values of the various parameters, collected from experts, are provided. Then, the existing methods cannot be used to solve fuzzy transportation problems and its various extensions.

To overcome this limitation, firstly, there is need to aggregate the provided values. But, as there exist several weighted averaging operators and weighted geometric operators for each extension of fuzzy set. Therefore, firstly, there is need to choose the appropriate weighted averaging operator and weighted geometric operator for each extension of fuzzy set.

After aggregating the provided values by the selected aggregation operator, researchers may use the recently proposed methods [48-51,136] for solving fuzzy transportation problems and its various extensions. However, after a deep study, some shortcomings have been observed in these methods. Therefore, it is scientifically incorrect to use these methods in their present form.

Keeping all above in mind, the aim of this thesis is

(i) To choose/propose an appropriate weighted averaging operator for various extensions of fuzzy set. Also, to show that the weighted geometric operator cannot be defined for fuzzy set and its extensions.

(ii) To point out the limitations as well as the flaws of the existing method [49]. Also, to propose a modified method overcome the limitations as well as to resolve the flaws of the existing method [49].

(iii) To propose a simplified approach for solving BFIFTPs as compared to the existing approach

[51]. Also, to generalize the proposed approach, with the help of the TrIFWAO, for solving such BFIFTPs in which the aggregated values of the parameters are not known.

(iv) To point out the flaws of the existing method [136] for transforming an unbalanced FIFTP into a BFIFTP as well as to propose a valid method for transforming an unbalanced FIFTP into a BFIFTP.

(v) To point out the flaws of the existing method [48] for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP as well as to propose a valid method for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP.

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8. Akansha Mishra, Amit Kumar, JMD method for transforming an unbalanced fully intuitionistic fuzzy transportation problem into a balanced fully intuitionistic fuzzy transportation problem (Communicated in Soft Computing).
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Chapter 1

Introduction

1.1 Introduction

In several real-life problems, the crisp weighted averaging operator (crisp weighted arithmetic mean) or crisp weighted geometric operator (crisp weighted geometric mean) is used to transform the data/ information, collected from various experts/decision makers, into a single value. Cost minimization transportation problem is one of such real-life problems. To solve a real-life cost minimization transportation problem, firstly, the information about its various parameters (the cost for transporting one unit quantity of the product from each source to each destination, the availability of the product at each source and the demand of the product at each destination) is collected from several experts. Then, the crisp weighted averaging operator or crisp weighted geometric operator is used to transform the collected data/ information, obtained from various experts/decision makers, into a single value.

It is pertinent to mention that the crisp weighted averaging operator and the crisp weighted geometric operator can be used only to aggregate the data of such transportation problems in which the collected data/ information can be represented as real numbers. However, in most of the real-life cost minimization transportation problems, the collected data/ information cannot be represented as real numbers due to various reasons e.g., the cost for transporting the unit quantity of the product from a source to a destination depends upon the various factors like route diversion due to unavoidable circumstances, variation in fuel price, weather condition, demand of vehicles etc. Hence, it is not always possible to represent the cost for transporting the unit quantity of the product from a source to a destination with a real number. Similarly, the availability of the product at a source and the demand of the product at a destination depends upon various factors. Hence, it is not always possible to represent the availability of the product at a source and the demand of the product at a destination with real numbers.

Due to the same reason, in the literature [4, 12, 14, 21-26, 37, 38, 44-51, 54, 55, 71, 81, 94, 98, 112-117, 131, 136, 149-151, 158, 159, 168, 174-176, 180, 183, 184, 219], fuzzy set [225] and its various extensions [20] have been used to represent the parameters of cost minimization transportation problems. Although, there exist several methods in the literature to solve fuzzy transportation problems and its various extensions. But, as all these methods have been proposed by considering the assumption that the aggregated value of all the parameters are known. Therefore, if instead of the aggregated values of the parameters, the values of the various parameters, collected from experts, are provided. Then, the existing methods cannot be used to solve fuzzy transportation problems and its various extensions.

To overcome this limitation, firstly, there is need to aggregate the provided values. But, as there exist several weighted averaging operators and weighted geometric operators [2, 6, 7, 27-31, 34-36, 39, 52, 53, 56-69, 87-93, 95, 96, 118-130, 132-135, 147, 152, 154, 155, 157, 165, 167, 178, 188-199, 201-209, 221-224, 226-233, 236] for each extension of fuzzy set. Therefore, firstly, there is need to choose the appropriate weighted averaging operator and/or weighted geometric operator for each extension of fuzzy set.

After aggregating the provided values by the selected aggregation operator, researchers may use the recently proposed methods [48-51,136] for solving fuzzy transportation problems and its various extensions. However, after a deep study, some shortcomings have been observed in these methods. Therefore, it is scientifically incorrect to use these methods in their present form.

Keeping all above in mind, the aim of this thesis is

(i) To choose/propose an appropriate weighted averaging operator for various extensions of fuzzy set. Also, to show that the weighted geometric operator cannot be defined for fuzzy set and its extensions.

(ii) To point out the limitations as well as the flaws of the existing method [49]. Also, to propose a modified method overcome the limitations as well as to resolve the flaws of the existing method [49].

(iii) To propose a simplified approach for solving BFIFTPs as compared to the existing approach [51]. Also, to generalize the proposed approach, with the help of the TrIFWAO, for solving such BFIFTPs in which the aggregated values of the parameters are not known.

(iv) To point out the flaws of the existing method [136] for transforming an unbalanced FIFTPs into a BFIFTPs as well as to propose a valid method for transforming an unbalanced FIFTP into a BFIFTP.

(v) To point out the flaws of the existing method [48] for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP as well as to propose a valid method for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP.

This chapter is organized as follows. Some existing fuzzy/IF/PF/IVIF/neutrosophic weighted averaging/geometric operators, proposed in the last few years, have been discussed in Section 1.2 to Section 1.5 of this chapter. The methods for comparing IFNs, PFNs, IVIFNs, SVNNs etc., used to prove some important properties of the fuzzy/IF/PF/IVIF/neutrosophic weighted aggregation operators, are discussed in Section 1.6. The chapter wise summary of the thesis is discussed in Section 1.7.

1.2 Some existing expressions to evaluate the sum and multiplication of IFNs as well as the corresponding IFW aggregation operators

It is pertinent to mention that the expressions to evaluate sum and multiplication of the real numbers are uniquely defined. However, there exist several expressions to evaluate sum and multiplication of IFNs. Since, an IFWAO operator fully depends upon the sum of IFNs and an IFWGO fully depends upon the multiplication of IFNs. Therefore, in the literature, there exist several IFWAO, based upon different types of sum of IFNs, as well as there exist several IFWGO, based upon different types multiplication of IFNs.

In this section, the existing expressions [28, 60, 65, 88, 204, 208, 223] to evaluate the sum and multiplication of IFNs as well as the corresponding IFW aggregation operators are discussed.

"An intuitionistic fuzzy set, over the universal set X, is defined as $\alpha = \{\langle x, \mu_\alpha(x), \nu_\alpha(x)\rangle \mid x \in X,\ 0 \le \mu_\alpha(x) \le 1,\ 0 \le \nu_\alpha(x) \le 1,\ \mu_\alpha(x) + \nu_\alpha(x) \le 1\}$. The values $\mu_\alpha(x)$, $\nu_\alpha(x)$ and $1 - \mu_\alpha(x) - \nu_\alpha(x)$ respectively are called the degree of membership, the degree of non-membership and the degree of hesitation for the element x. Also, the pair $\langle \mu_\alpha, \nu_\alpha\rangle$ is called an IFN. An IFN $\langle \mu_\alpha, \nu_\alpha\rangle$ may also be represented as $[\mu_\alpha, 1 - \nu_\alpha]$.

1.2.1 Xu and Yager's expression to evaluate the multiplication of IFNs as well as the corresponding IFWGO

Xu and Yager [208, Section 3.1, pp. 420] proposed the expression (1.2.1.1) to evaluate the multiplication of two IFNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle = [\mu_1, 1-\nu_1]$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle = [\mu_2, 1-\nu_2]$. Hence, the expression (1.2.1.2) to evaluate the positive power λ of an IFN $\alpha = \langle \mu, \nu \rangle = [\mu, 1-\nu]$.

$$\alpha_1 \otimes \alpha_2 = \langle \mu_1\mu_2, 1-(1-\nu_1)(1-\nu_2) \rangle = [\mu_1\mu_2, (1-\nu_1)(1-\nu_2)] \qquad (1.2.1.1)$$

$$\alpha^\lambda = \langle \mu^\lambda, 1-(1-\nu)^\lambda \rangle = \left[\mu^\lambda, (1-\nu)^\lambda\right] \qquad (1.2.1.2)$$

Also, using the expression (1.2.1.1) and the expression (1.2.1.2), Xu and Yager [208, Section 3.2, Theorem 3, pp. 422] proposed the IFWGO (1.2.1.3) and its extensions [208, Section 3.3, pp. 426; Section 3.4, pp. 428].

$$\otimes_{i=1}^{n} (\alpha_i)^{w_i} = \langle \prod_{i=1}^{n} \mu_i^{w_i}, 1-\prod_{i=1}^{n}(1-\nu_i)^{w_i} \rangle = \left[\prod_{i=1}^{n} \mu_i^{w_i}, \prod_{i=1}^{n}(1-\nu_i)^{w_i}\right] \qquad (1.2.1.3)$$

where,

(i) $\alpha_i = \langle \mu_i, \nu_i \rangle = [\mu_i, 1-\nu_i], i = 1, 2, \ldots, n$ are n IFNs.

(ii) w_i is the weight assigned to the i^{th} IFN $\alpha_i = \langle \mu_i, \nu_i \rangle = [\mu_i, 1-\nu_i]$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.2.2 Xu's expression to evaluate the sum of IFNs as well as the corresponding IFWAO

Xu [204, Definition 3.2, pp. 1182] proposed the expression (1.2.2.1) to evaluate the sum of two IFNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle = [\mu_1, 1-\nu_1]$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle = [\mu_2, 1-\nu_2]$. Hence, the expression (1.2.2.2) to evaluate the multiplication of a positive real number λ with an IFN $\alpha = \langle \mu, \nu \rangle = [\mu, 1-\nu]$.

$$\alpha_1 \oplus \alpha_2 = \langle 1-(1-\mu_1)(1-\mu_2), \nu_1\nu_2 \rangle = [1-(1-\mu_1)(1-\mu_2), 1-\nu_1\nu_2] \qquad (1.2.2.1)$$

$$\lambda\alpha = \langle 1-(1-\mu)^\lambda, \nu^\lambda \rangle = \left[1-(1-\mu)^\lambda, 1-\nu^\lambda\right] \qquad (1.2.2.2)$$

Also, using the expression (1.2.2.1) and the expression (1.2.2.2), Xu [204, Theorem 3.4, pp. 1183] proposed the IFWAO (1.2.2.3) and its extensions [204, Definition 3.4, pp. 1184; Definition 3.5, pp. 1185].

$$\oplus_{i=1}^{n}(w_i\alpha_i) = \langle 1-\prod_{i=1}^{n}(1-\mu_i)^{w_i}, \prod_{i=1}^{n}\nu_i^{w_i}\rangle = \left[1-\prod_{i=1}^{n}(1-\mu_i)^{w_i}, 1-\prod_{i=1}^{n}\nu_i^{w_i}\right] \quad (1.2.2.3)$$

where,

(i) $\alpha_i = \langle\mu_i, \nu_i\rangle = [\mu_i, 1-\nu_i], i = 1, 2, \ldots, n$ are n IFNs.

(ii) w_i is the weight assigned to the i^{th} IFN $\alpha_i = \langle\mu_i, \nu_i\rangle = [\mu_i, 1-\nu_i]$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.2.3 He et al.'s expression to evaluate the multiplication of IFNs as well as the corresponding IFWGO

He et al. [88, Section 3, Definition 5, pp. 144] proposed the expression (1.2.3.1) to evaluate the multiplication of two IFNs $\alpha_1 = \langle\mu_1, \nu_1\rangle$ and $\alpha_2 = \langle\mu_2, \nu_2\rangle$. Hence, the expression (1.2.3.2) to evaluate the positive power λ of an IFN $\alpha = \langle\mu, \nu\rangle$.

$$\alpha_1 \otimes \alpha_2 = \big\langle (1-\nu_1)(1-\nu_2) - \big(1-(\mu_1+\nu_1)\big)\big(1-(\mu_2+\nu_2)\big), 1-(1-\nu_1)(1-\nu_2)\big\rangle \quad (1.2.3.1)$$

$$\alpha^{\lambda} = \langle (1-\nu)^{\lambda} - \big(1-(\mu+\nu)\big)^{\lambda},\ 1-(1-\nu)^{\lambda}\rangle \quad (1.2.3.2)$$

Also, using the expression (1.2.3.1) and the expression (1.2.3.2), He et al. [88, Section 4.1, Theorem 5, pp. 148] proposed the IFWGO (1.2.3.3) and its extensions [88, Section 4.3, Theorem 8, pp. 150; Section 4.4, Theorem 11, pp. 152].

$$\otimes_{i=1}^{n}(\alpha_i)^{w_i} = \langle \prod_{i=1}^{n}(1-\nu_i)^{w_i} - \prod_{i=1}^{n}(1-(\mu_i+\nu_i))^{w_i}, 1-\prod_{i=1}^{n}(1-\nu_i)^{w_i}\rangle \quad (1.2.3.3)$$

where

(i) $\alpha_i = \langle\mu_i, \nu_i\rangle, i = 1, 2, \ldots, n$ are n IFNs.

(ii) w_i is the weight assigned to the i^{th} IFN $\alpha_i = \langle \mu_i, \nu_i \rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.2.4 Yu's expression to evaluate the sum of IFNs as well as the corresponding IFWAO

Yu [223, Section 3, Example 1, pp. 1019] pointed out the shortcomings of the existing IFWAO [Xu]. To resolve the shortcomings, Yu [223, Section 3, Definition 4, pp. 1020] proposed the expression (1.2.4.1) to evaluate the sum of two IFNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$. Hence, the expression (1.2.4.2) to evaluate the multiplication of a positive real number λ with an IFN $\alpha = \langle \mu, \nu \rangle$.

$$\alpha_1 \oplus \alpha_2 = \left\langle 1-(1-\mu_1)(1-\mu_2), (1-\mu_1)(1-\mu_2) - (1-(\mu_1+\nu_1))\left(1-(\mu_2+\nu_2)\right) \right\rangle \quad (1.2.4.1)$$

$$\lambda\alpha = \langle 1-(1-\mu)^{\lambda}, (1-\mu)^{\lambda} - \left(1-(\mu+\nu)\right)^{\lambda} \rangle \quad (1.2.4.2)$$

Also, using the expression (1.2.4.1) and the expression (1.2.4.2), Yu [223, Section 3.1, Theorem 2, pp. 1021] proposed the IFWAO (1.2.4.3) and its extensions [223, Section 3.2, Theorem 3, pp. 1023; Section 3.3, Theorem 4, pp. 1025].

$$\oplus_{i=1}^{n} w_i\alpha_i = \left\langle 1-\prod_{i=1}^{n}(1-\mu_i)^{w_i}, \prod_{i=1}^{n}(1-\mu_i)^{w_i} - \prod_{i=1}^{n}\left(1-(\mu_i+\nu_i)\right)^{w_i} \right\rangle \quad (1.2.4.3)$$

where,

(i) $\alpha_i = \langle \mu_i, \nu_i \rangle$, $i = 1, 2, \ldots, n$ are n IFNs.

(ii) w_i is the weight assigned to the i^{th} IFN $\alpha_i = \langle \mu_i, \nu_i \rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.2.5 Chen and Chang's expression to evaluate the multiplication of IFNs as well as the corresponding IFWGO

Chen and Chang [28, Example 3.1, pp. 138] pointed out the shortcomings of the existing IFWGO [88]. To resolve the shortcomings, Chen and Chang [28, Section 4, pp. 139] proposed the expression (1.2.5.1) to evaluate the multiplication of two IFNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$. Hence, the expression (1.2.5.2) to evaluate the positive power λ of an IFN $\alpha = \langle \mu, \nu \rangle$.

$$\alpha_1 \otimes \alpha_2 = \left\langle 1 - (1-\mu_1)(1-\mu_2), (1-\mu_1)(1-\mu_2) - \big(1 - (\mu_1 + \nu_1)\big)\big(1 - (\mu_2 + \nu_2)\big) \right\rangle$$

(1.2.5.1)

$$\alpha^{\lambda} = \langle 1 - (1-\mu)^{\lambda}, (1-\mu)^{\lambda} - \big(1 - (\mu + \nu)\big)^{\lambda} \rangle. \qquad (1.2.5.2)$$

Also, using the expression (1.2.5.1) and the expression (1.2.5.2), Chen and Chang [28, Section 4, Definition 4.1, pp. 142] proposed the IFWGO (1.2.5.3) and its extensions [28, Section 4; Definition 4.2, pp. 142; Definition 4.3, pp. 143].

$$\otimes_{i=1}^{n} (\alpha_i)^{w_i} = \left\langle 1 - \prod_{i=1}^{n} (1 - \mu_{\alpha_i})^{w_i}, \prod_{i=1}^{n} (1 - \mu_{\alpha_i})^{w_i} - \prod_{i=1}^{n} (1 - \mu_{\alpha_i} - \nu_{\alpha_i})^{w_i} \right\rangle \qquad (1.2.5.3)$$

where,

(i) $\alpha_i = \langle \mu_i, \nu_i \rangle$, $i = 1, 2, \dots, n$ are n IFNs.

(ii) w_i is the weight assigned to the i^{th} IFN $\alpha_i = \langle \mu_i, \nu_i \rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.2.6 Garg's expression to evaluate the sum of IFNs as well as the corresponding IFWAO

Garg [65, Section 2.2, pp. 166] pointed out the shortcomings of the existing IFWAOs [88, 191, 204, 208, 231]. To resolve the shortcomings, Garg [65. Section 3, Definition 3.1, pp. 166] proposed the expression (1.2.6.1) to evaluate the sum of two IFNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$. Hence, the expression (1.2.6.2) to evaluate the multiplication of a positive real number λ with an IFN $\alpha = \langle \mu, \nu \rangle$.

$$\alpha_1 \oplus \alpha_2 = \left\langle \frac{\prod_{i=1}^{2}(1+\mu_i)-\prod_{i=1}^{2}(1-\mu_i)}{\prod_{i=1}^{2}(1+\mu_i)+\prod_{i=1}^{2}(1-\mu_i)}, \frac{2\{\prod_{i=1}^{2}(1-\mu_i)-\prod_{i=1}^{2}(1-\mu_i-\nu_i)\}}{\prod_{i=1}^{2}(1+\mu_i)+\prod_{i=1}^{2}(1-\mu_i)} \right\rangle \tag{1.2.6.1}$$

$$\lambda\alpha = \left\langle \frac{(1+\mu)^{\lambda}-(1-\mu)^{\lambda}}{(1+\mu)^{\lambda}+(1-\mu)^{\lambda}}, \frac{2\{(1-\mu)^{\lambda}-(1-\mu-\nu)^{\lambda}\}}{(1+\mu)^{\lambda}+(1-\mu)^{\lambda}} \right\rangle. \tag{1.2.6.2}$$

Also, using the expression (1.2.6.1) and the expression (1.2.6.2), Garg [65, Section 3.1, Theorem 3.2, pp. 167] proposed the IFWAO (1.2.6.3) and its extensions [65, Section 3.2, Theorem 3.4, pp. 170; Section 3.3, Theorem 3.5, pp. 171].

$$\oplus_{i=1}^{n} (w_i\alpha_i) = \left\langle \frac{\prod_{i=1}^{n}(1+\mu_i)^{w_i}-\prod_{i=1}^{n}(1-\mu_i)^{w_i}}{\prod_{i=1}^{n}(1+\mu_i)^{w_i}+\prod_{i=1}^{n}(1-\mu_i)^{w_i}}, \frac{2\{\prod_{i=1}^{n}(1-\mu_i)^{w_i}-\prod_{i=1}^{n}(1-\mu_i-\nu_i)^{w_i}\}}{\prod_{i=1}^{n}(1+\mu_i)^{w_i}+\prod_{i=1}^{n}(1-\mu_i)^{w_i}} \right\rangle \tag{1.2.6.3}$$

where,

(i) $\alpha_i = \langle \mu_i, \nu_i \rangle, i = 1, 2, \ldots, n$ are n IFNs.

(ii) w_i is the weight assigned to the i^{th} IFN $\alpha_i = \langle \mu_i, \nu_i \rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.2.7 Garg's expression to evaluate the multiplication of IFNs as well as the corresponding IFWGO

Garg [60, Section 2.3, pp. 55] pointed out the shortcomings of the existing IFWGO [191, 204]. To resolve the shortcomings, Garg [60, Section 3.1, Definition 3.1, pp. 55] proposed the expression (1.2.7.1) to evaluate the multiplication of two IFNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$. Hence, the expression (1.2.7.2) to evaluate the positive power λ of an IFN $= \langle \mu, \nu \rangle$.

$$\alpha_1 \otimes \alpha_2 = \left\langle \frac{2\{\prod_{i=1}^{2}(1-\nu_i)-\prod_{i=1}^{2}(1-\mu_i-\nu_i)\}}{\prod_{i=1}^{2}(1+\nu_i)+\prod_{i=1}^{2}(1-\nu_i)}, \frac{\prod_{i=1}^{2}(1+\nu_i)-\prod_{i=1}^{2}(1-\nu_i)}{\prod_{i=1}^{2}(1+\nu_i)+\prod_{i=1}^{2}(1-\nu_i)} \right\rangle \tag{1.2.7.1}$$

$$\alpha^{\lambda} = \left\langle \frac{2\{(1-\nu)^{\lambda}-(1-\mu-\nu)^{\lambda}\}}{(1+\nu)^{\lambda}+(1-\nu)^{\lambda}}, \frac{(1+\nu)^{\lambda}-(1-\nu)^{\lambda}}{(1+\nu)^{\lambda}+(1-\nu)^{\lambda}} \right\rangle. \tag{1.2.7.2}$$

Also, using the expression (1.2.7.1) and the expression (1.2.7.2), Garg [60, Section 3.1, Theorem 3.1, pp. 56] proposed the IFWGO (1.2.7.3) and its extensions [60, Definition 3.1, pp. 56; Definition 3.2, pp. 57; Definition 3.3, pp. 58].

$$\otimes_{i=1}^{n} {\alpha_i}^{w_i} = \left\langle \frac{2\{\prod_{i=1}^{n}(1-\nu_i)^{w_i}-\prod_{i=1}^{n}(1-\mu_i-\nu_i)^{w_i}\}}{\prod_{i=1}^{n}(1+\nu_i)^{w_i}+\prod_{i=1}^{n}(1-\nu_i)^{w_i}}, \frac{\prod_{i=1}^{n}(1+\nu_i)^{w_i}-\prod_{i=1}^{n}(1-\nu_i)^{w_i}}{\prod_{i=1}^{n}(1+\nu_i)^{w_i}+\prod_{i=1}^{n}(1-\nu_i)^{w_i}} \right\rangle \tag{1.2.7.3}$$

where,

(i) $\alpha_i = \langle \mu_i, \nu_i \rangle, i = 1, 2, \ldots, n$ are n IFNs.

(ii) w_i is the weight assigned to the i^{th} IFN $\alpha_i = \langle \mu_i, \nu_i \rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.2.8 Some existing expressions to evaluate the sum and multiplication of IFMNs as well as the corresponding IFMW aggregation operators

There exists several expressions to evaluate sum and multiplication of IFMNs. Since, an IFMWAO operator fully depends upon the sum of IFMNs and an IFMWGO fully depends upon the multiplication of IFMNs. Therefore, in the literature, there exist several IFMWAO, based upon different types of sum of IFMNs, as well as there exist several IFMWGO, based upon different types multiplication of IFMNs.

In this section, the existing expressions [56, 66, 156] to evaluate the sum and multiplication of IFMNs as well as the corresponding IFMW aggregation operators are discussed.

"An IF multiplicative set, over the universal set X, is defined as $\alpha = \{\langle x, \mu_{\alpha}(x), \nu_{\alpha}(x) \rangle \mid x \in X, \frac{1}{q} \leq \mu_{\alpha}(x) \leq q, \frac{1}{q} \leq \nu_{\alpha}(x) \leq q, \mu_{\alpha}(x)\nu_{\alpha}(x) \leq 1, q > 1\}$. The values $\mu_{\alpha}(x)$, $\nu_{\alpha}(x)$ and $1 - \mu_{\alpha}(x) - \nu_{\alpha}(x)$ respectively are called the degree of membership, the degree of non-membership and the degree of hesitation for the element x. Also, the pair $\langle \mu_\alpha, \nu_\alpha \rangle$ is called an IFMN".

1.2.8.1 Garg's expression to evaluate the multiplication of IFMNs as well as the corresponding IFMWGO

Garg [56, Section 2.2, pp. 1078] pointed out the shortcomings of the existing IFMWGO [202]. To resolve the shortcomings, Garg [56, Section 3, pp. 1078] proposed the expression (1.2.8.1.1) to evaluate the multiplication of two IFMNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$. Hence, the expression (1.2.8.1.2) to evaluate the positive power λ of an IFMN $\alpha = \langle \mu, \nu \rangle$.

$$\alpha_1 \otimes \alpha_2 = \left\langle \frac{2\{1-(1-\mu_1\nu_1)(1-\mu_2\nu_2)\}}{(1+2\nu_1)(1+2\nu_1)-1}, \frac{(1+2\nu_1)(1+2\nu_2)-1}{2} \right\rangle, \tag{1.2.8.1.1}$$

$$\alpha^\lambda = \left\langle \frac{2\{1-(1-\mu\nu)^\lambda\}}{(1+2\nu)^\lambda-1}, \frac{(1+2\nu)^\lambda-1}{2} \right\rangle. \tag{1.2.8.1.2}$$

Also, using the expression (1.2.8.1.1) and the expression (1.2.8.1.2), Garg [56, Section 3.2, Theorem 1, pp. 1079] proposed the IFMGO (1.2.8.1.3) and its extensions [56, Definition 5, pp. 1082; Definition 6, pp. 1083; Definition 7, pp. 1084].

$$\otimes_{i=1}^{n} (\alpha_i)^{w_i} = \left\langle \frac{2\{1-\prod_{i=1}^{n}(1-\mu_i\nu_i)^{w_i}\}}{\prod_{i=1}^{n}(1+2\nu_i)^{w_i}-1}, \frac{\prod_{i=1}^{n}(1+2\nu_i)^{w_i}-1}{2} \right\rangle \tag{1.2.8.1.3}$$

where,

(i) $\alpha_i = \langle \mu_i, \nu_i \rangle$, $i = 1, 2, \ldots, n$ are n IFMNs.

(ii) w_i is the weight assigned to the i^{th} IFMN $\alpha_i = \langle \mu_i, \nu_i \rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.2.8.2 Garg's expression to evaluate the sum of IFMNs as well as the corresponding IFMWAO

Garg [66, Section 2, pp. 2122] pointed out the shortcomings of the existing

IFMWAO [200]. To resolve the shortcomings, Garg [66, Section 3, Definition 7, pp. 2123] proposed the expression (1.2.8.2.1) to evaluate the sum of two IFMNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$. Hence, the expression (1.2.8.2.2) to evaluate the multiplication of a positive real number λ with an IFMN $\alpha = \langle \mu, \nu \rangle$.

$$\alpha_1 \oplus \alpha_2 = \left\langle \frac{(1+2\mu_1)(1+2\mu_2)-1}{2}, \frac{2\{1-(1-\mu_1\nu_1)(1-\mu_2\nu_2)\}}{(1+2\mu_1)(1+2\mu_2)-1} \right\rangle \tag{1.2.8.2.1}$$

$$\lambda\alpha = \left\langle \frac{(1+2\mu)^{\lambda}-1}{2}, \frac{2\{1-(1-\mu\nu)^{\lambda}\}}{(1+2\mu)^{\lambda}-1} \right\rangle \tag{1.2.8.2.2}$$

Also, using the expression (1.2.8.2.1) and the expression (1.2.8.2.2), Garg [66, Section 3.1, Theorem 2, pp. 2123] proposed the IFMWAO (1.2.8.2.3) and its extensions [66, Section 3.2, pp. 2126].

$$\oplus_{i=1}^{n} w_i\alpha_i = \left\langle \frac{\prod_{i=1}^{n}(1+2\mu_i)^{w_i}-1}{2}, \frac{2\{1-\prod_{i=1}^{n}(1-\mu_i\nu_i)^{w_i}\}}{\prod_{i=1}^{n}(1+2\mu_i)^{w_i}-1} \right\rangle \tag{1.2.8.2.3}$$

where,

(i) $\alpha_i = \langle \mu_i, \nu_i \rangle$, $i = 1, 2, \ldots, n$ are n IFMNs.

(ii) w_i is the weight assigned to the i^{th} IFMN $\alpha_i = \langle \mu_i, \nu_i \rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.2.8.3 Qian and Niu's expression to evaluate the sum and multiplication of IFMNs as well as the corresponding IFMWAO and IFMWGO

Qian and Niu [156] pointed out the shortcomings of the existing IFMWAO [200]. To resolve the shortcomings, Qian and Niu [156, Section 3, Definition 3.1, pp. 2861] proposed

(i) The expression (1.2.8.3.1) to evaluate the sum of two IFMNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$. Hence, the expression (1.2.8.3.2) to evaluate the multiplication of a positive real number λ with an IFMN $\alpha = \langle \mu, \nu \rangle$.

$$\alpha_1 \oplus \alpha_2 = \left\langle q^{\frac{\log_q(\mu_1\mu_2)-(\log_q(\mu_1))(\log_q(\mu_2))+1}{2}}, q^{\frac{\log_q(\nu_1\nu_2)+(\log_q(\nu_1))(\log_q(\nu_2))-1}{2}} \right\rangle$$

(1.2.8.3.1)

$$\lambda\alpha = \left\langle q^{1-2\left(\frac{1-\log_q(\mu)}{2}\right)^{\lambda}}, q^{2\left(\frac{1+\log_q(\nu)}{2}\right)^{\lambda}-1} \right\rangle \tag{1.2.8.3.2}$$

(ii) The expression (1.2.8.3.3) to evaluate the multiplication of two IFMNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$. Hence, the expression (1.2.8.3.4) to evaluate the positive power λ of an IFMN $\alpha = \langle \mu, \nu \rangle$.

$$\alpha_1 \otimes \alpha_2 = \left\langle q^{\frac{\log_q(\mu_1\mu_2)+(\log_q(\mu_1))(\log_q(\mu_2))-1}{2}}, q^{\frac{\log_q(\nu_1\nu_2)+(\log_q(\nu_1))(\log_q(\nu_2))-1}{2}} \right\rangle$$

(1.2.8.3.3)

$$\alpha^{\lambda} = \left\langle q^{2\left(\frac{1+\log_q(\mu)}{2}\right)^{\lambda}-1}, q^{1-2\left(\frac{1-\log_q(\nu)}{2}\right)^{\lambda}} \right\rangle \tag{1.2.8.3.4}$$

Also, using the expression (1.2.8.3.1) and the expression (1.2.8.3.2), Qian and Niu [156, Section 4, Theorem 4.1, pp. 2863] proposed the IFMWAO (1.2.8.3.5).

$$\oplus_{i=1}^{n} w_i\alpha_i = \left\langle q^{1-2\prod_{i=1}^{n}\left(\frac{1-\log_q \mu}{2}\right)^{w_i}}, q^{2\prod_{i=1}^{n}\left(\frac{1+\log_q \nu}{2}\right)^{\lambda}-1} \right\rangle$$

(1.2.8.3.5)

Furthermore, using the expression (1.2.8.3.3) and the expression (1.2.8.3.4), Qian and Niu [156, Theorem 4.2, pp. 2863] proposed the IFMWGO (1.2.8.3.6).

$$\otimes_{i=1}^{n} {\alpha_i}^{w_i} = \left\langle q^{2\prod_{i=1}^{n}\left(\frac{1+\log_q(\mu)}{2}\right)^{w_i}-1}, q^{1-2\prod_{i=1}^{n}\left(\frac{1-\log_q(\nu)}{2}\right)^{w_i}} \right\rangle \tag{1.2.8.3.6}$$

where,

(i) $\alpha_i = \langle \mu_i, \nu_i \rangle$, $i = 1, 2, \ldots, n$ are n IFMNs.

w_i is the weight assigned to the i^{th} IFMN $\alpha_i = \langle \mu_i, \nu_i \rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.3 Some existing expressions to evaluate the sum and multiplication of PFNs as well as the corresponding PFW aggregation operators

The expressions to evaluate sum and multiplication of the real numbers are uniquely defined. However, there exists several expression to evaluate sum and multiplication of PFNs. Since, a PFWAO fully depends upon the sum of PFNs and a PFWGO fully depends upon the multiplication of PFNs. Therefore, there exist different PFWAOs, based upon different types of sum of PFNs, as well as there exist different PFWGOs, based upon different types of multiplication of PFNs, in the literature.

In this section, the existing expressions [58, 62, 63, 67, 135] to evaluate the sum and multiplication of PFNs as well as the corresponding PFWAOs and PFWGOs are discussed.

“A PF set, over the universal set X, is defined as $\alpha = \{\langle x, \mu_\alpha(x), \nu_\alpha(x)\rangle \mid x \in X,$ $0 \le \mu_\alpha(x) \le 1,\ 0 \le \nu_\alpha(x) \le 1,\ \left(\mu_\alpha(x)\right)^2 + \left(\nu_\alpha(x)\right)^2 \le 1\,\}$. The values $\mu_\alpha(x)$, $\nu_\alpha(x)$ and $\sqrt{1 - \left(\mu_\alpha(x)\right)^2 - \left(\nu_\alpha(x)\right)^2}$ respectively are called the degree of membership, the degree of non-membership and the degree of hesitation for the element x. Also, the pair $\langle \mu_\alpha, \nu_\alpha\rangle$ is called a PFN.”

1.3.1 Garg’s expression to evaluate the sum of PFNs as well as the corresponding PFWAO

Garg [58, Section 3, Definition 3.1, pp. 891] proposed the expression (1.3.1.1) to evaluate the sum of two PFNs $\alpha_1 = \langle \mu_1, \nu_1\rangle$ and $\alpha_2 = \langle \mu_2, \nu_2\rangle$ as well as the expression (1.3.1.2) to evaluate the multiplication of a positive power λ with a PFN $\alpha = \langle \mu, \nu\rangle$.

$$\alpha_1 \oplus \alpha_2 = \left\langle \sqrt{\frac{\mu_1^2+\mu_2^2}{1+\mu_1^2.\mu_2^2}}, \frac{\nu_1,\nu_2}{\sqrt{1+\left(1-\nu_1^2\right).\left(1-\nu_2^2\right)}}, \right\rangle \tag{1.3.1.1}$$

$$\lambda\alpha = \left\langle \sqrt{\frac{(1+\mu^2)^\lambda-(1-\mu^2)^\lambda}{(1+\mu^2)^\lambda+(1-\mu^2)^\lambda}}, \frac{\sqrt{2}\nu^\lambda}{\sqrt{(2-\nu^2)^\lambda+(\nu^2)^\lambda}} \right\rangle \tag{1.3.1.2}$$

Also, using the expression (1.3.1.1) and the expression (1.3.1.2), Garg [58, Section 4.1, Theorem 4.1, pp. 898] proposed the PFWAO (1.3.1.3) and its extensions [58, Section 4.2, pp. 906; Section 4.3, pp. 909; Section 4.4, pp. 912].

$$\oplus_{i=1}^{n}(w_i\alpha_i) = \left\langle \sqrt{\frac{\prod_{i=1}^{n}(1+\mu_i^2)^{w_i}-\prod_{i=1}^{n}(1-\mu_i^2)^{w_i}}{\prod_{i=1}^{n}(1+\mu_i^2)^{w_i}+\prod_{i=1}^{n}(1-\mu_i^2)^{w_i}}}, \frac{\sqrt{2}\prod_{i=1}^{n}\nu_i^{w_i}}{\sqrt{\prod_{i=1}^{n}(2-\nu_i^2)^{w_i}+\prod_{i=1}^{n}(\nu_i^2)^{w_i}}} \right\rangle \tag{1.3.1.3}$$

where,

(i) $\alpha_i = \langle\mu_i, \nu_i\rangle$, $i = 1, 2, \ldots, n$ are n PFNs.

(ii) w_i is the weight assigned to the i^{th} PFN $\alpha_i = \langle\mu_i, \nu_i\rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.3.2 Garg's expression to evaluate the multplication of PFNs as well as the corresponding PFWGO

Garg [63, Section 2, Definition 2.1, pp. 601] proposed the expression (1.3.2.1) to evaluate the multiplication of two PFNs $\alpha_1 = \langle\mu_1, \nu_1\rangle$ and $\alpha_2 = \langle\mu_2, \nu_2\rangle$ as well as the expression (1.3.2.2) to evaluate the positive power λ of a PFN $\alpha = \langle\mu, \nu\rangle$.

$$\alpha_1 \otimes \alpha_2 = \left\langle \frac{\mu_1, \mu_2}{\sqrt{1+(1-\mu_1^2)(1-\mu_2^2)}}, \sqrt{\frac{\nu_1^2+\nu_2^2}{1+\nu_1^2\nu_2^2}} \right\rangle, \tag{1.3.2.1}$$

$$\alpha^\lambda = \left\langle \frac{\sqrt{2}\mu^\lambda}{\sqrt{(2-\mu^2)^\lambda+(\mu^2)^\lambda}}, \sqrt{\frac{(1+\nu^2)^\lambda-(1-\nu^2)^\lambda}{(1+\nu^2)^\lambda+(1-\nu^2)^\lambda}} \right\rangle \tag{1.3.2.2}$$

Also, using the expression (1.3.2.1) and the expression (1.3.2.2), Garg [63, Section 3.1, Theorem 3.1, pp. 606] proposed the PFWGO (1.3.2.3) and its extensions [63, Section 3.2, pp. 616; Section 3.3, pp. 619; Section 3.4, pp. 622].

$$\otimes_{i=1}^{n} {\alpha_i}^{w_i} = \left\langle \frac{\sqrt{2}\prod_{i=1}^{n} {\mu_i}^{w_i}}{\sqrt{\prod_{i=1}^{n}(2-\mu_i^2)^{w_i}+\prod_{i=1}^{n}(\mu_i^2)^{w_i}}}, \sqrt{\frac{\prod_{i=1}^{n}(1+\nu_i^2)^{w_i}-\prod_{i=1}^{n}(1-\nu_i^2)^{w_i}}{\prod_{i=1}^{n}(1+\nu_i^2)^{w_i}+\prod_{i=1}^{n}(1-\nu_i^2)^{w_i}}} \right\rangle \tag{1.3.2.3}$$

where,

(i) $\alpha_i = \langle \mu_i, \nu_i \rangle$, $i = 1, 2, \ldots, n$ are n PFNs.

(ii) w_i is the weight assigned to the i^{th} PFN $\alpha_i = \langle \mu_i, \nu_i \rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.3.3 Garg's expression to evaluate the multiplication of PFNs as well as the corresponding PFWGO

Garg [67, Section 2.3] pointed out the shortcomings of the PFWGO [58, 63] proposed by himself. To resolve the shortcomings, Garg [67, Section 3.1, Definition 3.1] proposed the expression (1.3.3.1) to evaluate the multiplication of two PFNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$. Hence, the expression (1.3.3.2) to evaluate the positive power λ of a PFN $= \langle \mu, \nu \rangle$.

$$\alpha_1 \otimes \alpha_2 = \left\langle \sqrt{\frac{2\{\prod_{i=1}^{2}(1-\nu_i^2)-\prod_{i=1}^{2}(1-\mu_i^2-\nu_i^2)\}}{\prod_{i=1}^{2}(1+\nu_i^2)+\prod_{i=1}^{2}(1-\nu_i^2)}}, \sqrt{\frac{\prod_{i=1}^{2}(1+\nu_i^2)-\prod_{i=1}^{2}(1-\nu_i^2)}{\prod_{i=1}^{2}(1+\nu_i^2)+\prod_{i=1}^{2}(1-\nu_i^2)}} \right\rangle \tag{1.3.3.1}$$

$$\alpha^{\lambda} = \left\langle \sqrt{\frac{2\{(1-\nu^2)^{\lambda}-(1-\mu^2-\nu^2)^{\lambda}\}}{(1+\nu^2)^{\lambda}+(1-\nu^2)^{\lambda}}}, \sqrt{\frac{(1+\nu^2)^{\lambda}-(1-\nu^2)^{\lambda}}{(1+\nu^2)^{\lambda}+(1-\nu^2)^{\lambda}}} \right\rangle \tag{1.3.3.2}$$

Also, using the expressions (1.3.3.1) and the expression (1.3.3.2), Garg [67, Section 3.1, Theorem 3.2] proposed the PFWGO (1.3.3.3) and its extensions [67, Section 3.3, pp. 9; Section 3.4].

$$\otimes_{i=1}^{n} {\alpha_i}^{w_i} = \left\langle \sqrt{\frac{2\{\prod_{i=1}^{n}(1-\nu_i^2)^{w_i}-\prod_{i=1}^{n}(1-\mu_i^2-\nu_i^2)^{w_i}\}}{\prod_{i=1}^{n}(1+\nu_i^2)^{w_i}+\prod_{i=1}^{n}(1-\nu_i^2)^{w_i}}}, \sqrt{\frac{\prod_{i=1}^{n}(1+\nu_i^2)^{w_i}-\prod_{i=1}^{n}(1-\nu_i^2)^{w_i}}{\prod_{i=1}^{n}(1+\nu_i^2)^{w_i}+\prod_{i=1}^{n}(1-\nu_i^2)^{w_i}}} \right\rangle \tag{1.3.3.3}$$

where,

(i) $\alpha_i = \langle \mu_i, \nu_i \rangle$, $i = 1, 2, \ldots, n$ are n PFNs.

(ii) w_i is the weight assigned to the i^{th} PFN $\alpha_i = \langle \mu_i, \nu_i \rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i =$ 1.

1.3.4 Garg's PFWAO and PFWGO

Garg [62, Section 3, pp. 550] pointed out that all the aggregation operators have been proposed by considering the assumption that decision makers are surely familiar with the evaluated objects. But, it is not a realistic assumption. Therefore, to handle this situation, Garg [62, Section 3.1, Theorem 4, pp. 551; Section 3.3, Theorem 6, pp. 556] proposed the confidence level based PFWAO (1.3.4.1) and the confidence level based PFWGO (1.3.4.2) as well as the extensions of these operators [62, Section 3.2, Theorem 5, pp. 556; Section 3.4, Theorem 7, pp. 560].

$$\oplus_{j=1}^{n} w_j\big(\eta_j \alpha_j\big) = \left\langle \sqrt{1 - \prod_{j=1}^{n}(1-\mu_j^2)^{\eta_j w_j}}, \prod_{j=1}^{n}(\nu_j)^{\eta_j w_j} \right\rangle$$

(1.3.4.1)

$$\otimes_{j=1}^{n} (\alpha_j^{\eta_j})^{w_j} = \left\langle \prod_{j=1}^{n}(\mu_j)^{\eta_j w_j}, \sqrt{1 - \prod_{j=1}^{n}(1-\nu_j^2)^{\eta_j w_j}} \right\rangle$$

(1.3.4.2)

where,

(i) $0 \leq \eta_j \leq 1$ is confidence level of α_j.

(ii) w_j is the weight vector associated with α_j such that $w_j = [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

(iii) η_j is the confidence level of α_j such that $\eta_j = [0,1]$ and $\sum_{j=1}^{n} \eta_j = 1$.

1.3.5 Ma and Xu's expression to evaluate the multiplication of PFNs as well as the corresponding PFWAO

Ma and Xu [135, Section 4.1, pp. 1206] proposed the expression (1.3.5.1) to evaluate the sum of two PFNs $\alpha_1 = \langle \mu_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle$. Hence, the expression (1.3.5.2) to evaluate the multiplication of a positive real-number λ with a PFN

$\alpha = \langle \mu, \nu \rangle$.

$$\alpha_1 \oplus \alpha_2 = \left(\frac{\mu_1 \mu_2}{[(1-{\mu_1}^2)(1-{\mu_2}^2)+{\mu_1}^2{\mu_2}^2]^{\frac{1}{2}}}, \frac{\nu_1 \nu_2}{[(1-{\nu_1}^2)(1-{\nu_2}^2)+{\nu_1}^2{\nu_2}^2]^{\frac{1}{2}}} \right) \tag{1.3.5.1}$$

$$\lambda\alpha = \left(\frac{\mu^\lambda}{[(1-\mu^2)^\lambda+\mu^{2\lambda}]^{\frac{1}{2}}}, \frac{\nu^\lambda}{[(1-\nu^2)^\lambda+\nu^{2\lambda}]^{\frac{1}{2}}} \right) \tag{1.3.5.2}$$

Also, using the expression (1.3.5.1) and the expression (1.3.5.2), Ma and Xu [135, Section 4.1, Definition 11, pp. 1209] proposed the PFWAO (1.3.5.3) and its extensions [135, Section 4.2, pp. 1211].

$$\oplus_{i=1}^{n} w_i \alpha_i = \left(\frac{\prod_{i=1}^{n} {\mu_i}^{w_i}}{\left[\prod_{i=1}^{n}(1-\mu_i^2)^{w_i}+\prod_{i=1}^{n} {\mu_i}^{2w_i}\right]^{\frac{1}{2}}}, \frac{\prod_{i=1}^{n} {\nu_i}^{w_i}}{\left[\prod_{i=1}^{n}(1-\nu_i^2)^{w_i}+\prod_{i=1}^{n} {\nu_i}^{2w_i}\right]^{\frac{1}{2}}} \right)$$

(1.3.5.3)

where,

(i) $\alpha_i = (\mu_i, \nu_i), i = 1, 2, \ldots, n$ are n PFNs.

(ii) w_i is the weight assigned to the i^{th} PFN $\alpha_i = (\mu_i, \nu_i)$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.4 Some existing expressions to evaluate the sum and multiplication of IVIFNs as well as the corresponding IVIFW aggregation operators

The expressions to evaluate sum and multiplication of the real numbers are uniquely defined. However, there exist several expressions to evaluate sum and multiplication of the IVIFNs. Since, an IVIFWAO operator fully depends upon the sum of IVIFNs and an IVIFWGO fully depends upon the multiplication of IVIFNs. Therefore, there exist different IVIFWAOs, based upon different types of sum of IVIFNs, as well as there exist different IVIFWGOs, based upon different types of multiplication of IVIFNs, in the literature.

In this section, the existing expressions [61, 68] to evaluate the sum and multiplication of IVIFNs as well as the corresponding IVIFW aggregation operators are discussed.

An IVIF set, over the universal set X, is defined as $\alpha = \{\langle x, [\mu_\alpha^L(x), \mu_\alpha^U(x)], [\nu_\alpha^L(x), \nu_\alpha^U(x)]\rangle \mid x \in X,\ 0 \le \mu_\alpha^L(x) \le \mu_\alpha^U(x) \le 1,$ $0 \le \nu_\alpha^L(x) \le \nu_\alpha^U(x) \le 1, \mu_\alpha^U(x) + \nu_\alpha^U(x) \le 1\}$. The intervals $[\mu_\alpha^L(x), \mu_\alpha^U(x)]$, $[\nu_\alpha^L(x), \nu_\alpha^U(x)]$ and $[1 - \mu_\alpha^U(x) - \nu_\alpha^U(x), 1 - \mu_\alpha^L(x) - \nu_\alpha^L(x)]$ respectively are called the interval of degree of membership, the interval of degree of non-membership and the interval of degree of hesitation for the element x. Also, the pair $\langle[\mu_\alpha^L, \mu_\alpha^U], [\nu_\alpha^L, \nu_\alpha^U]\rangle$ is called an IVIFN."

1.4.1 Garg et al.'s expression to evaluate the sum of IVIFNs as well as the corresponding IVIFWAO

Garg et al. [68, Section 2.3, pp. 2584] pointed out the shortcomings of the existing IVIFWA operator [128]. To resolve the shortcomings, Garg et al. [68, Section 3, pp. 2584] proposed the expression (1.4.1.1) to evaluate the sum of two IVIFNs $\alpha_1 = \langle[a_1, b_1], [c_1, d_1]\rangle$ and $\alpha_2 = \langle[a_2, b_2], [c_2, d_2]\rangle$. Hence, the expression (1.4.1.2) to evaluate the multiplication of a positive real number λ with an IVIFN $\alpha = \langle[a, b], [c, d]\rangle$.

$$\alpha_1 \oplus \alpha_2 = \left\langle \left[\frac{\prod_{i=1}^2[1+(\gamma-1)a_i]-\prod_{i=1}^2(1-a_i)}{\prod_{i=1}^2[1+(\gamma-1)a_i]+(\gamma-1)\prod_{i=1}^2(1-a_i)}, \frac{\prod_{i=1}^2[1+(\gamma-1)b_i]-\prod_{i=1}^2(1-b_i)}{\prod_{i=1}^2[1+(\gamma-1)b_i]+(\gamma-1)\prod_{i=1}^2(1-b_i)}\right], \right.$$

$$\left. \left[\frac{\gamma\prod_{i=1}^2(1-a_i)-\gamma\prod_{i=1}^2[1-a_i-c_i]}{\prod_{i=1}^2[1+(\gamma-1)a_i]+(\gamma-1)\prod_{i=1}^2(1-a_i)}, \frac{\gamma\prod_{i=1}^2(1-b_i)-\gamma\prod_{i=1}^2[1-b_i-d_i]}{\prod_{i=1}^2[1+(\gamma-1)b_i]+(\gamma-1)\prod_{i=1}^2(1-b_i)}\right] \right\rangle \quad (1.4.1.1)$$

$$\lambda\alpha = \left\langle \left[\frac{[1+(\gamma-1)a]^\lambda-[1-a]^\lambda}{[1+(\gamma-1)a]^\lambda+(\gamma-1)[1-a]^\lambda}, \frac{[1+(\gamma-1)b]^\lambda-[1-b]^\lambda}{[1+(\gamma-1)b]^\lambda+(\gamma-1)[1-b]^\lambda}\right], \right.$$

$$\left. \left[\frac{\gamma[1-a]^\lambda-\gamma[1-a-c]^\lambda}{[1+(\gamma-1)a]^\lambda+(\gamma-1)[1-a]^\lambda}, \frac{\gamma[1-b]^\lambda-\gamma[1-b-d]^\lambda}{[1+(\gamma-1)b]^\lambda+(\gamma-1)[1-b]^\lambda}\right] \right\rangle \quad (1.4.1.2)$$

Also, using the expression (1.4.1.1) and the expression (1.4.1.2), Garg et al. [68, Section 3.1, Theorem 3.1, pp. 2585] proposed the IVIFWAO (1.4.1.3) and its extensions [68, Section 3.2, pp. 2594; Section 3.3, pp. 2595]

$$\oplus_{i=1}^{n}(w_i\alpha_i)=\left\langle\left[\frac{\prod_{i=1}^{n}[1+(\gamma-1)a_i]^{w_i}-\prod_{i=1}^{n}(1-a_i)^{w_i}}{\prod_{i=1}^{n}[1+(\gamma-1)a_i]^{w_i}+(\gamma-1)\prod_{i=1}^{n}(1-a_i)^{w_i}},\frac{\prod_{i=1}^{n}[1+(\gamma-1)b_i]^{w_i}-\prod_{i=1}^{n}(1-b_i)^{w_i}}{\prod_{i=1}^{n}[1+(\gamma-1)b_i]^{w_i}+(\gamma-1)\prod_{i=1}^{n}(1-b_i)^{w_i}}\right],\right.$$
$$\left.\left[\frac{\gamma\prod_{i=1}^{n}(1-a_i)^{w_i}-\gamma\prod_{i=1}^{n}[1-a_i-c_i]^{w_i}}{\prod_{i=1}^{n}[1+(\gamma-1)a_i]^{w_i}+(\gamma-1)\prod_{i=1}^{n}(1-a_i)^{w_i}},\frac{\gamma\prod_{i=1}^{n}(1-b_i)^{w_i}-\gamma\prod_{i=1}^{n}[1-b_i-d_i]^{w_i}}{\prod_{i=1}^{n}[1+(\gamma-1)b_i]^{w_i}+(\gamma-1)\prod_{i=1}^{n}(1-b_i)^{w_i}}\right]\right\rangle \qquad (1.4.1.3)$$

where,

(i) $\alpha_i=\langle[\mu_{i1},\mu_{i2}],[\nu_{i1},\nu_{i2}]\rangle, i=1,2,\dots,n$ are n IVIFNs.

(ii) w_i is the weight assigned to the i^{th} IVIFN $\alpha_i=\langle[\mu_{i1},\mu_{i2}],[\nu_{i1},\nu_{i2}]\rangle$ such that $w_i\geq 0$ and $\sum_{i=1}^{n}w_i=1$.

1.4.2 Garg's expression to evaluate the multiplication of IVIFNs as well as the corresponding IVIFWGO

Garg [61, Section 2.3, pp. 287] pointed out the shortcomings of the existing IVIFWGO [128]. To resolve the shortcomings, Garg 61, Section 3, Definition 3.1, pp. 287] proposed the expression (1.4.2.1) to evaluate the multiplication of two IVIFNs $\alpha_1=\langle[a_1,b_1],[c_1,d_1]\rangle$ and $\alpha_2=\langle[a_2,b_2],[c_2,d_2]\rangle$. Hence, the expression (1.4.2.2) to evaluate the positive power λ of an IVIFN $\alpha=\langle[a,b],[c,d]\rangle$.

$$\alpha_1\otimes\alpha_2=\left\langle\left[\frac{\gamma\prod_{i=1}^{2}(1-c_i)-\gamma\prod_{i=1}^{2}[1-a_i-c_i]}{\prod_{i=1}^{2}[1+(\gamma-1)c_i]+(\gamma-1)\prod_{i=1}^{2}(1-c_i)},\frac{\gamma\prod_{i=1}^{2}(1-d_i)-\gamma\prod_{i=1}^{2}[1-b_i-d_i]}{\prod_{i=1}^{2}[1+(\gamma-1)d_i]+(\gamma-1)\prod_{i=1}^{2}(1-d_i)}\right],\right.$$
$$\left.\left[\frac{\prod_{i=1}^{2}[1+(\gamma-1)c_i-\prod_{i=1}^{2}(1-c_i)]}{\prod_{i=1}^{2}[1+(\gamma-1)c_i]+(\gamma-1)\prod_{i=1}^{2}(1-c_i)},\frac{\prod_{i=1}^{2}[1+(\gamma-1)d_i-\prod_{i=1}^{2}(1-d_i)]}{\prod_{i=1}^{2}[1+(\gamma-1)b_i]+(\gamma-1)\prod_{i=1}^{2}(1-d_i)}\right]\right\rangle$$
$$(1.4.2.1)$$

$$\alpha^{\lambda}=\left\langle\left[\frac{\gamma[1-c]^{\lambda}-\gamma[1-a-c]^{\lambda}}{[1+(\gamma-1)c]^{\lambda}+(\gamma-1)[1-c]^{\lambda}},\frac{\gamma[1-d]^{\lambda}-\gamma[1-b-d]^{\lambda}}{[1+(\gamma-1)d]^{\lambda}+(\gamma-1)[1-d]^{\lambda}}\right],\right.$$

$$\left[\frac{[1+(\gamma-1)c]^{\lambda}-[1-c]^{\lambda}}{[1+(\gamma-1)c]^{\lambda}+(\gamma-1)[1-c]^{\lambda}},\frac{[1+(\gamma-1)d]^{\lambda}-[1-d]^{\lambda}}{[1+(\gamma-1)d]^{\lambda}+(\gamma-1)[1-d]^{\lambda}}\right]\right\rangle. \tag{1.4.2.2}$$

Also, using the expression (1.4.2.1) and the expression (1.4.2.2), Garg et al. [61, Section 3.1, Theorem 3.1, pp. 288] proposed the IVIFWGO (1.4.2.3) and its extensions [61, Section 3.2, pp. 299].

$$\otimes_{i=1}^{n} {\alpha_i}^{w_i} = \left\langle\left[\frac{\gamma\prod_{i=1}^{n}[1-c_i]^{w_i}-\gamma\prod_{i=1}^{n}(1-a_i-c_i)^{w_i}}{\prod_{i=1}^{n}[1+(\gamma-1)c_i]^{w_i}+(\gamma-1)\prod_{i=1}^{n}(1-c_i)^{w_i}},\frac{\gamma\prod_{i=1}^{n}[1-d_i]^{w_i}-\gamma\prod_{i=1}^{n}(1-b_i-d)^{w_i}}{\prod_{i=1}^{n}[1+(\gamma-1)d_i]^{w_i}+(\gamma-1)\prod_{i=1}^{n}(1-d_i)^{w_i}}\right],\right.$$

$$\left.\left[\frac{\prod_{i=1}^{n}(1+(\gamma-1)c_i)^{w_i}-\prod_{i=1}^{n}[1-c_i]^{w_i}}{\prod_{i=1}^{n}[1+(\gamma-1)c_i]^{w_i}+(\gamma-1)\prod_{i=1}^{n}(1-c_i)^{w_i}},\frac{\prod_{i=1}^{n}(1+(\gamma-1)d_i)^{w_i}-\prod_{i=1}^{n}[1-d_i]^{w_i}}{\prod_{i=1}^{n}[1+(\gamma-1)d_i]^{w_i}+(\gamma-1)\prod_{i=1}^{n}(1-d_i)^{w_i}}\right]\right\rangle \tag{1.4.2.3}$$

where,

(i) $\alpha_i = \langle[\mu_{i1},\mu_{i2}],[\nu_{i1},\nu_{i2}]\rangle i = 1,2,\ldots,n$ are n IVIFNs.

(ii) w_i is the weight assigned to the i^{th} IVIFN $\alpha_i = \langle[\mu_{i1},\mu_{i2}],[\nu_{i1},\nu_{i2}]\rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.5 Some existing expressions to evaluate the sum and multiplication of SVNNs as well as the corresponding SVNW aggregation operators

The expressions to evaluate sum and multiplication of the real numbers are uniquely defined. However, there exist several expressions to evaluate sum and multiplication of the SVNNs. Since, a SVNWAO operator fully depends upon the sum of SVNNs and a SVNWGO fully depends upon the multiplication of SVNNs. Therefore, there exist different SVNWAOs, based upon different types of sum of SVNNs as well as there exist different types of SVNWGOs, based upon different types of multiplication of SVNNs, in the literature.

In this section, the existing expression [134, 147] to evaluate the sum and multiplication of SVNNs as well as the corresponding SVNW aggregation operators are discussed.

“A single valued neutrosophic set, over the universal set X, is defined as $\alpha = \{\langle x, \mu_\alpha(x), \nu_\alpha(x), h_\alpha(x)\rangle \mid x \in X,\ 0 \le \mu_\alpha(x) \le 1,\ 0 \le \nu_\alpha(x) \le 1, 0 \le h_\alpha(x) \le 1, \mu_\alpha(x) + \nu_\alpha(x) + h_\alpha(x) \le 3\,\}$. The values $\mu_\alpha(x)$, $\nu_\alpha(x)$ and $h_\alpha(x)$ respectively are called the degree of membership, the degree of non-membership and the degree of indeterminacy for the element x. Also, the pair $\langle \mu_\alpha, \nu_\alpha, h_\alpha\rangle$ is called a SVNN.”

1.5.1 Nancy and Garg’s expression to evaluate sum and multiplication of SVNNs as well as the corresponding SVNWAO and SVNWGO

Nancy and Garg [147, Section 3, Definition 3.1, pp. 363] proposed

(i) The expression (1.5.1.1) to evaluate the sum of two SVNNs $\alpha_1 = \langle a_1, b_1, c_1\rangle$ and $\alpha_2 = \langle a_2, b_2, c_2\rangle$. Hence, the expression (1.5.1.2) to evaluate the multiplication of a positive real number n with a SVNN $\alpha = \langle a, b, c\rangle$.

$$\alpha_1 \oplus \alpha_2 = \left\langle 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-a_1}-1)(\lambda^{1-a_2}-1)}{\lambda-1}\right), \log_\lambda\left(1 + \frac{(\lambda^{b_1}-1)(\lambda^{b_2}-1)}{\lambda-1}\right), \log_\lambda\left(1 + \frac{(\lambda^{c_1}-1)(\lambda^{c_2}-1)}{\lambda-1}\right)\right\rangle, \lambda > 1, \qquad (1.5.1.1)$$

$$n\alpha = \left\langle 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-a}-1)^n}{(\lambda-1)^{n-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{b}-1)^n}{(\lambda-1)^{n-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{c}-1)^n}{(\lambda-1)^{n-1}}\right)\right\rangle, \lambda > 0, \qquad (1.5.1.2)$$

(ii) The expression (1.5.1.3) to evaluate the multiplication of two SVNNs $\alpha = \langle a, b, c\rangle$, $\alpha_1 = \langle a_1, b_1, c_1\rangle$ and $\alpha_2 = \langle a_2, b_2, c_2\rangle$. Hence, the expression (1.5.1.4) to evaluate the positive power n of a SVNN $\alpha = \langle a, b, c\rangle$.

$$\alpha_1 \otimes \alpha_2 = \left\langle \log_\lambda\left(1 + \frac{(\lambda^{a_1}-1)(\lambda^{a_2}-1)}{\lambda-1}\right), 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-b_1}-1)(\lambda^{1-b_2}-1)}{\lambda-1}\right), 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-c_1}-1)(\lambda^{1-c_2}-1)}{\lambda-1}\right)\right\rangle, \lambda > 1, \qquad (1.5.1.3)$$

$$\alpha^n = \left\langle \log_\lambda\left(1+\frac{(\lambda^a-1)^n}{(\lambda-1)^{n-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b}-1\right)^n}{(\lambda-1)^{n-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-c}-1\right)^n}{(\lambda-1)^{n-1}}\right)\right\rangle. \tag{1.5.1.4}$$

Also, using the expression (1.5.1.1) and the expression (1.5.1.2), Nancy and Garg [147, Section 3.1, Theorem 3.7, pp. 367] proposed the SVNWAO (1.5.1.5).

$$\oplus_{i=1}^{n}(w_i\alpha_i) = \left\langle 1-\log_\lambda\left(1+\prod_{i=1}^{n}(\lambda^{1-a_i}-1)^{w_i}\right), \log_\lambda\left(1+\prod_{i=1}^{n}\left(\lambda^{b_i}-1\right)^{w_i}\right), \log_\lambda\left(1+\prod_{i=1}^{n}(\lambda^{c_i}-1)^{w_i}\right)\right\rangle \tag{1.5.1.5}$$

Futhermore, using the expression (1.5.1.4) and the expression (1.5.1.5), Nancy and Garg [147, Section 3.2, Theorem 3.8, pp. 370] proposed the SVNWGO (1.5.1.6).

$$\otimes_{i=1}^{n}{\alpha_i}^{w_i} = \left\langle \log_\lambda\left(1+\prod_{i=1}^{n}(\lambda^{a_i}-1)^{w_i}\right), 1-\log_\lambda\left(1+\prod_{i=1}^{n}\left(\lambda^{1-b_i}-1\right)^{w_i}\right), 1-\log_\lambda\left(1+\prod_{i=1}^{n}\left(\lambda^{1-c_i}-1\right)^{w_i}\right)\right\rangle \tag{1.5.1.6}$$

where,

(i) $\alpha_i = \langle a_i, b_i, c_i\rangle$, $i = 1, 2, \ldots, n$ are n SVNNs.

(ii) w_i is the weight assigned to the i^{th} SVNNN $\alpha_i = \langle a_i, b_i, c_i\rangle$ such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

1.5.2 Liu and Luo's SVNHFOWAO

There exist several expressions to evaluate sum and multiplication of the SVNHFNs. Since, a SVNHFWAO operator fully depends upon the sum of SVNHFNs and a SVNHFWGO fully depends upon the multiplication of SVNHFNs. Therefore, there exist different SVNHFWAOs, based upon different types of sum of SVNHFNs as well as there exist different types of SVNHFWGOs, based upon different types of multiplication of SVNHFNs, in the literature.

“A single valued neutrosophic hesitant fuzzy set, over the universal set X, is defined as $\alpha = \{\langle x, \mu_\alpha(x), \nu_\alpha(x), h_\alpha(x)\rangle \mid x \in X,\ \mu_\alpha(x) = \{\gamma_j, j = 1,2,\dots,l: 0 \le \gamma_j \le 1\}$, $\nu_\alpha(x) = \{\delta_j, j = 1,2,\dots,p: 0 \le \delta_j \le 1\}, h_\alpha(x) = \{\zeta_j, j = 1,2,\dots,m: 0 \le \zeta_j \le 1\}, \max_{1\le j\le l}\{\gamma_j\} + \max_{1\le j\le p}\{\delta_j\} + \max_{1\le j\le m}\{\zeta_j\} \le 3\}$. The values $\mu_\alpha(x)$, $\nu_\alpha(x)$ and $h_\alpha(x)$ respectively are called the set of degree of membership, the set of degree of non-membership and the set of degree of indeterminacy for the element x. Also, the set $\{\mu_\alpha, \nu_\alpha, h_\alpha\}$ is called a SVNHFN.”

Liu and Luo [134, Section 2.3, Definition 4] proposed the SVNHFOWAO (1.5.2.1).

$$\oplus_{i=1}^{k} (w_i\alpha_i) = \bigcup_{\gamma_1\in\mu_1,\dots,\gamma_k\in\mu_k,\delta_1\in\nu_1,\dots,\delta_k\in\nu_k,\eta_1\in h_1,\dots,\eta_k\in h_k}\left\{\left\{1 \prod_{i=1}^{k}\left(1 - \gamma_{\sigma(i)}\right)^{\lambda_i}\right\}, \left\{\prod_{i=1}^{k}\left(\gamma_{\sigma(i)}\right)^{\lambda_i}\right\}, \left\{\prod_{i=1}^{k}\left(\eta_{\sigma(i)}\right)^{\lambda_i}\right\}\right\} \tag{1.5.2.1}$$

where,

(i) $\alpha_i = \{\mu_i, \nu_i, h_i\}$ $(i = 1, 2, \dots, k)$ is a collection of single valued neutrosophic hesitant fuzzy elements, where μ_i, ν_i and h_i are three sets of some values in $[0,1]$.

(ii) w_i is the weight associated with α_i $(i = 1, 2, \dots, k)$ satisfying $w_i > 0$, $\sum_{i=1}^{k} w_i = 1$.

(iii) $\sigma: \{1, 2, \dots, k\} \to \{1, 2, \dots, k\}$ is a permutation such that $\alpha_{\sigma(i)}$ is the largest number in $(\alpha_1, \alpha_2, \dots, \alpha_k)$.

1.6 Some existing methods for comparing different extensions of fuzzy set

If a and b are two real numbers then it can be easily verified that $a > b$ or $a < b$ or $a = b$. However, if α and β represent any of the extensions of fuzzy set then there is no unique way to check that $\alpha \succ \beta$ or $\alpha \prec \beta$ or $\alpha \approx \beta$. Therefore, various methods, for comparing different extensions of fuzzy set, have been proposed in the literature.

In this section, the existing methods for comparing different extensions of fuzzy set have been discussed.

1.6.1 Existing method for comparing IFNs

Xu and Yager [208, Definition 1, pp. 422], Xu [204, Definition 3.1, pp. 1181], He et al. [88, Definition 4, pp. 143], Yu [223, Definition 3, pp. 1019], Chen and Chang [28, Definition 2.4, pp. 136], Garg [60, Section 2.1, pp. 54] have used the following method for comparing IFNs to prove that their proposed IFWGO/IFWAO satisfies the boundedness and the monotonicity property.

If $\alpha_1 = \langle \mu_1, \nu_1 \rangle = [\mu_1, 1 - \nu_1]$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle = [\mu_2, 1 - \nu_2]$ are two IFNs. Then, use the following steps to check that $\alpha_1 \succ \alpha_2$ or $\alpha_1 \prec \alpha_2$ or $\alpha_1 = \alpha_2$.

Step 1: Find $S(\alpha_1) = \mu_1 - \nu_1$, $S(\alpha_2) = \mu_2 - \nu_2$ and check that $S(\alpha_1) > S(\alpha_2)$ or $S(\alpha_1) < S(\alpha_2)$ or $S(\alpha_1) = S(\alpha_2)$.

Case (i): If $S(\alpha_1) > S(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $S(\alpha_1) < S(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $S(\alpha_1) = S(\alpha_2)$ then go to Step 2.

Step 2: Find $H(\alpha_1) = \mu_1 + \nu_1$, $H(\alpha_2) = \mu_2 + \nu_2$ and check that $H(\alpha_1) > H(\alpha_2)$ or $H(\alpha_1) < H(\alpha_2)$ or $H(\alpha_1) = H(\alpha_2)$.

Case (i): If $H(\alpha_1) > H(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $H(\alpha_1) < H(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $H(\alpha_1) = H(\alpha_2)$ then $\alpha_1 = \alpha_2$.

1.6.2 Existing method for comparing IFMNs

Garg [56, Section 2.1, Definition 1, pp. 1077], Garg [66, Section 2, Definition 2, pp. 2122] and Qian and Niu [156, Section 2, Definition 2.3, pp. 2860] have used the following method for comparing IFMNs to prove that their proposed IFMWGO / IFMWAO satisfies the boundedness and the monotonicity property.

If $\alpha_1 = \langle\mu_1, \nu_1\rangle$ and $\alpha_2 = \langle\mu_2, \nu_2\rangle$ are two IFMNs. Then use the following steps to check that $\alpha_1 \succ \alpha_2$ or $\alpha_1 \prec \alpha_2$ or $\alpha_1 = \alpha_2$.

Step 1: Find $S(\alpha_1) = \frac{\mu_1}{\nu_1}$, $S(\alpha_2) = \frac{\mu_2}{\nu_2}$ and check that $S(\alpha_1) > S(\alpha_2)$ or $S(\alpha_1) < S(\alpha_2)$ or $S(\alpha_1) = S(\alpha_2)$.

Case (i): If $S(\alpha_1) > S(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $S(\alpha_1) < S(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $S(\alpha_1) = S(\alpha_2)$ then go to Step 2.

Step 2: Find $H(\alpha_1) = \mu_1\nu_1$, $H(\alpha_2) = \mu_2\nu_2$ and check that $H(\alpha_1) > H(\alpha_2)$ or $H(\alpha_1) < H(\alpha_2)$ or $H(\alpha_1) = H(\alpha_2)$.

Case (i): If $H(\alpha_1) > H(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $H(\alpha_1) < H(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $H(\alpha_1) = H(\alpha_2)$ then $\alpha_1 = \alpha_2$.

1.6.3 Existing method for comparing PFNs

Garg [58, Section 2, Definition 2.3, pp. 889], Garg [62, Section 2, Definition 2, pp. 548], Ma and Xu [135, Section 2.2, Definition 4, pp. 1201] have used the following method for comparing the PFNs to prove that their proposed PFWAO/PFWGO satisfies the boundedness and the monotonicity property.

If $\alpha_1 = \langle\mu_1, \nu_1\rangle$ and $\alpha_2 = \langle\mu_2, \nu_2\rangle$ are two PFNs. Then, use the following steps to check that $\alpha_1 \succ \alpha_2$ or $\alpha_1 \prec \alpha_2$ or $\alpha_1 = \alpha_2$.

Step 1: Find $S(\alpha_1) = {\mu_1}^2 - {\nu_1}^2$, $S(\alpha_2) = {\mu_2}^2 - {\nu_2}^2$ and check that $S(\alpha_1) > S(\alpha_2)$ or $S(\alpha_1) < S(\alpha_2)$ or $S(\alpha_1) = S(\alpha_2)$.

Case (i): If $S(\alpha_1) > S(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $S(\alpha_1) < S(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $S(\alpha_1) = S(\alpha_2)$ then go to Step 2.

Step 2: Find $H(\alpha_1) = {\mu_1}^2 + {\nu_1}^2$, $H(\alpha_2) = {\mu_1}^2 + {\nu_1}^2$ and check that $H(\alpha_1) > H(\alpha_2)$ or $H(\alpha_1) < H(\alpha_2)$ or $H(\alpha_1) = H(\alpha_2)$.

Case (i): If $H(\alpha_1) > H(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $H(\alpha_1) < H(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $H(\alpha_1) = H(\alpha_2)$ then $\alpha_1 = \alpha_2$.

1.6.4 Garg et al.'s method for comparing IVIFNs

Garg et al. [68, Section 2.1, pp. 2583] have used the following method for comparing IVIFNs to prove that their proposed IVIFWAO satisfies the boundedness and the monotonicity property.

If $\alpha_1 = \langle [a_1, b_1], [c_1, d_1] \rangle$ and $\alpha_2 = \langle [a_2, b_2], [c_2, d_2] \rangle$ are two IVIFNs. Then use the following steps to check that $\alpha_1 \succ \alpha_2$ or $\alpha_1 \prec \alpha_2$ or $\alpha_1 = \alpha_2$.

Step 1: Find $S(\alpha_1) = \frac{a_1 + b_1 - c_1 - d_1}{2}$, $S(\alpha_2) = \frac{a_2 + b_2 - c_2 - d_2}{2}$ and check that $S(\alpha_1) > S(\alpha_2)$ or $S(\alpha_1) < S(\alpha_2)$ or $S(\alpha_1) = S(\alpha_2)$.

Case (i): If $S(\alpha_1) > S(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $S(\alpha_1) < S(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $S(\alpha_1) = S(\alpha_2)$ then go to Step 2.

Step 2: Find $H(\alpha_1) = \frac{a_1 + b_1 + c_1 + d_1}{2}$, $H(\alpha_2) = \frac{a_1 + b_1 + c_1 + d_1}{2}$ and check that $H(\alpha_1) > H(\alpha_2)$ or $H(\alpha_1) < H(\alpha_2)$ or $H(\alpha_1) = H(\alpha_2)$.

Case (i): If $H(\alpha_1) > H(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $H(\alpha_1) < H(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $H(\alpha_1) = H(\alpha_2)$ then go to Step 3.

Step 3: Find $G(\alpha_1) = b_1 + d_1 - a_1 - c_1,\ G(\alpha_2) = b_2 + d_2 - a_2 - c_2$ and check that $G(\alpha_1) > G(\alpha_2)$ or $G(\alpha_1) < G(\alpha_2)$ or $G(\alpha_1) = G(\alpha_2)$.

Case (i): If $G(\alpha_1) > G(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $G(\alpha_1) < G(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $G(\alpha_1) = G(\alpha_2)$ then go to Step 4.

Step 4: Find $T(\alpha_1) = b_1 + c_1 - a_1 - d_1,\ T(\alpha_2) = b_2 + c_2 - a_2 - d_2$ and check that $T(\alpha_1) > T(\alpha_2)$ or $T(\alpha_1) < T(\alpha_2)$ or $T(\alpha_1) = T(\alpha_2)$.

Case (i): If $T(\alpha_1) > T(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $T(\alpha_1) < T(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $T(\alpha_1) = T(\alpha_2)$ then $\alpha_1 = \alpha_2$.

1.6.5 Garg 's method for comparing IVIFNs

Garg [61, Section 2.1, pp. 2582] have used the following method for comparing IVIFNs to prove that his proposed IVIFWGO satisfies the boundedness and the monotonicity property.

If $\alpha_1 = \langle [a_1, b_1], [c_1, d_1] \rangle$ and $\alpha_2 = \langle [a_2, b_2], [c_2, d_2] \rangle$ are two IVIFNs. Then use the following steps to check that $\alpha_1 \succ \alpha_2$ or $\alpha_1 \prec \alpha_2$ or $\alpha_1 = \alpha_2$.

Step 1: Find $S(\alpha_1) = \frac{a_1 + b_1 - c_1 - d_1}{2}$, $S(\alpha_2) = \frac{a_2 + b_2 - c_2 - d_2}{2}$ and check that $S(\alpha_1) > S(\alpha_2)$ or $S(\alpha_1) < S(\alpha_2)$ or $S(\alpha_1) = S(\alpha_2)$.

Case (i): If $S(\alpha_1) > S(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $S(\alpha_1) < S(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $S(\alpha_1) = S(\alpha_2)$ then go to Step 2.

Step 2: Find $H(\alpha_1) = \frac{a_1+ b_1+c_1+d_1}{2}$, $H(\alpha_2) = \frac{a_1+ b_1+c_1+d_1}{2}$ and check that $H(\alpha_1) > H(\alpha_2)$ or $H(\alpha_1) < H(\alpha_2)$ or $H(\alpha_1) = H(\alpha_2)$.

Case (i): If $H(\alpha_1) > H(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $H(\alpha_1) < H(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $H(\alpha_1) = H(\alpha_2)$ then $\alpha_1 = \alpha_2$.

1.6.6 Nancy and Garg's method for comparing SVNNs

Nancy and Garg [147, Section 2, Definition 2.3, pp. 362] have used the following method for comparing SVNNs to prove that their proposed IFWGO satisfies the boundedness and the monotonicity property.

If $\alpha_1 = \langle a_1, b_1, c_1 \rangle$ and $\alpha_2 = \langle a_2, b_2, c_2 \rangle$ are two SVNNs. Then, find $s(\alpha_1) = a_1 - b_1 - c_1$, $s(\alpha_2) = a_2 - b_2 - c_2$ and check that $s(\alpha_1) > s(\alpha_2)$ or $s(\alpha_1) < s(\alpha_2)$ or $s(\alpha_1) = s(\alpha_2)$.

Case (i): If $s(\alpha_1) > s(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $s(\alpha_1) < s(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $s(\alpha_1) = s(\alpha_2)$ then $\alpha_1 = \alpha_2$.

1.6.7 Liu and Luo's method for comparing SVNHFNs

Liu and Luo [134, Section 3.1, Definition 6] have used the following method, for comparing SVNHFNs to prove that their proposed SVNHFWAO satisfies the boundedness and the monotonicity property.

If $\alpha_1 = \{\mu_1, \nu_1, h_1\}$ and $\alpha_2 = \{\mu_2, \nu_2, h_2\}$ are two SVNHFNs. Then use the following steps to check that $\alpha_1 \succ \alpha_2$ or $\alpha_1 \prec \alpha_2$ or $\alpha_1 = \alpha_2$.

Step 1: Find $s(\alpha_1) = \frac{1}{3}\left(\frac{1}{l}\sum_{i=1}^{l}\gamma_{1_i} + \frac{1}{p}\sum_{i=1}^{p}\left(1-\delta_{1_i}\right) + \frac{1}{q}\sum_{i=1}^{q}\left(1-\eta_{1_i}\right)\right)$, $s(\alpha_2) = \frac{1}{3}\left(\frac{1}{l}\sum_{i=1}^{l}\gamma_{2_i} + \frac{1}{p}\sum_{i=1}^{p}\left(1-\delta_{2_i}\right) + \frac{1}{q}\sum_{i=1}^{q}\left(1-\eta_{2_i}\right)\right)$ and check that $s(\alpha_1) > s(\alpha_2)$ or $s(\alpha_1) < s(\alpha_2)$ or $s(\alpha_1) = s(\alpha_2)$.

Case (i): If $s(\alpha_1) > s(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $s(\alpha_1) < s(\alpha_2)$ then $n_1 \prec \alpha_2$.

Case (iii): If $s(\alpha_1) = s(\alpha_2)$ then go to Step 2.

Step 2: Find $a(\alpha_1) = \frac{1}{l}\sum_{i=1}^{l}\gamma_{1_i} - \frac{1}{q}\sum_{i=1}^{q}\left(1-\eta_{1_i}\right)$, $a(\alpha_2) = \frac{1}{l}\sum_{i=1}^{l}\gamma_{2_i} - \frac{1}{q}\sum_{i=1}^{q}\left(1-\eta_{2_i}\right)$ and check that $a(\alpha_1) > a(\alpha_2)$ or $a(\alpha_1) < a(\alpha_2)$ or $a(\alpha_1) = a(\alpha_2)$.

Case (i): If $a(\alpha_1) > a(\alpha_2)$ then $n_1 \succ \alpha_2$.

Case (ii): If $a(\alpha_1) < a(\alpha_2)$ then $n_1 \prec \alpha_2$.

Case (iii): If $a(\alpha_1) = a(\alpha_2)$ then go to Step 3.

Step 3: Find $c(\alpha_1) = \frac{1}{l}\sum_{i=1}^{l}\gamma_{1_i}$, $c(\alpha_2) = \frac{1}{l}\sum_{i=1}^{l}\gamma_{2_i}$ and check that $c(\alpha_1) > c(\alpha_2)$ or $c(n_1) < c(n_2)$ or $c(n_1) = c(n_2)$.

Case (i): If $c(\alpha_1) > c(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

Case (ii): If $c(\alpha_1) < c(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

Case (iii): If $c(\alpha_1) = c(\alpha_2)$ then $\alpha_1 = \alpha_2$.

1.7 Brief review about the work

The chapter wise summary of thesis is as follows:

Chapter 2

Appropriate Weighted Averaging Operator Under Intuitionistic Fuzzy Environment and Its Various Extensions

In this chapter,

(i) It is shown that the existing expressions to evaluate the sum and multiplication of IFNs as well as the corresponding IFWGOs and IFWAOs [28, 56, 60, 65, 66, 88, 156, 204, 208, 223], discussed in Section 1.2 of Chapter 1, are not valid.

(ii) It is shown that the existing expressions to evaluate the sum and multiplication of PFNs as well as the corresponding PFWGOs and PFWAOs [58, 62, 63, 67, 135], discussed in Section 1.3 of Chapter 1, are not valid.

(iii) It is shown that the existing expressions to evaluate the sum and multiplication of IVIFNs as well as the corresponding IVIFWGOs and IVIFFWAOs [61, 68], discussed in Section 1.4 of Chapter 1, are not valid.

(iv) It is shown that the existing expressions to evaluate the sum and multiplication of SVNNs as well as the corresponding SVNWGOs, SVNWAOs and SVNHFOWAOs [134, 147], discussed in Section 1.5 of Chapter 1, are not valid.

(v) With the help of the existing IVIFWAO [30], an IFWAO, a PFWAO, an IVIFWAO and a SVNWAO are proposed. Also, it is shown that the proposed IFWAO, PFWAO and SVNWAO are valid. Furthermore, it is shown that the weighted geometric operator cannot be defined for fuzzy set and its extensions.

Chapter 3

Mehar Method to Find the Unique Optimal Fuzzy Transportation Cost of Balanced Fully Fuzzy Transportation Problems with *LR* Flat Fuzzy Numbers

Ebrahimnejad [49, Section 4, pp. 113] claimed that on solving a BFFTP with *LR* FFNs (balanced transportation problems in which all the parameters are represented as *LR* FFNs) by Kumar and Kaur's method [115] more than one *LR* FFNs, representing the optimal fuzzy transportation cost, may be obtained, which is illogical. To resolve this flaw, Ebrahimnejad [49, Section 5, pp. 114] proposed a method for solving BFFTPs with LR FFNs.

In this chapter, it is shown that Ebrahimnejad's method can be used only if the aggregated value of the fuzzy transportation cost, fuzzy availability and fuzzy demand, provided by all the decision-makers, is available. However, if instead of the aggregated data, the data of each decision-maker is provided separately then Ebrahimnejad's method cannot be used to find the solution of a BFFTP. Also, it is shown that the flaw, pointed out by Ebrahimnejad in Kumar and Kaur's method is also occurring in Ebrahimnejad's method. Futhermore, to overcome the limitations of Ebrahimnejad's method, the method for aggregating the LR FFNs is discussed. Finally, to resolve the flaws of Ebrahimnejad's method [49], a new method (named as Mehar method) is proposed to solve the BFFTP with LR FFNs.

Chapter 4

Mehar Approach for Solving Balanced Fully Intuitionistic Fuzzy Transportation Problems

Ebrahimnejad and Verdegay [51, Section 5] proposed an approach for solving such BFIFTPs in which each parameter is represented as a TrIFN. In this chapter, it is shown that Ebrahimnejad and Verdegay's approach can be used only if the aggregated value of IF transportation cost, IF availability and IF demand, provided by all the decision-makers, is available. However, if instead of the aggregated data, the data of each decision-maker is provided separately then Ebrahimnejad and Verdegay's approach cannot be used to find the solution of a BFIFTP. Therefore, firstly, to overcome the limitation of Ebrahimnejad and Verdegay's approach, a method for aggregating the TrIFNs is discussed. Then, a new approach (named as Mehar approach) is proposed for solving BFIFTPs. It is shown that it is much easy to apply the proposed Mehar approach as compared to Ebrahimnejad and Verdegay's approach. Also, to illustrate the proposed Mehar approach, the existing BFIFTP [51] is solved.

Chapter 5

JMD Method for Transforming an Unbalanced Fully Intuitionistic Fuzzy Transportation Problem into a Balanced Fully Intuitionistic Fuzzy Transportation Problem

Ebrahimnejad and Verdegay's approach [51, Section 5] as well as Mehar approach (proposed in Chapter 4) cannot be used to solve an unbalanced FIFTP. Therefore, one may use the following methodology to solve an unbalanced FIFTP.

(1) Use the existing approach [136, Section 3] to transform an unbalanced FIFTP into a balanced FIFTP.

(2) Apply the Mehar approach, proposed in Chapter 4, to find the IF optimal solution of the BFIFTP.

However, after a deep study it is observed that the existing approach [136, Section 3] to transform an unbalanced FIFTP into a BFIFTP is not valid.

To validate this claim, the existing approach [136, Section 3] is applied on an unbalanced FIFTP and shown that the transformed FIFTP is not a BFIFTP. Furthermore, a new method (named as JMD method) is proposed to transform an unbalanced FIFTP into a BFIFTP.

Chapter 6

Mehar Method for Solving Unbalanced Generalized Interval-Valued Trapezoidal Fuzzy Number Transportation Problems

Ebrahimnejad [48, Section 5, pp. 304] proposed a method for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP as well as a method for solving a balanced GIVTrFNTP. In this chapter, it is shown that on applying Ebrahimnejad's method for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP, the obtained dummy supply and/or dummy demand is not a GIVTrFN and hence, this method is not valid. Also, a new method (named as Mehar method) is proposed to transform an unbalanced GIVTrFNTP into a balanced GIVTrFNTP. Furthermore, the validity of the proposed Mehar method is discussed.

Chapter 7

Future Scope

In this chapter, some open research problems are discussed.

Appendix A

Dhanasekar et al. [37] proposed a fuzzy Hungarian MODI algorithm to solve fully fuzzy transportation problems. Dhanasekar et al. have used the standard multiplication of TrFNs in their proposed method. In this appendix, it is pointed out that the method, proposed by Dhanasekar et al., is not valid for standard multiplication of TrFNs and is valid only if a special type of multiplication of TrFNs is used.

Chapter 2

Appropriate Weighted Averaging Operator Under Intuitionistic Fuzzy Environment and Its Various Extensions[*]

In this chapter,

(i) It is shown that the existing expressions to evaluate the sum and multiplication of IFNs as well as the corresponding IFWGOs and IFWAOs [28, 56, 60, 65, 66, 88, 156, 204, 208, 223], discussed in Section 1.2 of Chapter 1, are not valid.

(ii) It is shown that the existing expressions to evaluate the sum and multiplication of PFNs as well as the corresponding PFWGOs and PFWAOs [58, 62, 63, 67, 135], discussed in Section 1.3 of Chapter 1, are not valid.

(iii) It is shown that the existing expressions to evaluate the sum and multiplication of IVIFNs as well as the corresponding IVIFWGOs and IVIFFWAOs [61, 68], discussed in Section 1.4 of Chapter 1, are not valid.

(iv) It is shown that the existing expressions to evaluate the sum and multiplication of

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SVNNs as well as the corresponding SVNWGOs, SVNWAOs and SVNHFOWAOs [134, 147], discussed in Section 1.5 of Chapter 1, are not valid.

(v) With the help of the existing IVIFWAO [30], an IFWAO, a PFWAO, an IVIFWAO and a SVNWAO are proposed. Also, it is shown that the proposed IFWAO, PFWAO and SVNWAO are valid. Furthermore, it is shown that the weighted geometric operator cannot be defined for fuzzy set and its extensions.

This chapter is organized as follows:

(a) In Section 2.1, it is shown that the existing expressions to evaluate the sum of two IFNs, multiplication of two IFNs, multiplication of a positive real number with an IFN and positive power of an IFN [28, 56, 60, 65, 66, 88, 156, 204, 208, 223], discussed Section 1.2, are not valid.

(b) In Section 2.2, it is shown that the existing expressions to evaluate the sum of two PFNs, multiplication of two PFNs, multiplication of a positive real number with a PFN and positive power of a PFN [58, 62, 63, 67, 135], discussed Section 1.3, are not valid.

(c) In Section 2.3, it is shown that the existing expressions to evaluate the sum of two IVIFNs, multiplication of two IVIFNs, multiplication of a positive real number with an IVIFN and positive power of an IVIFN [61, 68], discussed in Section 1.4, are not valid.

(d) In Section 2.4, it is shown that the existing expressions to evaluate the sum of two SVNNs, multiplication of two SVNNs, multiplication of a positive real number with a SVNN and positive power of a SVNN [134, 147] are not valid.

(e) In Section 2.5, it is shown that the existing IFWAOs, IFWGOs, PFWAOs, PFWGOs, IVIFWAOs, IVIFWGOs IVIFWAO, SVNWAOs, SVNWGOs and

SVNHFOWAOs [28, 56, 58, 60-62, 63, 65-68, 88, 134, 147, 156, 204, 208, 223] are not valid.

(f) In Section 2.6, with the help of the existing IVIFWAO [30], an IFWAO, a PFWAO and a SVNWAO are proposed. Also,it is shown that the proposed IFWAO, PFWAO and SVNWAO are valid. Furthermore, it is shown that the weighted geometric operator cannot be defined for fuzzy set and its extensions.

2.1 Flaws of some existing expressions to evaluate the sum and multiplication of IFNs

In this section, the flaws of the existing expressions to evaluate the sum and multiplication of IFNs [28, 56, 60, 65, 66, 88, 156, 204, 208, 223], discussed in Section 1.2, are pointed out.

2.1.1 Flaws of Xu and Yager's expressions to evaluate the multiplication of IFNs

In this section, the flaws of Xu and Yager's expressions to evaluate the multiplication of IFNs [208], discussed in Section 1.2.1, are pointed out.

The expression (2.1.1.1) represents the generalized form of the existing expression (1.2.1.1).

$$\otimes_{i=1}^{n} \alpha_i = \otimes_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \langle \prod_{i=1}^{n} \mu_i , 1 - \prod_{i=1}^{n} (1 - \nu_i) \rangle \qquad (2.1.1.1)$$

The expression (2.1.1.1) and hence, the expression (1.2.1.1) as well as the expression (1.2.1.2) are not valid due to the following reasons:

(i) To obtain the expression (2.1.1.1), Xu and Yager [208] have assumed that $\langle \mu_i, \nu_i \rangle$ is equivalent to interval $[\mu_i, 1 - \nu_i]$.

Hence, $\otimes_{i=1}^{n} \alpha_i = \otimes_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \otimes_{i=1}^{n} [\mu_i, 1 - \nu_i] = [\prod_{i=1}^{n} \mu_i , \prod_{i=1}^{n} (1 - \nu_i)] =$

$\langle \prod_{i=1}^{n} \mu_i , 1 - \prod_{i=1}^{n} (1 - \nu_i) \rangle$.

However, it is mathematically incorrect to assume that the ordered pair $\langle \mu_i, \nu_i \rangle$ is equivalent to interval $[\mu_i, 1-\nu_i]$.

(ii) The expression (2.1.1.1) can also be represented as the expression (2.1.1.2).

$$\otimes_{i=1}^{n} \alpha_i = \otimes_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \langle \mu_p \times \prod_{\substack{i=1 \\ i \neq p}}^{n} \mu_i , 1 - \prod_{i=1}^{n} (1-\nu_i) \rangle \qquad (2.1.1.2)$$

If there exist an IFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\mu_p = 0$ then,

$$\otimes_{i=1}^{n} \alpha_i = \otimes_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \langle 0 \times \prod_{\substack{i=1 \\ i \neq p}}^{n} \mu_i , \prod_{i=1}^{n} (1-\nu_i) \rangle$$

$$= \langle 0, \prod_{i=1}^{n} (1-\nu_i) \rangle.$$

This clearly indicates that the membership value of the multiplication of IFNs is only depending upon the membership value of the p^{th} IFN $\alpha_p = \langle \mu_p, \nu_p \rangle = \langle 0, \nu_p \rangle$ and is independent from the membership values of the remaining $(n-1)$ IFNs, which is mathematically incorrect.

(iii) The expression (2.1.1.1) can also be represented as the expression (2.1.1.3).

$$\otimes_{i=1}^{n} \alpha_i = \otimes_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \langle \mu_p \prod_{\substack{i=1 \\ i \neq p}}^{n} \mu_i , 1 - \left(1-\nu_p\right) \prod_{\substack{i=1 \\ i \neq p}}^{n} (1-\nu_i) \rangle \qquad (2.1.1.3)$$

If there exist an IFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\mu_p = 0$ and $\nu_p = 1$ then,

$$\otimes_{i=1}^{n} \alpha_i = \otimes_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \langle 0 \times \prod_{\substack{i=1 \\ i \neq p}}^{n} \mu_i , 1 - (1-1) \times \prod_{\substack{i=1 \\ i \neq p}}^{n} (1-\nu_i) \rangle$$

$$= \langle 0,1 \rangle.$$

This clearly indicates that the multiplication of IFNs is only depending upon the p^{th} IFN $\alpha_p = \langle \mu_p, \nu_p \rangle = \langle 0,1 \rangle$ and is independent from the remaining $(n-1)$ IFNs, which is mathematically incorrect.

(iv) The expression (1.2.1.2) i.e., $\alpha^{\lambda} = \langle \mu^{\lambda}, 1-(1-\nu)^{\lambda} \rangle = \left[\mu^{\lambda}, (1-\nu)^{\lambda}\right]$ has been obtained by putting $n = \lambda$ and $\alpha_i = \alpha$ in the expression (2.1.1.1). However, as the expression (2.1.1.1) is not valid. Therefore, the expression (1.2.1.2) is also not valid.

(v) If $\alpha = \langle \mu, \nu \rangle = \langle 0,1 \rangle$ then using the expression (1.2.1.2) i.e., $\alpha^{\lambda} = \langle \mu^{\lambda}, 1 - (1-\nu)^{\lambda} \rangle = \left[\mu^{\lambda}, (1-\nu)^{\lambda}\right]$,

$\alpha^{\lambda} = \langle 0,1 \rangle = [0,0]$.

This clearly indicates that positive power λ of an IFN $\alpha = \langle \mu, \nu \rangle = \langle 0,1 \rangle$ is only depending upon the IFN $\alpha = \langle \mu, \nu \rangle = \langle 0,1 \rangle$ and is independent from the positive real number λ, which is mathematically incorrect.

2.1.2 Flaws of Xu's expressions to evaluate the sum of IFNs

In this section, the flaws of Xu's expressions to evaluate sum of IFNs [204], discussed in Section 1.2.2, are pointed out.

The expression (2.1.2.1) represents the generalized form of the existing expression (1.2.2.1).

$$\oplus_{i=1}^{n} \alpha_i = \oplus_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \langle 1 - \prod_{i=1}^{n}(1-\mu_i), \prod_{i=1}^{n} \nu_i \rangle \quad (2.1.2.1)$$

The expression (2.1.2.1) and hence, the expression (1.2.2.1) as well as the expression (1.2.2.2) are not valid due to the following reasons:

(i) To obtain the expression (2.1.2.1), Xu [204] has assumed that $\langle \mu_i, \nu_i \rangle$ is equivalent to the interval $[\nu_i, 1-\mu_i]$.

Hence, $\otimes_{i=1}^{n} \alpha_i = \otimes_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \otimes_{i=1}^{n} [\nu_i, 1-\mu_i] = [\prod_{i=1}^{n} \nu_i, \prod_{i=1}^{n}(1-\mu_i)] = \langle 1 - \prod_{i=1}^{n}(1-\mu_i), \prod_{i=1}^{n} \nu_i \rangle$.

However, it is mathematically incorrect to assume that the ordered pair $\langle \mu_i, \nu_i \rangle$ is equivalent to interval $[\nu_i, 1-\mu_i]$.

(ii) The expression (2.1.2.1) can also be represented as the expression (2.1.2.2).

$$\oplus_{i=1}^{n} \alpha_i = \oplus_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \langle 1 - \prod_{i=1}^{n}(1-\mu_i), \nu_p \times \prod_{\substack{i=1 \\ i \neq p}}^{n} \nu_i \rangle \qquad (2.1.2.2)$$

If there exist an IFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\nu_p = 0$ then,

$$\oplus_{i=1}^{n} \alpha_i = \oplus_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \langle 1 - \prod_{i=1}^{n}(1-\mu_i), 0 \times \prod_{\substack{i=1 \\ i \neq p}}^{n} \nu_i \rangle$$

$$= \langle 1 - \prod_{i=1}^{n}(1-\mu_i), 0 \rangle.$$

This clearly indicates that the non-membership value of the sum of IFNs is only depending upon the non-membership value of the p^{th} IFN $\alpha_p = \langle \mu_p, \nu_p \rangle = \langle \mu_p, 0 \rangle$ and is independent from the non-membership values of the remaining $(n-1)$ IFNs, which is mathematically incorrect.

(iii) The expression (2.1.2.1) can also be represented as the expression (2.1.2.3).

$$\oplus_{i=1}^{n} \alpha_i = \oplus_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \langle 1 - \left(1-\mu_p\right) \times \prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i), \nu_p \times \prod_{\substack{i=1 \\ i \neq p}}^{n} \nu_i \rangle \qquad (2.1.2.3)$$

If there exist an IFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\mu_p = 1$ and $\nu_p = 0$ then,

$$\oplus_{i=1}^{n} \alpha_i = \oplus_{i=1}^{n} \langle \mu_i, \nu_i \rangle = \langle 1 - (1-1) \times \prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i), \nu_p \times \prod_{\substack{i=1 \\ i \neq p}}^{n} \nu_i \rangle$$

$$= \langle 1,0 \rangle.$$

This clearly indicates that the sum of IFNs is only depending upon the p^{th} IFN $\alpha_p = \langle \mu_p, \nu_p \rangle = \langle 1,0 \rangle$ and is independent from the remaining $(n-1)$ IFNs, which is mathematically incorrect. Therefore, the expression (2.1.2.1) is not valid.

(iv) The expression (1.2.2.2) i.e., $\lambda \otimes \alpha = \langle 1-(1-\mu)^{\lambda}, \nu^{\lambda} \rangle = \left[1-(1-\mu)^{\lambda}, 1-\nu^{\lambda}\right]$ has been obtained by putting $n = \lambda$ and $\alpha_i = \alpha$ in the expression (2.1.2.1). However, as the expression (2.1.2.1) is not valid. Therefore, the expression (1.2.2.2) is also not valid.

(v) If $\alpha = \langle \mu, \nu \rangle = \langle 1,0 \rangle$ then using the expression (1.2.2.2) i.e., $\lambda \otimes \alpha = \langle 1 - (1-\mu)^{\lambda}, \nu^{\lambda} \rangle = \left[1-(1-\mu)^{\lambda}, 1-\nu^{\lambda}\right]$,

$\lambda \otimes \alpha = \langle 1,0 \rangle = [1,1]$.

This clearly indicates that the multiplication of a positive real number λ with an IFN $\alpha = \langle \mu, \nu \rangle = \langle 1,0 \rangle$ is only depending upon the IFN $\alpha = \langle \mu, \nu \rangle = \langle 1,0 \rangle$ and is independent from the positive real number λ, which is mathematically incorrect.

2.1.3 Flaws of He et al.’s expressions to evaluate the multiplication of IFNs

In this section, the flaws of He et al.’s expressions to evaluate the multiplication of IFNs [88], discussed in Section 1.2.3, are pointed out.

The expression (2.1.3.1) represents the generalized form of the existing expression (1.2.3.1).

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \prod_{i=1}^{n}(1-\nu_i) - \prod_{i=1}^{n}\left(1-(\mu_i+\nu_i)\right), 1-\prod_{i=1}^{n}(1-\nu_i) \right\rangle \quad (2.1.3.1)$$

The expression (2.1.3.1) and hence, the expression (1.2.3.1) as well as the expression (1.2.3.2) are not valid due to the following reasons:

(i) The expression (2.1.3.1) can also be represented as the expression (2.1.3.2).

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \prod_{i=1}^{n}(1-\nu_i) - \left(1-\left(\mu_p+\nu_p\right)\right) \times \prod_{\substack{i=1 \\ i\neq p}}^{n}\left(1-(\mu_i+\nu_i)\right), 1-\prod_{i=1}^{n}(1-\nu_i) \right\rangle \quad (2.1.3.2)$$

If there exist an IFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\mu_p + \nu_p = 1$ then,

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \prod_{i=1}^{n}(1-\nu_i) - (1-1) \times \prod_{\substack{i=1 \\ i\neq p}}^{n}\left(1-(\mu_i+\nu_i)\right), 1-\prod_{i=1}^{n}(1-\nu_i) \right\rangle$$

$$= \langle \prod_{i=1}^{n}(1-\nu_i)\ , 1-\prod_{i=1}^{n}(1-\nu_i)\rangle.$$

This clearly indicates that multiplication of IFNs is independent from the membership values of IFNs, which is mathematically incorrect.

(ii) The expression (2.1.3.1) can also be represented as the expression (2.1.3.3).

$$\otimes_{i=1}^{n}\alpha_i = \left\langle \left(1-\nu_p\right)\times\prod_{\substack{i=1\\ i\neq p}}^{n}(1-\nu_i)\ -\left(1-\left(\mu_p+\nu_p\right)\right)\times\prod_{\substack{i=1\\ i\neq p}}^{n}\left(1-(\mu_i+\nu_i)\right), 1-\left(1-\nu_p\right)\times\prod_{\substack{i=1\\ i\neq p}}^{n}(1-\nu_i)\right\rangle \qquad (2.1.3.3)$$

If there exist an IFN $\alpha_p = \langle\mu_p,\nu_p\rangle$ such that $\mu_p = 0$ and $\nu_p = 1$ then,

$$\otimes_{i=1}^{n}\alpha_i = \left\langle (1-1)\times\prod_{\substack{i=1\\ i\neq p}}^{n}(1-\nu_i)\ -\left(1-(0+1)\right)\times\prod_{\substack{i=1\\ i\neq p}}^{n}\left(1-(\mu_i+\nu_i)\right), 1-(1-1)\times\prod_{\substack{i=1\\ i\neq p}}^{n}(1-\nu_i)\right\rangle$$

$$= \langle 0,1\rangle.$$

This clearly indicates that multiplication of IFNs is only depending upon the p^{th} IFN $\alpha_p = \langle\mu_p,\nu_p\rangle = \langle 0,1\rangle$ and is independent from the remaining $(n-1)$ IFNs, which is mathematically incorrect.

(iii) If $\alpha = \langle\mu,\nu\rangle = \langle 0,1\rangle$ then using expression (1.2.3.2) i.e.,

$$\alpha^{\lambda} = \langle (1-\nu)^{\lambda} - (1-(\mu+\nu))^{\lambda}, 1-(1-\nu)^{\lambda}\rangle,$$

$$\alpha^{\lambda} = \langle 0,1\rangle.$$

This clearly indicates that the positive power λ of an IFN $\alpha = \langle\mu,\nu\rangle = \langle 0,1\rangle$ is only depending upon the IFN $\alpha = \langle\mu,\nu\rangle = \langle 0,1\rangle$ and is independent from the positive real number λ, which is mathematically incorrect.

2.1.4 Flaws of Yu's expressions to evaluate the sum of IFNs

In this section, the flaws of Yu's expressions to evaluate the sum of IFNs [223], discussed in Section 1.2.4, are pointed out.

The expression (2.1.4.1) represents the generalized form of the existing expression (1.2.4.1).

$$\oplus_{i=1}^{n}\alpha_i = \left\langle 1-\prod_{i=1}^{n}(1-\mu_i), \prod_{i=1}^{n}(1-\mu_i) - \prod_{i=1}^{n}\big(1-(\mu_i+\nu_i)\big)\right\rangle \quad (2.1.4.1)$$

The expression (2.1.4.1) and hence, the expression (1.2.4.1) as well as the expression (1.2.4.2) are not valid due to the following reasons:

(i) Thee expression (2.1.4.1) can also be represented as the expression (2.1.4.2).

$$\oplus_{i=1}^{n}\alpha_i = \left\langle 1-\prod_{i=1}^{n}(1-\mu_i), \prod_{i=1}^{n}(1-\mu_i) - \left(1-\left(\mu_p+\nu_p\right)\right)\times \prod_{\substack{i=1\\ i\neq p}}^{n}\big(1-(\mu_i+\nu_i)\big)\right\rangle \quad (2.1.4.2)$$

If there exist an IFN $\alpha_p = \langle \mu_p, \nu_p\rangle$ such that $\mu_p + \nu_p = 1$ then,

$$\oplus_{i=1}^{n}\alpha_i = \left\langle 1-\prod_{i=1}^{n}(1-\mu_i), \prod_{i=1}^{n}(1-\mu_i) - (1-1)\times \prod_{\substack{i=1\\ i\neq p}}^{n}\big(1-(\mu_i+\nu_i)\big)\right\rangle$$

$$= \left\langle 1-\prod_{i=1}^{n}(1-\mu_i), \prod_{i=1}^{n}(1-\mu_i)\right\rangle.$$

This clearly indicates that the sum of IFNs is is independent from the non-membership values of IFNs, which is mathematically incorrect.

(ii) The expression (2.1.4.1) can also be represented as the expression (2.1.4.3).

$$\oplus_{i=1}^{n}\alpha_i = \left\langle 1-\left(1-\mu_p\right)\times\prod_{\substack{i=1\\ i\neq p}}^{n}(1-\mu_i),\left(1-\mu_p\right)\times\prod_{\substack{i=1\\ i\neq p}}^{n}(1-\mu_i)\ -\right.$$

$$\left.\left(1-\left(\mu_p+\nu_p\right)\right)\times\prod_{\substack{i=1\\ i\neq p}}^{n}\left(1-(\mu_i+\nu_i)\right)\right\rangle \tag{2.1.4.3}$$

If there exist an IFN $\alpha_p = \langle\mu_p,\nu_p\rangle$ such that $\mu_p = 1$ and $\nu_p = 0$ then,

$$\oplus_{i=1}^{n}\alpha_i = \left\langle 1-(1-1)\times\prod_{\substack{i=1\\ i\neq p}}^{n}(1-\mu_i),(1-1)\times\prod_{\substack{i=1\\ i\neq p}}^{n}(1-\mu_i)-\left(1-(1+0)\right)\times\right.$$

$$\left.\prod_{\substack{i=1\\ i\neq p}}^{n}\left(1-(\mu_i+\nu_i)\right)\right\rangle = \langle 1,0\rangle.$$

This clearly indicates that the sum of IFNs is only depending upon the p^{th} IFN $\alpha_p = \langle\mu_p,\nu_p\rangle = \langle 0,1\rangle$ and is independent from the remaining $(n-1)$ IFNs, which is mathematically incorrect.

(iii) If $\alpha = \langle\mu,\nu\rangle = \langle 1,0\rangle$ then using the existing expression (1.2.4.2) i.e.,

$$\lambda\alpha = \langle 1-(1-\mu)^{\lambda},(1-\mu)^{\lambda}-\left(1-(\mu+\nu)\right)^{\lambda}\rangle = \langle 1-(1-1)^{\lambda},(1-1)^{\lambda}-$$

$$\left(1-(1+0)\right)^{\lambda}\rangle = \langle 1,0\rangle.$$

This clearly indicates that the multiplication of a positive real number λ with an IFN $\alpha = \langle\mu,\nu\rangle = \langle 1,0\rangle$ is only depending upon the IFN $\alpha = \langle\mu,\nu\rangle = \langle 1,0\rangle$ and is independent from the positive real number λ, which is mathematically incorrect.

2.1.5 Flaws of Chen and Chang’s expressions to evaluate the multiplication of IFNs

In this section, the flaws of Chen and Chang’s expressions to evaluate the multplication of IFNs [28], discussed in Section 1.2.5, are pointed out.

The expression (2.1.5.1) represents the generalized form of the existing expression (1.2.5.1).

$$\otimes_{i=1}^{n}\alpha_i=\left\langle 1-\prod_{i=1}^{n}(1-\mu_i),\prod_{i=1}^{n}(1-\mu_i)-\prod_{i=1}^{n}\big(1-(\mu_i+\nu_i)\big)\right\rangle \tag{2.1.5.1}$$

The expression (2.1.5.1) and hence, the expression (1.2.5.1) as well as the expression (1.2.5.2) are not valid due to the following reasons:

(i) The expression (2.1.5.1) can also be represented as the expression (2.1.5.2).

$$\otimes_{i=1}^{n}\alpha_i=\left\langle 1-\prod_{i=1}^{n}(1-\mu_i),\prod_{i=1}^{n}(1-\mu_i)-\Big(1-\big(\mu_p+\nu_p\big)\Big)\prod_{\substack{i=1\\i\neq p}}^{n}\big(1-(\mu_i+\nu_i)\big)\right\rangle \tag{2.1.5.2}$$

If there exist an IFN $\alpha_p=\langle\mu_p,\nu_p\rangle$ such that $\mu_p+\nu_p=1$ then,

$$\otimes_{i=1}^{n}\alpha_i=\left\langle 1-\prod_{i=1}^{n}(1-\mu_i),\prod_{i=1}^{n}(1-\mu_i)-(1-1)\prod_{\substack{i=1\\i\neq p}}^{n}\big(1-(\mu_i+\nu_i)\big)\right\rangle$$

$$=\langle 1-\prod_{i=1}^{n}(1-\mu_i),\prod_{i=1}^{n}(1-\mu_i)\rangle.$$

This clearly indicates that the multplication of IFNs is independent from the non-membership values of IFNs, which is mathematically incorrect.

(ii) The expression (2.1.5.1) can also be represented as the expression (2.1.5.3).

$$\otimes_{i=1}^{n}\alpha_i=\left\langle 1-\big(1-\mu_p\big)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\mu_i),\big(1-\mu_p\big)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\mu_i)-\Big(1-\big(\mu_p+\nu_p\big)\Big)\prod_{\substack{i=1\\i\neq p}}^{n}\big(1-(\mu_i+\nu_i)\big)\right\rangle \tag{2.1.5.3}$$

If there exist an IFN $\alpha_p=\langle\mu_p,\nu_p\rangle$ such that $\mu_p=1$ and $\nu_p=0$ then

$$\otimes_{i=1}^{n}\alpha_i=\left\langle 1-(1-1)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\mu_i),(1-1)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\mu_i)-\big(1-(1+0)\big)\prod_{\substack{i=1\\i\neq p}}^{n}\big(1-(\mu_i+\nu_i)\big)\right\rangle$$

$$= \langle 1,0 \rangle.$$

This clearly indicates that the multplication of IFNs is only depending upon the p^{th} IFN $\alpha_p = \langle \mu_p, \nu_p \rangle = \langle 0,1 \rangle$ and is independent from the remaining $(n-1)$ IFNs, which is mathematically incorrect.

(iii) If $\alpha = \langle \mu, \nu \rangle = \langle 1,0 \rangle$ then using the existing expression (1.2.5.2),

$\alpha^{\lambda} = \langle 1-(1-\mu)^{\lambda}, (1-\mu)^{\lambda} - \big(1-(\mu+\nu)\big)^{\lambda} \rangle = \langle 1-(1-1)^{\lambda}, (1-1)^{\lambda} - \big(1-(1+0)\big)^{\lambda} \rangle = \langle 1,0 \rangle.$

This clearly indicates that the positive power λ of an IFN $\alpha = \langle \mu, \nu \rangle = \langle 1,0 \rangle$ is only depending upon the IFN $\alpha = \langle \mu, \nu \rangle = \langle 1,0 \rangle$ and is independent from the positive real number λ, which is mathematically incorrect.

2.1.6 Flaws of Garg's expressions to evaluate the sum of IFNs

In this section, the flaws of Garg's expressions to evaluate the sum of IFNs [65], discussed in Section 1.2.6, are pointed out.

The expression (2.1.6.1) represents the generalized form of the existing expression (1.2.6.1).

$$\oplus_{i=1}^{n} \alpha_i = \left\langle \frac{\prod_{i=1}^{n}(1+\mu_i) - \prod_{i=1}^{n}(1-\mu_i)}{\prod_{i=1}^{n}(1+\mu_i) + \prod_{i=1}^{n}(1-\mu_i)}, \frac{2\{\prod_{i=1}^{n}(1-\mu_i) - \prod_{i=1}^{n}(1-\mu_i-\nu_i)\}}{\prod_{i=1}^{n}(1+\mu_i) + \prod_{i=1}^{n}(1-\mu_i)} \right\rangle \qquad (2.1.6.1)$$

The expression (2.1.6.1) and hence, the expression (1.2.6.1) as well as the expression (1.2.6.2) are not valid due to the following reasons:

(i) The expression (2.1.6.1) can also be represented as the expression (2.1.6.2).

$$\oplus_{i=1}^{n} \alpha_i =$$

$$= \left\langle \frac{\prod_{i=1}^{n}(1+\mu_i) - \prod_{i=1}^{n}(1-\mu_i)}{\prod_{i=1}^{n}(1+\mu_i) + \prod_{i=1}^{n}(1-\mu_i)}, \frac{2\left\{\prod_{i=1}^{n}(1-\mu_i) - (1-\mu_p-\nu_p)\prod_{\substack{i=1 \\ i\neq p}}^{n}(1-\mu_i-\nu_i)\right\}}{\prod_{i=1}^{n}(1+\mu_i) + \prod_{i=1}^{n}(1-\mu_i)} \right\rangle$$

(2.1.6.2)

If there exist an IFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\mu_p + \nu_p = 1$ then,

$$\oplus_{i=1}^{n} \alpha_i = \left\langle \frac{\prod_{i=1}^{n}(1+\mu_i) - \prod_{i=1}^{n}(1-\mu_i)}{\prod_{i=1}^{n}(1+\mu_i) + \prod_{i=1}^{n}(1-\mu_i)}, \frac{2\left\{\prod_{i=1}^{n}(1-\mu_i) - (1-1)\prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i-\nu_i)\right\}}{\prod_{i=1}^{n}(1+\mu_i) + \prod_{i=1}^{n}(1-\mu_i)} \right\rangle$$

$$= \left\langle \frac{\prod_{i=1}^{n}(1+\mu_i) - \prod_{i=1}^{n}(1-\mu_i)}{\prod_{i=1}^{n}(1+\mu_i) + \prod_{i=1}^{n}(1-\mu_i)}, \frac{2\{\prod_{i=1}^{n}(1-\mu_i)\}}{\prod_{i=1}^{n}(1+\mu_i) + \prod_{i=1}^{n}(1-\mu_i)} \right\rangle.$$

This clearly indicates that the sum of the IFNs is independent from the non-membership values of IFNs, which is mathematically incorrect.

(ii) The expression (2.1.6.1) can also be represented as the expression (2.1.6.3).

$\oplus_{i=1}^{n} \alpha_i =$

$$= \left\langle \frac{(\prod_{i=1}^{n}(1+\mu_i) - (1-\mu_p)\prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i)}{(\prod_{i=1}^{n}(1+\mu_i) + (1-\mu_p)\prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i)}, \frac{2\left\{(1-\mu_p)\prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i) - (1-\mu_p-\nu_p)\prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i-\nu_i)\right\}}{\prod_{i=1}^{n}(1+\mu_i) + \prod_{i=1}^{n}(1-\mu_i)} \right\rangle$$

(2.1.6.3)

If there exist an IFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\mu_p = 1$ and $\nu_p = 0$ then,

$$\oplus_{i=1}^{n} \alpha_i = \left\langle \frac{(\prod_{i=1}^{n}(1+\mu_i) - (1-1)\prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i)}{(\prod_{i=1}^{n}(1+\mu_i) + (1-1)\prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i)}, \frac{2\left\{(1-1)\prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i) - (1-1-0)\prod_{\substack{i=1 \\ i \neq p}}^{n}(1-\mu_i-\nu_i)\right\}}{\prod_{i=1}^{n}(1+\mu_i) + \prod_{i=1}^{n}(1-\mu_i)} \right\rangle$$

$$= \langle 1,0 \rangle.$$

This clearly indicates that the sum of the IFNs is only depending upon the p^{th} IFN $\alpha_p = \langle \mu_p, \nu_p \rangle = \langle 1,0 \rangle$ and is independent from the remaining $(n-1)$ IFNs, which is mathematically incorrect.

(iii) If $\alpha = \langle 1,0 \rangle$ then using the existing expression (1.2.6.2),

$$\lambda\alpha = \left\langle \frac{(1+\mu)^{\lambda} - (1-\mu)^{\lambda}}{(1+\mu)^{\lambda} + (1-\mu)^{\lambda}}, \frac{2\{(1-\mu)^{\lambda} - (1-\mu-\nu)^{\lambda}\}}{(1+\mu)^{\lambda} + (1-\mu)^{\lambda}} \right\rangle = \left\langle \frac{(1+1)^{\lambda-1} - (1-1)^{\lambda}}{(1+1)^{\lambda} + (1-1)^{\lambda}}, \frac{2\{(1-1)^{\lambda} - (1-1-0)^{\lambda}\}}{(1+1)^{\lambda} + (1-1)^{\lambda}} \right\rangle.$$

This clearly indicates that the multiplication of an IFN with a positive real number λ is only depending upon the IFN $\alpha = \langle 1,0 \rangle$ and is independent from the value of λ, which is mathematically incorrect.

(iv) Garg [65, Section 3.1, Theorem 3.1, pp. 166] has used the following method to obtain the non-membership value $\nu_{\alpha_1 \oplus \alpha_2} = \frac{2\{\prod_{i=1}^{n}(1-\mu_i) - \prod_{i=1}^{n}(1-\mu_i-\nu_i)\}}{\prod_{i=1}^{2}(1+\mu_i) + \prod_{i=1}^{2}(1-\mu_i)}$ of the existing expression (2.1.6.1)

$$0 \le \frac{1-\mu_i-\nu_i}{1-\mu_i} \le 1$$

$$\Rightarrow \quad 0 \le \prod_{i=1}^{n}\left(\frac{1-\mu_i-\nu_i}{1-\mu_i}\right) \le 1$$

$$\Rightarrow 0 \le 1 - \prod_{i=1}^{n}\left(\frac{1-\mu_i-\nu_i}{1-\mu_i}\right) \le 1 \tag{2.1}$$

Also, $0 \le \frac{1-\mu_i}{1+\mu_i} \le 1$

$$\Rightarrow \quad 0 \le \prod_{i=1}^{n}\left(\frac{1-\mu_i}{1+\mu_i}\right) \le 1$$

$$\Rightarrow \quad 1 \le \prod_{i=1}^{n}\left(\frac{1+\mu_i}{1-\mu_i}\right) < \infty$$

$$\Rightarrow \quad 2 \le 1 + \prod_{i=1}^{n}\left(\frac{1+\mu_i}{1-\mu_i}\right) < \infty$$

$$\Rightarrow \quad 0 < \frac{1}{1+\prod_{i=1}^{n}\left(\frac{1+\mu_i}{1-\mu_i}\right)} \le \frac{1}{2}$$

$$\Rightarrow 0 < \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1+\mu_i}{1-\mu_i}\right)} \le 1 \tag{2.2}$$

Multiplying (2.1) and (2.2),

$$0 \le \frac{2\left\{1-\prod_{i=1}^{n}\left(\frac{1-\mu_i-\nu_i}{1-\mu_i}\right)\right\}}{\frac{1}{1+\prod_{i=1}^{n}\left(\frac{1+\mu_i}{1-\mu_i}\right)}} \le 1$$

$$\Rightarrow 0 \le \frac{\frac{2\{\prod_{i=1}^{n}(1-\mu_i) - \prod_{i=1}^{n}(1-\mu_i-\nu_i)\}}{\prod_{i=1}^{n}(1-\mu_i)}}{\frac{\prod_{i=1}^{n}(1-\mu_i) + \prod_{i=1}^{n}(1+\mu_i)}{\prod_{i=1}^{n}(1-\mu_i)}} \le 1 \tag{2.3}$$

Cancelling the terms $\prod_{i=1}^{n}(1-\mu_i)$ in the expression (2.3), it will be transformed into the expression (2.4).

$$0 \le \frac{2\{\prod_{i=1}^{n}(1-\mu_i) - \prod_{i=1}^{n}(1-\mu_i-\nu_i)\}}{\prod_{i=1}^{n}(1-\mu_i) + \prod_{i=1}^{n}(1+\mu_i)} \le 1 \tag{2.4}$$

i.e., $0 \le \nu_{\alpha_1 \oplus \alpha_2} \le 1$.

However, this method is not valid due to the following reasons:

(a) To obtain the non-membership value $\nu_{\alpha_1 \oplus \alpha_2}$ of the existing expression (2.1.6.1), it is assumed that the terms $\prod_{i=1}^{n}(1-\mu_i)$, present in the numerator and denominator of the expression (2.3), can be cancelled from each other i.e., to obtain the non-membership value $\nu_{\alpha_1 \oplus \alpha_2}$ of the existing expression (2.1.6.1), it is assumed that $\frac{\prod_{i=1}^{n}(1-\mu_i)}{\prod_{i=1}^{n}(1-\mu_i)} = 1$. While, it is mathematically incorrect to assume the same as at $\mu_i = 1$, the value of the expression $\frac{\prod_{i=1}^{n}(1-\mu_i)}{\prod_{i=1}^{n}(1-\mu_i)}$ will be $\frac{0}{0}$ (an indeterminate value).

Therefore, the non-membership value $\nu_{\alpha_1 \oplus \alpha_2}$ of the existing expression (2.1.6.1) is not valid and hence, the expression (2.1.6.1) is not valid.

(b) It is obvious that to obtain the non-membership value $v_{\alpha_1 \oplus \alpha_2}$ of the existing expression (2.1.6.1), it is assumed that the relation $0 \le \frac{1-\mu_i-\nu_i}{1-\mu_i} \le 1$ is true for all the values of μ_i and ν_i where, $0 \le \mu_i \le 1$ and $0 \le \nu_i \le 1$.

However, if $\mu_i = 1, \nu_i = 0$ then the value of $\frac{1-\mu_i-\nu_i}{1-\mu_i}$ will be $\frac{0}{0}$ (an indeterminate value) which does not lie between 0 and 1. Therefore, the non-membership value $v_{\alpha_1 \oplus \alpha_2}$ of the existing expression (2.1.6.1) is not valid. Hence, the existing expression (2.1.6.1) is also not valid.

(c) It is obvious that the existing expression (1.2.6.2) is obtained by considering $\mu_i = \mu, \nu_i = \nu$ for all i and $n = \lambda$ in the existing expression (2.1.6.1). However, as discussed in this section that the existing expression (2.1.6.1) is not valid. Therefore, the existing expression (1.2.6.2) is also not valid.

2.1.7 Flaws of Garg's expressions to evaluate the multiplication of IFNs

In this section, the flaws of Garg's expressions to evaluate the multiplication of IFNs [60], discussed in Section 1.2.7, are pointed out.

(i) It is pertinent to mention that the membership value of the expression (1.2.7.1) is obtained by replacing μ_{i} with ν_{i} and vice-versa in the non-membership value of the expression (1.2.6.1). Also, the non-membership value of the expression (1.2.7.1) is obtained by replacing μ_{i} with ν_{i} and vice-versa in the membership value of the expression (1.2.6.1).

However, it is already pointed out that the membership value and the non-membership value of the expression (1.2.6.1) are not valid as to obtain these values some mathematical incorrect assumptions have been considered.

Therefore, the membership value and the non-membership value of the expression (1.2.7.1) are also not valid. Hence, the expression (1.2.7.1) is not valid.

(ii) The expression (1.2.7.2) is obtained by putting $\mu_{\mathrm{i}} = \mu$ and $\nu_{\mathrm{i}} = \nu$ in the existing expression (1.2.7.1). While, the expression (1.2.7.1) is not valid. Therefore, the expression (1.2.7.2) is also not valid.

2.1.8 Flaws of expressions to evaluate the sum and multiplication of IFMNs

In this section, with the help of numerical examples, the flaws of the expressions to evaluate sum and multiplication of IFMNs [56, 66], discussed in Section 1.2.8, are pointed out.

2.1.8.1 Flaws of Garg's expressions to evaluate the multiplication of IFMNs

In this section, with the help of numerical examples, the flaws of Garg's expressions to evaluate the multiplication of IFMNs [56], discussed in Section 1.2.8.1, are pointed out.

(a) Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \left\langle \frac{1}{4}, 3 \right\rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \left\langle \frac{1}{6}, 4 \right\rangle$ be two IFMNs. Then, on applying the existing expression (1.1.8.1.1),

$$\alpha_1 \otimes \alpha_2 = \left\langle \frac{2\{1-(1-\mu_1\nu_1)(1-\mu_2\nu_2)\}}{(1+2\nu_1)(1+2\nu_2)-1}, \frac{(1+2\nu_1)(1+2\nu_2)-1}{2} \right\rangle = \left\langle \frac{11}{372}, 31 \right\rangle.$$

It is well known fact that if $\alpha = \langle \mu, \nu \rangle$ is an IFMN then for μ and ν, the conditions $\frac{1}{9} \leq \mu, \nu \leq 9$ and $\mu\nu \leq 1$ should necessarily be satisfied. However, it is obvious that in $\alpha_1 \otimes \alpha_2 = \left\langle \frac{11}{372}, 31 \right\rangle$, the value of μ is $\frac{11}{372}$ which is less than $\frac{1}{9}$. Also, the value of ν is 31, which is greater than 9. Therefore, $\alpha_1 \otimes \alpha_2 = \left\langle \frac{11}{372}, 31 \right\rangle$ is not an IFMN. Hence, Garg's expression (1.2.8.1.1) to evaluate the multiplication of two IFMNs is not valid.

(b) The existing expression (1.2.8.1.2) i.e., $\alpha^\lambda = \left\langle \frac{2\{1-(1-\mu\nu)^\lambda\}}{(1+2\nu)^\lambda-1}, \frac{(1+2\nu)^\lambda-1}{2} \right\rangle$ is obtained by considering $\alpha_1 = \alpha_2$ in the existing expression (1.2.8.1.1) i.e., $\alpha_1 \otimes \alpha_2 = \left\langle \frac{2\{1-(1-\mu_1\nu_1)(1-\mu_2\nu_2)\}}{(1+2\nu_1)(1+2\nu_2)-1}, \frac{(1+2\nu_1)(1+2\nu_2)-1}{2} \right\rangle$.

However, as discussed in (a) that the existing expression (1.2.8.1.1) is not valid. Therefore, the existing expression (1.2.8.1.2) is also not valid.

2.1.8.2 Flaws of Garg's expressions to evaluate the sum of IFMNs

In this section, with the help of numerical examples, the flaws of Garg's expressions to evaluate sum of IFMNs [66], discussed in Section 1.2.8.2, are pointed out.

(a) Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \left\langle 3, \frac{1}{4} \right\rangle$ and $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \left\langle 4, \frac{1}{6} \right\rangle$ be two IFMNs. Then on applying the existing expression (1.2.8.2.1),

$$\alpha_1 \oplus \alpha_2 = \left\langle \frac{(1+2\mu_1)(1+2\mu_2)-1}{2}, \frac{2\{1-(1-\mu_1\nu_1)(1-\mu_2\nu_2)\}}{(1+2\mu_1)(1+2\mu_2)-1} \right\rangle = \left\langle 31, \frac{11}{372} \right\rangle.$$

It is well known fact that if $\alpha = \langle \mu, \nu \rangle$ is an IFMN then for μ and ν, the conditions $\frac{1}{9} \leq \mu, \nu \leq 9$ and $\mu\nu \leq 1$ should necessarily be satisfied. However, it is obvious that in, $\alpha_1 \oplus \alpha_2 = \langle \mu, \nu \rangle = \left\langle 31, \frac{11}{372} \right\rangle$, the value of μ is 31 which is greater than 9. Also, the value of ν is $\frac{11}{372}$, which is less than $\frac{1}{9}$. Therefore, $\alpha_1 \oplus \alpha_2 = \left\langle 31, \frac{11}{372} \right\rangle$ is not an IFMN. Hence, Garg's expression (1.2.8.2.1) i.e., $\alpha_1 \oplus \alpha_2 = \left\langle \frac{(1+2\mu_1)(1+2\mu_2)-1}{2}, \frac{2\{1-(1-\mu_1\nu_1)(1-\mu_2\nu_2)\}}{(1+2\mu_1)(1+2\mu_2)-1} \right\rangle$ is not valid.

(b) The existing expression (1.2.8.2.2) i.e., $\lambda\alpha = \left\langle \frac{(1+2\mu)^\lambda-1}{2}, \frac{2\{1-(1-\mu\nu)^\lambda\}}{(1+2\mu)^\lambda-1} \right\rangle$ is obtained by considering $\alpha_1 = \alpha_2$ in the existing expression (1.2.8.2.1) i.e., $\alpha_1 \oplus \alpha_2 = \left\langle \frac{(1+2\mu_1)(1+2\mu_2)-1}{2}, \frac{2\{1-(1-\mu_1\nu_1)(1-\mu_2\nu_2)\}}{(1+2\mu_1)(1+2\mu_2)-1} \right\rangle$.

However, as discussed that the existing expression (1.2.8.2.1) is not valid. Therefore, the existing expression (1.2.8.2.2) i.e., $\lambda\alpha = \left\langle \frac{(1+2\mu)^\lambda-1}{2}, \frac{2\{1-(1-\mu\nu)^\lambda\}}{(1+2\mu)^\lambda-1} \right\rangle$ is also not valid.

2.2 Flaws of existing expressions to evaluate the sum and multiplication of PFNs

In this section, the flaws of the expressions to evaluate the sum and multiplication of PFNs [58, 62, 63, 67, 135], discussed in Section 1.3, are pointed out.

2.2.1 Flaws of Garg's expressions to evaluate the sum of PFNs

In this section, the flaws of Garg's expressions to evaluate the sum of PFNs [58], discussed in Section 1.3.1, are pointed out.

The expression (2.2.1.1) represents the generalized form of the existing expression (1.3.1.1).

$$\oplus_{i=1}^{n} \alpha_i = \left\langle \sqrt{\frac{\sum_{i=1}^{n} \mu_i^2}{1+\prod_{i=1}^{n} \mu_i^2}}, \frac{\prod_{i=1}^{n} \nu_i}{\sqrt{1+\prod_{i=1}^{n}(1-\nu_i^2)}} \right\rangle \tag{2.2.1.1}$$

The expression (2.2.1.1) and hence, the expression (1.3.1.1) as well as the expression (1.3.1.2) are not valid due to the following reasons:

(i) The expression (2.2.1.1) can also be represented as the expression (2.2.1.2).

$$\oplus_{i=1}^{n} \alpha_i = \left\langle \sqrt{\frac{\sum_{i=1}^{n} \mu_i^2}{1+\prod_{i=1}^{n} \mu_i^2}}, \frac{\nu_p \prod_{\substack{i=1 \\ i\neq p}}^{n} \nu_i}{\sqrt{1+\prod_{i=1}^{n}(1-\nu_i^2)}} \right\rangle \tag{2.2.1.2}$$

If there exist a PFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\nu_p = 0$ then,

$$\oplus_{i=1}^{n} \alpha_i = \left\langle \sqrt{\frac{\sum_{i=1}^{n} \mu_i^2}{1+\prod_{i=1}^{n} \mu_i^2}}, \frac{(0) \prod_{\substack{i=1 \\ i\neq p}}^{n} \nu_i}{\sqrt{1+\prod_{i=1}^{n}(1-\nu_i^2)}} \right\rangle$$

$$= \left\langle \sqrt{\frac{\sum_{i=1}^{n} \mu_i^2}{1+\prod_{i=1}^{n} \mu_i^2}}, 0 \right\rangle$$

This clearly indicates that the non-membership value of the sum of PFNs is only depending upon the non-membership value of the PFN $\alpha_p = \langle \mu_p, 0 \rangle$ and is independent from the non-membership values of the remaining $(n-1)$ PFNs.

(ii) If $\alpha = \langle \mu, 0 \rangle$ then using the existing expression (1.3.1.2) i.e.,

$$\lambda\alpha = \left\langle \sqrt{\frac{(1+\mu^2)^\lambda-(1-\mu^2)^\lambda}{(1+\mu^2)^\lambda+(1-\mu^2)^\lambda}}, \frac{\sqrt{2}\nu^\lambda}{\sqrt{(2-\nu^2)^\lambda+(\nu^2)^\lambda}} \right\rangle,$$

$$\lambda\langle \mu, 0 \rangle = \left\langle \sqrt{\frac{(1+\mu^2)^\lambda-(1-\mu^2)^\lambda}{(1+\mu^2)^\lambda+(1-\mu^2)^\lambda}}, 0 \right\rangle.$$

This clearly indicates that the multiplication of a PFN with a positive real number λ is only depending upon the IFN $\alpha = \langle \mu, 0 \rangle$ and is independent from the value of λ, which is mathematically incorrect.

2.2.2 Flaws of Garg's expressions to evaluate the multiplication of PFNs

In this section, the flaws of the expressions to evaluate the multiplication of PFNs, discussed in Section 1.3.2, are pointed out.

The expression (2.2.2.1) represents the generalized form of the existing expression (1.3.2.1),

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \frac{\prod_{i=1}^{n} \mu_i}{\sqrt{1+\prod_{i=1}^{n}(1-\mu_i^2)}}, \sqrt{\frac{\sum_{i=1}^{n} \nu_i^2}{1+\prod_{i=1}^{n} \nu_i^2}} \right\rangle \tag{2.2.2.1}$$

The expression (2.2.2.1) and hence, the expression (1.3.2.1) as well as the expression (1.3.2.2) are not valid due to the following reasons:

(i) The expression (2.2.2.1) can also be represented as the expression (2.2.2.2).

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \frac{\mu_p \prod_{\substack{i=1 \\ i \neq p}}^{n} \mu_i}{\sqrt{1+\prod_{i=1}^{n}(1-\mu_i^2)}}, \sqrt{\frac{\sum_{i=1}^{n} \nu_i^2}{1+\prod_{i=1}^{n} \nu_i^2}} \right\rangle \tag{2.2.2.2}$$

If there exist a PFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\mu_p = 0$ then,

$$\otimes_{i=1}^{n} \alpha_i = \left\langle 0, \sqrt{\frac{\sum_{i=1}^{n} \nu_i^2}{1+\prod_{i=1}^{n} \nu_i^2}} \right\rangle.$$

This clearly indicates that the membership value of multiplication of PFNs is only depending upon the membership value of the PFN $\alpha_p = \langle 0, \nu_p \rangle$ and is independent from the membership values of tge remaining $(n-1)$ PFNs.

(ii) If $\alpha = \langle 0, \nu \rangle$ then using the existing expression (1.3.2.2) i.e.,

$$\alpha^{\lambda} = \left\langle \frac{\sqrt{2}\mu^{\lambda}}{\sqrt{(2-\mu^2)^{\lambda}+(\mu^2)^{\lambda}}}, \sqrt{\frac{(1+\nu^2)^{\lambda}-(1-\nu^2)^{\lambda}}{(1+\nu^2)^{\lambda}+(1-\nu^2)^{\lambda}}} \right\rangle, \alpha^{\lambda} = \left\langle 0, \frac{(1+2\nu)^{\lambda}-1}{2} \right\rangle.$$

This clearly indicates that the positive power λ of the PFN $\alpha = \langle 0, \nu \rangle$ is only depending upon the PFN $\alpha = \langle 0, \nu \rangle$ and is independent from the value of λ, which is mathematically incorrect.

2.2.3 Flaws of Garg's expressions to evaluate the multiplication of PFNs

In this section, the flaws of Garg's expressions to evaluate the multiplication of PFNs [67], discussed in Section 1.3.3, are pointed out.

The expression (2.2.3.1) represents the generalized form of the existing expression (1.3.3.1),

$$\alpha_1 \otimes \alpha_2 = \left\langle \sqrt{\frac{2\{\prod_{i=1}^{n}(1-\nu_i^2)-\prod_{i=1}^{n}(1-\mu_i^2-\nu_i^2)\}}{\prod_{i=1}^{n}(1+\nu_i^2)+\prod_{i=1}^{n}(1-\nu_i^2)}}, \sqrt{\frac{\prod_{i=1}^{n}(1+\nu_i^2)-\prod_{i=1}^{n}(1-\nu_i^2)}{\prod_{i=1}^{n}(1+\nu_i^2)+\prod_{i=1}^{n}(1-\nu_i^2)}} \right\rangle \tag{2.2.3.1}$$

The expression (2.2.3.1) and hence, the expression (1.3.3.1) as well as the expression (1.3.3.2) are not valid due to the following reasons:

(i) The expression (2.2.3.1) can also be represented as the expression (2.2.3.2).

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \sqrt{\frac{2\left\{(\prod_{i=1}^{n}(1-\nu_i^2)-(1-\mu_p^2-\nu_p^2)\prod_{\substack{i=1\\ i\neq p}}^{n}(1-\mu_i^2-\nu_i^2)\right\}}{\prod_{i=1}^{n}(1+\nu_i^2)+\prod_{i=1}^{n}(1-\nu_i^2)}}, \sqrt{\frac{\prod_{i=1}^{n}(1+\nu_i^2)-\prod_{i=1}^{n}(1-\nu_i^2)}{\prod_{i=1}^{n}(1+\nu_i^2)+\prod_{i=1}^{n}(1-\nu_i^2)}} \right\rangle \tag{2.2.3.2}$$

If there exist an PFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\mu_p^2 + \nu_p^2 = 1$ then,

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \sqrt{\frac{2\left\{(\prod_{i=1}^{n}(1-\nu_i^2)-(1-1)\prod_{\substack{i=1\\ i\neq p}}^{n}(1-\mu_i^2-\nu_i^2)\right\}}{\prod_{i=1}^{n}(1+\nu_i^2)+\prod_{i=1}^{n}(1-\nu_i^2)}}, \sqrt{\frac{\prod_{i=1}^{n}(1+\nu_i^2)-\prod_{i=1}^{n}(1-\nu_i^2)}{\prod_{i=1}^{n}(1+\nu_i^2)+\prod_{i=1}^{n}(1-\nu_i^2)}} \right\rangle$$

$$= \left\langle \sqrt{\frac{2\{(\prod_{i=1}^{n}(1-\nu_i^2)\}}{\prod_{i=1}^{n}(1+\nu_i^2)+\prod_{i=1}^{n}(1-\nu_i^2)}} \sqrt{\frac{\prod_{i=1}^{n}(1+\nu_i^2)-\prod_{i=1}^{n}(1-\nu_i^2)}{\prod_{i=1}^{n}(1+\nu_i^2)+\prod_{i=1}^{n}(1-\nu_i^2)}} \right\rangle.$$

This clearly indicates that the multiplication of PFNs is independent from the membership values of PFNs, which is mathematically incorrect.

(ii) The expression (2.2.3.1) can also be represented as the expression (2.2.3.3).

$$\otimes_{i=1}^{n} \alpha_i =$$

$$\left\langle \sqrt{\frac{2\left\{(1-\nu_p^2)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\nu_i^2)-(1-\mu_p^2-\nu_p^2)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\mu_i^2-\nu_i^2)\right\}}{\prod_{i=1}^{n}(1+\nu_i^2)+\prod_{i=1}^{n}(1-\nu_i^2)}}, \sqrt{\frac{\prod_{i=1}^{n}(1+\nu_i^2)-(1-\nu_p^2)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\nu_i^2)}{\prod_{i=1}^{n}(1+\nu_i^2)+(1-\nu_p^2)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\nu_i^2)}} \right\rangle$$

(2.2.3.2)

If there exist a PFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\mu_p = 0$ and $\nu_p = 1$ then,

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \sqrt{\frac{2\left\{(1-1)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\nu_i^2)-(1-0-1)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\mu_i^2-\nu_i^2)\right\}}{\prod_{i=1}^{n}(1+\nu_i^2)+\prod_{i=1}^{n}(1-\nu_i^2)}}, \sqrt{\frac{\prod_{i=1}^{n}(1+\nu_i^2)-(1-1)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\nu_i^2)}{\prod_{i=1}^{n}(1+\nu_i^2)+(1-1)\prod_{\substack{i=1\\i\neq p}}^{n}(1-\nu_i^2)}} \right\rangle$$

$$= \langle 0, 1 \rangle \, .$$

This clearly indicates that the multiplication of PFNs is only depending upon the PFN $\alpha_p = \langle \mu_p, \nu_p \rangle = \langle 0,1 \rangle$ and is independent from the remaining $(n-1)$ PFNs, which is mathematically incorrect.

(iii) If $\alpha = \langle 0,1 \rangle$ then using the existing expression (1.3.3.2),

$$\alpha^{\lambda} = \left\langle \sqrt{\frac{2\{(1-\nu^2)^{\lambda}-(1-\mu^2-\nu^2)^{\lambda}\}}{(1+\nu^2)^{\lambda}+(1-\nu^2)^{\lambda}}}, \sqrt{\frac{(1+\nu^2)^{\lambda}-(1-\nu^2)^{\lambda}}{(1+\nu^2)^{\lambda}+(1-\nu^2)^{\lambda}}} \right\rangle = \langle 0, 1 \rangle.$$

This clearly indicates that the positive power λ of the PFN $\alpha = \langle 0,1 \rangle$ is only depending upon the PFN $\alpha = \langle 0,1 \rangle$ and is independent from the value of λ, which is mathematically incorrect.

2.2.4 Flaws of Garg's expressions to evaluate the multiplication of PFNs

In this section, the flaws of Garg's expressions to evaluate the multiplication of PFNs [62], discussed in Section 1.3.4, are pointed out.

(i) If $\alpha = \langle 1,0 \rangle$ then using the existing expression (1.3.4.3),

$$\lambda\eta\alpha = \left\langle \sqrt{1-(1-\mu^2)^{\lambda\eta}},\ \nu^{\lambda\eta} \right\rangle,$$

$$= \left\langle \sqrt{1-(1-1^2)^{\lambda\eta}},\ 0^{\lambda\eta} \right\rangle$$

$$= \langle 1,0 \rangle.$$

This clearly indicates that the multiplication of the PFN $\alpha = \langle 1,0 \rangle$ with a positive real number λ is only depending upon the PFN $\alpha = \langle 1,0 \rangle$ and is independent from the value of λ, which is mathematically incorrect.

(ii) If $\alpha = \langle 0,1 \rangle$ then using the existing expression (1.3.3.4).

$$\alpha^{\lambda\eta} = \left\langle \mu^{\lambda\eta}, \sqrt{1-(1-\nu^2)^{\lambda\eta}} \right\rangle,$$

$$= \left\langle 0^{\lambda\eta}, \sqrt{1-(1-1^2)^{\lambda\eta}} \right\rangle$$

$$= \langle 0,1 \rangle.$$

This clearly indicates that the positive power λ of the PFN $\alpha = \langle 0,1 \rangle$ is only depending upon the PFN $\alpha = \langle 0,1 \rangle$ and is independent from the value of λ, which is mathematically incorrect.

2.2.5 Flaws of Ma and Xu's expressions to evaluate the multiplication of PFNs

In this section, with the help of numerical examples, the flaws of Ma and Xu's expressions to evaluate the multiplication of PFNs [135], discussed in Section 1.3.5, are pointed out.

The expression (2.2.5.1) represents the generalized form of existing expression (1.3.5.1).

$$\otimes_{i=1}^{n} \alpha_i = \left(\frac{\prod_{i=1}^{n} \mu_i}{\left[\prod_{i=1}^{n}(1-{\mu_i}^2)+\prod_{i=1}^{n} {\mu_i}^2\right]^{\frac{1}{2}}}, \frac{\prod_{i=1}^{n} \nu_i}{\left[\prod_{i=1}^{n}(1-{\nu_i}^2)+\prod_{i=1}^{n} {\nu_i}^2\right]^{\frac{1}{2}}} \right) \quad (2.2.5.1)$$

The expression (2.2.5.1) and hence, the expression (1.3.5.1) as well as the expression (1.3.5.2) are not valid due to the following reasons:

(i) The expression (2.2.5.1) can also be represented as (2.2.5.2).

$$\otimes_{i=1}^{n} \alpha_i = \left(\frac{\mu_p \prod_{\substack{i=1 \\ i\neq p}}^{n} \mu_i}{\left[\prod_{i=1}^{n}(1-{\mu_i}^2)+\prod_{i=1}^{n} {\mu_i}^2\right]^{\frac{1}{2}}}, \frac{\nu_p \prod_{\substack{i=1 \\ i\neq p}}^{n} \nu_i}{\left[\prod_{i=1}^{n}(1-{\nu_i}^2)+\prod_{i=1}^{n} {\nu_i}^2\right]^{\frac{1}{2}}} \right) \quad (2.2.5.2)$$

If there exist a PFN $\alpha_p = \langle \mu_p, \nu_p \rangle$ such that $\mu_p = 0$ and $\nu_p = 0$ then,

then $\otimes_{i=1}^{n} \alpha_i = \langle 0,0 \rangle$.

This clearly indicates that the the multiplication of PFNs is only depending upon the p^{th} PFN $\alpha_p = \langle 0,0 \rangle$ and is independent from the remaining $(n-1)$ PFNs.

(ii) If $\alpha = \langle 0,0 \rangle$ then using the existing expression (1.3.5.2)

$$\alpha^{\lambda} = \left(\frac{\mu^{\lambda}}{\left[(1-\mu^2)^{\lambda}+\mu^{2\lambda}\right]^{\frac{1}{2}}}, \frac{\nu^{\lambda}}{\left[(1-\nu^2)^{\lambda}+ \nu^{2\lambda}\right]^{\frac{1}{2}}} \right)$$

$$= \langle 0, 0 \rangle.$$

This clearly indicates that the positive power λ of the PFN $\alpha = \langle 0,0 \rangle$ is only depending upon the PFN $\alpha = \langle 0,0 \rangle$ and is independent from the value of λ, which is mathematically incorrect.

2.3 Flaws of existing expressions to evaluate the sum and multiplication of IVIFNs

In this section, the flaws of the expressions to evaluate the sum and multiplication of IVIFNs [61, 68], discussed in Section 1.4, are pointed out.

2.3.1 Flaws of Garg et al.'s expressions to evaluate the sum of IVIFNs

In this section, the flaws of Garg et al.'s expressions to evaluate sum of IVIFNs [68], discussed in Section 1.4.1, are pointed out.

(i) The expression (2.3.1.1) represents the generalized form of the existing expression (1.4.1.1).

$$\oplus_{i=1}^{n}\alpha_i = \left\langle \left[\frac{\prod_{i=1}^{n}[1+(\gamma-1)a_i]-\prod_{i=1}^{n}(1-a_i)}{\prod_{i=1}^{n}[1+(\gamma-1)a_i]+(\gamma-1)\prod_{i=1}^{n}(1-a_i)}, \frac{\prod_{i=1}^{n}[1+(\gamma-1)b_i]-\prod_{i=1}^{n}(1-b_i)}{\prod_{i=1}^{n}[1+(\gamma-1)b_i]+(\gamma-1)\prod_{i=1}^{n}(1-b_i)} \right], \right.$$

$$\left. \left[\frac{\gamma\prod_{i=1}^{n}(1-a_i)-\gamma\prod_{i=1}^{n}[1-a_i-c_i]}{\prod_{i=1}^{n}[1+(\gamma-1)a_i]+(\gamma-1)\prod_{i=1}^{n}(1-a_i)}, \frac{\gamma\prod_{i=1}^{2}(1-b_i)-\gamma\prod_{i=1}^{2}[1-b_i-d_i]}{\prod_{i=1}^{n}[1+(\gamma-1)b_i]+(\gamma-1)\prod_{i=1}^{n}(1-b_i)} \right] \right\rangle. \qquad (2.3.1.1)$$

The following clearly indicates that the expression (2.3.1.1) and hence, the expression (1.4.1.1) is not valid.

The expression (2.3.1.1) can also be represented as the expression (2.3.1.2).

$$\oplus_{i=1}^{n}\alpha_i =$$

$$\left\langle \left[\frac{\prod_{i=1}^{n}[1+(\gamma-1)a_i]-(1-a_p)\prod_{\substack{i=1\\i\neq p}}^{n}(1-a_i)}{\prod_{i=1}^{n}[1+(\gamma-1)a_i]+(\gamma-1)(1-a_p)\prod_{\substack{i=1\\i\neq p}}^{n}(1-a_i)}, \frac{\prod_{i=1}^{n}[1+(\gamma-1)b_i]-(1-b_p)\prod_{\substack{i=1\\i\neq p}}^{n}(1-b_i)}{\prod_{i=1}^{n}[1+(\gamma-1)b_i]+(\gamma-1)(1-b_p)\prod_{\substack{i=1\\i\neq p}}^{n}(1-b_i)} \right], \right.$$

$$\left. \left[\begin{array}{c} \frac{\gamma(1-a_p)\prod_{\substack{i=1\\i\neq p}}^{n}(1-a_i)-\gamma[1-a_p-c_p]\prod_{\substack{i=1\\i\neq p}}^{n}[1-a_i-c_i]}{\prod_{i=1}^{n}[1+(\gamma-1)a_i]+(\gamma-1)\prod_{i=1}^{n}(1-a_i)}, \\ \frac{\gamma(1-b_p)\prod_{\substack{i=1\\i\neq p}}^{n}(1-b_i)-\gamma[1-b_p-d_p]\prod_{\substack{i=1\\i\neq p}}^{n}[1-b_i-d_i]}{\prod_{i=1}^{n}[1+(\gamma-1)b_i]+(\gamma-1)\prod_{i=1}^{n}(1-b_i)} \end{array} \right] \right\rangle \qquad (2.3.1.2)$$

If there exist an IVIFN $\alpha_p = \langle [a_p, b_p], [c_p, d_p] \rangle$ such that $a_p = b_p = 1$ and $c_p = d_p = 0$ then $\oplus_{i=1}^{n}\alpha_i = \langle [1,1], [0,0] \rangle$.

This clearly indicates that the sum of IVIFNs is only depending upon the IVIFN $\alpha_p = \langle [1,1], [0,0] \rangle$ and is independent from the remaining $(n-1)$ IVIFNs.

(ii) If $\alpha = \langle [1,1], [0,0] \rangle$ then using the existing expression (1.3.1.2),

$$\lambda\alpha = \left\langle \left[\frac{[1+(\gamma-1)a]^{\lambda}-[1-a]^{\lambda}}{[1+(\gamma-1)a]^{\lambda}+(\gamma-1)[1-a]^{\lambda}}, \frac{[1+(\gamma-1)b]^{\lambda}-[1-b]^{\lambda}}{[1+(\gamma-1)b]^{\lambda}+(\gamma-1)[1-b]^{\lambda}} \right], \right.$$

$$\left[\frac{\gamma[1-a]^{\lambda}-\gamma[1-a-c]^{\lambda}}{[1+(\gamma-1)a]^{\lambda}+(\gamma-1)[1-a]^{\lambda}}, \frac{\gamma[1-b]^{\lambda}-\gamma[1-b-d]^{\lambda}}{[1+(\gamma-1)b]^{\lambda}+(\gamma-1)[1-b]^{\lambda}}\right]\Big\rangle$$

$$= \langle [1,1], [0,0] \rangle.$$

This clearly indicates that the multiplication of the IVIFN $\alpha = \langle [1,1], [0,0] \rangle$ with a positive real number λ is only depending upon IFN $\alpha = \langle [1,1], [0,0] \rangle$ and is independent from the value of λ, which is mathematically incorrect.

(iii) If $\gamma = 1$, $\lambda = 2$ and $n = 2$ then the existing expression (1.4.1.1) will be transformed into the expression (2.3.1.3).

$$\alpha_1 \oplus \alpha_2 = \Big\langle [1 - \prod_{i=1}^{2}(1 - a_i), 1 - \prod_{i=1}^{2}(1 - b_i)],$$
$$[\prod_{i=1}^{2}(1 - a_i) - \prod_{i=1}^{2}[1 - a_i - c_i], \prod_{i=1}^{2}(1 - b_i) - \prod_{i=1}^{2}[1 - b_i - d_i]]\Big\rangle \tag{2.3.1.3}$$

Now, let $\alpha_1 = \langle [0.65, 0.73], [0.17, 0.21] \rangle$ and $\alpha_2 = \langle [0.50, 0.60], [0.35, 0.40] \rangle$ be two IVIFNs. Then, using the expression (2.3.1.3),

$$\alpha_1 \oplus \alpha_2 = \langle [0.8250, 0.8920], [0.1480, 0.1080] \rangle$$

It is well-known fact that for an IVIFN, $\alpha = \langle [a, b], [c, d] \rangle$, the conditions $a \leq b$, $c \leq d$ and $b + d \leq 1$ should always be satisfied. However, it can be easily verified that for $\alpha_1 \oplus \alpha_2 = \langle [0.8250, 0.8920], [0.1480, 0.1080] \rangle$ the condition $c \leq d$ is not satisfying. Therefore, $\alpha_1 \oplus \alpha_2$, obtained by the expression (2.3.1.3), is not an IVIFN. Hence, the expression (2.3.1.1) is not valid.

2.3.2 Flaws of Garg et al.'s expressions to evaluate the multiplication of IVIFNs

In this section, the flaws of Garg et al.'s expressions to evaluate the multiplication of IVIFNs [61], discussed in Section 1.4.2, are pointed out.

(i) The expression (2.3.2.1) represents the generalized form of existing expression (1.4.2.1).

$$\otimes_{i=1}^{n}\alpha_i=\left\langle\left[\frac{\gamma\prod_{i=1}^{n}(1-a_i)-\gamma\prod_{i=1}^{n}[1-a_i-c_i]}{\prod_{i=1}^{n}[1+(\gamma-1)a_i]+(\gamma-1)\prod_{i=1}^{n}(1-a_i)},\frac{\gamma\prod_{i=1}^{n}(1-b_i)-\gamma\prod_{i=1}^{n}[1-b_i-d_i]}{\prod_{i=1}^{n}[1+(\gamma-1)b_i]+(\gamma-1)\prod_{i=1}^{n}(1-b_i)}\right]\right.$$

$$\left.\left[\frac{\prod_{i=1}^{n}[1+(\gamma-1)a_i-\prod_{i=1}^{n}(1-a_i)]}{\prod_{i=1}^{n}[1+(\gamma-1)a_i]+(\gamma-1)\prod_{i=1}^{n}(1-a_i)},\frac{\prod_{i=1}^{n}[1+(\gamma-1)b_i-\prod_{i=1}^{n}(1-b_i)]}{\prod_{i=1}^{n}[1+(\gamma-1)b_i]+(\gamma-1)\prod_{i=1}^{n}(1-b_i)}\right]\right\rangle \qquad (2.3.2.1)$$

The following clearly indicates that the expression (2.3.2.1) and hence, the expression (1.4.2.1) is not valid.

The expression (2.3..2.1) can also be represented as the expression (2.3..2.2).

$$\otimes_{i=1}^{n}\alpha_i=$$

$$\left\langle\left[\frac{\gamma(1-a_p)\prod_{\substack{i=1\\ i\neq p}}^{n}(1-a_i)-\gamma[1-a_p-c_p]\prod_{\substack{i=1\\ i\neq p}}^{n}[1-a_i-c_i]}{\prod_{i=1}^{n}[1+(\gamma-1)a_i]+(\gamma-1)\prod_{i=1}^{n}(1-a_i)},\frac{\gamma(1-b_p)\prod_{\substack{i=1\\ i\neq p}}^{n}(1-b_i)-\gamma[1-b_p-d_p]\prod_{\substack{i=1\\ i\neq p}}^{n}[1-b_i-d_i]}{\prod_{i=1}^{n}[1+(\gamma-1)b_i]+(\gamma-1)\prod_{i=1}^{n}(1-b_i)}\right]\right.$$

$$\left.\left[\frac{\prod_{i=1}^{n}[1+(\gamma-1)a_i-\prod_{i=1}^{n}(1-a_i)]}{\prod_{i=1}^{n}[1+(\gamma-1)a_i]+(\gamma-1)\prod_{i=1}^{n}(1-a_i)},\frac{\prod_{i=1}^{n}[1+(\gamma-1)b_i-\prod_{i=1}^{n}(1-b_i)]}{\prod_{i=1}^{n}[1+(\gamma-1)b_i]+(\gamma-1)\prod_{i=1}^{n}(1-b_i)}\right]\right\rangle \qquad (2.3.2.2)$$

If there exist an IVIFN $\alpha_p=\langle[a_p,b_p],[c_p,d_p]\rangle$ such that $a_p=b_p=0$ and $c_p=d_p=1$ then, $\otimes_{i=1}^{n}\alpha_i=\langle[0,0],[1,1]\rangle$.

This clearly indicates that the multiplication of IVIFNs is only depending upon the IVIFN $\alpha_p=\langle[0,0],[1,1]\rangle$ and is independent from the remaining $(n-1)$ IVIFNs, which is mathematically incorrect.

(ii) If $\alpha=\langle[0,0],[1,1]\rangle$ then using the existing expression (1.4.2.2),

$$\alpha^{\lambda}=\left\langle\left[\frac{\gamma[1-a]^{\lambda}-\gamma[1-a-c]^{\lambda}}{[1+(\gamma-1)a]^{\lambda}+(\gamma-1)[1-a]^{\lambda}},\frac{\gamma[1-b]^{\lambda}-\gamma[1-b-d]^{\lambda}}{[1+(\gamma-1)b]^{\lambda}+(\gamma-1)[1-b]^{\lambda}}\right],\right.$$

$$\left.\left[\frac{[1+(\gamma-1)a]^{\lambda}-[1-a]^{\lambda}}{[1+(\gamma-1)a]^{\lambda}+(\gamma-1)[1-a]^{\lambda}},\frac{[1+(\gamma-1)b]^{\lambda}-[1-b]^{\lambda}}{[1+(\gamma-1)b]^{\lambda}+(\gamma-1)[1-b]^{\lambda}}\right]\right\rangle$$

$$=\langle[0,0],[1,1]\rangle.$$

This clearly indicates that the positive power λ of the IVIFN $\alpha=\langle[0,0],[1,1]\rangle$ is only depending upon the IVIFN $\alpha=\langle[0,0],[1,1]\rangle$ and is independent from the value of λ, which is mathematically incorrect.

(iii) If $\gamma = 1$, $\lambda = 2$ and $n = 2$ then the existing expression (2.3.2.1) and (2.3.2.2) will be transformed into the expression (2.3.2.3) and the expression (2.3.2.4) respectively.

$$\alpha_1 \otimes \alpha_2 = \left\langle \left[\prod_{i=1}^{2}(1-a_i) - \prod_{i=1}^{2}[1-a_i-c_i], \prod_{i=1}^{2}(1-b_i) - \prod_{i=1}^{2}[1-b_i-d_i]\right], \left[1-\prod_{i=1}^{2}(1-a_i), 1-\prod_{i=1}^{2}(1-b_i)\right]\right\rangle \quad (2.3.2.3)$$

$$\alpha^2 = \langle [[1-a]^2 - [1-a-c]^2, [1-b]^2 - [1-b-d]^2], [1-[1-a]^2, 1-[1-b]^2]\rangle$$

(2.3.2.4)

Now, let $\alpha_1 = \langle [0.65, 0.73], [0.17, 0.21]\rangle$ and $\alpha_2 = \langle [0.50, 0.60], [0.35, 0.40]\rangle$ be two IVIFNs. Then,

$\alpha_1 \otimes \alpha_2 = \langle [0.1480, 0.1080], [0.8250, 0.8920]\rangle$.

${\alpha_1}^2 = \langle [\ 0.0901, 0.0693], [0.8775, 0.9271]\rangle$

It is well-known fact that for an IVIF number, $\alpha = \langle [a, b], [c, d]\rangle$, the conditions $a \leq b$, $c \leq d$ and $b + d \leq 1$ should always be satisfied. However, it is obvious that for $\alpha_1 \otimes \alpha_2 = \langle [0.1480, 0.1080], [0.8250, 0.8920]\rangle$ and ${\alpha_1}^2 = \langle [\ 0.0901, 0.0693], [0.8775, 0.9271]\rangle$, the condition $a \leq b$ is not satisfying. Therefore, $\alpha_1 \otimes \alpha_2$, obtained by the expression (2.3.2.3), and ${\alpha_1}^2 = \langle [\ 0.0901, 0.0693], [0.8775, 0.9271]\rangle$, obtained by the expression (2.3.2.4), are not IVIFNs. Hence, the expression (2.3.2.1) and the expression (2.3.2.2) are not valid.

2.4 Flaws of Nancy and Garg's expressions to evaluate the sum of SVNNs

In this section, the flaws of Nancy and Garg's expressions to evaluate the sum of SVNNs [147], discussed in Section 1.5.1, are pointed out.

The expression (2.4.1) represents the generalized form of the expression (1.5.1.1).

$$\oplus_{i=1}^{n}\alpha_i = \left\langle 1-\log_{\lambda}\left(1+\frac{\prod_{i=1}^{n}(\lambda^{1-a_i}-1)}{\lambda-1}\right), \log_{\lambda}\left(1+\frac{\prod_{i=1}^{n}(\lambda^{b_i}-1)}{\lambda-1}\right), \log_{\lambda}\left(1+\frac{\prod_{i=1}^{n}(\lambda^{c_i}-1)}{\lambda-1}\right)\right\rangle, \lambda > 1 \qquad (2.4.1)$$

The expression (2.4.1) and hence, the expression (1.5.1.1) as well as the expression (1.5.1.2) are not valid due to the following reasons:

(i) The expression (2.4.1) can also be represented as the expression (2.4.2).

$$\oplus_{i=1}^{n}\alpha_i = \left\langle 1- \log_{\lambda}\left(1+\frac{(\lambda^{1-a_p}-1)\prod_{i=1}^{n-1}(\lambda^{1-a_i}-1)}{\lambda-1}\right), \log_{\lambda}\left(1+\frac{\left(\lambda^{b_p}-1\right)\prod_{i=1}^{n-1}(\lambda^{b_i}-1)}{\lambda-1}\right), \log_{\lambda}\left(1+\frac{(\lambda^{c_p}-1)\prod_{i=1}^{n-1}(\lambda^{c_i}-1)}{\lambda-1}\right)\right\rangle, \lambda > 1. \quad (2.4.2)$$

If there exist a SVNN $\alpha_p = \langle [a_p, b_p, c_p\rangle$ such that $a_p = 1,\ b_p = 0$ and $c_p = 0$ then, $\oplus_{i=1}^{n}\alpha_i = \langle 1,0,0\rangle$.

This clearly indicates that the sum of SVNNs is only depending upon the SVNN $\alpha_p = \langle 1,0,0\rangle$ and is independent from the remaining $(n-1)$ SVNNs, which is mathematically incorrect.

(ii) If $\alpha = \langle 1,0,0\rangle$ then using the existing expression (1.5.1.2),

$$n\alpha = \left\langle 1-\log_{\lambda}\left(1+\frac{\left(\lambda^{1-a}-1\right)^{n}}{(\lambda-1)^{n-1}}\right), \log_{\lambda}\left(1+\frac{\left(\lambda^{b}-1\right)^{n}}{(\lambda-1)^{n-1}}\right), \log_{\lambda}\left(1+\frac{(\lambda^{c}-1)^{n}}{(\lambda-1)^{n-1}}\right)\right\rangle, \lambda > 0,$$

$$= \langle 1,0,0\rangle.$$

This clearly indicates that the multiplication of the SVNN $\alpha = \langle 1,0,0\rangle$ with a positive real number λ is only depending upon the SVNN $\alpha = \langle 1,0,0\rangle$ and is independent from the value of λ, which is mathematically incorrect.

Futhermore, the expression (2.4.3) represents the generalized form of the expression (1.5.1.3).

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \log_\lambda\left(1+\frac{\prod_{i=1}^{n}(\lambda^{a_i}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{\prod_{i=1}^{n}(\lambda^{1-b_i}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{\prod_{i=1}^{n}(\lambda^{1-c_i}-1)}{\lambda-1}\right)\right\rangle, \lambda > 1. \qquad (2.4.3)$$

The expression (2.4.3) and hence, the expression (1.5.1.3) as well as the expression (1.5.1.4) are not valid due to the following reasons:

(i) The expression (2.4.3) can also be represented as the expression (2.4.4).

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \log_\lambda\left(1+\frac{(\lambda^{a_p}-1)\prod_{i=1}^{n-1}(\lambda^{a_i}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_p}-1\right)\prod_{i=1}^{n-1}(\lambda^{1-b_i}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-c_p}-1)\prod_{i=1}^{n-1}(\lambda^{1-c_i}-1)}{\lambda-1}\right)\right\rangle, \lambda > 1. \qquad (2.4.4)$$

If there exist a SVNN$\alpha_p = \langle [a_p, b_p, c_p \rangle$ such that $a_p = 0$, $b_p = 1$ and $c_p = 1$ then,

$$\otimes_{i=1}^{n} \alpha_i = \left\langle \log_\lambda\left(1+\frac{(1-1)\prod_{i=1}^{n-1}(\lambda^{a_i}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{(1-1)\prod_{i=1}^{n-1}(\lambda^{1-b_i}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{(1-1)\prod_{i=1}^{n-1}(\lambda^{1-c_i}-1)}{\lambda-1}\right)\right\rangle = \langle 0,1,1\rangle$$

This clearly indicates that the sum of SVNNs is only depending upon the SVNN $\alpha_p = \langle 0,1,1\rangle$ and is independent from the remaining $(n-1)$ SVNNs, which is mathematically incorrect.

(iii) If $\alpha = \langle 0,1,1\rangle$ then using the existing expression (2.4.4),

$$\alpha^n = \left\langle \log_\lambda\left(1+\frac{(\lambda^{a}-1)^n}{(\lambda-1)^{n-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b}-1\right)^n}{(\lambda-1)^{n-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-c}-1\right)^n}{(\lambda-1)^{n-1}}\right)\right\rangle.$$

$$= \left\langle \log_\lambda\left(1+\frac{(1-1)^n}{(\lambda-1)^{n-1}}\right), 1-\log_\lambda\left(1+\frac{(1-1)^n}{(\lambda-1)^{n-1}}\right), 1-\log_\lambda\left(1+\frac{(1-1)^n}{(\lambda-1)^{n-1}}\right)\right\rangle$$

$$= \langle 0,1,1\rangle.$$

This clearly indicates that the positive power λ of a SVNN is only depending upon the SVNN $\alpha = \langle 0,1,1 \rangle$ and is independent from the value of λ, which is mathematically incorrect.

2.5 Invalidity of IFW, IFMW PFW, IVIFW, SVNW and SVNHFOW aggregation operators

In this section, it is shown that the IFW, IFMW, PFW, IVIFW, SVNW and SVNHFOW aggregation operators [28, 56, 60-62, 63, 65-68, 88, 134, 135, 147, 156, 204, 208, 223], discussed in Section 1.2 to Section 1.5, are not valid.

The following steps are used to define the IFWAO (IFWGO), IFMWAO (IFMWGO), PFWAO (PFWGO), IVIFWAO (IVIFWGO), SVNWAO and SVNHOFWAO discussed in Section 1.2.

Step 1: Define an expression to evaluate sum (multiplication) of n IFNs, IFMNs, PFNs, IVIFNs, SVNNs, SVNHFNs i.e., to evaluate $\oplus_{i=1}^{n}\alpha_i$ $(\otimes_{i=1}^{n} \alpha_i)$.

Step 2: Replacing α_i by α and n by λ in the expression $\oplus_{i=1}^{n}\alpha_i$ $(\otimes_{i=1}^{n} \alpha_i)$, the expression to evaluate $\lambda\alpha$ (α^{λ}) is obtained.

Step 3: Replacing λ by w_i and α by α_i in the expression to evaluate $\lambda\alpha$ (α^{λ}), the expression to evaluate $w_i\alpha_i$ $({\alpha_i}^{w_i})$ is obtained.

Step 4: Replacing α_i by $w_i\alpha_i$ $({\alpha_i}^{w_i})$ in the expression to evaluate $\oplus_{i=1}^{n}\alpha_i(\otimes_{i=1}^{n} \alpha_i)$, the IFWAO (IFWGO), IFMWAO (IFMWGO), PFWAO (PFWGO), IVIFWAO (IVIFWGO), SVNWAO and SVNHFOWAO, discussed in Section 1.2, are obtained.

It is obvious from Step 2 that the expression to evaluate $\lambda\alpha(\alpha^{\lambda})$ is obtained by replacing the natural number n with λ. Therefore, the expression to evaluate $\lambda\alpha$ (α^{λ})

cannot be used only if λ is not a natural number.

While, in Step 3 the expression to evaluate $\lambda\alpha$ (α^{λ}) has been used to evaluate $w_i\alpha_i$ (${\alpha_i}^{w_i}$), where, w_i is not a natural number. Therefore, the expression to evaluate $w_i\alpha_i$ (${\alpha_i}^{w_i}$), obtained in Step 3, is not valid.

Hence, the IFWAO (IFWGO), IFMWAO (IFMWGO), PFWAO (PFWGO), IVIFWAO (IVIFWGO), SVNWAO and SVNHFOWAO i.e., $\oplus_{i=1}^{n}\alpha_i w_i$ ($\otimes_{i=1}^{n}{\alpha_i}^{w_i}$), obtained in Step 4, are not valid.

Furthermore, it is pertinent to mention that any generalized expression of the crisp weighted averaging operator and the crisp weighted geometric operator will not be valid if the generalized expression does not satisfy the monotonicity property.

In this section, with the help of examples, it is shown that the aggregation operators of various extensions of FS, discussed in Section 1.2, are not satisfying the monotonicity property. Therefore, these aggregation operators are not valid.

(1) If a_i and b_i $(i = 1,2,\dots,n)$ are real numbers such that $a_i \le b_i$ for all $i = 1,2,\dots,n$, then the weighted arithmetic mean of a_i, $i = 1,2,\dots,n$ will always be less than or equal to the weighted arithmetic mean of b_i, $i = 1,2,\dots,n$ i.e., $\sum_{i=1}^{n} w_i a_i \le \sum_{i=1}^{n} w_i b_i$.

where, w_i represents the weight corresponding to i^{th} real numbers a_i, b_i and satisfies

the conditions $w_i \ge 0$, $\sum_{i=1}^{n} w_i = 1$.

This property is called the monotonicity property of crisp weighted averaging operator.

(2) If a_i and b_i $(i = 1,2,\dots,n)$ are positive real numbers such that $a_i \le b_i$ for all $i = 1,2,\dots,n$, then the crisp weighted geometric mean of a_i, $i = 1,2,\dots,n$ will

always be less than or equal to the crisp weighted geometric mean of b_i, $i = 1,2, \dots, n$ i.e.,

$$\prod_{i=1}^{n} a_i^{w_i} \leq \prod_{i=1}^{n} b_i^{w_i}.$$

where, w_i represents the weight corresponding to the i^{th} real numbers a_i, b_i and satisfies the conditions $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$.

This property is called the monotonicity property of the crisp aggregation operators (crisp weighted averaging operator and crisp weighted geometric operator).

If in the crisp weighted averaging operator and crisp weighted geometric operator, the real numbers a_i and b_i are replaced with

(i) The IFNs $\alpha_i = \langle \mu_i, \nu_i \rangle$ and $\beta_i = \langle \mu_i', \nu_i' \rangle$ then $\alpha_i \preccurlyeq \beta_i \Rightarrow \oplus_{i=1}^{n} w_i \alpha_i \preccurlyeq \oplus_{i=1}^{n} w_i \beta_i$ is called the monotonicity property for the IFWAO and $\alpha_i \preccurlyeq \beta_i \Rightarrow \otimes_{i=1}^{n} \alpha_i^{w_i} \preccurlyeq \otimes_{i=1}^{n} \beta_i^{w_i}$ is called the monotonicity property for the IFWGO.

(ii) The IFMNs $\alpha_i = \langle \mu_i, \nu_i \rangle$ and $\beta_i = \langle \mu_i', \nu_i' \rangle$ then $\alpha_i \preccurlyeq \beta_i \Rightarrow \oplus_{i=1}^{n} w_i \alpha_i \preccurlyeq \sum_{i=1}^{n} w_i \beta_i$ is called the monotonicity property for the IFMWAO and $\alpha_i \preccurlyeq \beta_i \Rightarrow \otimes_{i=1}^{n} \alpha_i^{w_i} \preccurlyeq \otimes_{i=1}^{n} \beta_i^{w_i}$ is called the monotonicity property for the IFMWGO.

(iii) The IVIFNs $\alpha_i = \langle [\mu_{i1}, \mu_{i2}], [\nu_{i1}, \nu_{i2}] \rangle$ and $\beta_i = \langle [\mu_{i1}', \mu_{i2}'], [\nu_{i1}', \nu_{i2}'] \rangle$ then $\alpha_i \preccurlyeq \beta_i$ $\Rightarrow \oplus_{i=1}^{n} w_i \alpha_i \preccurlyeq \sum_{i=1}^{n} w_i \beta_i$ is called the monotonicity property for the IVIFWAO and $\alpha_i \preccurlyeq \beta_i \Rightarrow \otimes_{i=1}^{n} \alpha_i^{w_i} \preccurlyeq \otimes_{i=1}^{n} \beta_i^{w_i}$ is called the monotonicity property for the IVIFWGO.

(iv) The PFNs $\alpha_i = \langle \mu_i, \nu_i \rangle$ and $\beta_i = \langle \mu_i', \nu_i' \rangle$ then $\alpha_i \preccurlyeq \beta_i \Rightarrow \oplus_{i=1}^{n} w_i \alpha_i \preccurlyeq \sum_{i=1}^{n} w_i \beta_i$ is called the monotonicity property for the PFWAO and $\alpha_i \preccurlyeq \beta_i \Rightarrow \otimes_{i=1}^{n} \alpha_i^{w_i} \preccurlyeq \otimes_{i=1}^{n} \beta_i^{w_i}$ is called the monotonicity property for the PFWGO.

(v) The SVNNs $\alpha_i = \langle T_i, I_i, F_i \rangle$ and $\beta_i = \langle T'_i, I'_i, F'_i \rangle$ then $\alpha_i \preccurlyeq \beta_i \Rightarrow \oplus_{i=1}^{n} w_i \alpha_i \preccurlyeq \sum_{i=1}^{n} w_i \beta_i$ is called the monotonicity property for the SVNWAO and $\alpha_i \preccurlyeq \beta_i \Rightarrow \otimes_{i=1}^{n} \alpha_i^{w_i} \preccurlyeq \otimes_{i=1}^{n} \beta_i^{w_i}$ is called the monotonicity property for the SVNWGO.

2.5.1 Invalidity of IFW aggregation operators

In this section, it is shown that the IFW aggregation operators, discussed in Section 1.2.1, are not satisfying the monotonicity property. Hence, the IFW aggregation operators, discussed in Section 1.2.1, are not valid.

2.5.1.1 Invalidity of Xu and Yager's IFWGO

Xu and Yager [208, Section 3.2, Theorem 6, pp. 425] stated the monotonicity property as follows:

"Let $\alpha_i = [\mu_{\alpha_i}, 1 - \nu_{\alpha_i}]$ and $\beta_i = [\mu_{\beta_i}, 1 - \nu_{\beta_i}]$, $i = 1,2,\dots,n$ be the collections of IFNs. If $\mu_{\alpha_i} \le \mu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$ for all $i = 1,2,\dots,n$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} \preccurlyeq \otimes_{i=1}^{n} \beta_i^{w_i}$".

This statement may be rewritten as:

"Let $\alpha_i = [\mu_{\alpha_i}, 1 - \nu_{\alpha_i}]$ and $\beta_i = [\mu_{\beta_i}, 1 - \nu_{\beta_i}]$, $i = 1,2,\dots,n$ be the collections of IFNs.

(i) If $\mu_{\alpha_i} \le \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ for $i \in S$, $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$ for $i \in T$, $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ for $i \in V$, where, $S \cup T \cup V = \{1,2,\dots,n\}$ and $S \cap T \cap V = \emptyset$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} \prec \otimes_{i=1}^{n} \beta_i^{w_i}$".

(ii) If $\mu_{\alpha_i} = \mu_{\beta_i}$ and $\nu_{\alpha_i} = \nu_{\beta_i}$ for all $i = 1,2,\dots,n$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} = \otimes_{i=1}^{n} \beta_i^{w_i}$.

In actual case, the monotonicity property is stated as follows:

(i) If $\alpha_i \prec \beta_i$ for all $i = 1,2,\dots,n$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} \prec \otimes_{i=1}^{n} \beta_i^{w_i}$".

(ii) If $\alpha_i = \beta_i$ for all $i = 1,2,\dots,n$ Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} = \otimes_{i=1}^{n} \beta_i^{w_i}$".

It is obvious from the above two statements that Xu and Yager [208] have assumed that the relation $\alpha_i \prec \beta_i$ hold if $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ i.e., the IFN $\alpha_i = [\mu_{\alpha_i}, 1 - \nu_{\alpha_i}]$ will be less than or equal to the IFN $\beta_i = [\mu_{\beta_i}, 1 - \nu_{\beta_i}]$ if $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$.

While, as discussed in Section 1.6.1, Xu and Yager [208] have stated that the relation, $\alpha_i \prec \beta_i$ hold if $\mu_{\alpha_i} - \nu_{\alpha_i} < \mu_{\beta_i} - \nu_{\beta_i}$or $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ if $\mu_{\alpha_i} - \nu_{\alpha_i} = \mu_{\beta_i} - \nu_{\beta_i}$ as there exist several IFNs which cannot be compared by considering the relation $\alpha_i \prec \beta_i$ if $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$e.g., if $\alpha_1 = [\mu_{\alpha_1}, 1 - \nu_{\alpha_1}] = [0.1, 0.7]$ and $\beta_1 = [\mu_{\beta_1}, 1 - \nu_{\beta_1}] = [0.2, 0.6]$ i.e., $\mu_{\alpha_1} = 0.1, \mu_{\beta_1} = 0.2, \nu_{\alpha_1} = 0.3$ and $\nu_{\beta_1} = 0.4$. Then, according to the relation, $\alpha_i \prec \beta_i$ if $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ neither $\alpha_1 \prec \beta_1$ nor $\alpha_1 \succ \beta_1$.

The above clearly indicates Xu and Yager [208] have used the following method to state the monotonicity property.

$\alpha_i \prec \beta_i \Rightarrow \mu_{\alpha_i} - \nu_{\alpha_i} < \mu_{\beta_i} - \nu_{\beta_i} \Rightarrow \mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$.

However, the following example clearly indicates that

$\mu_{\alpha_i} - \nu_{\alpha_i} < \mu_{\beta_i} - \nu_{\beta_i} \nRightarrow \mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$.

If $\alpha_1 = \left[\mu_{\alpha_1}, 1-\nu_{\alpha_1}\right] = [0.1, 0.9], \alpha_2 = \left[\mu_{\alpha_2}, 1-\nu_{\alpha_2}\right] = [0.3, 0.8]$ then, $\mu_{\alpha_1} - \nu_{\alpha_1} = 0.1 - 0.1 = 0$ is less than $\mu_{\alpha_2} - \nu_{\alpha_2} = 0.3 - 0.2 = 0.1$. While, $\mu_{\alpha_1} = 0.1$ is less than $\mu_{\alpha_2} = 0.3$ as well as $\nu_{\alpha_1} = 0.1$ is less than $\nu_{\alpha_2} = 0.2$.

The following example also validates that the monotonicity property is not satisfying for Xu and Yager's IFWGO [208]. Hence, Xu and Yager's IFWGO [208] is not valid.

Let $\alpha_1 = [\mu_1, 1-\nu_1] = [0,0.8]$, $\alpha_2 = [\mu_2, 1-\nu_2] = [0.1,0.9]$ and $\beta_1 = [\mu_1', 1-\nu_1'] = [0, 0.9]$ and $\beta_2 = [\mu_2', 1-\nu_2'] = [0.3, 0.8]$ be collection of IFNs.

Then, using the comparing method, discussed in Section 1.6.1,

(i) $\alpha_1 = [\mu_1, 1-\nu_1] = [0,0.8]$ is less than $\beta_1 = [\mu_1', 1-\nu_1'] = [0, 0.9]$ as $\mu_1 - \nu_1 = 0 - 0.2 = -0.2$ is less than $\mu_1' - \nu_1' = 0 - 0.1 = -0.1$.

(ii) $\alpha_2 = [\mu_2, 1-\nu_2] = [0.1,0.9]$ is less than $\beta_2 = [\mu_2', 1-\nu_2'] = [0.3,0.8]$ as $\mu_2 - \nu_2 = 0.1 - 0.9 = -0.8$ is less than $\mu_2' - \nu_2' = 0.3 - 0.8 = -0.5$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $\alpha_1^{w_1} \otimes \alpha_2^{w_2} \prec \beta_1^{w_1} \otimes \beta_2^{w_2}$ should hold.

While, if $w_1 = w_2 = 0.5$, then using the expression (1.1.1.3) i.e., $\otimes_{i=1}^{n} (\alpha_i)^{w_i} = \langle \prod_{i=1}^{n} \mu_i^{w_i}, 1 - \prod_{i=1}^{n} (1-\nu_i)^{w_i}] \rangle = \left[\prod_{i=1}^{n} \mu_i^{w_i}, \prod_{i=1}^{n} (1-\nu_i)^{w_i}\right]$,

$\alpha_1^{w_1} \otimes \alpha_2^{w_2} = \beta_1^{w_1} \otimes \beta_2^{w_2} = [0, (1-0.2)^{0.5}(1-0.1)^{0.5}] = [0, 0.84853]$.

This clearly indicates that for the IFWGO (1.1.1.3), proposed by Xu and Yager [208], the monotonicity property is not satisfying. Hence, the IFWGO (1.1.1.3), proposed by Xu and Yager [208], is not valid.

2.5.1.2 Invalidity of Xu's IFWAO

Xu [204, Theorem 3.5, pp. 1184] stated the monotonicity property as follows:

“Let $\alpha_i = [\mu_{\alpha_i}, 1-\nu_{\alpha_i}]$ and $\beta_i = [\mu_{\beta_i}, 1-\nu_{\beta_i}]$, $i = 1,2,\ldots,n$ be the collections of IFNs. If $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$ for all $i = 1,2,\ldots,n$. Then, $\oplus_{i=1}^{n}(w_i\alpha_i) \preccurlyeq \oplus_{i=1}^{n}(w_i\beta_i)$”.

This statement may be rewritten as:

“Let $\alpha_i = [\mu_{\alpha_i}, 1-\nu_{\alpha_i}]$ and $\beta_i = [\mu_{\beta_i}, 1-\nu_{\beta_i}]$, $i = 1,2,\ldots,n$ be the collections of IFNs.

(i) If $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ for $i \in S$, $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$ for $i \in T$, $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ for $i \in V$, where, $S \cup T \cup V = \{1,2,\ldots,n\}$ and $S \cap T \cap V = \emptyset$. Then, $\oplus_{i=1}^{n}(w_i\alpha_i) \prec \oplus_{i=1}^{n}(w_i\beta_i)$”.

(ii) If $\mu_{\alpha_i} = \mu_{\beta_i}$ and $\nu_{\alpha_i} = \nu_{\beta_i}$ for all $i = 1,2,\ldots,n$. Then, $\oplus_{i=1}^{n}(w_i\alpha_i) = \oplus_{i=1}^{n}(w_i\beta_i)$.

In actual case, the monotonicity property is stated as follows:

(i) If $\alpha_i \prec \beta_i$ for all $i = 1,2,\ldots,n$. Then, $\oplus_{i=1}^{n}(w_i\alpha_i) \prec \oplus_{i=1}^{n}(w_i\beta_i)$”.

(ii) If $\alpha_i = \beta_i$ for all $i = 1,2,\ldots,n$ Then, $\oplus_{i=1}^{n}(w_i\alpha_i) = \oplus_{i=1}^{n}(w_i\beta_i)$”.

It is obvious from the above two statements that Xu [204] has assumed that the relation $\alpha_i \prec \beta_i$ hold if $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ i.e., the IFN $\alpha_i = [\mu_{\alpha_i}, 1-\nu_{\alpha_i}]$ will be less than or equal to the IFN $\beta_i = [\mu_{\beta_i}, 1-\nu_{\beta_i}]$ if $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$.

While, as discussed in Section 1.6.1, Xu [204] has stated that the relation, $\alpha_i \prec \beta_i$ hold if $\mu_{\alpha_i} - \nu_{\alpha_i} < \mu_{\beta_i} - \nu_{\beta_i}$or $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ if $\mu_{\alpha_i} - \nu_{\alpha_i} = \mu_{\beta_i} - \nu_{\beta_i}$ as there exist several IFNs which cannot be compared by considering the relation $\alpha_i \prec \beta_i$ if

$\mu_{\alpha_i} \le \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ e.g., if $\alpha_1 = [\mu_{\alpha_1}, 1 - \nu_{\alpha_1}] = [0.1, 0.7]$ and $\alpha_2 = [\mu_{\alpha_2}, 1 - \nu_{\alpha_2}] = [0.2, 0.6]$ i.e., $\mu_{\alpha_1} = 0.1, \mu_{\alpha_2} = 0.2, \nu_{\alpha_1} = 0.3$ and $\nu_{\alpha_2} = 0.4$. Then, according to the relation, $\alpha_i \prec \beta_i$ if $\mu_{\alpha_i} \le \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ neither $\alpha_i \prec \beta_i$ nor $\alpha_i \succ \beta_i$

The above clearly indicates Xu [204] has used the following method to state the monotonicity property.

$\alpha_i \prec \beta_i \Rightarrow \mu_{\alpha_i} - \nu_{\alpha_i} < \mu_{\beta_i} - \nu_{\beta_i} \Rightarrow \mu_{\alpha_i} \le \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$.

However, the following example clearly indicates that

$\mu_{\alpha_i} - \nu_{\alpha_i} < \mu_{\beta_i} - \nu_{\beta_i} \nRightarrow \mu_{\alpha_i} \le \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$.

If $\alpha_1 = [\mu_{\alpha_1}, 1 - \nu_{\alpha_1}] = [0.1, 0.9], \alpha_2 = [\mu_{\alpha_2}, 1 - \nu_{\alpha_2}] = [0.3, 0.8]$ then, $\mu_{\alpha_1} - \nu_{\alpha_1} = 0.1 - 0.1 = 0$ is less than $\mu_{\alpha_2} - \nu_{\alpha_2} = 0.3 - 0.2 = 0.1$. While, $\mu_{\alpha_1} = 0.1$ is less than $\mu_{\alpha_2} = 0.3$ as well as $\nu_{\alpha_1} = 0.1$ is less than $\nu_{\alpha_2} = 0.2$.

The following example also validates that the monotonicity property is not satisfying for Xu's IFWAO [204]. Hence, Xu's IFWAO [204] is not valid.

Let $\alpha_1 = [\mu_1, 1 - \nu_1] = [0.2,1]$, $\alpha_2 = [\mu_2, 1 - \nu_2] = [0.4,0.6]$ and $\beta_1 = [\mu_1', 1 - \nu_1'] = [0.4, 0.9]$ and $\beta_2 = [\mu_2', 1 - \nu_2'] = [0.2, 1]$ be collection of IFNs. Furthermore, let $w_1 = w_2 = 0.5$.

Then, using the comparing method, discussed in Section 1.6.2,

(i) $\alpha_1 = [\mu_1, 1 - \nu_1] = [0.2,1]$ is less than $\beta_1 = [\mu_1', 1 - \nu_1'] = [0.4, 0.9]$ as $\mu_1 - \nu_1 = 0.2 - 0 = 0.2$ is less than $\mu_1' - \nu_1' = 0.4 - 0.1 = 0.3$.

(ii) $\alpha_2 = [\mu_2, 1 - \nu_2] = [0.4,0.6]$ is less than $\beta_2 = [\mu_2', 1 - \nu_2'] = [0.2, 1]$ as $\mu_2 - \nu_2 = 0.4 - 0.4 = 0$ is less than $\mu_2' - \nu_2' = 0.2 - 0.0 = 0.2$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$ should hold.

While, using the expression (1.2.2.3) i.e., $\oplus_{i=1}^{n} (w_i\alpha_i) = \langle 1 - \prod_{i=1}^{n}(1 - \mu_i)^{w_i}, \prod_{i=1}^{n} \nu_i^{w_i} \rangle = \left[1 - \prod_{i=1}^{n}(1 - \mu_i)^{w_i}, 1 - \prod_{i=1}^{n} \nu_i^{w_i}\right]$,

$w_1\alpha_1 \oplus w_2\alpha_2 = w_1\beta_1 \oplus w_2\beta_2 = [1 - (1 - 0.2)^{0.5}(1 - 0.4)^{0.5}, 1] =$

$[1 - (0.8)^{0.5}(0.6)^{0.5}, 1] = [0.30718, 1]$.

This clearly indicates that for the IFWAO (1.1.2.3), proposed by Xu [204], the monotonicity property is not satisfying. Hence, the IFWAO (1.1.1.3), proposed by Xu [204], is not valid.

2.5.1.3 Invalidity of He et al.'s IFWGO

He et al. [88, Section 4.2, Theorem 7, pp. 149] has stated the monotonicity property as follows:

"Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ and $\beta_i = \langle \mu_{\beta_i}, 1 - \nu_{\beta_i} \rangle$, $i = 1,2,\ldots,n$ be collections of IFNs. If $\mu_{\alpha_i} + \nu_{\alpha_i} \le \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$ for all $i = 1,2,\ldots,n$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} \preccurlyeq \otimes_{i=1}^{n} \beta_i^{w_i}$".

This statement may be rewritten as:

"Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ and $\beta_i = \langle \mu_{\beta_i}, 1 - \nu_{\beta_i} \rangle$, $i = 1,2,\ldots,n$ be collections of IFNs.

(i) If $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ for $i \in S$, $\mu_{\alpha_i} + \nu_{\alpha_i} \le \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ for $i \in T$, $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$ for $i \in V$, where, $S \cup T \cup V = \{1,2,\ldots,n\}$ and $S \cap T \cap V = \emptyset$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} \prec \otimes_{i=1}^{n} \beta_i^{w_i}$".

(ii) If $\mu_{\alpha_i} + \nu_{\alpha_i} = \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} = \nu_{\beta_i}$ for all $i = 1,2,\ldots,n$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} = \otimes_{i=1}^{n} \beta_i^{w_i}$".

In actual case, the monotonicity property is stated as follows:

(i) If $\alpha_i \prec \beta_i$ for all $i = 1,2,\dots,n$. Then, $\otimes_{i=1}^{n}\alpha_i^{w_i} \prec \otimes_{i=1}^{n}\beta_i^{w_i}$".

(ii) If $\alpha_i = \beta_i$ for all $i = 1,2,\dots,n$ Then, $\otimes_{i=1}^{n}\alpha_i^{w_i} = \otimes_{i=1}^{n}\beta_i^{w_i}$".

It is obvious from the above two statements that He et al. [88] have assumed that the relation $\alpha_i \prec \beta_i$ hold if $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} \le \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$and $\nu_{\alpha_i} \ge \nu_{\beta_i}$ i.e., the IFN $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ will be less than the IFN $\beta_i = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle$ if $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} \le \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$and $\nu_{\alpha_i} \ge \nu_{\beta_i}$.

While, as discussed in Section 1.6.3, He et al. [88] have stated that the relation, $\alpha_i \prec \beta_i$ hold if $\mu_{\alpha_i} - \nu_{\alpha_i} < \mu_{\beta_i} - \nu_{\beta_i}$or $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ if $\mu_{\alpha_i} - \nu_{\alpha_i} = \mu_{\beta_i} - \nu_{\beta_i}$ as there exist several IFNs which cannot be compared by considering the relation $\alpha_i \prec \beta_i$ if $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} \le \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$and $\nu_{\alpha_i} \ge \nu_{\beta_i}$. e.g., if $\alpha_1 = \langle \mu_{\alpha_1}, \nu_{\alpha_1} \rangle = \langle 0.1, 0.3 \rangle$ and $\alpha_2 = \langle \mu_{\alpha_2}, \nu_{\alpha_2} \rangle = \langle 0.2, 0.4 \rangle$ i.e., $\mu_{\alpha_1} = 0.1, \mu_{\alpha_2} = 0.2, \nu_{\alpha_1} = 0.3$ and $\nu_{\alpha_2} = 0.4$. Then, according to the relation, $\alpha_i \prec \beta_i$ if $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} \le \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$and $\nu_{\alpha_i} \ge \nu_{\beta_i}$ neither $\alpha_1 \prec \beta_1$ nor $\alpha_1 \succ \beta_1$.

The above clearly indicates He et al. [88] have used the following method to state the monotonicity property.

$\alpha_i \prec \beta_i \Rightarrow \mu_{\alpha_i} - \nu_{\alpha_i} < \mu_{\beta_i} - \nu_{\beta_i} \Rightarrow \mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} \le \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$and $\nu_{\alpha_i} \ge \nu_{\beta_i}$.

However, the following example clearly indicates that

$\mu_{\alpha_i} - \nu_{\alpha_i} < \mu_{\beta_i} - \nu_{\beta_i} \not\Rightarrow \mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} \leq \mu_{\beta_i} + \nu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} + \nu_{\alpha_i} < \mu_{\beta_i} + \nu_{\beta_i}$and $\nu_{\alpha_i} \geq \nu_{\beta_i}$.

If $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.1, 0.1 \rangle, \alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.3\ , 0.2 \rangle$ then, $\mu_1 - \nu_1 = 0.1 - 0.1 = 0$ is less than $\mu_2 - \nu_2 = 0.3 - 0.2 = 0.1$. While, $\mu_1 + \nu_1 = 0.1 + 0.1 = 0.2$ is less than $\mu_2 + \nu_2 = 0.3 + 0.2 = 0.5$ as well as $\nu_1 = 0.1$ is less than $\nu_2 = 0.2$.

The following example also validates that the monotonicity property is not satisfying for He et al.'s IFWGO [88]. Hence, He et al.'s IFWGO [88] is not valid.

Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.7, 0.3 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.4, 0.2 \rangle$ and $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.8, 0.2 \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.6, 0.3 \rangle$ be collection of IFNs. Furthermore, let $w_1 = w_2 = 0.5$.

Then, using the comparing method, discussed in Section 1.6.3,

(i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.7, 0.3 \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.8, 0.2 \rangle$ as $\mu_1 - \nu_1 = 0.7 - 0.3 = 0.4$ is less than $\mu_1' - \nu_1' = 0.8 - 0.2 = 0.6$.

(ii) $\alpha_2 = \alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.4, 0.2 \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.6, 0.3 \rangle$ as $\mu_2 - \nu_2 = 0.4 - 0.2 = 0.2$ is less than $\mu_2' - \nu_2' = 0.6 - 0.3 = 0.3$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $\alpha_1^{w_1} \otimes \alpha_2^{w_2} \prec \beta_1^{w_1} \otimes \beta_2^{w_2}$ should hold.

While, using the expression (1.2.3.3) i.e., $\otimes_{i=1}^{n} (\alpha_i)^{w_i} = \langle \prod_{i=1}^{n} (1 - \nu_i)^{w_i} - \prod_{i=1}^{n} \left(1 - (\mu_i + \nu_i)\right)^{w_i}, 1 - \prod_{i=1}^{n} (1 - \nu_i)^{w_i} \rangle$

$$\alpha_1^{w_1} \otimes \alpha_2^{w_2} = \beta_1^{w_1} \otimes \beta_2^{w_2} = \langle (1 - 0.3)^{0.5} (1 - 0.2)^{0.5}, 1 - (1 - 0.3)^{0.5} (1 - 0.2)^{0.5} \rangle =$$

$$\langle 0.74833, 0.25166 \rangle.$$

This clearly indicates that for the IFWGO (1.2.3.3), proposed by He et al. [88], the monotonicity property is not satisfying. Hence, the IFWGO (1.2.3.3), proposed by He et al. [88], is not valid.

2.5.1.4 Invalidity of Yu's IFWAO

Yu [223] has not stated the monotonicity property. However, the following example also validates that the monotonicity property is not satisfying for Yu's IFWAO [223]. Hence, Yu's IFWAO [223] is not valid.

Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.2, 0.8 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.3, 0.6 \rangle$ and $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.3, 0.7 \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.2, 0.4 \rangle$ be collection of IFNs.

Then, using the comparing method, discussed in Section 1.6.3,

(i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.2, 0.8 \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.3, 0.7 \rangle$ as $\mu_1 - \nu_1 = 0.2 - 0.8 = -0.6$ is less than $\mu_1' - \nu_1' = 0.3 - 0.7 = 0.4$.

(ii) $\alpha_2 = \alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.3, 0.6 \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.2, 0.4 \rangle$ as $\mu_2 - \nu_2 = 0.3 - 0.6 = -0.3$ is less than $\mu_2' - \nu_2' = 0.2 - 0.4 = -0.2$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$ should hold.

While, if $w_1 = w_2 = 0.5$ then using the expression (1.2.4.3) i.e., $\oplus_{i=1}^{n} w_i\alpha_i = \langle 1 - \prod_{i=1}^{n}(1-\mu_i)^{w_i}, \prod_{i=1}^{n}(1-\mu_i)^{w_i} - \prod_{i=1}^{n}\left(1-(\mu_i+\nu_i)\right)^{w_i} \rangle$,

$$w_1\alpha_1 \oplus w_2\alpha_2 = w_1\beta_1 \oplus w_2\beta_2 = \langle 1-(1-0.2)^{0.5}(1-0.3)^{0.5}, (1-0.2)^{0.5}(1-0.3)^{0.5} \rangle = \langle 0.25167, 0.74833 \rangle$$

This clearly indicates that for the IFWAO (1.2.4.3), proposed by Yu [223], the monotonicity property is not satisfying. Hence, the IFWGO (1.2.4.3), proposed by Yu [223], is not valid.

2.5.1.5 Invalidity of Chen and Cheng's IFWGO

Chen and Chang [28] has not stated the monotonicity property. However, the following example validates that the monotonicity property is not satisfying for Chen and Chang's IFWGO [28]. Hence, Chen and Chang's IFWGO [28] is not valid.

Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.2,0.8 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.3,0.6 \rangle$ and $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.3, 0.7 \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.2, 0.4 \rangle$ be collection of IFNs.

Then, using the comparing method, discussed in Section 1.6.5,

(i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.2,0.8 \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.3, 0.7 \rangle$ as $\mu_1 - \nu_1 = 0.2 - 0.8 = -0.6$ is less than $\mu_1' - \nu_1' = 0.3 - 0.7 = 0.4$.

(ii) $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.3,0.6 \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.2, 0.4 \rangle$ as $\mu_2 - \nu_2 = 0.3 - 0.6 = -0.3$ is less than $\mu_2' - \nu_2' = 0.2 - 0.4 = -0.2$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $\alpha_1^{w_1} \otimes \alpha_2^{w_2} \prec \beta_1^{w_1} \otimes \beta_2^{w_2}$ should hold.

While, if $w_1 = w_2 = 0.5$ then using the expression (1.1.5.3) i.e., $\otimes_{i=1}^{n} (\alpha_i)^{w_i} = \langle 1 - \prod_{i=1}^{n} (1 - \mu_i)^{w_i}, \prod_{i=1}^{n} (1 - \mu_i)^{w_i} - \prod_{i=1}^{n} \left(1 - (\mu_i + \nu_i)\right)^{w_i} \rangle$,

$$\alpha_1^{w_1} \otimes \alpha_2^{w_2} = \beta_1^{w_1} \otimes \beta_2^{w_2} = \langle 1 - (1 - 0.2)^{0.5} (1 - 0.3)^{0.5}, (1 - 0.2)^{0.5} (1 - 0.3)^{0.5} \rangle = \langle 0.25166, 0.74833 \rangle.$$

This clearly indicates that for the IFWGO (1.2.5.3), proposed by Chen and Chang [28], the monotonicity property is not satisfying. Hence, the IFWGO (1.2.5.3), proposed by Chen and Chang [28], is not valid.

2.5.1.6 Invalidity of Garg's IFWAO

Garg [65, Section 3, Property 3.3, pp. 169] has stated the monotonicity property but not proved it. However, the following example validates that the monotonicity property is not satisfying for Garg's IFWAO [65]. Hence, Garg's IFWAO [65] is not valid.

Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.5, 0.5 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.6, 0.3 \rangle$, $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.6, 0.4 \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.5, 0.1 \rangle$ be four IFNs. Then, using the comparing method, discussed in Section 1.5.1,

(i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.5, 0.5 \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.6, 0.4 \rangle$ as $\mu_1 - \nu_1 = 0.5 - 0.5 = 0.0$ is less than $\mu_1' - \nu_1' = 0.6 - 0.4 = 0.2$.

(ii) $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.6, 0.3 \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.5, 0.1 \rangle$ as $\mu_1 - \nu_1 = 0.6 - 0.3 = 0.3$ is less than to $\mu_1' - \nu_1' = 0.5 - 0.1 = 0.4$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$ should hold.

While, if $w_1 = w_2 = 0.5$ then using the expression (1.2.6.3) i.e., $\oplus_{i=1}^{n} (w_i\alpha_i) =$

$$\left\langle \frac{\prod_{i=1}^{n}(1+\mu_i)^{w_i} - \prod_{i=1}^{n}(1-\mu_i)^{w_i}}{\prod_{i=1}^{n}(1+\mu_i)^{w_i} + \prod_{i=1}^{n}(1-\mu_i)^{w_i}}, \frac{2\left\{\prod_{i=1}^{n}(1-\mu_i)^{w_i} - \prod_{i=1}^{n}(1-\mu_i-\nu_i)^{w_i}\right\}}{\prod_{i=1}^{n}(1+\mu_i)^{w_i} + \prod_{i=1}^{n}(1-\mu_i)^{w_i}} \right\rangle,$$

$w_1\alpha_1 \oplus w_2\alpha_2 = w_1\beta_1 \oplus w_2\beta_2 =$

$$\left\langle \frac{(1+0.5)^{0.5}(1+0.6)^{0.5} - (1-0.5)^{0.5}(1-0.6)^{0.5}}{(1+0.5)^{0.5}(1+0.6)^{0.5} + (1-0.5)^{0.5}(1-0.6)^{0.5}}, \frac{2\left\{(1-0.5)^{0.5}(1-0.6)^{0.5}\right\}}{(1+0.5)^{0.5}(1+0.6)^{0.5} + (1-0.5)^{0.5}(1-0.6)^{0.5}} \right\rangle =$$

$$\langle 0.55198, 0.44802 \rangle.$$

This clearly indicates that for IFWAO, proposed by Garg [65], the monotonicity property is not satisfying. Hence, the IFWAO, proposed by Garg [65], is not valid.

2.5.1.7 Invalidity of Garg's IFWGO

Garg [60] has not stated the monotonicity property. However, the following example validates that the monotonicity property is not satisfying for Garg's IFWGO [60]. Hence, Garg's IFWGO [60] is not valid.

Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.5, 0.5 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.1, 0.4 \rangle$, $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.6, 0.4 \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.3, 0.5 \rangle$ be four IFNs. Then, using the comparing method, discussed in Section 1.6.1,

(i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.5, 0.5 \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.6, 0.4 \rangle$ as $\mu_1 - \nu_1 = 0.5 - 0.5 = 0.0$ is less than $\mu_1' - \nu_1' = 0.6 - 0.4 = 0.2$.

(ii) $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.1, 0.4 \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.3, 0.5 \rangle$ as $\mu_2 - \nu_2 = 0.1 - 0.4 = -0.3$ is less than $\mu_2' - \nu_2' = 0.3 - 0.5 = -0.2$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $\alpha_1^{w_1} \otimes \alpha_2^{w_2} \prec \beta_1^{w_1} \otimes \beta_2^{w_2}$ should hold.

While, if $w_1 = w_2 = 0.5$ then using the expression (1.2.7.3) i.e., $\otimes_{i=1}^{n} {\alpha_i}^{w_i} =$

$$\left\langle \frac{2\{\prod_{i=1}^{n}(1-\nu_i)^{w_i} - \prod_{i=1}^{n}(1-\mu_i-\nu_i)^{w_i}\}}{\prod_{i=1}^{n}(1+\nu_i)^{w_i} + \prod_{i=1}^{n}(1-\nu_i)^{w_i}}, \frac{\prod_{i=1}^{n}(1+\nu_i)^{w_i} - \prod_{i=1}^{n}(1-\nu_i)^{w_i}}{\prod_{i=1}^{n}(1+\nu_i)^{w_i} + \prod_{i=1}^{n}(1-\nu_i)^{w_i}} \right\rangle$$

$$\alpha_1^{w_1} \otimes \alpha_2^{w_2} = \beta_1^{w_1} \otimes \beta_2^{w_2} =$$

$$\left\langle \frac{2\{(1-0.4)^{0.5}(1-0.5)^{0.5}\}}{(1+0.4)^{0.5}(1+0.5)^{0.5} + (1-0.4)^{0.5}(1-0.5)^{0.5}}, \frac{(1+0.4)^{0.5}(1+0.5)^{0.5} - (1-0.4)^{0.5}(1-0.5)^{0.5}}{(1+0.4)^{0.5}(1+0.5)^{0.5} + (1-0.4)^{0.5}(1-0.5)^{0.5}} \right\rangle =$$

$$\langle 0.54858, 0.45142 \rangle.$$

This clearly indicates that for IFWGO, proposed by Garg [60], the monotonicity property is not satisfying. Hence, the IFWGO, proposed by Garg [60], is not valid.

2.5.1.8 Invalidity of IFMW aggregation operators

In this section, it is shown that the IFMW aggregation operators [56, 66, 156], discussed in Section 1.2.7, are not valid.

2.5.1.8.1 Invalidity of Garg's IFMWGO

In Section 2.1.8.1, it is shown that the multiplication of two IFMNs is not necessarily an IFMN i.e., Garg's expression (1.2.8.1.1) to evaluate the multiplication of two IFMNs is not valid. Since, the IFMWGO (1.2.8.1.3) is fully based upon the expression (1.2.8.1.1). Therefore, the the IFMWGO (1.2.8.1.3) is also not valid.

2.5.1.8.2 Invalidity of Garg's IFMWAO

In Section 2.1.8.2, it is shown that the sum of two IFMNs is not necessarily an IFMN i.e., Garg's expression (1.2.8.2.1) to evaluate the multiplication of two IFMNs is not valid. Since, the IFMWGO (1.2.8.2.3) is fully based upon the expression (1.2.8.2.1). Therefore, the IFMWGO (1.2.8.2.3) is also not valid.

2.5.1.8.3 Invalidity of the IMWAO and the IMWGO proposed by Qian and Niu

Qian and Niu [156, Property 4.2, pp. 2864] has stated the monotonicity property as follows:

"Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ and $\beta_i = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle$, $i = 1,2,\dots,n$ be collections of IFMNs. If $\mu_{\alpha_i} \le \mu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$ for all $i = 1,2,\dots,n$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} \preccurlyeq \otimes_{i=1}^{n} \beta_i^{w_i}$".

This statement may be rewritten as:

"Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ and $\beta_i = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle$, $i = 1,2,\dots,n$ be collections of IFMNs

(i) If $\mu_{\alpha_i} \le \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ for $i \in S$, $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$ for $i \in T$, $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ for $i \in V$, where, $S \cup T \cup V = \{1,2,\dots,n\}$ and $S \cap T \cap V = \emptyset$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} \prec \otimes_{i=1}^{n} \beta_i^{w_i}$".

(ii) If $\mu_{\alpha_i} = \mu_{\beta_i}$ and $\nu_{\alpha_i} = \nu_{\beta_i}$ for all $i = 1,2,\dots,n$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} = \otimes_{i=1}^{n} \beta_i^{w_i}$".

In actual case, the monotonicity property is stated as follows:

(i) If $\alpha_i \prec \beta_i$ for all $i = 1,2,\dots,n$. Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} \prec \otimes_{i=1}^{n} \beta_i^{w_i}$".

(ii) If $\alpha_i = \beta_i$ for all $i = 1,2,\dots,n$ Then, $\otimes_{i=1}^{n} \alpha_i^{w_i} = \otimes_{i=1}^{n} \beta_i^{w_i}$".

It is obvious from the above two statements that Qian and Niu[156] have assumed that the relation $\alpha_i \prec \beta_i$ hold if $\mu_{\alpha_i} \le \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \ge \nu_{\beta_i}$ or

$\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ i.e., the IFMN $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ will be less than or equal to the IFMN $\beta_i = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle$ if $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$.

While, as discussed in Section 1.6.11, Qian and Niu [156] have stated that the relation, $\alpha_i \prec \beta_i$ hold if $\frac{\mu_{\alpha_i}}{\nu_{\alpha_i}} < \frac{\mu_{\beta_i}}{\nu_{\beta_i}}$ as there exist several IFMNs which cannot be compared by considering the relation $\alpha_i \prec \beta_i$ if $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ e.g., if $\alpha_1 = \langle \mu_{\alpha_1}, \nu_{\alpha_1} \rangle = \langle 0.1, 0.3 \rangle$ and $\beta_1 = \langle \mu_{\beta_1}, \nu_{\beta_1} \rangle = \langle 0.2, 0.4 \rangle$ i.e., $\mu_{\alpha_1} = 0.1, \mu_{\beta_1} = 0.2, \nu_{\alpha_1} = 0.3$ and $\nu_{\beta_1} = 0.4$. Then, according to the relation, $\alpha_i \prec \beta_i$ if $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$, neither $\alpha_1 \prec \beta_1$ nor $\alpha_1 \succ \beta_1$.

The above clearly indicates Qian and Niu [156] have used the following method to state the monotonicity property.

$\alpha_i \prec \beta_i \Rightarrow \frac{\mu_{\alpha_i}}{\nu_{\alpha_i}} < \frac{\mu_{\beta_i}}{\nu_{\beta_i}} \Rightarrow \mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$.

However, the following example clearly indicates that

$\frac{\mu_{\alpha_i}}{\nu_{\alpha_i}} < \frac{\mu_{\beta_i}}{\nu_{\beta_i}} \nRightarrow \mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$ or $\mu_{\alpha_i} < \mu_{\beta_i}$ and $\nu_{\alpha_i} > \nu_{\beta_i}$.

If $\alpha_1 = \langle \mu_{\alpha_1}, \nu_{\alpha_1} \rangle = \left\langle \frac{1}{4}, 1 \right\rangle, \beta_1 = \langle \mu_{\beta_1}, \nu_{\beta_1} \rangle = \left\langle \frac{1}{6}, \frac{1}{2} \right\rangle$ then, $\frac{\mu_{\alpha_1}}{\nu_{\alpha_1}} = \frac{1}{4}$ is less than $\frac{\mu_{\beta_1}}{\nu_{\beta_1}} = \frac{1}{3}$. While, $\mu_{\alpha_1} = \frac{1}{4}$ is greater than $\mu_{\beta_1} = \frac{1}{6}$ as well as $\nu_{\alpha_1} = 1$ is greater than $\nu_{\beta_1} = \frac{1}{2}$.

Therefore, the statement and hence the proof of the monotonicity property, proposed by Qian and Niu [156, Sec. 4, Property 4.2, pp. 2864], is not valid.

2.5.2 Invalidity of PFW aggregation operators

In this section, with the help of examples, it is shown that the PFW aggregation operators [58, 62, 63, 67, 135], discussed in Section 1.3, are not satisfying the monotonicity property. Hence, these aggregation operators are not valid.

2.5.2.1 Invalidity of Garg's PFWAO

Garg [58, Property 4.1, pp. 904] has stated the monotonicity property but not proved it. However, the following example clearly indicates that the monotonicity property is not satisfying for Garg's PFWAO (1.3.1.3). Hence, Garg's PFWAO (1.3.1.3) is not valid.

Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.1, 0.0 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.3, 0.4 \rangle$, $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.3, 0.0 \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.1, 0.2 \rangle$ be four PFNs. Then, using the comparing method, discussed in Section 1.6.9,

(i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.1, 0.0 \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.3, 0.0 \rangle$ as $\mu_1^2 - \nu_1^2 = 0.01 - 0.0 = 0.01$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.09 - 0.0 = 0.09$.

(ii) $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.3, 0.4 \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.1, 0.2 \rangle$ as $\mu_2^2 - \nu_2^2 = 0.09 - 0.16 = -0.07$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.01 - 0.04 = -0.03$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$ should hold. While, if $w_1 = w_2 = 0.5$ then using the expression (1.3.1.3) i.e.,

$$\oplus_{i=1}^{n} w_i\alpha_i = \left\langle \sqrt{\frac{\prod_{i=1}^{n}(1+\mu_i^2)^{w_i} - \prod_{i=1}^{n}(1-\mu_i^2)^{w_i}}{\prod_{i=1}^{n}(1+\mu_i^2)^{w_i} + \prod_{i=1}^{n}(1-\mu_i^2)^{w_i}}}, \frac{\sqrt{2}\prod_{i=1}^{n}\nu_i^{w_i}}{\sqrt{\prod_{i=1}^{n}(2-\nu_i^2)^{w_i} + \prod_{i=1}^{n}(\nu_i^2)^{w_i}}} \right\rangle$$

$$w_1\alpha_1 \oplus w_2\alpha_2 = w_1\beta_1 \oplus w_2\beta_2 = \langle \sqrt{\frac{((1+0.1)^2(1+0.3)^2)^{0.5}-((1-0.1)^2(1-0.3)^2)^{0.5}}{((1+0.1)^2(1+0.3)^2)^{0.5}+((1-0.1)^2(1-0.3)^2)^{0.5}}},\ 0\rangle =$$

$$\langle \sqrt{0.050080}, 0\rangle.$$

This clearly indicates that for PFWAO, proposed by Garg [58], the monotonicity property is not satisfying. Hence, the PFWAO, proposed by Garg [58], is not valid.

2.5.2.2 Invalidity of Garg's PFWGO

Garg [63, Property 3.1, (iii), pp. 614] has stated the monotonicity property but not proved it. However, the following example clearly indicates that the monotonicity property is not satisfying for Garg's PFWGO (1.3.1.3). Hence, Garg's PFWGO (1.3.1.3) is not valid.

Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.0, 0.2 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.4, 0.1 \rangle$, $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.0, 0.1 \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.5, 0.2 \rangle$ be four PFNs. Then, using the comparing method, discussed in Section 1.6.9,

(i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.0, 0.2 \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.0, 0.1 \rangle$ as $\mu_1^2 - \nu_1^2 = 0.0 - 0.04 = -0.04$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.0 - 0.01 = -0.01$.

(ii) $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.4, 0.1 \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.5, 0.2 \rangle$ as $\mu_2^2 - \nu_2^2 = 0.16 - 0.01 = 0.15$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.25 - 0.04 = 0.21$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $\alpha_1^{w_1} \otimes \alpha_2^{w_2} \prec \beta_1^{w_1} \otimes \beta_2^{w_2}$ should hold. While, if $w_1 = w_2 = 0.5$ then using the expression (1.3.2.3) i.e., $\otimes_{i=1}^{n} {\alpha_i}^{w_i} = \left\langle \frac{\sqrt{2}\prod_{i=1}^{n}\mu_i^{w_i}}{\sqrt{\prod_{i=1}^{n}(2-\mu_i^2)^{w_i}+\prod_{i=1}^{n}(\mu_i^2)^{w_i}}},\ \sqrt{\frac{\prod_{i=1}^{n}(1+\nu_i^2)^{w_i}-\prod_{i=1}^{n}(1-\nu_i^2)^{w_i}}{\prod_{i=1}^{n}(1+\nu_i^2)^{w_i}+\prod_{i=1}^{n}(1-\nu_i^2)^{w_i}}} \right\rangle$,

$$\alpha_1^{w_1} \otimes \alpha_2^{w_2} = \beta_1^{w_1} \otimes \beta_2^{w_2} = \langle 0, \sqrt{\frac{((1+0.1)^2(1+0.2)^2)^{0.5}-((1-0.1)^2(1-0.2)^2)^{0.5}}{((1+0.1)^2(1+0.2)^2)^{0.5}+((1-0.1)^2(1-0.2)^2)^{0.5}}}\rangle =$$

$$\langle 0, \sqrt{0.025006}\rangle.$$

This clearly indicates that for PFWGO, proposed by Garg [63], the monotonicity property is not satisfying. Hence, the PFWGO, proposed by Garg [63], is not valid.

2.5.2.3 Invalidity of Garg’s PFWGO

Garg [58, Property 4.1, (iii), pp. 904] has stated the monotonicity property but not proved it. However, the following example clearly indicates that the monotonicity property is not satisfying for Garg’s PFWGO (1.3.1.3). Hence, Garg’s PFWGO (1.3.1.3) is not valid.

Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle \sqrt{0.25}, \sqrt{0.75} \rangle$, $\alpha_2 = \langle 0.0, \sqrt{0.65} \rangle$, $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle \sqrt{0.35}, \sqrt{0.65} \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle \sqrt{0.25}, \sqrt{0.75} \rangle$ be four PFNs. Then, using the comparing method, discussed in Section 1.6.9,

(i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle \sqrt{0.25}, \sqrt{0.75} \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle \sqrt{0.35}, \sqrt{0.65} \rangle$ as $\mu_1^2 - \nu_1^2 = 0.25 - 0.75 = -0.50$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.35 - 0.65 = -0.30$.

(ii) $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.0, \sqrt{0.65} \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle \sqrt{0.25}, \sqrt{0.75} \rangle$ as $\mu_2^2 - \nu_2^2 = 0 - 0.65 = -0.65$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.25 - 0.75 = -0.50$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $\alpha_1^{w_1} \otimes \alpha_2^{w_2} \prec \beta_1^{w_1} \otimes \beta_2^{w_2}$ should hold. While, if $w_1 = w_2 = 0.5$ then using the expression (1.3.3.3) i.e. $\otimes_{i=1}^{n} {\alpha_i}^{w_i} = \left\langle \frac{2\{\prod_{i=1}^{n}(1-\nu_i^2)^{w_i} - \prod_{i=1}^{n}(1-\mu_i^2-\nu_i^2)^{w_i}\}}{\prod_{i=1}^{n}(1+\nu_i^2)^{w_i} + \prod_{i=1}^{n}(1-\nu_i^2)^{w_i}}, \frac{\prod_{i=1}^{n}(1+\nu_i^2)^{w_i} - \prod_{i=1}^{n}(1-\nu_i^2)^{w_i}}{\prod_{i=1}^{n}(1+\nu_i^2)^{w_i} + \prod_{i=1}^{n}(1-\nu_i^2)^{w_i}} \right\rangle$,

$\alpha_1^{w_1} \otimes \alpha_2^{w_2} = \beta_1^{w_1} \otimes \beta_2^{w_2} =$

$\left\langle \frac{2\{(1-0.75)^{0.5}(1-0.65)^{0.5}\}-0}{(1+0.75)^{0.5}(1+0.65)^{0.5}+(1-0.75)^{0.5}(1-0.65)^{0.5}}, \frac{(1+0.75)^{0.5}(1+0.65)^{0.5}-(1-0.75)^{0.5}(1-0.65)^{0.5}}{(1+0.75)^{0.5}(1+0.65)^{0.5}+(1-0.75)^{0.5}(1-0.65)^{0.5}} \right\rangle =$

$\langle \sqrt{0.29564}, \sqrt{0.70346} \rangle$

This clearly indicates that for PFWAO (1.3.3.3), proposed by Garg [58], the monotonicity property is not satisfying. Hence, the PFWAO (1.3.3.3), proposed by Garg, [58], is not valid.

2.5.2.4 Invalidity of Garg's PFWAO and PFWGO

Garg [62, Section 3.1, Property 3, pp. 555] has stated the monotonicity property but not proved it. However, the following example clearly indicates that the monotonicity property is neither satisfying for Garg's PFWAO (1.3.4.1) nor for Garg's PFWGO (1.3.4.2). Hence, Garg's PFWAO (1.2.4.1) and PFWGO (1.3.4.2) are not valid.

1. Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.1, 0.0 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.2, 0.6 \rangle$, $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.2, 0.0 \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.1, 0.4 \rangle$ be four PFNs. Then, using the comparing method, discussed in Section 1.6.9,

   (i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.1, 0.0 \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.2, 0.0 \rangle$ as $\mu_1^2 - \nu_1^2 = 0.01 - 0.0 = 0.01$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.04 - 0.0 = 0.04$.

   (ii) $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.2, 0.6 \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.1, 0.4 \rangle$ as $\mu_2^2 - \nu_2^2 = 0.04 - 0.36 = -0.32$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.01 - 0.16 = -0.15$.

   Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$ should hold. While, if $\eta_1 = \eta_2 = w_1 = w_2 = 0.5$ then using the expression (1.2.4.1) i.e., $\oplus_{j=1}^{n} w_j(\eta_j\alpha_j) = \left\langle \sqrt{1 - \prod_{j=1}^{n}(1 - \mu_j^2)^{\eta_j w_j}}, \prod_{j=1}^{n}(\nu_j)^{\eta_j w_j} \right\rangle$, $w_1\alpha_1 \oplus w_2\alpha_2 = w_1\beta_1 \oplus w_2\beta_2 = \langle \sqrt{1 - (1 - (0.2)^2)^{0.25}(1 - (0.1)^2)^{0.25}}, 0 \rangle = \langle 0.11242, 0 \rangle$.

   This clearly indicates that for PFWAO (1.3.4.1), proposed by Garg, the monotonicity property is not satisfying. Hence, the PFWAO (1.3.4.1), proposed by Garg [62], is not valid.

Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.0,0.2 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.4,0.1 \rangle$, $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.0, 0.1 \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.6, 0.2 \rangle$ be four PFNs. Then, using the comparing method, discussed in Section 1.6.9,

(i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.0,0.2 \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.0, 0.1 \rangle$ as $\mu_1^2 - \nu_1^2 = 0.0 - 0.04 = -0.04$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.0 - 0.01 = -0.01$.

(ii) $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.4,0.1 \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.6, 0.2 \rangle$ as $\mu_2^2 - \nu_2^2 = 0.16 - 0.01 = 0.15$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.36 - 0.04 = 0.32$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $\alpha_1^{w_1} \otimes \alpha_2^{w_2} \prec \beta_1^{w_1} \otimes \beta_2^{w_2}$ should hold.

While, if $\eta_1 = \eta_2 = w_1 = w_2 = 0.5$ then using the expression (1.2.4.2) i.e., $\otimes_{j=1}^{n} (\alpha_j^{\eta_j})^{w_j} = \left\langle \prod_{j=1}^{n} (\mu_j)^{\eta_j w_j}, \sqrt{1 - \prod_{j=1}^{n} (1 - \nu_j^2)^{\eta_j w_j}} \right\rangle$, $\alpha_1^{w_1} \otimes \alpha_2^{w_2} = \beta_1^{w_1} \otimes \beta_2^{w_2} = \langle 0, \sqrt{1 - (1 - (0.2)^2)^{0.25} (1 - (0.1)^2)^{0.25}} \rangle = \langle 0,\ 0.11242 \rangle$.

This clearly indicates that for PFWGO (1.3.4.2), proposed by Garg [62], the monotonicity property is not satisfying. Hence, the PFWGO (1.3.4.2), proposed by Garg [62], is not valid.

2.5.2.5 Invalidity of Ma and Xu's PFWAO

Ma and Xu [135, Proposition 1, (ii), pp. 1204] has stated the monotonicity property but not proved it. However, the following example clearly indicates that the monotonicity property is not satisfying for Ma and Xu's PFWAO (1.3.5.3). Hence, Ma and Xu's PFWAO (1.3.5.3) is not valid.

Let $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.2,0.0 \rangle$, $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0,0.4 \rangle$, $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.4, 0.0 \rangle$ and $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.0, 0.2 \rangle$ be four PFNs. Then, using the comparing method, discussed in Section 1.5.9,

(i) $\alpha_1 = \langle \mu_1, \nu_1 \rangle = \langle 0.2, 0.0 \rangle$ is less than $\beta_1 = \langle \mu_1', \nu_1' \rangle = \langle 0.4, 0.0 \rangle$ as $\mu_1^2 - \nu_1^2 = 0.04 - 0.0 = 0.04$ is less than $(\mu_1')^2 - (\nu_1')^2 = 0.16 - 0.0 = 0.16$.

(ii) $\alpha_2 = \langle \mu_2, \nu_2 \rangle = \langle 0.0, 0.4 \rangle$ is less than $\beta_2 = \langle \mu_2', \nu_2' \rangle = \langle 0.0, 0.2 \rangle$ as $\mu_2^2 - \nu_2^2 = 0.0 - 0.16 = -0.16$ is less than $(\mu_2')^2 - (\nu_2')^2 = 0 - 0.04 = -0.04$.

Since, $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. So, according to the monotonicity property, the relation $w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$ should hold.

While, if $w_1 = w_2 = 0.5$ then using the expression (1.3.5.3) i.e.,

$$\oplus_{i=1}^{n} w_i\alpha_i = \left(\frac{\prod_{i=1}^{n} \mu_i^{w_i}}{\left[\prod_{i=1}^{n}(1-\mu_i^2)^{w_i} + \prod_{i=1}^{n} \mu_i^{2w_i}\right]^{\frac{1}{2}}}, \frac{\prod_{i=1}^{n} \nu_i^{w_i}}{\left[\prod_{i=1}^{n}(1-\nu_i^2)^{w_i} + \prod_{i=1}^{n} \nu_i^{2w_i}\right]^{\frac{1}{2}}} \right),$$

$w_1\alpha_1 \oplus w_2\alpha_2 = w_1\beta_1 \oplus w_2\beta_2 = \langle 0,0 \rangle$

This clearly indicates that for PFWAO (1.3.5.3), proposed by Ma and Xu [135], the monotonicity property is not satisfying. Hence, the PFWAO (1.3.5.3), proposed by Ma and Xu [135], is not valid.

2.5.3 Invalidity of IVIFW aggregation operators

In Section 2.3, it is shown that the sum and multiplication of two IVIFNs is not necessarily an IVIFN i.e., Garg's expressions (1.4.1.1) and (1.4.2.1) to evaluate the sum and multiplication of two IVFNs are not valid. Since, the IVFWAO (1.4.1.3) and the IVIFWGO (1.4.2.3) are fully based upon the expression (1.4.1.1) and (1.4.2.1) respectively. Therefore, the IVFWAO (1.4.1.3) and the IVIFWGO (1.4.2.3) are also not valid.

2.5.4 Invalidity of SVNW aggregation operators

In this section, with the help of examples, it is shown that the SVN weighted aggregation operators [134, 147], discussed in Section 1.5, are not satisfying the monotonicity property. Hence, these aggregation operators are not valid.

2.5.4.1 Invalidity of SVNWAO and SVNWGO

Nancy and Garg [147, Property 3.2, pp. 369] has stated the monotonicity property as follows:

Let $\alpha_i = \langle a_i, b_i, c_i \rangle$ and $\alpha_i' = \langle a_i', b_i', c_i' \rangle$, $i = 1,2,\dots,n$ be collections of SVNNs. If $a_i \le a_i'$, $b_i \ge b_i'$ and $c_i \ge c_i'$ for all $i = 1,2,\dots,n$. Then, $\bigoplus_{i=1}^{n} (w_i \alpha_i) \prec \bigoplus_{i=1}^{n} (w_i \alpha_i')$ as well as $\bigotimes_{i=1}^{n} (\alpha_i)^{w_i} \prec \bigotimes_{i=1}^{n} (\alpha_i')^{w_i}$.

This statement may be rewritten as:

Let $\alpha_i = \langle a_i, b_i, c_i \rangle$ and $\alpha_i' = \langle a_i', b_i', c_i' \rangle$, $i = 1,2,\dots,n$ be collections of SVNNs.

(i) If $a_i < a_i'$, $b_i \ge b_i'$ and $c_i \ge c_i'$ for $i \in S_1$, $a_i \le a_i'$, $b_i > b_i'$ and $c_i \ge c_i'$ for $i \in S_2$, $a_i \le a_i'$, $b_i \ge b_i'$ and $c_i > c_i'$ for $i \in S_3$, $a_i < a_i'$, $b_i > b_i'$ and $c_i \ge c_i'$ for $i \in S_4$, $a_i < a_i'$, $b_i \ge b_i'$ and $c_i > c_i'$ for $i \in S_5$, $a_i \le a_i'$, $b_i > b_i'$ and $c_i > c_i'$ for $i \in S_6$, $a_i < a_i'$, $b_i > b_i'$ and $c_i > c_i'$ for $i \in S_7$, where, $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 = \{1,2,\dots,n\}$ and $S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5 \cap S_6 \cap S_7 = \emptyset$. Then, $\bigoplus_{i=1}^{n} (w_i \alpha_i) \prec \bigoplus_{i=1}^{n} (w_i \alpha_i')$ as well as $\bigotimes_{i=1}^{n} (\alpha_i)^{w_i} \prec \bigotimes_{i=1}^{n} (\alpha_i')^{w_i}$.

(ii) If $a_i = a_i'$, $b_i = b_i'$ and $c_i = c_i'$ for all $i = 1,2,\dots,n$. Then, $\bigoplus_{i=1}^{n} (w_i \alpha_i) \preccurlyeq \bigoplus_{i=1}^{n} (w_i \alpha_i')$ as well as $\bigotimes_{i=1}^{n} (\alpha_i)^{w_i} = \bigotimes_{i=1}^{n} (\alpha_i')^{w_i}$.

In actual case, the monotonicity property is stated as follows:

(i) If $\alpha_i \prec \alpha_i'$ for all $i = 1,2,\dots,n$. Then, $\bigoplus_{i=1}^{n} (w_i \alpha_i) \prec \bigoplus_{i=1}^{n} (w_i \alpha_i')$ as well as $\bigotimes_{i=1}^{n} (\alpha_i)^{w_i} \prec \bigotimes_{i=1}^{n} (\alpha_i')^{w_i}$.

(ii) If $\alpha_i = \alpha_i'$ for all $i = 1,2,\dots,n$ Then, $\bigoplus_{i=1}^{n} (w_i \alpha_i) = \bigoplus_{i=1}^{n} (w_i \alpha_i')$ as well as $\bigotimes_{i=1}^{n} (\alpha_i)^{w_i} = (\alpha_i')^{w_i}$.

It is obvious from the above two statements that Nancy and Garg [147] have assumed that the relation $\alpha_i \prec \alpha'_i$ hold if $a_i < a'_i$, $b_i \geq b'_i$ and $c_i \geq c'_i$ or $a_i \leq a'_i$, $b_i > b'_i$ and $c_i \geq c'_i$ or $a_i \leq a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ or $a_i < a'_i$, $b_i > b'_i$ and $c_i \geq c'_i$ or $a_i < a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ or $a_i < a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ i.e., the SVNN $\alpha_i = \langle a_i, b_i, c_i \rangle$ will be less than the SVNN $\alpha'_i = \langle a'_i, b'_i, c'_i \rangle$ if $a_i < a'_i$, $b_i \geq b'_i$ and $c_i \geq c'_i$ or $a_i \leq a'_i$, $b_i > b'_i$ and $c_i \geq c'_i$ or $a_i \leq a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ or $a_i < a'_i$, $b_i > b'_i$ and $c_i \geq c'_i$ or $a_i < a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ or $a_i < a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$.

While, as discussed in Section 1.6.1.4, Nancy and Garg [147] have stated that the relation, $\alpha_i \prec \alpha'_i$ hold if $a_1 - b_1 - c_1 < a'_1 - b'_1 - c'_1$ as there exist several SVNNs which cannot be compared by considering the relation $\alpha_i \prec \alpha'_i$ if $a_i < a'_i$, $b_i \geq b'_i$ and $c_i \geq c'_i$ or $a_i \leq a'_i$, $b_i > b'_i$ and $c_i \geq c'_i$ or $a_i \leq a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ or $a_i < a'_i$, $b_i > b'_i$ and $c_i \geq c'_i$ or $a_i < a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ or $a_i < a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ e.g., if $\alpha = \langle a, b, c \rangle = \langle 0.1, 0.3, 0.5 \rangle$ and $\alpha' = \langle a', b', c' \rangle = \langle 0.2, 0.1, 0.6 \rangle$ i.e., $a = 0.1, b = 0.3, c = 0.5, a' = 0.2, b' = 0.1$ and $c' = 0.6$. Then, according to the relation, $\alpha_i \prec \alpha'_i$ if $a_i < a'_i$, $b_i \geq b'_i$ and $c_i \geq c'_i$ or $a_i \leq a'_i$, $b_i > b'_i$ and $c_i \geq c'_i$ or $a_i \leq a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ or $a_i < a'_i$, $b_i > b'_i$ and $c_i \geq c'_i$ or $a_i < a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ or $a_i < a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$, neither $\alpha_i \prec \alpha'_i$ nor $\alpha_i \succ \alpha'_i$.

The above clearly indicates that Nancy and Garg [147] have used the following method to state the monotonicity property.

$\alpha_i \prec \alpha'_i \Rightarrow a_1 - b_1 - c_1 < a'_1 - b'_1 - c'_1 \Rightarrow a_i < a'_i$, $b_i \geq b'_i$ and $c_i \geq c'_i$ or $a_i \leq a'_i$, $b_i > b'_i$ and $c_i \geq c'_i$ or $a_i \leq a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ or $a_i < a'_i$, $b_i > b'_i$ and $c_i \geq c'_i$ or $a_i < a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$ or $a_i < a'_i$, $b_i \geq b'_i$ and $c_i > c'_i$.

However, the following example clearly indicates that

$a_1 - b_1 - c_1 < a_1' - b_1' - c_1' \nRightarrow a_i < a_i'$, $b_i \geq b_i'$ and $c_i \geq c_i'$ or $a_i \leq a_i'$, $b_i > b_i'$ and $c_i \geq c_i'$ or $a_i \leq a_i'$, $b_i \geq b_i'$ and $c_i > c_i'$ or $a_i < a_i'$, $b_i > b_i'$ and $c_i \geq c_i'$ or $a_i < a_i'$, $b_i \geq b_i'$ and $c_i > c_i'$ or $a_i < a_i'$, $b_i \geq b_i'$ and $c_i > c_i'$.

Let $\alpha = \langle a, b, c \rangle = \langle 0.1, 0.3, 0.5 \rangle$ and $\alpha' = \langle a', b', c' \rangle = \langle 0.2, 0.1, 0.6 \rangle$. Then, it can be easily verified that the relation $a - b - c \leq a' - b' - c'$ is satisfying but the relation $a_i < a_i'$, $b_i \geq b_i'$ and $c_i \geq c_i'$ or $a_i \leq a_i'$, $b_i > b_i'$ and $c_i \geq c_i'$ or $a_i \leq a_i'$, $b_i \geq b_i'$ and $c_i > c_i'$ or $a_i < a_i'$, $b_i > b_i'$ and $c_i \geq c_i'$ or $a_i < a_i'$, $b_i \geq b_i'$ and $c_i > c_i'$ or $a_i < a_i'$, $b_i \geq b_i'$ and $c_i > c_i'$ is not satisfying.

The following example also validates that the monotonicity property is not satisfying for Nancy and Garg's SVNWAO [147]. Hence, Nancy and Garg's SVNWAO [147] is not valid.

Let $\alpha_1 = \langle a_1, b_1, c_1 \rangle = \langle 0.2, 0.0, 0.0 \rangle$, $\alpha_2 = \langle a_2, b_2, c_2 \rangle = \langle 0.3, 0.4, 0.5 \rangle$ and $\alpha_1' = \langle a_1', b_1', c_1' \rangle = \langle 0.3, 0.0, 0.0 \rangle$ and $\alpha_2' = \langle a_2', b_2', c_2' \rangle = \langle 0.2, 0.3, 0.1 \rangle$ be collection of SVNNs.

Then, using the comparing method, discussed in Section 1.6.14,

(i) $\alpha_1 = \langle a_1, b_1, c_1 \rangle = \langle 0.2, 0, 0 \rangle$ is less than $\alpha_1' = \langle a_1', b_1', c_1' \rangle = \langle 0.3, 0.0, 0.0 \rangle$ as $a_1 - b_1 - c_1 = 0.2 - 0.0 - 0.0 = 0.2$ is less than $a_2 - b_2 - c_2 = 0.3 - 0.0 - 0.0 = 0.3$.

(ii) $\alpha_2 = \langle a_2, b_2, c_2 \rangle = \langle 0.3, 0.4, 0.5 \rangle$ is less than $\alpha_2' = \langle a_2', b_2', c_2' \rangle = \langle 0.2, 0.3, 0.1 \rangle$ as $a_2 - b_2 - c_2 = 0.3 - 0.4 - 0.5 = -0.6$ is less than $a_2' - b_2' - c_2' = 0.2 - 0.3 - 0.1 = -0.2$.

Since, $\alpha_1 \prec \alpha_1'$ and $\alpha_2 \prec \alpha_2'$. So, according to the monotonicity property, the relation $w_1 \alpha_1 \oplus w_2 \alpha_2 \prec w_1 \alpha_1' \oplus w_2 \alpha_2'$ should hold.

While, if $w_1 = w_2 = 0.5$ then using the expression (1.6.1.5) i.e., $\oplus_{i=1}^{n}(w_i\alpha_i) = \left\langle 1 - \log_\lambda(1 + \prod_{i=1}^{n}(\lambda^{1-a_i} - 1)^{w_i}), \log_\lambda\left(1 + \prod_{i=1}^{n}\left(\lambda^{b_i} - 1\right)^{w_i}\right), \log_\lambda(1 + \prod_{i=1}^{n}(\lambda^{c_i} - 1)^{w_i})\right\rangle$,

$w_1\alpha_1 \oplus w_2\alpha_2 = w_1\alpha'_1 \oplus w_2\alpha'_2 = \left\langle 1 - \log_\lambda\left(1 + \left(\lambda^{1-0.2} - 1\right)^{0.5}\left(\lambda^{1-0.3} - 1\right)^{0.5}\right), 0.0, 0.0\right\rangle$.

This clearly indicates that for the SVNWAO (1.6.1.5), proposed by Nancy and Garg [147], the monotonicity property is not satisfying. Hence, the SVNWAO (1.6.1.5), proposed by Nancy and Garg [147], is not valid.

The following example also validates that the monotonicity property is not satisfying for Nancy and Garg's SVNWGO [147]. Hence, Nancy and Garg's SVNWGO [147] is not valid.

Let $\alpha_1 = \langle a_1, b_1, c_1\rangle = \langle 0.1,1,1\rangle$, $\alpha_2 = \langle a_2, b_2, c_2\rangle = \langle 0.2,0.5,0.6\rangle$ and $\alpha'_1 = \langle a'_1, b'_1, c'_1\rangle = \langle 0.2,1,1\rangle$ and $\alpha'_2 = \langle a'_2, b'_2, c'_2\rangle = \langle 0.1,0.3,0.4\rangle$ be collection of SVNNs.

Then, using the comparing method, discussed in Section 1.6.14,

(i) $\alpha_1 = \langle a_1, b_1, c_1\rangle = \langle 0.1,1,1\rangle$ is less than $\alpha'_1 = \langle a'_1, b'_1, c'_1\rangle = \langle 0.2,1,1\rangle$ as $a_1 - b_1 - c_1 = 0.1 - 1 - 1 = -1.9$ is less than $a'_1 - b'_1 - c'_1 = 0.2 - 1 - 1 = -1.8$.

(ii) $\alpha_2 = \langle a_2, b_2, c_2\rangle = \langle 0.2,0.5,0.6\rangle$ is less than $\alpha'_2 = \langle a'_2, b'_2, c'_2\rangle = \langle 0.1,0.3,0.4\rangle$ as $a_2 - b_2 - c_2 = 0.2 - 0.5 - 0.6 = -1.9$ is less than $a'_2 - b'_2 - c'_2 = 0.1 - 0.3 - 0.4 = -0.6$.

Since, $\alpha_1 \prec \alpha'_1$ and $\alpha_2 \prec \alpha'_2$. So, according to the monotonicity property, the relation $(\alpha_1)^{w_1} \otimes (\alpha_2)^{w_2} \prec (\alpha'_1)^{w_1} \otimes (\alpha'_2)^{w_2}$ should hold.

While, if $w_1 = w_2 = 0.5$ then using the expression (1.4.1.6) i.e., $\otimes_{i=1}^{n} {\alpha_i}^{w_i} = \left\langle \log_\lambda(1 + \prod_{i=1}^{n}(\lambda^{a_i} - 1)^{w_i}), 1 - \log_\lambda\left(1 + \prod_{i=1}^{n}\left(\lambda^{1-b_i} - 1\right)^{w_i}\right), 1 - \log_\lambda\left(1 + \prod_{i=1}^{n}\left(\lambda^{1-c_i} - 1\right)^{w_i}\right)\right\rangle$,

$(\alpha_1)^{w_1} \otimes (\alpha_2)^{w_2} = (\alpha_1')^{w_1} \otimes (\alpha_2')^{w_2} = \left\langle \log_\lambda \left(1 + \left(\lambda^{0.1} - 1\right)^{0.5}\left(\lambda^{0.2} - 1\right)^{0.5}\right), 1, 1 \right\rangle$.

This clearly indicates that for the SVNWGO (1.6.1.6), proposed by Nancy and Garg [147], the monotonicity property is not satisfying. Hence, the SVNWGO (1.6.1.6), proposed by Nancy and Garg [147], is not valid.

2.5.4.2 Invalidity of SVNHFOWAO

Liu and Luo [134, Section 3.2, Theorem 3] has stated the monotonicity property but not proved it. However, the following example clearly indicates that the monotonicity property is neither satisfying for Liu and Luo's SVNHFOWAO (1.6.2.1). Hence, Liu and Luo's SVNHFOWAO (1.6.2.1) is not valid.

Let $n_1 = \{\{1\}, \{0\}, \{1\}\}$, $n_2 = \{\{0.1\}, \{0.2\}, \{0\}\}$ and $m_1 = \{\{1\}, \{0\}, \{0\}\}$, $m_2 = \{\{0.3\}, \{0.2\}, \{0\}\}$ be four SVNHFNs. Then, using the comparing method, discussed in Section 1.6.15,

(i) $n_1 = \{\{1\}, \{0\}, \{1\}\}$ is less than $m_1 = \{\{1\}, \{0\}, \{0\}\}$ as $s(n_1) = 0.6667$ is less than $s(m_1) = 1$.

(ii) $n_2 = \{\{0.1\}, \{0.2\}, \{0\}\}$ is less than $m_2 = \{\{0.3\}, \{0.2\}, \{0\}\}$ as $s(n_2) = 0.6333$ is less than $s(m_2) = 0.7$.

Since, $n_1 \prec m_1$ and $n_2 \prec m_2$. So, according to the monotonicity property, the relation $w_1 n_1 \oplus w_2 n_2 \prec w_1 m_1 \oplus w_2 m_2$ should hold. While, using the expression (1.6.2.1) i.e.,

$$\oplus_{i=1}^{k} (w_i \alpha_i) = \cup_{\gamma_1 \in \mu_1, \ldots, \gamma_k \in \mu_k, \delta_1 \in \nu_1, \ldots, \delta_k \in \nu_k, \eta_1 \in h_1, \ldots, \eta_k \in h_k} \left\{\left\{1 - \prod_{i=1}^{k}\left(1 - \gamma_{\sigma(i)}\right)^{\lambda_i}\right\}, \left\{\prod_{i=1}^{k}\left(\gamma_{\sigma(i)}\right)^{\lambda_i}\right\}, \left\{\prod_{i=1}^{k}\left(\eta_{\sigma(i)}\right)^{\lambda_i}\right\}\right\},$$

$w_1 n_1 \oplus w_2 n_2 = w_1 m_1 \oplus w_2 m_2 = \{\{1\}, \{0\}, \{0\}\}$

This clearly indicates that for the SVNHFOWAO (1.6.2.1), proposed by Liu and Luo [134], the monotonicity property is not satisfying. Hence, the SVNHFOWAO (1.6.2.1), proposed by Liu and Luo [134], is not valid.

2.6 Appropriate IFWAO

Chen et al. [30] proposed the IVIFWAO (2.6.1).

If $\alpha_i = \langle [\mu_{i1}, \mu_{i2}], [\nu_{i1}, \nu_{i2}] \rangle$, $i = 1,2,\ldots,n$, are n IVIFNs and $w_i, i = 1,2,\ldots,n$ are non-negative real numbers such that $w_i > 0$ for at least one i. Then,

$$\oplus_{i=1}^{n} (w_i \alpha_i) = \oplus_{i=1}^{n} \left(w_i \langle [\mu_{i1}, \mu_{i2}], [\nu_{i1}, \nu_{i2}] \rangle\right)$$

$$= \langle [\textstyle\sum_{i=1}^{n} w_i \, \mu_{i1}, \sum_{i=1}^{n} w_i \, \mu_{i2}], [\sum_{i=1}^{n} w_i \, \nu_{i1}, \sum_{i=1}^{n} w_i \, \nu_{i2}] \rangle$$

(2.6.1)

Using the same concept, the IFWAO (2.6.2) (weighted arithmetic mean of IFNs $\alpha_i = \langle \mu_i, \nu_i \rangle$ where, $0 \le \mu_i, \nu_i \le 1$, $\mu_i + \nu_i \le 1$) is defined.

$$\oplus_{i=1}^{n} (w_i \alpha_i) = \textstyle\sum_{i=1}^{n} (w_i \langle \mu_i, \nu_i \rangle)$$

$$= \langle \textstyle\sum_{i=1}^{n} w_i \mu_i \, , \sum_{i=1}^{n} w_i \nu_i \rangle. \qquad (2.6.2)$$

To prove that the proposed expression (2.6.2) is valid, it is shown that if the existing method for comparing IFNs, discussed in Section 1.6.1, is used. Then, the monotonicity property, the boundedness property, the idempotency property and the commutativity property will be satisfied for the proposed IFWAO.

2.6.1 Monotonicity property

In this section, it is proved that if $\alpha_i = \langle \mu_i, \nu_i \rangle$ and $\beta_i = \langle \mu_i', \nu_i' \rangle$, $i = 1,2,\ldots,n$ are collections of IFNs such that

(i) $\alpha_i = \beta_i \ \forall \ i = 1,2,\ldots,n$ then, $\oplus_{i=1}^{n} (w_i \alpha_i) = \oplus_{i=1}^{n} (w_i \beta_i)$.

(ii) $\alpha_i \prec \beta_i \ \forall \ i = 1,2,\ldots,n$ then, $\oplus_{i=1}^{n} (w_i \alpha_i) \prec \oplus_{i=1}^{n} (w_i \beta_i)$.

Case 1: Let $\alpha_i = \beta_i\ \forall\ i = 1,2,\dots,n$ then, according to comparing method, discussed in Section 1.6.1, $S(\alpha_i) = S(\beta_i)$ and $H(\alpha_i) = H(\beta_i)$.

$S(\alpha_i) = S(\beta_i) \Rightarrow S(\langle\mu_i, \nu_i\rangle) = S(\langle\mu_i', \nu_i'\rangle)$

$$\Rightarrow \mu_i - \nu_i = \mu_i' - \nu_i' \tag{2.6.3}$$

$H(\alpha_i) = H(\beta_i) \Rightarrow H(\langle\mu_i, \nu_i\rangle) = H(\langle\mu_i', \nu_i'\rangle)$

$$\Rightarrow \mu_i + \nu_i = \mu_i' + \nu_i' \tag{2.6.4}$$

Adding Equation (2.6.3) and Equation (2.6.4), $\mu_i = \mu_i'$.

Subtracting Equation (2.6.3) from Equation (2.6.4), $\nu_i = \nu_i'$.

Therefore, $\bigoplus_{i=1}^{n} w_i\alpha_i = \bigoplus_{i=1}^{n} (w_i\langle\mu_i, \nu_i\rangle)$

$= \bigoplus_{i=1}^{n} (w_i\langle\mu_i', \nu_i'\rangle)$

$= \bigoplus_{i=1}^{n} (w_i\beta_i)$.

Case 2: Let $\alpha_i \prec \beta_i\ \forall\ i = 1,2,\dots,n$ and there is need to prove that $\bigoplus_{i=1}^{n} (w_i\alpha_i) \prec \bigoplus_{i=1}^{n} (w_i\beta_i)$.

The result will be proved with the help of principle of mathematical induction i.e., the result will be proved for $n = 2$ and it will be assumed that the result is true for $n = k$ then, it will be proved that the result is also true for $n = k + 1$.

For $n = 2$, the result may be stated as,

$\alpha_1 \prec \beta_1, \alpha_2 \prec \beta_2 \Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$.

There may be following cases:

Case 2.1: Let $\alpha_1 \prec \beta_1, \alpha_2 \prec \beta_2 \Rightarrow S(\alpha_1) < S(\beta_1)$ and $S(\alpha_2) < S(\beta_2)$.

Case 2.2: Let $\alpha_1 \prec \beta_1, \alpha_2 \prec \beta_2 \Rightarrow S(\alpha_1) = S(\beta_1), H(\alpha_1) < H(\beta_1)$ and $S(\alpha_2) < S(\beta_2)$.

Case 2.3: Let $\alpha_1 \prec \beta_1, \alpha_2 \prec \beta_2 \Rightarrow S(\alpha_1) < S(\beta_1), S(\alpha_2) = S(\beta_2)$ and $H(\alpha_2) < H(\beta_2)$.

Case 2.4: Let $\alpha_1 \prec \beta_1$, $\alpha_2 \prec \beta_2 \Rightarrow S(\alpha_1) = S(\beta_1)$, $H(\alpha_1) < H(\beta_1)$ and $S(\alpha_2) = S(\beta_2)$, $H(\alpha_2) < H(\beta_2)$.

The result will be proved for all the four cases.

Case 2.1: $\alpha_1 \prec \beta_1, \alpha_2 \prec \beta_2$.

$\Rightarrow S(\alpha_1) < S(\beta_1)$ and $S(\alpha_2) < S(\beta_2)$.

$\Rightarrow \mu_1 - \nu_1 < \mu_1' - \nu_1'$ and $\mu_2 - \nu_2 < \mu_2' - \nu_2'$.

$$\Rightarrow w_1(\mu_1 - \nu_1) < w_1(\mu_1' - \nu_1') \quad (2.6.5)$$

$$\Rightarrow w_2(\mu_2 - \nu_2) < w_2(\mu_2' - \nu_2') \quad (2.6.6)$$

Adding Equation (2.6.5) and Equation (2.6.6),

$\Rightarrow w_1(\mu_1 - \nu_1) + w_2(\mu_2 - \nu_2) < w_1(\mu_1' - \nu_1') + w_2(\mu_2' - \nu_2')$.

$\Rightarrow (w_1\mu_1 + w_2\mu_2) - (w_1\nu_1 + w_2\nu_2) < (w_1\mu_1' + w_2\mu_2') - (w_1\nu_1' + w_2\nu_2')$.

$\Rightarrow S(w_1\langle\mu_1,\nu_1\rangle \oplus w_2\langle\mu_2,\nu_2\rangle) < S(w_1\langle\mu_1',\nu_1'\rangle \oplus w_2\langle\mu_2',\nu_2'\rangle)$.

$\Rightarrow w_1\langle\mu_1,\nu_1\rangle \oplus w_2\langle\mu_2,\nu_2\rangle \prec w_1\langle\mu_1',\nu_1'\rangle \oplus w_2\langle\mu_2',\nu_2'\rangle$.

$\Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$.

Case 2.2: $\alpha_1 \prec \beta_1, \alpha_2 \prec \beta_2$

$\Rightarrow S(\alpha_1) = S(\beta_1), H(\alpha_1) < H(\beta_1)$ and $S(\alpha_2) < S(\beta_2)$.

$\Rightarrow \mu_1 - \nu_1 = \mu_1' - \nu_1', \mu_1 + \nu_1 < \mu_1' + \nu_1'$ and $\mu_2 - \nu_2 < \mu_2' - \nu_2'$.

Case 2.2a: If $w_1 \neq 0$ and $w_2 \neq 0$ then,

$\Rightarrow \mu_1 - \nu_1 = \mu_1' - \nu_1'$ and $\mu_2 - \nu_2 < \mu_2' - \nu_2'$.

$$\Rightarrow w_1(\mu_1 - \nu_1) = w_1(\mu_1' - \nu_1') \quad (2.6.7)$$

$$\Rightarrow w_2(\mu_2 - \nu_2) < w_2(\mu_2' - \nu_2') \quad (2.6.8)$$

$\Rightarrow w_1(\mu_1 - \nu_1) + w_2(\mu_2 - \nu_2) < w_1(\mu_1' - \nu_1') + w_2(\mu_2' - \nu_2')$.

$\Rightarrow (w_1\mu_1 + w_2\mu_2) - (w_1\nu_1 + w_2\nu_2) < (w_1\mu_1' + w_2\mu_2') - (w_1\nu_1' + w_2\nu_2')$.

$\Rightarrow S(w_1\langle\mu_1,\nu_1\rangle \oplus w_2\langle\mu_2,\nu_2\rangle) < S(w_1\langle\mu_1',\nu_1'\rangle \oplus w_2\langle\mu_2',\nu_2'\rangle)$.

$\Rightarrow w_1\langle\mu_1,\nu_1\rangle \oplus w_2\langle\mu_2,\nu_2\rangle \prec w_1\langle\mu_1',\nu_1'\rangle \oplus w_2\langle\mu_2',\nu_2'\rangle$.

$\Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$.

Case 2.2b: If $w_1 = 1$ and $w_2 = 0$ then,

$$w_1\alpha_1 \oplus w_2\alpha_2 = w_1\alpha_1 = \alpha_1. \tag{2.6.9}$$

$$w_1\beta_1 \oplus w_2\beta_2 = w_1\beta_1 = \beta_1. \tag{2.6.10}$$

$\alpha_1 \prec \beta_1 \Rightarrow w_1\alpha_1 \prec w_1\beta_1$.

$\Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$.

Case 2.2c: If $w_1 = 0$ and $w_2 = 1$ then,

$$w_1\alpha_1 \oplus w_2\alpha_2 = w_2\alpha_2 = \alpha_2. \tag{2.6.11}$$

$$w_1\beta_1 \oplus w_2\beta_2 = w_2\beta_2 = \beta_2. \tag{2.6.12}$$

$\alpha_2 \prec \beta_2 \Rightarrow w_2\alpha_2 \prec w_2\beta_2$.

$\Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$.

Case 2.3: $\alpha_1 \prec \beta_1, \alpha_2 \prec \beta_2$

$\Rightarrow S(\alpha_1) < S(\beta_1), S(\alpha_2) = S(\beta_2)$ and $H(\alpha_2) < H(\beta_2)$.

$\Rightarrow \mu_1 - \nu_1 < \mu_1' - \nu_1', \mu_2 - \nu_2 < \mu_2' - \nu_2'$ and $\mu_2 + \nu_2 < \mu_2' + \nu_2'$.

Case 2.3a: If $w_1 \neq 0$ and $w_2 \neq 0$ then,

$\Rightarrow \mu_1 - \nu_1 < \mu_1' - \nu_1'$ and $\mu_2 - \nu_2 = \mu_2' - \nu_2'$.

$$\Rightarrow w_1(\mu_1 - \nu_1) < w_1(\mu_1' - \nu_1') \tag{2.6.13}$$

$$\Rightarrow w_2(\mu_2 - \nu_2) - w_2(\mu_2' - \nu_2') \tag{2.6.14}$$

$\Rightarrow w_1(\mu_1 - \nu_1) + w_2(\mu_2 - \nu_2) < w_1(\mu_1' - \nu_1') + w_2(\mu_2' - \nu_2')$.

$\Rightarrow (w_1\mu_1 + w_2\mu_2) - (w_1\nu_1 + w_2\nu_2) < (w_1\mu_1' + w_2\mu_2') - (w_1\nu_1' + w_2\nu_2')$.

$\Rightarrow S(w_1\langle\mu_1,\nu_1\rangle \oplus w_2\langle\mu_2,\nu_2\rangle) < S(w_1\langle\mu_1',\nu_1'\rangle \oplus w_2\langle\mu_2',\nu_2'\rangle)$.

$\Rightarrow\ w_1\langle\mu_1,\nu_1\rangle \oplus w_2\langle\mu_2,\nu_2\rangle \prec w_1\langle\mu_1',\nu_1'\rangle \oplus w_2\langle\mu_2',\nu_2'\rangle.$

$\Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2.$

Case 2.3b: If $w_1 = 1$ and $w_2 = 0$ then,

$$w_1\alpha_1 \oplus w_2\alpha_2 = w_1\alpha_1 = \alpha_1. \tag{2.6.15}$$

$$w_1\beta_1 \oplus w_2\beta_2 = w_1\beta_1 = \beta_1. \tag{2.6.16}$$

$\alpha_1 \prec \beta_1 \Rightarrow w_1\alpha_1 \prec w_1\beta_1.$

$\Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2.$

Case 2.3c: If $w_1 = 0$ and $w_2 = 1$ then,

$$w_1\alpha_1 \oplus w_2\alpha_2 = w_2\alpha_2 = \alpha_2. \tag{2.6.17}$$

$$w_1\beta_1 \oplus w_2\beta_2 = w_2\beta_2 = \beta_2. \tag{2.6.18}$$

$\alpha_2 \prec \beta_2 \Rightarrow w_2\alpha_2 \prec w_2\beta_2$.

$\Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2.$

Case 2.4: $\alpha_1 \prec \beta_1, \alpha_2 \prec \beta_2$

$\Rightarrow\ S(\alpha_1) = S(\beta_1), H(\alpha_1) < H(\beta_1)$ and $S(\alpha_2) < S(\beta_2).$

$\Rightarrow \mu_1 - \nu_1 = \mu_1' - \nu_1', \mu_1 + \nu_1 < \mu_1' + \nu_1', \mu_2 - \nu_2 < \mu_2' - \nu_2'$ and $\mu_2 + \nu_2 < \mu_2' + \nu_2'.$

$\Rightarrow w_1(\mu_1 - \nu_1) = w_1(\mu_1' - \nu_1').$

$\Rightarrow w_2(\mu_2 - \nu_2) = w_2(\mu_2' - \nu_2').$

$\Rightarrow w_1(\mu_1 + \nu_1) < w_1(\mu_1' + \nu_1').$

$\Rightarrow w_2(\mu_2 + \nu_2) < w_2(\mu_2' + \nu_2').$

$\Rightarrow w_1(\mu_1 - \nu_1) + w_2(\mu_2 - \nu_2) < w_1(\mu_1' - \nu_1') + w_2(\mu_2' - \nu_2').$

$\Rightarrow (w_1\mu_1 - w_2\mu_2) + (w_1\nu_1 - w_2\nu_2) = (w_1\mu_1' - w_2\mu_2') + (w_1\nu_1' - w_2\nu_2').$

$\Rightarrow (w_1\mu_1 + w_2\mu_2) + (w_1\nu_1 + w_2\nu_2) < (w_1\mu_1' + w_2\mu_2') + (w_1\nu_1' + w_2\nu_2').$

$\Rightarrow (w_1\mu_1 + w_2\mu_2) - (w_1\nu_1 + w_2\nu_2) = (w_1\mu_1' + w_2\mu_2') - (w_1\nu_1' + w_2\nu_2').$

$\Rightarrow (w_1\mu_1 + w_2\mu_2) + (w_1\nu_1 + w_2\nu_2) < (w_1\mu_1' + w_2\mu_2') + (w_1\nu_1' + w_2\nu_2')$.

$\Rightarrow S(w_1\alpha_1 \oplus w_2\alpha_2) = S(w_1\beta_1 \oplus w_2\beta_2)$.

$\Rightarrow H(w_1\alpha_1 \oplus w_2\alpha_2) < H(w_1\beta_1 \oplus w_2\beta_2)$.

$\Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \prec w_1\beta_1 \oplus w_2\beta_2$.

Let us consider that the result is true for $n = k$ i.e., $\oplus_{i=1}^{k} (w_i\alpha_i) \prec \oplus_{i=1}^{k} (w_i\beta_i)$, $i = 1,2, \dots, k$.

$\Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \oplus \dots \oplus w_k\alpha_k \prec w_1\beta_1 \oplus w_2\beta_2 \oplus \dots \oplus w_k\beta_k$.

Now there is need to prove that the result is true for $n = k + 1$ i.e., there is need to prove that $\alpha_i \prec \beta_i$, $i = 1,2, \dots, k + 1 \Rightarrow w_i\alpha_i \prec w_i\beta_i$, $i = 1,2, \dots, k + 1$.

$$(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \dots \oplus w_k\alpha_k) \oplus w_{k+1}\alpha_{k+1} = \left(\sum_{i=1}^{k} w_i\right) \otimes \frac{(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \dots \oplus w_k\alpha_k)}{\sum_{i=1}^{k} w_i} \oplus w_{k+1}\alpha_{k+1}$$

$$= \left(\sum_{i=1}^{k} w_i\right)\left(\frac{w_1\alpha_1}{\sum_{i=1}^{k} w_i} \oplus \frac{w_2\alpha_2}{\sum_{i=1}^{k} w_i} \oplus \dots \oplus \frac{w_k\alpha_k}{\sum_{i=1}^{k} w_i}\right) \oplus w_{k+1}\alpha_{k+1} \qquad = \left(\sum_{i=1}^{k} w_i\right)(W_1\alpha_1 \oplus W_2\alpha_2 \oplus \dots \oplus W_k\alpha_k)$$

$$\oplus\, w_{k+1}\alpha_{k+1}$$

where, $W_i = \frac{w_i}{\sum_{i=1}^{k} w_i}$.

Since, $W_1 + W_2 + \cdots + W_k = 1$. Therefore, $W_1\alpha_1 \oplus W_2\alpha_2 \oplus \dots \oplus W_k\alpha_k$ represents the aggregated value of α_1, α_2,…, α_k.

Assuming $W_1\alpha_1 \oplus W_2\alpha_2 \oplus \dots \oplus W_k\alpha_k = \delta$,

$(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \dots \oplus w_k\alpha_k) \oplus w_{k+1}\alpha_{k+1} = \left(\sum_{i=1}^{k} w_i\right)\delta \oplus w_{k+1}\alpha_{k+1}$.

Since, $\sum_{i=1}^{k} w_i + w_{k+1} = 1$. So, $\left(\sum_{i=1}^{k} w_i\right)\delta \oplus w_{k+1}\alpha_{k+1}$ represents the aggregated value of two IFNs δ and α_{k+1}. Also, $\delta = W_1\alpha_1 \oplus W_2\alpha_2 \oplus \dots \oplus W_k\alpha_k \prec W_1\beta_1 \oplus W_2\beta_2 \oplus \dots \oplus W_k\beta_k$, $\alpha_{k+1} \prec \beta_{k+1}$ and the result is valid for two IFNs. Therefore,

$\left(\sum_{i=1}^{k} w_i\right)(W_1\alpha_1 \oplus W_2\alpha_2 \oplus \ldots \oplus W_k\alpha_k) \oplus w_{k+1}\alpha_{k+1} \prec \left(\sum_{i=1}^{k} w_i\right)(W_1\beta_1 \oplus W_2\beta_2 \oplus \ldots \oplus W_k\beta_k) \oplus w_{k+1}\beta_{k+1}$.

$\Rightarrow w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots + w_k\alpha_k \oplus w_{k+1}\alpha_{k+1} \prec w_1\beta_1 \oplus w_2\beta_2 \oplus \ldots \oplus w_k\beta_k \oplus w_{k+1}\beta_{k+1}$.

2.6.2 Boundedness property

In this section, it is proved that $min_{1\le i\le n}\, \alpha_i \preccurlyeq w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n \preccurlyeq max_{1\le i\le n}\, \alpha_i$.

Proof: $S(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n)$

$$= S(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n)$$

$$= S(w_1\langle\mu_1,\nu_1\rangle \oplus w_2\langle\mu_2,\nu_2\rangle \oplus \ldots \oplus w_n\langle\mu_n,\nu_n\rangle)$$

$$= (w_1\mu_1 + w_2\mu_2 + \cdots + w_n\mu_n) - (w_1\nu_1 + w_2\nu_2 + \cdots + w_n\nu_n)$$

$$= w_1(\mu_1 - \nu_1) + w_2(\mu_2 - \nu_2) + \cdots + w_n(\mu_n - \nu_n)$$

$$= w_1S(\alpha_1) + w_2S(\alpha_2) + \cdots + w_nS(\alpha_n).$$

Since, $min_{1\le i\le n}S(\alpha_i) \le w_1S(\alpha_1) + w_2S(\alpha_2) + \cdots + w_nS(\alpha_n) \le max_{1\le i\le n}\, S(\alpha_i)$. So, $min_{1\le i\le n}S(\alpha_i) \le S(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n) \le max_{1\le i\le n}\, S(\alpha_i)$.

If $min_{1\le i\le n}S(\alpha_i) = S(\alpha_p)$ and $max_{1\le i\le n}S(\alpha_i) = S(\alpha_q)$ then,

$S(\alpha_p) \le S(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n) \le S(\alpha_q)$.

$\Rightarrow \alpha_p \preccurlyeq w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n \preccurlyeq \alpha_q$.

Similarly, $H(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n)$

$$= H(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n)$$

$$= H(w_1\langle\mu_1,\nu_1\rangle \oplus w_2\langle\mu_2,\nu_2\rangle \oplus \ldots \oplus w_n\langle\mu_n,\nu_n\rangle)$$

$$= (w_1\mu_1 + w_2\mu_2 + \cdots + w_n\mu_n) + (w_1\nu_1 + w_2\nu_2 + \cdots + w_n\nu_n)$$

$$= w_1(\mu_1 + \nu_1) + w_2(\mu_2 + \nu_2) + \cdots + w_n(\mu_n + \nu_n)$$

$$= w_1 H(\alpha_1) + w_2 H(\alpha_2) + \cdots + w_n H(\alpha_n).$$

Since, $min_{1\le i\le n} H(\alpha_i) \le w_1 H(\alpha_1) + w_2 H(\alpha_2) + \cdots + w_n H(\alpha_n) \le max_{1\le i\le n} H(\alpha_i)$.

So, $min_{1\le i\le n} H(\alpha_i) \le H(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n) \le max_{1\le i\le n} H(\alpha_i)$.

If $min_{1\le i\le n} H(\alpha_i) = H(\alpha_p)$ and $max_{1\le i\le n} H(\alpha_i) = H(\alpha_q)$ then,

$H(\alpha_p) \le H(w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n) \le H(\alpha_q)$.

$\Rightarrow \alpha_p \preccurlyeq w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_n\alpha_n \preccurlyeq \alpha_q$.

2.6.3 Idempotency property

In this section, it is proved that if $\alpha_i = \alpha = \langle \mu, \nu \rangle \; \forall \; i = 1,2,\ldots,n$.

Proof: $\oplus_{i=1}^{n} (w_i\alpha_i) = w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_{\mathrm{n}}\alpha_n$

$$= w_1\alpha \oplus w_2\alpha \oplus \ldots \oplus w_n\alpha$$

$$= (w_1 + w_2 + \cdots + w_n)\alpha$$

$$= \alpha$$

2.6.4 Commutativity property

It is obvious that if in the proposed IFWAO i.e., $w_1\alpha_1 \oplus w_2\alpha_2 \oplus \ldots \oplus w_p\alpha_p \oplus \ldots \oplus w_q\alpha_q \oplus \ldots \oplus w_n\alpha_n = \left\langle \begin{matrix} w_1\mu_1 \oplus w_2\mu_2 \oplus \ldots \oplus w_p\mu_p \oplus \ldots \oplus w_q\mu_q \oplus \ldots \oplus w_n\mu_n \\ , w_1\nu_1 \oplus w_2\nu_2 \oplus \ldots \oplus w_p\nu_p \oplus \ldots \oplus w_q\nu_q \oplus \ldots \oplus w_n\nu_n \end{matrix} \right\rangle$, the term $w_p\alpha_p$ is replaced by $w_q\alpha_q$ then there will be no change in the aggregated value. This indicates that the commutative property is satisfying for the proposed IFWAO.

2.7 Appropriate PFWAO

If $\alpha_i = \langle \mu_i, \nu_i \rangle$, $i = 1,2,\ldots,n$ are n PFNs then

$\oplus_{i=1}^{n}(w_i\alpha_i) = \oplus_{i=1}^{n}\left(w_i\langle\mu_i,\nu_i\rangle\right)$

$= \left\langle\sum_{i=1}^{n} w_i\,\mu_i^2, \sum_{i=1}^{n} w_i\nu_i^2\right\rangle.$

It can be easily verified that

(i) If $\alpha_p = \langle 1,0\rangle$ then $\sum_{i=1}^{n} w_i\alpha_i \neq \langle 1,0\rangle$.

(ii) If the following method is used for comparing the PFNs then the monotonicity property will be satisfied.

If $\alpha_1 = \langle\mu_1,\nu_1\rangle$ and $\alpha_2 = \langle\mu_2,\nu_2\rangle$ are two PFNs. Then use the following steps to check that $\alpha_1 > \alpha_2$ or $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$.

Step 1: Find $S(\alpha_1) = {\mu_1}^2 - {\nu_1}^2$, $S(\alpha_2) = {\mu_2}^2 - {\nu_2}^2$ and check that $S(\alpha_1) > S(\alpha_2)$ or $S(\alpha_1) < S(\alpha_2)$ or $S(\alpha_1) = S(\alpha_2)$.

Case (i): If $S(\alpha_1) > S(\alpha_2)$ then $\alpha_1 > \alpha_2$.

Case (ii): If $S(\alpha_1) < S(\alpha_2)$ then $\alpha_1 < \alpha_2$.

Case (iii): If $S(\alpha_1) = S(\alpha_2)$ then go to Step 2.

Step 2: Find $H(\alpha_1) = {\mu_1}^2 + {\nu_1}^2$, $H(\alpha_2) = {\mu_1}^2 + {\nu_1}^2$ and check that $H(\alpha_1) > H(\alpha_2)$ or $H(\alpha_1) < H(\alpha_2)$ or $H(\alpha_1) = H(\alpha_2)$.

Case (i): If $H(\alpha_1) > H(\alpha_2)$ then $\alpha_1 > \alpha_2$.

Case (ii): If $H(\alpha_1) < H(\alpha_2)$ then $\alpha_1 < \alpha_2$.

Case (iii): If $H(\alpha_1) = H(\alpha_2)$ then $\alpha_1 = \alpha_2$.

2.8 Appropriate SVNNWAO

If $\alpha_i = \langle T_i, I_i, F_i\rangle$, $i = 1,2,\ldots,n$ are n SVNNs then

$\oplus_{i=1}^{n}(w_i\alpha_i) = \oplus_{i=1}^{n}(w_i\langle T_i, I_i, F_i\rangle)$

$= \langle\sum_{i=1}^{n} w_i\,T_i, \sum_{i=1}^{n} w_i\,I_i, \sum_{i=1}^{n} w_i\,F_i\rangle.$

It can be easily verified that

(i) If $\alpha_p = \langle 1,0,0 \rangle$ then $\sum_{i=1}^{n} w_i \alpha_i \neq \langle 1,0,0 \rangle$.

(ii) If the following method is used for comparing the SVNNs then the monotonicity property will be satisfied.

If $\alpha_1 = \langle T_1, I_1, F_1 \rangle$ and $\alpha_2 = \langle T_2, I_2, F_2 \rangle$ are two SVNNs. Then use the following steps to check that $\alpha_1 > \alpha_2$ or $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$.

Step 1: Find $S(\alpha_1) = \frac{T_1 + 1 - I_1 + 1 - F_1}{3}$, $S(\alpha_2) = \frac{T_2 + 1 - I_2 + 1 - F_2}{3}$ and check that $S(\alpha_1) > S(\alpha_2)$ or $S(\alpha_1) < S(\alpha_2)$ or $S(\alpha_1) = S(\alpha_2)$.

Case (i): If $S(\alpha_1) > S(\alpha_2)$ then $\alpha_1 > \alpha_2$.

Case (ii): If $S(\alpha_1) < S(\alpha_2)$ then $\alpha_1 < \alpha_2$.

Case (iii): If $S(\alpha_1) = S(\alpha_2)$ then go to Step 2.

Step 2: Find $A(\alpha_1) = T_1 - F_1$, $A(\alpha_2) = T_2 - F_2$ and check that $A(\alpha_1) > A(\alpha_2)$ or $A(\alpha_1) < A(\alpha_2)$ or $A(\alpha_1) = A(\alpha_2)$.

Case (i): If $A(\alpha_1) > A(\alpha_2)$ then $\alpha_1 > \alpha_2$.

Case (ii): If $A(\alpha_1) < A(\alpha_2)$ then $\alpha_1 < \alpha_2$.

Case (iii): If $A(\alpha_1) = A(\alpha_2)$ then got to Step 3.

Step 3: Find $C(\alpha_1) = T_1$, $C(\alpha_2) = T_2$ and check that $C(\alpha_1) > C(\alpha_2)$ or $C(\alpha_1) < C(\alpha_2)$ or $C(\alpha_1) = C(\alpha_2)$.

Case (i): If $C(\alpha_1) > C(\alpha_2)$ then $\alpha_1 > \alpha_2$.

Case (ii): If $C(\alpha_1) < C(\alpha_2)$ then $\alpha_1 < \alpha_2$.

Case (iii): If $C(\alpha_1) = C(\alpha_2)$ then $\alpha_1 = \alpha_2$.

2.9 Non-existence of IFWGO, PFWGO, IVIFWGO

Motivated by the expressions (2.6.1) i.e., $\oplus_{i=1}^{n} (w_i \alpha_i) = \oplus_{i=1}^{n} \left(w_i \langle [\mu_{i1}, \mu_{i2}], [\nu_{i1}, \nu_{i2}] \rangle \right) = \left\langle \left[\frac{\sum_{i=1}^{n} w_i \mu_{i1}}{\sum_{i=1}^{n} w_i}, \frac{\sum_{i=1}^{n} w_i \mu_{i2}}{\sum_{i=1}^{n} w_i} \right], \left[\frac{\sum_{i=1}^{n} w_i \nu_{i1}}{\sum_{i=1}^{n} w_i}, \frac{\sum_{i=1}^{n} w_i \nu_{i2}}{\sum_{i=1}^{n} w_i} \right] \right\rangle$, one may assume that the expression $\otimes_{i=1}^{n}(\alpha_i)^{\frac{w_i}{\sum_{i=1}^{n}w_i}} = \otimes_{i=1}^{n}\left(\langle[\mu_{i1},\mu_{i2}],[\nu_{i1},\nu_{i2}]\rangle\right)^{\frac{w_i}{\sum_{i=1}^{n}w_i}} = \left\langle\left[\prod_{i=1}^{n}\left(\mu_{i1}^{\frac{w_i}{\sum_{i=1}^{n}w_i}}\right),\prod_{i=1}^{n}\left(\mu_{i2}^{\frac{w_i}{\sum_{i=1}^{n}w_i}}\right)\right],\left[\prod_{i=1}^{n}\left(\nu_{i1}^{\frac{w_i}{\sum_{i=1}^{n}w_i}}\right),\prod_{i=1}^{n}\left(\nu_{i2}^{\frac{w_i}{\sum_{i=1}^{n}w_i}}\right)\right]\right\rangle$ represents the IVIFWGO.

However, this assumption is not valid due to the following reason.

In the above expression, the terms $\prod_{i=1}^{n}\left(\mu_{i1}^{\frac{w_i}{\sum_{i=1}^{n}w_i}}\right)$; $\prod_{i=1}^{n}\left(\mu_{i2}^{\frac{w_i}{\sum_{i=1}^{n}w_i}}\right)$; $\prod_{i=1}^{n}\left(\nu_{i1}^{\frac{w_i}{\sum_{i=1}^{n}w_i}}\right)$ and $\prod_{i=1}^{n}\left(\nu_{i2}^{\frac{w_i}{\sum_{i=1}^{n}w_i}}\right)$ represents the geometric mean of $\mu_{i1}, i = 1,2,\ldots,n; \mu_{i2}, i = 1,2,\ldots,n;\ \nu_{i1},\ i = 1,2,\ldots,n$ and $\nu_{i2},\ i = 1,2,\ldots,n$ respectively. Also, it is well known fact that the geometric mean of n real numbers $a_1, a_2, \ldots, a_n$ exists only if $a_i > 0\ \forall\ i = 1,2,\ldots,n$. However, for the values of $\mu_{i1}, \mu_{i2}, \nu_{i1}$ and ν_{i2} the condition $\mu_{i1} > 0, \mu_{i2} > 0, \nu_{i1} > 0$ and $\nu_{i2} > 0$ will not necessarily be satisfied. Therefore, the geometric mean of μ_{i1}, the geometric mean of μ_{i2}, the geometric mean of ν_{i1} and the geometric mean of ν_{i2} may or may not exist.

Chapter 3

Mehar Method to Find the Unique Optimal Fuzzy Transportation Cost of Balanced Fully Fuzzy Transportation Problems with *LR* Flat Fuzzy Numbers[†]

Ebrahimnejad [49, Section 4, pp. 113] claimed that on solving a BFFTP with *LR* FFNs (balanced transportation problems in which all the parameters are represented as *LR* FFNs) by Kumar and Kaur's method [115] more than one *LR* FFNs, representing the optimal fuzzy transportation cost, may be obtained, which is illogical. To resolve this flaw, Ebrahimnejad [49, Section 5, pp. 114] proposed a method for solving BFFTPs with *LR* FFNs.

In this chapter, it is shown that Ebrahimnejad's method can be used only if the aggregated value of the fuzzy transportation cost, fuzzy availability and fuzzy demand, provided by all the decision-makers, is available. However, if instead of the aggregated data, the data of each decision-maker is provided separately then Ebrahimnejad's method cannot be used to find the solution of a BFFTP. Also, it is shown that the flaw, pointed out by Ebrahimnejad in Kumar and Kaur's method is also occurring in Ebrahimnejad's method. Futhermore, to overcome the limitations of Ebrahimnejad's method, the method

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[†] The contents of this chapter have been communicated in "Information Sciences" for the possible publication.

for aggregating the LR FFNs is discussed. Finally, to resolve the flaws of Ebrahimnejad's method [49], a new method (named as Mehar method) is proposed to solve the BFFTP with LR FFNs.

3.1 Preliminaries

In this section, some basic definitions, used in the further sections, are presented [49, Section 2, pp. 110].

Definition 3.1. A convex normalized fuzzy set having piecewise continuous membership function is called a fuzzy number.

Definition 3.2. A function $L: [0, \infty) \to [0,1]$ (or $R: [0, \infty) \to [0,1]$) is said to be the reference function of FNs if and only if (i) $L(0) = 1$ ($R(0) = 1$) (ii) L (or R) is non-increasing on $[0, \infty)$.

Definition 3.3. A FN $\tilde{A} = (a, b, c, d)_{LR}$ is said to be an LR FFN if its membership function $\mu_{\tilde{A}}(x)$ is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{b-x}{b-a}\right), & a \leq x < b, \\ 1, & b \leq x \leq c, \\ R\left(\frac{x-c}{d-c}\right), & c < x \leq d, \\ 0, & elsewhere. \end{cases}$$

Definition 3.4. If $L(x) = R(x) = max\{0, 1 - x\}$ then the LR FFN $\tilde{A} = (a, b, c, d)_{LR}$ is said to be a TrFN and it is denoted as $\tilde{A} = (a, b, c, d)$. The membership function of a TrFN $\tilde{A} = (a, b, c, d)$ is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x < b, \\ 1, & b \leq x \leq c, \\ \frac{d-x}{d-c}, & c < x \leq d, \\ 0, & elsewhere. \end{cases}$$

Definition 3.5. Two LR FFNs $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$ are said to be equal i.e. $\tilde{A}_1 = \tilde{A}_2$ if and only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$.

Definition 3.6. An LR FFN $\tilde{A} = (a, b, c, d)_{LR}$ is said to be non-negative LR FFN if and only if $a \geq 0$.

Definition 3.7. If $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$ are two LR FFNs then,

$\tilde{A}_1 + \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)_{LR}$.

Definition 3.8. If $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$ are two non-negative LR FFNs then, $\tilde{A}_1 \otimes \tilde{A}_2 = (a_1a_2, b_1b_2, c_1c_2, d_1d_2)_{LR}$.

3.2 Ranking of LR FFNs

If A_1 and A_2 are two distinct real numbers i.e., $A_1 \neq A_2$, then it can be easily concluded that $A_1 < A_2$ or $A_1 > A_2$. However, if $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$ are two distinct LR FFNs then it cannot easily concluded that $\tilde{A}_1 \prec \tilde{A}_2$ or $\tilde{A}_1 \succ \tilde{A}_2$. Different ranking methods have been proposed in the literature to find the ranking of two distinct LR FFNs.

In this section, the ranking methods, used by Kumar and Kaur [115, Section 6, pp. 88] and used by Ebrahimnejad [49], are discussed.

3.2.1 Ranking method used by Kumar and Kaur

Kumar and Kaur [115, Section 6, pp. 88] have used the following ranking method for comparing LR FFNs.

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$ be two distinct LR FFNs. Then,

(i) $\tilde{A}_1 \prec \tilde{A}_2$ if $Rank(\tilde{A}_1) < Rank(\tilde{A}_2)$.

(ii) $\tilde{A}_1 \succ \tilde{A}_2$ if $Rank(\tilde{A}_1) > Rank(\tilde{A}_2)$.

where,

$$Rank(\tilde{A}_i) = \frac{1}{2}\left\{\left(\int_0^1 b_i - (b_i - a_i)L^{-1}(\lambda)d\lambda\right) + \left(\int_0^1 c_i + (d_i - c_i)R^{-1}(\lambda)d\lambda\right)\right\}.$$

If $\tilde{A}_i = (a_i, b_i, c_i, d_i)$ is a TrFN i.e., $L(x) = R(x) = maximum\{0, 1 - x\}$ then, $L^{-1}(x) = R^{-1}(x) = 1 - x$ and hence, $Rank(\tilde{A}_i) = \frac{1}{2}\left\{\left(\int_0^1 b_i - (b_i - a_i)\,(1 - \lambda)d\lambda\right) + \left(\int_0^1 c_i + (d_i - c_i)\,(1 - \lambda)d\lambda\right)\right\} = \frac{a_i + b_i + c_i + d_i}{4}$.

3.2.2 Ranking method used by Ebrahimnejad

Ebrahimnejad [49] has used the following ranking method for comparing LR FFNs.

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$ be two distinct LR FFNs. Then,

(i) $\tilde{A}_1 \prec \tilde{A}_2$ if $a_1 < a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 < b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 < c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 < d_2$.

(ii) $\tilde{A}_1 \succ \tilde{A}_2$ if $a_1 > a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 > b_2$, $c_1 \geq c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 > c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, $d_1 > d_2$.

3.3 Fully fuzzy LPP of a BFFTP with *LR* FFNs

The fully fuzzy LPP $(P3.3.1)$ represents the fully fuzzy LPP of a BFFTP with LR FFNs [49, Section 3, pp. 112].

Fully fuzzy LPP $(P3.3.1)$

Minimize $\left[\sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4})_{LR} \otimes (x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4})_{LR}\right]$

Subject to

$\sum_{j=1}^{n}(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4})_{LR} = (a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4})_{LR}$, $i = 1,2,\dots,m;$

$\sum_{i=1}^{m}\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right)_{LR} = \left(b_{j,1}, b_{j,2}, b_{j,3}, b_{j,4}\right)_{LR}$, $j = 1,2, \ldots, n$;

$\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right)_{LR}$ is a non-negative LR FFN.

where,

(i) The LR FFN $\left(c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4}\right)_{LR}$ represents the fuzzy transportation cost for supplying the unit quantity of the product from the i^{th} source (S_i) to the j^{th} destination (D_j).

(ii) The LR FFN $\left(a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}\right)_{LR}$ represents the fuzzy availability of the product at i^{th} source (S_i).

(iii) The LR FFN $\left(b_{j,1}, b_{j,2}, b_{j,3}, b_{j,4}\right)_{LR}$ represents the fuzzy demand of the product at j^{th} destination (D_j).

(iv) m represents the number of sources and n represents the number of destinations.

(v) The LR FFN $\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right)_{LR}$ represents the quantity of the product to be supplied from the i^{th} source (S_i) to the j^{th} destination (D_j).

3.4. Existing methods for solving FFTPs with *LR* FFNs

The aim of this chapter is to point out the flaws of Ebrahimnejad's method [49, Section 5, pp. 114] and Kumar and Kaur's method [115, Section 6, pp. 88] as well as to resolve the flaws of these methods. Since, to point out the flaws of these methods, there is need to discuss these methods. Therefore, a brief review of these methods is presented in this section.

3.4.1 Kumar and Kaur's method

Kumar and Kaur [115, Section 6, pp. 88] have proposed the following method for solving the fully fuzzy LPP $(P1)$ of a BFFTP with LR FFNs.

Step 1: Using the multiplication, $(a_1, b_1, c_1, d_1)_{LR} \otimes (a_2, b_2, c_2, d_2)_{LR} = (a_1a_2, b_1b_2, c_1c_2, d_1d_2)_{LR}$, the fully fuzzy LPP $(P3.3.1)$ can be transformed into its equivalent fully fuzzy LPP $(P3.4.1.1)$.

Fully fuzzy LPP $(\boldsymbol{P3.4.1.1})$

Minimize $\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\right)_{LR}\right]$

Subject to

Constraints of the fully fuzzy LPP $(P3.3.1)$

Step 2: Using the relation, $\sum_{i=1}^{m}(a_i, b_i, c_i, d_i)_{LR} = (\sum_{i=1}^{m} a_i, \sum_{i=1}^{m} b_i, \sum_{i=1}^{m} c_i, \sum_{i=1}^{m} d_i)_{LR}$, the fully fuzzy LPP $(P3.4.1.1)$ can be transformed into its equivalent fully fuzzy LPP $(P3.4.1.2)$.

Fully fuzzy LPP $(\boldsymbol{P3.4.1.2})$

Minimize $\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\right)_{LR}\right]$

Subject to

$\left(\sum_{j=1}^{n} x_{ij,1}, \sum_{j=1}^{n} x_{ij,2}, \sum_{j=1}^{n} x_{ij,3}, \sum_{j=1}^{n} x_{ij,4}\right)_{LR} = \left(a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}\right)_{LR}; \quad i = 1,2, \dots, m,$

$\left(\sum_{i=1}^{m} x_{ij,1}, \sum_{i=1}^{m} x_{ij,2}, \sum_{i=1}^{m} x_{ij,3}, \sum_{i=1}^{m} x_{ij,4}\right)_{LR} = \left(b_{j,1}, b_{j,2}, b_{j,3}, b_{j,4}\right)_{LR}; \quad j = 1,2, \dots, n,$

$\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right)_{LR}$ is a non-negative LR FFN.

Step 3: Using the relation, $(a_1, b_1, c_1, d_1)_{LR} = (a_2, b_2, c_2, d_2)_{LR} \Rightarrow a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$ and the relation $(a, b, c, d)_{LR}$ is a non-negative LR FFN $\Rightarrow a \geq 0,\ b - a \geq$

$0\,, c-b \geq 0, d-c \geq 0$, the fully fuzzy LPP $(P3.4.1.2)$ can be transformed into its equivalent fuzzy LPP $(P3.4.1.3)$.

Fuzzy LPP $(\boldsymbol{P3.4.1.3})$

Minimize $\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\right)_{LR}\right]$

Subject to

$\sum_{j=1}^{n} x_{ij,1} = a_{i,1}, \sum_{j=1}^{n} x_{ij,2} = a_{i,2}, \sum_{j=1}^{n} x_{ij,3} = a_{i,3}, \sum_{j=1}^{n} x_{ij,4} = a_{i,4}; \quad i = 1,2,\ldots,m,$

$\sum_{i=1}^{m} x_{ij,1} = b_{j,1}\,, \sum_{i=1}^{m} x_{ij,2} = b_{j,2}\,, \sum_{i=1}^{m} x_{ij,3} = b_{j,3}\,, \sum_{i=1}^{m} x_{ij,4} = b_{j,4}\,;\ j = 1,2,\ldots,n,$

$x_{ij,1} \geq 0, x_{ij,2} - x_{ij,1} \geq 0, x_{ij,3} - x_{ij,2} \geq 0, x_{ij,4} - x_{ij,3} \geq 0.$

Step 4: The optimal solution of the fuzzy LPP $(P3.4.1.3)$ will be that feasible solution $\left\{x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right\}$ corresponding to which the value of the objective function $\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\right)_{LR}$ will be minimum.

Kumar and Kaur [115, Section 6, pp. 88] used the following method for finding the minimum of two LR FFNs $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$.

Find $minimum\left\{Rank\left(\tilde{A}_1\right), Rank\left(\tilde{A}_2\right)\right\}$ where, $Rank\left(\tilde{A}_1\right) = \frac{1}{2}\left\{\left(\int_0^1 b_1 - (b_1 - a_1)L^{-1}(\lambda)d\lambda\right) + \left(\int_0^1 c_1 + (d_1 - c_1)R^{-1}(\lambda)d\lambda\right)\right\}$ and $Rank\left(\tilde{A}_2\right) = \frac{1}{2}\left\{\left(\int_0^1 b_2 - (b_2 - a_2)L^{-1}(\lambda)d\lambda\right) + \left(\int_0^1 c_2 + (d_2 - c_2)R^{-1}(\lambda)d\lambda\right)\right\}$.

Case (i): If $minimum\left\{Rank\left(\tilde{A}_1\right), Rank\left(\tilde{A}_2\right)\right\} = Rank\left(\tilde{A}_1\right)$ then $minimum\left\{\tilde{A}_1, \tilde{A}_2\right\} = \tilde{A}_1$.

Case (ii): If $minimum\left\{Rank\left(\tilde{A}_1\right), Rank\left(\tilde{A}_2\right)\right\} = Rank\left(\tilde{A}_2\right)$ then $minimum\left\{\tilde{A}_1, \tilde{A}_2\right\} = \tilde{A}_2$.

Using this method, Kumar and Kaur [115, Section 6, pp. 88] transformed the fuzzy LPP $(P3.4.1.3)$ into its equivalent crisp LPP $(P3.4.1.4)$.

Crisp LPP $(\boldsymbol{P3.4.1.4})$

Minimize $\left[Rank\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\ \left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\right)_{LR}\right)\right]$

Subject to

Constraints of the fuzzy LPP $(P3.4.1.3)$.

Step 5: Using the relation, $k(\sum_{i=1}^{m}(a_i, b_i, c_i, d_i)_{LR}) = \sum_{i=1}^{m}(Rank(a_i, b_i, c_i, d_i)_{LR})$, the crisp LPP $(P3.4.1.4)$ can be transformed into its equivalent crisp LPP $(P3.4.1.5)$.

Crisp LPP $(\boldsymbol{P3.4.1.5})$

Minimize $\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\left(Rank\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\right)_{LR}\right)\right]$

Subject to

Constraints of the fuzzy LPP $(P3.4.1.3)$.

Step 6: Using the relation, $Rank((a, b, c, d)_{LR}) = \frac{1}{2}\left\{\left(\int_0^1 b - (b-a)L^{-1}(\lambda)d\lambda\right) + \left(\int_0^1 c + (d-c)R^{-1}(\lambda)d\lambda\right)\right\}$, the crisp LPP $(P3.4.1.5)$ can be transformed into its equivalent crisp LPP $(P3.4.1.6)$.

Crisp LPP $(\boldsymbol{P3.4.1.6})$

$$\text{Minimize}\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{2}\begin{pmatrix}\left(\int_0^1 c_{ij,2}x_{ij,2} - \left(c_{ij,2}x_{ij,2} - c_{ij,1}x_{ij,1}\right)L^{-1}(\lambda)d\lambda\right) + \\ \left(\int_0^1 c_{ij,3}x_{ij,3} + \left(c_{ij,4}x_{ij,4} - c_{ij,3}x_{ij,3}\right)R^{-1}(\lambda)d\lambda\right)\end{pmatrix}\right]$$

Subject to

Constraints of the fuzzy LPP $(P3.4.1.3)$

Step 7: Find the optimal solution $\{x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\}$ of the crisp LPP $(P3.4.1.6)$.

Step 8: Using the optimal solution $\{x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\}$, obtained in Step 7, find the fuzzy optimal solution $\left\{\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right)_{LR}\right\}$ and the optimal fuzzy transportation cost $\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\right)_{LR}$.

3.4.2 Ebrahimnejad's method

Ebrahimnejad [49, Section 5, pp. 114] proposed the following method for solving the fully fuzzy LPP $(P3.4.1.1)$ of a BFFTP with LR FFNs.

Step 1: Use Step 1 to Step 3 of Kumar and Kaur's method [115, Section 6, pp. 88] to transform the fully fuzzy LPP $(P3.4.1.1)$ into its equivalent fuzzy LPP $(P3.4.1.3)$.

Step 2: As discussed in Step 4 of Kumar and Kaur's method [115, Section 6, pp. 88], the optimal solution of the fuzzy LPP $(P3.4.1.3)$ will be that feasible solution $\{x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\}$ corresponding to which the value of the objective function $\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\right)_{LR}$ will be minimum.

Ebrahimnejad [49, Section 4, pp. 113] pointed out that Kumar and Kaur [115, Section 6, pp. 88] have assumed that if $\tilde{A}_1$ and $\tilde{A}_2$ are two distinct LR FFNs then $minimum\{Rank(\tilde{A}_1), Rank(\tilde{A}_2)\}$ will be either $Rank(\tilde{A}_1)$ or $Rank(\tilde{A}_2)$. However, there may exist two distinct LR FFNs $\tilde{A}_1$ and $\tilde{A}_2$ such that $Rank(\tilde{A}_1) = Rank(\tilde{A}_2)$ i.e., $minimum\{Rank(\tilde{A}_1), Rank(\tilde{A}_2)\}$ will be $Rank(\tilde{A}_1)$ as well as $Rank(\tilde{A}_2)$, which contradicts the well-known fact that the minimum of two fuzzy numbers should be a unique fuzzy number.

Keeping the same in mind, Ebrahimnejad [49, Section 5, pp. 114] used the following method to find the minimum of two LR FFNs $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$.

Check that $a_1 < a_2,\ b_1 \le b_2,\ c_1 \le c_2,\ d_1 \le d_2$ or $a_1 \le a_2,\ b_1 < b_2,\ c_1 \le c_2,\ d_1 \le d_2$ or $a_1 \le a_2, b_1 \le b_2, c_1 < c_2, d_1 \le d_2$ or $a_1 \le a_2, b_1 \le b_2, c_1 \le c_2, d_1 < d_2$ or $a_1 > a_2, b_1 \ge b_2, c_1 \ge c_2, d_1 \ge d_2$ or $a_1 \ge a_2, b_1 > b_2, c_1 \ge c_2, d_1 \ge d_2$ or $a_1 \ge a_2, b_1 \ge b_2, c_1 > c_2, d_1 \ge d_2$ or $a_1 \ge a_2, b_1 \ge b_2, c_1 \ge c_2, d_1 > d_2$.

Case (i): If $a_1 < a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 < b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 < c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 < d_2$ then $minimum\{\tilde{A}_1, \tilde{A}_2\} = \tilde{A}_1$.

Case (ii): If $a_1 > a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 > b_2$, $c_1 \geq c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 > c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, $d_1 > d_2$ then $minimum\{\tilde{A}_1, \tilde{A}_2\} = \tilde{A}_2$.

Using this method, Ebrahimnejad [49, Section 5, pp. 114] transformed the fuzzy LPP (P3.4.1.3) into its equivalent four crisp LPPs (P3.4.1.7), (P3.4.1.8), (P3.4.1.9) and (P3.4.1.10).

Crisp LPP ($\boldsymbol{P3.4.1.7}$)

Minimize$\left[\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,1}x_{ij,1}\right]$

Subject to

$\sum_{j=1}^{n} x_{ij,1} = a_{i,1}; \quad i = 1,2,\ldots,m,$

$\sum_{i=1}^{m} x_{ij,1} = b_{j,1}; \quad j = 1,2,\ldots,n,$

$x_{ij,1} \geq 0.$

Crisp LPP ($\boldsymbol{P3.4.1.8}$)

Minimize$\left[\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,2}x_{ij,2}\right]$

Subject to

$\sum_{j=1}^{n} x_{ij,2} = a_{i,2}; \quad i = 1,2,\ldots,m,$

$\sum_{i=1}^{m} x_{ij,2} = b_{j,2}; \quad j = 1,2,\ldots,n,$

$x_{ij,2} - x_{ij,1} \geq 0$, $x_{ij,2} \geq 0$, where, $x_{ij,1}$ is the optimal value of $x_{ij,1}$ obtained on solving the crisp LPP (P3.4.1.7).

Crisp LPP ($\boldsymbol{P3.4.1.9}$)

Minimize$\left[\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,3}x_{ij,3}\right]$

Subject to

$\sum_{j=1}^{n} x_{ij,3} = a_{i,3}; \quad i = 1,2,\ldots,m,$

$\sum_{i=1}^{m} x_{ij,3} = b_{j,3}; \quad j = 1,2,\ldots,n,$

Crisp LPP ($\boldsymbol{P3.4.1.10}$)

Minimize$\left[\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,4}x_{ij,4}\right]$

Subject to

$\sum_{j=1}^{n} x_{ij,4} = a_{i,4}; \quad i = 1,2,\ldots,m,$

$\sum_{i=1}^{m} x_{ij,4} = b_{j,4}; \quad j = 1,2,\ldots,n,$

$x_{ij,3} - x_{ij,2} \geq 0,\ x_{ij,3} \geq 0$, where, $x_{ij,2}$ is the optimal value of $x_{ij,2}$ obtained on solving the crisp LPP $(P3.4.1.8)$.

$x_{ij,4} - x_{ij,3} \geq 0,\ x_{ij,4} \geq 0$, where, $x_{ij,3}$ is the optimal value of $x_{ij,3}$ obtained on solving the crisp LPP $(P3.4.1.9)$.

Step 3: Find the optimal solutions $\{x_{ij,1}\}$, $\{x_{ij,2}\}$, $\{x_{ij,3}\}$ and $\{x_{ij,4}\}$ as well as the optimal values $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,1}x_{ij,1}$, $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,2}x_{ij,2}$, $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,3}x_{ij,3}$ and $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,4}x_{ij,4}$ of the crisp LPPs $(P3.4.1.7)$, $(P3.4.1.8)$, $(P3.4.1.9)$ and $(P3.4.1.10)$ respectively.

Step 4: Using the optimal solutions and optimal values, obtained in Step 3, find the fuzzy optimal solution $\left\{\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right)_{LR}\right\}$ and the optimal fuzzy transportation cost $\left(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,1}x_{ij,1}, \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,2}x_{ij,2}, \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,3}x_{ij,3}, \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij,4}x_{ij,4}\right)_{LR}$.

3.5 Limitations of Ebrahimnejad's method

To solve a real life transportation problem, the opinion of more than one expert about the parameters is collected. Then, all the collected information is aggregated to obtain a single value of each parameter. Since, Ebrahimnejad's method [49, Section 5, pp. 114] is proposed by considering that the aggregated value of each parameter is available. Therefore, Ebrahimnejad's method [49, Section 5, pp. 114] cannot be used to solve several real life FFTPs. For example, Ebrahimnejad's method [49, Section 5, pp. 14], cannot be used to solve the FFTP considered in Example 3.1.

Example 3.1 Let us consider, a product needs to be supplied from three sources to four destinations. For the same purpose, the information about each parameter is collected from two experts. If Table 3.1 represents the fuzzy transportation cost, the fuzzy availability and the fuzzy demand provided by the first decision-maker and if Table 3.2 represents the fuzzy transportation cost, the fuzzy availability and the fuzzy demand provided by the second decision-maker. Then, this FFTP cannot be solved by Ebrahimnejad's method [49, Section 5, pp. 114].

Table 3.1 Fuzzy data provided by the first decision-maker

Destination → Source ↓	D_1	D_2	Fuzzy availability
S_1	$(10,30,40,50)$	$(25,50,50,80)$	$(20,60,70,80)$
S_2	$(25,30,50,80)$	$(20,40,60,80)$	$(25,45,60,70)$
Fuzzy demand	$(40,60,80,100)$	$(40,45,60,70)$	

Table 3.2 Fuzzy data provided by the second decision-maker

Destination → Source ↓	D_1	D_2	Fuzzy availability
S_1	$(10,20,25,30)$	$(25,30,60,70)$	$(30,50,70,90)$
S_2	$(20,35,40,45)$	$(20,25,40,45)$	$(20,40,80,100)$
Fuzzy demand	$(20,55,60,100)$	$(40,45,50,80)$	

3.6 Modified Ebrahimnejad's method

The limitation of Ebrahimnejad's method [49, Section 5, pp. 114] can be overcome by including the following step (say Step 0) in Ebrahimnejad's method [49, Section 5, pp. 114].

Step 0: Use an appropriate aggregation operator to aggregate the data of all the decision-maker, For example, if

(i) w_k represents the normalized weight of the k^{th} decision-maker.

(ii) The LR FFN $\tilde{c}_{ij}^{k}=\left(c_{ij,1}^{k},c_{ij,2}^{k},c_{ij,3}^{k},c_{ij,4}^{k}\right)_{LR}$ represents the fuzzy transportation cost for supplying unit quantity of the product from i^{th} source to j^{th} destination, provided by the k^{th} decision-maker.

(iii) The LR FFN $\tilde{a}_{i}^{k}=\left(a_{i,1}^{k},a_{i,2}^{k},a_{i,3}^{k},a_{i,4}^{k}\right)_{LR}$ represents the fuzzy availability of the product at i^{th} source according to k^{th} decision-maker.

(iv) The LR FFN $\tilde{b}_j^k = \left(b_{j,1}^k, b_{j,2}^k, b_{j,3}^k, b_{j,4}^k\right)_{LR}$ represents the fuzzy demand of the product at j^{th} destination according to k^{th} decision-maker.

Then,

(i) $\sum_{k=1}^{p} w_k \tilde{c}_{ij}^k = \left(\sum_{k=1}^{p} w_k\, c_{ij,1}^k, \sum_{k=1}^{p} w_k\, c_{ij,2}^k, \sum_{k=1}^{p} w_k\, c_{ij,3}^k, \sum_{k=1}^{p} w_k\, c_{ij,4}^k\right)_{LR}$ will represent the aggregated value of the fuzzy transportation cost for transporting unit quantity of the product from i^{th} source to j^{th} destination.

(ii) The LR FFN $\sum_{k=1}^{p} w_k\, \tilde{a}_i^k = \left(\sum_{k=1}^{p} w_k\, a_{i,1}^k, \sum_{k=1}^{p} w_k\, a_{i,2}^k, \sum_{k=1}^{p} w_k\, a_{i,3}^k, \sum_{k=1}^{p} w_k\, a_{i,4}^k\right)_{LR}$ will represent the aggregated value of the fuzzy availability of the product at i^{th} source.

(iii) The LR FFN $\sum_{k=1}^{p} w_k\, \tilde{b}_j^k = \left(\sum_{k=1}^{p} w_k\, b_{j,1}^k, \sum_{k=1}^{p} w_k\, b_{j,2}^k, \sum_{k=1}^{p} w_k\, b_{j,3}^k, \sum_{k=1}^{p} w_k\, b_{j,4}^k\right)_{LR}$ will represent aggregated value of the fuzzy demand of the product at j^{th} destination.

For example, if in Example 3.1 the normalized weights of first and second decision-makers are 0.4 and 0.6 repsectively. Then, Table 3.3 will represent the aggregated fuzzy cost for supplying the unit quantity of the product, the fuzzy availability and the fuzzy demand.

Table 3.3 Aggregated fuzzy data of decision-makers

Destination → Source ↓	D_1	D_2	Fuzzy availability
S_1	$(10, 23, 29.5, 36)$	$(21.5, 43, 57, 80)$	$(27, 53, 70, 87)$
S_2	$(21.5, 33.5, 43, 55.5)$	$(20, 29.5, 46, 55.5)$	$(21.5, 41.5, 74, 91)$
Fuzzy demand	$(26, 56.5, 66, 100)$	$(26, 52, 53, 77)$	

3.7. Flaws of Ebrahimnejad's method

It is obvious from Step 2 of Ebrahimnejad's method [49, Section 5, pp. 114] that Ebrahimnejad [49, Section 5, pp. 114] has assumed that if $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$ are two distinct LR FFNs, then

Case (i): $minimum\{\tilde{A}_1, \tilde{A}_2\} = \tilde{A}_1$ if $a_1 < a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 < b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 < c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 < d_2$.

Case (ii): $minimum\{\tilde{A}_1, \tilde{A}_2\} = \tilde{A}_2$ if $a_1 > a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 > b_2$, $c_1 \geq c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 > c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, $d_1 > d_2$.

However, there may exist several LR FFNs for which neither the Case (i) nor the Case (ii) will be satisfied e.g.,

If $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR} = (1, 2, 4, 8)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR} = (1,3, 5, 7)_{LR}$, then it is obvious that neither Case (i) nor Case (ii) is satisfying i.e., $minimum\{\tilde{A}_1, \tilde{A}_2\}$ is neither $\tilde{A}_1$ nor $\tilde{A}_2$. Hence, both the LR FFNs $\tilde{A}_1$ and $\tilde{A}_2$ can be considered as minimum.

Due to this flaw, on solving a FFTP by Ebrahimnejad's method [49, Section 5, pp. 114] more than one fuzzy numbers, representing the optimal fuzzy transportation cost, may be obtained, which is ill-logical. To validate this statement, a BFFTP with TrFNs (LR FFNs with $L(x) = R(x) = maximum\,\{0, 1 - x\}$) having two sources S_1, S_2 and two destinations D_1, D_2, represented by Table 3.4, is considered.

Table 3.4 BFFTP with TrFNs

Destination → Source ↓	D_1	D_2	Fuzzy availability
S_1	$(20, 40, 60, 80)$	$(25, 50, 50, 80)$	$(40, 60, 80, 100)$
S_2	$(25, 50, 50, 80)$	$(20, 40, 60, 80)$	$(40, 60, 80, 100)$
Fuzzy demand	$(40, 60, 80, 100)$	$(40, 60, 80, 100)$	

Although, on solving the considered BFFTP, represented by Table 3.4, by Ebrahimnejad's method [49] the following fuzzy optimal solution and optimal fuzzy transportation cost is obtained.

Fuzzy optimal solution: $\tilde{x}_{11} = (40, 60, 60,60)$, $\tilde{x}_{12} = (0, 0, 20, 40)$, $\tilde{x}_{21} = (0, 0, 20, 40)$, $\tilde{x}_{22} = (40, 60, 60,60)$.

Optimal fuzzy transportation cost: $(1600, 4800, 9200, 16000)$.

However, it does not mean that the BFFTP, represented by Table 3.4, has a unique fuzzy optimal solution. The following clearly indicates that according to ranking method, used in in Step 2 of Ebrahimnejad's method [49, Section 5, pp. 114], $\tilde{x}_{11} = (40, 40, 40, 40)$, $\tilde{x}_{12} = (0, 20, 40, 60)$, $\tilde{x}_{21} = (0, 20, 40, 60)$, $\tilde{x}_{22} = (40, 40, 40, 40)$ is also a fuzzy optimal solution for the considered BFFTP.

It is well-known fact that every feasible solution (basic and non-basic) of a crisp transportation problem will be an optimal solution if corresponding to it, the obtained total transportation cost is minimum. On the same direction, every fuzzy feasible solution of a FFTP will be a fuzzy optimal solution if corresponding to it the obtained fuzzy transportation cost is minimum.

It can be easily verified that $\tilde{x}_{11} = (40, 40, 40, 40)$, $\tilde{x}_{12} = (0, 20, 40, 60)$, $\tilde{x}_{21} = (0, 20, 40, 60)$, $\tilde{x}_{22} = (40, 40, 40, 40)$ is a fuzzy feasible solution of the FFTP, represented by Table 3.4, as it is satisfying the following necessary constraints of the considered FFTP.

(i) $\tilde{x}_{11} + \tilde{x}_{12} = (40, 60, 80, 100)$,

(ii) $\tilde{x}_{21} + \tilde{x}_{22} = (40, 60, 80, 100)$,

(iii) $\tilde{x}_{11} + \tilde{x}_{21} = (40, 60, 80, 100)$,

(iv) $\tilde{x}_{12} + \tilde{x}_{22} = (40, 60, 80, 100)$,

(v) $\tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{21}, \tilde{x}_{22}$ are non-negative TrFNs.

Now, as discussed above this fuzzy feasible solution will be a fuzzy optimal solution of the BFFTP, represented by Table 3.4, if corresponding to it, the obtained fuzzy transportation cost i.e., $(20, 40, 60, 80) \otimes \tilde{x}_{11} + (25, 50, 50, 80) \otimes \tilde{x}_{12} + (25, 50, 50, 80) \otimes \tilde{x}_{21} + (20, 40, 60, 80) \otimes \tilde{x}_{22}$ will be minimum.

It can be easily verified that corresponding to the considered fuzzy feasible solution $\tilde{x}_{11} = (40, 40, 40, 40), \tilde{x}_{12} = (0, 20, 40, 60)$, $\tilde{x}_{21} = (0, 20, 40, 60)$, $\tilde{x}_{22} = (40, 40, 40, 40)$, the obtained fuzzy transportation cost is $(1600, 5200, 8800, 16000)$.

Now, as discussed in Step 2 of Ebrahimnejad’s method [49, Section 5, pp. 114] that if $\tilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)$ are two TrFNs then $\tilde{A}_1 < \tilde{A}_2$ i.e., $minimum\{\tilde{A}_1, \tilde{A}_2\} = \tilde{A}_1$ if $a_1 < a_2,\ b_1 \le b_2,\ c_1 \le c_2,\ d_1 \le d_2$ or $a_1 \le a_2,\ b_1 < b_2,$

$c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 < c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 < d_2$.

However, for the TrFN $(1600, 4800, 9200, 16000)$, representing the fuzzy optimal transportation cost of the considered fully fuzzy TP obtained by Ebrahimnejad's method [49, Section 5, pp. 114] and for the TrFN $(1600, 5200, 8800, 16000)$, representing the fuzzy optimal transportation cost of the considered BFFTP corresponding to the considered fuzzy feasible solution, none of the conditions $a_1 < a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 < b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 < c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 < d_2$ is satisfying.

Therefore, according to the ranking method, used in Step 2 of Ebrahimnejad's method [49, Section 5, pp. 114], it is not possible to conclude that the TrFN $(1600, 4800, 9200, 16000)$ is less than the TrFN $(1600, 5200, 8800, 16000)$ or the TrFN $(1600, 5200, 8800, 16000)$ is less than the TrFN $(1600, 4800, 9200, 16000)$.

Hence, according to ranking method, used in Step 2 of Ebrahimnejad's method [49, Section 5, pp. 114], both the TrFNs $(1600, 4800, 9200, 16000)$ and $(1600, 5200, 8800, 16000)$ represents the fuzzy optimal transportation cost of the BFFTP, represented by Table 3.4. Since, the physical meaning of the TrFNs $(1600, 4800, 9200, 16000)$ and $(1600, 5200, 8800, 16000)$ are different. So, it is inappropriate to use Ebrahimnejad's method [49, Section 5, pp. 114] for solving FFTPs.

3.8 Flaws of Kumar and Kaur's method

Ebrahimnejad [49, Section 4, pp. 113] pointed out that on applying Kumar and Kaur's method [115, Section 6, pp. 88] more than one *LR* FFNs, representing the optimal fuzzy transportation cost may be obtained, which is illogical. But, Ebrahimnejad [49] cannot find any FFTP to validate this claim.

However, it can be easily verified that on solving the FFTP, represented by Table 3.4, by Kumar and Kaur's method [115, Section 6, pp. 88], the following two distinct TrFN, representing the optimal fuzzy transportation cost, are obtained.

First TrFN representing the optimal fuzzy transportation cost: $(1600, 4800, 9200, 16000)$.

Second TrFN representing the optimal fuzzy transportation cost: $(1600, 5200, 8800, 16000)$.

The first TrFN, representing the optimal fuzzy transportation cost i.e., $(1600, 4800, 9200, 16000)$ is corresponding to the fuzzy optimal solution, $\tilde{x}_{11} = (40, 60, 60, 60)$, $\tilde{x}_{12} = (0, 0, 20, 40)$, $\tilde{x}_{21} = (0, 0, 20, 40)$, $\tilde{x}_{22} = (40, 60, 60, 60)$.

Whereas, the second TrFN, representing the optimal fuzzy transportation cost i.e., $(1600, 5200, 8800, 16000)$ is corresponding to the fuzzy optimal solution $\tilde{x}_{11} = (40, 40, 40, 40)$, $\tilde{x}_{12} = (0, 20, 40, 60)$, $\tilde{x}_{21} = (0, 20, 40, 60)$, $\tilde{x}_{22} = (40, 40, 40, 40)$.

It is pertinent to mention that as $Rank(1600, 4800, 9200, 16000) = Rank(1600, 5200, 8800, 16000)$. Therefore, according to the ranking method, used in Kumar and Kaur's method [115, Section 6, pp. 88], $(1600, 4800, 9200, 16000) = (1600, 5200, 8800, 16000)$

3.9 Reasons for the occurrence of flaws in Kumar and Kaur's method as well as Ebrahimnejad's method

In this section, the reasons for the occurrence of flaws in Kumar and Kaur's method [115, Section 6, pp. 88] as well as Ebrahimnejd's method [49, Section 5, pp. 114] are discussed.

3.9.1 Reasons for the occurrence of flaws in Kumar and Kaur's method

Kumar and Kaur [115, Section 6, pp. 88] used the ranking method, discussed in

Section 3.2.1, for the ranking of LR FFNs. However, it is not appropriate to use this method due to the following reason:

If $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$ are two distinct LR FNNs i.e., $a_1 \neq a_2$, $b_1 \neq b_2, c_1 \neq c_2, d_1 \neq d_2$ then, either $\tilde{A}_1 \succ \tilde{A}_2$ or $\tilde{A}_1 \prec \tilde{A}_2$. While, on applying the ranking method, used by Kumar and Kaur [49] for two distinct LR FNNs $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$, the relation $\tilde{A}_1 = \tilde{A}_2$ may be obtained, which is mathematically incorrect e.g., if $\tilde{A}_1 = (1, 2, 5, 6)$ and $\tilde{A}_2 = (1, 3, 4, 6)$ are two TrFNs then according to the ranking method, used by Kumar and Kaur [115] and discussed in Section 3.1, $Rank(\tilde{A}_1) = 3.5$, $Rank(\tilde{A}_2) = 3.5$. $Rank(\tilde{A}_1) = Rank(\tilde{A}_2) \Rightarrow \tilde{A}_1 = \tilde{A}_2$. But, it is obvious that $\tilde{A}_1 \neq \tilde{A}_2$. This clearly indicates that the ranking method, used by Kumar and Kaur [115, Section 6, pp. 88], is not valid.

3.9.2 Reasons for the occurrence of flaws in Ebrahimnejad's method

Ebrahimnejad [49, Section 5, pp. 114] has used the ranking method, discussed in Section 3.2, for the ranking of LR FFNs. However, it is not appropriate to use this method due to the following reason:

If $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$ are two distinct LR FFNs then, either $\tilde{A}_1 \succ \tilde{A}_2$ or $\tilde{A}_1 \prec \tilde{A}_2$. However, ranking method, used by Ebrahimnejad [49, Section 5, pp. 114], for two distinct LR FNNs $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$, fails to rank the $\tilde{A}_1$and $\tilde{A}_2$ e.g., if $\tilde{A}_1 = (1, 2, 5, 6)$ and $\tilde{A}_2 = (1, 3, 4, 6)$ are two TrFNs then for these LR FNNs none of the conditions $a_1 < a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 < b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 < c_2$, $d_1 \leq d_2$ or $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 < d_2$, $a_1 > a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 > b_2$, $c_1 \geq c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 > c_2$, $d_1 \geq d_2$ or $a_1 \geq a_2$, $b_1 \geq b_2$,

$c_1 \geq c_2,\ d_1 > d_2$ is satisfying. Therefore, according to the ranking method used by Ebrahimnejad [49] and discussed in Section 3.2, neither $\tilde{A}_1 \prec \tilde{A}_2$ nor $\tilde{A}_1 \succ \tilde{A}_2$ i.e., the ranking method, used by Ebrahimnejad [49, Section 5, pp. 114] fails to find the ranking of two distinct LR FNNs $\tilde{A}_1$ and $\tilde{A}_2$. This clearly indicates that the ranking method, used by Ebrahimnejad [49, Section 5, pp. 114], is not valid.

3.10 Advantages of RMDS ranking approach over the ranking approaches used by Kumar and Kaur as well as Ebrahimnejad

It is obvious from Section 3.9 that the flaws in Kumar and Kaur's method [115, Section 6, pp. 88] and Ebrahimnejad's method [49, Section 5, pp. 114] are occurring due to the following flaws of the ranking methods, used by Kumar and Kaur [115, Section 6, pp. 88] as well as Ebrahimnejad [49, Section 5, pp. 114], in their proposed methods.

(1) The ranking method, used by Kumar and Kaur [115, Section 6, pp. 88], indicates that two distinct LR FFNs are equal e.g., the ranking used by Kumar and Kaur [115, Section 6, pp. 88], indicates that the distinct TrFNs $\tilde{A}_1 = (1, 2, 5, 6)$ and $\tilde{A}_2 = (1, 3, 4, 6)$ are equal.

(2) The ranking method, used by Ebrahimnejad [49, Section 5, pp. 114], fails to find the ranking of two distinct LR FFNs e.g., the ranking method, used by Ebrahimnejad [49, Section 5, pp. 114], fails to find the ranking of two distinct LR FFNs $\tilde{A}_1 = (1, 2, 5, 6)_{LR}$ and $\tilde{A}_2 = (1, 3, 4, 6)_{LR}$.

It is pertinent to mention that these flaws are not occurring in the RMDS approach proposed by Kaur and Kumar [108] i.e., on applying the RMDS approach for two distinct LR FFNs, $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$, either the relation $\tilde{A}_1 \succ \tilde{A}_2$ or the relation $\tilde{A}_1 \prec \tilde{A}_2$ will be obtained. The relation $\tilde{A}_1 = \tilde{A}_2$ will be obtained only if in actual case $\tilde{A}_1 = \tilde{A}_2$ i.e., $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$.

Hence, the flaws of Kumar and Kaur's method [115, Section 6, pp. 88] and Ebrahimnejad's method [49, Section 5, pp. 114] can be resolved by using RMDS ranking approach [108] for finding the fuzzy optimal solution of fuzzy LPP $(P3.3.1)$.

3.10.1 RMDS ranking approach

The steps of the RMDS ranking approach [108] are as follows:

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1)_{LR}$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)_{LR}$ are two distinct LR FFNs.

Step 1: Find $Rank(\tilde{A}_1) = \frac{1}{2}\left\{\left(\int_0^1 b_1 - (b_1 - a_1)L^{-1}(\lambda)d\lambda\right) + \left(\int_0^1 c_1 + (d_1 - c_1)R^{-1}(\lambda)d\lambda\right)\right\}$,

$Rank(\tilde{A}_2) = \frac{1}{2}\left\{\left(\int_0^1 b_2 - (b_2 - a_2)L^{-1}(\lambda)d\lambda\right) + \left(\int_0^1 c_2 + (d_2 - c_2)R^{-1}(\lambda)d\lambda\right)\right\}$ and check that $Rank(\tilde{A}_1) > Rank(\tilde{A}_2)$ or $Rank(\tilde{A}_1) < Rank(\tilde{A}_2)$ or $Rank(\tilde{A}_1) = Rank(\tilde{A}_2)$.

Case (i): If $Rank(\tilde{A}_1) > Rank(\tilde{A}_2)$ then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (ii): If $Rank(\tilde{A}_1) < Rank(\tilde{A}_2)$ then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (iii): If $Rank(\tilde{A}_1) = Rank(\tilde{A}_2)$ then go to Step 2.

Step 2: Find $(\tilde{A}_1) = \frac{1}{2}\int_0^1 (b_1 + c_1)d\lambda$, $Mode(\tilde{A}_2) = \frac{1}{2}\int_0^1 (b_2 + c_2)d\lambda$ and check that $Mode(\tilde{A}_1) > Mode(\tilde{A}_2)$ or $Mode(\tilde{A}_1) < Mode(\tilde{A}_2)$ or $Mode(\tilde{A}_1) = Mode(\tilde{A}_2)$.

Case (i): If $Mode(\tilde{A}_1) > Mode(\tilde{A}_2)$ then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (ii): If $Mode(\tilde{A}_1) < Mode(\tilde{A}_2)$ then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (iii): If $Mode(\tilde{A}_1) = Mode(\tilde{A}_2)$ then go to Step 3.

Step 3: Find

$Div(\tilde{A}_1) = \int_0^1 c_1 \; d\lambda + \int_0^1 (d_1 - c_1)R^{-1}(\lambda) \; d\lambda - \int_0^1 b_1 \; d\lambda + \int_0^1 (b_1 - a_1)L^{-1}(\lambda) \; d\lambda$,

$Div(\tilde{A}_2) = \int_0^1 c_2 \; d\lambda + \int_0^1 (d_2 - c_2)R^{-1}(\lambda) \; d\lambda - \int_0^1 b_2 \; d\lambda + \int_0^1 (b_2 - a_2)L^{-1}(\lambda) \; d\lambda$ and check that $Div(\tilde{A}_1) > Div(\tilde{A}_2)$ or $Div(\tilde{A}_1) < Div(\tilde{A}_2)$ or $Div(\tilde{A}_1) = Div(\tilde{A}_2)$.

Case (i): If $Div(\tilde{A}_1) > Div(\tilde{A}_2)$ then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (ii): If $Div(\tilde{A}_1) < Div(\tilde{A}_2)$ then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (iii): If $Div(\tilde{A}_1) = Div(\tilde{A}_2)$ then go to Step 4.

Step 4: Find $Spread(\tilde{A}_1) = \int_0^1 (b_1 - a_1)L^{-1}(\lambda) \; d\lambda$, $Spread(\tilde{A}_2) = \int_0^1 (b_2 - a_2)L^{-1}(\lambda) \; d\lambda$ and check that $Spread(\tilde{A}_1) > Spread(\tilde{A}_2)$ or $Spread(\tilde{A}_1) < Spread(\tilde{A}_2)$ or $Spread(\tilde{A}_1) = Spread(\tilde{A}_2)$.

Case (i): If $Spread(\tilde{A}_1) > Spread(\tilde{A}_2)$ then $\tilde{A}_1 \succ \tilde{A}_2$.

Case (ii): If $Spread(\tilde{A}_1) < Spread(\tilde{A}_2)$ then $\tilde{A}_1 \prec \tilde{A}_2$.

Case (iii): If $Spread(\tilde{A}_1) = Spread(\tilde{A}_2)$ then $\tilde{A}_1 = \tilde{A}_2$.

Remark 3.1: If $\tilde{A} = (a, b, c, d)$ is a TrFN (LR FFN with $L(x) = R(x) = maximum\{0, 1 - x\}$) then

$Rank(\tilde{A}) = \frac{a+b+c+d}{4}$, $Mode(\tilde{A}) = \frac{b+c}{2}$, $Div(\tilde{A}) = \frac{c+d-a-b}{2}$, $Spread(\tilde{A}) = \frac{b-a}{2}$.

3.10.2 Validity of RMDS ranking approach

In Case (iii) of Step 4 of RMDS ranking approach [108], it is claimed that if $Rank(\tilde{A}_1) = Rank(\tilde{A}_2)$, $Mode(\tilde{A}_1) = Mode(\tilde{A}_2)$, $Div(\tilde{A}_1) = Div(\tilde{A}_2)$ and $Spread(\tilde{A}_1) = Spread(\tilde{A}_2)$ then $\tilde{A}_1 = \tilde{A}_2$ i.e., $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$.

The following clearly indicates that this claim is valid.

$Rank(\tilde{A}_1) = Rank(\tilde{A}_2) \Rightarrow \frac{1}{2}\left\{\left(\int_0^1 b_1 - (b_1 - a_1)L^{-1}(\lambda)d\lambda\right) + \left(\int_0^1 c_1 + (d_1 - c_1)R^{-1}(\lambda)d\lambda\right)\right\} = \frac{1}{2}\left\{\left(\int_0^1 b_2 - (b_2 - a_2)L^{-1}(\lambda)d\lambda\right) + \left(\int_0^1 c_2 + (d_2 - c_2)R^{-1}(\lambda)d\lambda\right)\right\}.$ (3.10.2.1)

$Mode(\tilde{A}_1) = Mode(\tilde{A}_2) \Rightarrow \frac{1}{2}\int_0^1 (b_1 + c_1)d\lambda = \frac{1}{2}\int_0^1 (b_2 + c_2)d\lambda.$ (3.10.2.2)

$Div(\tilde{A}_1) = Div(\tilde{A}_2) \Rightarrow \int_0^1 c_1 \, d\lambda + \int_0^1 (d_1 - c_1)R^{-1}(\lambda)\, d\lambda - \int_0^1 b_1 \, d\lambda + \int_0^1 (b_1 - a_1)L^{-1}(\lambda)\, d\lambda = \int_0^1 c_2 \, d\lambda + \int_0^1 (d_2 - c_2)R^{-1}(\lambda)\, d\lambda - \int_0^1 b_2 \, d\lambda + \int_0^1 (b_2 - a_2)L^{-1}(\lambda)\, d\lambda.$ (3.10.2.3)

$Spread(\tilde{A}_1) = Spread(\tilde{A}_2) \Rightarrow \int_0^1 (b_1 - a_1)L^{-1}(\lambda)\, d\lambda = \int_0^1 (b_2 - a_2)L^{-1}(\lambda)\, d\lambda.$ (3.10.2.4)

It can be easily verified that on solving equations (3.10.2.1), (3.10.2.2), (3.10.2.3) and (3.10.2.4), the obtained solution is $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$ i.e., $\tilde{A}_1 = \tilde{A}_2$.

3.11 Proposed Mehar method

In this section, using RMDS ranking approach [108], a method (named as Mehar method) is proposed for solving fully fuzzy LPP $(P3.3.1)$ of a BFFTP with LR FFNs.

The steps of the proposed Mehar method are as follows:

Step 1: Use Step 1 to Step 5 of Kumar and Kaur's method [115, Section 6, pp. 88] to transform the fully fuzzy LPP $(P3.4.1)$ into the crisp LPP $(P3.4.1.6)$.

Step 2: Find all the possible optimal basic feasible solutions $\{x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\}$ of the crisp LPP $(P3.4.1.6)$.

Step 3: Find the value of the objective function of the fuzzy LPP $(P3.4.1.3)$ i.e., $\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\right)_{LR}$ corresponding to each possible alternative optimal basic feasible solution obtained in Step 2.

Step 4: Find the $Mode$ of each LR FFN, representing the value of the objective function corresponding to an optimal basic feasible solution of the crisp LPP $(P3.4.1.6)$, obtained in Step 3.

Step 5: Find the minimum of all the real numbers (values of $Mode$), obtained in Step 4, and check that the obtained minimum value is corresponding to a unique LR FFN (representing the value of objective function of the fuzzy LPP $(P3.4.1.3)$ corresponding to an optimal basic feasible solution) or not.

Case (i): If the obtained minimum value is corresponding to a unique fuzzy LR FFN then that LR FFN will represent the optimal fuzzy transportation cost and $\left\{\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right)_{LR}\right\}$, obtained by the basic feasible solution corresponding to which this LR FFN is obtained, represents the fuzzy optimal solution of the BFFTP.

Case (ii): If Case (i) is not satisfied then go to Step 6.

Step 6: Find $Divergence$ of those LR FFNs, representing the value of objective function corresponding to an optimal basic feasible solution of the crisp LPP $(P3.4.1.6)$, corresponding to which the value of $Mode$ is minimum.

Step 7: Find the minimum of all the real numbers (values of $Divergence$), obtained in Step 6, and check that the obtained minimum value is corresponding to a unique LR FFN or not.

Case (i): If the obtained minimum value is corresponding to a unique LR FFN then that LR FFN will represent the optimal fuzzy transportation cost and $\left\{\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right)_{LR}\right\}$, obtained by the basic feasible solution corresponding to which this LR FFN is obtained, represents the fuzzy optimal solution of the BFFTP.

Case (ii): If Case (i) is not satisfied then go to Step 8.

Step 8: Find $Spread$ of those LR FFNs, representing the value of objective function corresponding to optimal basic feasible solution of the crisp LPP $(P3.4.1.6)$, corresponding to which the value of $Divergence$ is minimum.

Step 9: Find minimum of all the real numbers (values of $Spread$), obtained in Step 8. The minimum will occur corresponding to a unique LR FFN and hence, that LR FFN will represent the optimal fuzzy transportation cost as well as $\left\{\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right)_{LR}\right\}$, obtained by the basic feasible solution corresponding to which this LR FFN is obtained, represents the fuzzy optimal solution of the BFFTP.

3.12 Unique optimal fuzzy transportation cost of the considered FFTP

In Section 3.7 and Section 3.8, a FFTP, represented by Table 3.4, is solved by Ebrahimnejad's method [49, Section 5, pp. 114] and Kumar and Kaur's method [115, Section 6, pp. 88] respectively and shown that more than one TrFNs, representing the optimal fuzzy transportation cost, are obtained, which is mathematically incorrect.

In this section, the same FFTP is solved by proposed Mehar method and shown that a unique TrFN, representing the optimal fuzzy transportation cost, is obtained.

Using the proposed Mehar method, proposed in Section 3.11, a unique fuzzy optimal value, representing the optimal fuzzy transportation cost of the considered FFTP, of the fully fuzzy LPP $(P3.12.1)$ can be obtained as follows.

Fully fuzzy LPP $(\boldsymbol{P3.12.1})$

$$\text{Minimize}\begin{pmatrix} (20,40,60,80)\otimes\left(x_{11,1},x_{11,2},x_{11,3},x_{11,4}\right)+ \\ (25,50,50,80)\otimes(x_{12,1},x_{12,2},x_{12,3},x_{12,4}) \\ +(25,50,50,80)\otimes\left(x_{21,1},x_{21,2},x_{21,3},x_{21,4}\right)+ \\ (20,40,60,80)\otimes(x_{22,1},x_{22,2},x_{22,3},x_{22,4}) \end{pmatrix}$$

Subject to

$\left(x_{11,1},x_{11,2},x_{11,3},x_{11,4}\right)+(x_{12,1},x_{12,2},x_{12,3},x_{12,4})=(40,60,80,100)$,

$\left(x_{21,1},x_{21,2},x_{21,3},x_{21,4}\right)+(x_{22,1},x_{22,2},x_{22,3},x_{22,4})=(40,60,80,100)$,

$\left(x_{11,1},x_{11,2},x_{11,3},x_{11,4}\right)+\left(x_{21,1},x_{21,2},x_{21,3},x_{21,4}\right)=(40,60,80,100)$,

$(x_{12,1},x_{12,2},x_{12,3},x_{12,4})+(x_{22,1},x_{22,2},x_{22,3},x_{22,4})=(40,60,80,100)$,

$\left(x_{11,1},x_{11,2},x_{11,3},x_{11,4}\right)$, $\left(x_{12,1},x_{12,2},x_{12,3},x_{12,4}\right)$, $\left(x_{21,1},x_{21,2},x_{21,3},x_{21,4}\right)$ and $\left(x_{22,1},x_{22,2},x_{22,3},x_{22,4}\right)$ are non-negative TrFN.

Step 1: According to Step 1 of the proposed Mehar method, there is need to use Step 1 to Step 5 of Kumar and Kaur's method [115, Section 6, pp. 88] for transforming the fully fuzzy LPP $(P3.12.1)$ into a fuzzy LPP having fuzzy objective function and crisp constraints.

Using Kumar and Kaur's method [115, Section 6, pp. 88], discussed in Section 3.4.1, the fully fuzzy LPP $(P3.12.1)$ can be transformed as follows:

Step 1(a): Using Step 1 of Kumar and Kaur's method [115, Section 6, pp. 88], the fully fuzzy LPP $(P3.12.1)$ can be transformed into its equivalent fully fuzzy LPP $(P3.12.2)$.

Fully fuzzy LPP $(\boldsymbol{P3.12.2})$

$$\text{Minimize}\begin{pmatrix} \left(20x_{11,1},40x_{11,2},60x_{11,3},80x_{11,4}\right)+(25x_{12,1},50x_{12,2},50x_{12,3},80x_{12,4}) \\ +\left(25x_{21,1},50\,x_{21,2},50x_{21,3},80x_{21,4}\right)+(20x_{22,1},40x_{22,2},60x_{22,3},80x_{22,4}) \end{pmatrix}$$

Subject to

Constraints of the fully fuzzy LPP (P3.12.1).

Step 1(b): Using Step 2 of Kumar and Kaur's method [115, Section 6, pp. 88], the fully fuzzy LPP ($P3.12.2$) can be transformed into its equivalent fully fuzzy LPP ($P3.12.3$).

Fully fuzzy LPP ($\boldsymbol{P3.12.3}$)

$$\text{Minimize}\left(\begin{array}{c}(20x_{11,1}, 40x_{11,2}, 60x_{11,3}, 80x_{11,4}) + (25x_{12,1}, 50x_{12,2}, 50x_{12,3}, 80x_{12,4}) \\ +(25x_{21,1}, 50\,x_{21,2}, 50x_{21,3}, 80x_{21,4}) + (20x_{22,1}, 40x_{22,2}, 60x_{22,3}, 80x_{22,4})\end{array}\right)$$

Subject to

$(x_{11,1} + x_{12,1}, x_{11,2} + x_{12,2}, x_{11,3} + x_{12,3}, x_{11,4} + x_{12,4}) = (40, 60, 80, 100)$,

$(x_{21,1} + x_{22,1}, x_{21,2} + x_{22,2}, x_{21,3} + x_{22,3}, x_{21,4} + x_{22,4}) = (40, 60, 80, 100)$,

$(x_{11,1} + x_{21,1}, x_{11,2} + x_{21,2}, x_{11,3} + x_{21,3}, x_{11,4} + x_{21,4}) = (40, 60, 80, 100)$,

$(x_{12,1} + x_{22,1}, x_{12,2} + x_{22,2}, x_{12,3} + x_{22,3}, x_{12,4} + x_{22,4}) = (40, 60, 80, 100)$,

$(x_{11,1}, x_{11,2}, x_{11,3}, x_{11,4}), (x_{12,1}, x_{12,2}, x_{12,3}, x_{12,4}), (x_{21,1}, x_{21,2}, x_{21,3}, x_{21,4})$ and $(x_{22,1}, x_{22,2}, x_{22,3}, x_{22,4})$ are non-negative TrFNs.

Step 1(c): Using Step 3 of Kumar and Kaur's method [115, Section 6, pp. 88], the fully fuzzy LPP ($P3.12.3$) can be transformed into its equivalent fuzzy LPP ($P3.12.4$).

Fuzzy LPP ($\boldsymbol{P3.12.4}$)

$$\text{Minimize}\left(\begin{array}{c}(20x_{11,1}, 40x_{11,2}, 60x_{11,3}, 80x_{11,4}) + (25x_{12,1}, 50x_{12,2}, 50x_{12,3}, 80x_{12,4}) \\ +(25x_{21,1}, 50\,x_{21,2}, 50x_{21,3}, 80x_{21,4}) + (20x_{22,1}, 40x_{22,2}, 60x_{22,3}, 80x_{22,4})\end{array}\right)$$

Subject to

$x_{11,1} + x_{12,1} = 40$, $x_{21,1} + x_{22,1} = 40$, $x_{11,1} + x_{21,1} = 40$, $x_{12,1} + x_{22,1} = 40$, $x_{11,2} + x_{12,2} = 60$, $x_{21,2} + x_{22,2} = 60$, $x_{11,2} + x_{21,2} = 60$, $x_{12,2} + x_{22,2} = 60$, $x_{11,3} + x_{12,3} = 80$, $x_{21,3} + x_{22,3} = 80$, $x_{11,3} + x_{21,3} = 80$, $x_{12,3} + x_{22,3} = 100$, $x_{11,4} + x_{12,4} = 100$, $x_{21,4} + x_{22,4} = 100$, $x_{11,4} + x_{21,4} = 100$, $x_{12,4} + x_{22,4} = 100$, $x_{11,2} -$

$x_{11,1} \geq 0,\ x_{12,2} - x_{12,1} \geq 0, x_{21,2} - x_{21,1} \geq 0,\ x_{22,2} - x_{22,1} \geq 0,\ x_{11,3} - x_{11,2} \geq 0, x_{12,3} - x_{12,2} \geq 0, x_{21,3} - x_{21,2} \geq 0,\ x_{22,3} - x_{22,2} \geq 0,\ x_{11,4} - x_{11,3} \geq 0,\ x_{12,4} - x_{12,3} \geq 0, x_{21,4} - x_{21,3} \geq 0,\ x_{22,4} - x_{22,3} \geq 0,\ x_{11,1},\ x_{12,1},\ x_{21,1},\ x_{22,1} \geq 0.$

Step 1(d): Using Step 3 of Kumar and Kaur's method [115, Section 6, pp. 88], the fuzzy LPP $(P3.12.4)$ can be transformed into its equivalent crisp LPP $(P3.12.5)$.

Crisp LPP $(\boldsymbol{P3.12.5})$

$$\text{Minimize}\left(Rank\left(\begin{array}{c}(20x_{11,1}, 40x_{11,2}, 60x_{11,3}, 80x_{11,4}) + (25x_{12,1}, 50x_{12,2}, 50x_{12,3}, 80x_{12,4}) \\ +(25x_{21,1}, 50\ x_{21,2}, 50x_{21,3}, 80x_{21,4}) + (20x_{22,1}, 40x_{22,2}, 60x_{22,3}, 80x_{22,4})\end{array}\right)\right)$$

Subject to

Cosntraints of the fuzzy LPP $(P3.12.4)$.

Step 1(e): Using Step 5 of Kumar and Kaur's method [115, Section 6, pp. 88], the fuzzy LPP $(P3.12.5)$can be transformed into its equivalent crisp LPP $(P3.12.6)$.

Crisp LPP $(\boldsymbol{P3.12.6})$

$$\text{Minimize}\left(\frac{\begin{array}{c}20x_{11,1} + 40x_{11,2} + 60x_{11,3} + 80x_{11,4} + 25x_{12,1} + 50x_{12,2} + 50x_{12,3} + 80x_{12,4} \\ +25x_{21,1} + 50x_{21,2} + 50x_{21,3} + 80x_{21,4} + 20x_{22,1} + 40x_{22,2} + 60x_{22,3} + 80x_{22,4}\end{array}}{4}\right)$$

Subject to

Cosntraints of the fuzzy LPP $(P3.12.4)$.

Step 2: On solving the crisp LPP $(P3.12.6)$, the following two optimal basic feasible solutions are obtained.

(i) $x_{11,1} = 40, x_{11,2} = 40, x_{11,3} = 40, x_{11,4} = 40, x_{12,1} = 0, x_{12,2} = 20, x_{12,3} = 40, x_{12,4} = 60, x_{21,1} = 0, x_{21,2} = 20, x_{21,3} = 40, x_{21,4} = 60, x_{22,1} = 40, x_{22,2} = 40, x_{22,3} = 40, x_{22,4} = 40.$

(ii) $x_{11,1} = 40$, $x_{11,2} = 60$, $x_{11,3} = 60$, $x_{11,4} = 60$, $x_{12,1} = 0$, $x_{12,2} = 0$, $x_{12,3} = 20$, $x_{12,4} = 40$, $x_{21,1} = 0$, $x_{21,2} = 0$, $x_{21,3} = 20$, $x_{21,4} = 40$, $x_{22,1} = 40$, $x_{22,2} = 60$, $x_{22,3} = 60$, $x_{22,4} = 60$.

Step 3: Using these optimal basic feasible solutions, the following two fuzzy optimal basic feasible solutions are obtained:

(i) $\tilde{x}_{11} = (40, 40, 40, 40)$, $\tilde{x}_{12} = (0, 20, 40, 60)$, $\tilde{x}_{21} = (0, 20, 40, 60)$, $\tilde{x}_{22} = (40, 40, 40, 40)$.

(ii) $\tilde{x}_{11} = (40, 60, 60, 60)$, $\tilde{x}_{12} = (0, 0, 20, 40)$, $\tilde{x}_{21} = (0, 0, 20, 40)$, $\tilde{x}_{22} = (40, 60, 60, 60)$.

The fuzzy transportation cost $(20, 40, 60, 80) \otimes \left(x_{11,1}, x_{11,2}, x_{11,3}, x_{11,4}\right) + (25, 50, 50, 80) \otimes (x_{12,1}, x_{12,2}, x_{12,3}, x_{12,4})$ $+(25, 50, 50, 80) \otimes \left(x_{21,1}, x_{21,2}, x_{21,3}, x_{21,4}\right) + (20, 40, 60, 80) \otimes$ $(x_{22,1}, x_{22,2}, x_{22,3}, x_{22,4})$, corresponding to first and second optimal basic feasible solutions are $(1600, 5200, 8800, 16000)$ and $(1600, 4800, 9200, 16000)$ respectively.

Step 4: Using Step 4 of the proposed Mehar method,

$Mode(1600, 5200, 8800, 16000) = \frac{5200+ 8800}{2} = 7000$,

$Mode(1600, 4800, 9200, 16000) = \frac{4800+9200}{2} = 7000$.

Step 5: Using Step 5 of the proposed Mehar method, $minimum\{Mode(1600, 5200, 8800, 16000), Mode(1600, 4800, 9200, 16000)\} = minimum\{7000, 7000\} = 7000$.

Since, the obtained $minimum$ value is not corresponding to a unique number, so according to Case (ii) of Step 5 of the proposed Mehar method, there is need to go to Step 6.

Step 6: Using Step 6 of the proposed Mehar method,

$Div(1600, 5200, 8800, 16000) = \frac{16000+8800-1600-5200}{2} = 9000,$

$Div(1600, 4800, 9200, 16000) = \frac{16000+9200-1600-4800}{2} = 9400.$

Step 7: Using Step 7 of the proposed Mehar method,

$minimum\{Div(1600, 5200, 8800, 16000), Div(1600, 4800, 9200, 16000)\}$

$= minimum\{9000, 9400\} = 9000.$

Step 8: Since, $Div(1600, 5200, 8800, 16000) < Div(1600, 4800, 9200, 16000)$. So, according to Step 8 of the proposed Mehar method, the TrFN $(1600, 5200, 8800, 16000)$ represents the unique optimal fuzzy transportation cost of the considered FFTP. Furthermore, $\tilde{x}_{11} = (40, 40, 40, 40)$, $\tilde{x}_{12} = (0, 20, 40, 60)$, $\tilde{x}_{21} = (0, 20, 40, 60)$, $\tilde{x}_{22} = (40, 40, 40, 40)$ is the fuzzy optimal solution of the considered FFTP.

3.13 Conclusions

It is shown that on solving the fully fuzzy LPP $(P3.3.1)$ of a BFFTP by Kumar and Kaur's method [115, Section 6, pp. 88] and Ebrahimnejad's method [49, Section 5, pp. 114] more than one LR FFNs, representing the optimal fuzzy transportation cost, may be obtained, which is mathematically incorrect. Therefore, it is inappropriate to use Kumar and Kaur's method [115, Section 6, pp. 88] and Ebrahimnejad's method [49, Section 5, pp. 114] for solving BFFTPs. Also, a new method (named as Mehar method) is proposed for solving BFFTPs. Furthermore, it is shown that on solving the fully fuzzy LPP $(P3.3.1)$ of a BFFTP by the proposed Mehar method, always a unique LR FFN, representing the fuzzy optimal transportation cost, will be obtained. So, it is appropriate to use the proposed Mehar method for solving the fully fuzzy LPP $(P3.3.1)$ of a BFFTP.

Chapter 4

Mehar Approach for Solving Balanced Fully Intuitionistic Fuzzy Transportation Problems[‡]

Ebrahimnejad and Verdegay [51, Section 5] proposed an approach for solving such BFIFTPs in which each parameter is represented as a TrIFN. In this chapter, it is shown that Ebrahimnejad and Verdegay's approach can be used only if the aggregated value of IF transportation cost, IF availability and IF demand, provided by all the decision-makers, is available. However, if instead of the aggregated data, the data of each decision-maker is provided separately then Ebrahimnejad and Verdegay's approach cannot be used to find the solution of a BFIFTP. Therefore, firstly, to overcome the limitation of Ebrahimnejad and Verdegay's approach, a method for aggregating the TrIFNs is discussed. Then, a new approach (named as Mehar approach) is proposed for solving BFIFTPs. It is shown that it is much easy to apply the proposed Mehar approach as compared to Ebrahimnejad and Verdegay's approach. Also, to illustrate the proposed Mehar approach, the existing BFIFTP [51] is solved.

4.1 Preliminaries

In this section, some basic definitions and arithmetic operations are presented [51, Section 2].

[‡] The contents of this chapter have been communicated in "Fuzzy Optimization and Decision Making" for the possible publication.

4.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 4.1: An intuitionistic fuzzy, $\tilde{A}^I = \{\langle x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)\rangle : x \in \mathbb{R}\}$, defined over the set of real numbers $\mathbb{R}$, is called an IFN if the following holds:

(i) There exist $m \in \mathbb{R}$ such that $\mu_{\tilde{A}^I}(m) = 1$ and $\nu_{\tilde{A}^I}(m) = 0$, (m is called the mean value of $\tilde{A}^I$).

(ii) $\mu_{\tilde{A}^I}(x)$ and $\nu_{\tilde{A}^I}(x)$ are piecewise continuous mapping from $\mathbb{R}$ to the closed interval $[0,1]$ and the relation $0 \le \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x) \le 1$, $\forall\, x \in \mathbb{R}$ holds.

The membership and non-membership function of $\tilde{A}^I$ is of the following form:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} 0 & ; -\infty < x \le m - \alpha \\ f_1(x) & ; x \in (m - \alpha, m] \\ 1 & ; x = m \\ h_1(x) & ; x \in [m, m + \beta) \\ 0 & ; m + \beta \le x \le \infty \end{cases}$$

where $f_1(x)$ and $h_1(x)$ are strictly increasing and decreasing function in $(m - \alpha, m]$ and $[m, m + \beta)$ respectively.

$$\nu_{\tilde{A}^I}(x) = \begin{cases} 1 & ; -\infty < x \le m - \alpha' \\ f_2(x) & ; x \in (m - \alpha', m]; 0 \le f_1(x) + f_2(x) \le 1 \\ 0 & ; x = m \\ h_2(x) & ; x \in [m, m + \beta');\ 0 \le h_1(x) + h_2(x) \le 1 \\ 0 & ; m + \beta' \le x < \infty \end{cases}$$

where $f_2(x)$ and $h_2(x)$ are strictly increasing and decreasing function in $(m - \alpha', m]$ and $[m, m + \beta')$ respectively. Here α and β are called left and right spreads of membership function $\mu_{\tilde{A}^I}(x)$ respectively. α' and β' are called left and right spreads of non-membership function $\nu_{\tilde{A}^I}(x)$ respectively. The IFN $\tilde{A}^I$ is represented by $\tilde{A}^I = (m; \alpha, \beta; \alpha', \beta')$.

Definition 4.2: An IFN $\tilde{A}^I = (a_1, a_2, a_3; a_1', a_2, a_3')$ is said to be TIFN if its membership function $\mu_{\tilde{A}^I}(x)$ and non-membership function $\nu_{\tilde{A}^I}(x)$ is defined as

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & ; a_1 < x \le a_2 \\ \frac{a_1-x}{a_3-a_2} & ; a_2 < x \le a_3 \\ 0 & ; otherwise \end{cases} \quad \text{and} \quad \nu_{\tilde{A}^I}(x) = \begin{cases} \frac{a_1'-x}{a_2-a_1} & ; a_1' < x \le a_2 \\ \frac{x-a_2}{a_3'-a_2} & ; a_2 < x \le a_3' \\ 1 & ; otherwise \end{cases}$$

Definition 4.3: An IFN $\tilde{A}^I = (a_1, a_2, a_3, a_4; a_1', a_2', a_3', a_4')$ is said to be TrIFN if its membership function $\mu_{\tilde{A}^I}(x)$ and non-membership function $\nu_{\tilde{A}^I}(x)$ is defined as

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & ; a_1 \le x < a_2 \\ 1 & ; a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3} & ; a_3 < x \le a_4 \\ 0 & ; otherwise \end{cases} \quad \text{and} \quad \nu_{\tilde{A}^I}(x) = \begin{cases} \frac{a_2'-x}{a_2'-a_1'} & ; a_1' \le x < a_2' \\ 0 & ; a_2' \le x \le a_3' \\ \frac{a_3'-x}{a_3'-a_4'} & ; a_3' < x \le a_4' \\ 1 & ; otherwise \end{cases}$$

4.1.2 Arithmetic operations of IFNs

In this section, arithmetic operations of TIFNs and TrIFNs are presented.

4.1.2.1 Arithmetic operations of TIFNs

In this section, arithmetic operations of TIFNs are presented.

Let $\tilde{A}^I = (a_1, a_2, a_3; a_1', a_2, a_3')$ and $\tilde{B}^I = (b_1, b_2, b_3; b_1', b_2, b_3')$ be two TIFNs. Then,

(i) $\tilde{A}^I \oplus \tilde{B}^I = (a_1 + b_1, a_2 + b_2, a_3 + b_3; a_1' + b_1', a_2 + b_2, a_3' + b_3')$

(ii) $\tilde{A}^I \ominus \tilde{B}^I = (a_1 - b_3, a_2 - b_2, a_3 - b_1; a_1' - b_3', a_2 - b_2, a_3' - b_1')$

(iii) $\tilde{A}^I \otimes \tilde{B}^I = (m_1, m_2, m_3; m_1', m_2, m_3')$,

where, $m_1 = min\{a_1b_1, a_1b_3, a_3b_1, a_3b_3\}$, $m_2 = a_2b_2$, $m_3 = max\{a_1b_1, a_1b_3, a_3b_1, a_3b_3\}$, $m_1' = min\{a_1'b_1', a_1'b_3', a_3'b_1', a_3'b_3'\}$, $m_3' = max\{a_1'b_1', a_1'b_3', a_3'b_1', a_3'b_3'\}$.

(iv) $\lambda\tilde{A}^I = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3; \lambda a'_1, \lambda a_2, \lambda a'_3\ );\ \lambda \geq 0, \\ (\lambda a_3, \lambda a_2, \lambda a_1; \lambda a'_3, \lambda a_2, \lambda a'_1\ );\ \lambda < 0. \end{cases}$

4.1.2.2 Arithmetic operations of TrIFNs

In this section, arithmetic operations of TrIFNs are presented.

Let $\tilde{A}^I = (a_1, a_2, a_3, a_4; a'_1, a_2, a_3, a'_4\ )$ and $\tilde{B}^I = (b_1, b_2, b_3, b_4; b'_1, b_2, b_3, b'_4\ )$ be two TrIFNS. Then,

(i) $\tilde{A}^I \oplus \tilde{B}^I = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4; a'_1 + b'_1, a_2 + b_2, a_3 + b_3, a'_4 + b'_4)$

(ii) $\tilde{A}^I \ominus \tilde{B}^I = (a_1 - b_4, a_2 - b_4, a_3 - b_2, a_4 - b_1; a'_1 - b'_4, a_2 - b_2, a_3 - b_2, a'_4 - b'_1)$

(iii) $\tilde{A}^I \otimes \tilde{B}^I = (m_1, m_2, m_3, m_4; m'_1, m_2, m_3, m'_4)$,

where, $m_1 = min\{a_1b_1, a_1b_4, a_4b_1, a_4b_4\}$, $m_2 = min\{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}$, $m_3 = max\{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}$,$m_4 = max\{a_1b_1, a_1b_4, a_4b_1, a_4b_4\}$, $m'_1 = min\{a'_1b'_1, a'_1b'_4, a'_4b'_1, a'_4b'_4\}$, $m'_4 = max\{a'_1b'_1, a'_1b'_4, a'_4b'_1, a'_4b'_4\}$.

(iv) $\lambda\tilde{A}^I = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4; \lambda a'_1, \lambda a_2, \lambda a_3, \lambda a'_3\ );\ \lambda \geq 0, \\ (\lambda a_4, \lambda a_3, \lambda a_2, \lambda a_1; \lambda a'_4, \lambda a_3, \lambda a_2, \lambda a'_1\ );\ \lambda < 0. \end{cases}$

4.2 IFLPP of a BIFTP

Ebrahimnejad and Verdegay [51, Section 3.2] claimed that a BFIFTP can be transformed into the IFFLPP (P4.1).

IFFLPP ($\boldsymbol{P4.1}$)

Minimize$\left[\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij}^I \otimes \tilde{x}_{ij}^I\right]$

Subject to

$\sum_{j=1}^{n} \tilde{x}_{ij}^I = \tilde{a}_i^I\ ;\quad i = 1, 2, \dots, m,$

$\sum_{i=1}^{m} \tilde{x}_{ij}^I = \tilde{b}_j^I\ ;\ \ j = 1, 2, \dots, n,$

$\tilde{x}_{ij}^I$ is a non-negative IFN.

where,

(i) The TrIFN $\tilde{c}^I_{ij}$ represents the intuitionistic fuzzy transportation cost for supplying the unit quantity of the product from the i^{th} source (S_i) to the j^{th} destination $\left(D_j\right)$,

(ii) The TrIFN $\tilde{x}^I_{ij}$ represents the quantity of the product to be supplied from the i^{th} source (S_i) to the j^{th} destination $\left(D_j\right)$,

(iii) The TrIFN $\tilde{a}^I_i$ represents the availability of the product at the i^{th} source (S_i),

(iv) The TrIFN $\tilde{b}^I_j$ represents the demand of the product at the j^{th} destination $\left(D_j\right)$,

(v) $\sum_{i=1}^{m} \tilde{a}^I_i = \sum_{j=1}^{n} \tilde{b}^I_j$ represents that the total availability of the product at all the sources is equal to the total demand of the product at all the destinations.

4.3 Existing method for comparing TrIFNs

It is well-known fact that the optimal solution of a crisp LPP problem will be that feasible solution corresponding to which the value of the objective function will be minimum. On the same direction, the intuitionistic fuzzy optimal solution of the IFFLPP $(P4.1)$ will be that IF feasible solution corresponding to which the value of the objective function will be minimum. Since, in case of IFFLPP $(P4.1)$, the value of the objective function, corresponding to an intuitionistic fuzzy feasible solution, will be a TrIFN. Therefore, to find the intuitionistic fuzzy optimal solution of the IFLPP $(P4.1)$, there is need to find the minimum of TrIFNs i.e., there is need to compare the TrIFNs. In this section, the method for comparing the TrIFNs, used in Ebrahimnejad and Verdegay's approach [51, Section 5], is discussed.

Let $\tilde{A}^I = (a_1, a_2, a_3, a_4; a'_1, a'_2, a'_3, a'_4)$ and $\tilde{B}^I = (b_1, b_2, b_3, b_4; b'_1, b'_2, b'_3, b'_4)$ be two TrIFNs. Then,

(i) $\tilde{A}^I \succ \tilde{B}^I$ if $H\left(\tilde{A}^I\right) > H\left(\tilde{B}^I\right)$,

(ii) $\tilde{A}^I \prec \tilde{B}^I$ if $H(\tilde{A}^I) < H(\tilde{B}^I)$,

(iii) $\tilde{A}^I = \tilde{B}^I$ if $H(\tilde{A}^I) = H(\tilde{B}^I)$,

where, $H(\tilde{A}^I) = \frac{(a_1 + a_2 + a_3 + a_4) + (a_1' + a_2' + a_3' + a_4')}{8}$ and $H(\tilde{B}^I) = \frac{(b_1 + b_2 + b_3 + b_4) + (b_1' + b_2' + b_3' + b_4')}{8}$.

4.4 Ebrahimnejad and Verdegay's approach for solving BFIFTPs

The aim of this chapter is to propose such an approach for solving BFIFTPs which is much easy to apply as compared to Ebrahimnejad and Verdegay's approach [51, Section 5]. Since, to show that the proposed approach fulfills this criteria, there is need to discuss Ebrahimnejad and Verdegay's approach [51, Section 5]. Therefore, a brief review of Ebrahimnejad and Verdegay's approach [51, Section 5] is presented in this section.

Ebrahimnejad and Verdegay [51, Section 5] proposed the following approach for solving BFIFTPs.

Step 1: Replacing the parameters $\tilde{c}_{ij}^I$, $\tilde{x}_{ij}^I$, $\tilde{a}_i^I$ and $\tilde{b}_j^I$ with the TrIFNs $(c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4}\,;\, c_{ij,1}', c_{ij,2}', c_{ij,3}', c_{ij,4}')$, $(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\,;\, x_{ij,1}', x_{ij,2}', x_{ij,3}', x_{ij,4}')$, $(a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}\,;\, a_{i,1}', a_{i,2}', a_{i,3}', a_{i,4}')$ and $(b_{j,1}, b_{j,2}, b_{j,3}, b_{j,4}\,;\, b_{j,1}', b_{j,2}', b_{j,3}', b_{j,4}')$ respectively, the IFLPP $(P4.1)$ can be transformed into its equivalent IFFLPP $(P4.2)$.

IFFLPP $(P4.2)$

Minimize$\left[\sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4}\,;\, c_{ij,1}', c_{ij,2}', c_{ij,3}', c_{ij,4}') \otimes (x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\,;\, x_{ij,1}', x_{ij,2}', x_{ij,3}', x_{ij,4}')\right]$

Subject to

$\sum_{j=1}^{n}(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\,;\, x_{ij,1}', x_{ij,2}', x_{ij,3}', x_{ij,4}') = (a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}\,;\, a_{i,1}', a_{i,2}', a_{i,3}', a_{i,4}')\,;$ $\quad i = 1, 2, \ldots, m,$

$\sum_{i=1}^{m}\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\ ;\ x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4}\right) =$

$\left(b_{j,1}, b_{j,2}, b_{j,3}, b_{i,4}\ ;\ b'_{j,1}, b'_{j,2}, b'_{j,3}, b'_{j,4}\right);$ $\qquad j = 1, 2, \dots, n,$

$\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\ ;\ x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4}\right)$ is a non-negative TrIFN, $i = 1, 2, \dots, m,$ $j = 1, 2, \dots, n.$

Step 2: Using the existing relation [51, Section 2, Definition 9], $(a_1, a_2, a_3, a_4\ ;\ a'_1, a'_2, a'_3, a'_4) = (b_1, b_2, b_3, b_4\ ;\ b'_1, b'_2, b'_3, b'_4) \Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4, a'_1 = b'_1, a'_2 = b'_2, a'_3 = b'_3, a'_4 = b'_4$ and using the relation $(a_1, a_2, a_3, a_4\ ;\ a'_1, a'_2, a'_3, a'_4)$ is a non-negative TrIFN $\Rightarrow$ $a'_1 \geq 0,\ a_1 - a'_1 \geq 0,$ $a'_2 - a_1 \geq 0,\ a_2 - a'_2 \geq 0,\ a_3 - a_2 \geq 0,\ a'_3 - a_3 \geq 0,\ a_4 - a'_3 \geq 0,\ a'_4 - a_4 \geq 0,$ the IFFLPP $(P4.2)$ can be transformed into its equivalent IFLPP $(P4.3)$.

IFLPP $(\boldsymbol{P4.3})$

Minimize$\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4}\ ;\ c'_{ij,1}, c'_{ij,2}, c'_{ij,3}, c'_{ij,4}\right) \otimes \left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\ ;\ x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4}\right)\right]$

Subject to

$$\left.\begin{aligned}&\textstyle\sum_{j=1}^{n} x_{ij,1} = a_{i,1},\ \sum_{j=1}^{n} x_{ij,2} = a_{i,2}, \sum_{j=1}^{n} x_{ij,3} = a_{i,3}, \sum_{j=1}^{n} x_{ij,4} = a_{i,4},\\ &\textstyle\sum_{j=1}^{n} x'_{ij,1} = a'_{i,1}, \sum_{j=1}^{n} x'_{ij,2} = a'_{i,2},\ \sum_{j=1}^{n} x'_{ij,3} = a'_{i,3}, \sum_{j=1}^{n} x'_{ij,4} = a'_{i,4}\end{aligned}\right\},$$

$$i = 1,2, \dots, m;$$

$$\left.\begin{aligned}&\textstyle\sum_{i=1}^{m} x_{ij,1} = b_{j,1}, \sum_{i=1}^{m} x_{ij,2} = b_{j,2}, \sum_{i=1}^{m} x_{ij,3} = b_{j,3}, \sum_{i=1}^{m} x_{ij,4} = b_{j,4},\\ &\textstyle\sum_{i=1}^{m} x'_{ij,1} = b'_{j,1}, \sum_{i=1}^{m} x'_{ij,2} = b'_{j,2}, \sum_{i=1}^{m} x'_{ij,3} = b'_{j,3}, \sum_{i=1}^{m} x'_{ij,4} = b'_{j,4}\end{aligned}\right\},\ j = 1, 2, \dots, n;$$

$x'_{ij,1} \geq 0,\ x_{ij,1} - x'_{ij,1} \geq 0,\ x'_{ij,2} - x_{ij,1} \geq 0,\ x_{ij,2} - x'_{ij,2} \geq 0,\ x_{ij,3} - x_{ij,2} \geq 0,\ x'_{ij,3} - x_{ij,3} \geq 0, x_{ij,4} - x'_{ij,3} \geq 0, x'_{ij,4} - x_{ij,4} \geq 0,\ \ i = 1, 2, \dots, m, j = 1, 2, \dots, n.$

Step 3: Using the existing multiplication [51, Section 2, Definition 8], $(a_1, a_2, a_3, a_4\ ;\ a'_1, a'_2, a'_3, a'_4) \otimes (b_1, b_2, b_3, b_4\ ;\ b'_1, b'_2, b'_3, b'_4) =$

$(a_1b_1, a_2b_2, a_3b_3, a_4b_4\,;\, a'_1b'_1, a'_2b'_2, a'_3b'_3, a'_4b'_4)$, the IFLPP $(P4.3)$ can be transformed into its equivalent IFLPP $(P4.4)$.

IFLPP $(P4.4)$

Minimize$\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\,;\, c'_{ij,1}x'_{ij,1}, c'_{ij,2}x'_{ij,2}, c'_{ij,3}x'_{ij,3}, c'_{ij,4}x'_{ij,4}\right)\right]$

Subject to

Constraints of the IFLPP $(P4.3)$.

Step 4: Using the comparing method, discussed in Section 4.3, the IFLPP $(P4.4)$ can be transformed into its equivalent crisp LPP $(P4.5)$ and hence its equivalent crisp LPP $(P4.6)$.

Crisp LPP $(P4.5)$

Minimize $\left[\sum_{i=1}^{m}\sum_{j=1}^{n} H\begin{pmatrix} c_{ij,1}\,x_{ij,1}, c_{ij,2}\,x_{ij,2}, c_{ij,3}\,x_{ij,3}, c_{ij,4}x_{ij,4}\,;\, c'_{ij,1}\,x'_{ij,1}, c'_{ij,2}\,x'_{ij,2}, c'_{ij,3}\,x'_{ij,3} \\ , c'_{ij,4}\,x'_{ij,4} \end{pmatrix}\right]$

Subject to

Constraints of the IFLPP $(P4.3)$.

Crisp LPP $(P4.6)$

Minimize $\left[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}\,x_{ij,1} + c_{ij,2}\,x_{ij,2} + c_{ij,3}\,x_{ij,3} + c_{ij,4}x_{ij,4} + c'_{ij,1}\,x'_{ij,1} + \right.\right.$

$\left.\left. c'_{ij,2}\,x'_{ij,2} + c'_{ij,3}\,x'_{ij,3} + c'_{ij,4}\,x'_{ij,4}\right)\right]$ \

Subject to

Constraints of the IFLPP $(P4.3)$.

Step 5: Find the optimal solution $\{x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4}, x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\}$ of the crisp LPP $(P4.6)$.

Step 6: Using the optimal solution, obtained in Step 5, find the IF optimal solution $\{(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\,;\, x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4})\}$ and the IF optimal value

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\ ;\ c'_{ij,1}x'_{ij,1}, c'_{ij,2}x'_{ij,2}, c'_{ij,3}x'_{ij,3}, c'_{ij,4}x'_{ij,4}\right).$$

4.5 Limitations of Ebrahimnejad and Verdegay's approach

To solve a real life transportation problem, the opinion of more than one expert about the parameters is collected. Then, all the collected information is aggregated to obtain a single value of each parameter. Since, Ebrahimnejad and Verdegay's approach [51, Section 5] is proposed by considering that the aggregated value of each parameter is available. Therefore, Ebrahimnejad and Verdegay's approach [51, Section 5] cannot be used to solve several real life FIFTPs. For example, Ebrahimnejad and Verdegay's approach [51, Section 5], cannot be used to solve the FIFTP considered in Example 4.1.

Example 4.1 Let us consider, a product needs to be supplied from three sources to four destinations. For the same purpose, the information about each parameter is collected from two experts. If Table 4.1 represents the IF transportation cost, the IF availability and the IF demand provided by the first decision-maker and if Table 4.2 represents the IF transportation cost, the IF availability and the IF demand provided by the second decision-maker. Then, this FIFTP cannot be solved by Ebrahimnejad and Verdegay's approach [51, Section 5].

Table 4.1 IF data provided by the first decision-maker

Destination → Source ↓	D_1	D_2	IF availability
S_1	$\begin{pmatrix}10,30,40,50;\\ 5,15,45,55\end{pmatrix}$	$\begin{pmatrix}25,50,60,80;\\ 10,30,70,90\end{pmatrix}$	$\begin{pmatrix}20,60,70,80;\\ 15,50,75,85\end{pmatrix}$
S_2	$\begin{pmatrix}15,30,50,80;\\ 10,20,70,90\end{pmatrix}$	$\begin{pmatrix}20,40,60,80;\\ 15,35,70,85\end{pmatrix}$	$\begin{pmatrix}25,45,60,70;\\ 20,40,65,80\end{pmatrix}$
IF demand	$\begin{pmatrix}40,60,70,90;\\ 30,50,80,95\end{pmatrix}$	$\begin{pmatrix}10,45,55,70;\\ 5,30,60,80\end{pmatrix}$	

Table 4.2 IF data provided by the second decision-maker

Destination → Source ↓	D_1	D_2	IF availability
S_1	$\begin{pmatrix}15,35,45,55;\\10,20,50,60\end{pmatrix}$	$\begin{pmatrix}30,55,65,85;\\15,35,75,95\end{pmatrix}$	$\begin{pmatrix}25,65,75,85;\\20,55,80,90\end{pmatrix}$
S_2	$\begin{pmatrix}20,35,55,85;\\15,25,75,95\end{pmatrix}$	$\begin{pmatrix}25,45,65,85;\\20,40,75,90\end{pmatrix}$	$\begin{pmatrix}30,50,65,75;\\25,45,70,85\end{pmatrix}$
IF demand	$\begin{pmatrix}45,65,75,95;\\35,55,85,100\end{pmatrix}$	$\begin{pmatrix}15,50,60,75;\\10,35,65,85\end{pmatrix}$	

4.6 Modified Ebrahimnejad and Verdegay's approach

The limitation of Ebrahimnejad's approach [51, Section 5] can be overcome by including the following step (say Step 0) in Ebrahimnejad and Verdegay's approach [51, Section 5].

Step 0: Use an appropriate aggregation operator to aggregate the data of all the decision-maker, For example, if

(i) w_k represents the normalized weight of the k^{th} decision-maker.

(ii) The IFN $\tilde{c}_{ij}^{k}=\left(c_{ij,1}^{k},c_{ij,2}^{k},c_{ij,3}^{k},c_{ij,4}^{k};c_{ij,5}^{k},c_{ij,6}^{k},c_{ij,7}^{k},c_{ij,8}^{k}\right)$ represents the IF transportation cost for supplying unit quantity of the product from i^{th} source to j^{th} destination, provided by the k^{th} decision-maker.

(iii) The IFN $\tilde{a}_{i}^{k}=\left(a_{i,1}^{k},a_{i,2}^{k},a_{i,3}^{k},a_{i,4}^{k};a_{i,5}^{k},a_{i,6}^{k},a_{i,7}^{k},a_{i,8}^{k}\right)$ represents the IF availability of the product at i^{th} source according to k^{th} decision-maker.

(iv) The IFN $\tilde{b}_{j}^{k}=\left(b_{j,1}^{k},b_{j,2}^{k},b_{j,3}^{k},b_{j,4}^{k};b_{j5,}^{k},b_{j,6}^{k},b_{j,7}^{k},b_{j,8}^{k}\right)$ represents the IF demand of the product at j^{th} destination according to k^{th} decision-maker.

Then,

(i) $\sum_{k=1}^{p} w_k \tilde{c}_{ij}^{k} = \begin{pmatrix} \sum_{k=1}^{p} w_k\, c_{ij,1}^{k}, \sum_{k=1}^{p} w_k\, c_{ij,2}^{k}, \sum_{k=1}^{p} w_k\, c_{ij,3}^{k}, \sum_{k=1}^{p} w_k\, c_{ij,4}^{k}; \\ \sum_{k=1}^{p} w_k\, c_{ij,5}^{k}, \sum_{k=1}^{p} w_k\, c_{ij,6}^{k}, \sum_{k=1}^{p} w_k\, c_{ij,7}^{k}, \sum_{k=1}^{p} w_k\, c_{ij,8}^{k} \end{pmatrix}$ will represent the aggregated value of the IF transportation cost for transporting unit quantity of the product from i^{th} source to j^{th} destination.

(ii) The IFN $\sum_{k=1}^{p} w_k\, \tilde{a}_{i}^{k} = \begin{pmatrix} \sum_{k=1}^{p} w_k\, a_{i,1}^{k}, \sum_{k=1}^{p} w_k\, a_{i,2}^{k}, \sum_{k=1}^{p} w_k\, a_{i,3}^{k}, \sum_{k=1}^{p} w_k\, a_{i,4}^{k}; \\ \sum_{k=1}^{p} w_k\, a_{i,5}^{k}, \sum_{k=1}^{p} w_k\, a_{i,6}^{k}, \sum_{k=1}^{p} w_k\, a_{i,7}^{k}, \sum_{k=1}^{p} w_k\, a_{i,8}^{k} \end{pmatrix}$ will represent the aggregated value of the IF availability of the product at i^{th} source.

(iii) The IFN $\sum_{k=1}^{p} w_k\, \tilde{b}_{j}^{k} = \begin{pmatrix} \sum_{k=1}^{p} w_k\, b_{j,1}^{k}, \sum_{k=1}^{p} w_k\, b_{j,2}^{k}, \sum_{k=1}^{p} w_k\, b_{j,3}^{k}, \sum_{k=1}^{p} w_k\, b_{j,4}^{k}; \\ \sum_{k=1}^{p} w_k\, b_{j,5}^{k}, \sum_{k=1}^{p} w_k\, b_{j,6}^{k}, \sum_{k=1}^{p} w_k\, b_{j,7}^{k}, \sum_{k=1}^{p} w_k\, b_{j,8}^{k} \end{pmatrix}$ will represent aggregated value of the IF demand of the product at j^{th} destination.

For example, if in Example 4.1 the normalized weights of first and second decision-makers are 0.4 and 0.6 repsectively. Then, Table 4.3 will represent the aggregated IF cost for supplying the unit quantity of the product, the IF availability and the IF demand.

Table 4.3 Aggregated IF data of decision-makers

Destination → Source ↓	D_1	D_2	IF availability
S_1	$\begin{pmatrix} 13,33,43,53; \\ 8,18,48,58 \end{pmatrix}$	$\begin{pmatrix} 28,53,63,83; \\ 13,33,73,93 \end{pmatrix}$	$\begin{pmatrix} 23,63,73,83; \\ 18,53,78,88 \end{pmatrix}$
S_2	$\begin{pmatrix} 18,33,53,83; \\ 13,23,73,93 \end{pmatrix}$	$\begin{pmatrix} 23,43,63,83; \\ 18,38,73,88 \end{pmatrix}$	$\begin{pmatrix} 28,48,63,73; \\ 23,43,68,83 \end{pmatrix}$
IF demand	$\begin{pmatrix} 43,63,73,93; \\ 33,53,83,98 \end{pmatrix}$	$\begin{pmatrix} 15,50,65,75; \\ 8,33,58,83 \end{pmatrix}$	

4.7 Proposed Mehar approach

In this section, a new approach (named as Mehar approach) is proposed to find the IF optimal solution of such BFIFTPs in which each parameter is represented as a TrIFN.

The steps of the proposed Mehar approach are as follows:

Step 1: Using the known values of $\tilde{c}_i^I = \left(c_{i,1}, c_{i,2}, c_{i,3}, c_{i,4}\ ;\ c'_{i,1}, c'_{i,2}, c'_{i,3}, c'_{i,4}\right)$, $\tilde{a}_i^I = \left(a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}\ ;\ a'_{i,1}, a'_{i,2}, a'_{i,3}, a'_{i,4}\right)$ and $\tilde{b}_j^I = \left(b_{j,1}, b_{j,2}, b_{j,3}, b_{i,4}\ ;\ b'_{j,1}, b'_{j,2}, b'_{j,3}, b'_{j,4}\right)$, find the crisp optimal solutions $\{x'_{ij,1}\}$, $\{\alpha_{ij,1}\}$, $\{\alpha_{ij,2}\}$, $\{\alpha_{ij,3}\}$, $\{\alpha_{ij,4}\}$, $\{\alpha_{ij,5}\}$, $\{\alpha_{ij,6}\}$, $\{\alpha_{ij,7}\}$; $i = 1,2,\dots,m$; $j = 1,2,\dots,n$; of the crisp LPPs $(P4.7)$ to $(P4.14)$ respectively.

Crisp LPP $(\boldsymbol{P4.7})$

Minimize $\left[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1} + c_{ij,2} + c_{ij,3} + c_{ij,4} + c'_{ij,1} + c'_{ij,2} + c'_{ij,3} + c'_{ij,4}\right)x'_{ij,1}\right]$

Subject to

$\sum_{j=1}^{n} x'_{ij,1} = a'_{i,1},\ i = 1,2,\dots,m$;

$\sum_{i=1}^{m} x'_{ij,1} = b'_{j,1},\ j = 1,2,\dots,n,\ \ x'_{ij,1} \geq 0.$

Crisp LPP $(\boldsymbol{P4.8})$

Minimize $\left[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1} + c_{ij,2} + c_{ij,3} + c_{ij,4} + c'_{ij,2} + c'_{ij,3} + c'_{ij,4}\right)\alpha_{ij,1}\right]$

Subject to

$\sum_{j=1}^{n} \alpha_{ij,1} = a_{i,1} - a'_{i,1},\ i = 1,2,\dots,m$;

$\sum_{i=1}^{m} \alpha_{ij,1} = b_{j,1} - b'_{j,1},\ j = 1,2,\dots,n,\ \alpha_{ij,1} \geq 0.$

Crisp LPP $(\boldsymbol{P4.9})$

Minimize $\left[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,2} + c_{ij,3} + c_{ij,4} + c'_{ij,2} + c'_{ij,3} + c'_{ij,4}\right)\alpha_{ij,1}\right]$

Subject to

$\sum_{j=1}^{n} \alpha_{ij,2} = a'_{i,2} - a_{i,1}, i = 1,2, \dots, m$;

$\sum_{i=1}^{m} \alpha_{ij,2} = b'_{j,2} - b_{j,1}, j = 1,2, \dots, n, \ \alpha_{ij,2} \geq 0.$

Crisp LPP $(\boldsymbol{P4.10})$

Minimize $\left[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,2} + c_{ij,3} + c_{ij,4} + c'_{ij,3} + c'_{ij,4}\right)\alpha_{ij,2}\right]$

Subject to

$\sum_{j=1}^{n} \alpha_{ij,3} = a_{i,2} - a'_{i,2}, \ i = 1,2, \dots, m$;

$\sum_{i=1}^{m} \alpha_{ij,3} = b_{j,2} - b'_{j,2}, j = 1,2, \dots, n, \ \alpha_{ij,3} \geq 0.$

Crisp LPP $(\boldsymbol{P4.11})$

Minimize $\left[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,3} + c_{ij,4} + c'_{ij,3} + c'_{ij,4}\right)\alpha_{ij,2}\right]$

Subject to

$\sum_{j=1}^{n} \alpha_{ij,4} = a_{i,3} - a_{i,2}, i = 1,2, \dots, m$;

$\sum_{i=1}^{m} \alpha_{ij,4} = b_{j,3} - b_{j,2}, j = 1,2, \dots, n, \alpha_{ij,4} \geq 0.$

Crisp LPP $(\boldsymbol{P4.12})$

Minimize $\left[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,4} + c'_{ij,3} + c'_{ij,4}\right)\alpha_{ij,2}\right]$

Subject to

$\sum_{j=1}^{n} \alpha_{ij,5} = a'_{i,3} - a_{i,3}, \quad i = 1,2, \dots, m$;

$\sum_{i=1}^{m} \alpha_{ij,5} = b'_{j,3} - b_{j,3}, j = 1,2, \dots, n, \alpha_{ij,5} \geq 0.$

Crisp LPP $(\boldsymbol{P4.13})$

Minimize $\left[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,4} + c'_{ij,4}\right)\alpha_{ij,2}\right]$

Subject to

$\sum_{j=1}^{n} \alpha_{ij,6} = a_{i,4} - a'_{i,3}, i = 1,2, \dots, m$;

$\sum_{i=1}^{m} \alpha_{ij,6} = b_{j,4} - b'_{j,3}, j = 1,2, \dots, n, \alpha_{ij,6} \geq 0.$

Crisp LPP $(\boldsymbol{P4.14})$

Minimize $\left[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c'_{ij,4}\right)\alpha_{ij,2}\right]$

Subject to

$\sum_{j=1}^{n}\alpha_{ij,7}=a'_{i,4}-a_{i,4},\ i=1,2,\dots,m;$

$\sum_{i=1}^{m}\alpha_{ij,7}=b'_{j,4}-b_{j,4}, j=1,2,\dots,n, \alpha_{ij,7}\ge 0.$

Step 2: Using the optimal solutions, obtained in Step 1, find

(i) $x_{ij,1}=x'_{ij,1}+\alpha_{ij,1},\ i=1,2,\dots,m;\ j=1,2,\dots,n.$

(ii) $x_{ij,2}=x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3},\ i=1,2,\dots,m;\ j=1,2,\dots,n.$

(iii) $x_{ij,3}=\ x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4},\ i=1,2,\dots,m;\ j=1,2,\dots,n.$

(iv) $x_{ij,4}=x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}+\alpha_{ij,5}+\alpha_{ij,6},\quad i=1,2,\dots,m;$ $j=1,2,\dots,n.$

(v) $x'_{ij,2}=x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2},\ i=1,2,\dots,m;\ j=1,2,\dots,n.$

(vi) $x'_{ij,3}=x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}+\alpha_{ij,5},\ i=1,2,\dots,m;\ j=1,2,\dots,n.$

(vii) $x'_{ij,4}=x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}+\alpha_{ij,5}+\alpha_{ij,6}+\alpha_{ij,7}.\quad i=1,2,\dots,m;$ $j=1,2,\dots,n.$

Step 3: Using the optimal values of $x_{ij,k}$ and $x'_{ij,k};\ i=1,2,\dots,m;\ \ j=1,2,\dots,n;$ $k=1,2,3,4,$ obtained in Step 2, find the IF optimal solution $\{(x_{ij,1},x_{ij,2},x_{ij,3},x_{ij,4};x'_{ij,1},x'_{ij,2},x'_{ij,3},x'_{ij,4})\}$ and the IF optimal value $\sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij,1}x_{ij,1},c_{ij,2}x_{ij,2},c_{ij,3}x_{ij,3},c_{ij,4}x_{ij,4}\ ;\ c'_{ij,1}x'_{ij,1},c'_{ij,2}x'_{ij,2},c'_{ij,3}x'_{ij,3},c'_{ij,4}x'_{ij,4}).$

4.8 Proposed methodology

The proposed Mehar approach is obtained by considering the following methodology.

4.8.1 Mehar representation of a TrIFN

The aim of this chapter is to propose such an approach for solving BFIFTPs which is much easy to apply as compared to Ebrahimnejad and Verdegay's approach [51, Section 5]. After a deep study, it is observed that this objective cannot be achieved by considering the existing representation of a TrIFN [51, Section 2, Definition 7]. Keeping the same in mind, in this section, a new representation of a TrIFN (named as Mehar representation of a TrIFN) is proposed.

Let $\tilde{A}^I = (a_1, a_2, a_3, a_4; a_1', a_2', a_3', a_4')$ be a TrIFN, where $a_1' \leq a_1 \leq a_2' \leq a_2 \leq a_3 \leq a_3' \leq a_4 \leq a_4'$ then the Mehar representation of this TrIFN will be $\tilde{A}^I = (a_1', \beta_1, \beta_2, \beta_3;\ \beta_4, \beta_5, \beta_6, \beta_7)_M$ where, $\beta_1 = a_1 - a_1'$, $\beta_2 = a_2' - a_1$, $\beta_3 = a_2 - a_2'$, $\beta_4 = a_3 - a_2$, $\beta_5 = a_3' - a_3$, $\beta_6 = a_4 - a_3'$, $\beta_7 = a_4' - a_4$. e.g., the Mehar representation of the TrIFN $\tilde{A}^I = (20, 30, 40, 50; 15, 25, 45, 55)$ will be $\tilde{A}^I = (15,\ 20-15,\ 25-20,\ 30-25,\ 40-30,\ 45-40,\ 50-45,\ 55-50)_M = (15, 5, 5, 5, 10, 5, 5, 5)_M$.

4.8.2 Multiplication of a TrIFN in its existing representation with a TrIFN in its Mehar representation

In the Section 4.8.3, the origin of the proposed Mehar approach is discussed. Since, in Step 2 of Section 4.8.3, there is need to find the multiplication of a TrIFN in its existing representation [51, Section 2, Definition 7] with a TrIFN in its Mehar representation. Therefore, the same is discussed in this section.

Let $\left(c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4}; c_{ij,1}', c_{ij,2}', c_{ij,3}', c_{ij,4}'\right)$ and $\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}; x_{ij,1}', x_{ij,2}', x_{ij,3}', x_{ij,4}'\right)$ be two TrIFNs in its existing representation [51, Section 2, Definition 7]. Then, using the existing result [51, Section 2, Definition 8],

$\left(c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4}; c'_{ij,1}, c'_{ij,2}, c'_{ij,3}, c'_{ij,4}\right) \otimes$

$\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}; x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4}\right) =$

$\left(c_{ij,1}\, x_{ij,1}, c_{ij,2}\, x_{ij,2}, c_{ij,3}\, x_{ij,3}, c_{ij,4} x_{ij,4}\ ;\ c'_{ij,1}\, x'_{ij,1}, c'_{ij,2}\, x'_{ij,2}, c'_{ij,3}\, x'_{ij,3}, c'_{ij,4}\, x'_{ij,4}\right)$. (4.1)

Furthermore, using Section 4.8.1, the TrIFN, $\tilde{x}_{ij} = \left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}; x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4}\right)$ in its Mehar representation can be written as $\left(x'_{ij,1}, \alpha_{ij,1}, \alpha_{ij,2}, \alpha_{ij,3}, \alpha_{ij,4}, \alpha_{ij,5}, \alpha_{ij,6}, \alpha_{ij,7}\right)_M$, where

(i) $\alpha_{ij,1} = x_{ij,1} - x'_{ij,1} \Rightarrow x_{ij,1} = x'_{ij,1} + \alpha_{ij,1}$.

(ii) $\alpha_{ij,2} = x'_{ij,2} - x_{ij,1} \Rightarrow x'_{ij,2} = x_{ij,1} + \alpha_{ij,2} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2}$.

(iii) $\alpha_{ij,3} = x_{ij,2} - x'_{ij,2} \Rightarrow x_{ij,2} = x'_{ij,2} + \alpha_{ij,3} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3}$.

(iv) $\alpha_{ij,4} = x_{ij,3} - x_{ij,2} \Rightarrow x_{ij,3} = x_{ij,2} + \alpha_{ij,4} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4}$.

(v) $\alpha_{ij,5} = x'_{ij,3} - x_{ij,3} \Rightarrow x'_{ij,3} = x_{ij,3} + \alpha_{ij,5} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4} + \alpha_{ij,5}$.

(vi) $\alpha_{ij,6} = x_{ij,4} - x'_{ij,3} \Rightarrow x_{ij,4} = x'_{ij,3} + \alpha_{ij,6} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4} + \alpha_{ij,5} + \alpha_{ij,6}$.

(vii) $\alpha_{ij,7} = x'_{ij,4} - x_{ij,4} \Rightarrow x'_{ij,4} = x_{ij,4} + \alpha_{ij,7} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4} + \alpha_{ij,5} + \alpha_{ij,6} + \alpha_{ij,7}$.

Replacing the TrIFN $\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\ ; x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4}\right)$, present in left hand side of multiplication (4.1), with its Mehar representation $\left(x'_{ij,1}, \alpha_{ij,1}, \alpha_{ij,2}, \alpha_{ij,3}, \alpha_{ij,4}, \alpha_{ij,5}, \alpha_{ij,6}, \alpha_{ij,7}\right)_M$ and the values of $x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}, x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4}$, present in right hand side of multiplication (4.1), with $x'_{ij,1} + \alpha_{ij,1}, x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3}, x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4}, x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4} + \alpha_{ij,5} + \alpha_{ij,6}, x'_{ij,1}, x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2}, x'_{ij,1} +$

$\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}+\alpha_{ij,5}, x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}+\alpha_{ij,5}+\alpha_{ij,6}+\alpha_{ij,7}$ respectively, the multiplication (4.1) is transformed into its equivalent multiplication (4.2).

$$\left(c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4}; c'_{ij,1}, c'_{ij,2}, c'_{ij,3}, c'_{ij,4}\right) \otimes \left(x'_{ij,1}, \alpha_{ij,1}, \alpha_{ij,2}, \alpha_{ij,3}, \alpha_{ij,4}, \alpha_{ij,5}, \alpha_{ij,6}, \alpha_{ij,7}\right)_M$$

$$= \Big(c_{ij,1}\left(x'_{ij,1}+\alpha_{ij,1}\right), c_{ij,2}\left(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}\right), c_{ij,3}\left(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}\right), c_{ij,4}\left(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}+\alpha_{ij,5}+\alpha_{ij,6}\right); x'_{ij,1}, c'_{ij,2}\left(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}\right), c'_{ij,3}\left(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}+\alpha_{ij,5}\right), c'_{ij,4}\left(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}+\alpha_{ij,5}+\alpha_{ij,6}+\alpha_{ij,7}\right)\Big). \quad (4.2)$$

4.8.3 Origin of the proposed Mehar approach

In this section, the origin of the proposed Mehar approach is discussed.

Step 1: Replacing the parameters $\tilde{x}^I_{ij} = \left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\ ;\ x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4}\right)$, $\tilde{a}^I_i = \left(a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}\ ;\ a'_{i,1}, a'_{i,2}, a'_{i,3}, a'_{i,4}\right)$ and $\tilde{b}^I_j = \left(b_{j,1}, b_{j,2}, b_{j,3}, b_{j,4}\ ;\ b'_{j,1}, b'_{j,2}, b'_{j,3}, b'_{j,4}\right)$ with their Mehar representation,

$\tilde{x}^I_{ij} = \left(x'_{ij,1}, \alpha_{ij,1}, \alpha_{ij,2}, \alpha_{ij,3}, \alpha_{ij,4}, \alpha_{ij,5}, \alpha_{ij,6}, \alpha_{ij,7}\right)_M$,

$\tilde{a}^I_i = \left(a'_{i,1}, \beta_{i,1}, \beta_{i,2}, \beta_{i,3}, \beta_{i,4}, \beta_{i,5}, \beta_{i,6}, \beta_{i,7}\right)_M$ and

$\tilde{b}^I_j = \left(b'_{j,1}, \gamma_{j,1}, \gamma_{j,2}, \gamma_{j,3}, \gamma_{j,4}, \gamma_{j,5}, \gamma_{j,6}, \gamma_{j,7}\right)_M$ respectively, the IFFLPP $(P4.1)$ can be transformed into its equivalent IFFLPP $(P4.15)$.

IFFLPP $(\boldsymbol{P4.15})$

$$\text{Minimize}\begin{bmatrix}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4}\ ;\ c'_{ij,1}, c'_{ij,2}, c'_{ij,3}, c'_{ij,4}\right)\otimes\\ \left(x'_{ij,1}, \alpha_{ij,1}, \alpha_{ij,2}, \alpha_{ij,3}, \alpha_{ij,4}, \alpha_{ij,5}, \alpha_{ij,6}, \alpha_{ij,7}\right)_M\end{bmatrix}$$

Subject to

$\sum_{j=1}^{n}\left(x'_{ij,1},\alpha_{ij,1},\alpha_{ij,2},\alpha_{ij,3},\alpha_{ij,4},\alpha_{ij,5},\alpha_{ij,6},\alpha_{ij,7}\right)_M =$

$\left(a'_{i,1},\beta_{i,1},\beta_{i,2},\beta_{i,3},\beta_{i,4},\beta_{i,5},\beta_{i,6}\ ,\beta_{i,7}\right)_M,\ i = 1,2,\dots,m,$

$\sum_{i=1}^{m}\left(x'_{ij,1},\alpha_{ij,1},\alpha_{ij,2},\alpha_{ij,3},\alpha_{ij,4},\alpha_{ij,5},\alpha_{ij,6},\alpha_{ij,7}\right)_M =$

$\left(b'_{j,1},\gamma_{j,1},\gamma_{j,2},\gamma_{j,3},\gamma_{j,4},\gamma_{j,5},\gamma_{j,6},\gamma_{j,7}\right)_M,\ j = 1,2,\dots,n,$

$\left(x'_{ij,1},\alpha_{ij,1},\alpha_{ij,2},\alpha_{ij,3},\alpha_{ij,4},\alpha_{ij,5},\alpha_{ij,6},\alpha_{ij,7}\right)_M$ is a non-negative TrIFN.

where,

(i) $\alpha_{ij,1} = x_{ij,1} - x'_{ij,1}, \alpha_{ij,2} = x'_{ij,2} - x_{ij,1}, \alpha_{ij,3} = x_{ij,2} - x'_{ij,2}, \alpha_{i,4} = x_{ij,3} - x_{ij,2}, \alpha_{ij,5} = x'_{ij,3} - x_{ij,3}, \alpha_{ij,6} = x_{ij,4} - x'_{ij,3}\ , \alpha_{ij,7} = x'_{ij,4} - x_{ij,4}.$

(ii) $\beta_{i,1} = a_{i,1} - a'_{i,1}, \beta_{i,2} = a'_{i,2} - a_{i,1}, \beta_{i,3} = a_{i,2} - a'_{i,2},\ \ \beta_{i,4} = a_{i,3} - a_{i,2}, \beta_{i,5} = a'_{i,3} - a_{i,3}, \beta_{i,6} = a_{i4} - a'_{i,3}\ , \beta_{i,7} = a'_{i,4} - a_{i,4}.$

(iii) $\gamma_{j,1} = b_{j,1} - b'_{j,1}, \gamma_{j,2} = b'_{j,2} - b_{j,1}, \gamma_{j,3} = b_{j,2} - b'_{,2}, \gamma_{j,4} = b_{j,3} - b_{j,2}, \gamma_{j,5} = b'_{j,3} - b_{j,3}, \gamma_{j,6} = b_{j,4} - b'_{j,3}\ , \gamma_{j,7} = b'_{j,4} - b_{j,4}.$

Step 2: Using the multiplication (4.2), proposed is Section 4.8.2, the IFLPP $(P4.15)$ can be transformed into its equivalent IFFLPP $(P4.16)$.

IFFLPP $(\boldsymbol{P4.16})$

Minimize $\Big[\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(c_{ij,1}\left(x'_{ij,1} + \alpha_{ij,1}\right), c_{ij,2}\left(x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3}\right), c_{ij,3}\left(x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4}\right), c_{ij,4}\left(x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4} + \alpha_{ij,5} + \alpha_{ij,6}\right);\ x'_{ij,1},\ c'_{ij,2}\left(x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2}\right),\ c'_{ij,3}\left(x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4} + \alpha_{ij,5}\right), c'_{ij,4}\left(x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4} + \alpha_{ij,5} + \alpha_{ij,6} + \alpha_{ij,7}\right)\Big)\Big]$

Subject to

Constraints of the IFFLPP $(P4.15)$.

Step 3: Using the comparing method, discussed in Section 4.3, the IFLPP $(P4.16)$ can be transformed into its equivalent IFLPP $(P4.17)$.

IFLPP $(\boldsymbol{P4.17})$

$$\text{Minimize}\Big[\tfrac{1}{8}\textstyle\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(c_{ij,1}(x'_{ij,1}+\alpha_{ij,1})+\ c_{ij,2}(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3})+ c_{ij,3}(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4})+\ c_{ij,4}(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}+ \alpha_{ij,5}+\alpha_{ij,6})+\ x'_{ij,1}+\ c'_{ij,2}(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2})+\ c'_{ij,3}(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+ \alpha_{ij,4}+\ \alpha_{ij,5})+\ c'_{ij,4}(x'_{ij,1}+\alpha_{ij,1}+\alpha_{ij,2}+\alpha_{ij,3}+\alpha_{ij,4}+\ \alpha_{ij,5}+\alpha_{ij,6}+\alpha_{ij,7})\Big)\Big]$$

Subject to

Constraints of the IFLPP $(P4.15)$.

Step 4: Collecting the coefficients of $x'_{ij,1}, \alpha_{ij,1}, \alpha_{ij,2}, \alpha_{ij,3}, \alpha_{ij,4}, \alpha_{ij,5}, \alpha_{ij,6}, \alpha_{ij,7}$, the IFLPP $(P4.15)$ can be transformed into its equivalent IFLPP $(P4.18)$.

IFLPP $(\boldsymbol{P4.18})$

$$\text{Minimize}\Big[\tfrac{1}{8}\textstyle\sum_{i=1}^{m}\sum_{j=1}^{n}\Big((c_{ij,1}+\ c_{ij,2}+\ c_{ij,3}+\ c_{ij,4}+c'_{ij,1}+\ c'_{ij,2}+c'_{ij,3}+\ c'_{ij,4})\,x'_{ij,1}+ (c_{ij,1}+\ c_{ij,2}+\ c_{ij,3}+\ c_{ij,4}+\ c'_{ij,2},+c'_{ij,3}+\ c'_{ij,4})\alpha_{ij,1}+(\ c_{ij,2}+\ c_{ij,3}+\ c_{ij,4}+\ c'_{ij,2}+ c'_{ij,3}+\ c'_{ij,4})\alpha_{ij,2}+(\ c_{ij,2}+\ c_{ij,3}+\ c_{ij,4}+c'_{ij,3}+\ c'_{ij,4})\alpha_{ij,3}+(\ c_{ij,3}+\ c_{ij,4}+c'_{ij,3}+ c'_{ij,4})\alpha_{ij,4}\ +(c_{ij,4}+c'_{ij,3}+\ c'_{ij,4})\alpha_{ij,5}+(c_{ij,4}+c'_{ij,4})\alpha_{ij,6}+c'_{ij,4}\alpha_{ij,7})\Big]$$

Subject to

Constraints of the IFLPP $(P4.15)$.

Step 5: Using the relation $(a', \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7)_M = (b', \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7)_M \Rightarrow a' = b'$, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$, $\alpha_5 = \beta_5$,

$\alpha_6 = \beta_6, \alpha_7 = \beta_7$ and the relation $(a', \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7)_M$ is a non-negative TrIFN $\Rightarrow a' \geq 0,\ \alpha_1 \geq 0,\ \alpha_2 \geq 0, \alpha_3 \geq 0, \alpha_4 \geq 0, \alpha_5 \geq 0, \alpha_6 \geq 0, \alpha_7 \geq 0,$ the IFLPP $(P4.18)$ can be transformed into in its equivalent crisp LPP $(P4.19)$.

Crisp LPP $(\boldsymbol{P4.19})$

Minimize $\Big[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\big(c_{ij,1} + c_{ij,2} + c_{ij,3} + c_{ij,4} + c'_{ij,1} + c'_{ij,2} + c'_{ij,3} + c'_{ij,4}\big)x'_{ij,1} + \big(c_{ij,1} + c_{ij,2} + c_{ij,3} + c_{ij,4} + c'_{ij,2}, + c'_{ij,3} + c'_{ij,4}\big)\alpha_{ij,1} + \big(c_{ij,2} + c_{ij,3} + c_{ij,4} + c'_{ij,2} + c'_{ij,3} + c'_{ij,4}\big)\alpha_{ij,2} + \big(c_{ij,2} + c_{ij,3} + c_{ij,4} + c'_{ij,3} + c'_{ij,4}\big)\alpha_{ij,3} + \big(c_{ij,3} + c_{ij,4} + c'_{ij,3} + c'_{ij,4}\big)\alpha_{ij,4} + \big(c_{ij,4} + c'_{ij,3} + c'_{ij,4}\big)\alpha_{ij,5} + \big(c_{ij,4} + c'_{ij,4}\big)\alpha_{ij,6} + c'_{ij,4}\alpha_{ij,7}\Big]$

Subject to

$$\left.\begin{array}{c}\sum_{j=1}^{n} x'_{ij,1} = a'_{i,1}\,, \sum_{j=1}^{n}\alpha_{ij,1} = \beta_{i,1}\ \sum_{j=1}^{n}\alpha_{ij,2} = \beta_{i,2}\,, \sum_{j=1}^{n}\alpha_{ij,3} = \beta_{i,3}\,, \\ \sum_{j=1}^{n}\alpha_{ij,4} = \beta_{i,4}\,, \sum_{j=1}^{n}\alpha_{ij,5} = \beta_{i,5}\,, \sum_{j=1}^{n}\alpha_{ij,6} = \beta_{i,6}\,, \sum_{j=1}^{n}\alpha_{ij,7} = \beta_{i,7}\end{array}\right\},$$

$$i = 1, 2, \dots, m\ ;$$

$$\left.\begin{array}{c}\sum_{i=1}^{m}\alpha_{ij,1} = \gamma_{j,1}\,, \sum_{i=1}^{m}\alpha_{ij,2} = \gamma_{j,2}\,, \sum_{i=1}^{m}\alpha_{ij,3} = \gamma_{j,3}\,, \sum_{i=1}^{m}\alpha_{ij,4} = \gamma_{j,4}\,, \\ \sum_{i=1}^{m}\alpha_{ij,5} = \gamma_{j,5}\,, \sum_{i=1}^{m}\alpha_{ij,6} = \gamma_{j,6}, \sum_{i=1}^{m}\alpha_{ij,7} = \gamma_{j,7}\end{array}\right\} j = 1, 2, \dots, n\ ;$$

$x'_{ij,1} \geq 0,\ \ \alpha_{ij,1} \geq 0,\ \ \alpha_{ij,2} \geq 0,\ \ \alpha_{ij,3} \geq 0,\ \ \alpha_{ij,4} \geq 0,\ \ \alpha_{ij,5} \geq 0,\ \ \alpha_{ij,6} \geq 0,\ \ \alpha_{ij,7} \geq 0,$ $i = 1, 2, \dots, m, j = 1, 2, \dots, n.$

Step 6: To find the optimal solution $\{x'_{ij,1}, \alpha_{ij,1}, \alpha_{ij,2}, \alpha_{ij,3}, \alpha_{ij,4}, \alpha_{ij,5}, \alpha_{ij,6}, \alpha_{ij,7}\}$ of the crisp LPP $(P4.19)$ is equivalent to find the optimal solutions $\{x'_{ij,1}\}$, $\{\alpha_{ij,1}\}$, $\{\alpha_{ij,2}\}$, $\{\alpha_{ij,3}\}$, $\{\alpha_{ij,4}\}$, $\{\alpha_{ij,5}\}$, $\{\alpha_{ij,6}\}$, $\{\alpha_{ij,7}\}$ of the crisp LPPs $(P4.7)$ to $(P4.14)$ (already mentioned in Section 4.5) respectively.

Step 7: Using the optimal solutions, obtained in Step 6, find

(i) $x_{ij,1} = x'_{ij,1} + \alpha_{ij,1},\ i = 1,2, \dots, m\ ;\ j = 1,2, \dots, n.$

(ii) $x_{ij,2} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3}, i = 1,2, \dots, m\ ;\ j = 1,2, \dots, n.$

(iii) $x_{ij,3} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4},\ i = 1,2,\dots,m\ ;\ j = 1,2,\dots,n.$

(iv) $x_{ij,4} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4} + \alpha_{ij,5} + \alpha_{ij,6}, i = 1,2,\dots,m\ ; j = 1,2,\dots,n.$

(v) $x'_{ij,2} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2},\ i = 1,2,\dots,m\ ;\ j = 1,2,\dots,n.$

(vi) $x'_{ij,3} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4} + \alpha_{ij,5},\ i = 1,2,\dots,m\ ;\ j = 1,2,\dots,n.$

(vii) $x'_{ij,4} = x'_{ij,1} + \alpha_{ij,1} + \alpha_{ij,2} + \alpha_{ij,3} + \alpha_{ij,4} + \alpha_{ij,5} + \alpha_{ij,6} + \alpha_{ij,7}.$
$i = 1,2,\dots,m\ ;\ j = 1,2,\dots,n.$

Step 8: Using the optimal values of $x_{ij,k}$ and $x'_{ij,k};\ i = 1,2,\dots,m;\ j = 1,2,\dots,n; k = 1,2,3,4,$ obtained in Step 7, find the IF optimal solution $\{(x_{ij,1}, x_{ij,2}, x_{ij,3},\ x_{ij,4}; x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4})\}$ and the IF optimal value $\sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}, c_{ij,4}x_{ij,4}\ ;\ c'_{ij,1}x'_{ij,1}, c'_{ij,2}x'_{ij,2}, c'_{ij,3}x'_{ij,3}, c'_{ij,4}x'_{ij,4}).$

4.9 Advantages of the proposed Mehar approach

It is obvious from Step 4 of Ebrahimnejad and Verdegay approach [51, Section 5], discussed in Section 4.4, that to find the IF optimal solution of an IFTP having m sources and n destinations by Ebrahimnejad and Verdegay's approach [51, Section 5], there is need to solve a crisp LPP having

(i) The following $8m$ constraints:

$$\left.\begin{array}{l}\sum_{j=1}^{n} x_{ij,1} = a_{i,1}\,,\ \sum_{j=1}^{n} x_{ij,2} = a_{i,2}\,, \sum_{j=1}^{n} x_{ij,3} = a_{i,3}\,, \sum_{j=1}^{n} x_{ij,4} = a_{i,4},\\ \sum_{j=1}^{n} x'_{ij,1} = a'_{i,1}\,, \sum_{j=1}^{n} x'_{ij,2} = a'_{i,2}\,,\ \sum_{j=1}^{n} x'_{ij,3} = a'_{i,3}\,, \sum_{j=1}^{n} x'_{ij,4} = a'_{i,4}\end{array}\right\} i = 1,2,\dots,m.$$

(ii) The following $8n$ constraints:

(iii) $$\left.\begin{array}{l}\sum_{i=1}^{m} x_{ij,1} = b_{j,1}\,, \sum_{i=1}^{m} x_{ij,2} = b_{j,2}\,, \sum_{i=1}^{m} x_{ij,3} = b_{j,3}\,, \sum_{i=1}^{m} x_{ij,4} = b_{j,4}\,,\\ \sum_{i=1}^{m} x'_{ij,1} = b'_{j,1}\ , \sum_{i=1}^{m} x'_{ij,2} = b'_{j,2}\,, \sum_{i=1}^{m} x'_{ij,3} = b'_{j,3}\,, \sum_{i=1}^{m} x'_{ij,4} = b'_{j,4}\end{array}\right\} j = 1,2,\dots,n.$$

(iv) The following $8mn$ constraints:

$x'_{ij,1} \geq 0,\ x_{ij,1} - x'_{ij,1} \geq 0,\ x'_{ij,2} - x_{ij,1} \geq 0,\ x_{ij,2} - x'_{ij,2} \geq 0,\ x_{ij,3} - x_{ij,2} \geq 0,\ x'_{ij,3} - x_{ij,3} \geq 0, x_{ij,4} - x'_{ij,3} \geq 0, x'_{ij,4} - x_{ij,4} \geq 0,\ i = 1, 2, \dots, m, j = 1, 2, \dots, n.$

(v) The following $8mn$ variables:

$x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}, x'_{ij,1}, x'_{ij,2}, x'_{ij,3}, x'_{ij,4}, i = 1, 2, \dots, m, j = 1, 2, \dots, n.$

While, it is obvious from Step 1 of the Mehar approach, proposed in Section 4.7, that for solving the same problem by the proposed Mehar approach, there is need to solve 8 crisp LPPs, each having $(m + n + mn)$ constraints and mn variables.

Since, it is much easy to solve a crisp LPP having $(m + n + mn)$ constraints and mn variables as compared to solve a crisp LPP having $8(m + n + mn)$ constraints and $8mn$ variables. Therefore, it is much easy to apply the proposed Mehar approach as compared to Ebrahimnejad and Verdegay's approach [51, Section 5].

4.10 Illustrative example

Ebrahimnejad and Verdegay [51, Section 6.1] solved a BFIFTP having two sources O_1, O_2 and three destinations D_1, D_2 and D_3 by considering

$\tilde{c}^I_{11} = (10, 20, 30, 40;\ 5, 15, 35, 45),\ \tilde{c}^I_{12} = (50, 60, 70, 90;\ 45, 55, 75, 95),$

$\tilde{c}^I_{13} = (80, 90, 110, 120;\ 75, 85, 115, 125), \tilde{c}^I_{14} = (60, 70, 80, 90;\ 55, 65, 85, 95),$

$\tilde{c}^I_{22} = (70, 80, 100, 120;\ 65, 75, 115, 125),\ \tilde{c}^I_{23} = (20, 30, 50, 60;\ 15, 25, 35, 65),$

$\tilde{a}^I_1 = (60, 80, 100, 120;\ 50, 70, 110, 130), \tilde{a}^I_2 = (40, 60, 80, 100;\ 30, 50, 90, 110),$

$\tilde{b}^I_1 = (30, 50, 70, 90;\ 20, 40, 80, 100), \tilde{b}^I_2 = (20, 30, 40, 50;\ 15, 25, 45, 55),$

$\tilde{b}^I_3 = (50, 60, 70, 80;\ 45, 55, 75, 85).$

Using the proposed Mehar approach, the IF optimal solution of this BFIFTP can be obtained as follows:

Step 1: Since, $\tilde{a}_1^I = \left(a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}\,;\ a'_{1,1}, a'_{1,2}, a'_{1,3}, a'_{1,4}\right) =$ $(60, 80, 100, 120;\ 50, 70, 110, 130)$, $\tilde{a}_2^I = \left(a_{2,1}, a_{2,2}, a_{2,3}, a_{2,4}\,;\ a'_{2,1}, a'_{2,2}, a'_{2,3}, a'_{2,4}\right) =$ $(40, 60, 80, 100;\ 30, 50, 90, 110)$,

$\tilde{b}_1^I = \left(b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4}\,;\ b'_{1,1}, b'_{1,2}, b'_{1,3}, b'_{1,4}\right) = (30, 50, 70, 90;\ 20, 40, 80, 100)$,

$\tilde{b}_2^I = \left(b_{2,1}, b_{2,2}, b_{2,3}, b_{2,4}\,;\ b'_{2,1}, b'_{2,2}, b'_{2,3}, b'_{2,4}\right) = (20, 30, 40, 50;\ 15, 25, 45, 55)$,

$\tilde{b}_3^I = \left(b_{3,1}, b_{3,2}, b_{3,3}, b_{3,4}\,;\ b'_{3,1}, b'_{3,2}, b'_{3,3}, b'_{3,4}\right) = (50, 60, 70, 80;\ 45, 55, 75, 85)$.

So, according to Step 1 of the proposed Mehar approach, there is need to find the crisp optimal solutions $\{x'_{ij,1}\}$, $\{\alpha_{ij,1}\}$, $\{\alpha_{ij,2}\}$, $\{\alpha_{ij,3}\}$, $\{\alpha_{ij,4}\}$, $\{\alpha_{ij,5}\}$, $\{\alpha_{ij,6}\}$, $\{\alpha_{ij,7}\}$; $i = 1,2,3$; $j = 1,2$; of the crisp LPPs $(P4.20)$ to $(P4.27)$ respectively.

Crisp LPP $(\boldsymbol{P4.20})$

Minimize $\left[\frac{1}{8}\left(200\, x'_{11,1} + 540\, x'_{12,1} + 800\, x'_{13,1} + 600\, x'_{21,1} + 750\, x'_{22,1} + 300\, x'_{23,1}\right)\right]$

Subject to

$x'_{11,1} + x'_{12,1} + x'_{13,1} = 50,\ x'_{21,1} + x'_{22,1} + x'_{23,1} = 30,$

$x'_{11,1} + x'_{21,1} = 20,\ x'_{12,1} + x'_{22,1} = 15,\ x'_{13,1} + x'_{23,1} = 45,$

$x'_{11,1}, x'_{12,1}, x'_{13,1}, x'_{21,1}, x'_{22,1}, x'_{23,1} \geq 0.$

Crisp LPP $(\boldsymbol{P4.21})$

Minimize $\left[\frac{1}{8} 195\, \alpha_{11,1} + 495\, \alpha_{12,1} + 725\, \alpha_{13,1} + 545\, \alpha_{21,1} + 685\, \alpha_{22,1} + 285\, \alpha_{23,1}\right]$

Subject to

$\alpha_{11,1} + \alpha_{12,1} + \alpha_{13,1} = 10,\ \ \alpha_{21,1} + \alpha_{22,1} + \alpha_{23,1} = 10,$

$\alpha_{11,1} + \alpha_{21,1} = 10,\ \alpha_{12,1} + \alpha_{22,1} = 5,\ \alpha_{13,1} + \alpha_{23,1} = 5,$

$\alpha_{11,1}, \alpha_{12,1}, \alpha_{13,1}, \alpha_{21,1}, \alpha_{22,1}, \alpha_{23,1} \geq 0.$

Crisp LPP $(\boldsymbol{P4.22})$

Minimize $\left[\frac{1}{8}\left(185\ \alpha_{11,2} + 445\ \alpha_{12,2} + 645\ \alpha_{13,2} + 485\ \alpha_{21,2} + 615\ \alpha_{22,2} + 265\ \alpha_{23,2}\right)\right]$

Subject to

$\alpha_{11,2} + \alpha_{12,2} + \alpha_{13,2} = 10,\ \ \alpha_{21,2} + \alpha_{22,2} + \alpha_{23,2} = 10,$

$\alpha_{11,2} + \alpha_{21,2} = 10, \qquad \alpha_{12,2} + \alpha_{22,2} = 5, \qquad \alpha_{13,2} + \alpha_{23,2} = 5,$

$\alpha_{11,2}, \alpha_{12,2}, \alpha_{13,2}, \alpha_{21,2}, \alpha_{22,2}, \alpha_{23,2} \geq 0.$

Crisp LPP ($\boldsymbol{P4.23}$)

Minimize $\left[\frac{1}{8}\left(170\ \alpha_{11,3} + 390\ \alpha_{12,3} + 560\ \alpha_{13,3} + 420\ \alpha_{21,3} + 540\ \alpha_{22,3} + 240\ \alpha_{23,3}\right)\right]$

Subject to

$\alpha_{11,3} + \alpha_{12,3} + \alpha_{13,3} = 10, \qquad \alpha_{21,3} + \alpha_{22,3} + \alpha_{23,3} = 10,$

$\alpha_{11,3} + \alpha_{21,3} = 10, \qquad \alpha_{12,3} + \alpha_{22,3} = 5, \qquad \alpha_{13,3} + \alpha_{23,3} = 5,$

$\alpha_{11,3}, \alpha_{12,3}, \alpha_{13,3}, \alpha_{21,3}, \alpha_{22,3}, \alpha_{23,3} \geq 0.$

Crisp LPP ($\boldsymbol{P4.24}$)

Minimize $\left[\frac{1}{8}\left(150\ \alpha_{11,4} + 330\ \alpha_{12,4} + 470\ \alpha_{13,4} + 350\ \alpha_{21,4} + 460\ \alpha_{22,4} + 210\ \alpha_{23,4}\right)\right]$

Subject to

$\alpha_{11,4} + \alpha_{12,4} + \alpha_{13,4} = 20, \qquad \alpha_{21,4} + \alpha_{22,4} + \alpha_{23,4} = 20,$

$\alpha_{11,4} + \alpha_{21,4} = 20, \qquad \alpha_{12,4} + \alpha_{22,4} = 10, \qquad \alpha_{13,4} + \alpha_{23,4} = 10,$

$\alpha_{11,4}, \alpha_{12,4}, \alpha_{13,4}, \alpha_{21,4}, \alpha_{22,4}, \alpha_{23,4} \geq 0.$

Crisp LPP $(\boldsymbol{P4.25})$

Minimize $\left[\frac{1}{8}\left(120\ \alpha_{11,5} + 260\ \alpha_{12,5} + 360\ \alpha_{13,5} + 270\ \alpha_{21,5} + 360\ \alpha_{22,5} + 160\ \alpha_{23,5}\right)\right]$

Subject to

$\alpha_{11,5} + \alpha_{12,5} + \alpha_{13,5} = 10, \qquad \alpha_{21,5} + \alpha_{22,5} + \alpha_{23,5} = 10,$

$\alpha_{11,5} + \alpha_{21,5} = 10, \qquad \alpha_{12,5} + \alpha_{22,5} = 5, \qquad \alpha_{13,5} + \alpha_{23,5} = 5,$

$\alpha_{11,5}, \alpha_{12,5}, \alpha_{13,5}, \alpha_{21,5}, \alpha_{22,5}, \alpha_{23,5} \geq 0.$

Crisp LPP $(\boldsymbol{P4.26})$

Minimize $\left[\frac{1}{8}\left(85\ \alpha_{11,6} + 185\ \alpha_{12,6} + 245\ \alpha_{13,6} + 185\ \alpha_{21,6} + 245\ \alpha_{22,6} + 125\ \alpha_{23,6}\right)\right]$

Subject to

$\alpha_{11,6} + \alpha_{12,6} + \alpha_{13,6} = 10, \qquad \alpha_{21,6} + \alpha_{22,6} + \alpha_{23,6} = 10,$

$\alpha_{11,6} + \alpha_{21,6} = 10, \qquad \alpha_{12,6} + \alpha_{22,6} = 5, \alpha_{13,6} + \alpha_{23,6} = 5,$

$\alpha_{11,6}, \alpha_{12,6}, \alpha_{13,6}, \alpha_{21,6}, \alpha_{22,6}, \alpha_{23,6} \geq 0.$

Crisp LPP $(\boldsymbol{P4.27})$

Minimize $\left[\frac{1}{8}\left(45\ \alpha_{11,7} + 95\ \alpha_{12,7} + 125\ \alpha_{13,7} + 95\ \alpha_{21,7} + 125\ \alpha_{22,7} + 65\ \alpha_{23,7}\right)\right]$

Subject to

$\alpha_{11,7} + \alpha_{12,7} + \alpha_{13,7} = 10, \qquad \alpha_{21,7} + \alpha_{22,7} + \alpha_{23,7} = 10,$

$\alpha_{11,7} + \alpha_{21,7} = 10, \qquad \alpha_{12,7} + \alpha_{22,7} = 5, \qquad \alpha_{13,7} + \alpha_{23,7} = 5,$

$\alpha_{11,7}, \alpha_{12,7}, \alpha_{13,7}, \alpha_{21,7}, \alpha_{22,7}, \alpha_{23,7} \geq 0$

On solving the crisp LPPs $(P4.20)$ to $(P4.27)$, the following optimal solutions are obtained:

(i) Optimal solution of the crisp LPP $(P4.20)$ is

$x'_{11,1} = 20,\ x'_{12,1} = 15\ , x'_{13,1} = 15,\ x'_{21,1} = 0,\ x'_{22,1} = 0,\ x'_{23,1} = 30.$

(ii) Optimal solution of the crisp LPP $(P4.21)$ is

$\alpha_{11,1} = 10, \alpha_{12,1} = 0, \alpha_{13,1} = 0, \alpha_{21,1} = 0, \alpha_{22,1} = 5, \alpha_{23,1} = 5.$

(iii) Optimal solution of the crisp LPP $(P4.22)$ is

$\alpha_{11,2} = 10, \alpha_{12,2} = 0, \alpha_{13,2} = 0, \alpha_{21,2} = 0, \alpha_{22,2} = 5, \alpha_{23,2} = 5.$

(iv) Optimal solution of the crisp LPP $(P4.23)$ is

$\alpha_{11,3} = 10, \alpha_{12,3} = 0, \alpha_{13,3} = 0, \alpha_{21,3} = 0, \alpha_{22,3} = 5, \alpha_{23,3} = 5.$

(v) Optimal solution of the crisp LPP $(P4.24)$ is

$\alpha_{11,4} = 20, \alpha_{12,4} = 0, \alpha_{13,4} = 0, \alpha_{21,4} = 0, \alpha_{22,4} = 10, \alpha_{23,4} = 10.$

(vi) Optimal solution of the crisp LPP $(P4.25)$ is

$\alpha_{11,5} = 10, \alpha_{12,5} = 0, \alpha_{13,5} = 0, \alpha_{21,5} = 0, \alpha_{22,5} = 5, \alpha_{23,5} = 5.$

(vii) Optimal solution of the crisp LPP $(P4.26)$ is

$\alpha_{11,6} = 10, \alpha_{12,6} = 0, \alpha_{13,6} = 0, \alpha_{21,6} = 0, \alpha_{22,6} = 5, \alpha_{23,6} = 5.$

(viii) Optimal solution of the crisp LPP $(P4.27)$ is

$\alpha_{11,7} = 10, \alpha_{12,7} = 0, \alpha_{13,7} = 0, \alpha_{21,7} = 0, \alpha_{22,7} = 5, \alpha_{23,7} = 5.$

Step 2: Using Step 2, of the proposed Mehar approach,

(i) $x_{11,1} = 30,\quad x_{11,2} = 50,\quad x_{11,3} = 70, x_{11,4} = 90, x'_{11,1} = 20, x'_{11,2} = 40, x'_{11,3} = 80, x'_{11,4} = 100.$

(ii) $x_{12,2} = 15,\quad x_{12,2} = 15,\quad x_{12,3} = 15, x_{12,4} = 15, x'_{12,1} = 15, x'_{12,2} = 15, x'_{12,3} = 15, x'_{12,4} = 15.$

(iii) $x_{13,1} = 15,\quad x_{13,2} = 15,\quad x_{13,3} = 15, x_{13,4} = 15, x'_{13,1} = 15, x'_{13,2} = 15, x'_{13,3} = 15, x'_{13,4} = 15.$

(iv) $x_{21,1} = 0, x_{21,2} = 0, x_{21,3} = 0, x_{21,4} = 0, x'_{21,1} = 0, x'_{21,2} = 0, x'_{21,3} = 0, x'_{21,4} = 0.$

(v) $x_{22,1} = 5,$ $x_{22,2} = 15,$ $x_{22,3} = 25, x_{22,4} = 35, x'_{22,1} = 0, x'_{22,2} = 10, x'_{22,3} = 30, x'_{22,4} = 0.$

(vi) $x_{23,1} = 0,$ $x_{23,2} = 45,$ $x_{23,3} = 55, x_{23,4} = 65, x'_{23,1} = 30, x'_{23,2} = 40, x'_{23,3} = 60, x'_{23,4} = 70.$

Step 3: Using Step 3 of the proposed Mehar approach, the obtained IF optimal solution of the considered BFIFTP is $\tilde{x}^I_{11} = (30, 50, 70, 90; 20, 40, 80, 100)$, $\tilde{x}^I_{12} = (15, 15, 15, 15;\ 15, 15, 15, 15)$, $\tilde{x}^I_{13} = (15, 15, 15, 15;\ 15, 15, 15, 15), \tilde{x}^I_{21} = (0, 0, 0, 0;\ 0, 0, 0, 0), \tilde{x}^I_{22} = (5, 15, 25, 35;\ 0, 10, 30, 40)$, $\tilde{x}^I_{23} = (35, 45, 55, 65;\ 30, 40, 60, 70)$ and the obtained IF optimal value is $(3300, 5800, 9100, 13200;\ 2350, 4450, 11050, 15550)$.

4.11 Flaws of the existing and the proposed Mehar approach

Since, the proposed Mehar approach is an alternative simplified way, corresponding to existing approach [51, Section 5] for solving BFIFTP. So, in the proposed Mehar approach, the method, used in the existing approach for comparing TrIFNs, is used for comparing TrIFNs. However, this method for comparing TrIFNs is not valid due to the following reasons.

(1) There may exist two distinct TrIFNs $\tilde{A}^I$ and $\tilde{B}^I$ such that $H(\tilde{A}^I) = H(\tilde{B}^I)$. In this case, according to the existing approach for comparing TIFNs, the result $\tilde{A}^I = \tilde{B}^I$ is obtained which is mathematically incorrect. For example, if $\tilde{A}^I = (20, 30, 40, 50; 10, 25, 45, 55)$ and $\tilde{B}^I = (20, 28, 42, 50; 15, 23, 47, 55)$ then, $H(\tilde{A}^I) = \frac{20+30+40+50+10+25+45+55}{8}$ and $H(\tilde{B}^I) = \frac{20+28+42+50+15+23+47+55}{8}$.

Since, $H(\tilde{A}^I) = H(\tilde{B}^I)$. So, according to the existing method for comparing TrIFNs $\tilde{A}^I = \tilde{B}^I$. While, it is obvious that $\tilde{A}^I \neq \tilde{B}^I$. Due to the same reason, $\tilde{A}^I =$

$(20, 30, 40, 50; 10, 25, 45, 55)$ and $\tilde{B}^I = (20, 28, 42, 50; 15, 23, 47, 55)$ may be considered as the optimal IF transportation cost of a BFIFTP which is mathematically incorrect as the physical meaning of both TrIFNs is different.

4.12 Appropriate method for comparing TrIFNs

If instead of the method, discussed in Section 4.3, the following method is used for comparing TrIFNs, $\tilde{A}^I = (a_1, a_2, a_3, a_4; a_1', a_2', a_3', a_4')$ and $\tilde{B}^I = (b_1, b_2, b_3, b_4; b_1', b_2', b_3', b_4')$ for comparing TrIFNs. Then, always a unique TrIFN, representing the optimal IF transportation cost, will be obtained.

Step 1: Find $J(\tilde{A}^I) = \frac{a_1+a_2+a_3+a_4+a_1'+ a_2'+ a_3'+a_4'}{8}$ and $J(\tilde{B}^I) = \frac{b_1+b_2+b_3+b_4+b_1'+b_2'+b_3'+b_4'}{8}$ and check that $J(\tilde{A}^I) > J(\tilde{B}^I)$ or $J(\tilde{A}^I) < J(\tilde{B}^I)$ or $J(\tilde{A}^I) = J(\tilde{B}^I)$.

Case (i): If $J(\tilde{A}^I) > J(\tilde{B}^I)$ then $\tilde{A}^I \succ \tilde{B}^I$.

Case (ii): If $J(\tilde{A}^I) < J(\tilde{B}^I)$ then $\tilde{A}^I \prec \tilde{B}^I$.

Case (iii): If $J(\tilde{A}^I) = J(\tilde{B}^I)$ then go to Step 2.

Step 2: Find $A(\tilde{A}^I) = \frac{a_1-a_2+a_3+a_4+a_1'+ a_2'+ a_3'+a_4'}{8}$ and $A(\tilde{B}^I) = \frac{b_1-b_2+b_3+b_4+b_1'+b_2'+b_3'+b_4'}{8}$ and check that $A(\tilde{A}^I) > A(\tilde{B}^I)$ or $A(\tilde{A}^I) < A(\tilde{B}^I)$ or $A(\tilde{A}^I) = A(\tilde{B}^I)$.

Case (i): If $A(\tilde{A}^I) > A(\tilde{B}^I)$ then $\tilde{A}^I \succ \tilde{B}^I$.

Case (ii): If $A(\tilde{A}^I) < A(\tilde{B}^I)$ then $\tilde{A}^I \prec \tilde{B}^I$.

Case (iii): If $A(\tilde{A}^I) = A(\tilde{B}^I)$ then go to Step 3.

Step 3: Find $G(\tilde{A}^I) = \frac{a_1+a_2-a_3+a_4+a_1'+ a_2'+ a_3'+a_4'}{8}$ and $G(\tilde{B}^I) = \frac{b_1+b_2-b_3+b_4+b_1'+b_2'+b_3'+b_4'}{8}$ and check that $G(\tilde{A}^I) > G(\tilde{B}^I)$ or $G(\tilde{A}^I) < G(\tilde{B}^I)$ or $G(\tilde{A}^I) = G(\tilde{B}^I)$.

Case (i): If $G(\tilde{A}^I) > G(\tilde{B}^I)$ then $\tilde{A}^I > \tilde{B}^I$.

Case (ii): If $G(\tilde{A}^I) < G(\tilde{B}^I)$ then $\tilde{A}^I < \tilde{B}^I$.

Case (iii): If $G(\tilde{A}^I) = G(\tilde{B}^I)$ then go to Step 4.

Step 4: Find $D(\tilde{A}^I) = \frac{a_1+a_2+a_3-a_4+a_1'+ a_2'+ a_3'+a_4'}{8}$ and $D(\tilde{B}^I) = \frac{b_1+b_2+b_3-b_4+b_1'+b_2'+b_3'+b_4'}{8}$ and check that $D(\tilde{A}^I) > D(\tilde{B}^I)$ or $D(\tilde{A}^I) < D(\tilde{B}^I)$ or $D(\tilde{A}^I) = D(\tilde{B}^I)$.

Case (i): If $D(\tilde{A}^I) > D(\tilde{B}^I)$ then $\tilde{A}^I \succ \tilde{B}^I$.

Case (ii): If $D(\tilde{A}^I) < D(\tilde{B}^I)$ then $\tilde{A}^I \prec \tilde{B}^I$.

Case (iii): If $D(\tilde{A}^I) = D(\tilde{B}^I)$ then go to Step 5.

Step 5: Find $M(\tilde{A}^I) = \frac{a_1+a_2+a_3+a_4-a_1'+ a_2'+ a_3'+a_4'}{8}$ and $M(\tilde{B}^I) = \frac{b_1+b_2+b_3+b_4-b_1'+b_2'+b_3'+b_4'}{8}$ and check that $M(\tilde{A}^I) > M(\tilde{B}^I)$ or $M(\tilde{A}^I) < M(\tilde{B}^I)$ or $M(\tilde{A}^I) = M(\tilde{B}^I)$.

Case (i): If $M(\tilde{A}^I) > M(\tilde{B}^I)$ then $\tilde{A}^I \succ \tilde{B}^I$.

Case (ii): If $M(\tilde{A}^I) < M(\tilde{B}^I)$ then $\tilde{A}^I \prec \tilde{B}^I$.

Case (iii): If $M(\tilde{A}^I) = M(\tilde{B}^I)$ then go to Step 6.

Step 6: Find $B(\tilde{A}^I) = \frac{a_1+a_2+a_3+a_4+a_1'- a_2'+ a_3'+a_4'}{8}$, $B(\tilde{B}^I) = \frac{b_1+b_2+b_3+b_4+b_1'-b_2'+b_3'+b_4'}{8}$ and check that $B(\tilde{A}^I) > B(\tilde{B}^I)$ or $B(\tilde{A}^I) < B(\tilde{B}^I)$ or $B(\tilde{A}^I) = B(\tilde{B}^I)$.

Case (i): If $B(\tilde{A}^I) > B(\tilde{B}^I)$ then $\tilde{A}^I \succ \tilde{B}^I$.

Case (ii): If $B(\tilde{A}^I) < B(\tilde{B}^I)$ then $\tilde{A}^I \prec \tilde{B}^I$.

Case (iii): If $B(\tilde{A}^I) = B(\tilde{B}^I)$ then go to Step 7.

Step 7: Find $E(\tilde{A}^I) = \frac{a_1+a_2+a_3+a_4+a_1'+ a_2'- a_3'+a_4'}{8}$, $E(\tilde{B}^I) = \frac{b_1+b_2+b_3+b_4+b_1'+b_2'-b_3'+b_4'}{8}$ and check that $E(\tilde{A}^I) > E(\tilde{B}^I)$ or $E(\tilde{A}^I) < E(\tilde{B}^I)$ or $E(\tilde{A}^I) = E(\tilde{B}^I)$.

Case (i): If $E(\tilde{A}^I) > E(\tilde{B}^I)$ then $\tilde{A}^I \succ \tilde{B}^I$.

Case (ii): If $E(\tilde{A}^I) < E(\tilde{B}^I)$ then $\tilde{A}^I \prec \tilde{B}^I$.

Case (iii): If $E(\tilde{A}^I) = E(\tilde{B}^I)$ then go to Step 8.

Step 8: Find $S(\tilde{A}^I)=\frac{a_1+a_2+a_3+a_4+a_1'+a_2'+a_3'-a_4'}{8}$, $S(\tilde{B}^I)=\frac{b_1+b_2+b_3+b_4+b_1'+b_2'+b_3'-b_4'}{8}$ and check that $S(\tilde{A}^I)>S(\tilde{B}^I)$ or $S(\tilde{A}^I)<S(\tilde{B}^I)$ or $S(\tilde{A}^I)=S(\tilde{B}^I)$.

Case (i): If $S(\tilde{A}^I)>S(\tilde{B}^I)$ then $\tilde{A}^I\succ\tilde{B}^I$.

Case (ii): If $S(\tilde{A}^I)<S(\tilde{B}^I)$ then $\tilde{A}^I\prec\tilde{B}^I$.

Case (iii): If $S(\tilde{A}^I)=S(\tilde{B}^I)$ then $\tilde{A}^I=\tilde{B}^I$.

4.13 Modified Mehar approach

It is obvious from Section 4.11, that on applying the proposed Mehar approach, more than one TrIFNs, representing the optimal IF transportation cost, may be obtained. While, the TrIFN, representing optimal IF transportation cost should be unique. Therefore, there is need to incorporate the following modifications in the proposed Mehar approach.

(1) Using the proposed Mehar approach, find all the possible TrIFNs, representing the IFT cost.

(2) Apply the method, discussed in Section 4.12, to find the minimum of all these TrIFNs.

(3) The obtained minimum TrIFN will represent the optimal IF transportation cost and the solution corresponding to which this TrIFN exist, will represent the IF optimal solution of the BFIFTP.

4.14 Conclusions

A new approach (named as Mehar approach) is proposed to find the IF optimal solution of such BFIFTPs in which each parameter is represented as a TrIFN. Also, it is shown that it is much easy to apply the proposed Mehar approach as compared to Ebrahimnejad and Verdegay's approach [51, Section 5]. Furthermore, to illustrate the proposed Mehar approach a BFIFTP, solved by Ebrahimnejad and Verdegay [51, Section 5] to illustrate their proposed approach, is solved.

Chapter 5

JMD Method for Transforming an Unbalanced Fully Intuitionistic Fuzzy Transportation Problem into a Balanced Fully Intuitionistic Fuzzy Transportation Problem[§]

Ebrahimnejad and Verdegay's approach [51, Section 5] as well as Mehar approach (proposed in Chapter 4) cannot be used to solve an unbalanced FIFTP. Therefore, one may use the following methodology to solve an unbalanced FIFTP.

(1) Use the existing approach [136, Section 3] to transform an unbalanced FIFTP into a balanced FIFTP.

(2) Apply the Mehar approach, proposed in Chapter 4, to find the IF optimal solution of the BFIFTP.

However, after a deep study it is observed that the existing approach [136, Section 3] to transform an unbalanced FIFTP into a BFIFTP is not valid.

To validate this claim, the existing approach [136, Section 3] is applied on an unbalanced FIFTP and shown that the transformed FIFTP is not a BFIFTP. Furthermore, a new method (named as JMD method) is proposed to transform an unbalanced FIFTP into a BFIFTP.

[§] The contents of this chapter have been communicated in "Soft Computing" for the possible publication.

5.1 Subtraction of two TIFNs

Mahmoodirad et al. [136, Section 2, Definition 3] have stated that if $\tilde{A} = (a_1, a_2, a_3; a_1', a_2, a_3')$ and $\tilde{B} = (b_1, b_2, b_3; b_1', b_2, b_3')$ are two TIFNs. Then according to Singh and Yadav [176], $\tilde{A} \ominus \tilde{B} = (a_1 + b_3, a_2 + b_2, a_3 + b_1; a_1' + b_3', a_2 + b_2, a_3' + b_1')$. However, this claim is not valid as Singh and Yadav [176] have stated that if $\tilde{A} = (a_1, a_2, a_3; a_1', a_2, a_3')$ and $\tilde{B} = (b_1, b_2, b_3; b_1', b_2, b_3')$ are two TIFNs. Then, $\tilde{A} \ominus \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1; a_1' - b_3', a_2 - b_2, a_3' - b_1')$. It is pertinent to mention that if the arithmetic operation $\tilde{A} \ominus \tilde{B} = (a_1 + b_3, a_2 + b_2, a_3 + b_1; a_1' + b_3', a_2 + b_2, a_3' + b_1')$, defined by Mahmoodirad et al. [136, Section 2, Definition 3], will be used to find the subtraction of two TIFNs then, the obtained number will not necessarily be a TIFN i.e., this operation is not valid.

Let $\tilde{A} = (2,3,4; 1,3,6)$ and $\tilde{B} = (3,5,7; 1,5,9)$ be two TIFNs. Then, according to this arithmetic operation,

$\tilde{A} \ominus \tilde{B} = (A_1, A_2, A_3; A_1', A_2, A_3') = (9,8,7; 10,8,7)$ which is not a TIFN as the necessary condition $A_1' \leq A_1 \leq A_2 \leq A_3 \leq A_3'$ is not satisfying.

5.2 IFFLPP of a BFIFTP

Mahmoodirad et al. [136, Section 3] have solved the IFFLPP $(P5.1)$ to find the optimal solution of a BFIFTP represented by Table 5.1.

Table 5.1 Tabular representation of FIFTP

Destinations → Sources ↓	D_1	D_2	$\cdots$	D_j	$\cdots$	D_n	Availability
S_1	$\tilde{c}_{11}$	$\tilde{c}_{11}$	$\cdots$	$\tilde{c}_{1j}$	$\cdots$	$\tilde{c}_{1n}$	$\tilde{a}_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_i	$\tilde{c}_{i1}$	$\tilde{c}_{i2}$	$\cdots$	$\tilde{c}_{ij}$	$\cdots$	$\tilde{c}_{in}$	$\tilde{a}_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_m	$\tilde{c}_{m1}$	$\tilde{c}_{m2}$	$\cdots$	$\tilde{c}_{mj}$	$\cdots$	$\tilde{c}_{mn}$	$\tilde{a}_m$
Demand	$\tilde{b}_1$	$\tilde{b}_2$	$\cdots$	$\tilde{b}_j$	$\cdots$	$\tilde{b}_m$	$\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j$

IFFLPP $(\boldsymbol{P5.1})$

Minimize$\left[\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} \otimes \tilde{x}_{ij}\right]$

Subject to

$\sum_{j=1}^{n} \tilde{x}_{ij} = \tilde{a}_i\,,\ \ i = 1, 2, \dots, m,$

$\sum_{i=1}^{m} \tilde{x}_{ij} = \tilde{b}_j\,,\ j = 1, 2, \dots, n,$

$\tilde{x}_{ij}$ is a non-negative TIFN.

where,

(i) The TIFN $\tilde{c}_{ij} = \left(c_{ij,1}, c_{ij,2}, c_{ij,3}; c'_{ij,1}, c_{ij,2}, c'_{ij,3}\right)$ represents the IF transportation cost for supplying the unit quantity of the product from the i^{th} source (S_i) to the j^{th} destination $\left(D_j\right)$,

(ii) The TIFN $\tilde{x}_{ij} = \left(x_{ij,1}, x_{ij,2}, x_{ij,3}; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right)$ represents the quantity of the product to be supplied from the i^{th} source (S_i) to the j^{th} destination $\left(D_j\right)$,

(iii) The TIFN $\tilde{a}_i = \left(a_{i,1}, a_{i,2}, a_{i,3}; a'_{i,1}, a_{i,2}, a'_{i,3}\right)$ represents the IF availability of the product at the i^{th} source (S_i),

(iv) The TIFN $\tilde{b}_j = \left(b_{j,1}, b_{j,2}, b_{j,3}; b'_{j,1}, b_{j,2}, b'_{j,3}\right)$ represents the IF demand of the product at the j^{th} destination $\left(D_j\right)$,

(v) The TIFN $\tilde{x}_{ij}$ represents the quantity of the product to be supplied from the i^{th} source (S_i) to the j^{th} destination (D_j),

(vi) $\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j$ represents that the total availability of the product at all the sources is equal to the total demand of the product at all the destinations.

5.3 Existing method for comparing TIFNs

It is well-known fact that the optimal solution of a crisp LPP will be that feasible solution corresponding to which the value of the objective function will be minimum. On the same direction, the IF optimal solution of the IFFLPP $(P5.1)$ will be that IF feasible solution corresponding to which the value of the objective function will be minimum. Since, in case of IFFLPP $(P5.1)$, the value of the objective function, corresponding to an intuitionistic fuzzy feasible solution, will be a TIFN. Therefore, to find the IF optimal solution of the IFFLPP $(P5.1)$, there is need to find the minimum of TIFNs i.e., there is need to compare the TIFNs.

In this section, the method for comparing TIFNs, used by Mahmoodirad et al. [136, Section 2, Definition 5] in their proposed approach, is discussed.

Let $\tilde{A} = (a_1, a_2, a_3; a_1', a_2, a_3')$ and $\tilde{B} = (b_1, b_2, b_3; b_1', b_2, b_3')$ be two TIFNs. Then,

(i) $\tilde{A} > \tilde{B}$ if $Rank(\tilde{A}) > Rank(\tilde{B})$,

(ii) $\tilde{A} < \tilde{B}$ if $Rank(\tilde{A}) < Rank(\tilde{B})$,

(iii) $\tilde{A} = \tilde{B}^I$ if $Rank(\tilde{A}) = Rank(\tilde{B})$,

where, $Rank(\tilde{A}) = \frac{a_1+a_3+4a_2+a_1'+ a_3'}{8}$ and $Rank(\tilde{B}) = \frac{b_1+b_3+4b_2+b_1'+ b_3'}{8}$.

5.4 Mahmoodirad et al.'s approach for solving FFIFTPs

Mahmoodirad et al. [136, Section 4.2] proposed the following approach for solving FIFTPs.

Step 1: Check that the considered FIFTP is a balanced FIFTP or an unbalanced FIFTP i.e., check that $\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) = \left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right)$

or

$\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) \neq \left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right)$.

Case 1: If the considered FIFTP is a BFIFTP i.e., $\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) = \left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right)$ then go to Step 2.

Case 2: If the considered FIFTP is an unbalanced FIFTP i.e., $\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) \neq \left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right)$ then check that $\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) > \left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right)$

or

$\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) < \left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right)$.

Case 2(a): If $\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) > \left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right)$ then add a dummy destination with IF demand

$\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) -$

$\left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right)$ to transform the unbalanced FIFTP into a BFIFTP and go to Step 2.

Case 2(b): If $\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) <$ $\left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right)$ then add a dummy source with IF availability $\left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right) -$ $\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right)$ to transform the unbalanced FIFTP into a BFIFTP and go to Step 2.

Step 2: Replace the parameters $\tilde{c}_{ij}^{I}$, $\tilde{x}_{ij}^{I}$, $\tilde{a}_{i}^{I}$ and $\tilde{b}_{j}^{I}$ with the TIFNs $\left(c_{ij,1}, c_{ij,2}, c_{ij,3},; \; c'_{ij,1}, c_{ij,2}, c'_{ij,3}\right)$, $\left(x_{ij,1}, x_{ij,2}, x_{ij,3}; \; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right)$, $\left(a_{i,1}, a_{i,2}, a_{i,3}; \; a'_{i,1}, a_{i,2}, a'_{i,3}\right)$ and $\left(b_{j,1}, b_{j,2}, b_{j,3}; \; b'_{j,1}, b_{j,2}, b'_{j,3}\right)$ respectively, to transform the IFFLPP $(P5.1)$ into its equivalent IFFLPP $(P5.2)$.

IFFLPP $(\boldsymbol{P5.2})$

Minimize$\left[\sum_{i=1}^{m} \sum_{j=1}^{n} \left(c_{ij,1}, c_{ij,2}, c_{ij,3}; \; c'_{ij,1}, c_{ij,2}, c'_{ij,3}\right) \otimes \left(x_{ij,1}, x_{ij,2}, x_{ij,3}; \; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right)\right]$

Subject to

$\sum_{j=1}^{n} \left(x_{ij,1}, x_{ij,2}, x_{ij,3}; \; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right) = \left(a_{i,1}, a_{i,2}, a_{i,3}; \; a'_{i,1}, a_{i,2}, a'_{i,3}\right), \; i = 1, 2, \dots, m,$

$\sum_{i=1}^{m} \left(x_{ij,1}, x_{ij,2}, x_{ij,3}; \; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right) = \left(b_{j,1}, b_{j,2}, b_{j,3}; \; b'_{j,1}, b_{j,2}, b'_{j,3}\right), j = 1, 2, \dots, n,$

$\left(x_{ij,1}, x_{ij,2}, x_{ij,3}; \; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right)$ is a non-negative TIFN.

Step 3: Use the existing relation [136, Section 2, Theorem 2], $(a_1, a_2, a_3\ ;\ a'_1, a_2, a'_3) = (b_1, b_2, b_3\ ;\ b'_1, b_2, b'_3) \Rightarrow a_1 = b_1, a_3 = b_3, a'_1 = b'_1, a'_3 = b'_3$ as well as the relation $(a_1, a_2, a_3;\ a'_1, a_2, a'_3)$ is a non-negative TIFN $\Rightarrow a'_1 \geq 0$, $a_1 - a'_1 \geq 0$, $a_2 - a_1 \geq 0$, $a_3 - a_2 \geq 0$, $a'_3 - a_3 \geq 0$, to transform the IFFLPP $(P5.2)$ into its equivalent IFLPP $(P5.3)$.

IFLPP $(\boldsymbol{P5.3})$

Minimize $\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}, c_{ij,2}, c_{ij,3}\ ;\ c'_{ij,1}, c_{ij,2}, c'_{ij,3}\right) \otimes \left(x_{ij,1}, x_{ij,2}, x_{ij,3}\ ;\ x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right)\right]$

Subject to

$\sum_{j=1}^{n} x_{ij,1} = a_{i,1}$, $\sum_{j=1}^{n} x_{ij,2} = a_{i,2}$, $\sum_{j=1}^{n} x_{ij,3} = a_{i,3}$, $\sum_{j=1}^{n} x'_{ij,1} = a'_{i,1}$, $\sum_{j=1}^{n} x'_{ij,3} = a'_{i,3}$, $i = 1,2,\ldots,m,$

$\sum_{i=1}^{m} x_{ij,1} = b_{j,1}$, $\sum_{i=1}^{m} x_{ij,2} = b_{j,2}$, $\sum_{i=1}^{m} x_{ij,3} = b_{j,3}$, $\sum_{i=1}^{m} x'_{ij,1} = b'_{j,1}$, $\sum_{i=1}^{m} x'_{ij,3} = b'_{j,3}$, $j = 1, 2, \ldots, n,$

$x'_{ij,1} \geq 0, x_{ij,1} - x'_{ij,1} \geq 0, x_{ij,2} - x_{ij,1} \geq 0, x_{ij,3} - x_{ij,2} \geq 0, x'_{ij,3} - x_{ij,3} \geq 0, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$

Step 4: Use the existing multiplication [136, Section 2, Definition 3], $(a_1, a_2, a_3\ ;\ a'_1, a_2, a'_3) \otimes (b_1, b_2, b_3\ ;\ b'_1, b_2, b'_3,) = (a_1 b_1, a_2 b_2, a_3 b_3\ ;\ a'_1 b'_1, a_2 b_2, a'_3 b'_3)$, to transform the IFLPP $(P5.3)$ into its equivalent IFLPP $(P5.4)$.

IFLPP $(\boldsymbol{P5.4})$

Minimize$\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}\ ;\ c'_{ij,1}x'_{ij,1}c_{ij,2}x_{ij,2}, c'_{ij,3}x'_{ij,3}\right)\right]$

Subject to

Constraints of the IFLPP $(P5.3)$

Step 5: Use the comparing method, discussed in Section 5.4, to transform the IFLPP $(P5.4)$ into its equivalent crisp LPP $(P5.5)$ and hence its equivalent crisp LPP $(P5.6)$.

Crisp LPP $(\boldsymbol{P5.6})$

Minimize $\left[\sum_{i=1}^{m}\sum_{j=1}^{n} Rank\left(c_{ij,1}\, x_{ij,1}, c_{ij,2}\, x_{ij,2}, c_{ij,3}\, x_{ij,3}\, ;\, c'_{ij,1}\, x'_{ij,1}, c_{ij,2}\, x_{ij,2}, c'_{ij,3}\, x'_{ij,3}\right)\right]$

Subject to

Constraints of the IFLPP $(P5.3)$.

Crisp LPP $(\boldsymbol{P5.7})$

Minimize $\left[\frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}\, x_{ij,1} + c_{ij,3}\, x_{ij,3} + 4c_{ij,2}\, x_{ij,2} + c'_{ij,1}\, x'_{ij,1} + c'_{ij,3}\, x'_{ij,3}\right)\right]$

Subject to

Constraints of the IFLPP $(P5.3)$.

Step 6: Using the optimal solution, obtained in Step 5, find the IF optimal solution $\left\{\left(x_{ij,1}, x_{ij,2}, x_{ij,3}\, ; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right)\right\}$ and the IF optimal value $\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij,1}x_{ij,1}, c_{ij,2}x_{ij,2}, c_{ij,3}x_{ij,3}\, ;\, c'_{ij,1}x'_{ij,1}, c_{ij,2}x_{ij,2}, c'_{ij,3}x'_{ij,3}\right)$.

5.5 Ambiguity in Mahmoodirad et al.'s approach

There are the following ambiguities in Step 1 of Mahmoodirad et al.'s approach [136, Section 4.2].

(1) It is obvious from Case (i) of Step 1 of Mahmoodirad et al.'s approach [136], discussed in Section 5.4, that Mahmoodirad et al. (2018) have assumed that a FIFTP will be a BFIFTP if $\sum_{i=1}^{m}\tilde{a}_i = \sum_{j=1}^{n}\tilde{b}_j$.

Since, Mahmoodirad et al. [136, Section 3] have not specified the meaning of equality. So, one may interpret it in the following manners.

(i) $\sum_{i=1}^{m}(a_{i1}, a_{i2}, a_{i3}\ ;\ a'_{i1}, a_{i2}, a'_{i3}) = \sum_{j=1}^{n}\left(b_{j1}, b_{j2}, b_{j3}\ ;\ b'_{j1}, b_{j2}, b'_{j3}\right) \Rightarrow \sum_{i=1}^{m} a_{i1} = \sum_{j=1}^{n} b_{j1}, \sum_{i=1}^{m} a_{i2} = \sum_{j=1}^{n} b_{j2}, \sum_{i=1}^{m} a_{i3} = \sum_{j=1}^{n} b_{j3}, \sum_{i=1}^{m} a'_{i1} = \sum_{j=1}^{n} b'_{j1}, \sum_{i=1}^{m} a'_{i3} = \sum_{j=1}^{n} b'_{j3}$.

(ii) $\sum_{i=1}^{m}(a_{i1}, a_{i2}, a_{i3}\ ;\ a'_{i1}, a_{i2}, a'_{i3}) = \sum_{j=1}^{n}\left(b_{j1}, b_{j2}, b_{j3}\ ;\ b'_{j1}, b_{j2}, b'_{j3}\right)$ $\Rightarrow Rank(\sum_{i=1}^{m}(a_{i1}, a_{i2}, a_{i3}\ ;\ a'_{i1}, a_{i2}, a'_{i3})) = Rank\left(\sum_{j=1}^{n}\left(b_{j1}, b_{j2}, b_{j3}\ ;\ b'_{j1}, b_{j2}, b'_{j3}\right)\right)$.

However, the following clearly indicates that Mahmoodirad et al. [136, Section 3] have used the relation $\sum_{i=1}^{m}(a_{i1}, a_{i2}, a_{i3}\ ;\ a'_{i1}, a_{i2}, a'_{i3}) = \sum_{j=1}^{n}\left(b_{j1}, b_{j2}, b_{j3}\ ;\ b'_{j1}, b_{j2}, b'_{j3}\right) \Rightarrow \sum_{i=1}^{m} a_{i1} = \sum_{j=1}^{n} b_{j1}, \sum_{i=1}^{m} a_{i2} = \sum_{j=1}^{n} b_{j2}, \sum_{i=1}^{m} a_{i3} = \sum_{j=1}^{n} b_{j3}, \sum_{i=1}^{m} a'_{i1} = \sum_{j=1}^{n} b'_{j1}, \sum_{i=1}^{m} a'_{i3} = \sum_{j=1}^{n} b'_{j3}$ to check that a FIFTP is balanced or not.

(i) In Step 3 of Mahmoodirad et al.'s approach [136, Section 4.2], the relation $(a_1, a_2, a_3; a'_1, a_2, a'_3) = (b_1, b_2, b_3; b'_1, b_2, b'_3) \Rightarrow a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a'_1 = b'_1$, $a'_2 = b'_2$ is used to transform the FIFLPP $(P5.1)$ into IFFLPP $(P5.2)$.

(ii) If the relation, $\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j \Rightarrow Rank(\sum_{i=1}^{m} \tilde{a}_i) = Rank\left(\sum_{j=1}^{n} \tilde{b}_j\right)$ will be used to check that the considered FIFTP is a balanced or an unbalanced FIFTP . Then, in Step 3 of Mahmoodirad et al. approach [136, Section 4.2] the constraints of the IFFLPP $(P5.1)$ i.e., $\sum_{i=1}^{m}\left(x_{ij,1}, x_{ij,2}, x_{ij,3}\ ; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right) = \left(a_{i,1}, a_{i,2}, a_{i,3}\ ; a'_{i,1}, a_{i,2}, a'_{i,3}\right)$, $i = 1, 2, \ldots, m$ and $\sum_{j=1}^{n}\left(x_{ij,1}, x_{ij,2}, x_{ij,3}\ ; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right) = \left(b_{j,1}, b_{j,2}, b_{j,3}\ ; b'_{j,1}, b_{j,2}, b'_{j,3}\right), j = 1, 2, \ldots, n$, were replaced by

$Rank\left(\sum_{i=1}^{m}\left(x_{ij,1},x_{ij,2},x_{ij,3}\,;x'_{ij,1},x_{ij,2},x'_{ij,3}\right)\right)=$

$Rank\left(a_{i,1},a_{i,2},a_{i,3}\,;a'_{i,1},a_{i,2},a'_{i,3}\right)$ and

$Rank\left(\sum_{j=1}^{n}\left(x_{ij,1},x_{ij,2},x_{ij,3}\,;x'_{ij,1},x_{ij,2},x'_{ij,3}\right)\right)=$

$Rank\left(b_{j,1},b_{j,2},b_{j,3}\,;b'_{j,1},b_{j,2},b'_{j,3}\right)$ respectively.

(2) According to Case 2 of Step 1 of Mahmoodirad et al.'s approach [136, Section 4.2], discussed in Section 5.4, there is need to check that $\sum_{i=1}^{m}(a_{i1},a_{i2},a_{i3};a'_{i1},a_{i2},a'_{i3}) > \sum_{j=1}^{n}\left(b_{j1},b_{j2},b_{j3};b'_{j1},b_{j2},b'_{j3}\right)$ or $\sum_{i=1}^{m}(a_{i1},a_{i2},a_{i3};a'_{i1},a_{i2},a'_{i3}) < \sum_{j=1}^{n}\left(b_{j1},b_{j2},b_{j3};b'_{j1},b_{j2},b'_{j3}\right)$.

Although, Mahmoodirad et al. [136, Section 3] have not clarified the meaning of inequality. But, it is obvious from Section 5.3 that Mahmoodirad et al. [136, Section 3] have used the relations $Rank(\sum_{i=1}^{m}(a_{i1},a_{i2},a_{i3};a'_{i1},a_{i2},a'_{i3})) > Rank\left(\sum_{j=1}^{n}\left(b_{j1},b_{j2},b_{j3};b'_{j1},b_{j2},b'_{j3}\right)\right) \Rightarrow \sum_{i=1}^{m}(a_{i1},a_{i2},a_{i3};a'_{i1},a_{i2},a'_{i3}) > \sum_{j=1}^{n}\left(b_{j1},b_{j2},b_{j3};b'_{j1},b_{j2},b'_{j3}\right)$ and $Rank(\sum_{i=1}^{m}(a_{i1},a_{i2},a_{i3};a'_{i1},a_{i2},a'_{i3})) < Rank\left(\sum_{j=1}^{n}\left(b_{j1},b_{j2},b_{j3};b'_{j1},b_{j2},b'_{j3}\right)\right) \Rightarrow \sum_{i=1}^{m}(a_{i1},a_{i2},a_{i3};a'_{i1},a_{i2},a'_{i3}) < \sum_{j=1}^{n}\left(b_{j1},b_{j2},b_{j3};b'_{j1},b_{j2},b'_{j3}\right)$ in their proposed approach .

5.6 Limitations of Mahmoodirad et al.'s approach

All the FIFTPs, in which $Rank(\sum_{i=1}^{m}\tilde{a}_i) = Rank\left(\sum_{j=1}^{n}\tilde{b}_j\right)$ but $\sum_{i=1}^{m}\tilde{a}_i \neq \sum_{j=1}^{n}\tilde{b}_j$, cannot be transformed into a BFIFTP and hence, cannot be solved by Mahmoodirad et al.'s approach [136, Section 4.2] e.g., let us consider Table 5.2 represents a FIFTP with TFNs having two sources S_1, S_2 and two destinations D_1, D_2.

Table 5.2 FIFTP

Destinations → Sources ↓	D_1	D_2	Availability
S_1	$(2,4,9;1,4,12)$	$(4,13,14;3,13,15)$	$(2,3,6;0,3,12)$
S_2	$(1,3,8;0,3,11)$	$(3,6,7;0,6,14)$	$(2,3,10;1,3,15)$
Demand	$(2,3,4;1,3,5)$	$(3,4,11;2,4,16)$	

It is obvious that $\sum_{i=1}^{2} \tilde{a}_i = (4,6,16;1,6,27)$ and $\sum_{j=1}^{2} \tilde{b}_j = (5,7,15;3,7,21)$. Since, $\sum_{i=1}^{2} \tilde{a}_i \neq \sum_{j=1}^{2} \tilde{b}_j$. So, according to Case 2 of Step 1 of Mahmoodirad et al.'s approach [136, Section 4.2], discussed in Section 5.4, the considered FIFTP is an unbalanced FIFTP. Furthermore, since $Rank(4,6,16;1,6,27) = Rank(5,7,15;3,7,21)$ i.e., neither Case 2(a) nor Case 2(b) of Case 2 of Step 1 of Mahmoodirad et al.'s approach [136] is satisfying. So, the considered unbalanced FIFTP, represented by Table 5.2, cannot be transformed into a BFIFTP. Hence, the FIFTP, represented by Table 5.2, cannot be solved by Mahmoodirad et al.'s approach [136, Section 4.2].

5.7 Flaws of Mahmoodirad et al.'s approach

Mahmoodirad et al. [136, Section 3] have claimed that on applying Step 1 of their proposed approach, an unbalanced FIFTP will be transformed into a BFIFTP. To show that this claim is not valid, in this section, Step 1 of Mahmoodirad et al.'s approach [136, Section 3] is applied on the unbalanced FIFTP, represented by Table 5.3, and shown that the obtained FIFTP is not a BFIFTP.

Table 5.3 Unbalanced FIFTP

Destinations → Sources ↓	D_1	D_2	Availability
S_1	$(2,4,9; 1,4,12)$	$(4,13,14; 3,13,15)$	$(2,4,6; 0,4,12)$
S_2	$(1,3,8; 0,3,11)$	$(3,6,7; 0,6,14)$	$(2,4,10; 1,4,15)$
Demand	$(2,3,4; 1,3,5)$	$(3,4,11; 2,4,16)$	

It is obvious that

$\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) = (4,8,16; 1,8,27)$,

$\left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right) = (5,7,15; 3,7,21)$.

Since, $Rank\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) > Rank\left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right)$. So, the considered FIFTP is an unbalanced FIFTP problem. Now, according Case 2(a) of Step 2 of Mahmoodirad et al.'s approach [136, Section 3], the unbalanced FIFTP, represented by Table 5.3, can be transformed into a BFIFTP by adding a dummy destination (say D_3) having IF demand $\left(\sum_{i=1}^{m} a_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a_{i,3}; \sum_{i=1}^{m} a'_{i,1}, \sum_{i=1}^{m} a_{i,2}, \sum_{i=1}^{m} a'_{i,3}\right) - \left(\sum_{j=1}^{n} b_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b_{j,3}; \sum_{j=1}^{n} b'_{j,1}, \sum_{j=1}^{n} b_{j,2}, \sum_{j=1}^{n} b'_{j,3}\right) = (4-15, 8-7, 16-5; 1-21, 8-7, 27-3) = (-11, 1, 11; -20, 1, 24)$ with the assumption that the IF transportation cost for transporting one unit quantity of the product from the sources S_1 and S_2 to the dummy destination (D_3) are zero TIFNs $(0,0,0; 0,0,0)$ i.e., according to Case 2(a) of Step 1 of Mahmoodirad et al.'s approach [136, Section 3], the transformed FIFTP represented by Table 5.4, will be a BFIFTP.

Table 5.4 Transformed FIFTP

Destinations → Sources ↓	D_1	D_2	D_3	Availability
S_1	$(2,4,9;1,4,12)$	$(4,13,14;3,13,15)$	$(0,0,0;0,0,0)$	$(2,4,6;0,4,12)$
S_2	$(1,3,8;0,3,11)$	$(3,6,7;0,6,14)$	$(0,0,0;0,0,0)$	$(2,4,10;1,4,15)$
Demand	$(2,3,4;1,3,5)$	$(3,4,11;2,4,16)$	$(-11,1,11;-20,1,24)$	

While, it is obvious that total availability $\left(\sum_{i=1}^{2} a_{i,1}, \sum_{i=1}^{2} a_{i,2}, \sum_{i=1}^{2} a_{i,3}; \sum_{i=1}^{2} a'_{i,1}, \sum_{i=1}^{2} a_{i,2}, \sum_{i=1}^{2} a'_{i,3}\right) = (4,8,16;1,8,27)$ is not equal to total IF demand $\left(\sum_{j=1}^{2} b_{j,1}, \sum_{j=1}^{2} b_{j,2}, \sum_{j=1}^{2} b_{j,3}; \sum_{j=1}^{2} b'_{j,1}, \sum_{j=1}^{2} b_{j,2}, \sum_{j=1}^{2} b'_{j,3}\right) = (-6,8,26;-17,8,45)$. Therefore, the transformed FIFTP is not a BFIFTP.

5.8 Reason for the occurrence of the limitations

The limitations of Mahmoodirad et al.'s approach [136, Section 4.2], discussed in Section 5.6, are occurring due to the following reason.

It is obvious from Step 1 of Mahmoodirad et al.'s approach [136, Section 4.2] that Mahmoodirad et al. [136, Section 3] have assumed that if $\tilde{A}_1 = (a_1, a_2, a_3; a'_1, a_2, a'_3)$ and $\tilde{A}_2 = (b_1, b_2, b_3; b'_1, b_2, b'_3)$ are two distinct TIFNs, then either the relation $R(\tilde{A}_1) > R(\tilde{A}_2)$ or the relation $R(\tilde{A}_1) < R(\tilde{A}_2)$ will hold. However, in actual case, there may exist distinct TIFNs $\tilde{A}_1$ and $\tilde{A}_2$ such that $R(\tilde{A}_1) = R(\tilde{A}_2)$. e.g., the TIFNs $\tilde{A}_1 = (4,6,16;1,6,27)$ and $\tilde{A}_2 = (5,7,15;3,7,21)$ are two distinct TIFNs but it can be easily verified that $R(\tilde{A}_1) = R(\tilde{A}_2) = 9$.

5.9 Reasons for the occurrence of the flaws

It is obvious from Case 2 of Step 1 of Mahmoodirad et al.'s approach [136, Section 4.2] that Mahmoodirad et al. [136, Section 3] have assumed that if $\tilde{A}_1 =$

$(a_1, a_2, a_3; a_1', a_2, a_3')$ and $\tilde{A}_2 = (b_1, b_2, b_3; b_1', b_2, b_3')$ are two TIFNs such that $R(\tilde{A}_1) \neq R(\tilde{A}_2)$. Then

(i) If $R(\tilde{A}_1) > R(\tilde{A}_2)$ then on adding TIFN $(\tilde{A}_1 - \tilde{A}_2)$ in $\tilde{A}_2$, it will be equal to $\tilde{A}_1$ i.e., $\tilde{A}_2 + (\tilde{A}_1 - \tilde{A}_2) = \tilde{A}_1$.

(ii) If $R(\tilde{A}_1) < R(\tilde{A}_2)$ then on adding TIFN $(\tilde{A}_2 - \tilde{A}_1)$ in $\tilde{A}_1$, it will be equal to $\tilde{A}_2$ i.e., $\tilde{A}_1 + (\tilde{A}_2 - \tilde{A}_1) = \tilde{A}_2$.

However, the following example clearly indicates that in actual case this result will not hold.

For the TIFNs $\tilde{A}_1 = (5,7,15; 3,7,21)$ and $\tilde{A}_2 = (4,8,16; 1,8,27)$, the condition $R(\tilde{A}_1) < R(\tilde{A}_2)$ is satisfying. So, according to condition (ii), on adding $(\tilde{A}_2 - \tilde{A}_1) = (b_1 - a_3, b_2 - a_2, b_3 - a_1; b_1' - a_3', b_2 - a_2, b_3' - a_1') = (-11,1,11; -20,1,24)$ in $\tilde{A}_1$, the TIFN $\tilde{A}_2$ should be obtained. While, $\tilde{A}_1 + (\tilde{A}_2 - \tilde{A}_1) = (5,7,15; 3,7,21) + (-11,1,11; -20,1,24) = (-6,8,26; -17,8,45) \neq \tilde{A}_2$.

5.10 Proposed JMD method for transforming an unbalanced FIFTP into a BFIFTP

In this section, a new method (named as JMD method) is proposed to transform an unbalanced FIFTP into a BFIFTP.

The steps of the proposed JMD method are as follows:

Case 1: If $\sum_{i=1}^{m} a_{i1}' \leq \sum_{j=1}^{n} b_{j1}'$, $\sum_{i=1}^{m} a_{i1} - \sum_{i=1}^{m} a_{i1}' \leq \sum_{j=1}^{n} b_{j1} - \sum_{j=1}^{n} b_{j1}'$, $\sum_{i=1}^{m} a_{i2} - \sum_{i=1}^{m} a_{i1} \leq \sum_{j=1}^{n} b_{j2} - \sum_{j=1}^{n} b_{j1}$, $\sum_{i=1}^{m} a_{i3} - \sum_{i=1}^{m} a_{i2} \leq \sum_{j=1}^{n} b_{j3} - \sum_{j=1}^{n} b_{j2}$, $\sum_{i=1}^{m} a_{i3}' - \sum_{i=1}^{m} a_{i3} \leq \sum_{j=1}^{n} b_{j3}' - \sum_{j=1}^{n} b_{j3}$, then add a dummy source S_{m+1} having dummy supply $(A_{(m+1)1}, A_{(m+1)2}, A_{(m+1)3}; A'_{(m+1)1}, A_{(m+1)2}, A'_{(m+1)3})$ by considering the cost for supplying the unit quantity of the product from the dummy source S_{m+1} to all the destinations as a TIFN $\tilde{0} = (0,0,0; 0,0,0)$.

$A'_{(m+1)1} = \sum_{j=1}^{n} b'_{j1} - \sum_{i=1}^{m} a'_{i1}$ $A_{(m+1)1} = \sum_{j=1}^{n} b_{j1} - \sum_{i=1}^{m} a_{i1}$,

$A_{(m+1)2} = \sum_{j=1}^{n} b_{j2} - \sum_{i=1}^{m} a_{i2}$,

$A_{(m+1)3} = \sum_{j=1}^{n} b_{j3} - \sum_{i=1}^{m} a_{i3}$,

$A'_{(m+1)3} = \sum_{j=1}^{n} b'_{j3} - \sum_{i=1}^{m} a'_{j3}$.

Case 2: If $\sum_{j=1}^{n} b'_{j1} \leq \sum_{i=1}^{m} a'_{i1}$, $\sum_{j=1}^{n} b_{j1} - \sum_{j=1}^{n} b'_{j1} \leq \sum_{i=1}^{m} a_{i1} - \sum_{i=1}^{m} a'_{i1}$, $\sum_{j=1}^{n} b_{j2} - \sum_{j=1}^{n} b_{j1} \leq \sum_{i=1}^{m} a_{i2} - \sum_{i=1}^{m} a_{i1}$, $\sum_{j=1}^{n} b_{j3} - \sum_{j=1}^{n} b_{j2} \leq \sum_{i=1}^{m} a_{i3} - \sum_{i=1}^{m} a_{i2}$, $\sum_{j=1}^{n} b'_{j3} - \sum_{j=1}^{n} b_{j3} \leq \sum_{i=1}^{m} a'_{i3} - \sum_{i=1}^{m} a_{i3}$ then add a dummy destination D_{n+1} having dummy demand $\left(-A_{(m+1)1}, -A_{(m+1)2}, -A_{(m+1)3}; -A'_{(m+1)1}, -A_{(m+1)2}, -A'_{(m+1)3}\right)$ by considering the cost for supplying the unit quantity of the product from all the sources to the dummy destination D_{n+1} as a TIFN $\tilde{0} = (0,0,0; 0,0,0)$.

Case 3: If neither Case 1 nor Case 2 is satisfying then,

(1) Add a dummy source S_{m+1} having dummy supply $\left(A_{(m+1)1}, A_{(m+1)2}, A_{(m+1)3}; A'_{(m+1)1}, A_{(m+1)2}, A'_{(m+1)3}\right)$ by considering the cost for supplying the unit quantity of the product from the dummy source S_{m+1} to all the destinations as a TIFN $\tilde{0} = (0,0,0; 0,0,0)$.

$A'_{(m+1)1} = max\left\{0, \sum_{j=1}^{n} b'_{j1} - \sum_{i=1}^{m} a'_{i1}\right\}$, $A_{(m+1)1} = A'_{(m+1)1} + max\left\{0, \left(\sum_{j=1}^{n} b_{j1} - \sum_{j=1}^{n} b'_{j1}\right) - \left(\sum_{i=1}^{m} a_{i1} - \sum_{i=1}^{m} a'_{i1}\right)\right\}$, $A_{(m+1)2} = A_{(m+1)1} + max\left\{0, \left(\sum_{j=1}^{n} b_{j2} - \sum_{j=1}^{n} b_{j1}\right) - \left(\sum_{i=1}^{m} a_{i2} - \sum_{i=1}^{m} a_{i1}\right)\right\}$,

$A_{(m+1)3} = A_{(m+1)2} + max\left\{0, \left(\sum_{j=1}^{n} b_{j3} - \sum_{j=1}^{n} b_{j2}\right) - \left(\sum_{i=1}^{m} a_{i3} - \sum_{i=1}^{m} a_{i2}\right)\right\}$,

$A'_{(m+1)3} = A_{(m+1)3} + max\left\{0, \left(\sum_{j=1}^{n} b'_{j3} - \sum_{j=1}^{n} b_{j3}\right) - \left(\sum_{i=1}^{m} a'_{i3} - \sum_{i=1}^{m} a_{i3}\right)\right\}$.

(2) Add a dummy destination D_{n+1} having dummy demand

$\left(B_{(m+1)1}, B_{(m+1)2}, B_{(m+1)3}; B'_{(m+1)1}, B_{(m+1)2}, B'_{(m+1)3}\right)$ by considering the cost for supplying the unit quantity of the product from all the sources to the dummy destination D_{n+1} as a TIFN $\tilde{0} = (0,0,0;0,0,0)$.

$B'_{(m+1)1} = max\left\{0, \sum_{i=1}^{m} a'_{i1} - \sum_{j=1}^{n} b'_{j1}\right\}$, $B_{(m+1)1} = B'_{(m+1)1} + max\left\{0, (\sum_{i=1}^{m} a_{i1} - \sum_{i=1}^{m} a'_{i1}) - \left(\sum_{j=1}^{n} b_{j1} - \sum_{j=1}^{n} b'_{j1}\right)\right\}$, $B_{(m+1)2} = B_{(m+1)1} + max\left\{0, (\sum_{i=1}^{m} a_{i2} - \sum_{i=1}^{m} a_{i1}) - \left(\sum_{j=1}^{n} b_{j2} - \sum_{j=1}^{n} b_{j1}\right)\right\}$, $B_{(m+1)3} = B_{(m+1)2} + max\left\{0, (\sum_{i=1}^{m} a_{i3} - \sum_{i=1}^{m} a_{i2}) - \left(\sum_{j=1}^{n} b_{j3} - \sum_{j=1}^{n} b_{j2}\right)\right\}$, $B'_{(m+1)3} = B_{(m+1)3} + max\left\{0, (\sum_{i=1}^{m} a'_{i3} - \sum_{i=1}^{m} a_{i3}) - \left(\sum_{j=1}^{n} b'_{j3} - \sum_{j=1}^{n} b_{j3}\right)\right\}$.

5.11 Exact dummy supply and dummy demand for FIFTP

In Section 5.6 and Section 5.7, two FIFTPs were considered and shown that the Step 1 of the existing approach [136, Section 4.2] fails to transform the considered unbalanced FIFTP into a balanced FIFTP and hence, fails to find the IF optimal solution of the considered FIFTPs. In this section, JMD method, proposed in Section 5.10, is used to transform the unbalanced FIFTP into balanced FIFTP and then Step 2 to Step 7 of the existing approach[136, Section 4.2] is used to find the solution of transformed balanced FIFTP.

5.11.1 IF optimal solution of first FIFTP

Using the JMD method, proposed in Section 5.10, the unbalanced FIFTP, represented by Table 5.2, can be transformed into a BFIFTP as follows.

Step 1: $\sum_{i=1}^{2} \tilde{a}_i = (4,6,16;1,6,27)$, $\sum_{j=1}^{2} \tilde{b}_j = (5,7,15;3,7,21)$.

Since, Case 3 of JMD method, proposed in Section 5.10, is satisfying so, there is need to add a dummy source S_3 with IF availability $\tilde{a}_3 = (2,2,2;2,2,2)$ as well as a dummy destination D_3 with IF demand $\tilde{b}_3 = (1,3,3;0,3,8)$. Furthermore, there is need to assume that the IF transportation cost for transporting one unit of quantity of the product from the dummy source S_3 to all the destinations as a TIFN $(0,0,0;0,0,0)$ as well as there is need to assume the IF transportation cost for transporting one unit of quantity of the product from all the sources to the dummy destination D_3 as a TIFN $(0,0,0;0,0,0)$. By considering all these assumptions, the unbalanced FIFTP, represented by Table 5.2, will be transformed into a BFIFTP represented by Table 5.5.

Table 5.5 BFIFTP

Destinations → Sources ↓	D_1	D_2	D_3	Availability
S_1	$(2,4,9;1,4,12)$	$(4,13,14;3,13,15)$	$(0,0,0;0,0,0)$	$(2,3,6;0,3,12)$
S_2	$(1,3,8;0,3,11)$	$(3,6,7;0,6,14)$	$(0,0,0;0,0,0)$	$(2,3,10;1,3,15)$
S_3	$(0,0,0;0,0,0)$	$(0,0,0;0,0,0)$	$(0,0,0;0,0,0)$	$(2,2,2;2,2,2)$
Demand	$(2,3,4;1,3,5)$	$(3,4,11;2,4,16)$	$(1,1,3;0,1,8)$	

Step 2: Using Step 2 of Mahmoodirad et al.'s approach [136, Section 4.2], discussed in Section 5.4, the BFIFTP, represented by Table 5.5, can be transformed into IFFLPP $(P5.7)$.

IFFLPP ($\boldsymbol{P5.7}$)

$$\text{Minimize}\left[\begin{array}{c}
(2,4,9;1,4,12)\otimes\left(x_{11,1},x_{11,2},x_{11,3};x'_{11,1},x_{11,2},x'_{11,3}\right)\oplus\\
(4,13,14;3,13,15)\otimes\left(x_{12,1},x_{12,2},x_{12,3};x'_{12,1},x_{12,2},x'_{12,3}\right)\oplus\\
(0,0,0;0,0,0)\otimes\left(x_{13,1},x_{13,2},x_{13,3};x'_{13,1},x_{13,2},x'_{13,3}\right)\oplus\\
(1,3,8;0,3,11)\otimes\left(x_{21,1},x_{21,2},x_{21,3};x'_{21,1},x_{21,2},x'_{21,3}\right)\oplus\\
(3,6,7;0,6,14)\otimes\left(x_{22,1},x_{22,2},x_{22,3};x'_{22,1},x_{22,2},x'_{22,3}\right)\oplus\\
(0,0,0;0,0,0)\otimes\left(x_{23,1},x_{23,2},x_{23,3};x'_{23,1},x_{23,2},x'_{23,3}\right)\oplus\\
(0,0,0;0,0,0)\otimes\left(x_{31,1},x_{31,2},x_{31,3};x'_{31,1},x_{31,2},x'_{31,3}\right)\oplus\\
(0,0,0;0,0,0)\otimes\left(x_{32,1},x_{32,2},x_{32,3};x'_{32,1},x_{32,2},x'_{32,3}\right)\oplus\\
(0,0,0;0,0,0)\otimes\left(x_{33,1},x_{33,2},x_{33,3};x'_{33,1},x_{33,2},x'_{33,3}\right)
\end{array}\right]$$

Subject to

$$\left(x_{11,1},x_{11,2},x_{11,3};x'_{11,1},x_{11,2},x'_{11,3}\right)\oplus\left(x_{12,1},x_{12,2},x_{12,3};x'_{12,1},x_{12,2},x'_{12,3}\right)\oplus$$

$$\left(x_{13,1},x_{13,2},x_{13,3};x'_{13,1},x_{13,2},x'_{13,3}\right)=(2,3,6;0,3,12),$$

$$\left(x_{21,1},x_{21,2},x_{21,3};x'_{21,1},x_{21,2},x'_{21,3}\right)\oplus\left(x_{22,1},x_{22,2},x_{22,3};x'_{22,1},x_{22,2},x'_{22,3}\right)$$

$$\oplus\left(x_{23,1},x_{23,2},x_{23,3};x'_{23,1},x_{23,2},x'_{23,3}\right)=(2,3,10;1,3,15),$$

$$\left(x_{31,1},x_{31,2},x_{31,3};x'_{31,1},x_{31,2},x'_{31,3}\right)\oplus\left(x_{32,1},x_{32,2},x_{32,3};x'_{32,1},x_{32,2},x'_{32,3}\right)$$

$$\oplus\left(x_{33,1},x_{33,2},x_{33,3};x'_{33,1},x_{33,2},x'_{33,3}\right)=(2,2,2;2,2,2),$$

$$\left(x_{11,1},x_{11,2},x_{11,3};x'_{11,1},x_{11,2},x'_{11,3}\right)\oplus\left(x_{21,1},x_{21,2},x_{21,3};x'_{21,1},x_{21,2},x'_{21,3}\right)\oplus$$

$$\left(x_{31,1},x_{31,2},x_{31,3};x'_{31,1},x_{31,2},x'_{31,3}\right)=(2,3,4;1,3,5),$$

$$\left(x_{12,1},x_{12,2},x_{12,3};x'_{12,1},x_{12,2},x'_{12,3}\right)\oplus\left(x_{22,1},x_{22,2},x_{22,3};x'_{22,1},x_{22,2},x'_{22,3}\right)\oplus$$

$$\left(x_{32,1},x_{32,2},x_{32,3};x'_{32,1},x_{32,2},x'_{32,3}\right)=(3,4,11;2,4,16),$$

$$\left(x_{13,1}, x_{13,2}, x_{13,3}; x'_{13,1}, x_{13,2}, x'_{13,3}\right) \oplus \left(x_{23,1}, x_{23,2}, x_{23,3}; x'_{23,1}, x_{23,2}, x'_{23,3}\right)$$
$$\oplus \left(x_{33,1}, x_{33,2}, x_{33,3}; x'_{33,1}, x_{33,2}, x'_{33,3}\right) = (1,1,3; 0,1,8),$$

$\left(x_{ij,1}, x_{ij,2}, x_{ij,3}; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right)$ are non-negative TIFNs.

Step 3: Using Step 3 of Mahmoodirad et al.'s approach [136, Section 4.2], discussed in Section 5.4, the IFLPP $(P5.7)$ can be transformed into its equivalent IFLPP $(P5.8)$.

IFLPP $(\boldsymbol{P5.8})$

$$\text{Minimize}\begin{bmatrix} (2,4,9; 1,4,12) \otimes \left(x_{11,1}, x_{11,2}, x_{11,3}; x'_{11,1}, x_{11,2}, x'_{11,3}\right) \oplus \\ (4,13,14; 3,13,15) \otimes \left(x_{12,1}, x_{12,2}, x_{12,3}; x'_{12,1}, x_{12,2}, x'_{12,3}\right) \oplus \\ (0,0,0; 0,0,0) \otimes \left(x_{13,1}, x_{13,2}, x_{13,3}; x'_{13,1}, x_{13,2}, x'_{13,3}\right) \oplus \\ (1,3,8; 0,3,11) \otimes \left(x_{21,1}, x_{21,2}, x_{21,3}; x'_{21,1}, x_{21,2}, x'_{21,3}\right) \oplus \\ (3,6,7; 0,6,14) \otimes \left(x_{22,1}, x_{22,2}, x_{22,3}; x'_{22,1}, x_{22,2}, x'_{22,3}\right) \oplus \\ (0,0,0; 0,0,0) \otimes \left(x_{23,1}, x_{23,2}, x_{23,3}; x'_{23,1}, x_{23,2}, x'_{23,3}\right) \oplus \\ (0,0,0; 0,0,0) \otimes \left(x_{31,1}, x_{31,2}, x_{31,3}; x'_{31,1}, x_{31,2}, x'_{31,3}\right) \oplus \\ (0,0,0; 0,0,0) \otimes \left(x_{32,1}, x_{32,2}, x_{32,3}; x'_{32,1}, x_{32,2}, x'_{32,3}\right) \oplus \\ (0,0,0; 0,0,0) \otimes \left(x_{33,1}, x_{33,2}, x_{33,3}; x'_{33,1}, x_{33,2}, x'_{33,3}\right) \end{bmatrix}$$

Subject to

$x_{11,1} + x_{12,1} + x_{13,1} = 2, x_{11,2} + x_{12,2} + x_{13,2} = 3, x_{11,3} + x_{12,3} + x_{13,3} = 6,$

$x'_{11,1} + x'_{12,1} + x'_{13,1} = 0, x'_{11,3} + x'_{12,3} + x'_{13,3} = 12,$

$x_{21,1} + x_{22,1} + x_{23,1} = 2, x_{21,2} + x_{22,2} + x_{23,2} = 3, x_{21,3} + x_{22,3} + x_{23,3} = 10,$

$x'_{21,1} + x'_{22,1} + x'_{23,1} = 1, x'_{21,3} + x'_{22,3} + x'_{23,3} = 15,$

$x_{31,1} + x_{32,1} + x_{33,1} = 2, x_{31,2} + x_{32,2} + x_{33,2} = 2, x_{31,3} + x_{32,3} + x_{33,3} = 2,$

$x'_{31,1}+x'_{32,1}+x'_{33,1}=2, x'_{31,3}+x'_{32,3}+x'_{33,3}=2,$

$x_{11,1}+x_{21,1}+x_{31,1}=2, x_{11,2}+x_{21,2}+x_{31,2}=3, x_{11,3}+x_{21,3}+x_{31,3}=4,$

$x'_{11,1}+x'_{21,1}+x'_{31,1}=1, x'_{11,3}+x'_{21,3}+x'_{31,3}=5,$

$x_{12,1}+x_{22,1}+x_{32,1}=3, x_{12,2}+x_{22,2}+x_{32,1}=4, x_{12,3}+x_{22,3}+x_{32,3}=11,$

$x'_{12,1}+x'_{22,1}+x'_{32,1}=2, x'_{12,3}+x'_{22,3}+x'_{32,3}=16,$

$x_{13,1}+x_{23,1}+x_{33,1}=1, x_{13,2}+x_{23,2}+x_{33,2}=1, x_{13,3}+x_{23,3}+x_{33,3}=3,$

$x'_{13,1}+x'_{23,1}+x'_{33,1}=0, x'_{13,3}+x'_{23,3}+x'_{33,3}=8,$

$x'_{11,1}\geq 0, x_{11,1}-x'_{11,1}\geq 0, x_{11,2}-x_{11,1}\geq 0, x_{11,3}-x_{11,2}\geq 0, x'_{11,3}-x_{11,3}\geq 0, x'_{12,1}\geq$
$0, x_{12,1}-x'_{12,1}\geq 0, x_{12,2}-x_{12,1}\geq 0, x_{12,3}-x_{12,2}\geq 0, x'_{12,3}-x_{12,3}\geq 0, x'_{13,1}\geq$
$0, x_{13,1}-x'_{13,1}\geq 0, x_{13,2}-x_{13,1}\geq 0, x_{13,3}-x_{13,2}\geq 0, x'_{13,3}-x_{13,3}\geq 0, x'_{21,1}\geq$
$0, x_{21,1}-x'_{21,1}\geq 0, x_{21,2}-x_{21,1}\geq 0, x_{21,3}-x_{21,2}\geq 0, x'_{21,3}-x_{21,3}\geq 0, x'_{22,1}\geq$
$0, x_{22,1}-x'_{22,1}\geq 0, x_{22,2}-x_{22,1}\geq 0, x_{22,3}-x_{22,2}\geq 0, x'_{22,3}-x_{22,3}\geq 0, x'_{23,1}\geq$
$0, x_{23,1}-x'_{23,1}\geq 0, x_{23,2}-x_{23,1}\geq 0, x_{23,3}-x_{23,2}\geq 0, x'_{23,3}-x_{23,3}\geq 0, x'_{31,1}\geq$
$0, x_{31,1}-x'_{31,1}\geq 0, x_{31,2}-x_{31,1}\geq 0, x_{31,3}-x_{31,2}\geq 0, x'_{31,3}-x_{31,3}\geq 0, x'_{32,1}\geq$
$0, x_{32,1}-x'_{32,1}\geq 0, x_{32,2}-x_{32,1}\geq 0, x_{32,3}-x_{32,2}\geq 0, x'_{32,3}-x_{32,3}\geq 0, x'_{33,1}\geq$
$0, x_{33,1}-x'_{33,1}\geq 0, x_{33,2}-x_{33,1}\geq 0, x_{33,3}-x_{33,2}\geq 0, x'_{33,3}-x_{33,3}\geq 0.$

Step 4: Using Step 4 of Mahmoodirad et al.'s approach [136], discussed in Section 5.4, the IFLPP $(P5.8)$ can be transformed into its equivalent IFLPP $(P5.9)$.

IFLPP $(\boldsymbol{P5.9})$

$$\text{Minimize}\begin{bmatrix} \left(2x_{11,1}, 4x_{11,2}, 9x_{11,3}; 1x'_{11,1}, 4x_{11,2}, 12x'_{11,3}\right) \\ \oplus \left(4x_{12,1}, 13x_{12,2}, 14x_{12,3}; 3x'_{12,1}, 13x_{12,2}, 15x'_{12,3}\right) \\ \oplus \left(x_{21,1}, 3x_{21,2}, 8x_{21,3}; 0x'_{21,1}, 3x_{21,2}, 11x'_{21,3}\right) \\ \oplus \left(3x_{22,1}, 6x_{22,2}, 7x_{22,3}; 0x'_{22,1}, 6x_{22,2}, 14x'_{22,3}\right) \end{bmatrix}$$

Subject to

Constraints of the IFLPP (P5.8).

Step 5: Using Step 5 of Mahmoodirad et al.'s approach [136], discussed in Section 5.4, the IFLPP $(P5.9)$ can be transformed into its equivalent crisp LPP $(P5.10)$ and hence, its equivalent crisp LPP $(P5.11)$.

Crisp LPP $(\boldsymbol{P5.10})$

$$\text{Minimize}\begin{bmatrix} Rank\left(2x_{11,1}, 4x_{11,2}, 9x_{11,3}; 1x'_{11,1}, 4x_{11,2}, 12x'_{11,3}\right) + \\ Rank\left(4x_{12,1}, 13x_{12,2}, 14x_{12,3}; 3x'_{12,1}, 13x_{12,2}, 15x'_{12,3}\right) + \\ Rank\left(x_{21,1}, 3x_{21,2}, 8x_{21,3}; 0x'_{21,1}, 3x_{21,2}, 11x'_{21,3}\right) + \\ Rank\left(3x_{22,1}, 6x_{22,2}, 7x_{22,3}; 0x'_{22,1}, 6x_{22,2}, 14x'_{22,3}\right) \end{bmatrix}$$

Subject to

Constraints of the IFLPP (P5.8).

Crisp LPP $(\boldsymbol{P5.11})$

$$\text{Minimize}\begin{bmatrix} \frac{(2x_{11,1}+9x_{11,3}+16x_{11,2}+1x'_{11,1}+12x'_{11,3})}{8} + \frac{(4x_{12,1}+14x_{12,3}+52x_{12,2}+3x'_{12,1}+15x'_{12,3})}{8} \\ + \frac{(x_{21,1}+8x_{21,3}+12x_{21,2}+0x'_{21,1}+11x'_{21,3})}{8} + \frac{(3x_{22,1}+7x_{22,3}+24x_{22,2}+0x'_{22,1}+14x'_{22,3})}{8} \end{bmatrix}$$

Subject to

Constraints of the IFLPP (P5.8).

Step 6: On solving the crisp LPP $(P5.11)$ the obtained optimal solution is

$x_{11,1} = 1, x_{11,2} = 2, x_{11,3} = 3, x'_{11,1} = 0, x'_{11,3} = 3,$

$x_{12,1} = 0, x_{12,2} = 0, x_{12,3} = 0, x'_{12,1} = 0, x'_{12,3} = 1,$

$x_{13,1} = 1, x_{13,2} = 1, x_{13,3} = 3, x'_{13,1} = 0, x'_{13,3} = 8,$

$x_{21,1} = 1, x_{21,2} = 1, x_{21,3} = 1, x'_{21,1} = 1, x'_{21,3} = 2,$

$x_{22,1} = 1, x_{22,2} = 2, x_{22,3} = 9, x'_{22,1} = 0, x'_{22,3} = 13,$

$x_{23,1} = 0, x_{23,2} = 0, x_{23,3} = 0, x'_{23,1} = 0, x'_{23,3} = 0,$

$x_{31,1} = 0, x_{31,2} = 0, x_{31,3} = 0, x'_{31,1} = 0, x'_{31,3} = 0,$

$x_{32,1} = 2, x_{32,2} = 2, x_{32,3} = 2, x'_{32,1} = 2, x'_{32,3} = 2,$

$x_{33,1} = 0, x_{33,2} = 0, x_{33,3} = 0, x'_{33,1} = 0, x'_{33,3} = 0.$

Step 7: Using the optimal, obtained in Step 6, the IF optimal solution of the considered FIFTP, represented by Table 5.3, is

$\tilde{x}_{11} = (1,2,3; 0,2,3)$, $\tilde{x}_{12} = (0,0,0; 0,0,1)$, $\tilde{x}_{13} = (1,1,3; 0,1,8)$, $\tilde{x}_{21} = (1,1,1; 1,1,2)$, $\tilde{x}_{22} = (1,2,9; 0,2,13)$, $\tilde{x}_{23} = (0,0,0; 0,0,0)$, $\tilde{x}_{31} = (0,0,0; 0,0,0)$, $\tilde{x}_{32} = (2,2,2; 2,2,2)$, $\tilde{x}_{33} = (0,0,0; 0,0,0)$ and the IF optimal transportation cost is $(6,23,98; 0,23,255)$.

5.11.2 IF optimal solution of second FIFTP

Using the JMD method, proposed in Section 5.11.1, the FIFTP, represented by Table 5.3 can be transformed into a BFIFTP as follows.

Step 1: $\sum_{i=1}^{2} \tilde{a}_i = (4,8,16; 1,8,27)$, $\sum_{j=1}^{2} \tilde{b}_j = (5,7,15; 3,7,21)$.

Since, Case 3 of JMD method, proposed in Section 5.10 is satisfying so, there is need to add a dummy source S_3 with IF availability $\tilde{A}_3 = (2,2,2; 2,2,2)$ as well as a dummy destination D_3 with IF demand $\tilde{B}_3 = (1,3,3; 0,3,8)$. Furthermore, there is need to assume the IF transportation cost for transporting one unit of quantity of the product from dummy source S_3 to all the dummy destinations as TIFN $(0,0,0; 0,0,0)$ as well as there is need to assume the IF transportation cost for transporting one unit of quantity of the product from all the sources to the dummy destination D_3 as a TIFN $(0,0,0; 0,0,0)$. By considering all these assumptions, the unbalanced FIFTP, represented by Table 5.3, will be transformed into a BFIFTP represented by Table 5.6.

Table 5.6 BFIFTP

Destinations → Sources ↓	D_1	D_2	D_3	Availability
S_1	$(2,4,9;1,4,12)$	$(4,13,14;3,13,15)$	$(0,0,0;0,0,0)$	$(2,4,6;0,4,12)$
S_2	$(1,3,8;0,3,11)$	$(3,6,7;0,6,14)$	$(0,0,0;0,0,0)$	$(2,4,10;1,4,15)$
S_3	$(0,0,0;0,0,0)$	$(0,0,0;0,0,0)$	$(0,0,0;0,0,0)$	$(2,2,2;2,2,2)$
Demand	$(2,3,4;1,3,5)$	$(3,4,11;2,4,16)$	$(1,3,3;0,3,8)$	

Step 2: Using Step 2 of Mahmoodirad et al.'s approach [136, Section 4.2], discussed in Section 5.4, the BFIFTP, represented by Table 5.6, can be transformed into IFFLPP ($P5.12$).

IFFLPP ($\boldsymbol{P5.12}$)

$$\text{Minimize}\begin{bmatrix} (2,4,9;1,4,12)\otimes\left(x_{11,1},x_{11,2},x_{11,3};x'_{11,1},x_{11,2},x'_{11,3}\right)\oplus \\ (4,13,14;3,13,15)\otimes\left(x_{12,1},x_{12,2},x_{12,3};x'_{12,1},x_{12,2},x'_{12,3}\right)\oplus \\ (0,0,0;0,0,0)\otimes\left(x_{13,1},x_{13,2},x_{13,3};x'_{13,1},x_{13,2},x'_{13,3}\right)\oplus \\ (1,3,8;0,3,11)\otimes\left(x_{21,1},x_{21,2},x_{21,3};x'_{21,1},x_{21,2},x'_{21,3}\right)\oplus \\ (3,6,7;0,6,14)\otimes\left(x_{22,1},x_{22,2},x_{22,3};x'_{22,1},x_{22,2},x'_{22,3}\right)\oplus \\ (0,0,0;0,0,0)\otimes\left(x_{23,1},x_{23,2},x_{23,3};x'_{23,1},x_{23,2},x'_{23,3}\right)\oplus \\ (0,0,0;0,0,0)\otimes\left(x_{31,1},x_{31,2},x_{31,3};x'_{31,1},x_{31,2},x'_{31,3}\right)\oplus \\ (0,0,0;0,0,0)\otimes\left(x_{32,1},x_{32,2},x_{32,3};x'_{32,1},x_{32,2},x'_{32,3}\right)\oplus \\ (0,0,0;0,0,0)\otimes\left(x_{33,1},x_{33,2},x_{33,3};x'_{33,1},x_{33,2},x'_{33,3}\right) \end{bmatrix}$$

Subject to

$$\left(x_{11,1}, x_{11,2}, x_{11,3}; x'_{11,1}, x_{11,2}, x'_{11,3}\right) \oplus \left(x_{12,1}, x_{12,2}, x_{12,3}; x'_{12,1}, x_{12,2}, x'_{12,3}\right) \oplus \left(x_{13,1}, x_{13,2}, x_{13,3}; x'_{13,1}, x_{13,2}, x'_{13,3}\right) = (2,4,6; 0,4,12),$$

$$\left(x_{21,1}, x_{21,2}, x_{21,3}; x'_{21,1}, x_{21,2}, x'_{21,3}\right) \oplus \left(x_{22,1}, x_{22,2}, x_{22,3}; x'_{22,1}, x_{22,2}, x'_{22,3}\right) \oplus \left(x_{23,1}, x_{23,2}, x_{23,3}; x'_{23,1}, x_{23,2}, x'_{23,3}\right) = (2,4,10; 1,4,15),$$

$$\left(x_{31,1}, x_{31,2}, x_{31,3}; x'_{31,1}, x_{31,2}, x'_{31,3}\right) \oplus \left(x_{32,1}, x_{32,2}, x_{32,3}; x'_{32,1}, x_{32,2}, x'_{32,3}\right) \oplus \left(x_{33,1}, x_{33,2}, x_{33,3}; x'_{33,1}, x_{33,2}, x'_{33,3}\right) = (2,2,2; 2,2,2),$$

$$\left(x_{11,1}, x_{11,2}, x_{11,3}; x'_{11,1}, x_{11,2}, x'_{11,3}\right) \oplus \left(x_{21,1}, x_{21,2}, x_{21,3}; x'_{21,1}, x_{21,2}, x'_{21,3}\right) \oplus \left(x_{31,1}, x_{31,2}, x_{31,3}; x'_{31,1}, x_{31,2}, x'_{31,3}\right) = (2,3,4; 1,3,5),$$

$$\left(x_{12,1}, x_{12,2}, x_{12,3}; x'_{12,1}, x_{12,2}, x'_{12,3}\right) \oplus \left(x_{22,1}, x_{22,2}, x_{22,3}; x'_{22,1}, x_{22,2}, x'_{22,3}\right) \oplus \left(x_{32,1}, x_{32,2}, x_{32,3}; x'_{32,1}, x_{32,2}, x'_{32,3}\right) = (3,4,11; 2,4,16),$$

$$\left(x_{13,1}, x_{13,2}, x_{13,3}; x'_{13,1}, x_{13,2}, x'_{13,3}\right) \oplus \left(x_{23,1}, x_{23,2}, x_{23,3}; x'_{23,1}, x_{23,2}, x'_{23,3}\right) \oplus \left(x_{33,1}, x_{33,2}, x_{33,3}; x'_{33,1}, x_{33,2}, x'_{33,3}\right) = (1,3,3; 0,3,8),$$

$\left(x_{ij,1}, x_{ij,2}, x_{ij,3}; x'_{ij,1}, x_{ij,2}, x'_{ij,3}\right)$ are non-negative TIFNs.

Step 3: Using Step 3 of Mahmoodirad et al.'s approach [136, Section 4.2], discussed in Section 5.4, the IFFLPP $(P5.12)$ can be transformed into its equivalent IFLPP $(P5.13)$.

IFLPP $(\boldsymbol{P5.13})$

$$\text{Minimize}\begin{bmatrix} (2,4,9;1,4,12)\otimes\left(x_{11,1},x_{11,2},x_{11,3};x'_{11,1},x_{11,2},x'_{11,3}\right)\oplus \\ (4,13,14;3,13,15)\otimes\left(x_{12,1},x_{12,2},x_{12,3};x'_{12,1},x_{12,2},x'_{12,3}\right)\oplus \\ (0,0,0;0,0,0)\otimes\left(x_{13,1},x_{13,2},x_{13,3};x'_{13,1},x_{13,2},x'_{13,3}\right)\oplus \\ (1,3,8;0,3,11)\otimes\left(x_{21,1},x_{21,2},x_{21,3};x'_{21,1},x_{21,2},x'_{21,3}\right)\oplus \\ (3,6,7;0,6,14)\otimes\left(x_{22,1},x_{22,2},x_{22,3};x'_{22,1},x_{22,2},x'_{22,3}\right)\oplus \\ (0,0,0;0,0,0)\otimes\left(x_{23,1},x_{23,2},x_{23,3};x'_{23,1},x_{23,2},x'_{23,3}\right)\oplus \\ (0,0,0;0,0,0)\otimes\left(x_{31,1},x_{31,2},x_{31,3};x'_{31,1},x_{31,2},x'_{31,3}\right)\oplus \\ (0,0,0;0,0,0)\otimes\left(x_{32,1},x_{32,2},x_{32,3};x'_{32,1},x_{32,2},x'_{32,3}\right)\oplus \\ (0,0,0;0,0,0)\otimes\left(x_{33,1},x_{33,2},x_{33,3};x'_{33,1},x_{33,2},x'_{33,3}\right) \end{bmatrix}$$

Subject to

$x_{11,1}+x_{12,1}+x_{13,1}=2, x_{11,2}+x_{12,2}+x_{13,2}=4, x_{11,3}+x_{12,3}+x_{13,3}=6,$

$x'_{11,1}+x'_{12,1}+x'_{13,1}=0, x'_{11,3}+x'_{12,3}+x'_{13,3}=12,$

$x_{21,1}+x_{22,1}+x_{23,1}=2, x_{21,2}+x_{22,2}+x_{23,2}=4, x_{21,3}+x_{22,3}+x_{23,3}=10,$

$x'_{21,1}+x'_{22,1}+x'_{23,1}=1, x'_{21,3}+x'_{22,3}+x'_{23,3}=15,$

$x_{31,1}+x_{32,1}+x_{33,1}=2, x_{31,2}+x_{32,2}+x_{33,2}=2, x_{31,3}+x_{32,3}+x_{33,3}=2,$

$x'_{31,1}+x'_{32,1}+x'_{33,1}=2, x'_{31,3}+x'_{32,3}+x'_{33,3}=2,$

$x_{11,1}+x_{21,1}+x_{31,1}=2, x_{11,2}+x_{21,2}+x_{31,2}=3, x_{11,3}+x_{21,3}+x_{31,3}=4,$

$x'_{11,1}+x'_{21,1}+x'_{31,1}=1, x'_{11,3}+x'_{21,3}+x'_{31,3}=5,$

$x_{12,1}+x_{22,1}+x_{32,1}=3, x_{12,2}+x_{22,2}+x_{32,1}=4, x_{12,3}+x_{22,3}+x_{32,3}=11,$

$x'_{12,1}+x'_{22,1}+x'_{32,1}=2, x'_{12,3}+x'_{22,3}+x'_{32,3}=16,$

$x_{13,1} + x_{23,1} + x_{33,1} = 1, x_{13,2} + x_{23,2} + x_{33,2} = 3, x_{13,3} + x_{23,3} + x_{33,3} = 3,$

$x'_{13,1} + x'_{23,1} + x'_{33,1} = 0, x'_{13,3} + x'_{23,3} + x'_{33,3} = 8,$

$x'_{11,1} \geq 0, x_{11,1} - x'_{11,1} \geq 0, x_{11,2} - x_{11,1} \geq 0, x_{11,3} - x_{11,2} \geq 0, x'_{11,3} - x_{11,3} \geq 0,$

$x'_{12,1} \geq 0, x_{12,1} - x'_{12,1} \geq 0, x_{12,2} - x_{12,1} \geq 0, x_{12,3} - x_{12,2} \geq 0, x'_{12,3} - x_{12,3} \geq 0,$

$x'_{13,1} \geq 0, x_{13,1} - x'_{13,1} \geq 0, x_{13,2} - x_{13,1} \geq 0, x_{13,3} - x_{13,2} \geq 0, x'_{13,3} - x_{13,3} \geq 0,$

$x'_{21,1} \geq 0, x_{21,1} - x'_{21,1} \geq 0, x_{21,2} - x_{21,1} \geq 0, x_{21,3} - x_{21,2} \geq 0, x'_{21,3} - x_{21,3} \geq 0,$

$x'_{22,1} \geq 0, x_{22,1} - x'_{22,1} \geq 0, x_{22,2} - x_{22,1} \geq 0, x_{22,3} - x_{22,2} \geq 0, x'_{22,3} - x_{22,3} \geq 0,$

$x'_{23,1} \geq 0, x_{23,1} - x'_{23,1} \geq 0, x_{23,2} - x_{23,1} \geq 0, x_{23,3} - x_{23,2} \geq 0, x'_{23,3} - x_{23,3} \geq 0.$

Step 4: Using Step 4 of Mahmoodirad et al.'s approach [136, Section 4.2], discussed in Section 5.4, the IFLPP $(P5.13)$ can be transformed into its equivalent IFLPP $(P5.14)$.

IFLPP $(\boldsymbol{P5.14})$

$$\text{Minimize}\begin{bmatrix} (2x_{11,1}, 4x_{11,2}, 9x_{11,3}; 1x'_{11,1}, 4x_{11,2}, 12x'_{11,3}) \oplus \\ (4x_{12,1}, 13x_{12,2}, 14x_{12,3}; 3x'_{12,1}, 13x_{12,2}, 15x'_{12,3}) \oplus \\ (x_{21,1}, 3x_{21,2}, 8x_{21,3}; 0x'_{21,1}, 3x_{21,2}, 11x'_{21,3}) \oplus \\ (3x_{22,1}, 6x_{22,2}, 7x_{22,3}; 0x'_{22,1}, 6x_{22,2}, 14x'_{22,3}) \end{bmatrix}$$

Subject to

Constraints of the IFLPP $(P5.13)$.

Step 5: Using Step 5 of Mahmoodirad et al.'s approach [136, Section 4.2], discussed in Section 5.4, the IFLPP $(P5.14)$ can be transformed into its equivalent crisp LPP $(P5.15)$ and hence, its equivalent crisp LPP $(P5.16)$.

Crisp LPP $(\boldsymbol{P5.15})$

$$\text{Minimize}\begin{bmatrix} Rank\left(2x_{11,1}, 4x_{11,2}, 9x_{11,3}; 1x'_{11,1}, 4x_{11,2}, 12x'_{11,3}\right) + \\ Rank\left(4x_{12,1}, 13x_{12,2}, 14x_{12,3}; 3x'_{12,1}, 13x_{12,2}, 15x'_{12,3}\right) + \\ Rank\left(x_{21,1}, 3x_{21,2}, 8x_{21,3}; 0x'_{21,1}, 3x_{21,2}, 11x'_{21,3}\right) + \\ Rank\left(3x_{22,1}, 6x_{22,2}, 7x_{22,3}; 0x'_{22,1}, 6x_{22,2}, 14x'_{22,3}\right) \end{bmatrix}$$

Subject to

Constraints of the IFLPP $(P5.13)$.

Crisp LPP $(\boldsymbol{P5.16})$

$$\text{Minimize}\begin{bmatrix} \frac{\left(2x_{11,1}+9x_{11,3}+8x_{11,2}1x'_{11,1}+12x'_{11,3}\right)}{8} + \frac{\left(4x_{12,1}+14x_{12,3}+52x_{12,2}+3x'_{12,1}+15x'_{12,3}\right)}{8} \\ + \frac{\left(x_{21,1}+8x_{21,3}+6x_{21,2}+0x'_{21,1}+11x'_{21,3}\right)}{8} + \frac{\left(3x_{22,1}+7x_{22,3}+12x_{22,2}0x'_{22,1}+14x'_{22,3}\right)}{8} \end{bmatrix}$$

Subject to

Constraints of the IFLPP $(P5.13)$.

Step 6: On solving the crisp LPP $(P5.16)$ the obtained optimal solution is

$x_{11,1} = 1, x_{11,2} = 1, x_{11,3} = 2, x'_{11,1} = 0, x'_{11,3} = 2,$

$x_{12,1} = 0, x_{12,2} = 0, x_{12,3} = 1, x'_{12,1} = 0, x'_{12,3} = 2,$

$x_{13,1} = 1, x_{13,2} = 3, x_{13,3} = 3, x'_{13,1} = 0, x'_{13,3} = 8,$

$x_{21,1} = 1, x_{21,2} = 2, x_{21,3} = 2, x'_{21,1} = 1, x'_{21,3} = 3,$

$x_{22,1} = 1, x_{22,2} = 2, x_{22,3} = 8, x'_{22,1} = 0, x'_{22,3} = 12,$

$x_{23,1} = 0, x_{23,2} = 0, x_{23,3} = 0, x'_{23,1} = 0, x'_{23,3} = 0,$

$x_{31,1} = 0, x_{31,2} = 0, x_{31,3} = 0, x'_{31,1} = 0, x'_{31,3} = 0,$

$x_{32,1} = 2, x_{32,2} = 2, x_{32,3} = 2, x'_{32,1} = 2, x'_{32,3} = 2,$

$x_{33,1} = 0, x_{33,2} = 0, x_{33,3} = 0, x'_{33,1} = 0, x'_{33,3} = 0.$

Step 7: Using the optimal solution, obtained in Step 6, the IF optimal solution of the FIFTP, represented by Table 5.3, is

$\tilde{x}_{11} = (1,1,2; 0,1,2)$, $\tilde{x}_{12} = (0,0,1; 0,0,2)$, $\tilde{x}_{13} = (1,3,3; 0,3,8)$, $\tilde{x}_{21} = (1,2,2; 1,2,3)$, $\tilde{x}_{22} = (1,2,8; 0,2,12)$, $\tilde{x}_{23} = (0,0,0; 0,0,0)$, $\tilde{x}_{31} = (0,0,0; 0,0,0)$, $\tilde{x}_{32} = (2,2,2; 2,2,2)$, $\tilde{x}_{33} = (0,0,0; 0,0,0)$ and the IF optimal transportation cost is $(5,26,104; 0,26,255)$.

5.12 Conclusions

It is shown that the method, proposed in Step 1 of the existing approach [136, Section 3] for transforming an unbalanced FIFTP into a BFIFTP is not valid. Also, to resolve this flaw of the Step 1 of existing approach [136, Section 4.2], a new method (named as JMD method) is proposed to transform an unbalanced FIFTP into a BFIFTP.

Chapter 6

Mehar Method for Solving Unbalanced Generalized Interval-Valued Trapezoidal Fuzzy Number Transportation Problems[**]

Ebrahimnejad [48, Section 5, pp. 304] proposed a method for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP as well as a method for solving a balanced GIVTrFNTP. In this chapter, it is shown that on applying Ebrahimnejad's method for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP, the obtained dummy supply and/or dummy demand is not a GIVTrFN and hence, this method is not valid. Also, a new method (named as Mehar method) is proposed to transform an unbalanced GIVTrFNTP into a balanced GIVTrFNTP. Furthermore, the validity of the proposed Mehar method is discussed.

6.1 Ebrahimnejad's method for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP

Ebrahimnejad [48, Section 5, Step 1, pp. 305] has proposed the following method to transform an unbalanced GIVTrFNTP into a balanced GIVTrFNTP i.e., to transform $\sum_{i=1}^{m} \tilde{\tilde{a}}_i \neq \sum_{j=1}^{n} \tilde{\tilde{b}}_j$ into $\sum_{i=1}^{m} \tilde{\tilde{a}}_i = \sum_{j=1}^{n} \tilde{\tilde{b}}_j$.

where,

(i) m represents number of sources, n represents number of destinations.

[**] The contents of this chapter have been communicated in "Sadhna : Academy Proceedings in Engineering Sciences" for the possible publication.

(ii) The GIVTrFN

$\sum_{i=1}^{m} \tilde{\tilde{a}}_i =$

$\left\langle \left(\sum_{i=1}^{m} a_{i1}^{L}, \sum_{i=1}^{m} a_{i2}^{L}, \sum_{i=1}^{m} a_{i3}^{L}, \sum_{i=1}^{m} a_{i4}^{L}; \omega^{L}\right), \left(\sum_{i=1}^{m} a_{i1}^{U}, \sum_{i=1}^{m} a_{i2}^{U}, \sum_{i=1}^{m} a_{i3}^{U}, \sum_{i=1}^{m} a_{i4}^{U}; \omega^{U}\right)\right\rangle$

represents the total interval-valued fuzzy supply.

(iii) The GIVTrFN

$\sum_{j=1}^{n} \tilde{\tilde{b}}_j =$

$\left\langle \left(\sum_{j=1}^{n} b_{j1}^{L}, \sum_{j=1}^{n} b_{j2}^{L}, \sum_{j=1}^{n} b_{j3}^{L}, \sum_{j=1}^{n} b_{j4}^{L}; \omega^{L}\right), \left(\sum_{j=1}^{n} b_{j1}^{U}, \sum_{j=1}^{n} b_{j2}^{U}, \sum_{j=1}^{n} b_{j3}^{U}, \sum_{j=1}^{n} b_{j4}^{U}; \omega^{U}\right)\right\rangle$

represents the total interval-valued fuzzy demand.

(iv) The GIVTrFN $\tilde{\tilde{a}}_i = \left\langle \left(a_{i1}^{L}, a_{i2}^{L}, a_{i3}^{L}, a_{i4}^{L}; \omega^{L}\right), \left(a_{i1}^{U}, a_{i2}^{U}, a_{i3}^{U}, a_{i4}^{U}; \omega^{U}\right)\right\rangle$ represents the supply of the product at i^{th} source S_i.

(v) The GIVTrFN $\tilde{\tilde{b}}_j = \left\langle \left(b_{j1}^{L}, b_{j2}^{L}, b_{j3}^{L}, b_{j4}^{L}; \omega^{L}\right), \left(b_{j1}^{U}, b_{j2}^{U}, b_{j3}^{U}, b_{j4}^{U}; \omega^{U}\right)\right\rangle$ represents the demand of the product at j^{th} destination D_j.

Case 1: If $\sum_{i=1}^{m} a_{i1}^{L} \leq \sum_{j=1}^{n} b_{j1}^{L}$, $\sum_{i=1}^{m} a_{i2}^{L} \leq \sum_{j=1}^{n} b_{j2}^{L}$, $\sum_{i=1}^{m} a_{i3}^{L} \leq \sum_{j=1}^{n} b_{j3}^{L}$, $\sum_{i=1}^{m} a_{i4}^{L} \leq \sum_{j=1}^{n} b_{j4}^{L}$, $\sum_{i=1}^{m} a_{i1}^{U} \leq \sum_{j=1}^{n} b_{j1}^{U}$, $\sum_{i=1}^{m} a_{i2}^{U} \leq \sum_{j=1}^{n} b_{j2}^{U}$, $\sum_{i=1}^{m} a_{i3}^{U} \leq \sum_{j=1}^{n} b_{j3}^{U}$, $\sum_{i=1}^{m} a_{i4}^{U} \leq \sum_{j=1}^{n} b_{j4}^{U}$, then add a dummy source S_{m+1} having dummy supply $\left\langle \left(\sum_{j=1}^{n} b_{j1}^{L} - \sum_{i=1}^{m} a_{i1}^{L}, \sum_{j=1}^{n} b_{j2}^{L} - \sum_{i=1}^{m} a_{i2}^{L}, \sum_{j=1}^{n} b_{j3}^{L} - \sum_{i=1}^{m} a_{i3}^{L}, \sum_{j=1}^{n} b_{j4}^{L} - \sum_{i=1}^{m} a_{i4}^{L}; \omega^{L}\right), \left(\sum_{j=1}^{n} b_{j1}^{U} - \sum_{i=1}^{m} a_{i1}^{U}, \sum_{j=1}^{n} b_{j2}^{U} - \sum_{i=1}^{m} a_{i2}^{U}, \sum_{j=1}^{n} b_{j3}^{U} - \sum_{i=1}^{m} a_{i3}^{U}, \sum_{j=1}^{n} b_{j4}^{U} - \sum_{i=1}^{m} a_{i4}^{U}; \omega^{U}\right)\right\rangle$ by considering the cost for supplying the unit quantity of the product from the dummy source S_{m+1} to all the destinations as a GIVTrFN $\tilde{\tilde{0}} = \langle (0, 0, 0, 0; 1), (0, 0, 0, 0; 1)\rangle$.

Case 2: If $\sum_{j=1}^{n} b_{j1}^{L} \leq \sum_{i=1}^{m} a_{i1}^{L}$, $\sum_{j=1}^{n} b_{j2}^{L} \leq \sum_{i=1}^{m} a_{i2}^{L}$, $\sum_{j=1}^{n} b_{j3}^{L} \leq \sum_{i=1}^{m} a_{i3}^{L}$, $\sum_{j=1}^{n} b_{j4}^{L} \leq \sum_{i=1}^{m} a_{i4}^{L}$, $\sum_{j=1}^{n} b_{j1}^{U} \leq \sum_{i=1}^{m} a_{i1}^{U}$, $\sum_{j=1}^{n} b_{j2}^{U} \leq \sum_{i=1}^{m} a_{i2}^{U}$, $\sum_{j=1}^{n} b_{j3}^{U} \leq \sum_{i=1}^{m} a_{i3}^{U}$, $\sum_{j=1}^{n} b_{j4}^{U} \leq \sum_{i=1}^{m} a_{i4}^{U}$, then add a dummy destination D_{n+1} having dummy demand $\left\langle \left(\sum_{i=1}^{m} a_{i1}^{L} - \sum_{j=1}^{n} b_{j1}^{L}, \sum_{i=1}^{m} a_{i2}^{L} - \sum_{j=1}^{n} b_{j2}^{L}, \sum_{i=1}^{m} a_{i3}^{L} - \sum_{j=1}^{n} b_{j3}^{L}, \sum_{i=1}^{m} a_{i4}^{L} - \sum_{j=1}^{n} b_{j4}^{L}; \omega^{L}\right), \left(\sum_{i=1}^{m} a_{i1}^{U} -\right.\right.$

$\sum_{j=1}^{n} b_{j1}^{U}, \sum_{i=1}^{m} a_{i2}^{U} - \sum_{j=1}^{n} b_{j2}^{U}, \sum_{i=1}^{m} a_{i3}^{U} - \sum_{j=1}^{n} b_{j3}^{U}, \sum_{i=1}^{m} a_{i4}^{U} - \sum_{j=1}^{n} b_{j4}^{U}; \omega^{U}\big)\big\rangle$ by considering the cost for supplying the unit quantity of the product from all the sources to the dummy destination D_{n+1} as a GIVTrFN, $\tilde{\tilde{0}} = \langle(0,0,0,0;1),(0,0,0,0;1)\rangle$.

Case 3: If neither Case 1 nor Case 2 is satisfying then,

(i) Add a dummy source S_{m+1} having the dummy supply $\big\langle\big(A_{(m+1)1}^{L}, A_{(m+1)2}^{L}, A_{(m+1)3}^{L}, A_{(m+1)4}^{L}; \omega^{L}\big), \big(A_{(m+1)1}^{U}, A_{(m+1)2}^{U}, A_{(m+1)3}^{U}, A_{(m+1)4}^{U}; \omega^{U}\big)\big\rangle$ by considering the cost for supplying the unit quantity of the product from the dummy source S_{m+1} to all the destinations as a GIVTrFN $\tilde{\tilde{0}} = \langle(0,0,0,0;1),(0,0,0,0;1)\rangle$.

$A_{(m+1)1}^{L} = \left|\sum_{j=1}^{n} b_{j1}^{U} - \sum_{i=1}^{m} a_{i1}^{U}\right| + max\{0, \sum_{j=1}^{n} b_{j1}^{L} - \sum_{i=1}^{m} a_{i1}^{L}\}$,

$A_{(m+1)2}^{L} = A_{(m+1)1}^{L} + max\{0, \big(\sum_{j=1}^{n} b_{j2}^{L} - \sum_{j=1}^{n} b_{j1}^{L}\big) - (\sum_{i=1}^{m} a_{i2}^{L} - \sum_{i=1}^{m} a_{i1}^{L})\}$,

$A_{(m+1)3}^{L} = A_{(m+1)2}^{L} + max\{0, \big(\sum_{j=1}^{n} b_{j3}^{L} - \sum_{j=1}^{n} b_{j2}^{L}\big) - (\sum_{i=1}^{m} a_{i3}^{L} - \sum_{i=1}^{m} a_{i2}^{L})\}$,

$A_{(m+1)4}^{L} = A_{(m+1)3}^{L} + max\{0, \big(\sum_{j=1}^{n} b_{j4}^{L} - \sum_{j=1}^{n} b_{j3}^{L}\big) - (\sum_{i=1}^{m} a_{i4}^{L} - \sum_{i=1}^{m} a_{i3}^{L})\}$,

$A_{(m+1)1}^{U} = max\{0, \sum_{j=1}^{n} b_{j1}^{U} - \sum_{i=1}^{m} a_{i1}^{U}\}$,

$A_{(m+1)2}^{U} = \left|\sum_{j=1}^{n} b_{j1}^{U} - \sum_{i=1}^{m} a_{i1}^{U}\right| + max\{0, \sum_{j=1}^{n} b_{j1}^{U} - \sum_{i=1}^{m} a_{i1}^{U}\} + max\{0, \big(\sum_{j=1}^{n} b_{j2}^{U} - \sum_{j=1}^{n} b_{j1}^{U}\big) - \quad \big(\sum_{i=1}^{m} a_{i2}^{U} - \sum_{i=1}^{m} a_{i1}^{U}\big)\}$,

$A_{(m+1)3}^{U} = A_{(m+1)2}^{U} + max\{0, \big(\sum_{j=1}^{n} b_{j3}^{U} - \sum_{j=1}^{n} b_{j2}^{U}\big) - \big(\sum_{i=1}^{m} a_{i3}^{U} - \sum_{i=1}^{m} a_{i2}^{U}\big)\}$,

$A_{(m+1)4}^{U} = A_{(m+1)3}^{U} + max\{0, \big(\sum_{j=1}^{n} b_{j4}^{U} - \sum_{j=1}^{n} b_{j3}^{U}\big) - \big(\sum_{i=1}^{m} a_{i4}^{U} - \sum_{i=1}^{m} a_{i3}^{U}\big)\} + min\{0, A_{(m+1)3}^{U} + max\{0, \big(\sum_{j=1}^{n} b_{j4}^{U} - \sum_{j=1}^{n} b_{j3}^{U}\big) - \big(\sum_{i=1}^{m} a_{i4}^{U} - \sum_{i=1}^{m} a_{i3}^{U}\big)\} - A_{(m+1)4}^{L}\}$.

(ii) Add a dummy destination D_{n+1} having the dummy demand $\big\langle\big(B_{(n+1)1}^{L}, B_{(n+1)2}^{L}, B_{(n+1)3}^{L}, B_{(n+1)4}^{L}; \omega^{L}\big), \big(B_{(n+1)1}^{U}, B_{(n+1)2}^{U}, B_{(n+1)3}^{U}, B_{(n+1)4}^{U};\ \omega^{U}\big)\big\rangle$

by considering the cost for supplying the unit quantity of the product from all the sources to the dummy destination D_{n+1} as a GIVTrFN $\tilde{\tilde{0}} = \langle (0,0,0,0;1),(0,0,0,0;1) \rangle$.

$B_{(n+1)1}^{L} = \left|\sum_{j=1}^{n} b_{j1}^{U} - \sum_{i=1}^{m} a_{i1}^{U}\right| + max\left\{0, \sum_{i=1}^{m} a_{i1}^{L} - \sum_{j=1}^{n} b_{j1}^{L}\right\}$,

$B_{(n+1)2}^{L} = B_{(n+1)1}^{L} + max\left\{0, \left(\sum_{i=1}^{m} a_{i2}^{L} - \sum_{i=1}^{m} a_{i1}^{L}\right) - \left(\sum_{j=1}^{n} b_{j2}^{L} - \sum_{j=1}^{n} b_{j1}^{L}\right)\right\}$,

$B_{(n+1)3}^{L} = B_{(n+1)2}^{L} + max\left\{0, \left(\sum_{i=1}^{m} a_{i3}^{L} - \sum_{i=1}^{m} a_{i2}^{L}\right) - \left(\sum_{j=1}^{n} b_{j3}^{L} - \sum_{j=1}^{n} b_{j2}^{L}\right)\right\}$,

$B_{(n+1)4}^{L} = B_{(n+1)3}^{L} + max\left\{0, \left(\sum_{i=1}^{m} a_{i4}^{L} - \sum_{i=1}^{m} a_{i3}^{L}\right) - \left(\sum_{j=1}^{n} b_{j4}^{L} - \sum_{j=1}^{n} b_{j3}^{L}\right)\right\}$,

$B_{(n+1)1}^{U} = max\left\{0, \sum_{i=1}^{m} a_{i1}^{U} - \sum_{j=1}^{n} b_{j1}^{U}\right\}$,

$B_{(n+1)2}^{U} = B_{(n+1)1}^{U} + \left|\sum_{j=1}^{n} b_{j1}^{U} - \sum_{i=1}^{m} a_{i1}^{U}\right| + max\left\{0, \left(\sum_{i=1}^{m} a_{i2}^{U} - \sum_{i=1}^{m} a_{i1}^{U}\right) - \right.$

$\left.\left(\sum_{j=1}^{n} b_{j2}^{U} - \sum_{j=1}^{n} b_{j1}^{U}\right)\right\}$,

$B_{(n+1)3}^{U} = B_{(n+1)2}^{U} + max\left\{0, \left(\sum_{i=1}^{m} a_{i3}^{U} - \sum_{i=1}^{m} a_{i2}^{U}\right) - \left(\sum_{j=1}^{n} b_{j3}^{U} - \sum_{j=1}^{n} b_{j2}^{U}\right)\right\}$,

$B_{(n+1)4}^{U} = B_{(n+1)3}^{U} + max\left\{0, \left(\sum_{i=1}^{m} a_{i4}^{U} - \sum_{i=1}^{m} a_{i3}^{U}\right) - \left(\sum_{j=1}^{n} b_{j4}^{U} - \sum_{j=1}^{n} b_{j3}^{U}\right)\right\} +$

$min\left\{0, B_{(n+1)3}^{U} + max\left\{0, \left(\sum_{i=1}^{m} a_{i4}^{U} - \sum_{i=1}^{m} a_{i3}^{U}\right) - \left(\sum_{j=1}^{n} b_{j4}^{U} - \sum_{j=1}^{n} b_{j3}^{U}\right)\right\} - B_{(n+1)4}^{L}\right\}$.

6.2 Flaws of the Ebrahimnejad's method for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP

In this section, some numerical values are considered to show that Ebrahimnejad's method for transforming an unbalanced GTrFNTP into a balanced GIVTrFNTP [48, Section 5, Step 1, pp. 305] is not valid.

(1) Let

$\left\langle \left(\sum_{i=1}^{m} a_{i1}^{L}, \sum_{i=1}^{m} a_{i2}^{L}, \sum_{i=1}^{m} a_{i3}^{L}, \sum_{i=1}^{m} a_{i4}^{L}; \omega^{L}\right), \left(\sum_{i=1}^{m} a_{i1}^{U}, \sum_{i=1}^{m} a_{i2}^{U}, \sum_{i=1}^{m} a_{i3}^{U}, \sum_{i=1}^{m} a_{i4}^{U}; \omega^{U}\right) \right\rangle =$

$\langle (1,2,7,11;1),(0,3,8,7;1) \rangle$ and

$\left\langle\left(\sum_{j=1}^{n} b_{j1}^{L}, \sum_{j=1}^{n} b_{j2}^{L}, \sum_{j=1}^{n} b_{j3}^{L}, \sum_{j=1}^{n} b_{j4}^{L}; \omega^{L}\right), \left(\sum_{j=1}^{n} b_{j1}^{U}, \sum_{j=1}^{n} b_{j2}^{U}, \sum_{j=1}^{n} b_{j3}^{U}, \sum_{j=1}^{n} b_{j4}^{U}; \omega^{U}\right)\right\rangle = \langle(4, 8, 9, 13; 1), (1, 6, 10, 18; 1)\rangle$.

Since, $\sum_{i=1}^{m} a_{i1}^{L} \leq \sum_{j=1}^{n} b_{j1}^{L}$, $\sum_{i=1}^{m} a_{i2}^{L} \leq \sum_{j=1}^{n} b_{j2}^{L}$, $\sum_{i=1}^{m} a_{i3}^{L} \leq \sum_{j=1}^{n} b_{j3}^{L}$, $\sum_{i=1}^{m} a_{i4}^{L} \leq \sum_{j=1}^{n} b_{j4}^{L}$, $\sum_{i=1}^{m} a_{i1}^{U} \leq \sum_{j=1}^{n} b_{j1}^{U}$, $\sum_{i=1}^{m} a_{i2}^{U} \leq \sum_{j=1}^{n} b_{j2}^{U}$, $\sum_{i=1}^{m} a_{i3}^{U} \leq \sum_{j=1}^{n} b_{j3}^{U}$, $\sum_{i=1}^{m} a_{i4}^{U} \leq \sum_{j=1}^{n} b_{j4}^{U}$ i.e., Case 1 of Ebrahimnejad's method [48, Sec. 5, Step 1, pp. 305], discussed in Section 6.1, is satisfying. So, according to Ebrahimnejad's [48] method, the dummy supply will be $\langle(4-1, 8-2, 9-7, 13-11; 1), (1-0, 6-3, 10-8, 18-7; 1)\rangle = \langle(3, 6, 2, 2; 1), (1, 3, 2, 11; 1)\rangle$.

However, it is not a GIVTrFN due to the following reason:

It can be easily verified from the graphical representation of GIVTrFN [48, Section 2, Figure 1, pp. 302] as well as from the existing definition of a GIVTrFN [48, Section 2, Def. 2.3, pp. 301] that for a GIVTrFN $\langle(a_1^L, a_2^L, a_3^L, a_4^L; \omega^L), (a_1^U, a_2^U, a_3^U, a_4^U; \omega^U)\rangle$ the condition $a_1^U \leq a_1^L \leq a_2^U \leq a_2^L \leq a_3^L \leq a_3^U \leq a_4^L \leq a_4^U$ should always be satisfied. While, for the obtained dummy supply $\langle(3, 6, 2, 2; 1), (1, 3, 2, 11; 1)\rangle$, this condition is not satisfying. Hence, Ebrahimnejad's method [48, Section 5, Step 1, pp. 305] to obtain dummy supply is not valid.

(2) Let $\left\langle\left(\sum_{i=1}^{m} a_{i1}^{L}, \sum_{i=1}^{m} a_{i2}^{L}, \sum_{i=1}^{m} a_{i3}^{L}, \sum_{i=1}^{m} a_{i4}^{L}; \omega^{L}\right), \left(\sum_{i=1}^{m} a_{i1}^{U}, \sum_{i=1}^{m} a_{i2}^{U}, \sum_{i=1}^{m} a_{i3}^{U}, \sum_{i=1}^{m} a_{i4}^{U}; \omega^{U}\right)\right\rangle = \langle(4, 8, 9, 13; 1), (1, 6, 10, 18; 1)\rangle$ and $\left\langle\left(\sum_{j=1}^{n} b_{j1}^{L}, \sum_{j=1}^{n} b_{j2}^{L}, \sum_{j=1}^{n} b_{j3}^{L}, \sum_{j=1}^{n} b_{j4}^{L}; \omega^{L}\right), \left(\sum_{j=1}^{n} b_{j1}^{U}, \sum_{j=1}^{n} b_{j2}^{U}, \sum_{j=1}^{n} b_{j3}^{U}, \sum_{j=1}^{n} b_{j4}^{U}; \omega^{U}\right)\right\rangle = \langle(1,2, 7, 11; 1), (0,3, 8, 7; 1)\rangle$.

Since, $\sum_{j=1}^{n} b_{j1}^{L} \leq \sum_{i=1}^{m} a_{i1}^{L}$, $\sum_{j=1}^{n} b_{j2}^{L} \leq \sum_{i=1}^{m} a_{i2}^{L}$, $\sum_{j=1}^{n} b_{j3}^{L} \leq \sum_{i=1}^{m} a_{i3}^{L}$, $\sum_{j=1}^{n} b_{j4}^{L} \leq \sum_{i=1}^{m} a_{i4}^{L}$, $\sum_{j=1}^{n} b_{j1}^{U} \leq \sum_{i=1}^{m} a_{i1}^{U}$, $\sum_{j=1}^{n} b_{j2}^{U} \leq$

$\sum_{i=1}^{m} a_{i2}^{U}, \sum_{j=1}^{n} b_{j3}^{U} \leq \sum_{i=1}^{m} a_{i3}^{U}$, $\sum_{j=1}^{n} b_{j4}^{U} \leq \sum_{i=1}^{m} a_{i4}^{U}$ i.e., Case 2 of Ebrahimnejad's method [48, Section 5, Step 1, pp. 305], discussed in Section 6.2, is satisfying. So, according to Ebrahimnejad's method [48, Sec. 5, Step 1, pp. 305], the dummy demand will be $\langle(4-1, 8-2, 9-7, 13-11; 1), (1-0, 6-3, 10-8, 18-7; 1)\rangle = \langle(3, 6, 2, 2; 1), (1, 3, 2, 11; 1)\rangle$.

However, it is not a GIVTrFN due to the following reason:

It can be easily verified from the graphical representation of GIVTrFN [48, Section 2, Figure 1, pp. 302] as well as from the existing definition of a GIVTrFN [48, Section 2, Def. 2.3, pp. 301] that for a GIVTrFN $\langle(a_1^L, a_2^L, a_3^L, a_4^L; \omega^L), (a_1^U, a_2^U, a_3^U, a_4^U; \omega^U)\rangle$ the condition $a_1^U \leq a_1^L \leq a_2^U \leq a_2^L \leq a_3^L \leq a_3^U \leq a_4^L \leq a_4^U$ should always be satisfied. While, for the obtained dummy demand $\langle(3, 6, 2, 2; 1), (1, 3, 2, 11; 1)\rangle$, this condition is not satisfying. Hence, Ebrahimnejad's method [48, Section 5, Step 1, pp. 305] to obtain dummy demand is not valid.

(3) Let $\left\langle(\sum_{i=1}^{m} a_{i1}^{L}, \sum_{i=1}^{m} a_{i2}^{L}, \sum_{i=1}^{m} a_{i3}^{L}, \sum_{i=1}^{m} a_{i4}^{L}; \omega^L), \left(\sum_{i=1}^{m} a_{i1}^{U}, \sum_{i=1}^{m} a_{i2}^{U}, \sum_{i=1}^{m} a_{i3}^{U}, \sum_{i=1}^{m} a_{i4}^{U}; \omega^U\right)\right\rangle = \left\langle\left(110, 150, 160, 180; \frac{2}{3}\right), (100, 140, 170, 190; 1)\right\rangle$ and $\left\langle\left(\sum_{j=1}^{n} b_{j1}^{L}, \sum_{j=1}^{n} b_{j2}^{L}, \sum_{j=1}^{n} b_{j3}^{L}, \sum_{j=1}^{n} b_{j4}^{L}; \omega^L\right), \left(\sum_{j=1}^{n} b_{j1}^{U}, \sum_{j=1}^{n} b_{j2}^{U}, \sum_{j=1}^{n} b_{j3}^{U}, \sum_{j=1}^{n} b_{j4}^{U}; \omega^U\right)\right\rangle = \left\langle\left(90, 120, 140, 200; \frac{2}{3}\right), (75, 105, 155, 215; 1)\right\rangle$.

Since, Case 3 of Ebrahimnejad's method [48, Section 5, Step 1, pp. 305], discussed in Section 6.2, is satisfying. So, according to Ebrahimnejad's method [48, Section 5, Step 1, pp. 305], the dummy demand is

$\left\langle(45,55,55,55;\frac{2}{3}),(25,60,60,60;1)\right\rangle$ and the dummy supply is $\left\langle(25,25,35,75;\frac{2}{3}),(0,25,45,85;1)\right\rangle$.

However, the obtained dummy demand is not a GIVTrFN due to the following reason:

It can be easily verified from the graphical representation of GIVTrFN [48, Section 3, Figure 1, pp. 302] as well as from the existing definition of a GIVTrFN [48, Section 2, Def. 2.3, pp. 301] that for a GIVTrFN $\langle(a_1^L,a_2^L,a_3^L,a_4^L;\omega^L),(a_1^U,a_2^U,a_3^U,a_4^U;\omega^U)\rangle$ the condition $a_1^U \leq a_1^L \leq a_2^U \leq a_2^L \leq a_3^L \leq a_3^U \leq a_4^L \leq a_4^U$ should always be satisfied. While, for the obtained dummy demand $\left\langle(45,55,55,55;\frac{2}{3}),(25,60,60,60;1)\right\rangle$, this condition is not satisfying. Hence, Ebrahimnejad's method [48, Section 5, Step 1, pp. 305] to obtain dummy demand is not valid.

6.3 Invalidity of the existing result

Ebrahimnejad [48, Section 6, Example 6.1, pp. 308] considered a GIVTrFNTP having two sources S_1, S_2 and three destinations D_1, D_2, D_3 such that

(i) The GIVTrF supply at sources S_1 and S_2 are $\left\langle(70,90,90,100;\frac{2}{3}),(65,85,95,105;1)\right\rangle$ and $\left\langle(40,60,70,80;\frac{2}{3}),(35,55,75,85;1)\right\rangle$ respectively.

(ii) The GIVTrF demand at destinations D_1, D_2 and D_3 are $\left\langle(30,40,50,70;\frac{2}{3}),(25,35,55,75;1)\right\rangle$, $\left\langle(20,30,40,50;\frac{2}{3}),(15,25,45,55;1)\right\rangle$ and $\left\langle(40,50,50,80;\frac{2}{3}),(35,45,55,85;1)\right\rangle$ respectively.

Ebrahimnejad [48, Section 6, Example 6.1, pp. 308] claimed that as the total GIVTrF supply i.e., $\left\langle (70,90,90,100;\frac{2}{3}),(65,85,95,105;1)\right\rangle +$ $\left\langle (40,60,70,80;\frac{2}{3}),(35,55,75,85;1)\right\rangle = \left\langle (110,150,160,180\,;\frac{2}{3}),(35,55,75,85;1)\right\rangle$ is not equal to the total GIVTrF demand i.e., $\left\langle (30,40,50,70;\frac{2}{3}),(25,35,55,75;1)\right\rangle +$ $\left\langle (20,30,40,50;\frac{2}{3}),(15,25,45,55;1)\right\rangle + \left\langle (40,50,50,80;\frac{2}{3}),(35,45,55,85;1)\right\rangle =$ $\left\langle (190,120,140,200;\frac{2}{3}),(75,105,155,215;1)\right\rangle$.

Therefore, the considered GIVTrFNTP is an unbalanced GIVTrFNTP. So, there is need to add a dummy source S_3 having dummy GIVTrF supply $\left\langle (25,25,35,75;\ \frac{2}{3}),(0,25,45,85;1)\right\rangle$ and a dummy destination D_4 having dummy GIVTrF demand $\left\langle (45,55,55,55;\ \frac{2}{3}),(25,60,60,60;1)\right\rangle$ for solving the considered GIVTrFNTP.

However, the dummy GIVTrF demand $\left\langle (45,55,55,55;\ \frac{2}{3}),(25,60,60,60;1)\right\rangle$, obtained by Ebrahimnejad [48, Sec. 6, Example 6.1, pp. 308], is not a GIVTrFN due to the following reason:

It can be easily verified from the graphical representation of GIVTrFN [48, Section 2, Figure 1, pp. 302] as well as from the existing definition of a GIVTrFN [16, Sec. 2, Def. 2.4, pp. 301] that for a GIVTrFN $\langle (b_1^L,b_2^L,b_3^L,b_4^L;\omega^L),(b_1^U,b_2^U,b_3^U,b_4^U;\omega^U)\rangle$, the condition $b_1^U \le b_1^L \le b_2^U \le b_2^L \le b_3^L \le b_3^U \le b_4^L \le b_4^U$ should always be satisfied. While, it can be easily verified that if the GIVTrF demand $\left\langle (45,55,55,55;\frac{2}{3}),(25,60,60,60;1)\right\rangle$ is compared with a GIVTrFN

$\langle(b_1^L, b_2^L, b_3^L, b_4^L; \omega^L), (b_1^U, b_2^U, b_3^U, b_4^U; \omega^U)\rangle$ then $b_1^L = 45$, $b_2^L = 55$, $b_3^L = 55$, $b_4^L = 55$, $b_1^U = 25$, $b_2^U = 60$, $b_3^U = 60$, $b_4^U = 60$.

It is obvious that $b_2^U \geq b_2^L$ i.e., the necessary condition $b_2^U \leq b_2^L$ is not satisfying. Therefore, the obtained dummy demand is not a GIVTrFN. Hence, the dummy supply and dummy demand of this problem, obtained by Ebrahimnejad [48, Section 6, Example 6.1, pp. 308], are not correct.

6.4 Proposed Mehar method for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP

In this section, a new method (named as Mehar method) is proposed to transform an unbalanced GIVTrFNTP into a balanced GIVTrFNTP.

Using the proposed Mehar method, an unbalanced GIVTrFNTP can be transformed into a balanced GIVTrFNTP as follows:

Case 1: If $\sum_{i=1}^{m} a_{i1}^U \leq \sum_{j=1}^{n} b_{j1}^U$, $\sum_{i=1}^{m} a_{i1}^L - \sum_{i=1}^{m} a_{i1}^U \leq \sum_{j=1}^{n} b_{j1}^L - \sum_{j=1}^{n} b_{j1}^U$, $\sum_{i=1}^{m} a_{i2}^U - \sum_{i=1}^{m} a_{i1}^L \leq \sum_{j=1}^{n} b_{j2}^U - \sum_{j=1}^{n} b_{j1}^L$, $\sum_{i=1}^{m} a_{i2}^L - \sum_{i=1}^{m} a_{i2}^U \leq \sum_{j=1}^{n} b_{j2}^L - \sum_{j=1}^{n} b_{j2}^U$, $\sum_{i=1}^{m} a_{i3}^L - \sum_{i=1}^{m} a_{i2}^L \leq \sum_{j=1}^{n} b_{j3}^L - \sum_{j=1}^{n} b_{j2}^L$, $\sum_{i=1}^{m} a_{i3}^U - \sum_{i=1}^{m} a_{i3}^L \leq \sum_{j=1}^{n} b_{j3}^U - \sum_{j=1}^{n} b_{j3}^L$, $\sum_{i=1}^{m} a_{i4}^L - \sum_{i=1}^{m} a_{i3}^U \leq \sum_{j=1}^{n} b_{j4}^L - \sum_{j=1}^{n} b_{j3}^U$, $\sum_{i=1}^{m} a_{i4}^U - \sum_{i=1}^{m} a_{i4}^L \leq \sum_{j=1}^{n} b_{j4}^U - \sum_{j=1}^{n} b_{j4}^L$ then add a dummy source S_{m+1} having dummy supply $\left\langle\left(A_{(m+1)1}^L, A_{(m+1)2}^L, A_{(m+1)3}^L, A_{(m+1)4}^L;\ \omega^L\right), \left(A_{(m+1)1}^U, A_{(m+1)2}^U, A_{(m+1)3}^U, A_{(m+1)4}^U;\ \omega^U\right)\right\rangle$ by considering the cost for supplying the unit quantity of the product from the dummy source S_{m+1} to all the destinations as a GIVTrFN $\tilde{\tilde{0}} = \langle(0, 0, 0, 0; 1), (0, 0, 0, 0; 1)\rangle$.

$A_{(m+1)1}^U = \sum_{j=1}^{n} b_{j1}^U - \sum_{i=1}^{m} a_{i1}^U$, $A_{(m+1)1}^L = A_{(m+1)1}^U + \left(\sum_{j=1}^{n} b_{j1}^L - \sum_{j=1}^{n} b_{j1}^U\right) - \left(\sum_{i=1}^{m} a_{i1}^L - \sum_{i=1}^{m} a_{i1}^U\right)$, $A_{(m+1)2}^U = A_{(m+1)1}^L + \left(\sum_{j=1}^{n} b_{j2}^U - \sum_{j=1}^{n} b_{j1}^L\right) - \left(\sum_{i=1}^{m} a_{i2}^U - \sum_{i=1}^{m} a_{i1}^L\right)$,

$A^L_{(m+1)2} = A^U_{(m+1)2} + \left(\sum_{j=1}^n b^L_{j2} - \sum_{j=1}^n b^U_{j2}\right) - \left(\sum_{i=1}^m a^L_{i2} - \sum_{i=1}^m a^U_{i2}\right),$

$A^L_{(m+1)3} = A^L_{(m+1)2} + \left(\sum_{j=1}^n b^L_{j3} - \sum_{j=1}^n b^L_{j2}\right) - (\sum_{i=1}^m a^L_{i3} - \sum_{i=1}^m a^L_{i2}),$

$A^U_{(m+1)3} = A^L_{(m+1)3} + \left(\sum_{j=1}^n b^U_{j3} - \sum_{j=1}^n b^L_{j3}\right) - \left(\sum_{i=1}^m a^U_{i3} - \sum_{i=1}^m a^L_{i3}\right),$

$A^L_{(m+1)4} = A^U_{(m+1)3} + \left(\sum_{j=1}^n b^L_{j4} - \sum_{j=1}^n b^U_{j3}\right) - \left(\sum_{i=1}^m a^L_{i4} - \sum_{i=1}^m a^U_{i3}\right),$

$A^U_{(m+1)4} = A^L_{(m+1)4} + \left(\sum_{j=1}^n b^U_{j4} - \sum_{j=1}^n b^L_{j4}\right) - \left(\sum_{i=1}^m a^U_{i4} - \sum_{i=1}^m a^L_{i4}\right).$

Case 2: If $\sum_{j=1}^n b^U_{j1} \le \sum_{i=1}^m a^U_{i1}$, $\sum_{j=1}^n b^L_{j1} - \sum_{j=1}^n b^U_{j1} \le \sum_{i=1}^m a^L_{i1} - \sum_{i=1}^m a^U_{i1}$, $\sum_{j=1}^n b^U_{j2} - \sum_{j=1}^n b^L_{j1} \le \sum_{i=1}^m a^U_{i2} - \sum_{i=1}^m a^L_{i1}$, $\sum_{j=1}^n b^L_{j2} - \sum_{j=1}^n b^U_{j2} \le \sum_{i=1}^m a^L_{i2} - \sum_{i=1}^m a^U_{i2}$, $\sum_{j=1}^n b^L_{j3} - \sum_{j=1}^n b^L_{j2} \le \sum_{i=1}^m a^L_{i3} - \sum_{i=1}^m a^L_{i2}$, $\sum_{j=1}^n b^U_{j3} - \sum_{j=1}^n b^L_{j3} \le \sum_{i=1}^m a^U_{i3} - \sum_{i=1}^m a^L_{i3}$, $\sum_{j=1}^n b^L_{j4} - \sum_{j=1}^n b^U_{j3} \le \sum_{i=1}^m a^L_{i4} - \sum_{i=1}^m a^U_{i3}$, $\sum_{j=1}^n b^U_{j4} - \sum_{j=1}^n b^L_{j4} \le \sum_{i=1}^m a^U_{i4} - \sum_{i=1}^m a^L_{i4}$ then add a dummy destination D_{n+1} having dummy demand $\left\langle \left(B^L_{(n+1)1}, B^L_{(n+1)2}, B^L_{(n+1)3}, B^L_{(n+1)4}; \omega^L\right), \left(B^U_{(n+1)1}, B^U_{(n+1)2}, B^U_{(n+1)3}, B^U_{(n+1)4}; \omega^U\right) \right\rangle$ by considering the cost for supplying the unit quantity of the product from all the sources to the dummy destination D_{n+1} as a GIVTrFN $\tilde{\tilde{0}} = \langle (0, 0, 0, 0; 1), (0, 0, 0, 0; 1) \rangle$.

$B^U_{(n+1)1} = \sum_{i=1}^m a^U_{i1} - \sum_{j=1}^n b^U_{j1},$ $B^L_{(n+1)1} = B^U_{(n+1)1} + \left(\sum_{i=1}^m a^L_{i1} - \sum_{i=1}^m a^U_{i1}\right) - \left(\sum_{j=1}^n b^L_{j1} - \sum_{j=1}^n b^U_{j1}\right),$

$B^U_{(n+1)2} = B^L_{(n+1)1} + \left(\sum_{i=1}^m a^U_{i2} - \sum_{i=1}^m a^L_{i1}\right) - \left(\sum_{j=1}^n b^U_{j2} - \sum_{j=1}^n b^L_{j1}\right),$

$B^L_{(n+1)2} = B^U_{(n+1)2} + \left(\sum_{i=1}^m a^L_{i2} - \sum_{i=1}^m a^U_{i2}\right) - \left(\sum_{j=1}^n b^L_{j2} - \sum_{j=1}^n b^U_{j2}\right),$

$B^L_{(n+1)3} = B^L_{(n+1)2} + (\sum_{i=1}^m a^L_{i3} - \sum_{i=1}^m a^L_{i2}) - \left(\sum_{j=1}^n b^L_{j3} - \sum_{j=1}^n b^L_{j2}\right),$

$B^U_{(n+1)3} = B^L_{(n+1)3} + \left(\sum_{i=1}^m a^U_{i3} - \sum_{i=1}^m a^L_{i3}\right) - \left(\sum_{j=1}^n b^U_{j3} - \sum_{j=1}^n b^L_{j3}\right),$

$B^L_{(n+1)4} = B^U_{(n+1)3} + \left(\sum_{i=1}^m a^L_{i4} - \sum_{i=1}^m a^U_{i3}\right) - \left(\sum_{j=1}^n b^L_{j4} - \sum_{j=1}^n b^U_{j3}\right),$

$B^U_{(n+1)4} = B^L_{(n+1)4} + \left(\sum_{i=1}^m a^U_{i4} - \sum_{i=1}^m a^L_{i4}\right) - \left(\sum_{j=1}^n b^U_{j4} - \sum_{j=1}^n b^L_{j4}\right).$

Case 3: If neither Case 1 nor Case 2 is satisfying then,

(1) Add a dummy source S_{m+1} having dummy supply $\left\langle \left(A^L_{(m+1)1}, A^L_{(m+1)2}, A^L_{(m+1)3}, A^L_{(m+1)4};\ \omega^L\right), \left(A^U_{(m+1)1}, A^U_{(m+1)2}, A^U_{(m+1)3}, A^U_{(m+1)4};\ \omega^U\right)\right\rangle$ by considering the cost for supplying the unit quantity of the product from the dummy source S_{m+1} to all the destinations as a GIVTrFN $\tilde{\tilde{0}} = \langle (0,0,0,0;1), (0,0,0,0;1)\rangle$.

$A^U_{(m+1)1} = max\{0, \left(\sum_{j=1}^{n} b^U_{j1} - \sum_{i=1}^{m} a^U_{i1}\right)\}$,

$A^L_{(m+1)1} = A^U_{(m+1)1} + max\left\{0, \left(\left(\sum_{j=1}^{n} b^L_{j1} - \sum_{j=1}^{n} b^U_{j1}\right) - \left(\sum_{i=1}^{m} a^L_{i1} - \sum_{i=1}^{m} a^U_{i1}\right)\right)\right\}$,

$A^U_{(m+1)2} = A^L_{(m+1)1} + max\left\{0, \left(\left(\sum_{j=1}^{n} b^U_{j2} - \sum_{j=1}^{n} b^L_{j1}\right) - \left(\sum_{i=1}^{m} a^U_{i2} - \sum_{i=1}^{m} a^L_{i1}\right)\right)\right\}$,

$A^L_{(m+1)2} = A^U_{(m+1)2} + max\left\{0, \left(\left(\sum_{j=1}^{n} b^L_{j2} - \sum_{j=1}^{n} b^U_{j2}\right) - \left(\sum_{i=1}^{m} a^L_{i2} - \sum_{i=1}^{m} a^U_{i2}\right)\right)\right\}$,

$A^L_{(m+1)3} = A^L_{(m+1)2} + max\left\{0, \left(\left(\sum_{j=1}^{n} b^L_{j3} - \sum_{j=1}^{n} b^L_{j2}\right) - (\sum_{i=1}^{m} a^L_{i3} - \sum_{i=1}^{m} a^L_{i2})\right)\right\}$,

$A^U_{(m+1)3} = A^L_{(m+1)3} + max\left\{0, \left(\left(\sum_{j=1}^{n} b^U_{j3} - \sum_{j=1}^{n} b^L_{j3}\right) - \left(\sum_{i=1}^{m} a^U_{i3} - \sum_{i=1}^{m} a^L_{i3}\right)\right)\right\}$,

$A^L_{(m+1)4} = A^U_{(m+1)3} + max\left\{0, \left(\left(\sum_{j=1}^{n} b^L_{j4} - \sum_{j=1}^{n} b^U_{j3}\right) - \left(\sum_{i=1}^{m} a^L_{i4} - \sum_{i=1}^{m} a^U_{i3}\right)\right)\right\}$,

$A^U_{(m+1)4} = A^L_{(m+1)4} + max\left\{0, \left(\left(\sum_{j=1}^{n} b^U_{j4} - \sum_{j=1}^{n} b^L_{j4}\right) - \left(\sum_{i=1}^{m} a^U_{i4} - \sum_{i=1}^{m} a^L_{i4}\right)\right)\right\}$.

(2) Add a dummy destination D_{n+1} having the dummy demand $\left\langle \left(B^L_{(m+1)1}, B^L_{(m+1)2}, B^L_{(m+1)3}, B^L_{(m+1)4};\ \omega^L\right), \left(B^U_{(m+1)1}, B^U_{(m+1)2}, B^U_{(m+1)3}, B^U_{(m+1)4};\ \omega^U\right)\right\rangle$ by considering the cost for supplying the unit quantity of the product from all the sources to the dummy destination D_{n+1} as a GIVTrFN $\tilde{\tilde{0}} = \langle (0,0,0,0;1), (0,0,0,0;1)\rangle$.

$B^U_{(n+1)1} = max\{0, \left(\sum_{i=1}^{m} a^U_{i1} - \sum_{j=1}^{n} b^U_{j1}\right)\}$,

$B^L_{(n+1)1} = B^U_{(n+1)1} + max\left\{0, \left(\left(\sum_{i=1}^{m} a^L_{i1} - \sum_{i=1}^{m} a^U_{i1}\right) - \left(\sum_{j=1}^{n} b^L_{j1} - \sum_{j=1}^{n} b^U_{j1}\right)\right)\right\}$,

$$B^U_{(n+1)2} = B^L_{(n+1)1} + max\left\{0, \left(\left(\sum_{i=1}^m a^U_{i2} - \sum_{i=1}^m a^L_{i1}\right) - \left(\sum_{j=1}^n b^U_{j2} - \sum_{j=1}^n b^L_{j1}\right)\right)\right\},$$

$$B^L_{(n+1)2} = B^U_{(n+1)2} + max\left\{0, \left(\left(\sum_{i=1}^m a^L_{i2} - \sum_{i=1}^m a^U_{i2}\right) - \left(\sum_{j=1}^n b^L_{j2} - \sum_{j=1}^n b^U_{j2}\right)\right)\right\},$$

$$B^L_{(n+1)3} = B^L_{(n+1)2} + max\left\{0, \left(\left(\sum_{i=1}^m a^L_{i3} - \sum_{i=1}^m a^L_{i2}\right) - \left(\sum_{j=1}^n b^L_{j3} - \sum_{j=1}^n b^L_{j2}\right)\right)\right\},$$

$$B^U_{(n+1)3} = B^L_{(n+1)3} + max\left\{0, \left(\left(\sum_{i=1}^m a^U_{i3} - \sum_{i=1}^m a^L_{i3}\right) - \left(\sum_{j=1}^n b^U_{j3} - \sum_{j=1}^n b^L_{j3}\right)\right)\right\},$$

$$B^L_{(n+1)4} = B^U_{(n+1)3} + max\left\{0, \left(\left(\sum_{i=1}^m a^L_{i4} - \sum_{i=1}^m a^U_{i3}\right) - \left(\sum_{j=1}^n b^L_{j4} - \sum_{j=1}^n b^U_{j3}\right)\right)\right\},$$

$$B^U_{(n+1)4} = B^L_{(n+1)4} + max\left\{0, \left(\left(\sum_{i=1}^m a^U_{i4} - \sum_{i=1}^m a^L_{i4}\right) - \left(\sum_{j=1}^n b^U_{j4} - \sum_{j=1}^n b^L_{j4}\right)\right)\right\}.$$

6.5 Exact dummy supply and dummy demand for the existing GIVTrFNTP

In Section 6.4, it is shown that the dummy supply and dummy demand for the existing GIVTrFNTP [48, Section 6, Example 6.1, pp. 308] are not GIVTrFNs and hence, these are not valid. In this section, the exact dummy supply and dummy demand of the same problem is obtained by the proposed Mehar method.

Using the proposed Mehar method the exact dummy supply and dummy demand for the existing GIVTrFNTP [48, Section 6, Example 6.1, pp. 308] can be obtained as follows:

$$\sum_{i=1}^2 \tilde{\tilde{a}}_i =$$

$$\left\langle\left(\sum_{i=1}^2 a^L_{i1}, \sum_{i=1}^2 a^L_{i2}, \sum_{i=1}^2 a^L_{i3}, \sum_{i=1}^2 a^L_{i4}; \omega^L\right), \left(\sum_{i=1}^2 a^U_{i1}, \sum_{i=1}^2 a^U_{i2}, \sum_{i=1}^2 a^U_{i3}, \sum_{i=1}^2 a^U_{i4}; \omega^U\right)\right\rangle$$

$$= \left\langle(110, 150, 160, 180; \tfrac{2}{3}), (100, 140, 170, 190; 1)\right\rangle$$

and

$$\sum_{j=1}^3 \tilde{\tilde{b}}_j =$$

$$\left\langle\left(\sum_{j=1}^3 b^L_{j1}, \sum_{j=1}^3 b^L_{j2}, \sum_{j=1}^3 b^L_{j3}, \sum_{j=1}^3 b^L_{j4}; \omega^L\right), \left(\sum_{j=1}^3 b^U_{j1}, \sum_{j=1}^3 b^U_{j2}, \sum_{j=1}^3 b^U_{j3}, \sum_{j=1}^3 b^U_{j4}; \omega^U\right)\right\rangle$$

$$= \left\langle(90, 120, 140, 200; \tfrac{2}{3}), (75, 105, 155, 215; 1)\right\rangle.$$

Since, Case 3 of the proposed Mehar method is satisfying. So, according to Case 3 of the proposed Mehar method,

$$A_{31}^{U} = max\{0, \left(\sum_{j=1}^{3} b_{j1}^{U} - \sum_{i=1}^{2} a_{i1}^{U}\right)\} = max\{0, 75 - 100\} = 0,$$

$$A_{31}^{L} = A_{31}^{U} + max\left\{0, \left(\left(\sum_{j=1}^{3} b_{j1}^{L} - \sum_{j=1}^{3} b_{j1}^{U}\right) - \left(\sum_{i=1}^{2} a_{i1}^{L} - \sum_{i=1}^{2} a_{i1}^{U}\right)\right)\right\}$$

$$= 0 + max\{0, 15 - 10\} = 5,$$

$$A_{32}^{U} = A_{31}^{L} + max\left\{0, \left(\left(\sum_{j=1}^{3} b_{j2}^{U} - \sum_{j=1}^{3} b_{j1}^{L}\right) - \left(\sum_{i=1}^{2} a_{i2}^{U} - \sum_{i=1}^{2} a_{i1}^{L}\right)\right)\right\}$$

$$= 5 + max\{0, 15 - 30\} = 5,$$

$$A_{32}^{L} = A_{32}^{U} + max\left\{0, \left(\left(\sum_{j=1}^{3} b_{j2}^{L} - \sum_{j=1}^{3} b_{j2}^{U}\right) - \left(\sum_{i=1}^{2} a_{i2}^{L} - \sum_{i=1}^{2} a_{i2}^{U}\right)\right)\right\}$$

$$= 5 + max\{0, 15 - 10\} = 10,$$

$$A_{33}^{L} = A_{32}^{L} + max\left\{0, \left(\left(\sum_{j=1}^{3} b_{j3}^{L} - \sum_{j=1}^{3} b_{j2}^{L}\right) - \left(\sum_{i=1}^{2} a_{i3}^{L} - \sum_{i=1}^{2} a_{i2}^{L}\right)\right)\right\}$$

$$= 10 + max\{0, 20 - 10\} = 20,$$

$$A_{33}^{U} = A_{33}^{L} + max\left\{0, \left(\left(\sum_{j=1}^{3} b_{j3}^{U} - \sum_{j=1}^{3} b_{j3}^{L}\right) - \left(\sum_{i=1}^{2} a_{i3}^{U} - \sum_{i=1}^{2} a_{i3}^{L}\right)\right)\right\}$$

$$= 20 + max\{0, 15 - 10\} = 25,$$

$$A_{34}^{L} = A_{33}^{U} + max\left\{0, \left(\left(\sum_{j=1}^{3} b_{j4}^{L} - \sum_{j=1}^{3} b_{j3}^{U}\right) - \left(\sum_{i=1}^{2} a_{i4}^{L} - \sum_{i=1}^{2} a_{i3}^{U}\right)\right)\right\}$$

$$= 25 + max\{0, 45 - 10\} = 60,$$

$$A_{34}^{U} = A_{34}^{L} + max\left\{0, \left(\left(\sum_{j=1}^{3} b_{j4}^{U} - \sum_{j=1}^{3} b_{j4}^{L}\right) - \left(\sum_{i=1}^{2} a_{i4}^{U} - \sum_{i=1}^{2} a_{i4}^{L}\right)\right)\right\}$$

$$= 60 + max\{0, 15 - 10\} = 65,$$

$$B_{41}^{U} = max\{0, \left(\sum_{i=1}^{2} a_{i1}^{U} - \sum_{j=1}^{3} b_{j1}^{U}\right)\}$$

$$= max\{0, 100 - 75\} = 25$$

$$B_{41}^{L} = B_{41}^{U} + max\left\{0, \left(\left(\sum_{i=1}^{2} a_{i1}^{L} - \sum_{i=1}^{2} a_{i1}^{U}\right) - \left(\sum_{j=1}^{3} b_{j1}^{L} - \sum_{j=1}^{3} b_{j1}^{U}\right)\right)\right\},$$

$$= 25 + max\{0,10 - 15\} = 25,$$

$B^U_{42} = B^L_{41} + max\left\{0, \left((\sum_{i=1}^{2} a^U_{i2} - \sum_{i=1}^{2} a^L_{i1}) - (\sum_{j=1}^{3} b^U_{j2} - \sum_{j=1}^{3} b^L_{j1})\right)\right\}$

$= 25 + max\{0,30 - 15\} = 40,$

$B^L_{42} = B^U_{42} + max\left\{0, \left((\sum_{i=1}^{2} a^L_{i2} - \sum_{i=1}^{2} a^U_{i2}) - (\sum_{j=1}^{3} b^L_{j2} - \sum_{j=1}^{3} b^U_{j2})\right)\right\}$

$= 40 + max\{0,10 - 15\} = 40,$

$B^L_{43} = B^L_{42} + max\left\{0, \left((\sum_{i=1}^{2} a^L_{i3} - \sum_{i=1}^{2} a^L_{i2}) - (\sum_{j=1}^{3} b^L_{j3} - \sum_{j=1}^{3} b^L_{j2})\right)\right\}$

$= 40 + max\{0,10 - 20\} = 40,$

$B^U_{43} = B^L_{43} + max\left\{0, \left((\sum_{i=1}^{2} a^U_{i3} - \sum_{i=1}^{2} a^L_{i3}) - (\sum_{j=1}^{3} b^U_{j3} - \sum_{j=1}^{3} b^L_{j3})\right)\right\}$

$= 40 + max\{0,10 - 15\} = 40,$

$B^L_{44} = B^U_{43} + max\left\{0, \left((\sum_{i=1}^{2} a^L_{i4} - \sum_{i=1}^{2} a^U_{i3}) - (\sum_{j=1}^{3} b^L_{j4} - \sum_{j=1}^{3} b^U_{j3})\right)\right\}$

$= 40 + max\{0,10 - 45\} = 40,$

$B^U_{44} = B^L_{44} + max\left\{0, \left((\sum_{i=1}^{2} a^U_{i4} - \sum_{i=1}^{2} a^L_{i4}) - (\sum_{j=1}^{3} b^U_{j4} - \sum_{j=1}^{3} b^L_{j4})\right)\right\}$

$= 40 + max\{0,10 - 15\} = 40.$

Therefore, the exact dummy fuzzy supply $\langle(A^L_{(m+1)1}, A^L_{(m+1)2}, A^L_{(m+1)3}, A^L_{(m+1)4};\ \omega^L), (A^U_{(m+1)1}, A^U_{(m+1)2}, A^U_{(m+1)3}, A^U_{(m+1)4};\ \omega^U)\rangle = \left\langle(5, 10, 20, 60; \frac{2}{3}), (0, 5, 25, 65; 1)\right\rangle$ and the exact dummy fuzzy demand $\langle(B^L_{(n+1)1}, B^L_{(n+1)2}, B^L_{(n+1)3}, BF^L_{(n+1)4};\ \omega^L), (B^U_{(n+1)1}, B^U_{(n+1)2}, B^U_{(n+1)3}, B^U_{(n+1)4};\ \omega^U)\rangle = \left\langle(25, 40, 40, 40; \frac{2}{3}), (25, 40, 40, 40; 1)\right\rangle$.

6.6 Exact results of the existing GIVTrFNTP

It is pertinent to mention that the solution of the existing transportation problem [48, Section 6, Example 6.1, pp. 308] has been obtained with the help of the optimal solution of the existing linear programming problem [48, Section 6, Example 6.1, Step 3, pp. 310] having 56 constraints. Out of these 56 equality constraints, 8 constraints are corresponding to dummy GIVTrF supply $\left\langle(25,25,35,75;\frac{2}{3}),(0,25,45,85;1)\right\rangle$ and 8 constraints are corresponding to dummy GIVTrF demand $\left\langle(45,55,55,55;\frac{2}{3}),(25,60,60,60;1)\right\rangle$.

However, as discussed in Section 6.4 that the dummy GIVTrF supply and dummy GIVTrF demand, obtained by Ebrahimnejad [48, Section 6, Example 6.1, pp. 308], are not correct. Therefore, the result of this problem, obtained by Ebrahimnejad [48, Section 6, Example 6.1, pp. 308], is not correct.

Since, the exact dummy GIVTrF supply is $\left\langle(5,10,20,60;\frac{2}{3}),(0,5,25,65;1)\right\rangle$ and the exact dummy GIVTrF demand is $\left\langle(25,40,40,40;\frac{2}{3}),(25,40,40,40;1)\right\rangle$. So, to find the exact result of this problem, the equality constraints C_1 of the existing crisp LPP [48, Section 6, Example 6.1, Step 3, pp. 310], have been replaced by the equality constraints C_2.

$$\left.\begin{aligned}
x^L_{31,1}+x^L_{32,1}+x^L_{33,1}+x^L_{34,1}&=25,\\
x^L_{31,2}+x^L_{32,2}+x^L_{33,2}+x^L_{34,2}&=25,\\
x^L_{31,3}+x^L_{32,3}+x^L_{33,3}+x^L_{34,3}&=35,\\
x^L_{31,4}+x^L_{32,4}+x^L_{33,4}+x^L_{34,4}&=75,\\
x^U_{31,1}+x^U_{32,1}+x^U_{33,1}+x^U_{34,1}&=0,\\
x^U_{31,2}+x^U_{32,2}+x^U_{33,2}+x^U_{34,2}&=25,\\
x^U_{31,3}+x^U_{32,3}+x^U_{33,3}+x^U_{34,3}&=45,\\
x^U_{31,4}+x^U_{32,4}+x^U_{33,4}+x^U_{34,4}&=85,\\
x^L_{14,1}+x^L_{24,1}+x^L_{34,1}&=45,\\
x^L_{14,2}+x^L_{24,2}+x^L_{34,2}&=55,\\
x^L_{14,3}+x^L_{24,3}+x^L_{34,3}&=55,\\
x^L_{14,4}+x^L_{24,4}+x^L_{34,4}&=55,\\
x^U_{14,1}+x^U_{24,1}+x^U_{34,1}&=25,\\
x^U_{14,2}+x^U_{24,2}+x^U_{34,2}&=60,\\
x^U_{14,3}+x^U_{24,3}+x^U_{34,3}&=60,\\
x^U_{14,4}+x^U_{24,4}+x^U_{34,4}&=60.
\end{aligned}\right\}\qquad (C_1)$$

$$\left.\begin{aligned}
x^L_{31,1}+x^L_{32,1}+x^L_{33,1}+x^L_{34,1}&=5,\\
x^L_{31,2}+x^L_{32,2}+x^L_{33,2}+x^L_{34,2}&=10,\\
x^L_{31,3}+x^L_{32,3}+x^L_{33,3}+x^L_{34,3}&=20,\\
x^L_{31,4}+x^L_{32,4}+x^L_{33,4}+x^L_{34,4}&=60,\\
x^U_{31,1}+x^U_{32,1}+x^U_{33,1}+x^U_{34,1}&=0,\\
x^U_{31,2}+x^U_{32,2}+x^U_{33,2}+x^U_{34,2}&=5,\\
x^U_{31,3}+x^U_{32,3}+x^U_{33,3}+x^U_{34,3}&=25,\\
x^U_{31,4}+x^U_{32,4}+x^U_{33,4}+x^U_{34,4}&=65,\\
x^L_{14,1}+x^L_{24,1}+x^L_{34,1}&=25,\\
x^L_{14,2}+x^L_{24,2}+x^L_{34,2}&=40,\\
x^L_{14,3}+x^L_{24,3}+x^L_{34,3}&=40,\\
x^L_{14,4}+x^L_{24,4}+x^L_{34,4}&=40,\\
x^U_{14,1}+x^U_{24,1}+x^U_{34,1}&=25,\\
x^U_{14,2}+x^U_{24,2}+x^U_{34,2}&=40,\\
x^U_{14,3}+x^U_{24,3}+x^U_{34,3}&=40,\\
x^U_{14,4}+x^U_{24,4}+x^U_{34,4}&=40.
\end{aligned}\right\}\qquad (C_2)$$

On solving the existing crisp LPP [48, Section 6, Example 6.1, Step 3, pp. 310] with this modification, the obtained exact optimal solution of the existing GIVTrFNTP [48, Section 6, Example 6.1, pp. 308] is

$$\tilde{\tilde{x}}_{11}=\left\langle\left(30,40,40,50;\frac{2}{3}\right),(25,35,45,55;1)\right\rangle,$$

$$\tilde{\tilde{x}}_{12}=\left\langle\left(15,20,20,20;\frac{2}{3}\right),(15,20,20,20;1)\right\rangle,$$

$$\tilde{\tilde{x}}_{13}=\left\langle\left(0,0,0,0;\frac{2}{3}\right),(0,0,0,0;1)\right\rangle,\ \tilde{\tilde{x}}_{14}=\left\langle\left(25,30,30,30;\frac{2}{3}\right),(25,30,30,30;1)\right\rangle,$$

$$\tilde{\tilde{x}}_{21}=\left\langle\left(0,0,10,10;\frac{2}{3}\right),(0,0,10,10;1)\right\rangle,\tilde{\tilde{x}}_{22}=\left\langle\left(0,0,0,0;\frac{2}{3}\right),(0,0,0,0;1)\right\rangle,$$

$$\tilde{\tilde{x}}_{23}=\left\langle\left(40,50,50,50;\frac{2}{3}\right),(35,45,55,65;1)\right\rangle,$$

$$\tilde{\tilde{x}}_{24}=\left\langle\left(0,10,10,10;\frac{2}{3}\right),(0,10,10,10;1)\right\rangle,$$

$$\tilde{\tilde{x}}_{31}=\left\langle\left(0,0,0,10;\frac{2}{3}\right),(0,0,0,10;1)\right\rangle,\tilde{\tilde{x}}_{32}=\left\langle\left(5,10,20,30;\frac{2}{3}\right),(0,5,25,35;1)\right\rangle,$$

$$\tilde{\tilde{x}}_{33}=\left\langle\left(0,0,0,20;\frac{2}{3}\right),(0,0,0,20;1)\right\rangle,\tilde{\tilde{x}}_{34}=\left\langle\left(0,0,0,0;\frac{2}{3}\right),(0,0,0,0;1)\right\rangle.$$

6.7 Conclusions

It is shown that the existing method [48, Section 5, Step 1, pp. 304], for transforming an unbalanced GIVTrFNTP into a balanced GIVTrFNTP, is not valid. Also, a new method (named as Mehar method) is proposed for the same purpose and it is proved that the proposed Mehar method is valid. Furthermore, the exact result of the existing unbalanced GIVTrFNTP [48, Section 6, Example 6.1, pp. 308] is obtained.

Chapter 7
Future Scope

The following problems may be considered as challenging research problems:

(1) Mehar method for solving GIVTrFNTPs, proposed in Chapter 6, cannot be used to find the solution of such GIVTrFNTPs for which the data for more than one experts is available as in Chapter 6 no method has been discussed to aggregate GIVTrFNs. To overcome this limitation of Mehar method, there is need to use an appropriate aggregation operator. However, to the best of my knowledge, there is no such aggregation operator for GIVTrFNs which satisfies all the necessary properties (monotonicity, boundedness, idempotency and commutativity). Therefore, to propose an appropriate aggregation operator for GIVTrFNs and hence to overcome the limitations of Mehar method may be considered as challenging research problem.

(2) It can be easily verified that on applying the existing method [44] for solving the GTrFNTP, represented by Table 7.1, the GTrFNs $(15, 35, 40, 60; 0.4)$ and $(20, 30, 40, 60; 0.4)$, representing the optimal fuzzy transportation cost, are obtained.

Table 7.1 GIVTrFNTP

Destinations → Sources ↓	D_1	D_2	Availability
S_1	$(1, 3, 4, 7; 0.7)$	$(2, 3, 5, 5; 0.4)$	5
S_2	$(2, 3, 3, 7; 0.7)$	$(2, 4, 4, 5; 0.4)$	5
Demand	5	5	

Although, this flaw of the existing method can be easily resolved by considering the existing RMDS approach [108]. But, if the data for more than one experts are available then, a transporation problem with GTrFNs, cannot be solved by the existing method as to do the same, there is need to use an appropriate aggregation operators. While, to the best of my knowledge, there is no such aggregation operator for GTrFNs which satisfies all the necessary properties (monotonicity, boundedness, idempotency and commutativity). Therefore, to propose an appropriate aggregation operator for aggregating GTrFNs and hence to overcome the limitations of Mehar method may be considered as challenging research problem.

Appendix A

A note on “Fuzzy Hungarian MODI Algorithm to Solve Fully Fuzzy Transportation problems”[††]

Dhanasekar et al. [37, Section 3.3, pp. 1483] proposed a fuzzy Hungarian MODI algorithm to solve FFTPs. Dhanasekar et al. have used the standard multiplication of TrFNs in their proposed method. In this appendix, it is pointed out that the method, proposed by Dhanasekar et al., is not valid for standard multiplication of TrFNs and is valid only if a special type of multiplication of TrFNs is used.

A.1. Introduction

Dhanasekar at al. [37, Section 1, pp. 1480] pointed out the following limitations/flaws in the existing methods [23, 24, 106, 55, 151] for solving FFTPs.

(i) Either the solution turns out to be a crisp value or it does not guarantee the fuzzy solution to be positive.

(ii) The negative components exist in the obtained solutions which does not represent the solution of the real world fuzzy transportation problem.

(iii) There is no single algorithmic technique to handle the fuzzy transportation problem, unbalanced fuzzy transportation problem and the degenerate fuzzy transportation problems.

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To overcome/resolve these limitations/flaws, Dhanasekar et al. [37, Section 3.3, pp. 1483] proposed fuzzy Hungarian MODI algorithm to solve FFTPs. Dhanasekar et al. [37] claimed that their proposed method not only gives the optimal solution, but also satisfies the feasibility condition and retains the positive allocation of cells.

It is pertinent to mention that to find the optimal solution of FFTPs, Dhanasekar et al. [37] have assumed that the dual of fuzzy transportation problem (P1) will be problem (P2) and used the constraint $\tilde{u}_i + \tilde{v}_j \preceq \tilde{c}_{ij}\ ; i=1,2,\ldots,m,\ j=1,2,\ldots,n$ of the dual problem (P2) in the Step 10 of their proposed fuzzy Hungarian MODI algorithm.

$$\text{Minimize}\left(\sum_{\mathrm{i}=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}\otimes\ \tilde{x}_{ij}\right)$$

Subject to

$$\sum_{j=1}^{n}\tilde{x}_{ij}=\tilde{s}_i\ ,\ i=1,2,\ldots,m;$$
$$\sum_{i=1}^{m}\tilde{x}_{ij}=\tilde{d}_j\ ,\ j=1,2,\ldots,n;$$
$$\tilde{x}_{ij}\succeq\tilde{0}\ ,\ i=1,2,\ldots,m;\ j=1,2,\ldots,n.$$

(P1)

Here $\tilde{x}_{ij}$ is the number of units to be transported from i^{th} source to j^{th} destination, $\tilde{c}_{ij}$ is the cost of one unit transported from i^{th} source to j^{th} destination, $\tilde{s}_i$ is the number of units available in the i^{th} source and $\tilde{d}_j$ is the number of units required in the j^{th} destination.

$$\text{Maximize}\left(\sum_{i=1}^{m}\tilde{s}_i\otimes\tilde{u}_i\ \oplus\ \sum_{j=1}^{n}\tilde{d}_j\otimes\tilde{v}_j\right)$$

Subject to

(P2)

$$\tilde{u}_i+\tilde{v}_j\preceq\tilde{c}_{ij}\ ;\ i=1,2,\ldots,m,\ j=1,2,\ldots,n,$$

$\tilde{u}_i$ and $\tilde{v}_j$ are unrestricted TrFNs.

Furthermore, it is pertinent to mention that Dhanasekar et al. [37] have used the following standard multiplication of TrFNs.

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)$ be two TrFNs. Then,

$$\tilde{A}_1 \otimes \tilde{A}_2 = (a, b, c, d)$$

where,

$$a = \min\,(a_1a_2, a_1d_2, d_1a_2, d_1d_2),\ b = \min\,(b_1b_2, b_1c_2, c_1b_2, c_1c_2),$$
$$c = \max\,(b_1b_2, b_1c_2, c_1b_2, c_1c_2),\ d = \max\,(a_1a_2, a_1d_2, d_1a_2, d_1d_2).$$

Also, Dhanasekar et al. [37] have used the following method for comparing TrFNs.

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)$ be two TrFNs. Then,

(i) $\tilde{A}_1 \succ \tilde{A}_2$ if $R(\tilde{A}_1) > R(\tilde{A}_2)$.

(ii) $\tilde{A}_1 \prec \tilde{A}_2$ if $R(\tilde{A}_1) < R(\tilde{A}_2)$.

(iii) $\tilde{A}_1 \approx \tilde{A}_2$ if $R(\tilde{A}_1) = R(\tilde{A}_2)$.

where, $R(a, b, c, d) = \dfrac{a+b+c+d}{4}$.

In this paper, it is pointed out that the problem (P2) will be dual of the problem (P1) only and hence, the existing method [37] will be valid only if the special multiplication [113],

$$\tilde{A}_1 \otimes \tilde{A}_2 = (a, b, c, d),$$

where,

$$a = \frac{a_1(a_2+b_2+c_2+d_2)}{4},\ b = \frac{b_1(a_2+b_2+c_2+d_2)}{4},$$
$$c = \frac{c_1(a_2+b_2+c_2+d_2)}{4},\ d = \frac{d_1(a_2+b_2+c_2+d_2)}{4}.$$

is used instead of the standard multiplication.

A.2. Existing method for obtaining the dual problem

The dual problem (P2) corresponding to the problem (P1) is obtained as follows.

Step 1: Using the property, $\tilde{A}_1 \approx \tilde{A}_2$ if $R(\tilde{A}_1) = R(\tilde{A}_2)$. and $\tilde{A}_1 \succ \tilde{A}_2$ if $R(\tilde{A}_1) > R(\tilde{A}_2)$, the problem (P1) can be transformed into the problem (P3).

$$\text{Minimize } R\left(\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} \otimes \tilde{x}_{ij} \right)$$

Subject to

$$R\left(\sum_{j=1}^{n} \tilde{x}_{ij} \right) = R(\tilde{s}_i),\ i = 1,2,\ldots,m;$$

$$R\left(\sum_{i=1}^{m} \tilde{x}_{ij} \right) = R\left(\tilde{d}_j\right),\ j = 1,2,\ldots,n;$$

$$R\left(\tilde{x}_{ij}\right) \geq R\left(\tilde{0}\right),\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n.$$

(P3)

Step 2: Using the property, $\tilde{A}_1 \succ \tilde{A}_2$ if $R(\tilde{A}_1) > R(\tilde{A}_2)$ and the property $R\left(\sum_{i=1}^{n} \tilde{A}_i \right) = \sum_{i=1}^{n} R\left(\tilde{A}_i\right)$, the problem (P3) can be transformed into the problem (P4).

$$\text{Minimize } R\left(\left(\sum_{\mathrm{i}=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} \otimes \tilde{x}_{ij} \right) \right)$$

Subject to

(P4)

$$\sum_{j=1}^{n} R\left(\tilde{x}_{ij}\right) = R(\tilde{s}_i),\ i = 1,2,\ldots,m;$$

$$\sum_{i=1}^{m} R\left(\tilde{x}_{ij}\right) = R\left(\tilde{d}_j\right),\ j = 1,2,\ldots,n;$$

$$R\left(\tilde{x}_{ij}\right) \geq R\left(\tilde{0}\right),\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n.$$

Step 3: Using the property, $R(\tilde{A}_1 \otimes \tilde{A}_2) = R(\tilde{A}_1) \times R\left(\tilde{A}_2\right)$, the problem (P4) can be transformed into the problem (P5).

$$\text{Minimize}\left(\left(\sum_{i=1}^{m}\sum_{j=1}^{n} R(\tilde{c}_{ij}) \otimes R(\tilde{x}_{ij})\right)\right)$$

Subject to

$$\sum_{j=1}^{n} R(\tilde{x}_{ij}) = R(\tilde{s}_i),\ i = 1,2,\ldots,m; \tag{P5}$$

$$\sum_{i=1}^{m} R(\tilde{x}_{ij}) = R(\tilde{d}_j),\ j = 1,2,\ldots,n;$$

$$R(\tilde{x}_{ij}) \geq R(\tilde{0}),\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n.$$

Step 4: Since, $R\,(\text{fuzzy number}) = \text{real number}$. Therefore, assuming, $R(\tilde{c}_{ij}) = c_{ij}$, $R(\tilde{x}_{ij}) = x_{ij}$, $R(\tilde{s}_i) = s_i$, $R(\tilde{d}_j) = d_j$ and $R(\tilde{0}) = 0$, the problem (P5) can be transformed into the problem (P6).

$$\text{Minimize}\left(\left(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} \times x_{ij}\right)\right)$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = s_i,\ i = 1,2,\ldots,m; \tag{P6}$$

$$\sum_{i=1}^{m} x_{ij} = d_j,\ j = 1,2,\ldots,n;$$

$$x_{ij} \geq 0,\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n.$$

Step 5: The dual of the problem (P6) is the problem (P7).

$$\text{Maximize}\left(\sum_{i=1}^{m} s_i \otimes u_i \ \oplus\ \sum_{j=1}^{n} d_j \otimes v_j\right)$$

Subject to

$$u_i + v_j \preceq c_{ij}\ ;\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n, \tag{P7}$$

u_i and v_j are unrestricted real numbers.

Step 6: Replacing c_{ij}, s_i, d_j, u_i, v_j and 0 by $R(\tilde{c}_{ij})$, $R(\tilde{s}_i)$, $R(\tilde{d}_j)$, $R(\tilde{u}_i)$, $R(\tilde{v}_j)$ and $R(\tilde{0})$, the problem (P7) can be transformed into the problem (P8).

$$\text{Maximize}\left(\sum_{i=1}^{m} R(\tilde{s}_i)\otimes R(\tilde{u}_i) \oplus \sum_{j=1}^{n} R(\tilde{d}_j)\otimes R(\tilde{v}_j)\right)$$

subject to

$$R(\tilde{u}_i)+R(\tilde{v}_j)\leq R(\tilde{c}_{ij});\, i=1,2,\ldots,m,\, j=1,2,\ldots,n,$$

$R(\tilde{u}_i)$ and $R(\tilde{v}_j)$ are unrestricted real numbers.

(P8)

Step 7: Using the property, $R(\tilde{A}_1\otimes\tilde{A}_2)=R(\tilde{A}_2)\times R(\tilde{A}_2)$, the problem (P8) can be transformed into the problem (P9).

$$\text{Maximize}\left(\sum_{i=1}^{m} R(\tilde{s}_i\otimes\tilde{u}_i) \oplus \sum_{j=1}^{n} R(\tilde{d}_j\otimes\tilde{v}_j)\right)$$

Subject to

(P9)

$$R(\tilde{u}_i)+R(\tilde{v}_j)\leq R(\tilde{c}_{ij});\, i=1,2,\ldots,m,\, j=1,2,\ldots,n,$$

$R(\tilde{u}_i)$ and $R(\tilde{v}_j)$ are unrestricted real numbers.

Step 8: Using the property, $R\left(\sum_{i=1}^{n}\tilde{A}_i\right)=\sum_{i=1}^{n}R(\tilde{A}_i)$, the problem (P9) can be transformed into problem (P10).

$$\text{Maximize}\left(\sum_{i=1}^{m} R(\tilde{s}_i\otimes\tilde{u}_i) \oplus \sum_{j=1}^{n} R(\tilde{d}_j\otimes\tilde{v}_j)\right)$$

Subject to

(P10)

$$R(\tilde{u}_i\oplus\tilde{v}_j)\leq R(\tilde{c}_{ij});\, i=1,2,\ldots,m,\, j=1,2,\ldots,n,$$

$\tilde{u}_i$ and $\tilde{v}_j$ are unrestricted TrFNs.

Step 9: Using the property, $\tilde{A}_1\approx\tilde{A}_2$ if $R(\tilde{A}_1)=R(\tilde{A}_2)$, and $\tilde{A}_1\succ\tilde{A}_2$ if $R(\tilde{A}_1)>R(\tilde{A}_2)$, the problem (P10) can be transformed into problem (P11) (or (P2)).

Maximize$\left(\sum_{i=1}^{m} \tilde{s}_i \otimes \tilde{u}_i \oplus \sum_{j=1}^{n} \tilde{d}_j \otimes \tilde{v}_j \right)$

Subject to

(P11)

$\tilde{u}_i + \tilde{v}_j \preceq \tilde{c}_{ij}\ ; i = 1,2,\ldots,m,\ j = 1,2,\ldots,n,$

$\tilde{u}_i$ and $\tilde{v}_j$ are unrestricted TrFNs.

A.3. Mathematically incorrect assumption considered in writing the fuzzy dual

It can be easily verified that for any two trapezoidal fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$ the property "$R(\tilde{A}_1 \otimes \tilde{A}_2) = R(\tilde{A}_2) \times R(\tilde{A}_2)$" is not necessarily satisfied if multiplication of $\tilde{A}_1$ and $\tilde{A}_2$ is defined as $\tilde{A}_1 \otimes \tilde{A}_2 = (a, b, c, d)$

where,

$$a = \min\,(a_1a_2, a_1d_2, d_1a_2, d_1d_2),\ b = \min\,(b_1b_2, b_1c_2, c_1b_2, c_1c_2),$$
$$c = \max\,(b_1b_2, b_1c_2, c_1b_2, c_1c_2),\ d = \max\,(a_1a_2, a_1d_2, d_1a_2, d_1d_2).$$

However, Dhanasekar et al. [37], have used the property "$R(\tilde{A}_1 \otimes \tilde{A}_2) = R(\tilde{A}_1) \times R(\tilde{A}_2)$" at the following places for obtaining the dual problem (P2) corresponding to problem (P1).

(i) In Step 3 of the method, discussed in Section A.2, Dhanasekar et al. [37] have used the property "$R(\tilde{A}_1 \otimes \tilde{A}_2) = R(\tilde{A}_1) \times R(\tilde{A}_2)$" for transforming the problem (P4) into problem (P5).

(ii) In Step 7 of the method, discussed in Section A.2, Dhanasekar et al. [37] have used the property"$R(\tilde{A}_1 \otimes \tilde{A}_2) = R(\tilde{A}_1) \times R(\tilde{A}_2)$" for transforming the problem (P8) into problem (P9).

A.4. Validity of the existing results

It can be easily verified that if $\tilde{A}_1$ and $\tilde{A}_2$ are two TrFNs and the multiplication of $\tilde{A}_1$ and $\tilde{A}_2$ is defined as $\tilde{A}_1 \otimes \tilde{A}_2 = (a, b, c, d)$,

where,

$$a = \frac{a_1(a_2 + b_2 + c_2 + d_2)}{4},\ b = \frac{b_1(a_2 + b_2 + c_2 + d_2)}{4},$$
$$c = \frac{c_1(a_2 + b_2 + c_2 + d_2)}{4},\ d = \frac{d_1(a_2 + b_2 + c_2 + d_2)}{4}.$$

then, the property "$R(\tilde{A}_1 \otimes \tilde{A}_2) = R(\tilde{A}_1) \times R\left(\tilde{A}_2\right)$" will always be satisfied. Hence, the exiting fuzzy Hungarian MODI algorithm [37] is valid only for this special type of multiplication [113] of TrFNs.

A.5. Conclusion

On the basis of presented study, it can be concludeed that the existing fuzzy Hungarian MODI algorithm [37] is valid only for a special type of multiplication of TrFNs [113].

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