Some Best Proximity Point Problems and their Applications

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by

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Declaration

I, Shagun Sharma hereby declare that the work presented in this thesis entitled "Some Best Proximity Point Problems and their Applications" in fulfillment of the requirement for the award of degree of Doctor of Philosophy submitted at Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala is an authentic record of work carried out under supervision of Dr. Sumit Chandok, Associate Professor at Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala, India. The matter presented in this thesis has not been submitted either in part or full to any other university or institute for the award of any other degree.

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Certificate

It is certified that the work contained in the thesis titled "Some Best Proximity Point Problems and their Applications" by Ms. Shagun Sharma [902011009] has been carried out under my supervision and that this work has not been submitted elsewhere for any other degree.

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List of Publications

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4. Shagun Sharma and Sumit Chandok. Split fixed point problems for quasi-nonexpansive mappings in Hilbert spaces, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 86:109-118, 2024.

5. Shagun Sharma and Sumit Chandok. Strong convergence of best proximity points via projection operators with application, J. Anal., 32:2767-2788, 2024.

6. Shagun Sharma and Sumit Chandok. Strong convergence to a solution of nonlinear equations, Filomat, 38:8663-8675, 2024.

7. Shagun Sharma and Sumit Chandok. Existence of m-tuple best proximity point in metric spaces with applications, Appl. Math. E-Notes, 24: 296-316, 2024.

8. S. Sharma and S. Chandok. Global optimal solution for a system of differential equations via measure of noncompactness. Bol. Soc. Parana. Mat., 43:1-10, 2025.

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Abstract

Approximation theory is a subject with a long history and a huge importance in classical and contemporary research. Over the years, the theory has become so extensive that it intersects with every other branch of analysis. One of the problems in approximation theory is to detect a point that minimizes the distance between two subsets $\mathscr{E}_1, \mathscr{E}_2$ of a metric space $(\mathscr{W}, d)$. Once this is done, it motivates us to study the solution of minimization problem. For example,

$$\min_{\acute{k}\in\mathscr{E}_1} d(\acute{k}, \mathscr{B}\acute{k}),\ \min_{\acute{m}\in\mathscr{E}_2} d(\acute{m}, \mathscr{B}\acute{m}) \text{ and } \min_{(\acute{k},\acute{m})\in\mathscr{E}_1\times\mathscr{E}_2} d(\acute{k}, \acute{m}),$$

where $\mathscr{B}$ is a mapping on $\mathscr{E}_1 \cup \mathscr{E}_2$, such that $\mathscr{B}(\mathscr{E}_1) \subseteq \mathscr{E}_1$ and $\mathscr{B}(\mathscr{E}_2) \subseteq \mathscr{E}_1$. It is fascinating to arise a question whether is possible to find a pair $(\acute{k}, \acute{m}) \in \mathscr{E}_1 \times \mathscr{E}_2$ which is a solution of above problem that is, to find a pair $(\acute{k}, \acute{m}) \in \mathscr{E}_1 \times \mathscr{E}_2$ such that $\acute{k} = \mathscr{B}\acute{k}, \acute{m} = \mathscr{B}\acute{m}$ and $d(\acute{k}, \acute{m}) = d(\mathscr{E}_1, \mathscr{E}_2)$. If such a pair exists, it is called the best proximity pair for a mapping $\mathscr{B}$. If we take $\mathscr{B}$ a non-self mapping we find an approximate solution $\acute{k}$ such that the error $d(\acute{k}, \mathscr{B}\acute{k})$ is minimum. The existence of an approximate solution $\acute{k}$, called best proximity point, that is, to find $\acute{k} \in \mathscr{E}_1$ such that

$$d(\acute{k}, \mathscr{B}\acute{k}) = d(\mathscr{E}_1, \mathscr{E}_2) = \inf\left\{d(\acute{k}, \acute{m}) : \acute{k} \in \mathscr{E}_1, \acute{m} \in \mathscr{E}_2\right\}.$$

Approximation theory can be used to solve many kinds of problem such as systems of nonlinear matrices, integral and differential equations, fractals, split feasibility problems, and variational inequalities. The study of approximation theory is appropriately inspired by the fact that particular instances of approximation frequently arise from problems connected with science and technology.

The first aim of this project is construct algorithms for the existence and uniqueness of a best proximity point. Another aim of this project is to discuss some applications of best proximity points.

In the first chapter, we provide the supplementary material such as some definitions, preliminary results that are useful for upcoming chapters. It also includes the literature survey, thesis goals, as well as a synopsis of the information included in each of the thesis's chapters.

In the second chapter, we discuss the existence of best proximity points for non-self mappings satisfying some contractive conditions in the setting of metric spaces, relational metric spaces, quasi partial metric spaces, normed and binormed linear spaces. Furthermore, we discuss the existence of best proximity pair results using noncyclic contraction mapping at the end of this chapter.

Third chapter concerns with iterative schemes. In this chapter, we propose some algorithms that converge to a best proximity point and fixed point. Also, we introduce some algorithms which converge to solution of split fixed and split best proximity point problem.

The fourth chapter deals with applications of best proximity point problems. In this chapter, we present the solution for variational inequality problems in the framework of Hilbert spaces. We provide a solution for a system of differential equations in the context of metric spaces. We also solve the model that spreads a virus using a non-linear integral equation.

Notations and Abbreviations

$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	the set of natural numbers including zero
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}_+$	the set of positive real numbers
MS	Metric space
BS	Banach space
NE	Nonexpansive mapping
BPP	Best proximity point
FP	Fixed point
UCBS	uniformly convex Banach space
NBCC	non-empty, bounded, closed and convex subsets
$\mho$	measure of noncompactness
$\mathscr{W}$	non-empty set
$\\| . \\|$	norm
$\mathscr{E}_1, \mathscr{E}_2$	subsets of $\mathscr{W}$
$\mathscr{K}(\mathscr{W})$	family of bounded subsets of $\mathscr{W}$
$C(\mathscr{W})$	family of non-empty compact subsets of $\mathscr{W}$
$[a, b]$	a closed interval
$\mathtt{C}([a, b], \mathscr{W})$	the set of all real-valued and continuous functions from $[a, b]$ to $\mathscr{W}$
$\|\|\mathscr{E}_1 - \mathscr{E}_2\|\|$	$\inf \left\{ \|\|\acute{k}_1 - \acute{m}_1\|\| : \acute{k}_1 \in \mathscr{E}_1, \acute{m}_1 \in \mathscr{E}_2 \right\}$
$\mathscr{E}_{1_0}$	$\left\{ \acute{k}_1 \in \mathscr{E}_1 : \text{there exists some } \acute{m}_1 \in \mathscr{E}_2 \text{ such that } \|\|\acute{k}_1 - \acute{m}_1\|\| = \|\|\mathscr{E}_1 - \mathscr{E}_2\|\| \right\}$
$\mathscr{E}_{2_0}$	$\left\{ \acute{m}_1 \in \mathscr{E}_2 : \text{there exists some } \acute{k}_1 \in \mathscr{E}_1 \text{ such that } \|\|\acute{k}_1 - \acute{m}_1\|\| = \|\|\mathscr{E}_1 - \mathscr{E}_2\|\| \right\}$
$\|\|\acute{l} - \mathscr{E}_1\|\|$	$\inf \left\{ \|\|\acute{l} - \acute{k}\|\| : \acute{k} \in \mathscr{E}_1 \text{ and } \acute{l} \in \mathscr{W} \right\}$

$\mathscr{P}_{\mathscr{E}_1}(\acute{l})$	$\left\{\acute{k} \in \mathscr{E}_1 : \|\acute{k} - \acute{l}\| = \|\acute{l} - \mathscr{E}_1\| \text{ and } \acute{l} \in \mathscr{W}\right\}$
$\delta(\acute{l}, \mathscr{E}_1)$	$\sup\left\{\|\acute{l} - \acute{k}\| : \acute{k} \in \mathscr{E}_1\right\}; \acute{l} \in \mathscr{W}$
$\delta(\mathscr{E}_1, \mathscr{E}_2)$	$\sup\left\{\|\acute{k} - \acute{m}\| : \acute{k} \in \mathscr{E}_1 \text{ and } \acute{m} \in \mathscr{E}_2\right\}$
$\mathscr{F}(\mathscr{B})$	$\left\{\acute{k} \in \mathscr{E}_1 : \acute{k} = \mathscr{B}\acute{k} \text{ if } \mathscr{B} \text{ is a self map on } \mathscr{E}_1\right\}$
$Best_{\mathscr{E}_1}\mathscr{B}$	$\left\{\acute{k} \in \mathscr{E}_1 : \|\acute{k} - \mathscr{B}\acute{k}\| = \|\mathscr{E}_1 - \mathscr{E}_2\| \text{if } \mathscr{B} \text{ is a non-self map from } \mathscr{E}_1 \text{ to } \mathscr{E}_2\right\}$
$\overline{con}(\mathscr{E}_1)$	closed and convex hull of a set $\mathscr{E}_1$

Contents

Chapter 1

Introduction

The topic of approximation theory has a long history and is crucial to both traditional and modern research. The idea has expanded over time to the point where it touches on all other areas of study. Detect a point that minimizes the distance to a specific subset or point is one of the key problems in approximation theory, that is, given a subset $\mathscr{T}_1$ in a MS $(\mathscr{W}, d)$ and any point $\hat{o} \in \mathscr{W}$, determine a point of $\mathscr{T}_1$ that's close to $\hat{o}$ from all the points of $\mathscr{T}_1$. Approximation theory can be used to solve many kinds of problem such as systems of nonlinear matrices, integral and differential equations, fractals, split feasibility problems, and variational inequalities. The research area of approximation theory is appropriately inspired by the fact that particular instances of approximation frequently arise from problems connected with science and technology.

Fan [24] gave the classical best approximation theorem in a Hausdorff locally convex topological vector space using the compact convex set and continuous mapping. Using the approximately compact, convex sets which are weaker than compact sets, Reich [56] modified the Fan's Theorem and demonstrated the best approximation result. Prolla [54] gave the best approximation result for a pair of continuous mappings and compact convex subset in normed spaces. Singh and Watson [73] obtained a best approximation result for NEs using closed and convex subset in Hilbert spaces. Following that, numerous authors investigated the best approximation problems in a normed and MS (see [7, 55, 62, 79] and the listed references therein).

Minimizing the distance between two subsets in approximation theory is also attractive. Once this is done, it motivates us to study the solution of minimization problem. For example, we may study the particular minimization problem:

$$\min_{\hat{o}\in\mathscr{T}_1} d(\hat{o},\mathscr{B}\hat{o}),\ \min_{\hat{g}\in\mathscr{T}_2} d(\hat{g},\mathscr{B}\hat{g}) \text{ and } \min_{(\hat{o},\hat{g})\in\mathscr{T}_1\times\mathscr{T}_2} d(\hat{o},\hat{g}), \tag{1.1}$$

where $\mathscr{B}$ is a mapping on $\mathscr{T}_1\cup\mathscr{T}_2$ and $\mathscr{T}_1$, $\mathscr{T}_2$ are subsets of MS $(\mathscr{W},d)$. It is fascinating to arise a question whether is possible to find a pair $(\hat{o},\hat{g})\in\mathscr{T}_1\times\mathscr{T}_2$ which is a solution of (1.1) namely, to locate a pair $(\hat{o},\hat{g})\in\mathscr{T}_1\times\mathscr{T}_2$ such that $\hat{o}=\mathscr{B}\hat{o},\hat{g}=\mathscr{B}\hat{g}$ and $d(\hat{o},\hat{g})=d(\mathscr{T}_1,\mathscr{T}_2)$. If such a pair exists for a mapping $\mathscr{B}$, it is known as best proximity pair.
Several researchers have been actively working in this direction and have demonstrated the results for existence and convergence of minimization problem (1.1) solutions.

Eldred et al. [21] demonstrated in 2005 that there is a solution of (1.1) for relatively NEs in BSs that have a proximal normal structure. Some results on the existence of a solution to the minimization problem (1.1) were provided by Abkar and Gabeleh [1] in a MS. Espinola and Gabeleh [23] demonstrated that (1.1) has a solution for noncyclic relatively NEs using the structure of minimal sets. Gabeleh and Fernandes-Leon [25] proved the best proximity pair results using different types of noncyclic contraction mappings. Gabeleh [28] demonstrated that there is a solution of (1.1) for pointwise noncyclic mappings that have no proximal normal structure. Using the weak proximal normal structure, Digar et al. [18] proved that (1.1) has a solution for noncyclic mappings.

Non-self mapping research is extremely intriguing as, in this instance, we discover a point $\hat{o}$ such that $d(\hat{o},\mathscr{B}\hat{o})$ is minimum. If such a point exists, it is called BPP, namely, to locate a point $\hat{o}\in\mathscr{T}_1$ which satisfies

$$d(\hat{o},\mathscr{B}\hat{o})=d(\mathscr{T}_1,\mathscr{T}_2)=\inf\{d(\hat{o},\hat{g}):\hat{o}\in\mathscr{T}_1,\hat{g}\in\mathscr{T}_2\},$$

where $\mathscr{T}_1$, $\mathscr{T}_2$ are non-empty disjoint subsets of MS $(\mathscr{W},d)$ and $\mathscr{B}:\mathscr{T}_1\to\mathscr{T}_2$ is a non-self mapping. It puts up a question on the existence of such a point because such point may or may not exist. If there is $\hat{o}\in\mathscr{T}_1$ such that $d(\hat{o},\mathscr{B}\hat{o})=d(\mathscr{T}_1,\mathscr{T}_2)$, then we have a BPP in $\mathscr{T}_1$. It also raises the problem of uniqueness, since more than one

BPP may be possible.

Using the proximal normal structure, Eldred et al. [21] provided some BPP results in 2005 for relatively NEs in UCBS. Later, using the cyclic contraction, Eldred and Veeramani [22] obtained some BPP results in a MS. Bari, Suzuki and Vetro [6] obtained some BPP results using the cyclic Meir-Keeler contraction which were more general than cyclic contraction. A new class of cyclic ψ contraction mappings which contains cyclic contraction mappings, was introduced by Al-Thagafi and Shahzad [3] and provided some BPP results. Using non-self weakly contractive mappings and ₤-property Sankar Raj [55] gave some results on existence and uniqueness of BPPs. Karapınar [39] originated the idea of generalized cyclic contraction and obtained some BPP results in UCBS. Using proximal contraction, Gabeleh [26] provided some BPP results. Choudhury, Maity and Metiya [14] obtained some results on the existence of BPP using $\alpha - \psi$ proximal contraction.

Sintunavarat and Kumam [75] gave the idea of coupled BPP and property UC* in 2012 and provided some results for the existence of coupled BPPs. The idea of tripled BPP was originated by Cho et al. [13] in 2013, and demonstrated some results for the existence of tripled BPP using cyclic contraction mappings. The idea of quadruple BPP was originated by Hammad et al. [33] and shown that quadruple BPP exists and converges in a MS.

Approximating the FPs of contraction type mappings using different iterative methods have been developed by many researchers in nonlinear analysis, see Mann [48], Ishikawa [35], Agarwal et al. [2], Sahu [58], Kang et al. [38], Gopi and Pragadeeswarar [32], Khan [42], Ritika and Khan [57], Khan et al. [43] and so on.

In 2020, Dass et al. [17] gave the succeeding algorithm, to approximate common FPs of two self mappings in uniformly smooth BSs.

For arbitrary $\hat{o}_0 \in \mathscr{T}_1$. Define

$$\begin{cases} \hat{g}_n = & (1-\gamma_n')\hat{o}_n + \gamma_n'\mathscr{B}\hat{o}_n \\ \hat{i}_n = & (1-\delta_n')\hat{o}_n + \delta_n'\mathscr{B}_1\hat{g}_n \\ \hat{o}_{n+1} = & \mathscr{B}\hat{i}_n \end{cases} \qquad \text{Algorithm (D)}$$

where $\gamma'_n, \delta'_n \in (0,1]$, $n \in \mathbb{N}_0$.

Okeke and Ofem [50] proposed the following algorithm to approximate common FPs of two self mappings in BSs. For arbitrary $\hat{o}_0 \in \mathscr{T}_1$,

$$\begin{cases} \hat{g}_n = & (1-\gamma'_n)\hat{o}_n + \gamma'_n \mathscr{B}\hat{o}_n \\ \hat{i}_n = & (1-\eta'_n-\delta'_n)\hat{o}_n + \eta'_n \mathscr{B}_1 \hat{g}_n + \delta'_n \mathscr{B}_1 \hat{o}_n \\ \hat{o}_{n+1} = & \mathscr{B}\hat{i}_n \end{cases} \qquad \text{Algorithm (O)}$$

where $\gamma'_n, \eta'_n, \delta'_n \in (0,1]$, $n \in \mathbb{N}_0$.

Censor and Elfving [11] were the first to propose the split feasibility problem (SFP) in Euclidean spaces which is mathematically formulated as:

$$\text{find a point } \hat{o} \in \mathscr{T}_1 \text{ such that } \mathscr{L}\hat{o} \in \mathscr{T}_2, \tag{1.2}$$

where $\mathscr{T}_1$, $\mathscr{T}_2$ are non-empty convex and closed subsets of the Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$ respectively and $\mathscr{L} : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator.

The split FP problem for two operators was first investigated by Moudafi [77] which is mathematically formulated as:

$$\text{find a point } \hat{o} \in \mathscr{F}(\mathscr{B}) \text{ such that } \mathscr{L}\hat{o} \in \mathscr{F}(\mathscr{B}_1),$$

where $\mathscr{H}_1, \mathscr{H}_2$ are Hilbert spaces, $\mathscr{B} : \mathscr{H}_1 \to \mathscr{H}_1$, $\mathscr{B}_1 : \mathscr{H}_2 \to \mathscr{H}_2$ are mappings, $\mathscr{L} : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator and $\mathscr{F}(\mathscr{B}), \mathscr{F}(\mathscr{B}_1)$ denote the set of FP of the mappings $\mathscr{B}, \mathscr{B}_1$ respectively.

In 2020, Dadashi and Postolache [16] gave a iterative scheme for the FP problem using a maximal monotone operator, NEs and established that sequence converges strongly to FPs of a NE. In 2022, Yao et al. [83] studied the particular split equilibrium problem, for all $\hat{o} \in \mathscr{T}_1$

$$\{\hat{o} \in \mathscr{T}_1 : \hat{o} \in \acute{f}_1(\hat{o},\hat{o}) \geq 0 \ \cap \mathscr{F}(\mathscr{B}),\ \mathscr{L}\hat{o} \in \acute{g}_1(\hat{o},\hat{o}) \geq 0 \ \cap \mathscr{F}(\mathscr{B}_1)\},$$

where $\mathscr{H}_1, \mathscr{H}_2$ are Hilbert spaces, $\mathscr{B} : \mathscr{H}_1 \to \mathscr{H}_1$, $\mathscr{B}_1 : \mathscr{H}_2 \to \mathscr{H}_2$ are mappings, $\mathscr{L} : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator, $\acute{f}_1, \acute{g}_1$ are bifunctions and $\mathscr{F}(\mathscr{B}), \mathscr{F}(\mathscr{B}_1)$ denote the set of FP of the mappings $\mathscr{B}, \mathscr{B}_1$ respectively. They established that iterative scheme converges to solution of split equilibrium problems using bifunctions. Yao et al. [84] gave an iterative scheme for solving a split FP problem and demonstrate its weak convergence.

In 2021, Suantai and Tiammee [77] investigated the split BPP problem for two operators as:

$$\text{find a point } \hat{o} \in Best_{\mathscr{T}_1}\mathscr{B} \text{ such that } \mathscr{L}\hat{o} \in Best_{\mathscr{T}_2}\mathscr{B}_1, \tag{1.3}$$

where $\mathscr{T}_1$, $\mathscr{F}_1$, $\mathscr{T}_2$ and $\mathscr{F}_2$ are non-empty convex and closed subsets of the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ repectively, $\mathscr{B} : \mathscr{T}_1 \to \mathscr{F}_1$, $\mathscr{B}_1 : \mathscr{T}_2 \to \mathscr{F}_2$ are mappings, $\mathscr{L} : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator and $Best_{\mathscr{T}_1}\mathscr{B}, Best_{\mathscr{T}_2}\mathscr{B}_1$ denote the set of BPP of the mappings $\mathscr{B}, \mathscr{B}_1$ respectively.

In 2024, Husain et al. [34] gave an algorithm that converges to a solution of split BPP problem in the context of real Hilbert spaces.

The primary goal of this project is to construct algorithms for the existence and uniqueness of a BPP and introduce some algorithms which converges to BPP and FP. Also we introduce some algorithms which converge to solution of split fixed and split BPP problem. The discussion of some BPP applications is another goal of this research.

1.1 Preliminaries

Throughout the thesis $\mathscr{T}_1, \mathscr{T}_2$ are subsets of $\mathscr{W}$, solution sets for split FP problem and SBPP problem are denoted by

$$\mathscr{S} = \{\hat{o} \in \mathscr{F}(\mathscr{B}) : \mathscr{L}\hat{o} \in \mathscr{F}(\mathscr{B}_1)\} \text{ and } \mathscr{S}^* = \{\hat{o} \in Best_{\mathscr{T}_1}\mathscr{B} : \mathscr{L}\hat{o} \in Best_{\mathscr{T}_2}\mathscr{B}_1\}$$

respectively.

The basic definitions that will be utilized in the next chapters are covered in this section.

Definition 1.1.1. [15] Let $\mathscr{W}$ be a BS and $\hat{o}, \hat{g} \in \mathscr{W}$ such that $||\hat{o}|| = ||\hat{g}|| = 1$. Then $\mathscr{W}$ is a

(i) UCBS [15] if for each $0 \leq \epsilon \leq 2$ there is some $\delta \geq 0$ such that $||\hat{o} - \hat{g}|| \geq \epsilon$ implies $\left|\left|\frac{\hat{o}+\hat{g}}{2}\right|\right| \leq 1 - \delta$,

(ii) strictly convex BS [15] if $\hat{o} \neq \hat{g}$ implies $\left|\left|\frac{\hat{o}+\hat{g}}{2}\right|\right| < 1$;

Definition 1.1.2. [74] Suppose that ϱ is a mapping from MS $(\mathscr{W}, d)$ to $[0, \infty)$. Then ϱ said to be transitive if

$$\varrho(\hat{o}_1, \hat{o}_2) \geq 1, \ \varrho(\hat{o}_2, \hat{o}_3) \geq 1 \ \text{ implies } \ \varrho(\hat{o}_1, \hat{o}_3) \geq 1,$$

for all $\hat{o}_1, \hat{o}_2, \hat{o}_3 \in \mathscr{W}$.

Definition 1.1.3. [9] A function $\mathscr{L}$, which is defined on Hilbert space $\mathscr{H}$ is called a monotone if

$$\langle \mathscr{L}\hat{o} - \mathscr{L}\hat{g}, \hat{o} - \hat{g} \rangle \geq 0, \text{ for all } \hat{o}, \hat{g} \in \mathscr{H}.$$

Let $\mathscr{W}$ be a BS and $\mathscr{W}^*$ be a dual space of $\mathscr{W}$. Then $J : \mathscr{W} \to 2^{\mathscr{W}^*}$ is normalized duality mapping if

$$J(\hat{o}) = \left\{ e^* \in \mathscr{W}^* : ||e^*|| = ||\hat{o}||, \langle \hat{o}, e^* \rangle = ||\hat{o}||^2 \right\}, \text{ for all } \hat{o} \in \mathscr{W}, \tag{1.4}$$

Definition 1.1.4. Let $\mathscr{T}_1 \subseteq \mathscr{W}$ and $\mathscr{W}$ be a BS. A mapping $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_1$ is a
(i) quasi NE [20] if $\mathscr{F}(\mathscr{B}) \neq \emptyset$ and $||\mathscr{B}\hat{o} - \hat{\imath}|| \leq ||\hat{o} - \hat{\imath}||$ for all $\hat{o} \in \mathscr{W}, \hat{\imath} \in \mathscr{F}(\mathscr{B})$,
(ii) NE [73] if $||\mathscr{B}\hat{o} - \mathscr{B}\hat{\imath}|| \leq ||\hat{o} - \hat{\imath}||$,
(iii) pseudocontractive [41] if there is $j(\hat{o} - \hat{\imath}) \in J(\hat{o} - \hat{\imath})$ and

$$\left\langle \mathscr{B}\hat{o} - \mathscr{B}\hat{\imath}, j(\hat{o} - \hat{\imath}) \right\rangle \leq ||\hat{o} - \hat{\imath}||^2,$$

(iv) strongly pseudocontractive [44] if there is $j(\hat{o}-\hat{i}) \in J(\hat{o}-\hat{i})$ and

$$\left\langle \mathscr{B}\hat{o} - \mathscr{B}\hat{i}, j(\hat{o}-\hat{i}) \right\rangle \leq k||\hat{o}-\hat{i}||^2;\ k \in (0,1),$$

for all $\hat{o}, \hat{i} \in \mathscr{T}_1$.

Remark 1.1.1. [20] If $\mathscr{B} : \mathscr{W} \to \mathscr{W}$ is a continuous quasi NE, then $\mathscr{F}(\mathscr{B})$ is convex and closed.

Definition 1.1.5. Let $\mathscr{B}$ be a mapping defined on the MS $(\mathscr{W}, d)$. Then $\mathscr{B}$ is an

(i) $\alpha - \lambda$ contraction [45] if there are two functions $\alpha : \mathscr{W} \times \mathscr{W} \to [0, \infty)$ and $\lambda : \mathscr{W} \to [0, 1)$ such that $\lambda(\mathscr{B}(\hat{o}_1)) \leq \lambda(\hat{o}_1)$, $\limsup \lambda(\hat{o}_1) < 1$ for all $\hat{o}_1 \in \mathscr{W}$ and

$$\alpha(\hat{o}_1, \hat{o}_2)d(\mathscr{B}(\hat{o}_1), \mathscr{B}(\hat{o}_2)) \leq \lambda(\hat{o}_1)d(\hat{o}_1, \hat{o}_2),$$

(ii) $\alpha - \beta$ contraction [74] if there are two functions $\alpha : \mathscr{W} \times \mathscr{W} \to [0, \infty)$ and $\beta \in S$ such that

$$[\alpha(\hat{o}_1, \hat{o}_2) - 1 + \delta_*]^{d(\mathscr{B}\hat{o}_1, \mathscr{B}\hat{o}_2)} \leq \delta^{\beta(d(\hat{o}_1, \hat{o}_2))d(\hat{o}_1, \hat{o}_2)},$$

for all $\hat{o}_1, \hat{o}_2 \in \mathscr{W}$, $\delta_* > 1$, $1 < \delta \leq \delta_*$ and S denote the class of all functions $\beta : [0, \infty) \to [0, 1)$ such that $\lim\limits_{n\to\infty} \beta(\acute{t}_n) = 1$ implies that $\lim\limits_{n\to\infty} \acute{t}_n = 0$, where $\{\acute{t}_n\} \in \mathbb{R}_+ \cup \{0\}$.

Definition 1.1.6. [52] Suppose that $\mathscr{B} : \mathscr{T}_1^m \to \mathscr{T}_1$ is a mapping. A point $(\hat{o}_1, \hat{o}_2, \hat{o}_3, \ldots, \hat{o}_m) \in \mathscr{T}_1^m$ is a m-tuple FP of $\mathscr{B}$ if

$$\hat{o}_1 = d(\hat{o}_1, \mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m), \ldots, \hat{o}_m = d(\hat{o}_m, \mathscr{B}(\hat{o}_m, \hat{o}_1, \ldots, \hat{o}_{m-1})). \tag{1.5}$$

Definition 1.1.7. [21] A pair $(\mathscr{T}_1, \mathscr{T}_2)$ in norm space $\mathscr{W}$ is said to be a proximal pair if for each $(\hat{o}_1, \hat{g}_1) \in \mathscr{T}_1 \times \mathscr{T}_2$ there exists $(\hat{o}_2, \hat{g}_2) \in \mathscr{T}_1 \times \mathscr{T}_2$ such that

$$||\hat{o}_1 - \hat{g}_2|| = ||\hat{o}_2 - \hat{g}_1|| = d(\mathscr{T}_1, \mathscr{T}_2).$$

Definition 1.1.8. [21] In norm space $\mathscr{W}$, a convex pair $(\mathscr{T}_1, \mathscr{T}_2)$ is said to possess proximal normal structure whenever every closed, bounded, and convex proximal pair

$(\mathscr{I}, \mathscr{J}) \subseteq (\mathscr{T}_1, \mathscr{T}_2)$ for which $d(\mathscr{I}, \mathscr{J}) = d(\mathscr{T}_1, \mathscr{T}_2)$ and $d(\mathscr{I}, \mathscr{J}) < \delta(\mathscr{I}, \mathscr{J})$, there exists $(\hat{o}_1, \hat{g}_1) \in \mathscr{I} \times \mathscr{J}$ such that $\delta(\hat{o}_1, \mathscr{J}) < \delta(\mathscr{I}, \mathscr{J})$, $\delta(\hat{g}_1, \mathscr{I}) < \delta(\mathscr{I}, \mathscr{J})$.

Definition 1.1.9. [30] In a MS, a pair $(\mathscr{T}_1, \mathscr{T}_2)$ is considered proximinal if $\mathscr{T}_1 = \mathscr{T}_{1_0}$ and $\mathscr{T}_2 = \mathscr{T}_{2_0}$

Definition 1.1.10. Let $(\mathscr{W}, d)$ be a MS and $\mathscr{B}$ be a mapping defined on $\mathscr{T}_1 \cup \mathscr{T}_2$ with $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_2$, $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_1$. Then mapping $\mathscr{B}$ is a
(i) relatively NE [21] if $d(\mathscr{B}\hat{o}, \mathscr{B}\hat{g}) \leq d(\hat{o}, \hat{g})$,
(ii) cyclic contraction [22] if

$$d(\mathscr{B}\hat{o}, \mathscr{B}\hat{g}) \leq kd(\hat{o}, \hat{g}) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2),$$

for all $\hat{o} \in \mathscr{T}_1$, $\hat{g} \in \mathscr{T}_2$ and $k \in (0, 1)$.

Definition 1.1.11. ([77, 78]) Let $\mathscr{I}_1 \subseteq \mathscr{T}_1 \subseteq \mathscr{H}$ and $\mathscr{H}$ be a Hilbert space. A mapping $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is $\mathscr{I}_1$-NE if

$$||\mathscr{B}\hat{o} - \mathscr{B}\hat{k}|| \leq ||\hat{o} - \hat{k}|| \text{ for all } \hat{o} \in \mathscr{T}_1 \text{ and } \hat{k} \in \mathscr{I}_1.$$

If $\mathscr{I}_1 = Best_{\mathscr{T}_1}\mathscr{B}$, then $\mathscr{B}$ is a best proximally NE.

Remark 1.1.2. [78] If $\mathscr{B}$ is non-self NE, then $\mathscr{B}$ is $\mathscr{I}_1$-NE for every $\mathscr{I}_1 \subseteq \mathscr{T}_1$, and if $\mathscr{I}_1 = \mathscr{F}(\mathscr{B}) \neq \emptyset$, then every $\mathscr{I}_1$-NE is quasi-NE.

Definition 1.1.12. [63] Let $\mathscr{W}$ be a normed space. Then mapping $\mathscr{B}$ defined on $\mathscr{T}_1$ satisfy Condition (C) if there is a increasing function g on $[0, \infty)$ with $g(0) = 0, g(s) > 0$ for all $s \in (0, \infty)$, and $||\hat{o} - \mathscr{B}\hat{o}|| \geq g(||\hat{o} - \mathscr{F}(\mathscr{B})||)$ for all $\hat{o} \in \mathscr{T}_1$ where $||\hat{o} - \mathscr{F}(\mathscr{B})|| = \inf\{||\hat{o} - \acute{v}|| : \acute{v} \in \mathscr{F}(\mathscr{B})\}$.

Definition 1.1.13. [55]) A pair $(\mathscr{T}_1, \mathscr{T}_2)$ in MS $(\mathscr{W}, d)$ has £-property if and only if $d(\hat{o}_1, \hat{g}_1) = d(\hat{o}_2, \hat{g}_2) = d(\mathscr{T}_1, \mathscr{T}_2)$ then $d(\hat{o}_1, \hat{o}_2) = d(\hat{g}_1, \hat{g}_2)$, for all $\hat{o}_1, \hat{o}_2 \in \mathscr{T}_1$ and $\hat{g}_1, \hat{g}_2 \in \mathscr{T}_2$.

Remark 1.1.3. [55] In UCBS, every non-empty, convex and closed pair has the £-property.

Definition 1.1.14. [60] A pair $(\mathscr{T}_1, \mathscr{T}_2)$ in MS $(\mathscr{W}, d)$ has weak ₤-property if and only if $d(\hat{o}_1, \hat{g}_1) = d(\hat{o}_2, \hat{g}_2) = d(\mathscr{T}_1, \mathscr{T}_2)$ then $d(\hat{o}_1, \hat{o}_2) \leq d(\hat{g}_1, \hat{g}_2)$, for all $\hat{o}_1, \hat{o}_2 \in \mathscr{T}_1$ and $\hat{g}_1, \hat{g}_2 \in \mathscr{T}_2$.

Definition 1.1.15. [60] Let $(\mathscr{W}, d)$ be a MS and mapping $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ be modified ϱ proximal admissible if there is a mapping $\varrho : \mathscr{T}_2 \times \mathscr{T}_2 \to [0, \infty)$ with $\varrho(\mathscr{B}\hat{o}_0, \mathscr{B}\hat{o}_1) \geq 1$

$$
\begin{aligned}
& d(\hat{o}_1, \mathscr{B}\hat{o}_0) = d(\mathscr{T}_1, \mathscr{T}_2), \\
& d(\hat{o}_2, \mathscr{B}\hat{o}_1) = d(\mathscr{T}_1, \mathscr{T}_2), \quad \text{then} \\
& \varrho(\mathscr{B}\hat{o}_1, \mathscr{B}\hat{o}_2) \geq 1,
\end{aligned}
$$

for all $\hat{o}_0, \hat{o}_1, \hat{o}_1, \hat{o}_2 \in \mathscr{T}_1$.

Definition 1.1.16. [37] A mapping $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is said to be α proximal admissible if there is a mapping $\alpha : \mathscr{T}_1 \times \mathscr{T}_1 \to [0, \infty)$ with $\alpha(\hat{o}_0, \hat{o}_1) \geq 1$,

$$
\begin{aligned}
d(\hat{o}_1, \mathscr{B}\hat{o}_0) =& d(\mathscr{T}_1, \mathscr{T}_2), \\
d(\hat{o}_2, \mathscr{B}\hat{o}_1) =& d(\mathscr{T}_1, \mathscr{T}_2), \quad \text{then} \\
\alpha(\hat{o}_1, \hat{o}_2) \geq& 1,
\end{aligned}
$$

for all $\hat{o}_0, \hat{o}_1, \hat{o}_2 \in \mathscr{T}_1$.

Definition 1.1.17. A pair $(\mathscr{T}_1, \mathscr{T}_2)$ in MS $(\mathscr{W}, d)$ satisfies the property
(i) UC [79] if there exist $\{\hat{o}_n\}$, $\left\{\hat{k}_n\right\} \subset \mathscr{T}_1$ and $\{\hat{g}_n\} \subset \mathscr{T}_2$ such that $d(\hat{o}_n, \hat{g}_n) \to d(\mathscr{T}_1, \mathscr{T}_2)$ and $d(\hat{k}_n, \hat{g}_n) \to d(\mathscr{T}_1, \mathscr{T}_2)$, then $d(\hat{o}_n, \hat{k}_n) \to 0$ as $n \to \infty$,
(ii) UC* [13] if $(\mathscr{T}_1, \mathscr{T}_2)$ has the property UC and for each $\epsilon > 0$, there is $N_0 \in \mathbb{N}$ and $d(\hat{o}_m, \hat{g}_n) \leq d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon$ for all $m > n \geq N_0$. It implies $N_1 \in \mathbb{N}$ exists and $d(\hat{o}_m, \hat{k}_n) \leq \epsilon$ for all $m > n \geq N_1$.

Definition 1.1.18. [51] Assume that $\mathscr{W}$ is BS and $\mathscr{T}_1 \subseteq \mathscr{H}$. Then $\mathscr{T}_1$ satisfy Opial's condition if $\{\hat{o}_n\}$ converges to $p \in \mathscr{T}_1$ weakly and $\limsup_{n\to\infty} ||\hat{o}_n - p|| < \limsup_{n\to\infty} ||\hat{o}_n - l||$ for all $l \in \mathscr{T}_1$ with $p \neq l$.

Definition 1.1.19. [29] Let $\mathscr{W}$ be a MS and $\mathscr{B}$ be a mapping on $\mathscr{T}_1 \cup \mathscr{T}_2$ with $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_2$ and $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_1$ (respectively, $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_1$ and $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_2$). A mapping $\mathscr{B}$ is compact if $\mathscr{B}|_{\mathscr{T}_1}$ and $\mathscr{B}|_{\mathscr{T}_2}$ are compact.

Definition 1.1.20. [5] The measure of noncompactness (MNC) is a function $\mho : \mathscr{K}(\mathscr{W}) \to \mathbb{R}^+$ which met the requirements listed below:

(i') $\mathscr{T}_1$ is a relatively compact if and only if $\mho(\mathscr{T}_1) = 0$,

(i") $\mho(\mathscr{T}_1) = \mho(\overline{\mathscr{T}_1})$,

(i"') $\mho(\mathscr{T}_1 \cup \mathscr{T}_2) = \max\{\mho(\mathscr{T}_1), \mho(\mathscr{T}_2)\}$.

If $\mho$ is an MNC on $\mathscr{K}(\mathscr{W})$, then the the assertions listed below are true:

(a') $\mho(\mathscr{T}_1) = 0$ if and only if $\mathscr{T}_1$ is a finite set,

(b') $\mho(\mathscr{T}_1 \cap \mathscr{T}_2) = \min\{\mho(\mathscr{T}_1), \mho(\mathscr{T}_2)\}$,

(c') If $\lim_{n\to\infty}(\mathscr{E}_{1_n}) = 0$ for a non-increasing sequence $\{\mathscr{E}_{1_n}\}$ of NBCC subsets of $\mathscr{W}$ then $\mathscr{E}_{1_\infty} = \cap_{n\geq 1}\mathscr{E}_{1_n}$ is compact and non-empty,

(d') $\mathscr{T}_1 \subseteq \mathscr{T}_2$ implies $\mho(\mathscr{T}_1) \leq \mho(\mathscr{T}_2)$,

Also, the following axioms hold if $\mathscr{W}$ is a BS

(e') $\mho(\mathscr{T}_1) = \mho(\overline{con}\mathscr{T}_1)$,

(f') $\mho(t\mathscr{T}_1) = |t|\mho(\mathscr{T}_1)$, for any number t and $\mathscr{T}_1 \in \mathscr{K}(\mathscr{W})$,

(g') $\mho(\mathscr{T}_1 + \mathscr{T}_2) \leq \mho(\mathscr{T}_1) + \mho(\mathscr{T}_2)$.

In order to demonstrate our main theorems in the subsequent chapters, we require the following outcomes.

Proposition 1.1.1. [12] Suppose that $\mathscr{W}$ is a UCBS and for any $r > 0$, if $\hat{o}, \hat{z} \in \mathscr{W}$ with $||\hat{o}|| \leq r$, $||\hat{z}|| \leq r$, $||\hat{o} - \hat{z}|| \geq \epsilon$ then there is a $\delta^* = \delta^*(\frac{\epsilon}{r}) > 0$ such that $||(1-\eta')\hat{z} + \eta'\hat{o}|| \leq \Big(1 - 2\delta^*(\frac{\epsilon}{r})\min(\eta, 1-\eta)\Big)r$; where $\eta' \in (0,1)$ and $\epsilon > 0$.

Lemma 1.1.1. [78] Suppose that $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is a best proximally NE such that $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$. Then $\mathscr{P}_{\mathscr{T}_1}\mathscr{B}|_{\mathscr{T}_{1_0}}$ is a quasi-NE.

Lemma 1.1.2. [80] Let $\mathscr{T}_1 \subseteq \mathscr{H}$ which is non-empty convex and closed and $\mathscr{H}$ be Hilbert space. Then, for any $\hat{o}, \hat{g} \in \mathscr{H}$, the following assertions hold:

(i) $\left\langle \hat{o} - \mathscr{P}_{\mathscr{T}_2}\hat{o}, \hat{k} - \mathscr{P}_{\mathscr{T}_2}\hat{o} \right\rangle \geq 0$ for all $\hat{k} \in \mathscr{T}_1$;

(ii) $||\mathscr{P}_{\mathscr{T}_2}\hat{o} - \mathscr{P}_{\mathscr{T}_2}\hat{g}||^2 \leq \langle \mathscr{P}_{\mathscr{T}_2}\hat{o} - \mathscr{P}_{\mathscr{T}_2}\hat{g}, \hat{o} - \hat{g}\rangle$ for all $\hat{o}, \hat{g} \in \mathscr{H}$;

(iii) $||\mathscr{P}_{\mathscr{T}_2}\hat{o} - \hat{k}||^2 \leq ||\hat{o} - \hat{k}||^2 - ||\mathscr{P}_{\mathscr{T}_2}\hat{o} - \hat{o}||^2$ for all $\hat{k} \in \mathscr{T}_1$.

Lemma 1.1.3. [49] Let $\mathscr{B}$ be a quasi NE, and set $\{\mathscr{B}_{\eta'} := (1-\eta')I + \eta'\mathscr{B}\}$ for $\eta' \in (0, 1]$. Then, the assertions listed below are true:

(i) $\langle \hat{o} - \mathscr{B}\hat{o}, \hat{o} - \acute{q}\rangle \geq \frac{1}{2}||\hat{o} - \mathscr{B}\hat{o}||^2$ and $\langle \hat{o} - \mathscr{B}\hat{o}, \acute{q} - \mathscr{B}\hat{o}\rangle \leq \frac{1}{2}||\hat{o} - \mathscr{B}\hat{o}||^2$;

(ii) $||\mathscr{B}_{\eta'}\hat{o} - q||^2 \leq ||\hat{o} - \acute{q}||^2 - \eta'(1 - \eta')||\mathscr{B}\hat{o} - \hat{o}||^2$;

(iii) $\langle \hat{o} - \mathscr{B}_{\eta'}\hat{o}, \hat{o} - \acute{q}\rangle \geq \frac{\eta'}{2}||\hat{o} - \mathscr{B}\hat{o}||^2$,

for all $(\hat{o}, \acute{q}) \in \mathscr{H} \times \mathscr{F}(\mathscr{B})$.

Lemma 1.1.4. [22] Let $\mathscr{T}_2, \mathscr{T}_1$ be closed subsets of a UCBS $\mathscr{W}$ such that $\mathscr{T}_1$ convex. Let $\{\hat{o}_n\}$ and $\left\{\hat{k}_n\right\}$ be sequences in $\mathscr{T}_1$ and $\{\hat{g}_n\}$ be a sequence in $\mathscr{T}_2$ satisfying:

(i) $||\hat{o}_n - \hat{g}_n|| \to d(\mathscr{T}_1, \mathscr{T}_2)$,

(ii) for given $\epsilon > 0$ there is N_0 such that $||\hat{k}_n - \hat{g}_n|| \leq d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon$, for all $m > n \geq N_0$.

It implies $N_1 \in \mathbb{N}$ exists such that $||\hat{o}_n - \hat{k}_n|| \leq \epsilon$, for all $m > n \geq N_1$.

Lemma 1.1.5. [22] Assume that assumption (i) of Lemma 1.1.4 is true and $||\hat{k}_n - \hat{g}_n|| \to d(\mathscr{T}_1, \mathscr{T}_2)$, then $||\hat{o}_n - \hat{k}_n||$ converges to zero.

Lemma 1.1.6. [38] Let $J : \mathscr{W} \to 2^{\mathscr{W}^*}$ be the normalized duality mapping. Then for all $\hat{o}, \hat{\imath} \in \mathscr{W}$,

$$||\hat{o} + \hat{\imath}||^2 \leq ||\hat{o}||^2 + 2\left\langle \hat{\imath}, j(\hat{o} + \hat{\imath})\right\rangle, \forall j(\hat{o} + \hat{\imath}) \in J(\hat{o} + \hat{\imath}).$$

Lemma 1.1.7. [82] Suppose $\{\hat{o}_n\}$ and $\left\{\hat{i}_n\right\}$ are non-negative sequence such that

$$\hat{o}_{n+1} \leq (1-\eta')\hat{i}_n + \acute{c}_n,$$

where $\eta'_n \in (0,1)$, $\sum_{n=0}^{\infty} \eta' = \infty$ and $\lim\limits_{n\to\infty} \frac{\acute{c}_n}{\eta'_n} = 0$, then $\lim\limits_{n\to\infty} \hat{i}_n = 0$.

Lemma 1.1.8. [36] Let $\mathscr{W}$ be a BS, and $\mathscr{B} : [a,b] \to \mathscr{W}$ be a differentiable mapping. Let $\hat{o}, \hat{g} \in [a,b]$ with $\hat{o} < \hat{g}$. Then

$$\mathscr{B}(\hat{g}) - \mathscr{B}(\hat{o}) \in (\hat{g} - \hat{o})\overline{con}(\{\mathscr{B}(t) : t \in [\hat{o}, \hat{g}]\}).$$

Lemma 1.1.9. [82] Suppose $\{k_n\}$ and $\{l_n\}$ are non-negative sequence in BS $\mathscr{W}$ such that

$$k_{n+1} \leq (1-\eta')l_n + \acute{c}_n,$$

where $\eta'_n \in (0,1)$, $\sum_{n=0}^{\infty} \eta' = \infty$ and $\lim\limits_{n\to\infty} \frac{\acute{c}_n}{\eta'_n} = 0$, then $\lim\limits_{n\to\infty} l_n = 0$.

Lemma 1.1.10. [19] Suppose that $\mathscr{W}$ is a UCBS and $\left\{\acute{t}_n\right\}$ is a sequence in $[a,b]$ where $0 < a < b < 1$. Further, assume that $\{\hat{o}_n\}$, $\left\{\hat{i}_n\right\}$ are sequences in $\mathscr{W}$ such that $||\hat{o}_n|| \leq 1$, $||\hat{i}_n|| \leq 1$ for all $n \in \mathbb{N}$. Define $\{\acute{z}_n\}$ in $\mathscr{W}$ by $\acute{z}_n = (1-\acute{t}_n)\hat{o}_n + \acute{t}_n\hat{i}_n$. If $\lim\limits_{n\to\infty} ||\acute{z}_n|| = 1$, then $\lim\limits_{n\to\infty} ||\hat{o}_n - \hat{i}_n|| = 0$.

Lemma 1.1.11. [61] Suppose that $\mathscr{W}$ is a UCBS and $\left\{\acute{t}_n\right\}$ is a sequence in $[a,b]$ where $0 < a < b < 1$. Further, assume that $\{\hat{o}_n\}$, $\left\{\hat{i}_n\right\}$ are sequences in $\mathscr{W}$ such that $||\hat{o}_n|| \leq r$, $||\hat{i}_n|| \leq r$ for all $n \in \mathbb{N}, r \geq 0$. Define $\{\acute{z}_n\}$ in $\mathscr{W}$ by $\acute{z}_n = (1-\acute{t}_n)\hat{o}_n + \acute{t}_n\hat{i}_n$. If $\lim\limits_{n\to\infty} ||\acute{z}_n|| = r$, then $\lim\limits_{n\to\infty} ||\hat{o}_n - \hat{i}_n|| = 0$.

Theorem 1.1.1. [27] A relatively NE cyclic mapping $\mathscr{B}$ defined on $\mathscr{T}_1 \cup \mathscr{T}_2$ has a BPP if $(\mathscr{T}_1, \mathscr{T}_2)$ is a NBCC pair in a strictly convex BS $\mathscr{W}$, $\mathscr{T}_{1_0}$ is non-empty and $\mathscr{B}$ is compact.

1.2 Objectives of thesis

The main objectives of this research are

(I) To construct algorithms for the existence and uniqueness of a best proximity point.

(II) Study the real world application(s) of a best proximity point problems.

1.3 Summary of the thesis

The thesis has been divided into four chapters. The following is a chapter-wise summary of the thesis:

Chapter 1: Introduction

In this chapter, we provide the supplementary material such as some definitions, preliminary results that are useful for upcoming chapters. It also includes the literature survey, thesis goals, as well as a synopsis of the information included in each of the thesis's chapters.

Chapter 2: Existence results

In this chapter, we study some existence results on BPP and best proximity pair. There are two sections in this chapter. We provide some results on the existence of BPP in the context of a MS, relational MS, binormed linear space and quasi partial MS in the first section. The results of this section are published in Sharma and Chandok [64, 65, 66, 67]. We examine some findings on the existence of best proximity pairs in the context of a strictly convex BS in the last section. The findings of this part are accepted in Sharma and Chandok [71].

Chapter 3: Iterative convergence to best proximity points

In the optimization and approximation theory the convergence of iterative processes for BPPs has been an attractive problem in nonlinear analysis. Iterative convergence to BPPs is a crucial concept to find a point that minimizes the distance between two sets. As discussed in Chapter 1, there are many iterative schemes which converge to BPP of mapping. This chapter deals with iterative schemes. The structure of this chapter is: In the first section, we define a iterative scheme which converges to common

FPs of NEs and strongly pseudocontractive mappings. The findings from this part are published in Sharma and Chandok [70]. In the second section, we define some iterative algorithms that strongly converge to BPP using the class of NEs in the context of a UCBS. The outcomes of this part are published in Sharma and Chandok [69, 72]. In the third section, we propose a iterative scheme that converges to a solution of split common FP problem. This section's findings published in Sharma and Chandok [68]. In the last section, we define another algorithm using projection operator which converges to a solution of SBPP problem in context of Hilbert spaces. The findings of this part are published in Sharma and Chandok [68].

Chapter 4: Applications

Chapter 4 deals with applications of BPP problems. The structure of this chapter is: In the first section, we present the solution for variational inequality problems. This section's findings published in Sharma and Chandok [65]. In the second section, we provide a solution for a differential equations in the framework of a MS. This section's results accepted in Sharma and Chandok [71]. In the last section, we solve the model that spreads a virus with a cyclical variable periodic contract rate using a non-linear integral equation. The findings of this part are published in Sharma and Chandok [72].

Chapter 2

Existence results

The current chapter contains the study of some results on BPP and best proximity pair. The chapter has two sections. In the first section, we give some BPP results in the framework of a MS, relational MS, binormed linear spaces and quasi partial MS. The results of this section are published in Sharma and Chandok [64, 65, 66, 67]. We examine some findings on the existence of best proximity pairs in the scope of a strictly convex BS in the last section. The findings of this section are accepted in Sharma and Chandok [71].

2.1 Existence of best proximity points

This section split into five subsections. In the first subsection, we study some BPP results using new contraction mapping in the framework of a MS. The outcomes from this part are published in Sharma and Chandok [64, 67]. In the second subsection, we establish some BPP results in a relational MS. The findings from this subsection are published in Sharma and Chandok [64]. In the third subsection, we prove some BPP results in BSs using measure of noncompactness. The results of this section are accepted in Sharma and Chandok [71]. In the fourth subsection, we prove some BPP results in uniformly convex binormed linear spaces. The results of this section are published in Sharma and Chandok [65]. In the last subsection, we present some results on BPP in a quasi partial MS using $\alpha - \beta$ mapping. The outcomes of this part are published in Sharma and Chandok [66].

2.1.1 Best proximity points in a MS

In this subsection, first we define $\varrho-\vartheta$ contraction mapping. We find some BPP results for $\varrho-\vartheta$ contraction mappings with the help of modified ϱ admissible. Also, we give some illustrations that support our findings.

In the entire section $\mathscr{T}_1, \mathscr{T}_2$ are assumed non-empty closed subsets of a complete MS $(\mathscr{W}, d)$ and $\mathscr{T}_{1_0}$ is non-empty.

Let us first introduce the following definition:

Definition 2.1.1. A mapping $\mathscr{B}: \mathscr{T}_1 \to \mathscr{T}_2$ is $\varrho-\vartheta$ contraction mapping if there exist two functions $\vartheta: \mathscr{W} \to [0,1)$ and $\varrho: \mathscr{T}_2 \times \mathscr{T}_2 \to [0,\infty)$ for which $\vartheta(\mathscr{B}(\hat{o}_1)) \leq \vartheta(\hat{o}_1)$, $\limsup \vartheta(\hat{o}_1) < 1$ for all $\hat{o}_1 \in \mathscr{W}$ and

$$\varrho(\mathscr{B}(\hat{o}_1), \mathscr{B}(\hat{o}_2))d(\mathscr{B}(\hat{o}_1), \mathscr{B}(\hat{o}_2)) \leq \vartheta(\mathscr{B}\hat{o}_1)d(\hat{o}_1, \hat{o}_2),$$

for all $\hat{o}_1, \hat{o}_2 \in \mathscr{T}_1$.

Theorem 2.1.1. Assume that $\varrho: \mathscr{T}_2 \times \mathscr{T}_2 \to [0,\infty)$ is a transitive and $\mathscr{B}: \mathscr{T}_1 \to \mathscr{T}_2$ is a continuous mapping such that $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$ and satisfies the assumptions of Definitions 1.1.13, 1.1.15 and 2.1.1. Additionally, if there exist $\hat{o}_0, \hat{o}_1 \in \mathscr{T}_{1_0}$ and $\mathscr{B}\hat{o}_0 \in \mathscr{T}_{2_0}$ such that $d(\hat{o}_1, \mathscr{B}\hat{o}_0) = d(\mathscr{T}_1, \mathscr{T}_2)$ and $\varrho(\mathscr{B}\hat{o}_0, \mathscr{B}\hat{o}_1) \geq 1$. Then mapping $\mathscr{B}$ has a BPP.

Proof. By assumption, there exists $\hat{o}_0, \hat{o}_1 \in \mathscr{T}_{1_0} \subseteq \mathscr{T}_1$ and $\mathscr{B}\hat{o}_0 \in \mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0} \subseteq \mathscr{T}_2$ such that $d(\hat{o}_1, \mathscr{B}\hat{o}_0) = d(\mathscr{T}_1, \mathscr{T}_2)$ and $\varrho(\mathscr{B}\hat{o}_0, \mathscr{B}\hat{o}_1) \geq 1$.

Since $\hat{o}_1 \in \mathscr{T}_{1_0}$ then $\mathscr{B}\hat{o}_1 \in \mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$. By notation of $\mathscr{T}_{2_0}$ there exists $\hat{o}_2 \in \mathscr{T}_{1_0}$ such that $d(\hat{o}_2, \mathscr{B}\hat{o}_1) = d(\mathscr{T}_1, \mathscr{T}_2)$. Since $\mathscr{B}$ is modified ϱ proximal admissible and $\varrho(\mathscr{B}\hat{o}_0, \mathscr{B}\hat{o}_1) \geq 1$, we get

$$\begin{aligned} d(\hat{o}_1, \mathscr{B}\hat{o}_0) &= d(\mathscr{T}_1, \mathscr{T}_2), \\ d(\hat{o}_2, \mathscr{B}\hat{o}_1) &= d(\mathscr{T}_1, \mathscr{T}_2), \quad \text{then} \\ \varrho(\mathscr{B}\hat{o}_1, \mathscr{B}\hat{o}_2) &\geq 1. \end{aligned}$$

Again $\hat{o}_2 \in \mathscr{T}_{1_0}$ then $\mathscr{B}\hat{o}_2 \in \mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$. By notation of $\mathscr{T}_{2_0}$ there exists $\hat{o}_3 \in \mathscr{T}_{1_0}$ such that $d(\hat{o}_3, \mathscr{B}\hat{o}_2) = d(\mathscr{T}_1, \mathscr{T}_2)$. Since $\mathscr{B}$ is modified ϱ proximal admissible and $\varrho(\mathscr{B}\hat{o}_1, \mathscr{B}\hat{o}_2) \geq 1$, we have

$$\begin{aligned} d(\hat{o}_2, \mathscr{B}\hat{o}_1) =& d(\mathscr{T}_1, \mathscr{T}_2), \\ d(\hat{o}_3, \mathscr{B}\hat{o}_2) =& d(\mathscr{T}_1, \mathscr{T}_2), \quad \text{then} \\ \varrho(\mathscr{B}\hat{o}_2, \mathscr{B}\hat{o}_3) \geq& 1. \end{aligned}$$

Continuing in this fashion, we get

$$\begin{cases} d(\hat{o}_{n+1}, \mathscr{B}\hat{o}_n) = d(\mathscr{T}_1, \mathscr{T}_2), \\ d(\hat{o}_n, \mathscr{B}\hat{o}_{n-1}) = d(\mathscr{T}_1, \mathscr{T}_2), \\ \varrho(\mathscr{B}\hat{o}_n, \mathscr{B}\hat{o}_{n+1}) \geq 1. \end{cases} \tag{2.1}$$

As pair $(\mathscr{T}_1, \mathscr{T}_2)$ satisfies £- property, we have

$$d(\hat{o}_n, \hat{o}_{n+1}) = d(\mathscr{B}\hat{o}_{n-1}, \mathscr{B}\hat{o}_n). \tag{2.2}$$

For $n \in \mathbb{N}$, we have

$$\begin{aligned} d(\hat{o}_n, \hat{o}_{n+1}) = d(\mathscr{B}(\hat{o}_{n-1}), \mathscr{B}(\hat{o}_n)) &\leq \varrho(\mathscr{B}\hat{o}_{n-1}, \mathscr{B}\hat{o}_n)) d(\mathscr{B}\hat{o}_{n-1}, \mathscr{B}\hat{o}_n) \\ &\leq \vartheta(\mathscr{B}\hat{o}_{n-1}) d(\hat{o}_{n-1}, \hat{o}_n) \\ &\leq \vartheta(\hat{o}_{n-1}) d(\hat{o}_{n-1}, \hat{o}_n) \\ &< d(\hat{o}_{n-1}, \hat{o}_n), \end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$. Therefore, $\{d(\hat{o}_{n-1}, \hat{o}_n)\}$ is strictly non-increasing and

$$\lim_{n\to\infty} d(\hat{o}_{n-1}, \hat{o}_n) = r_1,$$

for some $r_1 \geq 0$. Assume that $r_1 > 0$. Taking $n \to \infty$ in (2.3), we get

$$\lim_{n\to\infty} \frac{d(\hat{o}_n, \hat{o}_{n+1})}{d(\hat{o}_{n-1}, \hat{o}_n)} \leq \vartheta(\hat{o}_{n-1}),$$
$$1 \leq \vartheta(\hat{o}_{n-1}),$$

which is a contradiction. So, $r_1 = 0$. By (2.1), (2.2) and triangle inequality, we obtain

$$\begin{aligned} d(\hat{o}_m, \hat{o}_n) &\leq d(\hat{o}_m, \hat{o}_{m+1}) + d(\hat{o}_{m+1}, \hat{o}_{n+1}) + d(\hat{o}_{n+1}, \hat{o}_n) \\ &= d(\hat{o}_m, \hat{o}_{m+1}) + d(\mathscr{B}\hat{o}_m, \mathscr{B}\hat{o}_n) + d(\hat{o}_{n+1}, \hat{o}_n) \\ &\leq \varrho(\mathscr{B}\hat{o}_m, \mathscr{B}\hat{o}_n) d(\hat{o}_m, \hat{o}_{m+1}) + \varrho(\mathscr{B}\hat{o}_m, \mathscr{B}\hat{o}_n) d(\mathscr{B}\hat{o}_m, \mathscr{B}\hat{o}_n) + \\ &\quad \varrho(\mathscr{B}\hat{o}_m, \mathscr{B}\hat{o}_n) d(\hat{o}_{n+1}, \hat{o}_n) \\ &\leq \varrho(\mathscr{B}\hat{o}_m, \mathscr{B}\hat{o}_n)[d(\hat{o}_m, \hat{o}_{m+1}) + d(\hat{o}_{n+1}, \hat{o}_n)] + \vartheta(\mathscr{B}\hat{o}_m) d(\hat{o}_m, \hat{o}_n) \\ &\leq \varrho(\mathscr{B}\hat{o}_m, \mathscr{B}\hat{o}_n)[d(\hat{o}_m, \hat{o}_{m+1}) + d(\hat{o}_{n+1}, \hat{o}_n)] + \vartheta(\hat{o}_m) d(\hat{o}_m, \hat{o}_n), \end{aligned}$$

for all $m, n \in \mathbb{N}$ and $m > n$. It implies that

$$(1 - \vartheta(\hat{o}_m)) d(\hat{o}_m, \hat{o}_n) \leq \varrho(\mathscr{B}\hat{o}_m, \mathscr{B}\hat{o}_n)[d(\hat{o}_m, \hat{o}_{m+1}) + d(\hat{o}_{n+1}, \hat{o}_n)]. \tag{2.4}$$

On taking $m, n \to \infty$ in (2.4), we get

$$\lim_{m,n\to\infty} d(\hat{o}_m, \hat{o}_n) = 0.$$

Therefore, $\{\hat{o}_n\}$ is a Cauchy sequence in $\mathscr{T}_1$. Given that $\mathscr{T}_1$ is closed, we get $\lim_{n\to\infty} \hat{o}_n = \hat{o}$, where $\hat{o} \in \mathscr{T}_1$. By continuity of $\mathscr{B}$, we obtain $\lim_{n\to\infty} \mathscr{B}(\hat{o}_n) = \mathscr{B}(\hat{o})$. By (2.1), we get

$$d(\hat{o}, \mathscr{B}\hat{o}) = \lim_{n\to\infty} d(\hat{o}_{n+1}, \mathscr{B}(\hat{o}_n)) = d(\mathscr{T}_1, \mathscr{T}_2).$$

Hence $\hat{o}$ is a BPP of $\mathscr{B}$. □

If we relax the continuity of mapping $\mathscr{B}$ then we need the following condition in Theorem 2.1.1.

($\Re$-property) Let $\{\hat{o}_n\}$ be a sequence in $\mathscr{W}$ such that $\varrho(\mathscr{B}\hat{o}_n, \mathscr{B}\hat{o}_{n+1}) \geq 1$ for all n and $\lim_{n\to\infty} \hat{o}_n = \hat{o} \in \mathscr{W}$. It follows that subsequence $\{\hat{o}_{n_{\hat{s}}}\}$ of $\{\hat{o}_n\}$ exists such that $\varrho(\mathscr{B}\hat{o}_{n_{\hat{s}}}, \mathscr{B}\hat{o}) \geq 1$, for all $\hat{s}$.

Theorem 2.1.2. Suppose that $\varrho : \mathscr{T}_2 \times \mathscr{T}_2 \to [0, \infty)$ is a transitive and $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is a mapping such that $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$ and satisfying the Definitions 1.1.13, 1.1.15 and 2.1.1. Further, suppose that ($\Re$-property) holds and there exist $\hat{o}_0, \hat{o}_1 \in \mathscr{T}_{1_0}$ and $\mathscr{B}\hat{o}_0 \in \mathscr{T}_{2_0}$ such that $d(\hat{o}_1, \mathscr{B}\hat{o}_0) = d(\mathscr{T}_1, \mathscr{T}_2)$ and $\varrho(\mathscr{B}\hat{o}_0, \mathscr{B}\hat{o}_1) \geq 1$. Then mapping $\mathscr{B}$ has a BPP.

Proof. By Theorem 2.1.1, we obtain that the sequence $\{\hat{o}_n\}$ is Cauchy sequence, which converges to $\hat{o} \in \mathscr{W}$. By assumption there is a subsequence $\{\hat{o}_{n_{\hat{s}}}\}$ of $\{\hat{o}_n\}$ which satisfies $\varrho(\mathscr{B}\hat{o}_{n_{\hat{s}}}, \mathscr{B}\hat{o}) \geq 1$, for all $\hat{s}$.

Now, we show that $\mathscr{B}$ has a BPP. Using (2.1) and triangle inequality, we obtain

$$\begin{aligned} d(\hat{o}_{n_{\hat{s}+1}}, \mathscr{B}\hat{o}) &\leq d(\hat{o}_{n_{\hat{s}+1}}, \mathscr{B}\hat{o}_{n_{\hat{s}}}) + d(\mathscr{B}\hat{o}_{n_{\hat{s}}}, \mathscr{B}\hat{o}), \\ &\leq d(\mathscr{T}_1, \mathscr{T}_2) + \varrho(\mathscr{B}\hat{o}_{n_{\hat{s}}}, \mathscr{B}\hat{o}) d(\mathscr{B}\hat{o}_{n_{\hat{s}}}, \mathscr{B}\hat{o}) \\ &\leq d(\mathscr{T}_1, \mathscr{T}_2) + \vartheta(\mathscr{B}\hat{o}_{n_{\hat{s}}}) d(\hat{o}_{n_{\hat{s}}}, \hat{o}) \\ &\leq d(\mathscr{T}_1, \mathscr{T}_2) + \vartheta(\hat{o}_{n_{\hat{s}}}) d(\hat{o}_{n_{\hat{s}}}, \hat{o}). \end{aligned} \tag{2.5}$$

By (2.5) and triangle inequality, we obtain

$$\begin{aligned} d(\mathscr{T}_1, \mathscr{T}_2) \leq d(\hat{o}, \mathscr{B}\hat{o}) &\leq d(\hat{o}, \hat{o}_{n_{\hat{s}+1}}) + d(\hat{o}_{n_{\hat{s}+1}}, \mathscr{B}\hat{o}) \\ &\leq d(\hat{o}, \hat{o}_{n_{\hat{s}+1}}) + d(\mathscr{T}_1, \mathscr{T}_2) + \vartheta(\hat{o}_{n_{\hat{s}}}) d(\hat{o}_{n_{\hat{s}}}, \hat{o}). \end{aligned} \tag{2.6}$$

Taking $\hat{s} \to \infty$ in (2.6), we have

$$d(\mathscr{T}_1, \mathscr{T}_2) = d(\hat{o}, \mathscr{B}\hat{o}).$$

It shows that $\mathscr{B}$ has a BPP. □

Remark 2.1.1. If $\varrho(\mathscr{B}\hat{o}, \mathscr{B}\hat{o}_1) \geq 1$ for all BPPs $\hat{o}, \hat{o}_1$ of $\mathscr{B}$ in Theorem 2.1.1 and Theorem 2.1.2, then we get unique BPP.

Proof. Assume that $\hat{o}, \hat{o}_1$ are two BPPs of $\mathscr{B}$ such that $\hat{o} \neq \hat{o}_1$. Since $d(\hat{o}, \mathscr{B}\hat{o}) = d(\mathscr{T}_1, \mathscr{T}_2), d(\hat{o}_1, \mathscr{B}\hat{o}_1) = d(\mathscr{T}_1, \mathscr{T}_2)$ by £-property and the condition $\varrho(\hat{o}, \hat{o}_1) \geq 1$ we have

$$\begin{aligned} d(\hat{o}, \hat{o}_1) =& d(\mathscr{B}\hat{o}, \mathscr{B}\hat{o}_1) \\ \leq& \varrho(\hat{o}, \hat{o}_1) d(\mathscr{B}\hat{o}, \mathscr{B}\hat{o}_1) \\ \leq& \vartheta(\hat{o}) d(\hat{o}, \hat{o}_1) < d(\hat{o}, \hat{o}_1), \end{aligned}$$

which is contradiction. Hence $d(\hat{o}, \hat{o}_1) = 0$. This shows $\mathscr{B}$ has a unique BPP. □

If we take $\mathscr{T}_1 = \mathscr{T}_2 = \mathscr{W}$ in Theorem 2.1.1 and Theorem 2.1.2, then $\varrho - \vartheta$ reduce to $\alpha - \lambda$ contraction and we get a few more outcomes:

Corollary 2.1.1. (Theorem 10, p-3, [45]) Let $\alpha : \mathscr{W} \times \mathscr{W} \to [0, \infty)$ be transitive, $\mathscr{B}$ be a continuous mapping on $\mathscr{W}$ which fulfill the assumptions of Definitions 1.1.5 (ii) and 1.1.16. Let us assume that $\hat{o}_0 \in \mathscr{T}_1$ exists in such a way that $\varrho(\hat{o}_0, \mathscr{B}\hat{o}_0) \geq 1$, then mapping $\mathscr{B}$ has a FP.

Corollary 2.1.2. (Theorem 12, p-4, [45]) Let $\alpha : \mathscr{W} \times \mathscr{W} \to [0, \infty)$ be a transitive and $\mathscr{B}$ be a mapping on $\mathscr{W}$ that satisfies the ($\Re$-property), Definitions 1.1.5(ii) and 1.1.16. Let us assume that $\hat{o}_0 \in \mathscr{T}_1$ exists in such a way that $\varrho(\hat{o}_0, \mathscr{B}\hat{o}_0) \geq 1$, then mapping $\mathscr{B}$ has a FP.

To demonstrate our findings, we now provide a few examples.

Example 2.1.1. *Suppose that $\mathscr{W} = \mathbb{R}^2$ induced with a metric*

$$d(\hat{o}^*, \hat{g}^*) = \sqrt{(\hat{o}_1 - \hat{g}_1)^2 + (\hat{o}_2 - \hat{g}_2)^2},$$

for all $\hat{o}^* = (\hat{o}_1, \hat{o}_2), \hat{g}^* = (\hat{g}_1, \hat{g}_2) \in \mathbb{R}^2$. Suppose $\mathscr{T}_1 = \left\{(\frac{1}{2}, \hat{o}_1) : 0 \leq \hat{o}_1 \leq 1\right\}$ and $\mathscr{T}_2 = \{(0, \hat{o}_1) : 0 \leq \hat{o}_1 \leq 1\}$. Also $d(\mathscr{T}_1, \mathscr{T}_2) = \frac{1}{2}$. Define a continuous $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ by

$$\mathscr{B}(\hat{o}_1) = \begin{cases} (0, 1), \hat{o}_1 = (\frac{1}{2}, 1), \\ (0, \frac{\acute{a}}{4}) : 0 \leq \acute{a} \leq \hat{o}_1, \ \textit{otherwise} \end{cases}$$

for all $\hat{o}_1 \in \mathscr{T}_1$. *Define* $\varrho : \mathscr{T}_2 \times \mathscr{T}_2 \to [0, \infty)$ *by*

$$\varrho((0, \hat{g}_1), (0, \hat{g}_2)) = \begin{cases} 1 \text{ if } \hat{o}_1, \hat{o}_2 \in [0, \frac{1}{4}] \\ 2 \text{ otherwise,} \end{cases}$$

then ϱ *is transitive. If* $\hat{i}_1 = (\frac{1}{2}, \hat{o}_1)$ *and* $\hat{i}_2 = (\frac{1}{2}, \hat{o}_2)$ *in* $\mathscr{T}_1$, *for* $\hat{o}_1, \hat{o}_2 \in [0, \frac{1}{2}]$. *Then*

$$\mathscr{B}\hat{i}_1 = \left\{(0, \frac{\acute{a}}{4}) : 0 \le \acute{a} \le \hat{o}_1\right\}, \quad \text{and}$$
$$\mathscr{B}\hat{i}_2 = \left\{(0, \frac{\acute{a}}{4}) : 0 \le \acute{a} \le \hat{o}_2\right\}.$$

Also $\varrho(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2) = 1$ *and* $d(\hat{o}_1, \mathscr{B}\hat{i}_1) = \frac{1}{2} = d(\mathscr{T}_1, \mathscr{T}_2)$ *and* $d(\hat{o}_2, \mathscr{B}\hat{i}_2) = \frac{1}{2} = d(\mathscr{T}_1, \mathscr{T}_2)$ *if and only if* $\hat{o}_1, \hat{o}_2 \in \left\{\left(\frac{1}{2}, \frac{\hat{k}}{4}\right); 0 \le \hat{k} \le \frac{1}{2}\right\}$. *Then* $\varrho(\mathscr{B}\hat{o}_1, \mathscr{B}\hat{o}_2) = 1$. *It implies* $\mathscr{B}$ *is a modified* ϱ *proximal admissible. Also* $\mathscr{T}_{1_0} = \mathscr{T}_1$, $\mathscr{T}_{2_0} = \mathscr{T}_2$, *so* $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$ *for all* $\hat{o}_1 \in \mathscr{T}_{1_0}$. *Let* $\vartheta(\acute{t}) = \frac{1}{3}$ *for all* $\acute{t} \in \mathscr{W}$. *Take* $\hat{i}_1 = (\frac{1}{2}, \hat{o}_1)$ *and* $\hat{i}_2 = (\frac{1}{2}, \hat{o}_2)$ *in* $\mathscr{T}_1$ *where* $0 \le \hat{o}_1, \hat{o}_2 \le \frac{1}{2}$. *Consider*

$$\begin{aligned} d(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2) =& \sqrt{(0-0)^2 + \left(\frac{\hat{o}_1}{4} - \frac{\hat{o}_2}{4}\right)^2} \\ =& \frac{1}{4}(\hat{o}_1 - \hat{o}_2) \le \frac{1}{3}(\hat{o}_1 - \hat{o}_2). \end{aligned}$$

It implies that, $\varrho(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2) d(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2) \le \vartheta(\hat{i}_1) d(\hat{i}_1, \hat{i}_2)$, *for all* $\hat{i}_1, \hat{i}_2 \in \mathscr{T}_1$. *Hence* $\mathscr{B}$ *is a* $\varrho - \vartheta$ *contraction. Therefore, all suppositions of Theorem 2.1.1 are hold and* $\mathscr{B}$ *has a BPP* $(\frac{1}{2}, 1)$.

Example 2.1.2. *Consider* $\mathscr{W} = \mathbb{R}^2$ *with a metric* $d(\hat{o}^*, \hat{g}^*) = \sqrt{(\hat{o}_1 - \hat{g}_1)^2 + (\hat{o}_2 - \hat{g}_2)^2}$, *for all* $\hat{o}^* = (\hat{o}_1, \hat{o}_2), \hat{g}^* = (\hat{g}_1, \hat{g}_2) \in \mathbb{R}^2$. *Suppose that*

$$\mathscr{T}_1 = \left\{(\hat{o}_1, \hat{o}_2) : \hat{o}_1^2 + \hat{o}_2^2 = 3 \text{ and } \hat{o}_2 \ge 0\right\}, \quad \text{and}$$
$$\mathscr{T}_2 = \left\{(\hat{g}_1, \hat{g}_2) : \hat{g}_1^2 + \hat{g}_2^2 = 1 \text{ and } \hat{g}_2 \ge 0\right\}.$$

Then $d(\mathscr{T}_1, \mathscr{T}_2) = 2$. *Define a continuous mapping* $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ *by* $\mathscr{B}(\hat{o}_1, \hat{o}_2) = \dfrac{(\hat{o}_1, \hat{o}_2)}{3}$, *for all* $(\hat{o}_1, \hat{o}_2) \in \mathscr{T}_1$. *Define* $\varrho : \mathscr{W} \times \mathscr{W} \to [0, \infty)$ *by* $\varrho((\hat{o}_1, \hat{o}_2), (\hat{g}_1, \hat{g}_2)) = 2$, *for all* $(\hat{o}_1, \hat{o}_2), (\hat{g}_1, \hat{g}_2) \in \mathscr{W}$. *Hence* ϱ *is transitive. If* $\hat{i}_1, \hat{i}_2 \in \mathscr{T}_1$, *then* $\varrho(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2) = 2 > 1, d(\hat{o}_1, \mathscr{B}\hat{i}_1) = 2 = d(\mathscr{T}_1, \mathscr{T}_2)$ *and* $d(\hat{o}_2, \mathscr{B}\hat{i}_2) = 2 = d(\mathscr{T}_1, \mathscr{T}_2)$ *if and only if* $\hat{o}_1, \hat{o}_2 \in \{(0,3), (3,0)\}$. *Therefore* $\varrho(\mathscr{B}\hat{o}_1, \mathscr{B}\hat{o}_2) = 2 > 1$. *It implies* $\mathscr{B}$ *is a modified* ϱ *proximal admissible. Also,* $\mathscr{T}_{1_0} = \mathscr{T}_1$, $\mathscr{T}_{2_0} = \mathscr{T}_2$, *so* $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$ *for all* $\hat{o}_1 \in \mathscr{T}_{1_0}$. *Let* $\vartheta(\acute{t}) = \dfrac{1}{2}$. *Take* $\hat{i}_1 = (\hat{o}_1, \hat{o}_2), \hat{i}_2 = (\hat{o}_3, \hat{o}_4)$ *in* $\mathscr{T}_1$. *Then*

$$\begin{aligned} d(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2) =& \sqrt{\left(\frac{\hat{o}_1}{3} - \frac{\hat{o}_3}{3}\right)^2 + \left(\frac{\hat{o}_2}{3} - \frac{\hat{o}_4}{3}\right)^2} \\ =& \frac{1}{3}\sqrt{(\hat{o}_1 - \hat{o}_3)^2 + (\hat{o}_2 - \hat{o}_4)^2} = \frac{1}{3} d(\hat{i}_1, \hat{i}_2). \end{aligned}$$

It implies that $\varrho(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2) d(\mathscr{B}\hat{i}_1, \mathscr{B}(\hat{i}_2) \leq \vartheta(\hat{i}_1) d(\hat{i}_1, \hat{i}_2)$, *for all* $\hat{i}_1, \hat{i}_2 \in \mathscr{W}$. *All the suppositions of Theorem 2.1.1 are met and* $\mathscr{B}$ *has a BPP* $(3, 0)$.

Next, we prove a very useful approximation result using cyclic contraction mappings as described below:

Proposition 2.1.1. Let $\mathscr{B}$ be a mapping on $\mathscr{T}_1 \cup \mathscr{T}_2$ that fulfilling the following hypotheses:

(*i*) $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_2$ and $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_1$,

(*ii*) there exist $\alpha', \beta', \gamma', \delta'$ are non-negative real numbers such that $\alpha' + \beta' + \gamma' + 2\delta' < 1$ and

$$\begin{aligned} d(\mathscr{B}\hat{o}, \mathscr{B}\hat{g}) \leq & \alpha' d(\hat{o}, \mathscr{B}\hat{o}) + \beta' d(\hat{g}, \mathscr{B}\hat{g}) + \gamma' d(\hat{o}, \hat{g}) + \delta' (d(\hat{o}, \mathscr{B}\hat{g}) + d(\hat{g}, \mathscr{B}\hat{o})) + \\ & (1 - \alpha' - \beta' - \gamma' - 2\delta') \, d(\mathscr{T}_1, \mathscr{T}_2), \end{aligned}$$

for all $\hat{o} \in \mathscr{T}_1$ and $\hat{g} \in \mathscr{T}_2$. If $\hat{o}_0 \in \mathscr{T}_1$ and $\hat{o}_{n+1} = \mathscr{B}\hat{o}_n$ where $n \in \mathbb{N}_0$, then $d(\hat{o}_n, \mathscr{B}\hat{o}_n) \to d(\mathscr{T}_1, \mathscr{T}_2)$.

Proof. Consider

$$d(\hat{o}_n, \hat{o}_{n+1}) = d(\mathscr{B}\hat{o}_{n-1}, \mathscr{B}\hat{o}_n)$$

$$\leq \alpha' d(\hat{o}_{n-1}, \mathscr{B}\hat{o}_{n-1}) + \beta' d(\hat{o}_n, \mathscr{B}\hat{o}_n) + \gamma' d(\hat{o}_{n-1}, \hat{o}_n) +$$
$$\delta' d(\hat{o}_{n-1}, \mathscr{B}\hat{o}_n) + \delta' d(\hat{o}_n, \mathscr{B}\hat{o}_{n-1}) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ d(\mathscr{T}_1, \mathscr{T}_2)$$
$$= \alpha' d(\hat{o}_{n-1}, \hat{o}_n) + \beta' d(\hat{o}_n, \hat{o}_{n+1}) + \gamma' d(\hat{o}_{n-1}, \hat{o}_n) + \delta' d(\hat{o}_{n-1}, \hat{o}_{n+1}) +$$
$$\delta' d(\hat{o}_n, \hat{o}_n) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ d(\mathscr{T}_1, \mathscr{T}_2)$$
$$\leq \alpha' d(\hat{o}_{n-1}, \hat{o}_n) + \beta' d(\hat{o}_n, \hat{o}_{n+1}) + \gamma' d(\hat{o}_{n-1}, \hat{o}_n) + \delta' d(\hat{o}_{n-1}, \hat{o}_n) +$$
$$\delta' d(\hat{o}_n, \hat{o}_{n+1}) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ d(\mathscr{T}_1, \mathscr{T}_2). \tag{2.7}$$

Rewriting (2.7), we have

$$\begin{aligned} d(\hat{o}_n, \hat{o}_{n+1}) &\leq \frac{\alpha' + \gamma' + \delta'}{(1 - \beta' - \delta')} d(\hat{o}_{n-1}, \hat{o}_n) + \frac{(1 - \alpha' - \beta' - \gamma' - 2\delta')}{(1 - \beta' - \delta')}\ d(\mathscr{T}_1, \mathscr{T}_2) \\ &\leq k d(\hat{o}_{n-1}, \hat{o}_n) + (1 - k)\ d(\mathscr{T}_1, \mathscr{T}_2)\ ; \quad k = \frac{\alpha' + \gamma' + \delta'}{1 - \beta' - \delta'} < 1, \\ &\leq k^2 d(\hat{o}_{n-2}, \hat{o}_{n-1}) + (1 - k^2)\ d(\mathscr{T}_1, \mathscr{T}_2) \\ &\leq k^3 d(\hat{o}_{n-2}, \hat{o}_{n-1}) + (1 - k^3)\ d(\mathscr{T}_1, \mathscr{T}_2) \\ &. \\ &. \\ &. \\ &\leq k^n d(\hat{o}_0, \hat{o}_1) + (1 - k^n)\ d(\mathscr{T}_1, \mathscr{T}_2). \end{aligned}$$

Since $k < 1$, $k^n \to 0$ as $n \to \infty$, we have

$$d(\hat{o}_n, \hat{o}_{n+1}) \to d(\mathscr{T}_1, \mathscr{T}_2).$$

□

Next, we prove an existence result for a BPP.

Theorem 2.1.3. Suppose that all the suppositions of Proposition 2.1.1 holds. Additionally, if $\{\hat{o}_{2n}\}$ has a convergent subsequence in $\mathscr{T}_1$ then $\mathscr{B}$ has a BPP.

Proof. Let $\{\hat{o}_{2n_{\hat{s}}}\}$ be a subsequence of $\{\hat{o}_{2n}\}$ which converges to a point $\hat{o} \in \mathscr{T}_1$. Now

$$d(\hat{o}, \hat{o}_{2n_{\hat{s}-1}}) \leq d(\hat{o}, \hat{o}_{2n_{\hat{s}}}) + d(\hat{o}_{2n_{\hat{s}}}, \hat{o}_{2n_{\hat{s}-1}}). \tag{2.8}$$

Taking $\hat{s} \to \infty$ in (2.8), we get

$$d(\hat{o}, \hat{o}_{2n_{\hat{s}-1}}) \to d(\mathscr{T}_1, \mathscr{T}_2).$$

Also

$$\begin{aligned}
&d(\hat{o}_{2n_{\hat{s}}}, \mathscr{B}\hat{o}) = d(\mathscr{B}\hat{o}_{2n_{\hat{s}-1}}, \mathscr{B}\hat{o}) \\
\leq&\alpha' d(\hat{o}_{2n_{\hat{s}-1}}, \mathscr{B}\hat{o}_{2n_{\hat{s}-1}}) + \beta' d(\hat{o}, \mathscr{B}\hat{o}) + \gamma' d(\hat{o}_{2n_{\hat{s}-1}}, \hat{o}) \\
&+ \delta'(d(\hat{o}_{2n_{\hat{s}-1}}, \mathscr{B}\hat{o}) + d(\hat{o}, \mathscr{B}\hat{o}_{2n_{\hat{s}-1}}))+ \\
&(1 - \alpha' - \beta' - \gamma' - 2\delta')\ d(\mathscr{T}_1, \mathscr{T}_2) \\
=&\alpha' d(\hat{o}_{2n_{\hat{s}-1}}, \hat{o}_{2n(k')})) + \beta' d(\hat{o}, \mathscr{B}\hat{o}) + \gamma' d(\hat{o}_{2n_{\hat{s}-1}}, \hat{o})+ \\
&\delta'(d(\hat{o}_{2n_{\hat{s}-1}}, \mathscr{B}\hat{o}) + d(\hat{o}, \hat{o}_{2n_{\hat{s}}})) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ d(\mathscr{T}_1, \mathscr{T}_2) \\
\leq&\alpha' d(\hat{o}_{2n_{\hat{s}-1}}, \hat{o}_{2n_{\hat{s}-1}})) + (\beta' + \delta')(d(\hat{o}, \hat{o}_{2n_{\hat{s}}}) + d(\hat{o}_{2n_{\hat{s}}}, \mathscr{B}\hat{o}))+ \\
&\gamma' d(\hat{o}_{2n_{\hat{s}-1}}, \hat{o}) + \delta' d(\hat{o}_{2n_{\hat{s}-1}}, \hat{o}) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ d(\mathscr{T}_1, \mathscr{T}_2)
\end{aligned} \tag{2.9}$$

Taking $\hat{s} \to \infty$ in (2.9), we have

$$\begin{aligned}
(1 - \beta' - \delta')d(\hat{o}, \mathscr{B}\hat{o}) \leq&(1 - \beta' - \delta')d(\mathscr{T}_1, \mathscr{T}_2) \\
d(\hat{o}, \mathscr{B}\hat{o}) \leq&d(\mathscr{T}_1, \mathscr{T}_2).
\end{aligned}$$

Since $d(\mathscr{T}_1, \mathscr{T}_2) \leq d(\hat{o}, \mathscr{B}\hat{o}) \leq d(\mathscr{T}_1, \mathscr{T}_2)$. Then $\mathscr{B}$ has a BPP. □

Now, we generalized BPP to m-tuple BPP in a MS. For this, we need to define m-tuple $(m \geq 1)$ BPP.

Definition 2.1.2. Let $\mathscr{B} : \mathscr{T}_1^m \to \mathscr{T}_2$ is a mapping the a point $(\hat{o}_1, \hat{o}_2, \hat{o}_3, \ldots, \hat{o}_m) \in$

$\mathscr{T}_1^m$ is a m-tuple BPP of $\mathscr{B}$ if

$$\begin{aligned}
&d(\hat{o}_1, \mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)) \\
=&d(\hat{o}_2, \mathscr{B}(\hat{o}_2, \hat{o}_3, \ldots, \hat{o}_1)) \\
&\vdots \\
=&d(\hat{o}_m, \mathscr{B}(\hat{o}_m, \hat{o}_1, \ldots, \hat{o}_{m-1})) \\
=&d(\mathscr{T}_1, \mathscr{T}_2).
\end{aligned}$$

If $m = 1, 2, 3 \cdots$, we get BPP, coupled BPP, triplet BPP and so on. If $\mathscr{T}_1 = \mathscr{T}_2$ in the above definition, then a m-tuple BPP reduces to a m-tuple FP.

Definition 2.1.3. The mappings $\mathscr{B} : \mathscr{T}_1^m \to \mathscr{T}_2$ and $\mathscr{B}_1 : \mathscr{T}_2^m \to \mathscr{T}_1$ are cyclic contractions if there exist $k \in [0, 1)$ and

$$\begin{aligned}
d(\mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m), \mathscr{B}_1(\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m)) \leq& \frac{k}{m}(d(\hat{o}_1, \hat{g}_1) + d(\hat{o}_2, \hat{g}_2) + \ldots + d(\hat{o}_m, \hat{g}_m) + \\
&(1-k)d(\mathscr{T}_1, \mathscr{T}_2),
\end{aligned} \tag{2.10}$$

for all $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m) \in \mathscr{T}_1^m$ and $\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m \in \mathscr{T}_2^m$.

Keep in mind that if a pair $(\mathscr{B}, \mathscr{B}_1)$ is a cyclic contraction, then so is the pair $(\mathscr{B}_1, \mathscr{B})$.

Example 2.1.3. *Consider $\mathscr{W} = \mathbb{R}^m$ endowed with*

$$d((\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m), (\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m)) = |\hat{o}_1 - \hat{g}_1| + |\hat{o}_2 - \hat{g}_2| + \ldots + |\hat{o}_m - \hat{g}_m|,$$

$(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m), (\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m) \in \mathscr{W}$. *Suppose that*

$$\mathscr{T}_1 = \{(\hat{o}_1, 0, \ldots, 0) \in \mathscr{W} : 0 \leq \hat{o}_1 \leq 1\}, \mathscr{T}_2 = \{(\hat{g}_1, 0, \ldots, 0) \in \mathscr{W} : 0 \leq \hat{g}_1 \leq 2\}.$$

Clearly, $d(\mathscr{T}_1, \mathscr{T}_2) = 0$. Define $\mathscr{B} : \mathscr{T}_1^m \to \mathscr{T}_2$ and $\mathscr{B}_1 : \mathscr{T}_2^m \to \mathscr{T}_1$ by

$$\mathscr{B}((\hat{o}_1, 0, \ldots, 0), (\hat{o}_2, 0, \ldots, 0), \ldots, (\hat{o}_m, 0, \ldots, 0)) = \left(\frac{\hat{o}_1 + \hat{o}_2 + \ldots + \hat{o}_m}{2m}, 0, 0, \ldots, 0\right),$$

and

$$\mathscr{B}_1((\hat{g}_1,0,\ldots,0),(\hat{g}_2,0,\ldots,0),\ldots,(\hat{g}_m,0,\ldots,0)) = \left(\frac{\hat{g}_1+\hat{g}_2+\ldots+\hat{g}_m}{2m},0,0,\ldots,0\right).$$

Then we obtain

$$\begin{aligned}
& d\begin{pmatrix}\mathscr{B}((\hat{o}_1,0,\ldots,0),(\hat{o}_2,0,\ldots,0),\ldots,(\hat{o}_m,0,\ldots,0)),\\ \mathscr{B}_1((\hat{g}_1,0,\ldots,0),(\hat{g}_2,0,\ldots,0),\ldots,(\hat{g}_m,0,\ldots,0))\end{pmatrix}\\
& d\left(\left(\frac{\hat{o}_1+\hat{o}_2+\ldots+\hat{o}_m}{2m},0,0,\ldots,0\right),\left(\frac{\hat{g}_1+\hat{g}_2+\ldots+\hat{g}_m}{2m},0,0,\ldots,0\right)\right)\\
=& \left|\frac{\hat{o}_1+\hat{o}_2+\ldots+\hat{o}_m}{2m}-\frac{\hat{g}_1+\hat{g}_2+\ldots+\hat{g}_m}{2m}\right|\\
=& \left|\frac{(\hat{o}_1-\hat{g}_1)+(\hat{o}_2-\hat{g}_2)+\ldots+(\hat{o}_m-\hat{g}_m)}{2m}\right|\\
\leq& \frac{|\hat{o}_1-\hat{g}_1|+|\hat{o}_2-\hat{g}_2|+\ldots+|\hat{o}_m-\hat{g}_m|}{2m}\\
\leq& \frac{k}{m}(d(\hat{o}_1,\hat{g}_1)+d(\hat{o}_2,\hat{g}_2)+\ldots+d(\hat{o}_m,\hat{g}_m))+(1-k)d(\mathscr{T}_1,\mathscr{T}_2); k=\frac{1}{2}.
\end{aligned}$$

Hence, pair $(\mathscr{B},\mathscr{B}_1)$ *is a cyclic contraction with* $k=\frac{1}{2}$.

Now, we prove some lemmas as described below:

Lemma 2.1.1. Suppose that $\mathscr{B}:\mathscr{T}_1^m\to\mathscr{T}_2$ and $\mathscr{B}_1:\mathscr{T}_2^m\to\mathscr{T}_1$ are two cyclic contraction mappings. If $(\hat{o}_1^0,\hat{o}_2^0,\ldots,\hat{o}_m^0)\in\mathscr{T}_1^m$ and the sequences $\{\hat{o}_1^n\}$,$\{\hat{o}_2^n\}$,…,$\{\hat{o}_m^n\}$ defined as:

$$\begin{cases}
\hat{o}_1^{2n+1} = & \mathscr{B}(\hat{o}_1^{2n},\hat{o}_2^{2n},\ldots,\hat{o}_m^{2n}), \hat{o}_1^{2n+2}=\mathscr{B}_1(\hat{o}_1^{2n+1},\hat{o}_2^{2n+1},\ldots,\hat{o}_m^{2n+1}),\\
\hat{o}_2^{2n+1} = & \mathscr{B}(\hat{o}_2^{2n},\hat{o}_3^{2n},\ldots,\hat{o}_1^{2n}), \hat{o}_2^{2n+2}=\mathscr{B}_1(\hat{o}_2^{2n+1},\hat{o}_3^{2n+1},\ldots,\hat{o}_1^{2n+1}),\\
 & \vdots\\
\hat{o}_m^{2n+1} = & \mathscr{B}(\hat{o}_m^{2n},\hat{o}_1^{2n},\hat{o}_2^{2n},\ldots,\hat{o}_{m-1}^{2n}), \hat{o}_m^{2n+2}=\mathscr{B}_1(\hat{o}_m^{2n+1},\hat{o}_1^{2n+1},\ldots,\hat{o}_{m-1}^{2n+1}),
\end{cases} \tag{2.11}$$

then

$$\begin{cases} d(\hat{o}_1^{2n}, \hat{o}_1^{2n+1}) \to d(\mathscr{T}_1, \mathscr{T}_2), d(\hat{o}_1^{2n+1}, \hat{o}_1^{2n+2}) \to d(\mathscr{T}_1, \mathscr{T}_2), \\ d(\hat{o}_2^{2n}, \hat{o}_2^{2n+1}) \to d(\mathscr{T}_1, \mathscr{T}_2), d(\hat{o}_2^{2n+1}, \hat{o}_2^{2n+2}) \to d(\mathscr{T}_1, \mathscr{T}_2), \\ \vdots \\ d(\hat{o}_m^{2n}, \hat{o}_m^{2n+1}) \to d(\mathscr{T}_1, \mathscr{T}_2), d(\hat{o}_m^{2n+1}, \hat{o}_m^{2n+2}) \to d(\mathscr{T}_1, \mathscr{T}_2), \text{ for all } n \geq 0. \end{cases}$$

Proof. Consider

$$\begin{aligned}
& d(\hat{o}_1^{2n}, \hat{o}_1^{2n+1}) \\
=& d(\hat{o}_1^{2n}, \mathscr{B}(\hat{o}_1^{2n}, \hat{o}_2^{2n}, \ldots, \hat{o}_m^{2n})) \\
=& d\left(\genfrac{}{}{0pt}{}{\mathscr{B}_1(\hat{o}_1^{2n-1}, \hat{o}_2^{2n-1}, \ldots, \hat{o}_m^{2n-1})}{\mathscr{B}\left(\mathscr{B}_1(\hat{o}_1^{2n-1}, \hat{o}_2^{2n-1}, \ldots, \hat{o}_m^{2n-1}), \mathscr{B}_1(\hat{o}_2^{2n-1}, \hat{o}_3^{2n-1}, \ldots, \hat{o}_1^{2n-1}), \ldots, \mathscr{B}_1(\hat{o}_m^{2n-1}, \hat{o}_1^{2n-1}, \ldots, \hat{o}_{m-1}^{2n-1})\right)}\right) \\
\leq& \frac{k}{m}\Big[d(\hat{o}_1^{2n-1}, \mathscr{B}_1(\hat{o}_1^{2n-1}, \hat{o}_2^{2n-1}, \ldots, \hat{o}_m^{2n-1})) + d(\hat{o}_2^{2n-1}, \mathscr{B}_1(\hat{o}_2^{2n-1}, \hat{o}_3^{2n-1}, \ldots, \hat{o}_1^{2n-1})) + \ldots + \\
& d(\hat{o}_m^{2n-1}, \mathscr{B}_1(\hat{o}_m^{2n-1}, \hat{o}_1^{2n-1}, \ldots, \hat{o}_{m-1}^{2n-1}))\Big] + (1-k)d(\mathscr{T}_1, \mathscr{T}_2) \\
=& \frac{k}{m}\left[d\left(\genfrac{}{}{0pt}{}{\mathscr{B}(\hat{o}_1^{2n-2}, \hat{o}_2^{2n-2}, \ldots, \hat{o}_m^{2n-2})}{\mathscr{B}_1\left(\mathscr{B}(\hat{o}_1^{2n-2}, \hat{o}_2^{2n-2}, \ldots, \hat{o}_m^{2n-2}), \mathscr{B}(\hat{o}_2^{2n-2}, \hat{o}_3^{2n-2}, \ldots, \hat{o}_1^{2n-2}), \ldots, \mathscr{B}(\hat{o}_m^{2n-2}, \hat{o}_1^{2n-2}, \ldots, \hat{o}_{m-1}^{2n-2})\right)}\right) + \right. \\
& d\left(\genfrac{}{}{0pt}{}{\mathscr{B}(\hat{o}_2^{2n-2}, \hat{o}_3^{2n-2}, \ldots, \hat{o}_1^{2n-2})}{\mathscr{B}_1\left(\mathscr{B}(\hat{o}_2^{2n-2}, \hat{o}_3^{2n-2}, \ldots, \hat{o}_1^{2n-2}), \mathscr{B}(\hat{o}_3^{2n-2}, \hat{o}_4^{2n-2}, \ldots, \hat{o}_2^{2n-2}), \ldots, \mathscr{B}(\hat{o}_1^{2n-2}, \hat{o}_2^{2n-2}, \ldots, \hat{o}_m^{2n-2})\right)}\right) \\
& \vdots \\
& \left. + d\left(\genfrac{}{}{0pt}{}{\mathscr{B}(\hat{o}_m^{2n-2}, \hat{o}_1^{2n-2}, \ldots, \hat{o}_{m-1}^{2n-2})}{\mathscr{B}_1\left(\mathscr{B}(\hat{o}_m^{2n-2}, \hat{o}_1^{2n-2}, \ldots, \hat{o}_{m-1}^{2n-2}), \mathscr{B}(\hat{o}_1^{2n-2}, \hat{o}_2^{2n-2}, \ldots, \hat{o}_m^{2n-2}), \ldots, \mathscr{B}(\hat{o}_{m-1}^{2n-2}, \hat{o}_m^{2n-2}, \ldots, \hat{o}_{m-2}^{2n-2})\right)}\right)\right].
\end{aligned}$$

Using (2.10), we get

$$\begin{aligned}
& d(\hat{o}_1^{2n}, \hat{o}_1^{2n+1}) \\
\leq& \frac{k}{m}\left[\frac{k}{m}(d(\hat{o}_1^{2n-2}, \mathscr{B}(\hat{o}_1^{2n-2}, \hat{o}_2^{2n-2}, \ldots, \hat{o}_m^{2n-2})) + d(\hat{o}_2^{2n-2}, \mathscr{B}(\hat{o}_2^{2n-2}, \hat{o}_3^{2n-2}, \ldots, \hat{o}_1^{2n-2}))\right. \\
& + \ldots + d(\hat{o}_m^{2n-2}, \mathscr{B}(\hat{o}_m^{2n-2}, \hat{o}_1^{2n-2}, \ldots, \hat{o}_{m-1}^{2n-2}))) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2) + \\
& \frac{k}{m}(d(\hat{o}_2^{2n-2}, \mathscr{B}(\hat{o}_2^{2n-2}, \hat{o}_3^{2n-2}, \ldots, \hat{o}_1^{2n-2})) + d(\hat{o}_3^{2n-2}, \mathscr{B}(\hat{o}_3^{2n-2}, \hat{o}_4^{2n-2}, \ldots, \hat{o}_2^{2n-2})) \\
& + \ldots + d(\hat{o}_1^{2n-2}, \mathscr{B}(\hat{o}_1^{2n-2}, \hat{o}_2^{2n-2}, \ldots, \hat{o}_m^{2n-2}))) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2)
\end{aligned}$$

$$\begin{aligned}
&\vdots\\
&+\frac{k}{m}(d(\hat{o}_m^{2n-2},\mathscr{B}(\hat{o}_m^{2n-2},\hat{o}_1^{2n-2},\ldots,\hat{o}_{m-1}^{2n-2}))+d(\hat{o}_1^{2n-2},\mathscr{B}(\hat{o}_1^{2n-2},\hat{o}_2^{2n-2},\ldots,\hat{o}_m^{2n-2}))\\
&+\ldots+d(\hat{o}_{m-1}^{2n-2},\mathscr{B}(\hat{o}_{m-1}^{2n-2},\hat{o}_m^{2n-2},\ldots,\hat{o}_{m-2}^{2n-2})))+(1-k)d(\mathscr{T}_1,\mathscr{T}_2)\Big]\\
=&\frac{k^2}{m}(d(\hat{o}_1^{2n-2},\mathscr{B}(\hat{o}_1^{2n-2},\hat{o}_2^{2n-2},\ldots,\hat{o}_m^{2n-2}))+d(\hat{o}_2^{2n-2},\mathscr{B}(\hat{o}_2^{2n-2},\hat{o}_3^{2n-2},\ldots,\hat{o}_1^{2n-2}))\\
&+\ldots+d(\hat{o}_m^{2n-2},\mathscr{B}(\hat{o}_m^{2n-2},\hat{o}_1^{2n-2},\ldots,\hat{o}_{m-1}^{2n-2})))+(1-k^2)d(\mathscr{T}_1,\mathscr{T}_2).
\end{aligned}$$

By mathematical induction, we get

$$\begin{aligned}
d(\hat{o}_1^{2n},\hat{o}_1^{2n+1})\leq&\frac{k^{2n}}{m}(d(\hat{o}_1^{0},\mathscr{B}(\hat{o}_1^{0},\hat{o}_2^{0},\ldots,\hat{o}_m^{0}))+d(\hat{o}_2^{0},\mathscr{B}(\hat{o}_2^{0},\hat{o}_3^{0},\ldots,\hat{o}_1^{0}))+\ldots+\\
&d(\hat{o}_m^{0},\mathscr{B}(\hat{o}_m^{0},\hat{o}_1^{0},\ldots,\hat{o}_{m-1}^{0})))+(1-k^{2n})d(\mathscr{T}_1,\mathscr{T}_2).
\end{aligned}$$

Taking $n\to\infty$, we have

$$d(\hat{o}_1^{2n},\hat{o}_1^{2n+1})\to d(\mathscr{T}_1,\mathscr{T}_2).$$

Similarly,

$$\begin{aligned}
&d(\hat{o}_2^{2n},\hat{o}_2^{2n+1})\to d(\mathscr{T}_1,\mathscr{T}_2),d(\hat{o}_2^{2n+1},\hat{o}_2^{2n+2})\to d(\mathscr{T}_1,\mathscr{T}_2),\ldots,\\
&d(\hat{o}_m^{2n},\hat{o}_m^{2n+1})\to d(\mathscr{T}_1,\mathscr{T}_2),d(\hat{o}_m^{2n+1},\hat{o}_m^{2n+2})\to d(\mathscr{T}_1,\mathscr{T}_2).
\end{aligned}$$

□

Lemma 2.1.2. Suppose that $(\mathscr{T}_1,\mathscr{T}_2)$,$(\mathscr{T}_2,\mathscr{T}_1)$ satisfy the property UC, $\mathscr{B}:\mathscr{T}_1^m\to\mathscr{T}_2$, $\mathscr{B}_1:\mathscr{T}_2^m\to\mathscr{T}_1$ are cyclic contraction mappings, $(\hat{o}_1^0,\hat{o}_2^0,\ldots,\hat{o}_m^0)\in\mathscr{T}_1^m$ and the sequences $\{\hat{o}_1^n\}$,…,$\{\hat{o}_m^n\}$ are defined as in (2.11). Then, there is $N_0>0$ for any $\epsilon>0$, so that

$$\frac{1}{m}\left(d(\hat{o}_1^{2m},\hat{o}_1^{2n+1})+d(\hat{o}_2^{2m},\hat{o}_2^{2n+1})+\ldots+d(\hat{o}_m^{2m},\hat{o}_m^{2n+1})\right)<d(\mathscr{T}_1,\mathscr{T}_2)+\epsilon, \tag{2.12}$$

for all $m>n\geq N_0$.

Proof. The Lemma 2.1.1 gives us

$$d(\hat{o}_1^{2n}, \hat{o}_1^{2n+1}) \to d(\mathscr{T}_1, \mathscr{T}_2), d(\hat{o}_1^{2n+1}, \hat{o}_1^{2n+2}) \to d(\mathscr{T}_1, \mathscr{T}_2), d(\hat{o}_2^{2n}, \hat{o}_2^{2n+1}) \to d(\mathscr{T}_1, \mathscr{T}_2),$$
$$d(\hat{o}_2^{2n+1}, \hat{o}_2^{2n+2}) \to d(\mathscr{T}_1, \mathscr{T}_2), \ldots, d(\hat{o}_m^{2n}, \hat{o}_m^{2n+1}) \to d(\mathscr{T}_1, \mathscr{T}_2), d(\hat{o}_m^{2n+1}, \hat{o}_m^{2n+2}) \to d(\mathscr{T}_1, \mathscr{T}_2).$$

Because a pair $(\mathscr{T}_1, \mathscr{T}_2)$ and $(\mathscr{T}_2, \mathscr{T}_1)$ has a property UC, we obtain

$$d(\hat{o}_1^{2n}, \hat{o}_1^{2n+2}) \to 0, d(\hat{o}_2^{2n}, \hat{o}_2^{2n+2}) \to 0, \ldots, d(\hat{o}_m^{2n}, \hat{o}_m^{2n+2}) \to 0,$$

and

$$d(\hat{o}_1^{2n+1}, \hat{o}_1^{2n+3}) \to 0, d(\hat{o}_2^{2n+1}, \hat{o}_2^{2n+3}) \to 0, \ldots, d(\hat{o}_m^{2n+1}, \hat{o}_m^{2n+3}) \to 0.$$

Assume that (2.12) is not true. For a given $\hat{s} \in \mathbb{N}$, with $m_{\hat{s}} > n_{\hat{s}} \geq \hat{s}$, there is $\epsilon > 0$ so that

$$\frac{1}{m}\Big(d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}}+1}) + d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2n_{\hat{s}}+1}) + \ldots + d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2n_{\hat{s}}+1})\Big) \geq d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon.$$

Suppose that $m_{\hat{s}}$ is the smallest integer greater than $n_{\hat{s}}$ to satisfy the above inequality. Now

$$\begin{aligned}&\frac{1}{m}\Big(d(\hat{o}_1^{2m_{\hat{s}}-2}, \hat{o}_1^{2n_{\hat{s}}+1}) + d(\hat{o}_2^{2m_{\hat{s}}-2}, \hat{o}_2^{2n_{\hat{s}}+1}) + \ldots + d(\hat{o}_m^{2m_{\hat{s}}-2}, \hat{o}_m^{2n_{\hat{s}}+1})\Big)\\ &< d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon.\end{aligned}$$

Consequently, we get

$$\begin{aligned}d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon \leq & \frac{1}{m}\Big(d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}}+1}) + d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2n_{\hat{s}}+1}) + \ldots + d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2n_{\hat{s}}+1})\Big)\\ \leq & \frac{1}{m}\Big(d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}}-2}) + d(\hat{o}_1^{2m_{\hat{s}}-2}, \hat{o}_1^{2n_{\hat{s}}+1}) + d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2m_{\hat{s}}-2}) +\\ & d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2n_{\hat{s}}+1}) + \ldots + d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2m_{\hat{s}}-2}) + d(\hat{o}_m^{2m_{\hat{s}}-2}, \hat{o}_m^{2n_{\hat{s}}+1})\Big)\\ < & \frac{1}{m}\Big(d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}}-2}) + d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2m_{\hat{s}}-2}) + \ldots + d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2m_{\hat{s}}-2})\Big) +\\ & d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon.\end{aligned}$$

As $\hat{s} \to \infty$, in above inequality, we obtain

$$\frac{1}{m}\left(d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}}) + d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2n_{\hat{s}+1}}) + \ldots + d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2n_{\hat{s}+1}})\right) \to d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon.$$

From triangle inequality, we get

$$\begin{aligned}
&\frac{1}{m}\left(d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}}) + d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2n_{\hat{s}+1}}) + \ldots + d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2n_{\hat{s}+1}})\right)\\
&\le \frac{1}{m}\Big[d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + d(\hat{o}_1^{2m_{\hat{s}+2}}, \hat{o}_1^{2m_{\hat{s}+3}}) + d(\hat{o}_1^{2m_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}})\\
&+ \ d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2m_{\hat{s}+2}}) + d(\hat{o}_2^{2m_{\hat{s}+2}}, \hat{o}_2^{2m_{\hat{s}+3}}) + d(\hat{o}_2^{2m_{\hat{s}+3}}, \hat{o}_2^{2n_{\hat{s}+1}}) + \ldots +\\
&d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2m_{\hat{s}+2}}) + d(\hat{o}_m^{2m_{\hat{s}+2}}, \hat{o}_m^{2m_{\hat{s}+3}}) + d(\hat{o}_m^{2m_{\hat{s}+3}}, \hat{o}_m^{2n_{\hat{s}+1}})\Big]\\
&= \frac{1}{m}\Big[d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + d(\mathscr{B}_1(\hat{o}_1^{2m_{\hat{s}+1}}, \ldots, \hat{o}_m^{2m_{\hat{s}+1}}), \mathscr{B}(\hat{o}_1^{2n_{\hat{s}+2}}, \ldots, \hat{o}_m^{2n_{\hat{s}+2}}))\\
&+ d(\hat{o}_1^{2m_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}})\Big] + \frac{1}{m}\Big[d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2m_{\hat{s}+2}}) + d(\mathscr{B}_1(\hat{o}_2^{2m_{\hat{s}+1}}, \ldots, \hat{o}_1^{2m_{\hat{s}+1}}),\\
&\mathscr{B}(\hat{o}_2^{2n_{\hat{s}+2}}, \ldots, \hat{o}_1^{2n_{\hat{s}+2}})) + d(\hat{o}_2^{2m_{\hat{s}+3}}, \hat{o}_2^{2n_{\hat{s}+1}})\Big] + \ldots + \frac{1}{m}\Big[d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2m_{\hat{s}+2}})\\
&+ d(\mathscr{B}_1(\hat{o}_m^{2m_{\hat{s}+1}}, \ldots, \hat{o}_{m-1}^{2m_{\hat{s}+1}}), \mathscr{B}(\hat{o}_m^{2n_{\hat{s}+2}}, \ldots, \hat{o}_{m-1}^{2n_{\hat{s}+2}})) + d(\hat{o}_m^{2m_{\hat{s}+3}}, \hat{o}_m^{2n_{\hat{s}+1}})\Big].
\end{aligned}$$

Applying (2.10), we have

$$\begin{aligned}
&\frac{1}{m}\left(d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}}) + d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2n_{\hat{s}+1}}) + \ldots + d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2n_{\hat{s}+1}})\right)\\
&\le \frac{1}{m}\Big[d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + \frac{k}{m}\Big(d(\hat{o}_1^{2m_{\hat{s}+1}}, \hat{o}_1^{2m_{\hat{s}+2}}) + d(\hat{o}_2^{2m_{\hat{s}+1}}, \hat{o}_2^{2m_{\hat{s}+2}}) + \ldots +\\
&d(\hat{o}_m^{2m_{\hat{s}+1}}, \hat{o}_m^{2m_{\hat{s}+2}})\Big) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2) + d(\hat{o}_1^{2m_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}}) + \ d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2m_{\hat{s}+2}})\\
&+ \frac{k}{m}\Big(d(\hat{o}_2^{2m_{\hat{s}+1}}, \hat{o}_2^{2m_{\hat{s}+2}}) + \ldots + d(\hat{o}_1^{2m_{\hat{s}+1}}, \hat{o}_1^{2m_{\hat{s}+2}})\Big) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2)\\
&+ d(\hat{o}_2^{2m_{\hat{s}+3}}, \hat{o}_2^{2n_{\hat{s}+1}}) + \ldots + \ d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2m_{\hat{s}+2}}) + \frac{k}{m}\Big(d(\hat{o}_m^{2m_{\hat{s}+1}}, \hat{o}_m^{2m_{\hat{s}+2}}) +\\
&d(\hat{o}_1^{2m_{\hat{s}+1}}, \hat{o}_1^{2m_{\hat{s}+2}}) + \ldots + d(\hat{o}_{m-1}^{2m_{\hat{s}+1}}, \hat{o}_{m-1}^{2m_{\hat{s}+2}})\Big) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2)\\
&+ d(\hat{o}_m^{2m_{\hat{s}+3}}, \hat{o}_m^{2n_{\hat{s}+1}})\Big]\\
&= \frac{1}{m}\Big[d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + d(\hat{o}_1^{2m_{\hat{s}+1}}, \hat{o}_1^{2m_{\hat{s}+3}}) + d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2m_{\hat{s}+2}}) + \ldots +\\
&d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2m_{\hat{s}+2}}) + d(\hat{o}_m^{2m_{\hat{s}+1}}, \hat{o}_m^{2m_{\hat{s}+3}})\Big] + \frac{k}{m}(d(\hat{o}_1^{2m_{\hat{s}+1}}, \hat{o}_1^{2m_{\hat{s}+2}}) +\\
&d(\hat{o}_2^{2m_{\hat{s}+1}}, \hat{o}_2^{2m_{\hat{s}+2}}) + \ldots + d(\hat{o}_m^{2m_{\hat{s}+1}}, \hat{o}_m^{2m_{\hat{s}+2}})) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{m}\Big(d(\hat{o}_1^{2m_{\hat{s}}},\hat{o}_1^{2n_{\hat{s}}+1})+d(\hat{o}_2^{2m_{\hat{s}}},\hat{o}_2^{2n_{\hat{s}}+1})+\ldots+d(\hat{o}_m^{2m_{\hat{s}}},\hat{o}_m^{2n_{\hat{s}}+1})\Big)\\
&\frac{1}{m}\Big[d(\hat{o}_1^{2m_{\hat{s}}},\hat{o}_1^{2m_{\hat{s}}+2})+d(\hat{o}_1^{2m_{\hat{s}}+1},\hat{o}_1^{2m_{\hat{s}}+3})+d(\hat{o}_2^{2m_{\hat{s}}},\hat{o}_2^{2m_{\hat{s}}+2})+\ldots+\\
&d(\hat{o}_m^{2m_{\hat{s}}},\hat{o}_m^{2m_{\hat{s}}+2})+d(\hat{o}_m^{2m_{\hat{s}}+1},\hat{o}_m^{2m_{\hat{s}}+3})\Big]+\frac{k}{m}\Big[d(\mathscr{B}(\hat{o}_1^{2m_{\hat{s}}},\ldots,\hat{o}_m^{2m_{\hat{s}}}),\\
&\mathscr{B}_1(\hat{o}_1^{2m_{\hat{s}}+1},\ldots,\hat{o}_m^{2m_{\hat{s}}+1}))+d(\mathscr{B}(\hat{o}_2^{2m_{\hat{s}}},\ldots,\hat{o}_1^{2m_{\hat{s}}}),\mathscr{B}_1(\hat{o}_2^{2m_{\hat{s}}+1},\ldots,\hat{o}_1^{2m_{\hat{s}}+1}))\\
&+\ldots+d(\mathscr{B}(\hat{o}_m^{2m_{\hat{s}}},\ldots,\hat{o}_{m-1}^{2m_{\hat{s}}}),\mathscr{B}_1(\hat{o}_m^{2m_{\hat{s}}+1},\ldots,\hat{o}_{m-1}^{2m_{\hat{s}}+1}))\Big]\\
\leq\ &\frac{1}{m}\Big[d(\hat{o}_1^{2m_{\hat{s}}},\hat{o}_1^{2m_{\hat{s}}+2})+d(\hat{o}_1^{2m_{\hat{s}}+1},\hat{o}_1^{2m_{\hat{s}}+3})+\ldots+d(\hat{o}_m^{2m_{\hat{s}}},\hat{o}_m^{2m_{\hat{s}}+2})+\\
&d(\hat{o}_m^{2m_{\hat{s}}+1},\hat{o}_m^{2m_{\hat{s}}+3})\Big]+\frac{k^2}{m}(d(\hat{o}_1^{2m_{\hat{s}}},\hat{o}_1^{2m_{\hat{s}}+1})+d(\hat{o}_2^{2m_{\hat{s}}},\hat{o}_2^{2m_{\hat{s}}+1})+\ldots+\\
&d(\hat{o}_m^{2m_{\hat{s}}},\hat{o}_m^{2m_{\hat{s}}+1}))+(1-k)d(\mathscr{T}_1,\mathscr{T}_2).
\end{aligned}$$

Take $\hat{s}\to\infty$, we obtain

$$d(\mathscr{T}_1,\mathscr{T}_2)+\epsilon\leq k^2(d(\mathscr{T}_1,\mathscr{T}_2)+\epsilon)+(1-k^2)d(\mathscr{T}_1,\mathscr{T}_2)=d(\mathscr{T}_1,\mathscr{T}_2)+k^2\epsilon,$$

which is a contradiction since $k<1$. Hence the result. □

Lemma 2.1.3. Under the suppositions of Lemma 2.1.2, $\{\hat{o}_1^{2n}\}$,$\{\hat{o}_2^{2n}\}$,. . . ,$\{\hat{o}_m^{2n}\}$ are Cauchy sequences.

Proof. The Lemma 2.1.1 gives us

$$d(\hat{o}_1^{2n},\hat{o}_1^{2n+1})\to d(\mathscr{T}_1,\mathscr{T}_2)\text{ and }d(\hat{o}_1^{2n+1},\hat{o}_1^{2n+2})\to d(\mathscr{T}_1,\mathscr{T}_2).$$

As $(\mathscr{T}_1,\mathscr{T}_2)$ satisfies the property UC, then $d(\hat{o}_1^{2n},\hat{o}_1^{2n+2})\to 0$. The UC property of $(\mathscr{T}_2,\mathscr{T}_1)$ shows we get $d(\hat{o}_1^{2n+1},\hat{o}_1^{2n+3})\to 0$. Now we claim that for each $\epsilon>0$, there is $N\in\mathbb{N}$ so that

$$d(\hat{o}_1^{2m_{\hat{s}}},\hat{o}_1^{2n_{\hat{s}}+1})\leq d(\mathscr{T}_1,\mathscr{T}_2)+\epsilon,\ \text{ for all } m>n\geq N. \tag{2.13}$$

Let us consider (2.13) is not true. For every $\hat{s}\in\mathbb{N}$, with $m_{\hat{s}}>n_{\hat{s}}\geq\hat{s}$, there is $\epsilon>0$

so that $d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}}) > d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon$. Suppose that $m_{\hat{s}}$ is the smallest integer greater than $n_{\hat{s}}$ to satisfy the above inequality. Now

$$\begin{aligned} d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon < d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}}) &\leq d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}}-2}) + d(\hat{o}_1^{2m_{\hat{s}}-2}, \hat{o}_1^{2n_{\hat{s}+1}}) \\ &\leq d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}}) + d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon. \end{aligned}$$

Taking $\hat{s} \to \infty$, we have $d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}}) \to d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon$. The Lemma 2.1.2 gives us

$$\frac{1}{m}\Big(d(\hat{o}_1^{2m}, \hat{o}_1^{2n+1}) + d(\hat{o}_2^{2m}, \hat{o}_2^{2n+1}) + \ldots + d(\hat{o}_m^{2m}, \hat{o}_m^{2n+1})\Big) < d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon.$$

Consider

$$\begin{aligned} & d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon \\ <& d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}}) \\ \leq& d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + d(\hat{o}_1^{2m_{\hat{s}+2}}, \hat{o}_1^{2n_{\hat{s}+3}}) + d(\hat{o}_1^{2n_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}}) \\ =& \ d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + d(\mathscr{B}_1(\hat{o}_1^{2m_{\hat{s}+1}}, \hat{o}_2^{2m_{\hat{s}+1}}, \ldots, \hat{o}_m^{2m_{\hat{s}+1}}), \\ & \ \mathscr{B}(\hat{o}_1^{2n_{\hat{s}+2}}, \hat{o}_2^{2n_{\hat{s}+2}}, \ldots, \hat{o}_m^{2n_{\hat{s}+2}})) + d(\hat{o}_1^{2n_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}}) \\ \leq& \ d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + \frac{k}{m}\Big(d(\hat{o}_1^{2m_{\hat{s}+1}}, \hat{o}_1^{2n_{\hat{s}+2}}) + d(\hat{o}_2^{2m_{\hat{s}+1}}, \hat{o}_2^{2n_{\hat{s}+2}}) + \ldots + \\ & \ d(\hat{o}_m^{2m_{\hat{s}+1}}, \hat{o}_m^{2n_{\hat{s}+2}})\Big) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2) + d(\hat{o}_1^{2n_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}}) \\ =& \frac{k}{m}\Big[d(\mathscr{B}(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_2^{2m_{\hat{s}}}, \ldots, \hat{o}_m^{2m_{\hat{s}}}), \mathscr{B}_1(\hat{o}_1^{2n_{\hat{s}+1}}, \hat{o}_2^{2n_{\hat{s}+1}}, \ldots, \hat{o}_m^{2n_{\hat{s}+1}})) + \\ & \ d(\mathscr{B}(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_3^{2m_{\hat{s}}}, \ldots, \hat{o}_1^{2m_{\hat{s}}}), \mathscr{B}_1(\hat{o}_2^{2n_{\hat{s}+1}}, \hat{o}_3^{2n_{\hat{s}+1}}, \ldots, \hat{o}_1^{2n_{\hat{s}+1}})) + \ldots + \\ & \ d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}}}, \ldots, \hat{o}_{m-1}^{2m_{\hat{s}}}), \mathscr{B}_1(\hat{o}_m^{2n_{\hat{s}+1}}, \hat{o}_1^{2n_{\hat{s}+1}}, \ldots, \hat{o}_{m-1}^{2n_{\hat{s}+1}})\Big] + \\ & \ (1-k)d(\mathscr{T}_1, \mathscr{T}_2) + d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + d(\hat{o}_1^{2n_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}}) \\ \leq& \frac{k}{m}\Bigg[\frac{k}{m}\Big(d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}}) + d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2n_{\hat{s}+1}}) + \ldots + \\ & \ d(\hat{o}_m^{2m_{\hat{s}+1}}, \hat{o}_m^{2n_{\hat{s}+2}})\Big) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2) + \frac{k}{m}\Big(d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2n_{\hat{s}+1}}) + \ldots + \\ & \ d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}})\Big) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2) + \ldots + \frac{k}{m}\Big(d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2n_{\hat{s}+1}}) + \ldots + \\ & d(\hat{o}_{m-1}^{2m_{\hat{s}}}, \hat{o}_{m-1}^{2n_{\hat{s}+1}})\Big) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2)\Bigg] + (1-k)d(\mathscr{T}_1, \mathscr{T}_2) + d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) \\ & \ + d(\hat{o}_1^{2n_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}}) \end{aligned}$$

$$
\begin{aligned}
=&\frac{k^2}{m}\Big(d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2n_{\hat{s}+1}}) + d(\hat{o}_2^{2m_{\hat{s}}}, \hat{o}_2^{2n_{\hat{s}+1}}) + \ldots + d(\hat{o}_m^{2m_{\hat{s}}}, \hat{o}_m^{2n_{\hat{s}+1}})\Big) + \\
&(1-k^2)d(\mathscr{T}_1, \mathscr{T}_2) + d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + d(\hat{o}_1^{2n_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}}) \\
<&k^2(d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon) + (1-k^2)d(\mathscr{T}_1, \mathscr{T}_2) + d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + d(\hat{o}_1^{2n_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}}) \\
=&k^2\epsilon + d(\mathscr{T}_1, \mathscr{T}_2) + d(\hat{o}_1^{2m_{\hat{s}}}, \hat{o}_1^{2m_{\hat{s}+2}}) + d(\hat{o}_1^{2n_{\hat{s}+3}}, \hat{o}_1^{2n_{\hat{s}+1}}).
\end{aligned}
$$

Letting $\hat{s} \to \infty$, we have

$$
d(\mathscr{T}_1, \mathscr{T}_2) + \epsilon \leq d(\mathscr{T}_1, \mathscr{T}_2) + k^2\epsilon,
$$

which is a contradiction. By inequality (2.13), $d(\hat{o}_1^{2n}, \hat{o}_1^{2n+1}) \to d(\mathscr{T}_1, \mathscr{T}_2)$ and from the property of UC*, we get $\{\hat{o}_1^{2n}\}$ is a Cauchy sequence. Similarly, $\{\hat{o}_2^{2n}\}$, $\{\hat{o}_3^{2n}\}$,…,$\{\hat{o}_m^{2n}\}$ are Cauchy sequences. □

In the next result, we discuss the existence and convergence of m-tuple BPP in a MS for cyclic contraction pairs using the property UC*.

Theorem 2.1.4. Let $(\mathscr{T}_1, \mathscr{T}_2)$,$(\mathscr{T}_2, \mathscr{T}_1)$ satisfy the property UC* and $\mathscr{B} : \mathscr{T}_1^m \to \mathscr{T}_2$, $\mathscr{B}_1 : \mathscr{T}_2^m \to \mathscr{T}_1$ be two cyclic contraction mappings. If $(\hat{o}_1^0, \hat{o}_2^0, \ldots, \hat{o}_m^0) \in \mathscr{T}_1^m$ and the sequences $\{\hat{o}_1^n\}$,…,$\{\hat{o}_m^n\}$ are defined as in (2.11). Then $\mathscr{B}$ has a m-tuple BPP $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m) \in \mathscr{T}_1^m$ and $\mathscr{B}_1$ has a m-tuple BPP $(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m) \in \mathscr{T}_2^m$. Moreover, we have

$$
\hat{o}_1^{2n} \to \hat{o}_1, \hat{o}_2^{2n} \to \hat{o}_2, \ldots, \hat{o}_m^{2n} \to \hat{o}_m \text{ and } \hat{o}_1^{2n+1} \to \hat{o}'_1, \hat{o}_2^{2n+1} \to \hat{o}'_2, \ldots, \hat{o}_m^{2n+1} \to \hat{o}'_m.
$$

In addition, if $\hat{o}_2 = \hat{o}_3 \ldots = \hat{o}_m$ and $\hat{o}'_2 = \hat{o}'_3 \ldots = \hat{o}'_m$ then

$$
d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m) = md(\mathscr{T}_1, \mathscr{T}_2). \tag{2.14}
$$

Proof. By Lemma 2.1.1, we obtain $d(\hat{o}_1^{2n}, \hat{o}_1^{2n+1}) \to d(\mathscr{T}_1, \mathscr{T}_2)$. From Lemma 2.1.2, we have $\{\hat{o}_2^{2n}\}$, $\{\hat{o}_3^{2n}\}$,…,$\{\hat{o}_m^{2n}\}$ are Cauchy sequences. Thus there are $\hat{o}_1, \ldots, \hat{o}_m \in \mathscr{T}_1^m$, so that $\hat{o}_1^{2n} \to \hat{o}_1$, $\hat{o}_2^{2n} \to \hat{o}_2$,$\ldots$,$\hat{o}_m^{2n} \to \hat{o}_m$. Hence, we have

$$d(\mathscr{T}_1, \mathscr{T}_2) \leq d(\hat{o}_1, \hat{o}_1^{2n-1}) \leq d(\hat{o}_1, \hat{o}_1^{2n}) + d(\hat{o}_1^{2n}, \hat{o}_1^{2n-1}). \tag{2.15}$$

Taking $n \to \infty$ in (2.15), we find that

$$d(\hat{o}_1, \hat{o}_1^{2n-1}) \to d(\mathscr{T}_1, \mathscr{T}_2).$$

In the similar way, we have

$$d(\hat{o}_2, \hat{o}_2^{2n-1}) \to d(\mathscr{T}_1, \mathscr{T}_2), d(\hat{o}_3, \hat{o}_3^{2n-1}) \to d(\mathscr{T}_1, \mathscr{T}_2), \ldots, d(\hat{o}_m, \hat{o}_m^{2n-1}) \to d(\mathscr{T}_1, \mathscr{T}_2).$$

Now consider

$$\begin{aligned} d(\hat{o}_1^{2n}, \mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)) =& d(\mathscr{B}_1(\hat{o}_1^{2n-1}, \hat{o}_2^{2n-1}, \ldots, \hat{o}_m^{2n-1}), \mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)) \\ \leq& \frac{k}{m}(d(\hat{o}_1^{2n-1}, \hat{o}_1) + \ldots + d(\hat{o}_m^{2n-1}, \hat{o}_m)) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2). \end{aligned}$$

Taking $n \to \infty$, we obtain

$$d(\hat{o}_1, \mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)) = d(\mathscr{T}_1, \mathscr{T}_2).$$

Analogously, we obtain

$$d(\hat{o}_2, \mathscr{B}(\hat{o}_2, \hat{o}_3, \ldots, \hat{o}_1)) = d(\mathscr{T}_1, \mathscr{T}_2), \ldots, d(\hat{o}_m, \mathscr{B}(\hat{o}_m, \hat{o}_1, \ldots, \hat{o}_{m-1})) = d(\mathscr{T}_1, \mathscr{T}_2).$$

Therefore, $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)$ is a m-tuple BPP of $\mathscr{B}$. Similarly, $\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m \in \mathscr{T}_2^m$ so that $\hat{o}_1^{2n+1} \to \hat{o}'_1, \hat{o}_2^{2n+1} \to \hat{o}'_2, \ldots, \hat{o}_m^{2n+1} \to \hat{o}'_m$. Moreover, we get

$$d(\hat{o}'_1, \mathscr{B}_1(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m)) = d(\mathscr{T}_1, \mathscr{T}_2), \ldots, d(\hat{o}'_m, \mathscr{B}_1(\hat{o}'_m, \hat{o}'_1, \ldots, \hat{o}'_{m-1})) = d(\mathscr{T}_1, \mathscr{T}_2).$$

Hence $(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m)$ is a m-tuple BPP of $\mathscr{B}_1$. Let $\hat{o}_2 = \hat{o}_3 = \ldots = \hat{o}_m$ and $\hat{o}'_2 = \hat{o}'_3 =$

$\ldots = \hat{o}'_m$. Now, we claim that (2.14) holds. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(\hat{o}_1^{2n}, \hat{o}_1^{2n+1}) =& d(\mathscr{B}_1(\hat{o}_1^{2n-1}, \hat{o}_2^{2n-1}, \ldots, \hat{o}_m^{2n-1}), \mathscr{B}(\hat{o}_1^{2n}, \hat{o}_2^{2n}, \ldots, \hat{o}_m^{2n})) \\ \leq& \frac{k}{m}(d(\hat{o}_1^{2n-1}, \hat{o}_1^{2n}) + d(\hat{o}_2^{2n-1}, \hat{o}_2^{2n}) + \ldots + d(\hat{o}_m^{2n-1}, \hat{o}_m^{2n})) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2). \end{aligned}$$

Taking $n \to \infty$, we get

$$d(\hat{o}_1, \hat{o}'_1) \leq \frac{k}{m}(d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m)) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2). \tag{2.16}$$

Also for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d(\hat{o}_2^{2n}, \hat{o}_2^{2n+1}) =& d(\mathscr{B}_1(\hat{o}_2^{2n-1}, \hat{o}_3^{2n-1}, \ldots, \hat{o}_1^{2n-1}), \mathscr{B}(\hat{o}_2^{2n}, \hat{o}_3^{2n}, \ldots, \hat{o}_1^{2n})) \\ \leq& \frac{k}{m}(d(\hat{o}_2^{2n-1}, \hat{o}_2^{2n}) + d(\hat{o}_3^{2n-1}, \hat{o}_3^{2n}) + \ldots + d(\hat{o}_1^{2n-1}, \hat{o}_1^{2n})) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2). \end{aligned}$$

Letting $n \to \infty$, we get

$$d(\hat{o}_2, \hat{o}'_2) \leq \frac{k}{m}(d(\hat{o}_2, \hat{o}'_2) + d(\hat{o}_3, \hat{o}'_3) + \ldots + d(\hat{o}_1, \hat{o}'_1)) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2). \tag{2.17}$$

Similarly, we obtain

$$d(\hat{o}_m, \hat{o}'_m) \leq \frac{k}{m}(d(\hat{o}_m, \hat{o}'_m) + d(\hat{o}_1, \hat{o}'_1) + \ldots + d(\hat{o}_{m-1}, \hat{o}'_{m-1})) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2). \tag{2.18}$$

By (2.16), (2.17) and (2.18), we get

$$\begin{aligned} d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m) \leq& k(d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m)) \\ &+ m(1-k)d(\mathscr{T}_1, \mathscr{T}_2). \end{aligned}$$

It implies

$$d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m) \leq m d(\mathscr{T}_1, \mathscr{T}_2). \tag{2.19}$$

Since $d(\mathscr{T}_1, \mathscr{T}_2) \leq d(\hat{o}_1, \hat{o}'_1)$, $d(\mathscr{T}_1, \mathscr{T}_2) \leq d(\hat{o}_2, \hat{o}'_2)$,..., $d(\mathscr{T}_1, \mathscr{T}_2) \leq d(\hat{o}_m, \hat{o}'_m)$, we have

$$d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m) \geq md(\mathscr{T}_1, \mathscr{T}_2). \tag{2.20}$$

By (2.19) and (2.20), we have $d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m) = md(\mathscr{T}_1, \mathscr{T}_2)$. □

Theorem 2.1.5. Let $\mathscr{B} : \mathscr{T}_1^m \to \mathscr{T}_2$, $\mathscr{B}_1 : \mathscr{T}_2^m \to \mathscr{T}_1$ be two cyclic contraction mappings. If $(\hat{o}_1^0, \hat{o}_2^0, \ldots, \hat{o}_m^0) \in \mathscr{T}_1^m$ and the sequences $\{\hat{o}_1^n\}$,$\{\hat{o}_2^n\}$,...,$\{\hat{o}_m^n\}$ are defined as in (2.11). Then $\mathscr{B}$ has a m-tuple BPP $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m) \in \mathscr{T}_1^m$ and $\mathscr{B}_1$ has a m-tuple BPP $(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m) \in \mathscr{T}_2^m$. Moreover, we have

$$\hat{o}_1^{2n} \to \hat{o}_1, \hat{o}_2^{2n} \to \hat{o}_2, \ldots, \hat{o}_m^{2n} \to \hat{o}_m \text{ and } \hat{o}_1^{2n+1} \to \hat{o}'_1, \hat{o}_2^{2n+1} \to \hat{o}'_2, \ldots, \hat{o}_m^{2n+1} \to \hat{o}'_m.$$

In addition, if $\hat{o}_2 = \hat{o}_3 \ldots = \hat{o}_m$ and $\hat{o}'_2 = \hat{o}'_3 \ldots = \hat{o}'_m$ then

$$d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m) = md(\mathscr{T}_1, \mathscr{T}_2).$$

Proof. Since $(\hat{o}_1^0, \hat{o}_2^0, \ldots, \hat{o}_m^0) \in \mathscr{T}_1^m$ and (2.11) holds for each $n \in \mathbb{N}_0$, we get

$$(\hat{o}_1^{2n}, \hat{o}_2^{2n}, \ldots, \hat{o}_m^{2n}) \in \mathscr{T}_1 \text{ and } (\hat{o}_1^{2n+1}, \hat{o}_2^{2n+1}, \ldots, \hat{o}_m^{2n+1}) \in \mathscr{T}_2.$$

The compactness of $\mathscr{T}_1$ implies that sequence $(\hat{o}_1^{2n}, \hat{o}_2^{2n}, \ldots, \hat{o}_m^{2n})$ have convergent sequence $(\hat{o}_1^{2n_{\hat{s}}}, \ldots, \hat{o}_m^{2n_{\hat{s}}})$, respectively, so that $\hat{o}_1^{2n_{\hat{s}}} \to \hat{o}_1, \hat{o}_2^{2n_{\hat{s}}} \to \hat{o}_2, \ldots, \hat{o}_m^{2n_{\hat{s}}} \to \hat{o}_m$. Following the same methodology as in the proof of Theorem 2.1.4, we get

$$d(\hat{o}_1, \mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)) = d(\mathscr{T}_1, \mathscr{T}_2), \ldots, d(\hat{o}_m, \mathscr{B}(\hat{o}_m, \hat{o}_1, \ldots, \hat{o}_{m-1})) = d(\mathscr{T}_1, \mathscr{T}_2).$$

Since $\mathscr{T}_2$ is compact, we have

$$d(\hat{o}'_1, \mathscr{B}_1(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m)) = d(\mathscr{T}_1, \mathscr{T}_2), \ldots, d(\hat{o}'_m, \mathscr{B}_1(\hat{o}'_m, \hat{o}'_1, \ldots, \hat{o}'_{m-1})) = d(\mathscr{T}_1, \mathscr{T}_2).$$

Hence $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)$ is a m-tuple BPP of $\mathscr{B}$ in $\mathscr{T}_1^m$, $(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m)$ is a m-tuple BPP of $\mathscr{B}_1$ in $\mathscr{T}_2^m$ and $d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m) = md(\mathscr{T}_1, \mathscr{T}_2)$. □

Example 2.1.4. *Consider* $\mathscr{W} = \mathbb{R}$ *endowed with* $d(\hat{o}_1, \hat{o}_2) = |\hat{o}_1 - \hat{o}_2|$. *Suppose that* $\mathscr{T}_1 = [1, 7]$ *and* $\mathscr{T}_2 = [-7, -1]$, *then* $d(\mathscr{T}_1, \mathscr{T}_2) = 2$. *Define* $\mathscr{B} : \mathscr{T}_1^m \to \mathscr{T}_2$ *and* $\mathscr{B}_1 : \mathscr{T}_2^m \to \mathscr{T}_1$ *by*

$$\mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m) = \left(\frac{-\hat{o}_1 - \hat{o}_2 - \ldots - \hat{o}_m - 3m}{4m}\right),$$

and

$$\mathscr{B}_1(\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m) = \left(\frac{-\hat{g}_1 - \hat{g}_2 - \ldots - \hat{g}_m + 3m}{4m}\right),$$

for all $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m) \in \mathscr{T}_1^m$ *and* $(\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m) \in \mathscr{T}_2^m$. *Then we obtain*

$$\begin{aligned}
& d(\mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m), \mathscr{B}_1((\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m)) \\
=& \left|\frac{-\hat{o}_1 - \hat{o}_2 - \ldots - \hat{o}_m - 3m}{4m} - \frac{-\hat{g}_1 - \hat{g}_2 - \ldots - \hat{g}_m + 3m}{4m}\right| \\
=& \frac{|\hat{o}_1 - \hat{g}_1| + |\hat{o}_2 - \hat{g}_2| + \ldots + |\hat{o}_m - \hat{g}_m|}{4m} + \frac{3}{2} \\
\leq& \frac{k}{m}(d(\hat{o}_1, \hat{g}_1) + d(\hat{o}_2, \hat{g}_2) + ... + d(\hat{o}_m, \hat{g}_m)) + (1-k)d(\mathscr{T}_1, \mathscr{T}_2),
\end{aligned}$$

It is a cyclic contraction with $k = \frac{1}{4}$. *As* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *are closed and convex subsets of a UCBS, the pairs* $(\mathscr{T}_1, \mathscr{T}_2)$ *and* $(\mathscr{T}_2, \mathscr{T}_1)$ *fulfills the property UC*. Therefore, all suppositions of Theorem 2.1.4 are hold. Hence,* $\mathscr{B}$ *and* $\mathscr{B}_1$ *have a* m*-tuple BPP say* $(1, 1, \ldots, 1) \in \mathscr{T}_1^m$ *and* $(-1, -1, \ldots, -1) \in \mathscr{T}_2^m$ *respectively.*

Example 2.1.5. *Consider* $\mathscr{W} = \mathbb{R}^m$ *endowed with*

$$d((\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m), (\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m)) = \max\{|\hat{o}_1 - \hat{g}_1|, |\hat{o}_2 - \hat{g}_2|, \ldots, |\hat{o}_m - \hat{g}_m|\},$$

$(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m), (\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m) \in \mathscr{W}$. *Suppose that* $\mathscr{T}_1 = \{(\hat{o}_1, 2, \ldots, 2) \in \mathscr{W} : 0 \leq \hat{o}_1 \leq 2\}$ *and* $\mathscr{T}_2 = \{(\hat{g}_1, 0, \ldots, 0) \in \mathscr{W} : 0 \leq \hat{g}_1 \leq 2\}$. *Clearly,* $d(\mathscr{T}_1, \mathscr{T}_2) = 2$. *Define* $\mathscr{B} : \mathscr{T}_1^m \to \mathscr{T}_2$ *and* $\mathscr{B}_1 : \mathscr{T}_2^m \to \mathscr{T}_1$ *by*

$$\mathscr{B}((\hat{o}_1, 2, \ldots, 2), (\hat{o}_2, 2, \ldots, 2), \ldots, (\hat{o}_m, 2, \ldots, 2)) = \left(\frac{\hat{o}_1 + \hat{o}_2 + \ldots + \hat{o}_m}{m}, 0, 0, \ldots, 0\right),$$

and

$$\mathscr{B}_1((\hat{g}_1,0,\ldots,0),(\hat{g}_2,0,\ldots,0),\ldots,(\hat{g}_m,0,\ldots,0)) = \left(\frac{\hat{g}_1+\hat{g}_2+\ldots+\hat{g}_m}{m},2,2,\ldots,2\right).$$

We obtain

$$\begin{aligned}
&d\begin{pmatrix}\mathscr{B}((\hat{o}_1,2,\ldots,2),(\hat{o}_2,2,\ldots,2),\ldots,(\hat{o}_m,2,\ldots,2)),\\ \mathscr{B}_1((\hat{g}_1,0,\ldots,0),(\hat{g}_2,0,\ldots,0),\ldots,(\hat{g}_m,0,\ldots,0))\end{pmatrix}\\
=&d\left(\left(\frac{\hat{o}_1+\hat{o}_2+\ldots+\hat{o}_m}{m},0,0,\ldots,0\right),\left(\frac{\hat{g}_1+\hat{g}_2+\ldots+\hat{g}_m}{m},2,2,\ldots,2\right)\right)\\
=&\max\left\{\left|\frac{\hat{o}_1+\hat{o}_2+\ldots+\hat{o}_m}{2m}-\frac{\hat{g}_1+\hat{g}_2+\ldots+\hat{g}_m}{2m}\right|,|2|,\ldots,|2|\right\}=2.
\end{aligned}$$

Also,

$$\begin{aligned}
&\frac{k}{m}\Big(d((\hat{o}_1,2,\ldots,2),(\hat{g}_1,0,\ldots,0))+d((\hat{o}_2,2,\ldots,2),(\hat{g}_2,0,\ldots,0))+\ldots+\\
&d((\hat{o}_m,2,\ldots,2),(\hat{g}_m,0,\ldots,0))\Big)+(1-k)d(\mathscr{T}_1,\mathscr{T}_2)\\
=&\frac{k}{m}\Bigg(\max\{|\hat{o}_1-\hat{g}_1|,|2|,\ldots,|2|\}+\max\{|\hat{o}_2-\hat{g}_2|,|2|,\ldots,|2|\}+\ldots+\\
&\max\{|\hat{o}_m-\hat{g}_m|,|2|,\ldots,|2|\}\Bigg)\\
=&\frac{k}{m}\times 2m+(1-k)2=2,
\end{aligned}$$

for any $k<1$. It shows that

$$\begin{aligned}
d(\mathscr{B}(\hat{o}_1,\hat{o}_2,\ldots,\hat{o}_m),\mathscr{B}_1((\hat{g}_1,\hat{g}_2,\ldots,\hat{g}_m))\leq&\frac{k}{m}(d(\hat{o}_1,\hat{g}_1)+...+d(\hat{o}_m,\hat{g}_m))\\
&+(1-k)d(\mathscr{T}_1,\mathscr{T}_2).
\end{aligned}$$

Since $\mathscr{T}_1$ and $\mathscr{T}_2$ are convex and closed subsets of a MS, pairs $(\mathscr{T}_1,\mathscr{T}_2)$ and $(\mathscr{T}_2,\mathscr{T}_1)$ fulfills the property UC. Therefore, all the suppositions of Theorem 2.1.4 are hold. Hence, $\mathscr{B}$ and $\mathscr{B}_1$ have m-tuple BPP, $(2,2,\ldots,2)\in\mathscr{T}_1^m$ and $(0,0,\ldots,0)\in\mathscr{T}_2^m$ respectively.*

Theorem 2.1.6. Let $\mathscr{B}:\mathscr{T}_1^m\to\mathscr{T}_2$, $\mathscr{B}_1:\mathscr{T}_2^m\to\mathscr{T}_1$ be two cyclic contraction mappings. If $(\hat{o}_1^0,\hat{o}_2^0,\ldots,\hat{o}_m^0)\in\mathscr{T}_1^m$ and the sequences $\{\hat{o}_1^n\}$,$\{\hat{o}_2^n\}$,…,$\{\hat{o}_m^n\}$ are defined as in (2.11). If $d(\mathscr{T}_1,\mathscr{T}_2)=0$, then $\mathscr{B}$ has a m-tuple FP $(\hat{o}_1,\hat{o}_2,\ldots,\hat{o}_m)\in\mathscr{T}_1^m$ and $\mathscr{B}_1$ has a m-tuple

FP $(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m) \in \mathscr{T}_2^m$. Moreover, we have

$$\hat{o}_1^{2n} \to \hat{o}_1, \hat{o}_2^{2n} \to \hat{o}_2, \ldots, \hat{o}_m^{2n} \to \hat{o}_m \text{ and } \hat{o}_1^{2n+1} \to \hat{o}'_1, \hat{o}_2^{2n+1} \to \hat{o}'_2, \ldots, \hat{o}_m^{2n+1} \to \hat{o}'_m.$$

In addition, if $\hat{o}_2 = \hat{o}_3 \ldots = \hat{o}_m$ and $\hat{o}_2 = \hat{o}_3 \ldots = \hat{o}_m$, then $\mathscr{B}$ and $\mathscr{B}_1$ has common m-tuple FP in $(\mathscr{T}_1 \cap \mathscr{T}_2)^m$.

Proof. Because $d(\mathscr{T}_1, \mathscr{T}_2) = 0$, we find that pairs $(\mathscr{T}_1, \mathscr{T}_2)$ and $(\mathscr{T}_2, \mathscr{T}_1)$ justify the property UC*. The Theorem 2.1.4 gives us

$$d(\hat{o}_1, \mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)) = d(\mathscr{T}_1, \mathscr{T}_2), \ldots, d(\hat{o}_m, \mathscr{B}(\hat{o}_m, \hat{o}_1, \ldots, \hat{o}_{m-1})) = d(\mathscr{T}_1, \mathscr{T}_2),$$

and

$$d(\hat{o}'_1, \mathscr{B}_1(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m)) = d(\mathscr{T}_1, \mathscr{T}_2), \ldots, d(\hat{o}'_m, \mathscr{B}_1(\hat{o}'_m, \hat{o}'_1, \ldots, \hat{o}'_{m-1})) = d(\mathscr{T}_1, \mathscr{T}_2).$$

Since $d(\mathscr{T}_1, \mathscr{T}_2) = 0$, we get

$$\hat{o}_1 = \mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m), \hat{o}_2 = \mathscr{B}(\hat{o}_2, \hat{o}_3, \ldots, \hat{o}_1), \ldots, \hat{o}_m = \mathscr{B}(\hat{o}_m, \hat{o}_1, \ldots, \hat{o}_{m-1}),$$

it means that $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)$ is a m-tuple FP of $\mathscr{B}$ in $\mathscr{T}_1^m$. Similarly,

$$\hat{o}'_1 = \mathscr{B}_1(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m), \hat{o}'_2 = \mathscr{B}_1(\hat{o}'_2, \hat{o}'_3, \ldots, \hat{o}'_1), \ldots, \hat{o}'_m = \mathscr{B}_1(\hat{o}'_m, \hat{o}'_1, \ldots, \hat{o}'_{m-1}).$$

It implies $(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m)$ is a m-tuple FP of $\mathscr{B}_1$ in $\mathscr{T}_2^m$. Let $\hat{o}_2 = \hat{o}_3 \ldots = \hat{o}_m$ and $\hat{o}_2 = \hat{o}_3 \ldots = \hat{o}_m$. So by Theorem 2.1.4, we obtain

$$d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m) = md(\mathscr{T}_1, \mathscr{T}_2).$$

Since $(\mathscr{T}_1, \mathscr{T}_2) = 0$, we have

$$d(\hat{o}_1, \hat{o}'_1) + d(\hat{o}_2, \hat{o}'_2) + \ldots + d(\hat{o}_m, \hat{o}'_m) = 0,$$

it follows that $\hat{o}_1 = \hat{o}'_1, \hat{o}_2 = \hat{o}'_2, \ldots, \hat{o}_m = \hat{o}'_m$. Hence $\mathscr{B}$ and $\mathscr{B}_1$ has common m-tuple FP $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m) \in (\mathscr{T}_1 \cap \mathscr{T}_2)^m$. □

If $\mathscr{T}_1 = \mathscr{T}_2$ in Theorem 2.1.6, We have a subsequent outcomes:

Corollary 2.1.3. Let $\mathscr{B} : \mathscr{T}_1^m \to \mathscr{T}_1$, $\mathscr{B}_1 : \mathscr{T}_1^m \to \mathscr{T}_1$ be two cyclic contraction mappings. If $(\hat{o}_1^0, \hat{o}_2^0, \ldots, \hat{o}_m^0) \in \mathscr{T}_1^m$ and the sequences $\{\hat{o}_1^n\}$,$\{\hat{o}_2^n\}$,...,$\{\hat{o}_m^n\}$ are defined as in (2.11). Then $\mathscr{B}$ has a m-tuple FP $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m) \in \mathscr{T}_1^m$ and $\mathscr{B}_1$ has a m-tuple FP $(\hat{o}'_1, \hat{o}'_2, \ldots, \hat{o}'_m) \in \mathscr{T}_1^m$. Moreover, we have

$$\hat{o}_1^{2n} \to \hat{o}_1, \hat{o}_2^{2n} \to \hat{o}_2, \ldots, \hat{o}_m^{2n} \to \hat{o}_m \text{ and } \hat{o}_1^{2n+1} \to \hat{o}'_1, \hat{o}_2^{2n+1} \to \hat{o}'_2, \ldots, \hat{o}_m^{2n+1} \to \hat{o}'_m.$$

In addition, if $\hat{o}_2 = \hat{o}_3 \ldots = \hat{o}_m$ and $\hat{o}_2 = \hat{o}_3 \ldots = \hat{o}_m$, then $\mathscr{B}$ and $\mathscr{B}_1$ has common m-tuple FP in $\mathscr{T}_1^m$.

Corollary 2.1.4. Suppose that $\mathscr{B} : \mathscr{T}_1^m \to \mathscr{T}_1$ is a mapping

$$d(\mathscr{B}(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m), \mathscr{B}(\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m)) \leq \frac{k}{m}(d(\hat{o}_1, \hat{g}_1) + d(\hat{o}_2, \hat{g}_2) + \ldots + d(\hat{o}_m, \hat{g}_m),$$

for all $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m)$, $\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m \in \mathscr{T}_1^m$ and $k \in (0, 1)$. Then $\mathscr{B}$ has a m-tuple FP $(\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m) \in \mathscr{T}_1^m$.

2.1.2 Best proximity points in a relational MS

Here, firstly, we define proximal comparative mapping and using this, we present some BPP results on a relational MS $(\mathscr{W}, d)$.

Throughout this section, we assume MS $(\mathscr{W}, d)$ equipped with an arbitrary binary relation ® and κ^* the symmetric relation attached to ®.

To start with, we present the definition that follows:

Definition 2.1.4. A mapping $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is called modified proximal comparative mapping if $\mathscr{B}\hat{o}_0 \ \kappa^* \ \mathscr{B}\hat{o}_1$, $d(\hat{k}_1, \mathscr{B}\hat{o}_0) = d(\hat{k}_2, \mathscr{B}\hat{o}_1) = d(\mathscr{T}_1, \mathscr{T}_2)$, then $\mathscr{B}\hat{k}_1 \ \kappa^* \ \mathscr{B}\hat{k}_2$, for all $\hat{o}_0, \hat{o}_1, \hat{k}_1, \hat{k}_2 \in \mathscr{T}_1$.

Theorem 2.1.7. Suppose that $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is a continuous mapping such that $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$ which satisfies the assumptions of Definitions 1.1.13, 2.1.4. If there exists $\vartheta : \mathscr{W} \to [0, 1)$ such that

$$d(\mathscr{B}\hat{o}_1, \mathscr{B}\hat{o}_2) \leq \vartheta(\hat{o}_1)(d(\hat{o}_1, \hat{o}_2)) \text{ for all } \hat{o}_1, \hat{o}_2 \in \mathscr{T}_1, \hat{o}_1 \ \kappa^* \ \hat{o}_2; \tag{2.21}$$

and $\hat{o}_3, \hat{o}_4 \in \mathscr{T}_{1_0}$ and $\mathscr{B}\hat{o}_3 \in \mathscr{T}_{2_0}$ such that $d(\hat{o}_4, \mathscr{B}\hat{o}_3) = d(\mathscr{T}_1, \mathscr{T}_2)$, $\hat{o}_3 \ \kappa^* \ \hat{o}_4$, then mapping $\mathscr{B}$ has a BPP.

Proof. Define a mapping $\varrho : \mathscr{T}_2 \times \mathscr{T}_2 \to \mathbb{R}$ by $\varrho(\mathscr{B}\hat{o}_1, \mathscr{B}\hat{o}_2) = \begin{cases} 1, \text{ if } \mathscr{B}\hat{o}_1 \ \kappa^* \ \mathscr{B}\hat{o}_2, \\ 2, \text{ otherwise.} \end{cases}$

Suppose that $\varrho(\mathscr{B}\hat{o}_3, \mathscr{B}\hat{o}_4) \geq 1$, such that

$$\begin{cases} d(\hat{o}_4, \mathscr{B}\hat{o}_3) = & d(\mathscr{T}_1, \mathscr{T}_2), \\ d(\hat{o}_5, \mathscr{B}\hat{o}_4) = & d(\mathscr{T}_1, \mathscr{T}_2), \end{cases}$$

hold for some $\hat{o}_3, \hat{o}_4, \hat{o}_5 \in \mathscr{T}_1$. By definition of ϱ, we get

$$\begin{aligned} &\mathscr{B}\hat{o}_3 \ \kappa^* \ \mathscr{B}\hat{o}_4, \\ &d(\hat{o}_4, \mathscr{B}\hat{o}_3) = d(\mathscr{T}_1, \mathscr{T}_2), \\ &d(\hat{o}_5, \mathscr{B}\hat{o}_4) = d(\mathscr{T}_1, \mathscr{T}_2). \end{aligned}$$

Definition 2.1.4, implies that $\mathscr{B}\hat{o}_4 \ \kappa^* \ \mathscr{B}\hat{o}_5$. By definition of ϱ, we get $\varrho(\mathscr{B}\hat{o}_4, \mathscr{B}\hat{o}_5) = 1$. It shows that $\mathscr{B}$ is a modified ϱ proximal admissible mapping. Also,

$$d(\hat{o}_4, \mathscr{B}\hat{o}_3) = d(\mathscr{T}_1, \mathscr{T}_2) \text{ and } \ \varrho(\mathscr{B}\hat{o}_3, \mathscr{B}\hat{o}_4) \geq 1.$$

By (2.21), we get $\varrho(\mathscr{B}\hat{o}_1, \mathscr{B}\hat{o}_2) d(\mathscr{B}\hat{o}_1, \mathscr{B}\hat{o}_2) \leq \vartheta(\hat{o}_1)(d(\hat{o}_1, \hat{o}_2))$, for all $\hat{o}_1, \hat{o}_2 \in \mathscr{T}_1$. Thus, all the suppositions of Theorem 2.1.1 are hold, and $\mathscr{B}$ has a BPP. □

If the mapping $\mathscr{B}$ continuity is relaxed then we need the following condition in Theorem 2.1.7:

($\Re^*$-property): If $\{\hat{o}_n\}$ is a sequence in $\mathscr{W}$ such that $\mathscr{B}\hat{o}_n \ \kappa^* \ \mathscr{B}\hat{o}_{n+1}$ for all n and

$\lim_{n\to\infty} \hat{o}_n = \hat{o} \in \mathscr{W}$. It follows that subsequence $\{\hat{o}_{n_{\hat{s}}}\}$ of $\{\hat{o}_n\}$ exists such that $\mathscr{B}\hat{o}_{n_{\hat{s}}}\ \kappa^*\ \mathscr{B}\hat{o}$, for all $\hat{s}$.

Theorem 2.1.8. Consider $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is a mapping such that $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$ and satisfies the assumptions of Definitions 1.1.13, 2.1.4 and (2.21). Further, assume that ($\mathfrak{R}^*$-property) holds and there exist $\hat{o}_3, \hat{o}_4 \in \mathscr{T}_{1_0}$ and $\mathscr{B}\hat{o}_3 \in \mathscr{T}_{2_0}$ such that $d(\hat{o}_4, \mathscr{B}\hat{o}_3) = \mathscr{B}(\mathscr{T}_1, \mathscr{T}_2)$, $\hat{o}_3\ \kappa^*\ \hat{o}_4$. Then mapping $\mathscr{B}$ has a BPP.

Proof. Noting that ($\mathfrak{R}^*$-property) implies ($\mathfrak{R}$-property), the result derives from Theorem 2.1.2. □

2.1.3 Best proximity points in normed linear spaces

This subsection contains the study of some sufficient conditions required for the existence of best proximity pairs using measure of noncompactness in the framework of a BS.

Throughout this section, we assume that $(\mathscr{T}_1, \mathscr{T}_2)$ is NBCC subset of a BS $\mathscr{W}$ and $\mathscr{T}_{1_0}$ is non-empty.

Definition 2.1.5. A mapping $\mathscr{B} : \mathscr{T}_1 \cup \mathscr{T}_2 \to \mathscr{T}_1 \cup \mathscr{T}_2$ is said to be ς-condensing cyclic (respectively, noncyclic), provided that for any NBCC proximal and $\mathscr{B}$ invariant pair $(\mathscr{I}_1, \mathscr{J}_1) \subseteq (\mathscr{T}_1, \mathscr{T}_2)$ of a BS $\mathscr{W}$ such that $||\mathscr{I}_1 - \mathscr{J}_1|| = ||\mathscr{T}_1 - \mathscr{T}_2||$ with $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_2$ and $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_1$ (respectively, $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_1$ and $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_2$), we have

$$\mho(\mathscr{B}(\mathscr{I}_1) \cup \mathscr{B}(\mathscr{J}_1)) \leq \varsigma(\mho(\mathscr{I}_1 \cup \mathscr{J}_1))\mho(\mathscr{I}_1 \cup \mathscr{J}_1),$$

where $\varsigma : [0, \infty) \to [0, 1)$ is a function satisfies $\limsup_{s\to t} \varsigma(s) < 1$ for all $t \in [0, \infty)$ and $\mho$ is an MNC on $\mathscr{W}$.

Definition 2.1.6. A mapping $\mathscr{B} : \mathscr{T}_1 \cup \mathscr{T}_2 \to \mathscr{T}_1 \cup \mathscr{T}_2$ is said to be ψ-condensing cyclic (respectively, noncyclic) if provided that for any NBCC proximal and $\mathscr{B}$ invariant pair $(\mathscr{I}_1, \mathscr{J}_1) \subseteq (\mathscr{T}_1, \mathscr{T}_2)$ of a BS $\mathscr{W}$ such that $||\mathscr{I}_1 - \mathscr{J}_1|| = ||\mathscr{T}_1 - \mathscr{T}_2||$, with $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_2$ and $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_1$ (respectively, $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_1$ and $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_2$), we have

$$\mho(\mathscr{B}(\mathscr{I}_1) \cup \mathscr{B}(\mathscr{J}_1)) \leq \mho(\mathscr{I}_1 \cup \mathscr{J}_1) - \psi(\mho(\mathscr{I}_1 \cup \mathscr{J}_1)),$$

where ψ is a continuous function on $[0, \infty)$ satisfies $\psi(t) = 0$ if and only if $t = 0$ and $\mho$ is an MNC on $\mathscr{W}$.

Theorem 2.1.9. Suppose that $\mathscr{B} : \mathscr{T}_1 \cup \mathscr{T}_2 \to \mathscr{T}_1 \cup \mathscr{T}_2$ is a relatively NE and satisfies Definition 2.1.5. Then $\mathscr{B}$ has a best proximity pair.

Proof. As $\mathscr{T}_{1_0}$ is non-empty, $(\mathscr{T}_{1_0}, \mathscr{T}_{2_0})$ is non-empty. Also, $(\mathscr{T}_{1_0}, \mathscr{T}_{2_0})$ is closed, convex, $\mathscr{B}$ -invariant and proximinal pair on $\mathscr{B}$. For $\hat{o} \in \mathscr{T}_{1_0}$, there is a $\hat{g} \in \mathscr{T}_{2_0}$ such that $||\hat{o} - \hat{g}|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Since $\mathscr{B}$ is relatively NE

$$||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| \leq ||\hat{o} - \hat{g}|| = ||\mathscr{T}_1 - \mathscr{T}_2||,$$

which gives $\mathscr{B}\hat{o} \in \mathscr{T}_{1_0}$, that is, $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{1_0}$. Similarly, $\mathscr{B}(\mathscr{T}_{2_0}) \subseteq \mathscr{T}_{2_0}$ and so $\mathscr{B}$ is noncyclic on $\mathscr{T}_{1_0} \cup \mathscr{T}_{2_0}$.

Define a pair $(\mathscr{Q}_n, \mathscr{R}_n)$ as $\mathscr{Q}_n = \overline{con}(\mathscr{B}(\mathscr{Q}_{n-1}))$ and $\mathscr{R}_n = \overline{con}(\mathscr{B}(\mathscr{R}_{n-1}))$, $n \geq 1$ with $\mathscr{Q}_0 = \mathscr{T}_{1_0}$ and $\mathscr{R}_0 = \mathscr{T}_{2_0}$. For all $n \in \mathbb{N}$, we have to show that $\mathscr{Q}_{n+1} \subseteq \mathscr{R}_n$ and $\mathscr{R}_n \subseteq \mathscr{Q}_{n-1}$. $\mathscr{R}_1 = \overline{con}(\mathscr{B}(\mathscr{R}_0)) = \overline{con}(\mathscr{B}(\mathscr{T}_{2_0})) = \overline{con}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{1_0} = \mathscr{Q}_0$. Therefore, $\mathscr{B}(\mathscr{R}_1) \subseteq \mathscr{B}(\mathscr{Q}_0)$. So $\mathscr{R}_2 = \overline{con}(\mathscr{B}(\mathscr{R}_1)) \subseteq \overline{con}(\mathscr{B}(\mathscr{Q}_0)) \subseteq \mathscr{Q}_1$. Continuing this process, we have $\mathscr{R}_n \subseteq \mathscr{Q}_{n-1}$ by induction. Using similar lines, we get $\mathscr{Q}_{n+1} \subseteq \mathscr{R}_n$. Thus $\mathscr{Q}_{n+2} \subset \mathscr{R}_{n+1} \subseteq \mathscr{Q}_n \subseteq \mathscr{R}_{n-1}$ for all $n \in \mathbb{N}$. So, $\{(\mathscr{Q}_{2n}, \mathscr{R}_{2n})\}$ non-increasing sequence of non-empty, closed and convex pairs in $\mathscr{T}_1 \times \mathscr{T}_2$. Also, $\mathscr{B}(\mathscr{R}_n) \subseteq \mathscr{B}(\mathscr{R}_{n-1}) \subseteq \overline{con}(\mathscr{B}(\mathscr{R}_{n-1})) = \mathscr{R}_n$ and $\mathscr{B}(\mathscr{Q}_n) \subseteq \mathscr{B}(\mathscr{Q}_{n-1}) \subseteq \overline{con}(\mathscr{B}(\mathscr{Q}_{n-1})) = \mathscr{Q}_n$. Hence a pair $(\mathscr{Q}_n, \mathscr{R}_n)$ is $\mathscr{B}$ -invariant for all $n \in \mathbb{N}$. If $(\hat{o}, \hat{g}) \in \mathscr{T}_1 \times \mathscr{T}_2$ is a proximinal pair then

$$d(\mathscr{Q}_{2n}, \mathscr{R}_{2n}) \leq ||\mathscr{B}^{2n}u - \mathscr{B}^{2n}v|| \leq ||\hat{o} - \hat{g}|| = ||\mathscr{T}_1 - \mathscr{T}_2||.$$

Since a pair $(\mathscr{Q}_0, \mathscr{R}_0)$ is proximinal, result is true for $n = 0$. Assume it is accurate for $n = \hat{s}$. Next, we have to show that it is accurate for $n = \hat{s} + 1$. Let $\mathscr{W}$ be an arbitrary member in $\mathscr{Q}_{\hat{s}+1} = \overline{con}(\mathscr{B}(\mathscr{Q}_{\hat{s}}))$. It can be written as $\hat{k} = \sum_{l=1}^{m'} \lambda_l \mathscr{B}(\hat{k}_l)$ with

$\hat{k}_l \in \mathscr{Q}_{\hat{s}}$, $m \in \mathbb{N}$, $\lambda_l \geq 0$ and $\sum_{l=1}^{m} \lambda_l = 1$. Since a pair $(\mathscr{Q}_{\hat{s}}, \mathscr{R}_{\hat{s}})$ is proximinal, there exists $\acute{f}_l \in \mathscr{R}_{\hat{s}}$ for $1 \leq l \leq m$ such that $\hat{k}_l - \acute{f}_l = ||\mathscr{Q}_{\hat{s}} - \mathscr{R}_{\hat{s}}|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Take $\acute{f} = \sum_{l=1}^{m} \lambda_l \mathscr{B}(\acute{f}_l)$. Then $\acute{f} \in \overline{con}(\mathscr{B}(\mathscr{R}_{\hat{s}})) = \mathscr{R}_{\hat{s}+1}$ and

$$||x - y|| = \left|\left|\sum_{l=1}^{m} \lambda_l \mathscr{B}(\hat{k}_l) - \sum_{l=1}^{m} \lambda_l \mathscr{B}(\acute{f}_l)\right|\right| \leq \sum_{l=1}^{m'} \lambda_l ||\hat{k}_l - \acute{f}_l|| = ||\mathscr{T}_1 - \mathscr{T}_2||.$$

It shows that $(\mathscr{Q}_{\hat{s}+1}, \mathscr{R}_{\hat{s}+1})$ is proximinal pair. By induction, we obtain that $(\mathscr{Q}_n, \mathscr{R}_n)$ is proximinal for all $n \in \mathbb{N}$. As is now known, there are two situations that can occur: either $\max\{\mho(\mathscr{Q}_{2\hat{s}}), \mho(\mathscr{R}_{2\hat{s}})\} = 0$ for some $\hat{s} \in \mathbb{N}$ or $\max\{\mho(\mathscr{Q}_{2n}), \mho(\mathscr{R}_{2n})\} > 0$. If $\max\{\mho(\mathscr{Q}_{\hat{s}}), \mho(\mathscr{R}_{\hat{s}})\} = 0$ for some $\hat{s} \in \mathbb{N}$, then $\mathscr{B} : \mathscr{Q}_{\hat{s}} \cup \mathscr{R}_{\hat{s}} \to \mathscr{Q}_{2\hat{s}} \cup \mathscr{R}_{2\hat{s}}$ is compact, by Theorem 1.1.1 we obtain the result. Consider $\max\{\mho(\mathscr{Q}_n), \mho(\mathscr{R}_n)\} > 0$ for all $n \in \mathbb{N}$. Since $\mathscr{Q}_{2n+1} \subseteq \mathscr{B}(\mathscr{Q}_{2n})$ and $\mathscr{R}_{2n+1} \subseteq \mathscr{B}(\mathscr{R}_{2n})$, we have

$$\begin{aligned}
\mho(\mathscr{Q}_{2n+1} \cup \mathscr{R}_{2n+1}) &= \max\{\mho(\mathscr{Q}_{2n+1}), \mho(\mathscr{R}_{2n+1})\} \\
&= \max\{\mho(\overline{con}(\mathscr{B}(\mathscr{Q}_{2n}))), \mho(\overline{con}(\mathscr{B}(\mathscr{R}_{2n})))\} \\
&= \max\{\mho(\mathscr{B}(\mathscr{Q}_{2n})), \mho(\mathscr{B}(\mathscr{R}_{2n}))\} \\
&= \mho(\mathscr{B}(\mathscr{Q}_{2n}) \cup \mathscr{B}(\mathscr{R}_{2n})) \\
&\leq \varsigma(\mho(\mathscr{Q}_n \cup \mathscr{R}_{2n}))\mho(\mathscr{Q}_n \cup \mathscr{R}_{2n}) \\
&\leq \mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n}).
\end{aligned}$$

Therefore, $\mho_n = \mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n})$ is a non-increasing sequence and there is $\alpha \geq 0$ in such way that $\mho_n \to \alpha$ as $n \to \infty$. Let $\alpha > 0$, for all $n \in \mathbb{N}$, we obtain

$$\frac{\mho(\mathscr{Q}_{2n+1} \cup \mathscr{R}_{2n+1})}{\mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n})} \leq \varsigma(\mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n})).$$

By above inequality $\varsigma(\mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n})) \geq 1$, which is a contradiction. So $\alpha = 0$ and $\mho_n = \mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n}) \to 0$ as $n \to \infty$. Now, let $\mathscr{Q}_\infty = \cap_{n=0}^{\infty} \mathscr{Q}_{2n}$ and $\mathscr{R}_\infty = \cap_{n=0}^{\infty} \mathscr{R}_{2n}$. By (d') of MNC, a pair $(\mathscr{Q}_\infty, \mathscr{R}_\infty)$ is $\mathscr{B}$ -invariant, non-empty, compact and convex with $||\mathscr{Q}_\infty - \mathscr{R}_\infty|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Therefore, $\mathscr{B}$ has a best proximity pair. □

Theorem 2.1.10. Suppose that $\mathscr{B} : \mathscr{T}_1 \cup \mathscr{T}_2 \to \mathscr{T}_1 \cup \mathscr{T}_2$ is a relatively NE and satisfies

Definition 2.1.5. Then $\mathscr{B}$ has a best proximity pair.

Proof. Using the similar lines of proof Theorem 2.1.9, we have $(\mathscr{Q}_n, \mathscr{R}_n)$ is proximinal for all $n \in \mathbb{N}$ with $||\mathscr{Q}_n - \mathscr{R}_n|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. If $\max\{\mho(\mathscr{Q}_{\hat{s}}), \mho(\mathscr{R}_{\hat{s}})\} = 0$ for some $\hat{s} \in \mathbb{N}$, then $\mathscr{B} : \mathscr{Q}_{\hat{s}} \cup \mathscr{R}_{\hat{s}} \to \mathscr{Q}_{2\hat{s}} \cup \mathscr{R}_{2\hat{s}}$ is compact, by Theorem 1.1.1 we obtain the result. Consider $\max\{\mho(\mathscr{Q}_{2n}), \mho(\mathscr{R}_{2n})\} > 0$. Since $\mathscr{Q}_{2n+1} \subseteq \mathscr{B}(\mathscr{Q}_{2n})$ and $\mathscr{R}_{2n+1} \subseteq \mathscr{B}(\mathscr{R}_{2n})$, we have

$$\begin{aligned}
\mho(\mathscr{Q}_{2n+1} \cup \mathscr{R}_{2n+1}) &= \max\{\mho(\mathscr{Q}_{2n+1}), \mho(\mathscr{R}_{2n+1})\} \\
&= \max\{\mho(\overline{con}(\mathscr{B}(\mathscr{Q}_{2n}))), \mho(\overline{con}(\mathscr{B}(\mathscr{R}_{2n})))\} \\
&= \max\{\mho(\mathscr{B}(\mathscr{Q}_{2n})), \mho(\mathscr{B}(\mathscr{R}_{2n}))\} \\
&= \mho(\mathscr{B}(\mathscr{Q}_{n}) \cup \mathscr{B}(\mathscr{R}_{2n})) \\
&\leq \mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n}) - \psi(\mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n})) \\
&\leq \mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n}).
\end{aligned}$$

Therefore, $\mho_n = \mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n})$ is a non-increasing sequence and there is $\alpha \geq 0$ in such way that $\mho_n \to \alpha$ as $n \to \infty$. Let $\alpha > 0$, for all $n \in \mathbb{N}$, we get

$$\mho(\mathscr{Q}_{2n+1} \cup \mathscr{R}_{2n+1}) - \mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n}) \leq -\psi(\mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n})).$$

Taking $n \to \infty$ in above inequality, we get $\psi(\alpha) = 0$, for $\alpha > 0$, which is contradiction. So $\alpha = 0$ and $\mho_n = \mho(\mathscr{Q}_{2n} \cup \mathscr{R}_{2n}) \to 0$ as $n \to \infty$. Now, let $\mathscr{Q}_\infty = \cap_{n=0}^{\infty} \mathscr{Q}_{2n}$ and $\mathscr{R}_\infty = \cap_{n=0}^{\infty} \mathscr{R}_{2n}$. By (d') of MNC, a pair $(\mathscr{Q}_\infty, \mathscr{R}_\infty)$ is $\mathscr{B}$ -invariant, non-empty, compact and convex with $||\mathscr{Q}_\infty - \mathscr{R}_\infty|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Therefore, $\mathscr{B}$ has a best proximity pair. □

2.1.4 Best proximity points in binormed spaces

In this section, we prove some BPP results in a uniformly convex binormed linear space using the Hardy Rogers type contraction mapping. We also give some numerical examples.

In the entire section, we assume that $\mathscr{T}_1, \mathscr{T}_2$ are non-empty closed subsets of a uniformly convex binormed linear space $(\mathscr{W}, ||.||_1, ||.||)$ with $||.||_1 \leq ||.||$.

Theorem 2.1.11. Suppose that a mapping $\mathscr{B} : \mathscr{T}_1 \cup \mathscr{T}_2 \to \mathscr{T}_1 \cup \mathscr{T}_2$ satisfies the following hypotheses:

(i) $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_2$ and $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_1$,

(ii) $\mathscr{W}$ is complete with respect to $||.||_1$,

(iii) there exist non-negative real numbers $\alpha', \beta', \gamma', \delta'$ such that $\alpha' + \beta' + \gamma' + 2\delta' < 1$ and

$$||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| \leq \alpha'||\hat{o} - \mathscr{B}\hat{o}|| + \beta'||\hat{g} - \mathscr{B}\hat{g}|| + \gamma'||\hat{o} - \hat{g}|| \\ + \delta'(||\hat{o} - \mathscr{B}\hat{g}|| + ||\hat{g} - \mathscr{B}\hat{o}||) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ ||\mathscr{T}_1 - \mathscr{T}_2||,$$

for all $\hat{o} \in \mathscr{T}_1$ and $\hat{g} \in \mathscr{T}_2$.
If $\hat{o}_0 \in \mathscr{T}_1$ and $\hat{o}_{n+1} = \mathscr{B}\hat{o}_n$, where $n \in \mathbb{N}$, then $\mathscr{B}$ has a BPP in $\mathscr{T}_1$.

Proof. By Proposition 2.1.1, we have

$$||\hat{o}_{2n} - \mathscr{B}\hat{o}_{2n}|| \to ||\mathscr{T}_1 - \mathscr{T}_2|| \text{ and } ||\hat{o}_{2n+1} - \mathscr{B}\hat{o}_{2n+1}|| \to ||\mathscr{T}_1 - \mathscr{T}_2||.$$

Since $\mathscr{W}$ is a UCBS by Lemma 1.1.5, we get

$$||\hat{o}_{2n} - \hat{o}_{2(n+1)}||_1 \to 0 \text{ and } ||\mathscr{B}\hat{o}_{2n+1} - \mathscr{B}\hat{o}_{2n}||_1 \to 0. \tag{2.22}$$

Now, we claim that for given $\epsilon > 0$, there is N_0 such that $||\hat{o}_{2m} - \mathscr{B}\hat{o}_{2n}||_1 < ||\mathscr{T}_1 - \mathscr{T}_2||_1 + \epsilon$, for all $m > n \geq N_0$. Suppose not, then there exist $\epsilon > 0$ such that for all $\hat{s} \in N$, there exists $m_{\hat{s}} > n_{\hat{s}} \geq \hat{s}$ for which $||\hat{o}_{2m_{\hat{s}}} - \mathscr{B}\hat{o}_{2n_{\hat{s}}}||_1 \geq ||\mathscr{T}_1 - \mathscr{T}_2||_1 + \epsilon$, suppose this $m_{\hat{s}}$ is the smallest integer greater than $n_{\hat{s}}$ to satisfy the above inequality. From triangle inequality, we get

$$||\mathscr{T}_1 - \mathscr{T}_2||_1 + \epsilon \leq ||\hat{o}_{2m_{\hat{s}}} - \mathscr{B}\hat{o}_{2n_{\hat{s}}}||_1$$

$$\leq ||\hat{o}_{2m_{\hat{s}}} - \hat{o}_{2(m_{\hat{s}-1})}||_1 + ||\hat{o}_{2(m_{\hat{s}-1})} - \mathscr{B}\hat{o}_{2n_{\hat{s}}}||_1$$
$$< ||\hat{o}_{2m_{\hat{s}}} - \hat{o}_{2(m_{\hat{s}-1})}||_1 + ||\mathscr{T}_1 - \mathscr{T}_2||_1 + \epsilon. \quad (2.23)$$

Using (2.22) and taking $\hat{s} \to \infty$ in (2.23), we have $||\hat{o}_{2m_{\hat{s}}} - \mathscr{B}\hat{o}_{2n_{\hat{s}}}||_1 = ||\mathscr{T}_1 - \mathscr{T}_2||_1 + \epsilon$. Again, consider

$$\begin{aligned}
&||\mathscr{T}_1 - \mathscr{T}_2||_1 + \epsilon \leq ||\hat{o}_{2m_{\hat{s}+1}} - \mathscr{B}\hat{o}_{2n_{\hat{s}+1}}||_1 = ||\mathscr{B}\hat{o}_{2m_{\hat{s}}} - \mathscr{B}\hat{o}_{2n_{\hat{s}+1}}||_1\\
\leq&\alpha'||\hat{o}_{2m_{\hat{s}}} - \mathscr{B}\hat{o}_{2m_{\hat{s}}}||_1 + \beta'||\hat{o}_{2n_{\hat{s}+1}} - \mathscr{B}\hat{o}_{2n_{\hat{s}+1}}||_1 + \gamma'||\hat{o}_{2m_{\hat{s}}} - \hat{o}_{2n_{\hat{s}+1}}||_1\\
&+ \delta'(||\hat{o}_{2m_{\hat{s}}} - \mathscr{B}\hat{o}_{2n_{\hat{s}+1}}||_1 + ||\mathscr{B}\hat{o}_{2m_{\hat{s}}} - \hat{o}_{2n_{\hat{s}+1}}||_1 + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ ||\mathscr{T}_1 - \mathscr{T}_2||_1\\
=&\alpha'||\hat{o}_{2m_{\hat{s}}} - \hat{o}_{2m_{\hat{s}+1}}||_1 + \beta'||\hat{o}_{2n_{\hat{s}+1}} - \hat{o}_{2n_{\hat{s}+2}}||_1 + \gamma'||\hat{o}_{2m_{\hat{s}}} - \mathscr{B}\hat{o}_{2n_{\hat{s}}}||_1\\
&+ \delta'(||\hat{o}_{2m_{\hat{s}}} - \mathscr{B}\hat{o}_{2n_{\hat{s}+1}}||_1 + ||\hat{o}_{2m_{\hat{s}+1}} - \hat{o}_{2n_{\hat{s}+1}}||_1 + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ ||\mathscr{T}_1 - \mathscr{T}_2||_1\\
\leq&\alpha'||\hat{o}_{2m_{\hat{s}}} - \hat{o}_{2m_{\hat{s}+1}}||_1 + \beta'||\hat{o}_{2n_{\hat{s}+1}} - \hat{o}_{2n_{\hat{s}+2}}||_1 + \gamma'||\hat{o}_{2m_{\hat{s}}} - \mathscr{B}\hat{o}_{2n_{\hat{s}}}||_1\\
&+ \delta'(||\hat{o}_{2m_{\hat{s}}} - \hat{o}_{2m_{\hat{s}-2}}||_1 + ||\hat{o}_{2m_{\hat{s}-2}} - \mathscr{B}\hat{o}_{2n_{\hat{s}+1}}||_1)\\
&+ \delta'(||\hat{o}_{2m_{\hat{s}+1}} - \hat{o}_{2m_{\hat{s}-1}}||_1 + ||\hat{o}_{2m_{\hat{s}-1}} - \mathscr{B}\hat{o}_{2n_{\hat{s}}}||_1) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ ||\mathscr{T}_1 - \mathscr{T}_2||_1.
\end{aligned} \quad (2.24)$$

Taking $\hat{s} \to \infty$ in (2.24), we get

$$||\mathscr{T}_1 - \mathscr{T}_2||_1 + \epsilon \leq ||\mathscr{T}_1 - \mathscr{T}_2||_1 + (\gamma' + 2\delta')\epsilon,$$

which is a contradiction. Hence, $\{\hat{o}_{2n}\}$ is a Cauchy sequence in $\mathscr{T}_1$ with respect $||.||_1$. By the closedness of $\mathscr{T}_1$, $\{\hat{o}_{2n}\}$ converges to a point $\hat{o}$ in $\mathscr{T}_1$. Therefore, by Theorem 2.1.3, $\mathscr{B}$ has a BPP in $\mathscr{T}_1$, that is, $||\hat{o} - \mathscr{B}\hat{o}||_1 = ||\mathscr{T}_1 - \mathscr{T}_2||_1$ in $\mathscr{T}_1$. □

By choosing different values of $\alpha', \beta', \gamma', \delta'$ in Theorem 2.1.11, we get many best proximity results as described below:

If $\alpha' = \beta'$ and $\gamma' = \delta' = 0$ in Theorem 2.1.11, We obtain a result as follows:

Corollary 2.1.5. Assume that $\mathscr{B}$ is fulfilling the hypotheses (i), (ii) of Theorem 2.1.11 and $||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| \leq \alpha'(||\hat{o} - \mathscr{B}\hat{o}|| + ||\hat{g} - \mathscr{B}\hat{g}||) + (1 - 2\alpha')\ ||\mathscr{T}_1 - \mathscr{T}_2||$, for all $\hat{o} \in \mathscr{T}_1, \hat{g} \in \mathscr{T}_2$ and α' is a non-negative real number with $2\alpha' < 1$. If $\hat{o}_0 \in \mathscr{T}_1$ and $\hat{o}_{n+1} = \mathscr{B}\hat{o}_n$ where $n \in \mathbb{N}$, then $\mathscr{B}$ has a BPP in $\mathscr{T}_1$.

If $\alpha' = \beta' = \delta' = 0$ in Theorem 2.1.11, We obtain a result as follows:

Corollary 2.1.6. Assume that $\mathscr{B}$ is fulfilling the hypotheses (i), (ii) of Theorem 2.1.11 and $||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| \leq \gamma'||\hat{o} - \hat{g}|| + (1-\gamma')\ ||\mathscr{T}_1 - \mathscr{T}_2||$, for all $\hat{o} \in \mathscr{T}_1, \hat{g} \in \mathscr{T}_2$ and γ' is a non-negative real number with $\gamma' < 1$. If $\hat{o}_0 \in \mathscr{T}_1$ and $\hat{o}_{n+1} = \mathscr{B}\hat{o}_n$ where $n \in \mathbb{N}$, then $\mathscr{B}$ has a BPP in $\mathscr{T}_1$.

If $\alpha' = \beta' = \gamma'$ and $\delta' = 0$ in Theorem 2.1.11,We obtain a result as follows:

Corollary 2.1.7. Assume that $\mathscr{B}$ is fulfilling the hypotheses (i), (ii) of Theorem 2.1.11 and $||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| \leq \alpha'(||\hat{o} - \mathscr{B}\hat{o}|| + ||\hat{g} - \mathscr{B}\hat{g}|| + ||\hat{o} - \hat{g}||) + (1-3\alpha')\ ||\mathscr{T}_1 - \mathscr{T}_2||$, for all $\hat{o} \in \mathscr{T}_1, \hat{g} \in \mathscr{T}_2$ and α' is a non-negative real number with $3\alpha' < 1$. If $\hat{o}_0 \in \mathscr{T}_1$ and $\hat{o}_{n+1} = \mathscr{B}\hat{o}_n$ where $n \in \mathbb{N}$, then $\mathscr{B}$ has a BPP in $\mathscr{T}_1$.

Remark 2.1.2. If we take $||.|| = ||.||_1$ in Corollaries 2.1.5, 2.1.6 and 2.1.7, then we get the corresponding results of Petric [53](Theorem 5, p-151), Eldred and Veeramani [22](Theorem 3.10, p-1005), Karapinar [39])(Theorem 10, p-1763).

If $\mathscr{T}_1 = \mathscr{T}_2 = \mathscr{W}$ in Theorem 2.1.11, we have FP results as follows:

Corollary 2.1.8. Assume that $\mathscr{B}$ is fulfilling the following hypotheses:

(i) $\mathscr{W}$ is complete with respect to $||.||_1$,

(ii) there exist $\alpha', \beta', \gamma', \delta'$ are non-negative real numbers such that $\alpha'+\beta'+\gamma'+2\delta' < 1$ and

$$||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| \leq \alpha'||\hat{o} - \mathscr{B}\hat{o}|| + \beta'||\hat{g} - \mathscr{B}\hat{g}|| + \gamma'||\hat{o} - \hat{g}|| \\ + \delta'(||\hat{o} - \mathscr{B}\hat{g}|| + ||\hat{g} - \mathscr{B}\hat{o}||), \tag{2.25}$$

for all $\hat{o}, \hat{g} \in \mathscr{W}$. If $\hat{o}_0 \in \mathscr{W}$ and $\hat{o}_{n+1} = \mathscr{B}\hat{o}_n$, where $n \in \mathbb{N}$, then $\mathscr{B}$ has a FP.

Corollary 2.1.9. (Theorem 1, p-139, [47]) Assume that $\mathscr{B}$ is fulfilling the condition (i) of Corollary 2.1.8 and $||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| \leq \gamma'||\hat{o} - \hat{g}||$, for all $\hat{o}, \hat{g} \in \mathscr{W}$ and γ' is a non-negative real number with $\gamma' < 1$. If $\hat{o}_0 \in \mathscr{W}$ and $\hat{o}_{n+1} = \mathscr{B}\hat{o}_n$ where $n \in \mathbb{N}$, then $\mathscr{B}$ has a FP.

Remark 2.1.3.
- If we take $\mathscr{T}_1 = \mathscr{T}_2 = \mathscr{W}$ and $||.|| = ||.||_1$ in Corollaries 2.1.5, 2.1.6 and 2.1.7, then we get the corresponding FP results of Banach [4](Theorem 6,

p-160), Karapinar and Erhan [40](Corollary 2.1, p-560), Karapinar and Erhan [40](Corollary 2.2, p-562).

Now, we provide some illustrations:

Example 2.1.6. *Consider* $\mathscr{W} = \mathbb{R}$*, define* $||.||, ||.||_1 : \mathscr{W} \to \mathbb{R}_+$ *by*

$$||\hat{o}|| = 2|\hat{o}| \text{ and } ||\hat{o}||_1 = |\hat{o}|$$

for all $\hat{o} \in \mathscr{W}$*. It is easy to see that* $||\hat{o}||_1 < ||\hat{o}||$*, for all* $\hat{o} \in \mathscr{W}$*. Suppose* $\mathscr{T}_1 = [\frac{1}{4}, \frac{1}{2}]$ *and* $\mathscr{T}_2 = [\frac{3}{4}, \frac{9}{8}]$ *are two subsets of* $\mathscr{W}$*, then* $||\mathscr{T}_1 - \mathscr{T}_2|| = 0.5$ *and* $||\mathscr{T}_1 - \mathscr{T}_2||_1 = 0.25$*.*

Define $\mathscr{B} : \mathscr{T}_1 \cup \mathscr{T}_2 \to \mathscr{T}_1 \cup \mathscr{T}_2$ *by* $\mathscr{B}(\hat{o}) = \begin{cases} \frac{\hat{o}}{32} + \frac{3}{4} \text{ if } \hat{o} \in [\frac{1}{4}, \frac{1}{2}) \\ \frac{3}{4} \text{ if } \hat{o} = \frac{1}{2} \\ \frac{1}{2} \text{ if } \hat{o} \in [\frac{3}{4}, \frac{9}{8}] \end{cases}$

for all $\hat{o} \in \mathscr{T}_1 \cup \mathscr{T}_2$*. Next, we have to prove that* $\mathscr{B}$ *satisfies the following inequality,*

$$\begin{aligned} ||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| \leq{}& \alpha'||\hat{o} - \mathscr{B}\hat{o}|| + \beta'||\hat{g} - \mathscr{B}\hat{g}|| + \gamma'||\hat{o} - \hat{g}|| \\ &+ \delta'(||\hat{o} - \mathscr{B}\hat{g}|| + ||\hat{g} - \mathscr{B}\hat{o}||) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ ||\mathscr{T}_1 - \mathscr{T}_2||, \end{aligned}$$

for all $\hat{o} \in \mathscr{T}_1$ *and* $\hat{g} \in \mathscr{T}_2$*. Let* $\alpha' = 0.98$*,* $\beta' = 0.005$*,* $\gamma' = 0.005$*,* $\delta' = 0$ *with* $\alpha' + \beta' + \gamma' + 2\delta' < 1$*. Consider*

$$\begin{aligned} ||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| &= \left|\left|\frac{\hat{o}}{32} + \frac{3}{4} - \frac{1}{2}\right|\right| \\ &= \left|\left|\frac{\hat{o}}{32} + \frac{1}{4}\right|\right|. \end{aligned}$$

If $\hat{o} \in \mathscr{T}_1$ *and* $\hat{g} \in \mathscr{T}_2$ *then* $\left|\left|\frac{\hat{o}}{32} + \frac{1}{4}\right|\right| \in [0.5156, 0.5312]$, $||\hat{o}-\hat{g}|| \in [0.5, 1.75]$, $||\hat{o}-\mathscr{B}\hat{o}|| \in [0.5312, 1.01]$ *and* $||\hat{g} - \mathscr{B}\hat{g}|| \in [0.5, 1.25]$*. It implies that*

$$\begin{aligned} ||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| \leq{}& \alpha'||\hat{o} - \mathscr{B}\hat{o}|| + \beta'||\hat{g} - \mathscr{B}\hat{g}|| + \gamma'||\hat{o} - \hat{g}|| \\ &+ \delta'(||\hat{o} - \mathscr{B}\hat{g}|| + ||\hat{g} - \mathscr{B}\hat{o}||) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ ||\mathscr{T}_1 - \mathscr{T}_2||, \end{aligned}$$

for all $\hat{o} \in \mathscr{T}_1$ *and* $\hat{g} \in \mathscr{T}_2$ *and* $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_2$, $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_1$. *Since* $||\hat{o}||_1 < ||\hat{o}||$, *for all* $\hat{o} \in \mathscr{W}$, *we have*

$$||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}||_1 \leq \alpha'||\hat{o} - \mathscr{B}\hat{o}||_1 + \beta'||\hat{g} - \mathscr{B}\hat{g}||_1 + \gamma'||\hat{o} - \hat{g}||_1 + \delta'(||\hat{o} - \mathscr{B}\hat{g}||_1 + ||\hat{g} - \mathscr{B}\hat{o}||_1) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ ||\mathscr{T}_1 - \mathscr{T}_2||_1.$$

Starting with point $\hat{o}_0 = \frac{1}{4} \in \mathscr{T}_1$, *we construct a sequence as*

$\hat{o}_{n+1}$	$\hat{o}_0$	$\hat{o}_1$	$\hat{o}_2$	$\hat{o}_3$	$\hat{o}_4$	$\hat{o}_5$	$\hat{o}_6$	$\hat{o}_7$	$\cdots$
$\mathscr{B}\hat{o}_n$	0.25	0.7578	0.50	0.75	0.50	0.75	0.50	0.75	$\cdots$

We found that $\{\hat{o}_{2n}\}$ *has a subsequence (0.25, 0.5, 0.5, 0.5,...), which converges to* $\frac{1}{2}$ *in* $\mathscr{T}_1$. *All the suppositions of Theorem 2.1.11 are hold, and* $\mathscr{B}$ *has a BPP* $\frac{1}{2}$.

Example 2.1.7. *Assume that* $\mathscr{W} = \mathbb{R}^2$ *with metric defined as*

$$||\hat{o}^*|| = ||\hat{o}^*||_1 = \sqrt{\hat{o}_1^2 + \hat{o}_2^2}$$

for all $\hat{o}^* = (\hat{o}_1, \hat{o}_2) \in \mathbb{R}^2$. *Suppose that* $\mathscr{T}_1 = \{(0, \hat{o}_1) : 0 \leq \hat{o}_1 \leq 3\}$ *and* $\mathscr{T}_2 = \{(0, \hat{o}_1) : 0 \leq \hat{o}_1 \leq 2\}$, *are two subsets of* $\mathbb{R}^2$, *then* $||\mathscr{T}_1 - \mathscr{T}_2|| = 0$. *Define* $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ *by*

$$\mathscr{B}(\hat{o}^*) = \frac{\hat{o}^*}{3}$$

for all $\hat{o}^* \in \mathscr{T}_1$. *Let* $\hat{o}^*, \hat{g}^* \in \mathscr{T}_1$. *Let* $\hat{i}_1 = (0, \hat{o}_1), \hat{i}_2 = (0, \hat{o}_2)$ *in* $\mathscr{T}_1$. *Next, we have to prove that* $\mathscr{B}$ *satisfies the following inequality,*

$$||\mathscr{B}\hat{i}_1 - \mathscr{B}\hat{i}_2|| < \alpha'||\hat{i}_1 - \mathscr{B}\hat{i}_1|| + \beta'||\hat{i}_2 - \mathscr{B}\hat{i}_2|| + \gamma'||\hat{i}_1 - \hat{i}_2|| + \delta(||\hat{i}_1 - \mathscr{B}\hat{i}_2|| + ||\hat{i}_2 - \mathscr{B}\hat{i}_1||) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ d(\mathscr{T}_1, \mathscr{T}_2),$$

for all $\hat{i}_1, \hat{i}_2 \in \mathscr{T}_1$. *Let* $\alpha' = \beta' = \delta' = 0$, $\gamma' = \frac{1}{2}$ *with* $\alpha' + \beta' + \gamma' + 2\delta' < 1$. *Consider*

$$||\mathscr{B}\hat{i}_1 - \mathscr{B}\hat{i}_2|| = \left|\left|\frac{\hat{i}_1}{3} - \frac{\hat{i}_1}{3}\right|\right| = \left|\left|\frac{(0, \hat{o}_1)}{3} - \frac{(0, \hat{o}_2)}{3}\right|\right|$$
$$= \sqrt{\left(\frac{\hat{o}_1}{3} - \frac{\hat{o}_2}{3}\right)^2} = \frac{1}{3}||\hat{i}_1 - \hat{i}_2||$$

It shows that

$$||\mathscr{B}\hat{i}_1 - \mathscr{B}\hat{i}_2|| < \alpha'||\hat{i}_1 - \mathscr{B}\hat{i}_1|| + \beta'||\hat{i}_2 - \mathscr{B}\hat{i}_2|| + \gamma'||\hat{i}_1 - \hat{i}_2|| + \delta(||\hat{i}_1 - \mathscr{B}\hat{i}_2|| + ||\hat{i}_2 - \mathscr{B}\hat{i}_1||) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\ d(\mathscr{T}_1, \mathscr{T}_2),$$

for all $\hat{k}_1, \hat{k}_2 \in \mathscr{T}_1$ *and* $\mathscr{B}(\mathscr{T}_1) \subseteq \mathscr{T}_2$, $\mathscr{B}(\mathscr{T}_2) \subseteq \mathscr{T}_1$. *Starting with point* $\hat{o}_0 = (0, 1) \in \mathscr{T}_1$, *we construct a sequence as*

$\hat{o}_{n+1}$	$\hat{o}_0$	$\hat{o}_1$	$\hat{o}_2$	$\hat{o}_3$	$\hat{o}_4$	$\hat{o}_5$	$\cdots$
$\mathscr{B}\hat{x}_n$	$(0, 1)$	$(0, 0.3333)$	$(0, 0.1111)$	$(0, 0.0366)$	$(0, 0.0122)$	$(0, 0.0040)$	$\cdots$

We found that $\{\hat{o}_{2n}\}$ *has a subsequence* $((0, 1)(0, 0.1111), (0, 0.0122), (0, 0.0013), \cdots)$, *which converges to* $(0, 0)$. *All the suppositions of Theorem 2.1.11 are hold, and* $\mathscr{B}$ *has a BPP* $(0, 0)$.

2.1.5 Best proximity points in a quasi partial MS

In this subsection, using the concept of α-proximal admissible we present some results on the existence of BPPs in a quasi partial MS.

In the entire section $\mathscr{T}_1, \mathscr{T}_2$ are assumed non-empty closed subsets of a complete quasi partial MS $(\mathscr{W}, d)$ and $\mathscr{T}_{1_0}$ is non-empty.

Theorem 2.1.12. Let $\alpha : \mathscr{T}_1 \times \mathscr{T}_1 \to [0, \infty)$ be a mapping such that $\alpha(\hat{o}_1, \hat{o}_2) \geq 1$, $\alpha(\hat{o}_2, \hat{k}_1) \geq 1$ implies $\alpha(\hat{o}_1, \hat{k}_1) \geq 1$, for all $\hat{o}_1, \hat{o}_2, \hat{k}_1 \in \mathscr{T}_1$ and $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ be a continuous mapping such that $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$ and satisfies the assumptions of Definitions 1.1.5(i), 1.1.14 and 1.1.16. Further, suppose that there exist $\hat{o}_0, \hat{o}_1 \in \mathscr{T}_{1_0}$ and $\mathscr{B}\hat{o}_0 \in \mathscr{T}_{2_0}$ such that $d(\hat{o}_1, \mathscr{B}\hat{o}_0) = d(\mathscr{T}_1, \mathscr{T}_2)$ and $\alpha(\hat{o}_0, \hat{o}_1) \geq 1$. Then mapping $\mathscr{B}$ has a BPP.

Proof. By assumption, there exists $\hat{o}_0, \hat{o}_1 \in \mathscr{T}_{1_0} \subseteq \mathscr{T}_1$ and $\mathscr{B}\hat{o}_0 \in \mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0} \subseteq \mathscr{T}_2$ such that $d(\hat{o}_1, \mathscr{B}\hat{o}_0) = d(\mathscr{T}_1, \mathscr{T}_2)$ and $\alpha(\hat{o}_0, \hat{o}_1) \geq 1$.

Since $\hat{o}_1 \in \mathscr{T}_{1_0}$ $\mathscr{B}\hat{o}_1 \in \mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$. By notation of $\mathscr{T}_{2_0}$, there exists $\hat{o}_2 \in \mathscr{T}_{2_0}$ such that $d(\hat{o}_2, \mathscr{B}\hat{o}_1) = d(\mathscr{T}_1, \mathscr{T}_2)$.

Further, as α is proximal admissible and $\alpha(\hat{o}_0, \hat{o}_1) \geq 1$, we have $\alpha(\hat{o}_1, \hat{o}_2) \geq 1$. Again $\hat{o}_2 \in \mathscr{T}_{1_0}$ then $\mathscr{B}\hat{o}_2 \in \mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$. By notation of $\mathscr{T}_{2_0}$, there exists $\hat{o}_3 \in \mathscr{T}_{1_0}$ such that $d(\hat{o}_3, \mathscr{B}\hat{o}_2) = d(\mathscr{T}_1, \mathscr{T}_2)$.

Similarly, as α is proximal admissible and $\alpha(\hat{o}_1, \hat{o}_2) \geq 1$, we get $\alpha(\hat{o}_2, \hat{o}_3) \geq 1$. Continuing in this manner, we obtain

$$d(\hat{o}_{n+1}, \mathscr{B}\hat{o}_n) = d(\mathscr{T}_1, \mathscr{T}_2). \tag{2.26}$$

Since α is proximal admissible and $\alpha(\hat{o}_{n-1}, \hat{o}_n) \geq 1$, we get $\alpha(\hat{o}_n, \hat{o}_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. As $(\mathscr{T}_1, \mathscr{T}_2)$ satisfies weak ₤-property. So, for any $m, n \in \mathbb{N}_0$, we have $d(\hat{o}_m, \hat{o}_n) \leq d(\mathscr{B}\hat{o}_{m-1}, \mathscr{B}\hat{o}_{n-1})$. If there is $s \in \mathbb{N}_0$ for which $d(\hat{o}_s, \hat{o}_{s+1}) = 0$ and $d(\hat{o}_{s+1}, \hat{o}_s) = 0$. Consider

$$d(\mathscr{T}_1, \mathscr{T}_2) \leq d(\hat{o}_s, \mathscr{B}\hat{o}_s) \leq d(\hat{o}_s, \hat{o}_{s+1}) + d(\hat{o}_{s+1}, \mathscr{B}\hat{o}_s) = d(\hat{o}_{s+1}, \mathscr{B}\hat{o}_s) = d(\mathscr{T}_1, \mathscr{T}_2).$$

Hence $d(\hat{o}_s, \mathscr{B}\hat{o}_s) = d(\mathscr{T}_1, \mathscr{T}_2)$. By assuming that $d(\hat{o}_n, \hat{o}_{n+1}) > 0$, for all $n \geq 0$, we deduce that $d(\mathscr{B}\hat{o}_{n-1}, \mathscr{B}\hat{o}_n) > 0$, for all $n \geq 0$. Using Definitions 1.1.5(i), we have

$$\begin{aligned}
\delta^{d(\mathscr{B}\hat{o}_{n-1}, \mathscr{B}\hat{o}_n)} &\leq \delta_*^{d(\mathscr{B}\hat{o}_{n-1}, \mathscr{B}\hat{o}_n)} \\
&\leq [\alpha(\hat{o}_{n-1}, \hat{o}_n) - 1 + \delta_*]^{d(\mathscr{B}\hat{o}_{n-1}, \mathscr{B}\hat{o}_n)} \\
&\leq \delta^{\beta(d(\hat{o}_{n-1}, \hat{o}_n))d(\hat{o}_{n-1}, \hat{o}_n)}.
\end{aligned}$$

It implies that

$$d(\hat{o}_n, \hat{o}_{n+1}) \leq d(\mathscr{B}\hat{o}_{n-1}, \mathscr{B}\hat{o}_n) \leq \beta(d(\hat{o}_{n-1}, \hat{o}_n))d(\hat{o}_{n-1}, \hat{o}_n) < d(\hat{o}_{n-1}, \hat{o}_n), \tag{2.27}$$

for all $n \in \mathbb{N}$. Therefore, $\{d(\hat{o}_{n-1}, \hat{o}_n)\}$ is strictly non-increasing and $\lim_{n\to\infty} d(\hat{o}_{n-1}, \hat{o}_n) = r_1$, for some $r_1 \geq 0$. Assume that $r_1 > 0$. On taking $n \to \infty$ in (2.27), we get $\lim_{n\to\infty} \beta(d(\hat{o}_{n-1}, \hat{o}_n)) = 1$. It gives

$$\lim_{n\to\infty} d(\hat{o}_{n-1}, \hat{o}_n) = 0, \tag{2.28}$$

which is a contradiction. So, $r_1 = 0$, and thus

$$\lim_{n\to\infty} d(\hat{o}_{n-1}, \hat{o}_n) = 0. \tag{2.29}$$

Suppose that $\{\hat{o}_n\}$ is not a Cauchy sequence. Then there is $\epsilon > 0$ and subsequence of integers $n_{\hat{s}}$ and $m_{\hat{s}}$ with $m_{\hat{s}} > n_{\hat{s}} \geq \hat{s}$ for which $d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{n_{\hat{s}}}) \geq \epsilon$, for all $\hat{s} \in \mathbb{N}$. Assume this $m_{\hat{s}}$ is the smallest integer greater than $n_{\hat{s}}$ to satisfy the above inequality. From triangle inequality, we get

$$\begin{aligned}
\epsilon \leq & d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{n_{\hat{s}}}) \\
\leq & d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{n_{\hat{s}-1}}) + d(\hat{o}_{n_{\hat{s}-1}}, \hat{o}_{n_{\hat{s}}}) - d(\hat{o}_{n_{\hat{s}-1}}, \hat{o}_{n_{\hat{s}-1}}) \\
\leq & d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{n_{\hat{s}-1}}) + d(\hat{o}_{n_{\hat{s}-1}}, \hat{o}_{n_{\hat{s}}}) \\
< & \epsilon + d(\hat{o}_{n_{\hat{s}-1}}, \hat{o}_{n_{\hat{s}}}).
\end{aligned}$$

Letting $\hat{s} \to \infty$ and using (2.29), we have

$$\lim_{n\to\infty} d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{n_{\hat{s}}}) = \epsilon > 0. \tag{2.30}$$

Using the assumption of α and $n_{\hat{s}} > m_{\hat{s}}$, we have

$$\alpha(\hat{o}_{m_{\hat{s}}}, \hat{o}_{n_{\hat{s}}}) \geq 1.$$

Using (2.26), (2.30) and triangle inequality, we have

$$\begin{aligned}
& \delta^{d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{n_{\hat{s}}})} \\
\leq & \delta^{d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{m_{\hat{s}+1}}) + d(\hat{o}_{m_{\hat{s}+1}}, \hat{o}_{n_{\hat{s}}}) - d(\hat{o}_{m_{\hat{s}+1}}, \hat{o}_{m_{\hat{s}+1}})} \\
\leq & \delta^{d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{m_{\hat{s}+1}}) + d(\hat{o}_{m_{\hat{s}+1}}, \hat{o}_{n_{\hat{s}+1}}) + d(\hat{o}_{n_{\hat{s}+1}}, \hat{o}_{n_{\hat{s}}}) - d(\hat{o}_{m_{\hat{s}+1}}, \hat{o}_{m_{\hat{s}+1}}) - d(\hat{o}_{n_{\hat{s}+1}}, \hat{o}_{n_{\hat{s}+1}})} \\
\leq & \delta^{d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{m_{\hat{s}+1}}) + d(\hat{o}_{m_{\hat{s}+1}}, \hat{o}_{n_{\hat{s}+1}}) + d(\hat{o}_{n_{\hat{s}+1}}, \hat{o}_{n_{\hat{s}}})} \\
\leq & \delta^{d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{m_{\hat{s}+1}}) + d(\mathscr{B}\hat{o}_{m_{\hat{s}}}, \mathscr{B}\hat{o}_{n_{\hat{s}}}) + d(\hat{o}_{n_{\hat{s}+1}}, \hat{o}_{n_{\hat{s}}})} \\
\leq & \delta^{d(\hat{o}_{m_{\hat{s}}}, \hat{o}_{m_{\hat{s}+1}}) + d(\hat{o}_{n_{\hat{s}+1}}, \hat{o}_{n_{\hat{s}}})} \delta^{d(\mathscr{B}\hat{o}_{m_{\hat{s}}}, \mathscr{B}\hat{o}_{n_{\hat{s}}})}
\end{aligned}$$

$$
\begin{aligned}
&\leq \delta^{d(\hat{o}_{m_{\hat{s}}},\hat{o}_{m_{\hat{s}+1}})+d(\hat{o}_{n_{\hat{s}+1}},\hat{o}_{n_{\hat{s}}})} \delta_*^{d(\mathscr{B}\hat{o}_{m_{\hat{s}}},\mathscr{B}\hat{o}_{n_{\hat{s}}})} \\
&\leq \delta^{d(\hat{o}_{m_{\hat{s}}},\hat{o}_{m_{\hat{s}+1}})+d(\hat{o}_{n_{\hat{s}+1}},\hat{o}_{n_{\hat{s}}})} \times [\alpha(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}}) - 1 + \delta_*]^{d(\mathscr{B}\hat{o}_{m_{\hat{s}}},\mathscr{B}\hat{o}_{n_{\hat{s}}})} \\
&\leq \delta^{d(\hat{o}_{m_{\hat{s}}},\hat{o}_{m_{\hat{s}+1}})+d(\hat{o}_{n_{\hat{s}+1}},\hat{o}_{n_{\hat{s}}})} \delta^{\beta(d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}}))d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}})} \\
&= \delta^{d(\hat{o}_{m_{\hat{s}}},\hat{o}_{m_{\hat{s}+1}})+d(\hat{o}_{n_{\hat{s}+1}},\hat{o}_{n_{\hat{s}}})+\beta(d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}}))d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}})}.
\end{aligned}
$$

It shows that

$$
d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}}) \leq d(\hat{o}_{m_{\hat{s}}},\hat{o}_{m_{\hat{s}+1}}) + d(\hat{o}_{n_{\hat{s}+1}},\hat{o}_{n_{\hat{s}}}) + \beta(d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}}))d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}}),
$$

that is,

$$
\frac{d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}}) - d(\hat{o}_{m_{\hat{s}}},\hat{o}_{m_{\hat{s}+1}}) - d(\hat{o}_{n_{\hat{s}+1}},\hat{o}_{n_{\hat{s}}})}{d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}})} \leq \beta(d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}})) < 1.
$$

On taking limit $\hat{s} \to \infty$ and using (2.28) and (2.30), we get

$$
\lim_{j\to\infty} \beta(d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}})) = 1.
$$

It implies

$$
\lim_{j\to\infty} d(\hat{o}_{m_{\hat{s}}},\hat{o}_{n_{\hat{s}}}) = 0,
$$

which contradicts with (2.30).

In the similar way, we get $\lim_{n\to\infty} d(\hat{o}_{n+1},\hat{o}_n) = 0$ and $\lim_{\hat{s}\to\infty} d(\hat{o}_{n_{\hat{s}}},\hat{o}_{m_{\hat{s}}}) = 0$. Therefore, $\{\hat{o}_n\}$ is a Cauchy sequence in $\mathscr{T}_1$. Given that $\mathscr{T}_1$ is closed, we get $\lim_{n\to\infty} \hat{o}_n = \hat{o}$, for some $\hat{o} \in \mathscr{W}$. Since $\mathscr{B}$ is continuous, we get $\lim_{n\to\infty} \mathscr{B}\hat{o}_n = \mathscr{B}\hat{o}$ for some $\mathscr{B}\hat{o} \in \mathscr{T}_2$. By (2.26) $d(\hat{o},\mathscr{B}\hat{o}) = \lim_{n\to\infty} d(\hat{o}_{n+1},\mathscr{B}\hat{o}_n) = d(\mathscr{T}_1,\mathscr{T}_2)$ and $d(\mathscr{B}\hat{o},\hat{o}) = \lim_{n\to\infty} d(\mathscr{B}\hat{o}_n,\hat{o}_{n+1}) = d(\mathscr{T}_2,\mathscr{T}_1)$. Hence $\hat{o}$ is a BPP of $\mathscr{B}$. □

In order to demonstrate our next finding, we substitute the $\Re_1$-property for the continuity hypothesis of $\mathscr{B}$ in Theorem 2.1.12:

($\Re_1$-property): If $\{\hat{o}_n\}$ is a sequence in $\mathscr{W}$ such that $\alpha(\hat{o}_n,\hat{o}_{n+1}) \geq 1$, for all n and $\lim_{n\to\infty} \hat{o}_n = \hat{o} \in \mathscr{W}$. It follows that subsequence $\{\hat{o}_{n_{\hat{s}}}\}$ of $\{\hat{o}_n\}$ exists and $\alpha(\hat{o}_{n_{\hat{s}}},\hat{o}) \geq 1$, for all $\hat{s}$.

Theorem 2.1.13. Let $\alpha : \mathscr{T}_1 \times \mathscr{T}_1 \to [0,\infty)$ be a mapping such that $\alpha(\hat{o}_1,\hat{o}_2) \geq 1$, $\alpha(\hat{o}_2,\hat{k}_1) \geq 1$ implies $\alpha(\hat{o}_1,\hat{k}_1) \geq 1$, for all $\hat{o}_1,\hat{o}_2,\hat{k}_1 \in \mathscr{T}_1$ and $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ be a mapping such that $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$ and satisfies the assumptions of Definitions 1.1.5(i), 1.1.14 and 1.1.16. Further, suppose that $\Re_1$-property holds and there exist $\hat{o}_0,\hat{o}_1 \in \mathscr{T}_{1_0}$ and $\mathscr{B}\hat{o}_0 \in \mathscr{T}_{2_0}$ such that $d(\hat{o}_1,\mathscr{B}\hat{o}_0) = d(\mathscr{T}_1,\mathscr{T}_2)$ and $\alpha(\hat{o}_0,\hat{o}_1) \geq 1$. Then mapping $\mathscr{B}$ has a BPP.

Proof. By Theorem 2.1.12, sequence $\hat{o}_n \to \hat{o} \in \mathscr{W}$ in $(\mathscr{W},d)$. By $\Re_1$-property and using (2.26), there is a subsequence $\{\hat{o}_{n_{\hat{s}}}\}$ of $\{\hat{o}_n\}$ and $\alpha(\hat{o}_{n_{\hat{s}}},\hat{o}) \geq 1$, for all $\hat{s}$. Now, we have to show that $\mathscr{B}$ has a BPP. Consider

$$\begin{aligned}
\delta^{d(\mathscr{B}\hat{o}_{n_{\hat{s}-1}},\mathscr{B}\hat{o})} \leq & \delta_*^{d(\mathscr{B}\hat{o}_{n_{\hat{s}-1}},\mathscr{B}\hat{o})} \\
\leq & [\alpha(\hat{o}_{n_{\hat{s}-1}},\hat{o}) - 1 + \delta_*]^{d(\mathscr{B}\hat{o}_{n_{\hat{s}-1}},\mathscr{B}\hat{o})} \\
\leq & \delta^{\beta(d(\hat{o}_{n_{\hat{s}-1}},\hat{o}))d(\hat{o}_{n_{\hat{s}-1}},\hat{o})}.
\end{aligned}$$

It implies that

$$d(\mathscr{B}\hat{o}_{n_{\hat{s}-1}},\mathscr{B}\hat{o}) \leq \beta(d(\hat{o}_{n_{\hat{s}-1}},\hat{o}))d(\hat{o}_{n_{\hat{s}-1}},\hat{o}). \tag{2.31}$$

From (2.26), (2.31) and triangle inequality, we get

$$\begin{aligned}
d(\hat{o}_{n_{\hat{s}+1}},\mathscr{B}\hat{o}) &\leq d(\hat{o}_{n_{\hat{s}+1}},\mathscr{B}\hat{o}_{n_{\hat{s}}}) + d(\mathscr{B}\hat{o}_{n_{\hat{s}}},\mathscr{B}\hat{o}) \\
&\leq d(\mathscr{T}_1,\mathscr{T}_2) + \beta(d(\hat{o}_{n_{\hat{s}}},\hat{o}))d(\hat{o}_{n_{\hat{s}}},\hat{o}).
\end{aligned} \tag{2.32}$$

Again by (2.32) and triangle inequality, we obtain

$$\begin{aligned}
d(\mathscr{T}_1,\mathscr{T}_2) \leq d(\hat{o},\mathscr{B}\hat{o}) &\leq d(\hat{o},\hat{o}_{n_{\hat{s}+1}}) + d(\hat{o}_{n_{\hat{s}+1}},\mathscr{B}\hat{o}) \\
&\leq d(\hat{o},\hat{o}_{n_{\hat{s}+1}}) + d(\mathscr{T}_1,\mathscr{T}_2) + \beta(d(\hat{o}_{n_{\hat{s}}},\hat{o}))d(\hat{o}_{n_{\hat{s}}},\hat{o}).
\end{aligned} \tag{2.33}$$

Taking $\hat{s} \to \infty$ in (2.33), we have $d(\hat{o},\mathscr{B}\hat{o}) = d(\mathscr{T}_1,\mathscr{T}_2)$. Using the similar arguments, we have $d(\mathscr{B}\hat{o},\hat{o}) = d(\mathscr{T}_2,\mathscr{T}_1)$. Hence $\hat{o}$ is a BPP of $\mathscr{B}$. □

To demonstrate our findings, we now provide a few examples.

Example 2.1.8. *Consider $\mathscr{W} = \mathbb{R}$ with metric defined as $d(\hat{o}_1, \hat{g}_1) = |\hat{o}_1 - \hat{g}_1| + \hat{o}_1$, for all $\hat{o}_1, \hat{g}_1 \in \mathscr{W}$ is a quasi partial MS not partial MS and MS. Suppose $\mathscr{T}_1 = [0,1]$ and $\mathscr{T}_2 = \left[0, \frac{1}{4}\right]$ then $d(\mathscr{T}_1, \mathscr{T}_2) = 0$ and $d(\mathscr{T}_2, \mathscr{T}_1) = 0$. Define $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ by $\mathscr{B}(\hat{o}_1) = \frac{\hat{o}_1}{4}$, for all $(\hat{o}_1, \hat{g}_1) \in \mathscr{T}_1$. Define $\alpha : \mathscr{T}_1 \times \mathscr{T}_1 \to [0, \infty)$ by $\alpha(\hat{o}_1, \hat{g}_1) = 2$, for all $\hat{o}_1, \hat{g}_1 \in \mathscr{T}_1$. Let $\hat{k}_1, \hat{k}_2 \in \mathscr{T}_1$ then $\alpha(\hat{i}_1, \hat{i}_2) = 2 > 1, d(\hat{o}_1, \mathscr{B}\hat{i}_1) = 0 = d(\mathscr{T}_1, \mathscr{T}_2)$ and $d(\hat{g}_1, \mathscr{B}\hat{i}_2) = 0 = d(\mathscr{T}_1, \mathscr{T}_2)$ if and only if $\hat{o}_1, \hat{g}_1 \in \{0\}$. Then $\alpha(\hat{o}_1, \hat{g}_1) = 2 > 1$. It gives that $\mathscr{B}$ is continuous and $\mathscr{B}$ is α proximal. As $\mathscr{T}_{1_0} = \mathscr{T}_{2_0} = 0$ then $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$ for each $\hat{o}_1 \in \mathscr{T}_{1_0}$. Also a pair $(\mathscr{T}_1, \mathscr{T}_2)$ satisfies weak £- property. Let $\zeta(\hat{t}) = \frac{\hat{t}}{2}$, then $\zeta \in S'$ and we get $\beta(\hat{t}) = \frac{1}{2}$. Take $\hat{i}_1, \hat{i}_2$ in $\mathscr{T}_1$, then*

$$
\begin{aligned}
d(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2) =& \left|\mathscr{B}(\hat{i}_1) - \mathscr{B}(\hat{i}_2)\right| + \mathscr{B}(\hat{i}_1) \\
=& \left|\frac{\hat{i}_1}{4} - \frac{\hat{i}_2}{4}\right| + \frac{\hat{i}_1}{4} \\
=& \frac{1}{4} d(\hat{i}_1, \hat{i}_2) \leq \beta(d(\hat{i}_1, \hat{i}_2)) d(\hat{i}_1, \hat{i}_2).
\end{aligned}
$$

It implies $[\alpha(\hat{i}_1, \hat{i}_2) - 1 + \delta_*]^{d(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2)} \leq \delta^{\beta(d(\hat{i}_1, \hat{i}_2)) d(\hat{i}_1, \hat{i}_2)}$, for all $\hat{i}_1, \hat{i}_2 \in \mathscr{T}_1$, where $1 < \delta \leq \delta_*$. Therefore, all suppositions of Theorem 2.1.12 are met and $\mathscr{B}$ has a BPP 0.

Example 2.1.9. *Suppose $\mathscr{W} = \mathbb{R}$ with metric defined as $d(\hat{o}_1, \hat{g}_1) = |\hat{o}_1^3 - \hat{g}_1^3| + \hat{o}_1^3$, for all $\hat{o}_1, \hat{g}_1 \in \mathscr{W}$, is a quasi partial MS not partial MS and MS. Suppose $\mathscr{T}_1 = \left\{0, 1, \sqrt{2}, \sqrt{3}, \sqrt{4} ...\right\}$ and $\mathscr{T}_2 = \left\{0, \frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{4}}{\sqrt{2}} ...\right\}$ then $d(\mathscr{T}_1, \mathscr{T}_2) = 0$ and $d(\mathscr{T}_2, \mathscr{T}_1) = 0$. Define $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ by $\mathscr{B}(\hat{o}_1) = \frac{\hat{o}_1}{\sqrt{2}}$, for all $(\hat{o}_1, \hat{g}_1) \in \mathscr{T}_1$. Define $\alpha : \mathscr{W} \times \mathscr{W} \to [0, \infty)$ by $\alpha(\hat{o}_1, \hat{g}_1) = 2.5$, for all $\hat{o}_1, \hat{g}_1 \in \mathscr{W}$. Let $\hat{i}_1, \hat{i}_2 \in \mathscr{T}_1$ then $\alpha(\hat{i}_1, \hat{i}_2) = 2.5 > 1, d(\hat{o}_1, \mathscr{B}\hat{i}_1) = 0 = d(\mathscr{T}_1, \mathscr{T}_2)$ and $d(\hat{g}_1, \mathscr{B}\hat{i}_2) = 0 = d(\mathscr{T}_1, \mathscr{T}_2)$ if and only if $\hat{o}_1, \hat{g}_1 \in \{0\}$. Then $\alpha(\hat{o}_1, \hat{g}_1) = 2.5 > 1$. It gives that $\mathscr{B}$ is continuous and $\mathscr{B}$ is α proximal. As $\mathscr{T}_{1_0} = \mathscr{T}_{2_0} = 0$ then $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$, for each $\hat{o}_1 \in \mathscr{T}_{1_0}$. Also a pair $(\mathscr{T}_1, \mathscr{T}_2)$ satisfies weak £- property. Let $\zeta(\hat{t}) = \frac{4\hat{t}}{5}$, then $\zeta \in S'$ and we get $\beta(\hat{t}) = \frac{4}{5}$. Take $\hat{i}_1, \hat{i}_2$ in $\mathscr{T}_1$, then*

$$
d(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2) = \left|(\mathscr{B}(\hat{i}_1))^3 - \mathscr{B}((\hat{i}_2))^3\right| + (\mathscr{B}(\hat{i}_1))^3
$$

$$= \left| \frac{\hat{i}_1^3}{\sqrt{2}} - \frac{\hat{i}_2^3}{\sqrt{2}} \right| + \frac{\hat{i}_1^3}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}} d(\hat{i}_1, \hat{i}_2) \leq \beta(d(\hat{i}_1, \hat{i}_2)) d(\hat{i}_1, \hat{i}_2).$$

It implies $[\alpha(\hat{i}_1, \hat{i}_2) - 1 + \delta_*]^{d(\mathscr{B}\hat{i}_1, \mathscr{B}\hat{i}_2)} \leq \delta^{\beta(d(\hat{i}_1,\hat{i}_2))d(\hat{i}_1,\hat{i}_2)}$, *for all* $\hat{i}_1, \hat{i}_2 \in \mathscr{T}_1$, *where* $1 < \delta \leq \delta_*$. *Therefore, all suppositions of Theorem 2.1.12 are met and* $\mathscr{B}$ *has a BPP* 0.

If we take $\alpha(\hat{o}_1, \hat{g}_1) = 1$, for all $\hat{o}_1, \hat{g}_1 \in \mathscr{W}$ in Theorem 2.1.12, we get a BPP result as described below:

Corollary 2.1.10. Let $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ be a mapping such that $d(\mathscr{B}(\hat{o}_1), \mathscr{B}(\hat{g}_1)) \leq \beta(d(\hat{o}_1, \hat{g}_1))d(\hat{o}_1, \hat{g}_1)$, for all $\hat{o}_1, \hat{g}_1 \in \mathscr{W}$ and $\beta \in S$. Suppose that a pair $(\mathscr{T}_1, \mathscr{T}_2)$ has the weak £-property and $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$. Then mapping $\mathscr{B}$ has a BPP.

If we take $\mathscr{T}_1 = \mathscr{T}_2 = \mathscr{W}$ in Theorem 2.1.12 and Theorem 2.1.13, we get the FP result as described below:

Corollary 2.1.11. Let $(\mathscr{W}, d)$ be a complete quasi partial MS, $\alpha : \mathscr{W} \times \mathscr{W} \to [0, \infty)$ be a mapping such that $\alpha(\hat{o}_1, \hat{g}_1) \geq 1$, $\alpha(\hat{g}_1, \hat{k}_1) \geq 1$ implies $\alpha(\hat{o}_1, \hat{k}_1) \geq 1$, for all $\hat{o}_1, \hat{g}_1, \hat{k}_1 \in \mathscr{W}$ and $\mathscr{B} : \mathscr{W} \to \mathscr{W}$ be a continuous mapping satisfies the assumptions of Definitions 1.1.5(i) and 1.1.16. Suppose that there exists $\hat{o}_0, \hat{o}_1 \in \mathscr{W}$ such that $\alpha(\hat{o}_1, \mathscr{B}\hat{o}_0) \geq 1$, then mapping $\mathscr{B}$ has a FP.

Corollary 2.1.12. Let $(\mathscr{W}, d)$ be a complete quasi partial MS, $\alpha : \mathscr{W} \times \mathscr{W} \to [0, \infty)$ be a mapping such that $\alpha(\hat{o}_1, \hat{g}_1) \geq 1$, $\alpha(\hat{g}_1, \hat{k}_1) \geq 1$ implies $\alpha(\hat{o}_1, \hat{k}_1) \geq 1$, for all $\hat{o}_1, \hat{g}_1, \hat{k}_1 \in \mathscr{W}$ and $\mathscr{B} : \mathscr{W} \to \mathscr{W}$ be a mapping satisfies the assumptions of Definitions 1.1.5(i) and 1.1.16. Suppose that $\Re_1$-property holds and there exists $\hat{o}_0, \hat{o}_1 \in \mathscr{W}$ such that $\alpha(\hat{o}_1, \mathscr{B}\hat{o}_0) \geq 1$, then mapping $\mathscr{B}$ has a FP.

Considering $\alpha(\hat{o}_1, \hat{g}_1) = 1$, for all $\hat{o}_1, \hat{g}_1 \in \mathscr{W}$ in Theorem 2.1.12, it reduces to Geraghty-type contraction mapping (Theorem 1.3, p-606, [31]).

Corollary 2.1.13. Let $(\mathscr{W}, d)$ be a complete quasi partial MS and $\mathscr{B} : \mathscr{W} \to \mathscr{W}$ be a mapping such that $d(\mathscr{B}(\hat{o}_1), \mathscr{B}(\hat{g}_1)) \leq \beta(d(\hat{o}_1, \hat{g}_1))d(\hat{o}_1, \hat{g}_1)$, for all $\hat{o}_1, \hat{g}_1 \in \mathscr{W}$ and $\beta \in S$, then $\mathscr{B}$ has a FP.

2.2 Existence of best proximity pair

In this section, we study some results for the existence of best proximity pairs using measure of noncompactness in framework of strictly convex BSs. Results of this section are accepted in Sharma and Chandok [71].

In the entire section, we assume that $(\mathscr{T}_1, \mathscr{T}_2)$ is NBCC subset of a strictly convex BS $\mathscr{W}$ and $\mathscr{T}_{1_0}$ is non-empty.

Theorem 2.2.1. Suppose that $\mathscr{B} : \mathscr{T}_1 \cup \mathscr{T}_2 \to \mathscr{T}_1 \cup \mathscr{T}_2$ is a relatively NE and satisfies Definition 2.1.5. Then $\mathscr{B}$ has a best proximity pair.

Proof. As $\mathscr{T}_{1_0}$ is non-empty, $(\mathscr{T}_{1_0}, \mathscr{T}_{2_0})$ is non-empty. Also one can show that $(\mathscr{T}_{1_0}, \mathscr{T}_{2_0})$ is $\mathscr{B}$ -invariant, proximinal pair, closed and convex considering conditions on $\mathscr{B}$. For $\hat{o} \in \mathscr{T}_{1_0}$, there is a $\hat{g} \in \mathscr{T}_{2_0}$ such that $||\hat{o} - \hat{g}|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Since $\mathscr{B}$ is relatively NE

$$||\mathscr{B}\hat{o} - \mathscr{B}\hat{g}|| \leq ||\hat{o} - \hat{g}|| = ||\mathscr{T}_1 - \mathscr{T}_2||,$$

which gives $\mathscr{B}\hat{o} \in \mathscr{T}_{1_0}$, that is, $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{1_0}$. Similarly, $\mathscr{B}(\mathscr{T}_{2_0}) \subseteq \mathscr{T}_{2_0}$ and so $\mathscr{B}$ is noncyclic on $\mathscr{T}_{1_0} \cup \mathscr{T}_{2_0}$. Using the similar lines of proof Theorem 2.1.9, we get $(\mathscr{Q}_n, \mathscr{R}_n)$ is proximinal. As is now known, there are two situations that can occur: either $\max\{\mho(\mathscr{Q}_{2\hat{s}}), \mho(\mathscr{R}_{2\hat{s}})\} = 0$ for some $\hat{s} \in \mathbb{N}$ or $\max\{\mho(\mathscr{Q}_{2n}), \mho(\mathscr{R}_{2n})\} > 0$ for all $n \in \mathbb{N}$. If $\max\{\mho(\mathscr{Q}_{\hat{s}}), \mho(\mathscr{R}_{\hat{s}})\} = 0$ for some $\hat{s} \in \mathbb{N}$, then $\mathscr{B} : \mathscr{Q}_{\hat{s}} \cup \mathscr{R}_{\hat{s}} \to \mathscr{Q}_{2\hat{s}} \cup \mathscr{R}_{2\hat{s}}$ is compact, by Theorem 1.1.1 we obtain the result. Consider $\max\{\mho(\mathscr{Q}_n), \mho(\mathscr{R}_n)\} > 0$. Since $\mathscr{Q}_{n+1} \subseteq \mathscr{B}(\mathscr{Q}_n)$ and $\mathscr{R}_{n+1} \subseteq \mathscr{B}(\mathscr{R}_n)$, we have

$$\begin{aligned}
\mho(\mathscr{Q}_{n+1} \cup \mathscr{R}_{n+1}) &= \max\{\mho(\mathscr{Q}_{n+1}), \mho(\mathscr{R}_{n+1})\} \\
&= \max\{\mho(\overline{con}(\mathscr{B}(\mathscr{Q}_n))), \mho(\overline{con}(\mathscr{B}(\mathscr{R}_n)))\} \\
&= \max\{\mho(\mathscr{B}(\mathscr{Q}_n)), \mho(\mathscr{B}(\mathscr{R}_n))\} \\
&= \mho(\mathscr{B}(\mathscr{Q}_n) \cup \mathscr{B}(\mathscr{R}_n)) \\
&\leq \varsigma(\mho(\mathscr{Q}_n \cup \mathscr{R}_n))\mho(\mathscr{Q}_n \cup \mathscr{R}_n) \\
&\leq \mho(\mathscr{Q}_n \cup \mathscr{R}_n).
\end{aligned}$$

$\mho_n = \mho(\mathscr{Q}_n \cup \mathscr{R}_n)$, is a non-increasing sequence. Thus there is $\alpha \geq 0$ in such way that $\mho_n \to \alpha$ as $n \to \infty$. Let $\alpha > 0$, for all $n \in \mathbb{N}$, we obtain

$$\frac{\mho(\mathscr{Q}_{n+1} \cup \mathscr{R}_{n+1})}{\mho(\mathscr{Q}_n \cup \mathscr{R}_n)} \leq \varsigma(\mho(\mathscr{Q}_n \cup \mathscr{R}_n)).$$

By above inequality $\varsigma(\mho(\mathscr{Q}_n \cup \mathscr{R}_n)) \geq 1$, which is contradiction. So $\alpha = 0$ and $\mho_n = \mho(\mathscr{Q}_n \cup \mathscr{R}_n) \to 0$ as $n \to \infty$. Now, let $\mathscr{Q}_\infty = \cap_{n=0}^{\infty} \mathscr{Q}_n$ and $\mathscr{R}_\infty = \cap_{n=0}^{\infty} \mathscr{R}_n$. By (d') of MNC, a pair $(\mathscr{Q}_\infty, \mathscr{R}_\infty)$ is $\mathscr{B}$ -invariant, non-empty, compact and convex with $||\mathscr{Q}_\infty - \mathscr{R}_\infty|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Therefore, $\mathscr{B}$ has a best proximity pair. □

Theorem 2.2.2. Suppose that $\mathscr{B} : \mathscr{T}_1 \cup \mathscr{T}_2 \to \mathscr{T}_1 \cup \mathscr{T}_2$ is a relatively NE and satisfies Definition 2.1.5. Then $\mathscr{B}$ has a best proximity pair

Proof. Using the similar lines of proof Theorem 2.1.9, we have $(\mathscr{Q}_n, \mathscr{R}_n)$ is proximinal for all $n \in \mathbb{N}$ with $||\mathscr{Q}_n - \mathscr{R}_n|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. If $\max\{\mho(\mathscr{Q}_{\hat{s}}), \mho(\mathscr{R}_{\hat{s}})\} = 0$ for some $\hat{s} \in \mathbb{N}$, then $\mathscr{B} : \mathscr{Q}_{\hat{s}} \cup \mathscr{R}_{\hat{s}} \to \mathscr{Q}_{2\hat{s}} \cup \mathscr{R}_{2\hat{s}}$ is compact, by Theorem 1.1.1 we get the result. Consider $\max\{\mho(\mathscr{Q}_n), \mho(\mathscr{R}_n)\} > 0$ for all $n \in \mathbb{N}$. Since $\mathscr{Q}_{n+1} \subseteq \mathscr{B}(\mathscr{Q}_n)$ and $\mathscr{R}_{n+1} \subseteq \mathscr{B}(\mathscr{R}_n)$, we have

$$\begin{aligned}
\mho(\mathscr{Q}_{n+1} \cup \mathscr{R}_{n+1}) &= \max\{\mho(\mathscr{Q}_{n+1}), \mho(\mathscr{R}_{n+1})\} \\
&= \max\{\mho(\overline{con}(\mathscr{B}(\mathscr{Q}_n))), \mho(\overline{con}(\mathscr{B}(\mathscr{R}_n)))\} \\
&= \max\{\mho(\mathscr{B}(\mathscr{Q}_n)), \mho(\mathscr{B}(\mathscr{R}_n))\} \\
&= \mho(\mathscr{B}(\mathscr{Q}_n) \cup \mathscr{B}(\mathscr{R}_n)) \\
&\leq \mho(\mathscr{Q}_n \cup \mathscr{R}_n) - \psi(\mho(\mathscr{Q}_n \cup \mathscr{R}_n)) \leq \mho(\mathscr{Q}_n \cup \mathscr{R}_n).
\end{aligned}$$

$\mho_n = \mho(\mathscr{Q}_n \cup \mathscr{R}_n)$, is a non-increasing sequence. So, there is $\alpha \geq 0$ in such way that $\mho_n \to \alpha$ as $n \to \infty$. Assume $\alpha > 0$, for all $n \in \mathbb{N}$, we obtain

$$\mho(\mathscr{Q}_{n+1} \cup \mathscr{R}_{n+1}) - \mho(\mathscr{Q}_n \cup \mathscr{R}_n) \leq -\psi(\mho(\mathscr{Q}_n \cup \mathscr{R}_n)).$$

Taking $n \to \infty$ in above inequality we get $\psi(\alpha) = 0$, for $\alpha > 0$, which is contradiction. So $\alpha = 0$ and $\mho_n = \mho(\mathscr{Q}_n \cup \mathscr{R}_n) \to 0$ as $n \to \infty$. Now, let $\mathscr{Q}_\infty = \cap_{n=0}^{\infty} \mathscr{Q}_n$ and

$\mathscr{R}_\infty = \cap_{n=0}^{\infty} \mathscr{R}_n$. By (d') of MNC, a pair $(\mathscr{Q}_\infty, \mathscr{R}_\infty)$ is $\mathscr{B}$ -invariant, non-empty, compact and convex with $||\mathscr{Q}_\infty - \mathscr{R}_\infty|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Therefore, $\mathscr{B}$ has a best proximity pair. $\square$

Chapter 3

Iterative convergence to best proximity points

In the optimization and approximation theory the convergence of iterative processes for BPPs has been an attractive problem in nonlinear analysis. Iterative convergence to BPPs is a crucial concept to find a point that minimizes the distance between two sets. As discussed in Chapter 1, there are many iterative schemes which converge to FPs of mapping. The structure of this chapter is as described below: In the first section, we define a iterative scheme that converges to common FPs of NEs and strongly pseudocontractive mappings. The results of this part are published in Sharma and Chandok [70]. Using the class of NEs in the setting of UCBS, we provide a few iterative techniques in the second section that converge strongly to BPP. The outcomes of this section are published in Sharma and Chandok [69, 72]. In the third section, we propose a three-step algorithm that converges to a solution of split common FP problem. The outcomes of this section are published in Sharma and Chandok [68]. In the last section, we define another algorithm using projection operator which converges to a solution of SBPP problem in the context of Hilbert spaces. The findings of this section are published in Sharma and Chandok [68].

3.1 Convergence to a solution of fixed point problem

We describe an iterative scheme in this section for estimating the common FPs of strongly pseudocontractive and NEs in real BS. We give some numerical examples to back up our assertions and demonstrate that our technique converges faster than well known algorithms in the literature. The results of this section are published in Sharma and Chandok [70].

In the entire section, we assume that $\mathscr{T}_1$ is a non-empty closed and convex subset of a UCBS $\mathscr{W}$. Suppose that $\mathscr{B}, \mathscr{B}_1 : \mathscr{T}_1 \to \mathscr{T}_1$ are self mappings. For an arbitrary element $\hat{o}_0 \in \mathscr{T}_1$, define a sequence $\{\hat{o}_n\}$ as described below:

$$\begin{cases} \hat{g}_n = & (1-\gamma_n')\mathscr{B}_1\hat{o}_n + \gamma_n'\hat{o}_n \\ \hat{\hat{i}}_n = & (1-\eta_n'-\delta_n'-\omega_n')\hat{o}_n + \omega_n'\mathscr{B}\hat{g}_n + \eta_n'\mathscr{B}_1\hat{g}_n + \delta_n'\mathscr{B}_1\hat{o}_n \\ \hat{o}_{n+1} = & \mathscr{B}\hat{\hat{i}}_n \end{cases} \qquad \text{Algorithm (1)}$$

where $\omega_n, \gamma_n', \eta_n', \delta_n' \in [0,1]$, $n \in \mathbb{N}_0$.

Theorem 3.1.1. Let $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_1$ be a NE and $\mathscr{B}_1 : \mathscr{T}_1 \to \mathscr{T}_1$ be a uniformly continuous strongly pseudocontractive mapping. Assume that $\mathscr{F} = \{\hat{o} \in \mathscr{T}_1 : \mathscr{B}\hat{o} = \mathscr{B}_1\hat{o} = \hat{o}\} \neq \emptyset$, $\mathscr{B}_1(\mathscr{T}_1)$ is bounded, and $\gamma_n', \eta_n', \delta_n', \omega_n'$ are real sequences in $[0,1]$ such that:

(i) $\gamma_n' + \eta_n' + \delta_n' + \omega_n' \leq 1$;

(ii) $\lim_{n\to\infty}(\omega_n' + \eta_n' + \delta_n') = 0 = \lim_{n\to\infty}(1-\gamma_n')$;

(iii) $\sum_{n=1}^{\infty}(\omega_n' + \eta_n' + \delta_n') = \infty$.

Then for any arbitrary $\hat{o}_0 \in \mathscr{T}_1$, sequence $\{\hat{o}_n\}$ defined by Algorithm (1) converges to a common FP $\hat{p} \in \mathscr{F}$.

Proof. Let $\hat{p} \in \mathscr{F}(\mathscr{B}) \cap \mathscr{F}(\mathscr{B}_1)$. Suppose that

$$\mathscr{M}_1 = \sup\left\{||\mathscr{B}_1\hat{o} - \mathscr{B}_1\hat{\hat{i}}|| : \hat{o}, \hat{\hat{i}} \in \mathscr{W}\right\}. \tag{3.1}$$

As $\mathscr{B}_1$ has bounded range, it shows $\mathscr{M}_1 < \infty$. Consider

$$\begin{aligned} ||\hat{g}_n - \hat{p}|| =& ||\gamma'_n \hat{o}_n + (1-\gamma'_n)\mathscr{B}_1\hat{o}_n - \hat{p}|| \\ =& ||\gamma'_n(\hat{o}_n - \hat{p}) + (1-\gamma'_n)(\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{p})|| \\ \leq& \gamma'_n||\hat{o}_n - \hat{p}|| + (1-\gamma'_n)||\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{p}|| \\ \leq& \gamma'_n||\hat{o}_n - \hat{p}|| + (1-\gamma'_n)\mathscr{M}_1. \end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned} &||\hat{i}_n - \hat{p}|| = ||(1-\eta'_n-\delta'_n-\omega'_n)\hat{o}_n + \omega'_n\mathscr{B}\hat{g}_n + \eta'_n\mathscr{B}_1\hat{g}_n + \delta'_n\mathscr{B}_1\hat{o}_n - \hat{p}|| \\ \leq& (1-\eta'_n-\delta'_n-\omega'_n)||\hat{o}_n - \hat{p}|| + \omega'_n||\mathscr{B}\hat{g}_n - \hat{p}|| + \eta'_n||\mathscr{B}_1\hat{g}_n - \hat{p}|| + \delta'_n||(\mathscr{B}_1\hat{o}_n - \hat{p})|| \\ =& (1-\eta'_n-\delta'_n-\omega'_n)||\hat{o}_n - \hat{p}|| + \omega'_n||\mathscr{B}\hat{g}_n - \mathscr{B}\hat{p}|| + \eta'_n||(\mathscr{B}_1\hat{g}_n - \mathscr{B}_1\hat{p})|| \\ &+ \delta'_n||(\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{p})||. \end{aligned} \tag{3.3}$$

We claim that $||\hat{o}_n - \hat{p}|| \leq \mathscr{M}_2$, where

$$\mathscr{M}_2 = \mathscr{M}_1 + ||\hat{o}_0 - \hat{p}||, n \geq 0 \text{ and } n \in \mathbb{N}_0. \tag{3.4}$$

Clearly, $\mathscr{M}_1 \leq \mathscr{M}_2$. It is true for $n = 0$. Let us consider it is accurate for $n = \hat{s}$. Now we have to show it is for $n = \hat{s}+1$. By Algorithm (1), (3.2) and (3.3), we have

$$\begin{aligned} &||\hat{o}_{\hat{s}+1} - \hat{p}|| \\ =& ||\mathscr{B}\hat{i}_{\hat{s}} - \hat{p}|| = ||\mathscr{B}\hat{i}_{\hat{s}} - \mathscr{B}\hat{p}|| \\ \leq& ||\hat{i}_{\hat{s}} - \hat{p}|| \\ \leq& (1-\eta'_{\hat{s}}-\delta'_{\hat{s}}-\omega'_{\hat{s}})||\hat{o}_{\hat{s}} - \hat{p}|| + \omega'_{\hat{s}}||\mathscr{B}\hat{g}_{\hat{s}} - \mathscr{B}\hat{p}|| + \eta'_{\hat{s}}||\mathscr{B}_1\hat{g}_{\hat{s}} - \mathscr{B}_1\hat{p}|| \\ &+ \delta'_{\hat{s}}||\mathscr{B}_1\hat{o}_{\hat{s}} - \mathscr{B}_1\hat{p}|| \\ \leq& (1-\eta'_{\hat{s}}-\delta'_{\hat{s}}-\omega'_{\hat{s}})||\hat{o}_{\hat{s}} - \hat{p}|| + \omega'_{\hat{s}}||\hat{g}_{\hat{s}} - \hat{p}|| + \eta'_{\hat{s}}||\mathscr{B}_1\hat{g}_{\hat{s}} - \mathscr{B}_1\hat{p}|| + \delta'_{\hat{s}}||\mathscr{B}_1\hat{o}_{\hat{s}} - \mathscr{B}_1\hat{p}|| \\ \leq& (1-\eta'_{\hat{s}}-\delta'_{\hat{s}}-\omega'_{\hat{s}})||\hat{o}_{\hat{s}} - \hat{p}|| + \omega'_{\hat{s}}(\gamma'_n||\hat{o}_{\hat{s}} - \hat{p}|| + (1-\gamma'_n)\mathscr{M}_1) + \eta'_{\hat{s}}||\mathscr{B}_1\hat{g}_{\hat{s}} - \mathscr{B}_1\hat{p}|| \\ &+ \delta'_{\hat{s}}||\mathscr{B}_1\hat{o}_{\hat{s}} - \mathscr{B}_1\hat{p}|| \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\eta'_{\hat{s}}-\delta'_{\hat{s}}-\omega'_{\hat{s}})\mathscr{M}_2+\omega'_{\hat{s}}(\gamma'_n\mathscr{M}_2+(1-\gamma'_n)\mathscr{M}_2)+\eta'_{\hat{s}}\mathscr{M}_2+\delta'_{\hat{s}}\mathscr{M}_2\\
&=(1-\eta'_{\hat{s}}-\delta'_{\hat{s}}-\omega'_{\hat{s}})\mathscr{M}_2+\omega'_{\hat{s}}\mathscr{M}_2+\eta'_{\hat{s}}\mathscr{M}_2+\delta'_{\hat{s}}\mathscr{M}_2\\
&=\mathscr{M}_2.
\end{aligned}
$$

This shows that $\{||\hat{o}_n-\hat{p}||\}$ is bounded. By (3.2) and (3.3), $\left\{||\hat{i}_n-\hat{p}||\right\}$ and $\{||\hat{g}_n-\hat{p}||\}$ are also bounded sequences. Assume that
$\mathscr{C}_1=\sup\{||\hat{o}_n-\hat{p}||:n\geq 0\}$, $\mathscr{C}_2=\sup\left\{||\hat{i}_n-\hat{p}||:n\geq 0\right\}$, $\mathscr{C}_3=\sup\{||\hat{g}_n-\hat{p}||:n\geq 0\}$.
Denote $\mathscr{M}=\mathscr{M}_2+\mathscr{C}_1+\mathscr{C}_2+\mathscr{C}_3$, then $\mathscr{M}<\infty$. Using Lemma 1.1.6, we have

$$
\begin{aligned}
&||\hat{i}_n-\hat{p}||^2=||(1-\eta'_n-\delta'_n-\omega'_n)\hat{o}_n+\omega'_n\mathscr{B}\hat{g}_n+\eta'_n\mathscr{B}_1\hat{g}_n+\delta'_n\mathscr{B}_1\hat{o}_n-\hat{p}||^2\\
&=||(1-\eta'_n-\delta'_n-\omega'_n)(\hat{o}_n-\hat{p})+\omega'_n(\mathscr{B}_1\hat{g}_n-\hat{p})+\eta'_n(\mathscr{B}_1\hat{g}_n-\hat{p})+\delta'_n(\mathscr{B}_1\hat{o}_n-\hat{p})||^2\\
&\leq(1-\eta'_n-\delta'_n-\omega'_n)^2||\hat{o}_n-\hat{p}||^2\\
&\quad+2\left\langle\omega'_n(\mathscr{B}\hat{g}_n-\hat{p})+\eta'_n(\mathscr{B}_1\hat{g}_n-\hat{p})+\delta'_n(\mathscr{B}_1\hat{o}_n-\hat{p}),j(\hat{i}_n-\hat{p})\right\rangle\\
&=(1-\eta'_n-\delta'_n-\omega'_n)^2||\hat{o}_n-\hat{p}||^2\\
&\quad+2\left\langle\omega'_n(\mathscr{B}\hat{g}_n-\mathscr{B}\hat{p})+\eta'_n(\mathscr{B}_1\hat{g}_n-\mathscr{B}_1\hat{p})+\delta'_n(\mathscr{B}_1\hat{o}_n-\mathscr{B}_1\hat{p}),j(\hat{i}_n-\hat{p})\right\rangle\\
&\leq(1-\eta'_n-\delta'_n-\omega'_n)^2||\hat{o}_n-\hat{p}||^2+2\omega'_n\left\langle\mathscr{B}\hat{g}_n-\mathscr{B}\hat{p},j(\hat{i}_n-\hat{p})\right\rangle\\
&\quad+2\eta'_n\left\langle\mathscr{B}_1\hat{g}_n-\mathscr{B}_1\hat{p},j(\hat{i}_n-\hat{p})\right\rangle+2\delta'_n\left\langle\mathscr{B}_1\hat{o}_n-\mathscr{B}_1\hat{p},j(\hat{i}_n-\hat{p})\right\rangle\\
&\leq(1-\eta'_n-\delta'_n-\omega'_n)^2||\hat{o}_n-\hat{p}||^2+2\omega'_n\left\langle\mathscr{B}\hat{g}_n-\mathscr{B}\hat{i}_n+\mathscr{B}\hat{i}_n-\mathscr{B}\hat{p},j(\hat{i}_n-\hat{p})\right\rangle\\
&\quad+2\eta'_n\left\langle\mathscr{B}_1\hat{g}_n-\mathscr{B}_1\hat{i}_n+\mathscr{B}_1\hat{i}_n-\mathscr{B}_1\hat{p},j(\hat{i}_n-\hat{p})\right\rangle\\
&\quad+2\delta'_n\left\langle\mathscr{B}_1\hat{o}_n-\mathscr{B}_1\hat{i}_n+\mathscr{B}_1\hat{i}_n-\mathscr{B}_1\hat{p},j(\hat{i}_n-\hat{p})\right\rangle\\
&\leq(1-\eta'_n-\delta'_n-\omega'_n)^2||\hat{o}_n-\hat{p}||^2+2\omega'_n\left\langle\mathscr{B}\hat{g}_n-\mathscr{B}\hat{i}_n,j(\hat{i}_n-\hat{p})\right\rangle\\
&\quad+2\omega'_n\left\langle\mathscr{B}\hat{i}_n-\mathscr{B}\hat{p},j(\hat{i}_n-\hat{p})\right\rangle+2\eta'_n\left\langle\mathscr{B}_1\hat{g}_n-\mathscr{B}_1\hat{i}_n,j(\hat{i}_n-\hat{p})\right\rangle\\
&\quad+2\eta'_n\left\langle\mathscr{B}_1\hat{i}_n-\mathscr{B}_1\hat{p},j(\hat{i}_n-\hat{p})\right\rangle+2\delta'_n\left\langle\mathscr{B}_1\hat{o}_n-\mathscr{B}_1\hat{i}_n,j(\hat{i}_n-\hat{p})\right\rangle\\
&\quad+2\delta'_n\left\langle\mathscr{B}_1\hat{i}_n-\mathscr{B}_1\hat{p},j(\hat{i}_n-\hat{p})\right\rangle\\
&\leq(1-\eta'_n-\delta'_n-\omega'_n)^2||\hat{o}_n-\hat{p}||^2+2\omega'_n||\mathscr{B}\hat{g}_n-\mathscr{B}\hat{i}_n||\ ||\hat{i}_n-\hat{p}||\\
&\quad+2\eta'_n||\mathscr{B}_1\hat{g}_n-\mathscr{B}_1\hat{i}_n||\ ||\hat{i}_n-\hat{p}||+2\delta'_n||\mathscr{B}_1\hat{o}_n-\mathscr{B}_1\hat{i}_n||\ ||\hat{i}_n-\hat{p}||\\
&\quad+2(\eta'_n+\delta'_n)\left\langle\mathscr{B}_1\hat{i}_n-\mathscr{B}_1\hat{p},j(\hat{i}_n-\hat{p})\right\rangle+2\omega'_n||\mathscr{B}\hat{i}_n-\mathscr{B}\hat{p}||||\hat{i}_n-\hat{p}||\\
&\leq(1-\eta'_n-\delta'_n-\omega'_n)^2||\hat{o}_n-\hat{p}||^2+2\omega'_n||\hat{g}_n-\hat{i}_n||\ ||\hat{i}_n-\hat{p}||
\end{aligned}
$$

$$
\begin{aligned}
&+2\eta_n'||\mathscr{B}_1\hat{g}_n-\mathscr{B}_1\hat{i}_n||\;||\hat{i}_n-\hat{p}||+2\delta_n'||\mathscr{B}_1\hat{o}_n-\mathscr{B}_1\hat{i}_n||\;||\hat{i}_n-\hat{p}||\\
&+2(\eta_n'+\delta_n')\left\langle\mathscr{B}_1\hat{i}_n-\mathscr{B}_1\hat{p},j(\hat{i}_n-\hat{p})\right\rangle+2\omega_n'||\hat{i}_n-\hat{p}||||\hat{i}_n-\hat{p}||\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2||\hat{o}_n-\hat{p}||^2+2\omega_n'||\hat{g}_n-\hat{i}_n||\;||\hat{i}_n-\hat{p}||\\
&+2\eta_n'||\mathscr{B}_1\hat{g}_n-\mathscr{B}_1\hat{i}_n||\;||\hat{i}_n-\hat{p}||+2\delta_n'||\mathscr{B}_1\hat{o}_n-\mathscr{B}_1\hat{i}_n||\;||\hat{i}_n-\hat{p}||\\
&+2(\eta_n'+\delta_n')\left\langle\mathscr{B}_1\hat{i}_n-\mathscr{B}_1\hat{p},j(\hat{i}_n-\hat{p})\right\rangle+2\omega_n'||\hat{i}_n-\hat{p}||^2\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2||\hat{o}_n-\hat{p}||^2+2\omega_n'\mathscr{M}||\hat{g}_n-\hat{i}_n||+2\eta_n'\mathscr{M}||\mathscr{B}_1\hat{g}_n-\mathscr{B}_1\hat{i}_n||\\
&+2\delta_n'\mathscr{M}||\mathscr{B}_1\hat{o}_n-\mathscr{B}_1\hat{i}_n||+2(\eta_n'+\delta_n')\left\langle\mathscr{B}_1\hat{i}_n-\mathscr{B}_1\hat{p},j(\hat{i}_n-\hat{p})\right\rangle+2\omega_n'||\hat{i}_n-\hat{p}||\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2||\hat{o}_n-\hat{p}||^2\\
&+2(\eta_n'+\delta_n'+\omega_n')\max\left\{||\hat{g}_n-\hat{i}_n||,||\mathscr{B}_1\hat{g}_n-\mathscr{B}_1\hat{i}_n||,||\mathscr{B}_1\hat{o}_n-\mathscr{B}_1\hat{i}_n||\right\}\\
&+2(\eta_n'+\delta_n')\left\langle\mathscr{B}_1\hat{i}_n-\mathscr{B}_1\hat{p},j(\hat{y}_n-\hat{p})\right\rangle+2\omega_n'||\hat{i}_n-\hat{p}||^2.
\end{aligned}
$$

From (3.4), we get

$$
\begin{aligned}
||\hat{g}_n-\hat{o}_n||=&||\gamma_n'\hat{o}_n+(1-\gamma_n')\mathscr{B}_1\hat{o}_n-\hat{o}_n||\\
=&||\gamma_n'(\hat{o}_n-\hat{o}_n)+(1-\gamma_n')(\mathscr{B}_1\hat{o}_n-\hat{o}_n)||\\
=&(1-\gamma_n')||\mathscr{B}_1\hat{o}_n-\hat{o}_n||\\
\leq&(1-\gamma_n')(||\mathscr{B}_1\hat{o}_n-\hat{p}||+||\hat{o}_n-\hat{p}||)\\
=&(1-\gamma_n')(||\mathscr{B}_1\hat{o}_n-\mathscr{B}_1\hat{p}||+||\hat{o}_n-\hat{p}||)\\
\leq&(1-\gamma_n')2\mathscr{M}_2.
\end{aligned}
$$

By Theorem 3.1.1 (ii), we obtain

$$
\lim_{n\to\infty}||\hat{g}_n-\hat{o}_n||=0. \tag{3.5}
$$

Again using (3.4), we have

$$
\begin{aligned}
&||\hat{i}_n-\hat{o}_n||=||(1-\eta_n'-\delta_n'-\omega_n')\hat{o}_n+\omega_n'\mathscr{B}\hat{g}_n+\eta_n'\mathscr{B}_1\hat{g}_n+\delta_n'\mathscr{B}_1\hat{o}_n-\hat{o}_n||\\
=&||(1-\eta_n'-\delta_n'-\omega_n')(\hat{o}_n-\hat{o}_n)+\omega_n'(\mathscr{B}\hat{g}_n-\hat{o}_n)+\eta_n'(\mathscr{B}_1\hat{g}_n-\hat{o}_n)+\delta_n'(\mathscr{B}_1\hat{o}_n-\hat{o}_n)||\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')||\hat{o}_n-\hat{o}_n||+\omega_n'||\mathscr{B}\hat{g}_n-\hat{o}_n||+\eta_n'||\mathscr{B}_1\hat{g}_n-\hat{o}_n||+\delta_n'||\mathscr{B}_1\hat{o}_n-\hat{o}_n)||
\end{aligned}
$$

$$
\begin{aligned}
&=\omega'_n||\mathscr{B}\hat{g}_n - \hat{o}_n|| + \eta'_n||\mathscr{B}_1\hat{g}_n - \hat{o}_n|| + \delta'_n||\mathscr{B}_1\hat{o}_n - \hat{o}_n)|| \\
&\leq\omega'_n(||\mathscr{B}\hat{g}_n - \hat{p}|| + ||\hat{p} - \hat{i}_n||) + \eta'_n(||\mathscr{B}_1\hat{g}_n - \hat{p}|| + ||\hat{p} - \hat{i}_n||) \\
&\quad + \delta'_n(||\mathscr{B}_1\hat{o}_n - \hat{p}|| + ||\hat{p} - \hat{i}_n||) \\
&=\omega'_n(||\hat{g}_n - \mathscr{B}\hat{p}|| + ||\hat{p} - \hat{i}_n||) + \eta'_n(||\mathscr{B}_1\hat{g}_n - \mathscr{B}_1\hat{p}|| + ||\hat{p} - \hat{i}_n||) \\
&\quad + \delta'_n(||\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{p}|| + ||\hat{p} - \hat{i}_n||) \\
&\leq 2\omega'_n\mathscr{M}_2 + 2\eta'_n\mathscr{M}_2 + 2\delta'_n\mathscr{M}_2 = 2(\omega'_n + \eta'_n + \delta'_n)\mathscr{M}_2.
\end{aligned}
$$

By Theorem 3.1.1 (ii), we obtain

$$
\lim_{n\to\infty} ||\hat{i}_n - \hat{o}_n|| = 0. \tag{3.6}
$$

Since $||\hat{g}_n - \hat{i}_n|| \leq ||\hat{g}_n - \hat{o}_n|| + ||\hat{o}_n - \hat{i}_n||$. By (3.5) and (3.6), we get

$$
\lim_{n\to\infty} ||\hat{g}_n - \hat{i}_n|| = 0.
$$

As $\mathscr{B}_1$ is uniformly continuous, we have

$$
\lim_{n\to\infty} ||\mathscr{B}_1\hat{g}_n - \mathscr{B}_1\hat{i}_n|| = 0 \text{ and } \lim_{n\to\infty} ||\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{i}_n|| = 0. \tag{3.7}
$$

Since $\mathscr{B}_1$ is a strongly pseudocontractive mapping, we get

$$
\begin{aligned}
||\hat{o}_{n+1} - \hat{p}||^2 \leq&(1 - \eta'_n - \delta'_n - \omega'_n)^2||\hat{i}_n - \hat{p}||^2 \\
&+ 2(\omega'_n + \eta'_n + \delta'_n)\max\left\{||\hat{g}_n - \hat{i}_n||, ||\mathscr{B}_1\hat{g}_n - \mathscr{B}_1\hat{i}_n||, ||\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{i}_n||\right\} \\
&+ 2(\eta'_n + \delta'_n)k||\hat{i}_n - \hat{p}||^2 + 2\omega'_n||\hat{i}_n - \hat{p}||^2 \\
=&\frac{(1 - \eta'_n - \delta'_n - \omega'_n)^2}{1 - 2(\omega'_n + (\eta'_n + \delta'_n)k)}||\hat{i}_n - \hat{p}||^2 \\
&+ \frac{2(\omega'_n + \eta'_n + \delta'_n)\max\left\{||\hat{g}_n - \hat{i}_n||, ||\mathscr{B}_1\hat{g}_n - \mathscr{B}_1\hat{i}_n||, ||\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{i}_n||\right\}}{1 - 2(\omega'_n + (\eta'_n + \delta'_n)k)}.
\end{aligned} \tag{3.8}
$$

Since $(\omega'_n + \eta'_n + \delta'_n) \to 0$ as $n \to \infty$, for all $n \geq n_0$, there is a number $n_0 \in \mathbb{N}$ such that

$$(\omega'_n + \eta'_n + \delta'_n) \leq \min\left\{\frac{1}{4k}, \frac{1-k}{(1-k)^2 + k^2}\right\};\ k < \frac{1}{2}.$$

It implies $\frac{1-(\omega'_n+\eta'_n+\delta'_n)}{1-2(\omega'_n+(\eta'_n+\delta'_n)k)k} \leq 1$ and $\frac{1}{1-2(\omega'_n+(\eta'_n+\delta'_n)k)k} \leq 2$. It now follows from (3.8) that

$$\begin{aligned} ||\hat{o}_{n+1} - \hat{p}||^2 \leq & (1 - \eta'_n - \delta'_n - \omega'_n)||\hat{i}_n - \hat{p}||^2 \\ & + 4(\omega'_n + \eta'_n + \delta'_n) \max\left\{||\hat{g}_n - \hat{i}_n||, ||\mathscr{B}_1\hat{g}_n - \mathscr{B}_1\hat{i}_n||, ||\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{i}_n||\right\}. \end{aligned}$$

Now, using (3.7) , we have $\max\left\{||\hat{g}_n - \hat{i}_n||, ||\mathscr{B}_1\hat{g}_n - \mathscr{B}_1\hat{i}_n||, ||\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{i}_n||\right\} \to 0$ as $n \to \infty$. Consider $||\hat{o}_{n+1} - \hat{p}||^2 = ||\mathscr{B}\hat{i}_n - \mathscr{B}_1\hat{p}||^2 \leq ||\hat{i}_n - \hat{p}||^2$. Therefore,

$$\begin{aligned} ||\hat{o}_{n+1} - \hat{p}||^2 \leq & (1 - \eta'_n - \delta'_n - \omega'_n)||\hat{o}_n - \hat{p}||^2 \\ & + 4(\omega'_n + \eta'_n + \delta'_n) \max\left\{||\hat{g}_n - \hat{i}_n||, ||\mathscr{B}_1\hat{g}_n - \mathscr{B}_1\hat{i}_n||, ||\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{i}_n||\right\}. \end{aligned}$$

Take $\alpha_n = ||\hat{o}_n - \hat{p}||, \beta_n = \omega'_n + \eta'_n + \delta'_n$ and

$$\beta'_n = \max\left\{||\hat{g}_n - \hat{i}_n||, ||\mathscr{B}_1\hat{g}_n - \mathscr{B}_1\hat{i}_n||, ||\mathscr{B}_1\hat{o}_n - \mathscr{B}_1\hat{i}_n||\right\},\ \text{for all } n \geq 1.$$

Using Lemma 1.1.9, we obtain $\lim_{n\to\infty} ||\hat{o}_n - \hat{p}|| = 0$. □

Remark 3.1.1. If we take $\gamma'_n = 1, \delta'_n, \omega'_n = 0$ in Algorithm (1), then it reduces to the algorithm presented by Kang et al. (Algorithm (12), p.2, [38]) as described below:

$$\begin{cases} \hat{i}_n = & (1 - \eta'_n)\hat{o}_n + \eta'_n\mathscr{B}_1\hat{o}_n \\ \hat{o}_{n+1} = & \mathscr{B}\hat{i}_n \end{cases} \qquad \text{Algorithm (HNS)}$$

Corollary 3.1.1. Let $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_1$ be a nonexpansive, $\mathscr{B}_1 : \mathscr{T}_1 \to \mathscr{T}_1$ be a strongly pseudocontractive mappings and $\mathscr{F}(\mathscr{B}) \cap \mathscr{F}(\mathscr{B}_1) = \{\hat{o} \in \mathscr{T}_1 : \mathscr{B}\hat{o} = \mathscr{B}_1\hat{o} = \hat{o}\} \neq \emptyset$. For any arbitrary $\hat{o}_0 \in \mathscr{T}_1$, sequence $\{\hat{o}_n\}$ defined by Algorithm (HNS) converges strongly to a $\hat{p} \in \mathscr{F}(\mathscr{B}_1)$.

Remark 3.1.2. If we take $\gamma'_n = 1, \delta'_n = 0, \mathscr{B} = \mathscr{B}_1$ in Algorithm (1), then it reduces to algorithm presented by Sahu (normal S-iteration process, p.193, [59]) as described below:

$$\begin{cases} \hat{i}_n = & (1-\eta'_n)\hat{o}_n + \eta'_n \mathscr{B}_1 \hat{o}_n \\ \hat{o}_{n+1} = & \mathscr{B}_1 \hat{i}_n \end{cases} \qquad \text{Algorithm (NS)}$$

Corollary 3.1.2. Let $\mathscr{B}_1 : \mathscr{T}_1 \to \mathscr{T}_1$ be a strongly pseudocontractive mapping and $\mathscr{F}(\mathscr{B}_1) = \{\hat{o} \in \mathscr{T}_1 : \mathscr{B}_1\hat{o} = \hat{o}\} \neq \emptyset$. Then for any arbitrary $\hat{o}_0 \in \mathscr{T}_1$, sequence $\{\hat{o}_n\}$ defined by Algorithm (NS) converges strongly to a $\hat{p} \in \mathscr{F}(\mathscr{B}_1)$.

Remark 3.1.3. If we take $\mathscr{B} = \mathscr{B}_1$, in Algorithm (1), we get the following algorithm:

$$\begin{cases} \hat{g}_n = & (1-\gamma'_n)\mathscr{B}\hat{o}_n + \gamma'_n \hat{o}_n \\ \hat{i}_n = & (1-\eta'_n-\delta'_n-\omega'_n)\hat{o}_n + (\omega'_n+\eta'_n)\mathscr{B}\hat{g}_n + \delta'_n \mathscr{B}\hat{o}_n \\ \hat{o}_{n+1} = & \mathscr{B}\hat{i}_n \end{cases} \qquad \text{Algorithm (S*)}$$

where $\gamma'_n, \omega'_n, \eta'_n, \delta'_n \in [0,1]$, $n \in \mathbb{N}_0$.

In Theorem 3.1.1, we assume that $\mathscr{B}_1$ is a NE, we have the following result, which is very important in the application section.

Corollary 3.1.3. Let $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_1$ be a NE and $\mathscr{F}(\mathscr{B}) = \{\hat{o} \in \mathscr{T}_1 : \mathscr{B}\hat{o} = \hat{o}\} \neq \emptyset$. Let $\gamma'_n, \omega'_n, \eta'_n, \delta'_n \in [0,1]$, such that:

(i) $\omega'_n + \gamma'_n + \eta'_n + \delta'_n \leq 1$;

(ii) $\lim_{n\to\infty}(\omega'_n + \eta'_n + \delta'_n) = 0 = \lim_{n\to\infty}(1-\gamma'_n)$;

(iii) $\sum_{n=1}^{\infty}(\omega'_n + \eta'_n + \delta'_n) = \infty$.

Then for any arbitrary $\hat{o}_0 \in \mathscr{T}_1$, sequence $\{\hat{o}_n\}$ defined by Algorithm (S*) converges strongly to a $\hat{p} \in \mathscr{F}(\mathscr{B}_1)$.

Example 3.1.1. *Let $\mathscr{W} = \mathbb{R}$ with a usual norm and $\mathscr{T}_1 = [0,\infty)$. Define $\mathscr{B}, \mathscr{B}_1$:*

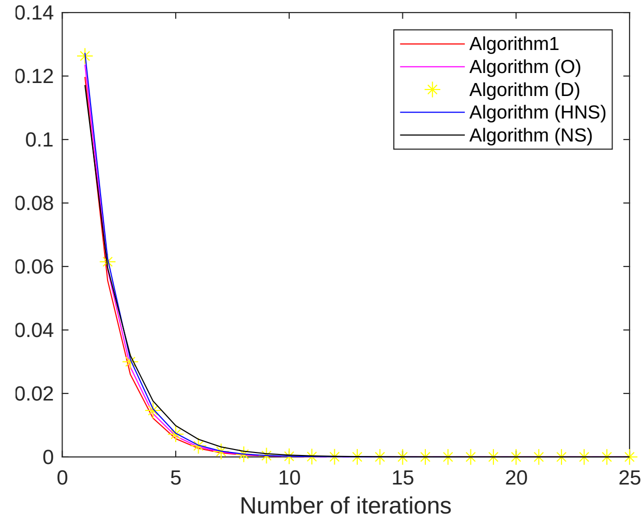

Figure 3-1

$\mathscr{T}_1 \to \mathscr{T}_1$ by

$$\mathscr{B}(\hat{o}) = \frac{\hat{o}}{2} \text{ and } \mathscr{B}_1(\hat{o}) = \frac{\hat{o}}{\sqrt{3}(1+\hat{o})},$$

for all $\hat{o} \in \mathscr{T}_1$. Clearly $\mathscr{B}$ is a NE. Also for all $\hat{o}, \hat{\imath} \in \mathscr{T}_1$, we get

$$\left\langle \mathscr{B}_1\hat{o} - \mathscr{B}_1\hat{\imath}, j(\hat{o} - \hat{\imath}) \right\rangle = \frac{1}{\sqrt{3}(1+\hat{o})(1+\hat{\imath})}(\hat{o} - \hat{\imath})^2 = k||\hat{o} - \hat{\imath}||^2,$$

where $k = \frac{1}{\sqrt{3}(1+\hat{x})(1+\hat{y})} < 1$. It shows that $\mathscr{B}_1$ is a strongly pseudocontractive mapping with bounded range and both $\mathscr{B}$, $\mathscr{B}_1$ are uniformly continuous on $\mathscr{T}_1$. It is easy to see that $\mathscr{B}$ and $\mathscr{B}_1$ has common FP say 0. All the suppositions of Theorem 3.1.1 are fulfilled, so sequence generated by Algorithm (1) converges to 0 see Table 3.1 and Figure 3-1. In Table 3.1, if we take $\gamma'_n = \frac{n}{n+25}$, $\omega'_n = \frac{1}{n+25}$, $\eta'_n = \frac{1}{n+25}$ and $\delta'_n = \frac{1}{n+25}$ which are close to 0 *and initial point* 0.26 *which is close to solution* 0 *then our Algorithm (1) converges to* 0 *in* 16 *iterations while Algorithms O, D, HNS, and NS converge to* 0 *in* 18, 18, 23 *and* 23 *iterations respectively see Figure 3-1(a). By Table 3.1 and Figure 3-1, it is easy to see that sequence defined by our algorithm converges to a solution much faster as compared to the algorithms present in the literature.*

$\hat{o}_{n+1}$	$Algorithm(1)$	$Algorithm(O)$	$Algorithm(D)$	$Algorithm(HNS)$	$Algorithm(NS)$
$\hat{o}_0$	0.26	0.26	0.26	0.26	0.26
$\hat{o}_2$	0.055625	0.059069	0.061500	0.062496	0.059575
$\hat{o}_4$	0.012207	0.013619	0.014646	0.015136	0.017619
$\hat{o}_6$	0.002715	0.003166	0.003504	0.003678	0.005557
$\hat{o}_8$	0.000610	0.000740	0.000842	0.000896	0.001789
$\hat{o}_{10}$	0.000138	0.000174	0.000203	0.000218	0.000580
$\hat{o}_{12}$	0.000031	0.000041	0.000049	0.000053	0.000189
$\hat{o}_{14}$	0.000007	0.000010	0.000012	0.000013	0.000062
$\hat{o}_{16}$	0.000000	0.000002	0.000003	0.000003	0.000020
$\hat{o}_{18}$	.	0.000000	0.000000	0.000001	0.000004
$\hat{o}_{23}$	.	0.000000	0.000000	0.000000	0.000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3.1: Comparison of convergence between different algorithms

Now, we give a new three-step algorithm and prove a convergence theorem. For an arbitrary element $\hat{o}_0 \in \mathscr{T}_1$, define sequence $\{\hat{o}_n\} \in \mathscr{T}_1$ as described below:

$$\begin{cases} \hat{o}_{n+1} = & (1-\eta'_n)\mathscr{B}\hat{g}_n + \eta'_n \mathscr{B}\hat{i}_n \\ \hat{i}_n = & \mathscr{B}((1-\delta'_n)\hat{g}_n + \delta'_n \mathscr{B}\hat{g}_n) \\ \hat{g}_n = & \mathscr{B}((1-\gamma'_n)\hat{o}_n + \gamma'_n \mathscr{B}\hat{o}_n) \end{cases} \qquad \text{Algorithm (S)}$$

where $\gamma'_n, \eta'_n, \delta'_n \in [0,1]$, $n \in \mathbb{N} \cup \{0\}$.

Theorem 3.1.2. Let $\mathscr{B}$ be a NE defined on $\mathscr{T}_1$. If $\mathscr{B}(\mathscr{T}_1)$ lies in a compact subset of $\mathscr{T}_1$, then sequence $\{\hat{o}_n\} \in \mathscr{T}_1$ defined by Algorithm S converges to a FP of $\mathscr{B}$ in $\mathscr{T}_1$ strongly.

Proof. By Browder's Theorem (see [8] , p. 1041), $\mathscr{B}$ has a FP. So, $\mathscr{F}(\mathscr{B}) \neq \emptyset$. Take

$\acute{z} \in \mathscr{F}(\mathscr{B})$. Consider

$$\begin{aligned}
||\hat{g}_n - \acute{z}|| =& ||\mathscr{B}((1-\gamma'_n)\hat{o}_n + \gamma'_n \mathscr{B}\hat{o}_n) - \acute{z}|| \\
=& ||\mathscr{B}((1-\gamma'_n)\hat{o}_n + \gamma'_n \mathscr{B}\hat{o}_n) - \mathscr{B}\acute{z}|| \\
\leq& ||(1-\gamma'_n)\hat{o}_n + \gamma'_n \mathscr{B}\hat{o}_n - \acute{z}|| \\
=& ||(1-\gamma'_n)(\hat{o}_n - \acute{z}) + \gamma'_n(\mathscr{B}\hat{o}_n - \acute{z})|| \\
=& ||(1-\gamma'_n)(\hat{o}_n - \acute{z}) + \gamma'_n(\mathscr{B}\hat{o}_n - \mathscr{B}\acute{z})|| \\
\leq& (1-\gamma'_n)||\hat{o}_n - \acute{z}|| + \gamma'_n||\mathscr{B}\hat{o}_n - \mathscr{B}\acute{z}|| \\
\leq& (1-\gamma'_n)||\hat{o}_n - \acute{z}|| + \gamma'_n||\hat{o}_n - \acute{z}|| \\
=& ||\hat{o}_n - \acute{z}||. \qquad (3.9)
\end{aligned}$$

Similarly,

$$\begin{aligned}
||\hat{i}_n - \acute{z}|| =& ||\mathscr{B}((1-\delta'_n)\hat{g}_n + \delta'_n \mathscr{B}\hat{g}_n) - \acute{z}|| \\
=& ||\mathscr{B}((1-\delta'_n)\hat{g}_n + \delta'_n \mathscr{B}\hat{g}_n) - \mathscr{B}\acute{z}|| \\
\leq& ||(1-\delta'_n)\hat{g}_n + \delta'_n \mathscr{B}\hat{g}_n - \acute{z}|| \\
=& ||(1-\delta'_n)(\hat{g}_n - \acute{z}) + \delta'_n(\mathscr{B}\hat{g}_n - \acute{z})|| \\
=& ||(1-\delta'_n)(\hat{g}_n - \acute{z}) + \delta'_n(\mathscr{B}\hat{g}_n - \mathscr{B}\acute{z})|| \\
\leq& (1-\delta'_n)||\hat{g}_n - \acute{z}|| + \delta'_n||\mathscr{B}\hat{g}_n - \mathscr{B}\acute{z}|| \\
\leq& (1-\delta'_n)||\hat{g}_n - \acute{z}|| + \delta'_n||\hat{g}_n - \acute{z}|| \\
=& ||\hat{g}_n - \acute{z}|| \leq ||\hat{o}_n - \acute{z}||. \qquad (3.10)
\end{aligned}$$

Using (3.9) and (3.10), we get

$$\begin{aligned}
||\hat{o}_{n+1} - \acute{z}|| =& ||(1-\eta'_n)\mathscr{B}\hat{g}_n + \eta'_n \mathscr{B}\hat{i}_n - \acute{z}|| \\
=& ||(1-\eta'_n)(\mathscr{B}\hat{g}_n - \acute{z}) + \eta'_n(\mathscr{B}\hat{i}_n - \acute{z})|| \\
=& ||(1-\eta'_n)(\mathscr{B}\hat{g}_n - \mathscr{B}\acute{z}) + \eta'_n(\mathscr{B}\hat{i}_n - \mathscr{B}\acute{z})|| \\
\leq& (1-\eta'_n)||\mathscr{B}\hat{g}_n - \mathscr{B}\acute{z}|| + \eta'_n||\mathscr{B}\hat{i}_n - \mathscr{B}\acute{z}|| \\
\leq& (1-\eta'_n)||\hat{g}_n - \acute{z}|| + \eta'_n||\hat{i}_n - \acute{z}||
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\eta'_n)||\hat{o}_n - \acute{z}|| + \eta'_n||\hat{o}_n - \acute{z}|| \\
&= ||\hat{o}_n - \acute{z}||.
\end{aligned}
$$

It signifies that sequence $\{||\hat{o}_n - \acute{z}||\}$ is a decreasing and bounded below. So, $\lim_{n\to\infty} ||\hat{o}_n - \acute{z}||$ exists. Suppose that $\lim_{n\to\infty} ||\hat{o}_n - \acute{z}|| = r$, for some $r \geq 0$.
Case (i): Let $r = 0$. Consider,

$$
\begin{aligned}
||\mathscr{B}\hat{o}_n - \hat{o}_n|| &\leq ||\mathscr{B}\hat{o}_n - \acute{z}|| + ||\hat{o}_n - \acute{z}|| \\
&= ||\mathscr{B}\hat{o}_n - \mathscr{B}\acute{z}|| + ||\hat{o}_n - \acute{z}|| \\
&\leq ||\hat{o}_n - \acute{z}|| + ||\hat{o}_n - \acute{z}|| \\
&= 2||\hat{o}_n - \acute{z}||.
\end{aligned}
$$

Taking $n \to \infty$ we get $||\mathscr{B}\hat{o}_n - \hat{o}_n|| \to 0$.
Case (ii): Let $r > 0$. Assume that an $\epsilon > 0$, $\{\hat{o}_n\}$ has a subsequence $\{\hat{o}_{n_{\hat{s}}}\}$ such that $||\mathscr{B}\hat{o}_{n_{\hat{s}}} - \hat{o}_{n_{\hat{s}}}|| \geq \epsilon > 0$ for all $\hat{s}$. We select $\xi > 0$ to be small since the modulus of convexity of δ^* of $\mathscr{W}$ is a increasing and continuous function. So, $\left(1 - c\delta^*\left(\frac{\epsilon}{r+\xi}\right)\right)(r+\xi) < r$, where $c > 0$. Now, we choose $\hat{s}$, such that $||\hat{o}_{n_{\hat{s}}} - \acute{z}|| \leq r + \xi$, $||\mathscr{B}\hat{o}_{n_{\hat{s}}} - \acute{z}|| = ||\mathscr{B}\hat{o}_{n_{\hat{s}}} - \mathscr{B}\acute{z}|| \leq ||\hat{o}_{n_{\hat{s}}} - \acute{z}|| \leq (r+\xi)$ and $||\mathscr{B}\hat{o}_{n_{\hat{s}}} - \hat{o}_{n_{\hat{s}}}|| \geq \epsilon$. Again using (3.9) and (3.10), we obtain

$$
\begin{aligned}
||\hat{o}_{n_{\hat{s}+1}} - \acute{z}|| &= ||(1-\eta'_{n_{\hat{s}}})\mathscr{B}\hat{g}_{n_{\hat{s}}} + \eta'_{n_{\hat{s}}}\mathscr{B}\hat{i}_{n_{\hat{s}}} - \acute{z}|| \\
&= ||(1-\eta'_{n_{\hat{s}}})(\mathscr{B}\hat{g}_{n_{\hat{s}}} - \acute{z}) + \eta'_{n_{\hat{s}}}(\mathscr{B}\hat{i}_{n_{\hat{s}}} - \acute{z})|| \\
&= ||(1-\eta'_{n_{\hat{s}}})(\mathscr{B}\hat{g}_{n_{\hat{s}}} - \mathscr{B}\acute{z}) + \eta'_{n_{\hat{s}}}(\mathscr{B}\hat{i}_{n_{\hat{s}}} - \mathscr{B}\acute{z})|| \\
&\leq (1-\eta'_{n_{\hat{s}}})||\mathscr{B}\hat{g}_{n_{\hat{s}}} - \mathscr{B}\acute{z}|| + \eta'_{n_{\hat{s}}}||\mathscr{B}\hat{i}_{n_{\hat{s}}} - \mathscr{B}\acute{z}|| \\
&\leq (1-\eta'_{n_{\hat{s}}})||\hat{g}_{n_{\hat{s}}} - \acute{z}|| + \eta'_{n_{\hat{s}}}||\hat{i}_{n_{\hat{s}}} - \acute{z}||. \qquad (3.11)
\end{aligned}
$$

By Proposition 1.1.1, we obtain

$$
\begin{aligned}
||\hat{g}_{n_{\hat{s}}} - \acute{z}|| &= ||((1-\gamma'_{n_{\hat{s}}})\hat{o}_{n_{\hat{s}}} + \gamma'_{n_{\hat{s}}}\mathscr{B}\hat{o}_{n_{\hat{s}}} - \acute{z}|| \\
&= ||(1-\gamma'_{n_{\hat{s}}})(\hat{o}_{n_{\hat{s}}} - \acute{z}) + \gamma'_{n_{\hat{s}}}(\mathscr{B}\hat{o}_{n_{\hat{s}}} - \acute{z})||
\end{aligned}
$$

$$\leq \left(1 - 2\delta^* \left(\frac{\epsilon}{r+\xi}\right) \min\left\{\gamma'_{n_{\hat{s}}}, 1 - \gamma'_{n_{\hat{s}}}\right\}\right)(r+\xi). \tag{3.12}$$

By (3.12), we have

$$\begin{aligned}
||\hat{i}_{n_{\hat{s}}} - \acute{z}|| =& ||\mathscr{B}((1-\delta'_{n_{\hat{s}}})\hat{g}_{n_{\hat{s}}} + \delta'_{n_{\hat{s}}}\mathscr{B}\hat{g}_{n_{\hat{s}}}) - \acute{z}|| \\
=& ||\mathscr{B}((1-\delta'_{n_{\hat{s}}})\hat{g}_{n_{\hat{s}}} + \delta'_{n_{\hat{s}}}\mathscr{B}\hat{g}_{n_{\hat{s}}}) - \mathscr{B}\acute{z}|| \\
\leq& ||(1-\delta'_{n_{\hat{s}}})\hat{g}_{n_{\hat{s}}} + \delta'_{n_{\hat{s}}}\mathscr{B}\hat{g}_{n_{\hat{s}}} - \acute{z}|| \\
=& ||(1-\delta'_{n_{\hat{s}}})(\hat{g}_{n_{\hat{s}}} - \acute{z}) + \delta'_{n_{\hat{s}}}(\mathscr{B}\hat{g}_{n_{\hat{s}}} - \acute{z})|| \\
=& ||(1-\delta'_{n_{\hat{s}}})(\hat{g}_{n_{\hat{s}}} - \acute{z}) + \delta'_{n_{\hat{s}}}(\mathscr{B}\hat{g}_{n_{\hat{s}}} - \mathscr{B}\acute{z})|| \\
\leq& (1-\delta'_{n_{\hat{s}}})||\hat{g}_{n_{\hat{s}}} - \acute{z}|| + \delta'_{n_{\hat{s}}}||\mathscr{B}\hat{g}_{n_{\hat{s}}} - \mathscr{B}\acute{z}|| \\
\leq& (1-\delta'_{n_{\hat{s}}})||\hat{g}_{n_{\hat{s}}} - \acute{z}|| + \delta'_{n_{\hat{s}}}||\hat{g}_{n_{\hat{s}}} - \acute{z}|| \\
=& ||\hat{g}_{n_{\hat{s}}} - \acute{z}|| \\
\leq& \left(1 - 2\delta^* \left(\frac{\epsilon}{r+\xi}\right) \min\left\{\gamma'_{n_{\hat{s}}}, 1 - \gamma'_{n_{\hat{s}}}\right\}\right)(r+\xi).
\end{aligned}$$

From (3.11), we get

$$||\acute{z} - \hat{o}_{n_{\hat{s}+1}}|| \leq \left(1 - 2\delta^* \left(\frac{\epsilon}{r+\xi}\right) \min\left\{\gamma'_{n_{\hat{s}}}, (1 - \gamma'_{n_{\hat{s}}})\right\}\right)(r+\xi).$$

Assume there is $l > 0$ such that $2\min\left\{\delta'_n\gamma'_{n_{\hat{s}}}, (1-\gamma'_{n_{\hat{s}}})\delta'_n\right\} \geq l$, and $\left(1 - 2\delta^* \left(\frac{\epsilon}{r+\xi}\right) \min\left\{\delta'_n\gamma'_{n_{\hat{s}}}, (1-\gamma'_{n_{\hat{s}}})\delta'_n\right\}\right)(r+\xi) \leq \left(1 - l\delta^*\left(\frac{\epsilon}{r+\xi}\right)\right)(r+\xi)$. Let $\xi > 0$ very small, we get $(r+\xi)\left(1 - l\delta^*\left(\frac{\epsilon}{r+\xi}\right)\right) < r$, which contradicts. Hence from the both cases, we get $\lim_{n\to\infty} ||\hat{o}_n - \mathscr{B}\hat{o}_n|| = 0$. Using Algorithm S, we get

$$\begin{aligned}
||\hat{g}_n - \mathscr{B}\hat{o}_n|| =& ||\mathscr{B}((1-\gamma'_n)\hat{o}_n + \gamma'_n\mathscr{B}\hat{o}_n) - \mathscr{B}\hat{o}_n|| \\
\leq& ||(1-\gamma'_n)\hat{o}_n + \gamma'_n\mathscr{B}\hat{o}_n - \hat{o}_n|| \\
=& ||(1-\gamma'_n)(\hat{o}_n - \hat{o}_n) + \gamma'_n(\mathscr{B}\hat{o}_n - \hat{o}_n|| \\
=& ||\gamma'_n(\mathscr{B}\hat{o}_n - \hat{o}_n)||
\end{aligned}$$

Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} ||\hat{g}_n - \mathscr{B}\hat{o}_n|| = 0. \tag{3.13}$$

Now, we have to prove that $||\mathscr{B}\hat{g}_n - \mathscr{B}\hat{i}_n|| \to 0$. Define $a_n = \dfrac{\hat{o}_{n+1} - \acute{z}}{||\hat{o}_n - \acute{z}||}$, $b_n = \dfrac{\mathscr{B}\hat{g}_n - \acute{z}}{||\hat{o}_n - \acute{z}||}$ and $c_n = \dfrac{\mathscr{B}\hat{i}_n - \acute{z}}{||\hat{o}_n - \acute{z}||}$. Now, using (3.9) and (3.10), we get $||\mathscr{B}\hat{g}_n - \acute{z}|| = ||\mathscr{B}\hat{g}_n - \mathscr{B}\acute{z}|| \leq ||\hat{g}_n - \acute{z}|| \leq ||\hat{o}_n - \acute{z}||$ and $||\mathscr{B}\hat{i}_n - \acute{z}|| = ||\mathscr{B}\hat{i}_n - \mathscr{B}\acute{z}|| \leq ||\hat{i}_n - \acute{z}|| \leq ||\hat{o}_n - \acute{z}||$. Therefore, $||b_n|| = \frac{||\mathscr{B}\hat{g}_n - z||}{||\hat{o}n - \acute{z}||} \leq \frac{||\hat{o}_n - \acute{z}||}{||\hat{o}_n - \acute{z}||} = 1$, $||c_n|| = \frac{||\mathscr{B}\hat{i}_n - \acute{z}||}{||\hat{o}_n - \acute{z}||} \leq \frac{||\hat{o}_n - \acute{z}||}{||\hat{o}_n - \acute{z}||} = 1$. By Algorithm S, we get $\hat{o}_{n+1} - \acute{z} = (1 - \eta'_n)(\mathscr{B}\hat{g}_n - \acute{z}) + \eta'_n(\mathscr{B}\hat{i}_n - \acute{z})$. Now, dividing by $||\hat{o}_n - \acute{z}||$, we obtain $\dfrac{\hat{o}_{n+1} - \acute{z}}{||\hat{o}_n - \acute{z}||} = (1 - \eta'_n)\dfrac{(\mathscr{B}\hat{g}_n - \acute{z})}{||\hat{o}_n - \acute{z}||} + \eta'_n \dfrac{(\mathscr{B}\hat{i}_n - \acute{z})}{||\hat{o}_n - \acute{z}||}$. Then $a_n = (1 - \eta'_n)b_n + \eta'_n c_n$. Next, we show that $||a_n|| \to 1$.

$$\lim_{n\to\infty} ||a_n|| = \lim_{n\to\infty} \frac{||\hat{o}_{n+1} - \acute{z}||}{||\hat{o}_n - \acute{z}||} = \frac{r}{r} = 1.$$

From Lemma 1.1.10, $||b_n - c_n|| \to 0$. So, $||\mathscr{B}\hat{g}_n - \mathscr{B}\hat{i}_n|| \to 0$.

Since $\lim_{n\to\infty} ||\hat{o}_n - \acute{z}|| = r$, and by (3.9) and (3.10), we get $\limsup_{n\to\infty} ||\hat{g}_n - \acute{z}|| \leq r$ and

$$\limsup_{n\to\infty} ||\hat{i}_n - \acute{z}|| \leq r. \tag{3.14}$$

Also, we have $||\mathscr{B}\hat{g}_n - \acute{z}|| = ||\mathscr{B}\hat{g}_n - \mathscr{B}\acute{z}|| \leq ||\hat{g}_n - \acute{z}||$. Take lim sup on both sides, we get

$$\limsup_{n\to\infty} ||\mathscr{B}\hat{g}_n - \acute{z}|| \leq r. \tag{3.15}$$

Now $||\hat{o}_{n+1} - \acute{z}|| = ||(1 - \eta'_n)(\mathscr{B}\hat{g}_n - \acute{z}) + \eta'_n(\mathscr{B}\hat{i}_n - \acute{z})|| \leq ||\mathscr{B}\hat{g}_n - \acute{z}|| + \eta'_n||\mathscr{B}\hat{i}_n - \mathscr{B}\hat{g}_n||$. On taking $n \to \infty$, and using $||\mathscr{B}\hat{i}_n - \mathscr{B}\hat{g}_n|| \to 0$, we get

$$r \leq \liminf_{n\to\infty} ||\mathscr{B}\hat{g}_n - \acute{z}||. \tag{3.16}$$

From (3.15) and (3.16), we obtain $\lim_{n\to\infty} ||\mathscr{B}\hat{g}_n - \acute{z}|| = r$. Now,

$$\begin{aligned} ||\mathscr{B}\hat{g}_n - \acute{z}|| \leq & ||\mathscr{B}\hat{g}_n - \mathscr{B}\hat{i}_n|| + ||\mathscr{B}\hat{i}_n - \acute{z}|| \\ \leq & ||\mathscr{B}\hat{g}_n - \mathscr{B}\hat{i}_n|| + ||\hat{i}_n - \acute{z}||, \end{aligned}$$

and it yields that

$$r \leq \liminf_{n\to\infty} ||\hat{i}_n - \acute{z}||. \tag{3.17}$$

By (3.14) and (3.17), we have $\lim_{n\to\infty} ||\hat{i}_n - \acute{z}|| = r$. Consider

$$\begin{aligned} ||\hat{i}_n - \acute{z}|| = & ||\mathscr{B}((1-\delta'_n)\hat{g}_n + \delta'_n \mathscr{B}\hat{g}_n) - \acute{z}|| \\ = & ||\mathscr{B}((1-\delta'_n)\hat{g}_n + \delta'_n \mathscr{B}\hat{g}_n) - \mathscr{B}\acute{z}|| \\ \leq & ||(1-\delta'_n)(\hat{g}_n - \acute{z}) + \delta'_n(\mathscr{B}\hat{g}_n - \acute{z})||. \end{aligned}$$

On taking $n \to \infty$, we obtain $r \leq ||(1-\delta'_n)(\hat{g}_n - \acute{z}) + \delta'_n(\mathscr{B}\hat{g}_n - \acute{z})||$ and $||(1-\delta'_n)(\hat{g}_n - \acute{z}) + \delta'_n(\mathscr{B}\hat{g}_n - \acute{z})|| \leq r$. It implies $||(1-\delta'_n)(\hat{g}_n - \acute{z}) + \delta'_n(\mathscr{B}\hat{g}_n - \acute{z})|| = r$. Using Lemma 1.1.11, we get

$$\lim_{n\to\infty} ||\hat{g}_n - \mathscr{B}\hat{g}_n|| = 0. \tag{3.18}$$

Also $||\mathscr{B}\hat{g}_n - \mathscr{B}\hat{o}_n|| \leq ||\mathscr{B}\hat{g}_n - \hat{g}_n|| + ||\hat{g}_n - \mathscr{B}\hat{o}_n||$. By (3.13) and (3.18) we get $\lim_{n\to\infty} ||\mathscr{B}\hat{g}_n - \mathscr{B}\hat{o}_n|| \to 0$. From the Algorithm (S), we have

$$\begin{aligned} ||\hat{i}_n - \mathscr{B}\hat{g}_n|| = & ||\mathscr{B}((1-\delta'_n)\hat{g}_n + \delta'_n \mathscr{B}\hat{g}_n) - \mathscr{B}\hat{g}_n|| \\ \leq & ||(1-\delta'_n)\hat{g}_n + \delta'_n \mathscr{B}\hat{g}_n - \hat{g}_n|| \\ = & ||(1-\delta'_n)(\hat{g}_n - \hat{g}_n) + \delta'_n(\mathscr{B}\hat{g}_n - \hat{g}_n|| \\ = & \delta'_n(\mathscr{B}\hat{g}_n - \hat{g}_n)||. \end{aligned} \tag{3.19}$$

Therefore, $||\hat{i}_n - \mathscr{B}\hat{g}_n|| \to 0$ as $n \to \infty$. Also $||\hat{i}_n - \hat{g}_n|| \leq ||\hat{i}_n - \mathscr{B}\hat{g}_n|| + ||\mathscr{B}\hat{g}_n - \hat{g}_n||$.

Using (3.18), we have $\lim_{n\to\infty} ||\hat{i}_n - \hat{g}_n|| \to 0$. Consider

$$\begin{aligned} ||\hat{o}_{n+1} - \hat{o}_n|| &= ||(1-\eta'_n)\mathscr{B}\hat{g}_n + \eta'_n\mathscr{B}\hat{i}_n - \hat{o}_n|| \\ &= ||\mathscr{B}\hat{g}_n - \hat{o}_n + \eta'_n(\mathscr{B}\hat{i}_n - \mathscr{B}\hat{g}_n)|| \\ &\leq ||\mathscr{B}\hat{g}_n - \hat{o}_n|| + \eta'_n||\mathscr{B}\hat{i}_n - \mathscr{B}\hat{g}_n|| \\ &\leq ||\mathscr{B}\hat{g}_n - \mathscr{B}\hat{o}_n|| + ||\mathscr{B}\hat{o}_n - \hat{o}_n|| + \eta'_n||\mathscr{B}\hat{i}_n - \mathscr{B}\hat{g}_n||. \end{aligned}$$

On taking $n \to \infty$, we get $||\hat{o}_{n+1} - \hat{o}_n|| \to 0$. Since $\mathscr{B}(\mathscr{T}_1)$ contained in a compact subset of $\mathscr{T}_1$, $\mathscr{B}(\hat{o}_n)$ having a subsequence $\{\mathscr{B}(\hat{o}_{n_{\hat{s}}})\}$ that converges to a point $\hat{o} \in \mathscr{T}_1$. Also, $\{\hat{o}_{n_{\hat{s}}}\}$ converges to a point $\hat{o}$. Now $||\mathscr{B}\hat{o} - \mathscr{B}\hat{o}_{n_{\hat{s}}}|| \leq ||\hat{o} - \hat{o}_{n_{\hat{s}}}||$ as $\hat{s} \to \infty$, we obtain $\mathscr{B}\hat{o} = \hat{o}$. It implies $\lim_{n\to\infty} ||\hat{o}_n - \hat{o}||$ exists, therefore $\lim_{n\to\infty} ||\hat{o}_n - \hat{o}|| = \lim_{\hat{s}\to\infty} ||\hat{o}_{n_{\hat{s}}} - \hat{o}|| = 0$. Hence, the result. □

3.2 Convergence to best proximity point problem

This section involves the introduction of two algorithms using the projection operator and gives some convergence results. The findings of this section are published in Sharma and Chandok [69, 72].

Throughout this section, we assume that $\mathscr{T}_1, \mathscr{T}_2$ be non-empty bounded closed convex subsets of a UCBS $\mathscr{W}$. Let $\hat{o}_0 \in \mathscr{T}_{1_0}$. Define

$$\begin{cases} \hat{g}_n = & (1-\gamma'_n)\hat{o}_n + \gamma'_n \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n, \\ \hat{i}_n = & (1-\delta'_n)\hat{o}_n + \delta'_n \mathscr{P}_{\mathscr{T}}\mathscr{B}\hat{g}_n \\ \hat{o}_{n+1} = & (1-\eta'_n)\hat{o}_n + \eta'_n \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n \text{ where } \gamma'_n, \eta'_n, \delta'_n \in (0,1]. \end{cases} \qquad \text{Algorithm S1}$$

Theorem 3.2.1. Let $\mathscr{B} : \mathscr{T}_{1_0} \to \mathscr{T}_{2_0}$ be a NE with a non-empty BPP set. If $\mathscr{P}_{\mathscr{T}_1}\mathscr{B}(\mathscr{T}_{1_0})$ lies in a compact subset of $\mathscr{T}_{1_0}$, then sequence $\{\hat{o}_n\}$ defined by Algorithm S1 converges to BPP of $\mathscr{B}$ in $\mathscr{T}_1$.

Proof. Assume that $||\mathscr{T}_1 - \mathscr{T}_2|| > 0$. Consider

$$\begin{aligned} ||\hat{g}_n - z|| =& ||(1-\gamma'_n)\hat{o}_n + \gamma'_n \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \acute{z}|| \\ =& ||(1-\gamma'_n)(\hat{o}_n - \acute{z}) + \gamma'_n(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \acute{z})|| \\ \leq& ||(1-\gamma'_n)(\hat{o}_n - \acute{z})|| + ||\gamma'_n(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \acute{z})|| \\ =& (1-\gamma'_n)||\hat{o}_n - \acute{z}|| + \gamma'_n||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \acute{z}||. \end{aligned} \tag{3.20}$$

Since $||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \mathscr{B}\hat{o}_n|| = ||\mathscr{T}_1 - \mathscr{T}_2||$ and $||\acute{z} - \mathscr{B}\acute{z}|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Also by Remark 1.1.3, a pair $(\mathscr{T}_1, \mathscr{T}_2)$ has ₤-property, we have

$$||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \acute{z}|| = ||\mathscr{B}\hat{o}_n - \mathscr{B}\acute{z}||. \tag{3.21}$$

Therefore, from (3.20), we get

$$\begin{aligned} ||\hat{g}_n - \acute{z}|| =& (1-\gamma'_n)||(\hat{o}_n - \acute{z})|| + \gamma'_n||\mathscr{B}\hat{o}_n - \mathscr{B}\acute{z}|| \\ \leq& (1-\gamma'_n)||\hat{o}_n - \acute{z}|| + \gamma'_n||\hat{o}_n - \acute{z}|| \\ =& ||\hat{o}_n - \acute{z}||. \end{aligned} \tag{3.22}$$

It shows $||\hat{g}_n - \acute{z}|| \leq ||\hat{o}_n - \acute{z}||$. In the same way, we obtain

$$\begin{aligned} ||\hat{i}_n - \acute{z}|| =& ||(1-\delta'_n)\hat{o}_n + \delta'_n \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n - \acute{z}|| \\ =& ||(1-\delta'_n)\hat{o}_n - \acute{z} + \delta'_n(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n - \acute{z})|| \\ \leq& ||(1-\delta'_n)\hat{o}_n - \acute{z}|| + ||\delta'_n(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n - \acute{z})|| \\ =& (1-\delta'_n)||\hat{o}_n - \acute{z}|| + \delta'_n||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n - \acute{z}|| \end{aligned} \tag{3.23}$$

Since $||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n - \mathscr{B}\hat{g}_n|| = ||\mathscr{T}_1 - \mathscr{T}_2||$ and $||\acute{z} - \mathscr{B}\acute{z}|| = ||\mathscr{T}_1 - \mathscr{T}_2||$, using ₤-property, we have

$$||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n - \acute{z}|| = ||\mathscr{B}\hat{g}_n - \mathscr{B}\acute{z}||. \tag{3.24}$$

Similarly,

$$||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n - \acute{z}|| = ||\mathscr{B}\hat{i}_n - \mathscr{B}\acute{z}||. \tag{3.25}$$

Using (3.24) in (3.23), we get

$$\begin{aligned}||\hat{i}_n - \acute{z}|| =&(1-\gamma_n')||\hat{o}_n - \acute{z}|| + \gamma_n'||\mathscr{B}\hat{g}_n - \mathscr{B}\acute{z}|| \\ \leq&(1-\gamma_n')||\hat{o}_n - \acute{z}|| + \gamma_n'||\hat{g}_n - \acute{z}|| \\ \leq&(1-\gamma_n')||\hat{o}_n - \acute{z}|| + \gamma_n'||\hat{o}_n - \acute{z}|| \\ =&||\hat{o}_n - \acute{z}||. \end{aligned} \tag{3.26}$$

It shows $||\hat{i}_n - \acute{z}|| \leq ||\hat{o}_n - \acute{z}||$. By (3.22), (3.24), (3.25) and (3.26), we have

$$\begin{aligned}||\hat{o}_{n+1} - \acute{z}|| =&||(1-\eta_n')\hat{o}_n + \eta_n'\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n - \acute{z}|| \\ \leq&||(1-\eta_n')\hat{o}_n - \acute{z}|| + ||\eta_n'(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n - \acute{z})|| \\ =&(1-\eta_n')||\hat{o}_n - \acute{z}|| + \eta_n'||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n - \acute{z}|| \\ =&(1-\eta_n')||\hat{o}_n - \acute{z}|| + \eta_n'||\mathscr{B}\hat{i}_n - \mathscr{B}\acute{z}|| \\ \leq&(1-\eta_n')||\hat{o}_n - \acute{z}|| + \eta_n'||\hat{i}_n - \acute{z}|| \\ \leq&(1-\eta_n')||\hat{o}_n - \acute{z}|| + \eta_n'||\hat{o}_n - \acute{z}|| \\ =&||\hat{o}_n - \acute{z}||. \end{aligned}$$

It signifies that sequence $\{||\hat{o}_n - \acute{z}||\}$ is a non-increasing and bounded below. So,

$$\lim_{n\to\infty} ||\hat{o}_n - \acute{z}|| = r. \tag{3.27}$$

Case (i): If $r = 0$. Since $||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \mathscr{B}\hat{o}_n|| = ||\mathscr{T}_1 - \mathscr{T}_2||$ and $||\acute{z} - \mathscr{B}\acute{z}|| = ||\mathscr{T}_1 - \mathscr{T}_2||$, using ₤-property we have $||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \acute{z}|| = ||\mathscr{B}\hat{o}_n - \mathscr{B}\acute{z}||$. Therefore,

$$\begin{aligned}||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \hat{o}_n|| \leq&||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \acute{z}|| + ||\hat{o}_n - \acute{z}|| \\ =&||\mathscr{B}\hat{o}_n - \mathscr{B}\acute{z}|| + ||\hat{o}_n - \acute{z}|| \\ \leq&||\hat{o}_n - \acute{z}|| + ||\hat{o}_n - \acute{z}|| \\ =&2||\hat{o}_n - \acute{z}||. \end{aligned}$$

Taking $n \to \infty$, we get $||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n - \hat{o}_n|| \to 0$.

Case (ii): If $r > 0$. Suppose there is a subsequence $\{\hat{o}_{n_{\hat{s}}}\}$ of $\{\hat{o}_n\}$ such that $||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_{n_{\hat{s}}} - \hat{o}_{n_{\hat{s}}}|| \geq \epsilon > 0$ for all $\hat{s}$. We select $\xi > 0$ to be small since the modulus of convexity of δ^* is a increasing and continuous mapping. So, $\left(1 - c\delta^*\left(\frac{\epsilon}{r+\xi}\right)\right)(r+\xi) < r$, where $c > 0$. Choose $\hat{s}$ in such a way that $||\hat{o}_{n_{\hat{s}}} - \acute{z}|| \leq r + \xi$. Using (3.21), we have

$$
\begin{aligned}
&||\hat{o}_{n_{\hat{s}+1}} - \acute{z}|| = ||((1-\eta'_{n_{\hat{s}}})\hat{o}_{n_{\hat{s}}} + \eta'_{n_{\hat{s}}}\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_{n_{\hat{s}}} - \acute{z}|| \\
\leq& ||((1-\eta'_{n_{\hat{s}}})\hat{o}_{n_{\hat{s}}} - \acute{z}|| + \eta'_{n_{\hat{s}}}||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_{n_{\hat{s}}} - \acute{z}|| \\
\leq& (1-\eta'_{n_{\hat{s}}})||\hat{o}_{n_{\hat{s}}} - \acute{z}|| + \eta'_{n_{\hat{s}}}||(1-\delta'_{n_{\hat{s}}})\hat{o}_{n_{\hat{s}}} + \delta'_{n_{\hat{s}}}\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_{n_{\hat{s}}} - \acute{z}|| \\
\leq& (1-\eta'_{n_{\hat{s}}})||\hat{o}_{n_{\hat{s}}} - \acute{z}|| + \eta'_{n_{\hat{s}}}(1-\delta'_{n_{\hat{s}}})||\hat{o}_{n_{\hat{s}}} - \acute{z}|| + \eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_{n_{\hat{s}}} - \acute{z}|| \\
=& (1-\eta'_{n_{\hat{s}}} + \eta'_{n_{\hat{s}}} - \eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}})||\hat{o}_{n_{\hat{s}}} - \acute{z}|| + \eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_{n_{\hat{s}}} - \acute{z}|| \\
=& (1-\eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}})||\hat{o}_{n_{\hat{s}}} - \acute{z}|| + \eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_{n_{\hat{s}}} - \acute{z}|| \\
\leq& (1-\eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}})||\hat{o}_{n_{\hat{s}}} - \acute{z}|| + \eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}||\hat{g}_{n_{\hat{s}}} - \acute{z}||. \qquad (3.28)
\end{aligned}
$$

Using Proposition 1.1.1, we have

$$
\begin{aligned}
||\hat{g}_{n_{\hat{s}}} - \acute{z}|| =& ||((1-\gamma'_{n_{\hat{s}}})\hat{o}_{n_{\hat{s}}} + \gamma'_{n_{\hat{s}}}\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_{n_{\hat{s}}}) - \acute{z}|| \\
=& ||(1-\gamma'_{n_{\hat{s}}})(\hat{o}_{n_{\hat{s}}} - \acute{z}) + \gamma'_{n_{\hat{s}}}(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_{n_{\hat{s}}} - \acute{z})|| \\
\leq& \left(1 - 2\delta^*\left(\frac{\epsilon}{r+\xi}\right)\min\left\{\gamma'_{n_{\hat{s}}}, 1 - -\gamma'_{n_{\hat{s}}}\right\}\right)(r+\xi).
\end{aligned}
$$

Therefore, by (3.28), we obtain

$$
\begin{aligned}
&||\hat{o}_{n_{\hat{s}+1}} - \acute{z}|| \\
\leq& \left((1-\eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}) + \eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}\left(1 - 2\delta^*\left(\frac{\epsilon}{r+\xi}\right)\min\left\{\gamma'_{n_{\hat{s}}}, 1-\gamma'_{n_{\hat{s}}}\right\}\right)\right)(r+\xi) \\
=& \left((1-\eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}) + \eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}} - 2\eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}\delta^*\left(\frac{\epsilon}{r+\xi}\right)\min\left\{\gamma'_{n_{\hat{s}}}, 1-\gamma'_{n_{\hat{s}}}\right\}\right)(r+\xi) \\
=& \left(1 - 2\delta^*\left(\frac{\epsilon}{r+\xi}\right)\min\left\{\gamma'_{n_{\hat{s}}}\eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}, \eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}(1-\gamma'_{n_{\hat{s}}})\right\}\right)(r+\xi).
\end{aligned}
$$

So, there is $l > 0$ such that $2\min\left\{\gamma'_{n_{\hat{s}}}\eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}, 1 - \gamma'_{n_{\hat{s}}}\eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}\right\} \geq l$ and $\left(1 - 2\delta^*\left(\frac{\epsilon}{r+\xi}\right)\min\left\{\gamma'_{n_{\hat{s}}}\eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}, 1 - \gamma'_{n_{\hat{s}}}\eta'_{n_{\hat{s}}}\delta'_{n_{\hat{s}}}\right\}\right)(r+\xi) \leq \left(1 - l\delta^*\left(\frac{\epsilon}{r+\xi}\right)\right)r + \xi.$

Let $\xi > 0$ very small, we get $(r+\xi)\left(1 - l\delta^*\left(\frac{\epsilon}{r+\xi}\right)\right) < r$, which contradicts. It gives $\lim_{n\to\infty} ||\hat{o}_n - \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n|| = 0$ and $||\hat{g}_n - \hat{o}_n|| \to 0$ as $n \to \infty$.
Now, we need to show that $||\hat{o}_{n+1} - \hat{o}_n|| \to 0$. Consider $a_n = \frac{\hat{o}_{n+1} - \acute{z}}{||\hat{o}_n - \acute{z}||}$, $b_n = \frac{\hat{o}_n - \acute{z}}{||\hat{o}_n - \acute{z}||}$ and $c_n = \frac{\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n - \acute{z}}{||\hat{o}_n - \acute{z}||}$. By (3.25), we get $||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n - \acute{z}|| \leq ||\hat{i}_n - \acute{z}|| \leq ||\hat{o}_n - \acute{z}||$. Therefore, $||b_n|| = \frac{||\hat{o}_n - \acute{z}||}{||\hat{o}_n - \acute{z}||} = 1$ and $||c_n|| = \frac{||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n - \acute{z}||}{||\hat{o}_n - \acute{z}||} \leq \frac{||\hat{o}_n - \acute{z}||}{||\hat{o}_n - \acute{z}||} = 1$. By Algorithm S1, we achieve $\hat{o}_{n+1} - \acute{z} = \eta'_n(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n - \acute{z}) + (1-\eta'_n)(\hat{o}_n - \acute{z})$. Now, by dividing $||\hat{o}_n - \acute{z}||$, we get $\frac{\hat{o}_{n+1} - \acute{z}}{||\hat{o}_n - \acute{z}||} = (1-\eta'_n)\frac{(\hat{o}_n - \acute{z})}{||\hat{o}_n - \acute{z}||} + \eta'_n\frac{(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n - \acute{z})}{||\hat{o}_n - \acute{z}||}$. So, $a_n = (1-\eta'_n)b_n + \eta'_n c_n$.
Now, we have to prove that $||a_n|| \to 1$ as $n \to \infty$.

$$\lim_{n\to\infty} ||a_n|| = \lim_{n\to\infty} \frac{||\hat{o}_{n+1} - \acute{z}||}{||\hat{o}_n - \acute{z}||} = \frac{r}{r} = 1.$$

By Lemma 1.1.10, $||b_n - c_n|| \to 0$. Therefore $||\hat{o}_n - \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n|| \to 0$. It implies $||\hat{o}_{n+1} - \hat{o}_n|| \to 0$. Now, define $\mathscr{B}_1 : \mathscr{T}_{1_0} \to \mathscr{T}_{1_0}$ as $\mathscr{B}_1(\hat{o}) = \mathscr{P}_{\mathscr{T}_1}\mathscr{B}(\hat{o})$ for all $\hat{o} \in \mathscr{T}_{1_0}$. Again using £-property, we get $\mathscr{B}_1$ is a nonexpansive and $\mathscr{F}(\mathscr{B}_1) = Best_{\mathscr{T}_1}\mathscr{B}$. Since $\mathscr{P}_{\mathscr{T}_1}\mathscr{B}(\mathscr{T}_{1_0}) = \mathscr{B}_1(\mathscr{T}_{1_0})$ lies in a compact subset of $\mathscr{T}_{1_0}$, $\mathscr{B}_1(\hat{o}_n)$ having a subsequence $\{\mathscr{B}_1(\hat{o}_{n_{\hat{s}}})\}$ which converges to a point $\hat{o} \in \mathscr{T}_{1_0}$. Also $\{\hat{o}_{n_{\hat{s}}}\}$ converges to a point $\hat{o}$. Therefore,

$$\lim_{n\to\infty} ||\hat{o}_{n_{\hat{s}}} - \hat{o}|| = 0. \tag{3.29}$$

Since $||\mathscr{B}_1\hat{o} - \hat{o}_{n_{\hat{s}}}|| \leq ||\mathscr{B}_1\hat{o} - \mathscr{B}_1(\hat{o}_{n_{\hat{s}}})|| + ||\mathscr{B}_1(\hat{o}_{n_{\hat{s}}}) - \hat{o}_{n_{\hat{s}}}||$. Taking $\hat{s} \to \infty$ we obtain $\mathscr{B}_1\hat{o} = \hat{o}$. It shows that $\hat{o}$ is a FP of $\mathscr{B}_1$, then $\hat{o}$ is a BPP of $\mathscr{B}$. Since $\hat{o}$ is a BPP of $\mathscr{B}$, by (3.27) $\lim_{n\to\infty} ||\hat{o}_n - \hat{o}||$ exists and is equal to r. So, every subsequence of $||\hat{o}_n - \hat{o}||$ converges to r. Therefore $\lim_{n\to\infty} ||\hat{o}_{n_{\hat{s}}} - \hat{o}|| = r$. By (3.29), we get $r = 0$. It shows that $\{\hat{o}_n\}$ defined by Algorithm S1 converges to a point $\hat{o} \in \mathscr{T}_{1_0}$, which satisfies $||\hat{o} - \mathscr{B}\hat{o}|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. □

Next, we propose another Algorithm using projection operator as described below:

Let $\hat{o}_0 \in \mathscr{T}_{1_0}$. Define

$$\begin{cases} \hat{g}_n = & (1-\gamma'_n)\hat{o}_n + \gamma'_n \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n \\ \hat{i}_n = & (1-\delta'_n)\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n + \delta'_n \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n \\ \hat{o}_{n+1} = & (1-\eta'_n)\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n + \eta'_n \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n, \text{ where } \gamma'_n, \eta'_n, \delta'_n \in (0,1]. \end{cases} \quad \text{Algorithm S2}$$

Theorem 3.2.2. Let $\mathscr{B} : \mathscr{T}_{1_0} \to \mathscr{T}_{2_0}$ be a NE with a non-empty BPP set. If $\mathscr{P}_{\mathscr{T}_1}\mathscr{B}(\mathscr{T}_{1_0})$ lies in a compact subset of $\mathscr{T}_{1_0}$, then sequence $\{\hat{o}_n\}$ defined by Algorithm S2 converges to BPP of $\mathscr{B}$ in $\mathscr{T}_1$.

Proof. Result follows from Theorem 3.2.1. □

Example 3.2.1. *Consider $\mathscr{W} = \mathbb{R}^2$ with norm*

$$||\hat{o}|| = \sqrt{|\hat{o}_1|^2 + |\hat{i}_1|^2}$$

for all $\hat{o} = (\hat{o}_1, \hat{i}_1) \in \mathbb{R}^2$. Suppose $\mathscr{T}_1 = \{(0,\hat{o}) : 0 \leq \hat{o} \leq 2\}$ and $\mathscr{T}_2 = \{(2,\hat{o}) : 0 \leq \hat{o} \leq 2\}$, such that $||\mathscr{T}_1 - \mathscr{T}_2|| = 2$. Since $\mathscr{T}_{1_0} = \mathscr{T}_1$ and $\mathscr{T}_{2_0} = \mathscr{T}_2$ then define $\mathscr{B} : \mathscr{T}_{1_0} \to \mathscr{T}_{2_0}$ by $\mathscr{B}(0,\hat{o}) = (2, 2-\hat{o})$ for all $(0,\hat{o}) \in \mathscr{T}_{1_0}$. Next we show that $\mathscr{B}$ is nonexpansive. Let $\acute{z}_1 = (0,\hat{o})$ and $\acute{z}_2 = (0,\hat{g})$ in $\mathscr{T}_{1_0}$ where $0 \leq \hat{o}, \hat{g} \leq 2$ then

$$\begin{aligned} ||\mathscr{B}\acute{z}_1 - \mathscr{B}\acute{z}_2|| &= \sqrt{(2-2)^2 + ((2-\hat{o}) - (2-\hat{g}))^2} \\ &= \sqrt{(\hat{o}-\hat{g})^2} = \hat{o} - \hat{g} = ||\acute{z}_1 - \acute{z}_2||. \end{aligned}$$

This implies

$$||\mathscr{B}\acute{z}_1 - \mathscr{B}\acute{z}_2|| \leq ||\acute{z}_1 - \acute{z}_2||.$$

for all $\acute{z}_1, \acute{z}_2 \in \mathscr{T}_{1_0}$. $\mathscr{B}$ is a nonexpansive. All the suppositions of Theorems 3.2.1 and 3.2.2 are hold and $\mathscr{B}$ has a BPP $(0,1)$. Since $\mathscr{P}_{\mathscr{T}_1}(\hat{o})$ contained in compact set $\mathscr{T}_{1_0}$ for all $\hat{o} \in \mathscr{T}_{1_0}$, sequence defined by Algorithm S1 and Algorithm S2 converges to BPP $(0,1)$ see Table 3.2.

$\hat{o}_{n+1}$	c_1	c_2	c_3	c_4
$\hat{o}_0$	$(0, 3/2)$	$(0, 3/2)$	$(0, 3/2)$	$(0, 3/2)$
$\hat{o}_2$	$(0, 1.0239)$	$(0, 1.2035)$	$(0, 1.0005)$	$(0, 1.1142)$
$\hat{o}_4$	$(0, 1.0011)$	$(0, 1.0828)$	$(0, 1.0000)$	$(0, 1.0261)$
$\hat{o}_6$	$(0, 1.0001)$	$(0, 1.0337)$	.	$(0, 1.0060)$
$\hat{o}_{10}$	.	$(0, 1.0056)$	.	$(0, 1.0003)$

Table 3.2: Camparison between Algorithm S1 and Algorithm S2

Table 3.2 shows a comparison between Algorithm S1 and Algorithm S2 for different values of $\eta'_n, \delta'_n, \gamma'_n$*. Here* c_1 *shows sequence defined by Algorithm S1 with* $\eta'_n = \delta'_n = \gamma'_n = \frac{3}{4}$*,* c_2 *shows sequence defined by Algorithm S1 with* $\eta'_n = \delta'_n = \gamma'_n = 0.9$*,* c_3 *shows sequence defined by Algorithm S2 with* $\eta'_n = \delta'_n = \gamma'_n = \frac{3}{4}$*,* c_4 *shows sequence defined by Algorithm S2 with* $\eta'_n = \delta'_n = \gamma'_n = 0.9$*.*

Next, we modify Algorithm S, for non-self mapping $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ using the projection operator as follows. For arbitrary $\hat{o}_0 \in \mathscr{T}_{1_0}$, define

$$\begin{cases} \hat{o}_{n+1} = & (1-\eta'_n)\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n + \eta'_n \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n \\ \hat{i}_n = & \mathscr{P}_{\mathscr{T}_1}\mathscr{B}((1-\delta'_n)\hat{g}_n + \delta'_n \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n) \\ \hat{g}_n = & \mathscr{P}_{\mathscr{T}_1}\mathscr{B}((1-\gamma'_n)\hat{o}_n + \gamma'_n \mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{o}_n) \end{cases} \qquad \text{Algorithm (S')}$$

where $\gamma'_n, \eta'_n, \delta'_n \in [0,1]$, $n \in \mathbb{N} \cup \{0\}$.

Theorem 3.2.3. Let $\mathscr{B} : \mathscr{T}_{1_0} \to \mathscr{T}_{2_0}$ is a NE. If $\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B}(\mathscr{T}_{1_0})$ contained in a compact subset of $\mathscr{T}_1$, then sequence $\{\hat{o}_n\}$ defined by Algorithm S' converges to a BPP of $\mathscr{B}$ strongly.

Proof. If $\hat{o}_0 \in \mathscr{T}_{1_0}$, then there exists $\mathscr{B}\hat{o}_1 \in \mathscr{T}_{2_0}$ such that $||\hat{o}_0 - \mathscr{B}\hat{o}_1|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Since $\mathscr{B}\hat{o}_1 \in \mathscr{T}_{2_0}$, then there exists $\mathscr{P}_{\mathscr{T}_{1_0}}\hat{\mathscr{B}(w_0)} \in \mathscr{T}_1$ such that

$$||\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B}\hat{o}_0 - \mathscr{B}\hat{o}_1|| = ||\mathscr{T}_1 - \mathscr{T}_2||. \tag{3.30}$$

If $y \in \mathscr{T}_1$ another point such that $||y - \mathscr{B}\hat{o}_1|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Using Remark 1.1.3, we obtain $||\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B}\hat{o}_0 - y|| = ||\mathscr{B}\hat{o}_1 - \mathscr{B}\hat{o}_1||$. Hence, $(\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B})\hat{o}_0 = y$. Similarly,

$$||\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B}\hat{o}_1 - \mathscr{B}\hat{o}_2|| = ||\mathscr{T}_1 - \mathscr{T}_2||. \tag{3.31}$$

Using (3.30), (3.31) and ₤- Property, we have

$$||\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B}\hat{o}_0 - \mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B}\hat{o}_1|| = ||\mathscr{B}\hat{o}_1 - \mathscr{B}\hat{o}_2||. \tag{3.32}$$

Next, we have to prove that $\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B} : \mathscr{T}_{1_0} \to \mathscr{T}_{1_0}$ is a NE. Using (3.32), we obtain

$$\begin{aligned} ||(\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B})\hat{o}_2 - (\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B})\hat{o}_3|| =& ||\mathscr{P}_{\mathscr{T}_{1_0}}(\mathscr{B}\hat{o}_2) - \mathscr{P}_{\mathscr{T}_{1_0}}(\mathscr{B}\hat{o}_3)|| \\ =& ||\mathscr{B}\hat{o}_2 - \mathscr{B}\hat{o}_3|| \\ \leq& ||\hat{o}_2 - \hat{o}_3||. \end{aligned}$$

Let $\acute{z}$ is BPP of $\mathscr{B}$. Thus $||\acute{z} - \mathscr{B}\acute{z}|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Since $\mathscr{B}\acute{z} \in \mathscr{T}_{2_0}$ then there exist $\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B} \in \mathscr{T}_1$ such that $||\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B}\acute{z} - \mathscr{B}\acute{z}|| = ||\mathscr{T}_1 - \mathscr{T}_2||$. Again using Remark 1.1.3, we get $\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B}\acute{z} = \acute{z}$. This implies $\acute{z} \in \mathscr{F}(\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B})$. Similarly, if $\acute{z} \in \mathscr{F}(\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B})$, then $\acute{z}$ is BPP of $\mathscr{B}$. It shows that the set of the FP of a mapping $\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B}$ is equal to the set of the BPP of a mapping $\mathscr{B}$. By Theorem 3.1.2, Algorithm S' converges to a FP of $\mathscr{P}_{\mathscr{T}_{1_0}}\mathscr{B}$ which is equal to a BPP of $\mathscr{B}$. □

3.3 Convergence to solution of split fixed problem

In the current section, we provide the iterative algorithm which converges to a solution of the split FP problem in Hilbert spaces. The results of this section are published in Sharma and Chandok [68].

Let $\mathscr{H}_1$, be a Hilbert space and $\hat{o}_0 \in \mathscr{H}_1$ be arbitrary. Define

$$\begin{cases} \hat{i}_n = & \hat{o}_n + \omega\beta\mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \\ \hat{g}_n = & (1-\gamma_n'-\delta_n')\hat{i}_n + (\gamma_n'+\delta_n')\mathscr{B}\hat{i}_n \\ \hat{o}_{n+1} = & (1-\eta_n')\hat{g}_n + \eta_n'\mathscr{B}\hat{g}_n \end{cases} \tag{SS}$$

where $\eta_n', \gamma_n', \delta_n', (\gamma_n'+\delta_n') \in (0,1)$, $n \in \mathbb{N}$, $\beta \in (0,1)$ and $\omega \in (0, \frac{1}{\kappa\beta})$ with κ being the spectral radius of $\mathscr{L}^*\mathscr{L}$.

Theorem 3.3.1. Let $\mathscr{H}_1$, $\mathscr{H}_2$ be real Hilbert spaces, $\mathscr{L} : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator and $\mathscr{B} : \mathscr{H}_1 \to \mathscr{H}_1$, $\mathscr{B}_1 : \mathscr{H}_2 \to \mathscr{H}_2$ be continuous quasi-NEs. If $\mathscr{S} = \{\hat{o} \in \mathscr{F}(\mathscr{B}) : \mathscr{L}\hat{o} \in \mathscr{F}(\mathscr{B}_1)\} \neq \emptyset$, and sequence $\{\hat{o}_n\}$ defined by Algorithm (SS), then

(i) $\{\hat{o}_n\}$ is Fejer monotone with respect to $\mathscr{S}$, that is, for every $z \in \mathscr{S}$,

$$||\hat{o}_{n+1} - z|| \leq ||\hat{o}_n - z||, n \in \mathbb{N}.$$

(ii) $\{\hat{o}_n\}$ converges weakly to a point $\hat{o} \in \mathscr{S}$.

Proof. By Remark 1.1.1, $\mathscr{F}(\mathscr{B})$ and $\mathscr{F}(\mathscr{B}_1)$ are closed and convex subsets of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Let $z \in \mathscr{S}$, by Lemma 1.1.3, we get

$$||\hat{o}_{n+1} - z||^2 \leq ||\hat{g}_n - z||^2 - \eta_n'(1-\eta_n')||\mathscr{B}\hat{g}_n - \hat{g}_n||^2.$$

Similarly,

$$||\hat{g}_n - z||^2 \leq ||\hat{i}_n - z||^2 - (\gamma_n'+\delta_n')(1-\gamma_n'-\delta_n')||\mathscr{B}\hat{i}_n - \hat{i}_n||^2. \tag{3.33}$$

On the other hand, we have

$$\begin{aligned} & ||\hat{i}_n - z||^2 \\ =& ||\hat{o}_n + \omega\beta\mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n - z||^2 \end{aligned}$$

$$
\begin{aligned}
=&||\hat{o}_n - z||^2 + \omega^2\beta^2||\mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2 + 2\omega\beta\left\langle \hat{o}_n - z, \mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \right\rangle \\
=&||\hat{o}_n - z||^2 + \omega^2\beta^2\left\langle (\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n, \mathscr{L}\mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \right\rangle + 2\omega\beta\left\langle \hat{o}_n - z, \mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \right\rangle.
\end{aligned} \tag{3.34}
$$

Since κ is the spectral radius of $\mathscr{L}^*\mathscr{L}$,

$$
\begin{aligned}
\omega^2\beta^2\left\langle (\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n, \mathscr{L}\mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \right\rangle &\leq \kappa\omega^2\beta^2\left\langle (\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n, (\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \right\rangle \\
&= \kappa\omega^2\beta^2||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2.
\end{aligned} \tag{3.35}
$$

By Lemma 1.1.3, we obtain

$$
\begin{aligned}
&2\omega\beta\left\langle \hat{o}_n - z, \mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \right\rangle \\
=&2\omega\beta\left\langle \mathscr{L}(\hat{o}_n - z), (\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \right\rangle \\
=&2\omega\beta\left\langle \mathscr{L}(\hat{o}_n - z) + (\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n - (\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n, (\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \right\rangle \\
=&2\omega\beta(\left\langle \mathscr{L}(\hat{o}_n - z) + (\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n, (\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \right\rangle - ||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2) \\
=&2\omega\beta(\left\langle \mathscr{B}_1\mathscr{L}\hat{o}_n - \mathscr{L}z, \mathscr{B}_1\mathscr{L}\hat{o}_n - \mathscr{L}\hat{o}_n \right\rangle - ||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2) \\
\leq&2\omega\beta\left(\frac{||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2}{2} - ||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2\right) \\
=&-\omega\beta||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2.
\end{aligned} \tag{3.36}
$$

Using (3.35) and (3.36) in (3.34), we obtain

$$
||\hat{i}_n - z||^2 \leq ||\hat{o}_n - z||^2 - \omega\beta\mathscr{L}(1 - \kappa\omega\beta)||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2. \tag{3.37}
$$

By (3.33) and (3.37), we get

$$
\begin{aligned}
||\hat{o}_{n+1} - z||^2 \leq&||\hat{o}_n - z||^2 - \omega\beta\mathscr{L}(1 - \kappa\omega\beta)||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2 - \\
&(\gamma_n' + \delta_n')(1 - \gamma_n' - \delta_n')||\mathscr{B}\hat{i}_n - \hat{i}_n||^2 - \eta_n'(1 - \eta_n')||\mathscr{B}\hat{g}_n - \hat{g}_n||^2 \\
\leq&||\hat{o}_n - z||^2 - \omega\beta\mathscr{L}(1 - \kappa\omega\beta)||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2 - \\
&(\gamma_n' + \delta_n')(1 - \gamma_n' - \delta_n')||\mathscr{B}\hat{i}_n - \hat{i}_n||^2.
\end{aligned} \tag{3.38}
$$

By (3.38), we get

$$||\hat{o}_{n+1} - z||^2 \leq ||\hat{o}_n - z||^2 - \omega\beta\mathscr{L}(1 - \kappa\omega\beta)||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n||^2. \tag{3.39}$$

It follows that $||\hat{o}_{n+1} - z|| \leq ||\hat{o}_n - z||$. Hence $\{\hat{o}_n\}$ is Fejer monotone with respect to $\mathscr{S}$ and $\{||\hat{o}_n - z||\}$ is monotonically decreasing. So, assume that $\lim_{n\to\infty} ||\hat{o}_n - z|| = l$ for some $l \geq 0$. It follows from (3.39) that

$$\lim_{n\to\infty} ||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n|| = 0. \tag{3.40}$$

Since sequence $\{\hat{o}_n\}$ is Fejer monotone, sequence $\{\hat{o}_n\}$ is bounded. As $\mathscr{F}(\mathscr{B})$ is compact, there exists a subsequence $\{\hat{o}_{n_{\hat{s}}}\}$ of $\{\hat{o}_n\}$ such that $\hat{o}_{n_{\hat{s}}} \to \hat{o} \in \mathscr{F}(\mathscr{B})$. Then it follows from (3.40) that $\mathscr{B}_1\mathscr{L}\hat{o} = \mathscr{L}\hat{o}$. This shows $\mathscr{L}\hat{o} \in \mathscr{F}(\mathscr{B}_1)$. On the other hand, by setting $\hat{i}_n = \hat{o}_n + \omega\beta\mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n$, we have

$$\begin{aligned} ||\hat{i}_{n_{\hat{s}}} - \hat{o}|| =& ||\hat{o}_{n_{\hat{s}}} + \omega\beta\mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_{n_{\hat{s}}} - \hat{o}|| \\ \leq& ||\hat{o}_{n_{\hat{s}}} - \hat{o}|| + \omega\beta||\mathscr{L}^*||||(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_{n_{\hat{s}}}|| \\ \to& \, 0 \text{ as } \hat{s} \to \infty. \end{aligned}$$

From (3.38), (3.39) and the convergence of sequence $\{||\hat{o}_n - z||\}$, we obtain

$$\lim_{n\to\infty} ||\mathscr{B}\hat{i}_{n_{\hat{s}}} - \hat{i}_{n_{\hat{s}}}|| = 0. \tag{3.41}$$

It follows that $\mathscr{B}\hat{o} = \hat{o}$. Hence $\hat{o} \in \mathscr{S}$. Assume that another subsequence $\{\hat{o}_{n_j}\}$ of $\{\hat{o}_n\}$ converges to $\hat{o}_1 \in \mathscr{S}$ such that $\hat{o} \neq \hat{o}_1$. By Opial's condition,

$$\begin{aligned} \lim_{n\to\infty} ||\hat{o}_n - \hat{o}|| &= \lim_{\hat{s}\to\infty} ||\hat{o}_{n_{\hat{s}}} - \hat{o}|| < \lim_{\hat{s}\to\infty} ||\hat{o}_{n_{\hat{s}}} - \hat{o}_1|| = \lim_{n\to\infty} ||\hat{o}_n - \hat{o}_1|| \\ &= \lim_{j\to\infty} ||\hat{o}_{n_j} - \hat{o}_1|| < \lim_{j\to\infty} ||\hat{o}_{n_j} - \hat{o}|| = \lim_{n\to\infty} ||\hat{o}_n - \hat{o}||, \end{aligned}$$

which is contradiction so $\hat{o} = \hat{o}_1$. Consequently, sequence $\{\hat{o}_n\}$ defined by Algorithm (SS) converges weakly to a point $\hat{o} \in \mathscr{S}$. □

Remark 3.3.1. If we take $(\gamma'_n + \delta'_n) = 0$ then Algorithm (SS) reduces to the algorithm presented by Moudafi (Algorithm (2.12), p.4085, [49]) as described below:

$$\begin{cases} \hat{i}_n = & \hat{o}_n + \omega\beta\mathscr{L}^*(\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n \\ \hat{o}_{n+1} = & (1-\eta'_n)\hat{i}_n + \eta'_n\mathscr{B}\hat{i}_n \end{cases} \tag{M}$$

where $\eta'_n \in (0,1)$, $n \in \mathbb{N}$, $\beta \in (0,1)$ and $\omega \in (0, \frac{1}{\kappa\beta})$ with κ being the spectral radius of $\mathscr{L}^*\mathscr{L}$.

Corollary 3.3.1. Under the assumptions of Theorem 3.3.1, sequence $\{\hat{o}_n\}$ defined by Algorithm (M). Then

(i) $\{\hat{o}_n\}$ is Fejer monotone with respect to $\mathscr{S}$, that is, for every $z \in \mathscr{S}$,

$$||\hat{o}_{n+1} - z|| \leq ||\hat{o}_n - z||, n \in \mathbb{N}.$$

(ii) $\{\hat{o}_n\}$ converges weakly to a point $\hat{o} \in \mathscr{S}$.

We give some examples to validate our results.

Example 3.3.1. *Let $\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{R}^2$ with Euclidean distance. Let $\mathscr{B} : \mathscr{H}_1 \to \mathscr{H}_2$ be defined by $\mathscr{B}(\hat{o}, \hat{g}) = (5 - \hat{o}, \hat{g})$, for all $(\hat{o}, \hat{g}) \in \mathscr{H}_1$. Let $\mathscr{B}_1 : \mathscr{H}_2 \to \mathscr{H}_2$ be defined by $\mathscr{B}(\hat{o}, \hat{g}) = (10 - \hat{o}, \frac{\hat{g}}{3})$ and $\mathscr{L} : \mathscr{H}_1 \to \mathscr{H}_2$ be defined by $\mathscr{L}(\hat{o}, \hat{g}) = (2\hat{o}, 2\hat{g})$ for all $(\hat{o}, \hat{g}) \in \mathscr{H}_1$. It is clear that both $\mathscr{B}$ and $\mathscr{B}_1$ are quasi-NEs and $\mathscr{B}$ has a FP $(\frac{5}{2}, 0)$ and $\mathscr{L}(\frac{5}{2}, 0) = (5, 0)$ is FP of $\mathscr{B}_1$. All the suppositions of Theorem 3.3.1 are hold, so sequence defined by Algorithm (SS) converges to $(\frac{5}{2}, 0)$.*

3.4 Convergence to solution of split best proximity point problem

In this section, we define a new algorithm using projection operators which converges to a solution of the SBPP problem in Hilbert spaces. We give a numerical example to back up our assertions and demonstrate that our technique converges to solution of

SBPP problem. The outcomes of this subsection are published in Sharma and Chandok [68].

Let $\hat{o}_0 \in \mathscr{T}_{1_0}$ arbitrary. Define

$$\begin{cases} \hat{i}_n = & \mathscr{P}_{\mathscr{T}_1}[\hat{o}_n + \omega\beta\mathscr{L}^*(\mathscr{P}_{\mathscr{T}_2}\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n] \\ \hat{g}_n = & (1-\gamma_n' - \delta_n')\hat{i}_n + (\gamma_n' + \delta_n')\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n \\ \hat{o}_{n+1} = & (1-\eta_n')\hat{g}_n + \eta_n'\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{g}_n \end{cases} \tag{SS'}$$

where $\eta_n', \gamma_n', \delta_n', (\gamma_n' + \delta_n') \in (0,1)$, $n \in \mathbb{N}$, $\beta \in (0,1)$ and $\omega \in (0, \frac{1}{\kappa\beta})$ with κ being the spectral radius of $\mathscr{L}^*\mathscr{L}$.

Theorem 3.4.1. Let $\mathscr{T}_1, \mathscr{F}_1$ and $\mathscr{T}_2, \mathscr{F}_2$ be non-empty closed convex subsets of Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Let $\mathscr{L} : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator such that $\mathscr{L}(\mathscr{T}_1) \subseteq \mathscr{T}_2$ and $\mathscr{B} : \mathscr{T}_1 \to \mathscr{F}_1$, $\mathscr{B}_1 : \mathscr{T}_2 \to \mathscr{F}_2$ be best proximally NEs with non-empty $Best_{\mathscr{T}_1}\mathscr{B}$ and $Best_{\mathscr{T}_2}(\mathscr{B})$. If $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{F}_{1_0}$, $\mathscr{B}_1(\mathscr{T}_{2_0}) \subseteq \mathscr{F}_{2_0}$, $\mathscr{S}^* = \{\hat{o} \in Best_{\mathscr{T}_1}\mathscr{B} : \mathscr{L}\hat{o} \in Best_{\mathscr{T}_2}\mathscr{B}_1\} \neq \emptyset$ and $\{\hat{o}_n\}$ is a sequence defined by Algorithm (SS'), then

(i) $\{\hat{o}_n\}$ is Fejer monotone with respect to $\mathscr{S}^*$, that is, for every $z \in \mathscr{S}^*$,

$$||\hat{o}_{n+1} - z|| \le ||\hat{o}_n - z||, n \in \mathbb{N}.$$

(ii) $\{\hat{o}_n\}$ converges weakly to a solution of SBPP $\hat{o} \in \mathscr{S}^*$.

Proof. Since $\mathscr{B} : \mathscr{T}_1 \to \mathscr{F}_1$ be best proximally NE and $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{F}_{1_0}$ by Lemma 1.1.1 $\mathscr{P}_{\mathscr{T}_1}\mathscr{B}$ is quasi-NE on $\mathscr{T}_{1_0}$. Next, we prove that set of FP of a mapping $\mathscr{P}_{\mathscr{T}_1}\mathscr{B}$ is equal to BPP of S. Let $\hat{o} \in \mathscr{F}(\mathscr{P}_{\mathscr{T}_1}\mathscr{B})$. Since $\mathscr{P}_{\mathscr{T}_1}\mathscr{B}$ is a quasi-NE on $\mathscr{T}_{1_0}$, we have $\mathscr{P}_{\mathscr{T}_1}\mathscr{B}(\hat{o}) \in \mathscr{T}_{1_0}$. So there exists $\mathscr{B}\hat{o} \in \mathscr{F}_1$ such that $||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}(\hat{o}) - \mathscr{B}\hat{o}|| = ||\mathscr{T}_1 - \mathscr{F}_1||$. If $\mathscr{B}\hat{x} \in \mathscr{F}_1$ is another point such that $||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}(\hat{o}) - \mathscr{B}\hat{x}|| = ||\mathscr{T}_1 - \mathscr{F}_1||$. Also by Remark 1.1.3, the pair $(\mathscr{T}_1, \mathscr{F}_1)$ has £-property we have $||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}(\hat{o}) - \mathscr{P}_{\mathscr{T}_1}(\hat{o})|| = ||\mathscr{B}\hat{o} - \mathscr{B}\hat{x}||$. This shows $\mathscr{B}\hat{o} = \mathscr{B}\hat{x}$. Thus

$$||\hat{o} - \mathscr{B}\hat{o}|| = ||\mathscr{P}_{\mathscr{T}_1}\mathscr{B}(\hat{o}) - \mathscr{B}\hat{o}|| = ||\mathscr{T}_1 - \mathscr{F}_1||,$$

where $\hat{o} \in \mathscr{F}(\mathscr{P}_{\mathscr{T}_1}\mathscr{B})$. This shows that $\mathscr{F}(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}) \subseteq Best_{\mathscr{T}_1}\mathscr{B}$. Similar arguments shows that $Best_{\mathscr{T}_1}\mathscr{B} \subseteq \mathscr{F}(\mathscr{P}_{\mathscr{T}_1}\mathscr{B})$. Hence $\mathscr{F}(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}) = Best_{\mathscr{T}_1}\mathscr{B}$. Similarly, we can prove that $\mathscr{P}_{\mathscr{T}_2}\mathscr{B}_1$ is quasi- NE and set of FP of a mapping $\mathscr{P}_{\mathscr{T}_2}\mathscr{B}_1$ is equal to BPP of $\mathscr{B}$. By Theorem 3.3.1, $\{\hat{o}_n\}$ is Fejer monotone with respect to $\mathscr{O}$ and converges weakly to a point $\hat{o} \in \mathscr{O}$ where $\mathscr{O} = \{\hat{o} \in \mathscr{F}(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}) : \mathscr{L}\hat{o} \in \mathscr{F}(\mathscr{P}_{\mathscr{T}_2}\mathscr{B}_1)\}$. Since $\mathscr{F}(\mathscr{P}_{\mathscr{T}_1}\mathscr{B}) = Best_{\mathscr{T}_1}\mathscr{B}$ and $\mathscr{F}(\mathscr{P}_{\mathscr{T}_2}\mathscr{B}_1) = Best_{\mathscr{T}_2}\mathscr{B}_1$. Therefore, $\{\hat{o}_n\}$ converges weakly to a point $\hat{o} \in \mathscr{S}^*$. □

Example 3.4.1. *Let $\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{R}^2$ induced by Euclidean distance. Let $\mathscr{T}_1 = [0,2] \times [0,2]$ and $\mathscr{F}_1 = [3,5] \times [0,2]$. Let $\mathscr{T}_2 = [0,4] \times [0,4]$ and $\mathscr{F}_2 = [6,10] \times [0,4]$. Let $\mathscr{B} : \mathscr{T}_1 \to \mathscr{F}_1$ be defined by $\mathscr{B}(\hat{o},\hat{g}) = \left(5 - \hat{o}, \frac{\hat{g}}{2}\right)$, for all $(\hat{o},\hat{g}) \in \mathscr{T}_1$. Let $\mathscr{B}_1 : \mathscr{T}_2 \to \mathscr{F}_2$ be defined by $\mathscr{B}_1(\hat{o},\hat{g}) = (10 - \hat{o}, \hat{g})$ for all $(\hat{o},\hat{g}) \in \mathscr{T}_2$. Let $\mathscr{L} : \mathscr{H}_1 \to \mathscr{H}_2$ be defined by $\mathscr{L}(\hat{o},\hat{g}) = (2\hat{o}, 2\hat{g})$ for all $(\hat{o},\hat{g}) \in \mathscr{H}_1$. It is clear that $\mathscr{T}_{1_0} = \{2\} \times [0,2], \mathscr{F}_{1_0} = \{3\} \times [0,2], \mathscr{T}_{2_0} = \{4\} \times [0,4], \mathscr{F}_{2_0} = \{6\} \times [0,4], d(\mathscr{T}_1,\mathscr{F}_1) = 1, d(\mathscr{T}_2,\mathscr{F}_1) = 2, Best_{\mathscr{T}_1}\mathscr{B} = \{(2,0)\}, Best_{\mathscr{T}_2}\mathscr{B}_1 = \{4\} \times [0,4], \mathscr{S} = (2,0)$ and $\mathscr{B}$, $\mathscr{B}_1$ are NEs such that $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{F}_{1_0}, \mathscr{B}_1(\mathscr{T}_{2_0}) \subseteq \mathscr{F}_{2_0}$. Clearly, $\mathscr{B}$ has BPP (2,0) and $\mathscr{L}(2,0) = (4,0)$ is a BPP of $\mathscr{B}_1$. All the suppositions of Theorem 3.4.1 are fulfilled, so sequence defined by Algorithm (SS') converges to (2,0), see Table 3.3. Take $\gamma_n' = \frac{4n+1}{12n}$ and $\eta_{n'} = \delta_n' = \frac{1}{n+25}$.*

n	$\hat{o}_n$	$\mathscr{L}\hat{o}_n$
0	$(2,2)$	$(4,4)$
1	$(2, 1.006173e+00)$	$(4, 2.0123)$
2	$(2, 5.473394e-01)$	$(4, 1.0947)$
⋮	⋮	⋮
39	$(2, 4.156611e-09)$	$(4, 0.0000)$

Table 3.3: Table

If we take $\mathscr{T}_1$ is compact in Theorem 3.4.1 then we have following result:

Theorem 3.4.2. Under the hypotheses of Theorem 3.4.1, suppose that $\mathscr{T}_1$ is compact and $\{\hat{o}_n\}$ is a sequence defined by Algorithm (SS'). Then

(i) $\{\hat{o}_n\}$ is Fejer monotone with respect to $\mathscr{S}^*$, that is, for every $z \in \mathscr{S}^*$,

$$||\hat{o}_{n+1} - z|| \leq ||\hat{o}_n - z||, n \in \mathbb{N}.$$

(ii) $\{\hat{o}_n\}$ converges strongly to a solution of SBPP $\hat{o} \in \mathscr{S}^*$.

Remark 3.4.1. If we take $(\gamma'_n + \delta'_n) = 0$ then Algorithm (SS') reduces to the algorithm presented by Suantai (Algorithm 3.1, p.2664, [77]) as described below:

$$\begin{cases} \hat{i}_n = & \mathscr{P}_{\mathscr{T}_1}[\hat{o}_n + \omega\beta\mathscr{L}^*(\mathscr{P}_{\mathscr{T}_2}\mathscr{B}_1 - I)\mathscr{L}\hat{o}_n] \\ \hat{o}_{n+1} = & (1-\eta'_n)\hat{i}_n + \eta'_n\mathscr{P}_{\mathscr{T}_1}\mathscr{B}\hat{i}_n \end{cases} \tag{SU}$$

where $\eta'_n \in (0,1)$, $n \in \mathbb{N}$, $\beta \in (0,1)$ and $\omega \in (0, \frac{1}{\kappa\beta})$ with κ being the spectral radius of $\mathscr{L}^*\mathscr{L}$.

Corollary 3.4.1. (Theorem 3.2, p. 2665[77]) Under the hypotheses of Theorem3.4.2, sequence defined by Algorithm (SU). Then

(i) $\{\hat{o}_n\}$ is Fejer monotone with respect to $\mathscr{S}^*$, that is, for every $z \in \mathscr{S}^*$,

$$||\hat{o}_{n+1} - z|| \leq ||\hat{o}_n - z||, n \in \mathbb{N}.$$

(ii) $\{\hat{o}_n\}$ converges strongly to a solution of SBPP $\hat{o} \in \mathscr{S}^*$.

Remark 3.4.2. Theorem 3.2 of ([77], p. 2665) is a straightforward consequence of Theorem 2.1 of Moudafi ([49], p. 4086).

Theorem 3.4.3. Let $\mathscr{T}_1, \mathscr{T}_2$ be two non-empty closed convex subsets of a real Hilbert space $\mathscr{H}$ with $\mathscr{T}_1$ is compact and $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$, $\mathscr{B}_1 : \mathscr{T}_1 \to \mathscr{T}_2$ be two best proximally NEs with non-empty $Best_{\mathscr{T}_1}\mathscr{B} \cap Best_{\mathscr{T}_1}(\mathscr{B}_1)$. Suppose that $\mathscr{B}(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$ and $\mathscr{B}_1(\mathscr{T}_{1_0}) \subseteq \mathscr{T}_{2_0}$. Then sequence $\{\hat{o}_n\}$ defined by Algorithm (SS') converges strongly to a common BPP $\hat{o} \in Best_{\mathscr{T}_1}\mathscr{B} \cap Best_{\mathscr{T}_1}(\mathscr{B}_1)$.

If we take $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$, $\mathscr{L} = I$, $\mathscr{T}_1 = \mathscr{F}_1$ and $\mathscr{T}_2 = \mathscr{F}_2$ in Theorem 3.4.3, then we get the following result:

Corollary 3.4.2. (Theorem 4.2, p. 2667, [77]) Under the hypotheses of Theorem 3.4.3, sequence $\{\hat{o}_n\}$ defined by Algorithm (SU) converges strongly to a common BPP $\hat{o} \in Best_{\mathscr{T}_1}\mathscr{B} \cap Best_{\mathscr{T}_1}(\mathscr{B}_1)$.

Chapter 4

Applications

This chapter deals with the applications of BPPs. It has been split into three sections. In the first section, we present the solutions for variational inequality problem (VIP) in Hilbert spaces. The findings of this section are published in Sharma and Chandok [65]. In the second section, we provide a solution for differential equations in the framework of BSs and the findings of this part are accepted in Sharma and Chandok [71]. In the last section, we consider the model that spreads a specific virus with a cyclical variable periodic contract rate using a non-linear integral equation. The findings of this section are published in Sharma and Chandok [72].

4.1 Variational inequality Problems

In this section, we provide the solutions for VIP in Hilbert space $\mathscr{H}_1$ endowed with two norms. Also, we prove that iterative scheme converges to solution of VIP.
The VIP was introduced by Stampacchia [76] in 1964, this gives us a tool for creating different kinds of optimization, FP, and equilibrium problems. VIP with respect to a subset $\mathscr{T}_1$ and a monotone operator $\mathscr{L} : \mathscr{T}_1 \to \mathscr{H}_1$ symbolized by $VIP(\mathscr{L}, \mathscr{T}_1)$, is to detect a point $\hat{o}^* \in \mathscr{T}_1$ such that

$$\langle \mathscr{L}\hat{o}^*, \hat{o} - \hat{o}^* \rangle \geq 0, \text{ for all } \hat{o} \in \mathscr{H}_1.$$

It is known that if $\omega > 0$, then $\hat{o}^* \in \mathscr{T}_1$ is a solution of $VIP(\mathscr{L}, \mathscr{T}_1)$ if and only if $\hat{o}^*$ is a solution of the FP problem

$$\hat{o} = \mathscr{P}_{\mathscr{T}_1}(I - \omega\mathscr{L})\hat{o},$$

where $\mathscr{P}_{\mathscr{T}_1}$ is projection operator onto $\mathscr{T}_1$. Byrne [10] has been proved, if $\mathscr{P}_{\mathscr{T}_1}(I-\omega\mathscr{L})$ and $(I - \omega\mathscr{L})$ are NEs, then the iterative algorithm defined by,

$$\hat{o}_{n+1} = \mathscr{P}_{\mathscr{T}_1}(I - \omega\mathscr{L})\hat{o}_n, n \in \mathbb{N},$$

converges weakly to a solution of $VIP(\mathscr{L}, \mathscr{T}_1)$, for any $\hat{o}_0 \in \mathscr{T}_1$.
Now, we generalize VIP in binormed spaces and prove the convergence of iterative algorithm to a solution of VIP.

Theorem 4.1.1. Assume that $\mathscr{T}_1$ is convex and closed subset of a $\mathscr{H}_1$ endowed with two norms $||.||$ and $||.||_1$ such that $||.||_1 \leq ||.||$ and $\mathscr{P}_{\mathscr{T}_1}(I - \omega\mathscr{L})$ satisfying (2.25) on $\mathscr{T}_1$, $\omega > 0$. If $VIP(\mathscr{L}, \mathscr{T}_1) \neq \emptyset$ then sequence generated by

$$\hat{o}_{n+1} = \mathscr{P}_{\mathscr{T}_1}(I - \omega\mathscr{L})\hat{o}_n,$$

converges to a unique solution $\hat{o}$ of the $VIP(\mathscr{L}, \mathscr{T}_1)$ for any $\hat{o}_0 \in \mathscr{T}_1$.

Proof. Given $\mathscr{T}_1$ is convex and closed, we can take $\mathscr{T}_1 = \mathscr{W}$ and $\mathscr{B} = \mathscr{P}_{\mathscr{T}_1}(I - \omega\mathscr{L})$ and using Corollary 2.1.8, we get result. □

4.2 System of differential equations

We give a solution to the system of differential equations in the context of a BS in this section. The findings of this section are accepted in Sharma and Chandok [71].

Definition 4.2.1. Let $\mathscr{U}_1 = B(\hat{o}_0, \acute{b})$, $\mathscr{U}_2 = B(\hat{g}_0, \acute{b})$ be closed balls in a BS $\mathscr{W}$, where $\acute{a}, \acute{b} \in \mathbb{R}^+$, $\acute{t}_0 \in \mathbb{R}$ and $\hat{o}_0, \hat{g}_0 \in \mathscr{W}$, G be a interval. Assume that $\Gamma : G \times \mathscr{U}_1 \to \mathscr{W}$ and $\Gamma_1 : G \times \mathscr{U}_2 \to \mathscr{W}$ are continuous mappings. Examine the differential equation system below:

$$\begin{cases} \hat{o}'(\acute{t}) = & \Gamma(\acute{t}, \hat{o}(\acute{t})), \quad \hat{o}(\acute{t}_0) = \hat{o}_0 \\ \hat{g}'(\acute{t}) = & \Gamma_1(\acute{t}, \hat{g}(\acute{t})), \quad \hat{g}(\acute{t}_0) = \hat{g}_0, \end{cases} \tag{DE}$$

defined on a closed real interval $G = [\acute{t}_0 - h, \acute{t}_0 + h]$ for some $h \in \mathbb{R}^+$. Take $\mathsf{C}(G, \mathscr{U}_1) = \left\{\hat{o} \in \mathsf{C}(G, \mathscr{W}) : \hat{o}(\acute{t}_0) = \hat{o}_0\right\}$, and $\mathsf{C}(G, \mathscr{U}_2) = \left\{\hat{g} \in \mathsf{C}(G, \mathscr{W}) : \hat{g}(\acute{t}_0) = \hat{g}_0\right\}$. Now

$$||\hat{o} - \hat{g}||_\infty = \sup_{\acute{t} \in G} ||\hat{o}(\acute{t}) - \hat{g}(\acute{t})|| \geq ||\hat{o}_0 - \hat{g}_0||, \text{ for all } (\hat{o}, \hat{g}) \in \mathsf{C}(G, \mathscr{U}_1) \times \mathsf{C}(G, \mathscr{U}_2),$$

and so,

$$||\mathsf{C}(G, \mathscr{U}_1) - \mathsf{C}(G, \mathscr{U}_2)|| = ||\hat{o}_0 - \hat{g}_0||. \tag{4.1}$$

Let $\mathscr{B} : \mathsf{C}(G, \mathscr{U}_1) \cup \mathsf{C}(G, \mathscr{U}_2) \to \mathsf{C}(G, \mathscr{W})$ be a mapping defined as

$$\begin{aligned} \mathscr{B}\hat{o}(\acute{t}) =& \hat{g}_0 + \int_{\acute{t}_0}^{t} \Gamma_1(\acute{s}, \hat{o}(\acute{s}))d\acute{s}, \ \hat{o} \in \mathsf{C}(G, \mathscr{U}_1) \\ \mathscr{B}\hat{g}(\acute{t}) =& \hat{o}_0 + \int_{\acute{t}_0}^{\acute{t}} \Gamma(\acute{s}, \hat{g}(\acute{s}))ds, \ \hat{g} \in \mathsf{C}(G, \mathscr{U}_2). \end{aligned}$$

We say that $\acute{z} \in \mathsf{C}(G, \mathscr{U}_1) \cup \mathsf{C}(G, \mathscr{U}_2)$ is a BPP for (DE), if $||\acute{z} - \mathscr{B}\acute{z}|| = ||\mathsf{C}(G, \mathscr{U}_1) - \mathsf{C}(G, \mathscr{U}_2)||$.

We demonstrate the succeeding theorem:

Theorem 4.2.1. Based on the definition 4.2.1 hypotheses, we assume that

(i) $\mho(\Gamma(G \times Q_2) \cup \Gamma_1(G \times Q_1)) \leq \varsigma(\mho(Q_1 \cup Q_2))\mho(Q_1 \cup Q_2)$;

(ii) $||\Gamma(\acute{t}, \hat{o}) - \Gamma_1(\acute{t}, \hat{g})|| \leq \frac{1}{h}(||\hat{o}(\acute{t}) - \hat{g}(\acute{t})|| - ||\hat{g}_0 - \hat{o}_0||)$,

where $\varsigma = \frac{1}{2\acute{b}}$ and for any $(Q_1, Q_2) \subseteq (\mathscr{U}_1, \mathscr{U}_2)$ and $h \leq \min\left\{\acute{a}, \acute{b}/I_1, \acute{b}/I_2, 1/2\acute{b}\right\}$, where $I_1 = \sup\left\{\Gamma(\acute{t}, \hat{g}) : (\acute{t}, \hat{o}) \in G' \times \mathscr{U}_1\right\}$, and $I_2 = \sup\left\{\Gamma_1(\acute{t}, \hat{o}) : (\acute{t}, \hat{g}) \in G' \times \mathscr{U}_2\right\}$. Then system (DE) has a BPP.

Proof. Notice that $(\mathsf{C}(G, \mathscr{U}_1), \mathsf{C}(G, \mathscr{U}_2))$ is a convex, closed, bounded pair in $\mathsf{C}(G, \mathscr{W})$

and $\mathscr{B}$ is cyclic on $(\mathsf{C}(G,\mathscr{U}_1)\cup\mathsf{C}(G,\mathscr{U}_2))$. Consider

$$\begin{aligned}||\mathscr{B}\hat{o}(\acute{t})|| = ||\hat{g}_0 + \int_{\acute{t}_0}^{\acute{t}}\Gamma_1(\acute{s},\hat{o}(\acute{s}))d\acute{s}|| &\leq ||\hat{g}_0|| + \int_{\acute{t}_0}^{\acute{t}}||\Gamma_1(\acute{s},\hat{o}(\acute{s}))||d\acute{s}\\ &\leq ||\hat{g}_0|| + I_2h \leq ||\hat{g}_0|| + \acute{b}.\end{aligned}$$

It shows the boundedness of $\mathscr{B}(\mathsf{C}(G,\mathscr{U}_1))$. Also,

$$\begin{aligned}||\mathscr{B}\hat{o}(\acute{t}) - \mathscr{B}\hat{o}(\acute{t}')|| &= \left|\left|\int_{\acute{t}_0}^{\acute{t}}\Gamma_1(\acute{s},\hat{o}(\acute{s}))d\acute{s} - \int_{\acute{t}_0}^{\acute{t}'}\Gamma_1(\acute{s},\hat{o}(\acute{s}))d\acute{s}\right|\right|\\ &\leq \int_{\acute{t}}^{\acute{t}'}||\Gamma_1(\acute{s},\hat{o}(\acute{s}))||d\acute{s} \leq I_2|\acute{t}-\acute{t}'|.\end{aligned}$$

It implies that $\mathscr{B}(\mathsf{C}(G,\mathscr{U}_1))$ is equicontinuous. Using similar lines, one can easily prove that $\mathscr{B}(\mathsf{C}(G,\mathscr{U}_2))$ is also bounded and equicontinuous. Accordingly, Arzela-Ascoli's theorem, $(\mathsf{C}(G,\mathscr{U}_1),\mathsf{C}(G,\mathscr{U}_2))$ is relatively compact. We have to claim that $\mathscr{B}$ is a ς cyclic mapping. Suppose that $(\mathscr{I},\mathscr{J})\subset(\mathsf{C}(G,\mathscr{U}_1),\mathsf{C}(G,\mathscr{U}_2))$ is a $\mathscr{B}$ invariant, closed, proximinal pair and convex. By (4.1), we have

$$||\mathscr{I}-\mathscr{J}|| = ||\mathsf{C}(G,\mathscr{U}_1)-\mathsf{C}(G,\mathscr{U}_2)|| = ||\hat{o}_0-\hat{g}_0||.$$

Using Theorem 2.11 of [81], we deduce that

$$\begin{aligned}\mho(\mathscr{B}(\mathscr{I})\cup\mathscr{B}(\mathscr{J})) &= \max\left\{\mho(\mathscr{B}(\mathscr{I}),\mho(\mathscr{B}(\mathscr{J}))\right\}\\ &= \max\left\{\sup_{\acute{t}\in G}\left\{\mho\left(\left\{\mathscr{B}\hat{o}(\acute{t}):\hat{o}\in\mathscr{I}\right\}\right)\right\},\sup_{\acute{t}\in G}\left\{\mho(\left\{\mathscr{B}\hat{o}(\acute{t}):\hat{o}\in\mathscr{J}\right\})\right\}\right\}\\ &= \max\left\{\sup_{\acute{t}\in G}\left\{\mho\left(\left\{\hat{g}_0+\int_{\acute{t}_0}^{\acute{t}}\Gamma_1(\acute{s},\hat{o}(\acute{s}))d\acute{s}:\hat{o}\in\mathscr{I}\right\}\right)\right\}\right\},\\ &\quad \sup_{\acute{t}\in G}\left\{\mho\left(\left\{\hat{o}_0+\int_{\acute{t}_0}^{\acute{t}}\Gamma(\acute{s},\hat{g}(\acute{s}))d\acute{s}:\hat{o}\in\mathscr{J}\right\}\right)\right\}.\end{aligned}$$

Now, using Lemma 1.1.8, we obtain

$$\begin{aligned}\hat{g}_0+\int_{\acute{t}_0}^{\acute{t}}\Gamma_1(\acute{s},\hat{o}(\acute{s}))d\acute{s} &\in \hat{g}_0+(\acute{t}-\acute{t}_0)\overline{con}\left(\left\{\Gamma_1(\acute{s},\hat{o}(\acute{s})):\acute{s}\in[\acute{t}_0,\acute{t}]\right\}\right),\\ \hat{o}_0+\int_{\acute{t}_0}^{\acute{t}}\Gamma(\acute{s},\hat{g}(\acute{s}))d\acute{s} &\in \hat{g}_0+(\acute{t}-\acute{t}_0)\overline{con}\left(\left\{\Gamma(\acute{s},\hat{g}(\acute{s})):\acute{s}\in[\acute{t}_0,\acute{t}]\right\}\right).\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\mho(\mathscr{B}(\mathscr{I})\cup\mathscr{B}(\mathscr{J}))\\
\leq&\max\left\{\sup_{\acute{t}\in G}\left\{\mho\Bigg(\left\{\hat{g}_0+(\acute{t}-\acute{t}_0)\overline{con}\left(\left\{\Gamma_1(\acute{s},\hat{o}(\acute{s}))):\acute{s}\in[\acute{t}_0,\acute{t}]\right\}\right)\right\}\right\}\right.,\\
&\left.\sup_{\acute{t}\in G}\left\{\mho\Bigg(\left\{\hat{o}_0+(\acute{t}-\acute{t}_0)\overline{con}\left(\left\{\Gamma(\acute{s},\hat{g}(\acute{s}))):\acute{s}\in[\acute{t}_0,\acute{t}]\right\}\right)\Bigg)\right\}\right\}\\
\leq&\sup_{0\leq\lambda\leq h}\left\{\mho\Bigg(\{\hat{g}_0+\lambda\overline{con}\left(\{\Gamma_1(G\times\mathscr{I})\}\right)\}\Bigg)\right\},\\
&\sup_{0\leq\lambda\leq h}\left\{\mho\Bigg(\{\hat{o}_0+\lambda\overline{con}\left(\{\Gamma(G\times\mathscr{J})\}\right)\}\Bigg)\right\}\\
=&\max\left\{h\mho(\Gamma_1(G\times\mathscr{I})),h\mho(\Gamma(G\times\mathscr{J}))\right\}\\
=&h\mho(\{(\Gamma_1(G\times\mathscr{I}))\cup(\Gamma(G\times\mathscr{J}))\})\\
\leq&\frac{1}{2\acute{b}}\mho(\{(\Gamma_1(G\times\mathscr{I}))\cup(\Gamma(J\times\mathscr{J}))\})\\
=&\varsigma(\mho(\mathscr{I}\cup\mathscr{J}))\mho(\mathscr{I}\cup\mathscr{J}).
\end{aligned}
$$

It implies that $\mathscr{B}$ is a ς cyclic condensing mapping. Next, we have to prove that $\mathscr{B}$ is cyclic relatively NE. For all $(\hat{o},\hat{g})\in\mathsf{C}(G,\mathscr{U}_1)\times\mathsf{C}(G,\mathscr{U}_1)$, we obtain

$$
\begin{aligned}
||\mathscr{B}\hat{o}(\acute{t})-\mathscr{B}\hat{g}(\acute{t})||=&\left|\left|\left(\hat{g}_0+\int_{\acute{t}_0}^{\acute{t}}\Gamma_1(\acute{s},\hat{o}(\acute{s})))d\acute{s}\right)-\left(\hat{o}_0+\int_{\acute{t}_0}^{\acute{t}}\Gamma(\acute{s},\hat{g}(\acute{s})))d\acute{s}\right)\right|\right|\\
\leq&||\hat{g}_0-\hat{o}_0||+\int_{\acute{t}_0}^{\acute{t}}||\Gamma_1(\acute{s},\hat{o}(\acute{s})))-\Gamma(\acute{s},\hat{g}(\acute{s})))||d\acute{s}\\
\leq&||\hat{g}_0-\hat{o}_0||+\frac{1}{h}\int_{\acute{t}_0}^{\acute{t}}(||\hat{o}(\acute{s}))-\hat{g}(\acute{s}))||-||\hat{g}_0-\hat{o}_0||)d\acute{s}\\
\leq&||\hat{g}_0-\hat{o}_0||+(||\hat{o}(\acute{s}))-\hat{g}(\acute{s}))||_\infty-||\hat{g}_0-\hat{o}_0||)=||\hat{o}-\hat{g}||_\infty.
\end{aligned}
$$

Thus, the outcome is derived from the Theorem 2.1.9. □

4.3 Model for transmission of virus

In this section, we solve the model that spreads a virus with a cyclical variable periodic contract rate using a non-linear integral equation. The succeeding equation can be seen as a model for the transmission of some virus diseases, whose cyclical variation in the periodic contract rate is seen (see [46]).

$$\hat{o}(t) = \int_{\acute{t}-\tau}^{\acute{t}} \Gamma(s, \hat{o}(s))ds, \tag{4.2}$$

where $\hat{o}(\acute{t})$ is how many people are infected with virus at time $\acute{t}$, $\Gamma(\acute{t}, \hat{o}(\acute{t}))$ is the proportion of new cases in a given time unit ($\Gamma(\acute{t}, 0) = 0$), and τ is how long a person can still spread the virus.

Assume that $\mathscr{W} = \mathtt{C}(\mathbb{R}, \mathbb{R})$ which are ϱ-periodic with maximum norm. Let $\mathscr{W}$ be a UCBS, K be a cone of non-negative functions in $\mathscr{W}$ and $\mathscr{B}$ be a defined on K as $\mathscr{B}\hat{o}(\acute{t}) = \int_{\acute{t}-\tau}^{\acute{t}} \Gamma(s, \hat{o}(s))ds$.

Theorem 4.3.1. Assume that the succeeding requirements true:

(i) $\Gamma : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is continuous,

(ii) $\Gamma(\acute{t}, \hat{o}) = \Gamma(\acute{t} + \varrho, \hat{o})$, for all $\acute{t} \in \mathbb{R}$, $\hat{o} \geq 0$,

(iii) $0 < \rho = \int_{\acute{t}-\tau}^{\acute{t}} ds < 1$,

(iv) $|\Gamma(\acute{t}, \hat{o}_1) - \Gamma(\acute{t}, \hat{o}_2)| \leq |\hat{o}_1(\acute{t}) - \hat{o}_2(\acute{t})|$, for all $\hat{o}_1, \hat{o}_2 \in \mathscr{W}$.

Then, the mathematical model (4.2) has a solution.

Further, if $\mathscr{B}(\mathscr{T}_1)$ lies in a compact subset of $\mathscr{T}_1$, then Algorithm (S) converges strongly to the solution of (4.2).

Proof. Consider

$$\begin{aligned}
||\mathscr{B}\hat{o}(\acute{t}) - \mathscr{B}\hat{i}(\acute{t})|| \leq & ||\mathscr{B}\hat{o}(\acute{t}) - \mathscr{B}\hat{i}(\acute{t})|| \\
= & \sup_{\acute{t}\in\mathbb{R}} |\mathscr{B}\hat{o}(\acute{t}) - \mathscr{B}\hat{i}(\acute{t})| \\
= & \sup_{\acute{t}\in\mathbb{R}} \left(|\Gamma(s, \hat{o}(s)) - \Gamma(s, \hat{i}(s))|\right) \int_{\acute{t}-\tau}^{\acute{t}} ds \\
\leq & \sup_{\acute{t}\in\mathbb{R}} \left(|\hat{o}(s) - \hat{i}(s)|\right) \int_{\acute{t}-\tau}^{\acute{t}} ds \\
= & \rho||\hat{o} - \hat{i}|| < ||\hat{o} - \hat{i}||.
\end{aligned}$$

It shows that $\mathscr{B}$ is a NE. By Browder's Theorem (Theorem 1, p.1041, [8]), mathematical model (4.2) has a solution. Thus, the outcome is derived from the Theorem 3.1.2. $\square$

Conclusion and Future Scope

In the thesis, we discuss some existence results on BPP in the framework of a MS, relational MS, binormed linear spaces and quasi partial MS using different types of contraction mapping. Moreover, we examined some findings on the existence of best proximity pairs in the scope of a strictly convex BS. We also introduce some iterative algorithms using projection operators that converge strongly to a best proximity point for non-self mappings. In addition, an algorithm for approximating the common FPs of NEs and strongly pseudocontractive mappings, a three-step algorithm that converges to a solution of split common FP problem and an algorithm that converges to a solution of SBPP problem. In the last we apply the theoretical results obtained in the thesis to variational inequality problems, system of differential equations, and non-linear integral equation models, including a virus transmission model.

This Ph.D. research lays a strong foundation for a range of future studies and advancements, outlined as follows:

1. To find sufficient conditions for the existence and convergence of solutions to minimization problem.

2. There are many approaches to construct an algorithm for existence of best proximity point. One of them is fixed point. We will try to construct an algorithm for best proximity point without using fixed point approach.

3. There are several other fields, in which we can extend the applications of best proximity point such as medical, economics, biology and chemistry to solve their problems.

4. To study variational best proximity point problems and define an algorithm using projection operator which converges to a solution of variational best proximity point problem for more general class of mappings.

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