Numerical Solutions of Some Differential Equations Using B-Spline Collocation Method

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under

the guidance of

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to the

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CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**Numerical Solutions of Some Differential Equations Using B-Spline Collocation Method**" which is being submitted for the award of degree of master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Ram Jiwari.

The matter presented in the thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

Chapter 1 is introductory in nature. Besides stating some numerical techniques like Finite Difference methods, Finite Element method, Finite Volume method and methods of weighted residuals it gives an introduction to B-Spline and existing literature review.

In chapter 2, we consider a second order singularly perturbed boundary value problem

$$Ly(x) = \varepsilon y''(x) + p(x)y'(x) - q(x)y(x) = r(x) \quad , \quad x \in [a,b]$$

where ε is a small positive parameter $(0 < \in < 1)$, p(x) ,q(x) and r(x) are sufficiently smooth functions. The boundary value problems under these assumptions have unique solution [1]. In general, as $\in$ tends to zero, the solution $y(x)$ exhibits an exponential boundary layer at the left end of the interval; that is, the domain of the differential equation contains a narrow region where the solution derivatives are extremely large. Regions like this frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in fluid mechanics, edge layers in solid mechanics, and skin layers in electrical applications. There are many physical situations in which the sharp changes occur inside the domain of interest, and the narrow regions across which these changes take place are usually referred to as shock layers in fluid and solid mechanics, transition points in quantum mechanics, and Strokes lines and surfaces in mathematics. These rapid changes cannot be handled by slow scales, but they can be handled by fast or magnified or stretched scales.

In this chapter, a numerical technique for a system of singularly perturbed boundary value problems using cubic-B-spline functions is derived. Simplicity of the

adaptation of B-splines and obtaining acceptable solutions can be noted as advantages of given numerical methods. The results obtained using the proposed method are good and maximum absolute errors are negligible.

In chapter 3, we consider the second order two point mixed boundary value problem

$$\begin{aligned} & y''(x) = F(x, y(x), y'(x)), \quad x \in [a,b], \\ & \alpha_1 y(a) + \beta_1 y'(a) = a_1, \\ & \alpha_2 y(b) + \beta_2 y'(b) = b_1 \end{aligned}$$

For simplicity such problems arise in many fields in engineering and science, such as heat transfer, deflection of plate, diffusion. So the problem such type has great importance in many real life fields.

In this chapter, a numerical scheme is proposed for the numerical solutions of the second order two point mixed boundary value problems based on quintic-B-spline functions. Two test problems have been considered to test the accuracy of the proposed method. The maximum absolute errors are decreasing as we increase the number of sub-intervals. This shows that the method is stable.

Chapter 1

Introduction

1.1 Numerical Solution of Differential Equations

Differential equations (PDE/ODEs) form the basis of very many mathematical models of physical, chemical and biological phenomena, and more recently their use has spread into economics, financial forecasting, image processing and other fields. It is not easy to get analytical solution treatment of these equations, so, to investigate the predictions of PDE models of such phenomena it is often necessary to approximate their solution numerically. In most cases, the approximate solution is represented by functional values at certain discrete points (grid points or mesh points). There seems a bridge between the derivatives in the PDE and the functional values at the grid points. The numerical technique is such a bridge, and the corresponding approximate solution is termed the numerical solution. Currently, there are a number of numerical methods available for finding the numerical solutions of differential equations. . Among them, the finite difference (FD), finite element (FE), and finite volume (FV) methods fall under the category of low order methods, whereas spectral and pseudo spectral methods are considered global methods. Sometimes the latter two methods are considered as subsets of the method of weighted residuals.

1.1.1 Finite Difference Methods

Finite difference methods are widely dominant in the numerical solution of DEs and their application. The finite difference (FD) methods are based on the Taylor series expansion or the polynomial approximation. A finite differential method proceeds by replacing the derivatives in the differential equations by finite difference approximation. This gives a large algebraic system of equations to be solved in place of the differential equation, which can be easily solved on computer. That is, the partial derivatives in differential equations are written in terms of discrete quantities of dependent and independent variables, resulting in simultaneous algebraic equations with all unknowns prescribed at discrete mesh points or grid points for the entire domain That is, the partial derivatives in PDEs are written in terms of discrete quantities of dependent and independent variables, resulting in simultaneous algebraic equations with all unknowns prescribed at discrete mesh points or grid points for the entire domain. Appropriate types of differencing schemes and suitable methods of solution are chosen in different applications. For example, in fluid dynamics applications, depending upon the particular physics of the flows, which may include in viscid, viscous, incompressible, compressible, irrigational, rotational, laminar, turbulent, supersonic, or hypersonic flows, finite difference schemes are written to conform to these different physical phenomena. The formulation of FD methods in one dimensional is simple but for multidimensional problems, meshes must be structured in either two or three dimensions. Curved meshes must be transformed into orthogonal Cartesian meshes. The challenge in analyzing finite difference methods for new classes of problems is often to find an appropriate definition of stability that allow one to prove convergence and to estimate the error in approximation.

Finite difference methods discredited the governing PDE directly using their strong form. Although it is most straight forward way to obtain the discrete system equations, but it is difficult to handle the typical boundary conditions. For a problem domain with complex geometry, the discretization of the geometry and the application of the natural and essential boundary conditions can seldom be done automatically by a computer program with no human involvement.

1.1.2 Finite Element Method

Another most popular method is Finite Element method. Finite element method (FEM) represents a powerful and general class of techniques for the approximate solution of partial differential equations. The basic idea in the FEM is to find the solution of a complicated problem by replacing it by a simpler one. Since the actual problem is replaced by a simpler one in finding the solution, we will be able to find only an approximate solution rather than the exact solution. This method is mostly used for the accurate solution of complex engineering problems with abundant software available commercially. Finite Element method was first developed in 1956 for the analysis of aircraft structural problems. Thereafter, within a decade, the potentialities of the method for the solution of different types of applied science and engineering problems were recognized. Over the years, the FEM technique has been so well established that today it is considered to be one of the best methods for solving a wide variety of practical problems efficiently .It has been applied to a number of physical problems, where the governing differential equations are available. In Finite Element method the domain is divided into a finite number of sub domains called elements and nodes are located at predetermined locations around the elements boundary. The elements, along with the nodes, form the mesh, which can be refined to provide any level of accuracy desired.

1.1.3 Finite Volume Method

The finite volume method is a discretizations method for the approximation of single or a system of differential equation. The Finite Volume method has been extensively used in several engineering fields such as fluid mechanics, heat and mass transfer or petroleum engineering. Some of the most features of finite volume method are similar to that element method. As in the finite element method a mesh is constructed, which consists in a partition of the domain where the space variable lives. The elements of the mesh are called control volumes. The integration of the PDE over each control volume results in a balance equation. The set of balance equations is then discretized with respect to a set of discrete unknowns. The main issue is the discretization of the fluxes at the boundaries of each control volume: in order for the FVM to be efficient, the numerical fluxes are generally *conservative*, *i.e.* the flux entering a control volume from its neighbour must be the opposite of the one entering the neighbour from the control volume, *consistent i.e.* the numerical flux of a regular function interpolation tends to the continuous flux as the mesh size vanishes.

1.1.4 Method of Weighted Residual

The methods of weighted residuals are the approximate methods which determine the solution of the differential equation in the form of functions which are closed in some sense to the exact solution. Consider a differential equation

$$\ell(u) = 0 \tag{1.1}$$

With initial condition, $I(u) = 0$, and boundary condition, $S(u) = 0$. The solution of differential equation, $U(x)$ is approximated by a finite series of functions $\phi_k(x)$ as follows:

$$U(x) = U_0(x) + \sum_{k=1}^{N} a_k \, \phi_k(x) \tag{1.2}$$

where $\phi_k(x)$ are the basis or trial functions, a_k are the coefficients to be determined that satisfy the differential equation, and *N* are the number of functions. The form of $U_0(x)$ is chosen to satisfy the boundary and the initial conditions exactly. There is another approach in which exact solutions of the differential equation are known and these are added together to satisfy the boundary conditions approximately. It is also possible to formulate a method in which the differential equation and the boundary conditions are satisfied approximately.

In general, the approximate solution does not satisfy the partial differential equation exactly, and substituting its value results in a residual, *R*,

$$R(x, a_1, a_2, \ldots, a_N) = \ell(U(x)) \tag{1.3}$$

which in turn is minimized in some sense. For a given N , $a_k\text{'}s$ are chosen by requiring that an integration of the weighted residual over the domain is zero. Thus

$$\langle W_k(x), R \rangle = 0. \tag{1.4}$$

By letting $k = 1,2,...N$ a system of equations involving only $a_k{}'s$ is obtained. For unsteady partial differential equation this would be a system of ordinary differential equations, for steady problems a system of algebraic equations obtained. Different choices of $W_k(x)$ give rise to the different methods within the class. Some of these methods are:

1.1.4.1 Galerkin Method

One of the most important weighted residual methods was invented by the Russian mathematician Boris Grigoryevich Galerkin. In the Galerkin method the weighting functions are chosen to be

$$W_k(x) = \phi_k(x) \tag{1.5}$$

i.e., the weighting functions are from the same family as the trial function in equation (1.2). Thus the residual becomes orthogonal to the space spanned by the trial functions.

In traditional Galerkin method each of the trial functions should satisfy the boundary condition but in spectral Tau method the trail functions need not satisfy the boundary condition instead, a supplementary set of equations is used to apply the boundary condition. A generalization of Galerkin method is Petrov-Galerkin method, in which, the weighting functions are different from trial functions.

1.1.4.2 Sub-domain Method

This method can be considered a modification of the collocation method. The idea is to force the weighted residual to zero not just at fixed points in the domain, but over various subsections of the domain. To accomplish this, the weight functions are set to unity, and the integral over the entire domain is broken into a number of sub domains sufficient to evaluate all unknown parameters.

1.1.4.3 Least Square Method

The basic idea of Least-Square is that the residual is minimized in a certain norm. The inner product of the governing equations is constructed, which are then differentiated with respect to the nodal values of the variables. A general Least-Square formulation is the following minimization problem:

$$S = \min \int_X \left| R(x, a_1, a_2, ..., a_N) \right|^2 dx \tag{1.6}$$

In order to achieve a minimum of this scalar function, the derivatives of S with respect to all unknown parameters must be zero. That is,

$$\frac{\partial S}{\partial a_k} = 2 \int_X R(x) \frac{\partial R}{\partial a_k} dx = 0 \tag{1.7}$$

or

$$\int_X R(x) \frac{\partial R}{\partial a_k} dx = 0 .$$

1.1.5 Collocation Method

Collocation method was developed to seek numerical solution of an initial or boundary value problem in form of linear combination of coordinate function with linear coefficients. In this method, we approximate a function by passing a polynomial through values of the function at selected points. The selected points are known as collocation points. Consider a differential equation

$$f(x,u,u_x)=0 \tag{1.8}$$

To be solved by collocation method .Let the function is defined in the domain [a,b] with boundary conditions given by

$$u(a)=g_0,\ u(b)=g_1 \tag{1.9}$$

The method begins with a proper choice of basic functions $\{\varphi_1,\phi_2,.....,\phi_n\}$ and a set of points $a=x_1<x_2<,......<x_n=b$ called nodes or collocation points in the domain $[a,b]$.

The approximate solution can be written in the form

$$U=\sum_{i=1}^{n} c_i\phi_i(x) \tag{1.10}$$

Here n can be thought of as the quality parameter, as n increases, the error in the approximation must reduces. This method requires that this approximation satisfies the given differential equation at each of the nodes and also satisfies the boundary conditions.

For this, the residual must be zero at selected points .the residual is defined as

$$r(x,u,u_x)=f(x,u,u_x)-f(x,U,U_x) \tag{1.11}$$

where U is the approximate solution. The choice of $\phi_i's$ can vary depending on the problem.

In the last few years another numerical technique has been increasingly used to solve mathematical models in engineering research, the B-spline Collocation Method. The B-spline Collocation Method has a few distinct advantages over the Finite Element and Finite Difference Methods. The advantage over the Finite Difference Method is that the B-spline Collocation Method provides a piecewise-continuous, closed form solution. An advantage over the Finite Element Method is that the B-spline collocation method procedure is simpler and easy to apply many problems involving differential equations.

Our aim is to explore the implementation of collocation method using B-spline basis function to solve initial and boundary value problems. We have used the collocation method with B-spline basis functions of third, fourth and fifth degree to find numerical solutions of some linear and non-linear equations. This chapter provides description of concept of spline and B-spline basis functions followed by the methodology adopted to solve the initial and boundary value problems using B-spline collocation method.

The term ‘spline’ is derived from the flexible device used by shipbuilder & draftsmen to draw a curve through pre-assigned points (knots) in such a way that not only the curve is continuous but also its slope and curvature are continuous functions. Draftsman attach the wooden or metal strip with weights called ducks, which can be adjusted to keep the strip in required shape. So weights are attached with the strip to keep it in the required shape.

1.2 Idea of Spline

In order to resolve the problem of working with higher degree polynomials the idea of piecewise polynomial come into existence .Instead of using polynomial for the entire domain ,the function can be approximated by several polynomials defined over the sub-domains.

A polynomial which is presented over a certain domain by means of several polynomials defined over its sub-domains called a piecewise polynomial. The piecewise polynomial approximation allows us to construct highly accurate approximations, but because some approximation functions are not smooth at the point connecting separate piecewise polynomial approximation. Sometimes, while the polynomial is continuous, it may not be continuously differentiable on the interval of approximation and the graph of the interpolant may not be smooth. Splines are an attempt to solve this problem.

So the basic idea of splines are to construct a piecewise polynomial approximation that not only interpolate the given data or function values but it is also smooth i.e. it must be continuously differentiable to some degree.

A spline is sufficiently smooth piecewise polynomial function.

Let us consider a uniform partition $x_0 < x_1 < ... < x_n$ the domain [a, b] with $x_0 = a$ and $x_n = b$ .The abscissas x_i are called the knots. A function $s(x)$ is called the spline of degree k if it is a k^{th} degree polynomial $p(x)$ in each of that interval $[x_i, x_{i+!}], i = 0,.......n-1$ with the property the $p(x)$ and its first $(k-1)$ derivative are continuous in the domain $[x_0, x_n]$.

Thus a spline $s(x)$ on domain $[x_0, x_n]$ can be defined as

$$s(x) = \sum_{i=1}^{n} p_i(x) \tag{1.12}$$

Here $p(x)$ is a k^{th} degree polynomial in each partition.

Since each segment (partition) is defined by k^{th} degree polynomial, the number of coefficients in each segment is $(k+1)$ and there are n segments.

Thus total number of coefficients becomes $n(k+1)$. So therefore to define a spline,

Number of required equations= $n(k+1)$ (1.13)

Using the continuity property of splines we have for a k^{th} degree spline the polynomial $p(x)$ with its $(k-1)$ derivatives is continuous at all interior knots, we can write

$$\begin{array}{cc}
0 & p(x_i) = p(x_{i+1}) \\
1 & p'(x_i) = p'(x_{i+1}) \\
: & \\
: & \\
k-1 & p^{k-1}(x_i) = p^{k-1}(x_{i+1})
\end{array}$$

where $i = 1,2,.......,n-1$. (i.e. internal knots).

As number of internal knots is $(n-1)$. So each of above relation provides $(n-1)$ equations. Thus, the total number of conditions is

$$(n-1)\{1+(k-1)\} = (n-1)k \tag{1.14}$$

Additional conditions

$$n(k+1)-(n-1)k=(n+k) \tag{1.15}$$

These additional conditions can be obtained from the boundary conditions.

1.3 B-Spline

“B-spline is a spline function that has minimal support with respect to given degree, smoothness and domain partition”

The first reference to the world B-spline function (‘B’ refers to basis) in the field of mathematics was given by Schonberg in 1946, who described it as a smooth piecewise polynomial approximation and is short for basis spline. A B –spline is defined as a spline function that has minimal support with respect to a given degree, smoothness, and domain partition.

The underlying core of the B-spline is its basis function. The defining feature of the basis function is knot sequence x_i. Let X be a set of N+1 non decreasing real numbers $x_0 \le x_1 \le x_2 \le .... \le x_{N-1} \le x_N$. Here $x_i's$ are called knots , the set X is the knot sequence which represents the active area of real numbers line that defines the B-spline basis ,and the half –open interval $[x_i, x_{i+1})$ the i^{th} knot span. If the knots are equally spaced ($i.e., x_{i+1} - x_i$ is a constant for $0 \le i \le N-1$), the knots vectors or the knot sequence is said to be uniform; otherwise, it is called non-uniform. Each B-spline function of degree k covers $k+1$ knots or k intervals.

In early 1970’s, a recurrence relation was independently established by Cox and Boor for the purpose of computing B-spline basis function. By applying the Leibniz’ theorem, Boor was able to drive the following formula for m^{th} B-spline basis function of k^{th} degree in a recursive manner as follow:

$$B_{m,k}(x) = V_{m,k}B_{m,k-1}(x) + (1 - V_{m+1,k})B_{m+1,k-1}(x) \tag{1.16}$$

where,

$$V_{m,k} = \left(\frac{x - x_m}{x_{m+k} - x_m} \right)$$

This formula is known as Cox de- Boor recursion formula. Here $B_{m,k}(x)$ define a m^{th} B-spline basis function of degree k,, $\{x_i\}$ is non –decreasing set of real numbers also called as the knot sequence and x is a parameter variable.

The recurrence relation starts with the first degree B-splines and builds the functions of successively higher orders. For degree $k \geq 1$ basis function $B_{m,k}(x)$ is a linear combination of two $(k-1)^{th}$ degree basis function.

1.3.1 Zero Degree B-Spline

For degree $k=0$, the basis function is just a step function. Thus, the zero degree B-spline is one of the simplest B-spline basis function and is given as

$$B_{m,0} = \begin{cases} 1, & x \in [x_m, x_{m+1}) \\ 0, & otherwise \end{cases} \tag{1.17}$$

Thus, a zero degree B-spline is equal to zero at all points expect on the half open interval [x_m ,x_{m+1}) and has the appearance as follow

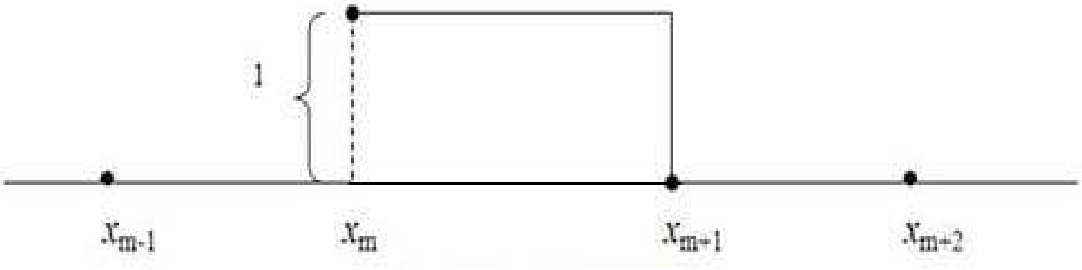

Figure 1.1: Pictorial representation of Zero Degree B-spline

1.3.2 First Degree B-Spline

The expression for the first degree B- spline, also called as linear B-spline can be obtained using the Cox and Boor recursion formula given by (1.16)

Put k=1 in (1.16) and use the definition of zero degree B –spline.The formula of the first degree B-spline basis function can be given as

$$B_{m,1} = \begin{cases} \dfrac{x - x_m}{x_{m+1} - x_m} & x \in [x_m, x_{m+1}) \\ \dfrac{x_{m+2} - x}{x_{m+2} - x_{m+1}} & x \in [x_{m+1}, x_{m+2}) \\ 0 & otherwise \end{cases} \tag{1.17}$$

The first degree B-spline is like a HAT or tent function which is non=zero for two knot spans $[x_m, x_{m+1})$ and $[x_{m+1}, x_{m+2})$ and can represented as

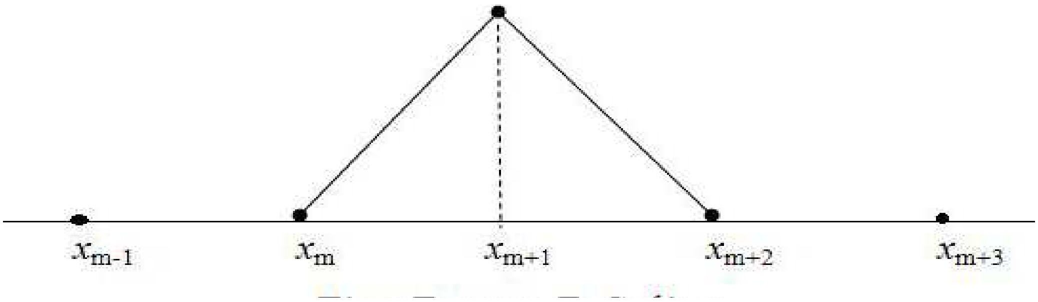

Figure 1.2: Pictorial representation of First Degree B-spline

1.3.3 Second Degree (Quadratic) B-Spline

The formula for the second degree B-spline also called as quadratic B-spline can be obtained by using the formula of linear B-spline basis function (1.17) and de-Boor recursion formula for k=2

The formula for the second degree B-spline can be given as

$$B_{m.2}(x)=\begin{cases} \dfrac{(x-x_m)^2}{(x_{m+2}-x_m)(x_{m+1}-x_m)} & x\in[x_m,x_{m+1}) \\ \dfrac{(x-x_m)(x_{m+2}-x)}{(x_{m+2}-x_m)(x_{m+2}-x_{m+1})}+\dfrac{(x_{m+2}-x)(x-x_{m+1})}{(x_{m+3}-x_{m+1})(x_{m+2}-x_{m+1})} & x\in[x_{m+1},x_{m+2}) \\ \dfrac{(x_{m+3}-x)^2}{(x_{m+3}-x_{m+1})(x_{m+3}-x_{m+2})} & x\in[x_{m+2},x_{m+3}) \\ 0 & otherwise \end{cases} \qquad (1.18)$$

The second degree B-spline basis function is non-zero between three knot spans and can be represented as

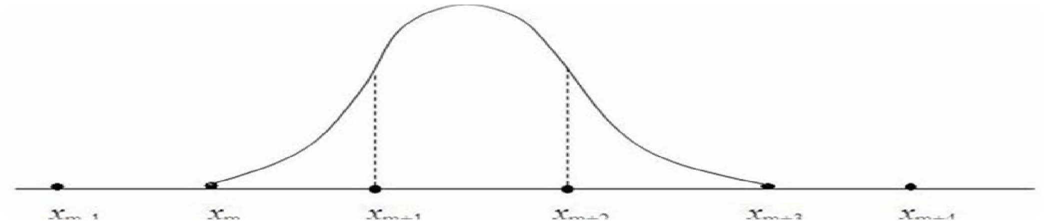

Figure 1.3: Pictorial representation of Second Degree B-spline

1.4 Collocation Method Using B –Spline Basis Function

The collocation method together with B-spline approximation represents an economical alternative, since it is based on evaluating the accuracy of a differential equation at a finite set of collocation points .the issue that effect the effectiveness and accuracy of B-spline collocation method for solving differential equations include, which points to use for collocation, what degree of B-spline to use and what level of continuity to maintain.

Let us consider

$$a = x_0 < x_1 < ....,x_{n-1} < x_n = b$$

as a uniform partition of the solution domain $[a,b]$ by the knots x_m with step-length $h = x_{m+1} - x_m$ where $m = 0,1,.....,n-1$.

Now to find the solution of differential equation using collocation method with B-spline basis function, the approximate solution $U(x)$ can be assumed as a linear combination of basis functions as

$$U(x) = \sum_{j=m-k+2}^{m+k-2} c_j B_j(x) \qquad (1.19)$$

Where, k is the degree of the B-spline, m is the number of nodes and c_m are the unknown constants to be determined from the boundary conditions and collocation from of the differential equation.

Let us now derive the approximate formula with the basis function of third, fourth and fifth degree B-spline.

1.4.1 Third Degree B-Spline

The third degree B-spline called as cubic B-spline basis function is given by formula

$$B_{m,3} = \frac{1}{h^3}\begin{cases} (x-x_{m-2})^3 & x\in[x_{m-2},x_{m-1}) \\ (x-x_{m-2})^3-4(x-x_{m-1})^3 & x\in[x_{m-1},x_m) \\ (x_{m+2}-x)^3-4(x_{m+1}-x)^3 & x\in[x_m,x_{m+1}) \\ (x_{m+2}-x)^3 & x\in[x_{m+1},x_{m+2}) \\ 0 & otherwise \end{cases} \tag{1.20}$$

This definition of cubic B-spline basis functions is given with *m x* as the middle knot and equal number of knots on the two sides. The third degree B-spline is non-zero on four knot spans and is represented as

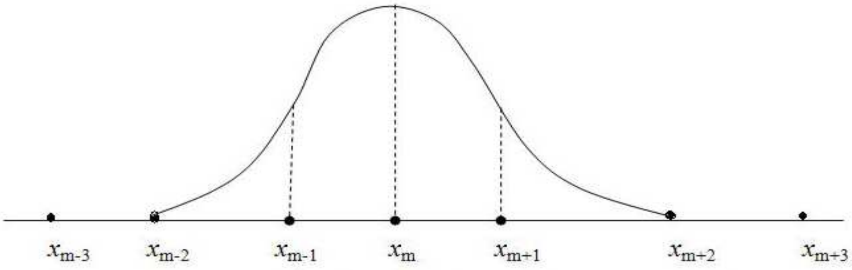

Figure 1.5: Pictorial representation of Third Degree B-spline

From the definition given by (1.20), the value of $B_{m,3}(x)$ at the nodal points can be obtained. On differentiating with respect to x we can obtain the value of first and second derivatives of $B_{m,3}(x)$ .the value of $B_{m,3}(x)$ and its first derivatives at the nodal points can be tabulated as in Table 1.1.

Table 1.1: value of $B_{m,3}(x)$ and its first derivatives at the nodal points

	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
$B_{m,3}(x)$	0	1	4	1	0
$B'_{m,3}(x)$	0	$\frac{3}{h}$	0	$\frac{-3}{h}$	0
$B''_{m,3}(x)$	0	$\frac{6}{h^2}$	$\frac{-12}{h^2}$	$\frac{6}{h^2}$	0

Now substitute k=3 in (1.20), we get

$$U(x)=\sum_{j=m-3+2}^{m+3-2} c_j B_j(x)$$

So the approximate solution can be written as

$$U(x)=\sum_{j=m-1}^{m+1} c_j B_j(x) \tag{1.21}$$

Without loss of generality, equation (1.21) can be expressed as $U(x_m)=c_{m-1}B_{m-1}(x_m)+c_m B_m(x_m)+c_{m+1}B_{m+1}(x_m)$

Or

$$U(x_m)=c_{m-1}B_m(x_{m+1})+c_m B_m(x_m)+c_{m+1}B_m(x_{m-1}) \tag{1.22}$$

As we are evaluating for cubic spline, so (1.22) can be written as

$$U(x_m)=c_{m-1}B_{m,3}(x_{m+1})+c_m B_{m,3}(x_m)+c_{m+1}B_{m,3}(x_{m-1}) \tag{1.23}$$

From here

$$U^{'}(x_m) = c_{m-1}B^{'}_{m,3}(x_{m+1}) + c_m B^{'}_{m,3}(x_m) + c_{m+1}B^{'}_{m,3}(x_{m-1})$$

$$U^{''}(x_m,.t) = c_{m-1}B^{''}_{m,3}(x_{m+1}) + c_m B^{''}_{m,3}(x_m) + c_{m+1}B^{''}_{m,3}(x_{m-1})$$

On substituting values of $B_{m,3}(x)$ at the knots from Table 1.1, we get

$$\begin{aligned} U(x_m) &= c_{m-1} + 4c_m + c_{m+1} \\ hU^{'}(x_m) &= 3(c_{m+1} - c_{m-1}) \\ h^2U^{''}(x_m,t) &= 6(c_{m-1} - 2c_m + c_{m+1}) \end{aligned} \tag{1.24}$$

1.4.1 Fourth Degree B-Spline

The B-spline basis function of fourth degree also called as quartic B-spline is given by

$$B_{m,4}(x) = \frac{1}{h^4}\begin{cases} (x-x_{m-2})^4 & x\in[x_{m-2},x_{m-1}) \\ (x-x_{m-2})^4 - 5(x-x_{m-1})^4 & x\in[x_{m-1},x_m) \\ (x-x_{m-2})^4 - 5(x-x_{m-1})^4 + 10(x-x_m)^4 & x\in[x_m,x_{m+1}) \\ (x_{m+3}-x)^4 - 5(x_{m+2}-x)^4 & x\in[x_{m+1},x_{m+2}) \\ (x_{m+3}-x)^4 & x\in[x_{m+2},x_{m+3}) \\ 0 & otherwise \end{cases} \tag{1.25}$$

This basis function is non zero on five knot spans. From definition given by (1.25) the value of $B_{m,4}(x)$ at the nodal points can be obtained .on differentiating with respect to x we can obtain the value of its three derivatives in similar way. The value of $B_{m,4}(x)$ and its derivatives at the nodal points may be tabulated as in the table 1.2.

Table 1.2: values of $B_{m,4}(x)$ and its derivatives at nodal points

	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}
$B_{m,4}(x)$	0	1	11	11	1	0
$B'_{m,4}(x)$	0	$\frac{4}{h}$	$\frac{12}{h}$	$\frac{-12}{h}$	$\frac{-4}{h}$	0
$B''_{m,4}(x)$	0	$\frac{12}{h^2}$	$\frac{-12}{h^2}$	$\frac{-12}{h^2}$	$\frac{12}{h^2}$	0
$B'''_{m,4}(x)$	0	$\frac{24}{h^3}$	$\frac{-72}{h^3}$	$\frac{72}{h^3}$	$\frac{-24}{h^3}$	0

By substituting k=4 in (1.19), we get

$$U(x) = \sum_{j=m-4+2}^{m+4-2} c_j B_j(x)$$

So the approximate solution can be written as

$$U(x) = \sum_{j=m-2}^{m+2} c_j B_j(x) \qquad (1.26)$$

Equation (1.26) can be expressed as

$$U(x_m) = c_{m-2}B_{m-2}(x_m) + c_{m-1}B_{m-1}(x_m) + c_m B_m(x_m) + c_{m+1}B_{m+1}(x_m) + c_{m+2}B_{m+2}(x_m)$$

$$U(x_m) = c_{m-2}B_m(x_{m+2}) + c_{m-1}B_m(x_{m+1}) + c_m B_m(x_m) + c_{m+1}B_m(x_{m-1}) + c_{m+2}B_m(x_{m-2})$$

As we are evaluating for quartic B-spline, so can be written as

$$U(x_m) = c_{m-2}B_{m,4}(x_{m+2}) + c_{m-1}B_{m,4}(x_{m+1}) + c_m B_{m,4}(x_m) + c_{m+1}B_{m,4}(x_{m-1}) + c_{m+2}B_{m,4}(x_{m-2}) \qquad (1.27)$$

So

$$U'(x_m) = c_{m-2}B'_{m,4}(x_{m+2}) + c_{m-1}B'_{m,4}(x_{m+1}) + c_m B'_{m,4}(x_m) + c_{m+1}B'_{m,4}(x_{m-1}) + c_{m+2}B'_{m,4}(x_{m-2})$$

On substituting the values of $B_{m,4}(x)$ at the nodes from Table 1.2 we get,

$$\begin{aligned} U(x_m) &= c_{m-2} + 11c_{m-1} + 11c_m + c_{m+1} \\ hU'(x_m) &= 4(-c_{m-2} - 3c_{m-1} + 3c_m + c_{m+1} \\ h^2U''(x_m) &= 12(c_{m-2} - c_{m-1} - c_m + c_{m+1}) \\ h^3U'''(x_m, t) &= 24(-c_{m-2} + 3c_{m-1} - 3c_m + c_{m+1}) \end{aligned} \qquad (1.28)$$

1.4.2 Fifth Degree B-Spline

The fifth degree B-spline, also called quintic B-spline basis function is given by formula

$$B_{m,5}(x) = \frac{1}{h^5}\begin{cases} (x - x_{m-3})^5 & x \in [x_{m-3}, x_{m-2}) \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 & x \in [x_{m-2}, x_{m-1}) \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 & x \in [x_{m-1}, x_m) \\ (x_{m+3} - x)^5 - 6(x_{m+2} - x) + 15(x_{m+1} - x)^5 & x \in [x_m, x_{m+1}) \\ (x_{m+3} - x)^5 - 6(x_{m+2} - x)^5 & x \in [x_{m+1}, x_{m+2}) \\ (x_{m+3} - x)^5 & x \in [x_{m+2}, x_{m+3}) \\ 0 & otherwise \end{cases} \qquad (1.29)$$

This basis function is non zero on six knot spans. From definition given by (1.29) the value of $B_{m,5}(x)$ at the nodal points can be obtained and on differentiating with respect to x the value of its four derivatives can be obtained in similar way. The value of $B_{m,5}(x)$ and its derivatives at the nodal points may be tabulated as in the table 1.3.

Table 1.3: Value of $B_{m,5}(x)$ for quintic B-spline and its derivatives at the nodal points.

	x_{m-3}	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}
$B_{m,5}(x)$	0	1	26	66	26	1	0
$B'_{m,5}(x)$	0	$\frac{5}{h}$	$\frac{50}{h}$	0	$\frac{-50}{h}$	$\frac{-5}{h}$	0
$B''_{m,5}(x)$	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$\frac{-120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0
$B'''_{m,5}(x)$	0	$\frac{60}{h^3}$	$\frac{-120}{h^3}$	0	$\frac{120}{h^3}$	$\frac{-60}{h^3}$	0
$B^{iv}_{m,5}(x)$	0	$\frac{120}{h^4}$	$\frac{-480}{h^4}$	$\frac{720}{h^4}$	$\frac{-480}{h^4}$	$\frac{120}{h^4}$	0

Substitute k=5 in (1.19)

$$U(x) = \sum_{j=m-5+2}^{m+5-2} c_j B_j(x)$$

We get the approximate solution

$$U(x) = \sum_{j=m-3}^{m+3} c_j B_j(x) \tag{1.31}$$

Equation (1.31) can be expressed as

$$U(x_m) = c_{m-3}B_{m-3}(x_m) + c_{m-2}B_{m-2}(x_m) + c_{m-1}B_{m-1}(x_m) + c_m B_m(x_m) + c_{m+1}B_{m+1}(x_m) + c_{m+2}B_{m+2}(x_m) + c_{m+3}B_{m+3}(x_m)$$

Or

$$U(x_m) = c_{m-3}B_m(x_{m+3}) + c_{m-2}B_m(x_{m+2}) + c_{m-1}B_m(x_{m+1}) + c_m B_m(x_m) + c_{m+1}B_m(x_{m-1}) + c_{m+2}B_m(x_{m-2}) + c_{m+3}B_m(x_{m-3})$$

As we are evaluating for quartic B-spline, so can be written as

$$\begin{aligned} U(x_m) = {} & c_{m-3}B_{m,5}(x_{m+3}) + c_{m-2}B_{m,5}(x_{m+2}) + c_{m-1}B_{m,5}(x_{m+1}) + c_m B_{m,5}(x_m) + c_{m+1}B_{m,5}(x_{m-1}) \\ & + c_{m+2}B_{m,5}(x_{m-2}) + c_{m+3}B_{m,5}(x_{m-3}) \end{aligned} \tag{1.32}$$

On substituting the values of $B_{m,5}(x)$ at the nodes from Table 1.3 we get

$$\begin{aligned} & U(x_m) = c_{m-2} + 26c_{m-1} + 66c_m + 26c_{m+1} + c_{m+2} \\ & hU(x_m) = 5(-c_{m-2} - 10c_{m-1} + 10c_{m+1} + c_{m+2} \\ & h^2U^{''}(x_m) = 20(c_{m-2} + 2c_{m-1} - 6c_m + 2c_{m+1} + c_{m+2} \\ & h^3U^{'''}(x_m) = 60(-c_{m-2} + 2c_{m-1} - 2c_{m+1} + c_{m+2}) \\ & h^4U^{iv}(x_m) = 120(c_{m-2} - 4c_{m-1} + 6c_m - 4c_{m+1} + c_{m+2}) \end{aligned} \tag{1.33}$$

1.5 Properties of B-Spline Basis Function

Some of the important properties of the B-spline basis functions are as follows:

1. $B_{m,k}(x)$ is a non-zero polynomial on $[x_m, x_{m+k+1})$ for degree $k \geq 0$.

2. On any span $[x,_m\ x_{m+1})$ at most k +1 basis functions of degree k are non-zero $B_{m-k,k}(x), B_{m-k+1,k}(x), B_{m-k+2,k}(x), ..., and\ B_{m.k}(x)$.

3. Non negativity

For all m, k and $x, B_{m,k}(x)$ is non-negative in the interval $[x_m, x_{m+1})$. The closed interval is called the support of $B_{m,k}(x)$.

4. Local knots

The m^{th} B-spline $B_{m,k}(x)$ depends only on the knots $x_m, x_{m+1}, x_{m+2}, ...,x_{m+k+1}$.

5. Local Support

If x is outside the interval $[x_m, x_{m+k+1})$ then $B_{m,k}(x) = 0$

Local support property indicates that each segment of a B-spline curve is influenced by only *k* control points or each control point affects only *k* curve segments.

1.6 Organization of Thesis

In this thesis an attempt has been made to solve some differential equations by using B-spline collocation methods. The chapter wise summary of the thesis is as follows.

In chapter 2, we consider a second order singularly perturbed boundary value problem

$$Ly(x) = \varepsilon y^{''}(x) + p(x)y^{'}(x) - q(x)y(x) = r(x) \quad , \quad x \in [a,b]$$

where ε is a small positive parameter $(0 < \in < 1)$, p(x) ,q(x) and r(x) are sufficiently smooth functions. The boundary value problems (2.1) under these assumptions have unique solution [1]. In general, as $\in$ tends to zero, the solution $y(x)$ exhibits an exponential boundary layer at the left end of the interval; that is, the domain of the differential equation contains a narrow region where the solution derivatives are extremely large. Regions like this frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in fluid mechanics, edge layers in solid mechanics, and skin layers in electrical applications. There are many physical situations in which the sharp changes occur inside the domain of interest, and the narrow regions across which these changes take place are usually referred to as shock layers in fluid and solid mechanics, transition points in quantum mechanics, and Strokes lines and surfaces in mathematics. These rapid changes cannot be handled by slow scales, but they can be handled by fast or magnified or stretched scales.

In this chapter, a numerical technique for a system of singularly perturbed boundary value problems using cubic-B-spline functions is derived. Simplicity of the adaptation of B-splines and obtaining acceptable solutions can be noted as advantages of given numerical methods. The results obtained using the proposed method are good and maximum absolute errors are negligible.

In chapter 3, we consider the second order two point mixed boundary value problem

$$\begin{aligned} &y''(x) = F(x, y(x), y'(x)), \quad x \in [a,b], \\ &\alpha_1 y(a) + \beta_1 y'(a) = a_1, \\ &\alpha_2 y(b) + \beta_2 y'(b) = b_1 \end{aligned}$$

For simplicity such problems arise in many fields in engineering and science, such as heat transfer, deflection of plate, diffusion. So the problem such type has great importance in many real life fields.

In this chapter, a numerical scheme is proposed for the numerical solutions of the second order two point mixed boundary value problems based on quintic-B-spline functions. Two test problems have been considered to test the accuracy of the proposed method. The maximum absolute errors are decreasing as we increase the number of sub-intervals. This shows that the method is stable.

CHAPTER 2

Cubic B-Spline Collocation Method for the Numerical Solution of Singularly Perturbed Two-Point Boundary Value Problem

2.1. Introduction

We consider a second order singularly perturbed boundary value problem:

$$Ly(x) = \varepsilon y''(x) + p(x)y'(x) - q(x)y(x) = r(x) \quad , \quad x \in [a,b] \tag{2.1a}$$

with boundary conditions

$$\begin{aligned} y(a) &= \alpha \\ y(b) &= \beta \end{aligned} \tag{2.1b}$$

where ε is a small positive parameter $(0 < \varepsilon < 1)$, p(x) ,q(x) and r(x) are sufficiently smooth functions. The boundary value problems (2.1a) under these assumptions have unique solution [1]. In general, as $\in$ tends to zero, the solution $y(x)$ exhibits an exponential boundary layer at the left end of the interval; that is, the domain of the differential equation contains a narrow region where the solution derivatives are extremely large. Regions like this frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in fluid mechanics, edge layers in solid mechanics, and skin layers in electrical applications. There are many physical situations in which the sharp changes occur inside the domain of interest, and the narrow regions across which these changes take place are usually referred to as shock layers in fluid and solid mechanics, transition points in quantum mechanics, and Strokes lines and surfaces in mathematics. These rapid changes cannot be handled by slow scales, but they can be handled by fast or magnified or stretched scales. For the analysis of this type of problems the readers can refer to the books by Bender and Orszag [2], O'Malley [3, 4], Nayfeh [5,6], Doolan [7], Miller [8] and Roos [9].

Due to the variation in the width of the layer with respect to the small perturbation parameter$\in$, several difficulties are experienced in solving the singular perturbation problems using standard numerical methods with uniform mesh. Then the mesh needs to be refined substantially to grasp the solution within the boundary layers. To avoid this, a piecewise-uniform mesh was first constructed by Shishkin [10]. Miller et al. [8] discussed the fitted numerical method with piecewise-uniform mesh for singularly perturbed boundary value problems. Pearson [11] was perhaps the first one who tried to solve SPPs (2.1a) numerically by taking net adjustments in finite difference method. This idea was further developed by Abrahamsson et al. [12] in their study of difference methods for a general class of SPPs. Their aim is to devise numerical schemes with constant mesh spacing $h \geq \in$ to yield accurate solutions in the outer region. However, they were able to show in general, that the accuracy of the scheme cannot be better than $O(h)$ in the interior. Kellogg and Tsan [13] and Berger et al. [14] have also analyzed some difference approximations of $O(h)$ and $O(h^2)$ for the SPPs (2.1a). Doolan et al. [7] have presented various exponentially fitted finite difference schemes for both initial and boundary value problems that are uniformly convergent in $\in$. Schatz and Wahlbin [15] used finite element techniques to solve such problems. Nitsche and Schatz [16] give the interior estimates for Ritz–Galerkin methods. Roos and Uzelac [17] used streamline-diffusion finite element method on a Shishkin mesh and have shown the method is convergent independently of the perturbation parameter.

The rest of the paper is organized as follows: In Section 2, we discuss preliminary results of cubic B-spline. In Section 3, the derivation of the B-spline collocation method has been discussed [18]. In Section 4, numerical results and comparison of approximate solutions based on B-spline collocation methods have been presented. Finally, Section 5 contains conclusion.

The cubic B-spline collocation method developed in this chapter has lower computational cost and its only requires solving $n+1$ linear or non-linear equations.

2.2. Preliminary Results of Cubic B-spline

Divide the interval $l=[a,b]$ into subintervals $l_i=[x_i,x_{i+1}]\,;i=0,1,......,n-1$ by the equidistant knots $x_i=a+ih$; $i=0,1,......,n-1$ where $h=\dfrac{b-a}{n}$. We define $S_3(\Pi)=\{p(t)\in C^2[0,1]\}$, such that *p(t)* reduces to cubic polynomial on each sub interval $[x_i,x_{i+1}]\ \forall i$. By the results in [17], we define the cubic B-spline by the following relationship,

$$B_{i,3}=\frac{1}{h^3}\begin{cases}(x-x_i+2h)^3 & x\in[x_{i-2},x_{i-1})\\ (x-x_i+2h)^3-4(x-x_{i1+h})^3 & x\in[x_{i-1},x_i)\\ (x_i+2h-x)^3-4(x_i+h-x)^3 & x\in[x_i,x_{i+1})\\ (x_i+2h\ -x)^3 & x\in[x_{i+1},x_{i+2})\\ 0 & otherwise\end{cases}$$

$B_i(x)$ is non negative and is locally supported on $[x_{i-2},x_{i+2}]$. Besides, it is easy to observe that $B_i(x)=B_{i+1}(x+h)(i=-1,......n+1)$ and $\sum_{i=-1}^{n+1}B_i(x)=1(x\in[a,b])$.

Furthermore, by some trivial computation, we obtain the values of $B_i^{(k)}(x)\ (i=-1,0,....,n+1 k=0,1,2)$ at the knots, which are listed in Table 2.1.

Table 2.1: Value of $B_{m,3}(x)$ and its derivatives at the nodal points

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_{i,3}(x)$	0	1	4	1	0
$B'_{\ i,3}(x)$	0	$\frac{3}{h}$	0	$\frac{-3}{h}$	0
$B''_{i,3}(x)$	0	$\frac{6}{h^2}$	$\frac{-12}{h^2}$	$\frac{6}{h^2}$	0

Given a sufficient smooth function y(x), there exists a unique quintic spline $S(x)=\sum_{i=-1}^{n+1} c_i B_i(x) \in C^2(R)$ satisfying the following interpolation condition:

$$\begin{aligned} &s(x_i)=y(x_i), && i=0,1,.....,n. \\ &s'(a)=y'(a), && s''(a)=y''(a), \\ &s'(b)=y'(b), && s''(b)=y''(b), \end{aligned} \tag{2.3}$$

Now from thae Table 2.1.,for the equidistant knots $x_i\,(j=0,1,.......,n)$,we

$$y(x_j)=s(x_j)=c_{j-1}+4c_j+c_{j+1} \tag{2.4}$$

$$y'(x_j)=s'(x_j)=\frac{3}{h}[c_{j-1}+c_{j+1}] \tag{2.5}$$

$$y''(x_j)=s''(x_j)=\frac{6}{h^3}[c_{j-1}-2c_j+c_{j+1}] \tag{2.6}$$

2.3. Cubic B-Spline Collocation Method

Consider the linear boundary value problem

$$\begin{aligned} &\varepsilon\, y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \ , \qquad x \in [a,b] \\ &y(a) = \alpha \\ &y(b) = \beta \end{aligned} \tag{2.7}$$

Let $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$ be the cubic solution of (2.7). Discretizing (2.7) at the knots $i = 0,1\,,2,.......,n$, we get

$$\varepsilon\, y''(x_i) + p(x_i)y'(x_i) + q(x_i)y(x_i) = r(x_i)$$

Now by (2.4), (2.5) and (2.6), we have

$$\frac{6\in}{h^2}[c_{j-1} - 2c_j + c_{j+1}] + p_i\frac{3}{h}[-c_{j-1} + c_{j+1}] + q_i[c_{j-1} + 4c_j + c\ \ ] = r_i\,; \forall\ i = 0,1,.......n$$

Where $p_i = p(x_i), q_i = q(x_i)$ and $r_i = r(x_i)$ be the values of $p(x), q(x)$ and $r(x)$ at the knots $x_i\,(i = 0,1,.....,n)$

$$[6\in c_{i-1} - 12c_i + 6c_{i+1}] + p_i h[-3c_{i-1} + 3c_{i+1}] + q_i h^2[c_{i-1} + 4c_i + c_{i+1}] = h^2 r_i$$

$$[6\in -3hp_i + q_i h^2]c_{i-1} + [-12\in +4q_i h^2]c_i + [6\in +3p_i h + q_i h^2]c_{i+1} = r_i h^2 \tag{2.8}$$

Now from the left boundary condition $y(a) = \alpha$, we get

$$s(a) = \alpha$$

$$c_{-1} + 4c_0 + c_1 = \alpha \tag{2.9}$$

Similarly from second boundary condition, we get

$$s(b)=\beta$$

$$c_{n-1}+4c_{n}+c_{n+1}=\beta \qquad (2.10)$$

(2.8), (2.9) and (2.10) leads to a $(n+30\times(n+3)$ system of equations with (n+3) unknowns $c=(c_{-1},c_{0},c_{1}.......c_{n+1})$.

We eliminate c_{-1} from (2.8) and (2.9), we have

$$[-36\in+12hp_{0}]c_{0}+(6hp_{0})c_{1}=r_{0}h^{2}-\alpha(6\in-3hp_{0}+q_{0}h^{2}) \qquad (2.11)$$

Similarly on eliminating c_{n+1} from (2.8) and (2.10) we get,

$$(-6p_{n}h)c_{n-1}+(-36\in-12p_{n}h)c_{n}=h^{2}r_{n}-\beta(6\in-3hp_{n}+q_{n}h^{2}) \qquad (2.12)$$

Coupling equation (2.11) and (2.12) with the second through $(n-1)st$ equations of (2.8), we lead to a system of(n+1) equations in (n+1) unknowns.

$$Ac=d \qquad (2.13)$$

where $c=(c_{0},c_{1},......c_{n})^{t}$ are unknowns real coefficients with the right hand side

$$d=[h^{2}r_{0}-\alpha(6\in-3hp_{0}+q_{0}h^{2}),h^{2}r_{1},h^{2}r_{2},..........h^{2}r_{n-1},h^{2}r_{n}-\beta(6\in-3hp_{n}+q_{n}h^{2})]^{t}$$

and the coefficient matrix A is given by

$$A=\begin{bmatrix} -36\in+12hp_0 & 6hp_0 & 0 & & \\ 6\in-3hp_1+q_1h^2 & -12\in+4q_1h^2 & 6\in+3p_1h+q_1h^2 & & \\ \ddots & \ddots & \ddots & & \\ & 6\in-3hp_1+q_1h^2 & -12\in+4q_1h^2 & 6\in+3p_1h^2+q_1h^2 & \\ & \ddots & \ddots & \ddots & \\ & & 6\in-3hp_1+q_1h^2 & -12\in+4q_1+q_1h^2 & 6\in+3p_1h+q_1h^2 \\ & & 0 & -6p_nh & -36\in-12p_nh \end{bmatrix}_{(n+1)\times(n+1)}$$

It can easily be seen that that the matrix A is strictly diagonally dominant and hence non-singular. Since A is non –singular, we can solve the system $Ac = d$ for $c_0, c_1, c_2, \ldots\ldots c_n$ and substitute into the boundary conditions (2.9) and (2.10) to obtain c_{-1} and c_{n+1}. Hence, the method of collocation using a basis of cubic B-splines applied to problem (2.1) has a unique solution $s(x)=\sum_{i=-1}^{n+1} c_i B_i(x)$.

For the derivation of uniform lemma convergence, we use the following lemma

Lemma 2.1: The B-spline $B_{-1}, B_0, ..., B_{n+1}$ satisfy the inequality

$$\sum_{i=-1}^{n+1} |B_i(x)| \le 10, \qquad 0 \le x \le 1.$$

Theorm 5.1: let $S(x)$ be the collocation approximation from the space of cubic splines $\phi_3(\Pi)$ to the solution of the boundary value problem

$$Ly(x) = \varepsilon y''(x) + p(x)y'(x) - q(x)y(x) \qquad x \in [0,1]$$

with

$$y(0)=\alpha, y(1)=\beta$$

where ε is a small positive parameter $(0 < \in < 1)$. If $f \in C^2[0,1]$, then the parameter-uniform error estimate is given by

$$\sup \max_{0<i<N} | y(x_i) - S(x_i) | \leq CN^{-2} (In\, N)^3$$

where C is a positive constant independent of $\in$ and N.

Proof: See proof [38].

2.4. Numerical Experiments

To demonstrate the applicability of the method, we have considered three linear singular perturbation problems. These examples have been chosen because they have been widely discussed in the literature and because exact solutions are available for comparison.

Example 1: Consider the singular perturbed boundary value problem

$$\begin{gathered} -\varepsilon y'' + y = -\left(\cos^2(\pi x) + 2\varepsilon\pi^2 \cos(2\pi x)\right), \qquad 0 \leq x \leq 1, \\ y(0) = 0, \;\; y(1) = 0. \end{gathered} \tag{2.14}$$

Its exact solution is given by

$$y(x) = \frac{\exp\left(-(1-x)/\sqrt{\varepsilon}\right) + \exp\left(-x/\sqrt{\varepsilon}\right)}{1 + \exp\left(-1/\sqrt{\varepsilon}\right)} - \cos^2(\pi x) \tag{2.15}$$

The numerical results of the example are presented in the Table 2.2 and Figures 1-3 for different values of ε and subintervals *n.* The Table shows as we increase the subintervals the maximum absolute error decreases. The Figure 1 shows the comparison of the exact and numerical solutions while Figures 2 and 3 show the physical behavior of the numerical solutions and absolute errors for different values of ε respectively.

Example 2: In this example, we consider the singular perturbed boundary value problem

$$\begin{gathered} -\varepsilon y'' + (1 + x(1-x))y = f(x), \qquad 0 \le x \le 1, \\ y(0) = 0, \ \ y(1) = 0, \end{gathered} \tag{2.16}$$

where

$$f(x) = 1 + x(1-x) + \left(2\sqrt{\varepsilon} - x^2(1-x)\right)\exp\left(-(1-x)/\sqrt{\varepsilon}\right) + \left(2\sqrt{\varepsilon} - x(1-x)^2\right)\exp\left(-x/\sqrt{\varepsilon}\right)$$

The exact solution is given by

$$y(x) = 1 + (x-1)\exp\left(-x/\sqrt{\varepsilon}\right) - x\exp\left(-(1-x)/\sqrt{\varepsilon}\right) \tag{2.17}$$

The numerical results of the example are presented in the Table 2.3 and Figures 4-6 for different values of ε and subintervals *n*. The Table shows as we increase the subintervals the maximum absolute error decreases. The Figure 4 shows the comparison of the exact and numerical solutions while Figures 5 and 6 show the physical behavior of the numerical solutions and absolute errors for different values of ε respectively.

Example 3: Consider the following singular perturbation problem

$$\varepsilon y''(x) + (1+x)y'(x) - y = (1+x)^2,$$

$$y(0) = 0, y(1) = 0$$

The numerical results of the example are presented in the Table 2.4 and Figures 7-9 for different values of ε and subintervals *n*. The Table shows as we increase the subintervals the maximum absolute error decreases. The Figure 7-9 shows the physical behavior of the numerical solutions of the problem.

2.5 Conclusion

In this paper, a numerical technique for a system of singularly perturbed boundary value problems using B-spline functions is derived. Simplicity of the adaptation of B-splines and obtaining acceptable solutions can be noted as advantages of given numerical methods. The results obtained using these methods are better than those using the stated existing methods with the same number of knots and values of ε.

Table 2.2: Maximum Absolute error of Example 1 at different values of subintervals.

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$
10^{-2}	1.8E-4	2.8E-5	3.9E-6	5.2E-7	6.2E-8	8.5E-9
10^{-4}	7.3E-5	3.5E-7	8.6E-8	2.2E-8	5.4E-9	1.4E-9

Table 2.3: Maximum Absolute error of Example 2 at different values of subintervals.

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$
¼	1.1E-4	2.7E-5	6.8E-6	1.7E-6	4.3E-7	1.1E-7
1/10	2.0E-4	5.1E-5	1.3E-5	3.2E-6	7.9E-7	2.0E-7

Table 2.4: Maximum error of Example 3 at different values of subintervals.

ε	$n=64$	$n=128$	$n=256$	$n=512$
2^{-1}	1.09E-05	2.72E-06	6.81E-07	1.70E-07
2^{-4}	1.37E-03	1.14E-02	8.54E-05	2.13E-05
2^{-8}	1.81E-03	1.54E-02	2.74E-04	1.15E-04
2^{-12}	1.53E-03	1.61E-02	1.75E-04	5.73E-05
2^{-16}	1.52E-03	1.61E-02	1.70E-04	5.37E-05
2^{-20}	1.52E-03	1.61E-02	1.70E-04	5.39E-05

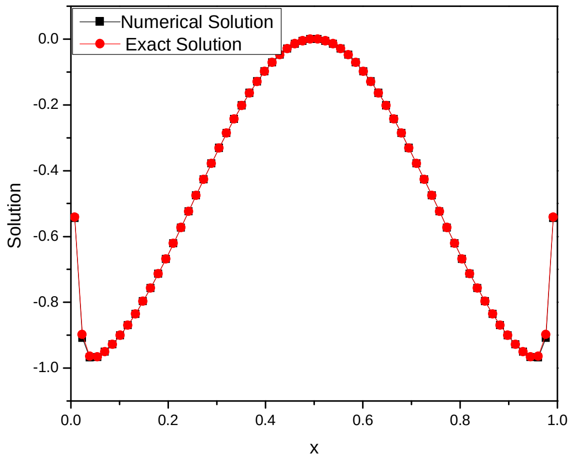

Figure 1: Comparison of exact and numerical solutions of Example for $n = 64,\ \varepsilon = 10^{-4}$.

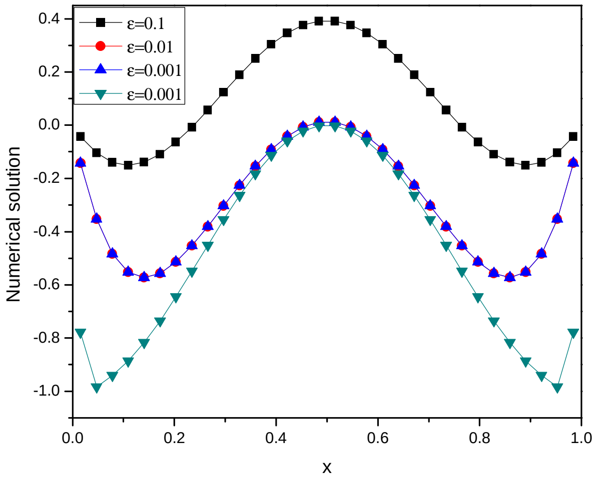

Figure 2: Numerical behavior of numerical solutions of Example 1 at different values of ε .

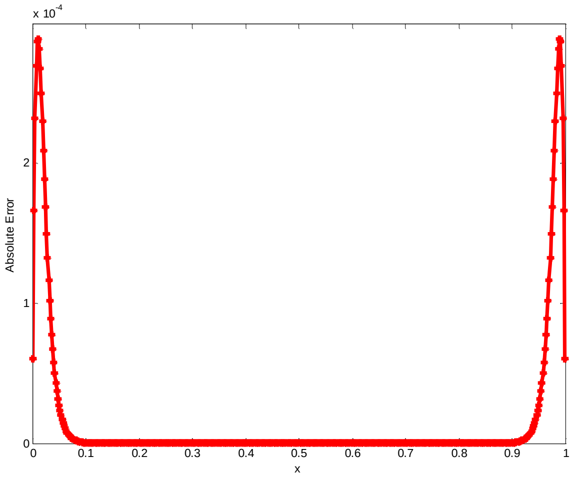

Figure 3: Behavior of absolute errors of Example 1 at $n = 256,\ \varepsilon = 10^{-4}$.

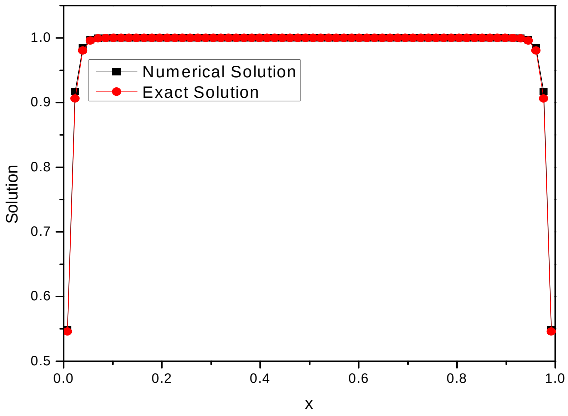

Figure 4: Comparison of exact and numerical solutions of Example 2 for $n = 32,\ \varepsilon = 10^{-4}$.

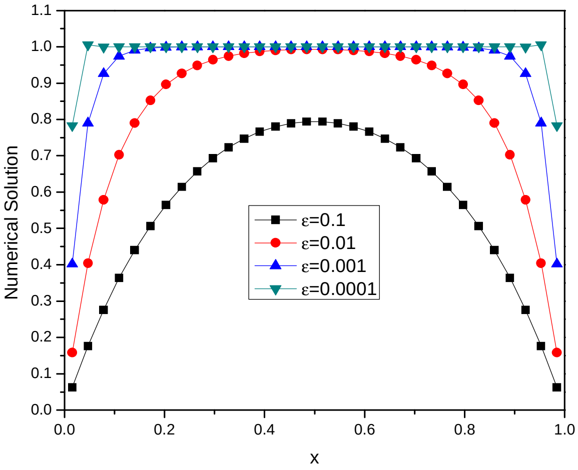

Figure 5: Numerical behavior of numerical solutions of Example 2 at different values of ε.

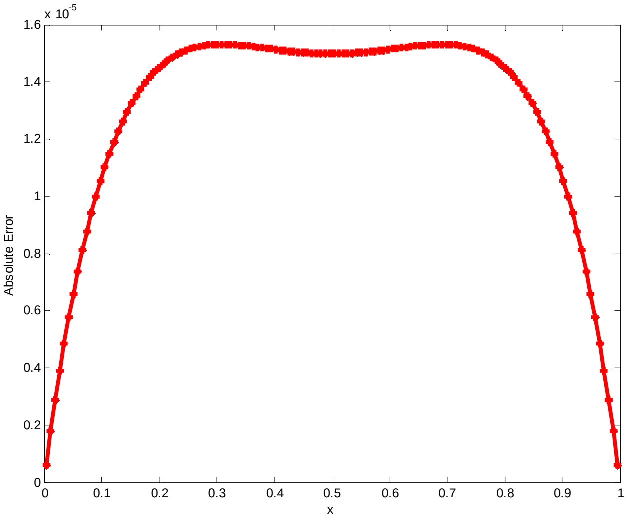

Figure 6: Behavior of absolute errors of Example 2 at $n = 64$, $\varepsilon = 1/16$.

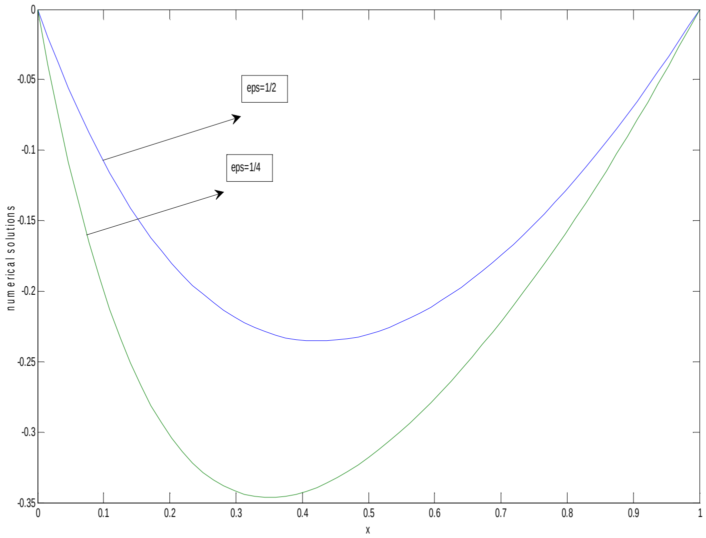

Figure 7: physical behaviour of Example 3 for two different values of $\in$.

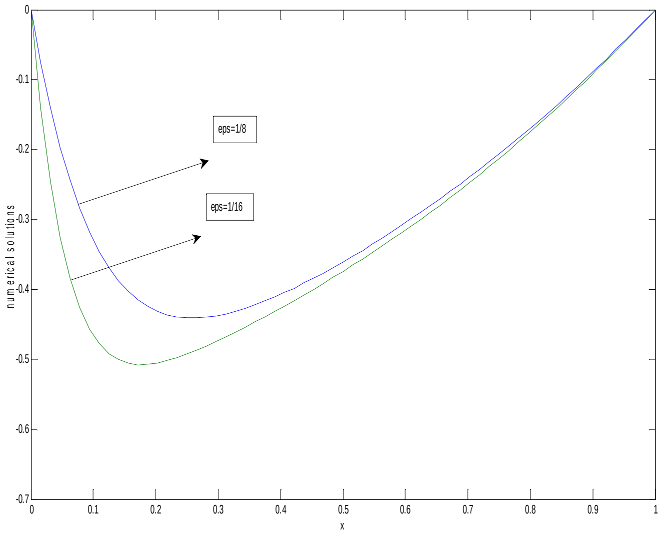

Figure 8: physical behaviour of Example 3 for two different values of $\in$.

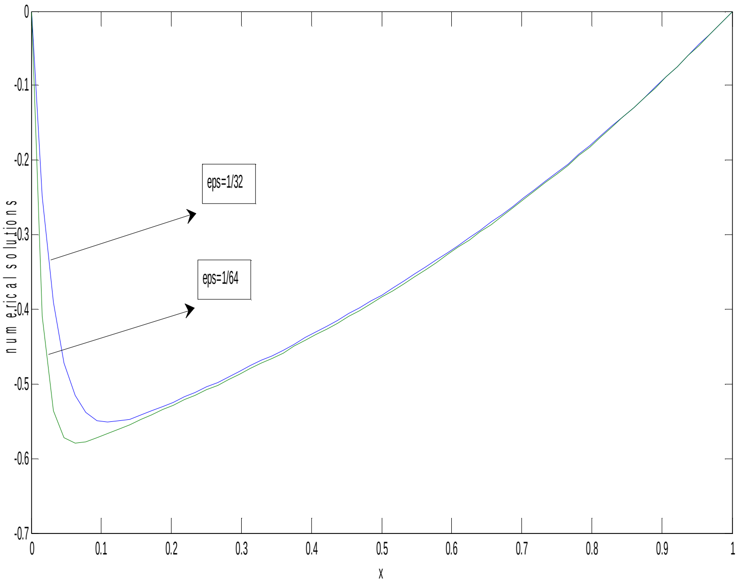

Figure 9: physical behaviour of Example 3 for two different values of $\in$.

Chapter 3

Quintic B-Spline Collocation Method for Second Order Mixed Boundary Value Problem

3.1 Introduction

We consider the second order two point mixed boundary value problem:

$$
\begin{aligned}
& y''(x) = F(x, y(x), y'(x)), \quad x \in [a,b], \\
& \alpha_1 y(a) + \beta_1 y'(a) = a_1, \\
& \alpha_2 y(b) + \beta_2 y'(b) = b_1
\end{aligned}
\tag{3.1}
$$

where a_1 and b_1 are given real constants, $F(x, y(x), y'(x))$ is differentiable on $[a,b]$. The aim of this chapter is to discuss variation quintic B-spline collocation method. We denote (3.1) by $D^2 y = F$ with the boundary conditions. For simplicity such problems arise in many fields in engineering and science, such as heat transfer, deflection of plate, diffusion, and so on,see [19-23]. Generally it is difficult to obtain the analytic solution of (3.1) for arbitrary $F(x, y(x), y'(x))$. Hence, numerical methods for (3.1) are required. There are a lot of papers on second order Dirichlet boundary value problem[24-32]. However, numerical analysis literature does not conation much on the numerical solution of Neumann boundary value problem,

$$
\begin{aligned}
& y''(x) = F(x, y(x), y'(x)), \quad x \in [a,b], \\
& y'(a) = a_1, \\
& y'(b) = b_1
\end{aligned}
\tag{3.2}
$$

Blue[33] obtain an $o(h^4)$ quintic spline method for (3.2), but this method required to solve 3n+3 linear or non linear equations, where n denotes the number of subinterval. In a recent paper [34], Ramadan et al. developed an $O(h^2)$ quadratic spline method QSM, an $O(h^2)$ cubic spline method CSM and an $O(h^3)$ non-polynomial spline method NSM. But, these methods in [34] for (3.2) are only applicable to the simple linear Neumann boundary value problems in the following form

$$\begin{aligned} &y''(x) + f(x)y(x) = g(x), \quad x \in [a,b] \\ &y'(a) = a_1, y'(b) = b_1 \end{aligned}$$

To overcome this drawback, in this paper we develop a new quintic B-spline collocation method for (3.1) and (3.2) . This method is fourth order convergent and has lower computational cost because it requires only solving n+5 linear or non linear equations.

The remainder of this paper is organized as follows. In Section 2, some preliminary results of quintic B-spline are given; Section 3 is devoted to the quintic B-spline collocation method for linear and nonlinear second order boundary value problem (3.1); in Section 4, some numerical examples are presented ; finally, Section6 contains conclusion.

3.2. Preliminary Results of Quintic B-Spline

Consider the interval l=[a,b], divide it into n subintervals $l_i = [x_i, x_{i+1}](i = 0,1,2......,n-1)$ by the equidistant knots $x_i = a + ih \ (i = 0,1,......n)$,where $h = \dfrac{b-a}{n}$. The quintic spline space is defined as follows:

$$S_5(l)=\{s(x)\in c^4(l)\,|\,s(x)_{|l_i}\in P_5, i=0,1,.......,n-1\},$$

Where $s(x)_{|l_i}$ denotes the restriction of $s(x)$ over l_i,and P_5 denotes the set of univariate quintic polynomials. The dimension of $S_5(l)$ is n+5. Extend $l=[a,b]$ to $l=[a-5h,b+5h]$ with the equidistant knots $x_i=a+ih$ $(i=-5,-4,.....n+5)$. By the result in [35], we obtain the explicit representation of the typical quintic B-spline $B_i(x)(i=-2,-1,,.....,n+2)$ as follows:

$$B_{m,5}(x)=\frac{1}{120h^5}\begin{cases}(x-x_i+3h)^5, & x\in[x_{i-3},x_{i-2}]\\ (x-x_i+3h)^5-6(x-x_i+2h)^5, & x\in[x_{i-2},x_{i-1}]\\ (x-x_i+3h)^5-6(x-x_i+2h)^5+15(x-x_i+h)^5 & x\in[x_{i-1},\ x_i]\\ (-x+x_i+3h)^5-6(-x+x_i+2h)^5+15(-x+x_i+h)^5 & x\in[x_i,\ x_{i+1}]\\ (-x+x_i+3h)^5-6(-x+x_i+2h)^5 & x\in[x_{i+1},\ x_{i+2}]\\ (-x+x_i+3h)^5 & x\in[x_{i+2},\ x_{i+3}]\\ 0 & else\end{cases} \qquad (3.3)$$

$B_i(x)$ is non negative and is locally supported on$[x_{i-3},x_{i+3}]$. Besides, it is easy to observe that $B_i(x)=B_{i+1}(x+h)(i=-2,-1,......n+1)$ and $\sum_{i=-2}^{n+2}B_i(x)=1(x\in[a,b])$. In addition $B_i(x)(i=-2,-1,....,n+2)$ are linearly independent, so they are the basis splines of $S_5(l)$.

Furthermore, by some trivial computation, we obtain the values of $B_i^{(k)}(x)$ $(i=-2,-1,....,n+2;k=0,1,2,3)$ at the knots, which are listed in Table 3.1.

Table 3.1: Value of $B_{m,5}(x)$ and its derivatives at the nodal points

	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	$else$
$B_{i,5}(x)$	0	$\frac{1}{120}$	$\frac{26}{120}$	$\frac{66}{120}$	$\frac{26}{120}$	$\frac{1}{120}$	0	0
$B^{'}_{i,5}(x)$	0	$\frac{1}{24h}$	$\frac{10}{24h}$	0	$-\frac{10}{24h}$	$-\frac{1}{24h}$	0	0
$B^{''}_{i,5}(x)$	0	$\frac{1}{6h^2}$	$\frac{2}{6h^2}$	$-\frac{6}{6h^2}$	$\frac{2}{6h^2}$	$\frac{1}{6h^2}$	0	0
$B^{'''}_{i,5}(x)$	0	$\frac{1}{2h^3}$	$\frac{-2}{2h^3}$	0	$\frac{2}{2h^3}$	$\frac{-1}{2h^3}$	0	0

Given a sufficient smooth function y(x), there exists a unique quintic spline $S(x)=\sum_{i=-2}^{n+2} c_i B_i(x) \in S_5(l)$ satisfying the following interpolation condition:

$$\begin{aligned} &s(x_i)=y(x_i), && i=0,1,.....,n. \\ &s'(a)=y^{'}(a), && s^{''}(a)=y^{''}(a), \\ &s^{'}(b)=y^{'}(b), && s^{''}(b)=y^{''}(b), \end{aligned}$$

From table 1 and result in [36,39],for the equidistant knots $x_j\,(j=0,1,.......n),$ we have

$$y(x_j) = s(x_j) = \frac{1}{120}(c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}), \tag{3.4}$$

$$y'(x_j) = s'(x_j) + O(h^6) = \frac{1}{24h}(-c_{j-2} - 10c_{j-1} + 10c_{j+1} + c_{j+2}) + O(h^6) \tag{3.5}$$

$$y''(x_j) = s''(x_j) + O(h^4) = \frac{1}{6h^2}(-c_{j-2} - 10c_{j-1} + 10c_{j+1} + c_{j+2}) + O(h^6) \tag{3.6}$$

$$y^{(3)}(x_j) = s^{(3)}(x_j) + O(h^4) = \frac{1}{2h^3}(-c_{j-2} + 2c_{j-1} - 2c_{j+1} + c_{j+2}) + O(h^4) \tag{3.7}$$

3.3. Quintic B-Spline Collocation Method

3.3.1. Linear case

Consider the linear boundary value problem

$$\begin{aligned} &y''(x) + p(x)y'(x) + q(x)y(x) = r(x), \qquad x \in [a,b], \\ &\alpha_1 y(a) + \beta_1 y'(a) = a_1, \\ &\alpha_2 y(b) + \beta_2 y'(b) = b_{1.} \end{aligned} \tag{3.8}$$

Let $s(x) = \sum_{i=-2}^{n+2} c_i B_i(x)$ be the quintic spline solution of (3.8), Discretizing (3.8) at the knots, we get $(i = 0,1,.......,n)$;

$$y''(x_i) + p(x_i)y'(x_i) + q(x_i)y(x_i) = r(x_i)$$

Now by using (3.4), (3.5) and (3.6), we have

$$\frac{1}{6h^2}[c_{i-2} + 2c_{i-1} - 6c_i + 2c_{i+1}] + \frac{1}{24h}[-c_{i-2} - 10c_{i-1} + 10c_{i+1} + c_{i+2}]p_i + \frac{1}{120}[c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}]q_i = r_i$$

$$[c_{i-2} + 2c_{i-1} - 6c_i + 2c_{i+1} + c_{i+2}] + \frac{h}{4}p_i[-c_{i-2} - 10c_{i-1} + 10c_{i+1} + c_{i+2}] + \frac{h^2}{20}q_i[c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}] = 6h^2 r_i \tag{3.9}$$

Now by boundary conditions,

$$\alpha_1 y(a) + \beta_1 y'(a) = a_1$$

$$\frac{\alpha_1}{120}[c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2] + \frac{\beta_1}{24h}[-c_{-2} - 10c_{-1} + 10c_1 + c_2] = a_1$$

$$\frac{\alpha_1 h}{5}[c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2] + \beta_1[-c_{-2} - 10c_{-1} + 10c_1 + c_2] = 24a_1 h \qquad (3.10)$$

Again , $\alpha_2 y(b) + \beta_2 y'(b) = b_1$

$$\alpha_2 \frac{1}{120}[c_{n-2} + 26c_{n-1} + 66c_n + 26c_{n+1} + c_{n+2}] + \beta_2 \frac{1}{24h}[-c_{n-2} - 10c_{n-1} + 10c_{n+1} + c_{n+2}] = b_1$$

$$\frac{\alpha_2 h}{5}[c_{n-2} + 26c_{n-1} + 66c_n + 26c_{n+1} + c_{n+2}] + \beta_2[-c_{n-2} - 10c_{n-1} + 10c_{n+1} + c_{n+2}] = 24hb_1 \qquad (3.11)$$

We still need two equations, from (3.8) ,we have

$$y''(x) = r(x) - p(x)y'(x) - q(x)y(x).$$

By using and differentiating it, we get

$$y^{(3)}(x) = r'(x) - p'(x)y'(x) - p(x)y''(x) - q'(x)y(x) - q(x)y'(x)$$

$$y^{(3)}(x) = (p^2(x) - p'(x) - q(x)y'(x) + (p(x)q(x) - q'(x))y(x) + r'(x) - p(x)r(x).$$

Let x=a ,we get

$$y^{(3)}(a) + \alpha_0 y(a) + \beta_0 y'(a) = \delta_0,$$

where $\alpha_0 = q'(a) - p(a)q(a)$, $\beta_0 = p'(a) + q(a) - p^2(a)$ and $\delta_0 = r'(a) - p(a)r(a)$.

By (3.4) ,(3.5) and (3.7),we have

$$\frac{1}{2h^3}(-c_{-2}+2c_{-1}-2c_1+c_2)+\frac{\alpha_0}{120}(c_{-2}+26c_{-1}+66c_0+26c_1+c_2)+\frac{\beta_0}{24h}(-c_{-2}-10c_{-1}+10c_1+c_2)=\delta_0$$

where the truncated error $O(h^4)$ is dropped . Hence ,we have

$$\begin{aligned}(-c_{-2}+2c_{-1}-2c_1+c_2)&+\frac{\alpha_0h^3}{60}(c_{-2}+26c_{-1}+66c_0+26c_1+c_2)\\&+\frac{\beta_0h^2}{12}(-c_{-2}-10c_{-1}+10c_1+c_2)=2\delta_0h^3\end{aligned} \tag{3.12}$$

Similarly, we have

$$\begin{aligned}&(-c_{n-2}+2c_{n-1}-2c_{n+1}+c_{n+2})+\frac{\alpha_nh^3}{60}(c_{n-2}+26c_{n-1}+66c_n+26c_{n+1}+c_{n+2})\\&+\frac{\beta_nh^2}{12}(-c_{n-2}-10c_{n-1}+10c_{n+1}+c_{n+2})\end{aligned} \tag{3.13}$$

where $\alpha_n=q^{'}(b)-p(b)q(b),\beta_n=p^{'}(b)+q(b)-p^2(b)$ and $\delta_n=r^{'}(b)-p(b)r(b).$

Take (3.9) , (3.10) ,(3.11) ,3.12) and (3.13) together we get n+5 linear equations with $c_i(i=-2\ ,-1\ ,......,n+2)$ as knowns. Let

$$C=[c_{-2,},c_{-1}\ ,......,c_{n+2}]^T\ ,$$
$$R=[24a_1h\ ,2\delta_0h^3\ ,6r_0h^2\ ,......6r_nh^2\ ,2\delta_nh^3\ ,24b_1h]^T$$

be two (n+5)dimensional column vectors. In matrix notations, the linear system can be written as

$$(A+\frac{1}{4}hPB_1+\frac{1}{20}h^3QB_2)C=R \tag{3.14}$$

where

$$
A=\begin{bmatrix}
-\beta_1 & -10\beta_1 & 0 & 10\beta_1 & \beta_1 & & \\
-1 & 2 & 0 & -2 & 1 & & \\
1 & 2 & -6 & 2 & 1 & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & 1 & 2 & -6 & 2 & 1 \\
 & & -1 & 2 & 0 & -2 & 1 \\
 & & -\beta_2 & -10\beta_2 & 0 & 10\beta_1 & \beta_2
\end{bmatrix}_{(n+5)\times(n+5)}
$$

$$
P=\begin{bmatrix}
\frac{4}{5}\alpha_1 & & & & & & \\
 & 0 & & & & & \\
 & & p_0 & & & & \\
 & & & \ddots & & & \\
 & & & & p_n & & \\
 & & & & & 0 & \\
 & & & & & & \frac{4}{5}\alpha_2
\end{bmatrix}_{(n+5)\times(n+5)}
$$

$$
B_1=\begin{bmatrix}
1 & 26 & 66 & 26 & 1 & & \\
0 & 0 & 0 & 0 & 0 & & \\
-1 & -10 & 0 & 10 & 1 & & \\
 & \ddots & \ddots & \ddots & \ddots & & \\
 & & -1 & -10 & 0 & 10 & 1 \\
 & & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 26 & 66 & 26 & 1
\end{bmatrix}_{(n+5)\times(n+5)}
$$

$$
Q=\begin{bmatrix}
0 & & & & & & \\
 & 1 & & & & & \\
 & & q_0 & & & & \\
 & & & \ddots & & & \\
 & & & & q_n & & \\
 & & & & & 1 & \\
 & & & & & & 0
\end{bmatrix}_{(n+5)\times(n+5)}
$$

$$
B_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & & \\
\frac{\alpha_0 h - 5\beta_0}{3} & \frac{26\alpha_0 h - 50\beta_0}{3} & 22\alpha_0 h & \frac{26\alpha_0 h + 50\beta_0}{3} & \frac{\alpha_0 h + 5\beta_0}{3} & & \\
1 & 26 & 66 & 26 & 1 & & \\
\ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & 1 & 26 & 66 & 26 & 1 & \\
 & \frac{\alpha_n h - 5\beta_n}{3} & \frac{26\alpha h - 50\beta_n}{3} & 22\alpha_n h & \frac{26\alpha_n h + 20\beta_n}{3} & \frac{\alpha_n h + 5\beta_n}{3} & \\
 & & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

After solving the linear system (1.14), we obtain the quintic spline approximation solution $S(x) = \sum_{i=-2}^{n+2} c_i B_i(x)$.

3.3.2 Nonlinear case

In this subsection, we briefly dicuss the nonlinear second order mixed boundary value problem in the form of (3.1). Similarly, let $S(x) = \sum_{i=-2}^{n+2} c_i B_i(x)$ be the quintic B-spline solution of(3.1). Discretizing (3.1) at the knots, i=0,1,2 …..,n., we get

$$
y''(x_i) = F(x_i, y(x_i), y'(x_i)).
$$

By (3.4), 3.5) and (3.6), we have

$$
\frac{c_{i-2} + 2c_{i-1} - 6c_i + 2c_{i+1} + c_{i+2}}{6h^2} = F\left(x_i, \frac{c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}}{120}, \frac{-c_{i-2} - 10c_{i-1} + 10c_{i+1} + c_{i+2}}{24h} \right)
$$

Where the truncated error $O(h^4)$ is dropped. Denote these nonlinear equations $(i = 0,1,.....n)$ as

$$\frac{c_{i-2} + 2c_{i-1} - 6c_i + 2c_{i+1} + c_{i+2}}{6h^2} = \phi(c_{i-1}, c_{i-1}, c_i, c_{i+1}, c_{i+2}) \tag{3.15}$$

Besides , we also have two linear equations obtained by the boundary conditions at $x = a$ and $x = b$,

$$\beta_1(-c_{-2} - 10c_{-1} + 10c_1 + c_2) + \frac{\alpha_1 h}{5}(c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2) = 24a_1 h \tag{3.16}$$

$$\begin{aligned}\beta_2(-c_{n-2} - 10c_{n-1} + 10c_{n+1} + c_{n+2})& \\ + \frac{\alpha_2 h}{5}(c_{n-2} + 26c_{n-1} + 66c_n + 26c_{n+1} + c_{n+2}) &= 24b_1 h\end{aligned} \tag{3.17}$$

Furthermore by differentiating (3.1), we have

$$y^{(3)}(x) = F_1^{'}(x, y(x), y^{'}(x)) + F_2^{'}(x, y(x), y^{'}(x))y^{'}(x) + F_3^{'}(x, y(x), y^{'}(x))y^{''}(x)$$

Let x=a, we have

$$y^{(3)}(a) = F_1^{'}(a, y(a), y^{'}(a)) + F_2^{'}(a, y(a), y^{'}(a))y^{'}(a) + F_3^{'}(a, y(a), y^{'}(a))y^{''}(a)$$

Let j=0, by (3.4) (3.5) and (3.6), we denote this nonlinear equation as

$$\frac{-c_{-2} + 2c_{-1} - 2c_1 + c_2}{2h3} = \phi_0(c_{-2}, c_{-1}, c_0, c_1, c_2). \tag{3.18}$$

Similarly at x=b, we get

$$\frac{-c_{n-2} + 2c_{n-1} - 2c_{n+1} + c_{n+2}}{2h3} = \phi_0(c_{n-2}, c_{n-1}, c_n, c_{n+1}, c_{n+2}). \tag{3.19}$$

Taking (3.15), (3.16), (3.17), (3.18) and (3.19) together, we get n+5 nonlinear equations with $c_i\,(i=-2,-1,.......,n+2)$ as unknowns. After solving the nonlinear system, we obtain the quintic spline approximation solution $s(x)=\sum_{i=-2}^{n+2} c_i B_i(x)$.

3.4 Numerical Experiments

In this section, we have considered two examples in order to check the accuracy of the presented method. The maximums absolute errors are calculated by the following formula

$L_\infty = \max_i |y_i - Y_i|$, where y_i and Y_i are appromate and exact solutions respectively.

Example 1: We have considered the following problem over the domain [0, 1]

$$y''(x)+(x^2-6x-1)y'(x)+(5x-x^2+6)y(x)=e^x-x^2+5x+6$$

subject to two kinds of boundary conditions

$BC_1=\{y'(0)=1\,,y'(1)=2e\}$ and

$BC_2=\{y(0)+y'(0)=2\ ,\ 2y(1)-y'(1)=2\}$,

where the analytic solution is given by $y(x)=xe^x+1$.

The numerical solutions of the example are presented in Table 3.2 in form of maximums absolute errors at different numbers of subintervals for both types of boundary conditions. The Table shows that the maximums absolute errors decrease as we increase the numbers of subintervals in both cases.

Example 2: We have considered the following problem over the domain [0, 4]

$$y''(x) = -\frac{1}{2} y(x) y'(x)$$

along with two kinds of boundary conditions

$$BC_1 : \{ y'(x) = -0.16, y'(4) = -4 \}$$

and $BC_2 = \{2y(0) - y'(0) = -1.44 \, , y(4) + 0.5y'(4) = -6\}$,

where the analytic solution is given by $y(x) = \dfrac{4}{x-5}$.

The numerical solutions of the example are presented in Table 3.2 in form of maximums absolute errors at different numbers of subintervals for both types of boundary conditions. The numerical results of this example have been taken from the paper [37].The Table shows that the maximums absolute errors decrease as we increase the numbers of subintervals in both cases.

Table 3.2: The maximum absolute errors L_∞ of Example 1 for different values of *n*.

n	$h=\frac{1}{n}$	$L_\infty[y(x_i)]of\ BC_1$	$L_\infty[y(x_i)]of\ BC_2$
5	0.2	2.885×10^{-6}	7.694×10^{-6}
10	0.1	1.835×10^{-7}	4.372×10^{-7}
20	0.05	1.151×10^{-8}	2.666×10^{-8}
40	0.025	7.202×10^{-10}	1.656×10^{-9}
80	0.0125	4.498×10^{-11}	1.031×10^{-10}

Table 3.3: The maximum absolute errors L_∞ of Example 2 for different values of *n*.

n	$h=\frac{1}{n}$	$L_\infty[y(x_i)]of\ BC_1$	$L_\infty[y(x_i)]of\ BC_2$
10	0.4	7.671×10^{-3}	3.663×10^{-3}
20	0.2	5.288×10^{-4}	2.642×10^{-4}
40	0.1	3.418×10^{-5}	1.777×10^{-5}
80	0.05	1.801×10^{-6}	1.125×10^{-6}
100	0.04	8.396×10^{-7}	4.617×10^{-7}