Some Remarks on Conjugate Closed Groups

Thesis submitted in the partial fulfillment requirement for

The award of the degree of

Masters of Science

In

Mathematics and Computing

Submitted By

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Under

the guidance of

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July 2012

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Dedicated to

God,

Parents and Teachers

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**Some remarks on Conjugate Closed Groups**" in partial fulfillment of the requirement for the award of the degree of Masters of Science, School of Mathematics and Computer applications, Thapar University, Patiala is an authentic record of my own work under the supervision of Dr. Deepak Gumber.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ACKNOWLEDGEMENTS

I feel privileged to express my deep sense of gratitude and respect to my guide ***Dr. Deepak Gumber,*** **Associate Professor, School of Mathematics and Computer Applications, Thapar University, Patiala,** for his keen interest and expert guidance, cool temperature, strong motivation, valuable suggestions and continuous encouragement throughout the course of the work. I thank him for his great patience, constructive criticism and myriad useful suggestions apart from invaluable guidance to me. I am sure that the knowledge gained through my association with my supervisor shall go a long way in helping me to realize my goals in life.

I am highly obliged to Prof. S.S. Bhatia, Head SMCA, Thapar University, Patiala, for their motivation and inspiration that trigged me for thesis work.

I would like to thank all the staff members and co-students who were always there at the need of the hour and provided with all the help and facilities, which I required to completion of my thesis.

I am also thankful to Mr. Hemant Kalra who helped me to complete my thesis.

I am also thankful to the authors whose works I have consulted and quoted in this work.

I also express my deepest gratitude to my **parents and family**, without whom I am nothing, to provide me great opportunities, everlasting support, big encouragement and lots of love.

Last but not the least i would like to thank God for not letting me down at the time of crisis and showing me the silver lining in the dark clouds.

Tina Verma

(Ms. Tina Verma)

Date: 12 July, 2012

Place: Thapar University, Patiala.

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Chapter-1

INTRODUCTION

Throughout this thesis, we denote $G', Z(G), x^G, x^H,$ the derived group, center, conjugacy class of x in G and conjugacy class of x in H, where H is a normal subgroup of G. The influence of arithmetic structure of conjugacy classes of G, like conjugacy class size, number of conjugacy class sizes on the structure of G is extensively studied. For more detail on this, reader can see the excellent survey article by Camina and Camina[1].

In this thesis, we study the structure of groups when $x^G = x^H$ for every $x \in H$.

A normal subgroup H of a group G is said to be conjugate closed if $x^G = x^H$ for every $x \in H$. A group G is said to be conjugate closed if every normal subgroup of G is conjugate closed. We abbreviate conjugate closed groups as $CC - Groups$.

A group G is called a $T - group$ if every normal subgroup of a normal subgroup of G is normal in G. It is easily seen that $CC - Groups$ form a proper subclass of $T - group$(Lemma 3.1.2). A group G is said to be perfect if $G' = G$. A group G is called semisimple if it has no non trivial normal abelian subgroup.

In section 1 of chapter 3 of this thesis we prove the following two theorems:

1) Every normal abelian subgroup of a $CC - Group$ is central.
2) Let G be a group such that $G = Z(G) \times H$. Then G is conjugate closed if and only if H is conjugate closed.

Our main result of this thesis is:

A finite group G is semisimple, conjugate closed and perfect if and only if $G = G_1 \times G_2 \times \dots G_r$, where each G_i is non abelian and simple subgroup of G.

Section 2 of chapter 3 is related to the structure of the $CC - Group\ G$, where G is nilpotent or solvable. We prove that a conjugate closed solvable group is abelian. In this section we also prove that for $n \geq 3$, the symmetric group S_n is not conjugate closed. All the main results in chapter 3 are proved in [3].

Chapter-2

NOTATIONS AND PRELIMINARIES

Definition 2.1 (Product of two subgroups)

Let H and K be two subgroups of a group G, then the set HK defined by

$HK = \{hk: \ \forall\ h \in H, k \in K\}$ is called the product of the subgroups H and K.

Definition 2.2 (Inverse of a subset of a group)

Let H be a subset of a group G, then the inverse of H is H^{-1} and is defined as

$H^{-1} = \{h^{-1}$: for all $h \in H\}$

Definition 2.3 (Centralizer of a subgroup)

Let $H \leq G$, if $x \in G$ s.t $xh = hx\ \forall\ h \in H$, then we say that x centralizes H. The centralizer of H in G is $C_G(H) = \{x \in G \mid xh = hx\ \forall\ h \in H\}$

For example, let S_3 be a symmetric group and $A_3 \leq S_3$ then $C_{S_3}(A_3) = A_3$.

Note: Let $H \leq G$, where G is an abelian group then $C_G(H) = G$.

Proposition 2.4 $C_G(H)$is a subgroup of G.

Proof: Let $x \in C_G(H)$. Then $xh = hx\ \ \forall\ h \in H$.

$\Rightarrow hx^{-1} = x^{-1}h\ \ \forall\ h \in H$

$\Rightarrow x^{-1} \in C_G(H)$

Let $x, y \in C_G(H)$. Then $xh = hx$ and $yh = hy\ \ \forall\ h \in H$

Now, $xyh = xhy = hxy\ \ \ \forall\ h \in H$

So $xy \in C_G(H)$

Therefore $C_G(H) \leq G$.

Definition 2.5 (Normalizer of a subgroup)

Let $H \leq G$, if $x \in G\ s.t\ xH = Hx$, then we say that x normalizes H. The normalizer of H in G is

$N_G(H) = \{x \in G | xH = Hx\}$

For example, $N_{S_3}(A_3) = S_3$.

Note: Let $H \leq G$, where G is an abelian group then $N_G(H) = G$.

Proposition 2.6 $N_G(H)$ is a subgroup of G.

Proof: Let $x \in N_G(H)$.

Then $xH = Hx \Rightarrow Hx^{-1} = x^{-1}H$

So $x^{-1} \in N_G(H)$.

Let $x, y \in N_G(H)$, then $xH = Hx$ and $yH = Hy$

Now $xyH = xHy = Hxy$. So $xy \in N_G(H)$.

Therefore $N_G(H) \leq G$.

Proposition2.7 If $H \leq G$, then H is normal in $N_G(H)$.

Proof: For each $a \in H$ we have $aH = Ha$.

Since $H \subseteq N_G(H)$ i.e. H is a subgroup of G contained in $N_G(H)$.

Also $\forall\ a \in N_G(H)$, we have $aH = Ha$.

So H is normal in $N_G(H)$.

Proposition 2.8 If H and K are subgroups of $G$and H is normal subgroup of K, then $K \subseteq N_G(H)$ i.e. $N_G(H)$ is the largest subgroup of G in which H is normal.

Proof: Let $H \subseteq K \subseteq G$ such that H is a normal subgroup of K and K is a subgroup of G.

Now we will show that $K \subseteq N_G(H)$.

Let $k \in K$ be any element, then $Hk = kH$,since H is normal subgroup of K so by the definition of $N_G(H)$, $k \in N_G(H)$.

Therefore, $k \in N_G(H)\ \ \forall\ k \in K$.

Hence $K \subseteq N_G(H)$.

Proposition 2.9 H is normal subgroup of G iff $N_G(H) = G$.

Proof: Firstly, let H is a normal subgroup of G.

Then $\forall g \in G$,we have $Hg = gH$. So $g \in N_G(H)$.

Therefore $G \subseteq N_G(H)$.

But $N_G(H) \subseteq G$ always, hence $G = N_G(H)$.

Conversely, let $N_G(H) = G$.

Therefore $N_G(H) = \{x \in G | xH = Hx\} = G$

So $xH = Hx\ \forall x \in G$.

Hence H is a normal subgroup of G.

Proposition 2.10 $C_G(H)$ is normal in $N_G(H)$.

Proof: Let $a \in C_G(H)$, then $ah = ha\ \forall\ h \in H$.

Let $g \in N_G(H)$, then $gH = Hg$. Therefore $g^{-1}H = Hg^{-1}$.

So $g^{-1}h = h_1 g^{-1}$ for some $h, h_1 \in H$

Now $(gag^{-1})h = gag^{-1}h = gah_1g^{-1} = gh_1ag^{-1} = h(gag^{-1})$.

So $C_G(H)$ is normal in $N_G(H)$.

Definition 2.11 (Center of a Group)

The center $Z(G)$, of a group G is the subset of elements in G that commute with every element of G. In symbols, $Z(G) = \{a \in G | ax = xa \ \forall \ x \in G\}$

For example, $Z(S_3) = \{1\}$

Remark: G is abelian iff $Z(G) = G$.

Proposition 2.12 $Z(G)$ is a normal subgroup of G.

Proof: Since $ex = xe, \forall \ x \in G \ \ \Rightarrow e \ \in Z(G)$.

Therefore $Z(G) \neq \varphi$.

Now we will show that $Z(G)$ is a subgroup of G.

Let $a, b \in Z(G)$, then $ax = xa, \forall \ x \in G$ and $bx = xb, \forall \ x \in G \ \Rightarrow \ x \, b^{-1} = b^{-1}x$

Now $x(ab^{-1})= (xa)b^{-1} = (ax)b^{-1} = a(xb^{-1}) = a(b^{-1}x) = (ab^{-1})x$

Therefore $ab^{-1} \in Z(G)$

So $Z(G) \leq G$.

Now we will show that $Z(G)$ is normal in G.

Let $a \in \ Z(G)$ and $g \in G$ be any element.

Now $gag^{-1} = (ga)g^{-1} = (ag)g^{-1} = a(gg^{-1})= ae = a \in Z(G)$.

So $Z(G)$ is normal in G.

Definition 2.13 (Quotient Group)

If H is a normal subgroup of a group G, then the group G/H of all right cosets of H in G under the composition $(Ha)(Hb) = Hab$ is called quotient group.

Proposition 2.13 If H is a subgroup of an abelian group G, then the group G/H of all right cosets of H in G forms an abelian group under composition defined by $Ha.Hb = Hab$.

Proof: If H is subgroup of an abelian group G, then H is a normal subgroup of G.

Therefore, G/H forms a quotient group.

Let $Ha, Hb \in G/$H so that $a, b \in G$.
$(Ha)(Hb) = Hab = Hba,$ since G is abelian therefore $ab = ba$

$=(Hb)(Ha)$

Hence G/H is an abelian group.

Definition 2.14 (Conjugate elements)

If a, b be any two elements of a group G, then b is said to be conjugate to a if $\exists$ an element $x \in G$ s.t $b = xax^{-1}$, and we write it as $b \sim a$.

Definition 2.15 (Conjugacy class)

Let G be a group and $x \in G$.The conjugacy class of x in G is defined as

$$x^G = \{gxg^{-1} | \, g \in G\}$$

It is also denoted as $cl(x)$.

Lemma 2.16 The relation of conjugacy in a group G is an equivalence relation.

Proof: Define a relation ~ on G as follows

$a \sim b$ iff $a = gbg^{-1}$ for some $g \in G$.

Let a, b, c be any arbitrary elements of G.

Since $a = eae^{-1}$. Thus $a \sim a$.

Therefore ~ is reflexive.

Let $a \sim b$. So there exists $g \in G$ such that $a = gbg^{-1}$.

Thus, $g^{-1}\, ag = g^{-1}a(g^{-1})^{-1}$=$b$. So $b \sim a$.

Therefore ~ is symmetric.

Let $a \sim b$ and $b \sim c$.So there exists $g, h \in G$ such that $a = gbg^{-1}$ and $b = hch^{-1}$.

Now $a = gbg^{-1} = g(hch^{-1})\, g^{-1} = (gh)c(gh)^{-1}$. So $a{\sim}c$.

Therefore ~ is transitive.

Hence the relation of conjugacy in a group G is an equivalence relation.

Lemma 2.17 Let G be a group. Then the set of conjugacy classes of G is a patition of G.

Proof: 1. Define a relation ~ on G as follows

$a{\sim}b$ iff $a = gbg^{-1}$ for some $g \in G$.

By lemma 2.16, ~ is an equivalence relation on G.

Let x^G denote the class of x under this relation.

Then x^G=$\{y \in G | x{\sim}y\}$

=$\{y \in G | x = gyg^{-1}\}$

Thus G is partitioned into conjugacy classes of G.

Lemma 2.18 (The number of conjugates of x)

Let G be a group and let x be an element of G. Then $|x^G|$=$[G: C_G(x)]$

Proof: Let $G/C_G(x)$ denote the set of all left cosets of $C_G(x)$ in G.

Define a map $\varphi : x^G \rightarrow G/C_G(x)$ by

$$\varphi(gxg^{-1}) = gC_G(x)$$

Any element of $G/C_G(x) is\ of\ the\ form\ gC_G(x)$ where $g \in G$ and clearly

$$\varphi(gxg^{-1}) = gC_G(x)$$

Therefore φ is onto.

Suppose $\varphi(gxg^{-1}) = \varphi(hxh^{-1})$

$\Rightarrow \quad gC_G(x) = hC_G(x)$

$\Rightarrow \quad g^{-1}h \in C_G(x)$

$\Rightarrow \quad (g^{-1}h)x = x(g^{-1}h)$

$\Rightarrow \quad hxh^{-1} = gxg^{-1}$

So φ is 1-1.

Hence $|x^G| = [G: C_G(x)]$.

Theorem 2.19 For any finite group G, $|G| = \sum_x [G: C_G(x)]$, where summation runs over exactly one element x from each conjugacy class of G.

Proof: Suppose G is finite, then $[G: C_G(x)] = |G|/|\ C_G(x)|$

Now by lemma 2.17, G partition into conjugacy classes of G.

Now $G = \bigcup_{x \in G} x^G$, a disjoint union of classes.

Therefore $|\ G| = \sum |x^G\ |$, where summation runs over exactly one element from each conjugacy class.

Also by lemma 2.18, $|x^G| = [G: C_G(x)]$

So $|G| = \sum [G: C_G(x)]$, where summation runs over exactly one element from each conjugacy class.

Definition 2.20 (The Class Equation)

Let G be a finite group. Then

$|G| = \sum |\ x^G|$, where summation runs over exactly one element from each conjugacy class.

If $x \in Z(G)$, then

$x^G = \{x\}$

And conversely if $x^G = \{x\}$ then $x \in Z(G)$

Therefore we can write

$|G| = |Z(G)| + \sum |x^G|$,where summation runs over exactly one element from each conjugacy class of order > 1

Theorem 2.21 If $\alpha, \sigma \in S_n$, then $\theta = \alpha\sigma\alpha^{-1}$ is the permutation obtained by applying α to the symbol in σ. Hence any two conjugate permutations in S_n have the same cyclic structure. Conversely, any two permutations in S_n with the same cyclic structure are conjugate.

Proof: Suppose $\sigma(i) = j$

Then$(\alpha\sigma\alpha^{-1})(\alpha(i)) = (\alpha\sigma)(i) = \alpha(j)$.

Thus if $(a_1 a_2 \cdots a_m)$ is a cycle in σ, then $(\alpha(a_1)\, \alpha(a_2) \cdots \alpha(a_m))$ is a cycle in θ.Therefore,

cycle decomposition of θ is obtained by substituting $\alpha(x)$ for every x in the cycle decomposition of σ.

Thus σ *and* θ have the same cyclic structure.

Conversely suppose that σ *and* θ have the same cyclic structure (p q ⋯ r). Then σ *and* θ have cycle decomposition

$\sigma = (a_1 a_2 \cdots a_p)\,(a_{p+1} a_{p+2} \cdots a_{p+q}) \cdots (a_{n-r+1}\, a_{n-r+2} \cdots a_n)$

$\theta = (b_1\, b_2 \cdots b_p)\,(b_{p+1}\, b_{p+2} \cdots b_{p+q}) \cdots (b_{n-r+1}\, b_{n-r+2} \cdots b_n)$

Define $\alpha \in S_n$ by $\alpha(a_i) = b_i,\ i = 1,2 \cdots \cdots n$

Clearly $(\alpha\sigma\alpha^{-1})(b_i) = \alpha\sigma(a_i) = \alpha(a_{i+1}) = b_{i+1} = \theta(b_i) \quad \forall\ b_i$

$\therefore$, $\alpha\sigma\alpha^{-1} = \theta$

Thus σ *and* θ are conjugate.

Definition 2.22 (Transposition)

A cyclic permutation of length 2 is called a transposition.

Definition 2.23 (Even or odd permutation)

A permutation $\sigma \in S_n$ is called even (or odd) permutation if it can b e wriiten as a product of even (or odd) number of transpositions.

Definition 2.24 (Commutator subgroup)

If $a, b \in G$, the *commutator* of a, b is denoted by $[a, b]$ and defined by $[a, b] = a^{-1}b^{-1}ab$. The commutator subgroup (or derived subgroup) of G, denoted by G', is the subgroup of G is generated by all the commutators. Thus $G' = \{[a, b] \mid a, b \in G\}$

Remark: G is abelian iff $G' = \{e\}$

Theorem 2.25 Let G be a group and G' be its commutator subgroup, then

1. G' is a normal subgroup of G.
2. G/G' is abelian.
3. For any normal subgroup H of G, G / H is an abelian group iff H contains G'.

Proof: 1. Let $x = a^{-1}b^{-1}ab$ be any commutator in G.Then $x^{-1} = b^{-1}a^{-1}ba$ is also a commutator. Moreover, for any $g \; in \; G$,

$$gxg^{-1} = (ga^{-1}g^{-1})\,(gb^{-1}g^{-1})\,(gag^{-1})\,(gbg^{-1})$$

$$= (gag^{-1})^{-1}(gbg^{-1})^{-1}\,(gag^{-1})(gbg^{-1}) \in G'.$$

Now any element $y \; in \; G'$ is a product of a finite number of commutators, say

$$y = x_1x_2 \cdots x_n,$$

where $x_1x_2 \cdots x_n$ are commutators. Then for $g \in G$,

$$gyg^{-1} = (gx_1g^{-1})\,(gx_2g^{-1}) \cdots (gx_ng^{-1}) \in G'.$$

Hence G' is a normal subgroup of G.

2. For all $a, b \in G$,

$$(aG')^{-1}(bG')^{-1}(aG')(bG') = (a^{-1}b^{-1}ab)\,G' = G'$$

Hence,

$(aG')(bG') = (bG')(aG')$. Therefore, G/G' is abelian.

3. Suppose G/H is abelian. Then for all $a, b \in G$,

$$(a^{-1}b^{-1}ab)\,H = (aH)^{-1}(bH)^{-1}(aH)(bH)$$

$$= (aH)^{-1}(aH)(bH)^{-1}(bH) = H$$

Hence, $a^{-1}b^{-1}ab \in H$. This proves that $G' \subset H$.

Conversely, let $G' \subset H$.Then $a^{-1}b^{-1}ab \in G'$ gives $a^{-1}b^{-1}ab \in H$ i.e. $abH = baH$.

Thus $(aH)(bH) = (bH)(aH)\ \forall\ aH, bH \in G/H$.

So G/H is abelian.

Theorem 2.26 The alternating group A_n is generated by the set of all 3-cycle in S_n.

Proof: for $n = 1,2$ result is trivially true.

Suppose $n \geq 3$

Clearly every 3-cycle is an even permuatation and therefore in A_n.

Now we prove that every $\sigma \in A_n$ is a product of 3-cycle.

We know that σ is a product of even number of transpositions. So we can pair the transformation in σ.

Choose one pair φ, θ.

Case 1: Suppose $\varphi\ and\ \theta$ are disjoint.

Let $\varphi = (ab)$ and $\theta = (cd)$

$\varphi\theta = (ab)(cd) = (abc)(bcd)$.

Case 2: Suppose φ, θ have one symbol in common.

Let $\varphi = (ab)$ and $\theta = (bc)$

Thus $\varphi\theta = (ab)(bc) = (abc)$

Hence, every $\sigma \in A_n$ is a product of 3-cycles.

Theorem 2.27 The derived group of S_n is A_n.

Proof: For n=1, 2; $S_n' = \{ I\} = A_n$

Suppose $n > 2$

Let $\alpha = (12)$ & $\beta = (123)$

Then $\alpha\beta\alpha^{-1}\beta^{-1}$=(12)(123)(12)(132)

$=(123) \in S_n'$.

But $S_n' \lhd S_n$

Therefore, every conjugate of (123) is in S_n'.

$\Rightarrow$ S_n' contains all the 3-cycles.

But A_n (By theorem 2.26) is generated by all the 3-cycles .

Therefore, $A_n \subseteq S_n'$.

On the other hand every commutator $[\alpha, \beta]$ in S_n' is an even permutation.

Therefore, $S_n' \subseteq A_n$

Hence $A_n = S_n'$

Definition 2.28 (Maximal subgroup)

Let G be a group. A subgroup N of G is called a maximal subgroup if

1. $N \neq G$.
2. $If\ N \leq H \leq G$, then $H = N\ or\ H = G$.

Definition 2.29 ($\boldsymbol{p - group}$)

Let p be a prime number. A group G is called $p -$ group if the order of every element in G is some power of p.

Theorem 2.30 Let G be a finite group of order p^n, where p is a prime number and $n > 0$. Then

(i) $Z(G)$ is non-trivial.

(ii) $Z(G) \cap N$ is non-trivial normal subgroup N of G.

(iii) If H is a proper subgroup of G, then H is properly contained in $N_G(H)$.

(iv)Every maximal subgroup of G is normal.

Proof: Consider the class equation of G.

$|G| = p^n = |Z(G)| + \sum_{x\in C}[G: C_G(x)]$, where summation runs over exactly one element from each conjugacy class of order >1. (1)

By lagrange's theorem; $|C_G(x)| \mid |G| \ \forall\, x \in G$

If $x \notin Z(G)$, then $|C_G(x)| < p^n$

$\Rightarrow\ p \mid [G: C_G(x)] \ \forall\, x \notin Z(G)$

$\Rightarrow p \mid \sum_{x\in C}[G: C_G(x)]$

Also $p \mid |G|$

Therefore by (1) $p \mid |Z(G)|$

$\Rightarrow Z(G)$ is non-trivial.

(ii) We have $G = Z(G) \bigcup_{x\in C} x^G$, where C is the set contains conjugacy classes of length > 1.

Now $N = N \cap G$

$= N \cap \{Z(G) \bigcup_{x\in C} x^G\}$

$= N \cap Z(G) \cup \{N \cap (\bigcup_{x\in C} x^G)\}$

Therefore, $|N| = |N \cap Z(G)| + |N \cap (\bigcup_{x\in C} x^G)|$

$= |N \cap Z(G)| + \sum_{x\in C} |N \cap x^G|$ (2)

If $x \in N$, then $x^G \subseteq N$

Therefore for any $x \in C, x^G \cap N = \varphi \ or\ x^G$

$\Rightarrow |x^G \cap N|$ is either 0 or $|x^G| = [G: C_G(x)]$

Now $p \mid |N|$ and $p \mid |x^G|$, therefore by (2); $p \mid |N \cap Z(G)|$

$\Rightarrow N \cap Z(G)$ is non trivial.

(iii) Let K be a maximal normal subgroup of G contained in H. Then the quotient group G/K is of order $p^r (r > 0)$

By (i) G/K has a non-trivial center say L/K.

Since $L/K \lhd G/K$, so $L \lhd G$.

Clearly L cannot be in H, because otherwise maximality of K will lost.

Let $h \in H, l \in L$, then $hK \in G/K$ and $lK \in L/K$

Because $L/K = Z(G/K)$, so$(lK)(hK) = (hK)(lK)$

$\Rightarrow lhK = hlK \Rightarrow l^{-1}hl \in HK \subset H$.

Therefore $L \subset N_G(H)$. This implies that $H \neq N_G(H)$.

Because $H \subset N_G(H)$, it follows that H is properly contained in $N_G(H)$.

(iv) If H is a maximal subgroup of G, then by (iii), $H < N_G(H)$ implies that $N_G(H) = G$; Therefore by proposition 2.9, $H \lhd G$.

Definition 2.31 (Group Homomorphism)

A homomorphism φ from a group G to a group $\bar{G}$ is a mapping from G to $\bar{G}$ that preserves the group operation; that is $\varphi(ab) = \varphi(a)\varphi(b)\ \forall\, a, b \in G$.

Definition 2.32 (Kernel of a Homomorphism)

The kernel of a homomorphism φ from a group G to a group $\bar{G}$ with identity e is the set $\{x \in G | \varphi(x) = e\}$. The kernel of φ is denoted by $Ker\varphi$.

Lemma 2.33 Let f be a group homomorphism from G to a group $\bar{G}$. Then Ker f is a normal subgroup of G.

Proof: For $e \in G$, we have $f(e) = e' \Rightarrow e \in$ Ker f

Also Ker$f \subseteq G$ $\Rightarrow$Ker f is non-empty.

Let $x, y \in$ Ker f be any two elements.

Now $f(xy^{-1}) = f(x)f(y^{-1}) = f(x)(f(y))^{-1} = e'.(e')^{-1} = e'e' = e'$

$\Rightarrow xy^{-1} \in$ Ker f

Thus $xy^{-1} \in$ Ker f, $\forall x, y \in$Ker f

Hence Ker f is a subgroup of G.

Further let $g \in G$ and $x \in$Ker f

Therefore $f(x) = e'$

Now $f(gxg^{-1}) = f(g)f(x)f(g^{-1}) = f(g).e'.(f(g))^{-1} = f(g)\big(f(g)\big)^{-1} = e'$

$\Rightarrow \; gxg^{-1} \in \text{Ker} f$

Thus $gxg^{-1} \in \text{Ker} f, \; \forall x \in K$ and $g \in G$

Hence, Ker $f \lhd G$.

Theorem 2.34 (First Isomorphism Theorem or Fundamental Theorem of Homomorphism)

Let $\varphi: G \to \bar{G}$ be a homomorphism of groups. Then $G/$ Ker $f \cong Im(\varphi)$.

Hence in particular, if φ is surjective then $G/$ Ker $f \cong \bar{G}$.

Proof: Consider the mapping $\psi: \frac{G}{K} \to Im(\varphi)$ given by $xK \mapsto \varphi(x)$, where K =Ker f for any $x, y \in G$.

Now $xK = yK \Leftrightarrow \varphi(y^{-1}x) = e' \Leftrightarrow \varphi(x) = \varphi(y)$

Hence ψ is well defined and injective.

Let $xK, yK \in G/K$.

Then $\psi(xKyK) = \psi(xyK) = \varphi(xy) = \varphi(x)\varphi(y) = \psi(xK)\psi(yK)$

Hence ψ is clearly Homomorphism.

ψ is clearly surjective, we conclude that ψ is an isomorphism of groups.

Theorem 2.35 (Second Isomorphism Theorem)

If N is a normal subgroup of G and H be any subgroup of G, then

$(HN)/N \cong H/(H \cap N)$

Proof: Since $N \lhd G$ and H be any subgroup of G

Therefore, $H \cap N$ is a normal subgroup of H and so $H/(H \cap N)$ is meaningful.

Also N is normal in $G \quad \Rightarrow Nx = xN, \; \forall x \in G$

$\Rightarrow Nx = xN, \quad \forall x \in H \subseteq G$

$\Rightarrow NH = HN$

$\Rightarrow HN$ is a subgroup of G

Thus we have $N \subseteq NH \subseteq G$

Again, N is normal in $G \quad \Rightarrow Nx = xN, \quad \forall x \in G$

$\Rightarrow N$ is a normal subgroup of HN.

Therefore, HN/N is meaningful.

Now, we define a map $f: H \rightarrow HN/N$ by

$f(x) = Nx, \ \forall x \in H$

We claim that f is homomorphism, onto with Kernel $H \cap N$.

Let $x, y \in H$ be any element.

Therefore, $f(xy) = Nxy = NxNy = f(x)f(y)$ [$\because N \lhd G$]

$\Rightarrow f$ is homomorphism

Now, $\forall Nx \in HN/N$, where $x \in HN$

Let $x = hn$, for some $h \in H, n \in N$

Since $HN = NH \quad \Rightarrow hn = n'h'$ for some $n' \in N, h' \in H$

$\therefore f(h') = Nh' = (Nn')h'$ [$\because n' \in N \Rightarrow Nn' = N$]

$= N(n'h')$

$= N(hn)$

$= Nx$

Thus, $\forall Nx \in HN/N, \exists h' \in H$ such that $f(h') = Nx$

$\therefore f$ is onto

Further, Ker $f = \{x \in H: f(x) = N\}$, where N is the identity element of HN/N

$= \{x \in H: Nx = N\}$

$= \{x \in H: x \in N\} = H \cap N$

Thus f is homomorphism, onto with Kernel $H \cap N$

∴ By the fundamental theorem of homomorphism (Theorem 2.34)

$$HN/N \cong H/H \cap N$$

Definition 2.36 (Inner automorphism)

The automorphism $f_a: G \to G$ given by $f_a(x) = axa^{-1}, \forall x \in G$, is called an inner automorphism of G determined by a.

The set of all **inner automorphism** of G is denoted by $In(G)$. So

$$In(G) = \{f_a : f_a G \to G \ s.t. f_a(x) = axa^{-1}, a \in G\}.$$

Theorem 2.37 For any group G, $In(G) \cong G/Z(G)$ where $Z(G)$ is the center of G.

Proof: Let us define a map $h: G \to In(G)$

by $h(a) = f_a, \forall a \in G$

Let $a, b \in G$ be any elements then

$h(ab) = f_{ab} = f_a o f_b = h(a)h(b)$

So h is homomorphism

And for any $f_a \in In(G)$, where $a \in G$ s.t $h(a) = f_a$

∴ h is onto

∴ By fundamental theorem of homomorphism

$$G/Ker\ h \cong In(G)$$

Now, Ker $h = \{a \in G: h(a) = i_e\}$, where i_e is the identity element of $In(G)$

$= \{a \in G: f_a = i_e\}$

$= \{a \in G: f_a(x) = i_e(x), \forall x \in G\}$

$= \{a \in G: axa^{-1} = exe^{-1} = x, \forall x \in G\}$

$= \{a \in G: ax = xa, \forall x \in G\}$

$= Z(G)$

Hence $G/Z(G) \cong In(G)$.

Definition 2.38 (Simple group)

A group G is simple if it has no proper normal subgroup.

Definition 2.39 (Normal Series)

A sequence $\{G_0, G_1 \dots \dots G_{r-1}, G_r\}$ of subgroups of a group G is called a normal series of G if $\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \dots \dots \dots \dots \dots \dots . \dots \lhd G_{r-1} \lhd G_r = G$.

The factors of a normal series are the quotient groups G/G_{i-1} , $1 \leq i \leq r$.

Definition 2.40 (Composition Series)

A composition series of a group G is a normal series (G_0 , $\dots \dots$, G_r) without repetition whose factors G_i/G_{i-1} are simple groups. The factors G_i/G_{i-1} are called composition factors of G.

Remark: For any group $G, \{e\} = G_0 \lhd G_1 = G$ is trivially normal series of G. If G is a simple group, then $\{e\} \lhd G$ is the only composition series.

Some examples of composition series:

1. $\{e\} \lhd \{e, (123), (132)\} \lhd S_3$ is a normal series of S_3.
2. $\{0\} \lhd \{0,9\} \lhd \{0,3,6,9,12,15\} \lhd \{0,1,2 \dots \dots \dots \dots .17\} = Z_{18}$ is a composition series of Z_{18}.

Definition 2.41 (Derived series)

For a group G, the series of subgroups $G = \delta_0(G) \supseteq \delta_1(G) \supseteq \delta_2(G) \supseteq \dots\dots$, of G defined inductively as $\delta_i(G) = \delta(\delta_{i-1}(G)) = [\delta_{i-1}(G), \delta_{i-1}(G)]$, is called the derived series of G.

We write $G = \delta_0(G)$.

Definition 2.42 (Upper central series)

For a group G, the sequence $Z_i(G)_{i\geq 0}$ of subgroups $Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots\ldots$, of G defined inductively by $Z_i/Z_{i-1} = Z(G/Z_{i-1})$, is called the upper central series for G. Here $Z = Z_1(G)$ and $Z_0 = \{1\}$, $Z_1 = Z\ (G)$, the center of G.

Definition 2.43 (Lower central series)

For a group G, the seriea of subgroups $G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \ \gamma_3(G) \supseteq \ \ldots.$, of G defined inductively by $\gamma_1(G) = [G, \gamma_{i-1}(G)]$, for each $i \geq 1$, is called the lower central series for G. We write $\gamma_2(G) = G'$.

Definition 2.44 (Solvable group)

A group G is solvable iff $\delta_m(G) = \{e\}$ for some positive integer m.

Remark: Every abelian group is solvable.

Example of solvable group

$S_3 = \{I, (12), (13), (23), (123), (132)\}$

$S_3' = A_3 = \{I, (123), (132)\}$

$S_3'' = \{I\}$

So S_3 is solvable.

Definition2.45 (Nilpotent Group)

A group G is nilpotent iff $Z_m = G$ for some positive integer m.

The smallest m such that $Z_m = G$ is called the class of nilpotency of G.

Or a group G is said to be nilpotent iff $\gamma_n(G) = \{e\}$ for some positive integer n.

If G is abelian group then $Z_1 = \ Z\ (G) = G$. Thus, trivially, every abelian group is nilpotent.

For examples,

A_3 is nilpotent of class 1.

Q_8 is nilpotent of class 2.

Theorem 2.46 Let G be a nilpotent group. Then every subgroup of G is nilpotent.

Proof: Let G be a nilpotent group of class m.

So $Z_m(G) = G$

Let $H \leq G$

Then clearly $H \cap Z(G) \leq Z(H)$

Now $\forall x \in Z(G), y \in G, [x, y] \in Z(G)$

Therefore $\forall\ x \in H \cap Z_2(G), y \in H, [x, y] \in H \cap Z(G) \leq Z(H)$

So $x \in Z_2(G)$

$\Rightarrow H \cap Z_2(G) \leq Z_2(H)$

Continuing like this we can show that $H \cap Z_i(G) \leq Z_i(H) \quad \forall\ i \geq 1$

In particular, $i = m$, we have

$H \cap Z_m(G) \leq Z_m(H)$

i.e. $H \leq Z_m(H)$

i.e. H is nilpotent.

Theorem 2.47 A finite group of order p^m(p is prime) is nilpotent.

Proof: Let G be a group such that $|G| = p^m$

So by theorem 2.30 (i), $Z_1(G)$ is non-trivial. If $Z_1(G) = G$ then we are through. Otherwise consider $G/Z_1(G)$ which is again a non trivial p-group.

$\Rightarrow Z\left(\frac{G}{Z_1(G)}\right) = \frac{Z_2(G)}{Z_1(G)}$

So $|Z_2(G)| \geq Z_1(G)$

If again $Z_2(G) = G$, then we are through.

Otherwise we continue this process.

So $\exists\ m$ such that $Z_m(G) = G$

Hence G is nilpotent.

Definition 2.48 (T-Groups)

As in [4], a group G is called T-group if normal subgroup of a normal subgroup is again normal in G.

Definition 2.49 (Direct product of groups)

Let H and K be two normal subgroups of a group G, then direct product is defined as

G=$H \times K$ iff $G = H \times K$ and $H \cap K = \{1\}$.

Lemma 2.50 Elements of direct product commute with each other.

Proof: Let $G = G_1 \times G_2$

Let $g_1 \in G_1$ and $g_2 \in G_2$.

As $G_1 \lhd G$ and $G_2 \lhd G$

So $g_1^{-1}g_2g_1 \in G_2$

Also $g_2^{-1}g_1^{-1}g_2g_1 \in G_2$

And $g_2^{-1}g_1^{-1}g_2g_1 \in G_1$

$\Rightarrow g_2^{-1}g_1^{-1}g_2g_1 \in G_1 \cap G_2 = \{1\}$

$\Rightarrow g_2^{-1}g_1^{-1}g_2g_1 = \{1\}$

$\Rightarrow g_1^{-1}g_2g_1 = g_2$

$\Rightarrow g_2g_1 = g_1g_2$

Hence proved.

Theorem 2.51 Let $G = H \times K$ be a group where H, K are subgroups of G. Then $G/H \cong K$.

Proof: Define a mapping $\phi: G \to K$ defined as

$\phi\ (hk) = k$

Let $h'k', h''k'' \in G$ be any elements.

So $\phi(h'k'h''k'') = \phi(h'h''k'k'')$ [by theorem 2.50]

$= k'k''$

$= \phi(h'k')\phi(h''k'')$

$\therefore$ ϕ is a homomorphism.

Clearly, for any $k \in K, \exists hk \in G$ s.t. $\phi(hk) = k$

So ϕ is onto.

$\therefore$ By fundamental theorem of isomorphism (Theorem 2.34)

$G/Ker\ \phi \cong K$

Now, for any $h \in H$

$\phi(h) = \phi(he) = e$, where e is the identity of K.

$\Rightarrow h \in Ker\ \phi$

$\Rightarrow H \leq Ker\ \phi$

Let $x \in Ker\ \phi$ be any element and Ker $\phi \leq G$.

So $x = h'k'$, where $h'k' \in G$

$\therefore$, $\phi(x) = \phi(h'k') = e$

But $\phi(h'k') = k'$

$\Rightarrow k' = e$

$\Rightarrow x = h'$

$\Rightarrow x \in H$

$\Rightarrow x \in Ker\ \phi \leq H$

$\Rightarrow Ker\ \phi \leq H$

$\therefore Ker\ \phi = H$

Hence, $G/H \cong K$

Definition2.52 (Perfect group)

A group G is called perfect if $G = G'$.

Theorem2.53 Direct product of Perfect groups is perfect.

Proof: Let $G = H \times K$, where H, K are perfect groups.

i.e. $H = H', K = K'$

To prove: $G = G'$

Let $[x, y] \in G'$

$\Rightarrow x = h_1k_1, y = h_2k_2$

$\therefore [x, y] = [h_1k_1, h_2k_2]$

$$= [h_1k_1, k_2][h_1k_1, h_2][h_1k_1, h_2, k_2] \qquad (1)$$

Now, $[h_1k_1, k_2] = [h_1, k_2][h_1, k_2, k_1][k_1, k_2]$

$= [k_1, k_2]$

And $[h_1k_1, h_2] = [h_1, h_2][h_1, h_2k_1][k_1h_2]$

$= [h_1, h_2]$

And $[h_1k_1, h_2, k_2] = [[h_1k_1, h_2], k_2]$

$= \big[[h_1, h_2], k_2\big] = I$

$\therefore (1) \Rightarrow$

$[x, y] = [h_1k_1, h_2k_2] = [k_1k_2][h_1, h_2]$

$$= [h_1, h_2][k_1k_2] \qquad (2)$$

As $H' = H, K' = k$

So let $x \in G$ be any element

$\therefore x = hk$, for some $h \in H, k \in K$

$= [h_1, h_2][k_1, k_2]$, for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$ as $H = H', K = K'$

$=[h_1k_1, h_2k_2] \in G'$ [By (2)]

$\Rightarrow G \subseteq G'$

But trivially $G' \subseteq G$

So $G = G'$

Hence, G is perfect

Theorem 2.54 (Correspondence Theorem) Suppoese G_1 and $G_2$are groups, $H \leq G_1, J \leq G_2$, and θ is a surjective homomorphism mapping G_1 to G_2 with $Ker\ \theta = K$. This implies $G_1\theta \cong G_2\theta$ and $K\theta = < e >$.

Definition 2.55 (Frattini subgroup)

Let G be a group , the intersection of all maximal subgroups of G is called the $Frattini$ $subgroup$ of G. It is denoted by $\phi(G)$. If G has no maximal subgroup we set $\phi(G) = G$.

Lemma2.56 $\phi(G)$ is normal subgroup of G.

Proof: If G has no maximal group ,then $\phi(G) = G$ and the result holds trivially.

Let H_i be the maximal subgroups of G. As $e \in H_i$ for all $i \in I$, we have $e \in \phi(G)$.

So $\phi(G)$ is non-empty.

Let $a, b \in \phi(G)$, then $a, b \in H_i$ for all i and each $H_i \leq G$ so $ab^{-1} \in H_i$ for all i.

Hence $ab^{-1} \in \phi(G)$.

So $\phi(G) \leq G$.

An automorphism maps a maximal subgroup to a maximal subgroup, this is consequence of correspondence theorem. Hence this result holds for all inner automorphisms, and normality follows.

Lemma 2.57 If G is finite, $H \leq G$ and $G = H\phi(G)$, then $H = G$.

Proof: If H $\neq$ G, then there exists a maximal subgroup J of G which contains H possibly H itself. Now ϕ(G) $\leq$ J by definition, therefore G =Hϕ(G) $\leq$ J , which is impossible. The result follows.

Definition 2.58 (Frattini Argument)
If G is a finite group, $K \leq G$ and P is a Sylow subgroup of K, then $G = N_G(P)K$

Theorem 2.59 If G is finite then ϕ(G) is nilpotent.

Proof: We shall show that all Sylow subgroups of $\phi(G)$ are normal in $\phi(G)$.

If $\phi(G) = \{e\}$, there is nothing to prove.

Suppose $\phi(G) > \{e\}$ and let P be a non-neutral sylow subgroup of $\phi(G)$.

Using Frattini argument we have $G = N_G(P)\phi(G)$, as $\phi(G) \lhd G$. Now applying lemma 2.50 with $H = N_G(P)$, we obtain $N_G(P) = G$, which gives $P \lhd \phi(G)$. The result follows as this holds for all sylow subgroups P of $\phi(G)$.

Definition 2.60 (Conjugate closed subgroup)

Let G be a group and H be a normal subgroup of G. For any $x \in H, x^G$ denotes the conjugacy class of x in G. A normal subgroup of G is said to be conjugate closed if $x^G = x^H$ for every $x \in H$.

Definition 2.61 (Conjugate closed group)

A group G is said to be conjugate closed if every normal subgroup of G is conjugate closed. It is abbreviated as $CC - Group$.

Note: Every abelian group and simple group are $CC - Groups$.

Definition 2.62 (Schreier Conjecture)

Let G be a group and Aut$(G), In(G)$ be the automorphism and inner automorphism of G.

Then $Out(G) = Aut(G)/In(G)$ is solvable.

Chapter-3

MAIN RESULT

3.1 General Properties of Conjugate Closed Groups

Lemma 3.1.1 Direct product of $CC - Groups$ is conjugate closed.

Proof: Let $G = G_1 \times G_2$, where G_1, G_2 are $CC - Groups$.

Let H be a normal subgroup of G.

So $H = H_1 \times H_2$ such that $H_1 \lhd G_1$ and $H_2 \lhd G_2$

To prove $x^G = x^H$ for every $x \in H$

Let $x \in H$ be any element.

So $x = h_1 h_2$ where $h_1 \in H_1, h_2 \in H_2$

So we will show $(h_1 h_2)^{H_1 \times H_2} = (h_1 h_2)^{G_1 \times G_2}$

Now $(h_1 h_2)^{G_1 \times G_2} = (g_1 g_2)^{-1}(h_1 h_2)(g_1 g_2)$ for some $g_1 \in G_1, g_2 \in G_2$

$= g_2^{-1} g_1^{-1} h_1 h_2 g_1 g_2$

$= g_1^{-1} h_1 g_1 g_2^{-1} h_2 g_2$ (1) {theorem 2.50}

So $g_1^{-1} h_1 g_1 \in H_1, g_2^{-1} h_2 g_2 \in H_2$

Also G_1 is a $CC - Group \Rightarrow h_1^{G_1} = h_1^{H_1}$

$\Rightarrow g_1^{-1} h_1 g_1 = h'^{-1} h_1 h'$, for some $h' \in H_1, g_1 \in G$

Similarly $h_2^{G_2} = h_2^{H_2}$

$\Rightarrow g_2^{-1} h_2 g_2 = h''^{-1} h_2 h''$, for some $h'' \in H_2, g_2 \in G$

So (1)

$\Rightarrow (h_1 h_2)^{G_1 \times G_2} = h'^{-1} h_1 h' h''^{-1} h_2 h''$

$=h'^{-1}h''^{-1}h_1h_2h'h''$

$=(h''h')^{-1}(h_1h_2)\ (h''h')$

$=\ (h_1h_2)^{H_1\times H_2}$

So G is a $CC-Group$.

Lemma 3.1.2 A conjugate closed group is a T-group. But converse is not true.

Proof: Let G be a $CC-Group$ and $H \lhd G$ and $K \lhd H$.

Since G is a $CC-Group$

Therefore, $x^G = x^H$ for every $x \in H$.

Now G will be $CC-Group$ if every normal subgroup of G is conjugate closed.

So if $K \lhd H$

$\Rightarrow x^K = x^H$, for every $x \in K$

$\Rightarrow x^K = x^G, x \in K$

$\Rightarrow K \lhd G$.

So G is a T-group.

Converse of above is not true.

For example, Let $G = S_3$

So normal subgroups of S_3 are I and A_3.

So $I \lhd A_3 \lhd S_3$

But $x^{A_3} \neq x^{S_3}, x \in A_3$

So S_3 is not a $CC-Group$.

Proposition 3.1.3 Every normal abelian subgroup of a $\mathrm{CC-Group}$ is central.

Proof: Let N be a normal abelian subgroup of G.

Let $x \in N$ and $g \in G$ be any element.

As G is a $CC-Group$

$\Rightarrow x^N = x^G$

$\Rightarrow n^{-1}xn = g^{-1}xg$

$\Rightarrow xn^{-1}n = g^{-1}xg$

$\Rightarrow \qquad x = g^{-1}xg$

$\Rightarrow gx = xg$

$\Rightarrow x \in Z(G)$

Now $N \subseteq Z(G)$

As N is abelian subgroup so $N = Z(N)$

Hence $N = Z(N) \subseteq Z(G)$.

Theorem 3.1.4 For a group G, the following are equivalent:

(1) G is conjugate closed.
(2) For every normal subgroup H of G, $G = C_G(x)H$, for every $x \in H$.
(3) For every normal subgroup H of G and every $x \in G$, the natural map, $f: C_G(x) \rightarrow G/H$ is an epimorphism.

Proof: (1)⇔(2)

Firstly suppose that G is a $CC-Group$ and H be any normal subgroup of G.

Therefore, for a given $g \in G$ and $x \in H$.

$x^G = x^H$

$\Rightarrow g^{-1}xg = h^{-1}xh$

$\Rightarrow gg^{-1}xg = gh^{-1}xh$

$\Rightarrow \ xg = gh^{-1}xh$

$\Rightarrow \ xgh^{-1} = gh^{-1}xhh^{-1}$

$\Rightarrow \ xgh^{-1} = gh^{-1}x$

$\Rightarrow gh^{-1} \in C_G(x)$

$\Rightarrow g \in C_G(x)h$ for some $h \in H$

$\Rightarrow G \leq C_G(x)H$

Trivially $C_G(x)H \leq G$

$\Rightarrow G = C_G(x)H$

Conversely suppose that $H \lhd G \; and \; G = C_G(x)H$

Let $g \in G$ be any element

$\Rightarrow g = zh; z \in C_G(x), h \in H$

To prove: $x^{G} = x^H$ for every $x \in H$

Let $x \in H$ be any element

Now $x^{G} = g^{-1}xg$

$= (zh)^{-1}x(zh)$

$= h^{-1}z^{-1}xzh$

$= h^{-1}xz^{-1}zh$

$= h^{-1}xh$

$= x^{H}$

So G is a $CC - Group$.

(2)⇔(3)

Firstly suppose that $H \lhd G \; and \; G = C_G(x)H$

To show: $f: C_G(x) \to G/H$ is an epimorphism

Now f is a natural map

$\Rightarrow f(g) = gH$

Let $g_1, g_2 \in C_G(x)$ be two elements.

$f(g_1g_2) = g_1g_2H$

$= g_1Hg_2H$

$= f(g_1)f(g_2)$

$\Rightarrow f$ is a homomorphism.

Now $G = C_G(x)H$

$\Rightarrow g = zh;\ z \in C_G(x), h \in H$

$\Rightarrow gH = zhH$

$= zH$

So $\forall\ zH \in C_G(x)H\ \ \exists\ z \in C_G(x)$ such that $f(z) = zH$

So f is onto.

Hence f is an epimorphism.

Conversely suppose that $f: C_G(x) \to G/H$ is an epimorphism

To prove: if $H \lhd G\ then\ G = C_G(x)H$

Since $f: C_G(x) \to G/H$ is a natural map

$\Rightarrow f(z) = zH$

Also natural map is onto. So for $gH \in G/H\ \exists\ z \in C_G(x)$ such that $f(z) = gH$

$\Rightarrow gH = zH$

$\Rightarrow z^{-1}gH = z^{-1}zH$

$\Rightarrow z^{-1}gH = H$

$\Rightarrow z^{-1}g \in$H

$\Rightarrow g \in zH$

$\Rightarrow G \leq C_G(x)H$

But $C_G(x)H \leq G$

$\Rightarrow G = C_G(x)H$

Hence proved.

Theorem 3.1.5 Let G be a group such that $G = Z(G) \times H$. Then G is conjugate closed if and only if H is conjugate closed.

Proof: Firstly suppose that H is conjugate closed.

To prove: $G = Z(G) \times H$ is conjugate closed

Since $Z(G) \lhd G$ and $Z(G)$ is an abelian group.

Hence $Z(G)$ is $CC - Group$. {every abelian group is conjugate closed}

Also H is conjugate closed.

$\Rightarrow Z(G) \times H$ is a $CC - Group$. BY using lemma 3.1.1 direct product of $CC - Group$ is a $CC - Group$.

$\Rightarrow G$ is conjugate closed.

Conversely suppose that $G = Z(G) \times H$ is conjugate closed

To prove : H is conjugate closed

Now $G = Z(G) \times H$

So by theorem 2.51, $G/Z(G) \cong H$

Now to prove H is conjugate closed; we will prove $G/Z(G)$ is conjugate closed.

Let $M/Z(G) \lhd G/Z(G)$

$\Rightarrow M \lhd G$

Let $N = Z(G)$

Now we have to show if $xN \in M/N \Rightarrow (xN)^{M/N} = (xN)^{G/N}$

$(xN)^{G/N}=(gN)^{-1}(xN)(gN)$

$=g^{-1}NxNgN$

$=g^{-1}xgN$

Since $G\ is\ a\ CC - Group$. So if $M \lhd G\ then\ x^{\ G}=x^M$ for every $x \in M$.

$\Rightarrow g^{-1}xg = m^{-1}xm$

So $(xN)^{G/N}=(m^{-1}xm)N$

$= m^{-1}NxNmN$

$=(mN)^{-1}(xN)(mN)$

$=(xN)^{M/N}$

So G/N is a $CC-Group$

⇒$G/Z(G)$ is a $CC-Group$.

⇒H is conjugate closed.

Theorem 3.1.6 A finite group G is semisimple, conjugate closed and perfect if and only if $G = G_1 \times G_2 \times \ldots G_r$, where each G_i is non abelian and simple subgroup of G.

Proof: Let G be a semisimple conjugate closed and perfect group. Since G is semisimple, G has no normal abelian subgroup and hence its center is trivial.

Let G_1 be a minimal normal(and so non-abelian) subgroup of G. As $G_1$is minimal normal subgroup so it has no non-trivial normal subgroup. ∴, $G_1$is simple and also non-abelian. Since G is conjugate closed and G_1 is non-abelian and simple group. Therefore $G/C_G(G_1) \cong$ a subgroup of $Aut(G_1)$ and in this isomorphism $G_1C_G(G_1)/C_G(G_1)$ corresponds to $Inn(G_1)$ as

$\frac{G_1C_G(G_1)}{C_G(G_1)} \cong \frac{G_1}{G_1 \cap C_G(G_1)}$ [∵ Second isomorphism theorem (Theorem 2.35)]

$= \frac{G_1}{Z(G)}$

$\cong In(G_1)$

So, $G_1C_G(G_1)/C_G(G_1) \cong In(G_1)$. Now G_1 is simple, therefore by Schreier Conjecture $Out(G_1) = Aut(G_1)/In(G_1)$ is solvable and hence $G/G_1C_G(G_1)$ is solvable as

$\frac{Aut(G_1)}{In(G_1)} = \frac{G/C_G(G_1)}{G_1C_G(G_1)/C_G(G_1)}$

$= G/G_1C_G(G_1)$

Since G is perfect so $G = G'$

Now, $\frac{G}{G_1C_G(G_1)}$ is solvable so ∃ a +ve integer n such that $(G/G_1C_G(G_1))^n = \{1\}$

Now, $\left(\frac{G}{G_1C_G(G_1)}\right)' = \frac{G'G_1C_G(G_1)}{G_1C_G(G_1)}$

$= \frac{GG_1C_G(G_1)}{G_1C_G(G_1)}$ [∵G is perfect]

$= \frac{G}{G_1C_G(G_1)}$

$\therefore$, $\left(\frac{G}{G_1C_G(G_1)}\right)' = \frac{G}{G_1C_G(G_1)}$

Continuing this way we get $\left(\frac{G}{G_1C_G(G_1)}\right)^n = 1$

Therefore, $G/G_1C_G(G_1)$ is trivial group.

But then $G = G' = G_1C_G(G_1)$.

As G is semisimple; $G_1 \cap C_G(G_1)$ is a normal abelian subgroup of Gas

Let $x, y \in G_1 \cap C_G(G_1)$

$\Rightarrow x \in G_1$ and $x \in C_G(G_1)$

also $y \in G_1$ and $y \in C_G(G_1)$

$\Rightarrow xy = yx$

But G is semisimple i.e. it does not contain non-trivial normal subgroup. Hence $G_1 \cap C_G(G_1)$ is trivial. But then $G = G_1 \times C_G(G_1)$. Now $o(C_G(G_1)) < o(G)$ and $C_G(G_1)$ is semisimple, conjugate closed and perfect. Therefore by induction, we have $C_G(G_1) = G_2 \times \ldots \times G_r$and hence $G = G_1 \times G_2 \times \ldots \times G_r$, where each G_i is non-abelian, simple subgroup of G.

Conversely suppose that $G = G_1 \times G_2 \times \ldots \times G_r$, where each G_i is non-abelian, simple subgroup of G and therefore each G_i is perfect and conjugate closed. From this it follows that G is perfect (by Theorem 2.53) and conjugate closed . Let H be a normal abelian subgroup of G. Then H is a direct product of some of $H_i's$, where for any i, H is a normal abelian subgroups of G_i. But then every $H_i = \{1\}$ and hence $H = \{1\}$. Therefore G is semisimple.

Theorem 3.1.7 Let G be a finite group such that $G = Z(G) \times H$ for some normal conjugate closed subgroup H of G. Then $G/Z(G)$ is perfect if and only if $G = Z(G) \times G_1 \times G_2 \times \ldots G_r$, where each G_i is non abelian and simple subgroup of G.

Proof: Firstly suppose that $G/Z(G)$ is perfect.

To prove: $G = Z(G) \times G_1 \times G_2 \times \ldots G_r$, where each G_i is non abelian and simple subgroup of G.

Now $G = Z(G) \times H$

So by theorem 2.51 $G/Z(G) \cong H$

As $G/Z(G)$ is perfect so H is perfect.

Also H is $CC - Group$.

First we prove H is semi-simple.

Since $Z(H)$ is trivial

Therefore, H cannot have normal abelian subgroup.

$\Rightarrow H$ is semi-simple.

Now H is semi-simple, perfect and $CC - Group$.

So by theorem 3.1.6, $H = G_1 \times G_2 \times \dots G_r$, where each G_i is non abelian and simple.

But $G = Z(G) \times H$

$\Rightarrow G = Z(G) \times G_1 \times G_2 \times \dots G_r$, where each G_i is non abelian and simple.

Conversely suppose that $G = Z(G) \times G_1 \times G_2 \times \dots G_r$, where each G_i is non abelian and simple.

To prove: $G/Z(G)$ is perfect.

Now $G = Z(G) \times G_1 \times G_2 \times \dots G_r$

Let $H = G_1 \times G_2 \times \dots G_r$

$\Rightarrow G = Z(G) \times H$

Since by theorem 2.53 direct product of perfect group is perfect.

$\Rightarrow G = Z(G) \times H$; where H is perfect

By theorem 2.51 $G/Z(G) \cong H$

Since H is perfect

$\Rightarrow G/Z(G)$ is perfect.

3. 2 Nilpotent and Solvable Groups

Theorem 3.2.1 For a $CC - Group\ G$, the lower central series of G coincides with the derived series of G.

Proof: Since the lower central series is

$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \gamma_3(G) \supseteq \ldots$, of G defined inductively by $\gamma_1(G) = [G, \gamma_{i-1}(G)]$, for each $i \geq 1$.

And the derived series is $G = \delta_0(G) \supseteq \delta_1(G) \supseteq \delta_2(G) \supseteq \ldots\ldots$, of G defined inductively as $\delta_i(G) = \delta(\delta_{i-1}(G)) = [\delta_{i-1}(G), \delta_{i-1}(G)]$.

Now we have to prove

$\gamma_i(G) = \delta_{i-1}(G)$

For i=2;

$\gamma_2(G) = [G, \gamma_1(G)] = [G, G] = G'$

And $\delta_1(G) = [\delta_0(G), \delta_0(G)] = [G, G] = G'$

$\Rightarrow \gamma_2(G) = \delta_1(G)$

So result is true for $i = 2$.

Let the result is true for i i.e. $\gamma_i(G) = \delta_{i-1}(G)$

Now we will prove that $\gamma_{i+1}(G) = \delta_i(G)$

Let $\gamma_i(G)$ is normal in G, for $x \in \gamma_i(G), g \in G, h \in \gamma_i(G)$.

As G is a $CC - Group, and\ \gamma_i(G) \lhd G$

$\Rightarrow$if $x \in \gamma_i(G)$ then $x^{\gamma_i(G)} = x^G$

$\Rightarrow h^{-1}xh = g^{-1}xg$, for some $h \in \gamma_i(G), g \in G$.

So, $[x, g] = x^{-1}g^{-1}xg = x^{-1}h^{-1}xh = [x, h] \in [\gamma_i(G), \gamma_i(G)] = [\delta_{i-1}(G), \delta_{i-1}(G)] = \delta_i(G)$

Thus, $\gamma_{i+1}(G) \subseteq \delta_i(G)$

Now $\delta_i(G) = [\delta_{i-1}(G), \delta_{i-1}(G)]$

$= [\gamma_i(G), \gamma_i(G)] \subseteq [G, \gamma_i(G)] = \gamma_{i+1}(G)$

$\Rightarrow \delta_i(G) \subseteq \gamma_{i+1}(G)$

$\Rightarrow \delta_i(G) = \gamma_{i+1}(G)$

Hence proved.

Theorem 3.2.2 A conjugate closed, nilpotent group is abelian.

Proof: Consider the upper central series $1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots\ldots \subseteq Z_m(G) = G$

Let $x \in Z_2(G)$ be any element.

$Z_2(G) = \{x \in G \mid [x, y] \in Z_1(G) \ \forall\, y \in G\}$

Consider $H = < x, Z_1(G) > = < x, Z(G) >$

So every element of H can be written as an integral power of $< x, Z_1(G) >$

Now we show H is an abelian group .

Let $g, h \in H$.

So $g = x^l z_1 z_2 \ldots \ldots z_m$ where $x \in x, z_i \in Z(G)$

lly, $h = x^m z_1' z_2' \ldots \ldots z_m'$

$gh = x^l z_1 z_2 \ldots \ldots z_m x^m z_1' z_2' \ldots \ldots z_m'$

$= x^{l+m} z_1' z_2' \ldots \ldots z_m' z_1 z_2 \ldots \ldots z_m$

$= x^{m+l} z_1' z_2' \ldots \ldots z_m'^{z_1 z_2} \ldots \ldots z_m$

$= x^m . x^l z_1' z_2' \ldots \ldots z_m'^{z_1 z_2} \ldots \ldots z_m$

$= x^m z_1' z_2' \ldots \ldots z_m' x^l z_1 z_2 \ldots \ldots z_m$

$= hg$

$\Rightarrow H$ is an abelian group of G containing $Z_1(G)$

Now let $hZ(G) \in H/Z(G)$ be any element.

Since $Z(G) \subseteq Z_2(G)$

Also $x \in Z_2(G)$

$\Rightarrow < x, Z(G) > = H \subseteq Z_2(G)$

Let $h \in H \subseteq Z_2(G)$

$\Rightarrow h \in Z_2(G)$

$\Rightarrow hZ(G) \in Z_2(G)/Z(G)$

$\Rightarrow H/Z(G) \subseteq Z_2(G)/Z(G)$=$Z(\frac{G}{Z(G)})$

$\Rightarrow H/Z(G) \subseteq Z\left(\frac{G}{Z(G)}\right)$

Thus, $H/Z(G)$ is central subset of $G/Z(G)$

$\Rightarrow H/Z(G) \lhd \left(\frac{G}{Z(G)}\right)$

$\Rightarrow H \lhd G$

Since H is normal abelian subgroup.

So by proposition 3.1.3;

$H \leq Z(G)$

$\Rightarrow x \in Z(G)$ for any element $x \in Z_2(G)$

$\Rightarrow Z_2(G) \subseteq Z(G)$

But $Z(G) \subseteq Z_2(G)$

Similarly by other elements $Z_m(G) = Z(G) = G$

$\Rightarrow G$ is abelian

Corollary 3.2.3 For a finite $CC - Group\ G$, the frattini subgroup $\phi(G)$ is central.

Proof: Let G be a finite $CC - group$.

Since $\phi(G)$ is a normal subgroup of G(Theorem2.56) and hence conjugate closed.

Also $\phi(G)$ is nilpotent subgroup of G(Theorem 2.59)

So by proposition 3.2.2; $\phi(G)$ is abelian.

Now $\phi(G)$ is abelian normal subgroup of G.

So by proposition 3.1.3, $\phi(G)$ is central.

Corollary 3.2.4 A conjugate closed solvable group is abelian.

Proof: Since G is solvable.

Therefore, its derived series terminates.

So ∃ a positive integer n such that

$G^n=1$

But G is $CC-group$. So by proposition 3.2.1

$\gamma_{n+1}(G) = \delta_n(G) = G^n = 1$

$\Rightarrow \gamma_{n+1} = 1$

Thus every solvable group is nilpotent.

So by proposition3.2.2,

G is abelian.

Remark: Every solvable group of odd order is solvable.[2]

Theorem 3.2.5 If G is a conjugate closed such that $G \neq G'$, then either G is abelian or $G = MG'$ for some maximal subgroup M of G.

Proof: Let G be a $CC-group$ such that $G \neq G'$. Let N be a normal subgroup of G.

If N is abelian then $N \subseteq Z(G)$

If $N \nsubseteq Z(G)$; then ∃ an element $x \in N$ such that $x \notin Z(G)$.

Now G is a $CC-Group$ and $N \lhd G$. Let $x \in N$ be any element.

Therefore $x^N = x^G$

$n^{-1}xn = g^{-1}xg$ for some $n \in N, g \in G$

$gn^{-1}xn = xg$

$gn^{-1}x = xgn^{-1}$

$\Rightarrow gn^{-1} \in C_G(x)$

$\Rightarrow g \in C_G(x)n$ for some $n \in N$

$\Rightarrow g \in C_G(x)N$

$\Rightarrow G \leq C_G(x)N$

But $C_G(x)N \leq G$

⇒G=$C_G(x)N$

⇒$C_G(x)N$ is a proper supplement of N in G.

Thus in a $CC - group$ either a normal subgroup N is central or it has a proper supplement.

Now in particular let $G' \lhd G$ and $G \neq G'$

So by above either $G' \subseteq Z(G)$ or G' has a proper supplement.

Therefore in first case G will be nilpotent and hence abelian. In the later G' has a proper supplement.

If G' is non-abelian

⇒$G' \nleq Z(G)$

By proposition 3.1.3, $\phi(G) \leq Z(G)$

⇒$G' \nleq \phi(G)$

Therefore ∃ a maximal subgroup say M such that $G' \nleq M$.

So $G'M = G$

⇒ $G'M = G$; for some maximal subgroup M of G.

Proposition 3.2.6 For $n \geq 3$, the symmetric group S_n is not conjugate closed.

Proof: Let $n \geq 3$, the symmetric group of degree n on the set $S = \{1,2, \dots n\}$. Two elements of S_n are conjugate iff they have the same cyclic structure. Let A_n be the alternating group of degree n. Take $G = S_n$ and $H = A_n$.

Now $|S_n| = n!$ and $|A_n| = n!/2$

Case 1: Suppose n is odd.

Now n is odd.

Then $(123 \dots n)$ is an even permutation.

Therefore, $(123 \dots n) \in A_n$.

As number of r-cycles in $S_n = \frac{1}{r} . \frac{n!}{(n-r)!}$

So number of n-cycles = $\frac{1}{n} \cdot \frac{n!}{(n-n)!} = (n-1)!$

$|(123 \ldots n)^{S_n}| = (n-1)!$

If possible let S_n be a $CC-group$.

$\Rightarrow (123 \ldots n)^{S_n} = (123 \ldots n)^{A_n}$

$\Rightarrow (n-1)! \,|\, (n!/2)$

Let $\frac{n!}{2} = (n-1)!\, k$ for some $k \in N$

$\Rightarrow \frac{n(n-1)!}{2} = (n-1)!\, k$

$\Rightarrow n = 2k$, which is contradiction as is odd.

Case 2: If n is even.

Now n is even.

Then $(123 \ldots n) \notin A_n$

$\Rightarrow (123 \ldots n-1) \in A_n$

Number of n-1 cycles= $\frac{1}{(n-1)!} \cdot \frac{n!}{(n-(n-1))!} = n(n-2)!$

$|(123 \ldots n-1)^{S_n}| = n(n-2)!$

Suppose S_n be a $CC-group$

$\Rightarrow |(123 \ldots n-1)^{S_n}| = |(123 \ldots n-1)^{A_n}$

$n(n-2)! \,|\, (n!/2)$

Let $\frac{n!}{2} = n(n-2)!\, k$ for some $k \in N$

$\Rightarrow \frac{n(n-1)!}{2} = n(n-2)!\, k$ for some $k \in N$

$\Rightarrow n = 2k+1$, which is a contradiction as n is even.

Hence S_n is not $CC-group$ for $n \geq 3$.

Proposition 3.2.7 If G is conjugate closed then G' is perfect and hence $G' = G'' =$ **…….**

Proof: Case 1: If G is perfect i.e. $G = G'$

Then $G'' = (G')' = G'$ which shows that G' is perfect.

Case 2: If G is not perfect then G/G'' is solvable.

Since G/G'' is conjugate closed.

Therefore by proposition 3.2.4 G/G'' is abelian.

$\Rightarrow G' \leq G''$

But $G'' \leq G'$

$\Rightarrow G' = G''$

$\Rightarrow G'$ is perfect.

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