SOME AGGREGATION OPERATORS & INFORMATION MEASURES FOR SOLVING DECISION-MAKING PROBLEMS

A Thesis

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**Some aggregation operators & information measures for solving decision-making problems**" in partial fulfillment of the requirement for the award of Degree of Philosophy and submitted in the School of Mathematics (SoM), Thapar Institute of Engineering & Technology, Patiala is an authentic record of my own, carried out during a period from January, 2017 to July, 2021 under the supervision of Dr. Harish Garg, Associate Professor, SoM, Thapar Institute of Engineering & Technology, Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

Multi criteria decision making (MCDM) techniques have wide applications in various areas such as decision theory, operation research, management research, social psychology etc. These methods enable us to find the most optimal alternative among the available choices which are characterized by different criteria. In MCDM processes, the judgements values corresponding to alternatives may not be expressed using crisp numbers always as uncertainty is present in almost every real world system. Therefore, in order to handle uncertain and fuzzy situations existing in the real world, the decision-makers need to have such theories using which they could consider fuzzy data values and maintain their decision-making (DM) criteria in accordance to the particular situation. In this direction, numerous models such as fuzzy sets, intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets have been designed and introduced so far. Under these disciplines, a number of researchers developed various methods for dealing with DM problems. Among these techniques, aggregation operators (AOs) and information measures are the basic and efficient tools for handling DM problems. AOs reduce a set of numbers into a unique representative one. Information measures such as similarity and distance process the uncertain information by calculating the degree of similarity and discrimination respectively among input arguments.

Although a number of DM techniques have been established so far under the above said models but these environments cannot handle time periodic problems and portray two dimensional information simultaneously in one set. So as to address this issue, a new model named as complex intuitionistic fuzzy set (CIFS) has been developed in 2012. CIFSs have the characteristic of portraying membership and non-membership degrees over the unit disc of the complex plane. Under this model, membership and non-membership degrees

are represented in polar form in which the amplitude terms corresponding to membership (non-membership) value explicit the intensity of belongingness (not-belongingness) of the element in a set and the phase terms provide additional periodic information.

The objective of this work is to develop new methodologies for handling cases which involve time periodic and two dimensional information. In order to achieve it, some information measures such as distance, similarity, correlation, entropy for processing CIFSs and various generalized AOs for aggregating dependent and independent input arguments are proposed. A number of properties of the proposed measures and operators are explored. Based on the presented measures and operators, MCDM techniques are developed for addressing such two dimensional problems which are either difficult or impossible to be solved using existing theories. The applicability of the proposed methods is demonstrated by applying them in several real life DM problems. The results of the proposed techniques are compared with several existing methods.

The presented thesis is organized into twelve chapters and the brief summary of these chapters is given as:

A brief account of the related work of various authors in the evaluation of decision making process by using several approaches is presented in the **Chapter 1**, while **Chapter 2** recapitulates the basic terms related with intuitionistic fuzzy sets, CIFSs, information measures, t-norms, conorms and AOs.

Chapter 3 introduces a series of distance measures by using Hamming, Euclidean, and Hausdorff metrics in order to find discrimination degree among CIFSs. Various desirable relations among proposed measures are explored. A DM method is developed for finding the best alternative under the set of feasible ones. Illustrative examples from the field of pattern recognition as well as the medical diagnosis are illustrated in order to show the applicability of the proposed work.

Chapter 4 presents correlation and weighted correlation coefficients under the CIFS environment. Some of the desirable properties of proposed measures are investigated. A MCDM approach for CIFSs is developed based on proposed correlation coefficients. Two illustrative examples are taken to demonstrate the efficiency of the proposed approach. Also, the obtained results are compared with several existing approaches results.

Chapter 5 introduces some novel formulae of information measures such as similarity measures, distance measures, entropy and inclusion measures. The transformation relationships among the presented measures are discussed in detail. The proposed similarity measures are applied on pattern recognition problem and a detailed comparative analysis is conducted with some of the existing measures. Further, algorithms based on proposed measures are developed for handing MCDM problems and their working is illustrated with the help of an example. Besides this, the practicality of the proposed similarity measure is demonstrated by developing a clustering algorithm under CIFS environment.

Chapter 6 puts forward some generalized weighted averaging and geometric AOs for aggregating the different complex intuitionistic fuzzy numbers (CIFNs) using archimedean t-conorm and t-norm operations. Some new operational laws of the CIFNs based on archimedean t-conorm and t-norm are defined and their fundamental properties are proved. Then, a series of weighted averaging and geometric AOs are developed based on proposed operations. Further, some desirable properties and special cases of the presented AOs are studied. Finally, a DM approach based on proposed AOs is developed in order to solve MCDM problems with CIFS information. A practical example is elaborated to illustrate the working of the proposed approach and its results are compared with some existing methods under CIFS and IFS studies.

Chapter 7 defines the possibility degree measure in order to rank the CIFNs and presents some novel operational laws and AOs for aggregating the various choices over CIFS environment. The properties of the proposed weighted averaging and geometric AOs are investigated. A MCDM approach is established for CIFSs, in which weights are derived objectively. A practical illustration is given to show the advantages of the proposed method and its results are compared with some existing methods.

Chapter 8 presents some new exponential, logarithmic and compensative exponential of logarithmic operational laws of CIFNs based on t-norm and co-norm. Using these laws, compensative operators namely generalized complex intuitionistic fuzzy compensative weighted averaging and generalized complex intuitionistic fuzzy compensative weighted geometric are developed. Some properties related to proposed operators are discussed. In light of the developed operators, a group DM method is put forward in which weights are determined objectively and is illustrated with the aid of an example. The reliability of the presented DM method is explored by comparing it with several prevailing studies. The influence of the parameters used in exponential and logarithmic operations on CIFNs is also discussed.

Chapter 9 considers dependency among CIFNs during aggregation process and fuses dependent CIFNs. It presents power averaging/geometric, power weighted averaging/geometric and power ordered weighted averaging/geometric AOs under CIFS environment. Based on the proposed operators, a MCDM approach is presented. An illustrative example related to the selection of the best alternative(s) is considered to demonstrate the efficiency of the proposed approach. The reliability of the presented method is explored by comparing its results with several existing studies.

Chapter 10 presents AOs namely generalized complex intuitionistic fuzzy Bonferroni mean and generalized complex intuitionistic fuzzy weighted Bonferroni mean which encapsulate the interaction among the criteria and preferences under CIFS conditions. Some properties related to proposed operators are investigated. Based on the developed operators, a DM method is put forward and is explained with the aid of an example. The reliability of the presented DM method is examined by comparing the results of the example with several predominating studies.

Chapter 11 aims to present some new prioritized AOs by considering priority degrees among priority orders for aggregating CIFNs. It presents prioritized averaging and geometric operators without priority degrees, prioritized averaging and geometric operators with priority degrees, prioritized ordered weighted averaging and geometric operators with priority degrees based on basic unit interval monotonic function for aggregating dependent CIFNs. A number of propositions related to proposed operators are proved. A group DM approach based on proposed operators is developed and is applied on a practical DM problem. The results and the characteristics of proposed method are compared with several existing studies. Besides this, the role of the priority degrees on aggregation result is also discussed in detail.

Chapter 12 outlines the summary and future scope of the presented work.

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List of Publications

Refereed Journals

(J1) Harish Garg, Dimple Rani, New prioritized aggregation operators with priority degrees among priority orders for complex intuitionistic fuzzy information, *Journal of Ambient Intelligence and Humanized Computing, Springer*, pp. 1 - 27, 2021, doi: 10.1007/s12652-021-03164-2 (**SCI: Impact Factor: 7.104**).

(J2) Harish Garg, Dimple Rani, Multi-criteria decision making method based on Bonferroni mean aggregation operators of complex intuitionistic fuzzy numbers, *Journal of Industrial and Management Optimization*, 17(5), 2279 - 2306, 2021, doi: http://dx.doi.org/10.3934/jimo.2020069 (**SCI: Impact Factor: 1.801**)

(J3) Harish Garg, Dimple Rani, Novel aggregation operators and ranking method for complex intuitionistic fuzzy sets and their applications to decision-making process, *Artificial Intelligence Review, Springer* 53(5), pp. 3595 - 3620, 2020 doi: 10.1007/s10462-019-09772-x (**SCI: Impact Factor: 8.139**).

(J4) Harish Garg, Dimple Rani, Generalized geometric aggregation operators based on t-norm operations for complex intuitionistic fuzzy sets and their application to decision making, *Cognitive Computation, Springer*, 12(3), pp. 679 - 698, 2020 (**SCI: Impact Factor: 5.418**).

(J5) Harish Garg, Dimple Rani, New generalized Bonferroni mean aggregation operators of complex intuitionistic fuzzy information based on Archimedean t-norm and t-conorm, *Journal of Experimental and Theoretical Artificial Intelligence*, 32(1), pp. 81-109, 2020 (**SCI: Impact Factor: 2.340**).

(J6) Harish Garg, Dimple Rani, Robust Averaging - Geometric aggregation operators for Complex Intuitionistic fuzzy sets and their applications to MCDM process, *Arabian Journal for Science and Engineering, Springer*, 45 (3), pp. 2017 - 2033, 2020, doi:10.1007/s13369-019-03925-4 (**SCI: Impact Factor: 2.334**)

(J7) Harish Garg, Dimple Rani, Some Results on Information measures for complex intuitionistic fuzzy sets, *International Journal of Intelligent Systems, Wiley*, 34(10): 2319 - 2363, 2019, doi: 10.1002/int.22127 (**SCI: Impact Factor: 8.709**)

(J8) Harish Garg, Dimple Rani, A robust correlation coefficient measure of complex intuitionistic fuzzy sets and their applications in decision-making, *Applied Intelligence, Springer*, 49(2), pp. 496 - 512, 2019 (**SCI: Impact Factor: 5.086**).

(J9) Harish Garg, Dimple Rani, Some generalized complex intuitionistic fuzzy aggregation operators and their application to multicriteria decision-making process, *Arabian Journal for Science and Engineering, Springer*, 44(3), 2679 - 2698, 2019 (**SCI: Impact Factor: 2.334**).

(J10) Harish Garg, Dimple Rani, Exponential, logarithmic and compensative generalized aggregation operators under complex intuitionistic fuzzy environment, *Group Decision and Negotiation, Springer* 28(5), 991 - 1050, 2019, doi: 10.1007/s10726-019-09631-8 (**SCI: Impact Factor: 2.648**).

(J11) Dimple Rani, Harish Garg, Complex intuitionistic fuzzy power aggregation operators and their applications in multi-criteria decision-making, *Expert Systems, Wiley*, 35(6), e12325, 2018, doi: 10.1111/exsy.12325 (**SCI: Impact Factor: 2.587**).

(J12) Dimple Rani, Harish Garg, Distance measures between the complex Intuitionistic Fuzzy sets and its applications to the decision making process, *International Journal for Uncertainty Quantification*, 7(5), 423 - 439, 2017 (**SCI: Impact Factor: 2.083**).

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Chapter 1

Introduction

Decision-making (DM) is one of the most crucial components involved in a large number of human activities. Models of DM have been successfully applied in various areas such as decision theory, operation research, management research, social psychology etc. DM is the cognitive process of determining the most optimal course of action among the available choices. Many a times, while making decisions, we come across situations where the available alternatives are characterized by different criteria (attributes). The process of DM which assesses alternatives by considering all the attributes simultaneously is termed as multiple criteria decision making (MCDM) and multiple attribute decision making (MADM). Recently, MCDM methods have been attaining more and more popularity due to their extensive applications in almost every area whether it is academic or businesses.

The general MCDM process involves a set of alternatives $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m\}$ which are characterized by another collection $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\}$ of criteria. The criteria set $\mathfrak{B}$ is partitioned into two disjoint subsets namely benefit and cost type criteria. Since all the criteria are not always equally important as the importance degree of criteria may differ. Therefore, the vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ assigns the weight to each criteria $\mathfrak{B}_v$ such that $\xi_v > 0 \ \forall \ v = 1, 2, \ldots, n$ and $\sum\limits_{v=1}^{n} \xi_v = 1$. In order to reach at conclusion, the alternatives $\mathcal{V}_u$ $(u = 1, 2, \ldots, m)$ are assessed under different criteria $\mathfrak{B}_v$ and are given preference values as $\mathcal{C}_{uv}$. Based on these assessment values, the decision matrix can be formulated as:

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \ldots & \mathcal{C}_{1n} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \ldots & \mathcal{C}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{m1} & \mathcal{C}_{m2} & \ldots & \mathcal{C}_{mn} \end{pmatrix} \end{array}$$

In MCDM problems, the alternatives may be assessed by a single decision maker or by a group of decision makers. The MCDM problems, in which decisions are made by a group of decision makers are termed as multi criteria group decision making (MCGDM) or multi attribute group decision making (MAGDM) problems. MCGDM is a tool in which more than one individuals make a decision regarding the available alternatives before them which are characterized by different criteria. The final decision made is then no longer attributable to any single individual of the group as all the members collectively contribute to the outcome.

In every real world system, due to undeniable presence of uncertainty, the decision makers may not give their assessment values using crisp numbers always. As the complexities in the socio-economic environments are increasing day by day therefore, dealing with uncertain information has become a basic concern. In order to incorporate uncertainty into system description efficiently, Zadeh [206] put forward the theory of fuzzy sets (FSs) in which a unique real number, belonging to $[0, 1]$, is assigned to each entity of the universal set and this number is called as membership degree (MD). The FS theory was put forward on the basis of the fundamental assumption that if any element of the universal set is assigned a MD, say $z \in [0, 1]$ then, automatically the non-membership degree (NMD) of that element is considered to be $1 - z$. However, Atanassov [10] pointed out that due to the presence of hesitation degree in human decisions and judgements, sum of MDs and NMDs is not necessarily one always. In light of this fact, Atanassov [10] introduced the concept of intuitionistic fuzzy set (IFS) theory characterized by NMDs, MDs, hesitation degrees and this theory proved quite successful in modeling uncertain and vague information more efficiently and effectively. Later on, it was analyzed that, with the growing complexities of the systems, it was not always possible for the decision-makers to give their preferences in terms of exact values. Therefore, Atanassov and Gargov [9] proposed the concept of interval valued IFSs (IVIFSs) in which MDs and NMDs can take values in terms of intervals with the restriction that the sum of the upper bounds of membership and non-membership intervals should not exceed one.

The FS, IFS and IVIFS theories express MDs and NMDs using real numbers belonging to the unit interval $[0, 1]$ and are inadequate to handle time periodic problems. Consequently, to portray periodic information in the judgement values, Ramot et al. [128] introduced the theory of complex fuzzy sets (CFSs). In this theory the membership function takes complex values with co-domain unit disc in the complex plane. Further, to incorporate the hesitation degree along with periodic information into the analysis, Alkouri and Salleh [5] proposed the theory of complex intuitionistic fuzzy sets (CIFSs). In this theory, MDs and NMDs take complex values lying in the unit disc and are expressed in polar form. The amplitude part of MDs (NMDs) presents the degree of belongingness (not belongingness) of the element in CIFS and the phase term provides additional periodic information regarding the component. Fascinated by the characteristics of CIFS theory, the main objective of this work is to develop some aggregation operators and information measures to solve MCDM and MCGDM problems under this environment.

1.1 Literature Review

In this section, we reviewed some of the existing techniques available in the literature for solving MCDM and MCGDM problems under IFS, IVIFS and CFS theories.

1.1.1 Review on Distance & Similarity measures

Distance and similarity measures are complement of each other and exhibit two aspects of the similar measure. The distance measure quantifies the degree of discrimination among two objects. In contrast to it, the similarity measure describes the intensity of similarity between the objects. Both of these measures quantify the uncertain and imperfect information and enable us to reach at optimal decision by processing the uncertainty existing in the data. The role of these measures in handling the uncertain information is huge and unavoidable due to their wide applications in various fields such as decision-making, pattern recognition, medical diagnosis, clustering analysis and image segmentation [20, 21, 23, 39, 41, 77, 99, 112, 116, 120, 122, 126, 134, 138, 145, 148, 155, 171–173]. Therefore, numerous researchers and scholars have worked in this direction. Szmidt and Kacprzyk [140] put forward the hamming and Euclidean distance formulas which calculate the degree of discrimination among IFSs. Further, Grzegorzewski [69] generalized the hamming and Euclidean distances to new distance measures using Hausdorff metric under IFS and IVIFS environment. Wang and Xin [160] introduced the axiomatic definitions of distance and similarity measures and put forward several novel formulas for measuring distance among IFSs. Dengfeng and Chuntian [36] proposed the similarity measures for discrete as well as continuous IFSs and applied them in pattern recognition problems. Later on, Mitchell [114] observed some instances in which the measure proposed by Dengfeng and Chuntian [36] does not fit well and therefore, made modifications in the measure given in [36]. Liang and Shi [98] put forward numerous cases where the prevailing measures fail in giving results and introduced novel similarity measures. Hung and Yang [86] utilized Hausdorff distance in formulating new distance and similarity measures under IFS environment and proved the related properties. Szmidt and Kacprzyk [142] gave new similarity measures for IFSs and applied them in medical diagnostic reasoning. Song et al. [136] proposed similarity and weighted similarity measures and showed their applications in pattern recognition problems. Chen [24] developed the notion of similarity between vague sets and proposed two similarity measures. But these measures [24] fail to give results in some instances and this drawback was analyzed by Hong and Kim [80]. Therefore, Hong and Kim [80] proposed a series of modified similarity measures and justified their presented measures by giving numerous examples. Liu [101] pointed out drawbacks of prevailing measures [25, 36, 80] and therefore, presented new similarity measures. Boran and Akay [18] developed similarity measure for IFSs based on two parameters. Khan and Lohani [91] put forward similarity measure for IFSs using distance measure of bounded variation and implemented it in clustering analysis. Ye [197] developed a cosine similarity measure under IFS environment and applied it in pattern recognition and medical diagnosis problems. Ngan et al. [119] developed a new measure namely H-max distance for

IFSs.

In addition to these, recently many researchers worked in the direction of formulating distance and similarity measures by transforming IFSs into triangular fuzzy numbers (TFNs). For example, Chen and Chang [26] developed similarity measure using transformation techniques among IFSs and TFNs and applied the presented measure in various pattern recognition problems. Jiang et al. [87] put forward a transformation technique among IFS and isosceles TFN and developed a distance measure based on the intersection of transformed TFNs. Chen, Cheng and Lan [28] transformed IFSs into right angled TFNs and developed similarity measure among IFSs using transformed numbers. Dhivya and Sridevi [37] developed similarity measure based on the mid-points of transformed TFNs and applied it to various pattern recognition and medical diagnosis problems existing in the literature.

The IVIFS theory, an extension of IFS environment, is one of the most successful mediums of handling uncertainty. Numerous researchers have developed distance and similarity measures under this environment. For instance, Park et al. [124] extended interval-valued fuzzy hamming and Euclidean distances to IVIFS environment. Xu [182] developed a new group DM method based on Euclidean distance under IVIFS theory. Wei, Wang and Zhang [164] proposed a technique for constructing similarity measure from entropy for IVIFSs and developed entropy and similarity measures. Ye [198] put forward cosine similarity measure for IVIFSs and developed MCDM technique based on it. Wu et al. [167] proposed a new similarity measure among IVIFSs by taking into account the hesitation degree and applied it in various pattern recognition and MCDM problems. Luo and Liang [110] developed a novel similarity measure by transforming IVIFSs to TFNs and applied it in pattern recognition and medical diagnosis problems. Tiwari and Gupta [150] proposed entropy measure to calculate fuzziness of IVIFSs and also derived its relation with similarity and distance measures. Liu and Jiang [107] developed a distance measure for IVIFSs by considering hesitation degree and all interval numbers. Dugenci [40] proposed a new generalized distance measure and applied it in MCGDM problems.

The IFSs and IVIFSs capture uncertainty using real values of MDs and NMDs. However, in CFS environment and its extensions, complex values of MDs are considered.

Keeping in view the advantages of CFS theory, various authors and researchers have investigated distance and similarity measures under this environment and its extensions. Zhang et al. [209] developed distance measure and used it to define $\delta-$ equalities of CFSs. Alkouri and Salleh [4] proposed hamming and Euclidean distances for CFSs. Further, Hu et al. [81] extended the measures proposed in [4] and developed minkowski distances for CFSs. Dai et al. [33] put forward hamming and Euclidean distance measures for interval-valued CFSs and applied them in a MCDM problem. Guo et al. [71] proposed cosine similarity measures under CFS environment and investigated their invariance properties. All of these works are done by taking into account MDs only. Further, considering NMD along with MD into the analysis, Alkouri and Salleh [6] introduced a formula to calculate distance among CIFSs and applied it in a MCDM problem. Kumar and Bajaj [93] presented distance and entropy measures for complex intuitionistic fuzzy (CIF) soft sets and showed their application in MCDM problem. Selvachandran et al. [132] initiated the idea of complex vague soft sets, discussed their basic operations and proposed distance measures under this environment. Further, Selvachandran et al. [131] put forward similarity measure for complex vague soft sets and applied it in pattern recognition problem. Ullah et al. [151] initiated the theory of complex pythagorean fuzzy sets and developed distance measures under this environment.

1.1.2 Review on Correlation Coefficients

The correlation coefficient (CC) is one of the most significant and frequently used measures in statistical analysis. It exhibits the degree of strength and the direction of relationship among two variables. It is employed to determine the measurement of dependency and the nature of relation between variables. This tool helps not only in comparing data entities with each other but additionally provides the association degree among them. Recently, many authors and scholars have worked in this direction and have applied CC in many practical fields such as DM, pattern recognition and medical diagnosis etc. In statistical theory, data observations are represented with the help of certain probability distributions. However, in real world system, due to unavoidable presence of uncertainty, data observations involve fuzziness. Therefore, to determine the degree of correlation among fuzzy observations, Gerstenkorn and Manko [63] developed the formula of CC under IFS environment. Later on, Hong and Hwang [79] extended the CC of IFSs into the probability spaces. Hung [84] developed the formulas of CC for IFSs using mathematical statistics and further extended it for IVIFSs. Mitchell [115] pointed out some instances in which the measure proposed by Hung [84] fails in giving satisfying results and therefore, put forward a new CC under IFS environment. Hung and Wu [85] employed the centroid concept to develop CC for IFSs. Zeng and Li [207] put forward formulas for calculating CC among IFSs by considering MDs, NMDs and hesitation degrees simultaneously. Szmidt and Kacprzyk [143] developed CC under IFS theory by taking into account membership, non-membership and hesitation margins and further proposed spearman CC in [144].

After the development of CCs, numerous researchers successfully applied them in various DM problems. Ye [195] put forward a MCDM method based on CC under IFS environment. Solanki et al. [135] developed a MCDM method using correlation under IFS theory and applied it in supplier selection problem. Zhao and Xu [213] proposed a new CC for IFSs and put forward a MCDM technique based on it. Tiwari [149] developed entropy measure for IFSs, further proposed weighted CC based on it and put forward a new MCDM method. Thao [146] developed a new formula for CC among IFSs and applied it in medical diagnosis problem. Further, Thao et al. [147] developed CC using variance and covariance among IFSs and applied it in pattern recognition problem and clustering analysis. Joshi and Kumar [89] proposed a MCDM approach based on weighted CC. Huang and Guo [82] developed an improved CC among IFSs and further applied it in medical diagnosis and clustering analysis.

As IVIFS theory is a successful medium of handling uncertainty therefore, numerous authors and researchers have explored correlation and CC under this environment also. Bustince and Burillo [19] initiated the theory of correlation and CC under IVIFS environment. Furthermore, several researchers [196, 199, 202] utilized weighted CC presented by Bustince and Burillo [19] and proposed a MCDM method by employing it under IVIFS theory. Hong [78] generalized the concept of CC of IVIFSs in probability space and extended the results proposed by Bustince and Burillo [19]. Park et al. [123] developed a new formula of CC for IVIFSs by adding hesitation degrees along with MDs and NMDs and presented a MCGDM technique based on the proposed measure. Xu [178] developed the CC under IFS and IVIFS environments and further applied them in medical diagnosis problem. Wei, Wang and Lin [165] developed a MCDM technique based on CC for IVIFSs in which criteria weights are determined objectively. Zeng and Wang [208] proposed a new formula for calculating CC among IVIFSs and showed that their presented CC defined for a finite set is identical with the cosine of the intersectional angle. Liu et al. [100] proposed a new CC for IFSs, further extended it under IVIFS environment and applied them in MCDM problems.

1.1.3 Review on traditional aggregation operators

The key perspective for most of the knowledge based frameworks, from image processing to DM, from pattern recognition to machine learning is data aggregation and fusion. In majority of the practical DM problems, it is always needed to fuse some numerical quantities and in such situations aggregation operators (AOs) assume a principal job. In general terms, it may be stated that the aggregation process uses different information pieces to make it possible to reach at some conclusion or decision. In our regular day to day existence, we encounter numerous circumstances, in which a mathematical function is needed possessing the capability of reducing a set of numbers into a unique representative one. Therefore, the study of AOs is a significant part of MCDM problems. Information aggregation has become basic concern in MCDM process. Recently, the issue of how to aggregate information has gained much attention of many authors due to their extensive uses in various fields for example pattern recognition, image processing and information retrieval. Xu and Yager [185] developed some geometric AOs to accumulate different judgement values of the decision-makers represented as intuitionistic fuzzy numbers (IFNs) and further, Xu [179] explored weighted averaging operators along with their desirable properties. Wang and Liu [156, 161] proposed averaging and geometric AOs based on einstein operational laws. Beliakov et al. [13] presented a method of constructing AOs based on archimedean triangular norm and conorm operations. Further, Xia et al. [169] studied the construction principle of AOs given by Beliakov et al. [13] and developed generalized averaging and geometric AOs using triangular norm and conorm operations.

Also, it is shown in this manuscript [169] that the AOs based on algebraic operations [179, 185], einstein operations [156, 161] and hamachar operations [83], Frank operations [125] can be acquired by taking different forms of additive generators in the operational laws proposed by Xia et al. [169]. Huang [83] gave the hamachar operational laws and developed averaging and ordered averaging AOs based on proposed operations.

In addition to the AOs based on basic algebraic, Einstein and hamachar operational laws, numerous researchers and scholars developed new operations for aggregating IFNs. In [72], authors proposed geometric interaction averaging AOs to accumulate IFNs and applied them in the MCDM process. Garg [44] put forward generalized interactive geometric interaction AOs for IFNs by utilizing einstein operations and further developed interactive AOs in [46]. Garg [47] improved the prevailing einstein operations and developed new AOs by using proposed improved operations. Chen and Chang [27] transformed IFNs to TFNs and proposed AOs for accumulating IFNs based on transformed numbers. Ye [201] developed hybrid AOs and gave a DM approach of mechanical design scheme. Goyal et al. [67] proposed weighted averaging operator by combining it with genetic algorithm and applied it in a DM problem based on e-learning. Zhou and Xu [217] developed four extreme intuitionistic fuzzy (IF) averaging and geometric AOs and also put forward their selection principles. Fan et al. [42] proposed a series of new AOs for fusing IFNs based on novel operations and developed a MCGDM technique using the presented AOs. Garg [52] improved the hamachar operations, developed new AOs and MCDM method based on proposed operators and entropy weights.

In the above studies, the AOs are developed by taking the bases to be IFNs and the weights to be real numbers. However, there may be certain situations in which the IFNs are taken in place of exponents (weights) and the real numbers are taken as bases. In this direction, Gou, Xu and Lei [65] proposed novel exponential operational laws (EOLs) for IFNs and developed operators for fusing IFS information. Luo et al. [111] employed t-norms and co-norms for investigating EOLs and proposed new generalized exponential AOs for accumulating IFNs. Gou and Xu [64] developed EOLs in which the bases are taken as interval numbers and applied them in MCDM problem. Garg [53] presented new EOLs in which exponents are intuitionistic multiplicative numbers and developed exponential

AOs based on them. Furthermore, keeping in view the importance and characteristics of logarithm function, Li and Wei [97] put forward logarithmic operational laws and developed AOs using the proposed operations by taking real numbers to be the logarithm bases.

Apart from IFS environment, numerous researchers have worked in the direction of developing AOs under the IVIFS environment also by accumulating the interval valued intuitionistic fuzzy numbers (IVIFNs). For instance, Xu [180] proposed operations of IVIFNs and developed weighted and geometric AOs using the presented operations. Further, Xu and Jian [184] developed ordered weighted averaging and hybrid averaging AOs for IVIFNs and presented a MCDM technique based on them. Xu and Chen [176] proposed ordered weighted geometric and hybrid geometric operators for IVIFNs and illustrated their applications by giving example. Wang et al. [159] developed weighted and ordered weighted AOs for accumulating IVIFNs using archimedean t-norm and conorm operations. Wang and Liu [157, 158] proposed arithmetic weighted averaging/geometric, ordered weighted averaging/geometric and hybrid weighted averaging/geometric AOs using einstein operations and further developed MCDM method under IVIFS environment. Liu [103] utilized hamachar operations for developing weighted averaging and geometric AOs for IVIFNs. Chen, Cheng and Tsai [30] proposed a technique of transforming IVIFNs to TFNs and then developed averaging geometric AOs based on transformed TFNs. Also, Chen, Cheng and Tsai [29] transformed IVIFNs to right angled TFNs and further developed weighted averaging AOs based on the transformed numbers. Garg [51] pointed out the drawbacks in prevailing hamachar geometric AOs proposed by Liu [103] and further proposed novel operations and AOs for IVIFNs. Gou, Xu and Liao [66] put forward EOLs for IVIFSs and developed AOs by utilizing the proposed operations. Garg [49] proposed EOLs for interval-valued pythagorean fuzzy numbers and developed AOs using the proposed operations.

1.1.4 Review on functional aspect aggregation operators

The work on the AOs, described in the above section, is done by assuming the arguments to be aggregated as independent. However, this may not be the case always. There may exist some kind of dependency in the form of supportive correlation, interrelationship and prioritization relationship among the arguments to be aggregated. In our day to day life, we encounter certain real-life problems in which some degree of interrelationship exists among the arguments and it is essential to take into account this interdependence during aggregation process to make an optimal decision. For handling such situations, Yager [190] introduced the idea of power averaging (PA) operator. PA operator considers the correlation between the arguments and in the process of aggregation using this operator the arguments reinforce each other. Keeping in view the characteristics of PA operator, recently many researchers and scholars have worked towards this direction. For instance, Xu and Yager [187] proposed power geometric (PG) operator, ordered PG operator using geometric mean and PA operator [190]. Further, Xu [175] proposed PA operator under IFS theory and applied the proposed operator in developing MCGDM approach and extended the proposed operator and DM technique for IVIFSs. Zhou et al. [214] developed generalized PA, ordered PA operators and applied them in developing MCGDM approach. Zhang [211] extended the PG operators proposed by Xu and Yager [187] to generalized PG operators under IFS environment. Zhang et al. [210] proposed PA, weighted PA, ordered PA and ordered weighted PA operators for IFNs using frank operations. He et al. [76] developed generalized PA operators by taking into account the interactions among the arguments to be aggregated. Jiang et al. [88] proposed weighted PA operator in which the weights are determined using entropy measure. He et al. [73] presented PA operators under IVIFS environment and developed MCGDM approach based on the proposed operators. Apart from these, to handle complex DM problems, several researchers developed hybrid AOs by combining the characteristics of PA operator with heronian mean [104], Bonferroni mean [105], maclaurin symmetric mean [108] and murihead mean [174]. Motivated by the features of CFS environment and its extensions, Garg et al. [55] presented PA, PG, weighted PA, weighted PG operators under complex q-rung orthopair fuzzy set theory and developed DM algorithm based on proposed AOs.

In addition to these, we encounter many situations in which there exists strict prioritization relationship among the arguments to be aggregated. To deal with such prioritized

MCDM issues, a series of AOs such as prioritized scoring, prioritized averaging, prioritized "and" and prioritized "or" operators were originally proposed by Yager [191]. Apart from these AOs, further Yager [193] developed prioritized ordered weighted averaging operator based on the basic unit interval monotonic (BUM) function. Yu [203] and Verma and Sharma [152] presented prioritized weighted averaging and geometric AOs using algebraic and einstein operational laws respectively for IFNs. Yu [204] presented generalized prioritized weighted geometric operators using t-norms and t-conorms under IFS theory. Further, Khan et al. [90] and Gao [43] developed prioritized averaging and geometric AOs respectively under pythagorean fuzzy set environment and gave their applications in MCDM problems. Li and Xu [94] introduced the concept of priority degrees among priority orders and developed prioritized AOs based on priority degrees. Yu et al. [205] developed prioritized weighted averaging and prioritized weighted geometric operators under IVIFS environment and developed a MCGDM approach based on the proposed AOs. Further, Li et al. [96] pointed out a few instances in which the prioritized operator proposed by Yu et al. [205] fails in giving results and therefore, modified the AO given in [205] and put forward an improved prioritized operator. Chen [31] presented a new prioritized AO to aggregate IVIFNs and applied to watershed site selection problems. Wu and Su [168] developed a new unit prioritized hybrid weighted AO which considers the prioritization relationship and the ordered positions of IVIFNs simultaneously

Further, to encapsulate the interconnection between each pair of arguments to be aggregated, Bonferroni [17] suggested a mean-type operator named as Bonferroni mean (BM). Beliakov et al. [14] presented the generalized BM operator by considering three arguments. Xu and Yager [188] developed a DM approach based on weighted BM operator under IFS environment. Although, intuitionistic fuzzy weighted BM operator was developed in [188] but it does not reduce into BM operator when all the weight values are taken same. Therefore, Xia et al. [170] resolved this issue and proposed new generalized weighted BM operator and extended the presented operator under IFS environment. Zhou and He [216] pointed out the drawbacks of prevailing operators [14, 17, 170, 188] and proposed normalized weighted BM and generalized normalized weighted BM operators. Further, Zhou and He [215] combined BM operator with geometric mean and put forward geometric BM and weighted BM operators. He, He and Chen [75] presented new interactive operational laws and developed BM and weighted BM operator using proposed interactive operations under IFS theory. Further, He and He [74] extended BM operator proposed in [75] and developed extended intuitionistic fuzzy interaction BM operator using t-norm and co-norm operations. Xu and Chen [177] initiated the idea of BM and weighted BM operators under IVIFS theory and developed MCDM technique based on weighted BM operator. Zhang [212] proposed the geometric BM and weighted geometric BM operators by combining the features of geometric mean and BM operators under IVIFS environment.

1.1.5 Review of CFS and its extensions

IFS and IVIFS theories capture uncertainty and fuzziness using real numbers lying between 0 and 1. Although numerous DM algorithms have been developed under these environments using AOs and information measures but these theories are inadequate for handling periodic information. With the motive of portraying periodic information, Ramot et al. [128] initiated the idea of CFSs and discussed its several properties in [127]. Further, Alkouri and Salleh [5] extended the theory of CFSs to CIFSs for incorporating the hesitation degree along with periodic information into the analysis and discussed several operations of CIFSs in [7]. As in real life problems, the judgement values cannot be always represented using exact numbers therefore, Greenfield et al. [68] proposed the concept of interval valued CFSs which allow the MDs to assume values in interval form. Selvachandran et al. [132] extended the range of truth and false membership functions from a subinterval of [0,1] to the unit circle in the complex plane by developing complex vague soft sets and also proposed the various distance measures under this environment. Dick et al. [38] investigated the relationship among pythagorean FS and CFS theories and gave interpretation of MDs in CFSs. Ali and Smarandache [3] developed the theory of complex neutrosophic sets to capture uncertain, incomplete, indeterminate and false information of periodic nature. Further, Ali et al. [2] initiated the idea of interval complex neutrosophic sets. Recently, Yazdanbakhsh and Dick [194] discussed numerous applications of CFS model. Gulistan and Khan [70] proposed the theory of complex neutrosophic cubic sets and gave its set theoretic operations.

In the direction of aggregating CFS information, Bi et al. [15] proposed geometric AOs and illustrated that the proposed operators can be utilized for fusing pythagorean fuzzy information also. Further, Bi et al. [16] presented weighted arithmetic and ordered weighted arithmetic AOs to fuse CFS information and explored them under pythagorean FS environment as well. Akram and Naz [1] explored the idea of complex pythagorean fuzzy graphs and proposed weighted averaging/geometric, ordered weighted averaging/geometric AOs to accumulate complex pythagorean fuzzy information. Dai et al. [34] developed weighted geometric and ordered weighted geometric AOs under interval- valued CFS theory and applied them in a MCDM problem. Liu et al. [106] proposed the notion of complex q-rung orthopair fuzzy set and investigated weighted averaging and geometric AOs under this environment. Garg et al. [55] developed PA, PG, weighted PA, weighted PG AOs under complex q-rung orthopair fuzzy set environment and presented MCDM method using proposed operators.

In addition to these a number of researchers have worked on information measures under CFS theory and its extensions. Zhang et al. [209] presented distance measure and defined $\delta-$ equalities of CFSs by using the proposed measure. Alkouri and Salleh [4] developed hamming and Euclidean distances to quantify the degree of discrimination among CFSs. Further, Hu et al. [81] generalized the measures given by Alkouri and Salleh [4] and put forward minkowski distances among CFSs. Dai et al. [33] developed hamming and Euclidean distances among interval-valued CFSs and showed their applicability by applying them in a MCDM problem. Guo et al. [71] presented cosine similarity measures for CFSs and investigated their invariance properties. Further, by incorporating NMD along with MD into the analysis, Alkouri and Salleh [6] developed a formula for calculating distance among CIFSs and gave its application in a MCDM problem. Kumar and Bajaj [93] introduced the notion of complex intuitionistic fuzzy soft sets and developed distance and entropy measures under this environment. Selvachandran et al. [131] explored a measure for calculating degree of similarity among complex vague soft sets and gave its application in pattern recognition problem. Ullah et al. [151] investigated complex pythagorean fuzzy set theory and developed distance measures under this environment. Ngan et al. [118]

presented a new representation of CIFSs using quaternion numbers and put forward a distance measure using it.

1.2 Gaps and motivation of the work towards CIFSs

The above sections reveal that a number of MCDM techniques have been developed under IFS and IVIFS theories. But, it is analyzed that, these theories can deal with only one dimensional DM problems. However, many real world complex problems include two dimensional data i.e., information related with the attributes and periodicity of the parameters concerned with the problem. In order to portray such two dimensional information using these theories, the decision-maker will have to consider two or more FSs/IFSs/IVIFSs which may increase execution time and the number of computations required while solving the problem. But CFSs [128] and CIFSs [5] have the ability of portraying two dimensional information together in one set. The fundamental gap in IFS theory is that it can deal with only uncertainty whereas CIFS theory fills this gap by handling uncertainty and periodicity simultaneously. In CIFS theory, MDs and NMDs take complex values and belong to the unit circle in the complex plane. The MDs and NMDs are represented in polar form in which the amplitude terms corresponding to MDs (NMDs) describe the extent of belongingness (not-belongingness) of the element in a set and the phase terms provide additional information which is generally related with periodicity. These phase terms differentiate the CIFS and IFS theories and avoid the case of information loss and project the complete information together in single set.

Further, to highlight the gaps in traditional FS/IFS/IVIFS theories and to illustrate the significance of CIFSs, we illustrate a real world example related with Indian economy (IE). From the economic perspective, IE is classified into three sectors namely $\mathcal{V}_1$: Primary Sector, $\mathcal{V}_2$: Secondary Sector and $\mathcal{V}_3$: Tertiary Sector. Suppose that we are interested in finding out the most important sector, out of these sectors $\mathcal{V}_u$ $(u = 1, 2, 3)$, which affected IE during a particular calender year (CY). The goal is to order these three sectors in descending order from the most important to least important that affected IE during that particular CY. The influence of a particular sector on IE does not always remain throughout the whole year. Some sectors affect economy for few months only and not for the whole year. Thus, this problem is two dimensional which involves the influence degree of a particular sector $\mathcal{V}_u$ on IE and the time span of this influence. Therefore, the most optimal way to model this problem is by utilizing CIFS theory. The amplitude term of CIFS may be employed to measure the influence degree of sector on IE while the phase term can be utilized to indicate the period of this influence. For modeling such kind of two dimensional problem using IFS theory, both the aspects of the information will have to be considered individually by defining two IFSs with the first set portraying the influence degree of a particular sector $\mathcal{V}_u$ on IE and the second set describing the time span of this influence. It may increase execution time and the number of calculations required while tackling with the problem. In this manner, the most ideal approach of portraying two dimensional information simultaneously in one set, is by utilizing CIFS environment. Besides this, the comparison of CIFS model with some existing models is shown in Table 1.1.

Table 1.1: Comparison of CIFS model with existing models in literature

Model	Uncertainty	Falsity	Hesitation	Periodicity	Ability to represent two-dimensional information
FS	✓	×	×	×	×
Interval-valued FS	✓	×	×	×	×
IFS	✓	✓	✓	×	×
IVIFS	✓	✓	✓	×	×
CFS	✓	×	×	✓	✓
Interval-valued CFS	✓	×	×	✓	✓
CIFS	✓	✓	✓	✓	✓

In addition to the above features of CIF model, from the literature review presented in the above sections, we analyzed a few gaps in MCDM process involving CIF values, which are described as:

(i) While investigating AOs, it is analyzed that, a number of AOs exist in the literature for fusing IF and interval-valued IF numbers but these operators cannot fuse CIF values. However, to tackle with complex MCDM problems, AOs are required which can accumulate a number of CIF values into a unique representative one. Therefore, there is a need to develop AOs under CIF environment.

(ii) In MCDM problems, there may exist some kind of dependency in the form of supportive correlation, interrelationship and prioritization relationship among the arguments to be aggregated. A number of AOs such as power, Bonferroni and prioritized operators have been developed to aggregate dependent IF and interval-valued IF values. However, no operator exists in the literature to fuse dependent CIF values. Therefore, this issue requires to be addressed.

(iii) In the present complex real world scenario, we encounter a number of pattern recognition and medical diagnosis problems in which judgement values are represented using CIFSs. In order to handle such problems, some information measures are required for processing CIF data values and comparing two or more CIFSs. Therefore, development of information measures under CIF environment is needed.

(iv) The decision outcomes are adversely affected by the criteria weight vector. In some situations, the attribute weights are known as prior whereas in many cases the weight values are partially known or completely unknown. In order to address such cases in which the weight vector is partially known or completely unknown, construction of effective models is required to obtain the optimal weight vector.

1.3 Objective of the Thesis

Motivated by the existing literature and to fill the the gaps in the prevailing MCDM techniques, the present work intends to develop some new AOs and information measures for solving DM problems under CIF environment. The complete objectives of this work are summarized as:

(O1) To develop some new generalized aggregation operators and/or information measures.

(O2) To develop some prioritized/dynamic aggregation operators.

(O3) To construct some non-linear mathematical models for some decision-making problems.

1.4 Structure of the Thesis

This thesis is assembled into twelve chapters including the present one that contains mainly the literature review. The rest of the chapters are described below:

In **Chapter 2** the basics and the preliminaries related to the IFSs and CIFSs which are to be used in the subsequent chapters are given.

Chapter 3 presents the series of the distance measures by using Hamming, Euclidean, and Hausdorff metrics between the pairs of the CIFNs. Based on these measures, various desirable relations have been studied in detail. Further, a decision-making method has been presented for finding the best alternative under the set of the feasible one. Illustrative examples from the field of pattern recognition as well as the medical diagnosis have been taken to validate the approach.

In **Chapter 4**, we develop correlation measures under the CIFS environment in which pairs of the membership degrees represent the two-dimensional information. To achieve it, we first define some operational laws, the informational energies and the covariance between the two CIFSs that involves both uncertainty and periodicity semantics. Some of the desirable properties of proposed measures are investigated. Further, based on these measures, a multicriteria decision-making approach is presented under the CIFS environment. The feasibility, as well as superiority of the approach, has been demonstrated through two numerical examples

In **Chapter 5**, we introduce some novel formulae of information measures (similarity measures, distance measures, entropy and inclusion measures) and discuss the transformation relationships among them. To demonstrate the efficiency of the proposed similarity measures, we apply it to pattern recognition problem and a detailed comparative analysis is conducted with some of the existing measures. Further, algorithms based on proposed measures are developed for handing multi-criteria decision-making problems and their working is illustrated with the help of an example. Besides this, the practicality of the proposed similarity measure is demonstrated by developing a clustering algorithm.

Chapter 6 presents some generalized operators for aggregating the different CIFNs using archimedean t-conorm and t-norm (ATT) operations. For it, some new operational laws of the CIFNs based on ATT are defined and their fundamental properties are proved. Then, a series of weighted averaging and geometric AOs are developed based on proposed operations. Further, some desirable properties and special cases of the presented AOs are studied. Finally, a decision-making approach based on proposed AOs is developed to solve MCDM problems with CIFS information.

Chapter 7 is focussed on some new AOs and a novel ranking method under CIF theory for handling the multi-dimensional complex data sets. For this, firstly a new possibility degree method is presented to rank CIF numbers. Then, some AOs namely CIF weighted averaging (CIFWA) and CIF weighted geometric (CIFWG) are proposed and some of their properties are also discussed. Furthermore, a novel DM methodology is presented by considering the multi-dimensional complex data sets in which weights are determined objectively.

In **Chapter 8**, an attempt has been made to present some new exponential, logarithmic and compensative exponential of logarithmic operational laws of CIFNs based on t-norm and co-norm. Based on these laws, compensative operators namely generalized CIF compensative weighted averaging and generalized CIF compensative weighted geometric are developed. In light of the developed operators, a group decision-making method is put forward in which weights are determined objectively and is illustrated with the aid of an example. The influence of the parameters used in exponential and logarithmic operations on CIF numbers is also discussed.

Chapter 9 considers the dependency among CIFNs during their fusion and hence presents some power AOs. In this chapter, we define some basic algebraic operational laws between the pairs of the CIFSs which involve both uncertainty and periodicity semantics and studied their properties. Then, based on these operations, we propose some power AOs named as CIF power averaging, CIF power geometric, CIF weighted power averaging and geometric as well as their corresponding ordered weighted operators to aggregate the different CIFNs. The results of the proposed method are compared with several existing studies under CIFS and IFS environment.

In **Chapter 10**, we discusses the idea of Bonferroni mean based operators for the CIFSs to aggregate the information. The major advantages of the proposed operator are that they have considered the interrelationships of aggregated values. Based on the developed operators, a decision-making method is put forward and is explained with the aid of an example. The reliability of the presented decision-making method is examined with the help of a validity test and by comparing the results of the example with several predominating studies.

In **Chapter 11**, we develop some new prioritized aggregation operators by considering priority degrees among priority orders for aggregating CIFNs. In it, we present prioritized averaging and geometric operators without priority degrees, prioritized averaging and geometric operators with priority degrees, prioritized ordered weighted averaging and geometric operators with priority degrees based on basic unit interval monotonic function.

Chapter 12 deals with the overall concluding observations of this study and a brief discussion on the scope for future work.

Chapter 2

Preliminaries

In this chapter we review the basic concepts related to IFSs and CIFSs defined on universal set $\mathcal{U}$. Also, some exiting work related with information measures and AOs is reviewed in this chapter.

2.1 Intuitionistic fuzzy set and its extensions

Definition 2.1.1. [10] An IFS, $\mathcal{I}$ on $\mathcal{U}$, is given as $\mathcal{I} = \{(x, \zeta(x), \vartheta(x)) \mid x \in \mathcal{U}\}$, where $\zeta, \vartheta : \mathcal{U} \to [0,1]$ represent the membership and non-membership functions respectively and satisfy the relation $0 \leq \zeta(x) + \vartheta(x) \leq 1 \; \forall \; x \in \mathcal{U}$ and $h(x) = 1 - \zeta(x) - \vartheta(x)$ represents the hesitation degree of x in $\mathcal{I}$. If $\mathcal{U}$ contains only one element, then, for convenience, IFS $\mathcal{I}$ over $\mathcal{U}$ is written as $\mathcal{I} = (\zeta, \vartheta)$ where $\zeta, \vartheta \in [0,1]$; $0 \leq \zeta + \vartheta \leq 1$ and is called IFN.

Definition 2.1.2. [10] For two IFNs $\mathcal{I}_1 = (\zeta_1, \vartheta_1)$ and $\mathcal{I}_2 = (\zeta_2, \vartheta_2)$, we have:

(i) $\mathcal{I}_1 \subseteq \mathcal{I}_2$ if $\zeta_1 \leq \zeta_2$ and $\vartheta_1 \geq \vartheta_2$.

(ii) $\mathcal{I}_1 = \mathcal{I}_2 \Leftrightarrow \mathcal{I}_1 \subseteq \mathcal{I}_2$ and $\mathcal{I}_2 \subseteq \mathcal{I}_1$.

(iii) $\mathcal{I}_1^c = (\vartheta_1, \zeta_1)$.

Definition 2.1.3. [11] For two IFNs $\mathcal{I}_1 = (\zeta_1, \vartheta_1)$ and $\mathcal{I}_2 = (\zeta_2, \vartheta_2)$, we have:

(i) $\mathcal{I}_1 \cup \mathcal{I}_2 = (\max(\zeta_1, \zeta_2), \min(\vartheta_1, \vartheta_2))$.

(ii) $\mathcal{I}_1 \cap \mathcal{I}_2 = (\min(\zeta_1, \zeta_2), \max(\vartheta_1, \vartheta_2))$.

(iii) $\mathcal{I}_1 \oplus \mathcal{I}_2 = (\zeta_1 + \zeta_2 - \zeta_1\zeta_2, \vartheta_1\vartheta_2)$.

(iv) $\mathcal{I}_1 \otimes \mathcal{I}_2 = (\zeta_1\zeta_2, \vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2)$.

Definition 2.1.4. [185] The score $\mathcal{S}$ and an accuracy $\mathcal{H}$ functions are given as

$$\mathcal{S}(\mathcal{I}_1) = \zeta_1 - \vartheta_1 \tag{2.1}$$

$$\mathcal{H}(\mathcal{I}_1) = \zeta_1 + \vartheta_1 \tag{2.2}$$

for an IFN $\mathcal{I}_1 = (\zeta_1, \vartheta_1)$. Further, depending on $\mathcal{S}$ and $\mathcal{H}$ functions, a comparison law between two IFNs $\mathcal{I}_1$ and $\mathcal{I}_2$ is defined as, if $\mathcal{S}(\mathcal{I}_1) > \mathcal{S}(\mathcal{I}_2)$ then, $\mathcal{I}_1 \succ \mathcal{I}_2$ and if $\mathcal{S}(\mathcal{I}_1) = \mathcal{S}(\mathcal{I}_2)$ then, calculate $\mathcal{H}(\mathcal{I}_1)$ and $\mathcal{H}(\mathcal{I}_2)$. If $\mathcal{H}(\mathcal{I}_1) > \mathcal{H}(\mathcal{I}_2)$ then, $\mathcal{I}_1 \succ \mathcal{I}_2$. Here, the symbol '$\succ$' stands for 'preferred to'.

Definition 2.1.5. [9] An IVIFS $\mathcal{I}$ defined on $\mathcal{U}$ is given by:

$$\mathcal{I} = \{(x, \zeta(x), \vartheta(x)) : x \in \mathcal{U}\} \tag{2.3}$$

where $\zeta(x) = [\zeta^-(x), \zeta^+(x)] \subseteq [0,1]$ and $\vartheta(x) = [\vartheta^-(x), \vartheta^+(x)] \subseteq [0,1]$ represent the interval-valued MDs and NMDs respectively such that $0 \leq \zeta^+(x) + \vartheta^+(x) \leq 1$ for all $x \in \mathcal{U}$. However, if $\mathcal{U}$ contains only one element, then, for convenience, IVIFS $\mathcal{I}$ over $\mathcal{U}$ is written as $\mathcal{I} = ([\zeta^-, \zeta^+], [\vartheta^-, \vartheta^+])$ where $0 \leq \zeta^-, \zeta^+, \vartheta^-, \vartheta^+, \zeta^+ + \vartheta^+ \leq 1$ and is called IVIFN..

Definition 2.1.6. [12] For two IVIFNs $\mathcal{I}_1 = \left([\zeta_1^-, \zeta_1^+], [\vartheta_1^-, \vartheta_1^+]\right)$ and $\mathcal{I}_2 = \left([\zeta_2^-, \zeta_2^+], [\vartheta_2^-, \vartheta_2^+]\right)$, we have:

(i) $\mathcal{I}_1 \subseteq \mathcal{I}_2$ if $\zeta_1^- \leq \zeta_2^-$, $\zeta_1^+ \leq \zeta_2^+$, $\vartheta_1^- \geq \vartheta_2^-$ and $\vartheta_1^+ \geq \vartheta_2^+$.

(ii) $\mathcal{I}_1 = \mathcal{I}_2 \Leftrightarrow \mathcal{I}_1 \subseteq \mathcal{I}_2$ and $\mathcal{I}_2 \subseteq \mathcal{I}_1$.

(iii) $\mathcal{I}_1^c = \left([\vartheta_1^-, \vartheta_1^+], [\zeta_1^-, \zeta_1^+]\right)$.

(iv) $\mathcal{I}_1 \cup \mathcal{I}_2 = \left(\left[\max\left(\zeta_1^-, \zeta_2^-\right), \max\left(\zeta_1^+, \zeta_2^+\right)\right], \left[\min\left(\vartheta_1^-, \vartheta_2^-\right), \min\left(\vartheta_1^+, \vartheta_2^+\right)\right]\right)$.

(v) $\mathcal{I}_1 \cap \mathcal{I}_2 = \left(\left[\min\left(\zeta_1^-, \zeta_2^-\right), \min\left(\zeta_1^+, \zeta_2^+\right)\right], \left[\max\left(\vartheta_1^-, \vartheta_2^-\right), \max\left(\vartheta_1^+, \vartheta_2^+\right)\right]\right)$.

(vi) $\mathcal{I}_1 \oplus \mathcal{I}_2 = \left(\left[\zeta_1^- + \zeta_2^- - \zeta_1^-\zeta_2^-, \zeta_1^+ + \zeta_2^+ - \zeta_1^+\zeta_2^+\right], \left[\vartheta_1^-\vartheta_2^-, \vartheta_1^+\vartheta_2^+\right]\right)$.

(vii) $\mathcal{I}_1 \otimes \mathcal{I}_2 = \left(\left[\zeta_1^- \zeta_2^-, \zeta_1^+ \zeta_2^+\right], \left[\vartheta_1^- + \vartheta_2^- - \vartheta_1^- \vartheta_2^-, \vartheta_1^+ + \vartheta_2^+ - \vartheta_1^+ \vartheta_2^+\right]\right)$.

Definition 2.1.7. [180] The score $\mathcal{S}$ and an accuracy $\mathcal{H}$ functions for an IVIFN $\mathcal{I}_1 = \left(\left[\zeta_1^-, \zeta_1^+\right], \left[\vartheta_1^-, \vartheta_1^+\right]\right)$ are given as

$$\mathcal{S}(\mathcal{I}_1) = \frac{\zeta_1^- + \zeta_1^+ - \vartheta_1^- - \vartheta_1^+}{2} \tag{2.4}$$

$$\mathcal{H}(\mathcal{I}_1) = \frac{\zeta_1^- + \zeta_1^+ + \vartheta_1^- + \vartheta_1^+}{2} \tag{2.5}$$

Further, depending on $\mathcal{S}$ and $\mathcal{H}$ functions, a comparison law between two IVIFNs $\mathcal{I}_1$ and $\mathcal{I}_2$ is defined as, if $\mathcal{S}(\mathcal{I}_1) > \mathcal{S}(\mathcal{I}_2)$ then, $\mathcal{I}_1 \succ \mathcal{I}_2$ and if $\mathcal{S}(\mathcal{I}_1) = \mathcal{S}(\mathcal{I}_2)$ then, calculate $\mathcal{H}(\mathcal{I}_1)$ and $\mathcal{H}(\mathcal{I}_2)$. If $\mathcal{H}(\mathcal{I}_1) > \mathcal{H}(\mathcal{I}_2)$ then, $\mathcal{I}_1 \succ \mathcal{I}_2$.

Definition 2.1.8. [128] A CFS $\mathcal{C}$ is defined as:

$$\mathcal{C} = \left\{(x, \mu(x)) : x \in \mathcal{U}\right\} \tag{2.6}$$

where $\mu : \mathcal{U} \to \{a : a \in \mathbb{C}, |a| \leq 1\}$ is a membership function and $\mu(x)$ is represented as: $\mu(x) = \zeta(x) e^{i2\pi\left(w_\zeta(x)\right)}$ $\forall\ x \in \mathcal{U}$ where $i = \sqrt{-1}$, $0 \leq \zeta(x), w_\zeta(x) \leq 1$.

Definition 2.1.9. [5] A CIFS $\mathcal{C}$ is defined as:

$$\mathcal{C} = \left\{(x, \mu(x), \nu(x)) : x \in \mathcal{U}\right\} \tag{2.7}$$

where $\mu, \nu : \mathcal{U} \to \{a : a \in C, |a| \leq 1\}$ are complex-valued membership and non-membership functions respectively and are represented as $\mu(x) = \zeta(x) e^{i2\pi\left(w_\zeta(x)\right)}$ and $\nu(x) = \vartheta(x) e^{i2\pi\left(w_\vartheta(x)\right)}$, where $0 \leq \zeta(x), \vartheta(x), w_\zeta(x), w_\vartheta(x), \zeta(x)+\vartheta(x), w_\zeta(x)+w_\vartheta(x) \leq 1$. For convenience, we write CIFS $\mathcal{C}$ over $\mathcal{U}$ as $\mathcal{C} = \{(x, (\zeta(x), w_\zeta(x)), (\vartheta(x), w_\vartheta(x))) \mid x \in \mathcal{U}\}$. However, if $\mathcal{U}$ contains only one element x then, for convenience, we write CIFS $\mathcal{C}$ over $\mathcal{U}$ as $\mathcal{C} = \Big((\zeta, w_\zeta), (\vartheta, w_\vartheta)\Big)$ and call it as complex IFN (CIFN), where $0 \leq \zeta, \vartheta, \zeta + \vartheta, w_\zeta, w_\vartheta, w_\zeta + w_\vartheta \leq 1$.

Definition 2.1.10. [5] For two CIFNs $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$ and $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$, we have

(i) $\mathcal{C}_1 \subseteq \mathcal{C}_2 \Leftrightarrow \zeta_1 \leq \zeta_2$, $\vartheta_1 \geq \vartheta_2$ and $w_{\zeta_1} \leq w_{\zeta_2}$, $w_{\vartheta_1} \geq w_{\vartheta_2}$.

(ii) $\mathcal{C}_1 = \mathcal{C}_2 \Leftrightarrow \mathcal{C}_1 \subseteq \mathcal{C}_2$ and $\mathcal{C}_2 \subseteq \mathcal{C}_1$.

(iii) $\mathcal{C}_1^c = ((\vartheta_1, w_{\vartheta_1}), (\zeta_1, w_{\zeta_1}))$.

(iv) $\mathcal{C}_1 \cup \mathcal{C}_2 = ((\max(\zeta_1, \zeta_2), \max(w_{\zeta_1}, w_{\zeta_2})), (\min(\vartheta_1, \vartheta_2), \min(w_{\vartheta_1}, w_{\vartheta_2})))$.

(v) $\mathcal{C}_1 \cap \mathcal{C}_2 = ((\min(\zeta_1, \zeta_2), \min(w_{\zeta_1}, w_{\zeta_2})), (\max(\vartheta_1, \vartheta_2), \max(w_{\vartheta_1}, w_{\vartheta_2})))$.

Definition 2.1.11. [7] For two CIFNs $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$, and $\mathcal{C}_2 = ((\zeta_2,\ w_{\zeta_2}), (\vartheta_2,\ w_{\vartheta_2}))$, we have

(i) $\mathcal{C}_1 \oplus \mathcal{C}_2 = ((\zeta_1 + \zeta_2 - \zeta_1\zeta_2, w_{\zeta_1} + w_{\zeta_2} - w_{\zeta_1}w_{\zeta_2}), (\vartheta_1\vartheta_2, w_{\vartheta_1}w_{\vartheta_2}))$.

(ii) $\mathcal{C}_1 \otimes \mathcal{C}_2 = ((\zeta_1\zeta_2, w_{\zeta_1}w_{\zeta_2}), (\vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2, w_{\vartheta_1} + w_{\vartheta_2} - w_{\vartheta_1}w_{\vartheta_2}))$.

2.2 Information measures

Let $\Phi(\mathcal{U})$ denotes the set of all IFSs defined on $\mathcal{U} = \{x_1, x_2, \ldots, x_n\}$ and $\mathcal{I}_1 = \{(x, \zeta_1(x), \vartheta_1(x)) \mid x \in \mathcal{U}\}$; $\mathcal{I}_2 = \{(x, \zeta_2(x), \vartheta_2(x)) \mid x \in \mathcal{U}\}$ be two IFSs defined on $\mathcal{U}$.

2.2.1 Distance measures

Definition 2.2.1. [160] For IFSs, distance measure $\mathcal{D} : \Phi(\mathcal{U}) \times \Phi(\mathcal{U}) \to [0,1]$ is a real-valued function satisfying the following properties:

(P1) $0 \le \mathcal{D}(\mathcal{I}_1, \mathcal{I}_2) \le 1$.

(P2) $\mathcal{D}(\mathcal{I}_1, \mathcal{I}_2) = 0$ iff $\mathcal{I}_1 = \mathcal{I}_2$.

(P3) $\mathcal{D}(\mathcal{I}_1, \mathcal{I}_2) = \mathcal{D}(\mathcal{I}_2, \mathcal{I}_1)$.

(P4) If $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_3$ then, $\mathcal{D}(\mathcal{I}_1, \mathcal{I}_3) \ge \mathcal{D}(\mathcal{I}_1, \mathcal{I}_2)$ and $\mathcal{D}(\mathcal{I}_1, \mathcal{I}_3) \ge \mathcal{D}(\mathcal{I}_2, \mathcal{I}_3)$ where $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \in \Phi(\mathcal{U})$.

Some of the existing distance measures for IFSs $\mathcal{I}_1$ and $\mathcal{I}_2$ are given below:

(i) Szmidt and Kacprzyk [140] proposed hamming ($\mathcal{D}_1$), normalized hamming ($\mathcal{D}_2$), Euclidean ($\mathcal{D}_3$) and normalized Euclidean ($\mathcal{D}_4$) distances which are given as:

$$\mathcal{D}_1(\mathcal{I}_1, \mathcal{I}_2) = \frac{1}{2}\sum_{j=1}^{n}\left(\left|\zeta_1(x_j) - \zeta_2(x_j)\right| + \left|\vartheta_1(x_j) - \vartheta_2(x_j)\right| + \left|h_1(x_j) - h_2(x_j)\right|\right)$$

$$\mathcal{D}_2(\mathcal{I}_1, \mathcal{I}_2) = \frac{1}{2n}\sum_{j=1}^{n}\left(\left|\zeta_1(x_j) - \zeta_2(x_j)\right| + \left|\vartheta_1(x_j) - \vartheta_2(x_j)\right| + \left|h_1(x_j) - h_2(x_j)\right|\right)$$

$$\mathcal{D}_3\left(\mathcal{I}_1,\mathcal{I}_2\right)=\sqrt{\tfrac{1}{2}\left[\sum_{j=1}^{n}\left(\left(\zeta_1\left(x_j\right)-\zeta_2\left(x_j\right)\right)^2+\left(\vartheta_1\left(x_j\right)-\vartheta_2\left(x_j\right)\right)^2+\left(h_1\left(x_j\right)-h_2\left(x_j\right)\right)^2\right)\right]}$$

$$\mathcal{D}_4\left(\mathcal{I}_1,\mathcal{I}_2\right)=\sqrt{\tfrac{1}{2n}\left[\sum_{j=1}^{n}\left(\left(\zeta_1\left(x_j\right)-\zeta_2\left(x_j\right)\right)^2+\left(\vartheta_1\left(x_j\right)-\vartheta_2\left(x_j\right)\right)^2+\left(h_1\left(x_j\right)-h_2\left(x_j\right)\right)^2\right)\right]}$$

(ii) Xu [181] generalized the above distance measures to the following formulas:

$$\mathcal{D}_5\left(\mathcal{I}_1,\mathcal{I}_2\right)=\left\{\frac{1}{2}\left[\sum_{j=1}^{n}\left(\begin{array}{c}\left|\zeta_1\left(x_j\right)-\zeta_2\left(x_j\right)\right|^{\alpha}+\left|\vartheta_1\left(x_j\right)-\vartheta_2\left(x_j\right)\right|^{\alpha}\\+\left|h_1\left(x_j\right)-h_2\left(x_j\right)\right|^{\alpha}\end{array}\right)\right]\right\}^{\frac{1}{\alpha}}$$

$$\mathcal{D}_6\left(\mathcal{I}_1,\mathcal{I}_2\right)=\left\{\frac{1}{2n}\left[\sum_{j=1}^{n}\left(\begin{array}{c}\left|\zeta_1\left(x_j\right)-\zeta_2\left(x_j\right)\right|^{\alpha}+\left|\vartheta_1\left(x_j\right)-\vartheta_2\left(x_j\right)\right|^{\alpha}\\+\left|h_1\left(x_j\right)-h_2\left(x_j\right)\right|^{\alpha}\end{array}\right)\right]\right\}^{\frac{1}{\alpha}}$$

where $\alpha>0$ and defined the weighted distance measure as:

$$\mathcal{D}_7\left(\mathcal{I}_1,\mathcal{I}_2\right)=\left\{\frac{1}{2}\left[\sum_{j=1}^{n}\xi_j\left(\begin{array}{c}\left|\zeta_1\left(x_j\right)-\zeta_2\left(x_j\right)\right|^{\alpha}+\left|\vartheta_1\left(x_j\right)-\vartheta_2\left(x_j\right)\right|^{\alpha}\\+\left|h_1\left(x_j\right)-h_2\left(x_j\right)\right|^{\alpha}\end{array}\right)\right]\right\}^{\frac{1}{\alpha}}$$

where ξ_j is the weight associated with $x_j\in\mathcal{U}$ such that $\xi_j>0$ and $\sum_{j=1}^{n}\xi_j=1$.

Definition 2.2.2. [6] Let $\mathcal{U}=\{x_1,x_2,\ldots,x_n\}$ be the universe of discourse and $\Psi(\mathcal{U})$ be the family of CIFSs. A real valued function $\mathcal{D}:\Psi(\mathcal{U})\times\Psi(\mathcal{U})\rightarrow[0,1]$ is called the distance measure, if $\mathcal{D}$ satisfies the following properties for $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3\in\Psi(\mathcal{U})$:

(P1) $0\leq\mathcal{D}\left(\mathcal{C}_1,\mathcal{C}_2\right)\leq 1$

(P2) $\mathcal{D}\left(\mathcal{C}_1,\mathcal{C}_2\right)=0$ if and only if $\mathcal{C}_1=\mathcal{C}_2$.

(P3) $\mathcal{D}\left(\mathcal{C}_1,\mathcal{C}_2\right)=\mathcal{D}\left(\mathcal{C}_2,\mathcal{C}_1\right)$

(P4) If $\mathcal{C}_1\subseteq\mathcal{C}_2\subseteq\mathcal{C}_3$, then $\mathcal{D}\left(\mathcal{C}_1,\mathcal{C}_3\right)\geq\mathcal{D}\left(\mathcal{C}_1,\mathcal{C}_2\right)$ and $\mathcal{D}\left(\mathcal{C}_1,\mathcal{C}_3\right)\geq\mathcal{D}\left(\mathcal{C}_2,\mathcal{C}_3\right)$

Alkouri and Salleh [6] defined the distance measure among CIFSs $\mathcal{C}_1=\{(x,(\zeta_1(x),w_{\zeta_1}(x)),(\vartheta_1(x),w_{\vartheta_1}(x)))\mid x\in\mathcal{U}\}$ and $\mathcal{C}_2=\{(x,(\zeta_2(x),w_{\zeta_2}(x)),(\vartheta_2(x),w_{\vartheta_2}(x)))\mid x\in\mathcal{U}\}$ as follows:

$$\mathcal{D}_8\left(\mathcal{C}_1,\mathcal{C}_2\right)=\frac{1}{2\times\sum_{j=1}^{n}\psi_j}\left\{\sum_{j=1}^{n}\left[\psi_j\left(\begin{array}{l}\alpha_1\times|\zeta_1(x_j)-\zeta_2(x_j)|+\beta_1\times|\vartheta_1(x_j)-\vartheta_2(x_j)|\\+\sigma_1\times\max\left\{|\zeta_1(x_j)-\zeta_2(x_j)|,|\vartheta_1(x_j)-\vartheta_2(x_j)|\right\}\\+\alpha_2\times|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|+\beta_2\times|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\\+\sigma_2\times\max\left\{|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|,|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\right\}\end{array}\right)\right]\right\}\tag{2.8}$$

where $\alpha_1, \beta_1, \sigma_1, \alpha_2, \beta_2, \sigma_2, \psi_j \in [0,1]$ such that $\alpha_1+\beta_1+\sigma_1 = 1$ and $\alpha_2+\beta_2+\sigma_2 = 1$.

2.2.2 Similarity measures

Definition 2.2.3. [160] For IFSs, a similarity measure $\mathcal{S}' : \Phi(\mathcal{U}) \times \Phi(\mathcal{U}) \to [0,1]$ is a real-valued function satisfying the following properties:

(P1) $0 \leq \mathcal{S}'(\mathcal{I}_1, \mathcal{I}_2) \leq 1$.

(P2) $\mathcal{S}'(\mathcal{I}_1, \mathcal{I}_2) = 1$ if $\mathcal{I}_1 = \mathcal{I}_2$.

(P3) $\mathcal{S}'(\mathcal{I}_1, \mathcal{I}_2) = \mathcal{S}'(\mathcal{I}_2, \mathcal{I}_1)$.

(P4) If $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_3$ then, $\mathcal{S}'(\mathcal{I}_1, \mathcal{I}_3) \leq \mathcal{S}'(\mathcal{I}_1, \mathcal{I}_2)$ and $\mathcal{S}'(\mathcal{I}_1, \mathcal{I}_3) \leq \mathcal{S}'(\mathcal{I}_2, \mathcal{I}_3)$ where $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \in \Phi(\mathcal{U})$.

Also, Wang and Xin [160] proved that the distance and similarity measures satisfy the relation $\mathcal{S}' = 1 - \mathcal{D}$. Some prevailing similarity measures among IFSs $\mathcal{I}_1$ and $\mathcal{I}_2$ are stated below.

(i) Xu [181] extended the distances $\mathcal{D}_5$ and $\mathcal{D}_6$ to the following similarity measures:

$$\begin{aligned}
\mathcal{S}'_1(\mathcal{I}_1, \mathcal{I}_2) &= 1 - \left\{ \frac{1}{2n} \left[\sum_{j=1}^{n} \left(\begin{array}{c} \left|\zeta_1(x_j) - \zeta_2(x_j)\right|^{\alpha} + \left|\vartheta_1(x_j) - \vartheta_2(x_j)\right|^{\alpha} \\ + \left|h_1(x_j) - h_2(x_j)\right|^{\alpha} \end{array} \right) \right] \right\}^{\frac{1}{\alpha}} \\
\mathcal{S}'_2(\mathcal{I}_1, \mathcal{I}_2) &= 1 - \left\{ \frac{1}{2} \left[\sum_{j=1}^{n} \xi_j \left(\begin{array}{c} \left|\zeta_1(x_j) - \zeta_2(x_j)\right|^{\alpha} + \left|\vartheta_1(x_j) - \vartheta_2(x_j)\right|^{\alpha} \\ + \left|h_1(x_j) - h_2(x_j)\right|^{\alpha} \end{array} \right) \right] \right\}^{\frac{1}{\alpha}}
\end{aligned}$$

(ii) Chen [25] proposed the following similarity measure:

$$\mathcal{S}'_3(\mathcal{I}_1, \mathcal{I}_2) = \frac{1}{n} \sum_{j=1}^{n} \left(1 - \frac{\left| \left(\zeta_1(x_j) - \zeta_2(x_j)\right) - \left(\vartheta_1(x_j) - \vartheta_2(x_j)\right) \right|}{2} \right)$$

(iii) Hong and Kim [80] defined the similarity measure as:

$$\mathcal{S}'_4(\mathcal{I}_1, \mathcal{I}_2) = 1 - \frac{\sum_{j=1}^{n} \left(|\zeta_1(x_j) - \zeta_2(x_j)| + |\vartheta_1(x_j) - \vartheta_2(x_j)| \right)}{2n}$$

2.2.3 Correlation coefficients

Definition 2.2.4. For IFSs, correlation coefficient $\mathcal{K} : \Phi(\mathcal{U}) \times \Phi(\mathcal{U}) \to [0,1]$ is a real-valued function satisfying the following properties:

(P1) $0 \leq \mathcal{K}(\mathcal{I}_1, \mathcal{I}_2) \leq 1$.

(P2) $\mathcal{K}(\mathcal{I}_1, \mathcal{I}_2) = \mathcal{K}(\mathcal{I}_2, \mathcal{I}_1)$.

(P3) $\mathcal{I}_1 = \mathcal{I}_2 \Leftrightarrow \mathcal{K}(\mathcal{I}_1, \mathcal{I}_2) = 1$.

For IFSs $\mathcal{I}_1$ and $\mathcal{I}_2$, some of the existing CCs are defined as:

(i) Gerstenkorn and Manko [63] initiated the formula of CC as:

$$\mathcal{K}_1(\mathcal{I}_1, \mathcal{I}_2) = \frac{\sum_{j=1}^{n} \left[\zeta_1(x_j)\,\zeta_2(x_j) + \vartheta_1(x_j)\,\vartheta_2(x_j)\right]}{\sqrt{\sum_{j=1}^{n} \left[\zeta_1^2(x_j) + \vartheta_1^2(x_j)\right] \times \sum_{j=1}^{n} \left[\zeta_2^2(x_j) + \vartheta_2^2(x_j)\right]}}$$

(ii) Ye [195] proposed the weighted CC as:

$$\mathcal{K}_2(\mathcal{I}_1, \mathcal{I}_2) = \frac{\sum_{j=1}^{n} \left\{\xi_j\left[\zeta_1(x_j)\,\zeta_2(x_j) + \vartheta_1(x_j)\,\vartheta_2(x_j)\right]\right\}}{\sqrt{\sum_{j=1}^{n} \left\{\xi_j\left[\zeta_1^2(x_j) + \vartheta_1^2(x_j)\right]\right\} \times \sum_{j=1}^{n} \left\{\xi_j\left[\zeta_2^2(x_j) + \vartheta_2^2(x_j)\right]\right\}}}$$

(iii) Zeng and Li [207] proposed the CC by taking into account hesitation degrees along with MDs and NMDs as:

$$\mathcal{K}_3(\mathcal{I}_1, \mathcal{I}_2) = \frac{\sum_{j=1}^{n} \left[\zeta_1(x_j)\,\zeta_2(x_j) + \vartheta_1(x_j)\,\vartheta_2(x_j) + h_1(x_j)\,h_2(x_j)\right]}{\sqrt{\sum_{j=1}^{n} \left[\zeta_1^2(x_j) + \vartheta_1^2(x_j) + h_1^2(x_j)\right] \times \sum_{j=1}^{n} \left[\zeta_2^2(x_j) + \vartheta_2^2(x_j) + h_2^2(x_j)\right]}}$$

2.2.4 Divergence measures

Definition 2.2.5. [117] A divergence measure $\mathcal{D}v : \Phi(\mathcal{U}) \times \Phi(\mathcal{U}) \to [0,1]$ is a real-valued function satisfying the following properties:

(P1) $\mathcal{D}v(\mathcal{I}_1, \mathcal{I}_2) = \mathcal{D}v(\mathcal{I}_2, \mathcal{I}_1)$.

(P2) $\mathcal{D}v\left(\mathcal{I}_1, \mathcal{I}_1\right) = 0$.

(P3) $\mathcal{D}v\left(\mathcal{I}_1 \cap \mathcal{I}_3, \mathcal{I}_2 \cap \mathcal{I}_3\right) \leq \mathcal{D}v\left(\mathcal{I}_1, \mathcal{I}_2\right)$.

(P4) $\mathcal{D}v\left(\mathcal{I}_1 \cup \mathcal{I}_3, \mathcal{I}_2 \cup \mathcal{I}_3\right) \leq \mathcal{D}v\left(\mathcal{I}_1, \mathcal{I}_2\right)$ where $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \in \Phi\left(\mathcal{U}\right)$.

Some prevailing divergence measure (DvMs) among IFSs $\mathcal{I}_1$ and $\mathcal{I}_2$ are listed as follows:

(i) Vlachos and Sergiadis [153] proposed the formula of DvM as:

$$\mathcal{D}v_1\left(\mathcal{I}_1, \mathcal{I}_2\right) = \frac{1}{n}\sum_{j=1}^{n}\left[\zeta_1\left(x_j\right)\log\left(\frac{2\zeta_1\left(x_j\right)}{\zeta_1\left(x_j\right)+\zeta_2\left(x_j\right)}\right) + \vartheta_1\left(x_j\right)\log\left(\frac{2\vartheta_1\left(x_j\right)}{\vartheta_1\left(x_j\right)+\vartheta_2\left(x_j\right)}\right)\right]$$

(ii) Garg et al. [54] defined generalized parametric divergence of order α and degree β as:

$$\begin{aligned}
&\mathcal{D}v_2\left(\mathcal{I}_1, \mathcal{I}_2\right)\\
= &\frac{\alpha}{n(2-\beta)}\sum_{j=1}^{n}\left[\begin{array}{l}\zeta_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)\log\left(\dfrac{\zeta_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)}{\lambda\zeta_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)+(1-\lambda)\zeta_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)}\right)+\vartheta_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)\log\left(\dfrac{\vartheta_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)}{\lambda\vartheta_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)+(1-\lambda)\vartheta_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)}\right)\\
\quad+h_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)\log\left(\dfrac{h_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)}{\lambda h_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)+(1-\lambda)h_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)}\right)\end{array}\right]\\
+ &\frac{\alpha}{n(2-\beta)}\sum_{j=1}^{n}\left[\begin{array}{l}\zeta_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)\log\left(\dfrac{\zeta_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)}{\lambda\zeta_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)+(1-\lambda)\zeta_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)}\right)+\vartheta_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)\log\left(\dfrac{\vartheta_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)}{\lambda\vartheta_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)+(1-\lambda)\vartheta_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)}\right)\\
\quad+h_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)\log\left(\dfrac{h_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)}{\lambda h_2^{\frac{\alpha}{2-\beta}}\left(x_j\right)+(1-\lambda)h_1^{\frac{\alpha}{2-\beta}}\left(x_j\right)}\right)\end{array}\right]
\end{aligned}$$

(iii) Ohlan [121] defined the exponential DvM as:

$$\mathcal{D}v_3\left(\mathcal{I}_1, \mathcal{I}_2\right) = \sum_{j=1}^{n}\left[\begin{array}{l}2-\left(1-\left(\dfrac{\left(\zeta_1\left(x_j\right)-\zeta_2\left(x_j\right)\right)-\left(\vartheta_1\left(x_j\right)-\vartheta_2\left(x_j\right)\right)}{2}\right)\right)e^{\frac{\left(\zeta_1\left(x_j\right)-\zeta_2\left(x_j\right)\right)-\left(\vartheta_1\left(x_j\right)-\vartheta_2\left(x_j\right)\right)}{2}}\\
-\left(1-\left(\dfrac{\left(\zeta_2\left(x_j\right)-\zeta_1\left(x_j\right)\right)-\left(\vartheta_2\left(x_j\right)-\vartheta_1\left(x_j\right)\right)}{2}\right)\right)e^{\frac{\left(\zeta_2\left(x_j\right)-\zeta_1\left(x_j\right)\right)-\left(\vartheta_2\left(x_j\right)-\vartheta_1\left(x_j\right)\right)}{2}}\end{array}\right]$$

2.3 Archimedean t-norms and t-conorms

Definition 2.3.1. [92] A t-norm (fuzzy intersection) is a binary function given as $\mathcal{T}$: $[0,1]\times[0,1]\rightarrow[0,1]$ and satisfying the following axioms $\forall a,b,c,d\in[0,1]$.

Axiom 1: (Boundary condition) $\mathcal{T}(0,0)=0$, $\mathcal{T}(a,1)=a$.

Axiom 2: (Monotonicity) If $a\leq b$, $c\leq d$ then, $\mathcal{T}(a,c)\leq\mathcal{T}(b,d)$

Axiom 3: (Commutativity) $\mathcal{T}(a,b) = \mathcal{T}(b,a)$

Axiom 4: (Associativity) $\mathcal{T}\big(a, \mathcal{T}(b,c)\big) = \mathcal{T}\big(\mathcal{T}(a,b), c\big)$

A t-norm is called Archimedean if

Axiom 5: (Continuity) $\mathcal{T}$ is continuous

Axiom 5: (Subidempotency) $\mathcal{T}(a,a) < a \;\forall\; a \in [0,1]$

An Archimedean t-norm (AT) is called strict if

Axiom 7: (Strictly increasing) $\mathcal{T}$ is Strictly increasing in $(0,1) \times (0,1)$

Definition 2.3.2. [92] A function $\mathcal{N} : [0,1] \times [0,1] \to [0,1]$ is called t-conorm (fuzzy union) if the following axioms are satisfied $\forall\; a,b,c,d \in [0,1]$

Axiom 1: (Boundary condition) $\mathcal{N}(1,1) = 1$, $\mathcal{N}(a,0) = a$.

Axiom 2: (Monotonicity) If $a \leq b$, $c \leq d$ then, $\mathcal{N}(a,c) \leq \mathcal{N}(b,d)$

Axiom 3: (Commutativity) $\mathcal{N}(a,b) = \mathcal{N}(b,a)$

Axiom 4: (Associativity) $\mathcal{N}\big(a, \mathcal{N}(b,c)\big) = \mathcal{N}\big(\mathcal{N}(a,b), c\big)$

A t-conorm is called Archimedean if

Axiom 5: (Continuity) $\mathcal{N}$ is continuous

Axiom 5: (Subidempotency) $\mathcal{N}(a,a) > a \;\forall\; a \in [0,1]$

An Archimedean t-conorm (AC) is called strict if

Axiom 7: (Strictly increasing) $\mathcal{N}$ is Strictly increasing in $(0,1) \times (0,1)$

AT and AC satisfy the following relation $\forall\; a,b \in [0,1]$.

$$\mathcal{N}(a,b) = 1 - \mathcal{T}(1-a, 1-b)$$

Furthermore, strict AT ($\mathcal{T}$) and AC ($\mathcal{N}$) can be be expressed using continuous functions $t, s : [0,1] \to [0,\infty]$ respectively as

$$\mathcal{T}(a,b) = t^{-1}\big(t(a) + t(b)\big) \quad \text{and} \quad \mathcal{N}(a,b) = s^{-1}\big(s(a) + s(b)\big)$$

where t is decreasing function with $t(1) = 0$; s is an increasing function with $s(0) = 0$ and $s(a) = t(1 - a)$. Some standard t-conorm (TC) and t-norm (TN) are defined as [92]:

(i) Standard union and intersection

$$\mathcal{N}(a,b) = \max(a,b) \qquad ; \qquad \mathcal{T}(a,b) = \min(a,b)$$

(ii) Algebraic sum and product

$$\mathcal{N}(a,b) = a + b - ab \qquad ; \qquad \mathcal{T}(a,b) = ab$$

(iii) Bounded sum and difference

$$\mathcal{N}(a,b) = \min(1, a+b) \qquad ; \qquad \mathcal{T}(a,b) = \max(0, a+b-1)$$

(iv) Drastic union and intersection

$$\mathcal{N}(a,b) = \begin{cases} a & ; \quad \text{when } b = 0 \\ b & ; \quad \text{when } a = 0 \\ 1 & ; \quad \text{otherwise} \end{cases} \qquad ; \qquad \mathcal{T}(a,b) = \begin{cases} a & ; \quad \text{when } b = 1 \\ b & ; \quad \text{when } a = 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(v) Yager class of t-conorm and t-norm

$$\mathcal{N}(a,b) = \min\left(1, (a^p + b^p)^{\frac{1}{p}}\right) \qquad ; \qquad \mathcal{T}(a,b) = 1 - \min\left(1, ((1-a)^p + (1-b)^p)^{\frac{1}{p}}\right)$$

where $p > 0$.

Besides these some commonly used TCs and TNs along with their corresponding generator are tabulated in Table 2.1.

Table 2.1: Some TCs and TNs with corresponding additive generator

Operation Name	TC	Additive generator	TN	Additive generator
	$\mathcal{N}(a,b)$	$s(x)$	$\mathcal{T}(a,b)$	$t(x)$
Algebraic	$a + b - ab$	$-\log(1-x)$	ab	$-\log(x)$
Einstein	$\frac{a+b}{1+ab}$	$\log\left(\frac{1+x}{1-x}\right)$	$\frac{ab}{1-(1-a)(1-b)}$	$\log\left(\frac{2-x}{x}\right)$
Hamachar $(\gamma > 0)$	$\frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab}$	$\log\left(\frac{\gamma+(1-\gamma)(1-x)}{1-x}\right)$	$\frac{ab}{\gamma+(1-\gamma)(a+b-ab)}$	$\log\left(\frac{\gamma+(1-\gamma)x}{x}\right)$

2.4 Aggregation Operator

Definition 2.4.1. [22] An AO is a function $\mathcal{A} : [0,1]^n \to [0,1]$ defined as

$$\mathcal{A}\left(a_1, a_2, \ldots, a_n\right) = a$$

which takes n arguments as input and returns a unique representative of input arguments as output. It must satisfy the two axioms stated as:

Axiom 1: (Boundary conditions) $\mathcal{A}\left(0, 0, \ldots, 0\right) = 0 \quad ; \quad \mathcal{A}\left(1, 1, \ldots, 1\right) = 1$;

Axiom 2: (Monotonicity) If $a_j \leq b_j$ for $j = 1, 2, \ldots, n$ then, $\mathcal{A}\left(a_1, a_2, \ldots, a_n\right) \leq \mathcal{A}(b_1, b_2, \ldots, b_n)$.

Definition 2.4.2. [179, 185] For IFNs $\mathcal{I}_1 = (\zeta_1, \vartheta_1)$ and $\mathcal{I}_2 = (\zeta_2, \vartheta_2)$, and real number $\rho > 0$, the following operational laws are defined:

(i) $\mathcal{I}_1 \oplus \mathcal{I}_2 = (\zeta_1 + \zeta_2 - \zeta_1\zeta_2, \vartheta_1\vartheta_2)$.

(ii) $\mathcal{I}_1 \otimes \mathcal{I}_2 = (\zeta_1\zeta_2, \vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2)$.

(iii) $\rho\mathcal{I}_1 = (1 - (1 - \zeta_1)^\rho, \vartheta_1^\rho)$

(iv) $\mathcal{I}_1^\rho = (\zeta_1^\rho, 1 - (1 - \vartheta_1)^\rho)$

Based on the above operational laws, several authors developed averaging and geometric AOs. Some of the AOs for aggregating $'n'$ independent IFNs $\mathcal{I}_j = (\zeta_j, \vartheta_j)$ $(j = 1, 2, \ldots, n)$ are listed below:

(i) Xu [179] proposed IF weighted averaging (IFWA) and IF ordered weighted averaging (IFOWA) operators as:

$$\text{IFWA}(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \left(1 - \prod_{j=1}^{n}(1 - \zeta_j)^{\xi_j}, \prod_{j=1}^{n}\vartheta_j^{\xi_j}\right) \tag{2.9}$$

$$\text{IFOWA}(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \left(1 - \prod_{j=1}^{n}(1 - \zeta_{\tau(j)})^{\psi_j}, \prod_{j=1}^{n}\vartheta_{\tau(j)}^{\psi_j}\right) \tag{2.10}$$

(ii) Xu and Yager [185] proposed IF weighted geometric (IFWG) and IF ordered weighted geometric (IFOWG) operators as:

$$\text{IFWG}(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \left(\prod_{j=1}^{n} \zeta_j^{\xi_j}, 1 - \prod_{j=1}^{n} (1-\vartheta_j)^{\xi_j}\right) \tag{2.11}$$

$$\text{IFOWG}(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \left(\prod_{j=1}^{n} \zeta_{\tau(j)}^{\psi_j}, 1 - \prod_{j=1}^{n} (1-\vartheta_{\tau(j)})^{\psi_j}\right) \tag{2.12}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ is the weight vector associated with IFNs $\mathcal{I}_j$ $(j = 1, 2, \ldots, n)$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T$ is the weight vector associated with IFOWA/IFOWG operators and these weights satisfy the conditions $\psi_j, \xi_j > 0$; $\sum\limits_{j=1}^{n} \psi_j = \sum\limits_{j=1}^{n} \xi_j = 1$. Here $\big(\tau(1), \tau(2), \ldots, \tau(n)\big)$ is an arrangement of $(1, 2, \ldots, n)$ such that $\mathcal{S}\left(\mathcal{I}_{\tau(j-1)}\right) \geq \mathcal{S}\left(\mathcal{I}_{\tau(j)}\right)$ for each $j = 2, 3, \ldots, n$.

Bonferroni [17] recommended the concept of BM operator in order to aggregate dependent arguments and defined it as:

Definition 2.4.3. [17] For non-negative real numbers a_j $(j = 1, 2, \ldots, n)$ and p, q if

$$\mathcal{B}^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{j,l=1 \\ j \neq l}}^{n} a_j^p a_l^q\right)^{\frac{1}{p+q}} \tag{2.13}$$

then $B^{p,q}$ is called BM operator.

Obviously, BM satisfies the following properties [192]:

(i) $\mathcal{B}^{p,q}(0, 0, \ldots, 0) = 0$.

(ii) $\mathcal{B}^{p,q}(a, a, \ldots, a) = a$ if $a_j = a$ for $j = 1, 2, \ldots, n$.

(iii) $\mathcal{B}^{p,q}$ is monotonic i.e., if $a_j \geq d_j$ for $j = 1, 2, \ldots, n$ then $\mathcal{B}^{p,q}(a_1, a_2, \ldots, a_n) \geq \mathcal{B}^{p,q}(d_1, d_2, \ldots, d_n)$.

(iv) $\min\limits_j\{a_j\} \leq \mathcal{B}^{p,q}(a_1, a_2, \ldots, a_n) \leq \max\limits_j\{a_j\}$

Xu and Yager [188] extended the idea of BM operator to IFS theory and defined it as:

Definition 2.4.4. [188] For IFNs $\mathcal{I}_j$ $(j = 1, 2, \ldots, n)$ and positive real numbers p, q, the IF bonferroni mean (IFBM) and IF weighted bonferroni mean (IFWBM) operators are defined as:

$$\text{IFBM}^{p,q}(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{j,l=1 \\ j \neq l}}^{n} \left(\mathcal{I}_j^p \otimes \mathcal{I}_l^q \right) \right) \right)^{\frac{1}{p+q}}$$

and $$\text{IFWBM}^{p,q}(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{j,l=1 \\ j \neq l}}^{n} \left((\xi_j \mathcal{I}_j)^p \otimes (\xi_l \mathcal{I}_l)^q \right) \right) \right)^{\frac{1}{p+q}}$$

Yager [190] proposed the PA operator as:

Definition 2.4.5. [190] For a collection of "n" data a_j, the PA operator is defined as:

$$PA(a_1, a_2, \ldots, a_n) = \frac{\sum_{j=1}^{n} \Big(1 + T(a_j)\Big) a_j}{\sum_{j=1}^{n} \Big(1 + T(a_j)\Big)} \tag{2.14}$$

where $T(a_j) = \sum_{\substack{l=1 \\ l \neq j}}^{n} \text{Sup}(a_j, a_l)$ and $\text{Sup}(a_j, a_l)$ is the support of a_j from a_l satisfying the following properties:

(i) $\text{Sup}(a_j, a_l) \in [0, 1]$.

(ii) $\text{Sup}(a_j, a_l) = \text{Sup}(a_l, a_j)$.

(iii) $\text{Sup}(a_j, a_l) \geq \text{Sup}(a_{j1}, a_{l1})$ if $|a_j - a_l| < |a_{j1} - a_{l1}|$

Here, $\text{Sup}(a_j, a_l)$ is a similarity index.

Xu [175] extended the idea of PA operator to IFS environment and defined PA operator as:

Definition 2.4.6. [175] For IFNs $\mathcal{I}_j$ $(j = 1, 2, \ldots, n)$, the IF weighted power averaging (IFWPA) operator is defined as:

$$\text{IFWPA}\left(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\right) = \frac{\bigoplus_{j=1}^{n} \left[\xi_j \left(1 + T(\mathcal{I}_j)\right) \mathcal{I}_j\right]}{\sum_{j=1}^{n} \left[\xi_j \left(1 + T(\mathcal{I}_j)\right)\right]}$$

Yager [191] introduced the prioritized operator by considering the prioritization relationship among the arguments and defined it as:

Definition 2.4.7. [191] Let $a = \{a_1, a_2, \ldots, a_n\}$ be a collection of criteria and there is a prioritization among the criteria expressed by the strict priority orders $a_1 \succ a_2 \succ \ldots \succ a_n$ where $a_j \succ a_{j+1}$ indicates that the criteria a_j has a higher priority than the criteria a_{j+1} for $j \in \{1, 2, \ldots, n-1\}$. For any alternative x, the value $a_j(x)$ gives the performance of x under the criteria a_j and $a_j(x) \in [0, 1]$. Let

$$\text{PrA}\big(a_1, a_2, \ldots, a_n\big) = \sum_{j=1}^{n} \xi_j a_j(x) \tag{2.15}$$

where $\xi_j = \frac{T'_j}{\sum\limits_{j=1}^{n} T'_j}$, $T'_j = \prod\limits_{l=1}^{j-1} a_l(x)$, $j = 2, 3, \ldots, n$ and $T'_1 = 1$. Then PrA is known as prioritized averaging operator.

Yu [203] extended the idea of PrA operator under IFS theory and defined it as:

Definition 2.4.8. [203] For a collection of IFNs $\mathcal{I}_j$ $(j = 1, 2, \ldots, n)$ the IF prioritized averaging (IFPrA) operator is defined as:

$$\text{IFPrA}\big(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\big) = \left(1 - \prod_{j=1}^{n} (1-\zeta_j)^{\xi_j}, \prod_{j=1}^{n} (\vartheta_j)^{\xi_j}\right) \tag{2.16}$$

where $\xi_j = \frac{T'_j}{\sum\limits_{j=1}^{n} T'_j}$, $T'_j = \prod\limits_{l=1}^{j-1} \mathcal{S}(\mathcal{I}_l)$, $j \in \{2, 3, \ldots, n\}$ and $T_1 = 1$.

2.5 Description of decision making model

In this thesis, we have developed various DM techniques based on AOs and information measures under CIFS environment. The brief description of the MCDM/MCGDM problem is given as follows:

Consider a set $\{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m\}$ of 'm' alternatives characterized by another collection $\{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\}$ of 'n' criteria is to be evaluated with the corresponding criteria weight vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ such that $\xi_v > 0$ and $\sum\limits_{v=1}^{n} \xi_v = 1$. Further, the alternatives $\mathcal{V}_u$ are evaluated by 'k' expert(s) $\big\{\mathcal{E}^{(1)}, \mathcal{E}^{(2)}, \ldots, \mathcal{E}^{(k)}\big\}$ who give their rating values in terms of CIFNs $\mathcal{C}_{uv}^{(z)} = \left(\left(\zeta_{uv}^{(z)}, w_{\zeta_{uv}}^{(z)}\right), \left(\vartheta_{uv}^{(z)}, w_{\vartheta_{uv}}^{(z)}\right)\right)$ where $0 \leq \zeta_{uv}^{(z)}, w_{\zeta_{uv}}^{(z)}, \vartheta_{uv}^{(z)}, w_{\vartheta_{uv}}^{(z)}, \zeta_{uv}^{(z)} + w_{\zeta_{uv}}^{(z)}, \vartheta_{uv}^{(z)} + w_{\vartheta_{uv}}^{(z)} \leq 1$ for each $z = 1, 2, \ldots, k$; $u = 1, 2, \ldots, m$ and $v = 1, 2, \ldots n$. The terms $\zeta_{uv}^{(z)}$, $w_{\zeta_{uv}}^{(z)}$ and $\vartheta_{uv}^{(z)}$, $w_{\vartheta_{uv}}^{(z)}$ exhibit satisfaction and dissatisfaction degrees respectively corresponding to alternative $\mathcal{V}_u$ under the criteria $\mathfrak{B}_v$ given by the expert $\mathcal{E}^{(z)}$. However if there is only one expert i.e., $k = 1$, then the problem may be reffered to as MCDM problem and if there are more than one experts i.e., $k > 1$ then the problem is called as MCGDM problem. The main objective of the problem is to order the alternatives from the most favorable to the least favorable ones. The general procedure of attaining the most favorable alternative involves the following steps:

Step 1: Construct the CIF decision matrices $\mathcal{M}^{(z)} = \left(\mathcal{C}_{uv}^{(z)}\right)_{m \times n}$ representing information related to all alternatives for the different criteria, given by 'k' experts, as:

$$\mathcal{M}^{(z)} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{C}_{11}^{(z)} & \mathcal{C}_{12}^{(z)} & \ldots & \mathcal{C}_{1n}^{(z)} \\ \mathcal{C}_{21}^{(z)} & \mathcal{C}_{22}^{(z)} & \ldots & \mathcal{C}_{2n}^{(z)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{m1}^{(z)} & \mathcal{C}_{m2}^{(z)} & \ldots & \mathcal{C}_{mn}^{(z)} \end{pmatrix} \end{array} \tag{2.17}$$

Step 2: If $k > 1$ then, accumulate the different decision matrices $\mathcal{M}^{(z)}$ into a single one $\mathcal{M}$ by utilizing some appropriate technique i.e.,

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \ldots & \mathcal{C}_{1n} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \ldots & \mathcal{C}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{m1} & \mathcal{C}_{m2} & \ldots & \mathcal{C}_{mn} \end{pmatrix} \end{array} \tag{2.18}$$

Step 3: If the criteria $\mathfrak{B}_v$ are all of benefit type, then there is no need of normalization. However, if there are different types of criteria namely benefit and cost then, convert the cost type into profit type by using the following normalization formula:

$$\mathcal{C}_{uv} = \begin{cases} \left(\left(\zeta_{uv}, w_{\zeta_{uv}}\right), \left(\vartheta_{uv}, w_{\vartheta_{uv}}\right)\right) & ; \quad \text{if } \mathfrak{B}_v \text{ is of benefit type} \\ \left(\left(\vartheta_{uv}, w_{\vartheta_{uv}}\right), \left(\zeta_{uv}, w_{\zeta_{uv}}\right)\right) & ; \quad \text{if } \mathfrak{B}_v \text{ is of cost type} \end{cases} \tag{2.19}$$

Step 4: Obtain a unique representative of the judgement values corresponding to each alternative by utilizing some appropriate technique.

Step 5: Obtain the crisp value corresponding to each alternative by utilizing some appropriate defuzzification method.

Step 6: Rank the alternatives in accordance with defuzzified values and obtain the most favorable one.

Chapter 3

Distance measures between the complex intuitionistic fuzzy sets and its applications to the decision-making process[1]

In this chapter, some series of the distance measures by using Hamming, Euclidean, and Hausdorff metrics have been presented. Based on these measures, various desirable relations have been studied in detail. Further, based on these distance measures, a decision-making method has been presented for finding the best alternative under the set of the feasible one. Illustrative examples from the field of pattern recognition as well as the medical diagnosis have been taken to validate the approach.

3.1 Introduction

Distance or similarity measures are of key importance in a number of theoretical and applied statistical inferences and data processing problems. Furthermore, it has been deduced from the studies that the similarity, entropy and divergence measures could be induced by the normalized distance measure based on their axiomatic definitions. Further, CIFSs have great powerful ability to express the uncertainty and fuzzy decision process more precisely and objectively and hence is a useful tool for handling the imprecise and

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ambiguous information under the uncertain environment. By keeping the advantages of these as well as motivated by the characteristics of the CIFS model, this chapter is focused on exploring the structural characteristics of CIFSs and its information measures for the handling of multi-dimensional complex data sets. In the present chapter, by incorporating the concept of the complex numbers and IFSs, the axiomatic definition of the distance measures between CIFSs is given, and subsequently used to introduce a series of the functions that measures the distance between two CIFSs. Based on these distance measures, by using Hamming, Euclidean, and Hausdorff metrics, some desirable relations between them have been investigated in detail. Furthermore, efforts have been put forth to solve the pattern recognition as well as the medical diagnosis problems by considering the multi-dimensional complex data sets. The advantages of the proposed measures over the existing measure in the CIFSs environment have been discussed. To the best of our knowledge, no work has been carried out in the direction of the distance measures under CIFSs environment. Due to this fact as well as the consideration to handle the periodicity that exists in two-dimensional information during the decision-making process, it is necessary to develop such distance measures.

3.2 Proposed distance Measures between CIFSs

In this section, we present the Hamming and the Euclidean distances between CIFSs which can be used in real scientific and engineering applications.

Let $\Psi(\mathcal{U})$ be the class of CIFSs over the universal set $\mathcal{U} = \{x_1, x_2, \ldots, x_n\}$. Then we define the distances for CIFSs $\mathcal{C}_1 = \{(x, (\zeta_1(x), w_{\zeta_1}(x)), (\vartheta_1(x), w_{\vartheta_1}(x))) \mid x \in \mathcal{U}\}$; $\mathcal{C}_2 = \{(x, (\zeta_2(x), w_{\zeta_2}(x)), (\vartheta_2(x), w_{\vartheta_2}(x))) \mid x \in \mathcal{U}\}$ as follows.

(i) Hamming distance measure

$$\mathcal{D}_1(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{4} \sum_{j=1}^{n} \left\{ \begin{array}{l} \mid \zeta_1(x_j) - \zeta_2(x_j) \mid + \mid \vartheta_1(x_j) - \vartheta_2(x_j) \mid \\ + \mid w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j) \mid + \mid w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j) \mid \end{array} \right\} \tag{3.1}$$

(ii) Normalized Hamming distance

$$\mathcal{D}_2(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{4n} \sum_{j=1}^{n} \left\{ \begin{array}{l} |\zeta_1(x_j) - \zeta_2(x_j)| + |\vartheta_1(x_j) - \vartheta_2(x_j)| \\ + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)| + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)| \end{array} \right\} \tag{3.2}$$

(iii) Euclidean distance

$$\mathcal{D}_3\left(\mathcal{C}_1, \mathcal{C}_2\right)=\left[\frac{1}{4} \sum_{j=1}^{n}\left\{\begin{array}{l}|\zeta_1(x_j)-\zeta_2(x_j)|^2+|\vartheta_1(x_j)-\vartheta_2(x_j)|^2 \\ +|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^2+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^2\end{array}\right\}\right]^{\frac{1}{2}} \tag{3.3}$$

(iv) Normalized Euclidean distance

$$\mathcal{D}_4\left(\mathcal{C}_1, \mathcal{C}_2\right)=\left[\frac{1}{4n} \sum_{j=1}^{n}\left\{\begin{array}{l}|\zeta_1(x_j)-\zeta_2(x_j)|^2+|\vartheta_1(x_j)-\vartheta_2(x_j)|^2 \\ +|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^2+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^2\end{array}\right\}\right]^{\frac{1}{2}} \tag{3.4}$$

Then, based on the distance properties as defined in Definition 2.2.2, we can obtain the following results:

Theorem 3.2.1. The above defined distance $\mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_2\right)$, between two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfies the following properties (P1)–(P4):

(P1) $0 \leq \mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_2\right) \leq 1$;

(P2) $\mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_2\right)=0$ if and only if $\mathcal{C}_1=\mathcal{C}_2$;

(P3) $\mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_2\right)=\mathcal{D}_2(\mathcal{C}_2, \mathcal{C}_1)$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then $\mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_3\right) \geq \mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_2\right)$ and $\mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_3\right) \geq \mathcal{D}_2\left(\mathcal{C}_2, \mathcal{C}_3\right)$, where $\mathcal{C}_3 \in \Psi(\mathcal{U})$.

Proof. For any two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$, we have

(P1) Since $0 \leq \zeta_1(x_j), \zeta_2(x_j) \leq 1$ and $0 \leq \vartheta_1(x_j), \vartheta_2(x_j) \leq 1$. It implies that $0 \leq |\zeta_1(x_j)-\zeta_2(x_j)| \leq 1$ and $0 \leq |\vartheta_1(x_j)-\vartheta_2(x_j)| \leq 1$. Also, $0 \leq w_{\zeta_1}(x_j), w_{\zeta_2}(x_j) \leq 1$ and $0 \leq w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j) \leq 1$ which implies that $0 \leq |w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)| \leq 1$ and $0 \leq |w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)| \leq 1$. Thus, we get $|\zeta_1(x_j)-\zeta_2(x_j)|+|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)| \leq 4$ which implies that $\mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_2\right) \leq 1$. Hence, $0 \leq \mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_2\right) \leq 1$.

(P2) For any two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$,

$$\begin{aligned}
& \mathcal{D}_2(\mathcal{C}_1, \mathcal{C}_2) = 0 \\
\Leftrightarrow\ & \frac{1}{4n} \sum_{j=1}^{n} \left\{ \begin{array}{l} |\zeta_1(x_j) - \zeta_2(x_j)| + |\vartheta_1(x_j) - \vartheta_2(x_j)| \\ + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)| + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)| \end{array} \right\} = 0 \\
\Leftrightarrow\ & \zeta_1(x_j) - \zeta_2(x_j) = 0, \vartheta_1(x_j) - \vartheta_2(x_j) = 0, \\
& w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j) = 0, \text{and } w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j) = 0 \text{ for all } j \\
\Leftrightarrow\ & \zeta_1(x_j) = \zeta_2(x_j), \vartheta_1(x_j) = \vartheta_2(x_j), w_{\zeta_1}(x_j) = w_{\zeta_2}(x_j), w_{\vartheta_1}(x_j) = w_{\vartheta_2}(x_j) \text{ for all } j \\
\Leftrightarrow\ & \mathcal{C}_1 = \mathcal{C}_2
\end{aligned}$$

(P3) Since for any two numbers a and b, we have $|a-b| = |b-a|$. Thus, we have

$$\begin{aligned}
\mathcal{D}_2(\mathcal{C}_1, \mathcal{C}_2) &= \frac{1}{4n} \sum_{j=1}^{n} \left\{ \begin{array}{l} |\zeta_1(x_j) - \zeta_2(x_j)| + |\vartheta_1(x_j) - \vartheta_2(x_j)| + \\ |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)| + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)| \end{array} \right\} \\
&= \frac{1}{4n} \sum_{j=1}^{n} \left\{ \begin{array}{l} |\zeta_2(x_j) - \zeta_1(x_j)| + |\vartheta_2(x_j) - \vartheta_1(x_j)| + \\ |w_{\zeta_2}(x_j) - w_{\zeta_1}(x_j)| + |w_{\vartheta_2}(x_j) - w_{\vartheta_1}(x_j)| \end{array} \right\} \\
&= \mathcal{D}_2(\mathcal{C}_2, \mathcal{C}_1)
\end{aligned}$$

Hence, $\mathcal{D}_2(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{D}_2(\mathcal{C}_2, \mathcal{C}_1)$.

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$, then $\zeta_1(x_j) \leq \zeta_2(x_j) \leq \zeta_3(x_j)$, $\vartheta_1(x_j) \geq \vartheta_2(x_j) \geq \vartheta_3(x_j)$ and $w_{\zeta_1}(x_j) \leq w_{\zeta_2}(x_j) \leq w_{\zeta_3}(x_j)$, $w_{\vartheta_1}(x_j) \geq w_{\vartheta_2}(x_j) \geq w_{\vartheta_3}(x_j)$ for all $x_j \in \mathcal{U}$. It implies that $|\zeta_1(x_j) - \zeta_3(x_j)| \geq |\zeta_1(x_j) - \zeta_2(x_j)|$, $|\vartheta_1(x_j) - \vartheta_3(x_j)| \geq |\vartheta_1(x_j) - \vartheta_2(x_j)|$ and $|w_{\zeta_1}(x_j) - w_{\zeta_3}(x_j)| \geq |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|$, $|w_{\vartheta_1}(x_j) - w_{\vartheta_3}(x_j)| \geq |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|$ $\Rightarrow |\zeta_1(x_j) - \zeta_3(x_j)| + |\vartheta_1(x_j) - \vartheta_3(x_j)| + |w_{\zeta_1}(x_j) - w_{\zeta_3}(x_j)| + |w_{\vartheta_1}(x_j) - w_{\vartheta_3}(x_j)| \geq |\zeta_1(x_j) - \zeta_2(x_j)| + |\vartheta_1(x_j) - \vartheta_2(x_j)| + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)| + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|$ which implies that $\mathcal{D}_2(\mathcal{C}_1, \mathcal{C}_3) \geq \mathcal{D}_2(\mathcal{C}_1, \mathcal{C}_2)$. Similarly, $\mathcal{D}_2(\mathcal{C}_1, \mathcal{C}_3) \geq \mathcal{D}_2(\mathcal{C}_2, \mathcal{C}_3)$.

Hence, $\mathcal{D}_2$ is a valid distance measure. $\square$

Theorem 3.2.2. For any two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$, the distance measure $\mathcal{D}_4(\mathcal{C}_1, \mathcal{C}_2)$ satisfies the properties (P1)-(P4) as described in Definition 2.2.2.

Proof. For any two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$, we have

(P1) Since $0 \leq \zeta_1(x_j), \zeta_2(x_j), \vartheta_1(x_j), \vartheta_2(x_j) \leq 1$ which implies that $0 \leq |\zeta_1(x_j)-\zeta_2(x_j)|^2 \leq 1$ and $0 \leq |\vartheta_1(x_j) - \vartheta_2(x_j)|^2 \leq 1$. Also, $0 \leq w_{\zeta_1}(x_j), w_{\zeta_2}(x_j), w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j) \leq 1$ which implies that $0 \leq |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|^2 \leq 1$ and $0 \leq |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|^2 \leq 1$. Hence, $|\zeta_1(x_j) - \zeta_2(x_j)|^2 + |\vartheta_1(x_j) - \vartheta_2(x_j)|^2 + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|^2 + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|^2 \leq 4$ and thus, we have $\mathcal{D}_4(\mathcal{C}_1, \mathcal{C}_2) \leq 1$. Also, $\mathcal{D}_4(\mathcal{C}_1, \mathcal{C}_2) \geq 0$ and therefore, we get $0 \leq \mathcal{D}_4(\mathcal{C}_1, \mathcal{C}_2) \leq 1$.

(P2) For $\mathcal{D}_4(\mathcal{C}_1, \mathcal{C}_2) = 0 \Leftrightarrow \left[\frac{1}{4n}\sum_{j=1}^{n}\left\{|\zeta_1(x_j) - \zeta_2(x_j)|^2 + |\vartheta_1(x_j) - \vartheta_2(x_j)|^2 + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|^2 + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|^2\right\}\right]^{\frac{1}{2}} = 0 \Leftrightarrow \zeta_1(x_j) - \zeta_2(x_j) = 0$, $\vartheta_1(x_j) - \vartheta_2(x_j) = 0$, $w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j) = 0$ and $w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j) = 0$ for all $j = 1, 2, \ldots, n \Leftrightarrow \zeta_1(x_j) = \zeta_2(x_j)$, $\vartheta_1(x_j) = \vartheta_2(x_j)$, $w_{\zeta_1}(x_j) = w_{\zeta_2}(x_j)$, $w_{\vartheta_1}(x_j) = w_{\vartheta_2}(x_j)$ for all $j = 1, 2, \ldots, n$ $\Leftrightarrow \mathcal{C}_1 = \mathcal{C}_2$.

(P3) Since for any two numbers a and b, we have $|a - b| = |b - a|$. Thus, we have $\mathcal{D}_4(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{D}_4(\mathcal{C}_2, \mathcal{C}_1)$.

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$, then $\zeta_1(x_j) \leq \zeta_2(x_j) \leq \zeta_3(x_j)$, $\vartheta_1(x_j) \geq \vartheta_2(x_j) \geq \vartheta_3(x_j)$ and $w_{\zeta_1}(x_j) \leq w_{\zeta_2}(x_j) \leq w_{\zeta_3}(x_j)$, $w_{\vartheta_1}(x_j) \geq w_{\vartheta_2}(x_j) \geq w_{\vartheta_3}(x_j)$ for all $x_j \in \mathcal{U}$. Thus, we have $|\zeta_1(x_j)-\zeta_3(x_j)|^2 \geq |\zeta_1(x_j)-\zeta_2(x_j)|^2$, $|\vartheta_1(x_j)-\vartheta_3(x_j)|^2 \geq |\vartheta_1(x_j)-\vartheta_2(x_j)|^2$ and $|w_{\zeta_1}(x_j) - w_{\zeta_3}(x_j)|^2 \geq |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|^2$, $|w_{\vartheta_1}(x_j) - w_{\vartheta_3}(x_j)|^2 \geq |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|^2$ which further implies that $\sum_{j=1}^{n}\left\{|\zeta_1(x_j) - \zeta_3(x_j)|^2 + |\vartheta_1(x_j) - \vartheta_3(x_j)|^2 + |w_{\zeta_1}(x_j) - w_{\zeta_3}(x_j)|^2 + |w_{\vartheta_1}(x_j) - w_{\vartheta_3}(x_j)|^2\right\} \geq \sum_{j=1}^{n}\left\{|\zeta_1(x_j) - \zeta_2(x_j)|^2 + |\vartheta_1(x_j) - \vartheta_2(x_j)|^2 + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|^2 + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|^2\right\}$ and hence $\mathcal{D}_4(\mathcal{C}_1, \mathcal{C}_3) \geq \mathcal{D}_4(\mathcal{C}_1, \mathcal{C}_2)$. Similarly, we have $\mathcal{D}_4(\mathcal{C}_1, \mathcal{C}_3) \geq \mathcal{D}_4(\mathcal{C}_2, \mathcal{C}_3)$.

Hence, $\mathcal{D}_4$ is a valid distance measure. □

Example 3.2.1. Consider two sets $\mathcal{C}_1$ and $\mathcal{C}_2$ which are represented in the form of the CIFSs on a universe $\mathcal{U}$ as given by $\mathcal{C}_1 = \{(x_1, (0.4, 0.3), (0.3, 0.2)), (x_2, (0.7, 0.5), (0.1, 0.4))\}$

and $\mathcal{C}_2 = \{(x_1, (0.7, 0.5), (0.1, 0.3)),\ (x_2, (0.4, 0.6),\ (0.5, 0.2))\}$, then

$$\begin{aligned}
\mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_2\right) &= \frac{1}{4\times 2}\Bigg[|0.4-0.7| + |0.3-0.1| + |0.3-0.5| + |0.2-0.3| \\
&\quad + |0.7-0.4| + |0.1-0.5| + |0.5-0.6| + |0.4-0.2|\Bigg] \\
&= 0.2250
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}_4\left(\mathcal{C}_1, \mathcal{C}_2\right) &= \Bigg[\frac{1}{4\times 2}\Bigg(|0.4-0.7|^2 + |0.3-0.1|^2 + |0.3-0.5|^2 + |0.2-0.3|^2 \\
&\quad + |0.7-0.4|^2 + |0.1-0.5|^2 + |0.5-0.6|^2 + |0.4-0.2|^2\Bigg)\Bigg]^{\frac{1}{2}} \\
&= 0.2449
\end{aligned}$$

Theorem 3.2.3. For any two CIFSs, measures $\mathcal{D}_1$ and $\mathcal{D}_3$ satisfy the following properties:

(i) $0 \leq \mathcal{D}_1 \leq n$

(ii) $0 \leq \mathcal{D}_3 \leq \sqrt{n}$

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ are two CIFSs defined on $\mathcal{U}$. Clearly $\mathcal{D}_1\left(\mathcal{C}_1, \mathcal{C}_2\right) = n\mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_2\right)$ and by Theorem 3.2.1, we have $0 \leq \mathcal{D}_2\left(\mathcal{C}_1, \mathcal{C}_2\right) \leq 1$. It implies $0 \leq \frac{\mathcal{D}_1(\mathcal{C}_1, \mathcal{C}_2)}{n} \leq 1$. Thus, we obtain $0 \leq \mathcal{D}_1\left(\mathcal{C}_1, \mathcal{C}_2\right) \leq n$. Similarly, we can obtain $0 \leq \mathcal{D}_3\left(\mathcal{C}_1, \mathcal{C}_2\right) \leq \sqrt{n}$. $\square$

Theorem 3.2.4. For any two CIFSs, the measures $\mathcal{D}_1, \mathcal{D}_3$ and $\mathcal{D}_2, \mathcal{D}_4$ satisfy the following inequalities:

(i) $\mathcal{D}_3 \leq \sqrt{\mathcal{D}_1}$

(ii) $\mathcal{D}_4 \leq \sqrt{\mathcal{D}_2}$

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two CIFSs. Since $0 \leq \zeta_1(x_j), \zeta_2(x_j) \leq 1, 0 \leq \vartheta_1(x_j), \vartheta_2(x_j) \leq 1$, $0 \leq w_{\zeta_1}(x_j)$, $w_{\zeta_2}(x_j) \leq 1$ and $0 \leq w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j) \leq 1$. Then by using the property that $a^2 \leq a$ for any $a \in [0, 1]$ we have, $\mid \zeta_1(x_j) - \zeta_2(x_j) \mid^2 \leq \mid \zeta_1(x_j) - \zeta_2(x_j) \mid$, $\mid \vartheta_1(x_j) - \vartheta_2(x_j) \mid^2 \leq \mid \vartheta_1(x_j) - \vartheta_2(x_j) \mid$, $\mid w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j) \mid^2 \leq \mid w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j) \mid$ and $\mid w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j) \mid^2 \leq \mid w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j) \mid$ which implies that $\mid \zeta_1(x_j) - \zeta_2(x_j) \mid^2 + \mid \vartheta_1(x_j) - \vartheta_2(x_j) \mid^2 + \mid w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j) \mid^2 + \mid w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j) \mid^2 \leq \mid \zeta_1(x_j) - \zeta_2(x_j) \mid + \mid \vartheta_1(x_j) - \vartheta_2(x_j) \mid$

$+ \mid w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j) \mid + \mid w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j) \mid$ and hence $\frac{1}{4}\sum_{j=1}^{n}\Big\{ \mid \zeta_1(x_j) - \zeta_2(x_j) \mid^2$ $+ \mid \vartheta_1(x_j) - \vartheta_2(x_j) \mid^2 + \mid w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j) \mid^2 + \mid w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j) \mid^2 \leq \frac{1}{4}\sum_{j=1}^{n}\Big\{ \mid$ $\zeta_1(x_j) - \zeta_2(x_j) \mid + \mid \vartheta_1(x_j) - \vartheta_2(x_j) \mid + \mid w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j) \mid + \mid w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j) \mid \Big\}$. Therefore, $\mathcal{D}_3(\mathcal{C}_1, \mathcal{C}_2) \leq \sqrt{\mathcal{D}_1(\mathcal{C}_1, \mathcal{C}_2)}$. Since, $\mathcal{C}_1$ and $\mathcal{C}_2$ are arbitrary CIFSs and thus $\mathcal{D}_3 \leq \sqrt{\mathcal{D}_1}$ is true for all CIFSs.

Similarly, we can prove the other part. □

However, in many practical situations, there are various problems in which the different set may have taken different weights and thus, the weight ξ_j with $\xi_j > 0, \sum_{j=1}^{n} \xi_j = 1$ of the element $x_j \in \mathcal{U}$ come into the existence. Based on it, in the following, we define a weighted Hamming and Euclidean distances between the two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$.

(i) Weighted Hamming distance

$$\mathcal{D}_5(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{4}\sum_{j=1}^{n}\left\{\xi_j \begin{pmatrix} |\zeta_1(x_j) - \zeta_2(x_j)| + |\vartheta_1(x_j) - \vartheta_2(x_j)| \\ + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)| + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)| \end{pmatrix}\right\} \tag{3.5}$$

(ii) Weighted Euclidean distance

$$\mathcal{D}_6(\mathcal{C}_1, \mathcal{C}_2) = \left[\frac{1}{4}\sum_{j=1}^{n}\left\{\xi_j \begin{pmatrix} |\zeta_1(x_j) - \zeta_2(x_j)|^2 + |\vartheta_1(x_j) - \vartheta_2(x_j)|^2 \\ + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|^2 + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|^2 \end{pmatrix}\right\}\right]^{\frac{1}{2}} \tag{3.6}$$

Especially, when $\xi_j = 1/n$, for $j = 1, 2, \ldots, n$, then Eqs. (3.5) and (3.6) reduce to Eqs. (3.2) and (3.4) respectively.

Theorem 3.2.5. The weighted measures $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2), (r = 5, 6)$ also satisfy the following properties for $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \Psi(\mathcal{U})$:

(P1) $0 \leq \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2) \leq 1$;

(P2) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2) = 0 \Leftrightarrow \mathcal{C}_1 = \mathcal{C}_2$;

(P3) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_1)$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2) \leq \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_3)$ and $\mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_3) \leq \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_3)$, for $\mathcal{C}_3 \in \Psi(\mathcal{U})$.

Proof. Since $\xi_j > 0$ and $\sum\limits_{j=1}^{n} \xi_j = 1$, then we can easily get $0 \le \mathcal{D}_5(\mathcal{C}_1, \mathcal{C}_2) \le \mathcal{D}_1(\mathcal{C}_1, \mathcal{C}_2) \le 1$. Thus, $\mathcal{D}_5(\mathcal{C}_1, \mathcal{C}_2)$ satisfies (P1). The proof of the properties (P2)-(P4) are similar to the Theorem 3.2.1. Similarly, for $\mathcal{D}_6$. □

Theorem 3.2.6. The distance measures $\mathcal{D}_5$ and $\mathcal{D}_1$ satisfy the relation $\mathcal{D}_5 \le \mathcal{D}_1$.

Proof. Consider two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$ and $\xi_j > 0$, $(j = 1, 2, \ldots, n)$ be the weight vector of the elements of $\mathcal{U}$ such that $\sum_{j=1}^{n} \xi_j = 1$. Then, we have $\mathcal{D}_5(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{4}\sum\limits_{j=1}^{n}\Bigg\{\xi_j\Bigg(|\zeta_1(x_j) - \zeta_2(x_j)| + |\vartheta_1(x_j) - \vartheta_2(x_j)| + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)| + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|\Bigg)\Bigg\} = \frac{1}{4}\Bigg\{\xi_1\Bigg(|\zeta_1(x_1) - \zeta_2(x_1)| + |\vartheta_1(x_1) - \vartheta_2(x_1)| + |w_{\zeta_1}(x_1) - w_{\zeta_2}(x_1)| + |w_{\vartheta_1}(x_1) - w_{\vartheta_2}(x_1)|\Bigg)\Bigg\} + \frac{1}{4}\Bigg\{\xi_2\Bigg(|\zeta_1(x_2) - \zeta_2(x_2)| + |\vartheta_1(x_2) - \vartheta_2(x_2)| + |w_{\zeta_1}(x_2) - w_{\zeta_2}(x_2)| + |w_{\vartheta_1}(x_2) - w_{\vartheta_2}(x_2)|\Bigg\}\Bigg) + \ldots + \frac{1}{4}\Bigg\{\xi_n\Bigg(|\zeta_1(x_n) - \zeta_2(x_n)| + |\vartheta_1(x_n) - \vartheta_2(x_n)| + |w_{\zeta_1}(x_n) - w_{\zeta_2}(x_n)| + |w_{\vartheta_1}(x_n) - w_{\vartheta_2}(x_n)|\Bigg\}\Bigg) \le \frac{1}{4}\sum\limits_{j=1}^{n}\Bigg\{|\zeta_1(x_j) - \zeta_2(x_j)| + |\vartheta_1(x_j) - \vartheta_2(x_j)| + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)| + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|\Bigg\} = \mathcal{D}_1(\mathcal{C}_1, \mathcal{C}_2)$, i.e., $\mathcal{D}_5(\mathcal{C}_1, \mathcal{C}_2) \le \mathcal{D}_1(\mathcal{C}_1, \mathcal{C}_2)$. As $\mathcal{C}_1$ and $\mathcal{C}_2$ are arbitrary, so we get $\mathcal{D}_5 \le \mathcal{D}_1$. □

Theorem 3.2.7. The distance measures $\mathcal{D}_6$ and $\mathcal{D}_3$ satisfy the relation $\mathcal{D}_6 \le \mathcal{D}_3$.

Proof. Proof follows from the Theorem 3.2.6, so we omit here. □

Theorem 3.2.8. The distance measures $\mathcal{D}_6$ and $\mathcal{D}_1$ satisfy the relation $\mathcal{D}_6 \le \sqrt{\mathcal{D}_1}$

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two CIFSs then for all j, we have $0 \le \zeta_1(x_j)$, $\zeta_2(x_j) \le 1$ and hence $0 \le |\zeta_1(x_j) - \zeta_2(x_j)| \le 1$. Now for any real number $a \in [0, 1]$, we know that $|\ a\ |^2 \le |\ a\ |$. Thus, it follows that $|\zeta_1(x_j) - \zeta_2(x_j)|^2 \le |\zeta_1(x_j) - \zeta_2(x_j)|$. Similarly, we can get $|\vartheta_1(x_j) - \vartheta_2(x_j)|^2 \le |\vartheta_1(x_j) - \vartheta_2(x_j)|$. Also, $0 \le w_{\zeta_1}(x_j)$, $w_{\zeta_2}(x_j) \le 1$. It implies $0 \le |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)| \le 1$, which follows that $|w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|^2 \le |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|$. Similarly, $|w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|^2 \le |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|$. Now, for $\xi_j > 0$ such that $\sum_{j=1}^{n} \xi_j = 1$, we have $\mathcal{D}_6(\mathcal{C}_1, \mathcal{C}_2) = \Bigg[\frac{1}{4}\sum\limits_{j=1}^{n}\Bigg\{\xi_j\Bigg(|\zeta_1(x_j) - \zeta_2(x_j)|^2 + |\vartheta_1(x_j) - \vartheta_2(x_j)|^2 + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|^2 + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|^2\Bigg)\Bigg\}\Bigg]^{\frac{1}{2}} \le \Bigg[\frac{1}{4}\sum\limits_{j=1}^{n}\Bigg\{|\zeta_1(x_j) - \zeta_2(x_j)| + |\vartheta_1(x_j) -$

$\vartheta_2(x_j)| + +|w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)| + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)| \Big\}\Big]^{\frac{1}{2}} = \sqrt{\mathcal{D}_1\left(\mathcal{C}_1, \mathcal{C}_2\right)}$, i.e., $\mathcal{D}_6\left(\mathcal{C}_1, \mathcal{C}_2\right) \leq \sqrt{\mathcal{D}_1\left(\mathcal{C}_1, \mathcal{C}_2\right)}$. As $\mathcal{C}_1$ and $\mathcal{C}_2$ are arbitrary, so we get $\mathcal{D}_6 \leq \sqrt{\mathcal{D}_1}$. □

Theorem 3.2.9. The distance measure $\mathcal{D}_6$ and $\mathcal{D}_5$ satisfy the relation $\mathcal{D}_6 \leq \sqrt{\mathcal{D}_5}$.

Proof. The proof is similar to the proof of the Theorem 3.2.8, so we omit here. □

Hausdorff distance between two non-empty closed and bounded sets is a measure of resemblance between them. For example, consider $\mathcal{C}_1 = [x_1, x_2]$ and $\mathcal{C}_2 = [y_1, y_2]$ in the Euclidean domain R, the Hausdorff distance in additive set environment is given by

$$\mathcal{D}^H\left(\mathcal{C}_1, \mathcal{C}_2\right) = \max\left\{ \mid x_1 - y_1 \mid, \mid x_2 - y_2 \mid \right\}$$

Now, for any two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$ over $\mathcal{U} = \{x_1, x_2, \ldots, x_n\}$, we propose the following Hausdorff distance measures:

(i) Hausdorff Hamming distance

$$\mathcal{D}_1^H\left(\mathcal{C}_1, \mathcal{C}_2\right) = \frac{1}{4}\sum_{j=1}^{n}\left[\begin{array}{l} \max\Big(|\zeta_1(x_j) - \zeta_2(x_j)|, |\vartheta_1(x_j) - \vartheta_2(x_j)|\Big) \\ + \max\Big(|w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|, |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|\Big) \end{array}\right] \tag{3.7}$$

(ii) Hausdorff Normalized Hamming distance

$$\mathcal{D}_2^H\left(\mathcal{C}_1, \mathcal{C}_2\right) = \frac{1}{4n}\sum_{j=1}^{n}\left[\begin{array}{l} \max\Big(|\zeta_1(x_j) - \zeta_2(x_j)|, |\vartheta_1(x_j) - \vartheta_2(x_j)|\Big) \\ + \max\Big(|w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|, |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|\Big) \end{array}\right] \tag{3.8}$$

(iii) Hausdorff Euclidean distance

$$\mathcal{D}_3^H\left(\mathcal{C}_1, \mathcal{C}_2\right) = \left[\frac{1}{4}\sum_{j=1}^{n}\left\{\begin{array}{l} \max\Big(|\zeta_1(x_j) - \zeta_2(x_j)|^2, |\vartheta_1(x_j) - \vartheta_2(x_j)|^2\Big) \\ + \max\Big(|w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|^2, |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|^2\Big) \end{array}\right\}\right]^{\frac{1}{2}} \tag{3.9}$$

(iv) Hausdorff Normalized Euclidean distance

$$\mathcal{D}_4^H\left(\mathcal{C}_1, \mathcal{C}_2\right) = \left[\tfrac{1}{4n}\sum_{j=1}^{n}\left\{\begin{array}{l} \max\Big(|\zeta_1(x_j) - \zeta_2(x_j)|^2, |\vartheta_1(x_j) - \vartheta_2(x_j)|^2\Big) \\ + \max\Big(|w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|^2, |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|^2\Big) \end{array}\right\}\right]^{\frac{1}{2}} \tag{3.10}$$

(v) Hausdorff Weighted Hamming distance

$$\mathcal{D}_5^H(\mathcal{C}_1,\mathcal{C}_2)=\frac{1}{4}\sum_{j=1}^{n}\left\{\xi_j\left(\begin{array}{c}\max\left(|\zeta_1(x_j)-\zeta_2(x_j)|,|\vartheta_1(x_j)-\vartheta_2(x_j)|\right)\\+\max\left(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|,|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\right)\end{array}\right)\right\} \tag{3.11}$$

(vi) Hausdorff Weighted Euclidean distance

$$\mathcal{D}_6^H(\mathcal{C}_1,\mathcal{C}_2)=\left[\frac{1}{4}\sum_{j=1}^{n}\left\{\xi_j\left(\begin{array}{c}\max\left(|\zeta_1(x_j)-\zeta_2(x_j)|^2,|\vartheta_1(x_j)-\vartheta_2(x_j)|^2\right)\\+\max\left(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^2,|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^2\right)\end{array}\right)\right\}\right]^{\frac{1}{2}} \tag{3.12}$$

Theorem 3.2.10. For $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3\in\Psi(\mathcal{U})$, the distance measures $\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2),(r=2,4)$ defined in Eqs. (3.8) and (3.10) also satisfy the following properties.

(P1) $0\leq\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2)\leq 1$;

(P2) $\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2)=0\Leftrightarrow\mathcal{C}_1=\mathcal{C}_2$;

(P3) $\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2)=\mathcal{D}_r^H(\mathcal{C}_2,\mathcal{C}_1)$;

(P4) If $\mathcal{C}_1\subseteq\mathcal{C}_2\subseteq\mathcal{C}_3$ then $\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2)\leq\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_3)$ and $\mathcal{D}_r^H(\mathcal{C}_2,\mathcal{C}_3)\leq\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_3)$.

Proof. Let $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3\in\Psi(\mathcal{U})$ be any three CIFSs. Then for $q=1,2$, we have

(P1) Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are any two CIFSs, so by the definitions of CIFSs, we have $0\leq|\zeta_1(x_j)-\zeta_2(x_j)|^q\leq 1$, $0\leq|\vartheta_1(x_j)-\vartheta_2(x_j)|^q\leq 1$, $0\leq|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^q\leq 1$ and $0\leq|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^q\leq 1$ which implies that $0\leq\max\left(|\zeta_1(x_j)-\zeta_2(x_j)|^q,|\vartheta_1(x_j)-\vartheta_2(x_j)|^q\right)+\max\left(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^q,|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^q\right)\leq 4$. Thus, $0\leq\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2)\leq 1$.

(P2) Assume that $\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2)=0\Leftrightarrow\frac{1}{4n}\sum_{j=1}^{n}\Big[\max\left(|\zeta_1(x_j)-\zeta_2(x_j)|^q,|\vartheta_1(x_j)-\vartheta_2(x_j)|^q\right)+\max\left(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^q,|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^q\right)\Big]=0\Leftrightarrow\max\left(|\zeta_1(x_j)-\zeta_2(x_j)|,|\vartheta_1(x_j)-\vartheta_2(x_j)|\right)=0$ and $\max\left(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|,|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\right)=0\Leftrightarrow\zeta_1(x_j)-\zeta_2(x_j)=0,\vartheta_1(x_j)-\vartheta_2(x_j)=0$ and $w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)=0,w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)=0$ for all $j=1,2,\ldots,n\Leftrightarrow\zeta_1(x_j)=\zeta_2(x_j),\vartheta_1(x_j)=\vartheta_2(x_j)$ and $w_{\zeta_1}(x_j)=w_{\zeta_2}(x_j),w_{\vartheta_1}(x_j)=w_{\vartheta_2}(x_j)$ for all $j\Leftrightarrow\mathcal{C}_1=\mathcal{C}_2$.

(P3) For any two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$, we have $\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2) = \left[\frac{1}{4n}\sum_{j=1}^{n}\left\{\max\left(|\zeta_1(x_j) - \zeta_2(x_j)|^q, |\vartheta_1(x_j)-\vartheta_2(x_j)|^q\right) + \max\left(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^q, |w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^q\right)\right\}\right]^{1/q}$ $= \left[\frac{1}{4n}\sum_{j=1}^{n}\left\{\max\left(|\zeta_2(x_j)-\zeta_1(x_j)|^q, |\vartheta_2(x_j)-\vartheta_1(x_j)|^q\right) + \max\left(|w_{\zeta_2}(x_j)-w_{\zeta_1}(x_j)|^q, |w_{\vartheta_2}(x_j) - w_{\vartheta_1}(x_j)|^q\right)\right\}\right]^{1/q} = \mathcal{D}_r^H(\mathcal{C}_2,\mathcal{C}_1)$. Hence, $\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2) = \mathcal{D}_r^H(\mathcal{C}_2,\mathcal{C}_1)$

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ which implies that $|\zeta_1(x_j) - \zeta_3(x_j)|^q \geq |\zeta_1(x_j) - \zeta_2(x_j)|^q$, $|\vartheta_1(x_j) - \vartheta_3(x_j)|^q \geq |\vartheta_1(x_j)-\vartheta_2(x_j)|^q$ and $|w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)|^q \geq |w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^q$, $|w_{\vartheta_1}(x_j)- w_{\vartheta_3}(x_j)|^q \geq |w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^q$. Thus, $\max\left(|\zeta_1(x_j)-\zeta_3(x_j)|^q, |\vartheta_1(x_j)-\vartheta_3(x_j)|^q\right) \geq \max\left(|\zeta_1(x_j)-\zeta_2(x_j)|^q, |\vartheta_1(x_j)-\vartheta_2(x_j)|^q\right)$ and $\max\left(|w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)|^q, |w_{\vartheta_1}(x_j) - w_{\vartheta_3}(x_j)|^q\right) \geq \max\left(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^q, |w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^q\right)$. Therefore,

$$\begin{aligned}
\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_3) &= \left[\frac{1}{4n}\sum_{j=1}^{n}\left\{\max\left(|\zeta_1(x_j)-\zeta_3(x_j)|^q, |\vartheta_1(x_j)-\vartheta_3(x_j)|^q\right)\right.\right.\\
&\qquad \left.\left. + \max\left(|w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)|^q, |w_{\vartheta_1}(x_j)-w_{\vartheta_3}(x_j)|^q\right)\right\}\right]^{1/q}\\
&\geq \left[\frac{1}{4n}\sum_{j=1}^{n}\left\{\max\left(|\zeta_1(x_j)-\zeta_2(x_j)|^q, |\vartheta_1(x_j)-\vartheta_2(x_j)|^q\right)\right.\right.\\
&\qquad \left.\left. + \max\left(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^q, |w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^q\right)\right\}\right]^{1/q}\\
&= \mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2)
\end{aligned}$$

Hence, $\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_3) \geq \mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_2)$. Similarly $\mathcal{D}_r^H(\mathcal{C}_1,\mathcal{C}_3) \geq \mathcal{D}_r^H(\mathcal{C}_2,\mathcal{C}_3)$.

□

Theorem 3.2.11. The measures $\mathcal{D}_5^H$ and $\mathcal{D}_6^H$ are also the valid distance measures.

Proof. The proof is similar to the proof of the Theorem 3.2.10. □

Theorem 3.2.12. The Hausdorff distance measures $\mathcal{D}_1^H$ and $\mathcal{D}_5^H$ satisfy the relation $\mathcal{D}_5^H \leq \mathcal{D}_1^H$.

Proof. For any two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$ and $\xi_j > 0$ such that $\sum_{j=1}^{n} \xi_j = 1$. Then, we have

$$\begin{aligned}
\mathcal{D}_5^H(\mathcal{C}_1, \mathcal{C}_2) &= \frac{1}{4}\sum_{j=1}^{n}\Bigg\{\xi_j\Bigg(\max\Big(|\zeta_1(x_j)-\zeta_2(x_j)|, |\vartheta_1(x_j)-\vartheta_2(x_j)|\Big) \\
&\qquad + \max\Big(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|, |w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\Big)\Bigg)\Bigg\} \\
&\leq \frac{1}{4}\sum_{j=1}^{n}\Bigg[\max\Big(|\zeta_1(x_j)-\zeta_2(x_j)|, |\vartheta_1(x_j)-\vartheta_2(x_j)|\Big) \\
&\qquad + \max\Big(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|, |w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\Big)\Bigg] \\
&= \mathcal{D}_1^H(\mathcal{C}_1, \mathcal{C}_2)
\end{aligned}$$

As $\mathcal{C}_1$ and $\mathcal{C}_2$ are arbitrary CIFSs, so we get $\mathcal{D}_5^H \leq \mathcal{D}_1^H$. □

Theorem 3.2.13. The measures $\mathcal{D}_3^H$ and $\mathcal{D}_6^H$ satisfy the relation $\mathcal{D}_6^H \leq \mathcal{D}_3^H$.

Proof. The proof is similar to the proof of the Theorem 3.2.12. □

Theorem 3.2.14. The measure $\mathcal{D}_1^H$, $\mathcal{D}_2^H$, $\mathcal{D}_3^H$ and $\mathcal{D}_4^H$ satisfy the following relations

(i) $\mathcal{D}_3^H \leq \sqrt{\mathcal{D}_1^H}$

(ii) $\mathcal{D}_4^H \leq \sqrt{\mathcal{D}_2^H}$

Proof. We will prove (i) part only, while other can be proven similarly. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two CIFSs and for any number $a \in [0,1]$, we know that $| a |^2 \leq | a |$. Thus, by the definition of $\mathcal{D}_3^H$, we have $\mathcal{D}_3^H(\mathcal{C}_1, \mathcal{C}_2) = \left[\frac{1}{4}\sum\limits_{j=1}^{n}\left\{\max\Big(|\zeta_1(x_j)-\zeta_2(x_j)|^2, |\vartheta_1(x_j)-\vartheta_2(x_j)|^2\Big)+\max\Big(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|^2, |w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|^2\Big)\right\}\right]^{\frac{1}{2}} \leq \left[\frac{1}{4}\sum\limits_{j=1}^{n}\left\{\max\Big(|\zeta_1(x_j)-\zeta_2(x_j)|, |\vartheta_1(x_j)-\vartheta_2(x_j)|\Big)+\max\Big(|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|, |w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\Big)\right\}\right]^{\frac{1}{2}} = \sqrt{\mathcal{D}_1^H(\mathcal{C}_1, \mathcal{C}_2)}$.
As $\mathcal{C}_1$ and $\mathcal{C}_2$ are arbitrary, so we get $\mathcal{D}_3^H \leq \sqrt{\mathcal{D}_1^H}$. □

Theorem 3.2.15. The distance measures $\mathcal{D}_2$ and $\mathcal{D}_2^H$ satisfy the relation $\mathcal{D}_2^H \leq \mathcal{D}_2$.

Proof. Since for any positive numbers $a_j (j = 1, 2, \ldots, n)$, we know that $\max_j\{a_j\} \leq \sum_{j=1}^{n} a_j$ and hence by the definition of $\mathcal{D}_2^H$, we get

$$\begin{aligned}
\mathcal{D}_2^H(\mathcal{C}_1, \mathcal{C}_2) &= \frac{1}{4n}\sum_{j=1}^{n}\Bigg[\max\Big(|\zeta_1(x_j) - \zeta_2(x_j)|, |\vartheta_1(x_j) - \vartheta_2(x_j)|\Big) \\
&\qquad + \max\Big(|w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)|, |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|\Big)\Bigg] \\
&\leq \frac{1}{4n}\sum_{j=1}^{n}\Bigg[|\zeta_1(x_j) - \zeta_2(x_j)| + |\vartheta_1(x_j) - \vartheta_2(x_j)| \\
&\qquad + |w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)| + |w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)|\Bigg] \\
&= \mathcal{D}_2(\mathcal{C}_1, \mathcal{C}_2)
\end{aligned}$$

Hence, $\mathcal{D}_2^H \leq \mathcal{D}_2$. □

Theorem 3.2.16. The distance measures $\mathcal{D}_r^H$ and $\mathcal{D}_r$, $(r = 1, 2, \ldots, 6)$ satisfies the relation $\mathcal{D}_r^H \leq \mathcal{D}_r$.

Proof. Proof follows from Theorem 3.2.15 and hence we omit here. □

Theorem 3.2.17. The distance measures $\mathcal{D}_5^H$ and $\mathcal{D}_1$ satisfy the relation $\mathcal{D}_5^H \leq \mathcal{D}_1$.

Proof. Since from the Theorems 3.2.16 and 3.2.6, we get $\mathcal{D}_5^H \leq \mathcal{D}_5$ and $\mathcal{D}_5 \leq \mathcal{D}_1$. Therefore, $\mathcal{D}_5^H \leq \mathcal{D}_1$. □

Theorem 3.2.18. The distance measures $\mathcal{D}_6^H$ and $\mathcal{D}_3$ satisfy the relation $\mathcal{D}_6^H \leq \mathcal{D}_3$.

Proof. From Theorem 3.2.16, $\mathcal{D}_6^H \leq \mathcal{D}_6$ and from Theorem 3.2.7, $\mathcal{D}_6 \leq \mathcal{D}_3$. Hence, $\mathcal{D}_6^H \leq \mathcal{D}_3$. □

Theorem 3.2.19. Distance measures $\mathcal{D}_5$, $\mathcal{D}_1$, $\mathcal{D}_1^H$ and $\mathcal{D}_3$, $\mathcal{D}_6$, $\mathcal{D}_3^H$ satisfy the following relations.

(i) $\mathcal{D}_1 \geq \frac{\mathcal{D}_5 + \mathcal{D}_1^H}{2}$ and $\mathcal{D}_1 \geq \sqrt{\mathcal{D}_5 \cdot \mathcal{D}_1^H}$

(ii) $\mathcal{D}_3 \geq \frac{\mathcal{D}_6 + \mathcal{D}_3^H}{2}$ and $\mathcal{D}_3 \geq \sqrt{\mathcal{D}_6 \cdot \mathcal{D}_3^H}$

Proof. Since $\mathcal{D}_1^H \leq \mathcal{D}_1$ and $\mathcal{D}_5 \leq \mathcal{D}_1$. So, by adding these inequalities, we get $\frac{\mathcal{D}_5+\mathcal{D}_1^H}{2} \leq \mathcal{D}_1$. On the other hand, by multiplying these, we get $\sqrt{\mathcal{D}_5 \cdot \mathcal{D}_1^H} \leq \mathcal{D}_1$.

Also, $\mathcal{D}_3^H \leq \mathcal{D}_3$ and $\mathcal{D}_6 \leq \mathcal{D}_3$. So, by adding these inequalities, we get $\frac{\mathcal{D}_6+\mathcal{D}_3^H}{2} \leq \mathcal{D}_3$. On the other hand, by multiplying these, we get $\sqrt{\mathcal{D}_6 \cdot \mathcal{D}_3^H} \leq \mathcal{D}_3$. □

3.3 Decision making approach based on the proposed distances

In this section, a decision-making method by using distance measures for CIFSs has been presented followed by an illustrative example for demonstrating the approach.

3.3.1 Decision-making approach

The general description of DM problem is summarized in Section 2.5 of Chapter 2. Suppose that an expert evaluated the alternatives $\mathcal{V}_u$ under the criteria $\mathfrak{B}_v$ and gave their judgement values as $\mathcal{V}_u = \{(\mathfrak{B}_v, (\zeta_{uv}, w_{\zeta_{uv}}), (\vartheta_{uv}, w_{\vartheta_{uv}})) \,|v = 1, 2, \ldots, n\} \;\forall\; u = 1, 2, \ldots, m$ where $0 \leq \zeta_{uv}, \vartheta_{uv}, w_{\zeta_{uv}}, w_{\vartheta_{uv}}, \zeta_{uv} + \vartheta_{uv}, w_{\zeta_{uv}} + w_{\vartheta_{uv}} \leq 1$. For convenience, we express the rating values, given by expert, for alternative $\mathcal{V}_u$ under criteria $\mathfrak{B}_v$ as CIFN $\mathcal{C}_{uv} = ((\zeta_{uv}, w_{\zeta_{uv}}), (\vartheta_{uv}, w_{\vartheta_{uv}}))$. Let $\xi_v (v = 1, 2, \ldots, n)$ be the weight of the criteria $\mathfrak{B}_v$ such that $\xi_v > 0$ and $\sum\limits_{v=1}^{n} \xi_v = 1$. In the following, we develop an approach based on the proposed measures with complex intuitionistic fuzzy information, which involve the following steps:

Step 1: Collect all the information corresponding to each alternative in terms of CIFNs and hence an overall decision matrix $\mathcal{M}$ is expressed as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \ldots & \mathcal{C}_{1n} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \ldots & \mathcal{C}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{m1} & \mathcal{C}_{m2} & \ldots & \mathcal{C}_{mn} \end{pmatrix} \end{array} \tag{3.13}$$

Step 2: Utilize the proposed distance measures '$\mathcal{D}$' to compute the distance between the alternatives.

Step 3: Rank the alternatives by finding the minimum value of "$\arg\min\{\mathcal{D}\}$" and select the best one(s).

3.3.2 Illustrative Examples

In order to illustrate the performance and validity of the above proposed distance measures, an illustrative example has been taken into account from the field of the pattern recognition, as well as medical diagnosis, which can be read as below.

Example 3.3.1. Consider four unknown patterns $\mathcal{V}_1, \mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$ which are represented by the following CIFSs in a given universe $\mathcal{U} = \{x_1, x_2, x_3, x_4, x_5\}$ as

$$\begin{aligned}
\mathcal{V}_1 &= \Big\{(x_1, (0.8, 0.9), (0.1, 0.1)), (x_2, (0.8, 0.5), (0.1, 0.4)), (x_3, (0.6, 0.6), (0.2, 0.2)), \\
&\qquad (x_4, (0.8, 0.7), (0.1, 0.2)), (x_5, (0.6, 0.4), (0.2, 0.5))\Big\} \\
\mathcal{V}_2 &= \Big\{(x_1, (0.7, 0.6), (0.3, 0.3)), (x_2, (0.4, 0.9), (0.2, 0.1)), (x_3, (0.7, 0.7), (0.2, 0.3)), \\
&\qquad (x_4, (0.4, 0.6), (0.3, 0.1)), (x_5, (0.5, 0.3), (0.3, 0.6))\Big\} \\
\mathcal{V}_3 &= \Big\{(x_1, (0.3, 0.4), (0.6, 0.4)), (x_2, (0.6, 0.6), (0.3, 0.4)), (x_3, (0.3, 0.4), (0.5, 0.6)), \\
&\qquad (x_4, (0.7, 0.7), (0.1, 0.1)), (x_5, (0.7, 0.8), (0.2, 0.1))\Big\} \\
\mathcal{V}_4 &= \Big\{(x_1, (0.2, 0.8), (0.5, 0.1)), (x_2, (0.7, 0.3), (0.3, 0.3)), (x_3, (0.6, 0.5), (0.1, 0.3)), \\
&\qquad (x_4, (0.6, 0.5), (0.3, 0.4)), (x_5, (0.4, 0.8), (0.4, 0.1))\Big\}
\end{aligned}$$

Consider the known pattern $\mathcal{V} \in \Psi(\mathcal{U})$ which will be recognized, where

$$\begin{aligned}
\mathcal{V} &= \Big\{(x_1, (0.7, 0.5), (0.1, 0.3)), (x_2, (0.4, 0.6), (0.5, 0.2)), (x_3, (0.5, 0.5), (0.3, 0.1)), \\
&\qquad (x_4, (0.8, 0.7), (0.2, 0.1)), (x_5, (0.6, 0.9), (0.1, 0.1))\Big\}
\end{aligned}$$

then the target of this problem is to classify the pattern $\mathcal{V}$ in one of the classes $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ and $\mathcal{V}_4$. For it, proposed distance measures, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_1^H, \mathcal{D}_2^H, \mathcal{D}_3^H, \mathcal{D}_4^H$ from $\mathcal{V}$ to

$\mathcal{V}_u(u = 1, 2, 3, 4)$ are computed and the results are given in Table 3.1 in which "$\succ$" refers to the "preferred to". From this table, it has been clearly seen that the the pattern $\mathcal{V}$ should belong to pattern $\mathcal{V}_3$. Further, it has been seen that the different distance measures have different ranking order but the best alternative remains same. This is due to the fact that the distance measures $\mathcal{D}_1$ and $\mathcal{D}_2$ are based on the hamming distances while $\mathcal{D}_3$ and $\mathcal{D}_4$ are based on the geometric mean distance measures. Therefore, based on the preference of the decision makers, they can choose their corresponding ranking order and select the best alternatives for the desired one.

On the other hand, if we assume that the weight of x_1, x_2, x_3, x_4 and x_5 is 0.20, 0.25, 0.20, 0.15, 0.20 respectively, then we utilize the distance measures $\mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_5^H$ and $\mathcal{D}_6^H$ for obtaining the most suitable pattern and their corresponding results are summarized in Table 3.2. Thus, from the ranking order of these four patterns, we get $\mathcal{V}_3$ is the most desirable pattern to be classified with $\mathcal{V}$. Furthermore, it can be easily verified that these results validate the above-proposed theorems on the distance measures.

Now, if we apply the distance measure $\mathcal{D}$ as defined in Alkouri and Salleh [6] corresponding to the considered data, then their measurement values are $\mathcal{D}(\mathcal{V}_1, \mathcal{V}) = 0.1915$, $\mathcal{D}(\mathcal{V}_2, \mathcal{V}) = 0.2117$, $\mathcal{D}(\mathcal{V}_3, \mathcal{V}) = 0.1755$ and $\mathcal{D}(\mathcal{V}_4, \mathcal{V}) = 0.2245$. Therefore, $\mathcal{D}(\mathcal{V}_3, \mathcal{V}) \leq \mathcal{D}(\mathcal{V}_r, \mathcal{V})$ for all $r = 1, 2, 3, 4$ and hence we conclude that pattern $\mathcal{V}$ should belong to the pattern $\mathcal{V}_3$ which coincides with the proposed measure results.

Example 3.3.2. Consider a set of diseases $\mathcal{V} = \{\mathcal{V}_1(\text{Viral fever}), \mathcal{V}_2(\text{Malaria}), \mathcal{V}_3(\text{Typhoid}), \mathcal{V}_4(\text{Stomach Problem})\}$ and a set of symptoms $S = \{s_1(\text{Temperature}),\ s_2\ (\text{HeadAche}), s_3(\text{Stomach Pain}),\ s_4\ (\text{Cough})\}$. Suppose a patient $\mathfrak{P}$, with respect to all the symptoms, can be represented by the following CIFSs:

$$\mathfrak{P}(\text{Patient}) = \left\{ \begin{array}{c} (s_1, (0.8, 0.6), (0.1, 0.2)), (s_2, (0.9, 0.7), (0.1, 0.2)), \\ (s_3, (0.7, 0.8), (0.2, 0.1)), (s_4, (0.6, 0.5), (0.2, 0.4)) \end{array} \right\}$$

and each diseases $\mathcal{V}_r(r = 1, 2, 3, 4)$ are as follows:

$$\mathcal{V}_1 = \left\{ \begin{array}{c} (s_1, (0.8, 0.7), (0.1, 0.2)), (s_2, (0.9, 0.6), (0.1, 0.2)), \\ (s_3, (0.7, 0.8), (0.2, 0.1)), (s_4, (0.8, 0.7), (0.2, 0.1)) \end{array} \right\}$$

$$\mathcal{V}_2 = \left\{ \begin{array}{c} (s_1, (0.6, 0.4), (0.1, 0.5)), (s_2, (0.4, 0.9), (0.5, 0.1)), \\ (s_3, (0.5, 0.5), (0.3, 0.3)), (s_4, (0.4, 0.9), (0.5, 0.1)) \end{array} \right\}$$

$$\mathcal{V}_3 = \left\{ \begin{array}{c} (s_1, (0.3, 0.8), (0.3, 0.1)), (s_2, (0.8, 0.3), (0.1, 0.6)), \\ (s_3, (0.7, 0.6), (0.2, 0.2)), (s_4, (0.2, 0.7), (0.8, 0.2)) \end{array} \right\}$$

$$\mathcal{V}_4 = \left\{ \begin{array}{c} (s_1, (0.5, 0.3), (0.4, 0.6)), (s_2, (0.3, 0.1), (0.6, 0.3)), \\ (s_3, (0.8, 0.3), (0.1, 0.5)), (s_4, (0.1, 0.3), (0.6, 0.5)) \end{array} \right\}$$

Now, the target of this problem is to diagnose the disease of patient $\mathfrak{P}$ among $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$, and $\mathcal{V}_4$. For it, the proposed distance measures, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_1^H, \mathcal{D}_2^H, \mathcal{D}_3^H$ and $\mathcal{D}_4^H$ have been computed from $\mathfrak{P}$ to $\mathcal{V}_r (r = 1, 2, 3, 4)$ and then their corresponding results are given in Table 3.3. From these results, it has been seen the patient $\mathfrak{P}$ suffers from disease $\mathcal{V}_1$.

On the other hand, if we considered the weight vector of s_1, s_2, s_3, s_4 as $0.3, 0.2, 0.1, 0.4$ respectively Then, we compute the distance measures $\mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_5^H$ and $\mathcal{D}_6^H$ and hence their corresponding measurement values are summarized in Table 3.4. Thus, the ranking orders corresponding to these weighted distance measures are $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$ and hence we conclude that the patient $\mathfrak{P}$ suffer from the disease $\mathcal{V}_1$, i.e., Viral fever.

If we apply the existing distance measure ($\mathcal{D}$) defined in Alkouri and Salleh [6] then, their measurement values are $\mathcal{D}(\mathcal{V}_1, \mathfrak{P}) = 0.0870$, $\mathcal{D}(\mathcal{V}_2, \mathfrak{P}) = 0.2635$, $\mathcal{D}(\mathcal{V}_3, \mathfrak{P}) = 0.2815$ and $\mathcal{D}(\mathcal{V}_4, \mathfrak{P}) = 0.3440$. Thus $\mathcal{D}(\mathcal{V}_1, \mathfrak{P}) \leq \mathcal{D}(\mathcal{V}_r, \mathfrak{P})$ for all $r = 2, 3, 4$. From it, we conclude that patient $\mathfrak{P}$ suffers from disease $\mathcal{V}_1$.

3.4 Conclusion

The key contribution of this chapter is summarized as follows:

1) An attempt has been made to present some families of distance measures based on the Hamming, Euclidean, and Hausdorff metric for the collections of the complex intuitionistic fuzzy sets on the finite universe of discourse.

2) Under IFS environment, various scholars have presented several kinds of the distance measures in which range of their corresponding membership degrees is the subset of

the real numbers. As a supplement, this condition has been relaxed in this chapter by extending their ranges to unit disc is of the form $(\zeta, 2\pi w_\zeta)$ where $\zeta, w_\zeta \in [0,1]$.

3) A concept of the phase angle, for representing the periodicity of the data, has been added into the analysis and hence by keeping the advantages of these sets, in the present work, the authors have presented some series of the distances measure for ranking the different sets in which the preferences related to the different objects are summarized in the form of the CIFSs. Various desirable relations between the measures have been proposed and are validated these through illustrative examples.

4) A decision-making method has been proposed based on the suggested distance measures. To demonstrate the efficiency of the proposed measures, numerical examples of pattern recognition as well as the medical diagnosis have been taken. From their comparative study, it has been concluded that the proposed measures place an alternative way for solving the decision-making problems and hence a new and easy way to the handle the uncertainties where the existing theories have failed to handle the situation in a more precise way.

Table 3.1: Computed results by the proposed distance measures

Measures	Measures values of pattern $\mathcal{V}$ from				Ranking order
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
$\mathcal{D}_1$	0.8500	0.9250	0.8000	1.0500	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$\mathcal{D}_2$	0.1700	0.1850	0.1600	0.2100	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$\mathcal{D}_3$	0.5148	0.5500	0.4848	0.5431	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
$\mathcal{D}_4$	0.2302	0.2460	0.2168	0.2429	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
$\mathcal{D}_1^H$	0.5250	0.6500	0.5000	0.6750	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$\mathcal{D}_2^H$	0.1050	0.1300	0.1000	0.1350	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$\mathcal{D}_3^H$	0.4093	0.4690	0.4062	0.4555	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
$\mathcal{D}_4^H$	0.1830	0.2098	0.1817	0.2037	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$

Table 3.2: Computed results by the proposed weighted distance measures

Measures	Measures values of pattern $\mathcal{V}$ from				Ranking order
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
$\mathcal{D}_5$	0.1813	0.1862	0.1650	0.2112	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$\mathcal{D}_6$	0.2395	0.2462	0.2197	0.2442	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
$\mathcal{D}_5^H$	0.1100	0.1313	0.1038	0.1362	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$\mathcal{D}_6^H$	0.1891	0.2101	0.1841	0.2052	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$

Table 3.3: Computed results by the proposed distance measures for Example 4.2

Distances	Measures values of patient $\mathfrak{P}$ from disease				Ranking order
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
$\mathcal{D}_1$	0.2250	0.9750	0.9000	1.3500	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$\mathcal{D}_2$	0.0450	0.1950	0.1800	0.2700	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$\mathcal{D}_3$	0.2179	0.5454	0.5745	0.7583	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
$\mathcal{D}_4$	0.0975	0.2439	0.2569	0.3391	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
$\mathcal{D}_1^H$	0.1750	0.6000	0.5500	0.8000	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$\mathcal{D}_2^H$	0.0350	0.1200	0.1100	0.1600	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
$\mathcal{D}_3^H$	0.1936	0.4472	0.4743	0.6164	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
$\mathcal{D}_4^H$	0.0866	0.2000	0.2121	0.2757	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$

Table 3.4: Computed results by the proposed weighted distance measures for Example 4.2

Distances	Measures values of patient $\mathfrak{P}$ from disease				Ranking order
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
$\mathcal{D}_5$	0.0825	0.2525	0.2675	0.3350	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
$\mathcal{D}_6$	0.1351	0.2797	0.3213	0.3715	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
$\mathcal{D}_5^H$	0.0625	0.1550	0.1625	0.1975	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
$\mathcal{D}_6^H$	0.1194	0.2291	0.2669	0.3004	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$

Chapter 4

A robust correlation coefficient measure of complex intuitionistic fuzzy sets and their applications in decision-making[1]

In this chapter, we develop correlation and weighted CCs under the CIFS environment in which pairs of the membership degrees represent the two-dimensional information. Some of the desirable properties of proposed measures are investigated. Further, based on these measures, a multicriteria decision-making approach is presented under the CIFS environment. Two illustrative examples are taken to demonstrate the efficiency of the proposed approach and validate it with the existing approaches.

4.1 Introduction

Correlation measure is one of the most important measures which helps not only in comparing one data entity with other but also show the extent of association between them and their direction. In statistical analysis, one of the important measures is CC which gives us an idea of the strength and direction of a linear relationship between the pairs of two variables. On the other hand, in fuzzy set theory, these measures determine the degree of the dependency between the two fuzzy sets. The extensive literature review

[1]The content of this chapter is published as "A robust correlation coefficient measure of complex intuitionistic fuzzy sets and their applications in decision-making", *Applied Intelligence, Springer*, 49(2), pp. 496 - 512, 2019 (**SCI: Impact Factor: 5.086**).

related to correlation measure has been done in Section 1.1.2 of Chapter 1. In the existing studies under the FSs, IFSs or its generalizations, it has been analyzed that they can handle the uncertainty and vagueness that exists in the data only. None of these existing models (FSs/IFSs/IVIFSs) are able to represent the uncertainty and periodicity factors simultaneously. To overcome it, Ramot et al. [128] initiated CFS theory in which the range of membership function is extended from real number to the complex number with the unit disc. Furthermore, Alkouri and Salleh [5] extended the concepts of CFS to CIFS by adding the degree of complex non-membership functions and studied their basic operations. CIFSs have powerful ability to model the imprecise and ambiguous information in real-world applications in a better way than the existing theories such as FSs, IFSs and CFSs.

Therefore, keeping the advantages of this set and taking the importance of correlation measure, this chapter presents the theory of the CCs among the CIFSs. To achieve it, we first define some operational laws, the informational energies and the covariance between the two CIFSs that involves both uncertainty and periodicity semantics. Then, based on these, we propose some CCs for CIFSs and investigate their properties. Further, some weighted CCs are proposed to address the situations where the attributes have different importance values. Furthermore, we propose a decision-making approach based on the proposed CCs for CIFSs. The feasibility, as well as superiority of the approach, has been demonstrated through two numerical examples.

4.2 Correlation Coefficient for CIFSs

In this section, we propose some CCs for the CIFSs which can be applied in numerous engineering and scientific fields to rank the objects. For it, throughout this chapter, we shall use $\mathcal{U} = \{x_1, x_2, \ldots x_n\}$ as the universe of discourse.

Let $\mathcal{C}_1 = \{(x, (\zeta_1(x), w_{\zeta_1}(x)), (\vartheta_1(x), w_{\vartheta_1}(x))) \mid x \in \mathcal{U}\}$ and $\mathcal{C}_2 = \{(x, (\zeta_2(x), w_{\zeta_2}(x)),$ $(\vartheta_2(x), w_{\vartheta_2}(x))) \mid x \in \mathcal{U}\}$ be two CIFSs defined on $\mathcal{U}$. Then, the informational energies of two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined as

$$\mathfrak{T}(\mathcal{C}_1) = \sum_{j=1}^{n} \left(\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j)\right), \tag{4.1}$$

$$\mathfrak{T}(\mathcal{C}_2) = \sum_{j=1}^{n} \left(\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j)\right). \tag{4.2}$$

The correlation of the CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined as

$$\mathfrak{C}\left(\mathcal{C}_1, \mathcal{C}_2\right) = \sum_{j=1}^{n} \left[\begin{array}{l} \zeta_1(x_j)\zeta_2(x_j) + w_{\zeta_1}(x_j)w_{\zeta_2}(x_j) \\ + \vartheta_1(x_j)\vartheta_2(x_j) + w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j) \end{array}\right]. \tag{4.3}$$

From Eq. (4.3), it is clearly seen that correlation of CIFSs satisfies the following properties:

(P1) $\mathfrak{C}\left(\mathcal{C}_1, \mathcal{C}_2\right) = \mathfrak{C}\left(\mathcal{C}_2, \mathcal{C}_1\right)$

(P2) $\mathfrak{C}\left(\mathcal{C}_1, \mathcal{C}_1\right) = \mathfrak{T}(\mathcal{C}_1)$

Then, based on these, we defined the CC between CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$, as follows:

Definition 4.2.1. If $\mathcal{C}_1 = \{(x, (\zeta_1(x), w_{\zeta_1}(x)), (\vartheta_1(x), w_{\vartheta_1}(x))) \mid x \in \mathcal{U}\}$ and $\mathcal{C}_2 = \{(x, (\zeta_2(x), w_{\zeta_2}(x)), (\vartheta_2(x), w_{\vartheta_2}(x))) \mid x \in \mathcal{U}\}$ be two CIFSs defined on $\mathcal{U}$, then the CC between them is denoted by $\mathcal{K}_1\left(\mathcal{C}_1, \mathcal{C}_2\right)$ and is defined as

$$\begin{aligned} \mathcal{K}_1\left(\mathcal{C}_1, \mathcal{C}_2\right) &= \frac{\mathfrak{C}\left(\mathcal{C}_1, \mathcal{C}_2\right)}{\sqrt{\mathfrak{T}(\mathcal{C}_1) \times \mathfrak{T}(\mathcal{C}_2)}} \\ &= \frac{\sum_{j=1}^{n} \left(\begin{array}{l} \zeta_1(x_j)\zeta_2(x_j) + w_{\zeta_1}(x_j)w_{\zeta_2}(x_j) \\ + \vartheta_1(x_j)\vartheta_2(x_j) + w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j) \end{array}\right)}{\left\{\begin{array}{l} \sqrt{\sum_{j=1}^{n} \left(\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j)\right)} \\ \times \sqrt{\sum_{j=1}^{n} \left(\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j)\right)} \end{array}\right\}} \end{aligned} \tag{4.4}$$

Theorem 4.2.1. The CC '$\mathcal{K}_1$' between two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$ defined on $\mathcal{U}$ satisfies the following properties:

(P1) $0 \leq \mathcal{K}_1\left(\mathcal{C}_1, \mathcal{C}_2\right) \leq 1$;

(P2) $\mathcal{K}_1\left(\mathcal{C}_1, \mathcal{C}_2\right) = \mathcal{K}_1\left(\mathcal{C}_2, \mathcal{C}_1\right)$;

(P3) $\mathcal{K}_1\left(\mathcal{C}_1, \mathcal{C}_2\right) = 1$, if $\mathcal{C}_1 = \mathcal{C}_2$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{K}_1(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{K}_1(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{K}_1(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{K}_1(\mathcal{C}_2, \mathcal{C}_3)$ for CIFS $\mathcal{C}_3$ defined on $\mathcal{U}$.

Proof. Let $\mathcal{C}_1 = \{(x, (\zeta_1(x), w_{\zeta_1}(x)), (\vartheta_1(x), w_{\vartheta_1}(x))) \mid x \in \mathcal{U}\}$ and $\mathcal{C}_2 = \{(x, (\zeta_2(x), w_{\zeta_2}(x)), (\vartheta_2(x), w_{\vartheta_2}(x))) \mid x \in \mathcal{U}\}$ be two CIFSs defined on $\mathcal{U}$. Then, we have

(P1) The inequality $\mathcal{K}_1(\mathcal{C}_1, \mathcal{C}_2) \geq 0$ is obvious due to $\mathfrak{C}(\mathcal{C}_1, \mathcal{C}_2) \geq 0$ is obtained from the Eq. (4.3). Now we shall prove $\mathcal{K}_1(\mathcal{C}_1, \mathcal{C}_2) \leq 1$. For it, based on the Definition 4.2.1, we get

$$\mathcal{K}_1(\mathcal{C}_1, \mathcal{C}_2) = \frac{\mathfrak{C}(\mathcal{C}_1, \mathcal{C}_2)}{\sqrt{\mathfrak{C}(\mathcal{C}_1, \mathcal{C}_1) \times \mathfrak{C}(\mathcal{C}_2, \mathcal{C}_2)}}$$

$$= \frac{\sum_{j=1}^{n} \Big(\zeta_1(x_j)\zeta_2(x_j) + w_{\zeta_1}(x_j)w_{\zeta_2}(x_j) + \vartheta_1(x_j)\vartheta_2(x_j) + w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j) \Big)}{\left\{ \begin{array}{c} \sqrt{\sum_{j=1}^{n} \Big(\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j) \Big)} \\ \times \sqrt{\sum_{j=1}^{n} \Big(\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j) \Big)} \end{array} \right\}}$$

$$= \frac{\left(\sum_{j=1}^{n} \zeta_1(x_j)\zeta_2(x_j) + \sum_{j=1}^{n} w_{\zeta_1}(x_j)w_{\zeta_2}(x_j) + \sum_{j=1}^{n} \vartheta_1(x_j)\vartheta_2(x_j) + \sum_{j=1}^{n} w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j) \right)}{\left\{ \begin{array}{c} \sqrt{\sum_{j=1}^{n} \Big(\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j) \Big)} \\ \times \sqrt{\sum_{j=1}^{n} \Big(\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j) \Big)} \end{array} \right\}}$$

Now, by using Cauchy-Schwarz inequality which states that, in Euclidean space R^n with standard inner product, we have:

$$\left(\sum_{j=1}^{n} \mathfrak{u}_j \mathfrak{v}_j \right)^2 \leq \left(\sum_{j=1}^{n} \mathfrak{u}_j^2 \right) \left(\sum_{j=1}^{n} \mathfrak{v}_j^2 \right)$$

where $\mathfrak{u} = (\mathfrak{u}_1, \mathfrak{u}_2, \ldots \mathfrak{u}_n)$ and $\mathfrak{v} = (\mathfrak{v}_1, \mathfrak{v}_2, \ldots \mathfrak{v}_n) \in R^n$ and equality holds if and only if $\mathfrak{u}$ and $\mathfrak{v}$ are linearly dependent vectors. Therefore,

$$\mathcal{K}_1\left(\mathcal{C}_1,\mathcal{C}_2\right)\leq\frac{\left(\begin{array}{c}\sqrt{\sum_{j=1}^{n}\zeta_1^2(x_j)}\sqrt{\sum_{j=1}^{n}\zeta_2^2(x_j)}+\sqrt{\sum_{j=1}^{n}w_{\zeta_1}^2(x_j)}\sqrt{\sum_{j=1}^{n}w_{\zeta_2}^2(x_j)}\\ \sqrt{\sum_{j=1}^{n}\vartheta_1^2(x_j)}\sqrt{\sum_{j=1}^{n}\vartheta_2^2(x_j)}+\sqrt{\sum_{j=1}^{n}w_{\vartheta_1}^2(x_j)}\sqrt{\sum_{j=1}^{n}w_{\vartheta_2}^2(x_j)}\end{array}\right)}{\left\{\begin{array}{c}\sqrt{\sum_{j=1}^{n}\left(\zeta_1^2(x_j)+\vartheta_1^2(x_j)+w_{\zeta_1}^2(x_j)+w_{\vartheta_1}^2(x_j)\right)}\\ \times\sqrt{\sum_{j=1}^{n}\left(\zeta_2^2(x_j)+\vartheta_2^2(x_j)+w_{\zeta_2}^2(x_j)+w_{\vartheta_2}^2(x_j)\right)}\end{array}\right\}}$$

By taking the notations, $\sum_{j=1}^{n}\zeta_1^2(x_j)=a$, $\sum_{j=1}^{n}\zeta_2^2(x_j)=b$, $\sum_{j=1}^{n}\vartheta_1^2(x_j)=c$, $\sum_{j=1}^{n}\vartheta_2^2(x_j)=d$, $\sum_{j=1}^{n}w_{\zeta_1}^2(x_j)=p$, $\sum_{j=1}^{n}w_{\zeta_2}^2(x_j)=q$, $\sum_{j=1}^{n}w_{\vartheta_1}^2(x_j)=r$ and $\sum_{j=1}^{n}w_{\vartheta_2}^2(x_j)=s$, the above inequality reduces to

$$\mathcal{K}_1\left(\mathcal{C}_1,\mathcal{C}_2\right)\leq\frac{\sqrt{ab}+\sqrt{cd}+\sqrt{pq}+\sqrt{rs}}{\sqrt{(a+c+p+r)(b+d+q+s)}}$$

Therefore,

$$\begin{aligned}
\left(\mathcal{K}_1\left(\mathcal{C}_1,\mathcal{C}_2\right)\right)^2-1 &\leq \frac{\left(\sqrt{ab}+\sqrt{cd}+\sqrt{pq}+\sqrt{rs}\right)^2}{(a+c+p+r)(b+d+q+s)}-1\\
&= \frac{\left(\sqrt{ab}+\sqrt{cd}+\sqrt{pq}+\sqrt{rs}\right)^2-(a+c+p+r)(b+d+q+s)}{(a+c+p+r)(b+d+q+s)}\\
&= \frac{\left(\begin{array}{l}ab+cd+pq+rs+2\sqrt{abcd}+2\sqrt{pqrs}+2\sqrt{abpq}+2\sqrt{abrs}\\ +2\sqrt{cdpq}+2\sqrt{cdrs}-ab-ad-aq-as-cb-cd-cq\\ -cs-pb-pd-pq-ps-rb-rd-rq-rs\end{array}\right)}{(a+c+p+r)(b+d+q+s)}\\
&= -\left(\frac{\begin{array}{c}(\sqrt{ad}-\sqrt{bc})^2+(\sqrt{ps}-\sqrt{qr})^2+(\sqrt{aq}-\sqrt{bp})^2\\ +(\sqrt{as}-\sqrt{br})^2+(\sqrt{cq}-\sqrt{dp})^2+(\sqrt{cs}-\sqrt{dr})^2\end{array}}{(a+c+p+r)(b+d+q+s)}\right)\\
&\leq 0
\end{aligned}$$

Hence, $\mathcal{K}_1^2(\mathcal{C}_1,\mathcal{C}_2) \leq 1$ which implies $\mathcal{K}_1(\mathcal{C}_1,\mathcal{C}_2) \leq 1$. So, $0 \leq \mathcal{K}_1(\mathcal{C}_1,\mathcal{C}_2) \leq 1$.

(P2) For any two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$, we have

$$\begin{aligned}
\mathcal{K}_1(\mathcal{C}_1,\mathcal{C}_2) &= \frac{\sum_{j=1}^{n}\Big(\zeta_1(x_j)\zeta_2(x_j)+w_{\zeta_1}(x_j)w_{\zeta_2}(x_j)+\vartheta_1(x_j)\vartheta_2(x_j)+w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j)\Big)}{\left\{\begin{array}{l}\sqrt{\sum_{j=1}^{n}\Big(\zeta_1^2(x_j)+\vartheta_1^2(x_j)+w_{\zeta_1}^2(x_j)+w_{\vartheta_1}^2(x_j)\Big)}\\ \times\sqrt{\sum_{j=1}^{n}\Big(\zeta_2^2(x_j)+\vartheta_2^2(x_j)+w_{\zeta_2}^2(x_j)+w_{\vartheta_2}^2(x_j)\Big)}\end{array}\right\}}\\
&= \frac{\sum_{j=1}^{n}\Big(\zeta_2(x_j)\zeta_1(x_j)+w_{\zeta_2}(x_j)w_{\zeta_1}(x_j)+\vartheta_2(x_j)\vartheta_1(x_j)+w_{\vartheta_2}(x_j)w_{\vartheta_1}(x_j)\Big)}{\left\{\begin{array}{l}\sqrt{\sum_{j=1}^{n}\Big(\zeta_2^2(x_j)+\vartheta_2^2(x_j)+w_{\zeta_2}^2(x_j)+w_{\vartheta_2}^2(x_j)\Big)}\\ \times\sqrt{\sum_{j=1}^{n}\Big(\zeta_1^2(x_j)+\vartheta_1^2(x_j)+w_{\zeta_1}^2(x_j)+w_{\vartheta_1}^2(x_j)\Big)}\end{array}\right\}}\\
&= \mathcal{K}_1(\mathcal{C}_2,\mathcal{C}_1)
\end{aligned}$$

(P3) If $\mathcal{C}_1 = \mathcal{C}_2$ this implies that $\zeta_1(x_j) = \zeta_2(x_j)$, $\vartheta_1(x_j) = \vartheta_2(x_j)$, $w_{\zeta_1}(x_j) = w_{\zeta_2}(x_j)$ and $w_{\vartheta_1}(x_j) = w_{\vartheta_2}(x_j)$ for all j and thus from Eq. (4.4), it follows that $\mathcal{K}_1(\mathcal{C}_1,\mathcal{C}_2) = 1$.

(P4) Geometrically, if $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$, then the angle between $\mathcal{C}_1$ and $\mathcal{C}_3$ should be larger than the angle between $\mathcal{C}_2$ and $\mathcal{C}_3$ for any element x_j and $\cos\theta$ is decreasing function within interval $[0, \frac{\pi}{2}]$. Therefore, $\mathcal{K}_1(\mathcal{C}_1,\mathcal{C}_3) \leq \mathcal{K}_1(\mathcal{C}_1,\mathcal{C}_2)$ and $\mathcal{K}_1(\mathcal{C}_1,\mathcal{C}_3) \leq \mathcal{K}_1(\mathcal{C}_2,\mathcal{C}_3)$.

Hence, the theorem holds. □

Example 4.2.1. Let $\mathcal{U} = \{x_1, x_2, x_3\}$ be the universal set and $\mathcal{C}_1 = \big\{(x_1$,(0.6,0.7), (0.2,0.2)), (x_2,(0.7,0.5),(0.3,0.4)), (x_3, (0.5,0.4),(0.4,0.1))$\big\}$ and $\mathcal{C}_2 = \big\{(x_1$,(0.5,0.6), (0.1,0.2)), (x_2, (0.7,0.4), (0.1,0.4)),(x_3,(0.6,0.5),(0.3,0.4))$\big\}$ are two CIFSs defined on the universal set $\mathcal{U}$. By applying Eq. (4.1), we obtain the informational energy of $\mathcal{C}_1$ as:

$$\mathfrak{T}(\mathcal{C}_1) = \sum_{j=1}^{n}\left(\zeta_1^2(x_j)+\vartheta_1^2(x_j)+w_{\zeta_1}^2(x_j)+w_{\vartheta_1}^2(x_j)\right)$$

$$\begin{aligned}
&= (0.6)^2 + (0.2)^2 + (0.7)^2 + (0.2)^2 + (0.7)^2 + (0.3)^2 + (0.5)^2 + (0.4)^2 \\
&+ (0.5)^2 + (0.4)^2 + (0.4)^2 + (0.1)^2 \\
&= 0.36 + 0.04 + 0.49 + 0.04 + 0.49 + 0.09 + 0.25 + 0.16 + 0.25 \\
&\quad +0.16 + 0.16 + 0.01 \\
&= 2.5
\end{aligned}$$

Similarly, the informational energy of CIFS $\mathcal{C}_2$ is:

$$\begin{aligned}
\mathfrak{T}(\mathcal{C}_2) &= \sum_{j=1}^{n} \left(\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j)\right) \\
&= (0.5)^2 + (0.1)^2 + (0.6)^2 + (0.2)^2 + (0.7)^2 + (0.1)^2 + (0.4)^2 + (0.4)^2 \\
&\quad +(0.6)^2 + (0.3)^2 + (0.5)^2 + (0.4)^2 \\
&= 0.25 + 0.01 + 0.36 + 0.04 + 0.49 + 0.01 + 0.16 + 0.16 + \\
&\quad 0.36 + 0.09 + 0.25 + 0.16 \\
&= 2.34
\end{aligned}$$

By using Eq. (4.3), the correlation between CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$ is computed as:

$$\begin{aligned}
\mathfrak{C}(\mathcal{C}_1, \mathcal{C}_2) &= \sum_{j=1}^{n} \begin{pmatrix} \zeta_1(x_j)\zeta_2(x_j) + w_{\zeta_1}(x_j)w_{\zeta_2}(x_j) \\ +\vartheta_1(x_j)\vartheta_2(x_j) + w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j) \end{pmatrix} \\
&= 0.6 \times 0.5 + 0.2 \times 0.1 + 0.7 \times 0.6 + 0.2 \times 0.2 + 0.7 \times 0.7 + 0.3 \times 0.1 \\
&\quad +0.5 \times 0.4 + 0.4 \times 0.4 + 0.5 \times 0.6 + 0.4 \times 0.3 + 0.4 \times 0.5 + 0.1 \times 0.4 \\
&= 0.30 + 0.02 + 0.42 + 0.04 + 0.49 + 0.03 + 0.20 + 0.16 + 0.30 \\
&\quad +0.12 + 0.20 + 0.04 \\
&= 2.32
\end{aligned}$$

Thus, the CC between $\mathcal{C}_1$ and $\mathcal{C}_2$ is given by Eq. (4.4) as

$$\begin{aligned}
\mathcal{K}_1(\mathcal{C}_1, \mathcal{C}_2) &= \frac{\mathfrak{C}(\mathcal{C}_1, \mathcal{C}_2)}{\sqrt{\mathfrak{T}(\mathcal{C}_1) \times \mathfrak{T}(\mathcal{C}_2)}} \\
&= \frac{2.32}{\sqrt{2.5 \times 2.34}} = 0.9592
\end{aligned}$$

Definition 4.2.2. Let $\mathcal{C}_1 = \{(x, (\zeta_1(x), w_{\zeta_1}(x)), (\vartheta_1(x), w_{\vartheta_1}(x))) \mid x \in \mathcal{U}\}$ and $\mathcal{C}_2 = \{(x, (\zeta_2(x), w_{\zeta_2}(x)), (\vartheta_2(x), w_{\vartheta_2}(x))) \mid x \in \mathcal{U}\}$ be two CIFSs defined on $\mathcal{U}$. Then, the CC, denoted by $\mathcal{K}_2$, is defined as

$$\begin{aligned}
\mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_2) &= \frac{\mathfrak{C}(\mathcal{C}_1, \mathcal{C}_2)}{\max\{\mathfrak{T}(\mathcal{C}_1), \mathfrak{T}(\mathcal{C}_2)\}} \\
&= \frac{\sum_{j=1}^{n} \Big(\zeta_1(x_j)\zeta_2(x_j) + w_{\zeta_1}(x_j)w_{\zeta_2}(x_j) + \vartheta_1(x_j)\vartheta_2(x_j) + w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j) \Big)}{\max \left\{ \begin{array}{c} \sum\limits_{j=1}^{n} \left(\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j) \right), \\ \sum\limits_{j=1}^{n} \left(\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j) \right) \end{array} \right\}}
\end{aligned} \tag{4.5}$$

Theorem 4.2.2. The CC of two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$, as defined in Eq. (4.5), satisfies the following properties:

(P1) $0 \leq \mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_2) \leq 1$

(P2) $\mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{K}_2(\mathcal{C}_2, \mathcal{C}_1)$

(P3) $\mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_2) = 1$ if $\mathcal{C}_1 = \mathcal{C}_2$

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{K}_2(\mathcal{C}_2, \mathcal{C}_3)$ for any CIFS $\mathcal{C}_3$ defined on $\mathcal{U}$.

Proof. Since $\mathcal{C}_1, \mathcal{C}_2$ are CIFSs, then $0 \leq \zeta_1(x_j), \vartheta_1(x_j) \leq 1$, $0 \leq w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j) \leq 1$ and $\zeta_1(x_j) + \vartheta_1(x_j) \leq 1$ and $w_{\zeta_1}(x_j) + w_{\vartheta_1}(x_j) \leq 1$ for all $x_j \in \mathcal{U}$ and thus from Eq. (4.5), we can obtain the inequality $\mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_2) \geq 0$. The inequality $\mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_2) \leq 1$ can be proven directly by using the well-known Cauchy-Schwarz inequality:

$$\sum_{j=1}^{n} a_j b_j \leq \sqrt{\left(\sum_{j=1}^{n} a_j^2\right) \cdot \left(\sum_{j=1}^{n} b_j^2\right)} \tag{4.6}$$

with equality if and only if the two vectors $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are linearly dependent.

In fact, by Eq. (4.6), we have

$$\begin{aligned}\sum_{j=1}^{n} a_j b_j &\leq \sqrt{\left(\sum_{j=1}^{n} a_j^2\right) \cdot \left(\sum_{j=1}^{n} b_j^2\right)} \leq \sqrt{\left(\max\left\{\sum_{j=1}^{n} a_j^2, \sum_{j=1}^{n} b_j^2\right\}\right)^2} \\ &= \max\left\{\sum_{j=1}^{n} a_j^2, \sum_{j=1}^{n} b_j^2\right\}\end{aligned}$$

and from Eq. (4.5), it follows that $0 \leq \mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_2) \leq 1$ which completes the proof of (P1). In addition, by Eq. (4.5), we have

$$\begin{aligned}\mathcal{K}_2(\mathcal{C}_1, \mathcal{C}_2) &= \frac{\sum_{j=1}^{n}\Big(\zeta_1(x_j)\zeta_2(x_j) + w_{\zeta_1}(x_j)w_{\zeta_2}(x_j) + \vartheta_1(x_j)\vartheta_2(x_j) + w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j)\Big)}{\max\left\{\begin{array}{c}\sum_{j=1}^{n}\left(\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j)\right), \\ \sum_{j=1}^{n}\left(\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j)\right)\end{array}\right\}} \\ &= \frac{\sum_{j=1}^{n}\Big(\zeta_2(x_j)\zeta_1(x_j) + w_{\zeta_2}(x_j)w_{\zeta_1}(x_j) + \vartheta_2(x_j)\vartheta_1(x_j) + w_{\vartheta_2}(x_j)w_{\vartheta_1}(x_j)\Big)}{\max\left\{\begin{array}{c}\sum_{j=1}^{n}\left(\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j)\right), \\ \sum_{j=1}^{n}\left(\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j)\right)\end{array}\right\}} \\ &= \mathcal{K}_2(\mathcal{C}_2, \mathcal{C}_1)\end{aligned}$$

Thus, (P2) also holds. Similarly, we can complete the proofs of (P3) and (P4).

Hence, the theorem holds. □

From Definition 4.2.1 and Definition 4.2.2, we observe that the CC defined by Eq. (4.4) uses the geometric mean of the informational energies of the CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$, and the CC defined by Eq. (4.5) applies the maximum between them. For the optimistic decision makers, they tend to use the CC defined by Eq. (4.4). Contrary to the optimistic decision makers, the pessimistic decision makers tend to apply the CC defined by Eq. (4.5).

In the above defined formulas for calculating coefficient of correlation, equal importance is given to all the elements of the universal set. But in real life situations, this may not be always possible. Some elements in the universal set are more important than the others. So we must take into account the proper weightage given to the various elements of the universal set. In the following, we propose a weighted CC between CIFSs. Let $\xi = (\xi_1, \xi_2, \ldots \xi_n)^T$ be the weight vector corresponding to the elements x_j $(j = 1, 2, \ldots, n)$ with $\xi_j > 0$ and $\sum_{j=1}^{n} \xi_j = 1$. Then, we extend the above defined CCs $\mathcal{K}_1$ and $\mathcal{K}_2$ to weighted CCs, $\mathcal{K}_3$ and $\mathcal{K}_4$ respectively, as follows:

$$\begin{aligned}\mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_2) &= \frac{\mathfrak{C}_w(\mathcal{C}_1, \mathcal{C}_2)}{\sqrt{\mathfrak{T}_w(\mathcal{C}_1) \times \mathfrak{T}_w(\mathcal{C}_2)}} \\ &= \frac{\sum_{j=1}^{n} \left(\xi_j \begin{bmatrix} \zeta_1(x_j)\zeta_2(x_j) + w_{\zeta_1}(x_j)w_{\zeta_2}(x_j) \\ +\vartheta_1(x_j)\vartheta_2(x_j) + w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j) \end{bmatrix} \right)}{\left\{ \begin{array}{l} \sqrt{\sum_{j=1}^{n} \left(\xi_j \left[\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j) \right] \right)} \\ \times \sqrt{\sum_{j=1}^{n} \left(\xi_j \left[\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j) \right] \right)} \end{array} \right\}}\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\mathcal{K}_4(\mathcal{C}_1, \mathcal{C}_2) &= \frac{\mathfrak{C}_w(\mathcal{C}_1, \mathcal{C}_2)}{\max\{\mathfrak{T}_w(\mathcal{C}_1), \mathfrak{T}_w(\mathcal{C}_2)\}} \\ &= \frac{\sum_{j=1}^{n} \left(\xi_j \begin{bmatrix} \zeta_1(x_j)\zeta_2(x_j) + w_{\zeta_1}(x_j)w_{\zeta_2}(x_j) \\ +\vartheta_1(x_j)\vartheta_2(x_j) + w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j) \end{bmatrix} \right)}{\max \left\{ \begin{array}{l} \sum_{j=1}^{n} \left(\xi_j \left[\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j) \right] \right), \\ \sum_{j=1}^{n} \left(\xi_j \left[\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j) \right] \right) \end{array} \right\}}\end{aligned} \tag{4.8}$$

It can be easily verify that, if $\xi = (1/n, 1/n, \ldots, 1/n)^T$ then Eqs. (4.7) and (4.8) reduce to the CCs given in Eqs. (4.4) and (4.5) respectively. Further, it can be deduced that the CCs $\mathcal{K}_3$ and $\mathcal{K}_4$ between CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$ also satisfies the property of $0 \leq \mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_2) \leq 1$ and $0 \leq \mathcal{K}_4(\mathcal{C}_1, \mathcal{C}_2) \leq 1$.

Theorem 4.2.3. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two CIFSs defined on $\mathcal{U}$. If $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ be the weight vector corresponding to x_j, $(j = 1, 2, \ldots, n)$ with $\xi_j > 0$ and $\sum_{j=1}^{n} \xi_j = 1$ then the weighted CC $\mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_2)$ between the two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$ defined in Eq. (4.7), satisfies the following properties:

(P1) $0 \leq \mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_2) \leq 1$

(P2) $\mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{K}_3(\mathcal{C}_2, \mathcal{C}_1)$

(P3) $\mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_2) = 1$ if $\mathcal{C}_1 = \mathcal{C}_2$

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{K}_3(\mathcal{C}_2, \mathcal{C}_3)$ for any CIFS $\mathcal{C}_3$ defined on $\mathcal{U}$.

Proof. The properties (P2)-(P4) are straightforward, so we omit their proof here. Now, we shall prove only (P1) property. For it, let $\mathcal{C}_1 = \{(x, (\zeta_1(x), w_{\zeta_1}(x)), (\vartheta_1(x), w_{\vartheta_1}(x))) \mid x \in \mathcal{U}\}$ and $\mathcal{C}_2 = \{(x, (\zeta_2(x), w_{\zeta_2}(x)), (\vartheta_2(x), w_{\vartheta_2}(x))) \mid x \in \mathcal{U}\}$ be two CIFSs defined on $\mathcal{U}$. From the Eq. (4.7), it is clearly seen that, $\mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_2) \geq 0$. So, we will prove only $\mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_2) \leq 1$.

$$
\begin{aligned}
& \mathcal{K}_3(\mathcal{C}_1, \mathcal{C}_2) \\
= \; & \frac{\mathfrak{C}_w(\mathcal{C}_1, \mathcal{C}_2)}{\sqrt{\mathfrak{T}_w(\mathcal{C}_1) \times \mathfrak{T}_w(\mathcal{C}_2)}} \\
= \; & \frac{\sum_{j=1}^{n} \left(\xi_j \left[\zeta_1(x_j)\zeta_2(x_j) + w_{\zeta_1}(x_j) w_{\zeta_2}(x_j) + \vartheta_1(x_j)\vartheta_2(x_j) + w_{\vartheta_1}(x_j) w_{\vartheta_2}(x_j) \right] \right)}{\left\{ \begin{array}{c} \sqrt{\sum_{j=1}^{n} \left(\xi_j \left[\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j) \right] \right)} \\ \times \sqrt{\sum_{j=1}^{n} \left(\xi_j \left[\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j) \right] \right)} \end{array} \right\}} \\
= \; & \frac{\left(\begin{array}{c} \sum_{j=1}^{n} \left(\sqrt{\xi_j} \zeta_1(x_j) \right) \left(\sqrt{\xi_j} \zeta_2(x_j) \right) + \sum_{j=1}^{n} \left(\sqrt{\xi_j} w_{\zeta_1}(x_j) \right) \left(\sqrt{\xi_j} w_{\zeta_2}(x_j) \right) \\ + \sum_{j=1}^{n} \left(\sqrt{\xi_j} \vartheta_1(x_j) \right) \left(\sqrt{\xi_j} \vartheta_2(x_j) \right) + \sum_{j=1}^{n} \left(\sqrt{\xi_j} w_{\vartheta_1}(x_j) \right) \left(\sqrt{\xi_j} w_{\vartheta_2}(x_j) \right) \end{array} \right)}{\left\{ \begin{array}{c} \sqrt{\sum_{j=1}^{n} \left(\xi_j \left[\zeta_1^2(x_j) + \vartheta_1^2(x_j) + w_{\zeta_1}^2(x_j) + w_{\vartheta_1}^2(x_j) \right] \right)} \\ \times \sqrt{\sum_{j=1}^{n} \left(\xi_j \left[\zeta_2^2(x_j) + \vartheta_2^2(x_j) + w_{\zeta_2}^2(x_j) + w_{\vartheta_2}^2(x_j) \right] \right)} \end{array} \right\}}
\end{aligned}
$$

By using Cauchy-Schwarz inequality, we obtain

$$\mathcal{K}_3\left(\mathcal{C}_1,\mathcal{C}_2\right) \leq \frac{\left(\begin{array}{c}\sqrt{\sum_{j=1}^{n}\xi_j\zeta_1^2(x_j)}\sqrt{\sum_{j=1}^{n}\xi_j\zeta_2^2(x_j)}+\sqrt{\sum_{j=1}^{n}\xi_j\left(w_{\zeta_1}(x_j)\right)^2}\sqrt{\sum_{j=1}^{n}\xi_j\left(w_{\zeta_2}(x_j)\right)^2}\\ +\sqrt{\sum_{j=1}^{n}\xi_j\vartheta_1^2(x_j)}\sqrt{\sum_{j=1}^{n}\xi_j\vartheta_2^2(x_j)}+\sqrt{\sum_{j=1}^{n}\xi_j\left(w_{\vartheta_1}(x_j)\right)^2}\sqrt{\sum_{j=1}^{n}\xi_j\left(w_{\vartheta_2}(x_j)\right)^2}\end{array}\right)}{\left\{\begin{array}{c}\sqrt{\sum_{j=1}^{n}\left(\xi_j\left[\zeta_1^2(x_j)+\vartheta_1^2(x_j)+w_{\zeta_1}^2(x_j)+w_{\vartheta_1}^2(x_j)\right]\right)}\\ \times\sqrt{\sum_{j=1}^{n}\left[\xi_j\left(\zeta_2^2(x_j)+\vartheta_2^2(x_j)+w_{\zeta_2}^2(x_j)+w_{\vartheta_2}^2(x_j)\right)\right]}\end{array}\right\}}$$

By using the following notations

$$\begin{array}{c}\sum_{j=1}^{n}\xi_j\zeta_1^2(x_j)=a,\sum_{j=1}^{n}\xi_j\zeta_2^2(x_j)=b,\sum_{j=1}^{n}\xi_jw_{\zeta_1}^2(x_j)=p,\sum_{j=1}^{n}\xi_j\vartheta_1^2(x_j)=c,\\ \sum_{j=1}^{n}\xi_j\vartheta_2^2(x_j)=d,\sum_{j=1}^{n}\xi_jw_{\zeta_2}^2(x_j)=q,\sum_{j=1}^{n}\xi_jw_{\vartheta_1}^2(x_j)=r,\sum_{j=1}^{n}\xi_jw_{\vartheta_2}^2(x_j)=s,\end{array}$$

we can reduce the above inequality to

$$\mathcal{K}_3\left(\mathcal{C}_1,\mathcal{C}_2\right)\leq\frac{\sqrt{ab}+\sqrt{cd}+\sqrt{pq}+\sqrt{rs}}{\sqrt{(a+c+p+r)(b+d+q+s)}}$$

Therefore,

$$\begin{aligned}\left(\mathcal{K}_3\left(\mathcal{C}_1,\mathcal{C}_2\right)\right)^2-1 &\leq \frac{\left(\sqrt{ab}+\sqrt{cd}+\sqrt{pq}+\sqrt{rs}\right)^2}{(a+c+p+r)(b+d+q+s)}-1\\ &= \frac{\left(\sqrt{ab}+\sqrt{cd}+\sqrt{pq}+\sqrt{rs}\right)^2-(a+c+p+r)(b+d+q+s)}{(a+c+p+r)(b+d+q+s)}\\ &= -\left(\frac{\begin{array}{c}(\sqrt{ad}-\sqrt{bc})^2+(\sqrt{ps}-\sqrt{qr})^2+(\sqrt{aq}-\sqrt{bp})^2\\ +(\sqrt{as}-\sqrt{br})^2+(\sqrt{cq}-\sqrt{dp})^2+(\sqrt{cs}-\sqrt{dr})^2\end{array}}{(a+c+p+r)(b+d+q+s)}\right)\\ &\leq 0\end{aligned}$$

which implies that $\mathcal{K}_3\left(\mathcal{C}_1,\mathcal{C}_2\right)\leq 1$. Hence, $0\leq\mathcal{K}_3\left(\mathcal{C}_1,\mathcal{C}_2\right)\leq 1$. □

Theorem 4.2.4. The CC of two CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$, as defined in Eq. (4.8) i.e., $\mathcal{K}_4$, satisfies the same properties as those in Theorem 4.2.2.

Proof. The proof is similar to Theorem 4.2.2, so we omit here. □

4.3 MCDM approach based on the proposed correlation coefficients

In this section, we utilize the proposed CCs of CIFSs to present the multicriteria decision making method. The general description of DM problem is summarized in Section 2.5 of Chapter 2. The rating values corresponding to each alternative are represented as follows

$$\mathcal{V}_u = \{(\mathfrak{B}_v, (\zeta_{uv}(\mathfrak{B}_v), w_{\zeta_{uv}}(\mathfrak{B}_v)), (\vartheta_{uv}(\mathfrak{B}_v), w_{\vartheta_{uv}}(\mathfrak{B}_v))) \mid v = 1, 2, \ldots, n\}, u = 1, 2, \ldots, m.$$

For convenience, we denote the rating values of alternative $\mathcal{V}_u$ under criteria $\mathfrak{B}_v$ by $\mathcal{C}_{uv} = ((\zeta_{uv}, w_{\zeta_{uv}}), (\vartheta_{uv}, w_{\vartheta_{uv}}))$ where $\zeta_{uv}, \vartheta_{uv}, w_{\zeta_{uv}}, w_{\vartheta_{uv}} \in [0, 1]$ and $\zeta_{uv} + \vartheta_{uv}; w_{\zeta_{uv}} + w_{\vartheta_{uv}} \leq 1$ for $u = 1, 2, \ldots, m; v = 1, 2, \ldots, n$ and call it as CIFNs. Then, we utilize the following steps based on the proposed CCs for solving the MCDM problems under the CIFSs environment.

Step 1: Construct the ideal reference set to find the best alternative in the decision set whose rating values are taken under the CIFSs environment. We denote such reference set by $\mathcal{V}$.

Step 2: Construct the decision matrix based on the collective information of the alternatives $\mathcal{V}_u(u = 1, 2, \ldots, m)$ under the set of criteria $\mathfrak{B}_v(v = 1, 2, \ldots, n)$ as provided by an expert in terms of CIFNs $\mathcal{C}_{uv} = ((\zeta_{uv}, w_{\zeta_{uv}}), (\vartheta_{uv}, w_{\vartheta_{uv}}))$. We denote such matrix as $\mathcal{M} = (\mathcal{C}_{uv})_{m \times n}$ which can be represented as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \ldots & \mathcal{C}_{1n} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \ldots & \mathcal{C}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{m1} & \mathcal{C}_{m2} & \ldots & \mathcal{C}_{mn} \end{pmatrix} \end{array}$$

Step 3: Calculate the CC between the alternatives $\mathcal{V}_u(u = 1, 2, \ldots, m)$ and the reference set $\mathcal{V}$ by using either $\mathcal{K}_1$ or $\mathcal{K}_2$ or by using weighted CCs $\mathcal{K}_3$ or $\mathcal{K}_4$, to compute the degree of the relationship among the alternatives.

Step 4: Rank the alternatives based on the index values of CCs as obtained from $\arg\max\{\mathcal{K}\}$. As larger the value of CCs, the better is the alternative $\mathcal{V}_u(u = 1, 2, \ldots, m)$.

4.4 Illustrative Examples

In order to demonstrate the above mentioned approach based on CCs, we present two illustrative examples which are described as follows.

4.4.1 Example 1: Decision-making problem

An earthquake of 7.8 magnitude, also called as Gorkha earthquake, racked Nepal on 25 April 2015 at a depth of approximately 15km and lasted nearly fifty seconds and its epicenter was about 21 miles east southest of Lamjung and 48 miles northwest of Kathmandu and its focus was 9.3 miles underground and it destroyed thousands of houses across many districts of the country with entire villages flattened especially near the epicenter. An aftershock occurred on 12 May 2015 in Nepal which heightened the fears and tensions among the affected people. The two earthquakes together resulted in many damages, economic losses in almost 35 districts out of which 5 regions were severely affected namely: $\mathcal{V}_1$: Lalitpur, $\mathcal{V}_2$: Kathmandu, $\mathcal{V}_3$: Gorkha, $\mathcal{V}_4$: Bhaktapur and $\mathcal{V}_5$: Makwanpur. An earthquake relief camp decided to help victims of the earthquake in these five different regions $\mathcal{V}_u(u = 1, 2, \ldots, 5)$ of the affected area with a different intensity of an earthquake. The panel decided that they will plan their budget by considering the four basic needs of victims, considered as criterion, namely $\mathfrak{B}_1$(Food), $\mathfrak{B}_2$(Shelter), $\mathfrak{B}_3$(Clothes) and $\mathfrak{B}_4$(Medical requirements) and decided to allocate the budget firstly to the most affected region so that by initial efforts only, a large strata of people get relief. The weight vector corresponding to these basic needs is taken as $\xi = (0.30, 0.25, 0.15, 0.30)^T$. Clearly, according to the intensity of the earthquake, the basic needs of victims will be affected and changed. The target of this problem is to find the most affected region out of $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_5$ so as to allocate the proper budget and all the necessary facility to them. To achieve it, we utilize the developed approach to rank the regions and the best one(s) can be found by implementing the steps of the proposed approach as follows:

Step 1: Assume that an expert gives their preference in terms of maximum possible needs of all the five regions over the each basic need $\mathfrak{B}_v(v = 1, 2, 3, 4)$ as a CIFS. The rating values of this set, denoted by $\mathcal{V}$ and called as a "reference set" in order to evaluate the given five regions are summarized as below:

$$\mathcal{V} = \left\{ \begin{array}{l} (\mathfrak{B}_1, (0.7, 0.5), (0.1, 0.3)), (\mathfrak{B}_2, (0.4, 0.6), (0.5, 0.2)), \\ (\mathfrak{B}_3, (0.5, 0.5), (0.3, 0.1)), (\mathfrak{B}_4, (0.8, 0.7), (0.2, 0.1)) \end{array} \right\}$$

Step 2: An expert evaluates the each region $\mathcal{V}_u(u = 1, 2, \ldots, 5)$ individually and estimates the requirements under the set of criteria $\mathfrak{B}_v(v = 1, 2, 3, 4)$. Their rating values towards each region is expressed in terms of CIFNs whose values are summarized as follows.

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \mathcal{V}_3 \\ \mathcal{V}_4 \\ \mathcal{V}_5 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathfrak{B}_1\quad & \quad\mathfrak{B}_2\quad & \quad\mathfrak{B}_3\quad & \quad\mathfrak{B}_4\quad \end{array} \\ \begin{pmatrix} ((0.6,0.7),(0.1,0.2)) & ((0.9,0.8),(0.1,0.1)) & ((0.5,0.4),(0.3,0.4)) & ((0.6,0.4),(0.2,0.1)) \\ ((0.4,0.2),(0.3,0.1)) & ((0.5,0.3),(0.1,0.1)) & ((0.6,0.4),(0.2,0.3)) & ((0.8,0.6),(0.1,0.2)) \\ ((0.7,0.7),(0.1,0.2)) & ((0.4,0.6),(0.3,0.1)) & ((0.7,0.7),(0.1,0.1)) & ((0.6,0.5),(0.3,0.4)) \\ ((0.7,0.6),(0.3,0.3)) & ((0.4,0.9),(0.2,0.1)) & ((0.7,0.7),(0.2,0.3)) & ((0.5,0.3),(0.3,0.6)) \\ ((0.2,0.8),(0.5,0.1)) & ((0.7,0.3),(0.3,0.3)) & ((0.6,0.5),(0.1,0.3)) & ((0.6,0.5),(0.3,0.4)) \end{pmatrix} \end{array}$$

Step 3a: By applying the CC $\mathcal{K}_1$ as given in Eq. (4.4) between the alternatives $\mathcal{V}_u(u = 1, 2, 3, 4, 5)$ and the reference set $\mathcal{V}$, we can obtain their measurement values as $\mathcal{K}_1(\mathcal{V}_1, \mathcal{V}) = 0.8936$, $\mathcal{K}_1(\mathcal{V}_2, \mathcal{V}) = 0.9068$, $\mathcal{K}_1(\mathcal{V}_3, \mathcal{V}) = 0.9424$, $\mathcal{K}_1(\mathcal{V}_4, \mathcal{V}) = 0.8830$ and $\mathcal{K}_1(\mathcal{V}_5, \mathcal{V}) = 0.8450$. On the other hand, by utilizing the CC $\mathcal{K}_2$ given in Eq. (4.5), their corresponding measurement values are $\mathcal{K}_2(\mathcal{V}_1, \mathcal{V}) = 0.8722$, $\mathcal{K}_2(\mathcal{V}_2, \mathcal{V}) = 0.7522$, $\mathcal{K}_2(\mathcal{V}_3, \mathcal{V}) = 0.9316$, $\mathcal{K}_2(\mathcal{V}_4, \mathcal{V}) = 0.8228$ and $\mathcal{K}_2(\mathcal{V}_5, \mathcal{V}) = 0.8251$.

Step 3b: If we assign the weight vector $\xi = (0.30, 0.25, 0.15, 0.30)^T$ to the criteria, then by utilizing a weighted CC $\mathcal{K}_3$ as given in Eq. (4.7) to compute the measurement values between the alternatives $\mathcal{V}_u(u = 1, 2, \ldots, 5)$ and set $\mathcal{V}$, we get $\mathcal{K}_3(\mathcal{V}_1, \mathcal{V}) = 0.8965$, $\mathcal{K}_3(\mathcal{V}_2, \mathcal{V}) = 0.9087$, $\mathcal{K}_3(\mathcal{V}_3, \mathcal{V}) = 0.9439$, $\mathcal{K}_3(\mathcal{V}_4, \mathcal{V}) = 0.8747$ and $\mathcal{K}_3(\mathcal{V}_5, \mathcal{V}) = 0.8351$. Similarly, by using CC $\mathcal{K}_4$, we get their corresponding results are $\mathcal{K}_4(\mathcal{V}_1, \mathcal{V}) = 0.8920$, $\mathcal{K}_4(\mathcal{V}_2, \mathcal{V}) = 0.7379$, $\mathcal{K}_4(\mathcal{V}_3, \mathcal{V}) = 0.9299$, $\mathcal{K}_4(\mathcal{V}_4, \mathcal{V}) = 0.8429$ and $\mathcal{K}_4(\mathcal{V}_5, \mathcal{V}) = 0.8058$.

Step 4: From these computed measurement values, we conclude that the ranking order of the regions $\mathcal{V}_u(u = 1, 2, \ldots, 5)$ is $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_5$, where "$\succ$" stands for "preferred to" when $\mathcal{K}_1$ CC index has been used while $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2$ when $\mathcal{K}_2$ index has been used. Similarly, the ranking order of the region by considering the weight factor into the account is $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_5$ and $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_5 \succ \mathcal{V}_2$ respectively, when either the CC $\mathcal{K}_3$ or $\mathcal{K}_4$ is utilized. It is observed from this analysis that the ranking order of the region is different for the different indices. However, the best alternative i.e., the most affected area remains same ($\mathcal{V}_3$) while the worst changes according to the optimistic to pessimistic behavior. Hence, based on the behavior of the decision makers' toward the ranking order related to optimistic and pessimistic behavior, they can choose the desired one accordingly. For instance, related to an optimistic decision makers behavior, they tend to prefer $\mathcal{V}_4$ over $\mathcal{V}_5$ due to ranking order $\mathcal{V}_4 \succ \mathcal{V}_5$ while for pessimistic attitude choice towards the region, they tend to choose $\mathcal{V}_5$ over $\mathcal{V}_4$ regions to allocate the funds due to $\mathcal{V}_5 \succ \mathcal{V}_4$. Therefore, based on the attitude behavioral characteristic of the decision maker, they can use the best and worst region to allocate the budget.

4.4.2 Example 2: Medical Diagnosis

Consider the medical diagnosis problem as stated in Example 3.3.2 of Chapter 3. The target of this decision-making problem is to diagnose the disease of the patient $\mathfrak{P}$ among $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4$. For it, we utilized the steps of the proposed approach to obtain the suitable ranking of the diagnoses and are summarized as follows:

Step 1: The patient $\mathfrak{P}$ is treated as a reference set and expert gave their preferences with respect to all the symptoms as:

$$\mathfrak{P} = \left\{ \begin{array}{l} (s_1, (0.8, 0.6), (0.1, 0.2)), (s_2, (0.9, 0.7), (0.1.0.2)), \\ (s_3, (0.7, 0.8), (0.2, 0.1)), (s_4, (0.6, 0.5), (0.2, 0.4)) \end{array} \right\}$$

Step 2: The rating values of each diagnosis $\mathcal{V}_u (u = 1, 2, 3, 4)$ are expressed by a doctor (called as an expert) under the set of symptoms $s_v (v = 1, 2, 3, 4)$ and are summarized as a complex intuitionistic fuzzy decision matrix $\mathcal{M}$ as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \mathcal{V}_3 \\ \mathcal{V}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad s_1 \quad\quad\quad & \quad\quad s_2 \quad\quad\quad & \quad\quad s_3 \quad\quad\quad & \quad\quad s_4 \end{array} \\ \begin{pmatrix} ((0.8, 0.7), (0.1, 0.2)) & ((0.9, 0.6), (0.1, 0.2)) & ((0.7, 0.8), (0.2, 0.1)) & ((0.8, 0.7), (0.2, 0.1)) \\ ((0.6, 0.4), (0.1, 0.5)) & ((0.4, 0.9), (0.5, 0.1)) & ((0.5, 0.5), (0.3, 0.3)) & ((0.4, 0.9), (0.5, 0.1)) \\ ((0.3, 0.8), (0.3, 0.1)) & ((0.8, 0.3), (0.1, 0.6)) & ((0.7, 0.6), (0.2, 0.2)) & ((0.2, 0.7), (0.8, 0.2)) \\ ((0.5, 0.3), (0.4, 0.6)) & ((0.3, 0.1), (0.6, 0.3)) & ((0.8, 0.3), (0.1, 0.5)) & ((0.1, 0.3), (0.6, 0.5)) \end{pmatrix} \end{array}$$

Step 3a: By applying the CC $\mathcal{K}_1$ between the set $\mathcal{V}_u(u = 1,2,3,4)$ and the patient $\mathfrak{P}$, we get their measurement values are $\mathcal{K}_1(\mathcal{V}_1,\mathfrak{P}) = 0.9800$, $\mathcal{K}_1(\mathcal{V}_2,\mathfrak{P}) = 0.8582$, $\mathcal{K}_1(\mathcal{V}_3,\mathfrak{P}) = 0.8446$ and $\mathcal{K}_1(\mathcal{V}_4,\mathfrak{P}) = 0.7037$. On the other hand, by using CC $\mathcal{K}_2$, then these values are $\mathcal{K}_2(\mathcal{V}_1,\mathfrak{P}) = 0.9412$, $\mathcal{K}_2(\mathcal{V}_2,\mathfrak{P}) = 0.8109$, $\mathcal{K}_3(\mathcal{V}_3,\mathfrak{P}) = 0.8132$ and $\mathcal{K}_4(\mathcal{V}_4,\mathfrak{P}) = 0.5923$.

Step 3b: If we assign the weightage to the set of symptoms $s_v(v = 1,2,3,4)$ as $\xi = (0.30, 0.20, 0.10, 0.40)^T$ then the measurement values by CCs $\mathcal{K}_3$ and $\mathcal{K}_4$ are $\mathcal{K}_3(\mathcal{V}_1,\mathfrak{P}) = 0.9696$, $\mathcal{K}_3(\mathcal{V}_2,\mathfrak{P}) = 0.8486$, $\mathcal{K}_3(\mathcal{V}_3,\mathfrak{P}) = 0.8008$, $\mathcal{K}_3(\mathcal{V}_4,\mathfrak{P}) = 0.6980$ and $\mathcal{K}_4(\mathcal{V}_1,\mathfrak{P}) = 0.9015$, $\mathcal{K}_4(\mathcal{V}_2,\mathfrak{P}) = 0.8433$, $\mathcal{K}_4(\mathcal{V}_3,\mathfrak{P}) = 0.7935$, $\mathcal{K}_4(\mathcal{V}_4,\mathfrak{P}) = 0.5969$.

Step 4: Based on the optimal values of the diseases, we conclude that its ranking order is $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$ when the $\mathcal{K}_1$ CC index has been used, while $\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$ when $\mathcal{K}_2$ index has been used. From this analysis, it is concluded that the patient $\mathfrak{P}$ suffer from the $\mathcal{V}_1$ diseases. Further, from these ranking orders, we observe that when decision maker utilize the $\mathcal{K}_1$ CC by keeping his mind towards the optimistic view, then the second most diseases affected to the patient is $\mathcal{V}_2$. On the other hand, if decision makers attitude towards the diseases is pessimistic in nature, they will tend towards $\mathcal{V}_3$ be the second most diseases affected to the patient $\mathfrak{P}$. Similarly, we get the ranking order of the diseases affected to the patient corresponding to the utilization of $\mathcal{K}_3$ and $\mathcal{K}_4$ as $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$.

4.5 Comparative analysis

In this section, we compare the performance of the proposed measures with some of the existing approaches under the CIFS as well as IFS environment. The detailed analysis of the above considered examples is explained as below.

4.5.1 Comparative studies of Example 1 under CIFS environment

In order to compare the proposed approach results with some of the existing approaches [6, 129] under the CIFS environment, an analysis has been conducted for the considered data. The results corresponding to these approaches are summarized as follows:

(i) If we utilize the distance measure, denoted by $\mathcal{D}_1$, as proposed by Alkouri and Salleh [6] to the considered data, then corresponding to each region the measurement values from the reference set $\mathcal{V}$ are $\mathcal{D}_1(\mathcal{V}_1,\mathcal{V}) = 0.1817$, $\mathcal{D}_1(\mathcal{V}_2,\mathcal{V}) = 0.1917$, $\mathcal{D}_1(\mathcal{V}_3,\mathcal{V}) = 0.1400$, $\mathcal{D}_1(\mathcal{V}_4,\mathcal{V}) = 0.2167$ and $\mathcal{D}_1(\mathcal{V}_5,\mathcal{V}) = 0.2600$. Since $\mathcal{D}_1(\mathcal{V}_3,\mathcal{V}) < \mathcal{D}_1(\mathcal{V}_1,\mathcal{V}) < \mathcal{D}_1(\mathcal{V}_2,\mathcal{V}) < \mathcal{D}_1(\mathcal{V}_4,\mathcal{V}) < \mathcal{D}_1(\mathcal{V}_5,\mathcal{V})$ and hence the ranking order of the given regions is $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_5$. Thus, it has been computed that that $\mathcal{V}_3$ is the most affected region.

(ii) If we utilize the weighted Euclidean distance measure, denoted by $\mathcal{D}_2$, as defined by Rani and Garg [129] to the considered problem, then the measurement values of each region are computed as $\mathcal{D}_2(\mathcal{V}_1,\mathcal{V}) = 0.1871$, $\mathcal{D}_2(\mathcal{V}_2,\mathcal{V}) = 0.1803$, $\mathcal{D}_2(\mathcal{V}_3,\mathcal{V}) = 0.1374$, $\mathcal{D}_2(\mathcal{V}_4,\mathcal{V}) = 0.2086$ and $\mathcal{D}_2(\mathcal{V}_5,\mathcal{V}) = 0.2225$. Since $\mathcal{D}_2(\mathcal{V}_3,\mathcal{V}) < \mathcal{D}_2(\mathcal{V}_2,\mathcal{V}) < \mathcal{D}_2(\mathcal{V}_1,\mathcal{V}) < \mathcal{D}_2(\mathcal{V}_4,\mathcal{V}) < \mathcal{D}_2(\mathcal{V}_5,\mathcal{V})$ and hence ranking order of the region is $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_5$. Therefore, we conclude that $\mathcal{V}_3$ is again the most affected region.

From these comparative studies, it is concluded that the best region obtained from the proposed measure coincides with the existing measures and hence it validate the feasibility of the approach under the CIFS environment.

4.5.2 Comparative studies of Example 1 under IFS environment

In order to compare the proposed approach results with the results obtained under IFSs environment, we conducted an analysis by executing some of the existing approaches [100, 109, 197, 207] to the considered data. Since IFS is a special case of the CIFS, so we firstly convert the CIFS environment data into the IFS environment data by setting phase term corresponding to each criteria to 0 in every CIFN. Then, based on these existing approaches, we conduct the following analysis to compute the most affected region(s) under the IFS environment.

(i) If we apply the CC, denoted by $\mathcal{K}_1'$, as defined by Zeng and Li [207] to the considered problem, then we get the measurement value of each region is $\mathcal{K}_1'(\mathcal{V}_1,\mathcal{V}) = 0.8740$, $\mathcal{K}_1'(\mathcal{V}_2,\mathcal{V}) = 0.8874$, $\mathcal{K}_1'(\mathcal{V}_3,\mathcal{V}) = 0.9442$, $\mathcal{K}_1'(\mathcal{V}_4,\mathcal{V}) = 0.8822$ and $\mathcal{K}_1'(\mathcal{V}_5,\mathcal{V}) = 0.8262$.

Since $\mathcal{K}_1'(\mathcal{V}_3,\mathcal{V}) > \mathcal{K}_1'(\mathcal{V}_2,\mathcal{V}) > \mathcal{K}_1'(\mathcal{V}_4,\mathcal{V}) > \mathcal{K}_1'(\mathcal{V}_1,\mathcal{V}) > \mathcal{K}_1'(\mathcal{V}_5,\mathcal{V})$ and hence we conclude that $\mathcal{V}_3$ is the most affected region.

(ii) If we apply the CC, denoted by $\mathcal{K}_2'$, as defined by Ye [197] on the reduced data, then we get $\mathcal{K}_2'(\mathcal{V}_1,\mathcal{V}) = 0.8808$, $\mathcal{K}_2'(\mathcal{V}_2,\mathcal{V}) = 0.9106$, $\mathcal{K}_2'(\mathcal{V}_3,\mathcal{V}) = 0.9551$, $\mathcal{K}_2'(\mathcal{V}_4,\mathcal{V}) = 0.9246$ and $\mathcal{K}_2'(\mathcal{V}_5,\mathcal{V}) = 0.8250$. Since, $\mathcal{K}_2'(\mathcal{V}_3,\mathcal{V}) > \mathcal{K}_2'(\mathcal{V}_4,\mathcal{V}) > \mathcal{K}_2'(\mathcal{V}_2,\mathcal{V}) > \mathcal{K}_2'(\mathcal{V}_1,\mathcal{V}) > \mathcal{K}_2'(\mathcal{V}_5,\mathcal{V})$ and hence ranking order is $\mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_5$. Thus, we conclude that $\mathcal{V}_3$ is the most affected region which coincides with the proposed one.

(iii) If we utilize CC ($\mathcal{K}_3'$) as presented by Liu et al. [100], then the corresponding indices to each region are computed as $\mathcal{K}_3'(\mathcal{V}_1,\mathcal{V}) = -0.4603$, $\mathcal{K}_3'(\mathcal{V}_2,\mathcal{V}) = 0.0000$, $\mathcal{K}_3'(\mathcal{V}_3,\mathcal{V}) = 0.5198$, $\mathcal{K}_3'(\mathcal{V}_4,\mathcal{V}) = 0.1143$ and $\mathcal{K}_3'(\mathcal{V}_5,\mathcal{V}) = -0.6336$. Since, $\mathcal{K}_3'(\mathcal{V}_3,\mathcal{V}) > \mathcal{K}_3'(\mathcal{V}_4,\mathcal{V}) > \mathcal{K}_3'(\mathcal{V}_2,\mathcal{V}) > \mathcal{K}_3'(\mathcal{V}_1,\mathcal{V}) > \mathcal{K}_3'(\mathcal{V}_5,\mathcal{V})$ and hence again the most affected region is $\mathcal{V}_3$ and it coincides with the proposed measure results.

(iv) By utilizing the similarity measure [109], denoted by $\mathcal{K}_4'$, on the considered data, we get their measurement values are $\mathcal{K}_4'(\mathcal{V}_1,\mathcal{V}) = 0.8221$, $\mathcal{K}_4'(\mathcal{V}_2,\mathcal{V}) = 0.8527$, $\mathcal{K}_4'(\mathcal{V}_3,\mathcal{V}) = 0.8827$, $\mathcal{K}_4'(\mathcal{V}_4,\mathcal{V}) = 0.8492$ and $\mathcal{K}_4'(\mathcal{V}_5,\mathcal{V}) = 0.7562$. From it, we get the ranking order of the regions is $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_5$ and hence we conclude that the most affected region is again $\mathcal{V}_3$.

Thus, from this analysis, it has been clearly seen that the results computed by the existing approaches coincides with the proposed one and hence it supports the proposed results.

4.5.3 Comparative studies of Example 2 under CIFS environment

To compare the performance of the proposed approach with some of the existing approaches under the CIFSs environment, an analysis has been done and their results are summarized as follows.

(i) By applying the approach of Alkouri and Salleh [6] using distance measures, denoted by $\mathcal{D}_1$ to the considered data, we get the measurement values of each disease as $\mathcal{D}_1(\mathcal{V}_1,\mathfrak{P}) = 0.0967$, $\mathcal{D}_1(\mathcal{V}_2,\mathfrak{P}) = 0.2717$, $\mathcal{D}_1(\mathcal{V}_3,\mathfrak{P}) = 0.2867$ and $\mathcal{D}_1(\mathcal{V}_4,\mathfrak{P}) =$

0.3550. From these values, we observed that $\mathcal{D}_1(\mathcal{V}_1, \mathfrak{P}) < \mathcal{D}_1(\mathcal{V}_2, \mathfrak{P}) < \mathcal{D}_1(\mathcal{V}_3, \mathfrak{P}) < \mathcal{D}_1(\mathcal{V}_4, \mathfrak{P})$ and hence conclude that the patient $\mathfrak{P}$ suffers from disease $\mathcal{V}_1$.

(ii) By utilizing the distance measure ($\mathcal{D}_2$) as proposed by Rani and Garg [129] to the considered problem, then the measurement values for each disease are computed as $\mathcal{D}_2(\mathcal{V}_1, \mathfrak{P}) = 0.1194$, $\mathcal{D}_2(\mathcal{V}_2, \mathfrak{P}) = 0.2291$, $\mathcal{D}_2(\mathcal{V}_3, \mathfrak{P}) = 0.2669$ and $\mathcal{D}_2(\mathcal{V}_4, \mathfrak{P}) = 0.3004$. Since measurement value of $\mathcal{V}_1$ is minimum among all these and hence we conclude that patient $\mathfrak{P}$ suffers from disease $\mathcal{V}_1$ which again coincides with the proposed measure results.

4.5.4 Comparative studies of Example 2 under IFS environment

In order to validate the efficiency of the proposed approach under the IFS environment, we conducted an analysis based on some of the existing CCs [100, 109, 197, 207]. The results corresponding to its are summarized as follows.

(i) If we utilize CC ($\mathcal{K}_1'$) as proposed by Zeng and Li [207], then their measurement values for each diagnosis are summarized as $\mathcal{K}_1'(\mathcal{V}_1, \mathfrak{P}) = 0.9856$, $\mathcal{K}_1'(\mathcal{V}_2, \mathfrak{P}) = 0.8461$, $\mathcal{K}_1'(\mathcal{V}_3, \mathfrak{P}) = 0.7959$ and $\mathcal{K}_1'(\mathcal{V}_4, \mathfrak{P}) = 0.7258$. From it, we conclude that $\mathcal{K}_1'(\mathcal{V}_1, \mathfrak{P}) > \mathcal{K}_1'(\mathcal{V}_2, \mathfrak{P}) > \mathcal{K}_1'(\mathcal{V}_3, \mathfrak{P}) > \mathcal{K}_1'(\mathcal{V}_4, \mathfrak{P})$ and hence the patient $\mathfrak{P}$ suffers from disease $\mathcal{V}_1$.

(ii) By applying the CC($\mathcal{K}_2'$) as defined by Ye [197] to the considered data then we get $\mathcal{K}_2'(\mathcal{V}_1, \mathfrak{P}) = 0.9912$, $\mathcal{K}_2'(\mathcal{V}_2, \mathfrak{P}) = 0.8585$, $\mathcal{K}_2'(\mathcal{V}_3, \mathfrak{P}) = 0.7265$ and $\mathcal{K}_2'(\mathcal{V}_4, \mathfrak{P}) = 0.6645$. Thus $\mathcal{K}_2'(\mathcal{V}_1, \mathfrak{P}) > \mathcal{K}_2'(\mathcal{V}_2, \mathfrak{P}) > \mathcal{K}_2'(\mathcal{V}_3, \mathfrak{P}) > \mathcal{K}_2'(\mathcal{V}_4, \mathfrak{P})$. From it, we conclude that patient $\mathfrak{P}$ suffers from disease $\mathcal{V}_1$.

(iii) If we apply the CC($\mathcal{K}_3'$) as proposed by Liu et al. [100] to the data, then the measurement values are obtained as $\mathcal{K}_3'(\mathcal{V}_1, \mathfrak{P}) = 0.8485$, $\mathcal{K}_3'(\mathcal{V}_2, \mathfrak{P}) = 0.1907$, $\mathcal{K}_3'(\mathcal{V}_3, \mathfrak{P}) = 0.6608$ and $\mathcal{K}_3'(\mathcal{V}_4, \mathfrak{P}) = -0.0690$. Thus $\mathcal{K}_3'(\mathcal{V}_1, \mathfrak{P}) > \mathcal{K}_3'(\mathcal{V}_3, \mathfrak{P}) > \mathcal{K}_3'(\mathcal{V}_2, \mathfrak{P}) > \mathcal{K}_3'(\mathcal{V}_4, \mathfrak{P})$ which implies that the ranking order of $\mathcal{V}_p (p = 1, 2, 3, 4)$ is $\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$. From it, we conclude that patient $\mathfrak{P}$ suffers from disease $\mathcal{V}_1$.

(iv) On applying the similarity measure ($\mathcal{K}_4'$) as defined by Luo and Ren [109] on the considered information, then we get $\mathcal{K}_4'(\mathcal{V}_1, \mathfrak{P}) = 0.9642$, $\mathcal{K}_4'(\mathcal{V}_2, \mathfrak{P}) = 0.7394$, $\mathcal{K}_4'(\mathcal{V}_3, \mathfrak{P}) =$

0.7725 and $\mathcal{K}_4'(\mathcal{V}_4, \mathfrak{P}) = 0.6538$. As, $\mathcal{K}_4'(\mathcal{V}_1, \mathfrak{P}) > \mathcal{K}_4'(\mathcal{V}_3, \mathfrak{P}) > \mathcal{K}_4'(\mathcal{V}_2, \mathfrak{P}) > \mathcal{K}_4'(\mathcal{V}_4, \mathfrak{P})$ and hence we conclude that patient $\mathfrak{P}$ suffers from disease $\mathcal{V}_1$.

Thus, from the above analysis, it has been seen that results computed by the proposed approach coincides with the existing approaches which validates the feasibility of the proposed approach.

4.5.5 Advantages of the proposed approach

From the existing studies and the proposed measures, we address the following merits of the proposed method to solve the decision-making problem under the CIFS environment.

(i) A CIFS is a generalization of the existing studies such as CFSs [128], IFSs [10], FSs [206] by considering much more information related to an object during the process and to handle the two-dimensional information in a single set. For instance, CIFS contains information (both the membership and non-membership degrees are complex valued) with amplitude and phase terms than the CFS (contains only complex valued membership degree), IFS (with a real-valued membership and non-membership degrees and only considered amplitude term), FS (with only crisp membership degrees with amplitude term only). Thus, the proposed CCs under CIFSs environment are more generalized than the existing CCs [8, 19, 45, 48, 50, 63, 79, 85, 100, 109, 197, 207].

(ii) It is revealed from the present study that the CCs under IFSs, FSs [8, 19, 45, 48, 50, 63, 79, 85, 100, 109, 197, 207] are the special cases of the proposed measures. Thus, the proposed CCs can be equivalently utilized to solve the MCDM problem under these existing environment by setting phase term to be zero while the existing measures [19, 48, 63, 85, 197, 207] are unable to solve the problems under the environment considered in the present chapter.

(iii) The major advantages of the proposed decision-making approach are to consider the much more information to the alternative to reduce the information loss. Further, the CCs based on the optimistic and pessimistic with or without weighting factor will help the decision maker to select the best alternative(s) more accurately. In other words, we can say that the proposed CCs will give the various choices to the decision makers based on their optimists and pessimists behavior towards the decision-making process.

4.6 Conclusion

The main contribution of this chapter is outlined as follows:

1) An attempt has been made to present different kinds of the CCs for decision-making process under the CIFS environment. Various CCs have been already defined under the IFSs environment in the literature where the range of their corresponding membership and non-membership degrees is the subset of the real numbers. But this condition has been relaxed in the present chapter by considering the CIFSs where the ranges of the membership degrees are extended from the real numbers to the complex numbers with the unit disc.

2) CIFS environment models the information in a better way for time-periodic problems and has the ability to handle two-dimensional information in the single set. Besides this, if we set phase terms in the proposed measures equal to zero, then the presented measures can be applied on IFS data as well.

3) A decision-making approach is presented to find the best alternative in the CIFSs environment. Two numerical examples are taken for illustrating the developed approach and their results are compared with some of the existing correlation measures to show the validity of it. From the studies, we conclude that the proposed approach can be efficiently used in decision-making problems where two-dimensional information is clubbed in a single set. Also, it is observed that the existing correlation measures under the IFSs environment can be taken as a special case of the proposed measure.

Chapter 5

Some results on information measures for complex intuitionistic fuzzy sets[1]

In this chapter, we introduce some novel formulae of information measures (similarity measures, distance measures, entropy and inclusion measures) and discuss the transformation relationships among them. To demonstrate the efficiency of the proposed similarity measures, we apply it to pattern recognition problem and a detailed comparative analysis is conducted with some of the existing measures. Further, algorithms based on proposed measures are developed for handing multi-criteria decision-making problems and their working is illustrated with the help of an example. Besides this, the practicality of the proposed similarity measure is demonstrated by developing a clustering algorithm under CIFS environment.

5.1 Introduction

The detailed literature review on various information measures has been done in Sections 1.1.1, 1.1.2 of Chapter 1. From the existing literature, it can be worth noticed that similarity, distance, entropy and inclusion measures are important tools for measuring the uncertainty associated with FS and IFS. Similarity measure and distance measure are

[1]The content of this chapter is published as "Some Results on Information measures for complex intuitionistic fuzzy sets", *International Journal of Intelligent Systems, Wiley*, 34(10): 2319 - 2363, 2019, doi: 10.1002/int.22127 (**SCI: Impact Factor: 8.709**).

functions that give the degree of similarity and discrimination respectively among two objects. Entropy measure quantifies the degree of fuzziness or uncertain information and the inclusion measure between two sets gives the extent to which a set is contained in another set. All these measures of information have been extensively explored by many researchers and scholars as vital topics. For instance, Szmidt [139] presented distance and similarity measures for IFSs and applied them to various DM problems. Shen et al. [133] proposed a novel distance measure and discussed its various properties and further developed a technique for order preference by similarity to ideal solution (TOPSIS) based on the proposed measure. Ye [200] put forward two similarity measures based on cosine functions and illustrated their effectiveness by comparing them with existing trigonometric similarity measures. Song et al. [137] proposed a measure of similarity based on the direct operation of the membership, non-membership and hesitation degrees of IFSs and applied it to medical diagnosis problem. Garg and Kumar [56] defined the similarity measures based on the connection numbers of the set pair analysis theory to solve the decision making problems under the IFS environment. Szmidt and Kacprzyk [141] gave the axiomatic definition of entropy measure under IFS theory based on the geometrical interpretation of IFSs. Meng and Chen [113] proposed a novel entropy measure for IFSs and presented a method to derive the similarity measure using entropy. Liu and Ren [102] put forward a novel formula for entropy measure under IFS theory and based on it, developed two methods for weight determination. Da-Zeng Tian and Zhong-Tang Yang [32] proposed an exponential entropy for IFSs and verified its effectiveness by applying it to an MCDM problem. Garg [48] presented improved cosine similarity measures for IFSs. IFS theory deals with only one dimension at a time which results in information loss in some instances. However, in real life, we come across complex natural phenomena where it becomes essential to add the second dimension to the expression of membership and non-membership grades. By introducing this second dimension, the complete information can be projected in one set and hence, loss of information can be avoided.

The focus of this chapter is to present information measures such as similarity, distance, entropy and inclusion measures under CIF environment for handling the multi-dimensional complex data sets. Transformation relationships among proposed measures are also given.

Furthermore, efforts have been put forth to solve DM problems by considering the multi-dimensional complex data sets and a clustering algorithm is also developed. Because of the discussed advantages of information measures and CIFSs, it is essential to develop information measures under this environment for addressing those problems, which are either impossible to handle or difficult to be handled with one-dimensional grades of membership.

5.2 Information Measures for CIFSs

In this section, we define some information measures such as similarity, distance, entropy and inclusion measures under CIF environment and discuss their transformation relationships. Throughout the chapter, we shall use the term 'Crisp set' for CIFS $\{(x, (\zeta(x), w_\zeta(x)), (\vartheta(x), w_\vartheta(x))) : x \in \mathcal{U}\}$ which satisfies $\zeta(x) = 0$ or $\vartheta(x) = 0$ and $w_\zeta(x) = 0$ or $w_\vartheta(x) = 0$ and $\zeta(x) + \vartheta(x) = 1$, $w_\zeta(x) + w_\vartheta(x) = 1$ for $x \in \mathcal{U}$. Also, we shall use $\Psi(\mathcal{U})$ as the set of all non-zero and non-crisp CIFSs defined on finite universal set $\mathcal{U} = \{x_1, x_2, \ldots, x_n\}$ in this chapter.

5.2.1 Similarity Measures for CIFSs

Definition 5.2.1. For two sets $\mathcal{C}_1$ and $\mathcal{C}_2 \in \Psi(\mathcal{U})$, similarity measure $\mathcal{S}' : \Psi(\mathcal{U}) \times \Psi(\mathcal{U}) \to [0, 1]$ is a real-valued function satisfying the following properties:

(P1) $0 \leq \mathcal{S}'(\mathcal{C}_1, \mathcal{C}_2) \leq 1$;

(P2) $\mathcal{S}'(\mathcal{C}_1, \mathcal{C}_2) = 1$ if $\mathcal{C}_1 = \mathcal{C}_2$;

(P3) $\mathcal{S}'(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{S}'(\mathcal{C}_2, \mathcal{C}_1)$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{S}'(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{S}'(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{S}'(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{S}'(\mathcal{C}_2, \mathcal{C}_3)$ where $\mathcal{C}_3 \in \Psi(\mathcal{U})$.

Definition 5.2.2. For two CIFSs $\mathcal{C}_1 = \{(x, (\zeta_1(x), w_{\zeta_1}(x)), (\vartheta_1(x), w_{\vartheta_1}(x))) : x \in \mathcal{U}\}$ and $\mathcal{C}_2 = \{(x, (\zeta_2(x), w_{\zeta_2}(x)), (\vartheta_2(x), w_{\vartheta_2}(x))) : x \in \mathcal{U}\}$ defined on $\mathcal{U}$, we define a series of similarity measures as follows:

(i)

$$\mathcal{S}_1'(\mathcal{C}_1,\mathcal{C}_2)=\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\begin{pmatrix}\min\big(\zeta_1(x_j),\zeta_2(x_j)\big)+\min\big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\big)\\+\min\big(\vartheta_1(x_j),\vartheta_2(x_j)\big)+\min\big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\big)\end{pmatrix}}{\begin{pmatrix}\max\big(\zeta_1(x_j),\zeta_2(x_j)\big)+\max\big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\big)\\+\max\big(\vartheta_1(x_j),\vartheta_2(x_j)\big)+\max\big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\big)\end{pmatrix}}\right) \tag{5.1}$$

(ii)

$$\mathcal{S}_2'(\mathcal{C}_1,\mathcal{C}_2)=\frac{\sum_{j=1}^{n}\begin{pmatrix}\min\big(\zeta_1(x_j),\zeta_2(x_j)\big)+\min\big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\big)\\+\min\big(\vartheta_1(x_j),\vartheta_2(x_j)\big)+\min\big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\big)\end{pmatrix}}{\sum_{j=1}^{n}\begin{pmatrix}\max\big(\zeta_1(x_j),\zeta_2(x_j)\big)+\max\big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\big)\\+\max\big(\vartheta_1(x_j),\vartheta_2(x_j)\big)+\max\big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\big)\end{pmatrix}} \tag{5.2}$$

(iii)

$$\mathcal{S}_3'(\mathcal{C}_1,\mathcal{C}_2)=\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\begin{pmatrix}\min\Big(\zeta_1(x_j),\zeta_2(x_j)\Big)+\min\Big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\Big)\\+\min\Big(1-\vartheta_1(x_j),1-\vartheta_2(x_j)\Big)+\min\Big(1-w_{\vartheta_1}(x_j),1-w_{\vartheta_2}(x_j)\Big)\end{pmatrix}}{\begin{pmatrix}\max\Big(\zeta_1(x_j),\zeta_2(x_j)\Big)+\max\Big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\Big)\\+\max\Big(1-\vartheta_1(x_j),1-\vartheta_2(x_j)\Big)+\max\Big(1-w_{\vartheta_1}(x_j),1-w_{\vartheta_2}(x_j)\Big)\end{pmatrix}}\right) \tag{5.3}$$

(iv)

$$\mathcal{S}_4'(\mathcal{C}_1,\mathcal{C}_2)=\frac{\sum_{j=1}^{n}\begin{pmatrix}\min\Big(\zeta_1(x_j),\zeta_2(x_j)\Big)+\min\Big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\Big)\\+\min\Big(1-\vartheta_1(x_j),1-\vartheta_2(x_j)\Big)+\min\Big(1-w_{\vartheta_1}(x_j),1-w_{\vartheta_2}(x_j)\Big)\end{pmatrix}}{\sum_{j=1}^{n}\begin{pmatrix}\max\Big(\zeta_1(x_j),\zeta_2(x_j)\Big)+\max\Big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\Big)\\+\max\Big(1-\vartheta_1(x_j),1-\vartheta_2(x_j)\Big)+\max\Big(1-w_{\vartheta_1}(x_j),1-w_{\vartheta_2}(x_j)\Big)\end{pmatrix}} \tag{5.4}$$

(v)

$$\mathcal{S}_5'(\mathcal{C}_1,\mathcal{C}_2)=1-\frac{1}{4n}\sum_{j=1}^{n}\begin{pmatrix}|\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|\\+|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{pmatrix} \tag{5.5}$$

(vi)

$$\mathcal{S}_6'(\mathcal{C}_1,\mathcal{C}_2)=1-\frac{1}{4}\begin{pmatrix}\max_j|\zeta_1(x_j)-\zeta_2(x_j)|+\max_j|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|\\+\max_j|\vartheta_1(x_j)-\vartheta_2(x_j)|+\max_j|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{pmatrix} \tag{5.6}$$

(vii)

$$\mathcal{S}'_7(\mathcal{C}_1,\mathcal{C}_2) = 1 - \frac{\sum_{j=1}^{n}\left(\begin{array}{l}|\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{array}\right)}{\sum_{j=1}^{n}\left(\begin{array}{l}|\zeta_1(x_j)+\zeta_2(x_j)|+|w_{\zeta_1}(x_j)+w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)+\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)+w_{\vartheta_2}(x_j)|\end{array}\right)} \tag{5.7}$$

Theorem 5.2.1. The similarity measures $\mathcal{S}'_r$ $(r = 1, 2, \ldots, 7)$ satisfy the following properties:

(P1) $0 \le \mathcal{S}'_r(\mathcal{C}_1,\mathcal{C}_2) \le 1$;

(P2) $\mathcal{S}'_r(\mathcal{C}_1,\mathcal{C}_2) = 1$ if $\mathcal{C}_1 = \mathcal{C}_2$;

(P3) $\mathcal{S}'_r(\mathcal{C}_1,\mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_2,\mathcal{C}_1)$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{S}'_r(\mathcal{C}_1,\mathcal{C}_3) \le \mathcal{S}'_r(\mathcal{C}_1,\mathcal{C}_2)$ and $\mathcal{S}'_r(\mathcal{C}_1,\mathcal{C}_3) \le \mathcal{S}'_r(\mathcal{C}_2,\mathcal{C}_3)$ where $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3 \in \Psi(\mathcal{U})$.

Proof. Since, the conditions $(P1)-(P3)$ are straightforward for similarity measures $\mathcal{S}'_r$ $(r = 1, 2, \ldots, 7)$. Therefore, we only prove the condition $(P4)$ for similarity measures $\mathcal{S}'_1$, $\mathcal{S}'_5$ and $\mathcal{S}'_7$ as for others this property can be proved similarly. For this, let $\mathcal{C}_3 = \big\{\big(x, \big(\zeta_3(x), w_{\zeta_3}(x)\big), \big(\vartheta_3(x), w_{\vartheta_3}(x)\big)\big) : x \in \mathcal{U}\big\}$. As $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$. It implies that $\zeta_1(x_j) \le \zeta_2(x_j) \le \zeta_3(x_j)$; $\vartheta_1(x_j) \ge \vartheta_2(x_j) \ge \vartheta_3(x_j)$ and $w_{\zeta_1}(x_j) \le w_{\zeta_2}(x_j) \le w_{\zeta_3}(x_j)$; $w_{\vartheta_1}(x_j) \ge w_{\vartheta_2}(x_j) \ge w_{\vartheta_3}(x_j)$.

(i) $$\mathcal{S}'_1(\mathcal{C}_1,\mathcal{C}_2) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\left(\begin{array}{l}\min\big(\zeta_1(x_j),\zeta_2(x_j)\big)+\min\big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\big)\\ +\min\big(\vartheta_1(x_j),\vartheta_2(x_j)\big)+\min\big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\big)\end{array}\right)}{\left(\begin{array}{l}\max\big(\zeta_1(x_j),\zeta_2(x_j)\big)+\max\big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\big)\\ +\max\big(\vartheta_1(x_j),\vartheta_2(x_j)\big)+\max\big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\big)\end{array}\right)}\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\zeta_1(x_j)+\vartheta_2(x_j)+w_{\zeta_1}(x_j)+w_{\vartheta_2}(x_j)}{\zeta_2(x_j)+\vartheta_1(x_j)+w_{\zeta_2}(x_j)+w_{\vartheta_1}(x_j)}\right)$$

and

$$\begin{aligned}
\mathcal{S}'_1(\mathcal{C}_1,\mathcal{C}_3) &= \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\begin{pmatrix} \min\big(\zeta_1(x_j),\zeta_3(x_j)\big)+\min\big(w_{\zeta_1}(x_j),w_{\zeta_3}(x_j)\big) \\ +\min\big(\vartheta_1(x_j),\vartheta_3(x_j)\big)+\min\big(w_{\vartheta_1}(x_j),w_{\vartheta_3}(x_j)\big)\end{pmatrix}}{\begin{pmatrix} \max\big(\zeta_1(x_j),\zeta_3(x_j)\big)+\max\big(w_{\zeta_1}(x_j),w_{\zeta_3}(x_j)\big) \\ +\max\big(\vartheta_1(x_j),\vartheta_3(x_j)\big)+\max\big(w_{\vartheta_1}(x_j),w_{\vartheta_3}(x_j)\big)\end{pmatrix}}\right) \\
&= \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\zeta_1(x_j)+\vartheta_3(x_j)+w_{\zeta_1}(x_j)+w_{\vartheta_3}(x_j)}{\zeta_3(x_j)+\vartheta_1(x_j)+w_{\zeta_3}(x_j)+w_{\vartheta_1}(x_j)}\right)
\end{aligned}$$

As $\vartheta_3(x_j) \le \vartheta_2(x_j)$; $w_{\vartheta_3}(x_j) \le w_{\vartheta_2}(x_j)$ and $\zeta_2(x_j) \le \zeta_3(x_j)$; $w_{\zeta_2}(x_j) \le w_{\zeta_3}(x_j)$. Therefore, $\mathcal{S}'_1(\mathcal{C}_1,\mathcal{C}_3) \le \mathcal{S}'_1(\mathcal{C}_1,\mathcal{C}_2)$. Similarly, we can obtain that $\mathcal{S}'_1(\mathcal{C}_1,\mathcal{C}_3) \le \mathcal{S}'_1(\mathcal{C}_2,\mathcal{C}_3)$.

(v) $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ gives $|\zeta_1(x_j)-\zeta_2(x_j)| \le |\zeta_1(x_j)-\zeta_3(x_j)|$; $|\vartheta_1(x_j)-\vartheta_2(x_j)| \le |\vartheta_1(x_j)-\vartheta_3(x_j)|$ and $|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)| \le |w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)|$; $|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)| \le |w_{\vartheta_1}(x_j)-w_{\vartheta_3}(x_j)|$. Therefore,

$$\begin{aligned}
\mathcal{S}'_5(\mathcal{C}_1,\mathcal{C}_3) &= 1-\frac{1}{4n}\sum_{j=1}^{n}\begin{pmatrix} |\zeta_1(x_j)-\zeta_3(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)| \\ +|\vartheta_1(x_j)-\vartheta_3(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_3}(x_j)|\end{pmatrix} \\
&\le 1-\frac{1}{4n}\sum_{j=1}^{n}\begin{pmatrix} |\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)| \\ +|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{pmatrix} \\
&= \mathcal{S}'_5(\mathcal{C}_1,\mathcal{C}_2)
\end{aligned}$$

Hence, $\mathcal{S}'_5(\mathcal{C}_1,\mathcal{C}_3) \le \mathcal{S}'_5(\mathcal{C}_1,\mathcal{C}_2)$. In the similar manner, it can be proved that $\mathcal{S}'_5(\mathcal{C}_1,\mathcal{C}_3) \le \mathcal{S}'_5(\mathcal{C}_2,\mathcal{C}_3)$.

(vii) We have,

$$\begin{pmatrix} |\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)| \\ +|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{pmatrix} \le \begin{pmatrix} |\zeta_1(x_j)-\zeta_3(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)| \\ +|\vartheta_1(x_j)-\vartheta_3(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_3}(x_j)|\end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{\begin{pmatrix} |\zeta_1(x_j)-\zeta_3(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)| \\ +|\vartheta_1(x_j)-\vartheta_3(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_3}(x_j)|\end{pmatrix}} \le \frac{1}{\begin{pmatrix} |\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)| \\ +|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{pmatrix}}$$

$$\Rightarrow \quad \frac{2\times\left(\zeta_1(x_j)+\vartheta_3(x_j)+\left(w_{\zeta_1}(x_j)+w_{\vartheta_3}(x_j)\right)\right)}{\begin{pmatrix}|\zeta_1(x_j)-\zeta_3(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_3(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_3}(x_j)|\end{pmatrix}} \leq \frac{2\times\left(\zeta_1(x_j)+\vartheta_2(x_j)+\left(w_{\zeta_1}(x_j)+w_{\vartheta_2}(x_j)\right)\right)}{\begin{pmatrix}|\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{pmatrix}}$$

$$\Rightarrow \quad 1+\frac{2\times\left(\zeta_1(x_j)+\vartheta_3(x_j)+\left(w_{\zeta_1}(x_j)+w_{\vartheta_3}(x_j)\right)\right)}{\begin{pmatrix}|\zeta_1(x_j)-\zeta_3(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_3(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_3}(x_j)|\end{pmatrix}} \leq 1+\frac{2\times\left(\zeta_1(x_j)+\vartheta_2(x_j)+\left(w_{\zeta_1}(x_j)+w_{\vartheta_2}(x_j)\right)\right)}{\begin{pmatrix}|\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{pmatrix}}$$

$$\Rightarrow \quad \frac{\begin{pmatrix}|\zeta_1(x_j)+\zeta_3(x_j)|+|w_{\zeta_1}(x_j)+w_{\zeta_3}(x_j)|\\ +|\vartheta_1(x_j)+\vartheta_3(x_j)|+|w_{\vartheta_1}(x_j)+w_{\vartheta_3}(x_j)|\end{pmatrix}}{\begin{pmatrix}|\zeta_1(x_j)-\zeta_3(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_3(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_3}(x_j)|\end{pmatrix}} \leq \frac{\begin{pmatrix}|\zeta_1(x_j)+\zeta_2(x_j)|+|w_{\zeta_1}(x_j)+w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)+\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)+w_{\vartheta_2}(x_j)|\end{pmatrix}}{\begin{pmatrix}|\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{pmatrix}}$$

$$\Rightarrow \quad 1-\frac{\begin{pmatrix}|\zeta_1(x_j)-\zeta_3(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_3}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_3(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_3}(x_j)|\end{pmatrix}}{\begin{pmatrix}|\zeta_1(x_j)+\zeta_3(x_j)|+|w_{\zeta_1}(x_j)+w_{\zeta_3}(x_j)|\\ +|\vartheta_1(x_j)+\vartheta_3(x_j)|+|w_{\vartheta_1}(x_j)+w_{\vartheta_3}(x_j)|\end{pmatrix}} \leq 1-\frac{\begin{pmatrix}|\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{pmatrix}}{\begin{pmatrix}|\zeta_1(x_j)+\zeta_2(x_j)|+|w_{\zeta_1}(x_j)+w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)+\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)+w_{\vartheta_2}(x_j)|\end{pmatrix}}$$

Hence, $\mathcal{S}'_7(\mathcal{C}_1,\mathcal{C}_3) \leq \mathcal{S}'_7(\mathcal{C}_1,\mathcal{C}_2)$. Similarly, we can prove that $\mathcal{S}'_7(\mathcal{C}_1,\mathcal{C}_3) \leq \mathcal{S}'_7(\mathcal{C}_2,\mathcal{C}_3)$.

□

Further, we observe that these proposed similarity measures satisfy certain properties which are discussed as follows:

Property 5.2.1. For $\mathcal{C}_1,\mathcal{C}_2 \in \Psi(\mathcal{U})$ and $r = 1,2,\dots 7$, we have

(i) $\mathcal{S}'_r(\mathcal{C}_1,\mathcal{C}_2^c) = \mathcal{S}'_r(\mathcal{C}_1^c,\mathcal{C}_2)$, $r \neq 3,4$;

(ii) $\mathcal{S}'_r(\mathcal{C}_1^c,\mathcal{C}_2^c) = \mathcal{S}'_r(\mathcal{C}_1,\mathcal{C}_2)$, $r \neq 3,4$;

(iii) $\mathcal{S}'_r(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_1\cup\mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_1,\mathcal{C}_2)$.

Proof. Here we prove the part (i) only, while others can be proved similarly. For this, let $\mathcal{C}_1 = \big\{\big(x,\big(\zeta_1(x),\ w_{\zeta_1}(x)\big),\big(\vartheta_1(x)\ ,w_{\vartheta_1}(x)\big)\big) : x \in \mathcal{U}\big\}$ and $\mathcal{C}_2 = \big\{\big(x,\big(\zeta_2(x),w_{\zeta_2}(x)\big),\big(\vartheta_2(x),w_{\vartheta_2}(x)\big)\big) : x \in \mathcal{U}\big\}$.

(i) By using similarity measure $\mathcal{S}'_1$, defined in Eq. (5.1), we have

$$\mathcal{S}'_1(\mathcal{C}_1, \mathcal{C}_2^c) = \frac{1}{n}\sum_{j=1}^{n} \frac{\begin{pmatrix} \min\big(\zeta_1(x_j), \vartheta_2(x_j)\big) + \min\big(w_{\zeta_1}(x_j), w_{\vartheta_2}(x_j)\big) \\ + \min\big(\vartheta_1(x_j), \zeta_2(x_j)\big) + \min\big(w_{\vartheta_1}(x_j), w_{\zeta_2}(x_j)\big) \end{pmatrix}}{\begin{pmatrix} \max\big(\zeta_1(x_j), \vartheta_2(x_j)\big) + \max\big(w_{\zeta_1}(x_j), w_{\vartheta_2}(x_j)\big) \\ + \max\big(\vartheta_1(x_j), \zeta_2(x_j)\big) + \max\big(w_{\vartheta_1}(x_j), w_{\zeta_2}(x_j)\big) \end{pmatrix}}$$
$$= \mathcal{S}'_1(\mathcal{C}_1^c, \mathcal{C}_2)$$

Hence, $\mathcal{S}'_1(\mathcal{C}_1, \mathcal{C}_2^c) = \mathcal{S}'_1(\mathcal{C}_1^c, \mathcal{C}_2)$. Similarly, we can obtain that $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2^c) = \mathcal{S}'_r(\mathcal{C}_1^c, \mathcal{C}_2)$ for $r = 2, 5, 6, 7$.

$\square$

Property 5.2.2. For $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$ and $r = 5, 6$, we have

(i) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_1 \cup \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)$;

(ii) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_1 \cup \mathcal{C}_2)$;

(iii) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_1 \oplus \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_1 \otimes \mathcal{C}_2)$;

(iv) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_1 \otimes \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$.

Proof. Here we prove the parts (i) and (iii) only, while others can be proved similarly.

(i) Let $\mathcal{C}_1 = \big\{\big(x, \big(\zeta_1(x), w_{\zeta_1}(x)\big), \big(\vartheta_1(x), w_{\vartheta_1}(x)\big)\big) : x \in \mathcal{U}\big\}$ and $\mathcal{C}_2 = \big\{\big(x, \big(\zeta_2(x), w_{\zeta_2}(x)\big), \big(\vartheta_2(x), w_{\vartheta_2}(x)\big)\big) : x \in \mathcal{U}\big\}$. Now, by using similarity measure $\mathcal{S}'_5$, defined in Eq. (5.5), we obtain

$$\mathcal{S}'_5(\mathcal{C}_1, \mathcal{C}_1 \cup \mathcal{C}_2)$$
$$= 1 - \frac{1}{4n}\sum_{j=1}^{n} \begin{pmatrix} \big|\zeta_1(x_j) - \max\big(\zeta_1(x_j), \zeta_2(x_j)\big)\big| + \big|w_{\zeta_1}(x_j) - \max\big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\big)\big| \\ + \big|\vartheta_1(x_j) - \min\big(\vartheta_1(x_j), \vartheta_2(x_j)\big)\big| + \big|w_{\vartheta_1}(x_j) - \min\big(w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j)\big)\big| \end{pmatrix}$$
$$= 1 - \frac{1}{4n}\sum_{j=1}^{n} \begin{pmatrix} \big|\min\big(0, \zeta_1(x_j) - \zeta_2(x_j)\big)\big| + \big|\min\big(0, w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)\big)\big| \\ + \big|\max\big(0, \vartheta_1(x_j) - \vartheta_2(x_j)\big)\big| + \big|\max\big(0, w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)\big)\big| \end{pmatrix}$$

and

$$\begin{aligned}
&\mathcal{S}_5'(\mathcal{C}_2,\mathcal{C}_1\cap\mathcal{C}_2)\\
=\;& 1-\frac{1}{4n}\sum_{j=1}^{n}\begin{pmatrix}\left|\zeta_2(x_j)-\min\left(\zeta_1(x_j),\zeta_2(x_j)\right)\right|+\left|w_{\zeta_2}(x_j)-\min\left(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\right)\right|\\ +\left|\vartheta_2(x_j)-\max\left(\vartheta_1(x_j),\vartheta_2(x_j)\right)\right|+\left|w_{\vartheta_2}(x_j)-\max\left(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\right)\right|\end{pmatrix}\\
=\;& 1-\frac{1}{4n}\sum_{j=1}^{n}\begin{pmatrix}\left|\max\left(0,\zeta_2(x_j)-\zeta_1(x_j)\right)\right|+\left|\max\left(0,w_{\zeta_2}(x_j)-w_{\zeta_1}(x_j)\right)\right|\\ +\left|\min\left(0,\vartheta_2(x_j)-\vartheta_1(x_j)\right)\right|+\left|\min\left(0,w_{\vartheta_2}(x_j)-w_{\vartheta_1}(x_j)\right)\right|\end{pmatrix}\\
=\;& 1-\frac{1}{4n}\sum_{j=1}^{n}\begin{pmatrix}\left|\min\left(0,\zeta_1(x_j)-\zeta_2(x_j)\right)\right|+\left|\min\left(0,w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)\right)\right|\\ +\left|\max\left(0,\vartheta_1(x_j)-\vartheta_2(x_j)\right)\right|+\left|\max\left(0,w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)\right)\right|\end{pmatrix}\\
=\;& \mathcal{S}_5'(\mathcal{C}_1,\mathcal{C}_1\cup\mathcal{C}_2)
\end{aligned}$$

Hence, $\mathcal{S}_5'(\mathcal{C}_1,\mathcal{C}_1\cup\mathcal{C}_2)=\mathcal{S}_5'(\mathcal{C}_2,\mathcal{C}_1\cap\mathcal{C}_2)$. Similarly, we can obtain that $\mathcal{S}_6'(\mathcal{C}_1,\mathcal{C}_1\cup\mathcal{C}_2)=\mathcal{S}_6'(\mathcal{C}_2,\mathcal{C}_1\cap\mathcal{C}_2)$.

(iii) Again, by using Eq. (5.5), we obtain

$$\begin{aligned}
&\mathcal{S}_5'(\mathcal{C}_1,\mathcal{C}_1\oplus\mathcal{C}_2)\\
=\;& 1-\frac{1}{4n}\sum_{j=1}^{n}\begin{pmatrix}\Big|\zeta_1(x_j)-(\zeta_1(x_j)+\zeta_2(x_j)-\zeta_1(x_j)\zeta_2(x_j))\Big|\\ +\Big|\vartheta_1(x_j)-\vartheta_1(x_j)\vartheta_2(x_j)\Big|\\ +|w_{\zeta_1}(x_j)-(w_{\zeta_1}(x_j)+w_{\zeta_2}(x_j)-w_{\zeta_1}(x_j)w_{\zeta_2}(x_j))|\\ +|w_{\vartheta_1}(x_j)-w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j)|\end{pmatrix}\\
=\;& 1-\frac{1}{4n}\sum_{j=1}^{n}\begin{pmatrix}\Big|\zeta_2(x_j)\big(1-\zeta_1(x_j)\big)\Big|+\Big|\vartheta_1(x_j)\big(1-\vartheta_2(x_j)\big)\Big|\\ +(|w_{\zeta_2}(x_j)\,(1-w_{\zeta_1}(x_j))|+|w_{\vartheta_1}(x_j)\,(1-w_{\vartheta_2}(x_j))|)\end{pmatrix}
\end{aligned}$$

and $\mathcal{S}_5'(\mathcal{C}_2,\mathcal{C}_1\mathcal{C}_2)$

$$= 1-\frac{1}{4n}\sum_{j=1}^{n}\begin{pmatrix}\Big|\zeta_2(x_j)-\zeta_1(x_j)\zeta_2(x_j)\Big|+\\ \Big|\vartheta_2(x_j)-(\vartheta_1(x_j)+\vartheta_2(x_j)-\vartheta_1(x_j)\vartheta_2(x_j))\Big|\\ +|w_{\zeta_2}(x_j)-w_{\zeta_1}(x_j)w_{\zeta_2}(x_j)|\\ +|w_{\vartheta_2}(x_j)-(w_{\vartheta_1}(x_j)+w_{\vartheta_2}(x_j)-w_{\vartheta_1}(x_j)w_{\vartheta_2}(x_j))|\end{pmatrix}$$

$$
\begin{aligned}
&= 1-\frac{1}{4n}\sum_{j=1}^{n}\left(\begin{array}{c}\Big|\zeta_2(x_j)\big(1-\zeta_1(x_j)\big)\Big|+\Big|\vartheta_1(x_j)\big(1-\vartheta_2(x_j)\big)\Big| \\ +\left(\left|w_{\zeta_2}(x_j)\left(1-w_{\zeta_1}(x_j)\right)\right|+\left|w_{\vartheta_1}(x_j)\left(1-w_{\vartheta_2}(x_j)\right)\right|\right)\end{array}\right) \\
&= \mathcal{S}_5'(\mathcal{C}_1,\mathcal{C}_1\oplus\mathcal{C}_2)
\end{aligned}
$$

Hence, $\mathcal{S}_5'(\mathcal{C}_1,\mathcal{C}_1\oplus\mathcal{C}_2)=\mathcal{S}_5'(\mathcal{C}_2,\mathcal{C}_1\otimes\mathcal{C}_2)$. Similarly, we can obtain that $\mathcal{S}_6'(\mathcal{C}_1,\mathcal{C}_1\oplus\mathcal{C}_2)=\mathcal{S}_6'(\mathcal{C}_2,\mathcal{C}_1\otimes\mathcal{C}_2)$.

□

Next, we propose a novel CIF exponential similarity measure and discuss some of its properties as follows:

Definition 5.2.3. For two CIFSs $\mathcal{C}_1=\Big\{\big(x,\big(\zeta_1(x),w_{\zeta_1}(x)\big),\big(\vartheta_1(x),w_{\vartheta_1}(x)\big)\big):x\in\mathcal{U}\Big\}$ and $\mathcal{C}_2=\Big\{\big(x,\big(\zeta_2(x),\,w_{\zeta_2}(x)\big),\big(\vartheta_2(x),\,w_{\vartheta_2}(x)\big)\big):x\in\mathcal{U}\Big\}$ defined on $\mathcal{U}$, we define the exponential similarity measure as follows:

$$
\begin{aligned}
&\mathcal{S}_8'(\mathcal{C}_1,\mathcal{C}_2) \\
=\ &\frac{1}{4n(1-e^{-1})}\sum_{j=1}^{n}\left(\begin{array}{c}\Big(1-t_j(\mathcal{C}_1,\mathcal{C}_2)\Big)\exp\Big(t_j(\mathcal{C}_1,\mathcal{C}_2)\Big)+\Big(1+t_j(\mathcal{C}_1,\mathcal{C}_2)\Big)\exp\Big(-t_j(\mathcal{C}_1,\mathcal{C}_2)\Big) \\ +\Big(1-s_j(\mathcal{C}_1,\mathcal{C}_2)\Big)\exp\Big(s_j(\mathcal{C}_1,\mathcal{C}_2)\Big)+\Big(1+s_j(\mathcal{C}_1,\mathcal{C}_2)\Big)\exp\Big(-s_j(\mathcal{C}_1,\mathcal{C}_2)\Big)-4e^{-1}\end{array}\right)
\end{aligned}
\tag{5.8}
$$

where $t_j(\mathcal{C}_1,\mathcal{C}_2)=\left(\frac{\big(\zeta_1(x_j)-\zeta_2(x_j)\big)-\big(\vartheta_1(x_j)-\vartheta_2(x_j)\big)}{2}\right)$ and

$s_j(\mathcal{C}_1,\mathcal{C}_2)=\left(\frac{\big(w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)\big)-\big(w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)\big)}{2}\right)$. Also from the defining of t_j and s_j, it is evident that $t_j(\mathcal{C}_1,\mathcal{C}_2)=-t_j(\mathcal{C}_2,\mathcal{C}_1)$ and $s_j(\mathcal{C}_1,\mathcal{C}_2)=-s_j(\mathcal{C}_2,\mathcal{C}_1)$.

Lemma 5.2.1. Let $f(y)=(1-y)\exp(y)+(1+y)\exp(-y)-2\exp(-1)$ be a function, where $y\in[-1,1]$. Then,

$$0\leq f(y)\leq 2-2\exp(-1)$$

Proof. Since $f(y)=(1-y)\exp(y)+(1+y)\exp(-y)-2\exp(-1)$. It gives that $f'(y)=-y\big(\exp(y)+\exp(-y)\big)$ which follows that $f(y)$ is increasing in $[-1,0]$ and decreasing in $[0,1]$. Therefore, when $y\in[-1,0]$, $f(-1)\leq f(y)\leq f(0)$ i.e, $0\leq f(y)\leq 2-2\exp(-1)$ and similarly for $y\in[0,1]$, $f(1)\leq f(y)\leq f(0)$ i.e, $0\leq f(y)\leq 2-2\exp(-1)$. Hence, for $y\in[-1,1]$, we have $0\leq f(y)\leq 2-2\exp(-1)$.

□

Lemma 5.2.2. For $y \in [-1, 0]$, the functions $f_1(y) = (1-y)\exp(y)$ and $f_2(y) = (1+y)\exp(-y)$ are increasing functions.

Proof. Since $f_1(y) = (1-y)\exp(y)$. For $y \in [-1,0]$, $f_1'(y) = -y\exp(y) \geq 0$ which gives that $f_1(y)$ is an increasing function. Similarly, we can prove that $f_2(y)$ is an increasing function for $y \in [-1, 0]$. □

Theorem 5.2.2. The similarity measure $\mathcal{S}_8'$ satisfies the following properties:

(P1) $0 \leq \mathcal{S}_8'(\mathcal{C}_1, \mathcal{C}_2) \leq 1$;

(P2) $\mathcal{S}_8'(\mathcal{C}_1, \mathcal{C}_2) = 1$ if $\mathcal{C}_1 = \mathcal{C}_2$;

(P3) $\mathcal{S}_8'(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{S}_8'(\mathcal{C}_2, \mathcal{C}_1)$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{S}_8'(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{S}_8'(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{S}_8'(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{S}_8'(\mathcal{C}_2, \mathcal{C}_3)$ where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \Psi(\mathcal{U})$.

Proof. Let $\mathcal{C}_3 = \Big\{\big(x, \big(\zeta_3(x), w_{\zeta_3}(x)\big), \big(\vartheta_3(x), w_{\vartheta_3}(x)\big)\big) : x \in \mathcal{U}\Big\}$.

(i) Since $0 \leq \zeta_1(x_j), \zeta_2(x_j), \vartheta_1(x_j), \vartheta_2(x_j) \leq 1$. It implies that $-1 \leq \zeta_1(x_j) - \zeta_2(x_j) \leq 1$; $-1 \leq \vartheta_1(x_j) - \vartheta_2(x_j) \leq 1$ which gives that $-2 \leq \big(\zeta_1(x_j) - \zeta_2(x_j)\big) - \big(\vartheta_1(x_j) - \vartheta_2(x_j)\big) \leq 2$ and hence, $-1 \leq t_j(\mathcal{C}_1, \mathcal{C}_2) \leq 1$. Similarly, we can prove that, $-1 \leq s_j(\mathcal{C}_1, \mathcal{C}_2) \leq 1$. Then, by using the above Lemma 5.2.1, we obtain that $0 \leq \Big(1 - t_j(\mathcal{C}_1, \mathcal{C}_2)\Big)\exp\Big(t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) + \Big(1 + t_j(\mathcal{C}_1, \mathcal{C}_2)\Big)\exp\Big(-t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) - 2\exp(-1) \leq 2 - 2\exp(-1)$ and $0 \leq \Big(1 - s_j(\mathcal{C}_1, \mathcal{C}_2)\Big)\exp\Big(s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) + \Big(1 + s_j(\mathcal{C}_1, \mathcal{C}_2)\Big)\exp\Big(-s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) - 2\exp(-1) \leq 2 - 2\exp(-1)$ which gives that $0 \leq \Big(1 - t_j(\mathcal{C}_1, \mathcal{C}_2)\Big)\exp\Big(t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) + \Big(1 + t_j(\mathcal{C}_1, \mathcal{C}_2)\Big)\exp\Big(-t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) + \Big(1 - s_j(\mathcal{C}_1, \mathcal{C}_2)\Big)\exp\Big(s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) + \Big(1 + s_j(\mathcal{C}_1, \mathcal{C}_2)\Big)\exp\Big(-s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) - 4\exp(-1) \leq 4 - 4\exp(-1)$. Hence, $0 \leq \mathcal{S}_8'(\mathcal{C}_1, \mathcal{C}_2) \leq 1$.

(ii) For $\mathcal{C}_1 = \mathcal{C}_2$, we have $\zeta_1(x_j) = \zeta_2(x_j)$, $\vartheta_1(x_j) = \vartheta_2(x_j)$, $w_{\zeta_1}(x_j) = w_{\zeta_2}(x_j)$ and $w_{\vartheta_1}(x_j) = w_{\vartheta_2}(x_j)$ for all j which gives that $t_j(\mathcal{C}_1, \mathcal{C}_2)$, $s_j(\mathcal{C}_1, \mathcal{C}_2) = 0$. It implies that $\mathcal{S}_8'(\mathcal{C}_1, \mathcal{C}_2) = 1$.

(iii) It is obvious.

(iv) Since $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$. It implies that $\zeta_1(x_j) \leq \zeta_2(x_j) \leq \zeta_3(x_j)$ and $\vartheta_1(x_j) \geq \vartheta_2(x_j) \geq \vartheta_3(x_j)$ which gives that $\zeta_1(x_j) - \zeta_3(x_j) \leq \zeta_1(x_j) - \zeta_2(x_j) \leq 0$ and $0 \leq \vartheta_1(x_j) - \vartheta_2(x_j) \leq \vartheta_1(x_j) - \vartheta_3(x_j)$. It follows that $t_j(\mathcal{C}_1, \mathcal{C}_3) \leq t_j(\mathcal{C}_1, \mathcal{C}_2) \leq 0$. Also, $t_j(\mathcal{C}_1, \mathcal{C}_3), t_j(\mathcal{C}_1, \mathcal{C}_2) \geq -1$. Then, by using Lemma 5.2.2, we have

$\Big(1 - t_j(\mathcal{C}_1, \mathcal{C}_3)\Big) \exp\Big(t_j(\mathcal{C}_1, \mathcal{C}_3)\Big) \leq \Big(1 - t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) \exp\Big(t_j(\mathcal{C}_1, \mathcal{C}_2)\Big)$ and

$\Big(1 + t_j(\mathcal{C}_1, \mathcal{C}_3)\Big) \exp\Big(-t_j(\mathcal{C}_1, \mathcal{C}_3)\Big) \leq \Big(1 + t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) \exp\Big(-t_j(\mathcal{C}_1, \mathcal{C}_2)\Big)$.

Similarly, we can prove that

$\Big(1 - s_j(\mathcal{C}_1, \mathcal{C}_3)\Big) \exp\Big(s_j(\mathcal{C}_1, \mathcal{C}_3)\Big) \leq \Big(1 - s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) \exp\Big(s_j(\mathcal{C}_1, \mathcal{C}_2)\Big)$ and

$\Big(1 + s_j(\mathcal{C}_1, \mathcal{C}_3)\Big) \exp\Big(-s_j(\mathcal{C}_1, \mathcal{C}_3)\Big) \leq \Big(1 + s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) \exp\Big(-s_j(\mathcal{C}_1, \mathcal{C}_2)\Big)$.

Hence, $\mathcal{S}'_8(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{S}'_8(\mathcal{C}_1, \mathcal{C}_2)$. In the similar manner, we can obtain that $\mathcal{S}'_8(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{S}'_8(\mathcal{C}_2, \mathcal{C}_3)$.

□

Based on the CIF exponential similarity measure $\mathcal{S}'_8$, we propose two exponential similarity measures under CIF environment as follows.

Definition 5.2.4. For $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$, we define exponential similarity measures as:

(i) $\mathcal{S}'_9(\mathcal{C}_1, \mathcal{C}_2) = \frac{\mathcal{S}'_8(\mathcal{C}_1, \mathcal{C}_2)}{2 - \mathcal{S}'_8(\mathcal{C}_1, \mathcal{C}_2)}$

(ii) $\mathcal{S}'_{10}(\mathcal{C}_1, \mathcal{C}_2) = \frac{e^{\mathcal{S}'_8(\mathcal{C}_1, \mathcal{C}_2) - 1} - e^{-1}}{1 - e^{-1}}$

Theorem 5.2.3. The CIF exponential similarity measures $\mathcal{S}'_r$ $(r = 9, 10)$ satisfy the following properties:

(P1) $0 \leq \mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2) \leq 1$;

(P2) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2) = 1$ if $\mathcal{C}_1 = \mathcal{C}_2$;

(P3) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_1)$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_3) \leq \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_3)$ where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \Psi(\mathcal{U})$.

Proof. Its proof follows from the Theorem 5.2.2. So, we omit here. $\square$

Further, we analyze that these proposed exponential similarity measures also satisfy certain properties which are explained as follows:

Property 5.2.3. For $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$ and $r = 8, 9, 10$, we have

(i) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2^c) = \mathcal{S}'_r(\mathcal{C}_1^c, \mathcal{C}_2)$;

(ii) $\mathcal{S}'_r(\mathcal{C}_1^c, \mathcal{C}_2^c) = \mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2)$;

(iii) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_1 \cup \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)$;

(iv) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_1 \cup \mathcal{C}_2)$;

(v) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_1 \oplus \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_1 \otimes \mathcal{C}_2)$;

(vi) $\mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_1 \otimes \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$.

Proof. It can be proved as similar to the Property 5.2.1 and Property 5.2.2. So, we omit here. $\square$

Besides this, we prove some more properties related to proposed exponential similarity measures and for their proofs we consider two subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ of universal set $\mathcal{U}$ as: $\mathcal{U}_1 = \{x_j \mid \mathcal{C}_1(x_j) \subseteq \mathcal{C}_2(x_j)\}$ and $\mathcal{U}_2 = \{x_j \mid \mathcal{C}_2(x_j) \subseteq \mathcal{C}_1(x_j)\}$ where $\mathcal{C}_1(x_j) = \{\big(x_j, \big(\zeta_1(x_j), w_{\zeta_1}(x_j)\big), \big(\vartheta_1(x_j), w_{\vartheta_1}(x_j)\big)\big)\}$ and $\mathcal{C}_2(x_j) = \{\big(x_j, \big(\zeta_2(x_j), w_{\zeta_2}(x_j)\big), \big(\vartheta_2(x_j), w_{\vartheta_2}(x_j)\big)\big)\}$.

Property 5.2.4. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3 \in \Psi(\mathcal{U})$. Suppose that for each $x_j \in \mathcal{U}$, either $\mathcal{C}_1(x_j) \subseteq \mathcal{C}_2(x_j)$ or $\mathcal{C}_2(x_j) \subseteq \mathcal{C}_1(x_j)$. Then, for $r = 8, 9, 10$, we have

(i) $\mathcal{S}'_r(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2)$;

(ii) $\mathcal{S}'_r(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3) + \mathcal{S}'_r(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_3) = \mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_3) + \mathcal{S}'_r(\mathcal{C}_2, \mathcal{C}_3)$;

(iii) $\mathcal{S}'_r(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1) + \mathcal{S}'_r(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_1) = 1 + \mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2)$;

(iv) $\mathcal{S}'_r(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_2) + \mathcal{S}'_r(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_2) = 1 + \mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2)$.

Proof. Let $\mathcal{C}_1 = \{(x, (\zeta_1(x), w_{\zeta_1}(x)), (\vartheta_1(x), w_{\vartheta_1}(x))) : x \in \mathcal{U}\}$, $\mathcal{C}_2 = \{(x, (\zeta_2(x), w_{\zeta_2}(x)), (\vartheta_2(x), w_{\vartheta_2}(x))) : x \in \mathcal{U}\}$ and $\mathcal{C}_3 = \{(x, (\zeta_3(x), w_{\zeta_3}(x)), (\vartheta_3(x), w_{\vartheta_3}(x))) : x \in \mathcal{U}\}$.

(i) Using Eq. (5.8), we have

$$
\begin{aligned}
& \mathcal{S}'_8(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2) \\
= \; & \frac{1}{4n(1-e^{-1})} \sum_{j=1}^{n} \begin{pmatrix} \Big(1 - t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} + \Big(1 + t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{-t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} \\ + \Big(1 - s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} + \Big(1 + s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{-s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} - 4e^{-1} \end{pmatrix} \\
= \; & \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{U}_1} \begin{pmatrix} \Big(1 - t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} + \Big(1 + t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{-t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} \\ + \Big(1 - s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} + \Big(1 + s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{-s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} - 4e^{-1} \end{pmatrix} \\
+ \; & \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{U}_2} \begin{pmatrix} \Big(1 - t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} + \Big(1 + t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{-t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} \\ + \Big(1 - s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{s_j(\mathcal{C}_1 \cup B, \mathcal{C}_1 \cap \mathcal{C}_2)} + \Big(1 + s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)\Big) e^{-s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)} - 4e^{-1} \end{pmatrix} \\
= \; & \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{U}_1} \begin{pmatrix} \Big(1 - t_j(\mathcal{C}_2, \mathcal{C}_1)\Big) e^{t_j(\mathcal{C}_2, \mathcal{C}_1)} + \Big(1 + t_j(\mathcal{C}_2, \mathcal{C}_1)\Big) e^{-t_j(\mathcal{C}_2, \mathcal{C}_1)} \\ + \Big(1 - s_j(\mathcal{C}_2, \mathcal{C}_1)\Big) e^{s_j(\mathcal{C}_2, \mathcal{C}_1)} + \Big(1 + s_j(\mathcal{C}_2, \mathcal{C}_1)\Big) e^{-s_j(\mathcal{C}_2, \mathcal{C}_1)} - 4e^{-1} \end{pmatrix} \\
+ \; & \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{U}_2} \begin{pmatrix} \Big(1 - t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{t_j(\mathcal{C}_1, \mathcal{C}_2)} + \Big(1 + t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{-t_j(\mathcal{C}_1, \mathcal{C}_2)} \\ + \Big(1 - s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{s_j(\mathcal{C}_1, \mathcal{C}_2)} + \Big(1 + s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{-s_j(\mathcal{C}_1, \mathcal{C}_2)} - 4e^{-1} \end{pmatrix} \\
= \; & \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{U}_1} \begin{pmatrix} \Big(1 + t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{-t_j(\mathcal{C}_1, \mathcal{C}_2)} + \Big(1 - t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{t_j(\mathcal{C}_1, \mathcal{C}_2)} \\ + \Big(1 + s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{-s_j(\mathcal{C}_1, \mathcal{C}_2)} + \Big(1 - s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{s_j(\mathcal{C}_1, \mathcal{C}_2)} - 4e^{-1} \end{pmatrix} \\
+ \; & \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{U}_2} \begin{pmatrix} \Big(1 - t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{t_j(\mathcal{C}_1, \mathcal{C}_2)} + \Big(1 + t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{-t_j(\mathcal{C}_1, \mathcal{C}_2)} \\ + \Big(1 - s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{s_j(\mathcal{C}_1, \mathcal{C}_2)} + \Big(1 + s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{-s_j(\mathcal{C}_1, \mathcal{C}_2)} - 4e^{-1} \end{pmatrix} \\
= \; & \frac{1}{4n(1-e^{-1})} \sum_{j=1}^{n} \begin{pmatrix} \Big(1 - t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{t_j(\mathcal{C}_1, \mathcal{C}_2)} + \Big(1 + t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{-t_j(\mathcal{C}_1, \mathcal{C}_2)} \\ + \Big(1 - s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{s_j(\mathcal{C}_1, \mathcal{C}_2)} + \Big(1 + s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) e^{-s_j(\mathcal{C}_1, \mathcal{C}_2)} - 4e^{-1} \end{pmatrix} \\
= \; & \mathcal{S}'_8(\mathcal{C}_1, \mathcal{C}_2)
\end{aligned}
$$

Hence, $\mathcal{S}'_8(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2) = \mathcal{S}'_8(\mathcal{C}_1, \mathcal{C}_2)$. Similarly, we can obtain that $\mathcal{S}'_r(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2) = \mathcal{S}'_r(\mathcal{C}_1, \mathcal{C}_2)$ for $r = 9, 10$.

(ii) Again, using Eq. (5.8) we have

$$
\begin{aligned}
& \mathcal{S}'_8(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3) + \mathcal{S}'_8(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_3) \\
= \; & \frac{1}{4n(1-e^{-1})} \sum_{j=1}^{n} \begin{pmatrix} \Big(1 - t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3)\Big) e^{t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3)} + \Big(1 + t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3)\Big) e^{-t_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3)} \\ + \Big(1 - s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3)\Big) e^{s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3)} + \Big(1 + s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3)\Big) e^{-s_j(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3)} - 4e^{-1} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{4n(1-e^{-1})}\sum_{j=1}^{n}\begin{pmatrix}\Big(1-t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}\\ +\Big(1-s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}-4e^{-1}\end{pmatrix}\\
&= \quad \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{U}_1}\begin{pmatrix}\Big(1-t_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)\Big)e^{t_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+t_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-t_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)}\\ +\Big(1-s_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)\Big)e^{s_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+s_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-s_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)}-4e^{-1}\end{pmatrix}\\
&+ \quad \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{U}_2}\begin{pmatrix}\Big(1-t_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)\Big)e^{t_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+t_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-t_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)}\\ +\Big(1-s_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)\Big)e^{s_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+s_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-s_j(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)}-4e^{-1}\end{pmatrix}\\
&+ \quad \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{U}_1}\begin{pmatrix}\Big(1-t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}\\ +\Big(1-s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}-4e^{-1}\end{pmatrix}\\
&+ \quad \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{U}_2}\begin{pmatrix}\Big(1-t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-t_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}\\ +\Big(1-s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-s_j(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)}-4e^{-1}\end{pmatrix}\\
&= \quad \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{U}_1}\begin{pmatrix}\Big(1-t_j(\mathcal{C}_2,\mathcal{C}_3)\Big)e^{t_j(\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+t_j(\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-t_j(\mathcal{C}_2,\mathcal{C}_3)}\\ +\Big(1-s_j(\mathcal{C}_2,\mathcal{C}_3)\Big)e^{s_j(\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+s_j(\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-s_j(\mathcal{C}_2,\mathcal{C}_3)}-4e^{-1}\end{pmatrix}\\
&+ \quad \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{U}_2}\begin{pmatrix}\Big(1-t_j(\mathcal{C}_1,\mathcal{C}_3)\Big)e^{t_j(\mathcal{C}_1,\mathcal{C}_3)}+\Big(1+t_j(\mathcal{C}_1,\mathcal{C}_3)\Big)e^{-t_j(\mathcal{C}_1,\mathcal{C}_3)}\\ +\Big(1-s_j(\mathcal{C}_1,\mathcal{C}_3)\Big)e^{s_j(\mathcal{C}_1,\mathcal{C}_3)}+\Big(1+s_j(\mathcal{C}_1,\mathcal{C}_3)\Big)e^{-s_j(\mathcal{C}_1,\mathcal{C}_3)}-4e^{-1}\end{pmatrix}\\
&+ \quad \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{U}_1}\begin{pmatrix}\Big(1-t_j(\mathcal{C}_1,\mathcal{C}_3)\Big)e^{t_j(\mathcal{C}_1,\mathcal{C}_3)}+\Big(1+t_j(\mathcal{C}_1,\mathcal{C}_3)\Big)e^{-t_j(\mathcal{C}_1,\mathcal{C}_3)}\\ +\Big(1-s_j(\mathcal{C}_1,\mathcal{C}_3)\Big)e^{s_j(\mathcal{C}_1,\mathcal{C}_3)}+\Big(1+s_j(\mathcal{C}_1,\mathcal{C}_3)\Big)e^{-s_j(\mathcal{C}_1,\mathcal{C}_3)}-4e^{-1}\end{pmatrix}\\
&+ \quad \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{U}_2}\begin{pmatrix}\Big(1-t_j(\mathcal{C}_2,\mathcal{C}_3)\Big)e^{t_j(\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+t_j(\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-t_j(\mathcal{C}_2,\mathcal{C}_3)}\\ +\Big(1-s_j(\mathcal{C}_2,\mathcal{C}_3)\Big)e^{s_j(\mathcal{C}_2,\mathcal{C}_3)}+\Big(1+s_j(\mathcal{C}_2,\mathcal{C}_3)\Big)e^{-s_j(\mathcal{C}_2,\mathcal{C}_3)}-4e^{-1}\end{pmatrix}\\
&= \quad \mathcal{S}_8'(\mathcal{C}_1,\mathcal{C}_3)+\mathcal{S}_8'(\mathcal{C}_2,\mathcal{C}_3)
\end{aligned}
$$

Hence, $\mathcal{S}_8'(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)+\mathcal{S}_8'(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)=\mathcal{S}_8'(\mathcal{C}_1,\mathcal{C}_3)+\mathcal{S}_8'(\mathcal{C}_2,\mathcal{C}_3)$. Similarly we can obtain that $\mathcal{S}_r'(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_3)+\mathcal{S}_r'(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_3)=\mathcal{S}_r'(\mathcal{C}_1,\mathcal{C}_3)+\mathcal{S}_r'(\mathcal{C}_2,\mathcal{C}_3)$ for $r=9,10$.

(iii) Replacing $\mathcal{C}_3$ by $\mathcal{C}_1$ in the above part, we obtain $\mathcal{S}_r'(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_1)+\mathcal{S}_r'(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_1)=\mathcal{S}_r'(\mathcal{C}_1,\mathcal{C}_1)+\mathcal{S}_r'(\mathcal{C}_2,\mathcal{C}_1)$ which gives that $\mathcal{S}_r'(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_1)+\mathcal{S}_r'(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_1)=1+\mathcal{S}_r'(\mathcal{C}_1,\mathcal{C}_2)$.

(iv) Replacing $\mathcal{C}_3$ by $\mathcal{C}_2$ in the part (ii), we obtain $\mathcal{S}_r'(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_2)+\mathcal{S}_r'(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_2)=\mathcal{S}_r'(\mathcal{C}_1,\mathcal{C}_2)+\mathcal{S}_r'(\mathcal{C}_2,\mathcal{C}_2)$ which gives that $\mathcal{S}_r'(\mathcal{C}_1\cup\mathcal{C}_2,\mathcal{C}_2)+\mathcal{S}_r'(\mathcal{C}_1\cap\mathcal{C}_2,\mathcal{C}_2)=1+\mathcal{S}_r'(\mathcal{C}_1,\mathcal{C}_2)$.

$\square$

5.2.2 Distance Measures for CIFSs

Definition 5.2.5. [6] For $\mathcal{C}_1$ and $\mathcal{C}_2\in\Psi(\mathcal{U})$, distance measure $\mathcal{D}:\Psi(\mathcal{U})\times\Psi(\mathcal{U})\to[0,1]$

is a real-valued function satisfying the following properties:

(P1) $0 \leq \mathcal{D}(\mathcal{C}_1, \mathcal{C}_2) \leq 1$;

(P2) $\mathcal{D}(\mathcal{C}_1, \mathcal{C}_2) = 0$ if $\mathcal{C}_1 = \mathcal{C}_2$;

(P3) $\mathcal{D}(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{D}(\mathcal{C}_2, \mathcal{C}_1)$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{D}(\mathcal{C}_1, \mathcal{C}_3) \geq \mathcal{D}(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{D}(\mathcal{C}_1, \mathcal{C}_3) \geq \mathcal{D}(\mathcal{C}_2, \mathcal{C}_3)$ where $\mathcal{C}_3 \in \Psi(\mathcal{U})$.

Definition 5.2.6. For two CIFSs $\mathcal{C}_1 = \Big\{\big(x, \big(\zeta_1(x), w_{\zeta_1}(x)\big), \big(\vartheta_1(x), w_{\vartheta_1}(x)\big)\big) : x \in \mathcal{U}\Big\}$ and $\mathcal{C}_2 = \Big\{\big(x, \big(\zeta_2(x),\ w_{\zeta_2}(x)\big), \big(\vartheta_2(x),\ w_{\vartheta_2}(x)\big)\big) : x \in \mathcal{U}\Big\}$ defined on $\mathcal{U}$, we define a series of distance measures as follows:

(i)

$$\mathcal{D}_1(\mathcal{C}_1, \mathcal{C}_2) = 1 - \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\left(\begin{array}{l}\min\big(\zeta_1(x_j), \zeta_2(x_j)\big) + \min\big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\big)\\ + \min\big(\vartheta_1(x_j), \vartheta_2(x_j)\big) + \min\big(w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j)\big)\end{array}\right)}{\left(\begin{array}{l}\max\big(\zeta_1(x_j), \zeta_2(x_j)\big) + \max\big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\big)\\ + \max\big(\vartheta_1(x_j), \vartheta_2(x_j)\big) + \max\big(w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j)\big)\end{array}\right)}\right) \tag{5.9}$$

(ii)

$$\mathcal{D}_2(\mathcal{C}_1, \mathcal{C}_2) = 1 - \frac{\sum_{j=1}^{n}\left(\begin{array}{l}\min\big(\zeta_1(x_j), \zeta_2(x_j)\big) + \min\big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\big)\\ + \min\big(\vartheta_1(x_j), \vartheta_2(x_j)\big) + \min\big(w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j)\big)\end{array}\right)}{\sum_{j=1}^{n}\left(\begin{array}{l}\max\big(\zeta_1(x_j), \zeta_2(x_j)\big) + \max\big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\big)\\ + \max\big(\vartheta_1(x_j), \vartheta_2(x_j)\big) + \max\big(w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j)\big)\end{array}\right)} \tag{5.10}$$

(iii)

$$\mathcal{D}_3(\mathcal{C}_1, \mathcal{C}_2) = 1 - \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\left(\begin{array}{l}\min\Big(\zeta_1(x_j), \zeta_2(x_j)\Big) + \min\Big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\Big)\\ + \min\Big(1 - \vartheta_1(x_j), 1 - \vartheta_2(x_j)\Big) + \min\Big(1 - w_{\vartheta_1}(x_j), 1 - w_{\vartheta_2}(x_j)\Big)\end{array}\right)}{\left(\begin{array}{l}\max\Big(\zeta_1(x_j), \zeta_2(x_j)\Big) + \max\Big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\Big)\\ + \max\Big(1 - \vartheta_1(x_j), 1 - \vartheta_2(x_j)\Big) + \max\Big(1 - w_{\vartheta_1}(x_j), 1 - w_{\vartheta_2}(x_j)\Big)\end{array}\right)}\right) \tag{5.11}$$

(iv)

$$\mathcal{D}_4(\mathcal{C}_1,\mathcal{C}_2) = 1 - \frac{\sum_{j=1}^{n}\left(\begin{array}{l}\min\Big(\zeta_1(x_j),\zeta_2(x_j)\Big)+\min\Big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\Big)\\ +\min\Big(1-\vartheta_1(x_j),1-\vartheta_2(x_j)\Big)+\min\Big(1-w_{\vartheta_1}(x_j),1-w_{\vartheta_2}(x_j)\Big)\end{array}\right)}{\sum_{j=1}^{n}\left(\begin{array}{l}\max\Big(\zeta_1(x_j),\zeta_2(x_j)\Big)+\max\Big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\Big)\\ +\max\Big(1-\vartheta_1(x_j),1-\vartheta_2(x_j)\Big)+\max\Big(1-w_{\vartheta_1}(x_j),1-w_{\vartheta_2}(x_j)\Big)\end{array}\right)} \tag{5.12}$$

(v)

$$\mathcal{D}_5(\mathcal{C}_1,\mathcal{C}_2) = \frac{1}{4n}\sum_{j=1}^{n}\left(\begin{array}{l}|\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{array}\right) \tag{5.13}$$

(vi)

$$\mathcal{D}_6(\mathcal{C}_1,\mathcal{C}_2) = \frac{1}{4}\left(\begin{array}{l}\max_j|\zeta_1(x_j)-\zeta_2(x_j)|+\max_j|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|\\ +\max_j|\vartheta_1(x_j)-\vartheta_2(x_j)|+\max_j|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{array}\right) \tag{5.14}$$

(vii)

$$\mathcal{D}_7(\mathcal{C}_1,\mathcal{C}_2) = \frac{\sum_{j=1}^{n}\left(\begin{array}{l}|\zeta_1(x_j)-\zeta_2(x_j)|+|w_{\zeta_1}(x_j)-w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)-\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)-w_{\vartheta_2}(x_j)|\end{array}\right)}{\sum_{j=1}^{n}\left(\begin{array}{l}|\zeta_1(x_j)+\zeta_2(x_j)|+|w_{\zeta_1}(x_j)+w_{\zeta_2}(x_j)|\\ +|\vartheta_1(x_j)+\vartheta_2(x_j)|+|w_{\vartheta_1}(x_j)+w_{\vartheta_2}(x_j)|\end{array}\right)} \tag{5.15}$$

Theorem 5.2.4. The distance measures $\mathcal{D}_r(\mathcal{C}_1,\mathcal{C}_2)$ $(r = 1,2,\ldots,7)$ satisfy the following properties:

(P1) $0 \le \mathcal{D}_r(\mathcal{C}_1,\mathcal{C}_2) \le 1$;

(P2) $\mathcal{D}_r(\mathcal{C}_1,\mathcal{C}_2) = 0$ if $\mathcal{C}_1 = \mathcal{C}_2$;

(P3) $\mathcal{D}_r(\mathcal{C}_1,\mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2,\mathcal{C}_1)$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{D}_r(\mathcal{C}_1,\mathcal{C}_3) \ge \mathcal{D}_r(\mathcal{C}_1,\mathcal{C}_2)$ and $\mathcal{D}_r(\mathcal{C}_1,\mathcal{C}_3) \ge \mathcal{D}_r(\mathcal{C}_2,\mathcal{C}_3)$ where $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3 \in \Psi(\mathcal{U})$.

Proof. It can be proved as similar to the Theorem 5.2.1. So, we omit here. $\square$

Further, we observe that these proposed distance measures satisfy the certain properties which are discussed as follows:

Property 5.2.5. For $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$ and $r = 1, 2, \ldots 7$, we have

(i) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2^c) = \mathcal{D}_r(\mathcal{C}_1^c, \mathcal{C}_2)$, $r \neq 3, 4$;

(ii) $\mathcal{D}_r(\mathcal{C}_1^c, \mathcal{C}_2^c) = \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2)$, $r \neq 3, 4$;

(iii) $\mathcal{D}_r(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_1 \cup \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2)$.

Proof. It can be proved as similar to the Property 5.2.1. So, we omit here. □

Property 5.2.6. For $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$ and $r = 5, 6$, we have

(i) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_1 \cup \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)$;

(ii) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_1 \cup \mathcal{C}_2)$;

(iii) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_1 \oplus \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_1 \otimes \mathcal{C}_2)$;

(iv) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_1 \otimes \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$.

Proof. It can be obtained as similar to the Property 5.2.2. So, we omit here. □

Next, we define exponential distance measures under CIF theory and discuss some of its properties as follows:

Definition 5.2.7. For two CIFSs $\mathcal{C}_1 = \big\{\big(x, \big(\zeta_1(x), w_{\zeta_1}(x)\big), \big(\vartheta_1(x), w_{\vartheta_1}(x)\big)\big) : x \in \mathcal{U}\big\}$ and $\mathcal{C}_2 = \big\{\big(x, \big(\zeta_2(x), w_{\zeta_2}(x)\big), \big(\vartheta_2(x), w_{\vartheta_2}(x)\big)\big) : x \in \mathcal{U}\big\}$ defined on $\mathcal{U}$, we define the exponential distance measure as follows:

$$\begin{aligned}
&\mathcal{D}_8(\mathcal{C}_1, \mathcal{C}_2)\\
= \;& \frac{1}{4n(1-e^{-1})} \sum_{j=1}^{n} \begin{pmatrix} 4 - \Big(1 - t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) \exp\Big(t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) - \Big(1 + t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) \exp\Big(-t_j(\mathcal{C}_1, \mathcal{C}_2)\Big) \\ - \Big(1 - s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) \exp\Big(s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) - \Big(1 + s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) \exp\Big(-s_j(\mathcal{C}_1, \mathcal{C}_2)\Big) \end{pmatrix}
\end{aligned} \tag{5.16}$$

where $t_j(\mathcal{C}_1, \mathcal{C}_2)$ and $s_j(\mathcal{C}_1, \mathcal{C}_2)$ are same as defined in Definition 5.2.3.

Theorem 5.2.5. The CIF exponential distance measure $\mathcal{D}_8$ satisfies the following properties:

(P1) $0 \leq \mathcal{D}_8(\mathcal{C}_1, \mathcal{C}_2) \leq 1$;

(P2) $\mathcal{D}_8(\mathcal{C}_1, \mathcal{C}_2) = 0$ if $\mathcal{C}_1 = \mathcal{C}_2$;

(P3) $\mathcal{D}_8(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{D}_8(\mathcal{C}_2, \mathcal{C}_1)$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{D}_8(\mathcal{C}_1, \mathcal{C}_3) \geq \mathcal{D}_8(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{D}_8(\mathcal{C}_1, \mathcal{C}_3) \geq \mathcal{D}_8(\mathcal{C}_2, \mathcal{C}_3)$ where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \Psi(\mathcal{U})$.

Proof. It can be proved in the similar manner as Theorem 5.2.2. So, we omit here. □

Based on the proposed CIF exponential distance measure $\mathcal{D}_8$, we introduce two exponential distance measures as follows:

Definition 5.2.8. For $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$, we define two exponential distance measures as:

(i) $\mathcal{D}_9(\mathcal{C}_1, \mathcal{C}_2) = \frac{2\mathcal{D}_8(\mathcal{C}_1, \mathcal{C}_2)}{1+\mathcal{D}_8(\mathcal{C}_1, \mathcal{C}_2)}$;

(ii) $\mathcal{D}_{10}(\mathcal{C}_1, \mathcal{C}_2) = \frac{1-e^{-\mathcal{D}_8(\mathcal{C}_1, \mathcal{C}_2)}}{1-e^{-1}}$.

Theorem 5.2.6. The CIF exponential distance measures $\mathcal{D}_r$ $(r = 9, 10)$ satisfy the following properties:

(P1) $0 \leq \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2) \leq 1$;

(P2) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2) = 0$ if $\mathcal{C}_1 = \mathcal{C}_2$;

(P3) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_1)$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_3) \geq \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_3) \geq \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_3)$ where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \Psi(\mathcal{U})$.

Proof. Its proof follows from the Theorem 5.2.5. So, we omit here. □

Moreover, it is observed that these proposed exponential distance measures satisfy certain properties which are explained as follows:

Property 5.2.7. For $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$ and $r = 8, 9, 10$, we have

(i) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2^c) = \mathcal{D}_r(\mathcal{C}_1^c, \mathcal{C}_2)$;

(ii) $\mathcal{D}_r(\mathcal{C}_1^c, \mathcal{C}_2^c) = \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2)$;

(iii) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_1 \cup \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2)$;

(iv) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_1 \cup \mathcal{C}_2)$;

(v) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_1 \oplus \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_1 \otimes \mathcal{C}_2)$;

(vi) $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_1 \otimes \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$.

Proof. It can be proved as similar to the Property 5.2.1 and Property 5.2.2. So, we omit here. □

Property 5.2.8. Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \Psi(\mathcal{U})$ and for each $x_j \in \mathcal{U}$, either $\mathcal{C}_1(x_j) \subseteq \mathcal{C}_2(x_j)$ or $\mathcal{C}_2(x_j) \subseteq \mathcal{C}_1(x_j)$. Then, for $r = 8, 9, 10$, we have

(i) $\mathcal{D}_r(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2)$;

(ii) $\mathcal{D}_r(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_3) + \mathcal{D}_r(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_3) = \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_3) + \mathcal{D}_r(\mathcal{C}_2, \mathcal{C}_3)$;

(iii) $\mathcal{D}_r(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_1) + \mathcal{D}_r(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_1) = \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2)$;

(iv) $\mathcal{D}_r(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{C}_2) + \mathcal{D}_r(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_2) = \mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2)$.

Proof. It can be obtained as similar to the Property 5.2.4. So, we omit here. □

5.2.3 Entropy Measure for CIFSs

Definition 5.2.9. For any set $\mathcal{C}_1 \in \Psi(\mathcal{U})$, entropy measure $\mathcal{E}n : \Psi(\mathcal{U}) \to [0, 1]$ is a real valued function satisfying the following properties:

(P1) $0 \leq \mathcal{E}n(\mathcal{C}_1) \leq 1$;

(P2) $\mathcal{E}n(\mathcal{C}_1) = 0$ if $\mathcal{C}_1$ is crisp set;

(P3) $\mathcal{E}n(\mathcal{C}_1) = 1 \Leftrightarrow \zeta_1(x_j) = \vartheta_1(x_j)$ and $w_{\zeta_1}(x_j) = w_{\vartheta_1}(x_j)$;

(P4) $\mathcal{E}n(\mathcal{C}_1) = \mathcal{E}n(\mathcal{C}_1^c)$;

(P5) $\mathcal{E}n(\mathcal{C}_1) \leq \mathcal{E}n(\mathcal{C}_2)$ if either $\mathcal{C}_1 \subseteq \mathcal{C}_2$ with $\zeta_2(x_j) \leq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \leq w_{\vartheta_2}(x_j)$ or $\mathcal{C}_2 \subseteq \mathcal{C}_1$ with $\zeta_2(x_j) \geq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \geq w_{\vartheta_2}(x_j)$ where $\mathcal{C}_2 \in \Psi(\mathcal{U})$.

Definition 5.2.10. For CIFS $\mathcal{C}_1 = \left\{\left(x, \left(\zeta_1(x), w_{\zeta_1}(x)\right), \left(\vartheta_1(x), w_{\vartheta_1}(x)\right)\right) : x \in \mathcal{U}\right\}$ we define series of entropy measures as follows:

(i)

$$\mathcal{E}n_1(\mathcal{C}_1) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\min\left(\zeta_1(x_j), \vartheta_1(x_j)\right) + \min\left(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\right)}{\max\left(\zeta_1(x_j), \vartheta_1(x_j)\right) + \max\left(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\right)}\right) \tag{5.17}$$

(ii)

$$\mathcal{E}n_2(\mathcal{C}_1) = \frac{\sum\limits_{j=1}^{n}\left(\min\left(\zeta_1(x_j), \vartheta_1(x_j)\right) + \min\left(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\right)\right)}{\sum\limits_{j=1}^{n}\left(\max\left(\zeta_1(x_j), \vartheta_1(x_j)\right) + \max\left(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\right)\right)} \tag{5.18}$$

(iii)

$$\mathcal{E}n_3(\mathcal{C}_1) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\left(\begin{array}{l}\min\left(\zeta_1(x_j), \vartheta_1(x_j)\right) + \min\left(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\right)\\ + \min\left(1-\zeta_1(x_j), 1-\vartheta_1(x_j)\right) + \min\left(1-w_{\zeta_1}(x_j), 1-w_{\vartheta_1}(x_j)\right)\end{array}\right)}{\left(\begin{array}{l}\max\left(\zeta_1(x_j), \vartheta_1(x_j)\right) + \max\left(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\right)\\ + \max\left(1-\zeta_1(x_j), 1-\vartheta_1(x_j)\right) + \max\left(1-w_{\zeta_1}(x_j), 1-w_{\vartheta_1}(x_j)\right)\end{array}\right)}\right) \tag{5.19}$$

(iv)

$$\mathcal{E}n_4(\mathcal{C}_1) = \frac{\sum\limits_{j=1}^{n}\left(\begin{array}{l}\min\left(\zeta_1(x_j), \vartheta_1(x_j)\right) + \min\left(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\right)\\ + \min\left(1-\zeta_1(x_j), 1-\vartheta_1(x_j)\right) + \min\left(1-w_{\zeta_1}(x_j), 1-w_{\vartheta_1}(x_j)\right)\end{array}\right)}{\sum\limits_{j=1}^{n}\left(\begin{array}{l}\max\left(\zeta_1(x_j), \vartheta_1(x_j)\right) + \max\left(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\right)\\ + \max\left(1-\zeta_1(x_j), 1-\vartheta_1(x_j)\right) + \max\left(1-w_{\zeta_1}(x_j), 1-w_{\vartheta_1}(x_j)\right)\end{array}\right)} \tag{5.20}$$

(v)

$$\mathcal{E}n_5(\mathcal{C}_1) = \frac{1}{2n}\sum_{j=1}^{n}\left(2 - \left(|\zeta_1(x_j) - \vartheta_1(x_j)| + |w_{\zeta_1}(x_j) - w_{\vartheta_1}(x_j)|\right)\right) \tag{5.21}$$

(vi)

$$\mathcal{E}n_6(\mathcal{C}_1) = 1 - \frac{1}{2}\left(\max_j |\zeta_1(x_j) - \vartheta_1(x_j)| + \max_j |w_{\zeta_1}(x_j) - w_{\vartheta_1}(x_j)|\right) \tag{5.22}$$

(vii)

$$\mathcal{E}n_7(\mathcal{C}_1) = 1 - \frac{\sum\limits_{j=1}^{n} \left(|\zeta_1(x_j) - \vartheta_1(x_j)| + |w_{\zeta_1}(x_j) - w_{\vartheta_1}(x_j)|\right)}{\sum\limits_{j=1}^{n} \left(|\zeta_1(x_j) + \vartheta_1(x_j)| + |w_{\zeta_1}(x_j) + w_{\vartheta_1}(x_j)|\right)} \tag{5.23}$$

Theorem 5.2.7. The CIF exponential entropy measures $\mathcal{E}n_r$ $(r = 1, 2, \ldots, 7)$ satisfy the following properties:

(P1) $0 \leq \mathcal{E}n_r(\mathcal{C}_1) \leq 1$;

(P2) $\mathcal{E}n_r(\mathcal{C}_1) = 0$ if $\mathcal{C}_1$ is crisp set;

(P3) $\mathcal{E}n_r(\mathcal{C}_1) = 1 \Leftrightarrow \zeta_1(x_j) = \vartheta_1(x_j)$ and $w_{\zeta_1}(x_j) = w_{\vartheta_1}(x_j)$;

(P4) $\mathcal{E}n_r(\mathcal{C}_1) = \mathcal{E}n_r(\mathcal{C}_1^c)$;

(P5) $\mathcal{E}n_r(\mathcal{C}_1) \leq \mathcal{E}n_r(\mathcal{C}_2)$ if either $\mathcal{C}_1 \subseteq \mathcal{C}_2$ with $\zeta_2(x_j) \leq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \leq w_{\vartheta_2}(x_j)$ or $\mathcal{C}_2 \subseteq \mathcal{C}_1$ with $\zeta_2(x_j) \geq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \geq w_{\vartheta_2}(x_j)$ where $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$.

Proof. It can be proved as similar to the Theorem 5.2.1. So, we omit here. $\square$

Next, we propose CIF exponential entropy measure as follows:

Definition 5.2.11. For CIFS $\mathcal{C}_1 = \left\{\left(x, \left(\zeta_1(x), w_{\zeta_1}(x)\right), \left(\vartheta_1(x), w_{\vartheta_1}(x)\right)\right) : x \in \mathcal{U}\right\}$ defined on $\mathcal{U}$, we define the exponential entropy measure as follows:

$$\begin{aligned}
&\mathcal{E}n_8(\mathcal{C}_1) \\
= \quad &\frac{1}{4n(1-e^{-1})} \sum_{j=1}^{n} \left(\begin{array}{l}
\Big(1 - \big(\zeta_1(x_j) - \vartheta_1(x_j)\big)\Big) \exp\Big(\zeta_1(x_j) - \vartheta_1(x_j)\Big) \\
+ \Big(1 + \big(\zeta_1(x_j) - \vartheta_1(x_j)\big)\Big) \exp\Big(\vartheta_1(x_j) - \zeta_1(x_j)\Big) \\
+ \left(1 - (w_{\zeta_1}(x_j) - w_{\vartheta_1}(x_j))\right) \exp\left(w_{\zeta_1}(x_j) - w_{\vartheta_1}(x_j)\right) \\
+ \left(1 + (w_{\zeta_1}(x_j) - w_{\vartheta_1}(x_j))\right) \exp\left(w_{\vartheta_1}(x_j) - w_{\zeta_1}(x_j)\right) - 4e^{-1}
\end{array} \right)
\end{aligned} \tag{5.24}$$

Theorem 5.2.8. The CIF exponential entropy measure $\mathcal{E}n_8$ satisfies the following properties:

(P1) $0 \leq \mathcal{E}n_8(\mathcal{C}_1) \leq 1$;

(P2) $\mathcal{E}n_8(\mathcal{C}_1) = 0$ if $\mathcal{C}_1$ is crisp set;

(P3) $\mathcal{E}n_8(\mathcal{C}_1) = 1 \Leftrightarrow \zeta_1(x_j) = \vartheta_1(x_j)$ and $w_{\zeta_1}(x_j) = w_{\vartheta_1}(x_j)$;

(P4) $\mathcal{E}n_8(\mathcal{C}_1) = \mathcal{E}n_8(\mathcal{C}_1^c)$;

(P5) $\mathcal{E}n_8(\mathcal{C}_1) \leq \mathcal{E}n_8(\mathcal{C}_2)$ if either $\mathcal{C}_1 \subseteq \mathcal{C}_2$ with $\zeta_2(x_j) \leq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \leq w_{\vartheta_2}(x_j)$ or $\mathcal{C}_2 \subseteq \mathcal{C}_1$ with $\zeta_2(x_j) \geq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \geq w_{\vartheta_2}(x_j)$ where $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$.

Proof. It can be obtained as similar to the Theorem 5.2.2. So, we omit here. □

Now, based on the proposed CIF exponential entropy measure, we propose the exponential entropy measures as follows:

Definition 5.2.12. For $\mathcal{C}_1 \in \Psi(\mathcal{U})$, we define the exponential entropy measure as follows:

(i) $\mathcal{E}n_9(\mathcal{C}_1) = \frac{\mathcal{E}n_8(\mathcal{C}_1)}{2-\mathcal{E}n_8(\mathcal{C}_1)}$;

(ii) $\mathcal{E}n_{10}(\mathcal{C}_1) = \frac{e^{\mathcal{E}n_8(\mathcal{C}_1)-1} - e^{-1}}{1-e^{-1}}$.

Theorem 5.2.9. The CIF exponential entropy measures $\mathcal{E}n_r$ $(r = 9, 10)$ satisfy the following properties:

(P1) $0 \leq \mathcal{E}n_r(\mathcal{C}_1) \leq 1$;

(P2) $\mathcal{E}n_r(\mathcal{C}_1) = 0$ if $\mathcal{C}_1$ is crisp set;

(P3) $\mathcal{E}n_r(\mathcal{C}_1) = 1 \Leftrightarrow \zeta_1(x_j) = \vartheta_1(x_j)$ and $w_{\zeta_1}(x_j) = w_{\vartheta_1}(x_j)$;

(P4) $\mathcal{E}n_r(\mathcal{C}_1) = \mathcal{E}n_r(\mathcal{C}_1^c)$;

(P5) $\mathcal{E}n_r(\mathcal{C}_1) \leq \mathcal{E}n_r(\mathcal{C}_2)$ if either $\mathcal{C}_1 \subseteq \mathcal{C}_2$ with $\zeta_2(x_j) \leq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \leq w_{\vartheta_2}(x_j)$ or $\mathcal{C}_2 \subseteq \mathcal{C}_1$ with $\zeta_2(x_j) \geq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \geq w_{\vartheta_2}(x_j)$ where $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$.

Proof. Its proof follows from the Theorem 5.2.8. So, we omit here. □

5.2.4 Inclusion Measure for CIFSs

Definition 5.2.13. For two sets $\mathcal{C}_1$ and $\mathcal{C}_2 \in \Psi(\mathcal{U})$, inclusion measure $\mathcal{I}n : \Psi(\mathcal{U})\times\Psi(\mathcal{U}) \to [0,1]$ is a real valued function satisfying the following properties:

(P1) $0 \leq \mathcal{I}n(\mathcal{C}_1, \mathcal{C}_2) \leq 1$;

(P2) $\mathcal{I}n(\mathcal{C}_1, \mathcal{C}_2) = 1 \Leftrightarrow \mathcal{C}_1 \subseteq \mathcal{C}_2$;

(P3) $\mathcal{I}n(\mathcal{U}, \phi) = 0$;

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{I}n(\mathcal{C}_3, \mathcal{C}_1) \leq \mathcal{I}n(\mathcal{C}_3, \mathcal{C}_2)$ and $\mathcal{I}n(\mathcal{C}_3, \mathcal{C}_1) \leq \mathcal{I}n(\mathcal{C}_2, \mathcal{C}_1)$.

where $\phi = \Big\{\big(x, (0,0), (1,1)\big) : x \in \mathcal{U}\Big\}$ and $\mathcal{U} = \Big\{\big(x, (1,1), (0,0)\big) : x \in \mathcal{U}\Big\}$.

Theorem 5.2.10. Let $\mathcal{C}_1 = \Big\{\big(x, \big(\zeta_1(x), w_{\zeta_1}(x)\big), \big(\vartheta_1(x), w_{\vartheta_1}(x)\big)\big) : x \in \mathcal{U}\Big\}$ and $\mathcal{C}_2 = \Big\{\big(x, \big(\zeta_2(x), w_{\zeta_2}(x)\big), \big(\vartheta_2(x), w_{\vartheta_2}(x)\big)\big) : x \in \mathcal{U}\Big\}$ be two CIFSs defined on $\mathcal{U}$. Then, $\mathcal{I}n_r(\mathcal{C}_1, \mathcal{C}_2)$ $(r = 1, 2, \ldots, 7)$ satisfy the properties as stated in Definition 5.2.13.

(i)

$$\mathcal{I}n_1(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\min\Big(\zeta_1(x_j),\zeta_2(x_j)\Big)+\vartheta_1(x_j)+\min\Big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\Big)+w_{\vartheta_1}(x_j)}{\zeta_1(x_j)+\max\Big(\vartheta_1(x_j),\vartheta_2(x_j)\Big)+w_{\zeta_1}(x_j)+\max\Big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\Big)}\right) \tag{5.25}$$

(ii)

$$\mathcal{I}n_2(\mathcal{C}_1, \mathcal{C}_2) = \frac{\sum_{j=1}^{n}\Big(\min\Big(\zeta_1(x_j),\zeta_2(x_j)\Big)+\vartheta_1(x_j)+\min\Big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\Big)+w_{\vartheta_1}(x_j)\Big)}{\sum_{j=1}^{n}\Big(\zeta_1(x_j)+\max\Big(\vartheta_1(x_j),\vartheta_2(x_j)\Big)+w_{\zeta_1}(x_j)+\max\Big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\Big)\Big)} \tag{5.26}$$

(iii)

$$\mathcal{I}n_3(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{n}\sum_{j=1}^{n}\frac{\begin{pmatrix}\min\Big(\zeta_1(x_j), \zeta_2(x_j)\Big) + \min\Big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\Big) \\ + \min\Big(1-\vartheta_1(x_j), 1-\vartheta_2(x_j)\Big) + \min\Big(1-w_{\vartheta_1}(x_j), 1-w_{\vartheta_2}(x_j)\Big)\end{pmatrix}}{\zeta_1(x_j)+1-\vartheta_1(x_j)+\Big(w_{\zeta_1}(x_j)+1-w_{\vartheta_1}(x_j)\Big)} \tag{5.27}$$

(iv)

$$\mathcal{I}n_4(\mathcal{C}_1, \mathcal{C}_2) = \frac{\sum_{j=1}^{n}\begin{pmatrix}\min\Big(\zeta_1(x_j), \zeta_2(x_j)\Big) + \min\Big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\Big) \\ + \min\Big(1-\vartheta_1(x_j), 1-\vartheta_2(x_j)\Big) + \min\Big(1-w_{\vartheta_1}(x_j), 1-w_{\vartheta_2}(x_j)\Big)\end{pmatrix}}{\sum_{j=1}^{n}\Big(\zeta_1(x_j)+1-\vartheta_1(x_j)+\Big(w_{\zeta_1}(x_j)+1-w_{\vartheta_1}(x_j)\Big)\Big)} \tag{5.28}$$

(v)

$$\mathcal{I}n_5(\mathcal{C}_1,\mathcal{C}_2) = 1 - \frac{1}{4n}\sum_{j=1}^{n}\left(\begin{array}{c} \left|\max\left(0, \zeta_1(x_j) - \zeta_2(x_j)\right)\right| + \left|\max\left(0, w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)\right)\right| \\ + \left|\min\left(0, \vartheta_1(x_j) - \vartheta_2(x_j)\right)\right| + \left|\min\left(0, w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)\right)\right| \end{array}\right) \tag{5.29}$$

(vi)

$$\mathcal{I}n_6(\mathcal{C}_1,\mathcal{C}_2) = 1 - \frac{1}{4}\left(\begin{array}{c} \max_j \left|\max\left(0, \zeta_1(x_j) - \zeta_2(x_j)\right)\right| + \max_j \left|\max\left(0, w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)\right)\right| \\ + \max_j \left|\min\left(0, \vartheta_1(x_j) - \vartheta_2(x_j)\right)\right| + \max_j \left|\min\left(0, w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)\right)\right| \end{array}\right) \tag{5.30}$$

(vii)

$$\mathcal{I}n_7(\mathcal{C}_1,\mathcal{C}_2) = 1 - \frac{\sum_{j=1}^{n}\left(\begin{array}{c} \left|\max\left(0, \zeta_1(x_j) - \zeta_2(x_j)\right)\right| + \left|\max\left(0, w_{\zeta_1}(x_j) - w_{\zeta_2}(x_j)\right)\right| \\ + \left|\min\left(0, \vartheta_1(x_j) - \vartheta_2(x_j)\right)\right| + \left|\min\left(0, w_{\vartheta_1}(x_j) - w_{\vartheta_2}(x_j)\right)\right| \end{array}\right)}{\sum_{j=1}^{n}\left(\begin{array}{c} \left|\min\left(2\times\zeta_1(x_j), \zeta_1(x_j) + \zeta_2(x_j)\right)\right| + \left|\min\left(2\times w_{\zeta_1}(x_j), w_{\zeta_1}(x_j) + w_{\zeta_2}(x_j)\right)\right| \\ + \left|\max\left(2\times\vartheta_1(x_j), \vartheta_1(x_j) + \vartheta_2(x_j)\right)\right| + \left|\max\left(2\times w_{\vartheta_1}(x_j), w_{\vartheta_1}(x_j) + w_{\vartheta_2}(x_j)\right)\right| \end{array}\right)} \tag{5.31}$$

Proof. It can be obtained as similar to the Theorem 5.2.1. So, we omit here. □

5.2.5 Transformation relationships among proposed information measures

Theorem 5.2.11. Let $\mathcal{S}'(\mathcal{C}_1,\mathcal{C}_2)$ denotes the similarity measure for $\mathcal{C}_1,\mathcal{C}_2 \in \Psi(\mathcal{U})$. Then, $\mathcal{D}(\mathcal{C}_1,\mathcal{C}_2) = 1 - \mathcal{S}'(\mathcal{C}_1,\mathcal{C}_2)$ is the distance measure for CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$.

Proof. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are CIFSs.

(i) As $0 \leq \mathcal{S}'(\mathcal{C}_1,\mathcal{C}_2) \leq 1$. Therefore, $0 \leq \mathcal{D}(\mathcal{C}_1,\mathcal{C}_2) \leq 1$

(ii) If $\mathcal{C}_1 = \mathcal{C}_2$ then, $\mathcal{S}'(\mathcal{C}_1,\mathcal{C}_2) = 1$. It implies that $\mathcal{D}(\mathcal{C}_1,\mathcal{C}_2) = 0$

(iii) $\mathcal{S}'(\mathcal{C}_1,\mathcal{C}_2) = \mathcal{S}'(\mathcal{C}_2,\mathcal{C}_1)$ gives that $\mathcal{D}(\mathcal{C}_1,\mathcal{C}_2) = \mathcal{D}(\mathcal{C}_2,\mathcal{C}_1)$

(iv) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{S}'(\mathcal{C}_1,\mathcal{C}_3) \leq \mathcal{S}'(\mathcal{C}_1,\mathcal{C}_2)$ and $\mathcal{S}'(\mathcal{C}_1,\mathcal{C}_3) \leq \mathcal{S}'(\mathcal{C}_2,\mathcal{C}_3)$ which gives that $\mathcal{D}(\mathcal{C}_1,\mathcal{C}_3) \geq \mathcal{D}(\mathcal{C}_1,\mathcal{C}_2)$ and $\mathcal{D}(\mathcal{C}_1,\mathcal{C}_3) \geq \mathcal{D}(\mathcal{C}_2,\mathcal{C}_3)$.

□

In order to validate the Theorem 5.2.11, we show that similarity measure $\mathcal{S}'_1$ and distance measure $\mathcal{D}_1$ can be related with the relation $\mathcal{D}_1 = 1 - \mathcal{S}'_1$.

Example 5.2.1. For $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$, using similarity measure $\mathcal{S}_1'$ as defined in Eq. (5.1), we have

$$\mathcal{S}_1'(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\begin{pmatrix} \min\big(\zeta_1(x_j), \zeta_2(x_j)\big) + \min\big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\big) \\ + \min\big(\vartheta_1(x_j), \vartheta_2(x_j)\big) + \min\big(w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j)\big) \end{pmatrix}}{\begin{pmatrix} \max\big(\zeta_1(x_j), \zeta_2(x_j)\big) + \max\big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\big) \\ + \max\big(\vartheta_1(x_j), \vartheta_2(x_j)\big) + \max\big(w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j)\big) \end{pmatrix}}\right)$$

Further, from the distance measure $\mathcal{D}_1$ as defined in Eq. (5.9), we have

$$\mathcal{D}_1(\mathcal{C}_1, \mathcal{C}_2) = 1 - \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\begin{pmatrix} \min\big(\zeta_1(x_j), \zeta_2(x_j)\big) + \min\big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\big) \\ + \min\big(\vartheta_1(x_j), \vartheta_2(x_j)\big) + \min\big(w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j)\big) \end{pmatrix}}{\begin{pmatrix} \max\big(\zeta_1(x_j), \zeta_2(x_j)\big) + \max\big(w_{\zeta_1}(x_j), w_{\zeta_2}(x_j)\big) \\ + \max\big(\vartheta_1(x_j), \vartheta_2(x_j)\big) + \max\big(w_{\vartheta_1}(x_j), w_{\vartheta_2}(x_j)\big) \end{pmatrix}}\right)$$

Thus $\mathcal{D}_1(\mathcal{C}_1, \mathcal{C}_2) = 1 - \mathcal{S}_1'(\mathcal{C}_1, \mathcal{C}_2)$. Similarly, we can prove that $\mathcal{D}_r(\mathcal{C}_1, \mathcal{C}_2) = 1 - \mathcal{S}_r'(\mathcal{C}_1, \mathcal{C}_2)$ for $r = 2, 3, \ldots 10$.

Theorem 5.2.12. Let $\mathcal{S}'(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{D}(\mathcal{C}_1, \mathcal{C}_2)$ denotes the similarity and distance measures respectively for $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$. Then, $\mathcal{E}n(\mathcal{C}_1) = \mathcal{S}'(\mathcal{C}_1, \mathcal{C}_1^c) = 1 - \mathcal{D}(\mathcal{C}_1, \mathcal{C}_1^c)$ is the entropy measure for CIFS $\mathcal{C}_1$.

Proof. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are CIFSs.

(i) As $0 \leq \mathcal{S}'(\mathcal{C}_1, \mathcal{C}_2) \leq 1$. Therefore, $0 \leq \mathcal{E}n(\mathcal{C}_1) \leq 1$.

(ii) If $\mathcal{C}_1$ is crisp set then, $\mathcal{S}'(\mathcal{C}_1, \mathcal{C}_1^c) = 0 \Rightarrow \mathcal{E}n(\mathcal{C}_1) = 0$.

(iii) $\mathcal{E}n(\mathcal{C}_1) = 1 \Leftrightarrow \mathcal{S}'(\mathcal{C}_1, \mathcal{C}_1^c) = 1 \Leftrightarrow \mathcal{C}_1 = \mathcal{C}_1^c \Leftrightarrow \zeta_1(x_j) = \vartheta_1(x_j)$ and $w_{\zeta_1}(x_j) = w_{\vartheta_1}(x_j)$

(iv) As $\mathcal{E}n(\mathcal{C}_1) = \mathcal{S}'(\mathcal{C}_1, \mathcal{C}_1^c)$ therefore, $\mathcal{E}n(\mathcal{C}_1) = \mathcal{E}n(\mathcal{C}_1^c)$.

(v) If $\mathcal{C}_1 \subseteq \mathcal{C}_2$ with $\zeta_2(x_j) \leq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \leq w_{\vartheta_2}(x_j)$ then, it implies that $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_2^c \subseteq \mathcal{C}_1^c$ which gives that $\mathcal{S}'(\mathcal{C}_1, \mathcal{C}_1^c) \leq \mathcal{S}'(\mathcal{C}_2, \mathcal{C}_1^c) \leq \mathcal{S}'(\mathcal{C}_2, \mathcal{C}_2^c)$. Hence, $\mathcal{E}n(\mathcal{C}_1) \leq \mathcal{E}n(\mathcal{C}_2)$. Similarly, if $\mathcal{C}_2 \subseteq \mathcal{C}_1$ with $\zeta_2(x_j) \geq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \geq w_{\vartheta_2}(x_j)$ then, we can obtain $\mathcal{S}'(\mathcal{C}_1^c, \mathcal{C}_1) \leq \mathcal{S}'(\mathcal{C}_2^c, \mathcal{C}_1) \leq \mathcal{S}'(\mathcal{C}_2^c, \mathcal{C}_2)$. Hence, $\mathcal{E}n(\mathcal{C}_1) \leq \mathcal{E}n(\mathcal{C}_2)$.

□

For validating the Theorem 5.2.12, we show that similarity measure $\mathcal{S}_1'$ and entropy measure $\mathcal{E}n_1$ can be related with the relation $\mathcal{E}n_1(\mathcal{C}_1) = 1 - \mathcal{S}_1'(\mathcal{C}_1, \mathcal{C}_1^c)$.

Example 5.2.2. For $\mathcal{C}_1 \in \Psi(\mathcal{U})$, using similarity measure as given in Eq. (5.1) we have

$$\begin{aligned}
\mathcal{S}_1'(\mathcal{C}_1, \mathcal{C}_1^c) &= \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\begin{pmatrix}\min\big(\zeta_1(x_j), \vartheta_1(x_j)\big) + \min\big(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\big)\\ + \min\big(\vartheta_1(x_j), \zeta_1(x_j)\big) + \min\big(w_{\vartheta_1}(x_j), w_{\zeta_1}(x_j)\big)\end{pmatrix}}{\begin{pmatrix}\max\big(\zeta_1(x_j), \vartheta_1(x_j)\big) + \max\big(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\big)\\ + \max\big(\vartheta_1(x_j), \zeta_1(x_j)\big) + \max\big(w_{\vartheta_1}(x_j), w_{\zeta_1}(x_j)\big)\end{pmatrix}}\right)\\
&= \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\min\big(\zeta_1(x_j), \vartheta_1(x_j)\big) + \min\big(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\big)}{\max\big(\zeta_1(x_j), \vartheta_1(x_j)\big) + \max\big(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\big)}\right)\\
&= \mathcal{E}n_1(\mathcal{C}_1)
\end{aligned}$$

Thus $\mathcal{E}n_1(\mathcal{C}_1) = \mathcal{S}_1'(\mathcal{C}_1, \mathcal{C}_1^c)$. In the similar manner, we can obtain that $\mathcal{E}n_r(\mathcal{C}_1) = \mathcal{S}_r'(\mathcal{C}_1, \mathcal{C}_1^c)$ for $r = 2, 3, \ldots, 10$.

Theorem 5.2.13. Let $\mathcal{S}'(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{D}(\mathcal{C}_1, \mathcal{C}_2)$ denote the similarity and distance measures respectively for $\mathcal{C}_1, \mathcal{C}_2 \in \Psi(\mathcal{U})$. Then, $\mathcal{I}n(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{S}'(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{C}_2) = 1 - \mathcal{D}(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{C}_2)$ is the inclusion measure for CIFSs $\mathcal{C}_1$ and $\mathcal{C}_2$.

Proof. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are CIFSs.

(P1) Since, $0 \le \mathcal{S}'(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{C}_2) \le 1$ therefore, $0 \le \mathcal{I}n(\mathcal{C}_1, \mathcal{C}_2) \le 1$.

(P2) $\mathcal{I}n(\mathcal{C}_1, \mathcal{C}_2) = 1 \Leftrightarrow \mathcal{S}'(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{C}_2) = 1 \Leftrightarrow \mathcal{C}_1 = \mathcal{C}_1 \cap \mathcal{C}_2 \Leftrightarrow \mathcal{C}_1 \subseteq \mathcal{C}_2$.

(P3) $\mathcal{I}n(\mathcal{U}, \phi) = \mathcal{S}'(\mathcal{U}, \mathcal{U} \cap \phi) = \mathcal{S}'(\mathcal{U}, \phi) = 0$.

(P4) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$ then, $\mathcal{S}'(\mathcal{C}_1, \mathcal{C}_3) \le \mathcal{S}'(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{S}'(\mathcal{C}_1, \mathcal{C}_3) \le \mathcal{S}'(\mathcal{C}_2, \mathcal{C}_3)$. It gives that $\mathcal{I}n(\mathcal{C}_3, \mathcal{C}_1) = \mathcal{S}'(\mathcal{C}_3, \mathcal{C}_3 \cap \mathcal{C}_1) = \mathcal{S}'(\mathcal{C}_3, \mathcal{C}_1) \le \mathcal{S}'(\mathcal{C}_3, \mathcal{C}_2) = \mathcal{S}'(\mathcal{C}_3, \mathcal{C}_3 \cap \mathcal{C}_2) = \mathcal{I}n(\mathcal{C}_3, \mathcal{C}_2)$. Thus, $\mathcal{I}n(\mathcal{C}_3, \mathcal{C}_1) \le \mathcal{I}n(\mathcal{C}_3, \mathcal{C}_2)$. Similarly, we can prove that $\mathcal{I}n(\mathcal{C}_3, \mathcal{C}_1) \le \mathcal{I}n(\mathcal{C}_2, \mathcal{C}_1)$.

□

In order to validate the Theorem 5.2.13, we show that similarity measure $\mathcal{S}_1'$ and inclusion measure $\mathcal{I}n_1$ can be related with the relation $\mathcal{I}n_1(\mathcal{C}_1,\mathcal{C}_2)=\mathcal{S}_1'(\mathcal{C}_1,\mathcal{C}_1\cap\mathcal{C}_2)$.

Example 5.2.3. For $\mathcal{C}_1,\mathcal{C}_2\in\Psi(\mathcal{U})$, using similarity measure as defined in Eq. (5.1) we have

$$\begin{aligned}
&\mathcal{S}_1'(\mathcal{C}_1,\mathcal{C}_1\cap\mathcal{C}_2)\\
=\;&\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\begin{pmatrix}\min\Big(\zeta_1(x_j),\min\big(\zeta_1(x_j),\zeta_2(x_j)\big)\Big)+\min\Big(w_{\zeta_1}(x_j),\min\big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\big)\Big)\\+\min\Big(\vartheta_1(x_j),\max\big(\vartheta_1(x_j),\vartheta_2(x_j)\big)\Big)+\min\Big(w_{\vartheta_1}(x_j),\max\big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\big)\Big)\end{pmatrix}}{\begin{pmatrix}\max\Big(\zeta_1(x_j),\min\big(\zeta_1(x_j),\zeta_2(x_j)\big)\Big)+\max\Big(w_{\zeta_1}(x_j),\min\big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\big)\Big)\\+\max\Big(\vartheta_1(x_j),\max\big(\vartheta_1(x_j),\vartheta_2(x_j)\big)\Big)+\max\Big(w_{\vartheta_1}(x_j),\max\big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\big)\Big)\end{pmatrix}}\right)\\
=\;&\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\min\big(\zeta_1(x_j),\zeta_2(x_j)\big)+\vartheta_1(x_j)+\Big(\min\big(w_{\zeta_1}(x_j),w_{\zeta_2}(x_j)\big)+w_{\vartheta_1}(x_j)\Big)}{\zeta_1(x_j)+\max\big(\vartheta_1(x_j),\vartheta_2(x_j)\big)+\Big(w_{\zeta_1}(x_j)+\max\big(w_{\vartheta_1}(x_j),w_{\vartheta_2}(x_j)\big)\Big)}\right)\\
=\;&\mathcal{I}n_1(\mathcal{C}_1,\mathcal{C}_2)
\end{aligned}$$

Thus, $\mathcal{I}n_1(\mathcal{C}_1,\mathcal{C}_2)=\mathcal{S}_1'(\mathcal{C}_1,\mathcal{C}_1\cap\mathcal{C}_2)$. Similarly, we can obtain that $\mathcal{I}n_r(\mathcal{C}_1,\mathcal{C}_2)=\mathcal{S}_r'(\mathcal{C}_1,\mathcal{C}_1\cap\mathcal{C}_2)$ for $r=2,3,\ldots,7$.

Theorem 5.2.14. Let $\mathcal{I}n(\mathcal{C}_1,\mathcal{C}_2)$ denotes the inclusion measure for $\mathcal{C}_1,\mathcal{C}_2\in\Psi(\mathcal{U})$. Then, $\mathcal{E}n(\mathcal{C}_1)=\mathcal{I}n(\mathcal{C}_1\cup\mathcal{C}_1^c,\mathcal{C}_1\cap\mathcal{C}_1^c)$ is the entropy measure for CIFS $\mathcal{C}_1$.

Proof. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are CIFSs.

(P1) Since $0\le\mathcal{I}n(\mathcal{C}_1\cup\mathcal{C}_1^c,\mathcal{C}_1\cap\mathcal{C}_1^c)\le1$. Therefore, $0\le\mathcal{E}n(\mathcal{C}_1)\le1$.

(P2) If $\mathcal{C}_1$ is crisp set then, $\mathcal{C}_1\cup\mathcal{C}_1^c=\mathcal{U}$ and $\mathcal{C}_1\cap\mathcal{C}_1^c=\phi$ which implies that $\mathcal{I}n(\mathcal{C}_1\cup\mathcal{C}_1^c,\mathcal{C}_1\cap\mathcal{C}_1^c)=0\Rightarrow\mathcal{E}n(\mathcal{C}_1)=0$.

(P3) $\mathcal{E}n(\mathcal{C}_1)=1\Leftrightarrow\mathcal{I}n(\mathcal{C}_1\cup\mathcal{C}_1^c,\mathcal{C}_1\cap\mathcal{C}_1^c)=1\Leftrightarrow\mathcal{C}_1\cup\mathcal{C}_1^c\subseteq\mathcal{C}_1\cap\mathcal{C}_1^c\Leftrightarrow\mathcal{C}_1=\mathcal{C}_1^c\Leftrightarrow\zeta_1(x_j)=\vartheta_1(x_j)$ and $w_{\zeta_1}(x_j)=w_{\vartheta_1}(x_j)$

(P4) $\mathcal{E}n(\mathcal{C}_1)=\mathcal{I}n(\mathcal{C}_1\cup\mathcal{C}_1^c,\mathcal{C}_1\cap\mathcal{C}_1^c)=\mathcal{I}n(\mathcal{C}_1^c\cup\mathcal{C}_1,\mathcal{C}_1^c\cap\mathcal{C}_1)=\mathcal{E}n(\mathcal{C}_1^c)$.

(P5) If $\mathcal{C}_1\subseteq\mathcal{C}_2$ with $\zeta_2(x_j)\le\vartheta_2(x_j)$ and $w_{\zeta_2}(x_j)\le w_{\vartheta_2}(x_j)$ then, it implies that $\mathcal{C}_1\subseteq\mathcal{C}_2\subseteq\mathcal{C}_2^c\subseteq\mathcal{C}_1^c$ which gives that $\mathcal{I}n(\mathcal{C}_1^c,\mathcal{C}_1)\le\mathcal{I}n(\mathcal{C}_1^c,\mathcal{C}_2)\le\mathcal{I}n(\mathcal{C}_2^c,\mathcal{C}_2)$ which follows that $\mathcal{I}n(\mathcal{C}_1\cup\mathcal{C}_1^c,\mathcal{C}_1\cap\mathcal{C}_1^c)\le\mathcal{I}n(\mathcal{C}_2\cup\mathcal{C}_2^c,\mathcal{C}_2\cap\mathcal{C}_2^c)$. Hence, $\mathcal{E}n(\mathcal{C}_1)\le\mathcal{E}n(\mathcal{C}_2)$. Similarly, if $\mathcal{C}_2 \subseteq \mathcal{C}_1$ with $\zeta_2(x_j) \geq \vartheta_2(x_j)$ and $w_{\zeta_2}(x_j) \geq w_{\vartheta_2}(x_j)$ then, we can obtain $\mathcal{I}n(\mathcal{C}_1^c \cup \mathcal{C}_1, \mathcal{C}_1^c \cap \mathcal{C}_1) \leq \mathcal{I}n(\mathcal{C}_2^c \cup \mathcal{C}_2, \mathcal{C}_2^c \cap \mathcal{C}_2)$. Hence, $\mathcal{E}n(\mathcal{C}_1) \leq \mathcal{E}n(\mathcal{C}_2)$.

□

In order to validate the Theorem 5.2.14, we show that entropy measure $\mathcal{E}n_1$ and inclusion measure $\mathcal{I}n_1$ can be related with the relation $\mathcal{E}n_1(\mathcal{C}_1) = \mathcal{I}n_1(\mathcal{C}_1 \cup \mathcal{C}_1^c, \mathcal{C}_1 \cap \mathcal{C}_1^c)$.

Example 5.2.4. For $\mathcal{C}_1 \in \Psi(\mathcal{U})$, using inclusion measure $\mathcal{I}n_1$ as given in Eq. (5.25) , we have

$$
\begin{aligned}
& \mathcal{I}n_1(\mathcal{C}_1 \cup \mathcal{C}_1^c, \mathcal{C}_1 \cap \mathcal{C}_1^c) \\
= \ & \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\left(\begin{array}{c}\min\big(\zeta_1(x_j), \vartheta_1(x_j)\big) + \min\big(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\big) \\ + \min\big(\zeta_1(x_j), \vartheta_1(x_j)\big) + \min\big(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\big)\end{array}\right)}{\left(\begin{array}{c}\max\big(\zeta_1(x_j), \vartheta_1(x_j)\big) + \max\big(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\big) \\ + \max\big(\zeta_1(x_j), \vartheta_1(x_j)\big) + \max\big(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\big)\end{array}\right)}\right) \\
= \ & \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\min\big(\zeta_1(x_j), \vartheta_1(x_j)\big) + \min\big(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\big)}{\max\big(\zeta_1(x_j), \vartheta_1(x_j)\big) + \max\big(w_{\zeta_1}(x_j), w_{\vartheta_1}(x_j)\big)}\right) \\
= \ & \mathcal{E}n_1(\mathcal{C}_1)
\end{aligned}
$$

Thus, $\mathcal{E}n_1(\mathcal{C}_1) = \mathcal{I}n_1(\mathcal{C}_1 \cup \mathcal{C}_1^c, \mathcal{C}_1 \cap \mathcal{C}_1^c)$. In the similar manner, we can prove that $\mathcal{E}n_r(\mathcal{C}_1) = \mathcal{I}n_r(\mathcal{C}_1 \cup \mathcal{C}_1^c, \mathcal{C}_1 \cap \mathcal{C}_1^c)$ for $r = 2, 3, \ldots, 7$.

5.3 Applications of the proposed similarity measures

In this section, we show the superiority of proposed similarity measures by comparing their results with some of the existing approaches.

5.3.1 Comparative study

For demonstrating the superiority of the proposed similarity measure $\mathcal{S}_1'$, a comparative study between some prevailing similarity measures and proposed similarity measure $\mathcal{S}_1'$ is conducted. A list of some of the existing similarity measures is given in Table 5.1. We consider $\mathcal{U} = \{x_1\}$ and we shall denote any CIFS $\mathcal{V} = \big\{\big(x_1, \big(0.2, 0.4\big), \big(0.2, 0.1\big)\big)\big\}$

Table 5.1: Existing similarity measures

Authors	Prevailing similarity measures between IFSs $\mathcal{C}_1 = \Big\{\big(x, \zeta_1(x), \vartheta_1(x)\big) \mid x \in \mathcal{U}\Big\}$ and $\mathcal{C}_2 = \Big\{\big(x, \zeta_2(x), \vartheta_2(x)\big) \mid x \in \mathcal{U}\Big\}$
Boran and Akay [18]	$\mathcal{S}'_{BA}\left(\mathcal{C}_1, \mathcal{C}_2\right) = 1 - \left(\sum\limits_{j=1}^{n} \left\{ \begin{array}{c} \lvert t\big(\zeta_1(x_j) - \zeta_2(x_j)\big) - \big(\vartheta_1(x_j) - \vartheta_2(x_j)\big)\rvert \\ + \lvert t\big(\vartheta_1(x_j) - \vartheta_2(x_j)\big) - \big(\zeta_1(x_j) - \zeta_2(x_j)\big)\rvert \end{array} \right\}\right)^{\frac{1}{p}}$
Chen [25]	$\mathcal{S}'_{C}\left(\mathcal{C}_1, \mathcal{C}_2\right) = \frac{1}{n} \sum\limits_{j=1}^{n} \left\{1 - \frac{\left\lvert\big(\zeta_1(x_j) - \zeta_2(x_j)\big) - \big(\vartheta_1(x_j) - \vartheta_2(x_j)\big)\right\rvert}{2}\right\}$
Hung and Yang [86]	$\mathcal{S}'_{HY1}\left(\mathcal{C}_1, \mathcal{C}_2\right) = 1 - \frac{\sum\limits_{j=1}^{n}\left\{\max\Big(\lvert\zeta_1(x_j) - \zeta_2(x_j)\rvert, \lvert\vartheta_1(x_j) - \vartheta_2(x_j)\rvert\Big)\right\}}{n}$, $\mathcal{S}'_{HY2}\left(\mathcal{C}_1, \mathcal{C}_2\right) = \frac{e^{\mathcal{S}'_{HY1}(\mathcal{C}_1, \mathcal{C}_2) - 1} - e^{-1}}{1 - e^{-1}}$, $\mathcal{S}'_{HY3}\left(\mathcal{C}_1, \mathcal{C}_2\right) = \frac{\mathcal{S}'_{HY1}(\mathcal{C}_1, \mathcal{C}_2)}{2 - \mathcal{S}'_{HY1}(\mathcal{C}_1, \mathcal{C}_2)}$
Dengfeng and Chuntian [36]	$\mathcal{S}'_{DC}\left(\mathcal{C}_1, \mathcal{C}_2\right) = 1 - \left(\frac{1}{n} \sum\limits_{j=1}^{n} \left\{\left\lvert\frac{\big(\zeta_1(x_j) - \vartheta_1(x_j)\big) - \big(\zeta_2(x_j) - \vartheta_2(x_j)\big)}{2}\right\rvert^{p}\right\}\right)^{\frac{1}{p}}$
Mitchell [114]	$\mathcal{S}'_{M}\left(\mathcal{C}_1, \mathcal{C}_2\right) = 1 - \frac{1}{2}\left(\frac{\sum\limits_{j=1}^{n}\{\lvert\zeta_1(x_j) - \zeta_2(x_j)\rvert^p + \lvert\vartheta_1(x_j) - \vartheta_2(x_j)\rvert^p\}}{n}\right)^{\frac{1}{p}}$
Ye [197]	$\mathcal{S}'_{Y}\left(\mathcal{C}_1, \mathcal{C}_2\right) = \frac{1}{n} \sum\limits_{j=1}^{n} \left\{\frac{\zeta_1(x_j)\zeta_2(x_j) + \vartheta_1(x_j)\vartheta_2(x_j)}{\sqrt{\zeta_1^2(x_j) + \vartheta_1^2(x_j)}\sqrt{\zeta_2^2(x_j) + \vartheta_2^2(x_j)}}\right\}$
Ye [200]	$\mathcal{S}'_{Y1}\left(\mathcal{C}_1, \mathcal{C}_2\right) = \dfrac{1}{n} \sum\limits_{j=1}^{n} \left\{\cos\left[\frac{\pi}{2}\left(\max\left\{\begin{array}{c} \lvert\zeta_1(x_j) - \zeta_2(x_j)\rvert, \lvert\vartheta_1(x_j) - \vartheta_2(x_j)\rvert, \\ \lvert h_P(x_j) - h_Q(x_j)\rvert \end{array}\right\}\right)\right]\right\}$
Hong and Kim [80]	$\mathcal{S}'_{HK}\left(\mathcal{C}_1, \mathcal{C}_2\right) = 1 - \frac{\sum\limits_{j=1}^{n}\left\{\lvert\zeta_1(x_j) - \zeta_2(x_j)\rvert + \lvert\vartheta_1(x_j) - \vartheta_2(x_j)\rvert\right\}}{2n}$
Liang and Shi [98]	$\mathcal{S}'_{LS}\left(\mathcal{C}_1, \mathcal{C}_2\right) = 1 - \frac{1}{n^{\frac{1}{p}}}\left(\sum\limits_{j=1}^{n}\left(\frac{\lvert\zeta_1(x_j) - \zeta_2(x_j)\rvert + \lvert\vartheta_1(x_j) - \vartheta_2(x_j)\rvert}{2}\right)^p\right)^{\frac{1}{p}}$
Xu [181]	$\mathcal{S}'_{X}\left(\mathcal{C}_1, \mathcal{C}_2\right) = 1 - \left(\frac{\sum\limits_{j=1}^{n}\left\{\lvert\zeta_1(x_j) - \zeta_2(x_j)\rvert^p + \lvert\vartheta_1(x_j) - \vartheta_2(x_j)\rvert^p + \lvert h_P(x_j) - h_Q(x_j)\rvert^p\right\}}{\sum\limits_{j=1}^{n}\left\{\lvert\zeta_1(x_j) + \zeta_2(x_j)\rvert^p + \lvert\vartheta_1(x_j) + \vartheta_2(x_j)\rvert^p + \lvert h_P(x_j) + h_Q(x_j)\rvert^p\right\}}\right)^{\frac{1}{p}}$

defined on $\mathcal{U}$ as $\mathcal{V} = \Big(\big(0.2, 0.4\big), \big(0.2, 0.1\big)\Big)$. Consider here five pairs of CIFSs as given in Table 5.2. In order to compare the performance of the proposed similarity measure with some of the existing measures [18, 25, 36, 80, 86, 98, 114, 181, 197, 200], we firstly convert CIFS information into IFSs by setting phase terms corresponding to each CIFS equal to zero. The results corresponding to these approaches along with the proposed one are summarized in Table 5.2. From the results of this table, it can be analyzed that the proposed similarity measure $\mathcal{S}'_1$ and the similarity measure $\mathcal{S}'_{BA}$ [18] can overcome the drawbacks of existing similarity measures [25, 36, 80, 86, 98, 114, 181, 197, 200].

However, similarity measure $\mathcal{S}_{BA}$ [18] also has shortcoming of giving unreasonable results in some special cases which are shown in Table 5.3. In this table, another five pairs of CIFSs are taken to compare the results of the proposed similarity measure with the existing measures [18, 25, 36, 80, 86, 98, 114, 181, 197, 200]. From the results tabulated in Table 5.3, it can be analyzed that the proposed similarity measure $\mathcal{S}_1$ can overcome the drawbacks of existing similarity measures [18, 25, 36, 80, 86, 98, 114, 181, 197, 200].

Table 5.2: Comparative study results of similarity measures

	Case 1	Case 2	Case 3	Case 4	Case 5
$\mathcal{V}_1$	$((0.2, 0.4), (0.2, 0.1))$	$((0.4, 0.8), (0.2, 0.1))$	$((1.0, 0.7), (0.0, 0.1))$	$((0.2, 0.5), (0.2, 0.4))$	$((0.6, 0.6), (0.1, 0.2))$
$\mathcal{V}_2$	$((0.3, 0.8), (0.3, 0.1))$	$((0.2, 0.7), (0.4, 0.2))$	$((0.0, 0.9), (0.0, 0.0))$	$((0.0, 0.8), (0.0, 0.2))$	$((0.7, 0.9), (0.1, 0.1))$
$\mathcal{S}'_{BA}$[18]	0.9667	0.8000	0.5000	0.9333	0.9500
$\mathcal{S}'_{C}$[25]	**1.0000**	0.8000	0.5000	**1.0000**	0.9500
$\mathcal{S}'_{HY1}$[86]	**0.9000**	**0.8000**	0.0000	**0.8000**	**0.9000**
$\mathcal{S}'_{HY2}$[86]	**0.8495**	**0.7132**	0.0000	**0.7132**	**0.8495**
$\mathcal{S}'_{HY3}$[86]	**0.8182**	**0.6667**	0.0000	**0.6667**	**0.8182**
$\mathcal{S}'_{DC}$[36]	**1.0000**	0.8000	0.5000	**1.0000**	0.9500
$\mathcal{S}'_{M}$[114]	0.9000	**0.8000**	0.5000	**0.8000**	0.9500
$\mathcal{S}'_{Y}$[197]	1.0000	0.8000	**NaN**	**NaN**	0.9997
$\mathcal{S}'_{Y1}$[200]	**0.9511**	**0.9511**	0.0000	**0.9511**	0.9877
$\mathcal{S}'_{HK}$[80]	0.9000	**0.8000**	0.5000	**0.8000**	0.9500
$\mathcal{S}'_{LS}$[98]	0.9000	**0.8000**	0.5000	**0.8000**	0.9500
$\mathcal{S}'_{X}$[181]	**0.8000**	**0.8000**	0.0000	0.6000	0.9000
$\mathcal{S}'_1$ (Proposed)	0.6000	0.6667	0.3500	0.4375	0.7368

Bold denotes unreasonable results and NaN denotes that similarity cannot be computed due to division by zero problem

Here: $p = 1, t = 2$ in $\mathcal{S}'_{BA}$ and $p = 1$ in $\mathcal{S}'_{DC}, \mathcal{S}'_M, \mathcal{S}'_{LS}, \mathcal{S}'_X$

Table 5.3: Comparative study results of similarity measures

	Case 1	Case 2	Case 3	Case 4	Case 5
$\mathcal{V}_1$	$((0.7, 0.2), (0.3, 0.4))$	$((0.0, 0.1), (0.77, 0.5))$	$((0.5, 0.3), (0.17, 0.1))$	$((0.125, 0.1), (0.075, 0.2))$	$((0.4, 0.3), (0.35, 0.2))$
$\mathcal{V}_2$	$((0.0, 0.1), (0.0, 0.3))$	$((0.18, 0.4), (0.45, 0.3))$	$((0.18, 0.3), (0.45, 0.3))$	$((0.175, 0.2), (0.025, 0.1))$	$((0.45, 0.4), (0.3, 0.2))$
$\mathcal{S}'_{BA}$[18]	0.8000	0.7500	0.7000	**0.9500**	**0.9500**
$\mathcal{S}'_{C}$[25]	0.8000	0.7500	0.7000	**0.9500**	**0.9500**
$\mathcal{S}'_{HY1}$[86]	0.3000	**0.6800**	**0.6800**	**0.9500**	**0.9500**
$\mathcal{S}'_{HY2}$[86]	0.2036	**0.5668**	**0.5668**	**0.9228**	**0.9228**
$\mathcal{S}'_{HY3}$[86]	0.1765	**0.5152**	**0.5152**	**0.9048**	**0.9048**
$\mathcal{S}'_{DC}$[36]	0.8000	0.7500	0.7000	**0.9500**	**0.9500**
$\mathcal{S}'_{M}$[114]	0.5000	0.7500	0.7000	**0.9500**	**0.9500**
$\mathcal{S}'_{Y}$[197]	**NaN**	0.9285	0.6505	0.9216	0.9915
$\mathcal{S}'_{Y1}$[200]	0.4540	**0.8763**	**0.8763**	**0.9969**	**0.9969**
$\mathcal{S}'_{HK}$[80]	0.5000	0.7500	0.7000	**0.9500**	**0.9500**
$\mathcal{S}'_{LS}$[98]	0.5000	0.7500	0.7000	**0.9500**	**0.9500**
$\mathcal{S}'_{X}$[181]	0.0000	**0.6800**	**0.6800**	**0.9500**	**0.9500**
$\mathcal{S}'_1$ (Proposed)	0.2500	0.4595	0.4839	0.5385	0.8571

Bold denotes unreasonable results and NaN denotes that similarity cannot be computed due to division by zero problem

Here, $p = 1, t = 2$ in S_{BA} and $p = 1$ in $\mathcal{S}'_{DC}, \mathcal{S}'_M, \mathcal{S}'_{LS}, \mathcal{S}'_X$

5.3.2 Pattern recognition problem

Example 5.3.1. Consider four known patterns $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$ which are represented by the following CIFSs in the universe of discourse $\mathcal{U} = \{x_1, x_2, x_3\}$ as:

$$\begin{aligned}
\mathcal{V}_1 &= \Big\{\Big(x_1, (0.2, 0.3), (0.2, 0.1)\Big), \Big(x_2, (0.5, 0.4), (0.0, 0.1)\Big), \Big(x_3, (0.1, 0.3), (0.5, 0.4)\Big)\Big\}\\
\mathcal{V}_2 &= \Big\{\Big(x_1, (0.4, 0.2), (0.2, 0.2)\Big), \Big(x_2, (0.7, 0.8), (0.0, 0.1)\Big), \Big(x_3, (0.1, 0.1), (0.5, 0.3)\Big)\Big\}\\
\mathcal{V}_3 &= \Big\{\Big(x_1, (0.4, 0.5), (0.2, 0.1)\Big), \Big(x_2, (0.5, 0.7), (0.0, 0.2)\Big), \Big(x_3, (0.1, 0.2), (0.5, 0.3)\Big)\Big\}\\
\mathcal{V}_4 &= \Big\{\Big(x_1, (0.6, 0.4), (0.3, 0.1)\Big), \Big(x_2, (0.4, 0.3), (0.0, 0.1)\Big), \Big(x_3, (0.1, 0.3), (0.5, 0.4)\Big)\Big\}
\end{aligned}$$

Consider the unknown pattern $\mathcal{V}$ on $\mathcal{U}$ as $\mathcal{V} = \{\big(x_1, \big(0.3, 0.5\big), \big(0.2, 0.2\big)\big), \big(x_2, \big(0.6, 0.3\big), \big(0.0, 0.2\big)\big), \big(x_3, \big(0.2, 0.1\big), \big(0.5, 0.4\big)\big)\}$.

The target of the problem is to find out to which class $\mathcal{V}$ belongs to. For this, the proposed similarity measure $\mathcal{S}_1'$ is computed between $\mathcal{V}$ and $\mathcal{V}_u$ ($u = 1, 2, 3, 4$) and the results are obtained as: $\mathcal{S}_1'(\mathcal{V}_1, \mathcal{V}) = 0.7341$, $\mathcal{S}_1'(\mathcal{V}_2, \mathcal{V}) = 0.7046$, $\mathcal{S}_1'(\mathcal{V}_3, \mathcal{V}) = 0.7607$ and $\mathcal{S}_1'(\mathcal{V}_4, \mathcal{V}) = 0.7127$. Hence, the unknown pattern $\mathcal{V}$ is classified with pattern $\mathcal{V}_3$. Further, a comparative study of the classification result of the computed similarity measure $\mathcal{S}_1'$ is conducted with existing similarity measures [18, 25, 36, 80, 86, 98, 114, 181, 197, 200] and the results are tabulated in Table 5.4. From the results of this table, it reveals that prevailing similarity measures [18, 25, 36, 80, 86, 98, 114, 181, 197, 200] cannot classify the unknown pattern. Hence, the proposed similarity measure $\mathcal{S}_1'$ can overcome the drawbacks of existing similarity measures [18, 25, 36, 80, 86, 98, 114, 181, 197, 200].

5.4 Decision-making algorithm based on proposed measures

Now, based on proposed information measures, multi-criteria decision-making methods are presented for evaluating the available alternatives under CIFS environment followed by an illustrative example.

5.4.1 Description of MCDM problem

Assume that, a set of different alternatives $\{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m\}$ characterized by another set of n criteria $\{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\}$ is to be evaluated by an expert who gave their preferences in terms of the CIFSs as follows $\mathcal{V}_u = \{(\mathfrak{B}_v, (\zeta_{uv}, w_{\zeta_{uv}}), (\vartheta_{uv}, w_{\vartheta_{uv}})) \mid v = 1, 2, \ldots, n\}$; $u = 1, 2, \ldots, m$. For convenience, the preferences of alternative $\mathcal{V}_u$ for criteria $\mathfrak{B}_v$ are represented as: $\mathcal{C}_{uv} = ((\zeta_{uv}, w_{\zeta_{uv}}), (\vartheta_{uv}, w_{\vartheta_{uv}}))$ where $u = 1, 2, \ldots, m$; $v = 1, 2, \ldots n$ such that $0 \leq \zeta_{uv}, \vartheta_{uv} \leq 1$, $0 \leq \zeta_{uv} + \vartheta_{uv} \leq 1$ and $0 \leq w_{\zeta_{uv}}, w_{\vartheta_{uv}} \leq 1$, $0 \leq w_{\zeta_{uv}} + w_{\vartheta_{uv}} \leq 1$. This information related to all alternatives for the different criteria can be represented in terms of CIF decision-matrix, $\mathcal{M} = (\mathcal{C}_{uv})_{m\times n}$. The target of the expert is to choose the most optimal alternative among the feasible ones.

5.4.2 Determination of ideal alternatives

Ideal alternative (IA) is the most desirable alternative which can be taken as $\mathcal{V}^* = \{(\mathfrak{B}_v, (1, 1), (0, 0)) \mid v = 1, 2, \ldots, n\}$. Furthermore, instead of fixing IA, it can be computed from the CIF decision matrix $\mathcal{M}$ as:

$$\mathcal{V}^* = \{(\mathfrak{B}_v, (\zeta_v, w_{\zeta_v}), (\vartheta_v, w_{\vartheta_v})) \mid v = 1, 2, \ldots, n\} \tag{5.32}$$

where $\zeta_v = \max\limits_u \{\zeta_{uv}\}$; $\vartheta_v = \min\limits_u \{\vartheta_{uv}\}$; $w_{\zeta_v} = \max\limits_u \{w_{\zeta_{uv}}\}$ and $w_{\vartheta_v} = \min\limits_u \{w_{\vartheta_{uv}}\}$ $\forall$ $v = 1, 2, \ldots, n$.

Also, the expert may provide preferences of the IA in accordance with the problem.

5.4.3 Algorithms based on proposed measures

To determine the most desirable alternative(s), we present the following Algorithms 5.1, 5.2, 5.3 and 5.4 based on proposed measures.

Algorithm 5.1 Algorithm based on similarity measure

1: Input CIF decision-matrix $\mathcal{M} = (\mathcal{C}_{uv})_{m\times n}$.
2: Identify IA either by using Eq. (5.32) or depending upon the information given in the problem.
3: Compute the similarity measure $\mathcal{S}'_r(\mathcal{V}_u, \mathcal{V}^*)$ for $r \in \{1, 2, \ldots 10\}$.
4: Determine the ranking order of the alternatives according to the decreasing values of $\mathcal{S}'_r(\mathcal{V}_u, \mathcal{V}^*)$.

To demonstrate the working of the proposed algorithm, we illustrate an example here.

Example 5.4.1. Suppose an entrepreneur decides to purchase new machine for his company from a machine maker X. The machine maker provides information on five models

Algorithm 5.2 Algorithm based on distance measure

1: It is similar to steps 1-2 in Algorithm 5.1.
2: Compute the distance measure $\mathcal{D}_r(\mathcal{V}_u, \mathcal{V}^*)$ for $r \in \{1, 2, \ldots 10\}$.
3: Determine the ranking order of the alternatives according to the increasing values of $\mathcal{D}_r(\mathcal{V}_u, \mathcal{V}^*)$.

Algorithm 5.3 Algorithm based on entropy measure

1: It is similar to step 1 in Algorithm 5.1.
2: Compute the entropy measure $\mathcal{E}n_r(\mathcal{V}_u)$ for $r \in \{1, 2, \ldots 10\}$.
3: Determine the ranking order of the alternatives according to the increasing values of $\mathcal{E}n_r(\mathcal{V}_u)$.

Algorithm 5.4 Algorithm based on inclusion measure

1: It is similar to steps 1-2 in Algorithm 5.1.
2: Compute the inclusion measure $\mathcal{I}n_r(\mathcal{V}_u, \mathcal{V}^*)$ for $r \in \{1, 2, \ldots 7\}$.
3: Determine the ranking order of the alternatives according to the decreasing values of $\mathcal{I}n_r(\mathcal{V}_u, \mathcal{V}^*)$.

of machine $\mathcal{V}_u$ ($u = 1, 2 \ldots, 5$) with different production dates for each model. The entrepreneur decides to consider four criteria namely $\mathfrak{B}_1$: Reliability, $\mathfrak{B}_2$: Safety, $\mathfrak{B}_3$: Flexibility and $\mathfrak{B}_4$: Productivity for selecting machine. According to the changes in production date for the same model of machines, these criteria will also affect and change. The purpose of the entrepreneur is to select the best model of machine and its production date simultaneously. Thus the problem is two dimensional here namely: (i) Model of machine (ii) Production date of machine. To fulfil this purpose, he consults an expert who evaluates the available alternatives and gives their preferences in terms of CIFSs. The main procedure steps, for fulfilling the required purpose, by using the proposed Algorithm 5.1 are elaborated as:

Step 1: The expert has given their preferences to each alternative with respect to the four major criteria in the form of CIFSs which are summarized in Table 5.5.

Step 2: The preferences of IA are given by expert as:

$$\mathcal{V}^* = \left\{ \begin{array}{c} \Big(\mathfrak{B}_1, (0.7, 0.8), (0.1, 0.2)\Big), \Big(\mathfrak{B}_2, (0.9, 0.8), (0.1, 0.1)\Big), \\ \Big(\mathfrak{B}_3, (0.8, 0.9), (0.2, 0.1)\Big), \Big(\mathfrak{B}_4, (0.8, 0.8), (0.1, 0.2)\Big) \end{array} \right\}$$

Step 3: The obtained similarity measures values $\mathcal{S}'_r(\mathcal{V}_u, \mathcal{V}^*)$ are tabulated in Table 5.6.

Step 4: Ranking order of the alternatives is tabulated in last column of Table 5.6.

5.5 Clustering algorithm based on proposed measures

In this section, we propose a clustering algorithm under CIF environment.

Definition 5.5.1. For a collection of CIFSs, $\mathcal{V}_u$ $(u = 1, 2, \ldots, m)$, a matrix $\mathscr{C} = (c_{uv})_{m\times m}$, is called a similarity matrix where $c_{uv} = \mathcal{S}'(\mathcal{V}_u, \mathcal{V}_v)$ denotes the similarity measure between CIFSs $\mathcal{V}_u$ and $\mathcal{V}_v$ and the matrix $\mathscr{C}$ satisfies:

(i) $0 \leq c_{uv} \leq 1$.

(ii) $c_{uu} = 1$.

(iii) $c_{uv} = c_{vu}$ where $u, v = 1, 2, \ldots, m$.

Definition 5.5.2. [183] For a similarity matrix $\mathscr{C}$, the composition matrix $\mathscr{C}^2$ is defined as: $\mathscr{C}^2 = \mathscr{C} o \mathscr{C} = (\bar{c}_{uv})_{m\times m}$ where $\bar{c}_{uv} = \max\limits_{y}\big(\min(c_{uy}, c_{yv})\big)$

Definition 5.5.3. [183] For a similarity matrix $\mathscr{C}$, if $\mathscr{C}^2 \subseteq \mathscr{C}$ i.e. $\max\limits_{y}\big(\min(c_{uy}, c_{yv}\big) \leq c_{uv}$ $\forall\ u, v = 1, 2, \ldots, m$ then $\mathscr{C}^2$ is called an equivalent similarity matrix.

Theorem 5.5.1. [183] For a similarity matrix $\mathscr{C}$, the composition matrix $\mathscr{C}^2 = \mathscr{C} o \mathscr{C} = (\bar{c}_{uv})_{m\times m}$ is also a similarity matrix.

Theorem 5.5.2. [183] Let $\mathscr{C} = (c_{uv})_{m\times m}$ be a similarity matrix. Then, after the finite times of compositions: $\mathscr{C} \to \mathscr{C}^2 \to \mathscr{C}^4 \to \ldots \to \mathscr{C}^{2^z} \to \ldots$, there must exist a positive integer z such that $\mathscr{C}^{2^z} = \mathscr{C}^{2^{z+1}}$ and $\mathscr{C}^{2^z}$ is also an equivalent similarity matrix.

Definition 5.5.4. [183] For an equivalent similarity matrix $\mathscr{C} = (c_{uv})_{m\times m}$, the matrix $\mathscr{C}_\lambda = (c^\lambda_{uv})_{m\times m}$ is called $\lambda-$cutting matrix of $\mathscr{C}$, where

$$c^\lambda_{uv} = \begin{cases} 1 & \text{if} \quad c_{uv} \geq \lambda \\ 0 & \text{if} \quad c_{uv} < \lambda \end{cases} \tag{5.33}$$

and $\lambda \in [0, 1]$ is the confidence level.

5.5.1 Proposed Clustering Algorithm

Consider m alternatives $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ which are characterized by n attributes $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$. Suppose that an expert assess these alternatives and give their preferences in terms of CIFSs. The target of the expert is to cluster these alternatives $\mathcal{V}_u$ $(u = 1, 2, \ldots, m)$. The steps of the algorithm for fulfilling the required purpose are summarized as:

Algorithm 5.5 Clustering Algorithm for CIFSs

1: Construct similarity matrix $\mathscr{C} = (c_{uv})_{m\times m}$ where $c_{uv} = \mathcal{S}_5'(\mathcal{V}_u, \mathcal{V}_v)$.
2: Construct equivalent similarity matrix $\mathscr{C}^{2^z}$, where z is a positive integer, until $\mathscr{C}^{2^z} = \mathscr{C}^{2^{z+1}}$.
3: Construct $\lambda-$cutting matrix $\mathscr{C}_\lambda = (c_{uv}^\lambda)_{m\times m}$, for confidence level λ, using Definition 5.5.4.
4: Classify $\mathcal{V}_u$ and $\mathcal{V}_v$ of same type if the corresponding elements of uth row(column) and vth row(column) of matrix $\mathscr{C}_\lambda$ are identical.

5.5.2 Illustrative Example

A regional space center namely, North-Eastern Space Applications Centre (NESAC), is an outcome of a collaborative initiative of Department of Science, Government of India with North-Eastern (NE) council. The main goal of NESAC is to explore innovative methods of utilizing the natural resources with the help of technically advanced tools and methods such as remote sensing. This ameliorates the scope of satellite services to be accessed by the NE states and as an outcome of this space research is promoted in these areas. Likewise, Unmanned Aerial Vehicle (UAV) remote sensing (RS) is a new technical advancement to NESAC for large scale mapping as well as application monitoring of various activities. UAV, also known as drone, is basically a flying robot. In more simple terms, it is just an aircraft which can either be remote controlled by a human operator or autonomously by an onboard computer. The main task of the NESAC is to assemble 10 new UAVs for large scale mapping in NE region. For this, NESAC wants to cluster ten data processing and analysis softwares $\mathcal{V}_u$ $(u = 1, 2, \ldots, 10)$, which are required for processing the images and videos captured by UAVs and these softwares should support latest version of the operating system. The softwares $\mathcal{V}_u$ have different software versions. NESAC consults an expert, who evaluate the softwares on the basis of four criteria namely $\mathfrak{B}_1$: image processing capability, $\mathfrak{B}_2$: Measurement tools for co-ordinate/distance/area/volume, $\mathfrak{B}_3$: Generation of contour lines using Digital elevation models (DEM)/Digital surface models (DSM) and $\mathfrak{B}_4$: Generation of 3D modeling/texturing capabilities. Obviously, these criteria would be affected with the changes in software version. The expert assesses softwares $\mathcal{V}_u$'s on the basis of the type of software and its version and recorded their preferences in terms of CIFSs given in Table 5.7, as the CIF model handles two-dimensional information simultaneously. The rating values for $\mathcal{V}_1$ at $\mathfrak{B}_1$ are given as $\Big((0.5, 0.4), (0.4, 0.5)\Big)$ which describes that the expert is 50% agreed with the suitability of $\mathcal{V}_1$ at $\mathfrak{B}_1$ and 40% disagrees. The phase term that represents the version of software is given as: the expert is 40% satisfied with the suitability of software version at $\mathfrak{B}_1$ and 50% is not satisfied. In the similar manner, all data of Table 5.7 can be interpreted. The main procedure steps, for fulfilling the required purpose, by using the proposed Algorithm 5.5 are elaborated as:

Step 1: Calculate $\mathcal{S}'_5(\mathcal{V}_u, \mathcal{V}_v)$ for $u, v = 1, 2, \ldots, 10$ and then construct similarity matrix:

$$\mathscr{C} = \begin{pmatrix}
1.0000 & 0.9000 & 0.8937 & 0.9500 & 0.9188 & 0.9250 & 0.9563 & 0.9312 & 0.9063 & 0.9500 \\
0.9000 & 1.0000 & 0.9187 & 0.9125 & 0.8813 & 0.9250 & 0.9312 & 0.9063 & 0.9437 & 0.8875 \\
0.8937 & 0.9187 & 1.0000 & 0.9187 & 0.8375 & 0.9063 & 0.9000 & 0.9250 & 0.9000 & 0.8938 \\
0.9500 & 0.9125 & 0.9187 & 1.0000 & 0.9188 & 0.9250 & 0.9563 & 0.9312 & 0.9188 & 0.9750 \\
0.9188 & 0.8813 & 0.8375 & 0.9188 & 1.0000 & 0.9063 & 0.9375 & 0.8875 & 0.9125 & 0.9437 \\
0.9250 & 0.9250 & 0.9063 & 0.9250 & 0.9063 & 1.0000 & 0.9187 & 0.9063 & 0.9562 & 0.9250 \\
0.9563 & 0.9312 & 0.9000 & 0.9563 & 0.9375 & 0.9187 & 1.0000 & 0.9125 & 0.9375 & 0.9437 \\
0.9312 & 0.9063 & 0.9250 & 0.9312 & 0.8875 & 0.9063 & 0.9125 & 1.0000 & 0.9000 & 0.9187 \\
0.9063 & 0.9437 & 0.9000 & 0.9188 & 0.9125 & 0.9562 & 0.9375 & 0.9000 & 1.0000 & 0.9312 \\
0.9500 & 0.8875 & 0.8938 & 0.9750 & 0.9437 & 0.9250 & 0.9437 & 0.9187 & 0.9312 & 1.0000
\end{pmatrix}$$

Step 2: Calculate

$$\mathscr{C}^2 = \mathscr{C} o \mathscr{C} = \begin{pmatrix}
1.0000 & 0.9312 & 0.9250 & 0.9563 & 0.9437 & 0.9250 & 0.9563 & 0.9312 & 0.9375 & 0.9500 \\
0.9312 & 1.0000 & 0.9187 & 0.9312 & 0.9312 & 0.9437 & 0.9375 & 0.9187 & 0.9437 & 0.9312 \\
0.9250 & 0.9187 & 1.0000 & 0.9250 & 0.9187 & 0.9187 & 0.9187 & 0.9250 & 0.9187 & 0.9187 \\
0.9563 & 0.9312 & 0.9250 & 1.0000 & 0.9437 & 0.9250 & 0.9563 & 0.9312 & 0.9375 & 0.9750 \\
0.9437 & 0.9312 & 0.9187 & 0.9437 & 1.0000 & 0.9250 & 0.9437 & 0.9188 & 0.9375 & 0.9437 \\
0.9250 & 0.9437 & 0.9187 & 0.9250 & 0.9250 & 1.0000 & 0.9375 & 0.9250 & 0.9562 & 0.9312 \\
0.9563 & 0.9375 & 0.9187 & 0.9563 & 0.9437 & 0.9375 & 1.0000 & 0.9312 & 0.9375 & 0.9563 \\
0.9312 & 0.9187 & 0.9250 & 0.9312 & 0.9188 & 0.9250 & 0.9312 & 1.0000 & 0.9188 & 0.9312 \\
0.9375 & 0.9437 & 0.9187 & 0.9375 & 0.9375 & 0.9562 & 0.9375 & 0.9188 & 1.0000 & 0.9375 \\
0.9500 & 0.9312 & 0.9187 & 0.9750 & 0.9437 & 0.9312 & 0.9563 & 0.9312 & 0.9375 & 1.0000
\end{pmatrix}$$

Since $\mathscr{C}^2 \neq \mathscr{C}$. Therefore, we compute $\mathscr{C}^4$.

$$\mathscr{C}^4 = \mathscr{C}^2 o \mathscr{C}^2 = \begin{pmatrix} 1.0000 & 0.9375 & 0.9250 & 0.9563 & 0.9437 & 0.9375 & 0.9563 & 0.9312 & 0.9375 & 0.9563 \\ 0.9375 & 1.0000 & 0.9250 & 0.9375 & 0.9375 & 0.9437 & 0.9375 & 0.9312 & 0.9437 & 0.9375 \\ 0.9250 & 0.9250 & 1.0000 & 0.9250 & 0.9250 & 0.9250 & 0.9250 & 0.9250 & 0.9250 & 0.9250 \\ 0.9563 & 0.9375 & 0.9250 & 1.0000 & 0.9437 & 0.9375 & 0.9563 & 0.9312 & 0.9375 & 0.9750 \\ 0.9437 & 0.9375 & 0.9250 & 0.9437 & 1.0000 & 0.9375 & 0.9437 & 0.9312 & 0.9375 & 0.9437 \\ 0.9375 & 0.9437 & 0.9250 & 0.9375 & 0.9375 & 1.0000 & 0.9375 & 0.9312 & 0.9562 & 0.9375 \\ 0.9563 & 0.9375 & 0.9250 & 0.9563 & 0.9437 & 0.9375 & 1.0000 & 0.9312 & 0.9375 & 0.9563 \\ 0.9312 & 0.9312 & 0.9250 & 0.9312 & 0.9312 & 0.9312 & 0.9312 & 1.0000 & 0.9312 & 0.9312 \\ 0.9375 & 0.9437 & 0.9250 & 0.9375 & 0.9375 & 0.9562 & 0.9375 & 0.9312 & 1.0000 & 0.9375 \\ 0.9563 & 0.9375 & 0.9250 & 0.9750 & 0.9437 & 0.9375 & 0.9563 & 0.9312 & 0.9375 & 1.0000 \end{pmatrix}$$

Since $\mathscr{C}^4 \neq \mathscr{C}^2$. Therefore, we compute $\mathscr{C}^8$.

$$\mathscr{C}^8 = \mathscr{C}^4 o \mathscr{C}^4 = \begin{pmatrix} 1.0000 & 0.9375 & 0.9250 & 0.9563 & 0.9437 & 0.9375 & 0.9563 & 0.9312 & 0.9375 & 0.9563 \\ 0.9375 & 1.0000 & 0.9250 & 0.9375 & 0.9375 & 0.9437 & 0.9375 & 0.9312 & 0.9437 & 0.9375 \\ 0.9250 & 0.9250 & 1.0000 & 0.9250 & 0.9250 & 0.9250 & 0.9250 & 0.9250 & 0.9250 & 0.9250 \\ 0.9563 & 0.9375 & 0.9250 & 1.0000 & 0.9437 & 0.9375 & 0.9563 & 0.9312 & 0.9375 & 0.9750 \\ 0.9437 & 0.9375 & 0.9250 & 0.9437 & 1.0000 & 0.9375 & 0.9437 & 0.9312 & 0.9375 & 0.9437 \\ 0.9375 & 0.9437 & 0.9250 & 0.9375 & 0.9375 & 1.0000 & 0.9375 & 0.9312 & 0.9562 & 0.9375 \\ 0.9563 & 0.9375 & 0.9250 & 0.9563 & 0.9437 & 0.9375 & 1.0000 & 0.9312 & 0.9375 & 0.9563 \\ 0.9312 & 0.9312 & 0.9250 & 0.9312 & 0.9312 & 0.9312 & 0.9312 & 1.0000 & 0.9312 & 0.9312 \\ 0.9375 & 0.9437 & 0.9250 & 0.9375 & 0.9375 & 0.9562 & 0.9375 & 0.9312 & 1.0000 & 0.9375 \\ 0.9563 & 0.9375 & 0.9250 & 0.9750 & 0.9437 & 0.9375 & 0.9563 & 0.9312 & 0.9375 & 1.0000 \end{pmatrix}$$

As $\mathscr{C}^8 = \mathscr{C}^4$. Therefore, $\mathscr{C}^8$ is an equivalent similarity matrix.

Step 3: Without loss of generality, considering confidence level $\lambda = 0.9375$, $\lambda-$ cutting matrix $\mathscr{C}_\lambda$ is obtained using Definition 5.5.4 as:

$$\mathscr{C}_\lambda = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \tag{5.34}$$

Step 4: From the matrix, given in Eq. (5.34), we observe that, alternatives $\mathcal{V}_u$ are classified into the following three types as:

$$\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_9, \mathcal{V}_{10}\}, \{\mathcal{V}_3\}, \{\mathcal{V}_8\}$$

Since different values of λ will give different $\lambda-$ cutting matrices and hence we will obtain different clustering results. Therefore, a detailed sensitivity analysis with respect to confidence level λ is given in Table 5.8. We choose the value of confidence level λ from the smallest one to the biggest one. By analyzing the obtained results, corresponding to different values of λ, we obtain that as the value of λ increases more and more patterns get differentiated. Besides this, for a particular cluster number, there is only one case. For example, if the softwares $\mathcal{V}_u$ are categorized into four types then, the obtained results can only be $\{\mathcal{V}_1, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_7, \mathcal{V}_{10}\}$, $\{\mathcal{V}_2, \mathcal{V}_6, \mathcal{V}_9\}$, $\{\mathcal{V}_3\}$, $\{\mathcal{V}_8\}$. This is useful in taking final decision as it reduces uncertainty in choosing λ.

Furthermore, the clustering distribution of ten softwares $\mathcal{V}_u$ is shown in Figure 5.1. This figure depicts that the softwares $\mathcal{V}_u$ are mainly divided into two trends which are: $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_8, \mathcal{V}_9, \mathcal{V}_{10}\}$, $\{\mathcal{V}_3\}$. Moreover, when the confidence level stays in the relaxed level, the overall trend can be judged by means of the Figure 5.1. As the above discussion reveals that with the change in the value of λ, the clustering results will also differ. This impact of confidence level λ on the clustering results makes our proposed clustering algorithm more flexible as the decision-maker(s) can choose the value of λ according to their preferences and practical situations.

5.6 Conclusion

The key contribution of this chapter is outlined as follows:

1) Keeping in view the superiority of CIFS model to handle time-periodic problems and the ability to represent two-dimensional information in a single set, in this chapter, an attempt has been made to present various information measures for CIFSs. Earlier, measures of information had been defined under the IFS environment where the membership and non-membership degrees can take real values with a corresponding range

set [0,1]. But, in CIF environment the range of membership and non-membership degrees is extended to unit disc and the concept of the phase angle, for representing the periodicity of the data, is considered.

2) Various information measures such as similarity, distance, entropy and inclusion measures under CIF environment have been proposed and their transformation relationships have been discussed in detail. The proposed measures can be utilized for handling IFS data as well by setting the phase term equal to zero. Therefore, the measures presented in this chapter are more generalized and can be applied on IFS as well as CIFS data.

3) A detailed comparative study has been conducted to justify the superiority of proposed measures and algorithms for handling the MCDM problems and clustering analysis have been developed and illustrated with the aid of examples. Thus, the proposed approach can be efficiently used for solving DM problems where uncertainties and vagueness in the data occur concurrently with changes to the phase (periodicity).

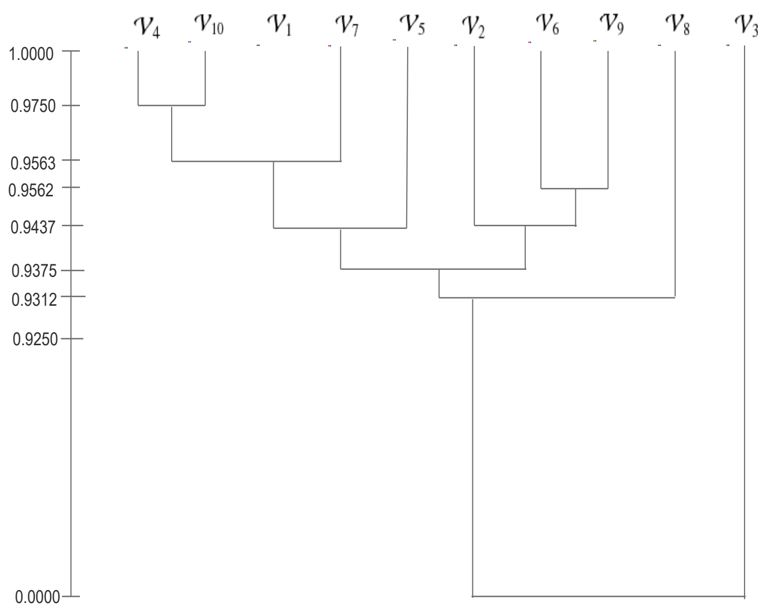

Figure 5.1: The clustering effect diagram of ten softwares $\mathcal{V}_u$

Table 5.4: Comparative study results of pattern recognition problem

	$\mathcal{S}'(\mathcal{V}_1, \mathcal{V})$	$\mathcal{S}'(\mathcal{V}_2, \mathcal{V})$	$\mathcal{S}'(\mathcal{V}_3, \mathcal{V})$	$\mathcal{S}'(\mathcal{V}_4, \mathcal{V})$	Classification
$\mathcal{S}'_{BA}$[18]	**0.9500**	**0.9500**	**0.9500**	0.9167	Cannot be recognized
$\mathcal{S}'_{C}$[25]	**0.9500**	**0.9500**	**0.9500**	0.9167	Cannot be recognized
$\mathcal{S}'_{HY1}$[86]	**0.9000**	**0.9000**	**0.9000**	0.8000	Cannot be recognized
$\mathcal{S}'_{HY2}$[86]	**0.8495**	**0.8495**	**0.8495**	0.7132	Cannot be recognized
$\mathcal{S}'_{HY3}$[86]	**0.8182**	**0.8182**	**0.8182**	0.6667	Cannot be recognized
$\mathcal{S}'_{DC}$[36]	**0.9500**	**0.9500**	**0.9500**	0.9167	Cannot be recognized
$\mathcal{S}'_{M}$[114]	**0.9500**	**0.9500**	**0.9500**	0.8833	Cannot be recognized
$\mathcal{S}'_{Y}$[197]	0.9880	**0.9919**	**0.9919**	**0.9919**	Cannot be recognized
$\mathcal{S}'_{Y1}$[200]	**0.9877**	**0.9877**	**0.9877**	0.9433	Cannot be recognized
$\mathcal{S}'_{HK}$[80]	**0.9500**	**0.9500**	**0.9500**	0.8833	Cannot be recognized
$\mathcal{S}'_{LS}$[98]	**0.9500**	**0.9500**	**0.9500**	0.8833	Cannot be recognized
$\mathcal{S}'_{X}$[181]	**0.9000**	**0.9000**	**0.9000**	0.7667	Cannot be recognized
$\mathcal{S}'_1$ (Proposed)	0.7341	0.7046	0.7607	0.7127	$\mathcal{V}_3$

Bold denotes unreasonable results

Table 5.5: Input information of machines in the form of the CIFNs

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.6, 0.7), (0.2, 0.1)\big)$	$\big((0.5, 0.7), (0.3, 0.2)\big)$	$\big((0.7, 0.5), (0.2, 0.3)\big)$	$\big((0.7, 0.5), (0.1, 0.4)\big)$
$\mathcal{V}_2$	$\big((0.7, 0.9), (0.3, 0.1)\big)$	$\big((0.7, 0.5), (0.3, 0.3)\big)$	$\big((0.6, 0.5), (0.4, 0.4)\big)$	$\big((0.4, 0.6), (0.5, 0.4)\big)$
$\mathcal{V}_3$	$\big((0.3, 0.6), (0.5, 0.3)\big)$	$\big((0.4, 0.4), (0.3, 0.5)\big)$	$\big((0.3, 0.4), (0.5, 0.3)\big)$	$\big((0.5, 0.5), (0.3, 0.1)\big)$
$\mathcal{V}_4$	$\big((0.4, 0.5), (0.3, 0.5)\big)$	$\big((0.5, 0.4), (0.3, 0.3)\big)$	$\big((0.6, 0.3), (0.3, 0.5)\big)$	$\big((0.8, 0.7), (0.2, 0.1)\big)$
$\mathcal{V}_5$	$\big((0.8, 0.7), (0.1, 0.2)\big)$	$\big((0.7, 0.5), (0.2, 0.3)\big)$	$\big((0.7, 0.7), (0.2, 0.1)\big)$	$\big((0.9, 0.8), (0.1, 0.1)\big)$

Table 5.6: Similarity measures values and ranking order of alternatives

Similarity measures	Similarity measures values of $\mathcal{V}^*$ and					Ranking order
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	
$\mathcal{S}'_1$	0.7055	0.6246	0.4733	0.5825	0.8203	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
$\mathcal{S}'_2$	0.7024	0.6170	0.4681	0.5714	0.8148	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
$\mathcal{S}'_3$	0.8156	0.7345	0.6304	0.7107	0.8914	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
$\mathcal{S}'_4$	0.8148	0.7353	0.6296	0.7111	0.8905	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
$\mathcal{S}'_5$	0.8438	0.7750	0.6875	0.7563	0.9063	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
$\mathcal{S}'_6$	0.7000	0.6250	0.5500	0.6000	0.8000	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
$\mathcal{S}'_7$	0.8252	0.7632	0.6377	0.7273	0.8980	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
$\mathcal{S}'_8$	0.9730	0.9508	0.9161	0.9377	0.9893	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
$\mathcal{S}'_9$	0.9475	0.9062	0.8453	0.8826	0.9787	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
$\mathcal{S}'_{10}$	0.9579	0.9241	0.8728	0.9044	0.9831	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$

Table 5.7: Input information of software models in the form of the complex intuitionistic fuzzy decision-matrix

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.5,0.4),(0.4,0.5)\big)$	$\big((0.6,0.5),(0.3,0.4)\big)$	$\big((0.8,0.6),(0.2,0.3)\big)$	$\big((0.7,0.5),(0.1,0.3)\big)$
$\mathcal{V}_2$	$\big((0.6,0.4),(0.3,0.5)\big)$	$\big((0.7,0.5),(0.3,0.3)\big)$	$\big((0.4,0.5),(0.4,0.3)\big)$	$\big((0.4,0.6),(0.5,0.2)\big)$
$\mathcal{V}_3$	$\big((0.3,0.6),(0.5,0.3)\big)$	$\big((0.4,0.4),(0.3,0.5)\big)$	$\big((0.3,0.4),(0.5,0.3)\big)$	$\big((0.5,0.5),(0.3,0.1)\big)$
$\mathcal{V}_4$	$\big((0.4,0.5),(0.3,0.5)\big)$	$\big((0.5,0.4),(0.3,0.3)\big)$	$\big((0.6,0.3),(0.3,0.5)\big)$	$\big((0.8,0.7),(0.2,0.1)\big)$
$\mathcal{V}_5$	$\big((0.8,0.9),(0.1,0.1)\big)$	$\big((0.8,0.1),(0.1,0.4)\big)$	$\big((0.6,0.6),(0.2,0.2)\big)$	$\big((0.8,0.7),(0.1,0.2)\big)$
$\mathcal{V}_6$	$\big((0.7,0.6),(0.3,0.3)\big)$	$\big((0.4,0.9),(0.2,0.1)\big)$	$\big((0.7,0.7),(0.2,0.3)\big)$	$\big((0.4,0.6),(0.3,0.1)\big)$
$\mathcal{V}_7$	$\big((0.6,0.6),(0.2,0.2)\big)$	$\big((0.6,0.6),(0.3,0.1)\big)$	$\big((0.5,0.8),(0.3,0.1)\big)$	$\big((0.7,0.7),(0.1,0.2)\big)$
$\mathcal{V}_8$	$\big((0.2,0.8),(0.5,0.1)\big)$	$\big((0.7,0.3),(0.3,0.3)\big)$	$\big((0.6,0.5),(0.1,0.1)\big)$	$\big((0.6,0.5),(0.3,0.4)\big)$
$\mathcal{V}_9$	$\big((0.6,0.5),(0.2,0.4)\big)$	$\big((0.5,0.3),(0.1,0.6)\big)$	$\big((0.5,0.4),(0.2,0.4)\big)$	$\big((0.4,0.3),(0.4,0.4)\big)$
$\mathcal{V}_{10}$	$\big((0.4,0.2),(0.3,0.1)\big)$	$\big((0.5,0.3),(0.1,0.1)\big)$	$\big((0.6,0.4),(0.2,0.3)\big)$	$\big((0.8,0.6),(0.1,0.2)\big)$

Table 5.8: Clustering results for different confidence levels

Class	Confidence level	Clustering results
8	$0.9750 < \lambda \leq 1.0000$	$\{\mathcal{V}_1\}$, $\{\mathcal{V}_2\}$, $\{\mathcal{V}_3\}$, $\{\mathcal{V}_4\}$, $\{\mathcal{V}_5\}$, $\{\mathcal{V}_6\}$, $\{\mathcal{V}_7\}$, $\{\mathcal{V}_8\}$, $\{\mathcal{V}_9\}$, $\{\mathcal{V}_{10}\}$
7	$0.9563 < \lambda \leq 0.9750$	$\{\mathcal{V}_1\}$, $\{\mathcal{V}_2\}$, $\{\mathcal{V}_3\}$, $\{\mathcal{V}_4, \mathcal{V}_{10}\}$, $\{\mathcal{V}_5\}$, $\{\mathcal{V}_6\}$, $\{\mathcal{V}_7\}$, $\{\mathcal{V}_8\}$, $\{\mathcal{V}_9\}$
6	$0.9562 < \lambda \leq 0.9563$	$\{\mathcal{V}_1, \mathcal{V}_4, \mathcal{V}_7, \mathcal{V}_{10}\}$, $\{\mathcal{V}_2\}$, $\{\mathcal{V}_3\}$, $\{\mathcal{V}_5\}$, $\{\mathcal{V}_6\}$, $\{\mathcal{V}_8\}$, $\{\mathcal{V}_9\}$
5	$0.9437 < \lambda \leq 0.9562$	$\{\mathcal{V}_1, \mathcal{V}_4, \mathcal{V}_7, \mathcal{V}_{10}\}$, $\{\mathcal{V}_2\}$, $\{\mathcal{V}_3\}$, $\{\mathcal{V}_5\}$, $\{\mathcal{V}_6, \mathcal{V}_9\}$ $\{\mathcal{V}_8\}$
4	$0.9375 < \lambda \leq 0.9437$	$\{\mathcal{V}_1, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_7, \mathcal{V}_{10}\}$, $\{\mathcal{V}_2, \mathcal{V}_6, \mathcal{V}_9\}$, $\{\mathcal{V}_3\}$, $\{\mathcal{V}_8\}$
3	$0.9312 < \lambda \leq 0.9375$	$\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_9, \mathcal{V}_{10}\}$, $\{\mathcal{V}_3\}$, $\{\mathcal{V}_8\}$
2	$0.9250 < \lambda \leq 0.9312$	$\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_8, \mathcal{V}_9, \mathcal{V}_{10}\}$, $\{\mathcal{V}_3\}$
1	$0.0000 \leq \lambda \leq 0.9250$	$\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_8, \mathcal{V}_9, \mathcal{V}_{10}\}$

Chapter 6

Generalized aggregation operators based on t-norm operations for complex intuitionistic fuzzy sets[1]

The objective of this chapter is to present some generalized weighted averaging and geometric AOs for aggregating the different CIFNs using archimedean t-conorm and t-norm (ATT) operations. For it, some new operational laws of the CIFNs based on ATT are defined and their fundamental properties are proved. Then, a series of weighted averaging and geometric AOs are developed based on proposed operations. Further, some desirable properties and special cases of the presented AOs are studied. Finally, a decision-making approach based on proposed AOs is developed in order to solve MCDM problems with CIF information. A practical example is elaborated to illustrate the working of the proposed approach and its results are compared with some existing methods under CIFS and IFS studies.

6.1 Introduction

In most of the practical MCDM problems, it is often required to accumulate some numerical values and this is when AOs play a fundamental role. More generally, it can be said

[1]The content of this chapter is published as "Generalized geometric aggregation operators based on t-norm operations for complex intuitionistic fuzzy sets and their application to decision making", *Cognitive Computation, Springer*, 12(3), pp. 679 - 698, 2020 (**SCI: Impact Factor: 5.418**) and "Some generalized complex intuitionistic fuzzy aggregation operators and their application to multicriteria decision-making process", *Arabian Journal for Science and Engineering, Springer*, 44(3), 2679 - 2698, 2019 (**SCI: Impact Factor: 2.334**).

that the process of aggregation makes use of distinct information pieces to make it possible to reach at some conclusion or decision. In this direction, Yager [189] proposed the OWA operator by giving weights to all the inputs according to their ranking positions. Xu and Yager [185] presented geometric AO while Xu [179] presented weighted averaging operator for aggregating the different IFNs. Wang and Liu [156] presented some aggregation operations using Einstein norm operations. He, Chen, Zhau, Liu and Tao [72] presented some interactive averaging AOs to solve the MCDM problems. Garg [44, 47] presented some improved interactive AOs for different IFNs. Wei and Wang [166] developed ordered weighted AOs for IFNs. Ye [201] presented some hybrid averaging and geometric AOs under the IFS environment to solve the MCDM problems. The extensive literature review on various AOs has been done in section 1.1.3 of Chapter 1.

In CIFS hypothesis, complex estimations of membership and NMDs are considered and are communicated in polar form. Therefore, keeping in view the advantages of this set and taking the importance of AOs, this chapter presents the theory of the weighted averaging and geometric AOs in order to accumulate CIF information. As per our knowledge, in the aforementioned studies, the operators cannot be utilized to handle the CIFS information. Thus, in order to achieve it, we first define some operational laws for CIFNs which involve both uncertainty and periodicity semantics and study their properties. Then, based on these, we propose some generalized ATT based AOs named as CIF weighted averaging/geometric (CIFWA/CIFWG), CIF ordered weighted averaging/geometric (CIFOWA/CIFOWG) and CIF hybrid averaging/geometric (CIFHA/CIFHG) to aggregate the different CIFNs. The various properties of these operators are investigated in details. Furthermore, we propose an MCDM approach based on the proposed operators for CIFSs. The feasibility as well as superiority of the approach has been demonstrated through an illustrative example.

6.2 New Operational laws and Operators of CIFNs

In this section, some elementary operational laws of the CIFNs and some series of the averaging and geometric operators are defined.

6.2.1 Score and Accuracy functions

Definition 6.2.1. For CIFN $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1.w_{\vartheta_1}))$, the score function is defined as

$$\mathcal{S}(\mathcal{C}_1) = \zeta_1 - \vartheta_1 + w_{\zeta_1} - w_{\vartheta_1}, \tag{6.1}$$

and an accuracy function $\mathcal{H}$ of $\mathcal{C}_1$ is stated as

$$\mathcal{H}(\mathcal{C}_1) = \zeta_1 + \vartheta_1 + w_{\zeta_1} + w_{\vartheta_1}. \tag{6.2}$$

It is clear that $\mathcal{S}(\mathcal{C}_1) \in [-2, 2]$ and $\mathcal{H}(\mathcal{C}_1) \in [0, 2]$.

Based on these functions, an order relation between two CIFNs $\mathcal{C}_1$ and $\mathcal{C}_2$ is stated as

(a) if $\mathcal{S}(\mathcal{C}_1) > \mathcal{S}(\mathcal{C}_2)$ then $\mathcal{C}_1 \succ \mathcal{C}_2$.

(b) if $\mathcal{S}(\mathcal{C}_1) = \mathcal{S}(\mathcal{C}_2)$

  (i) if $\mathcal{H}(\mathcal{C}_1) > \mathcal{H}(\mathcal{C}_2)$ then $\mathcal{C}_1 \succ \mathcal{C}_2$

  (ii) if $\mathcal{H}(\mathcal{C}_1) = \mathcal{H}(\mathcal{C}_2)$ then $\mathcal{C}_1$ and $\mathcal{C}_2$ represent the same information.

Here the symbol $\succ$ stands for "preferred to". Further, to study the properties of score function and accuracy function, we propose the following results.

Theorem 6.2.1. (Monotonicity of score function). Let $\mathcal{C} = ((\zeta, w_\zeta), (\vartheta, w_\vartheta))$ is CIFN, the score function $\mathcal{S}(\mathcal{C}) = \zeta - \vartheta + w_\zeta - w_\vartheta$ is a monotonic increasing function with ζ, w_ζ, and a monotone decreasing function with ϑ, w_ϑ.

Proof. Omitted. □

Theorem 6.2.2. (Symmetry of score function) Let $\mathcal{C}_j = \left((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\right)$, $j = 1, 2$ be two CIFNs, $\mathcal{C}_j^c = \left((\vartheta_j, w_{\vartheta_j}), (\zeta_j, w_{\zeta_j})\right)$, $j = 1, 2$ be their associated inverse function, respectively, then we have the following conclusion $\mathcal{S}(\mathcal{C}_1) \leq \mathcal{S}(\mathcal{C}_2) \Leftrightarrow \mathcal{S}(\mathcal{C}_1^c) \geq \mathcal{S}(\mathcal{C}_2^c)$.

Proof. By the definition of score function for CIFNs $\mathcal{C}_j (j = 1, 2)$ we obtain

$$\mathcal{S}(\mathcal{C}_1) = \zeta_1 - \vartheta_1 + w_{\zeta_1} - w_{\vartheta_1} \text{ and } \mathcal{S}(\mathcal{C}_2) = \zeta_2 - \vartheta_2 + w_{\zeta_2} - w_{\vartheta_2}$$

Since $\mathcal{S}(\mathcal{C}_1) \leq \mathcal{S}(\mathcal{C}_2)$, then $\zeta_1 - \vartheta_1 + w_{\zeta_1} - w_{\vartheta_1} \leq \zeta_2 - \vartheta_2 + w_{\zeta_2} - w_{\vartheta_2}$ which implies that $-\zeta_1 + \vartheta_1 + -w_{\zeta_1} + w_{\vartheta_1} \geq -\zeta_2 + \vartheta_2 + -w_{\zeta_2} + w_{\vartheta_2}$. Hence, $\mathcal{S}(\mathcal{C}_1^c) \geq \mathcal{S}(\mathcal{C}_2^c)$.

□

Theorem 6.2.3. (Monotonicity of accuracy function) Let $\mathcal{C} = ((\zeta, w_\zeta), (\vartheta, w_\vartheta))$ is CIFN, the accuracy function $\mathcal{S}(\mathcal{C}) = \zeta + \vartheta + w_\zeta + w_\vartheta$ is a monotonic increasing function with ζ, w_ζ, ϑ and w_ϑ.

Theorem 6.2.4. (Symmetry of accuracy function) Let $\mathcal{C} = ((\zeta, w_\zeta), (\vartheta, w_\vartheta))$ be CIFN and $\mathcal{C}^c = ((\vartheta, w_\vartheta), (\zeta, w_\zeta))$ be their associated complement, then we have $\mathcal{H}(\mathcal{C}) = \mathcal{H}(\mathcal{C}^c)$.

Proof. Omitted. $\square$

6.2.2 Operational laws of CIFNs

Next, we define the basic operational laws of CIFNs based on the Archimedean t-norm operations as follows.

Definition 6.2.2. Let $\mathcal{C}_j = \left((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\right)$, $j = 1, 2$ be any two CIFNs and let $\rho > 0$ be any real number. Then, we have

(i) $\mathcal{C}_1 \oplus \mathcal{C}_2 = \left(\left(s^{-1}\Big(s(\zeta_1) + s(\zeta_2)\Big), s^{-1}\Big(s(w_{\zeta_1}) + s(w_{\zeta_2})\Big)\right), \left(t^{-1}\Big(t(\vartheta_1) + t(\vartheta_2)\Big), t^{-1}\Big(t(w_{\vartheta_1}) + t(w_{\vartheta_2})\Big)\right) \right)$

(ii) $\mathcal{C}_1 \otimes \mathcal{C}_2 = \left(\left(t^{-1}\Big(t(\zeta_1) + t(\zeta_2)\Big), t^{-1}\Big(t(w_{\zeta_1}) + t(w_{\zeta_2})\Big)\right), \left(s^{-1}\Big(s(\vartheta_1) + s(\vartheta_2)\Big), s^{-1}\Big(s(w_{\vartheta_1}) + s(w_{\vartheta_2})\Big)\right) \right)$

(iii) $\rho\mathcal{C}_1 = \left(\left(s^{-1}\Big(\rho s(\zeta_1)\Big), s^{-1}\Big(\rho s(w_{\zeta_1})\Big)\right),\ \left(t^{-1}\Big(\rho t(\vartheta_1)\Big), t^{-1}\Big(\rho t(w_{\vartheta_1})\Big)\right)\right)$

(iv) $(\mathcal{C}_1)^\rho = \left(\left(t^{-1}\Big(\rho t(\zeta_1)\Big), t^{-1}\Big(\rho t(w_{\zeta_1})\Big)\right),\ \left(s^{-1}\Big(\rho s(\vartheta_1)\Big), s^{-1}\Big(\rho s(w_{\vartheta_1})\Big)\right)\right)$

Theorem 6.2.5. If $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two CIFNs and $\rho > 0$ be any real number, then $\mathcal{C}_1 \oplus \mathcal{C}_2$, $\rho\mathcal{C}_1$, $\mathcal{C}_1 \otimes \mathcal{C}_2$ and $\mathcal{C}_1^\rho$ are also CIFNs.

Proof. Let $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$ and $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$ are two CIFNs. So, by definition of CIFN, we have $\zeta_1, \zeta_2, \vartheta_1, \vartheta_2, w_{\zeta_1}, w_{\zeta_2}, w_{\vartheta_1}, w_{\vartheta_2}, \zeta_1+\vartheta_1, \zeta_2+\vartheta_2, w_{\zeta_1}+w_{\vartheta_1}, w_{\zeta_2}+w_{\vartheta_2} \in [0,1]$. Now, by using Definition 6.2.2, we obtain $\mathcal{C}_1 \oplus \mathcal{C}_2 = ((\zeta_3, w_{\zeta_3}), (\vartheta_3, w_{\vartheta_3}))$ where $\zeta_3 = s^{-1}\big(s(\zeta_1) + s(\zeta_2)\big)$, $\vartheta_3 = t^{-1}\big(t(\vartheta_1) + t(\vartheta_2)\big)$, $w_{\zeta_3} = s^{-1}\big(s(w_{\zeta_1}) + s(w_{\zeta_2})\big)$ and $w_{\vartheta_3} = t^{-1}\big(t(w_{\vartheta_1}) + t(w_{\vartheta_2})\big)$. In order to show $\mathcal{C}_1 \oplus \mathcal{C}_2$ is CIFN, it is enough to show that $\zeta_3, \vartheta_3, w_{\zeta_3}, w_{\vartheta_3}, \zeta_3 + \vartheta_3, w_{\zeta_3} + w_{\vartheta_3} \in [0,1]$. Since $t, s : [0,1] \to [0,\infty)$ are the continuous function with $t(1) = 0$ and $s(a) = t(1-a)$, so it is clearly seen that $\zeta_3, \vartheta_3, w_{\zeta_3}, w_{\vartheta_3} \in [0,1]$.

Further, using the conditions $\zeta_j + \vartheta_j \leq 1$ for $j = 1, 2$ and s is an increasing function, we have

$$\begin{aligned}
\zeta_3 + \vartheta_3 &= s^{-1}\Big(s(\zeta_1) + s(\zeta_2)\Big) + t^{-1}\Big(t(\vartheta_1) + t(\vartheta_2)\Big) \\
&\leq s^{-1}\Big(s(1-\vartheta_1) + s(1-\vartheta_2)\Big) + t^{-1}\Big(t(\vartheta_1) + t(\vartheta_2)\Big) \\
&= 1 - t^{-1}\Big(t(\vartheta_1) + t(\vartheta_2)\Big) + t^{-1}\Big(t(\vartheta_1) + t(\vartheta_2)\Big) \\
&= 1
\end{aligned}$$

Thus, $\zeta_3 + \vartheta_3 \leq 1$. Also, $\zeta_3 + \vartheta_3 \geq 0$ as $\zeta_3, \vartheta_3 \geq 0$. Hence, $0 \leq \zeta_3 + \vartheta_3 \leq 1$. Similarly, $0 \leq w_{\zeta_3} + w_{\vartheta_3} \leq 1$. Therefore, $\mathcal{C}_1 \oplus \mathcal{C}_2$ is a CIFN. Similarly, we can prove that $\mathcal{C}_1 \otimes \mathcal{C}_2$, $\mathcal{C}_1^{\rho}$, $\rho\mathcal{C}_1$ are also CIFNs. □

Theorem 6.2.6. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two CIFNs and ρ, ρ_1, $\rho_2 > 0$ be three real numbers. Then,

(i) $\mathcal{C}_1 \oplus \mathcal{C}_2 = \mathcal{C}_2 \oplus \mathcal{C}_1$;

(ii) $\mathcal{C}_1 \otimes \mathcal{C}_2 = \mathcal{C}_2 \otimes \mathcal{C}_1$;

(iii) $\rho\big(\mathcal{C}_1 \oplus \mathcal{C}_2\big) = \rho\mathcal{C}_1 \oplus \rho\mathcal{C}_2$;

(iv) $\big(\mathcal{C}_1 \otimes \mathcal{C}_2\big)^{\rho} = \mathcal{C}_1^{\rho} \otimes \mathcal{C}_2^{\rho}$;

(v) $\rho_1\mathcal{C}_1 \oplus \rho_2\mathcal{C}_1 = \big(\rho_1 + \rho_2\big)\mathcal{C}_1$;

(vi) $\mathcal{C}_1^{\rho_1} \otimes \mathcal{C}_1^{\rho_2} = \mathcal{C}_1^{\rho_1+\rho_2}$.

Proof. Here we have proved the parts (i), (iii) and (v), while others can be deduced similarly for two CIFNs $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$ and $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$.

(i) By Definition 6.2.2, we have

$$\begin{aligned}
\mathcal{C}_1 \oplus \mathcal{C}_2 &= \left(\left(s^{-1}\Big(s(\zeta_1) + s(\zeta_2)\Big), s^{-1}\Big(s(w_{\zeta_1}) + s(w_{\zeta_2})\Big)\right), \left(t^{-1}\Big(t(\vartheta_1) + t(\vartheta_2)\Big), t^{-1}\Big(t(w_{\vartheta_1}) + t(w_{\vartheta_2})\Big)\right)\right) \\
&= \left(\left(s^{-1}\Big(s(\zeta_2) + s(\zeta_1)\Big), s^{-1}\Big(s(w_{\zeta_2}) + s(w_{\zeta_1})\Big)\right), \left(t^{-1}\Big(t(\vartheta_2) + t(\vartheta_1)\Big), t^{-1}\Big(t(w_{\vartheta_2}) + t(w_{\vartheta_1})\Big)\right)\right) \\
&= \mathcal{C}_2 \oplus \mathcal{C}_1
\end{aligned}$$

(iii) Since $\mathcal{C}_1$, $\mathcal{C}_2$ are CIFNs and $\rho > 0$ is a real number. So, by Definition 6.2.2, we have

$$
\begin{aligned}
& \rho\big(\mathcal{C}_1 \oplus \mathcal{C}_2\big) \\
= \; & \rho\Big(\Big(s^{-1}\Big(s(\zeta_1)+s(\zeta_2)\Big), s^{-1}\Big(s(w_{\zeta_1})+s(w_{\zeta_2})\Big)\Big), \Big(t^{-1}\Big(t(\vartheta_1)+t(\vartheta_2)\Big), t^{-1}\Big(t(w_{\vartheta_1})+t(w_{\vartheta_2})\Big)\Big)\Big) \\
= \; & \left(\begin{pmatrix} s^{-1}\left(\rho s\left(s^{-1}\big(s(\zeta_1)+s(\zeta_2)\big)\right)\right), \\ s^{-1}\left(\rho s\left(s^{-1}\big(s(w_{\zeta_1})+s(w_{\zeta_2})\big)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\rho t\left(t^{-1}\left(t(\vartheta_1)+t(\vartheta_2)\right)\right)\right), \\ t^{-1}\left(\rho t\left(t^{-1}\left(t(w_{\vartheta_1})+t(w_{\vartheta_2})\right)\right)\right) \end{pmatrix}\right) \\
= \; & \left(\begin{pmatrix} s^{-1}\big(\rho s(\zeta_1)+\rho s(\zeta_2)\big), \\ s^{-1}\left(\rho s\left(w_{\zeta_1}\right)+\rho s\left(w_{\zeta_2}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\big(\rho t(\vartheta_1)+\rho t(\vartheta_2)\big), \\ t^{-1}\left(\rho t\left(w_{\vartheta_1}\right)+\rho t\left(w_{\vartheta_2}\right)\right) \end{pmatrix}\right) \\
= \; & \left(\begin{pmatrix} s^{-1}\left(s\left(s^{-1}\big(\rho s(\zeta_1)\big)\right)+s\left(s^{-1}\big(\rho s(\zeta_2)\big)\right)\right), \\ s^{-1}\left(s\left(s^{-1}\left(\rho s\left(w_{\zeta_1}\right)\right)\right)+s\left(s^{-1}\left(\rho s\left(w_{\zeta_2}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(t\left(t^{-1}\big(\rho t(\vartheta_1)\big)\right)+t\left(t^{-1}\big(\rho t(\vartheta_2)\big)\right)\right), \\ t^{-1}\left(t\left(t^{-1}\left(\rho t\left(w_{\vartheta_1}\right)\right)\right)+t\left(t^{-1}\left(\rho t\left(w_{\vartheta_2}\right)\right)\right)\right) \end{pmatrix}\right) \\
= \; & \left(\left(s^{-1}\big(\rho s(\zeta_1)\big), s^{-1}\left(\rho s\left(w_{\zeta_1}\right)\right)\right), \left(t^{-1}\left(\rho t(\vartheta_1)\right), t^{-1}\left(\rho t\left(w_{\vartheta_1}\right)\right)\right)\right) \\
\oplus \; & \left(\left(s^{-1}\big(\rho s(\zeta_2)\big), s^{-1}\left(\rho s\left(w_{\zeta_2}\right)\right)\right), \left(t^{-1}\big(t(\vartheta_2)\big), t^{-1}\left(\rho t\left(w_{\vartheta_2}\right)\right)\right)\right) \\
= \; & \rho\mathcal{C}_1 \oplus \rho\mathcal{C}_2
\end{aligned}
$$

Hence, $\rho\big(\mathcal{C}_1 \oplus \mathcal{C}_2\big) = \rho\mathcal{C}_1 \oplus \rho\mathcal{C}_2$.

(v) Since $\mathcal{C}_1$ is CIFN and $\rho_1, \rho_2 > 0$ are real numbers.

$$
\begin{aligned}
& \rho_1\mathcal{C}_1 \oplus \rho_2\mathcal{C}_1 \\
= \; & \left(\left(s^{-1}\left(\rho_1 s(\zeta_1)\right), s^{-1}\left(\rho_1 s\left(w_{\zeta_1}\right)\right)\right), \left(t^{-1}\left(\rho_1 t(\vartheta_1)\right), t^{-1}\left(\rho_1 t\left(w_{\vartheta_1}\right)\right)\right)\right) \\
\oplus \; & \left(\left(s^{-1}\left(\rho_2 s(\zeta_1)\right), s^{-1}\left(\rho_2 s\left(w_{\zeta_1}\right)\right)\right), \left(t^{-1}\left(\rho_2 t(\vartheta_1)\right), t^{-1}\left(\rho_2 t\left(w_{\vartheta_1}\right)\right)\right)\right) \\
= \; & \left(\begin{pmatrix} s^{-1}\left(s\left(s^{-1}\left(\rho_1 s(\zeta_1)\right)\right)+s\left(s^{-1}\left(\rho_2 s(\zeta_1)\right)\right)\right), \\ s^{-1}\left(s\left(s^{-1}\left(\rho_1 s\left(w_{\zeta_1}\right)\right)\right)+s\left(s^{-1}\left(\rho_2 s\left(w_{\zeta_1}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(t\left(t^{-1}\left(\rho_1 t(\vartheta_1)\right)\right)+t\left(t^{-1}\left(\rho_2 t(\vartheta_1)\right)\right)\right), \\ t^{-1}\left(t\left(t^{-1}\left(\rho_1 t\left(w_{\vartheta_1}\right)\right)\right)+t\left(t^{-1}\left(\rho_2 t\left(w_{\vartheta_1}\right)\right)\right)\right) \end{pmatrix}\right) \\
= \; & \left(\begin{pmatrix} s^{-1}\left(\left(\rho_1+\rho_2\right)s(\zeta_1)\right), \\ s^{-1}\left(\left(\rho_1+\rho_2\right)s\left(w_{\zeta_1}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\left(\rho_1+\rho_2\right)t(\vartheta_1)\right), \\ t^{-1}\left(\left(\rho_1+\rho_2\right)t\left(w_{\vartheta_1}\right)\right) \end{pmatrix}\right) \\
= \; & \left(\rho_1+\rho_2\right)\mathcal{C}_1
\end{aligned}
$$

Hence, $\rho_1\mathcal{C}_1 \oplus \rho_2\mathcal{C}_1 = \big(\rho_1+\rho_2\big)\mathcal{C}_1$.

$\square$

Theorem 6.2.7. Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}$ be three CIFNs and $\rho > 0$ be any real number. Then, we have

(i) $\big(\mathcal{C}^c\big)^\rho = \big(\rho\mathcal{C}\big)^c$;

(ii) $\rho\big(\mathcal{C}^c\big) = \big(\mathcal{C}^\rho\big)^c$;

(iii) $(\mathcal{C}_1 \oplus \mathcal{C}_2)^c = \mathcal{C}_1^c \otimes \mathcal{C}_2^c$;

(iv) $(\mathcal{C}_1 \otimes \mathcal{C}_2)^c = \mathcal{C}_1^c \oplus \mathcal{C}_2^c$.

Proof. Let $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$, $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$ and $\mathcal{C} = ((\zeta, w_\zeta), (\vartheta, w_\vartheta))$ be three CIFNs. Then, $\mathcal{C}^c = ((\vartheta, w_\vartheta), (\zeta, w_\zeta))$. Therefore, we have

(i) $(\mathcal{C}^c)^\rho = \big(\big(t^{-1}(\rho t(\vartheta)), t^{-1}(\rho t(w_\vartheta))\big), \big(s^{-1}(\rho s(\zeta)), s^{-1}(\rho s(w_\zeta))\big)\big) = (\rho\mathcal{C})^c$

(ii) $\rho(\mathcal{C}^c) = \big(\big(s^{-1}(\rho s(\vartheta)), s^{-1}(\rho s(w_\vartheta))\big), \big(t^{-1}(\rho t(\zeta)), t^{-1}(\rho t(w_\zeta))\big)\big) = (\mathcal{C}^\rho)^c$

(iii) $$(\mathcal{C}_1 \oplus \mathcal{C}_2)^c = \left(\begin{pmatrix} t^{-1}(t(\vartheta_1) + t(\vartheta_2)), \\ t^{-1}(t(w_{\vartheta_1}) + t(w_{\vartheta_2})) \end{pmatrix}, \begin{pmatrix} s^{-1}(s(\zeta_1) + s(\zeta_2)), \\ s^{-1}(s(w_{\zeta_1}) + s(w_{\zeta_2})) \end{pmatrix}\right) = \mathcal{C}_1^c \otimes \mathcal{C}_2^c$$

(iv) $$(\mathcal{C}_1 \otimes \mathcal{C}_2)^c = \left(\begin{pmatrix} s^{-1}(s(\vartheta_1) + s(\vartheta_2)), \\ s^{-1}(s(w_{\vartheta_1}) + s(w_{\vartheta_2})) \end{pmatrix}, \begin{pmatrix} t^{-1}(t(\zeta_1) + t(\zeta_2)), \\ t^{-1}(t(w_{\zeta_1}) + t(w_{\zeta_2})) \end{pmatrix}\right) = \mathcal{C}_1^c \oplus \mathcal{C}_2^c$$

$\square$

Remark 6.2.1. Consider here some special cases of CIFN $\mathcal{C} = ((\zeta, w_\zeta), (\vartheta, w_\vartheta))$ and any positive real number ρ.

(i) If $\mathcal{C} = ((1,1),(0,0))$ then

$$\rho\mathcal{C} = \big(\big(s^{-1}(\rho s(1)), s^{-1}(\rho s(1))\big), \big(t^{-1}(\rho t(0)), t^{-1}(\rho t(0))\big)\big) = ((1,1),(0,0))$$

(ii) If $\mathcal{C} = ((0,0),(1,1))$ then

$$\rho\mathcal{C} = \big(\big(s^{-1}(\rho s(0)), s^{-1}(\rho s(0))\big), \big(t^{-1}(\rho t(1)), t^{-1}(\rho t(1))\big)\big) = ((0,0),(1,1))$$

(iii) If $\mathcal{C} = ((0,0),(0,0))$ then

$$\rho\mathcal{C} = \big(\big(s^{-1}(\rho s(0)), s^{-1}(\rho s(0))\big), \big(t^{-1}(\rho t(0)), t^{-1}(\rho t(0))\big)\big) = ((0,0),(0,0))$$

(iv) If $\mathcal{C} = ((\zeta, w_\zeta), (\vartheta, w_\vartheta))$ and $\rho \to 0$ then

$$\rho\mathcal{C} = \big(\big(s^{-1}(\rho s(\zeta)), s^{-1}(\rho s(w_\zeta))\big), \big(t^{-1}(\rho t(\vartheta)), t^{-1}(\rho t(w_\vartheta))\big)\big) \to ((0,0),(1,1))$$

(v) If $\mathcal{C} = ((\zeta, w_\zeta), (\vartheta, w_\vartheta))$ and $\rho \to \infty$ then

$$\rho\mathcal{C} = \left(\left(s^{-1}\left(\rho s(\zeta)\right), s^{-1}\left(\rho s\left(w_\zeta\right)\right)\right), \left(t^{-1}\left(\rho t(\vartheta)\right), t^{-1}\left(\rho t\left(w_\vartheta\right)\right)\right)\right) \to ((1,1),(0,0))$$

(vi) If $\mathcal{C} = ((\zeta, w_\zeta), (\vartheta, w_\vartheta))$ and $\rho = 1$ then

$$\rho\mathcal{C} = \left(\left(s^{-1}\left(\rho s(\zeta)\right), s^{-1}\left(\rho s\left(w_\zeta\right)\right)\right), \left(t^{-1}\left(\rho t(\vartheta)\right), t^{-1}\left(\rho t\left(w_\vartheta\right)\right)\right)\right) = ((\zeta, w_\zeta), (\vartheta, w_\vartheta))$$

Next, based on the above defined operational laws of CIFNs, we propose some new averaging and geometric AOs named as CIFWA, CIFOWA, CIFHA, CIFWG, CIFOWG and CIFHG under CIF environment.

6.2.3 Weighted averaging and geometric operators

In this section, weighted averaging and geometric AOs for a collection of CIFNs are defined.

Definition 6.2.3. Let Ω is the collection of all CIFNs $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ with corresponding weights $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ such that $\xi_j > 0$ and $\sum\limits_{j=1}^{n} \xi_j = 1$. If CIFWA: $\Omega^n \to \Omega$, is a mapping defined by

$$\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \ldots \oplus \xi_n\mathcal{C}_n \tag{6.3}$$

then, CIFWA is called CIF weighted averaging operator.

Theorem 6.2.8. For any collection of CIFNs $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right)$ $(j = 1, 2, \ldots, n)$, the combined value obtained by using CIFWA operator is still CIFN and is given as

$$\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s(\zeta_j)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s\left(w_{\zeta_j}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t(\vartheta_j)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t\left(w_{\vartheta_j}\right)\right) \end{pmatrix}\right) \tag{6.4}$$

Proof. The fact that, the value obtained after applying CIFWA operator is still CIFN, follows from the Theorem 6.2.5. Now, by making use of mathematical induction, we will show that Eq. (6.4) holds.

Since for each j, $\mathcal{C}_j$ is a CIFN and for real numbers $\xi_j > 0$, we have $\xi_j\mathcal{C}_j$ is also CIFN by using Theorem 6.2.5. Then, by using mathematical induction we have:

Step 1: For $n=2$, we get $\mathcal{C}_1=\left(\left(\zeta_1, w_{\zeta_1}\right),\left(\vartheta_1, w_{\vartheta_1}\right)\right)$ and $\mathcal{C}_2=\left(\left(\zeta_2, w_{\zeta_2}\right),\left(\vartheta_2, w_{\vartheta_2}\right)\right)$. Thus, by the operation laws of CIFNs, we get

$$\begin{aligned} \xi_1\mathcal{C}_1 &= \left(\left(s^{-1}\left(\xi_1 s(\zeta_1)\right), s^{-1}\left(\xi_1 s\left(w_{\zeta_1}\right)\right)\right), \left(t^{-1}\left(\xi_1 t(\vartheta_1)\right), t^{-1}\left(\xi_1 t\left(w_{\vartheta_1}\right)\right)\right)\right) \\ \text{and}\quad \xi_2\mathcal{C}_2 &= \left(\left(s^{-1}\left(\xi_2 s(\zeta_2)\right), s^{-1}\left(\xi_2 s\left(w_{\zeta_2}\right)\right)\right), \left(t^{-1}\left(\xi_2 t(\vartheta_2)\right), t^{-1}\left(\xi_2 t\left(w_{\vartheta_2}\right)\right)\right)\right) \end{aligned}$$

Hence, by addition law of CIFNs, we get

$$\begin{aligned} \mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2) &= \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \\ &= \left(\begin{array}{c} \left(\begin{array}{l} s^{-1}\left(s\left(s^{-1}\left(\xi_1 s(\zeta_1)\right)\right)+s\left(s^{-1}\left(\xi_2 s(\zeta_2)\right)\right)\right), \\ s^{-1}\left(s\left(s^{-1}\left(\xi_1 s\left(w_{\zeta_1}\right)\right)\right)+s\left(s^{-1}\left(\xi_2 s\left(w_{\zeta_2}\right)\right)\right)\right) \end{array}\right), \\ \left(\begin{array}{l} t^{-1}\left(t\left(t^{-1}\left(\xi_1 t(\vartheta_1)\right)\right)+t\left(t^{-1}\left(\xi_2 t(\vartheta_2)\right)\right)\right), \\ t^{-1}\left(t\left(t^{-1}\left(\xi_1 t\left(w_{\vartheta_1}\right)\right)\right)+t\left(t^{-1}\left(\xi_2 t\left(w_{\vartheta_2}\right)\right)\right)\right) \end{array}\right) \end{array}\right) \\ &= \left(\left(\begin{array}{l} s^{-1}\left(\sum_{j=1}^{2}\xi_j s(\zeta_j)\right), \\ s^{-1}\left(\sum_{j=1}^{2}\xi_j s\left(w_{\zeta_j}\right)\right) \end{array}\right), \left(\begin{array}{l} t^{-1}\left(\sum_{j=1}^{2}\xi_j t(\vartheta_j)\right), \\ t^{-1}\left(\sum_{j=1}^{2}\xi_j t\left(w_{\vartheta_j}\right)\right) \end{array}\right)\right) \end{aligned}$$

Thus, results holds for $n=2$.

Step 2: If Eq. (6.4) holds for $n=m$, where m is any natural number, then

$$\mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_m) = \left(\left(\begin{array}{l} s^{-1}\left(\sum_{j=1}^{m}\xi_j s(\zeta_j)\right), \\ s^{-1}\left(\sum_{j=1}^{m}\xi_j s\left(w_{\zeta_j}\right)\right) \end{array}\right), \left(\begin{array}{l} t^{-1}\left(\sum_{j=1}^{m}\xi_j t(\vartheta_j)\right), \\ t^{-1}\left(\sum_{j=1}^{m}\xi_j t\left(w_{\vartheta_j}\right)\right) \end{array}\right)\right)$$

then for $n=m+1$, we have

$$\begin{aligned} \mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_{m+1}) &= \mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_m)\oplus \xi_{m+1}\mathcal{C}_{m+1} \\ &= \left(\left(\begin{array}{l} s^{-1}\left(\sum_{j=1}^{m}\xi_j s(\zeta_j)\right), \\ s^{-1}\left(\sum_{j=1}^{m}\xi_j s\left(w_{\zeta_j}\right)\right) \end{array}\right), \left(\begin{array}{l} t^{-1}\left(\sum_{j=1}^{m}\xi_j t(\vartheta_j)\right), \\ t^{-1}\left(\sum_{j=1}^{m}\xi_j t\left(w_{\vartheta_j}\right)\right) \end{array}\right)\right) \end{aligned}$$

$$\oplus \left(\begin{pmatrix} s^{-1}\Big(\xi_{m+1}s(\zeta_{m+1})\Big), \\ s^{-1}\left(\xi_{m+1}s\left(w_{\zeta_{m+1}}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\xi_{m+1}t(\vartheta_{m+1})\right), \\ t^{-1}\left(\xi_{m+1}t\left(w_{\vartheta_{m+1}}\right)\right) \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{m+1}\xi_j s(\zeta_j)\right), \\ s^{-1}\left(\sum_{j=1}^{m+1}\xi_j s\left(w_{\zeta_j}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{m+1}\xi_j t(\vartheta_j)\right), \\ t^{-1}\left(\sum_{j=1}^{m+1}\xi_j t\left(w_{\vartheta_j}\right)\right) \end{pmatrix} \right)$$

Thus, the result is true for $n = m+1$ and hence, the Eq. (6.4) holds for all natural numbers n.

□

Definition 6.2.4. For a collection of "n" CIFNs $\mathcal{C}_j$, a mapping CIFWG: $\Omega^n \to \Omega$, given by

$$\text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathcal{C}_1^{\xi_1} \otimes \mathcal{C}_2^{\xi_2} \otimes \ldots \otimes \mathcal{C}_n^{\xi_n}. \tag{6.5}$$

is called as CIF weighted geometric operator.

Theorem 6.2.9. For a collection of CIFNs $\mathcal{C}_j = \Big((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\Big)$ with corresponding weights $\xi_j > 0$ and $\sum\limits_{j=1}^{n} \xi_j = 1$, the combined value obtained by using Definition 6.2.4 is still CIFN and is given as

$$\text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\zeta_j)\right), \\ t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(w_{\zeta_j}\right)\right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\vartheta_j)\right), \\ s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(w_{\vartheta_j}\right)\right) \end{pmatrix} \right) \tag{6.6}$$

Proposition 6.2.1. For $w_{\zeta_j}, w_{\vartheta_j} = 0$ for all j, CIFWA operator reduces to IFWA operator in IFS environment.

Proof. Since $w_{\zeta_j}, w_{\vartheta_j} = 0$. Therefore, Eq. (6.4) reduces to:

$$\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \;=\; \left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_j)\right), t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_j)\right) \right)$$

which is the weighted averaging operator in IFS environment. Hence, the proposed CIFWA operator is an extension of existing IFWA operator. □

Proposition 6.2.2. If for all j, $w_{\zeta_j}, w_{\vartheta_j} = 0$ and $\zeta_j + \vartheta_j = 1$ then, the CIFWA operator reduces to weighted operator in fuzzy environment.

Proof. Similar to the above Proposition. $\square$

The working of the proposed CIFWA AO is explained with a numerical example as follows:

Example 6.2.1. Let $\mathcal{C}_1 = ((0.6, 0.8), (0.2, 0.1))$, $\mathcal{C}_2 = ((0.8, 0.7), (0.2, 0.1))$, $\mathcal{C}_3 = ((0.5, 0.6), ,$ $(0.3, 0.4))$, $\mathcal{C}_4 = ((0.6, 0.7),\ (0.3, 0.2))$ be four CIFNs and $\xi = (0.35, 0.3, 0.1, 0.25)^T$ be the corresponding weight vector of $\mathcal{C}_j (j = 1, 2, 3, 4)$. Without loss of generality, we consider the additive generators $t(a) = -\log a$ if $0 < a \leq 1$ with $t(0) = \infty$ and $s(a) = -\log(1 - a)$ if $0 \leq a < 1$ with $s(1) = \infty$ corresponding to t-norm and t-conorm respectively. Then, Eq. (6.4) becomes

$$\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} 1 - \prod_{j=1}^{n} (1 - \zeta_j)^{\xi_j}, \\ 1 - \prod_{j=1}^{n} \left(1 - w_{\zeta_j}\right)^{\xi_j} \end{pmatrix}, \begin{pmatrix} \prod_{j=1}^{n} \vartheta_j^{\xi_j}, \\ \prod_{j=1}^{n} \left(w_{\vartheta_j}\right)^{\xi_j} \end{pmatrix} \right) \tag{6.7}$$

Now, based on the given information, we have

$$\begin{aligned} \prod_{j=1}^{4} (1 - \zeta_j)^{\xi_j} &= (1 - 0.6)^{0.35} \times (1 - 0.8)^{0.3} \times (1 - 0.5)^{0.1} \times (1 - 0.6)^{0.25} = 0.3322 \\ \prod_{j=1}^{4} \vartheta_j^{\xi_j} &= (0.2)^{0.35} \times (0.2)^{0.3} \times (0.3)^{0.1} \times (0.3)^{0.25} = 0.2305 \\ \prod_{j=1}^{4} \left(1 - w_{\zeta_j}\right)^{\xi_j} &= (1 - 0.8)^{0.35} \times (1 - 0.7)^{0.3} \times (1 - 0.6)^{0.1} \times (1 - 0.7)^{0.25} = 0.2679 \\ \prod_{j=1}^{4} \left(w_{\vartheta_j}\right)^{\xi_j} &= (0.1)^{0.35} \times (0.1)^{0.3} \times (0.4)^{0.1} \times (0.2)^{0.25} = 0.1366 \end{aligned}$$

Thus, by using Eq. (6.7), we get

$$\begin{aligned} \text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4) &= \left((1 - 0.3322, 1 - 0.2679), (0.2305, 0.1366)\right) \\ &= ((0.6678, 0.7321), (0.2305, 0.1366)) \end{aligned}$$

Based on the Theorem 6.2.8, it is observed that the CIFWA and CIFWG operators satisfy some properties which are elaborated for CIFWA operator as follows:

Property 6.2.1. Let $\mathcal{C}_0$ be CIFN and if $\mathcal{C}_j = \mathcal{C}_0$ for all $j = 1, 2, \ldots, n$, then, we have

$$\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2 \ldots \mathcal{C}_n) = \mathcal{C}_0$$

This property is called Idempotency.

Proof. Let $\mathcal{C}_0 = \left(\left(\zeta_0, w_{\zeta_0}\right), \left(\vartheta_0, w_{\vartheta_0}\right)\right)$ and $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right)$ $(j = 1, 2, \ldots, n)$ be CIFNs such that $\mathcal{C}_j = \mathcal{C}_0$ which implies that $\zeta_j = \zeta_0$, $\vartheta_j = \vartheta_0$, $w_{\zeta_j} = w_{\zeta_0}$ and $w_{\vartheta_j} = w_{\vartheta_0}$ for all j. Also, we have $\xi_j > 0$ such that $\sum\limits_{j=1}^{n} \xi_j = 1$. Then, by using Theorem 6.2.8, we get

$$\begin{aligned}
& \text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2 \ldots \mathcal{C}_n) \\
= & \left(\begin{pmatrix} s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s(\zeta_0)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s\left(w_{\zeta_0}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t(\vartheta_0)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t\left(w_{\vartheta_0}\right)\right) \end{pmatrix} \right) \\
= & \left(\left(s^{-1}\big(s(\zeta_0)\big), s^{-1}\left(s\left(w_{\zeta_0}\right)\right)\right), \left(t^{-1}\big(t(\vartheta_0)\big), t^{-1}\left(t\left(w_{\vartheta_0}\right)\right)\right)\right) \\
= & \left(\left(\zeta_0, w_{\zeta_0}\right), \left(\vartheta_0, w_{\vartheta_0}\right)\right) \\
= & \mathcal{C}_0
\end{aligned}$$

Hence, $\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2 \ldots \mathcal{C}_n) = \mathcal{C}_0$. $\square$

Property 6.2.2. For two collections of CIFN $\mathcal{C}_j = ((\zeta_{\mathcal{C}_j}, w_{\zeta_{\mathcal{C}_j}}), (\vartheta_{\mathcal{C}_j}, w_{\vartheta_{\mathcal{C}_j}}))$ and $\mathcal{Z}_j = ((\zeta_{\mathcal{Z}_j}, w_{\zeta_{\mathcal{Z}_j}}), (\vartheta_{\mathcal{Z}_j}, w_{\vartheta_{\mathcal{Z}_j}}))$ satisfying $\zeta_{\mathcal{C}_j} \leq \zeta_{\mathcal{Z}_j}$, $\vartheta_{\mathcal{C}_j} \geq \vartheta_{\mathcal{Z}_j}$, $w_{\zeta_{\mathcal{C}_j}} \leq w_{\zeta_{\mathcal{Z}_j}}$ and $w_{\vartheta_{\mathcal{C}_j}} \geq w_{\vartheta_{\mathcal{Z}_j}}$ for all j. Then, we have

$$\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \text{CIFWA}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n).$$

This property is called Monotonicity.

Proof. For two CIFNs $\mathcal{C}_j = ((\zeta_{\mathcal{C}_j}, w_{\zeta_{\mathcal{C}_j}}), (\vartheta_{\mathcal{C}_j}, w_{\vartheta_{\mathcal{C}_j}}))$ and $\mathcal{Z}_j = ((\zeta_{\mathcal{Z}_j}, w_{\zeta_{\mathcal{Z}_j}}), (\vartheta_{\mathcal{Z}_j}, w_{\vartheta_{\mathcal{Z}_j}}))$ such that $\zeta_{\mathcal{C}_j} \leq \zeta_{\mathcal{Z}_j}$, $\vartheta_{\mathcal{C}_j} \geq \vartheta_{\mathcal{Z}_j}$, $w_{\zeta_{\mathcal{C}_j}} \leq w_{\zeta_{\mathcal{Z}_j}}$, $w_{\vartheta_{\mathcal{C}_j}} \geq w_{\vartheta_{\mathcal{Z}_j}}$ for all j which implies that $\mathcal{C}_j \subseteq \mathcal{Z}_j$. Further, t and s are decreasing and increasing functions respectively, then we have

$$s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_{\mathcal{C}_j}\right) \leq s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_{\mathcal{Z}_j}))\right)$$

$$t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_{\mathcal{C}_j}\right) \geq t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_{\mathcal{Z}_j}))\right)$$

$$s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(w_{\zeta_{\mathcal{C}_j}}\right)\right) \leq s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(w_{\zeta_{\mathcal{Z}_j}}\right)\right)$$

$$\text{and} \quad t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(w_{\vartheta_{\mathcal{C}_j}}\right)\right) \geq t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(w_{\vartheta_{\mathcal{Z}_j}}\right)\right)$$

Hence, $\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \text{CIFWA}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n)$. □

Property 6.2.3. For a collection of CIFNs $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right)$ $(j = 1, 2, \ldots n)$, let $\mathcal{C}^- = ((\zeta^-, w_\zeta^-), (\vartheta^+, w_\vartheta^+))$ and $\mathcal{C}^+ = ((\zeta^+, w_\zeta^+), (\vartheta^-, w_\vartheta^-))$ where $\zeta^- = \min_j\{\zeta_j\}$, $w_\zeta^- = \min_j\{w_{\zeta_j}\}$, $\zeta^+ = \max_j\{\zeta_j\}$, $w_\zeta^+ = \max_j\{w_{\zeta_j}\}$, $\vartheta^- = \min_j\{\vartheta_j\}$, $w_\vartheta^- = \min_j\{w_{\vartheta_j}\}$, $\vartheta^+ = \max_j\{\vartheta_j\}$, $w_\vartheta^+ = \max_j\{w_{\vartheta_j}\}$. Then, we have

$$\mathcal{C}^- \subseteq \text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2 \ldots \mathcal{C}_n) \subseteq \mathcal{C}^+$$

This property is called Boundedness.

Proof. Let $\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2 \ldots \mathcal{C}_n) = ((\zeta_S, w_{\zeta_S}), (\vartheta_S, w_{\vartheta_S}))$. For a CIFN $\mathcal{C}_j$, we have $\min_j\{\zeta_j\} \leq \zeta_j \leq \max_j\{\zeta_j\}$ and $\min_j\{\vartheta_j\} \leq \vartheta_j \leq \max_j\{\vartheta_j\}$. Since t, s are decreasing and increasing functions respectively, therefore,

$$s^{-1}\left(\sum_{j=1}^{n}\xi_j s\Big(\min_j\{\zeta_j\}\Big)\right) \leq s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_j)\right) \leq s^{-1}\left(\sum_{j=1}^{n}\xi_j s\Big(\max_j\{\zeta_j\}\Big)\right)$$

$$\text{and} \quad t^{-1}\left(\sum_{j=1}^{n}\xi_j t\Big(\min_j\{\vartheta_j\}\Big)\right) \leq t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_j)\right) \leq t^{-1}\left(\sum_{j=1}^{n}\xi_j t\Big(\max_j\{\vartheta_j\}\Big)\right)$$

which implies that $\min_j\{\zeta_j\} \leq \zeta_S \leq \max_j\{\zeta_j\}$ and $\min_j\{\vartheta_j\} \leq \vartheta_S \leq \max_j\{\vartheta_j\}$, i.e., $\zeta_j^- \leq \zeta_S \leq \zeta_j^+$ and $\vartheta_j^- \leq \vartheta_S \leq \vartheta_j^+$. Similarly, we can obtain $w_{\zeta_j}^- \leq w_{\zeta_S} \leq w_{\zeta_j}^+$ and $w_{\vartheta_j}^- \leq w_{\vartheta_S} \leq w_{\vartheta_j}^+$. Therefore,

$$\mathcal{C}^- \subseteq \text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \mathcal{C}^+$$

□

Property 6.2.4. For a collection of CIFNs $\mathcal{C}_j(j = 1, 2, \ldots, n)$ and CIFN $\mathcal{Z}$, we have

$$\text{CIFWA}(\mathcal{C}_1 \oplus \mathcal{Z}, \mathcal{C}_2 \oplus \mathcal{Z}, \ldots, \mathcal{C}_n \oplus \mathcal{Z}) = \text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \oplus \mathcal{Z}$$

This property is called Shift Invariance.

Proof. Let $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right)$ and $\mathcal{Z} = ((\zeta_{\mathcal{Z}}, w_{\zeta_{\mathcal{Z}}}), (\vartheta_{\mathcal{Z}}, w_{\vartheta_{\mathcal{Z}}}))$ be CIFNs. Then, by using addition law for CIFNs for all j, we get

$$\mathcal{C}_j \oplus \mathcal{Z} = \left(\begin{pmatrix} s^{-1}\left(s(\zeta_j) + s(\zeta_{\mathcal{Z}})\right), \\ s^{-1}\left(s\left(w_{\zeta_j}\right) + s\left(w_{\zeta_{\mathcal{Z}}}\right)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(t(\vartheta_j) + t(\vartheta_{\mathcal{Z}})\right), \\ t^{-1}\left(t\left(w_{\vartheta_j}\right) + t\left(w_{\vartheta_{\mathcal{Z}}}\right)\right)\end{pmatrix}\right)$$

Now, using Eq. (6.4), we get

$$\begin{aligned}
&\text{CIFWA}(\mathcal{C}_1 \oplus \mathcal{Z}, \mathcal{C}_2 \oplus \mathcal{Z}, \ldots, \mathcal{C}_n \oplus \mathcal{Z}) \\
= &\left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(s^{-1}\left(s(\zeta_j) + s(\zeta_{\mathcal{Z}})\right)\right)\right), \\ s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(s^{-1}\left(s\left(w_{\zeta_j}\right) + s\left(w_{\zeta_{\mathcal{Z}}}\right)\right)\right)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(t^{-1}\left(t(\vartheta_j) + t(\vartheta_{\mathcal{Z}})\right)\right)\right), \\ t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(t^{-1}\left(t\left(w_{\vartheta_j}\right) + t\left(w_{\vartheta_{\mathcal{Z}}}\right)\right)\right)\right)\end{pmatrix}\right) \\
= &\left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{n} \xi_j \left(s(\zeta_j) + s(\zeta_{\mathcal{Z}})\right)\right), \\ s^{-1}\left(\sum_{j=1}^{n} \xi_j \left(s\left(w_{\zeta_j}\right) + s\left(w_{\zeta_{\mathcal{Z}}}\right)\right)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{n} \xi_j \left(t(\vartheta_j) + t(\vartheta_{\mathcal{Z}})\right)\right), \\ t^{-1}\left(\sum_{j=1}^{n} \xi_j \left(t\left(w_{\vartheta_j}\right) + t\left(w_{\vartheta_{\mathcal{Z}}}\right)\right)\right)\end{pmatrix}\right) \\
= &\left(\begin{pmatrix} s^{-1}\left(\left(\sum_{j=1}^{n} \xi_j s(\zeta_j)\right) + s(\zeta_{\mathcal{Z}})\right), \\ s^{-1}\left(\left(\sum_{j=1}^{n} \xi_j s\left(w_{\zeta_j}\right)\right) + s\left(w_{\zeta_{\mathcal{Z}}}\right)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\left(\sum_{j=1}^{n} \xi_j t(\vartheta_j)\right) + t(\vartheta_{\mathcal{Z}})\right), \\ t^{-1}\left(\left(\sum_{j=1}^{n} \xi_j t\left(w_{\vartheta_j}\right)\right) + t\left(w_{\vartheta_{\mathcal{Z}}}\right)\right)\end{pmatrix}\right) \\
= &\left(\begin{pmatrix} s^{-1}\left(s\left(s^{-1}\left(\sum_{j=1}^{n} \xi_j s(\zeta_j)\right)\right) + s(\zeta_{\mathcal{Z}})\right), \\ s^{-1}\left(s\left(s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(w_{\zeta_j}\right)\right)\right) + s\left(w_{\zeta_{\mathcal{Z}}}\right)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(t\left(t^{-1}\left(\sum_{j=1}^{n} \xi_j t(\vartheta_j)\right)\right) + t(\vartheta_{\mathcal{Z}})\right), \\ t^{-1}\left(t\left(t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(w_{\vartheta_j}\right)\right)\right) + t\left(w_{\vartheta_{\mathcal{Z}}}\right)\right)\end{pmatrix}\right) \\
= &\ \text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \oplus \mathcal{Z}
\end{aligned}$$

which completes the proof of the theorem. □

Property 6.2.5. For a collection of CIFNs $\mathcal{C}_j$ and any real number $\rho > 0$, we have

$$\text{CIFWA}(\rho\mathcal{C}_1, \rho\mathcal{C}_2, \ldots, \rho\mathcal{C}_n) = \rho\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)$$

This property is called Homogeneity.

Proof. For a collection of CIFNs $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right) (j = 1, 2, \ldots, n)$ and real number $\rho > 0$, we have

$$\rho\mathcal{C}_j = \left(\left(s^{-1}\left(\rho s(\zeta_j)\right), s^{-1}\left(\rho s\left(w_{\zeta_j}\right)\right)\right), \left(t^{-1}\left(\rho t(\vartheta_j)\right), t^{-1}\left(\rho t\left(w_{\vartheta_j}\right)\right)\right)\right)$$

Now, using Eq. (6.4), we get

$$\begin{aligned}
& \text{CIFWA}(\rho\mathcal{C}_1, \rho\mathcal{C}_2, \ldots, \rho\mathcal{C}_n) \\
= & \left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(s^{-1}\left(\rho s(\zeta_j)\right)\right)\right), \\ s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(s^{-1}\left(\rho s\left(w_{\zeta_j}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(t^{-1}\left(\rho t(\vartheta_j)\right)\right)\right), \\ t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(t^{-1}\left(\rho t\left(w_{\vartheta_j}\right)\right)\right)\right) \end{pmatrix}\right) \\
= & \left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{n} \xi_j \rho s(\zeta_j)\right), \\ s^{-1}\left(\sum_{j=1}^{n} \xi_j \rho s\left(w_{\zeta_j}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{n} \xi_j \rho t(\vartheta_j)\right), \\ t^{-1}\left(\sum_{j=1}^{n} \xi_j \rho t\left(w_{\vartheta_j}\right)\right) \end{pmatrix}\right) \\
= & \left(\begin{pmatrix} s^{-1}\left(\rho s\left(s^{-1}\left(\sum_{j=1}^{n} \xi_j s(\zeta_j)\right)\right)\right), \\ s^{-1}\left(\rho s\left(s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(w_{\zeta_j}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\rho t\left(t^{-1}\left(\sum_{j=1}^{n} \xi_j t(\vartheta_j)\right)\right)\right), \\ t^{-1}\left(\rho t\left(t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(w_{\vartheta_j}\right)\right)\right)\right) \end{pmatrix}\right) \\
= & \rho\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)
\end{aligned}$$

which completes the proof. □

Furthermore, some of the special cases of the AOs are obtained from the proposed CIFWA and CIFWG operators by assigning the different form of the generator t with $t(0) = \infty$ as follows:

(i) If $t(a) = -\log a$, $0 < a \leq 1$ then Eqs. (6.4) and (6.6) become

$$
\begin{aligned}
&\mathrm{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
= &\left(\left(1 - \prod_{j=1}^{n} (1-\zeta_j)^{\xi_j}, 1 - \prod_{j=1}^{n} (1 - w_{\zeta_j})^{\xi_j} \right), \left(\prod_{j=1}^{n} \left(\vartheta_j^{\xi_j} \right), \prod_{j=1}^{n} (w_{\vartheta_j})^{\xi_j} \right) \right) \\
&\mathrm{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
= &\left(\left(\prod_{j=1}^{n} \left(\zeta_j^{\xi_j} \right), \prod_{j=1}^{n} (w_{\zeta_j})^{\xi_j} \right), \left(1 - \prod_{j=1}^{n} (1-\vartheta_j)^{\xi_j}, 1 - \prod_{j=1}^{n} (1 - w_{\vartheta_j})^{\xi_j} \right) \right)
\end{aligned}
$$

and are called as CIF Archimedean weighted averaging and CIF Archimedean weighted geometric operators respectively.

(ii) If $t(a) = \log\left(\frac{2-a}{a}\right)$, then Eqs. (6.4) and (6.6) become

$$
\begin{aligned}
&\mathrm{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
= &\left(\left(\begin{array}{c} \dfrac{\prod_{j=1}^{n} (1+\zeta_j)^{\xi_j} - \prod_{j=1}^{n} (1-\zeta_j)^{\xi_j}}{\prod_{j=1}^{n} (1+\zeta_j)^{\xi_j} + \prod_{j=1}^{n} (1-\zeta_j)^{\xi_j}}, \\ \dfrac{\prod_{j=1}^{n} (1+w_{\zeta_j})^{\xi_j} - \prod_{j=1}^{n} (1-w_{\zeta_j})^{\xi_j}}{\prod_{j=1}^{n} (1+w_{\zeta_j})^{\xi_j} + \prod_{j=1}^{n} (1-w_{\zeta_j})^{\xi_j}} \end{array} \right), \left(\begin{array}{c} \dfrac{2\prod_{j=1}^{n} \vartheta_j^{\xi_j}}{\prod_{j=1}^{n} (2-\vartheta_j)^{\xi_j} + \prod_{j=1}^{n} \vartheta_j^{\xi_j}}, \\ \dfrac{2\prod_{j=1}^{n} (w_{\vartheta_j})^{\xi_j}}{\prod_{j=1}^{n} (2-w_{\vartheta_j})^{\xi_j} + \prod_{j=1}^{n} (w_{\vartheta_j})^{\xi_j}} \end{array} \right) \right) \\
&\mathrm{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
= &\left(\left(\begin{array}{c} \dfrac{2\prod_{j=1}^{n} \zeta_j^{\xi_j}}{\prod_{j=1}^{n} (2-\zeta_j)^{\xi_j} + \prod_{j=1}^{n} \zeta_j^{\xi_j}}, \\ \dfrac{2\prod_{j=1}^{n} (w_{\zeta_j})^{\xi_j}}{\prod_{j=1}^{n} (2-w_{\zeta_j})^{\xi_j} + \prod_{j=1}^{n} (w_{\zeta_j})^{\xi_j}} \end{array} \right), \left(\begin{array}{c} \dfrac{\prod_{j=1}^{n} (1+\vartheta_j)^{\xi_j} - \prod_{j=1}^{n} (1-\vartheta_j)^{\xi_j}}{\prod_{j=1}^{n} (1+\vartheta_j)^{\xi_j} + \prod_{j=1}^{n} (1-\vartheta_j)^{\xi_j}}, \\ \dfrac{\prod_{j=1}^{n} (1+w_{\vartheta_j})^{\xi_j} - \prod_{j=1}^{n} (1-w_{\vartheta_j})^{\xi_j}}{\prod_{j=1}^{n} (1+w_{\vartheta_j})^{\xi_j} + \prod_{j=1}^{n} (1-w_{\vartheta_j})^{\xi_j}} \end{array} \right) \right)
\end{aligned}
$$

and are called as CIF Einstein weighted averaging(CIFEWA) and CIF Einstein weighted geometric(CIFEWG) operators respectively.

(iii) If $t(a) = \log\left(\frac{\gamma+(1-\gamma)a}{a}\right)$, $\gamma \in (0,\infty)$ then Eqs. (6.4) and (6.6) become

$$\text{CIFWA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)$$
$$= \left(\left(\begin{array}{c}\frac{\prod_{j=1}^{n}(1+(\gamma-1)\zeta_j)^{\xi_j}-\prod_{j=1}^{n}(1-\zeta_j)^{\xi_j}}{\prod_{j=1}^{n}(1+(\gamma-1)\zeta_j)^{\xi_j}+\prod_{j=1}^{n}(1-\zeta_j)^{\xi_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+(\gamma-1)w_{\zeta_j}\right)^{\xi_j}-\prod_{j=1}^{n}\left(1-w_{\zeta_j}\right)^{\xi_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)w_{\zeta_j}\right)^{\xi_j}+\prod_{j=1}^{n}\left(1-w_{\zeta_j}\right)^{\xi_j}}\end{array}\right), \left(\begin{array}{c}\frac{\gamma\prod_{j=1}^{n}\vartheta_j^{\xi_j}}{\prod_{j=1}^{n}(1+(\gamma-1)(1-\vartheta_j))^{\xi_j}+(\gamma-1)\prod_{j=1}^{n}\vartheta_j^{\xi_j}}, \\ \frac{\gamma\prod_{j=1}^{n}\left(w_{\vartheta_j}\right)^{\xi_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(1-w_{\vartheta_j}\right)\right)^{\xi_j}+(\gamma-1)\prod_{j=1}^{n}\left(w_{\vartheta_j}\right)^{\xi_j}}\end{array}\right)\right)$$

$$\text{CIFWG}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)$$
$$= \left(\left(\begin{array}{c}\frac{\gamma\prod_{j=1}^{n}\zeta_j^{\xi_j}}{\prod_{j=1}^{n}(1+(\gamma-1)(1-\zeta_j))^{\xi_j}+(\gamma-1)\prod_{j=1}^{n}\zeta_j^{\xi_j}}, \\ \frac{\gamma\prod_{j=1}^{n}\left(w_{\zeta_j}\right)^{\xi_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(1-w_{\zeta_j}\right)\right)^{\xi_j}+(\gamma-1)\prod_{j=1}^{n}\left(w_{\zeta_j}\right)^{\xi_j}}\end{array}\right), \left(\begin{array}{c}\frac{\prod_{j=1}^{n}(1+(\gamma-1)\vartheta_j)^{\xi_j}-\prod_{j=1}^{n}(1-\vartheta_j)^{\xi_j}}{\prod_{j=1}^{n}(1+(\gamma-1)\vartheta_j)^{\xi_j}+\prod_{j=1}^{n}(1-\vartheta_j)^{\xi_j}}, \\ \frac{\prod_{j=1}^{n}\left(1+(\gamma-1)w_{\vartheta_j}\right)^{\xi_j}-\prod_{j=1}^{n}\left(1-w_{\vartheta_j}\right)^{\xi_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)w_{\vartheta_j}\right)^{\xi_j}+\prod_{j=1}^{n}\left(1-w_{\vartheta_j}\right)^{\xi_j}}\end{array}\right)\right)$$

and are called as CIF Hammacher weighted averaging(CIFHWA) and CIF Hammacher weighted geometric(CIFHWG) operators respectively.

6.2.4 Ordered weighted averaging and geometric operators

In this section, we propose a new operator named as CIF ordered weighted averaging (CIFOWA) operator.

Definition 6.2.5. Let Ω be a collection of CIFNs. We define a map CIFOWA : $\Omega^n \to \Omega$ by

$$\text{CIFOWA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n) = \xi_1\mathcal{C}_{\tau(1)} \oplus \xi_2\mathcal{C}_{\tau(2)} \oplus \ldots \oplus \xi_n\mathcal{C}_{\tau(n)}$$

for all $\mathcal{C}_j \in \Omega$ where $\big(\tau(1),\tau(2),\ldots,\tau(n)\big)$ is a permutation of $(1,2,\ldots,n)$ such that $\mathcal{S}\left(\mathcal{C}_{\tau(j-1)}\right) \geq \mathcal{S}\left(\mathcal{C}_{\tau(j)}\right)$ for $j = 2,3,\ldots,n$ and $\xi = (\xi_1,\xi_2,\ldots,\xi_n)^T$ is the weight vector of $\mathcal{C}_j$ with $\xi_j > 0$ and $\sum_{j=1}^{n}\xi_j = 1$. Then, CIFOWA is called CIF ordered weighted averaging operator.

Theorem 6.2.10. The aggregated value by using CIFOWA operator for a collection of

CIFNs $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right)$ $(j = 1, 2, \ldots n)$ is still a CIFN and is given by

$$\begin{aligned} &\mathrm{CIFOWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\ &= \left(\left(\begin{array}{c} s^{-1}\left(\sum_{j=1}^{n} \xi_j s(\zeta_{\tau(j)})\right), \\ s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(w_{\zeta_{\tau(j)}}\right)\right) \end{array}\right), \left(\begin{array}{c} t^{-1}\left(\sum_{j=1}^{n} \xi_j t(\vartheta_{\tau(j)})\right), \\ t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(w_{\vartheta_{\tau(j)}}\right)\right) \end{array}\right)\right) \end{aligned} \tag{6.8}$$

In particular, if $w_{\zeta_j}, w_{\vartheta_j} = 0 \ \forall \ j$ then, Eq. (6.8) reduces to

$$\mathrm{CIFOWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\left(s^{-1}\left(\sum_{j=1}^{n} \xi_j s(\zeta_{\tau(j)})\right)\right), \left(t^{-1}\left(\sum_{j=1}^{n} \xi_j t(\vartheta_{\tau(j)})\right)\right)\right)$$

which is an intuitionistic fuzzy OWA operator.

Proof. Proof is similar to Theorem 6.2.8. $\square$

Definition 6.2.6. Let Ω be a collection of CIFNs. We define a map CIFOWG : $\Omega^n \to \Omega$ by

$$\mathrm{CIFOWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathcal{C}_{\tau(1)}^{\xi_1} \otimes \mathcal{C}_{\tau(2)}^{\xi_2} \otimes \ldots \otimes \mathcal{C}_{\tau(n)}^{\xi_n}$$

Then, CIFOWG is called CIF ordered weighted geometric operator.

Theorem 6.2.11. The aggregated value by using CIFOWG operator for a collection of CIFNs $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right)$ $(j = 1, 2, \ldots n)$ is still a CIFN and is given by

$$\mathrm{CIFOWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\left(\begin{array}{c} t^{-1}\left(\sum_{j=1}^{n} \xi_j t(\zeta_{\tau(j)})\right), \\ t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(w_{\zeta_{\tau(j)}}\right)\right) \end{array}\right), \left(\begin{array}{c} s^{-1}\left(\sum_{j=1}^{n} \xi_j s(\vartheta_{\tau(j)})\right), \\ s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(w_{\vartheta_{\tau(j)}}\right)\right) \end{array}\right)\right) \tag{6.9}$$

Example 6.2.2. Let $\mathcal{C}_1 = \big((0.6, 0.8), (0.2, 0.1)\big)$, $\mathcal{C}_2 = \big((0.8, 0.7), (0.2, 0.1)\big)$, $\mathcal{C}_3 = \big((0.5, 0.6), (0.3, 0.4)\big)$, $\mathcal{C}_4 = \big((0.6, 0.7), (0.3, 0.2)\big)$ be four CIFNs and $\xi = (0.35, 0.3, 0.1, 0.25)^T$ be the associated weight vector. Then, score function of each CIFN is calculated as $\mathcal{S}(\mathcal{C}_1) = 1.1$, $\mathcal{S}(\mathcal{C}_2) = 1.2$, $\mathcal{S}(\mathcal{C}_3) = 0.4$ and $\mathcal{S}(\mathcal{C}_4) = 0.8$. Since $\mathcal{S}(\mathcal{C}_2) > \mathcal{S}(\mathcal{C}_1) > \mathcal{S}(\mathcal{C}_4) > \mathcal{S}(\mathcal{C}_3)$ and hence by permutation, we have $\mathcal{C}_{\tau(1)} = \big((0.8, 0.7), (0.2, 0.1)\big)$, $\mathcal{C}_{\tau(2)} = \big((0.6, 0.8), (0.2, 0.1)\big)$,

$\mathcal{C}_{\tau(3)} = \big((0.6, 0.7), (0.3, 0.2)\big)$ and $\mathcal{C}_{\tau(4)} = \big((0.5, 0.6), (0.3, 0.4)\big)$. Without loss of generality, we consider the additive generator $t(a) = -\log a$ if $0 < a \leq 1$ with $t(0) = \infty$ corresponding to t-norm. Then, Eq. (6.8) becomes

$$\mathrm{CIFOWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{array}{c} \left(1 - \prod_{j=1}^{n} \left(1 - \zeta_{\tau(j)}\right)^{\xi_j}, 1 - \prod_{j=1}^{n} \left(1 - w_{\zeta_{\tau(j)}}\right)^{\xi_j} \right), \\ \left(\prod_{j=1}^{n} \vartheta_{\tau(j)}^{\xi_j}, \prod_{j=1}^{n} \left(w_{\vartheta_{\tau(j)}}\right)^{\xi_j} \right) \end{array} \right) \tag{6.10}$$

Therefore,

$$\begin{aligned} \prod_{j=1}^{4} \left(1 - \zeta_{\tau(j)}\right)^{\xi_j} &= (1 - 0.8)^{0.35} \times (1 - 0.6)^{0.3} \times (1 - 0.6)^{0.1} \times (1 - 0.5)^{0.25} = 0.3318 \\ \prod_{j=1}^{4} \vartheta_{\tau(j)}^{\xi_j} &= (0.2)^{0.35} \times (0.2)^{0.3} \times (0.3)^{0.1} \times (0.3)^{0.25} = 0.2305 \\ \prod_{j=1}^{4} \left(1 - w_{\zeta_{\tau(j)}}\right)^{\xi_j} &= (1 - 0.7)^{0.35} \times (1 - 0.8)^{0.3} \times (1 - 0.7)^{0.1} \times (1.0.6)^{0.25} = 0.2854 \\ \prod_{j=1}^{4} \left(w_{\vartheta_{\tau(j)}}\right)^{\xi_j} &= (0.1)^{0.35} \times (0.1)^{0.3} \times (0.2)^{0.1} \times (0.4)^{0.25} = 0.1516 \end{aligned}$$

Thus, by using Eq. (6.10), we obtain

$$\begin{aligned} \mathrm{CIFOWA}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4) &= \left((1 - 0.3318, 1 - 0.2854), (0.2305, 0.1516)\right) \\ &= \left((0.6682, 0.7146), (0.2305, 0.1516)\right) \end{aligned}$$

Also, it is analyzed that CIFOWA and CIFOWG operators satisfy the properties of idempotency, monotonicity and boundedness. Besides these properties, CIFOWA operator satisfies the additional properties which are stated as below.

Property 6.2.6. Let $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ be the collection of CIFNs. Then,

(i) If $\xi = (1, 0, \ldots, 0)^T$ then, $\mathrm{CIFOWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \max\left\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\right\}$

(ii) If $\xi = (0, 0, \ldots, 1)^T$ then, $\mathrm{CIFOWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \min\left\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\right\}$

(iii) If $\xi_j = 1$ and $\xi_l = 0$ for $l \neq j$ then, $\mathrm{CIFOWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathcal{C}_{\tau(j)}$ where $\mathcal{C}_{\tau(j)}$ is the jth largest of $\mathcal{C}_j$.

6.2.5 Hybrid averaging and geometric operators

In this section, by taking the advantages of both ordered weighted and weighted averaging operators, we propose a hybrid averaging AO for a collection of CIFNs.

Definition 6.2.7. Let Ω be a collection of CIFNs. A map CIFHA : $\Omega^n \to \Omega$, having associated weight vector $\psi = \left(\psi_1, \psi_2, \ldots, \psi_n\right)^T$ with $\psi_j > 0$ and $\sum_{j=1}^{n} \psi_j = 1$, defined by

$$\text{CIFHA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \psi_1 \dot{\mathcal{C}}_{\tau(1)} \oplus \psi_2 \dot{\mathcal{C}}_{\tau(2)} \oplus \ldots \oplus \psi_n \dot{\mathcal{C}}_{\tau(n)}$$

where $\dot{\mathcal{C}}_j = n\xi_j \mathcal{C}_j$, $j = 1, 2, \ldots, n$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ is the weight vector of $\mathcal{C}_j$ with $\xi_j > 0$ and $\sum_{j=1}^{n} \xi_j = 1$, is called CIF hybrid averaging (CIFHA) operator.

Theorem 6.2.12. The combined value obtained after applying CIFHA operator for a collection of CIFNs $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right) \right)$ $(j = 1, 2, \ldots n)$ is still a CIFN and is given by

$$\text{CIFHA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{n} \psi_j s(\dot{\zeta}_{\tau(j)}) \right), \\ s^{-1}\left(\sum_{j=1}^{n} \psi_j s\left(\dot{w}_{\zeta_{\tau(j)}} \right) \right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{n} \psi_j t(\dot{\vartheta}_{\tau(j)}) \right), \\ t^{-1}\left(\sum_{j=1}^{n} \psi_j t\left(\dot{w}_{\vartheta_{\tau(j)}} \right) \right) \end{pmatrix} \right) \tag{6.11}$$

Proof. Similar to the Theorem 6.2.8. □

Definition 6.2.8. Let Ω be a collection of CIFNs. A map CIFHG : $\Omega^n \to \Omega$, having associated weight vector $\psi = \left(\psi_1, \psi_2, \ldots, \psi_n\right)^T$ with $\psi_j > 0$ and $\sum_{j=1}^{n} \psi_j = 1$, defined by

$$\text{CIFHG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\dot{\mathcal{C}}_{\tau(1)}\right)^{\psi_1} \otimes \left(\dot{\mathcal{C}}_{\tau(2)}\right)^{\psi_2} \otimes \ldots \otimes \left(\dot{\mathcal{C}}_{\tau(n)}\right)^{\psi_n}$$

where $\dot{\mathcal{C}}_j = (\mathcal{C}_j)^{n\xi_j}$, $j = 1, 2, \ldots, n$ and is called CIF hybrid geometric (CIFHG) operator.

Theorem 6.2.13. The combined value obtained after applying CIFHG operator for a collection of CIFNs $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right) \right)$ $(j = 1, 2, \ldots n)$ is still a CIFN and is given by

$$\text{CIFHG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} t^{-1}\left(\sum_{j=1}^{n} \psi_j t(\dot{\zeta}_{\tau(j)}) \right), \\ t^{-1}\left(\sum_{j=1}^{n} \psi_j t\left(\dot{w}_{\zeta_{\tau(j)}} \right) \right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\sum_{j=1}^{n} \psi_j s(\dot{\vartheta}_{\tau(j)}) \right), \\ s^{-1}\left(\sum_{j=1}^{n} \psi_j s\left(\dot{w}_{\vartheta_{\tau(j)}} \right) \right) \end{pmatrix} \right) \tag{6.12}$$

The proposed CIFHA and CIFHG operators also satisfy the above stated properties.

Theorem 6.2.14. If $\psi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$ then, CIFHA operator becomes CIFWA operator.

Proof. Since, $\dot{\mathcal{C}}_{\tau(j)} = n\xi_j \mathcal{C}_j$ and $\psi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$ which implies that $\psi_j \dot{\mathcal{C}}_{\tau(j)} = \xi_j \mathcal{C}_j$. Therefore,

$$\begin{aligned} \mathrm{CIFHA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) &= \psi_1 \dot{\mathcal{C}}_{\tau(1)} \oplus \psi_2 \dot{\mathcal{C}}_{\tau(2)} \oplus \ldots \oplus \psi_n \dot{\mathcal{C}}_{\tau(n)} \\ &= \xi_1 \mathcal{C}_1 \oplus \xi_2 \mathcal{C}_2 \oplus \ldots \oplus \xi_n \mathcal{C}_n \\ &= \mathrm{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \end{aligned}$$

Hence, $\mathrm{CIFHA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathrm{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)$. □

Theorem 6.2.15. If $\xi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$ then, CIFHA operator becomes CIFOWA operator.

Proof. Similar to the proof of the above theorem. □

6.2.6 Fundamental properties related to AOs

In this section, we investigated the several relations between the proposed AOs and study their some fundamental properties by taking generator $t(a) = -\log(a)$, $a > 0$, as follows.

Theorem 6.2.16. For two CIFNs $\mathcal{C}_1$ and $\mathcal{C}_2$ we have, $\mathcal{C}_1 \oplus \mathcal{C}_2 \supseteq \mathcal{C}_1 \otimes \mathcal{C}_2$.

Proof. Let $\mathcal{C}_1 = \Big((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1})\Big)$ and $\mathcal{C}_2 = \Big((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2})\Big)$ be two CIFNs. Then, by using Definition 6.2.2, we get

$$\mathcal{C}_1 \oplus \mathcal{C}_2 = \left(\left(1 - \prod_{j=1}^{2}(1-\zeta_j), 1 - \prod_{j=1}^{2}(1 - w_{\zeta_j})\right), \left(\prod_{j=1}^{2}\vartheta_j, \prod_{j=1}^{2} w_{\vartheta_j}\right)\right)$$

and

$$\mathcal{C}_1 \otimes \mathcal{C}_2 = \left(\left(\prod_{j=1}^{2}\zeta_j, \prod_{j=1}^{2} w_{\zeta_j}\right), \left(1 - \prod_{j=1}^{2}(1-\vartheta_j), 1 - \prod_{j=1}^{2}(1 - w_{\vartheta_j})\right)\right)$$

Since for any two non-negative real numbers, their arithmetic mean is greater than or equal to their geometric mean therefore, $\frac{\zeta_1 + \zeta_2}{2} \geq \sqrt{\zeta_1 \zeta_2} \geq \zeta_1 \zeta_2$, which follows that

$\zeta_1+\zeta_2-\zeta_1\zeta_2 \geq \zeta_1\zeta_2$ which gives $1-\prod_{j=1}^{2}\left(1-\zeta_j\right) \geq \prod_{j=1}^{2}\zeta_j$. Similarly, $\prod_{j=1}^{2}\vartheta_j \leq 1-\prod_{j=1}^{2}\left(1-\vartheta_j\right)$, $1-\prod_{j=1}^{2}\left(1-w_{\zeta_j}\right) \geq \prod_{j=1}^{2}w_{\zeta_j}$ and $\prod_{j=1}^{2}w_{\vartheta_j} \leq 1-\prod_{j=1}^{2}\left(1-w_{\vartheta_j}\right)$. Hence, by using Definition 2.1.10, we get $\mathcal{C}_1 \oplus \mathcal{C}_2 \supseteq \mathcal{C}_1 \otimes \mathcal{C}_2$. □

Theorem 6.2.17. For any CIFN $\mathcal{C}$ and real number $\rho > 0$, $\rho\mathcal{C} \supseteq \mathcal{C}^{\rho}$ if and only if $\rho \geq 1$ and $\rho\mathcal{C} \subseteq \mathcal{C}^{\rho}$ if and only if $0 < \rho \leq 1$.

Proof. As similar to the above theorem. So, we omit the proof here. □

Lemma 6.2.1. For $a_j \geq 0$ and $b_j > 0$, then we have $\prod_{j=1}^{n} a_j^{b_j} \leq \sum_{j=1}^{n} b_j a_j$ and the equality holds iff $a_1 = a_2 = \ldots = a_n$.

Lemma 6.2.2. Let $0 \leq a, b \leq 1$, and $0 \leq x \leq 1$, then $0 \leq ax + b(1-x) \leq 1$.

Lemma 6.2.3. For $0 \leq a, b \leq 1$, we have $(1-a)(1-b) + ab \leq 1$.

Theorem 6.2.18. For a collection of "n" CIFNs $\mathcal{C}_j$, we have

$$\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \supseteq \text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n),$$

Proof. For "n" CIFNs $\mathcal{C}_j = \left((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\right)$ and normalized weight vector $\xi_j > 0$, we have

$$\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\left(1-\prod_{j=1}^{2}\left(1-\zeta_j\right)^{\xi_j}, 1-\prod_{j=1}^{2}\left(1-w_{\zeta_j}\right)^{\xi_j}\right), \left(\prod_{j=1}^{2}\left(\vartheta_j\right)^{\xi_j}, \prod_{j=1}^{2}\left(w_{\vartheta_j}\right)^{\xi_j}\right)\right)$$

and

$$\text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\left(\prod_{j=1}^{2}\left(\zeta_j\right)^{\xi_j}, \prod_{j=1}^{2}\left(w_{\zeta_j}\right)^{\xi_j}\right), \left(1-\prod_{j=1}^{2}\left(1-\vartheta_j\right)^{\xi_j}, 1-\prod_{j=1}^{2}\left(1-w_{\vartheta_j}\right)^{\xi_j}\right)\right)$$

For $\xi_j > 0$, $\zeta_j \in [0,1]$ and by Lemma 6.2.1, we have $\prod_{j=1}^{n}\left(1-\zeta_j\right)^{\xi_j} \leq \sum_{j=1}^{n}\xi_j(1-\zeta_j) = 1-\sum_{j=1}^{n}\xi_j\zeta_j = 1-\prod_{j=1}^{n}\left(\zeta_j\right)^{\xi_j}$. Hence, $1-\prod_{j=1}^{n}\left(1-\zeta_j\right)^{\xi_j} \geq \prod_{j=1}^{n}\left(\zeta_j\right)^{\xi_j}$. Similarly, we can obtain $1-\prod_{j=1}^{n}\left(1-\vartheta_j\right)^{\xi_j} \geq \prod_{j=1}^{n}\left(\vartheta_j\right)^{\xi_j}$, $1-\prod_{j=1}^{n}\left(1-w_{\zeta_j}\right)^{\xi_j} \geq \prod_{j=1}^{n}\left(w_{\zeta_j}\right)^{\xi_j}$ and $1-\prod_{j=1}^{n}\left(1-w_{\vartheta_j}\right)^{\xi_j} \geq \prod_{j=1}^{n}\left(w_{\vartheta_j}\right)^{\xi_j}$. Hence, by using Definition 2.1.10, we get the result. □

Theorem 6.2.19. Let $\mathcal{C}_j$ $(j = 1(1)n)$ be the collection of CIFNs and let $\mathcal{C}$ is also CIFN. If $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ be the weight vector corresponding to them such that $\xi_j > 0$ and $\sum\limits_{j=1}^{n} \xi_j = 1$, then

(i) $\text{CIFWG}(\mathcal{C}_1 \oplus \mathcal{C}, \mathcal{C}_2 \oplus \mathcal{C}, \ldots, \mathcal{C}_n \oplus \mathcal{C}) \supseteq \text{CIFWG}(\mathcal{C}_1 \otimes \mathcal{C}, \mathcal{C}_2 \otimes \mathcal{C}, \ldots, \mathcal{C}_n \otimes \mathcal{C})$;

(ii) $\text{CIFWG}(\mathcal{C}_1 \oplus \mathcal{C}, \mathcal{C}_2 \oplus \mathcal{C}, \ldots, \mathcal{C}_n \oplus \mathcal{C}) \supseteq \text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \otimes \mathcal{C}$;

(iii) $\text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \oplus \mathcal{C} \supseteq \text{CIFWG}(\mathcal{C}_1 \otimes \mathcal{C}, \mathcal{C}_2 \otimes \mathcal{C}, \ldots, \mathcal{C}_n \otimes \mathcal{C})$;

(iv) $\text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \oplus \mathcal{C} \supseteq \text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \otimes \mathcal{C}$;

(v) $\text{CIFWA}(\mathcal{C}_1 \oplus \mathcal{C}, \mathcal{C}_2 \oplus \mathcal{C}, \ldots, \mathcal{C}_n \oplus \mathcal{C}) \supseteq \text{CIFWA}(\mathcal{C}_1 \otimes \mathcal{C}, \mathcal{C}_2 \otimes \mathcal{C}, \ldots, \mathcal{C}_n \otimes \mathcal{C})$;

(vi) $\text{CIFWA}(\mathcal{C}_1 \oplus \mathcal{C}, \mathcal{C}_2 \oplus \mathcal{C}, \ldots, \mathcal{C}_n \oplus \mathcal{C}) \supseteq \text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \otimes \mathcal{C}$;

(vii) $\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \oplus \mathcal{C} \supseteq \text{CIFWA}(\mathcal{C}_1 \otimes \mathcal{C}, \mathcal{C}_2 \otimes \mathcal{C}, \ldots, \mathcal{C}_n \otimes \mathcal{C})$;

(viii) $\text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \oplus \mathcal{C} \supseteq \text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \otimes \mathcal{C}$.

Proof. Here, we will prove the parts (i) and (ii) while others can be deduced similarly. For this, let $\mathcal{C}_j = \Big((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\Big)$ and $\mathcal{C} = \Big((\zeta, w_\zeta), (\vartheta, w_\vartheta)\Big)$.

(i) Since $\mathcal{C}_j$ and $\mathcal{C}$ are CIFNs.

$$\begin{aligned} &\text{CIFWG}(\mathcal{C}_1 \oplus \mathcal{C}, \mathcal{C}_2 \oplus \mathcal{C}, \ldots, \mathcal{C}_n \oplus \mathcal{C}) \\ = &\left(\begin{pmatrix} \prod\limits_{j=1}^{n} \Big(1 - (1-\zeta_j)(1-\zeta)\Big)^{\xi_j}, \\ \prod\limits_{j=1}^{n} \Big(1 - (1-w_{\zeta_j})(1-w_\zeta)\Big)^{\xi_j} \end{pmatrix}, \begin{pmatrix} 1 - \prod\limits_{j=1}^{n} (1-\vartheta_j\vartheta)^{\xi_j}, \\ 1 - \prod\limits_{j=1}^{n} (1-w_{\vartheta_j}w_\vartheta)^{\xi_j} \end{pmatrix} \right) \end{aligned} \tag{6.13}$$

and
$$\begin{aligned} &\text{CIFWG}(\mathcal{C}_1 \otimes \mathcal{C}, \mathcal{C}_2 \otimes \mathcal{C}, \ldots, \mathcal{C}_n \otimes \mathcal{C}) \\ = &\left(\begin{pmatrix} \prod\limits_{j=1}^{n} (\zeta_j\zeta)^{\xi_j}, \\ \prod\limits_{j=1}^{n} (w_{\zeta_j}w_\zeta)^{\xi_j} \end{pmatrix}, \begin{pmatrix} 1 - \prod\limits_{j=1}^{n} \Big((1-\vartheta_j)(1-\vartheta)\Big)^{\xi_j}, \\ 1 - \prod\limits_{j=1}^{n} \Big((1-w_{\vartheta_j})(1-w_\vartheta)\Big)^{\xi_j} \end{pmatrix} \right) \end{aligned} \tag{6.14}$$

Since for any two non-negative numbers, their arithmetic mean is greater than or equal to their geometric mean therefore, $\frac{\zeta_j+\zeta}{2} \geq \sqrt{\zeta_j\zeta} \geq \zeta_j\zeta$, which follows that

$\zeta_j + \zeta - \zeta_j\zeta \geq \zeta_j\zeta$ which gives $1-(1-\zeta_j)(1-\zeta) \geq \zeta_j\zeta$. Hence, $\prod\limits_{j=1}^{n}\Big(1-\big(1-\zeta_j\big)\big(1-\zeta\big)\Big)^{\xi_j} \geq \prod\limits_{j=1}^{n}(\zeta_j\zeta)^{\xi_j}$. Similarly, $\prod\limits_{j=1}^{n}\Big(1-\big(1-w_{\zeta_j}\big)\big(1-w_{\zeta}\big)\Big)^{\xi_j} \geq \prod\limits_{j=1}^{n}(w_{\zeta_j}w_{\zeta})^{\xi_j}$. Further, for $\vartheta_j, \vartheta \in [0,1]$ and by Lemma 6.2.3, we have $(1-\vartheta_j)(1-\vartheta)+\vartheta_j\vartheta \leq 1$ which implies that $(1-\vartheta_j)(1-\vartheta) \leq 1-\vartheta_j\vartheta$. Hence,

$$\begin{aligned} & ((1-\vartheta_j)(1-\vartheta))^{\xi_j} \leq (1-\vartheta_j\vartheta)^{\xi_j} \\ \Rightarrow\;\; & 1-\prod_{j=1}^{n}((1-\vartheta_j)(1-\vartheta))^{\xi_j} \geq 1-\prod_{j=1}^{n}(1-\vartheta_j\vartheta)^{\xi_j} \end{aligned}$$

Similarly, we have

$$1-\prod_{j=1}^{n}\big((1-w_{\vartheta_j})(1-w_{\vartheta})\big)^{\xi_j} \geq 1-\prod_{j=1}^{n}\big(1-w_{\vartheta_j}w_{\vartheta}\big)^{\xi_j}.$$

Therefore, from Eqs. (6.13), Eq. (6.14) and by Definition 2.1.10, we get

$$\text{CIFWG}(\mathcal{C}_1\oplus\mathcal{C}, \mathcal{C}_2\oplus\mathcal{C}, \ldots, \mathcal{C}_n\oplus\mathcal{C}) \supseteq \text{CIFWG}(\mathcal{C}_1\otimes\mathcal{C}, \mathcal{C}_2\otimes\mathcal{C}, \ldots, \mathcal{C}_n\otimes\mathcal{C}).$$

(iii) Since $\mathcal{C}_j$ and $\mathcal{C}$ are CIFNs and $\xi_j > 0$ such that $\sum\limits_{j=1}^{n}\xi_j = 1$. Now, using Theorems 6.2.16, 6.2.17 and 6.2.18, we get

$$\begin{aligned} & \text{CIFWG}(\mathcal{C}_1\oplus\mathcal{C}, \mathcal{C}_2\oplus\mathcal{C}, \ldots, \mathcal{C}_n\oplus\mathcal{C}) \\ = \;& (\mathcal{C}_1\oplus\mathcal{C})^{\xi_1}\otimes(\mathcal{C}_2\oplus\mathcal{C})^{\xi_2}\otimes\ldots\otimes(\mathcal{C}_n\oplus\mathcal{C})^{\xi_n} \\ \supseteq \;& \xi_1(\mathcal{C}_1\oplus\mathcal{C})\oplus\xi_2(\mathcal{C}_2\oplus\mathcal{C})\oplus\ldots\oplus\xi_n(\mathcal{C}_n\oplus\mathcal{C}) \\ = \;& \xi_1\mathcal{C}_1\oplus\xi_1\mathcal{C}\oplus\xi_2\mathcal{C}_2\oplus\xi_2\mathcal{C}\oplus\ldots\oplus\xi_n\mathcal{C}_n\oplus\xi_n\mathcal{C} \\ = \;& \big(\xi_1\mathcal{C}_1\oplus\xi_2\mathcal{C}_2\oplus\ldots\oplus\xi_n\mathcal{C}_n\big)\oplus\big(\xi_1\mathcal{C}\oplus\xi_2\mathcal{C}\oplus\ldots\oplus\xi_n\mathcal{C}\big) \\ = \;& \text{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)\oplus\mathcal{C} \\ \supseteq \;& \text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)\otimes\mathcal{C} \end{aligned}$$

Hence, the result.

□

Theorem 6.2.20. Let $\mathcal{C}_j$ $(j = 1(1)n)$ be the collection of CIFNs and $\mathcal{C}$ is also CIFN. For a real number $\rho > 0$, we have

(i) $\text{CIFWG}(\rho\mathcal{C}_1 \oplus \mathcal{C}, \rho\mathcal{C}_2 \oplus \mathcal{C}, \ldots, \rho\mathcal{C}_n \oplus \mathcal{C}) \supseteq \text{CIFWG}(\mathcal{C}_1^\rho \otimes \mathcal{C}, \mathcal{C}_2^\rho \otimes \mathcal{C}, \ldots, \mathcal{C}_n^\rho \otimes \mathcal{C})$.

(ii) $\text{CIFWG}(\mathcal{C}_1^\rho \oplus \mathcal{C}, \mathcal{C}_2^\rho \oplus \mathcal{C}, \ldots, \mathcal{C}_n^\rho \oplus \mathcal{C}) \supseteq \text{CIFWG}(\rho\mathcal{C}_1 \otimes \mathcal{C}, \rho\mathcal{C}_2 \otimes \mathcal{C}, \ldots, \rho\mathcal{C}_n \otimes \mathcal{C})$ if $0 < \rho \leq 1$;

(iii) $\text{CIFWA}(\rho\mathcal{C}_1 \oplus \mathcal{C}, \rho\mathcal{C}_2 \oplus \mathcal{C}, \ldots, \rho\mathcal{C}_n \oplus \mathcal{C}) \supseteq \text{CIFWA}(\mathcal{C}_1^\rho \otimes \mathcal{C}, \mathcal{C}_2^\rho \otimes \mathcal{C}, \ldots, \mathcal{C}_n^\rho \otimes \mathcal{C})$.

(iv) $\text{CIFWA}(\mathcal{C}_1^\rho \oplus \mathcal{C}, \mathcal{C}_2^\rho \oplus \mathcal{C}, \ldots, \mathcal{C}_n^\rho \oplus \mathcal{C}) \supseteq \text{CIFWA}(\rho\mathcal{C}_1 \otimes \mathcal{C}, \rho\mathcal{C}_2 \otimes \mathcal{C}, \ldots, \rho\mathcal{C}_n \otimes \mathcal{C})$ if $0 < \rho \leq 1$.

Proof. Here, we will prove the part (i) only, while others can be deduced similarly. For this, let $\mathcal{C}_j = \Big((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\Big)$ and $\mathcal{C} = \Big((\zeta, w_\zeta), (\vartheta, w_\vartheta)\Big)$.

(i) Since $\mathcal{C}_j$ and $\mathcal{C}$ are CIFNs and $\rho > 0$ is any real number.

$$\begin{aligned}
&\text{CIFWG}(\rho\mathcal{C}_1 \oplus \mathcal{C}, \rho\mathcal{C}_2 \oplus \mathcal{C}, \ldots, \rho\mathcal{C}_n \oplus \mathcal{C}) \\
&= \left(\begin{pmatrix} \prod_{j=1}^{n} \Big(1 - (1-\zeta_j)^\rho (1-\zeta)\Big)^{\xi_j}, \\ \prod_{j=1}^{n} \Big(1 - (1-w_{\zeta_j})^\rho (1-w_\zeta)\Big)^{\xi_j} \end{pmatrix}, \begin{pmatrix} 1 - \prod_{j=1}^{n} \Big(1 - (\vartheta_j)^\rho \vartheta\Big)^{\xi_j}, \\ 1 - \prod_{j=1}^{n} \Big(1 - (w_{\vartheta_j})^\rho w_\vartheta\Big)^{\xi_j} \end{pmatrix} \right)
\end{aligned}$$

and

$$\begin{aligned}
&\text{CIFWG}(\mathcal{C}_1^\rho \otimes \mathcal{C}, \mathcal{C}_2^\rho \otimes \mathcal{C}, \ldots, \mathcal{C}_n^\rho \otimes \mathcal{C}) \\
&= \left(\begin{pmatrix} \prod_{j=1}^{n} \left(\zeta(\zeta_j)^\rho\right)^{\xi_j}, \\ \prod_{j=1}^{n} \left(w_\zeta(w_{\zeta_j})^\rho\right)^{\xi_j} \end{pmatrix}, \begin{pmatrix} 1 - \prod_{j=1}^{n} \Big((1-\vartheta_j)^\rho (1-\vartheta)\Big)^{\xi_j}, \\ 1 - \prod_{j=1}^{n} \Big((1-w_{\vartheta_j})^\rho (1-w_\vartheta)\Big)^{\xi_j} \end{pmatrix} \right)
\end{aligned}$$

As $\vartheta_j, \vartheta \in [0,1]$ and $\rho > 0$ be a real number. Thus, $(1-\vartheta_j)^\rho, (\vartheta_j)^\rho \in [0,1]$. So, by using Lemma 6.2.2, we have $(\vartheta_j)^\rho\vartheta + (1-\vartheta_j)^\rho(1-\vartheta) \leq 1$ which implies that

$$\begin{aligned}
&(1-\vartheta_j)^\rho(1-\vartheta) \leq 1 - (\vartheta_j)^\rho\vartheta \\
\Rightarrow\ & \prod_{j=1}^{n} ((1-\vartheta_j)^\rho(1-\vartheta))^{\xi_j} \leq \prod_{j=1}^{n} (1-(\vartheta_j)^\rho\vartheta)^{\xi_j} \\
\Rightarrow\ & 1 - \prod_{j=1}^{n} ((1-\vartheta_j)^\rho(1-\vartheta))^{\xi_j} \geq 1 - \prod_{j=1}^{n} (1-(\vartheta_j)^\rho\vartheta)^{\xi_j} \qquad (6.15)
\end{aligned}$$

Further, for $\zeta_j, \zeta \in [0,1]$, we have $(1-\zeta_j)^\rho, (\zeta_j)^\rho \in [0,1]$. This, again by Lemma 6.2.2, we have

$$\begin{aligned} & (\zeta_j)^\rho \zeta + (1-\zeta_j)^\rho (1-\zeta) \leq 1 \\ \Rightarrow \quad & (\zeta_j)^\rho \zeta \leq 1 - (1-\zeta_j)^\rho (1-\zeta) \\ \Rightarrow \quad & \prod_{j=1}^{n} ((\zeta_j)^\rho \zeta)^{\xi_j} \leq \prod_{j=1}^{n} (1-(1-\zeta_j)^\rho(1-\zeta))^{\xi_j} \end{aligned} \tag{6.16}$$

Similarly, we can obtain that

$$1 - \prod_{j=1}^{n} \left((1-w_{\vartheta_j})^\rho (1-w_\vartheta)\right)^{\xi_j} \geq 1 - \prod_{j=1}^{n} \left(1-(w_{\vartheta_j})^\rho w_\vartheta\right)^{\xi_j} \tag{6.17}$$

$$\text{and} \qquad \prod_{j=1}^{n} \left((w_{\zeta_j})^\rho w_\zeta\right)^{\xi_j} \leq \prod_{j=1}^{n} \left(1-(1-w_{\zeta_j})^\rho (1-w_\zeta)\right)^{\xi_j} \tag{6.18}$$

Thus, based on inequalities (6.15) - (6.18) and by Definition 2.1.10 of Chapter 2, we get

$$\text{CIFWG}(\rho\mathcal{C}_1 \oplus \mathcal{C}, \rho\mathcal{C}_2 \oplus \mathcal{C}, \ldots, \rho\mathcal{C}_n \oplus \mathcal{C}) \supseteq \text{CIFWG}(\mathcal{C}_1^\rho \otimes \mathcal{C}, \mathcal{C}_2^\rho \otimes \mathcal{C}, \ldots, \mathcal{C}_n^\rho \otimes \mathcal{C}).$$

□

Theorem 6.2.21. Let $\mathcal{C}_j$ $(j = 1(1)n)$ be the collection of CIFNs and let $\mathcal{C}$ is also CIFN. For a real number $\rho > 0$, we have

(i) $\text{CIFWG}(\rho\mathcal{C}_1 \oplus \mathcal{C}, \rho\mathcal{C}_2 \oplus \mathcal{C}, \ldots, \rho\mathcal{C}_n \oplus \mathcal{C}) \supseteq \text{CIFWG}(\mathcal{C}_1^\rho \oplus \mathcal{C}, \mathcal{C}_2^\rho \oplus \mathcal{C}, \ldots, \mathcal{C}_n^\rho \oplus \mathcal{C})$ if and only if $\rho \geq 1$; $\text{CIFWG}(\rho\mathcal{C}_1 \oplus \mathcal{C}, \rho\mathcal{C}_2 \oplus \mathcal{C}, \ldots, \rho\mathcal{C}_n \oplus \mathcal{C}) \subseteq \text{CIFWG}(\mathcal{C}_1^\rho \oplus \mathcal{C}, \mathcal{C}_2^\rho \oplus \mathcal{C}, \ldots, \mathcal{C}_n^\rho \oplus \mathcal{C})$ if and only if $0 < \rho \leq 1$;

(ii) $\text{CIFWG}(\rho\mathcal{C}_1 \otimes \mathcal{C}, \rho\mathcal{C}_2 \otimes \mathcal{C}, \ldots, \rho\mathcal{C}_n \otimes \mathcal{C}) \supseteq \text{CIFWG}(\mathcal{C}_1^\rho \otimes \mathcal{C}, \mathcal{C}_2^\rho \otimes \mathcal{C}, \ldots, \mathcal{C}_n^\rho \otimes \mathcal{C})$ if and only if $\rho \geq 1$; $\text{CIFWG}(\rho\mathcal{C}_1 \otimes \mathcal{C}, \rho\mathcal{C}_2 \otimes \mathcal{C}, \ldots, \rho\mathcal{C}_n \otimes \mathcal{C}) \subseteq \text{CIFWG}(\mathcal{C}_1^\rho \otimes \mathcal{C}, \mathcal{C}_2^\rho \otimes \mathcal{C}, \ldots, \mathcal{C}_n^\rho \otimes \mathcal{C})$ if and only if $0 < \rho \leq 1$;

(iii) $\text{CIFWA}(\rho\mathcal{C}_1 \oplus \mathcal{C}, \rho\mathcal{C}_2 \oplus \mathcal{C}, \ldots, \rho\mathcal{C}_n \oplus \mathcal{C}) \supseteq \text{CIFWA}(\mathcal{C}_1^\rho \oplus \mathcal{C}, \mathcal{C}_2^\rho \oplus \mathcal{C}, \ldots, \mathcal{C}_n^\rho \oplus \mathcal{C})$ if and only if $\rho \geq 1$; $\text{CIFWA}(\rho\mathcal{C}_1 \oplus \mathcal{C}, \rho\mathcal{C}_2 \oplus \mathcal{C}, \ldots, \rho\mathcal{C}_n \oplus \mathcal{C}) \subseteq \text{CIFWA}(\mathcal{C}_1^\rho \oplus \mathcal{C}, \mathcal{C}_2^\rho \oplus \mathcal{C}, \ldots, \mathcal{C}_n^\rho \oplus \mathcal{C})$ if and only if $0 < \rho \leq 1$;

(iv) $\text{CIFWA}(\rho\mathcal{C}_1\otimes\mathcal{C},\rho\mathcal{C}_2\otimes\mathcal{C},\ldots,\rho\mathcal{C}_n\otimes\mathcal{C})\supseteq\text{CIFWA}(\mathcal{C}_1^\rho\otimes\mathcal{C},\mathcal{C}_2^\rho\otimes\mathcal{C},\ldots,\mathcal{C}_n^\rho\otimes\mathcal{C})$ if and only if $\rho\geq 1$; $\text{CIFWA}(\rho\mathcal{C}_1\otimes\mathcal{C},\rho\mathcal{C}_2\otimes\mathcal{C},\ldots,\rho\mathcal{C}_n\otimes\mathcal{C})\subseteq\text{CIFWA}(\mathcal{C}_1^\rho\otimes\mathcal{C},\mathcal{C}_2^\rho\otimes\mathcal{C},\ldots,\mathcal{C}_n^\rho\otimes\mathcal{C})$ if and only if $0<\rho\leq 1$.

Proof. Here, we will prove the part (i) only, while others can be deduced similarly. For this, let $\mathcal{C}_j=\Big((\zeta_j,w_{\zeta_j}),(\vartheta_j,w_{\vartheta_j})\Big)$ and $\mathcal{C}=\Big((\zeta,w_\zeta),(\vartheta,w_\vartheta)\Big)$.

(i) Since $\mathcal{C}_j$ and $\mathcal{C}$ are CIFNs and $\rho>0$ is any real number. Now, by Theorem 6.2.17, we get

$$\begin{aligned}
&\text{CIFWG}(\rho\mathcal{C}_1\oplus\mathcal{C},\rho\mathcal{C}_2\oplus\mathcal{C},\ldots,\rho\mathcal{C}_n\oplus\mathcal{C})\\
=\;&(\rho\mathcal{C}_1\oplus\mathcal{C})^{\xi_1}\otimes(\rho\mathcal{C}_2\oplus\mathcal{C})^{\xi_2}\otimes\ldots\otimes(\rho\mathcal{C}_n\oplus\mathcal{C})^{\xi_n}\\
\supseteq\;&\xi_1(\rho\mathcal{C}_1\oplus\mathcal{C})\oplus\xi_2(\rho\mathcal{C}_2\oplus\mathcal{C})\oplus\ldots\oplus\xi_n(\rho\mathcal{C}_n\oplus\mathcal{C})\\
=\;&\Big(\xi_1(\rho\mathcal{C}_1)\oplus\xi_2(\rho\mathcal{C}_2)\oplus\ldots\oplus\xi_n(\rho\mathcal{C}_n)\Big)\oplus\Big(\xi_1\mathcal{C}\oplus\xi_2\mathcal{C}\oplus\ldots\oplus\xi_n\mathcal{C}\Big)\\
\supseteq\;&(\xi_1(\mathcal{C}_1)^\rho\oplus\xi_2(\mathcal{C}_2)^\rho\oplus\ldots\oplus\xi_n(\mathcal{C}_n)^\rho)\oplus\mathcal{C}\qquad\text{if and only if}\quad\rho\geq 1\\
=\;&\text{CIFWA}(\mathcal{C}_1^\rho,\mathcal{C}_2^\rho,\ldots,\mathcal{C}_n^\rho)\oplus\mathcal{C}\\
\supseteq\;&\text{CIFWG}(\mathcal{C}_1^\rho,\mathcal{C}_2^\rho,\ldots,\mathcal{C}_n^\rho)\oplus\mathcal{C}\qquad\text{by Theorem 6.2.18}\\
=\;&\text{CIFWG}(\mathcal{C}_1^\rho\oplus\mathcal{C},\mathcal{C}_2^\rho\oplus\mathcal{C},\ldots,\mathcal{C}_n^\rho\oplus\mathcal{C})
\end{aligned}$$

Hence, $\text{CIFWG}(\rho\mathcal{C}_1\oplus\mathcal{C},\rho\mathcal{C}_2\oplus\mathcal{C},\ldots,\rho\mathcal{C}_n\oplus\mathcal{C})\supseteq\text{CIFWG}(\mathcal{C}_1^\rho\oplus\mathcal{C},\mathcal{C}_2^\rho\oplus\mathcal{C},\ldots,\mathcal{C}_n^\rho\oplus\mathcal{C})$ if and only if $\rho\geq 1$. Similarly, $\text{CIFWG}(\rho\mathcal{C}_1\oplus\mathcal{C},\rho\mathcal{C}_2\oplus\mathcal{C},\ldots,\rho\mathcal{C}_n\oplus\mathcal{C})\subseteq\text{CIFWG}(\mathcal{C}_1^\rho\oplus\mathcal{C},\mathcal{C}_2^\rho\oplus\mathcal{C},\ldots,\mathcal{C}_n^\rho\oplus\mathcal{C})$ if and only if $0<\rho\leq 1$. □

6.3 MCDM approach using proposed averaging operators

The general description of DM problem is summarized in Section 2.5 of Chapter 2. Suppose that an expert evaluated the alternatives $\mathcal{V}_u$ under the criteria $\mathfrak{B}_v$ and gave their judgement values in terms of CIFNs as $\mathcal{C}_{uv}=((\zeta_{uv},w_{\zeta_{uv}}),(\vartheta_{uv},w_{\vartheta_{uv}}))$. Then, to determine the most desirable alternative(s), the proposed operators are utilized to develop a MCDM method with CIF information, which involves the following steps:

Step 1: Collect the CIF decision matrix $\mathcal{M} = (\mathcal{C}_{uv})_{m\times n}$ corresponding to the rating values of each alternative as

$$\mathcal{M} = \begin{array}{c c} & \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \dots & \mathfrak{B}_n \end{array} \\ \begin{array}{c} \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} & \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \dots & \mathcal{C}_{1n} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \dots & \mathcal{C}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{m1} & \mathcal{C}_{m2} & \dots & \mathcal{C}_{mn} \end{pmatrix} \end{array}$$

Step 2: Aggregate the collective rating values $\mathcal{C}_{uv}$ of the alternative $\mathcal{V}_u(u = 1, 2, \dots, m)$ into the overall assessment value $\mathcal{C}_u = ((\zeta_u, w_{\zeta_u}), (\vartheta_u, w_{\vartheta_u}))$ based on the either of Eqs. (6.4), (6.8) and (6.11). For instance, if we utilize CIFWA operator to aggregate each rating value of the alternative $\mathcal{V}_u$, then we get $\mathcal{C}_u(u = 1, 2, \dots, m)$ as

$$\begin{aligned} \mathcal{C}_u &= \mathrm{CIFWA}(\mathcal{C}_{u1}, \mathcal{C}_{u2}, \dots, \mathcal{C}_{un}) \\ &= \left(\begin{pmatrix} s^{-1}\left(\sum_{v=1}^{n} \xi_v s(\zeta_{uv})\right), \\ s^{-1}\left(\sum_{v=1}^{n} \xi_v s\left(w_{\zeta_{uv}}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{v=1}^{n} \xi_v t(\vartheta_{uv})\right), \\ t^{-1}\left(\sum_{v=1}^{n} \xi_v t\left(w_{\vartheta_{uv}}\right)\right) \end{pmatrix} \right) \end{aligned} \quad (6.19)$$

or by using CIFOWA operator as follows

$$\begin{aligned} \mathcal{C}_u &= \mathrm{CIFOWA}(\mathcal{C}_{u1}, \mathcal{C}_{u2}, \dots, \mathcal{C}_{un}) \\ &= \left(\begin{pmatrix} s^{-1}\left(\sum_{v=1}^{n} \xi_v s(\zeta_{u\tau(v)})\right), \\ s^{-1}\left(\sum_{v=1}^{n} \xi_v s\left(w_{\zeta_{u\tau(v)}}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{v=1}^{n} \xi_v t(\vartheta_{u\tau(v)})\right), \\ t^{-1}\left(\sum_{v=1}^{n} \xi_v t\left(w_{\vartheta_{u\tau(v)}}\right)\right) \end{pmatrix} \right) \end{aligned} \quad (6.20)$$

or by using CIFHA operator as follows

$$\begin{aligned} \mathcal{C}_u &= \mathrm{CIFHA}(\mathcal{C}_{u1}, \mathcal{C}_{u2}, \dots, \mathcal{C}_{un}) \\ &= \left(\begin{pmatrix} s^{-1}\left(\sum_{v=1}^{n} \psi_v s(\dot{\zeta}_{u\tau(v)})\right), \\ s^{-1}\left(\sum_{v=1}^{n} \psi_v s\left(\dot{w}_{\zeta_{u\tau(v)}}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{v=1}^{n} \psi_v t(\dot{\vartheta}_{u\tau(v)})\right), \\ t^{-1}\left(\sum_{v=1}^{n} \psi_v t\left(\dot{w}_{\vartheta_{u\tau(v)}}\right)\right) \end{pmatrix} \right) \end{aligned} \quad (6.21)$$

where τ is the permutation map of $\{1,2,\ldots,n\}$ such that $\mathcal{S}\left(\mathcal{C}_{u\tau(v-1)}\right) \geq \mathcal{S}\left(\mathcal{C}_{u\tau(v)}\right)$ for $v = 2,3,\ldots,n$ and t,s respectively, be the decreasing and increasing t-norm function such that $s(a) = t(1-a)$ for $a \in [0,1]$, ψ_v be the standardized weight vector associated with CIFHA operator and $\dot{\mathcal{C}}_{uv} = n\xi_v\mathcal{C}_{uv}$.

Step 3: Compute the score values of the overall aggregated values $\mathcal{C}_u = ((\zeta_u, w_{\zeta_u}), (\vartheta_u, w_{\vartheta_u}))$ $(u = 1,2,\ldots,m)$ by using equation

$$\mathcal{S}(\mathcal{C}_u) = \zeta_u - \vartheta_u + w_{\zeta_u} - w_{\vartheta_u}.$$

If there is no difference between the score values of aggregated numbers, then we need to calculate the accuracy values of the alternatives as

$$\mathcal{H}(\mathcal{C}_u) = \zeta_u + \vartheta_u + w_{\zeta_u} + w_{\vartheta_u}.$$

Step 4: Rank all the feasible alternatives $\mathcal{V}_u(u = 1,2,\ldots,m)$ according to the Definition 6.2.1 and hence select the most desirable alternative(s).

6.4 Illustrative example

In order to demonstrate the above-mentioned approach, we illustrate it with a numerical example which is stated as below:

6.4.1 Case Study

Consider the DM problem as described in Example 5.4.1 of Chapter 5. The considered weight vectors corresponding to four preferences factors is $\xi = (0.4, 0.25, 0.15, 0.2)^T$ while the positional weight vectors of the factors is $\psi = (0.35, 0.3, 0.1, 0.25)^T$. In what follows, we utilize the MCDM method proposed in above section to determine the most desirable alternative(s) under CIF environment.

Step 1: The given expert evaluate each model of the machine taken as an alternative with respect to the four criteria under the CIFS environment and their corresponding rating values are summarized in the decision matrices represented in Table

6.1. In this table, for instance, the rating value for a model of machine $\mathcal{V}_1$ under "reliability"($\mathfrak{B}_1$) criteria is given as $((0.7, 0.9), (0.1, 0.1))$ by an expert which describes that the expert is agreed 70% with the suitability of the model $\mathcal{V}_1$ at $\mathfrak{B}_1$ while disagree with 10%. On the other hand, the same expert satisfied 90% with the suitability of production date of the model at $\mathfrak{B}_1$ and not satisfied with the 10%. In a similar manner, the other data values can be interpreted.

Step 2: Without loss of generality, we take a generator $t(a) = -\log(a)$. Now, by utilizing CIFWA operator defined in Eq. (6.19) corresponding to the weight vectors $\xi = (0.4, 0.25, 0.15, 0.2)^T$, we get the aggregated values $\mathcal{C}_u(u = 1, 2, 3, 4, 5)$ corresponding to each alternative $\mathcal{V}_u(u = 1, 2, 3, 4, 5)$ as

$$
\begin{aligned}
\mathcal{C}_1 &= ((0.7170, 0.7707), (0.1469, 0.1803)), \quad \mathcal{C}_2 = ((0.5902, 0.7291), (0.2551, 0.1830)),\\
\mathcal{C}_3 &= ((0.4863, 0.5280), (0.3431, 0.3222)), \quad \mathcal{C}_4 = ((0.5422, 0.6230), (0.3121, 0.2048)),\\
\mathcal{C}_5 &= ((0.8217, 0.7112), (0.1320, 0.1464)).
\end{aligned}
$$

Step 3: The score values of the alternative $\mathcal{V}_u(u = 1, 2, 3, 4, 5)$ are obtained based on the overall assessment values $\mathcal{C}_u(u = 1, 2, 3, 4, 5)$ as $\mathcal{S}(\mathcal{C}_1) = 1.1605$, $\mathcal{S}(\mathcal{C}_2) = 0.8812$, $\mathcal{S}(\mathcal{C}_3) = 0.3491$, $\mathcal{S}(\mathcal{C}_4) = 0.6484$, $\mathcal{S}(\mathcal{C}_5) = 1.2545$.

Step 4: Since $\mathcal{S}(\mathcal{C}_5) > \mathcal{S}(\mathcal{C}_1) > \mathcal{S}(\mathcal{C}_2) > \mathcal{S}(\mathcal{C}_4) > \mathcal{S}(\mathcal{C}_3)$ and hence based on it, the ranking of all the feasible alternatives $\mathcal{V}_u(u = 1, 2, 3, 4, 5)$ is given as

$$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3,$$

where the symbol "$\succ$" means "preferred to". Thus, we conclude that the best alternative is $\mathcal{V}_5$, i.e., $\mathcal{V}_5$ is the most optimal model.

On the other hand, if we utilize CIFOWA operator instead of CIFWA operator to aggregate the given preferences, then the following steps of the proposed approach are executed to find the best alternative(s) as below

Step 1: The information is summarized in Table 6.1.

Step 2: Compute the score values of each CIFN and based on the ordering relation between them as defined in Definition 6.2.1, we get the permutation rating values of each alternative under different criteria. These values corresponding to each alternative are summarized in Table 6.2. Now, take a generator $t(a) = -\log(a)$ and weight vector $\xi = (0.4, 0.25, 0.15, 0.2)^T$, we aggregate the preferences of the alternatives by using CIFOWA operator as defined in Eq. (6.20). The collective values corresponding to each alternative $\mathcal{V}_u (u = 1, 2, 3, 4, 5)$ are obtained as

$$\mathcal{C}_1 = ((0.7010, 0.7789), (0.1639, 0.1682)), \quad \mathcal{C}_2 = ((0.5453, 0.7862), (0.2305, 0.1552)),$$
$$\mathcal{C}_3 = ((0.5663, 0.5891), (0.2376, 0.2491)), \quad \mathcal{C}_4 = ((0.5567, 0.5818), (0.2197, 0.2415)),$$
$$\mathcal{C}_5 = ((0.8062, 0.7298), (0.1275, 0.1366)).$$

Step 3: The score values of these aggregated numbers are computed by using Eq. (6.1) as

$$\mathcal{S}(\mathcal{C}_1) = 1.1478, \quad \mathcal{S}(\mathcal{C}_2) = 0.9458, \quad \mathcal{S}(\mathcal{C}_3) = 0.6687, \quad \mathcal{S}(\mathcal{C}_4) = 0.6774, \quad \mathcal{S}(\mathcal{C}_5) = 1.2719$$

Step 4: The ranking order of these alternatives, based on optimal values of the score values is given as

$$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3,$$

and hence we obtain $\mathcal{V}_5$ is the best alternative for the required machine.

From these computed results, we conclude that the best alternative by both the operator remains same, but the computational procedure is entirely different. In CIFWA operator, the weight vector is assigned directly to the CIFNs and then collect the aggregated values. On the other hand, in CIFOWA operator, firstly the importance of the numbers are arranged sequentially based on the importance of the numbers and then aggregate the different values in the collective one. Apart from the above analysis, if we utilize the different t-norm generators such as $t(a) = -\log(a)$ or $t(a) = \log\left(\frac{2-a}{a}\right)$ for $a \in (0, 1]$; $t(0) = \infty$ and by using CIFWA, CIFOWA and CIFHA operator to the considered data then the overall score values of the alternatives $\mathcal{V}_u (u = 1, 2, 3, 4, 5)$ along with the final ranking order of the alternatives are summarized in Table 6.3. From this table, it is clearly seen that the ranking order is preserved by all the operators and hence it shows the stability of the result. Further, by taking the importance of the corresponding AO, the decision maker can choose the appropriate one and hence select the best alternative for the required task.

6.4.2 Validity Test

Since the different MCDM methods may give different evaluations(ranking) when applied to same decision-making problem, which leads to uncertain results. Therefore, there is a need to validate the results obtained from their corresponding approach to explain the reliability of the approach. For it, Wang and Triantaphyllou [162] presented the following three test criteria to validate the MCDM approach.

Test criterion 1: "An MCDM method is effective if on replacing a non-optimal alternative by another worse alternative without changing the relative importance of each decision-criteria, the indication of the best alternative remains same."

Test criterion 2: "An effective MCDM method should follow transitive property."

Test criterion 3: "An MCDM method is effective if on decomposing the MCDM problem into smaller problems and by applying the same MCDM method to these sub-problems for ranking the alternatives, the combined ranking of the alternatives remains same to the ranking of the original problem."

Now, we validate these criteria on our proposed MCDM approach as follows:

Validity test by applying criterion 1

For testing the validity of proposed approach under the criterion 1, we replace the non optimal alternative $\mathcal{V}_3$ with the worse alternative $\mathcal{V}_3'$ in original decision matrix of the expert with their rating values is summarized in Table 6.4. Now by utilizing the CIFWA operator in Step 2 of the proposed approach corresponding to generator $t(a) = -\log(a)$ to this modified data, we get the collective values of each alternative are

$$\mathcal{C}_1 = ((0.7170, 0.7707), (0.1469, 0.1803)), \quad \mathcal{C}_2 = ((0.5902, 0.7291), (0.2551, 0.1830)),$$
$$\mathcal{C}_3 = ((0.2617, 0.3075), (0.4895, 0.5370)), \quad \mathcal{C}_4 = ((0.5422, 0.6230), (0.3121, 0.2048))$$
$$\mathcal{C}_5 = ((0.8217, 0.7112), (0.1320, 0.1464))$$

The score values of the alternatives corresponding to these values are computed by using Eq. (6.1) and are obtained as

$\mathcal{S}(\mathcal{C}_1) = 1.1605, \quad \mathcal{S}(\mathcal{C}_2) = 0.8812, \quad \mathcal{S}(\mathcal{C}_3) = -0.4573, \quad \mathcal{S}(\mathcal{C}_4) = 0.6484, \quad \mathcal{S}(\mathcal{C}_5) = 1.2545$

Therefore, the final ranking order of the alternatives is $\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3'$ which indicates that $\mathcal{V}_5$ is still the best alternative. Thus, the proposed MCDM method satisfies *test criterion 1.*

Validity test by using criteria 2 and 3

Under these test, if we decompose original problem in four subparts which are: $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$, $\{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5\}$, $\{\mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_1\}$ and $\{\mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_1, \mathcal{V}_2\}$ and then applying the proposed approach individually to these subproblems, we get the final ranking order of the alternatives corresponding to each subproblem is $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$, $\mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$, $\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_3$ and $\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$ respectively . Thus, the overall combined ranking order of the alternative is $\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$ which is same as un-decomposed problem and it shows transitive property. Hence, the proposed MCDM approach is valid under the criteria 2 and 3.

6.5 Comparative study

In order to validate the performance of the proposed MCDM approach with some of the existing approaches, an investigation has been done where we compare the results obtained by the proposed approach with existing approaches under CIFS as well as IFS environment.

6.5.1 Comparative studies under CIFS environment

Firstly, an analysis has been conducted under the CIFS environment by applying the existing approaches [6, 129] to the considered data. The results corresponding to these approaches from its ideal alternative $\mathcal{V}^* = ((\zeta_v, w_{\zeta_v}), (\vartheta_v, w_{\vartheta_v}))$; $v = 1, 2, \ldots, n$ where $\zeta_v = \max\limits_{1\le u\le m}\{\zeta_{uv}\}$; $\vartheta_v = \min\limits_{1\le u\le m}\{\vartheta_{uv}\}$ and $w_{\zeta_u} = \max\limits_{1\le u\le m}\{w_{\zeta_{uv}}\}$; $w_{\vartheta_v} = \min\limits_{1\le u\le m}\{w_{\vartheta_{uv}}\}$ are computed and summarized as below:

(i) By applying the approach of Alkouri and Salleh [6] using distance measures, denoted by $\mathcal{D}_1$ to the considered data, we get the measurement values of each alternative from its ideal values as $\mathcal{D}_1(\mathcal{V}_1, \mathcal{V}^*) = 0.1110$, $\mathcal{D}_1(\mathcal{V}_2, \mathcal{V}^*) = 0.1758$, $\mathcal{D}_1(\mathcal{V}_3, \mathcal{V}^*) = 0.3245$, $\mathcal{D}_1(\mathcal{V}_4, \mathcal{V}^*) = 0.2503$ and $\mathcal{D}_1(\mathcal{V}_5, \mathcal{V}^*) = 0.0700$. From these values, we observed that $\mathcal{D}_1(\mathcal{V}_5, \mathcal{V}^*) < \mathcal{D}_1(\mathcal{V}_1, \mathcal{V}^*) < \mathcal{D}_1(\mathcal{V}_2, \mathcal{V}^*) < \mathcal{D}_1(\mathcal{V}_4, \mathcal{V}^*) < \mathcal{D}_1(\mathcal{V}_3, \mathcal{V}^*)$ and hence conclude that the best alternative is $\mathcal{V}_5$.

(ii) By utilizing the distance measure ($\mathcal{D}_2$) as proposed by Rani and Garg [129] to the considered problem, then the measurement values for each alternative are computed as $\mathcal{D}_2(\mathcal{V}_1, \mathcal{V}^*) = 0.1593$, $\mathcal{D}_2(\mathcal{V}_2, \mathcal{V}^*) = 0.2107$, $\mathcal{D}_2(\mathcal{V}_3, \mathcal{V}^*) = 0.3655$, $\mathcal{D}_2(\mathcal{V}_4, \mathcal{V}^*) = 0.2964$ and $\mathcal{D}_2(\mathcal{V}_5, \mathcal{V}^*) = 0.0975$. Since measurement value of $\mathcal{V}_5$ is minimum among all these and hence we conclude that the most optimal alternative is $\mathcal{V}_5$ which again coincides with the proposed results.

6.5.2 Comparative studies under IFS environment

In this section, we compare the performance of the proposed MCDM approach with some of the existing approaches [27, 44, 47, 67, 72, 83, 156, 179, 185, 186, 201, 217] under an intuitionistic fuzzy set theory. For it, firstly the considered preferences of the expert are converted into the intuitionistic fuzzy numbers by taking the phase terms corresponding to each CIFN is zero. Then, based on this information, we applied the existing AOs based approaches to the considered data and hence their results are summarized in Table 6.5. From this table, it is concluded that the best alternative obtained from the final ranking of the alternative remains same but the preferences of the other alternatives are different. This is mainly due to the changeable decision environment. For instance, in [27, 44, 47, 67, 72, 83, 156, 179, 185, 186, 201, 217] approaches, weighted averaging and geometric AOs were introduced by taking into account only one dimensional grades of membership and non-membership.

It is noted from the study that the computational procedure of the proposed approach is entirely different from the existing approaches under CIFS as well as IFS environment, but the proposed result in this chapter is more rational to reality in decision-making process due to the consideration of the two-dimensional information simultaneously into a single set. In addition, the advantages of the proposed operators are that it is based on the generalized t-norm operations and hence by assigning a different function "t" and their equivalent "s", we can get the more generalized aggregation operations for different CIFNs.

Also, it is revealed that in IFS, the information contains a real-valued membership and non-membership degrees and only considered amplitude term which causes loss of information during the execution. On the other hand, a complex intuitionistic fuzzy set is a generalization of the existing studies such as complex fuzzy sets [128], intuitionistic fuzzy sets [10], fuzzy set [206] by considering much more information related to an object during the process and to handle the two-dimensional information in a single set.

6.5.3 Advantages of the proposed approach

From the existing studies and the proposed operators, we address the following merits of the proposed method to solve the decision-making problem under the CIFS environment.

1) A CIFS is a generalization of the existing studies such as CFS [128], IFS [10], FS [206] by considering much more information related to an object during the process and to handle the two-dimensional information in a single set. For instance, CIFS contains information (both the membership and non-membership degrees are complex valued) with amplitude and phase terms than the CFS (contains only complex valued membership degree), IFS (with a real-valued membership and non-membership degrees and only considered amplitude term), FS (with only crisp membership degrees with amplitude term only). Thus, the proposed AOs under CIFSs environment are more generalized than the existing operators.

2) It is revealed from the present study that the AOs under IFSs, FSs [27, 44, 47, 67, 72, 83, 156, 179, 185, 186, 201, 217] are the special cases of the proposed operators. Thus, the proposed operators can be equivalently utilized to solve the MCDM problem under these existing environment by setting phase term to be zero while the existing operators are unable to solve the problems under the environment considered in the present chapter.

3) The major advantages of the proposed decision-making approach are to consider the much more information to assess the alternative and to reduce the information loss. Further, by utilizing the various expressions to the t-norm and its equivalent s -norm will help the decision maker to select the best alternative(s) more accurately. In other words, we can say that the proposed generalized AOs will give the various choices to the decision makers towards the decision-making process.

6.6 Conclusion

The key contribution of this chapter is described as follows:

1) Some new operational laws based on ATT operations for CIFNs are developed and their properties are investigated in detail. A series of generalized weighted averaging operators and geometric operators namely CIFWA, CIFOWA, CIFHA, CIFWG, CIFOWG and CIFHG are developed. The fundamental properties of these AOs are investigated.

2) Some special cases of the proposed operators are discussed in detail. It is analyzed that the existing AOs such as IFWA [179], IFOWA [179], IFHA [179], IFWG [185], IFOWG [185], IFHG [185] can easily be deduced from the proposed ones and hence the proposed operators are the more generalized to the existing theories and the operators.

3) A MCDM approach is presented to solve the DM problems and is demonstrated with a numerical example. A detailed analysis with some existing approaches is executed to show the advantages and superiority of presented MCDM method. In addition to these, the advantages of the proposed method are demonstrated through the validity test and characteristic comparison.

Table 6.1: Input information in the form of the complex intuitionistic fuzzy decision-matrix

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$((0.7,0.9),(0.1,0.1))$	$((0.8,0.5),(0.1,0.4))$	$((0.6,0.6),(0.3,0.2))$	$((0.7,0.7),(0.3,0.2))$
$\mathcal{V}_2$	$((0.7,0.6),(0.3,0.3))$	$((0.4,0.9),(0.2,0.1))$	$((0.7,0.7),(0.2,0.3))$	$((0.4,0.6),(0.3,0.1))$
$\mathcal{V}_3$	$((0.3,0.4),(0.6,0.4))$	$((0.6,0.6),(0.3,0.4))$	$((0.3,0.4),(0.5,0.6))$	$((0.7,0.7),(0.1,0.1))$
$\mathcal{V}_4$	$((0.4,0.8),(0.5,0.1))$	$((0.7,0.3),(0.3,0.3))$	$((0.6,0.5),(0.1,0.3))$	$((0.5,0.5),(0.3,0.4))$
$\mathcal{V}_5$	$((0.9,0.7),(0.1,0.2))$	$((0.7,0.7),(0.2,0.1))$	$((0.7,0.6),(0.2,0.2))$	$((0.8,0.8),(0.1,0.1))$

Table 6.2: Ordering position data

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$((0.7,0.9),(0.1,0.1))$	$((0.7,0.7),(0.3,0.2))$	$((0.8,0.5),(0.1,0.4))$	$((0.6,0.6),(0.3,0.2))$
$\mathcal{V}_2$	$((0.4,0.9),(0.2,0.1))$	$((0.7,0.7),(0.2,0.3))$	$((0.7,0.6),(0.3,0.3))$	$((0.4,0.6),(0.3,0.1))$
$\mathcal{V}_3$	$((0.7,0.7),(0.1,0.1))$	$((0.6,0.6),(0.3,0.4))$	$((0.3,0.4),(0.6,0.4))$	$((0.3,0.4),(0.5,0.6)))$
$\mathcal{V}_4$	$((0.6,0.5),(0.1,0.3))$	$((0.4,0.8),(0.5,0.1))$	$((0.7,0.3),(0.3,0.3))$	$((0.5,0.5),(0.3,0.4))$
$\mathcal{V}_5$	$((0.8,0.8),(0.1,0.1))$	$((0.9,0.7),(0.1,0.2))$	$((0.7,0.7),(0.2,0.1))$	$((0.7,0.6),(0.2,0.2))$

Table 6.3: Ranking of the alternatives based on the proposed operators

Additive	Proposed	Score values of the alternatives					Ranking
Generators	Operators	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	
$t(a) = -\log(a)$	CIFWA	1.1605	0.8812	0.3491	0.6484	1.2545	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
	CIFOWA	1.1478	0.9458	0.6687	0.6774	1.2719	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
	CIFHA	1.3249	1.0153	0.4290	0.7917	1.3421	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
$t(a) = \log\left(\frac{2-a}{a}\right)$	CIFEWA	1.1468	0.8650	0.3096	0.6193	1.2501	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
	CIFEOWA	1.1359	0.9306	0.6339	0.6539	1.2675	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
	CIFEHA	1.3352	1.0264	0.3598	0.8005	1.3611	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$

Table 6.4: Transformed input information

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$((0.7,0.9),(0.1,0.1))$	$((0.8,0.5),(0.1,0.4))$	$((0.6,0.6),(0.3,0.2))$	$((0.7,0.7),(0.3,0.2))$
$\mathcal{V}_2$	$((0.7,0.6),(0.3,0.3))$	$((0.4,0.9),(0.2,0.1))$	$((0.7,0.7),(0.2,0.3))$	$((0.4,0.6),(0.3,0.1))$
$\mathcal{V}_3'$	$((0.1,0.3),(0.7,0.5))$	$((0.3,0.3),(0.5,0.6))$	$((0.2,0.2),(0.6,0.8))$	$((0.5,0.4),(0.2,0.4))$
$\mathcal{V}_4$	$((0.4,0.8),(0.5,0.1))$	$((0.7,0.3),(0.3,0.3))$	$((0.6,0.5),(0.1,0.3))$	$((0.5,0.5),(0.3,0.4))$
$\mathcal{V}_5$	$((0.9,0.7),(0.1,0.2))$	$((0.7,0.7),(0.2,0.1))$	$((0.7,0.6),(0.2,0.2))$	$((0.8,0.8),(0.1,0.1))$

Table 6.5: Comparative study with some existing approaches

Existing Methods	Score values					Ranking
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	
Xu and Yager [185]	-0.1459	-0.2007	-0.2343	-0.1767	-0.0731	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3$
Xu [179]	-0.1073	-0.1482	-0.0732	-0.0931	-0.0368	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_2$
Wang and Liu [156]	0.5670	0.3276	0.1183	0.2181	0.6871	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
Garg [44]	0.6563	0.4787	0.0142	0.2849	0.7193	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
Xu and Yager [186]	-0.3968	-0.5370	-0.6319	-0.5754	-0.3136	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
He et al. [72]	0.6484	0.4768	-0.0085	0.2707	0.7172	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
Huang [83]	0.5658	0.3241	0.1064	0.2127	0.6860	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
Chen and Chang [27]	0.4339	0.1804	0.1000	0.0845	0.6435	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$
Goyal et al. [67]	0.7982	0.6623	0.3109	0.4510	0.8604	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
Ye [201]	0.5506	0.3084	0.0596	0.1876	0.6715	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
Zhou and Xu [217]	0.5868	0.3824	0.3288	0.3776	0.6979	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
Garg [47]	0.4316	0.1669	0.0809	0.0743	0.6392	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$

Chapter 7

Novel aggregation operators and ranking method for complex intuitionistic fuzzy sets and their applications to decision-making process[1]

The aim of this chapter is classified into two turns: (i) to define the possibility degree measure in order to rank the CIFNs and (ii) to define some novel operational laws and AOs for aggregating the various choices over CIFS environment. The properties of the proposed weighted averaging and geometric AOs are investigated. Finally, a decision-making approach is established for the MCDM problems with CIF information, in which weights are derived objectively.

7.1 Introduction

The detailed literature review on various AOs has been done in section 1.1.3. From these works, it is evident that the two major aspects of MCDM process are: (i) how to aggregate the information (ii) how to rank the numbers. Recently, a great attention is paid towards aggregating uncertain information using AOs. Furthermore, to rank the given alternatives,

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there is a need to defuzzify the values by using either a score function or a possibility degree measure (PDM). Thus, for this, researchers have developed various kinds of measures to rank the numbers under the IFSs environment. For example, Xu and Yager [185] presented the score function for ranking IFNs. Wei and Tang [163] presented the PDM methods in order to rank the IFNs. Wan and Dong [154] presented PDM based method to solve the MCDM problem. Garg and Kumar [57] presented an improved PDM by overcoming the shortcomings of existing PDM [163] to rank the IFNs. Dammak et al. [35] compared the several existing PDMs over the IFSs and interval-valued IFSs.

After reviewing the existing AOs under IFS environment, it is noted that most of the fusion processes are based on simple algebraic operations. Therefore, the task of developing new AOs is still a meaningful and challenging task. The development of new operations may provide more choice to decision maker during the process of aggregation in order to take a sound decision. Besides this from the above prevailing studies, it is observed that the decision-making problems(DMPs) have been investigated for FS, IFS theories or their extensions. These models are unable to handle time-periodic problems and two-dimensional information together in one set. However, in real life, we come across complex natural phenomena where it becomes essential to add the second dimension to the expression of membership and non-membership grades. By introducing this second dimension, the complete information can be projected in one set and hence, loss of information can be avoided. CIFS theory has the feature of portraying two dimensional information together in one set.

Thus, inspired by the features of the CIFS model, importance of information aggregation and ranking techniques, this chapter is focussed on some new AOs and a novel ranking method under CIF theory for handling the multi-dimensional complex data sets. For this, firstly a new possibility degree method is presented in order to rank CIF numbers. Then, some AOs namely CIF weighted averaging(CIFWA) and CIF weighted geometric(CIFWG) are proposed and some of their properties are also discussed. Furthermore, a novel DM methodology is presented by considering the multi-dimensional complex data sets in which weights are determined objectively. The proposed DM method is validated by using validity test and by performing comparative studies with existing theories.

7.2 Proposed Possibility degree measure for CIFNs

In this section, a concept of new PDM for comparing CIFNs is proposed.

Definition 7.2.1. For two CIFNs $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$, $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$, let $h_{\mathcal{C}_1} = 2 - (\zeta_1 + \vartheta_1) - (w_{\zeta_1} + w_{\vartheta_1})$ and $h_{\mathcal{C}_2} = 2 - (\zeta_2 + \vartheta_2) - (w_{\zeta_2} + w_{\vartheta_2})$ be their hesitation degrees respectively. A possibility degree of $\mathcal{C}_1 \succ \mathcal{C}_2$ is defined as:

$$\chi(\mathcal{C}_1 \succ \mathcal{C}_2) = \min\left(\max\left(\frac{2 + \zeta_1 - 2\zeta_2 - \vartheta_2 + w_{\zeta_1} - 2w_{\zeta_2} - w_{\vartheta_2}}{h_{\mathcal{C}_1} + h_{\mathcal{C}_2}}, 0\right), 1\right) \tag{7.1}$$

provided either $h_{\mathcal{C}_1} \neq 0$ or $h_{\mathcal{C}_2} \neq 0$. However, if $h_{\mathcal{C}_1} = h_{\mathcal{C}_2} = 0$ then,

$$\chi(\mathcal{C}_1 \succ \mathcal{C}_2) = \begin{cases} 1; & \zeta_1 + w_{\zeta_1} > \zeta_2 + w_{\zeta_2} \\ 0.5; & \zeta_1 + w_{\zeta_1} = \zeta_2 + w_{\zeta_2} \\ 0; & \zeta_1 + w_{\zeta_1} < \zeta_2 + w_{\zeta_2} \end{cases} \tag{7.2}$$

Theorem 7.2.1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be CIFNs, we have

i) $0 \leq \chi(\mathcal{C}_1 \succ \mathcal{C}_2) \leq 1$;

ii) $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) = 0.5$ if $\mathcal{C}_1 = \mathcal{C}_2$;

iii) $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) + \chi(\mathcal{C}_2 \succ \mathcal{C}_1) = 1$.

Proof. Let either $h_{\mathcal{C}_1} \neq 0$ or $h_{\mathcal{C}_2} \neq 0$ for the CIFNs $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$ and $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$. The other cases are trivial from it.

i) $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) \geq 0$ is trivial. Hence, we shall prove only $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) \leq 1$. For it, consider $y = \dfrac{2 + \zeta_1 - 2\zeta_2 - \vartheta_2 + w_{\zeta_1} - 2w_{\zeta_2} - w_{\vartheta_2}}{h_{\mathcal{C}_1} + h_{\mathcal{C}_2}}$ such that either of the following cases appear.

a) If $y \leq 0$ then, by Eq. (7.1), $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) = 0$.

b) If $y \geq 1$ then, by Eq. (7.1), $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) = 1$.

c) If $0 < y < 1$ then, by Eq. (7.1), $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) = y$.

Hence, by all, we obtain $0 \leq \chi(\mathcal{C}_1 \succ \mathcal{C}_2) \leq 1$.

ii) If $\mathcal{C}_1 = \mathcal{C}_2$ then, $\zeta_1 = \zeta_2$, $\vartheta_1 = \vartheta_2$, $w_{\zeta_1} = w_{\zeta_2}$ and $w_{\vartheta_1} = w_{\vartheta_2}$. Now,

$$\begin{aligned}
\chi(\mathcal{C}_1 \succ \mathcal{C}_2) &= \min\left(\max\left(\frac{2+\zeta_2-2\zeta_2-\vartheta_2+w_{\zeta_2}-2w_{\zeta_2}-w_{\vartheta_2}}{h_{\mathcal{C}_2}+h_{\mathcal{C}_2}},0\right),1\right)\\
&= \min\left(\max\left(\frac{2-\zeta_2-\vartheta_2-w_{\zeta_2}-w_{\vartheta_2}}{2h_{\mathcal{C}_2}},0\right),1\right)\\
&= \min\left(\max\left(\frac{h_{\mathcal{C}_2}}{2h_{\mathcal{C}_2}},0\right),1\right)\\
&= 0.5
\end{aligned}$$

iii) Let $y = \dfrac{2+\zeta_1-2\zeta_2-\vartheta_2+w_{\zeta_1}-2w_{\zeta_2}-w_{\vartheta_2}}{h_{\mathcal{C}_1}+h_{\mathcal{C}_2}}$ and

$z = \dfrac{2+\zeta_2-2\zeta_1-\vartheta_1+w_{\zeta_2}-2w_{\zeta_1}-w_{\vartheta_1}}{h_{\mathcal{C}_1}+h_{\mathcal{C}_2}}$. Then,

$$\begin{aligned}
y+z &= \frac{2+\zeta_1-2\zeta_2-\vartheta_2+w_{\zeta_1}-2w_{\zeta_2}-w_{\vartheta_2}}{h_{\mathcal{C}_1}+h_{\mathcal{C}_2}} + \frac{2+\zeta_2-2\zeta_1-\vartheta_1+w_{\zeta_2}-2w_{\zeta_1}-w_{\vartheta_1}}{h_{\mathcal{C}_1}+h_{\mathcal{C}_2}}\\
&= \frac{(2-(\zeta_1+\vartheta_1)-(w_{\zeta_1}+w_{\vartheta_1}))+(2-(\zeta_2+\vartheta_2)-(w_{\zeta_2}+w_{\vartheta_2}))}{h_{\mathcal{C}_1}+h_{\mathcal{C}_2}}\\
&= 1
\end{aligned}$$

Now, if $y \leq 0, z \geq 1$ or $y \geq 1, z \leq 0$ then in such cases, we have $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) + \chi(\mathcal{C}_2 \succ \mathcal{C}_1) = 0 + 1 = 1$. On the other hand, if $0 < y, z < 1$ then $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) + \chi(\mathcal{C}_2 \succ \mathcal{C}_1) = y + z = 1$. Hence, the result.

$\square$

Theorem 7.2.2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be CIFNs

i) $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) = 1$ if $\zeta_1 + w_{\zeta_1} - h_{\mathcal{C}_1} \geq \zeta_2 + w_{\zeta_2}$;

ii) $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) = 0$ if $\zeta_2 + w_{\zeta_2} - h_{\mathcal{C}_2} \geq \zeta_1 + w_{\zeta_1}$.

Proof. We shall discuss these cases when either $h_{\mathcal{C}_1} \neq 0$ or $h_{\mathcal{C}_2} \neq 0$ for CIFNs $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$ and $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$.

i) Since $\zeta_1 + w_{\zeta_1} - h_{\mathcal{C}_1} \geq \zeta_2 + w_{\zeta_2}$. Then, we have

$$y = \frac{2+\zeta_1-2\zeta_2-\vartheta_2+w_{\zeta_1}-2w_{\zeta_2}-w_{\vartheta_2}}{h_{\mathcal{C}_1}+h_{\mathcal{C}_2}}$$

$$
\begin{aligned}
&= \frac{\zeta_1 - \zeta_2 + (2 - (\zeta_2 + \vartheta_2) - (w_{\zeta_2} + w_{\vartheta_2})) + w_{\zeta_1} - w_{\zeta_2}}{h_{\mathcal{C}_1} + h_{\mathcal{C}_2}} \\
&\geq \frac{h_{\mathcal{C}_1} + h_{\mathcal{C}_2}}{h_{\mathcal{C}_1} + h_{\mathcal{C}_2}} \\
&= 1
\end{aligned}
$$

which implies that $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) = 1$.

ii) Since $\zeta_2 + w_{\zeta_2} - h_{\mathcal{C}_2} \geq \zeta_1 + w_{\zeta_1}$. Then, we have

$$
\begin{aligned}
y &= \frac{2 + \zeta_1 - 2\zeta_2 - \vartheta_2 + w_{\zeta_1} - 2w_{\zeta_2} - w_{\vartheta_2}}{h_{\mathcal{C}_1} + h_{\mathcal{C}_2}} \\
&= \frac{\zeta_1 - \zeta_2 + (2 - (\zeta_2 + \vartheta_2) - (w_{\zeta_2} + w_{\vartheta_2})) + w_{\zeta_1} - w_{\zeta_2}}{h_{\mathcal{C}_1} + h_{\mathcal{C}_2}} \\
&= \frac{\zeta_1 - \zeta_2 + h_{\mathcal{C}_2} + w_{\zeta_1} - w_{\zeta_2}}{h_{\mathcal{C}_1} + h_{\mathcal{C}_2}} \\
&\leq 0
\end{aligned}
$$

which gives that $\chi(\mathcal{C}_1 \succ \mathcal{C}_2) = 0$.

□

To rank the "n" CIFNs $\mathcal{C}_j$ by using PDM, a possibility degree matrix $\mathcal{P} = (\chi_{jl})_{n\times n}$ is constructed where $\chi_{jl} = \chi(\mathcal{C}_j \succ \mathcal{C}_l)$ is computed by Definition 7.2.1. Based on it, the optimal degree $\mathcal{R}_j$ of the alternative $\mathcal{C}_j$ is calculated by

$$
\mathcal{R}_j = \frac{1}{n(n-1)} \left(\sum_{l=1}^{n} \chi_{jl} + \frac{n}{2} - 1 \right) \tag{7.3}
$$

From the descending values of $\mathcal{R}_j$, rank the given CIFNs.

Example 7.2.1. Let $\mathcal{C}_1 = ((0.5, 0.4), (0.3, 0.1))$, $\mathcal{C}_2 = ((0.4, 0.3), (0.2, 0.5))$, $\mathcal{C}_3 = ((0.3, 0.4), (0.2, 0.2))$ and $\mathcal{C}_4 = ((0.2, 0.1), (0.4, 0.5))$ be four CIFNs. By Definition 7.2.1, the possibility matrix is constructed as

$$
\mathcal{P} = \begin{pmatrix}
0.5000 & 0.6154 & 0.6875 & 0.9333 \\
0.3846 & 0.5000 & 0.6000 & 0.8571 \\
0.3125 & 0.4000 & 0.5000 & 0.7059 \\
0.0667 & 0.1429 & 0.2941 & 0.5000
\end{pmatrix}
$$

Thus, the values of $\mathcal{R}_j$ are computed by Eq. (7.3) and are obtained as $\mathcal{R}_1 = 0.3114$, $\mathcal{R}_2 = 0.2785$, $\mathcal{R}_3 = 0.2432$ and $\mathcal{R}_4 = 0.1670$. From it, we conclude that $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \mathcal{C}_3 \succ \mathcal{C}_4$.

7.3 Proposed Operational laws and AOs

Let Ω denotes the collection of CIFNs. Then, in this section, some robust operational laws and the AOs on Ω are presented.

7.3.1 Operational laws of CIFNs

Definition 7.3.1. For two CIFNs $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$, $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$ and positive real number ρ, the operational laws between them are defined as

(i) $$\mathcal{C}_1 \oplus \mathcal{C}_2 = \left(\left(1 - \frac{\prod_{j=1}^{2}\left(1-\zeta_j\right)}{1-\prod_{j=1}^{2}\zeta_j}, 1 - \frac{\prod_{j=1}^{2}\left(1-w_{\zeta_j}\right)}{1-\prod_{j=1}^{2}w_{\zeta_j}}\right), \left(\frac{\prod_{j=1}^{2}\vartheta_j}{1-\prod_{j=1}^{2}\left(1-\vartheta_j\right)}, \frac{\prod_{j=1}^{2}w_{\vartheta_j}}{1-\prod_{j=1}^{2}\left(1-w_{\vartheta_j}\right)}\right)\right).$$

(ii) $$\mathcal{C}_1 \otimes \mathcal{C}_2 = \left(\left(\frac{\prod_{j=1}^{2}\zeta_j}{1-\prod_{j=1}^{2}\left(1-\zeta_j\right)}, \frac{\prod_{j=1}^{2}w_{\zeta_j}}{1-\prod_{j=1}^{2}\left(1-w_{\zeta_j}\right)}\right), \left(1 - \frac{\prod_{j=1}^{2}\left(1-\vartheta_j\right)}{1-\prod_{j=1}^{2}\vartheta_j}, 1 - \frac{\prod_{j=1}^{2}\left(1-w_{\vartheta_j}\right)}{1-\prod_{j=1}^{2}w_{\vartheta_j}}\right)\right).$$

(iii) $$\rho\mathcal{C}_1 = \left(\left(1 - \frac{1-\zeta_1}{1+(\rho-1)\zeta_1}, 1 - \frac{1-w_{\zeta_1}}{1+(\rho-1)w_{\zeta_1}}\right), \left(\frac{\vartheta_1}{\rho-(\rho-1)\vartheta_1}, \frac{w_{\vartheta_1}}{\rho-(\rho-1)w_{\vartheta_1}}\right)\right).$$

(iv) $$\mathcal{C}_1^{\rho} = \left(\left(\frac{\zeta_1}{\rho-(\rho-1)\zeta_1}, \frac{w_{\zeta_1}}{\rho-(\rho-1)w_{\zeta_1}}\right), \left(1 - \frac{1-\vartheta_1}{1+(\rho-1)\vartheta_1}, 1 - \frac{1-w_{\vartheta_1}}{1+(\rho-1)w_{\vartheta_1}}\right)\right).$$

Remark 7.3.1. For simplification, we can rewrite the above operations as

(i) $\mathcal{C}_1 \oplus \mathcal{C}_2 = \left((\mathcal{A}(\zeta_1, \zeta_2), \mathcal{A}(w_{\zeta_1}, w_{\zeta_2})), (\mathcal{B}(\vartheta_1, \vartheta_2), \mathcal{B}(w_{\vartheta_1}, w_{\vartheta_2}))\right)$

(ii) $\mathcal{C}_1 \otimes \mathcal{C}_2 = \left((\mathcal{B}(\zeta_1, \zeta_2), \mathcal{B}(w_{\zeta_1}, w_{\zeta_2})), (\mathcal{A}(\vartheta_1, \vartheta_2), \mathcal{A}(w_{\vartheta_1}, w_{\vartheta_2}))\right)$

where $\mathcal{A}(x, y) = 1 - \frac{(1-x)(1-y)}{1-xy}$ and $\mathcal{B}(x, y) = \frac{xy}{1-(1-x)(1-y)}$. Also, $\mathcal{A}(x, y) = 1 - \mathcal{B}(1-x, 1-y)$.

Remark 7.3.2. The functions $\mathcal{A}(x, y)$ and $\mathcal{B}(x, y)$ are undefined when $x = y = 1$ and $x = y = 0$ respectively. However, it can be easily proved that $\lim\limits_{(x,y)\to(1,1)} \mathcal{A}(x, y) = 1$ and $\lim\limits_{(x,y)\to(0,0)} \mathcal{B}(x, y) = 0$. Therefore, we have

(i) $\big((1,0),(1,0)\big) \oplus \big((1,0),(1,0)\big) = \big((1,0),(1,0)\big)$;

(ii) $\big((0,0),(\vartheta_1, w_{\vartheta_1})\big) \otimes \big((0,0),(\vartheta_2, w_{\vartheta_2})\big) = \big((0,0),(\vartheta_3, w_{\vartheta_3})\big)$ where $\vartheta_3 = \mathcal{A}(\vartheta_1, \vartheta_2)$ and $w_{\vartheta_3} = \mathcal{A}\left(w_{\vartheta_1}, w_{\vartheta_2}\right)$.

Theorem 7.3.1. All the operations defined in Definition 7.3.1 for CIFNs are also CIFNs.

Proof. Let $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}),(\vartheta_1, w_{\vartheta_1}))$, $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}),(\vartheta_2, w_{\vartheta_2}))$ be CIFNS such that $0 \leq \zeta_j, \vartheta_j, \zeta_j + \vartheta_j \leq 1$ and $0 \leq w_{\zeta_j}, w_{\vartheta_j}, w_{\zeta_j} + w_{\vartheta_j} \leq 1$ for $j = 1, 2$. Now, by Definition 7.3.1, we get $\mathcal{C}_3 = \mathcal{C}_1 \oplus \mathcal{C}_2 = ((\zeta_3, w_{\zeta_3}),\ (\vartheta_3, w_{\vartheta_3}))$. Further, we get $1 - \zeta_1\zeta_2 \geq 0$ and $(1-\zeta_1)(1-\zeta_2) \geq 0$ and hence $1 - \zeta_3 \geq 0 \Rightarrow \zeta_3 \leq 1$. Also, $(1-\zeta_1)(1-\zeta_2) = 1 - \zeta_1 - \zeta_2 + \zeta_1\zeta_2 \leq 1 - \zeta_1\zeta_2 - \zeta_1\zeta_2 + \zeta_1\zeta_2 = 1 - \zeta_1\zeta_2 \Rightarrow \frac{(1-\zeta_1)(1-\zeta_2)}{1-\zeta_1\zeta_2} \leq 1 \Rightarrow 1 - \zeta_3 \leq 1 \Rightarrow \zeta_3 \geq 0$. Therefore, $0 \leq \zeta_3 \leq 1$. Also, $\vartheta_1\vartheta_2 \geq 0$, $1 - (1-\vartheta_1)(1-\vartheta_2) \geq 0$ and $1 - (1-\vartheta_1)(1-\vartheta_2) = \vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2 \geq \vartheta_1\vartheta_2 + \vartheta_1\vartheta_2 - \vartheta_1\vartheta_2 = \vartheta_1\vartheta_2 \Rightarrow \frac{\vartheta_1\vartheta_2}{1-(1-\vartheta_1)(1-\vartheta_2)} \leq 1$. Hence, $0 \leq \vartheta_3 \leq 1$. Now,

$$\begin{aligned}
\zeta_3 + \vartheta_3 &= 1 - \frac{\big(1-\zeta_1\big)\big(1-\zeta_2\big)}{1-\zeta_1\zeta_2} + \frac{\vartheta_1\vartheta_2}{1-\big(1-\vartheta_1\big)\big(1-\vartheta_2\big)} \\
&= 1 - \frac{\big(1-\zeta_1\big)\big(1-\zeta_2\big)}{1-\zeta_1\zeta_2} + \frac{1}{\frac{1}{\vartheta_2} + \frac{1}{\vartheta_1} - 1} \\
&\leq 1 - \frac{\big(1-\zeta_1\big)\big(1-\zeta_2\big)}{1-\zeta_1\zeta_2} + \frac{1}{\frac{1}{1-\zeta_2} + \frac{1}{1-\zeta_1} - 1} \\
&= 1 - \frac{\big(1-\zeta_1\big)\big(1-\zeta_2\big)}{1-\zeta_1\zeta_2} + \frac{\big(1-\zeta_1\big)\big(1-\zeta_2\big)}{1-\zeta_1\zeta_2} \\
&= 1.
\end{aligned}$$

Hence, $0 \leq \zeta_3 + \vartheta_3 \leq 1$. Also, $0 \leq w_{\zeta_3}, w_{\vartheta_3}, w_{\zeta_3} + w_{\vartheta_3} \leq 1$. Thus, $\mathcal{C}_1 \oplus \mathcal{C}_2$ is CIFN. Similarly, $\rho\mathcal{C}_1$, $\mathcal{C}_1 \otimes \mathcal{C}_2$ and $\mathcal{C}_1^{\rho}$ are also CIFNs. □

Theorem 7.3.2. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be CIFNs. Then, the following equalities hold

(i) $\mathcal{C}_1 \oplus \mathcal{C}_2 = \mathcal{C}_2 \oplus \mathcal{C}_1$.

(ii) $\mathcal{C}_1 \otimes \mathcal{C}_2 = \mathcal{C}_2 \otimes \mathcal{C}_1$.

(iii) $\mathcal{C}_1 \oplus (\mathcal{C}_2 \oplus \mathcal{C}_3) = (\mathcal{C}_1 \oplus \mathcal{C}_2) \oplus \mathcal{C}_3$.

(iv) $\mathcal{C}_1 \otimes (\mathcal{C}_2 \otimes \mathcal{C}_3) = (\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes \mathcal{C}_3$.

Proof. It can be easily derived from the definition. $\square$

Theorem 7.3.3. For CIFNs $\mathcal{C}_1$ and $\mathcal{C}_2$,

(i) $(\mathcal{C}_1 \oplus \mathcal{C}_2)^c = \mathcal{C}_1^c \otimes \mathcal{C}_2^c$.

(ii) $(\mathcal{C}_1 \otimes \mathcal{C}_2)^c = \mathcal{C}_1^c \oplus \mathcal{C}_2^c$.

Proof. We shall prove only (i) part, rest are similar. For $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$ and $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$,

$$\begin{aligned}
(\mathcal{C}_1 \oplus \mathcal{C}_2)^c &= \left(\left(\frac{\prod_{j=1}^{2} \vartheta_j}{1-\prod_{j=1}^{2}\left(1-\vartheta_j\right)}, \frac{\prod_{j=1}^{2} w_{\vartheta_j}}{1-\prod_{j=1}^{2}\left(1-w_{\vartheta_j}\right)} \right), \left(1 - \frac{\prod_{j=1}^{2}\left(1-\zeta_j\right)}{1-\prod_{j=1}^{2}\zeta_j}, 1 - \frac{\prod_{j=1}^{2}\left(1-w_{\zeta_j}\right)}{1-\prod_{j=1}^{2} w_{\zeta_j}} \right) \right) \\
&= ((\vartheta_1, w_{\vartheta_1}), (\zeta_1, w_{\zeta_1})) \otimes ((\vartheta_2, w_{\vartheta_2}), (\zeta_2, w_{\zeta_2})) \\
&= \mathcal{C}_1^c \otimes \mathcal{C}_2^c
\end{aligned}$$

$\square$

Theorem 7.3.4. Let $\mathbf{1} = \big((1,1),(0,0)\big)$ and $\mathbf{0} = \big((0,0),(1,1)\big)$. Then, for any $\mathcal{C} = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$, we have

(i) $\mathcal{C} \oplus \mathbf{1} = \mathbf{1}$ if ϑ_1, $w_{\vartheta_1} \neq 0$.

(ii) $\mathcal{C} \otimes \mathbf{0} = \mathbf{0}$ if ζ_1, $w_{\zeta_1} \neq 0$.

(iii) $\mathcal{C} \oplus \mathbf{0} = \mathcal{C}$.

(iv) $\mathcal{C} \otimes \mathbf{1} = \mathcal{C}$.

Proof. Follows from Definition 7.3.1 and Remark 7.3.2. $\square$

Theorem 7.3.5. For CIFNs $\mathcal{C}_j$ and $\mathcal{Z}_j$, $j = 1, 2$ satisfying $\mathcal{C}_j \subseteq \mathcal{Z}_j$ and a real number $0 < \rho \leq 1$, we have

(i) $\mathcal{C}_1 \oplus \mathcal{C}_2 \subseteq \mathcal{Z}_1 \oplus \mathcal{Z}_2$.

(ii) $\mathcal{C}_1 \otimes \mathcal{C}_2 \subseteq \mathcal{Z}_1 \otimes \mathcal{Z}_2$.

(iii) $\rho\mathcal{C}_j \subseteq \rho\mathcal{Z}_j$.

(iv) $\mathcal{C}_j^{\rho} \subseteq \mathcal{Z}_j^{\rho}$.

Proof. We shall prove only (i) part while rest are similar.

Let $\mathcal{C}_j = ((\zeta_{\mathcal{C}_j}, w_{\zeta_{\mathcal{C}_j}}), (\vartheta_{\mathcal{C}_j}, w_{\vartheta_{\mathcal{C}_j}}))$ and $\mathcal{Z}_j = ((\zeta_{\mathcal{Z}_j}, w_{\zeta_{\mathcal{Z}_j}}), (\vartheta_{\mathcal{Z}_j}, w_{\vartheta_{\mathcal{Z}_j}}))$ for $j = 1, 2$. Then, $\mathcal{C}_j \subseteq \mathcal{Z}_j$ implies that $\zeta_{\mathcal{C}_j} \leq \zeta_{\mathcal{Z}_j}$; $\vartheta_{\mathcal{C}_j} \geq \vartheta_{\mathcal{Z}_j}$; $w_{\zeta_{\mathcal{C}_j}} \leq w_{\zeta_{\mathcal{Z}_j}}$ and $w_{\vartheta_{\mathcal{C}_j}} \geq w_{\vartheta_{\mathcal{Z}_j}}$. Further let, $\mathcal{A}(x, y) = 1 - \frac{(1-x)(1-y)}{1-xy}$ and $\mathcal{B}(x, y) = \frac{xy}{1-(1-x)(1-y)}$. Then, differentiating $\mathcal{A}, \mathcal{B}$ partially with respect to x and y, we obtain $\frac{\partial \mathcal{A}}{\partial x} = \frac{(1-y)^2}{(1-xy)^2} \geq 0$; $\frac{\partial \mathcal{A}}{\partial y} = \frac{(1-x)^2}{(1-xy)^2} \geq 0$; $\frac{\partial \mathcal{B}}{\partial x} = \frac{y^2}{(x+y-xy)^2} \geq 0$ and $\frac{\partial \mathcal{B}}{\partial y} = \frac{x^2}{(x+y-xy)^2} \geq 0$. Thus, $\mathcal{A}(x, y)$ and $\mathcal{B}(x, y)$ are monotonically increasing functions with x and y. So, $\mathcal{A}(\zeta_{\mathcal{C}_1}, \zeta_{\mathcal{C}_2}) \leq \mathcal{A}(\zeta_{\mathcal{Z}_1}, \zeta_{\mathcal{Z}_2})$; $\mathcal{A}\left(w_{\zeta_{\mathcal{C}_1}}, w_{\zeta_{\mathcal{C}_2}}\right) \leq \mathcal{A}\left(w_{\zeta_{\mathcal{Z}_1}}, w_{\zeta_{\mathcal{Z}_2}}\right)$; $\mathcal{B}(\vartheta_{\mathcal{C}_1}, \vartheta_{\mathcal{C}_2}) \geq \mathcal{B}(\vartheta_{\mathcal{Z}_1}, \vartheta_{\mathcal{Z}_2})$ and $\mathcal{B}\left(w_{\vartheta_{\mathcal{C}_1}}, w_{\vartheta_{\mathcal{C}_2}}\right) \geq \mathcal{B}\left(w_{\vartheta_{\mathcal{Z}_1}}, w_{\vartheta_{\mathcal{Z}_2}}\right)$. Thus, we obtain that

$$\begin{aligned}
&1 - \frac{(1-\zeta_{\mathcal{C}_1})(1-\zeta_{\mathcal{C}_2})}{1-\zeta_{\mathcal{C}_1}\zeta_{\mathcal{C}_2}} \leq 1 - \frac{(1-\zeta_{\mathcal{Z}_1})(1-\zeta_{\mathcal{Z}_2})}{1-\zeta_{\mathcal{Z}_1}\zeta_{\mathcal{Z}_2}};\\
&\frac{\vartheta_{\mathcal{C}_1}\vartheta_{\mathcal{C}_2}}{1-(1-\vartheta_{\mathcal{C}_1})(1-\vartheta_{\mathcal{C}_2})} \geq \frac{\vartheta_{\mathcal{Z}_1}\vartheta_{\mathcal{Z}_2}}{1-(1-\vartheta_{\mathcal{Z}_1})(1-\vartheta_{\mathcal{Z}_2})};\\
&1 - \frac{\left(1-w_{\zeta_{\mathcal{C}_1}}\right)\left(1-w_{\zeta_{\mathcal{C}_2}}\right)}{1-w_{\zeta_{\mathcal{C}_1}}w_{\zeta_{\mathcal{C}_2}}} \leq 1 - \frac{\left(1-w_{\zeta_{\mathcal{Z}_1}}\right)\left(1-w_{\zeta_{\mathcal{Z}_2}}\right)}{1-w_{\zeta_{\mathcal{Z}_1}}w_{\zeta_{\mathcal{Z}_2}}};\\
&\frac{w_{\vartheta_{\mathcal{C}_1}}w_{\vartheta_{\mathcal{C}_2}}}{1-\left(1-w_{\vartheta_{\mathcal{C}_1}}\right)\left(1-w_{\vartheta_{\mathcal{C}_2}}\right)} \geq \frac{w_{\vartheta_{\mathcal{Z}_1}}w_{\vartheta_{\mathcal{Z}_2}}}{1-\left(1-w_{\vartheta_{\mathcal{Z}_1}}\right)\left(1-w_{\vartheta_{\mathcal{Z}_2}}\right)}
\end{aligned}$$

Hence, $\mathcal{C}_1 \oplus \mathcal{C}_2 \subseteq \mathcal{Z}_1 \oplus \mathcal{Z}_2$. □

7.3.2 Weighted Averaging & Geometric operators

For "n" CIFNs $\mathcal{C}_j = ((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j}))$, with $\vartheta_j, w_{\vartheta_j} \neq 0\ \forall j$ and $\xi_j > 0$, $\sum_{j=1}^{n} \xi_j = 1$ be weight vector of them, we define averaging AO as follows.

Definition 7.3.2. A map CIFWA : $\Omega^n \to \Omega$, defined by:

$$\mathrm{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \ldots \oplus \xi_n\mathcal{C}_n \tag{7.4}$$

is called as CIF weighted averaging (CIFWA) operator.

Theorem 7.3.6. For CIFNs $\mathcal{C}_j$, the collective value by Definition 7.3.2 is CIFN and given as

$$\mathrm{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \begin{pmatrix} (1-\mathfrak{s}(\zeta_1, \zeta_2, \ldots, \zeta_n), 1-\mathfrak{s}(w_{\zeta_1}, w_{\zeta_2} \ldots, w_{\zeta_n})), \\ (\mathfrak{t}(\vartheta_1, \vartheta_2 \ldots, \vartheta_n), \mathfrak{t}(w_{\vartheta_1}, w_{\vartheta_2}, \ldots, w_{\vartheta_n})) \end{pmatrix} \tag{7.5}$$

where $\mathfrak{s}(x_1,x_2,\ldots,x_n) = \dfrac{\prod\limits_{j=1}^{n}(1-x_j)}{\sum\limits_{j=1}^{n}\left(\xi_j\left(\prod\limits_{\substack{l=1\\l\neq j}}^{n}(1-x_l)\right)\right)}$; $\mathfrak{t}(x_1,x_2,\ldots,x_n) = \dfrac{\prod\limits_{j=1}^{n}x_j}{\sum\limits_{j=1}^{n}\left(\xi_j\left(\prod\limits_{\substack{l=1\\l\neq j}}^{n}x_l\right)\right)}$.

Proof. For any CIFN $\mathcal{C}_j$, weight ξ_j, using Theorem 7.3.1, we get $\xi_j\mathcal{C}_j$ is CIFN. Thus, collective value of CIFWA is also CIFN.

For $n = 2$, we have $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$ and $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$ and $\xi_1, \xi_2 > 0$ such that $\sum\limits_{j=1}^{2}\xi_j = 1$. Thus,

$$\xi_1\mathcal{C}_1 = \left(\begin{pmatrix} 1-\dfrac{1-\zeta_1}{1+(\xi_1-1)\zeta_1}, \\ 1-\dfrac{1-w_{\zeta_1}}{1+(\xi_1-1)w_{\zeta_1}} \end{pmatrix}, \begin{pmatrix} \dfrac{\vartheta_1}{\xi_1-(\xi_1-1)\vartheta_1}, \\ \dfrac{w_{\vartheta_1}}{\xi_1-(\xi_1-1)w_{\vartheta_1}} \end{pmatrix}\right)$$

$$\text{and} \quad \xi_2\mathcal{C}_2 = \left(\begin{pmatrix} 1-\dfrac{1-\zeta_2}{1+(\xi_2-1)\zeta_2}, \\ 1-\dfrac{1-w_{\zeta_2}}{1+(\xi_2-1)w_{\zeta_2}} \end{pmatrix}, \begin{pmatrix} \dfrac{\vartheta_2}{\xi_2-(\xi_2-1)\vartheta_2}, \\ \dfrac{w_{\vartheta_2}}{\xi_2-(\xi_2-1)w_{\vartheta_2}} \end{pmatrix}\right)$$

Hence,

$$\begin{aligned}
&\text{CIFWA}(\mathcal{C}_1,\mathcal{C}_2) = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \\
&= \left(\begin{pmatrix} 1-\dfrac{\left(\frac{1-\zeta_1}{1+(\xi_1-1)\zeta_1}\right)\times\left(\frac{1-\zeta_2}{1+(\xi_2-1)\zeta_2}\right)}{1-\left(1-\frac{1-\zeta_1}{1+(\xi_1-1)\zeta_1}\right)\times\left(1-\frac{1-\zeta_2}{1+(\xi_2-1)\zeta_2}\right)}, \\ 1-\dfrac{\left(\frac{1-w_{\zeta_1}}{1+(\xi_1-1)w_{\zeta_1}}\right)\times\left(\frac{1-w_{\zeta_2}}{1+(\xi_2-1)w_{\zeta_2}}\right)}{1-\left(1-\frac{1-w_{\zeta_1}}{1+(\xi_1-1)w_{\zeta_1}}\right)\times\left(1-\frac{1-w_{\zeta_2}}{1+(\xi_2-1)w_{\zeta_2}}\right)} \end{pmatrix}, \begin{pmatrix} \dfrac{\left(\frac{\vartheta_1}{\xi_1-(\xi_1-1)\vartheta_1}\right)\times\left(\frac{\vartheta_2}{\xi_2-(\xi_2-1)\vartheta_2}\right)}{1-\left(1-\frac{\vartheta_1}{\xi_1-(\xi_1-1)\vartheta_1}\right)\times\left(1-\frac{\vartheta_2}{\xi_2-(\xi_2-1)\vartheta_2}\right)}, \\ \dfrac{\left(\frac{w_{\vartheta_1}}{\xi_1-(\xi_1-1)w_{\vartheta_1}}\right)\times\left(\frac{w_{\vartheta_2}}{\xi_2-(\xi_2-1)w_{\vartheta_2}}\right)}{1-\left(1-\frac{w_{\vartheta_1}}{\xi_1-(\xi_1-1)w_{\vartheta_1}}\right)\times\left(1-\frac{w_{\vartheta_2}}{\xi_2-(\xi_2-1)w_{\vartheta_2}}\right)} \end{pmatrix}\right) \\
&= \left(\begin{pmatrix} 1-\dfrac{\prod\limits_{j=1}^{2}(1-\zeta_j)}{1+(\xi_1-1)\zeta_1+(\xi_2-1)\zeta_2+(1-\xi_1-\xi_2)\zeta_1\zeta_2}, \\ 1-\dfrac{\prod\limits_{j=1}^{2}(1-w_{\zeta_j})}{1+(\xi_1-1)w_{\zeta_1}+(\xi_2-1)w_{\zeta_2}+(1-\xi_1-\xi_2)w_{\zeta_1}w_{\zeta_2}} \end{pmatrix}, \begin{pmatrix} \dfrac{\prod\limits_{j=1}^{2}\vartheta_j}{\vartheta_1\xi_2+\vartheta_2\xi_1+(1-\xi_1-\xi_2)\vartheta_1\vartheta_2}, \\ \dfrac{\prod\limits_{j=1}^{2}w_{\vartheta_j}}{\xi_2 w_{\vartheta_1}+\xi_1 w_{\vartheta_2}+(1-\xi_1-\xi_2)w_{\vartheta_1}w_{\vartheta_2}} \end{pmatrix}\right) \\
&= \left(\left(1-\dfrac{\prod\limits_{j=1}^{2}(1-\zeta_j)}{\sum\limits_{j=1}^{2}\left(\xi_j\left(\prod\limits_{\substack{l=1\\l\neq j}}^{2}(1-\zeta_l)\right)\right)},\ 1-\dfrac{\prod\limits_{j=1}^{2}(1-w_{\zeta_j})}{\sum\limits_{j=1}^{2}\left(\xi_j\left(\prod\limits_{\substack{l=1\\l\neq j}}^{2}(1-w_{\zeta_l})\right)\right)}\right), \left(\dfrac{\prod\limits_{j=1}^{2}\vartheta_j}{\sum\limits_{j=1}^{2}\left(\xi_j\left(\prod\limits_{\substack{l=1\\l\neq j}}^{2}\vartheta_l\right)\right)},\ \dfrac{\prod\limits_{j=1}^{2}w_{\vartheta_j}}{\sum\limits_{j=1}^{2}\left(\xi_j\left(\prod\limits_{\substack{l=1\\l\neq j}}^{2}w_{\vartheta_l}\right)\right)}\right)\right) \\
&= \left(\left(1-\mathfrak{s}(\zeta_1,\zeta_2), 1-\mathfrak{s}(w_{\zeta_1},w_{\zeta_2})\right), \left(\mathfrak{t}(\vartheta_1,\vartheta_2), \mathfrak{t}(w_{\vartheta_1},w_{\vartheta_2})\right)\right)
\end{aligned}$$

For $n = 3$, we have another $\mathcal{C}_3 = ((\zeta_3, w_{\zeta_3}), (\vartheta_3, w_{\vartheta_3}))$ and $\xi_1, \xi_2, \xi_3 > 0$ such that

$\sum\limits_{j=1}^{3} \xi_j = 1$. Thus,

$$\xi_3 \mathcal{C}_3 = \left(\begin{pmatrix} 1 - \dfrac{1-\zeta_3}{1+(\xi_3-1)\zeta_3}, \\ 1 - \dfrac{1-w_{\zeta_3}}{1+(\xi_3-1)w_{\zeta_3}} \end{pmatrix}, \begin{pmatrix} \dfrac{\vartheta_3}{\xi_3-(\xi_3-1)\vartheta_3}, \\ \dfrac{w_{\vartheta_3}}{\xi_3-(\xi_3-1)w_{\vartheta_3}} \end{pmatrix} \right)$$

Now,

$$\begin{aligned}
&\mathrm{CIFWA}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = (\xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2) \oplus \xi_3\mathcal{C}_3 \\
&= \left(\begin{pmatrix} 1 - \dfrac{\prod\limits_{j=1}^{2}(1-\zeta_j)}{1+(\xi_1-1)\zeta_1+(\xi_2-1)\zeta_2+(1-\xi_1-\xi_2)\zeta_1\zeta_2}, \\ 1 - \dfrac{\prod\limits_{j=1}^{2}(1-w_{\zeta_j})}{1+(\xi_1-1)w_{\zeta_1}+(\xi_2-1)w_{\zeta_2}+(1-\xi_1-\xi_2)w_{\zeta_1}w_{\zeta_2}} \end{pmatrix}, \begin{pmatrix} \dfrac{\prod\limits_{j=1}^{2}\vartheta_j}{\vartheta_1\xi_2+\vartheta_2\xi_1+(1-\xi_1-\xi_2)\vartheta_1\vartheta_2}, \\ \dfrac{\prod\limits_{j=1}^{2}w_{\vartheta_j}}{\xi_2 w_{\vartheta_1}+\xi_1 w_{\vartheta_2}+(1-\xi_1-\xi_2)w_{\vartheta_1}w_{\vartheta_2}} \end{pmatrix} \right) \\
&\oplus \left(\begin{pmatrix} 1 - \dfrac{1-\zeta_3}{1+(\xi_3-1)\zeta_3}, \\ 1 - \dfrac{1-w_{\zeta_3}}{1+(\xi_3-1)w_{\zeta_3}} \end{pmatrix}, \begin{pmatrix} \dfrac{\vartheta_3}{\xi_3-(\xi_3-1)\vartheta_3}, \\ \dfrac{w_{\vartheta_3}}{\xi_3-(\xi_3-1)w_{\vartheta_3}} \end{pmatrix} \right) \\
&= \left(\left(1 - \dfrac{\prod\limits_{j=1}^{2}(1-R_j)}{1-\prod\limits_{j=1}^{2}R_j}, 1 - \dfrac{\prod\limits_{j=1}^{2}(1-w_{R_j})}{1-\prod\limits_{j=1}^{2}w_{R_j}} \right), \left(\dfrac{\prod\limits_{j=1}^{2}K_j}{1-\prod\limits_{j=1}^{2}(1-K_j)}, \dfrac{\prod\limits_{j=1}^{2}w_{K_j}}{1-\prod\limits_{j=1}^{2}(1-w_{K_j})} \right) \right)
\end{aligned}$$

where $R_1 = 1 - \frac{\prod\limits_{j=1}^{2}(1-\zeta_j)}{1+(\xi_1-1)\zeta_1+(\xi_2-1)\zeta_2+(1-\xi_1-\xi_2)\zeta_1\zeta_2}$, $R_2 = 1 - \frac{1-\zeta_3}{1+(\xi_3-1)\zeta_3}$, $K_2 = \frac{\vartheta_3}{\xi_3-(\xi_3-1)\vartheta_3}$, $K_1 = \dfrac{\prod\limits_{j=1}^{2}\vartheta_j}{\vartheta_1\xi_2+\vartheta_2\xi_1+(1-\xi_1-\xi_2)\vartheta_1\vartheta_2}$, $w_{R_1} = 1 - \frac{\prod\limits_{j=1}^{2}\left(1-w_{\zeta_j}\right)}{1+(\xi_1-1)w_{\zeta_1}+(\xi_2-1)w_{\zeta_2}+(1-\xi_1-\xi_2),w_{\zeta_1}w_{\zeta_2}}$, $w_{R_2} = 1 - \frac{1-w_{\zeta_3}}{1+(\xi_3-1)w_{\zeta_3}}$, $w_{K_1} = \frac{\prod\limits_{j=1}^{2}w_{\vartheta_j}}{\xi_2 w_{\vartheta_1}+\xi_1 w_{\vartheta_2}+(1-\xi_1-\xi_2)w_{\vartheta_1}w_{\vartheta_2}}$ and $w_{K_2} = \frac{w_{\vartheta_3}}{\xi_3-(\xi_3-1)w_{\vartheta_3}}$.

Thus,

$$\begin{aligned}
1 - \dfrac{\prod\limits_{j=1}^{2}\left(1-R_j\right)}{1-\prod\limits_{j=1}^{2}R_j} &= 1 - \dfrac{\prod\limits_{j=1}^{3}(1-\zeta_j)}{\begin{pmatrix} 1+(\xi_1-1)\zeta_1+(\xi_2-1)\zeta_2+(\xi_3-1)\zeta_3+(1-\xi_1-\xi_2)\zeta_1\zeta_2 \\ +(1-\xi_2-\xi_3)\zeta_2\zeta_3+(1-\xi_1-\xi_3)\zeta_1\zeta_3-(1-\xi_1-\xi_2-\xi_3)\zeta_1\zeta_2\zeta_3 \end{pmatrix}} \\
&= 1 - \dfrac{\prod\limits_{j=1}^{3}(1-\zeta_j)}{\xi_1(1-\zeta_2)(1-\zeta_3)+\xi_2(1-\zeta_1)(1-\zeta_3)+\xi_3(1-\zeta_1)(1-\zeta_2)}
\end{aligned}$$

$$= 1 - \frac{\prod_{j=1}^{3} (1-\zeta_j)}{\sum_{j=1}^{3} \left(\xi_j \left(\prod_{\substack{l=1 \\ l \neq j}}^{3} (1-\zeta_l) \right) \right)}$$

$$= 1 - \mathfrak{s}(\zeta_1, \zeta_2, \zeta_3)$$

Further,

$$\frac{\prod_{j=1}^{2} K_j}{1 - \prod_{j=1}^{2} \left(1 - K_j\right)} = \frac{\left(\frac{\prod_{j=1}^{2} \vartheta_j}{\vartheta_1\xi_2 + \vartheta_2\xi_1 + (1-\xi_1-\xi_2)\vartheta_1\vartheta_2} \right) \times \left(\frac{\vartheta_3}{\xi_3 - (\xi_3 - 1)\vartheta_3} \right)}{1 - \left(1 - \frac{\prod_{j=1}^{2} \vartheta_j}{\vartheta_1\xi_2 + \vartheta_2\xi_1 + (1-\xi_1-\xi_2)\vartheta_1\vartheta_2} \right) \times \left(1 - \frac{\vartheta_3}{\xi_3 - (\xi_3 - 1)\vartheta_3} \right)}$$

$$= \frac{\prod_{j=1}^{3} \vartheta_j}{\vartheta_1\vartheta_2\xi_3 + \vartheta_2\vartheta_3\xi_1 + \vartheta_1\vartheta_3\xi_2}$$

$$= \frac{\prod_{j=1}^{3} \vartheta_j}{\sum_{j=1}^{3} \left(\xi_j \left(\prod_{\substack{l=1 \\ l \neq j}}^{3} \vartheta_l \right) \right)}$$

$$= \mathfrak{t}(\vartheta_1, \vartheta_2, \vartheta_3)$$

Similarly, we derive that $1 - \frac{\prod_{j=1}^{2}\left(1 - w_{R_j}\right)}{1 - \prod_{j=1}^{2} w_{R_j}} = 1 - \mathfrak{s}(w_{\zeta_1}, w_{\zeta_2}, w_{\zeta_3})$ and $\frac{\prod_{j=1}^{2} w_{K_j}}{1 - \prod_{j=1}^{2}\left(1 - w_{K_j}\right)} = \mathfrak{t}(w_{\vartheta_1}, w_{\vartheta_2}, w_{\vartheta_3})$. Hence, Eq. (7.5) holds for $n = 3$. Continuing in this way, we can derive that Eq. (7.5) holds for all n. $\square$

The working of the above theorem is explained with an example as below.

Example 7.3.1. Let $\mathcal{C}_1 = ((0.6, 0.8), (0.2, 0.1))$, $\mathcal{C}_2 = ((0.8, 0.7), (0.2, 0.1))$, $\mathcal{C}_3 = ((0.5, 0.6), (0.3, 0.4))$, $\mathcal{C}_4 = ((0.6, 0.7), (0.3, 0.2))$ be four CIFNs with $\xi = (0.15, 0.30, 0.25, 0.30)^T$ as their weight vector. Based on these values, $\prod_{j=1}^{4} (1 - \zeta_j) = 0.0160$, $\sum_{j=1}^{4} \left(\xi_j \left(\prod_{\substack{l=1 \\ l \neq j}}^{4} (1 - \zeta_l) \right) \right) = 0.05$, $\prod_{j=1}^{4} \left(1 - w_{\zeta_j}\right) = 0.0072$, $\sum_{j=1}^{4} \left(\xi_j \left(\prod_{\substack{l=1 \\ l \neq j}}^{4} (1 - w_{\zeta_l}) \right) \right) = 0.0243$, $\prod_{j=1}^{4} \vartheta_j = 0.0036$,

$$\sum_{j=1}^{4}\left(\xi_j\left(\prod_{\substack{l=1\\l\neq j}}^{4}\vartheta_l\right)\right)=0.0147,\ \prod_{j=1}^{4}w_{\vartheta_j}=0.0008,\ \sum_{j=1}^{4}\left(\xi_j\left(\prod_{\substack{l=1\\l\neq j}}^{4}w_{\vartheta_l}\right)\right)=0.0053.$$

Thus, we get

$$\mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4) = \Big(\big(0.6800, 0.7037\big), \big(0.2449, 0.1509\big)\Big)$$

Some desirable properties of CIFWA operator are stated as

Theorem 7.3.7. (Idempotency) Let $\mathcal{C}_0$ be another CIFN such that $\mathcal{C}_j = \mathcal{C}_0$ for all j. Then,

$$\mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2\ldots\mathcal{C}_n)=\mathcal{C}_0$$

Proof. Let $\mathcal{C}_0=\Big(\big(\zeta_0,w_{\zeta_0}\big),\big(\vartheta_0,w_{\vartheta_0}\big)\Big)$ with $\mathcal{C}_j=\mathcal{C}_0$. Therefore for all j, $\zeta_j=\zeta_0$, $\vartheta_j=\vartheta_0$, $w_{\zeta_j}=w_{\zeta_0}$, $w_{\vartheta_j}=w_{\vartheta_0}$ and by Eq. (7.5), we get $\mathfrak{s}(\zeta_0,\zeta_0,\ldots,\zeta_0)=\dfrac{\prod_{j=1}^{n}(1-\zeta_0)}{\sum_{j=1}^{n}\left(\xi_j\left(\prod_{\substack{l=1\\l\neq j}}^{n}(1-\zeta_0)\right)\right)}=\dfrac{(1-\zeta_0)^n}{\sum_{j=1}^{n}\left(\xi_j(1-\zeta_0)^{n-1}\right)}=\dfrac{(1-\zeta_0)^n}{\sum_{j=1}^{n}\left(\xi_j(1-\zeta_0)^{n-1}\right)}=1-\zeta_0$ and $\mathfrak{t}(\vartheta_0,\vartheta_0,\ldots,\vartheta_0)=\dfrac{\vartheta_0^n}{\sum_{j=1}^{n}\left(\xi_j(\vartheta_0)^{n-1}\right)}=\vartheta_0$. Similarly, $\mathfrak{s}(w_{\zeta_0},w_{\zeta_0},\ldots,w_{\zeta_0})=1-w_{\zeta_0}$ and $\mathfrak{t}(w_{\vartheta_0},w_{\vartheta_0},\ldots,w_{\vartheta_0})=w_{\vartheta_0}$. Hence, $\mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2\ldots\mathcal{C}_n)=\mathcal{C}_0$. □

Theorem 7.3.8. (Monotonicity) For collections of CIFNs $\mathcal{C}_j$ and $\mathcal{Z}_j$ satisfying $\mathcal{C}_j\subseteq\mathcal{Z}_j$, we have

$$\mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)\subseteq\mathrm{CIFWA}(\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\mathcal{Z}_n).$$

Proof. As $\mathcal{C}_j\subseteq\mathcal{Z}_j\ \forall\ j$. Thus, by using results of Theorem 7.3.5 we get $\xi_1\mathcal{C}_1\oplus\xi_2\mathcal{C}_2\oplus\ldots\oplus\xi_n\mathcal{C}_n\subseteq\xi_1\mathcal{Z}_1\oplus\xi_2\mathcal{Z}_2\oplus\ldots\oplus\xi_n\mathcal{Z}_n$. Hence, $\mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)\subseteq\mathrm{CIFWA}(\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\mathcal{Z}_n)$. □

Theorem 7.3.9. (Boundedness) Let $\mathcal{C}^-$ and $\mathcal{C}^+$ are lower and upper bounds of CIFNs $\mathcal{C}_j$. Then,

$$\mathcal{C}^-\subseteq\mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2\ldots\mathcal{C}_n)\subseteq\mathcal{C}^+$$

Proof. Since, $\mathcal{C}^-\subseteq\mathcal{C}_j\subseteq\mathcal{C}^+\ \forall\ j$. Thus, by Theorem 7.3.8, $\mathrm{CIFWA}(\mathcal{C}^-,\mathcal{C}^-,\ldots,\mathcal{C}^-)\subseteq\mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)\subseteq\mathrm{CIFWA}(\mathcal{C}^+,\mathcal{C}^+,\ldots,\mathcal{C}^+)$. Further, by idempotency property, we obtain $\mathcal{C}^-\subseteq\mathrm{CIFWA}(\mathcal{C}_1,\mathcal{C}_2\ldots\mathcal{C}_n)\subseteq\mathcal{C}^+$. Hence, the result. □

Next, some new geometric AOs for CIFNs $\mathcal{C}_j = ((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j}))$, with $\zeta_j, w_{\zeta_j} \neq 0$ $\forall\ j$ are defined.

Definition 7.3.3. A function CIFWG : $\Omega^n \to \Omega$, defined by:

$$\text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \quad = \quad \mathcal{C}_1^{\xi_1} \otimes \mathcal{C}_2^{\xi_2} \otimes \ldots \otimes \mathcal{C}_n^{\xi_n} \tag{7.6}$$

is called as CIF weighted geometric (CIFWG) AO.

Theorem 7.3.10. For "n" CIFNs $\mathcal{C}_j$, the value acquired by applying Definition 7.3.3 is CIFN and is given by

$$\text{CIFWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \begin{pmatrix} (\mathfrak{t}(\zeta_1, \zeta_2, \ldots, \zeta_n), \mathfrak{t}(w_{\zeta_1}, w_{\zeta_2} \ldots, w_{\zeta_n}))\,, \\ (1 - \mathfrak{s}(\vartheta_1, \vartheta_2 \ldots, \vartheta_n), 1 - \mathfrak{s}(w_{\vartheta_1}, w_{\vartheta_2}, \ldots, w_{\vartheta_n})) \end{pmatrix} \tag{7.7}$$

Proof. Follows as similar to Theorem 7.3.6. □

7.4 Proposed MCDM method

In this section, an efficient approach for solving the MCDM problems is presented along with a method to determine the criteria weights.

7.4.1 Brief description of the problem

The general description of MCDM problem is given in Section 2.5 of Chapter 2. Suppose that an expert gave their rating towards $\mathcal{V}_u$, $(u = 1, 2, \ldots, m)$ under $\mathfrak{B}_v$, $(v = 1, 2, \ldots, n)$ in terms of CIFNs $\mathcal{C}_{uv} = \Big((\zeta_{uv}, w_{\zeta_{uv}}), (\vartheta_{uv}, w_{\vartheta_{uv}})\Big)$ such that $0 \leq \zeta_{uv}, \vartheta_{uv} \leq 1$, $\zeta_{uv} + \vartheta_{uv} \leq 1$ and $0 \leq w_{\zeta_{uv}}, w_{\vartheta_{uv}} \leq 1$, $w_{\zeta_{uv}} + w_{\vartheta_{uv}} \leq 1$. The collective information of the expert is represented as a decision matrix $\mathcal{M}$ given by

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \ldots & \mathcal{C}_{1n} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \ldots & \mathcal{C}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{m1} & \mathcal{C}_{m2} & \ldots & \mathcal{C}_{mn} \end{pmatrix} \end{array} \tag{7.8}$$

7.4.2 Weight determination

In real-life DMPs, it is quite obvious that all the criteria do not pay equal attention during the process. For instance, in some certain problem, reliability pays more attention than the cost, manufacturing time of the product while for some different problems, some other parameters pay more attention. Thus, choosing the proper weight to the criteria is very important. To determine these, two methods based on the situation available are presented below:

(i) *If partial information is known about the criteria weight:*

Let $\triangle$ denotes the partial information about the criteria weights. To compute these weights, an optimization model by using the concept of the distance measures between the alternatives $\mathcal{V}_u$ from its positive and negative ideal alternatives, denoted by PIA and NIA is constructed. To it, we consider PIA as $\mathcal{V}^+ = (\mathcal{C}_v^+)_{1\times n}$ and NIA as $\mathcal{V}^- = (\mathcal{C}_v^-)_{1\times n}$, where

$$\begin{aligned} \mathcal{C}_v^+ &= \Big((\zeta_v^+, w_{\zeta_v}^+), (\vartheta_v^+, w_{\vartheta_v}^+)\Big) \\ &= \Big(\Big(\max_u\{\zeta_{uv}\}, \max_u\{w_{\zeta_{uv}}\}\Big), \Big(\min_u\{\vartheta_{uv}\}, \min_u\{w_{\vartheta_{uv}}\}\Big)\Big) \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_v^- &= \Big(\Big(\zeta_v^-, w_{\zeta_v}^-\Big), (\vartheta_v^-, w_{\vartheta_v}^-)\Big) \\ &= \Big(\Big(\min_u\{\zeta_{uv}\}, \min_u\{w_{\zeta_{uv}}\}\Big), \Big(\max_u\{\vartheta_{uv}\}, \max_u\{w_{\vartheta_{uv}}\}\Big)\Big) \end{aligned}$$

Thus, by Eq. (3.6) of Chapter 3, we have

$$\mathcal{D}(\mathcal{V}_u, \mathcal{V}^+) = \left[\frac{1}{4}\sum_{v=1}^{n}\left\{\xi_v\begin{pmatrix} |\zeta_{uv} - \zeta_v^+|^2 + |w_{\zeta_{uv}} - w_{\zeta_v}^+|^2 \\ + |\vartheta_{uv} - \vartheta_v^+|^2 + |w_{\vartheta_{uv}} - w_{\vartheta_v}^+|^2 \end{pmatrix}\right\}\right]^{\frac{1}{2}} \tag{7.9}$$

and

$$\mathcal{D}(\mathcal{V}_u, \mathcal{V}^-) = \left[\frac{1}{4}\sum_{v=1}^{n}\left\{\xi_v\begin{pmatrix} |\zeta_{uv} - \zeta_v^-|^2 + |w_{\zeta_{uv}} - w_{\zeta_v}^-|^2 \\ + |\vartheta_{uv} - \vartheta_v^-|^2 + |w_{\vartheta_{uv}} - w_{\vartheta_v}^-|^2 \end{pmatrix}\right\}\right]^{\frac{1}{2}} \tag{7.10}$$

It is evident that the smaller $\mathcal{D}(\mathcal{V}_u, \mathcal{V}^+)$ or larger $\mathcal{D}(\mathcal{V}_u, \mathcal{V}^-)$, the better the alternative $\mathcal{V}_u$. However, the individual distance does not give the optimal decision and hence from Eqs. (7.9) and (7.10), we define the satisfaction degree of $\mathcal{V}_u$ as

$$\eta(\mathcal{V}_u) = \frac{\mathcal{D}(\mathcal{V}_u, \mathcal{V}^-)}{\mathcal{D}(\mathcal{V}_u, \mathcal{V}^-) + \mathcal{D}(\mathcal{V}_u, \mathcal{V}^+)}; \qquad \mathcal{D}(\mathcal{V}_u, \mathcal{V}^+) \neq 0 \tag{7.11}$$

where $\eta(\mathcal{V}_u) \in [0, 1]$. Now based on these values, the multi-objective optimization model to determine the criteria weight is constructed as

$$\text{(M-1:)} \qquad \begin{cases} \max \Big(\eta(\mathcal{V}_1), \eta(\mathcal{V}_2), \ldots, \eta(\mathcal{V}_m)\Big) \\ \text{subject to} \quad \xi_v \in \triangle, \sum\limits_{v=1}^{n} \xi_v = 1 \end{cases} \tag{7.12}$$

Further, M-1 is converted into M-2 as

$$\text{(M-2:)} \qquad \begin{cases} \max \sum\limits_{u=1}^{m} \eta(\mathcal{V}_u) \\ \text{subject to} \quad \xi_v \in \triangle, \sum\limits_{v=1}^{n} \xi_v = 1 \end{cases} \tag{7.13}$$

By solving the M-2, the optimal weight as $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ can be obtained.

(ii) *if information is completely unknown about the criteria weight:*

For CIFN $\mathcal{C}_{uv}$, the hesitation degree of it is defined as

$$h_{\mathcal{C}_{uv}} = 2 - (\zeta_{uv} + \vartheta_{uv}) - (w_{\zeta_{uv}} + w_{\vartheta_{uv}}) \tag{7.14}$$

During the real DMPs, it is always preferable to design an algorithm which have less hesitancy degree. In other words, the role of the criteria during the decision process is entirely based on the degree of hesitancy and hence smaller the hesitation degree, the more important is the object. Thus, based on it, the hesitancy matrix $\mathcal{V}$ for the given alternatives is constructed as

$$\mathcal{V} = \begin{pmatrix} h_{\mathcal{C}_{11}} & h_{\mathcal{C}_{12}} & \ldots & h_{\mathcal{C}_{1n}} \\ h_{\mathcal{C}_{21}} & h_{\mathcal{C}_{22}} & \ldots & h_{\mathcal{C}_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\mathcal{C}_{m1}} & h_{\mathcal{C}_{m2}} & \ldots & h_{\mathcal{C}_{mn}} \end{pmatrix} \tag{7.15}$$

where $h_{\mathcal{C}_{uv}}$ is calculated from Eq. (7.14). Hence, the weight vector ξ_v is determined as

$$\xi_v = \frac{2 - \left(\frac{1}{m} \sum\limits_{u=1}^{m} h_{\mathcal{C}_{uv}}\right)}{\sum\limits_{v=1}^{n} \left(2 - \left(\frac{1}{m} \sum\limits_{u=1}^{m} h_{\mathcal{C}_{uv}}\right)\right)} \tag{7.16}$$

7.4.3 Aggregation and Ranking phase

By utilizing the weight ξ_v, aggregate the values of $\mathcal{C}_{uv}$ into $\mathcal{C}_u$ by using either CIFWA operator i.e.,

$$\mathcal{C}_u = \begin{pmatrix} (1-\mathfrak{s}(\zeta_{u1},\zeta_{u2},\ldots,\zeta_{un}), 1-\mathfrak{s}(w_{\zeta_{u1}},w_{\zeta_{u2}},\ldots,w_{\zeta_{un}})), \\ (\mathfrak{t}(\vartheta_{u1},\vartheta_{u2},\ldots,\vartheta_{un}), \mathfrak{t}(w_{\vartheta_{u1}},w_{\vartheta_{u2}},\ldots,w_{\vartheta_{un}})) \end{pmatrix} \tag{7.17}$$

or by CIFWG operator i.e.,

$$\mathcal{C}_u = \begin{pmatrix} (\mathfrak{t}(\zeta_{u1},\zeta_{u2},\ldots,\zeta_{un}), \mathfrak{t}(w_{\zeta_{u1}},w_{\zeta_{u2}},\ldots,w_{\zeta_{un}})), \\ (1-\mathfrak{s}(\vartheta_{u1},\vartheta_{u2},\ldots,\vartheta_{un}), 1-\mathfrak{s}(w_{\vartheta_{u1}},w_{\vartheta_{u2}},\ldots,w_{\vartheta_{un}})) \end{pmatrix} \tag{7.18}$$

To rank the given alternatives, construct the possibility degree matrix $\mathcal{P} = (\chi_{ul})_{m\times m}$ by utilizing the proposed PDM χ and get

$$\mathcal{P} = \begin{pmatrix} \chi_{11} & \chi_{12} & \cdots & \chi_{1m} \\ \chi_{21} & \chi_{22} & \cdots & \chi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{m1} & \chi_{m2} & \cdots & \chi_{mm} \end{pmatrix} \tag{7.19}$$

where $\chi_{ul} = \chi(\mathcal{C}_u \succ \mathcal{C}_l)$, $u, l \in \{1, 2, \ldots, m\}$ is calculated as:

(i) If either $h_{\mathcal{C}_u} \neq 0$ or $h_{\mathcal{C}_l} \neq 0$ then,

$$\chi(\mathcal{C}_u \succ \mathcal{C}_l) = \min\left(\max\left(\frac{2+\zeta_u-2\zeta_l-\vartheta_l+w_{\zeta_u}-2w_{\zeta_l}-w_{\vartheta_l}}{h_{\mathcal{C}_u}+h_{\mathcal{C}_l}}, 0\right), 1\right) \tag{7.20}$$

(ii) If $h_{\mathcal{C}_u} = h_{\mathcal{C}_l} = 0$ then,

$$\chi(\mathcal{C}_u \succ \mathcal{C}_l) = \begin{cases} 1; & \zeta_u + w_{\zeta_u} > \zeta_l + w_{\zeta_l} \\ 0.5; & \zeta_u + w_{\zeta_u} = \zeta_l + w_{\zeta_l} \\ 0; & \zeta_u + w_{\zeta_u} < \zeta_l + w_{\zeta_l} \end{cases} \tag{7.21}$$

Finally, for each alternative $\mathcal{V}_u$, the optimal degree of it is calculated by Eq. (7.22) as

$$\mathcal{R}_u = \frac{1}{m(m-1)}\left(\sum_{l=1}^{m}\chi_{ul} + \frac{m}{2} - 1\right) \tag{7.22}$$

and hence, select the desired one(s).

In short, we compile the preceding analysis into the subsequent steps:

Step 1: Arrange the collective information of alternatives in the matrix $\mathcal{M}$ as given in Eq. (7.8).

Step 2: Compute the criteria weight based on the information and using the Eq. (7.13) or Eq. (7.16).

Step 3: Utilize Eq. (7.17) or Eq. (7.18) to aggregate the preference values.

Step 4: Construct the possibility matrix $\mathcal{P} = (\chi_{ul})_{m\times m}$ by using Eqs. (7.20) and Eq. (7.21).

Step 5: Rank the alternatives $\mathcal{V}_u$ based on descending values of $\mathcal{R}_u$ computed from Eq. (7.22).

7.5 A case study

The presented approach has been demonstrated with a numerical example whose description is given as follows.

A regional space center namely, North-Eastern Space Applications Centre (NESAC), is an outcome of a collaborative initiative of Department of Science, Government of India with North-Eastern (NE) council. The main goal of NESAC is to explore innovative methods of utilizing natural resources with the help of technically advanced tools and methods such as remote sensing. This ameliorates the scope of satellite services to be accessed by the NE states and as an outcome of this space, research is promoted in these areas. Likewise, Unmanned Aerial Vehicle (UAV) remote sensing is a new technological advancement to NESAC for large scale mapping as well as application monitoring of various activities. UAV, also known as a drone, is basically a flying robot. In more simple terms, it is just an aircraft which can either be remotely controlled by a human operator or autonomously by an onboard computer. The main task of the NESAC is to assemble ten new UAVs for large scale mapping in the NE region. For this, NESAC is looking for data processing and analysis software in order to process the images and videos captured by UAVs and this software should support the latest version of the operating system. For the installation of required software, NESAC consults an Information Technology (IT) company which provides the information on five models of software viz: $\mathcal{V}_u$ $(u = 1, 2, 3, 4, 5)$ with different software versions. NESAC consults an expert who evaluates the software on the basis of four criteria namely $\mathfrak{B}_1$: Image processing capability, $\mathfrak{B}_2$: Measurement tools for co-ordinate/distance/area/volume, $\mathfrak{B}_3$: Generation of contour lines using DEM/DSM and $\mathfrak{B}_4$: Generation of 3D modeling/texturing capabilities. Obviously, the changes in software version will affect these criteria. The primary goal of the NESAC is to select the best software(s) and its version simultaneously. Thus the problems are two dimensional namely, type of software and software version. Therefore, the decision-makers express their rating values in terms of CIFNs as the CIF model handles two-dimensional information simultaneously. The rating values of the expert for $\mathcal{V}_1$ at $\mathfrak{B}_1$ are $\Big((0.6, 0.9), (0.1, 0.1)\Big)$ which depict that the expert is 60% satisfied for $\mathcal{V}_1$ at $\mathfrak{B}_1$ and 10% disagrees. The phase term that represents the version of the software is given as the expert is 90% satisfied with the suitability of software version at $\mathfrak{B}_1$ and 10% is not satisfied. Similarly, all data can be decoded. The preferences of the expert towards each $\mathcal{V}_u$ is represented in Table 7.1.

7.5.1 If information about the weight is partially known

Assume that given partial information of the weight is

$$\triangle = \begin{Bmatrix} 0.25 \leq \xi_1 \leq 0.4, & 0.1 \leq \xi_2 \leq 0.5, & 0.2 \leq \xi_3 \leq 0.5, \\ 0.2 \leq \xi_4 \leq 0.45, & \xi_4 \leq \xi_1 + \xi_2, & \xi_1 \leq \xi_2 \end{Bmatrix}$$

Then, the procedure steps of the proposed method are implemented as follows.

Step 1: Rating values are given in Table 7.1.

Step 2: Based on information $\triangle$, formulate an optimization model (7.13) as

$$\begin{aligned} \max \quad & \eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5 \\ \text{subject to} \quad & 0.25 \leq \xi_1 \leq 0.40, \quad 0.10 \leq \xi_2 \leq 0.50, \\ & 0.20 \leq \xi_3 \leq 0.50, \quad 0.20 \leq \xi_4 \leq 0.45, \\ & \xi_4 \leq \xi_1 + \xi_2, \quad \xi_1 \leq \xi_2, \xi_1 + \xi_2 + \xi_3 + \xi_4 = 1. \end{aligned}$$

where

$$\eta_1 = \frac{\sqrt{0.2325\xi_1 + 0.1250\xi_2 + 0.0500\xi_3 + 0.1125\xi_4}}{\sqrt{\begin{pmatrix} 0.2325\xi_1 + 0.1250\xi_2 + \\ 0.0500\xi_3 + 0.1125\xi_4 \end{pmatrix}} + \sqrt{\begin{pmatrix} 0.0025\xi_1 + 0.1825\xi_2 + \\ 0.0175\xi_3 + 0.0025\xi_4 \end{pmatrix}}}$$

$$\eta_2 = \frac{\sqrt{0.1175\xi_1 + 0.2250\xi_2 + 0.0625\xi_3 + 0.0625\xi_4}}{\sqrt{\begin{pmatrix} 0.1175\xi_1 + 0.2250\xi_2 + \\ 0.0625\xi_3 + 0.0625\xi_4 \end{pmatrix}} + \sqrt{\begin{pmatrix} 0.0425\xi_1 + 0.0425\xi_2 + \\ 0.0150\xi_3 + 0.0525\xi_4 \end{pmatrix}}}$$

$$\eta_3 = \frac{\sqrt{0.1250\xi_1 + 0.1300\xi_2 + 0.1025\xi_3 + 0.0950\xi_4}}{\sqrt{\begin{pmatrix} 0.1250\xi_1 + 0.1300\xi_2 + \\ 0.1025\xi_3 + 0.0950\xi_4 \end{pmatrix}} + \sqrt{\begin{pmatrix} 0.0300\xi_1 + 0.0425\xi_2 + \\ 0.0200\xi_3 + 0.0050\xi_4 \end{pmatrix}}}$$

$$\eta_4 = \frac{\sqrt{0.1250\xi_1 + 0.0750\xi_2 + 0.0625\xi_3 + 0.0225\xi_4}}{\sqrt{\begin{pmatrix} 0.1250\xi_1 + 0.0750\xi_2 + \\ 0.0625\xi_3 + 0.0225\xi_4 \end{pmatrix}} + \sqrt{\begin{pmatrix} 0.1050\xi_1 + 0.1125\xi_2 + \\ 0.0250\xi_3 + 0.0525\xi_4 \end{pmatrix}}}$$

$$\text{and} \quad \eta_5 = \frac{\sqrt{0.0250\xi_1 + 0.0025\xi_2 + 0.0200\xi_3 + 0.0100\xi_4}}{\sqrt{\begin{pmatrix} 0.0250\xi_1 + 0.0025\xi_2 + \\ 0.0200\xi_3 + 0.0100\xi_4 \end{pmatrix}} + \sqrt{\begin{pmatrix} 0.1850\xi_1 + 0.2950\xi_2 + \\ 0.1025\xi_3 + 0.1000\xi_4 \end{pmatrix}}}$$

After solving, we get $\xi = (0.25, 0.25, 0.30, 0.20)^T$.

Step 3: Aggregate the values of each $\mathcal{V}_u$ under different $\mathfrak{B}_v$ by considering CIFWA operator, as given in Eq. (7.17). The resultant values obtained from them are given as

$$\mathcal{C}_1 = \Big((0.7241, 0.7616), (0.1176, 0.1778)\Big), \mathcal{C}_2 = \Big((0.6129, 0.7838), (0.2353, 0.1579)\Big),$$
$$\mathcal{C}_3 = \Big((0.6026, 0.7073), (0.1967, 0.1290)\Big), \mathcal{C}_4 = \Big((0.5826, 0.6164), (0.2000, 0.1463)\Big),$$
$$\mathcal{C}_5 = \Big((0.5758, 0.2588), (0.2124, 0.4444)\Big).$$

Step 4: By Eq. (7.19), the matrix $\mathcal{P}$ is

$$\mathcal{P} = \begin{pmatrix} 0.5000 & 0.6974 & 0.9261 & 1.0000 & 1.0000 \\ 0.3026 & 0.5000 & 0.7852 & 0.9812 & 1.0000 \\ 0.0739 & 0.2148 & 0.5000 & 0.6906 & 1.0000 \\ 0.0000 & 0.0188 & 0.3094 & 0.5000 & 0.9064 \\ 0.0000 & 0.0000 & 0.0000 & 0.0936 & 0.5000 \end{pmatrix}$$

Step 5: By Eq. (7.22), $\mathcal{R}_1 = 0.2812$, $\mathcal{R}_2 = 0.2534$, $\mathcal{R}_3 = 0.1990$, $\mathcal{R}_4 = 0.1617$ and $\mathcal{R}_5 = 0.1047$. Thus, from them, we have $\mathcal{V}_1$ is the most desirable alternative.

7.5.2 If information is completely unknown

The steps of the approach are executed as below.

Step 1: Data is summarized in Table 7.1.

Step 2: By Eq. (7.16), we get $\xi = (0.2638, 0.2301, 0.2485, 0.2577)^T$.

Step 3: Aggregate the values of Table 7.1 by Eq. (7.17) and get

$$\mathcal{C}_1 = \Big((0.7312, 0.7714), (0.1142, 0.1741)\Big), \mathcal{C}_2 = \Big((0.6033, 0.7744), (0.2421, 0.1518)\Big),$$
$$\mathcal{C}_3 = \Big((0.6140, 0.7003), (0.1821, 0.1353)\Big), \mathcal{C}_4 = \Big((0.5767, 0.6241), (0.2156, 0.1530)\Big),$$
$$\mathcal{C}_5 = \Big((0.5706, 0.2624), (0.2279, 0.4506)\Big).$$

Step 4: The possibility matrix $\mathcal{P}$ is computed as

$$\mathcal{P} = \begin{pmatrix} 0.5000 & 0.8072 & 0.9637 & 1.0000 & 1.0000 \\ 0.1928 & 0.5000 & 0.7236 & 0.9219 & 1.0000 \\ 0.0363 & 0.2764 & 0.5000 & 0.6810 & 1.0000 \\ 0.0000 & 0.0781 & 0.3190 & 0.5000 & 0.9316 \\ 0.0000 & 0.0000 & 0.0000 & 0.0684 & 0.5000 \end{pmatrix}$$

Step 5: Based on matrix $\mathcal{P}$, we obtain $\mathcal{R}_1 = 0.2885$, $\mathcal{R}_2 = 0.2419$, $\mathcal{R}_3 = 0.1997$, $\mathcal{R}_4 = 0.1664$ and $\mathcal{R}_5 = 0.1034$. Hence, ranking order of alternatives is $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_5$ which gives that $\mathcal{V}_1$ is the best.

7.5.3 Validity Test

To examine the authenticity and legality of the method, some test criteria are presented in [162], which are asserted as:

Test with criterion 1: In it, we replace only the ratings of $\mathcal{V}_5$ with $\mathcal{V}_5'$= { ((0.3,0.2),(0.6,0.7)), ((0.2,0.1),(0.7,0.5)), ((0.5,0.2), (0.4,0.6)), ((0.1,0.1), (0.7,0.7))} and hence steps of the proposed approach are executed. The final optimal value of alternatives is computed as $\mathcal{R}_1 = 0.2812$, $\mathcal{R}_2 = 0.2534$, $\mathcal{R}_3 = 0.1990$, $\mathcal{R}_4 = 0.1664$ and $\mathcal{R}_5 = 0.1000$. From these, we conclude $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_5'$ and hence $\mathcal{V}_1$ is still best one. Therefore, "*test criterion 1*" validates.

Test with criteria 2 & 3: In it, the considered alternatives of DMP is split into four subproblems as $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$, $\{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5\}$, $\{\mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_1\}$ and $\{\mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_1, \mathcal{V}_2\}$. Now for each subproblems, steps of proposed approach are executed and hence obtain the final ranking order as $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$, $\mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_5$, $\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_5$ and $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_5$ respectively. Hence, the overall ranking is $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_5$ which states that it validates the "*test-criteria 2 and 3*".

7.6 Comparative studies

To manifest the fulfillment of the intended approach with some comprehensive studies under CIFS [6, 59, 129, 130] and IFS [27, 44, 47, 72, 83, 156, 201, 217] environment, the analysis has been conducted by considering the weight of the criteria as $\chi = (0.25, 0.25, 0.30, 0.20)^T$. The discussion of them is compiled as follows.

7.6.1 With CIFS studies

The performance of the presented work is tested with some existing distance measures [6, 129] and averaging operators [59, 130]. The obtained performance by them is given as

1) By applying the weighted Euclidean measure presented by Rani and Garg [129] on to the considered information, we get the collective measurement values of each alternative as $\mathcal{D}(\mathcal{V}_1, \mathcal{V}^+) = 0.2280$, $\mathcal{D}(\mathcal{V}_2, \mathcal{V}^+) = 0.1904$, $\mathcal{D}(\mathcal{V}_3, \mathcal{V}^+) = 0.1585$, $\mathcal{D}(\mathcal{V}_4, \mathcal{V}^+) = 0.2690$ and $\mathcal{D}(\mathcal{V}_5, \mathcal{V}^+) = 0.4132$. Here $\mathcal{V}^+$ and $\mathcal{V}^-$ are PIA and NIA respectively. Based on

them, we get ordering values as $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_5$. Hence, the most optimal is $\mathcal{V}_3$.

2) Alkouri and Salleh [6] defined the distance measure given in Eq. (2.8). Under it, by taking $\alpha_1 = \beta_1 = \sigma_1 = \alpha_2 = \beta_2 = \sigma_2 = \frac{1}{3}$, the measurement value of the given alternatives is $\mathcal{D}(\mathcal{V}_1, \mathcal{V}^+) = 0.1342$, $\mathcal{D}(\mathcal{V}_2, \mathcal{V}^+) = 0.1625$, $\mathcal{D}(\mathcal{V}_3, \mathcal{V}^+) = 0.1350$, $\mathcal{D}(\mathcal{V}_4, \mathcal{V}^+) = 0.2325$ and $\mathcal{D}(\mathcal{V}_5, \mathcal{V}^+) = 0.3717$. Hence, the ordering position of the alternatives is $\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_5$ which suggest that $\mathcal{V}_3$ is the suitable one.

3) If we aggregate the given information with weighted power averaging operator [130], we get the overall score values of the alternatives as $S(\mathcal{V}_1) = 1.0622$, $S(\mathcal{V}_2) = 0.9052$, $S(\mathcal{V}_3) = 0.9356$, $S(\mathcal{V}_4) = 0.7025$ and $S(\mathcal{V}_5) = 0.0713$. Thus, using Definition 6.2.1, we have $\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_5$ which gives that $\mathcal{V}_1$ is the required choice.

4) By implementing the averaging operator [59], with generator $t(a) = -\log(a)$ (For more details, refer to [59]), the score value of the alternatives is $S(\mathcal{V}_1) = 1.0570$, $S(\mathcal{V}_2) = 0.9077$, $S(\mathcal{V}_3) = 0.9353$, $S(\mathcal{V}_4) = 0.7060$ and $S(\mathcal{V}_5) = 0.0721$. Thus, the ordering position becomes $\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_5$ and hence, $\mathcal{V}_1$ is the most preferable option.

From the implementation of these approaches, we conclude that the alternative $\mathcal{V}_3$ is obtained as a successful one from the existing distance measures proposed in [129]. On the other hand, $\mathcal{V}_1$ is the best alternative by the existing AOs [59, 130] and distance measure [6]. This change in their optimal one is due to the entirely differ computational procedure. For example, in [6, 129] approaches, the optimality of the alternative depends on PIA alone, which itself is not sufficient to conclude the desirable alternatives. However, in reality, several situations may occur where there is a need to examine both the PIA and NIA to avoid the loss of suitable information based on the small distance or large similarity. Since the suitable and desirable alternative remains at a larger distance from its NIA and the shortest distance from PIA, therefore both PIA and NIA are simultaneously considered in the present work with a satisfaction function defined in Eq. (7.11). Hence, the presented approach is more suitable and reliable than the existing approaches [6, 129]. Furthermore, it is concluded from the ordering of the alternatives by approaches mentioned in [59, 130] that $\mathcal{V}_3 \succ \mathcal{V}_2$, i.e., $\mathcal{V}_3$ is more superior than $\mathcal{V}_2$, whereas by proposed approach we get $\mathcal{V}_2 \succ \mathcal{V}_3$. These changes occur due to the selection of the ranking procedure. For instance, in [59, 130] approaches, alternatives are ranked by score functions and by Definition 6.2.1 whereas, in the proposed approach, PDM is utilized for the same.

7.6.2 With IFS studies

From the study it is proved that IFS is an exceptional case of the CIFS by fixing phase terms to be zero, thus, we examine the appearance of the intended results with some of the existing approaches [27, 44, 47, 72, 83, 156, 201, 217] under the IFS environment also. The results corresponding to them are summarized in Table 7.2. From this table, it is seen that the best alternative comes out to be $\mathcal{V}_1$ which is the same as the proposed method. However, the ordering of the alternative between $\mathcal{V}_2$ and $\mathcal{V}_3$ changes with from the proposed one. This change in the ranking order is quite reasonable because in the existing approaches [83, 156, 201, 217], only amplitude values of MDs and NMDs are aggregated and ignore the influence of the fluctuation term of the data. However, the proposed MCDM approach considers the complex-valued MDs and NMDs and hence aggregated the two-dimensional information simultaneously in a single set. This analysis leads to the conclusion that if the person wants to consider only the models of the software and not their versions, then $\mathcal{V}_3$ is preferable over $\mathcal{V}_2$. However, if the software versions are also taken into the account in the evaluation process then a person can prefer $\mathcal{V}_2$ over $\mathcal{V}_3$.

Further, as compared to the other existing AOs proposed in [27, 44, 47, 72], we conclude that the best alternative remains $\mathcal{V}_1$ with the proposed and the existing ones. However, the ordering of other alternatives also changes. This is due to the fact that the operators defined in [27, 44, 47, 72] pay more attention to the pessimistic nature of the decision makers. However, the results computed by the presented method using CIFWA operator shows optimistic nature to the expert. Also, if by utilizing CIFWG operator to the consider problem then the results computed from them will be more pessimistic in nature. Therefore, both the optimistic and pessimistic nature are considered in a presented MCDM method. Finally, it is observed that the AOs presented by Fan et al. [42] is considered as a special case of the proposed AOs. Hence, the proposed AOs are the more generalized ones than the existing ones and successfully handle the DMPs under both IFS and CIFS environment.

7.6.3 Characteristic comparison

Apart from the above advantages, some characteristic advantages of the proposed method over the several existing ones [27, 44, 47, 72, 83, 156, 201, 217] are presented in Table 7.3. In this table, the symbol '✓' represents that the associated characteristic is satisfied by the method while '×' means it does not. In the presented approach as well as in [72] method, the criteria weights are determined by some suitable method while in other approaches [27, 44, 47, 83, 156, 201, 217], weights are taken subjectively without impacts on the rating information. As the random weights may lead to wrong interpretation to the final ordering of the alternatives, thus computation of weight vector objectively is more important than subjectively. Hence, the presented approach is more suitable than existing approaches [27, 44, 47, 83, 156, 201, 217]. In addition to them, the DMPs presented in [27, 44, 47, 72] emphasis on geometric AOs while approaches in [83, 156, 217] are based on averaging AOs. On the other hand, the presented approach discuss both of them and the decision maker may choose them according to their choices during the aggregation process and hence discuss the optimistic/pessimistic nature towards the problem.

Finally, the approaches [27, 44, 47, 72, 83, 156, 201, 217] fail to model the complex problems whereas, in the proposed method, the ranges of MD and NMD are extended from real set to unit disc in the complex plane. This extensions of the ranges will enable the proposed approach to deal with one-dimensional problems also as described in [27, 44, 47, 72, 83, 156, 201, 217].

7.7 Conclusion

The main contribution of this chapter is summarized as follows:

1) Considering the characteristics of the CIFS, a PDM is presented in this chapter in order to compare two or more CIFNs. Some desirable properties of PDM are also investigated.

2) New operational laws for CIFNs are presented and their properties are investigated. Based on developed operational laws, a series of weighted averaging and geometric AOs namely CIFWA and CIFWG for a collection of CIFN is given. IFS is a particular case of CIFS due to which the work done in this chapter can effectively manage the data under both IFS and CIFS conditions.

3) A non-linear optimization model is established in order to determine attribute weights. This model is established with the main objective of maximizing the distance of each alternative from NIA and minimizing the distance from PIA simultaneously.

4) Based on the proposed operators and ranking method, a MCDM approach has been presented for solving DM problems in a more efficient way under the CIF environment and for demonstrating the working of the proposed method an example has been illustrated. Also, the proposed approach has been validated by comparing the results of the example with existing studies and by performing validity test-criteria. Thus, the proposed MCDM method can be efficiently used for solving complex time-periodic DM problems.

Table 7.1: Preferences given by expert in terms of CIFN

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.6,0.9),(0.1,0.1)\big)$	$\big((0.8,0.1),(0.1,0.4)\big)$	$\big((0.6,0.6),(0.2,0.2)\big)$	$\big((0.8,0.7),(0.1,0.2)\big)$
$\mathcal{V}_2$	$\big((0.7,0.6),(0.3,0.3)\big)$	$\big((0.4,0.9),(0.2,0.1)\big)$	$\big((0.7,0.7),(0.2,0.3)\big)$	$\big((0.4,0.6),(0.3,0.1)\big)$
$\mathcal{V}_3$	$\big((0.6,0.6),(0.2,0.2)\big)$	$\big((0.6,0.6),(0.3,0.1)\big)$	$\big((0.5,0.8),(0.3,0.1)\big)$	$\big((0.7,0.7),(0.1,0.2)\big)$
$\mathcal{V}_4$	$\big((0.2,0.8),(0.5,0.1)\big)$	$\big((0.7,0.3),(0.3,0.3)\big)$	$\big((0.6,0.5),(0.1,0.1)\big)$	$\big((0.6,0.5),(0.3,0.4)\big)$
$\mathcal{V}_5$	$\big((0.5,0.3),(0.4,0.6)\big)$	$\big((0.3,0.1),(0.6,0.3)\big)$	$\big((0.7,0.3),(0.1,0.5)\big)$	$\big((0.6,0.3),(0.3,0.5)\big)$

Table 7.2: Comparative study $\big(\gamma = 3$ in [83] and $\lambda = 0.5$ in [201] $\big)$

Ref.	Score values					Ranking
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	
Wang and Liu [156]	0.5806	0.3427	0.3749	0.2954	0.2601	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_5$
He et al. [72]	0.5755	0.5067	0.3540	0.3878	0.1398	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_5$
Huang [83]	0.5792	0.3391	0.3734	0.2867	0.2517	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_5$
Garg [44]	0.5772	0.5087	0.3584	0.4051	0.1642	$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_5$
Chen and Chang [27]	0.5764	0.1804	0.3606	0.1147	0.2306	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
Ye [201]	0.5678	0.3234	0.3627	0.2376	0.2050	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_5$
Zhou and Xu [217]	0.6119	0.3502	0.4214	0.3334	0.2997	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_5$
Garg [47]	0.5721	0.1669	0.3581	0.0940	0.2183	$\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_4$

Table 7.3: The characteristic comparison of different approaches

Method	Determination of weights objectively	No unknown parameter to choose for aggregation	Pessimistic approach	Optimistic approach	Ability to handle time-periodic problems	Ability to handle two-dimensional information
Wang and Liu [156]	×	✓	×	✓	×	×
He et al. [72]	✓	✓	✓	×	×	×
Huang [83]	×	×	×	✓	×	×
Garg [44]	×	✓	✓	×	×	×
Chen and Chang [27]	×	✓	✓	×	×	×
Ye [201]	×	×	✓	✓	×	×
Zhou and Xu [217]	×	✓	×	✓	×	×
Garg [47]	×	✓	✓	×	×	×
The proposed approach	✓	✓	✓	✓	✓	✓

Chapter 8

Exponential, logarithmic and compensative generalized aggregation operators[1]

This chapter presents some new exponential, logarithmic and compensative exponential of logarithmic operational laws of CIFNs based on t-norm and co-norm. Based on these laws, compensative operators namely generalized CIF compensative weighted averaging and generalized CIF compensative weighted geometric are developed. Some properties related to proposed operators are discussed. In light of the developed operators, a group decision-making method is put forward in which weights are determined objectively and is illustrated with the aid of an example. The reliability of the presented decision-making method is explored by comparing it with several prevailing studies. The influence of the parameters used in exponential and logarithmic operations on CIF numbers is also discussed.

8.1 Introduction

The detailed literature review on AOs is done in Section 1.1.3 of Chapter 1. These works depict that most of the existing AOs under IFS environment are developed using algebraic, Einstein and Hamachar laws of operation and are such that the weights in these operators

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are real numbers and the bases are IFNs. However, we may come across situations where the exponents (weights) are IFNs and the bases are real numbers. In that direction, Gou, Xu and Lei [65] put forward new EOLs under IFS theory along with their various properties and gave a method for aggregating IFS information. Further, Gou, Xu and Liao [66] presented EOLs for interval-valued IFNs and AOs using these laws. Luo et al. [111] investigated EOLs and new AOs based on them using t-norms and co-norms under IFS environment. Garg [49] presented AOs using EOLs under interval-valued pythagorean fuzzy set theory. Furthermore, as logarithm is an important mathematical operation therefore, Li and Wei [97] developed logarithmic operational laws and AOs based on them under IFS theory in which bases to the logarithms are real numbers.

All the existing approaches of DM, based on exponential and logarithmic AOs, in FS and IFS theories, deal with real values in $[0, 1]$. In CIFS theory, complex values of membership and NMDs are considered and are expressed in polar form. Inspired by the features of CIFS model and keeping in view the ultimate objective to deal with those circumstances where bases are crisp numbers and the exponents are CIF numbers, we develop generalized EOLs using t-norm and co-norm in this chapter. As logarithm and exponents of same base are inverses of each other therefore, along with EOLs, we investigate logarithmic operational laws also. Furthermore, we develop compensative exponential of logarithmic operations and some AOs based on them. In addition to these, a novel DM methodology is presented by considering the multi-dimensional complex data sets in which the criteria weights are determined objectively. The proposed DM method is validated by performing comparative studies with existing theories.

8.2 Operational laws and aggregation operators of CIFNs

In this section, we introduce exponential, logarithmic and compensative exponential of logarithmic operational laws of CIFNs using t-norm and co-norm. Depending on the proposed operations, we develop generalized CIF compensative weighted averaging/geometric (GCIFCWA/GCIFCWG) and ordered weighted averaging/geometric operators (GCIF-COWA/GCIFCOWG). Also, we investigate some of their properties.

8.2.1 Exponential operational laws of CIFNs

Definition 8.2.1. Let $\mathcal{C} = \Big((\zeta, w_\zeta), (\vartheta, w_\vartheta)\Big)$ be a CIFN and $\delta > 0$ be a real number. Then, EOL of $\mathcal{C}$, denoted by $\delta^{\mathcal{C}}$, is stated as:

$$\delta^{\mathcal{C}} = \begin{cases} \left(\left(\begin{array}{c} t^{-1}\left((1-\zeta)t(\delta)\right), \\ t^{-1}\left((1-w_\zeta)\,t(\delta)\right) \end{array}\right), \left(\begin{array}{c} s^{-1}\left(\vartheta t(\delta)\right), \\ s^{-1}\left(w_\vartheta t(\delta)\right) \end{array}\right)\right) & ;\delta \in (0,1) \\ \left(\left(\begin{array}{c} t^{-1}\left((1-\zeta)t\left(\frac{1}{\delta}\right)\right), \\ t^{-1}\left((1-w_\zeta)\,t\left(\frac{1}{\delta}\right)\right) \end{array}\right), \left(\begin{array}{c} s^{-1}\left(\vartheta t\left(\frac{1}{\delta}\right)\right), \\ s^{-1}\left(w_\vartheta t\left(\frac{1}{\delta}\right)\right) \end{array}\right)\right) & ;\delta \geq 1 \end{cases} \tag{8.1}$$

Theorem 8.2.1. For CIFN $\mathcal{C}$ and real number $\delta > 0$, $\delta^{\mathcal{C}}$ is also CIFN.

Proof. We shall prove the theorem for $\delta \in (0,1)$ as for the other case it can be proved similarly. For this, let $\mathcal{C} = \Big((\zeta, w_\zeta), (\vartheta, w_\vartheta)\Big)$. Then, by definition of CIFN we have, $0 \leq \zeta, \vartheta, \zeta + \vartheta \leq 1$ and $0 \leq w_\zeta, w_\vartheta, w_\zeta + w_\vartheta \leq 1$. Further, let $\delta^{\mathcal{C}} = \Big((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1})\Big)$ where $\zeta_1 = t^{-1}\Big((1-\zeta)t(\delta)\Big)$, $\vartheta_1 = s^{-1}\Big(\vartheta t(\delta)\Big)$, $w_{\zeta_1} = \Big(t^{-1}\Big((1-w_\zeta)\,t(\delta)\Big)\Big)$ and $w_{\vartheta_1} = \Big(s^{-1}\Big(w_\vartheta t(\delta)\Big)\Big)$. In order to prove that $\delta^{\mathcal{C}}$ is CIFN, it is sufficient to show that $0 \leq \zeta_1, \vartheta_1, \zeta_1 + \vartheta_1 \leq 1$ and $0 \leq w_{\zeta_1}, w_{\vartheta_1}, w_{\zeta_1} + w_{\vartheta_1} \leq 1$.

Since $t^{-1}, s^{-1} : [0, \infty) \to [0,1]$. Therefore, we obtain that $0 \leq \zeta_1, \vartheta_1 \leq 1$ and $0 \leq w_{\zeta_1}, w_{\vartheta_1} \leq 1$. Further, using the conditions that $0 \leq \zeta + \vartheta \leq 1$ and $t(a) = s(1-a)$, we obtain that

$$\begin{aligned} \zeta_1 + \vartheta_1 &= t^{-1}\Big((1-\zeta)t(\delta)\Big) + s^{-1}\Big(\vartheta t(\delta)\Big) \\ &\leq t^{-1}\Big((1-\zeta)t(\delta)\Big) + s^{-1}\Big((1-\zeta)t(\delta)\Big) \\ &= 1 - s^{-1}\Big((1-\zeta)t(\delta)\Big) + s^{-1}\Big((1-\zeta)t(\delta)\Big) \\ &= 1 \end{aligned}$$

Thus, $\zeta_1 + \vartheta_1 \leq 1$. Also, $\zeta_1 + \vartheta_1 \geq 0$ as $\zeta_1, \vartheta_1 \geq 0$. Hence, $0 \leq \zeta_1 + \vartheta_1 \leq 1$. Similarly, we can obtain that $0 \leq w_{\zeta_1} + w_{\vartheta_1} \leq 1$. Hence, $\delta^{\mathcal{C}}$ is a CIFN. □

Theorem 8.2.2. For CIFNs $\mathcal{C}_1$, $\mathcal{C}_2$ and real number $\delta \in (0,1)$, we have

(i) $\delta^{\mathcal{C}_1} \oplus \delta^{\mathcal{C}_2} = \delta^{\mathcal{C}_2} \oplus \delta^{\mathcal{C}_1}$;

(ii) $\delta^{\mathcal{C}_1} \otimes \delta^{\mathcal{C}_2} = \delta^{\mathcal{C}_2} \otimes \delta^{\mathcal{C}_1}$;

(iii) $\left(\delta^{\mathcal{C}_1} \oplus \delta^{\mathcal{C}_2}\right) \oplus \delta^{\mathcal{C}_3} = \delta^{\mathcal{C}_1} \oplus \left(\delta^{\mathcal{C}_2} \oplus \delta^{\mathcal{C}_3}\right)$;

(iv) $\left(\delta^{\mathcal{C}_1} \otimes \delta^{\mathcal{C}_2}\right) \otimes \delta^{\mathcal{C}_3} = \delta^{\mathcal{C}_1} \otimes \left(\delta^{\mathcal{C}_2} \otimes \delta^{\mathcal{C}_3}\right)$.

Proof. Here, just the part (i) is demonstrated while the remaining can be gotten likewise. For this, let $\mathcal{C}_j = \Big(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j,\, w_{\vartheta_j}\right)\Big)$ $(j = 1, 2)$. On utilizing Definition 8.2.1, we obtain that

$$\delta^{\mathcal{C}_1} = \Big(\Big(t^{-1}\left((1-\zeta_1)t(\delta)\right),\quad t^{-1}\left(\left(1-w_{\zeta_1}\right)t(\delta)\right)\Big), \Big(s^{-1}\left(\vartheta_1 t(\delta)\right),\quad s^{-1}\left(w_{\vartheta_1} t(\delta)\right)\Big)\Big)$$

and $$\delta^{\mathcal{C}_2} = \Big(\Big(t^{-1}\left((1-\zeta_2)t(\delta)\right),\quad t^{-1}\left(\left(1-w_{\zeta_2}\right)t(\delta)\right)\Big), \Big(s^{-1}\left(\vartheta_2 t(\delta)\right),\quad s^{-1}\left(w_{\vartheta_2} t(\delta)\right)\Big)\Big)$$

Now, using the Definition 6.2.2 of Chapter 6, we have

$$\begin{aligned}
&\delta^{\mathcal{C}_1} \oplus \delta^{\mathcal{C}_2}\\
=&\left(\begin{array}{r}\left(\begin{array}{l} s^{-1}\Big(s\Big(t^{-1}\big((1-\zeta_1)t(\delta)\big)\Big) + s\Big(t^{-1}\big((1-\zeta_2)t(\delta)\big)\Big)\Big),\\ s^{-1}\Big(s\Big(t^{-1}\Big(\left(1-w_{\zeta_1}\right)t(\delta)\Big)\Big) + s\Big(t^{-1}\Big(\left(1-w_{\zeta_2}\right)t(\delta)\Big)\Big)\Big)\end{array}\right),\\ \left(\begin{array}{l} t^{-1}\Big(t\Big(s^{-1}\big(\vartheta_1 t(\delta)\big)\Big) + t\Big(s^{-1}\big(\vartheta_2 t(\delta)\big)\Big)\Big),\\ t^{-1}\Big(t\Big(s^{-1}\Big(w_{\vartheta_1} t(\delta)\Big)\Big) + t\Big(s^{-1}\Big(w_{\vartheta_2} t(\delta)\Big)\Big)\Big)\end{array}\right)\end{array}\right)\\
=&\left(\begin{array}{r}\left(\begin{array}{l} s^{-1}\Big(s\Big(t^{-1}\big((1-\zeta_2)t(\delta)\big)\Big) + s\Big(t^{-1}\big((1-\zeta_1)t(\delta)\big)\Big)\Big),\\ s^{-1}\Big(s\Big(t^{-1}\Big(\left(1-w_{\zeta_2}\right)t(\delta)\Big)\Big) + s\Big(t^{-1}\Big(\left(1-w_{\zeta_1}\right)t(\delta)\Big)\Big)\Big)\end{array}\right),\\ \left(\begin{array}{l} t^{-1}\Big(t\Big(s^{-1}\big(\vartheta_2 t(\delta)\big)\Big) + t\Big(s^{-1}\big(\vartheta_1 t(\delta)\big)\Big)\Big),\\ t^{-1}\Big(t\Big(s^{-1}\Big(w_{\vartheta_2} t(\delta)\Big)\Big) + t\Big(s^{-1}\Big(w_{\vartheta_1} t(\delta)\Big)\Big)\Big)\end{array}\right)\end{array}\right)\\
=&\ \delta^{\mathcal{C}_2} \oplus \delta^{\mathcal{C}_1}
\end{aligned}$$

□

Theorem 8.2.3. For CIFNs $\mathcal{C}_j$, real numbers ρ_j $(j = 1, 2)$ and $\delta \in (0, 1)$, we have

(i) $\rho_1\left(\delta^{\mathcal{C}_1} \oplus \delta^{\mathcal{C}_2}\right) = \rho_1\delta^{\mathcal{C}_1} \oplus \rho_1\delta^{\mathcal{C}_2}$;

(ii) $\left(\delta^{\mathcal{C}_1} \otimes \delta^{\mathcal{C}_2}\right)^{\rho_1} = \left(\delta^{\mathcal{C}_1}\right)^{\rho_1} \otimes \left(\delta^{\mathcal{C}_2}\right)^{\rho_1}$;

(iii) $\rho_1 \delta^{\mathcal{C}_1} \oplus \rho_2 \delta^{\mathcal{C}_1} = (\rho_1 + \rho_2)\, \delta^{\mathcal{C}_1}$;

(iv) $\left(\delta^{\mathcal{C}_1}\right)^{\rho_1} \otimes \left(\delta^{\mathcal{C}_1}\right)^{\rho_2} = \left(\delta^{\mathcal{C}_1}\right)^{\rho_1+\rho_2}$.

Proof. Here, just the parts (i) and (iii) are demonstrated while the remaining can be gotten likewise. For this, let $\mathcal{C}_j = \Big(\big(\zeta_j, w_{\zeta_j}\big), \big(\vartheta_j,\, w_{\vartheta_j}\big)\Big)$ $(j = 1, 2)$.

(i) By using the Definitions 6.2.2 and 8.2.1, we have

$$\begin{aligned}
& \rho_1 \left(\delta^{\mathcal{C}_1} \oplus \delta^{\mathcal{C}_2}\right) \\
= \;& \rho_1 \left(\begin{pmatrix} s^{-1}\Big(s\Big(t^{-1}((1-\zeta_1)t(\delta))\Big) + s\Big(t^{-1}((1-\zeta_2)t(\delta))\Big)\Big), \\ s^{-1}\Big(s\Big(t^{-1}\Big((1-w_{\zeta_1})\,t(\delta)\Big)\Big) + s\Big(t^{-1}\Big((1-w_{\zeta_2})\,t(\delta)\Big)\Big)\Big) \end{pmatrix}, \right. \\
& \left. \begin{pmatrix} t^{-1}\Big(t\Big(s^{-1}(\vartheta_1 t(\delta))\Big) + t\Big(s^{-1}(\vartheta_2 t(\delta))\Big)\Big), \\ t^{-1}\Big(t\Big(s^{-1}\Big(w_{\vartheta_1} t(\delta)\Big)\Big) + t\Big(s^{-1}\Big(w_{\vartheta_2} t(\delta)\Big)\Big)\Big) \end{pmatrix} \right) \\
= \;& \left(\begin{pmatrix} s^{-1}\Big(\rho_1\Big(s\Big(t^{-1}((1-\zeta_1)t(\delta))\Big) + s\Big(t^{-1}((1-\zeta_2)t(\delta))\Big)\Big)\Big), \\ s^{-1}\Big(\rho_1\Big(s\Big(t^{-1}\Big((1-w_{\zeta_1})\,t(\delta)\Big)\Big) + s\Big(t^{-1}\Big((1-w_{\zeta_2})\,t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix}, \right. \\
& \left. \begin{pmatrix} t^{-1}\Big(\rho_1\Big(t\Big(s^{-1}(\vartheta_1 t(\delta))\Big) + t\Big(s^{-1}(\vartheta_2 t(\delta))\Big)\Big)\Big), \\ t^{-1}\Big(\rho_1\Big(t\Big(s^{-1}\Big(w_{\vartheta_1} t(\delta)\Big)\Big) + t\Big(s^{-1}\Big(w_{\vartheta_2} t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix} \right) \\
= \;& \left(\begin{pmatrix} s^{-1}\Big(\rho_1\Big(s\Big(t^{-1}((1-\zeta_1)t(\delta))\Big)\Big)\Big), \\ s^{-1}\Big(\rho_1\Big(s\Big(t^{-1}\Big((1-w_{\zeta_1})\,t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix}, \begin{pmatrix} t^{-1}\Big(\rho_1\Big(t\Big(s^{-1}(\vartheta_1 t(\delta))\Big)\Big)\Big), \\ t^{-1}\Big(\rho_1\Big(t\Big(s^{-1}\Big(w_{\vartheta_1} t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix} \right) \\
\oplus \;& \left(\begin{pmatrix} s^{-1}\Big(\rho_1\Big(s\Big(t^{-1}((1-\zeta_2)t(\delta))\Big)\Big)\Big), \\ s^{-1}\Big(\rho_1\Big(s\Big(t^{-1}\Big((1-w_{\zeta_2})\,t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix}, \begin{pmatrix} t^{-1}\Big(\rho_1\Big(t\Big(s^{-1}(\vartheta_2 t(\delta))\Big)\Big)\Big), \\ t^{-1}\Big(\rho_1\Big(t\Big(s^{-1}\Big(w_{\vartheta_2} t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix} \right) \\
= \;& \rho_1 \Big(\Big(t^{-1}\left((1-\zeta_1)t(\delta)\right), t^{-1}\left((1-w_{\zeta_1})\,t(\delta)\right)\Big), \Big(s^{-1}\left(\vartheta_1 t(\delta)\right), s^{-1}\left(w_{\vartheta_1} t(\delta)\right)\Big)\Big) \\
\oplus \;& \rho_1 \Big(\Big(t^{-1}\left((1-\zeta_2)t(\delta)\right), t^{-1}\left((1-w_{\zeta_2})\,t(\delta)\right)\Big), \Big(s^{-1}\left(\vartheta_2 t(\delta)\right), s^{-1}\left(w_{\vartheta_2} t(\delta)\right)\Big)\Big) \\
= \;& \rho_1 \delta^{\mathcal{C}_1} \oplus \rho_1 \delta^{\mathcal{C}_2}
\end{aligned}$$

(iii) Again by using the Definitions 6.2.2 and 8.2.1, we have

$$
\begin{aligned}
& \rho_1 \delta^{\mathcal{C}_1} \oplus \rho_2 \delta^{\mathcal{C}_1} \\
= \;& \rho_1 \left(\left(t^{-1}\left((1-\zeta_1)t(\delta)\right), t^{-1}\left(\left(1-w_{\zeta_1}\right)t(\delta)\right)\right), \left(s^{-1}\left(\vartheta_1 t(\delta)\right), s^{-1}\left(w_{\vartheta_1} t(\delta)\right)\right)\right) \\
\oplus \;& \rho_2 \left(\left(t^{-1}\left((1-\zeta_1)t(\delta)\right), t^{-1}\left(\left(1-w_{\zeta_1}\right)t(\delta)\right)\right), \left(s^{-1}\left(\vartheta_1 t(\delta)\right), s^{-1}\left(w_{\vartheta_1} t(\delta)\right)\right)\right) \\
= \;& \left(\begin{pmatrix} s^{-1}\Big(\rho_1\Big(s\Big(t^{-1}\big((1-\zeta_1)t(\delta)\big)\Big)\Big)\Big), \\ s^{-1}\Big(\rho_1\Big(s\Big(t^{-1}\Big(\left(1-w_{\zeta_1}\right)t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix}, \begin{pmatrix} t^{-1}\Big(\rho_1\Big(t\Big(s^{-1}\big(\vartheta_1 t(\delta)\big)\Big)\Big)\Big), \\ t^{-1}\Big(\rho_1\Big(t\Big(s^{-1}\Big(w_{\vartheta_1} t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix} \right) \\
\oplus \;& \left(\begin{pmatrix} s^{-1}\Big(\rho_2\Big(s\Big(t^{-1}\big((1-\zeta_1)t(\delta)\big)\Big)\Big)\Big), \\ s^{-1}\Big(\rho_2\Big(s\Big(t^{-1}\Big(\left(1-w_{\zeta_1}\right)t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix}, \begin{pmatrix} t^{-1}\Big(\rho_2\Big(t\Big(s^{-1}\big(\vartheta_1 t(\delta)\big)\Big)\Big)\Big), \\ t^{-1}\Big(\rho_2\Big(t\Big(s^{-1}\Big(w_{\vartheta_1} t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix} \right) \\
= \;& \left(\begin{pmatrix} s^{-1}\Big((\rho_1+\rho_2)\Big(s\Big(t^{-1}\big((1-\zeta_1)t(\delta)\big)\Big)\Big)\Big), \\ s^{-1}\Big((\rho_1+\rho_2)\Big(s\Big(t^{-1}\Big(\left(1-w_{\zeta_1}\right)t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix}, \begin{pmatrix} t^{-1}\Big((\rho_1+\rho_2)\Big(t\Big(s^{-1}\big(\vartheta_1 t(\delta)\big)\Big)\Big)\Big), \\ t^{-1}\Big((\rho_1+\rho_2)\Big(t\Big(s^{-1}\Big(w_{\vartheta_1} t(\delta)\Big)\Big)\Big)\Big) \end{pmatrix} \right) \\
= \;& (\rho_1+\rho_2)\, \delta^{\mathcal{C}_1}
\end{aligned}
$$

□

Theorem 8.2.4. Let $\mathcal{C}$ be a CIFN and δ_1, δ_2 be two positive real numbers such that $\delta_1 \geq \delta_2$. Then,

(i) $\delta_2^{\mathcal{C}} \subseteq \delta_1^{\mathcal{C}}$ if $\delta_1, \delta_2 \in (0,1)$;

(ii) $\delta_1^{\mathcal{C}} \subseteq \delta_2^{\mathcal{C}}$ if $\delta_1, \delta_2 \geq 1$.

Proof. Here, just the part (i) is demonstrated while the remaining can be gotten likewise.

(i) Let $\mathcal{C} = \Big((\zeta, w_\zeta), (\vartheta, w_\vartheta)\Big)$. Then, on utilizing Definition 8.2.1, we obtain that

$$
\begin{aligned}
\delta_1^{\mathcal{C}} &= \left(\left(t^{-1}\left((1-\zeta)t(\delta_1)\right), t^{-1}\left(\left(1-w_\zeta\right)t(\delta_1)\right)\right), \left(s^{-1}\left(\vartheta t(\delta_1)\right), s^{-1}\left(w_\vartheta t(\delta_1)\right)\right)\right) \\
\text{and} \quad \delta_2^{\mathcal{C}} &= \left(\left(t^{-1}\left((1-\zeta)t(\delta_2)\right), t^{-1}\left(\left(1-w_\zeta\right)t(\delta_2)\right)\right), \left(s^{-1}\left(\vartheta t(\delta_2)\right), s^{-1}\left(w_\vartheta t(\delta_2)\right)\right)\right)
\end{aligned}
$$

Now, as $\delta_1 \geq \delta_2$ and 's' and 't' are increasing and decreasing functions respectively. Therefore, we obtain that

$$t^{-1}\left((1-\zeta)t(\delta_2)\right) \leq t^{-1}\left((1-\zeta)t(\delta_1)\right) \quad ; \quad s^{-1}\left(\vartheta t(\delta_2)\right) \geq s^{-1}\left(\vartheta t(\delta_1)\right)$$
$$t^{-1}\left((1-w_\zeta)\, t(\delta_2)\right) \leq t^{-1}\left((1-w_\zeta)\, t(\delta_1)\right) \quad ; \quad s^{-1}\left(w_\vartheta t(\delta_2)\right) \geq s^{-1}\left(w_\vartheta t(\delta_1)\right)$$

Hence, by using the Definition 2.1.10, we obtain that $\delta_2^{\mathcal{C}} \subseteq \delta_1^{\mathcal{C}}$.

□

8.2.2 Logarithmic operational laws of CIFNs

Definition 8.2.2. For CIFN $\mathcal{C} = \Big((\zeta, w_\zeta), (\vartheta, w_\vartheta)\Big)$ and positive real number $\delta \neq 1$, we define logarithmic operation of $\mathcal{C}$, denoted by $\log_\delta(\mathcal{C})$, as:

$$\log_\delta(\mathcal{C}) = \begin{cases} \left(\left(1-\frac{t(\zeta)}{t(\delta)},\ 1-\frac{t(w_\zeta)}{t(\delta)}\right), \left(\frac{s(\vartheta)}{t(\delta)},\ \frac{s(w_\vartheta)}{t(\delta)}\right)\right) & ; 0<\delta\leq \min\{\zeta, w_\zeta\}\leq 1 \\ \left(\left(1-\frac{t(\zeta)}{t\left(\frac{1}{\delta}\right)},\ 1-\frac{t(w_\zeta)}{t\left(\frac{1}{\delta}\right)}\right), \left(\frac{s(\vartheta)}{t\left(\frac{1}{\delta}\right)},\ \frac{s(w_\vartheta)}{t\left(\frac{1}{\delta}\right)}\right)\right) & ; 0<\frac{1}{\delta}\leq \min\{\zeta, w_\zeta\}\leq 1 \end{cases} \tag{8.2}$$

Theorem 8.2.5. Let $\mathcal{C}$ be a CIFN and $\delta > 0$ be a real number such that $\delta \neq 1$. Then, $\log_\delta(\mathcal{C})$ is also CIFN.

Proof. Let $\mathcal{C} = \Big((\zeta, w_\zeta), (\vartheta, w_\vartheta)\Big)$ and $\log_\delta(\mathcal{C}) = \Big((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1})\Big)$. Here, the proof is demonstrated for $0 < \delta \leq \min\{\zeta, w_\zeta\} \leq 1$ while for the other case it can be obtained likewise. Now, on utilizing Definition 8.2.2, we obtain that $\zeta_1 = 1 - \frac{t(\zeta)}{t(\delta)}$; $\vartheta_1 = \frac{s(\vartheta)}{t(\delta)}$; $w_{\zeta_1} = \left(1 - \frac{t(w_\zeta)}{t(\delta)}\right)$ and $w_{\vartheta_1} = \left(\frac{s(w_\vartheta)}{t(\delta)}\right)$. In order to prove that $\log_\delta(\mathcal{C})$ is CIFN, it is sufficient to show that $0 \leq \zeta_1, \vartheta_1, \zeta_1 + \vartheta_1 \leq 1$ and $0 \leq w_{\zeta_1}, w_{\vartheta_1}, w_{\zeta_1} + w_{\vartheta_1} \leq 1$. Since, $\delta \leq \min\{\zeta, w_\zeta\}$, $\delta \neq 1$ and t is decreasing function. Therefore, $0 \leq \frac{t(\zeta)}{t(\delta)} \leq 1$ which gives that $0 \leq 1-\zeta_1 \leq 1 \Rightarrow 0 \leq \zeta_1 \leq 1$. Further, as $\delta \leq \zeta \leq 1-\vartheta \Rightarrow t(\delta) \geq t(1-\vartheta) \Rightarrow \frac{s(\vartheta)}{t(\delta)} \leq 1$. Also, since the co-domain of t and s is $[0, \infty)$ therefore, $\frac{s(\vartheta)}{t(\delta)} \geq 0$. Hence, $0 \leq \vartheta_1 \leq 1$. Finally, by using $\zeta + \vartheta \leq 1$, $s(a) = t(1-a)$ and t is decreasing function, we have

$$\zeta_1 + \vartheta_1 \quad = \quad 1 - \frac{t(\zeta)}{t(\delta)} + \frac{s(\vartheta)}{t(\delta)} = 1 - \frac{t(\zeta)}{t(\delta)} + \frac{t(1-\vartheta)}{t(\delta)} \leq 1 - \frac{t(\zeta)}{t(\delta)} + \frac{t(\zeta)}{t(\delta)} = 1$$

Hence, $0 \leq \zeta_1 + \vartheta_1 \leq 1$. Similarly, we can prove that $0 \leq w_{\zeta_1}, w_{\vartheta_1}, w_{\zeta_1} + w_{\vartheta_1} \leq 1$. Hence, $\log_\delta(\mathcal{C})$ is also CIFN. □

Theorem 8.2.6. Let $\mathcal{C} = \Big((\zeta, w_\zeta), (\vartheta, w_\vartheta)\Big)$ be a CIFN and $0 < \delta \leq \min\{\zeta, w_\zeta\} \leq 1$ be such that $\delta \neq 1$. Then,

(i) $\delta^{\log_\delta(\mathcal{C})} = \mathcal{C}$;

(ii) $\log_\delta(\delta)^{\mathcal{C}} = \mathcal{C}$.

Proof. Since, $0 < \delta \leq \min\{\zeta, w_\zeta\} \leq 1$ and $\delta \neq 1$.

(i) By using the Definitions 8.2.1 and 8.2.2, we have

$$\begin{aligned}
\log_\delta(\mathcal{C}) &= \left(\left(1 - \frac{t(\zeta)}{t(\delta)}, 1 - \frac{t\left(w_\zeta\right)}{t(\delta)}\right), \left(\frac{s(\vartheta)}{t(\delta)}, \frac{s\left(w_\vartheta\right)}{t(\delta)}\right)\right) \\
\delta^{\log_\delta(\mathcal{C})} &= \left(\left(\begin{array}{l} t^{-1}\left(\dfrac{t(\zeta)t(\delta)}{t(\delta)}\right), \\ t^{-1}\left(\dfrac{t\left(w_\zeta\right)t(\delta)}{t(\delta)}\right) \end{array}\right), \left(\begin{array}{l} s^{-1}\left(\dfrac{s(\vartheta)t(\delta)}{t(\delta)}\right), \\ s^{-1}\left(\dfrac{s\left(w_\vartheta\right)t(\delta)}{t(\delta)}\right) \end{array}\right)\right) \\
&= \Big(\left(\zeta, w_\zeta\right), \left(\vartheta, w_\vartheta\right)\Big) = \mathcal{C}
\end{aligned}$$

(ii) Again, by using the Definitions 8.2.1 and 8.2.2, we have

$$\begin{aligned}
\delta^{\mathcal{C}} &= \Big(\Big(t^{-1}\left((1-\zeta)t(\delta)\right), t^{-1}\left(\left(1-w_\zeta\right)t(\delta)\right)\Big), \Big(s^{-1}\left(\vartheta t(\delta)\right), s^{-1}\left(w_\vartheta t(\delta)\right)\Big)\Big) \\
\log_\delta(\delta)^{\mathcal{C}} &= \left(\left(1 - \frac{(1-\zeta)t(\delta)}{t(\delta)}, 1 - \frac{\left(1-w_\zeta\right)t(\delta)}{t(\delta)}\right), \left(\frac{kt(\delta)}{t(\delta)}, \frac{\left(w_\vartheta\right)t(\delta)}{t(\delta)}\right)\right) \\
&= \Big(\left(\zeta, w_\zeta\right), \left(\vartheta, w_\vartheta\right)\Big) = \mathcal{C}
\end{aligned}$$

□

Theorem 8.2.7. Let $\mathcal{C}_j = \Big(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\Big)$ $(j = 1, 2)$ be two CIFNs and $0 < \delta \leq \min\{\zeta_1, \zeta_2, w_{\zeta_1}, w_{\zeta_2}\} \leq 1$ be such that $\delta \neq 1$. Then,

(i) $\log_\delta(\mathcal{C}_1) \oplus \log_\delta(\mathcal{C}_2) = \log_\delta(\mathcal{C}_2) \oplus \log_\delta(\mathcal{C}_1)$;

(ii) $\log_\delta(\mathcal{C}_1) \otimes \log_\delta(\mathcal{C}_2) = \log_\delta(\mathcal{C}_2) \otimes \log_\delta(\mathcal{C}_1)$.

Theorem 8.2.8. Let $\mathcal{C}_j = \Big(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\Big)$ $(j = 1, 2, 3)$ be three CIFNs and $0 < \delta \leq \min\Big\{\zeta_1, \zeta_2, \zeta_3, w_{\zeta_1}, w_{\zeta_2}, w_{\zeta_3}\Big\} \leq 1$ be such that $\delta \neq 1$. Then,

(i) $\big(\log_\delta(\mathcal{C}_1) \oplus \log_\delta(\mathcal{C}_2)\big) \oplus \log_\delta(\mathcal{C}_3) = \log_\delta(\mathcal{C}_1) \oplus \big(\log_\delta(\mathcal{C}_2) \oplus \log_\delta(\mathcal{C}_3)\big)$;

(ii) $\big(\log_\delta(\mathcal{C}_1) \otimes \log_\delta(\mathcal{C}_2)\big) \otimes \log_\delta(\mathcal{C}_3) = \log_\delta(\mathcal{C}_1) \otimes \big(\log_\delta(\mathcal{C}_2) \otimes \log_\delta(\mathcal{C}_3)\big)$.

Proof. Just part (i) is demonstrated here as the remaining can be gotten likewise.

(i) By using the Definitions 8.2.2 and 6.2.2, we have

$$
\begin{aligned}
&\big(\log_\delta(\mathcal{C}_1) \oplus \log_\delta(\mathcal{C}_2)\big) \oplus \log_\delta(\mathcal{C}_3) \\
= &\left(\begin{pmatrix} s^{-1}\left(s\left(1-\dfrac{t(\zeta_1)}{t(\delta)}\right)+s\left(1-\dfrac{t(\zeta_2)}{t(\delta)}\right)\right), \\ s^{-1}\left(s\left(1-\dfrac{t\left(w_{\zeta_1}\right)}{t(\delta)}\right)+s\left(1-\dfrac{t\left(w_{\zeta_2}\right)}{t(\delta)}\right)\right) \end{pmatrix}, \atop \begin{pmatrix} t^{-1}\left(t\left(\dfrac{s(\vartheta_1)}{t(\delta)}\right)+t\left(\dfrac{s(\vartheta_2)}{t(\delta)}\right)\right), \\ t^{-1}\left(t\left(\dfrac{s\left(w_{\vartheta_1}\right)}{t(\delta)}\right)+t\left(\dfrac{s\left(w_{\vartheta_2}\right)}{t(\delta)}\right)\right) \end{pmatrix}\right) \\
\oplus &\left(\left(1-\frac{t(\zeta_3)}{t(\delta)},\ 1-\frac{t\left(w_{\zeta_3}\right)}{t(\delta)}\right),\left(\frac{s(\vartheta_3)}{t(\delta)},\ \frac{s\left(w_{\vartheta_3}\right)}{t(\delta)}\right)\right) \\
= &\left(\begin{pmatrix} s^{-1}\left(\left(s\left(1-\frac{t(\zeta_1)}{t(\delta)}\right)+s\left(1-\frac{t(\zeta_2)}{t(\delta)}\right)\right)+s\left(1-\frac{t(\zeta_3)}{t(\delta)}\right)\right), \\ s^{-1}\left(\left(s\left(1-\frac{t\left(w_{\zeta_1}\right)}{t(\delta)}\right)+s\left(1-\frac{t\left(w_{\zeta_2}\right)}{t(\delta)}\right)\right)+s\left(1-\frac{t\left(w_{\zeta_3}\right)}{t(\delta)}\right)\right) \end{pmatrix}, \atop \begin{pmatrix} t^{-1}\left(\left(t\left(\frac{s(\vartheta_1)}{t(\delta)}\right)+t\left(\frac{s(\vartheta_2)}{t(\delta)}\right)\right)+t\left(\frac{s(\vartheta_3)}{t(\delta)}\right)\right), \\ t^{-1}\left(\left(t\left(\frac{s\left(w_{\vartheta_1}\right)}{t(\delta)}\right)+t\left(\frac{s\left(w_{\vartheta_2}\right)}{t(\delta)}\right)\right)+t\left(\frac{s\left(w_{\vartheta_3}\right)}{t(\delta)}\right)\right) \end{pmatrix}\right) \\
= &\left(\begin{pmatrix} s^{-1}\left(s\left(1-\frac{t(\zeta_1)}{t(\delta)}\right)+\left(s\left(1-\frac{t(\zeta_2)}{t(\delta)}\right)+s\left(1-\frac{t(\zeta_3)}{t(\delta)}\right)\right)\right), \\ s^{-1}\left(s\left(1-\frac{t\left(w_{\zeta_1}\right)}{t(\delta)}\right)+\left(s\left(1-\frac{t\left(w_{\zeta_2}\right)}{t(\delta)}\right)+s\left(1-\frac{t\left(w_{\zeta_3}\right)}{t(\delta)}\right)\right)\right) \end{pmatrix}, \atop \begin{pmatrix} t^{-1}\left(t\left(\frac{s(\vartheta_1)}{t(\delta)}\right)+\left(t\left(\frac{s(\vartheta_2)}{t(\delta)}\right)+t\left(\frac{s(\vartheta_3)}{t(\delta)}\right)\right)\right), \\ t^{-1}\left(t\left(\frac{s\left(w_{\vartheta_1}\right)}{t(\delta)}\right)+\left(t\left(\frac{s\left(w_{\vartheta_2}\right)}{t(\delta)}\right)+t\left(\frac{s\left(w_{\vartheta_3}\right)}{t(\delta)}\right)\right)\right) \end{pmatrix}\right) \\
= &\left(\begin{pmatrix} 1-\dfrac{t(\zeta_1)}{t(\delta)}, \\ 1-\dfrac{t\left(w_{\zeta_1}\right)}{t(\delta)} \end{pmatrix}, \begin{pmatrix} \dfrac{s(\vartheta_1)}{t(\delta)}, \\ \dfrac{s\left(w_{\vartheta_1}\right)}{t(\delta)} \end{pmatrix}\right) \\
\oplus &\left(\begin{pmatrix} s^{-1}\left(s\left(1-\frac{t(\zeta_2)}{t(\delta)}\right)+s\left(1-\frac{t(\zeta_3)}{t(\delta)}\right)\right), \\ s^{-1}\left(s\left(1-\frac{t\left(w_{\zeta_2}\right)}{t(\delta)}\right)+s\left(1-\frac{t\left(w_{\zeta_3}\right)}{t(\delta)}\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(t\left(\frac{s(\vartheta_2)}{t(\delta)}\right)+t\left(\frac{s(\vartheta_3)}{t(\delta)}\right)\right), \\ t^{-1}\left(t\left(\frac{s\left(w_{\vartheta_2}\right)}{t(\delta)}\right)+t\left(\frac{s\left(w_{\vartheta_3}\right)}{t(\delta)}\right)\right) \end{pmatrix}\right) \\
= &\log_\delta(\mathcal{C}_1) \oplus \big(\log_\delta(\mathcal{C}_2) \oplus \log_\delta(\mathcal{C}_3)\big).
\end{aligned}
$$

$\square$

Theorem 8.2.9. Let $\mathcal{C}_j = \Big(\big(\zeta_j, w_{\zeta_j}\big), \big(\vartheta_j, w_{\vartheta_j}\big)\Big)$ $(j = 1, 2)$ be two CIFNs and δ, ρ be positive real numbers such that $\delta \leq \min\{\zeta_1, \zeta_2, w_{\zeta_1}, w_{\zeta_2}\} \leq 1$; $\delta \neq 1$. Then,

(i) $\rho\big(\log_\delta(\mathcal{C}_1) \oplus \log_\delta(\mathcal{C}_2)\big) = \rho \log_\delta(\mathcal{C}_1) \oplus \rho \log_\delta(\mathcal{C}_2)$;

(ii) $\big(\log_\delta(\mathcal{C}_1) \otimes \log_\delta(\mathcal{C}_2)\big)^\rho = \big(\log_\delta(\mathcal{C}_1)\big)^\rho \otimes \big(\log_\delta(\mathcal{C}_2)\big)^\rho$.

Proof. Just part (i) is demonstrated here as the remaining can be gotten likewise.

(i) By using the Definitions 6.2.2 and 8.2.2, we have

$$
\begin{aligned}
& \rho\big(\log_\delta(\mathcal{C}_1) \oplus \log_\delta(\mathcal{C}_2)\big) \\
= \; & \rho \left(\begin{array}{c} \left(\begin{array}{l} s^{-1}\left(s\left(1 - \dfrac{t(\zeta_1)}{t(\delta)}\right) + s\left(1 - \dfrac{t(\zeta_2)}{t(\delta)}\right)\right), \\ s^{-1}\left(s\left(1 - \dfrac{t\left(w_{\zeta_1}\right)}{t(\delta)}\right) + s\left(1 - \dfrac{t\left(w_{\zeta_2}\right)}{t(\delta)}\right)\right) \end{array} \right), \\ \left(\begin{array}{l} t^{-1}\left(t\left(\dfrac{s(\vartheta_1)}{t(\delta)}\right) + t\left(\dfrac{s(\vartheta_2)}{t(\delta)}\right)\right), \\ t^{-1}\left(t\left(\dfrac{s\left(w_{\vartheta_1}\right)}{t(\delta)}\right) + t\left(\dfrac{s\left(w_{\vartheta_2}\right)}{t(\delta)}\right)\right) \end{array} \right) \end{array} \right) \\
= \; & \left(\begin{array}{c} \left(\begin{array}{l} s^{-1}\left(\rho\left(s\left(1 - \frac{t(\zeta_1)}{t(\delta)}\right)\right) + \rho\left(s\left(1 - \frac{t(\zeta_2)}{t(\delta)}\right)\right)\right), \\ s^{-1}\left(\rho\left(s\left(1 - \frac{t\left(w_{\zeta_1}\right)}{t(\delta)}\right)\right) + \rho\left(s\left(1 - \frac{t\left(w_{\zeta_2}\right)}{t(\delta)}\right)\right)\right) \end{array} \right), \\ \left(\begin{array}{l} t^{-1}\left(\rho\left(t\left(\frac{s(\vartheta_1)}{t(\delta)}\right)\right) + \rho\left(t\left(\frac{s(\vartheta_2)}{t(\delta)}\right)\right)\right), \\ t^{-1}\left(\rho\left(t\left(\frac{s\left(w_{\vartheta_1}\right)}{t(\delta)}\right)\right) + \rho\left(t\left(\frac{s\left(w_{\vartheta_2}\right)}{t(\delta)}\right)\right)\right) \end{array} \right) \end{array} \right) \\
= \; & \left(\left(\begin{array}{l} s^{-1}\left(\rho\left(s\left(1 - \frac{t(\zeta_1)}{t(\delta)}\right)\right)\right), \\ s^{-1}\left(\rho\left(s\left(1 - \frac{t\left(w_{\zeta_1}\right)}{t(\delta)}\right)\right)\right) \end{array} \right), \left(\begin{array}{l} t^{-1}\left(\rho\left(t\left(\frac{s(\vartheta_1)}{t(\delta)}\right)\right)\right), \\ t^{-1}\left(\rho\left(t\left(\frac{s\left(w_{\vartheta_1}\right)}{t(\delta)}\right)\right)\right) \end{array} \right) \right) \\
\oplus \; & \left(\left(\begin{array}{l} s^{-1}\left(\rho\left(s\left(1 - \dfrac{t(\zeta_2)}{t(\delta)}\right)\right)\right), \\ s^{-1}\left(\rho\left(s\left(1 - \dfrac{t\left(w_{\zeta_2}\right)}{t(\delta)}\right)\right)\right) \end{array} \right), \left(\begin{array}{l} t^{-1}\left(\rho\left(t\left(\dfrac{s(\vartheta_2)}{t(\delta)}\right)\right)\right), \\ t^{-1}\left(\rho\left(t\left(\dfrac{s\left(w_{\vartheta_2}\right)}{t(\delta)}\right)\right)\right) \end{array} \right) \right) \\
= \; & \rho \log_\delta(\mathcal{C}_1) \oplus \rho \log_\delta(\mathcal{C}_2)
\end{aligned}
$$

□

Theorem 8.2.10. Let $\mathcal{C} = \Big(\big(\zeta, w_\zeta\big), \big(\vartheta, w_\vartheta\big)\Big)$ be a CIFN and δ, ρ_1, ρ_2 be three positive real numbers such that $\delta \le \min\{\zeta, w_\zeta\} \le 1$; $\delta \ne 1$. Then,

(i) $\rho_1 \log_\delta(\mathcal{C}) \oplus \rho_2 \log_\delta(\mathcal{C}) = \big(\rho_1 + \rho_2\big) \log_\delta(\mathcal{C})$;

(ii) $\left(\log_\delta(\mathcal{C})\right)^{\rho_1} \otimes \left(\log_\delta(\mathcal{C})\right)^{\rho_2} = \left(\log_\delta(\mathcal{C})\right)^{\rho_1+\rho_2}$.

Proof. Just part (i) is demonstrated here as the remaining can be gotten likewise.

(i) By using the Definitions 6.2.2 and 8.2.2, we have

$$\begin{aligned}
&\rho_1 \log_\delta(\mathcal{C}) \oplus \rho_2 \log_\delta(\mathcal{C}) \\
=& \left(\begin{pmatrix} s^{-1}\left(\rho_1\left(s\left(1-\frac{t(\zeta)}{t(\delta)}\right)\right)\right), \\ s^{-1}\left(\rho_1\left(s\left(1-\frac{t(w_\zeta)}{t(\delta)}\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\rho_1\left(t\left(\frac{s(\vartheta)}{t(\delta)}\right)\right)\right), \\ t^{-1}\left(\rho_1\left(t\left(\frac{s(w_\vartheta)}{t(\delta)}\right)\right)\right) \end{pmatrix} \right) \\
\oplus& \left(\begin{pmatrix} s^{-1}\left(\rho_2\left(s\left(1-\frac{t(\zeta)}{t(\delta)}\right)\right)\right), \\ s^{-1}\left(\rho_2\left(s\left(1-\frac{t(w_\zeta)}{t(\delta)}\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\rho_2\left(t\left(\frac{s(\vartheta)}{t(\delta)}\right)\right)\right), \\ t^{-1}\left(\rho_2\left(t\left(\frac{s(w_\vartheta)}{t(\delta)}\right)\right)\right) \end{pmatrix} \right) \\
=& \left(\begin{pmatrix} s^{-1}\left(\rho_1\left(s\left(1-\frac{t(\zeta)}{t(\delta)}\right)\right)+\rho_2\left(s\left(1-\frac{t(\zeta)}{t(\delta)}\right)\right)\right), \\ s^{-1}\left(\rho_1\left(s\left(1-\frac{t(w_\zeta)}{t(\delta)}\right)\right)+\rho_2\left(s\left(1-\frac{t(w_\zeta)}{t(\delta)}\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\rho_1\left(t\left(\frac{s(\vartheta)}{t(\delta)}\right)\right)+\rho_2\left(t\left(\frac{s(\vartheta)}{t(\delta)}\right)\right)\right), \\ t^{-1}\left(\rho_1\left(t\left(\frac{s(w_\vartheta)}{t(\delta)}\right)\right)+\rho_2\left(t\left(\frac{s(w_\vartheta)}{t(\delta)}\right)\right)\right) \end{pmatrix} \right) \\
=& \left(\begin{pmatrix} s^{-1}\left((\rho_1+\rho_2)\left(s\left(1-\frac{t(\zeta)}{t(\delta)}\right)\right)\right), \\ s^{-1}\left((\rho_1+\rho_2)\left(s\left(1-\frac{t(w_\zeta)}{t(\delta)}\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left((\rho_1+\rho_2)\left(t\left(\frac{s(\vartheta)}{t(\delta)}\right)\right)\right), \\ t^{-1}\left((\rho_1+\rho_2)\left(t\left(\frac{s(w_\vartheta)}{t(\delta)}\right)\right)\right) \end{pmatrix} \right) \\
=& \left(\rho_1+\rho_2\right)\log_\delta(\mathcal{C})
\end{aligned}$$

□

Theorem 8.2.11. Let $\mathcal{C} = \left((\zeta, w_\zeta), (\vartheta, w_\vartheta)\right)$ be a CIFN and $0 \leq \delta_1 \leq \delta_2 \leq \min\{\zeta, w_\zeta\} \leq 1$; $\delta_1, \delta_2 \neq 1$. Then, $\log_{\delta_2}(\mathcal{C}) \subseteq \log_{\delta_1}(\mathcal{C})$.

Proof. By using Definition 8.2.2, we obtain that

$$\begin{aligned}
\log_{\delta_1}(\mathcal{C}) &= \left(\left(1-\frac{t(\zeta)}{t(\delta_1)},\ 1-\frac{t(w_\zeta)}{t(\delta_1)}\right), \left(\frac{s(\vartheta)}{t(\delta_1)},\ \frac{s(w_\vartheta)}{t(\delta_1)}\right)\right) \\
\text{and}\quad \log_{\delta_2}(\mathcal{C}) &= \left(\left(1-\frac{t(\zeta)}{t(\delta_2)},\ 1-\frac{t(w_\zeta)}{t(\delta_2)}\right), \left(\frac{s(\vartheta)}{t(\delta_2)},\ \frac{s(w_\vartheta)}{t(\delta_2)}\right)\right)
\end{aligned}$$

Since $\delta_1 \leq \delta_2 \leq \min\{\zeta, w_\zeta\}$ and s, t are increasing and decreasing functions respectively. So, we get that

$$1-\frac{t(\zeta)}{t(\delta_2)} \leq 1-\frac{t(\zeta)}{t(\delta_1)}; \frac{s(\vartheta)}{t(\delta_2)} \geq \frac{s(\vartheta)}{t(\delta_1)}; 1-\frac{t(w_\zeta)}{t(\delta_2)} \leq 1-\frac{t(w_\zeta)}{t(\delta_1)}; \frac{s(w_\vartheta)}{t(\delta_2)} \geq \frac{s(w_\vartheta)}{t(\delta_1)}.$$

Hence, by using Definition 2.1.10, we have $\log_{\delta_2}(\mathcal{C}) \subseteq \log_{\delta_1}(\mathcal{C})$. □

Theorem 8.2.12. Let $\mathcal{C}_j = \Big(\big(\zeta_j, w_{\zeta_j}\big), \big(\vartheta_j, w_{\vartheta_j}\big)\Big)$ $(j = 1, 2)$ be two CIFNs such that $\zeta_1 \leq \zeta_2$; $w_{\zeta_1} \leq w_{\zeta_2}$; $\vartheta_1 \geq \vartheta_2$; $w_{\vartheta_1} \geq w_{\vartheta_2}$ and $0 < \delta \leq \min\{\zeta_1, \zeta_2, w_{\zeta_1}, w_{\zeta_2}\} \leq 1$; $\delta \neq 1$. Then, $\log_\delta(\mathcal{C}_1) \subseteq \log_\delta(\mathcal{C}_2)$.

Proof. By using Definition 8.2.2, we have

$$\begin{aligned}
\log_\delta(\mathcal{C}_1) &= \left(\left(1 - \frac{t(\zeta_1)}{t(\delta)},\ \ 1 - \frac{t\left(w_{\zeta_1}\right)}{t(\delta)}\right), \left(\frac{s(\vartheta_1)}{t(\delta)},\ \ \frac{s\left(w_{\vartheta_1}\right)}{t(\delta)}\right)\right) \\
\text{and} \quad \log_\delta(\mathcal{C}_2) &= \left(\left(1 - \frac{t(\zeta_2)}{t(\delta)},\ \ 1 - \frac{t\left(w_{\zeta_2}\right)}{t(\delta)}\right), \left(\frac{s(\vartheta_2)}{t(\delta)},\ \ \frac{s\left(w_{\vartheta_2}\right)}{t(\delta)}\right)\right)
\end{aligned}$$

We have $\zeta_1 \leq \zeta_2$, $\vartheta_1 \geq \vartheta_2$, $w_{\zeta_1} \leq w_{\zeta_2}$ and $w_{\vartheta_1} \geq w_{\vartheta_2}$. Also, as t and s are decreasing and increasing functions respectively and $\delta \leq \min\{\zeta_1, \zeta_2, w_{\zeta_1}, w_{\zeta_2}\}$ therefore, we have

$$1 - \frac{t(\zeta_1)}{t(\delta)} \leq 1 - \frac{t(\zeta_2)}{t(\delta)}; \frac{s(\vartheta_1)}{t(\delta)} \geq \frac{s(\vartheta_2)}{t(\delta)}; 1 - \frac{t\left(w_{\zeta_1}\right)}{t(\delta)} \leq 1 - \frac{t\left(w_{\zeta_2}\right)}{t(\delta)}; \frac{s\left(w_{\vartheta_1}\right)}{t(\delta)} \geq \frac{s\left(w_{\vartheta_2}\right)}{t(\delta)}.$$

Hence, again by using Definition 2.1.10, we have $\log_\delta(\mathcal{C}_1) \subseteq \log_\delta(\mathcal{C}_2)$. $\square$

8.2.3 Exponential of logarithmic operational laws

Definition 8.2.3. For a CIFN $\mathcal{C} = \Big(\big(\zeta, w_\zeta\big), \big(\vartheta, w_\vartheta\big)\Big)$ and real numbers $\delta, \delta' > 0$ such that $\delta' \neq 1$, we define the exponential of logarithmic of $\mathcal{C}$, denoted by $\delta^{\log_{\delta'}(\mathcal{C})}$, as:

$$\delta^{\log_{\delta'}(\mathcal{C})} = \begin{cases}
\left(\begin{pmatrix} t^{-1}\left(\dfrac{t(\zeta)\cdot t(\delta)}{t(\delta')}\right), \\ t^{-1}\left(\dfrac{t\left(w_\zeta\right)\cdot t(\delta)}{t(\delta')}\right)\end{pmatrix}, \begin{pmatrix} s^{-1}\left(\dfrac{s(\vartheta)\cdot t(\delta)}{t(\delta')}\right), \\ s^{-1}\left(\dfrac{s\left(w_\vartheta\right)\cdot t(\delta)}{t(\delta')}\right)\end{pmatrix}\right) & ; \delta, \delta' \in (0,1) \\
\left(\begin{pmatrix} t^{-1}\left(\dfrac{t(\zeta)\cdot t\left(\frac{1}{\delta}\right)}{t(\delta')}\right), \\ t^{-1}\left(\dfrac{t\left(w_\zeta\right)\cdot t\left(\frac{1}{\delta}\right)}{t(\delta')}\right)\end{pmatrix}, \begin{pmatrix} s^{-1}\left(\dfrac{s(\vartheta)\cdot t\left(\frac{1}{\delta}\right)}{t(\delta')}\right), \\ s^{-1}\left(\dfrac{s\left(w_\vartheta\right)\cdot t\left(\frac{1}{\delta}\right)}{t(\delta')}\right)\end{pmatrix}\right) & ; \delta \geq 1, \delta' \in (0,1) \\
\left(\begin{pmatrix} t^{-1}\left(\dfrac{t(\zeta)\cdot t(\delta)}{t\left(\frac{1}{\delta'}\right)}\right), \\ t^{-1}\left(\dfrac{t\left(w_\zeta\right)\cdot t(\delta)}{t\left(\frac{1}{\delta'}\right)}\right)\end{pmatrix}, \begin{pmatrix} s^{-1}\left(\dfrac{s(\vartheta)\cdot t(\delta)}{t\left(\frac{1}{\delta'}\right)}\right), \\ s^{-1}\left(\dfrac{s\left(w_\vartheta\right)\cdot t(\delta)}{t\left(\frac{1}{\delta'}\right)}\right)\end{pmatrix}\right) & ; \delta \in (0,1), \delta' > 1 \\
\left(\begin{pmatrix} t^{-1}\left(\dfrac{t(\zeta)\cdot t\left(\frac{1}{\delta}\right)}{t\left(\frac{1}{\delta'}\right)}\right), \\ t^{-1}\left(\dfrac{t\left(w_\zeta\right)\cdot t\left(\frac{1}{\delta}\right)}{t\left(\frac{1}{\delta'}\right)}\right)\end{pmatrix}, \begin{pmatrix} s^{-1}\left(\dfrac{s(\vartheta)\cdot t\left(\frac{1}{\delta}\right)}{t\left(\frac{1}{\delta'}\right)}\right), \\ s^{-1}\left(\dfrac{s\left(w_\vartheta\right)\cdot t\left(\frac{1}{\delta}\right)}{t\left(\frac{1}{\delta'}\right)}\right)\end{pmatrix}\right) & ; \delta \geq 1, \delta' > 1
\end{cases} \tag{8.3}$$

Theorem 8.2.13. Let $\mathcal{C}$ be a CIFN and $\delta, \delta' > 0$ be real numbers such that $\delta' \neq 1$. Then, $\delta^{\log_{\delta'}(\mathcal{C})}$ is also CIFN.

Proof. We shall prove the theorem for $\delta, \delta' \in (0,1)$ as for the other cases it can be proved similarly. For this, let $\mathcal{C} = \Big(\big(\zeta, w_\zeta\big), \big(\vartheta, w_\vartheta\big)\Big)$. Then, by definition of CIFN we have, $0 \leq \zeta, \vartheta, \zeta + \vartheta \leq 1$ and $0 \leq w_\zeta, w_\vartheta, w_\zeta + w_\vartheta \leq 1$.

Take $\delta^{\log_{\delta'}(\mathcal{C})} = \Big(\big(\zeta_1, w_{\zeta_1}\big), \big(\vartheta_1, w_{\vartheta_1}\big)\Big)$ where $\zeta_1 = t^{-1}\left(\frac{t(\zeta)\cdot t(\delta)}{t(\delta')}\right)$, $\vartheta_1 = s^{-1}\left(\frac{s(\vartheta)\cdot t(\delta)}{t(\delta')}\right)$, $w_{\zeta_1} = t^{-1}\left(\frac{t(w_\zeta)\cdot t(\delta)}{t(\delta')}\right)$ and $w_{\vartheta_1} = s^{-1}\left(\frac{s(w_\vartheta)\cdot t(\delta)}{t(\delta')}\right)$. In order to prove that $\delta^{\log_{\delta'}(\mathcal{C})}$ is CIFN, it is sufficient to show that $0 \leq \zeta_1, \vartheta_1, \zeta_1 + \vartheta_1 \leq 1$ and $0 \leq w_{\zeta_1}, w_{\vartheta_1}, w_{\zeta_1} + w_{\vartheta_1} \leq 1$.

Since $t^{-1}, s^{-1} : [0, \infty) \to [0,1]$. Therefore, we obtain that $0 \leq \zeta_1, \vartheta_1 \leq 1$ and $0 \leq w_{\zeta_1}, w_{\vartheta_1} \leq$. Further, using the conditions that $0 \leq \zeta + \vartheta \leq 1$, $t(a) = s(1-a)$ and t is a decreasing function, we have:

$$\begin{aligned}
\zeta_1 + \vartheta_1 &= t^{-1}\left(\frac{t(\zeta)\cdot t(\delta)}{t(\delta')}\right) + s^{-1}\left(\frac{s(\vartheta)\cdot t(\delta)}{t(\delta')}\right) \\
&= t^{-1}\left(\frac{t(\zeta)\cdot t(\delta)}{t(\delta')}\right) + 1 - t^{-1}\left(\frac{t(1-\vartheta)\cdot t(\delta)}{t(\delta')}\right) \\
&\leq t^{-1}\left(\frac{t(\zeta)\cdot t(\delta)}{t(\delta')}\right) + 1 - t^{-1}\left(\frac{t(\zeta)\cdot t(\delta)}{t(\delta')}\right) \\
&= 1
\end{aligned}$$

Thus, $\zeta_1 + \vartheta_1 \leq 1$. Also, $\zeta_1 + \vartheta_1 \geq 0$ as $\zeta_1, \vartheta_1 \geq 0$. Hence, $0 \leq \zeta_1 + \vartheta_1 \leq 1$. Similarly, we can obtain that $0 \leq w_{\zeta_1} + w_{\vartheta_1} \leq 1$. Hence, $\delta^{\log_{\delta'}(\mathcal{C})}$ is also a CIFN. □

Theorem 8.2.14. For CIFNs $\mathcal{C}_j$ and positive real numbers $\delta_j, \delta'_j \in (0,1)$ $(j = 1,2,3)$, we have

(i) $\delta_1^{\log_{\delta'_1}(\mathcal{C}_1)} \oplus \delta_2^{\log_{\delta'_2}(\mathcal{C}_2)} = \delta_2^{\log_{\delta'_2}(\mathcal{C}_2)} \oplus \delta_1^{\log_{\delta'_1}(\mathcal{C}_1)}$;

(ii) $\delta_1^{\log_{\delta'_1}(\mathcal{C}_1)} \otimes \delta_2^{\log_{\delta'_2}(\mathcal{C}_2)} = \delta_2^{\log_{\delta'_2}(\mathcal{C}_2)} \otimes \delta_1^{\log_{\delta'_1}(\mathcal{C}_1)}$;

(iii) $\left(\delta_1^{\log_{\delta'_1}(\mathcal{C}_1)} \oplus \delta_2^{\log_{\delta'_2}(\mathcal{C}_2)}\right) \oplus \delta_3^{\log_{\delta'_3}(\mathcal{C}_3)} = \delta_1^{\log_{\delta'_1}(\mathcal{C}_1)} \oplus \left(\delta_2^{\log_{\delta'_2}(\mathcal{C}_2)} \oplus \delta_3^{\log_{\delta'_3}(\mathcal{C}_3)}\right)$;

(iv) $\left(\delta_1^{\log_{\delta'_1}(\mathcal{C}_1)} \otimes \delta_2^{\log_{\delta'_2}(\mathcal{C}_2)}\right) \otimes \delta_3^{\log_{\delta'_3}(\mathcal{C}_3)} = \delta_1^{\log_{\delta'_1}(\mathcal{C}_1)} \otimes \left(\delta_2^{\log_{\delta'_2}(\mathcal{C}_2)} \otimes \delta_3^{\log_{\delta'_3}(\mathcal{C}_3)}\right)$.

Proof. Just parts (i) and (iii) are demonstrated here as the remaining can be gotten likewise.

(i) Let $\mathcal{C}_j = \left((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\right)$. Then, by using Definitions 6.2.2 and 8.2.3, we have

$$
\begin{aligned}
&\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)} \oplus \delta_2^{\log_{\delta_2'}(\mathcal{C}_2)} \\
=& \left(\begin{pmatrix} s^{-1}\left(s\left(t^{-1}\left(\dfrac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)+s\left(t^{-1}\left(\dfrac{t(\zeta_2)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)\right), \\ s^{-1}\left(s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)+s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_2}\right)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)\right) \end{pmatrix}, \right. \\
&\left. \begin{pmatrix} t^{-1}\left(t\left(s^{-1}\left(\dfrac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)+t\left(s^{-1}\left(\dfrac{s(\vartheta_2)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)\right), \\ t^{-1}\left(t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)+t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_2}\right)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)\right) \end{pmatrix} \right) \\
=& \left(\begin{pmatrix} s^{-1}\left(s\left(t^{-1}\left(\dfrac{t(\zeta_2)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)+s\left(t^{-1}\left(\dfrac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right), \\ s^{-1}\left(s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_2}\right)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)+s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right) \end{pmatrix}, \right. \\
&\left. \begin{pmatrix} t^{-1}\left(t\left(s^{-1}\left(\dfrac{s(\vartheta_2)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)+t\left(s^{-1}\left(\dfrac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right), \\ t^{-1}\left(t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_2}\right)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)+t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right) \end{pmatrix} \right) \\
=& \delta_2^{\log_{\delta_2'}(\mathcal{C}_2)} \oplus \delta_1^{\log_{\delta_1'}(\mathcal{C}_1)}
\end{aligned}
$$

(iii) Again, by using Definition 6.2.2 and Definition 8.2.3, we have

$$
\begin{aligned}
&\left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)} \oplus \delta_2^{\log_{\delta_2'}(\mathcal{C}_2)}\right) \oplus \delta_3^{\log_{\delta_3'}(\mathcal{C}_3)} \\
=& \left(\begin{pmatrix} s^{-1}\left(s\left(t^{-1}\left(\dfrac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)+s\left(t^{-1}\left(\dfrac{t(\zeta_2)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)\right), \\ s^{-1}\left(s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)+s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_2}\right)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)\right) \end{pmatrix}, \right. \\
&\left. \begin{pmatrix} t^{-1}\left(t\left(s^{-1}\left(\dfrac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)+t\left(s^{-1}\left(\dfrac{s(\vartheta_2)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)\right), \\ t^{-1}\left(t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)+t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_2}\right)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)\right) \end{pmatrix} \right) \\
\oplus& \left(\begin{pmatrix} t^{-1}\left(\dfrac{t(\zeta_3)\cdot t(\delta_3)}{t(\delta_3')}\right), \\ t^{-1}\left(\dfrac{t\left(w_{\zeta_3}\right)\cdot t(\delta_3)}{t(\delta_3')}\right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\dfrac{s(\vartheta_3)\cdot t(\delta_3)}{t(\delta_3')}\right), \\ s^{-1}\left(\dfrac{s\left(w_{\vartheta_3}\right)\cdot t(\delta_3)}{t(\delta_3')}\right) \end{pmatrix} \right) \\
=& \left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{3}\left(s\left(t^{-1}\left(\frac{t(\zeta_j)\cdot t(\delta_j)}{t(\delta_j')}\right)\right)\right)\right), \\ s^{-1}\left(\sum_{j=1}^{3}\left(s\left(t^{-1}\left(\frac{t\left(w_{\zeta_j}\right)\cdot t(\delta_j)}{t(\delta_j')}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{3}\left(t\left(s^{-1}\left(\frac{s(\vartheta_j)\cdot t(\delta_j)}{t(\delta_j')}\right)\right)\right)\right), \\ t^{-1}\left(\sum_{j=1}^{3}\left(t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_j}\right)\cdot t(\delta_j)}{t(\delta_j')}\right)\right)\right)\right) \end{pmatrix} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\begin{pmatrix} t^{-1}\left(\dfrac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta_1')}\right), \\ t^{-1}\left(\dfrac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\dfrac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta_1')}\right), \\ s^{-1}\left(\dfrac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right) \end{pmatrix} \right) \\
&\oplus \left(\begin{pmatrix} s^{-1}\left(s\left(t^{-1}\left(\dfrac{t(\zeta_2)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)+s\left(t^{-1}\left(\dfrac{t(\zeta_3)\cdot t(\delta_3)}{t(\delta_3')}\right)\right)\right), \\ s^{-1}\left(s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_2}\right)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)+s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_3}\right)\cdot t(\delta_3)}{t(\delta_3')}\right)\right)\right) \end{pmatrix}, \right. \\
&\qquad \left. \begin{pmatrix} t^{-1}\left(t\left(s^{-1}\left(\dfrac{s(\vartheta_2)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)+t\left(s^{-1}\left(\dfrac{s(\vartheta_3)\cdot t(\delta_3)}{t(\delta_3')}\right)\right)\right), \\ t^{-1}\left(t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_2}\right)\cdot t(\delta_2)}{t(\delta_2')}\right)\right)+t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_3}\right)\cdot t(\delta_3)}{t(\delta_3')}\right)\right)\right) \end{pmatrix} \right) \\
&= \delta_1^{\log_{\delta_1'}(\mathcal{C}_1)} \oplus \left(\delta_2^{\log_{\delta_2'}(\mathcal{C}_2)} \oplus \delta_3^{\log_{\delta_3'}(\mathcal{C}_3)}\right)
\end{aligned}
$$

□

Theorem 8.2.15. For CIFNs $\mathcal{C}_j$ and positive real numbers $\delta_j, \delta_j' \in (0,1)$ $(j = 1,2,3)$, ρ, ρ_1, ρ_2 we have

(i) $\rho\left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)} \oplus \delta_2^{\log_{\delta_2'}(\mathcal{C}_2)}\right) = \rho\left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)}\right) \oplus \rho\left(\delta_2^{\log_{\delta_2'}(\mathcal{C}_2)}\right)$;

(ii) $\left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)} \otimes \delta_2^{\log_{\delta_2'}(\mathcal{C}_2)}\right)^{\rho} = \left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)}\right)^{\rho} \otimes \left(\delta_2^{\log_{\delta_2'}(\mathcal{C}_2)}\right)^{\rho}$;

(iii) $\rho_1\left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)}\right) \oplus \rho_2\left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)}\right) = (\rho_1+\rho_2)\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)}$;

(iv) $\left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)}\right)^{\rho_1} \otimes \left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)}\right)^{\rho_2} = \left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)}\right)^{\rho_1+\rho_2}$.

Proof. Just parts (i) and (iii) are demonstrated here as the remaining can be gotten likewise.

(i) Let $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right)$. Then, by using Definition 6.2.2 and Definition 8.2.3, we have

$$
\begin{aligned}
&\rho\left(\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)} \oplus \delta_2^{\log_{\delta_2'}(\mathcal{C}_2)}\right) \\
&= \rho\left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{2}\left(s\left(t^{-1}\left(\frac{t(\zeta_j)\cdot t(\delta_j)}{t(\delta_j')}\right)\right)\right)\right), \\ s^{-1}\left(\sum_{j=1}^{2}\left(s\left(t^{-1}\left(\frac{t\left(w_{\zeta_j}\right)\cdot t(\delta_j)}{t(\delta_j')}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{2}\left(t\left(s^{-1}\left(\frac{s(\vartheta_j)\cdot t(\delta_j)}{t(\delta_j')}\right)\right)\right)\right), \\ t^{-1}\left(\sum_{j=1}^{2}\left(t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_j}\right)\cdot t(\delta_j)}{t(\delta_j')}\right)\right)\right)\right) \end{pmatrix} \right)
\end{aligned}
$$

$$= \left(\begin{pmatrix} s^{-1}\left(\rho\left(\sum_{j=1}^{2}\left(s\left(t^{-1}\left(\frac{t(\zeta_j)\cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right)\right)\right), \\ s^{-1}\left(\rho\left(\sum_{j=1}^{2}\left(s\left(t^{-1}\left(\frac{t\left(w_{\zeta_j}\right)\cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\rho\left(\sum_{j=1}^{2}\left(t\left(s^{-1}\left(\frac{s(\vartheta_j)\cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right)\right)\right), \\ \left(t^{-1}\left(\rho\left(\sum_{j=1}^{2}\left(t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_j}\right)\cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right)\right)\right)\right) \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} s^{-1}\left(\rho\left(s\left(t^{-1}\left(\frac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right), \\ s^{-1}\left(\rho\left(s\left(t^{-1}\left(\frac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\rho\left(t\left(s^{-1}\left(\frac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right), \\ t^{-1}\left(\rho\left(t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right) \end{pmatrix} \right)$$

$$\oplus \left(\begin{pmatrix} s^{-1}\left(\rho\left(s\left(t^{-1}\left(\frac{t(\zeta_2)\cdot t(\delta_2)}{t(\delta'_2)}\right)\right)\right)\right), \\ s^{-1}\left(\rho\left(s\left(t^{-1}\left(\frac{t\left(w_{\zeta_2}\right)\cdot t(\delta_2)}{t(\delta'_2)}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\rho\left(t\left(s^{-1}\left(\frac{s(\vartheta_2)\cdot t(\delta_2)}{t(\delta'_2)}\right)\right)\right)\right), \\ t^{-1}\left(\rho\left(t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_2}\right)\cdot t(\delta_2)}{t(\delta'_2)}\right)\right)\right)\right) \end{pmatrix} \right)$$

$$= \rho\left(\delta_1^{\log_{\delta'_1}(\mathcal{C}_1)}\right) \oplus \rho\left(\delta_2^{\log_{\delta'_2}(\mathcal{C}_2)}\right)$$

(iii) Again, by using Definition 6.2.2 and Definition 8.2.3, we have

$$\rho_1\left(\delta_1^{\log_{\delta'_1}(\mathcal{C}_1)}\right) \oplus \rho_2\left(\delta_1^{\log_{\delta'_1}(\mathcal{C}_1)}\right)$$

$$= \left(\begin{pmatrix} s^{-1}\left(\rho_1\left(s\left(t^{-1}\left(\frac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right), \\ s^{-1}\left(\rho_1\left(s\left(t^{-1}\left(\frac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\rho_1\left(t\left(s^{-1}\left(\frac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right), \\ t^{-1}\left(\rho_1\left(t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right) \end{pmatrix} \right)$$

$$\oplus \left(\begin{pmatrix} s^{-1}\left(\rho_2\left(s\left(t^{-1}\left(\frac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right), \\ s^{-1}\left(\rho_2\left(s\left(t^{-1}\left(\frac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\rho_2\left(t\left(s^{-1}\left(\frac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right), \\ t^{-1}\left(\rho_2\left(t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta'_1)}\right)\right)\right)\right) \end{pmatrix} \right)$$

$$
\begin{aligned}
&= \left(\begin{pmatrix} s^{-1}\left(\rho_1\left(s\left(t^{-1}\left(\dfrac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)+\rho_2\left(s\left(t^{-1}\left(\dfrac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)\right), \\ s^{-1}\left(\rho_1\left(s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)+\rho_2\left(s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)\right) \end{pmatrix}, \atop \begin{pmatrix} t^{-1}\left(\rho_1\left(t\left(s^{-1}\left(\dfrac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)+\rho_2\left(t\left(s^{-1}\left(\dfrac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)\right), \\ t^{-1}\left(\rho_1\left(t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)+\rho_2\left(t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)\right) \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} s^{-1}\left((\rho_1+\rho_2)\left(s\left(t^{-1}\left(\frac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)\right), \\ s^{-1}\left((\rho_1+\rho_2)\left(s\left(t^{-1}\left(\frac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left((\rho_1+\rho_2)\left(t\left(s^{-1}\left(\frac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)\right), \\ t^{-1}\left((\rho_1+\rho_2)\left(t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right)\right)\right)\right) \end{pmatrix} \right) \\
&= (\rho_1+\rho_2)\left(\begin{pmatrix} t^{-1}\left(\dfrac{t(\zeta_1)\cdot t(\delta_1)}{t(\delta_1')}\right), \\ t^{-1}\left(\dfrac{t\left(w_{\zeta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\dfrac{s(\vartheta_1)\cdot t(\delta_1)}{t(\delta_1')}\right), \\ s^{-1}\left(\dfrac{s\left(w_{\vartheta_1}\right)\cdot t(\delta_1)}{t(\delta_1')}\right) \end{pmatrix} \right) \\
&= (\rho_1+\rho_2)\delta_1^{\log_{\delta_1'}(\mathcal{C}_1)}
\end{aligned}
$$

□

Theorem 8.2.16. For CIFN $\mathcal{C}$ and positive real numbers δ and δ' with $\delta' \neq 1$, we have

(i) If $\delta < \delta'$ then, $\delta^{\log_{\delta'}(\mathcal{C})} \subseteq \mathcal{C}$;

(ii) If $\delta > \delta'$ then, $\mathcal{C} \subseteq \delta^{\log_{\delta'}(\mathcal{C})}$.

Proof. Let $\mathcal{C} = \Big(\big(\zeta, w_\zeta\big), \big(\vartheta, w_\vartheta\big)\Big)$.

(i) Since, $\delta < \delta'$ and s and t are increasing and decreasing functions respectively. It implies that $\frac{t(\delta)}{t(\delta')} > 1$. Therefore, we get that

$t^{-1}\left(\frac{t(\zeta)\cdot t(\delta)}{t(\delta')}\right) < \zeta \quad ; \quad s^{-1}\left(\frac{s(\vartheta)\cdot t(\delta)}{t(\delta')}\right) > \vartheta \quad ; \quad t^{-1}\left(\frac{t\left(w_\zeta\right)\cdot t(\delta)}{t(\delta')}\right) < w_\zeta \quad ; \quad s^{-1}\left(\frac{s(w_\vartheta)\cdot t(\delta)}{t(\delta')}\right) > w_\vartheta.$

Hence, by using the Definition 2.1.10, we have $\delta^{\log_{\delta'}(\mathcal{C})} \subseteq \mathcal{C}$.

(ii) Since, $\delta > \delta'$ and s and t are increasing and decreasing functions respectively. It implies that $\frac{t(\delta)}{t(\delta')} < 1$. Therefore, we get that

$t^{-1}\left(\frac{t(\zeta)\cdot t(\delta)}{t(\delta')}\right) > \zeta \quad ; \quad s^{-1}\left(\frac{s(\vartheta)\cdot t(\delta)}{t(\delta')}\right) < \vartheta \quad ; \quad t^{-1}\left(\frac{t\left(w_\zeta\right)\cdot t(\delta)}{t(\delta')}\right) > w_\zeta \quad ; \quad s^{-1}\left(\frac{s(w_\vartheta)\cdot t(\delta)}{t(\delta')}\right) < w_\vartheta.$

Hence, by using the Definition 2.1.10, we have $\mathcal{C} \subseteq \delta^{\log_{\delta'}(\mathcal{C})}$.

□

8.2.4 Compensative weighted averaging operator

Here, we develop GCIFCWA operator and investigate its properties. For this, throughout this section, we consider CIFNs $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ having associated weights $\xi_j > 0$ satisfying $\sum_{j=1}^{n} \xi_j = 1$ and Ω as a set of all CIFNs.

Definition 8.2.4. A map, GCIFCWA : $\Omega^n \to \Omega$, defined by

$$\text{GCIFCWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \xi_1 \left(\delta_1^{\log_{\delta'_1}(\mathcal{C}_1)} \right) \oplus \xi_2 \left(\delta_2^{\log_{\delta'_2}(\mathcal{C}_2)} \right) \oplus \ldots \oplus \xi_n \left(\delta_n^{\log_{\delta'_n}(\mathcal{C}_n)} \right) \quad (8.4)$$

is called as GCIFCWA operator, where δ_j, δ'_j are positive real numbers and $\delta'_j \neq 1 \ \forall \ j$.

Theorem 8.2.17. The aggregated value acquired on applying GCIFCWA operator remains CIFN and is given as

$$\text{GCIFCWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)$$
$$= \begin{cases} \left(\left(\begin{array}{c} s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t(\zeta_j) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t\left(w_{\zeta_j}\right) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right) \end{array} \right), \left(\begin{array}{c} t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s(\vartheta_j) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_j}\right) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right) \end{array} \right) \right) & ; \delta_j, \delta'_j \in (0,1) \\ \left(\left(\begin{array}{c} s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t(\zeta_j) \cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t\left(w_{\zeta_j}\right) \cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\right) \end{array} \right), \left(\begin{array}{c} t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s(\vartheta_j) \cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_j}\right) \cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\right) \end{array} \right) \right) & ; \delta_j \geq 1, \delta'_j \in (0,1) \\ \left(\left(\begin{array}{c} s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t(\zeta_j) \cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t\left(w_{\zeta_j}\right) \cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right) \end{array} \right), \left(\begin{array}{c} t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s(\vartheta_j) \cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_j}\right) \cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right) \end{array} \right) \right) & ; \delta_j \in (0,1), \delta'_j > 1 \\ \left(\left(\begin{array}{c} s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t(\zeta_j) \cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t\left(w_{\zeta_j}\right) \cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right) \end{array} \right), \left(\begin{array}{c} t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s(\vartheta_j) \cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_j}\right) \cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right) \end{array} \right) \right) & ; \delta_j \geq 1, \delta'_j > 1 \end{cases} \quad (8.5)$$

Proof. The first part of the result follows from the Theorems 8.2.13 and 6.2.5. Now, we shall prove that the Eq. (8.5) holds by using mathematical induction on n. Without loss of generality (WLOG), we show the validness of Eq. (8.5) for the case $\delta_j, \delta'_j \in (0, 1)$ as for the other cases it can be proved similarly.

Since, $\mathcal{C}_j$ is a CIFN and δ_j, δ'_j are positive real numbers with $\delta'_j \neq 1$ for each j. Therefore, $\delta_j^{\log_{\delta'_j}(\mathcal{C}_j)}$ is also CIFN by using Theorem 8.2.13. Further, as $\xi_j > 0$ so, on utilizing Theorem 6.2.5, we obtain that $\xi_j\left(\delta_j^{\log_{\delta'_j}(\mathcal{C}_j)}\right)$ is also CIFN $\forall\ j$. The rest of the part can be obtained from the operational laws of the CIFNs as defined in Definition 6.2.2 of Chapter 6. □

Example 8.2.1. Let $\mathcal{C}_1 = ((0.5, 0.4), (0.2, 0.2))$, $\mathcal{C}_2 = ((0.3, 0.2), (0.2, 0.2))$, $\mathcal{C}_3 = ((0.6, 0.3), (0.3, 0.1))$, $\mathcal{C}_4 = ((0.5, 0.2), (0.2, 0.3))$ be four CIFNs and let $\xi = (0.30, 0.20, 0.35, 0.15)^T$ be the weight vector corresponding to $\mathcal{C}_j$ $(j = 1, 2, 3, 4)$. WLOG take $t(a) = -\log a$ if $0 < a \leq 1$ and $t(0) = \infty$; $\delta_j = 0.5$; $\delta'_j = 4$ for $j = 1, 2, 3, 4$. Then, using Eq. (8.5), we have

$$\begin{aligned}
&\text{GCIFCWA}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4)\\
&= \left(\begin{pmatrix} 1-\prod_{j=1}^{4}\left(1-(0.5)^{\log_{(0.25)}(\zeta_j)}\right)^{\xi_j}, \\ 1-\prod_{j=1}^{4}\left(1-(0.5)^{\log_{(0.25)}\left(w_{\zeta_j}\right)}\right)^{\xi_j} \end{pmatrix}, \begin{pmatrix} \prod_{j=1}^{4}\left(1-(0.5)^{\log_{(0.25)}(1-\vartheta_j)}\right)^{\xi_j}, \\ \prod_{j=1}^{4}\left(1-(0.5)^{\log_{(0.25)}\left(1-w_{\vartheta_j}\right)}\right)^{\xi_j} \end{pmatrix}\right)
\end{aligned} \tag{8.6}$$

Based on these information, we have

$$\begin{aligned}
\prod_{j=1}^{4}\left(1-(0.5)^{\log_{(0.25)}(\zeta_j)}\right)^{\xi_j} &= \left(1-(0.5)^{\log_{(0.25)}(0.5)}\right)^{0.30} \times \left(1-(0.5)^{\log_{(0.25)}(0.3)}\right)^{0.20}\\
&\quad \times \left(1-(0.5)^{\log_{(0.25)}(0.6)}\right)^{0.35} \times \left(1-(0.5)^{\log_{(0.25)}(0.5)}\right)^{0.15}\\
&= 0.2915\\
\prod_{j=1}^{4}\left(1-(0.5)^{\log_{(0.25)}(1-\vartheta_j)}\right)^{\xi_j} &= \left(1-(0.5)^{\log_{(0.25)}(0.8)}\right)^{0.30} \times \left(1-(0.5)^{\log_{(0.25)}(0.8)}\right)^{0.20}\\
&\quad \times \left(1-(0.5)^{\log_{(0.25)}(0.7)}\right)^{0.35} \times \left(1-(0.5)^{\log_{(0.25)}(0.8)}\right)^{0.15}\\
&= 0.1230\\
\prod_{j=1}^{4}\left(1-(0.5)^{\log_{(0.25)}\left(w_{\zeta_j}\right)}\right)^{\xi_j} &= \left(1-(0.5)^{\log_{(0.25)}(0.4)}\right)^{0.30} \times \left(1-(0.5)^{\log_{(0.25)}(0.2)}\right)^{0.20}\\
&\quad \times \left(1-(0.5)^{\log_{(0.25)}(0.3)}\right)^{0.35} \times \left(1-(0.5)^{\log_{(0.25)}(0.2)}\right)^{0.15}\\
&= 0.4559\\
\prod_{j=1}^{4}\left(1-(0.5)^{\log_{(0.25)}\left(1-w_{\vartheta_j}\right)}\right)^{\xi_j} &= \left(1-(0.5)^{\log_{(0.25)}(0.8)}\right)^{0.30} \times \left(1-(0.5)^{\log_{(0.25)}(0.8)}\right)^{0.20}\\
&\quad \times \left(1-(0.5)^{\log_{(0.25)}(0.9)}\right)^{0.35} \times \left(1-(0.5)^{\log_{(0.25)}(0.7)}\right)^{0.15}\\
&= 0.0876
\end{aligned}$$

Now, by using Eq. (8.6), we have

$$\begin{aligned}\mathrm{GCIFCWA}(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4) &= \left(\begin{pmatrix}1-0.2915,\\(1-0.4559)\end{pmatrix},\begin{pmatrix}0.1230,\\(0.0876)\end{pmatrix}\right)\\ &= \Big(\big(0.7085,(0.5441)\big),\big(0.1230,(0.0876)\big)\Big)\end{aligned}$$

Further, we analyze that certain properties hold under GCIFCWA operator and these are explained as follows:

Property 8.2.1. (Idempotency) Consider a CIFN $\mathcal{C}_0$ and positive real numbers δ_j,δ'_j such that $\mathcal{C}_j=\mathcal{C}_0$; $\delta_j=\delta'_j$ and $\delta'_j\neq 1\ \forall\ j$. Then,

$$\mathrm{GCIFCWA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)=\mathcal{C}_0$$

Proof. Since $\delta_j=\delta'_j$. Therefore, either $\delta_j,\delta'_j\in(0,1)$ or $\delta_j,\delta'_j>1$. Without loss of generality, we prove this property for the case when $\delta_j,\delta'_j\in(0,1)$ as for the other case, the proof is similar. For this let, $\mathcal{C}_j=\Big(\big(\zeta_j,w_{\zeta_j}\big),\big(\vartheta_j,w_{\vartheta_j}\big)\Big)$ and $\mathcal{C}_0=\Big(\big(\zeta_0,w_{\zeta_0}\big),\big(\vartheta_0,w_{\vartheta_0}\big)\Big)$. Then, $\mathcal{C}_j=\mathcal{C}_0$ implies that $\zeta_j=\zeta_0$, $\vartheta_j=\vartheta_0$, $w_{\zeta_j}=w_{\zeta_0}$ and $w_{\vartheta_j}=w_{\vartheta_0}$ for all j. Further, on utilizing Eq. (8.5) and $\sum\limits_{j=1}^{n}\xi_j=1$, we have

$$\begin{aligned}&\mathrm{GCIFCWA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)\\ &=\left(\begin{pmatrix}s^{-1}\left(\sum\limits_{j=1}^{n}\xi_j s\left(t^{-1}\left(\dfrac{t(\zeta_0)\cdot t(\delta_j)}{t(\delta_j)}\right)\right)\right),\\ s^{-1}\left(\sum\limits_{j=1}^{n}\xi_j s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_0}\right)\cdot t(\delta_j)}{t(\delta_j)}\right)\right)\right)\end{pmatrix},\begin{pmatrix}t^{-1}\left(\sum\limits_{j=1}^{n}\xi_j t\left(s^{-1}\left(\dfrac{s(\vartheta_0)\cdot t(\delta_j)}{t(\delta_j)}\right)\right)\right),\\ t^{-1}\left(\sum\limits_{j=1}^{n}\xi_j t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_0}\right)\cdot t(\delta_j)}{t(\delta_j)}\right)\right)\right)\end{pmatrix}\right)\\ &=\left(\begin{pmatrix}s^{-1}\big(s(\zeta_0)\big),\\ s^{-1}\left(s\left(w_{\zeta_0}\right)\right)\end{pmatrix},\begin{pmatrix}t^{-1}\big(t(\vartheta_0)\big),\\ t^{-1}\left(t\left(w_{\vartheta_0}\right)\right)\end{pmatrix}\right)\\ &=\Big(\big(\zeta_0,w_{\zeta_0}\big),\big(\vartheta_0,w_{\vartheta_0}\big)\Big)\\ &=\mathcal{C}_0.\end{aligned}$$

□

Property 8.2.2. (Monotonicity) Consider CIFNs $\mathcal{C}_j=\Big(\big(\zeta_{\mathcal{C}_j},w_{\zeta_{\mathcal{C}_j}}\big),\big(\vartheta_{\mathcal{C}_j},w_{\vartheta_{\mathcal{C}_j}}\big)\Big)$ and $\mathcal{Z}_j=\Big(\big(\zeta_{\mathcal{Z}_j},w_{\zeta_{\mathcal{Z}_j}}\big),\big(\vartheta_{\mathcal{Z}_j},w_{\vartheta_{\mathcal{Z}_j}}\big)\Big)$ $(j=1,2,\ldots,n)$ having weights $\xi_j>0$ satisfying $\sum\limits_{j=1}^{n}\xi_j=$

1 and $\zeta_{\mathcal{C}_j} \leq \zeta_{\mathcal{Z}_j}$, $\vartheta_{\mathcal{C}_j} \geq \vartheta_{\mathcal{Z}_j}$, $w_{\zeta_{\mathcal{C}_j}} \leq w_{\zeta_{\mathcal{Z}_j}}$, $w_{\vartheta_{\mathcal{C}_j}} \geq w_{\vartheta_{\mathcal{Z}_j}}$ $\forall$ j. Then, we have

$$\text{GCIFCWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \text{GCIFCWA}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n).$$

Proof. WLOG, this property is demonstrated for the case when $\delta_j, \delta'_j \in (0,1)$ while, the proof can be obtained likewise for remaining cases. We have, $\zeta_{\mathcal{C}_j} \leq \zeta_{\mathcal{Z}_j}$, $\vartheta_{\mathcal{C}_j} \geq \vartheta_{\mathcal{Z}_j}$, $w_{\zeta_{\mathcal{C}_j}} \leq w_{\zeta_{\mathcal{Z}_j}}$ and $w_{\vartheta_{\mathcal{C}_j}} \geq w_{\vartheta_{\mathcal{Z}_j}}$. Further, as s and t are increasing and decreasing functions respectively, so, we have

$$\begin{aligned}
& s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t(\zeta_{\mathcal{C}_j}) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right) \leq s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t(\zeta_{\mathcal{Z}_j}) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right); \\
& t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s(\vartheta_{\mathcal{C}_j}) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right) \geq t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s(\vartheta_{\mathcal{Z}_j}) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right); \\
& s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t\left(w_{\zeta_{\mathcal{C}_j}}\right) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right) \leq s^{-1}\left(\sum_{j=1}^{n} \xi_j s\left(t^{-1}\left(\frac{t\left(w_{\zeta_{\mathcal{Z}_j}}\right) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right); \\
\text{and} \quad & t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_{\mathcal{C}_j}}\right) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right) \geq t^{-1}\left(\sum_{j=1}^{n} \xi_j t\left(s^{-1}\left(\frac{s\left(w_{\vartheta_{\mathcal{Z}_j}}\right) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right)
\end{aligned}$$

Then, by using the Definition 2.1.10, we obtain the result. $\square$

Property 8.2.3. (Boundedness) Let $\mathcal{C}_j = \big((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\big)$, $\mathcal{C}^- = \big(\big(\min_j\{\zeta_j\}, \min_j\{w_{\zeta_j}\}\big), \big(\max_j\{\vartheta_j\}, \max_j\{w_{\vartheta_j}\}\big)\big)$ and $\mathcal{C}^+ = \big(\big(\max_j\{\zeta_j\}, \max_j\{w_{\zeta_j}\}\big), \big(\min_j\{\vartheta_j\}, \min_j\{w_{\vartheta_j}\}\big)\big)$ and $\delta_j = \delta'_j$. Then,

$$\mathcal{C}^- \subseteq \text{GCIFCWA}(\mathcal{C}_1, \mathcal{C}_2 \ldots \mathcal{C}_n) \subseteq \mathcal{C}^+$$

Proof. Since, $\mathcal{C}^- \subseteq \mathcal{C}_j \subseteq \mathcal{C}^+$ $\forall$ j. Then, by using Property 8.2.2, we obtain that $\text{GCIFCWA}(\mathcal{C}^-, \mathcal{C}^-, \ldots, \mathcal{C}^-) \subseteq \text{GCIFCWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \text{GCIFCWA}(\mathcal{C}^+, \mathcal{C}^+, \ldots, \mathcal{C}^+)$. Further, as $\delta_j = \delta'_j$ therefore, by using property 8.2.1, we get that $\mathcal{C}^- \subseteq \text{GCIFCWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \mathcal{C}^+$, which completes the proof. $\square$

Furthermore, it is analyzed that, by taking $\delta_j = \delta'_j$ $\forall$ j, the proposed AO GCIFCWA becomes CIF weighted averaging operator [59]. Hence, the presented GCIFCWA operator is more generalized.

8.2.5 Compensative ordered weighted averaging operator

This section introduces GCIFCOWA AO for CIFNs $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ having associated weights $\xi_j > 0$ such that $\sum\limits_{j=1}^{n} \xi_j = 1$ and some of its properties are discussed here.

Definition 8.2.5. A map, GCIFCOWA : $\Omega^n \to \Omega$, defined by

$$\begin{aligned}
&\text{GCIFCOWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
&= \xi_1 \left(\delta_1^{\log_{\delta'_1}\left(\mathcal{C}_{\tau(1)}\right)}\right) \oplus \xi_2 \left(\delta_2^{\log_{\delta'_2}\left(\mathcal{C}_{\tau(2)}\right)}\right) \oplus \ldots \oplus \xi_n \left(\delta_n^{\log_{\delta'_n}\left(\mathcal{C}_{\tau(n)}\right)}\right)
\end{aligned} \tag{8.7}$$

is called as GCIFCOWA operator where $\delta_j, \delta'_j > 0$ are real numbers with the condition that $\delta'_j \neq 1 \ \forall \ j$; $\big(\tau(1), \tau(2), \ldots, \tau(n)\big)$ is a permutation of $(1, 2, \ldots, n)$ in such a way that $\mathcal{S}\left(\mathcal{C}_{\tau(j-1)}\right) \geq \mathcal{S}\left(\mathcal{C}_{\tau(j)}\right)$ for $j = 2, 3, \ldots, n$ and Ω is the set of all CIFNs.

Theorem 8.2.18. The aggregated value acquired on applying GCIFCOWA operator remains CIFN and is given as

$$\begin{aligned}
&\text{GCIFCOWA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
&= \begin{cases}
\left(\begin{pmatrix} s^{-1}\left(\sum\limits_{j=1}^{n}\xi_j s\left(t^{-1}\left(\dfrac{t(\zeta_{\tau(j)})\cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\xi_j s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_{\tau(j)}}\right)\cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum\limits_{j=1}^{n}\xi_j t\left(s^{-1}\left(\dfrac{s(\vartheta_{\tau(j)})\cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\xi_j t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_{\tau(j)}}\right)\cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\right)\end{pmatrix}\right) & ; \delta_j, \delta'_j \in (0,1) \\
\left(\begin{pmatrix} s^{-1}\left(\sum\limits_{j=1}^{n}\xi_j s\left(t^{-1}\left(\dfrac{t(\zeta_{\tau(j)})\cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\xi_j s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_{\tau(j)}}\right)\cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum\limits_{j=1}^{n}\xi_j t\left(s^{-1}\left(\dfrac{s(\vartheta_{\tau(j)})\cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\xi_j t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_{\tau(j)}}\right)\cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\right)\end{pmatrix}\right) & ; \delta_j \geq 1, \delta'_j \in (0,1) \\
\left(\begin{pmatrix} s^{-1}\left(\sum\limits_{j=1}^{n}\xi_j s\left(t^{-1}\left(\dfrac{t(\zeta_{\tau(j)})\cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\xi_j s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_{\tau(j)}}\right)\cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum\limits_{j=1}^{n}\xi_j t\left(s^{-1}\left(\dfrac{s(\vartheta_{\tau(j)})\cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\xi_j t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_{\tau(j)}}\right)\cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right)\end{pmatrix}\right) & ; \delta_j \in (0,1), \delta'_j > 1 \\
\left(\begin{pmatrix} s^{-1}\left(\sum\limits_{j=1}^{n}\xi_j s\left(t^{-1}\left(\dfrac{t(\zeta_{\tau(j)})\cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\xi_j s\left(t^{-1}\left(\dfrac{t\left(w_{\zeta_{\tau(j)}}\right)\cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum\limits_{j=1}^{n}\xi_j t\left(s^{-1}\left(\dfrac{s(\vartheta_{\tau(j)})\cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\xi_j t\left(s^{-1}\left(\dfrac{s\left(w_{\vartheta_{\tau(j)}}\right)\cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\right)\end{pmatrix}\right) & ; \delta_j \geq 1, \delta'_j > 1
\end{cases}
\end{aligned} \tag{8.8}$$

Proof. It is same as the Theorem 8.2.17. Therefore, the proof is ommitted here. $\square$

Furthermore, it is noticed that as GCIFCWA operator, GCIFCOWA operator also satisfy the aforementioned properties. Also, by taking $\delta_j = \delta'_j \ \forall \ j$, GCIFCOWA operator reduces to existing CIF ordered weighted averaging operator [59].

8.2.6 Compensative weighted geometric operators

This section introduces GCIFCWG and GCIFCOWG operators for CIFNs $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ having associated weights ξ_j such that $\xi_j > 0$; $\sum\limits_{j=1}^{n} \xi_j = 1$.

Definition 8.2.6. A map, GCIFCWG : $\Omega^n \to \Omega$, given as

$$\begin{aligned}&\text{GCIFCWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)\\ =\;& \left(\delta_1^{\log_{\delta'_1}(\mathcal{C}_1)}\right)^{\xi_1} \otimes \left(\delta_2^{\log_{\delta'_2}(\mathcal{C}_2)}\right)^{\xi_2} \otimes \ldots \otimes \left(\delta_n^{\log_{\delta'_n}(\mathcal{C}_n)}\right)^{\xi_n}\end{aligned} \tag{8.9}$$

is called GCIFCWG operator where $\delta_j, \delta'_j > 0$ are real numbers satisfying $\delta'_j \neq 1\ \forall\ j$ and Ω is the collection of all CIFNs.

Theorem 8.2.19. The aggregated value acquired on utilizing GCIFCWG operator remains CIFN and is given as

$$\begin{aligned}&\text{GCIFCWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)\\ =\;&\begin{cases}\left(\left(\begin{array}{c} t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t(\zeta_j)\cdot t(\delta_j)}{t(\delta'_j)}\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t\left(w_{\zeta_j}\right)\cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\end{array}\right), \left(\begin{array}{c} s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s(\vartheta_j)\cdot t(\delta_j)}{t(\delta'_j)}\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s\left(w_{\vartheta_j}\right)\cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\end{array}\right)\right) & ; \delta_j, \delta'_j \in (0,1)\\[2ex] \left(\left(\begin{array}{c} t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t(\zeta_j)\cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t\left(w_{\zeta_j}\right)\cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\end{array}\right), \left(\begin{array}{c} s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s(\vartheta_j)\cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\left(\xi_j \cdot \dfrac{s\left(w_{\vartheta_j}\right)\cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\end{array}\right)\right) & ; \delta_j \geq 1, \delta'_j \in (0,1)\\[2ex] \left(\left(\begin{array}{c} t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t(\zeta_j)\cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t\left(w_{\zeta_j}\right)\cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\end{array}\right), \left(\begin{array}{c} s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s(\vartheta_j)\cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s\left(w_{\vartheta_j}\right)\cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\end{array}\right)\right) & ; \delta_j \in (0,1), \delta'_j > 1\\[2ex] \left(\left(\begin{array}{c} t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t(\zeta_j)\cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t\left(w_{\zeta_j}\right)\cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\end{array}\right), \left(\begin{array}{c} s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s(\vartheta_j)\cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s\left(w_{\vartheta_j}\right)\cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\end{array}\right)\right) & ; \delta_j \geq 1, \delta'_j > 1\end{cases}\end{aligned} \tag{8.10}$$

Proof. It is omitted here because it is same as the Theorem 8.2.17. □

Definition 8.2.7. A GCIFCOWG operator is a map, GCIFCOWG : $\Omega^n \to \Omega$, defined by

$$\begin{aligned}&\text{GCIFCOWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)\\ =\;& \left(\delta_1^{\log_{\delta'_1}\left(\mathcal{C}_{\tau(1)}\right)}\right)^{\xi_1} \otimes \left(\delta_2^{\log_{\delta'_2}\left(\mathcal{C}_{\tau(2)}\right)}\right)^{\xi_2} \otimes \ldots \otimes \left(\delta_n^{\log_{\delta'_n}\left(\mathcal{C}_{\tau(n)}\right)}\right)^{\xi_n}\end{aligned} \tag{8.11}$$

where $\delta_j, \delta'_j > 0$ are real numbers satisfying $\delta'_j \neq 1 \ \forall \ j$; $\big(\tau(1), \tau(2), \ldots, \tau(n)\big)$ is an rearrangement of $(1, 2, \ldots, n)$ with the condition that $\mathcal{S}\left(\mathcal{C}_{\tau(j-1)}\right) \geq \mathcal{S}\left(\mathcal{C}_{\tau(j)}\right)$ for $j = 2, 3, \ldots, n$ and Ω is set of all CIFNs.

Theorem 8.2.20. The aggregated value acquired on utilizing GCIFCOWG operator remains CIFN and is given as

$$
\begin{aligned}
&\mathrm{GCIFCOWG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
&= \begin{cases}
\left(\left(\begin{array}{l} t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t(\zeta_{\tau(j)}) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t\left(w_{\zeta_{\tau(j)}}\right) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\end{array}\right), \left(\begin{array}{l} s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s(\vartheta_{\tau(j)}) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s\left(w_{\vartheta_{\tau(j)}}\right) \cdot t(\delta_j)}{t(\delta'_j)}\right)\right)\end{array}\right)\right) & ; \delta_j, \delta'_j \in (0,1) \\
\left(\left(\begin{array}{l} t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t\left(\zeta_{\tau(j)}\right) \cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t\left(w_{\zeta_{\tau(j)}}\right) \cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\end{array}\right), \left(\begin{array}{l} s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s(\vartheta_{\tau(j)}) \cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\left(\xi_j \cdot \dfrac{s\left(w_{\vartheta_{\tau(j)}}\right) \cdot t\left(\frac{1}{\delta_j}\right)}{t(\delta'_j)}\right)\right)\end{array}\right)\right) & ; \delta_j \geq 1, \delta'_j \in (0,1) \\
\left(\left(\begin{array}{l} t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t(\zeta_{\tau(j)}) \cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t\left(w_{\zeta_{\tau(j)}}\right) \cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\end{array}\right), \left(\begin{array}{l} s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s(\vartheta_{\tau(j)}) \cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s\left(w_{\vartheta_{\tau(j)}}\right) \cdot t(\delta_j)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\end{array}\right)\right) & ; \delta_j \in (0,1), \delta'_j > 1 \\
\left(\left(\begin{array}{l} t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t(\zeta_{\tau(j)}) \cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot t\left(w_{\zeta_{\tau(j)}}\right) \cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\end{array}\right), \left(\begin{array}{l} s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s(\vartheta_{\tau(j)}) \cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{n}\left(\dfrac{\xi_j \cdot s\left(w_{\vartheta_{\tau(j)}}\right) \cdot t\left(\frac{1}{\delta_j}\right)}{t\left(\frac{1}{\delta'_j}\right)}\right)\right)\end{array}\right)\right) & ; \delta_j \geq 1, \delta'_j > 1
\end{cases}
\end{aligned}
\tag{8.12}
$$

Proof. It is omitted here because it is same as the Theorem 8.2.17. □

Furthermore, it is analyzed that GCIFCWG and GCIFCOWG operators also satisfy the properties of idempotency, monotonicity and boundedness.

8.3 Multi-criteria decision-making approach

This section presents an MCDM approach and the method for determining the weights corresponding to criteria objectively under CIFS environment.

8.3.1 Description of MCDM problem

The general description of MCDM problem is given in Section 2.5 of Chapter 2. The alternatives are assessed by a collection of 'k' experts $\mathcal{E} = \left\{\mathcal{E}^{(1)}, \mathcal{E}^{(2)} \ldots, \mathcal{E}^{(k)}\right\}$ who gave their assessment results as CIFNs, which are represented as: $\mathcal{C}_{uv}^{(z)} = \left(\left(\zeta_{uv}^{(z)}, w_{\zeta_{uv}}^{(z)}\right), \left(\vartheta_{uv}^{(z)}, w_{\vartheta_{uv}}^{(z)}\right)\right)$ where $z = 1, 2, \ldots, k$; $u = 1, 2, \ldots, m$; $v = 1, 2, \ldots n$; $\zeta_{uv}^{(z)}, \vartheta_{uv}^{(z)}, \zeta_{uv}^{(z)} + \vartheta_{uv}^{(z)} \in [0, 1]$ and $w_{\zeta_{uv}}^{(z)}, w_{\vartheta_{uv}}^{(z)} w_{\zeta_{uv}}^{(z)} + w_{\vartheta_{uv}}^{(z)} \in [0, 1]$. The weight vector associated with the set of experts is given as: $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_k)^T$ such that $\kappa_z > 0$ and $\sum\limits_{z=1}^{k} \kappa_z = 1$. This information related to all alternatives for the different criteria, given by 'k' experts, may be expressed as CIF matrices given as follows:

$$\mathcal{M}^{(z)} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{C}_{11}^{(z)} & \mathcal{C}_{12}^{(z)} & \ldots & \mathcal{C}_{1n}^{(z)} \\ \mathcal{C}_{21}^{(z)} & \mathcal{C}_{22}^{(z)} & \ldots & \mathcal{C}_{2n}^{(z)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{m1}^{(z)} & \mathcal{C}_{m2}^{(z)} & \ldots & \mathcal{C}_{mn}^{(z)} \end{pmatrix} \end{array} \tag{8.13}$$

On the other hand, the exponent and logarithmic base indices are denoted by δ_{uv} and δ'_{uv} respectively where $\delta'_{uv} \neq 1$. These indices are summarized in matrices $\wedge = \left(\delta_{uv}\right)_{m\times n}$ and $\wedge' = \left(\delta'_{uv}\right)_{m\times n}$ as:

$$\wedge = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \delta_{11} & \delta_{12} & \ldots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \ldots & \delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{m1} & \delta_{m2} & \ldots & \delta_{mn} \end{pmatrix} \end{array} \tag{8.14}$$

and

$$\wedge' = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \delta'_{11} & \delta'_{12} & \ldots & \delta'_{1n} \\ \delta'_{21} & \delta'_{22} & \ldots & \delta'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta'_{m1} & \delta'_{m2} & \ldots & \delta'_{mn} \end{pmatrix} \end{array} \tag{8.15}$$

8.3.2 Aggregation of decision matrices $\mathcal{M}^{(z)}$ into one matrix $\mathcal{M}$

In order to aggregate all the individual CIF decision matrices $\mathcal{M}^{(z)} = \left(\mathcal{C}_{uv}^{(z)}\right)_{m\times n}$ into collective one $\mathcal{M} = \left(\mathcal{C}_{uv}\right)_{m\times n}$, where $\mathcal{C}_{uv} = \Big(\big(\zeta_{uv}, w_{\zeta_{uv}}\big), \big(\vartheta_{uv}, w_{\vartheta_{uv}}\big)\Big)$, utilize either GCIF-COWA operator i.e.,

$$\mathcal{C}_{uv} = \text{GCIFCOWA}\left(\mathcal{C}_{uv}^{(1)}, \mathcal{C}_{uv}^{(2)}, \ldots, \mathcal{C}_{uv}^{(k)}\right) \tag{8.16}$$

or GCIFCOWG operator i.e.,

$$\mathcal{C}_{uv} = \text{GCIFCOWG}\left(\mathcal{C}_{uv}^{(1)}, \mathcal{C}_{uv}^{(2)}, \ldots, \mathcal{C}_{uv}^{(k)}\right) \tag{8.17}$$

8.3.3 Determination of weights corresponding to criteria

In MCDM problems, the criteria weights play a significant part in the evaluation of alternatives. Therefore, the procedure of choosing and determining the criteria weights is very important. Motivated by it, we propose a method for determining weights corresponding to criteria for the case when the partial information regarding the weights is given.

In order to determine the partially known weights objectively, firstly we apply the exponential of logarithmic operation, as given in Eq. (8.3), on each CIFN $\mathcal{C}_{uv}$ and then, obtain the score matrix $\mathscr{S} = (\mathcal{S}_{uv})_{m\times n}$ given as:

$$\mathscr{S} = \begin{pmatrix} \mathcal{S}\left(\delta_{11}^{\log_{\delta'_{11}}(\mathcal{C}_{11})}\right) & \mathcal{S}\left(\delta_{12}^{\log_{\delta'_{12}}(\mathcal{C}_{12})}\right) & \cdots & \mathcal{S}\left(\delta_{1n}^{\log_{\delta'_{1n}}(\mathcal{C}_{1n})}\right) \\ \mathcal{S}\left(\delta_{21}^{\log_{\delta'_{21}}(\mathcal{C}_{21})}\right) & \mathcal{S}\left(\delta_{22}^{\log_{\delta'_{22}}(\mathcal{C}_{22})}\right) & \cdots & \mathcal{S}\left(\delta_{2n}^{\log_{\delta'_{2n}}(\mathcal{C}_{2n})}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{S}\left(\delta_{m1}^{\log_{\delta'_{m1}}(\mathcal{C}_{m1})}\right) & \mathcal{S}\left(\delta_{m2}^{\log_{\delta'_{m2}}(\mathcal{C}_{m2})}\right) & \cdots & \mathcal{S}\left(\delta_{mn}^{\log_{\delta'_{mn}}(\mathcal{C}_{mn})}\right) \end{pmatrix} \tag{8.18}$$

where $\mathcal{S}_{uv} = \mathcal{S}\left(\delta_{uv}^{\log_{\delta'_{uv}}(\mathcal{C}_{uv})}\right)$ is calculated using Eq. (6.1).

Now, based on the entries of score matrix, compute the weighted sum of scores of every alternative $\mathcal{V}_u$ $(u = 1, 2, \ldots, m)$, called as suitability function $Q(\mathcal{V}_u)$, given as

$Q(\mathcal{V}_u) = \sum\limits_{v=1}^{n} (\xi_v \mathcal{S}_{uv})$. Now, in order to determine the optimal weight vector, we construct a mathematical model given as follows:

$$\begin{aligned} &\max \quad \sum_{u=1}^{m} Q(\mathcal{V}_u) \\ &\text{subject to} \quad \xi_v \in \triangle; \\ &\xi_v \geq 0 \quad ; \quad \sum_{v=1}^{n} \xi_v = 1. \end{aligned} \tag{8.19}$$

where $\triangle$ is the set containing partial information about the weights associated with criteria. By solving the mathematical model, given in Eq. (8.19), we can obtain the optimal weight vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$.

8.3.4 Aggregation of rating values corresponding to criteria and ranking of alternatives

Aggregate the preferences $\mathcal{C}_{uv}$ into collective one $\mathcal{C}_u = \left((\zeta_u, w_{\zeta_u}), (\vartheta_u, w_{\vartheta_u})\right)$ of alternatives $\mathcal{V}_u$ $(u = 1, 2, \ldots, m)$ either by utilizing proposed GCIFCWA operator i.e.,

$$\begin{aligned} \mathcal{C}_u &= \text{GCIFCWA}(\mathcal{C}_{u1}, \mathcal{C}_{u2}, \ldots, \mathcal{C}_{un}) \\ &= \begin{cases} \left(\left(\begin{array}{l} s^{-1}\left(\sum\limits_{v=1}^{n}\xi_v s\left(t^{-1}\left(\frac{t(\zeta_{uv})\cdot t(\delta_{uv})}{t(\delta'_{uv})}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{v=1}^{n}\xi_v s\left(t^{-1}\left(\frac{t(w_{\zeta_{uv}})\cdot t(\delta_{uv})}{t(\delta'_{uv})}\right)\right)\right) \end{array}\right), \left(\begin{array}{l} t^{-1}\left(\sum\limits_{v=1}^{n}\xi_v t\left(s^{-1}\left(\frac{s(\vartheta_{uv})\cdot t(\delta_{uv})}{t(\delta'_{uv})}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{v=1}^{n}\xi_v t\left(s^{-1}\left(\frac{s(w_{\vartheta_{uv}})\cdot t(\delta_{uv})}{t(\delta'_{uv})}\right)\right)\right) \end{array}\right)\right) & ; \delta_{uv}, \delta'_{uv} \in (0,1) \\ \left(\left(\begin{array}{l} s^{-1}\left(\sum\limits_{v=1}^{n}\xi_v s\left(t^{-1}\left(\frac{t(\zeta_{uv})\cdot t\left(\frac{1}{\delta_{uv}}\right)}{t(\delta'_{uv})}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{v=1}^{n}\xi_v s\left(t^{-1}\left(\frac{t(w_{\zeta_{uv}})\cdot t\left(\frac{1}{\delta_{uv}}\right)}{t(\delta'_{uv})}\right)\right)\right) \end{array}\right), \left(\begin{array}{l} t^{-1}\left(\sum\limits_{v=1}^{n}\xi_v t\left(s^{-1}\left(\frac{s(\vartheta_{uv})\cdot t\left(\frac{1}{\delta_{uv}}\right)}{t(\delta'_{uv})}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{v=1}^{n}\xi_v t\left(s^{-1}\left(\frac{s(w_{\vartheta_{uv}})\cdot t\left(\frac{1}{\delta_{uv}}\right)}{t(\delta'_{uv})}\right)\right)\right) \end{array}\right)\right) & ; \delta_{uv} \geq 1, \delta'_{uv} \in (0,1) \\ \left(\left(\begin{array}{l} s^{-1}\left(\sum\limits_{v=1}^{n}\xi_v s\left(t^{-1}\left(\frac{t(\zeta_{uv})\cdot t(\delta_{uv})}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{v=1}^{n}\xi_v s\left(t^{-1}\left(\frac{t(w_{\zeta_{uv}})\cdot t(\delta_{uv})}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right)\right) \end{array}\right), \left(\begin{array}{l} t^{-1}\left(\sum\limits_{v=1}^{n}\xi_v t\left(s^{-1}\left(\frac{s(\vartheta_{uv})\cdot t(\delta_{uv})}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{v=1}^{n}\xi_v t\left(s^{-1}\left(\frac{s(w_{\vartheta_{uv}})\cdot t(\delta_{uv})}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right)\right) \end{array}\right)\right) & ; \delta_{uv} \in (0,1), \delta'_{uv} > 1 \\ \left(\left(\begin{array}{l} s^{-1}\left(\sum\limits_{v=1}^{n}\xi_v s\left(t^{-1}\left(\frac{t(\zeta_{uv})\cdot t\left(\frac{1}{\delta_{uv}}\right)}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right)\right), \\ s^{-1}\left(\sum\limits_{v=1}^{n}\xi_v s\left(t^{-1}\left(\frac{t(w_{\zeta_{uv}})\cdot t\left(\frac{1}{\delta_{uv}}\right)}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right)\right) \end{array}\right), \left(\begin{array}{l} t^{-1}\left(\sum\limits_{v=1}^{n}\xi_v t\left(s^{-1}\left(\frac{s(\vartheta_{uv})\cdot t\left(\frac{1}{\delta_{uv}}\right)}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right)\right), \\ t^{-1}\left(\sum\limits_{v=1}^{n}\xi_v t\left(s^{-1}\left(\frac{s(w_{\vartheta_{uv}})\cdot t\left(\frac{1}{\delta_{uv}}\right)}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right)\right) \end{array}\right)\right) & ; \delta_{uv} \geq 1, \delta'_{uv} > 1 \end{cases} \end{aligned} \tag{8.20}$$

or by using GCIFCWG operator i.e.,

$$
\begin{aligned}
\mathcal{C}_u &= \text{GCIFCWG}(\mathcal{C}_{u1}, \mathcal{C}_{u2}, \ldots, \mathcal{C}_{un}) \\
&= \begin{cases}
\left(\begin{pmatrix} t^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot t(\zeta_{uv}) \cdot t(\delta_{uv})}{t(\delta'_{uv})}\right)\right), \\ t^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot t\left(w_{\zeta_{uv}}\right) \cdot t(\delta_{uv})}{t(\delta'_{uv})}\right)\right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot s(\vartheta_{uv}) \cdot t(\delta_{uv})}{t(\delta'_{uv})}\right)\right), \\ s^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot s\left(w_{\vartheta_{uv}}\right) \cdot t(\delta_{uv})}{t(\delta'_{uv})}\right)\right) \end{pmatrix} \right) & ; \delta_{uv}, \delta'_{uv} \in (0,1) \\
\left(\begin{pmatrix} t^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot t(\zeta_{uv}) \cdot t\left(\frac{1}{\delta_{uv}}\right)}{t(\delta'_{uv})}\right)\right), \\ t^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot t\left(w_{\zeta_{uv}}\right) \cdot t\left(\frac{1}{\delta_{uv}}\right)}{t(\delta'_{uv})}\right)\right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot s(\vartheta_{uv}) \cdot t\left(\frac{1}{\delta_{uv}}\right)}{t(\delta'_{uv})}\right)\right), \\ s^{-1}\left(\sum\limits_{v=1}^{n}\left(\xi_v \cdot \frac{s\left(w_{\vartheta_{uv}}\right) \cdot t\left(\frac{1}{\delta_{uv}}\right)}{t(\delta'_{uv})}\right)\right) \end{pmatrix} \right) & ; \delta_{uv} \geq 1, \delta'_{uv} \in (0,1) \\
\left(\begin{pmatrix} t^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot t(\zeta_{uv}) \cdot t(\delta_{uv})}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right), \\ t^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot t\left(w_{\zeta_{uv}}\right) \cdot t(\delta_{uv})}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot s(\vartheta_{uv}) \cdot t(\delta_{uv})}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right), \\ s^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot s\left(w_{\vartheta_{uv}}\right) \cdot t(\delta_{uv})}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right) \end{pmatrix} \right) & ; \delta_{uv} \in (0,1), \delta'_{uv} > 1 \\
\left(\begin{pmatrix} t^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot t(\zeta_{uv}) \cdot t\left(\frac{1}{\delta_{uv}}\right)}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right), \\ t^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot t\left(w_{\zeta_{uv}}\right) \cdot t\left(\frac{1}{\delta_{uv}}\right)}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot s(\vartheta_{uv}) \cdot t\left(\frac{1}{\delta_{uv}}\right)}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right), \\ s^{-1}\left(\sum\limits_{v=1}^{n}\left(\frac{\xi_v \cdot s\left(w_{\vartheta_{uv}}\right) \cdot t\left(\frac{1}{\delta_{uv}}\right)}{t\left(\frac{1}{\delta'_{uv}}\right)}\right)\right) \end{pmatrix} \right) & ; \delta_{uv} \geq 1, \delta'_{uv} > 1
\end{cases}
\end{aligned} \tag{8.21}
$$

Further, calculate score values of CIFNs $\mathcal{C}_u$ $(u = 1, 2, \ldots, m)$ using the Eq. (8.22) given as:

$$\mathcal{S}(\mathcal{C}_u) = \zeta_u - \vartheta_u + w_{\zeta_u} - w_{\vartheta_u}. \tag{8.22}$$

However, if any two of the score values of these aggregated numbers are equal then, compute their corresponding accuracy values using the Eq. (8.23) stated as:

$$\mathcal{S}(\mathcal{C}_u) = \zeta_u + \vartheta_u + w_{\zeta_u} + w_{\vartheta_u}. \tag{8.23}$$

Finally, obtain the ordering position of alternatives $\mathcal{V}_u$ using Definition 6.2.1.

In a nutshell, we summarize the above explained procedure of solving MCDM problems in the following steps:

Step 1: Collect the preferences, given by 'k' experts, as CIF matrices $\mathcal{M}^{(z)} = \left(\mathcal{C}_{uv}^{(z)}\right)_{m \times n}$ as given in Eq. (8.13) and formulate the matrices $\wedge$ and $\wedge'$ as stated in Eqs. (8.14) and (8.15) respectively.

Step 2: Aggregate the decision matrices $\mathcal{M}^{(z)}$ into one matrix $\mathcal{M} = (\mathcal{C}_{uv})_{m \times n}$ by utilizing either Eq. (8.16) or Eq. (8.17).

Step 3: Convert the matrix $\mathcal{M}$ into its equivalent score matrix $\mathscr{S} = \mathcal{S}\left(\delta_{uv}^{\log_{\delta'_{uv}}(\mathcal{C}_{uv})}\right)$ as stated in Eq. (8.18).

Step 4: Obtain the criteria weights by applying the mathematical model given in Eq. (8.19).

Step 5: Aggregate the preferences $\mathcal{C}_{uv}$ into collective one $\mathcal{C}_u$ of alternatives $\mathcal{V}_u$ by utilizing either Eq. (8.20) or Eq. (8.21).

Step 6: Calculate score and accuracy values of CIFNs $\mathcal{C}_u$ by utilizing Eqs. (8.22) and (8.23) and hence, rank the alternatives using Definition 6.2.1.

8.4 Illustrative example

This section demonstrates the functionality of the presented MCDM method via a case whose results are further compared with several prevailing studies. The description of the problem is as follows:

State Bank of India (SBI) is a prominent financial services providing government-owned Indian bank, having its head office in Mumbai, Maharashtra. SBI invited tenders for purchasing of petrol and diesel commercial cars (only air conditioned) for the use of SBI local head office New Delhi and its branches in Delhi. The bidder should be recognized by the department of India, government of India/state government/any other reputed public institution and must have experience in supplying vehicles/cars for the last two years to any central/state government organization. Out of the applicants, SBI selected one travel agency for the purchasing of cars. The selected travel agency provides information to SBI regarding five models of cars $\mathcal{V}_u$ $(u = 1, 2, \ldots, 5)$ along with their different manufacturing dates. The goal of the SBI is to find out the most optimal car model and its production date simultaneously. For this, the SBI consults three experts $\mathcal{E}^{(z)}$ $(z = 1, 2, 3)$, who assess the available alternatives $\mathcal{V}_u$ on the basis of four criteria namely $\mathfrak{B}_1$: Reliability, $\mathfrak{B}_2$: Maximum speed, $\mathfrak{B}_3$: Durability and $\mathfrak{B}_4$: Maximum payload. Obviously, the changes in the assembling date for similar model of cars will influence the criteria. Therefore, this problem has two dimensions which are: model of cars and the corresponding manufacturing date. In light of this, the experts provide their assessment values as CIFNs because the CIF model handles two-dimensional information simultaneously. The rating values of the $\mathcal{E}^{(1)}$ for $\mathcal{V}_1$ at $\mathfrak{B}_1$ are given as $\Big((0.6, 0.3), (0.1, 0.2)\Big)$ which describes that the expert $\mathcal{E}^{(1)}$ is 60% agreed with the suitability of $\mathcal{V}_1$ at $\mathfrak{B}_1$ and 10% disagrees. The phase term that represents the production date of cars is given as: $\mathcal{E}^{(1)}$ is 30% satisfied with manufacturing date at $\mathfrak{B}_1$ and 20% is dissatisfied. In the similar manner, all data of Tables 8.1, 8.2 and 8.3 can be interpreted. The weight vector corresponding to three experts is $\kappa = (0.40, 0.25, 0.35)^T$.

The main procedure steps in order to obtain the most desirable alternative(s), using proposed MCDM method described in the above section, are summarized as follows:

Step 1: The rating values of every alternative, given by three experts, are tabulated in Tables 8.1, 8.2 and 8.3. Furthermore, the matrices $\wedge = (\delta_{uv})_{m\times n}$ and $\wedge' = (\delta'_{uv})_{m\times n}$ are summarized as:

$$\wedge = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \mathcal{V}_3 \\ \mathcal{V}_4 \\ \mathcal{V}_5 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \mathfrak{B}_3 & \mathfrak{B}_4 \end{array} \\ \begin{pmatrix} 0.40 & 0.30 & 0.20 & 0.70 \\ 0.50 & 0.20 & 0.70 & 0.90 \\ 0.20 & 0.30 & 0.10 & 0.60 \\ 0.40 & 0.70 & 0.20 & 0.80 \\ 0.20 & 0.40 & 0.80 & 0.10 \end{pmatrix} \end{array} \quad \text{and} \quad \wedge' = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \mathcal{V}_3 \\ \mathcal{V}_4 \\ \mathcal{V}_5 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \mathfrak{B}_3 & \mathfrak{B}_4 \end{array} \\ \begin{pmatrix} 0.25 & 0.40 & 0.80 & 0.65 \\ 0.26 & 0.27 & 0.50 & 0.40 \\ 0.60 & 0.80 & 0.10 & 0.12 \\ 0.30 & 0.35 & 0.40 & 0.70 \\ 0.20 & 0.10 & 0.50 & 0.70 \end{pmatrix} \end{array}$$

Step 2: WLOG, by taking generator $t(a) = -\log(a)$, the Eq. (8.16) reduces to Eq. (8.24) given as:

$$\begin{aligned} \mathcal{C}_{uv} &= \text{GCIFCOWA}\left(\mathcal{C}_{uv}^{(1)}, \mathcal{C}_{uv}^{(2)}, \ldots, \mathcal{C}_{uv}^{(k)}\right) \qquad (8.24) \\ &= \left(\begin{pmatrix} 1 - \prod_{z=1}^{k} \left(1 - \delta_{uv}^{\log_{\delta'_{uv}}\left(\zeta_{uv}^{(\tau(z))}\right)}\right)^{\kappa_z}, \\ 1 - \prod_{z=1}^{k} \left(1 - \delta_{uv}^{\log_{\delta'_{uv}}\left(w_{\zeta_{uv}}^{(\tau(z))}\right)}\right)^{\kappa_z} \end{pmatrix}, \begin{pmatrix} \prod_{z=1}^{k} \left(1 - \delta_{uv}^{\log_{\delta'_{uv}}\left(1-\vartheta_{uv}^{(\tau(z))}\right)}\right)^{\kappa_z}, \\ \prod_{z=1}^{k} \left(1 - \delta_{uv}^{\log_{\delta'_{uv}}\left(1-w_{\vartheta_{uv}}^{(\tau(z))}\right)}\right)^{\kappa_z} \end{pmatrix} \right) \end{aligned}$$

Now, utilize Eq. (8.24) in order to accumulate the CIFNs $\mathcal{C}_{uv}^{(z)}$ into collective one $\mathcal{C}_{uv}$. The collective decision-matrix is summarized in Table 8.4.

Step 3: The score matrix is obtained as:

$$\mathscr{S} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \mathcal{V}_3 \\ \mathcal{V}_4 \\ \mathcal{V}_5 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathfrak{B}_1 & \quad\quad\mathfrak{B}_2 & \quad\quad\mathfrak{B}_3 & \quad\quad\mathfrak{B}_4 \end{array} \\ \begin{pmatrix} 0.9762 & 0.4855 & -1.4088 & 0.9188 \\ 0.8650 & 0.0282 & 0.9420 & 1.0539 \\ 0.5323 & -0.5565 & -0.5166 & 1.4594 \\ 0.9862 & 1.2751 & 0.8768 & 1.0871 \\ 1.0672 & 1.3941 & 1.2532 & -1.5981 \end{pmatrix} \end{array}$$

Step 4: Consider that the partial information regarding the weights associated with $\mathfrak{B}_v$ $(v = 1, 2, 3, 4)$ is: $\triangle = \Big\{0.10 \leq \xi_1 \leq 0.20, 0.22 \leq \xi_2 \leq 0.30, 0.18 \leq \xi_3 \leq 0.30, 0.35 \leq \xi_4 \leq 0.50, \xi_1 - \xi_2 \leq 0, \xi_1 + \xi_3 - \xi_4 \leq 0\Big\}$. Then, utilizing Eq. (8.19), a linear programming problem is formulated as:

$$\begin{aligned} \max \quad & 4.4269\xi_1 + 2.6264\xi_2 + 1.1466\xi_3 + 2.9212\xi_4 \\ \text{subject to} \quad & 0.10 \leq \xi_1 \leq 0.20, \quad 0.22 \leq \xi_2 \leq 0.30, \\ & 0.18 \leq \xi_3 \leq 0.30, \quad 0.35 \leq \xi_4 \leq 0.50, \\ & \xi_1 - \xi_2 \leq 0, \quad \xi_1 + \xi_3 - \xi_4 \leq 0, \\ & \sum_{v=1}^{4} \xi_v = 1 \end{aligned} \tag{8.25}$$

By solving this linear programming problem, we obtain $\xi = (0.20, 0.22, 0.18, 0.40)^T$.

Step 5: WLOG, by taking generator $t(a) = -\log(a)$, the Eq. (8.20) reduces to Eq. (8.26) stated as:

$$\begin{aligned} \mathcal{C}_u &= \text{GCIFCWA}(\mathcal{C}_{u1}, \mathcal{C}_{u2}, \ldots, \mathcal{C}_{un}) \qquad (8.26) \\ &= \left(\begin{pmatrix} 1 - \prod_{v=1}^{n} \left(1 - \delta_{uv}^{\log_{\delta'_{uv}}(\zeta_{uv})}\right)^{\xi_v}, \\ 1 - \prod_{v=1}^{n} \left(1 - \delta_{uv}^{\log_{\delta'_{uv}}(w_{\zeta_{uv}})}\right)^{\xi_v} \end{pmatrix}, \begin{pmatrix} \prod_{v=1}^{n} \left(1 - \delta_{uv}^{\log_{\delta'_{uv}}(1-\vartheta_{uv})}\right)^{\xi_v}, \\ \prod_{v=1}^{n} \left(1 - \delta_{uv}^{\log_{\delta'_{uv}}(1-w_{\vartheta_{uv}})}\right)^{\xi_v} \end{pmatrix} \right) \end{aligned}$$

Now, utilize Eq. (8.26) in order to aggregate the CIFNs $\mathcal{C}_{uv}$ into $\mathcal{C}_u$. The results corresponding to them are obtained as:

$$\mathcal{C}_1 = \big((0.7385, 0.6758), (0.1469, 0.1980)\big), \mathcal{C}_2 = \big((0.6704, 0.5978), (0.1509, 0.1645)\big),$$
$$\mathcal{C}_3 = \big((0.7672, 0.6167), (0.1431, 0.1609)\big), \mathcal{C}_4 = \big((0.9049, 0.8536), (0.0722, 0.1030)\big),$$
$$\mathcal{C}_5 = \big((0.8251, 0.7630), (0.1211, 0.1584)\big).$$

Step 6: The score values of aggregated CIFNs $\mathcal{C}_u$ are: $\mathcal{S}(\mathcal{C}_1) = 1.0694$, $\mathcal{S}(\mathcal{C}_2) = 0.9529$, $\mathcal{S}(\mathcal{C}_3) = 1.0798$, $\mathcal{S}(\mathcal{C}_4) = 1.5833$ and $\mathcal{S}(\mathcal{C}_5) = 1.3085$. Therefore, the corresponding ordering is $\mathcal{V}_4 \succ \mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2$. Hence, the most preferable alternative is $\mathcal{V}_4$.

Further, the impact of the different AOs, utilized for aggregating the estimation values of experts and criteria in Step 2 and Step 5 respectively, on the ranking position of alternatives, is studied and tabulated in Table 8.5. The initial column of this table exhibits the AO utilized in order to accumulate the individual rating values of three decision-makers into a single CIFN whereas, the second column depicts AO applied for combining the criteria values, acquired in Step 2, into a unique one. An optimistic approach is followed by taking compensative ordered weighted averaging operator for accumulation of the rating estimations of three experts in the first two rows of this Table. On the other hand, in the next two rows of the Table 8.5, the preferences of three decision-makers are accumulated with pessimistic attitude by opting compensative ordered weighted geometric operator. The tabulated values depict that the final score values of alternatives are greater when optimistic approach is followed in Step 2 than the values obtained using pessimistic approach. Further, it is analyzed that on using compensative weighted averaging operator in Step 5 for criteria aggregation, the ranking order of alternative remains same and by utilizing compensative weighted geometric operator in Step 5, the ordering position of the alternatives remains identical irrespective of the operator used in Step 2.

8.5 Comparative studies

In light of showing the superiority of presented method, here the results of developed approach are compared with prevailing CIFS studies [6, 59, 129, 130] as well as IFS studies [27, 44, 46, 47, 67, 72, 83, 156, 201].

8.5.1 With CIFS studies

The presented method results are compared with results acquired using distance measures defined in [6, 129] and averaging operators proposed in [59, 130]. In order to compare the results with distance measures given in [6, 129], we take the the PIA ($\mathcal{V}^+$) as ideal alternative whose preferences are: $\mathcal{V}^+ = \{\mathcal{C}_1^+, \mathcal{C}_2^+, \ldots, \mathcal{C}_n^+\}$ where $\mathcal{C}_v^+ = \big((\max\limits_{1\leq u\leq m}\{\zeta_{uv}\}, \max\limits_{1\leq u\leq m}\{w_{\zeta_{uv}}\}), (\min\limits_{1\leq u\leq m}\{\vartheta_{uv}\}, \min\limits_{1\leq u\leq m}\{w_{\vartheta_{uv}}\})\big)$. Then, the results obtained by utilizing measures defined in [6, 129] and aggregation operators proposed in [59, 130] are summarized as follows:

(i) On utilizing the weighted Euclidean distance measure, defined by Rani and Garg [129], as given in Eq. (3.6), we obtain the measure values as $\mathcal{D}(\mathcal{V}_1, \mathcal{V}^+) = 0.3073$, $\mathcal{D}(\mathcal{V}_2, \mathcal{V}^+) = 0.5206$, $\mathcal{D}(\mathcal{V}_3, \mathcal{V}^+) = 0.3283$, $\mathcal{D}(\mathcal{V}_4, \mathcal{V}^+) = 0.0993$ and $\mathcal{D}(\mathcal{V}_5, \mathcal{V}^+) = 0.4725$. Thus, the ordering of $\mathcal{V}_u$ $(u = 1, 2, \ldots, 5)$ is: $\mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_2$. Hence, $\mathcal{V}_4$ is the most optimal alternative.

(ii) On utilizing the distance measure, defined by Alkouri and Salleh [6], as given in Eq. (2.8), we obtain the measure values as $\mathcal{D}(\mathcal{V}_1, \mathcal{V}^+) = 0.2066$, $\mathcal{D}(\mathcal{V}_2, \mathcal{V}^+) = 0.5092$, $\mathcal{D}(\mathcal{V}_3, \mathcal{V}^+) = 0.2290$, $\mathcal{D}(\mathcal{V}_4, \mathcal{V}^+) = 0.0671$ and $\mathcal{D}(\mathcal{V}_5, \mathcal{V}^+) = 0.3480$. These measure values are obtained by taking $\alpha_1 = \beta_1 = \sigma_1 = \alpha_2 = \beta_2 = \sigma_2 = \frac{1}{3}$. From the measure values, we get that the ordering position of $\mathcal{V}_u$ is: $\mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_2$. Hence, $\mathcal{V}_4$ is the most preferable choice.

(iii) On utilizing the CIF weighted power averaging operator, presented by Rani and Garg [130], we obtain the score values of the alternatives as: $\mathcal{S}(\mathcal{V}_1) = 1.0479$, $\mathcal{S}(\mathcal{V}_2) = -0.0293$, $\mathcal{S}(\mathcal{V}_3) = 1.0349$, $\mathcal{S}(\mathcal{V}_4) = 1.4188$ and $\mathcal{S}(\mathcal{V}_5) = 1.1266$. Hence, their corresponding ranking order becomes: $\mathcal{V}_4 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2$ which gives that $\mathcal{V}_4$ is the required choice.

(iv) By utilizing CIF weighted averaging operator, proposed by Garg and Rani [59], with additive generator $t(a) = -\log(a)$ (For more details, refer to [59]), we have: $\mathcal{S}(\mathcal{V}_1) = 1.0340$, $\mathcal{S}(\mathcal{V}_2) = -0.0890$, $\mathcal{S}(\mathcal{V}_3) = 1.0048$, $\mathcal{S}(\mathcal{V}_4) = 1.4142$ and $\mathcal{S}(\mathcal{V}_5) = 1.0402$. Thus, the corresponding ordering position of alternatives becomes: $\mathcal{V}_4 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2$. Hence, $\mathcal{V}_4$ is the most preferable option.

From the above computed results, we observe that the best and the worst alternatives obtained on utilizing existing distance measures [6, 129] and aggregation operators [59, 130] are same as that of our proposed approach results. However, the ranking order of alternatives $\mathcal{V}_1$, $\mathcal{V}_3$ and $\mathcal{V}_5$ changes. The reason behind this change is that the proposed approach results are obtained based on the exponential of logarithmic compensative operators in which exponential and logarithmic base indices, named as δ_{uv} and δ'_{uv}, are used. On the other hand, there is no role of δ_{uv} and δ'_{uv} during aggregation process in the existing studies [6, 59, 129, 130]. Therefore, while using prevailing approaches [6, 59, 129, 130] on the data set of the proposed example, δ_{uv} and δ'_{uv} are not considered. Moreover, by taking $\delta_{uv} = \delta'_{uv}$ for all p and q, the proposed GCIFCWA and GCIFCOWA operators reduce to existing CIF weighted averaging and CIF ordered weighed averaging operators [59] respectively. Therefore, the presented operators are more generalized than the existing ones.

8.5.2 With IFS environment

In light of comparing the presented approach results with prevailing studies under IFS theory, the phase term corresponding to each CIFN has been set to zero. Then, a comparative analysis is conducted based on different existing aggregation operators [27, 44, 46, 47, 67, 72, 83, 156, 201] under IFS theory and is tabulated in Table 8.6.

From the above calculated results, we analyze that on applying the operators [44, 72] in Step 5 of the presented MCDM method, ranking order of alternatives $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$ coincides with our results whereas the ordering position of alternatives $\mathcal{V}_1$ and $\mathcal{V}_5$ changes. On the other hand, by using the operator, given by Chen and Chang [27], in Step 5 of the presented approach, ordering position of alternatives $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_4$ remains same and $\mathcal{V}_3$, $\mathcal{V}_5$ changes. Furthermore, on utilizing aggregation operators given in [46, 47, 67, 83, 156, 201], the best and the worst alternative remains identical with the developed method results and the position of other alternatives changes. This change in the positioning of the alternatives is due to the fact that the operators used in prevailing studies [27, 44, 46, 47, 67, 72, 83, 156, 201] aggregate real valued MDs and NMDs and tackle with only one dimensional problem. On the other hand, the presented work fuses complex membership and non-membership values and handles more than one dimensional problems. Besides this, the proposed approach aggregates criteria information using GCIFCWA operator in which exponent and logarithmic indices are used whereas, the results tabulated in Table 8.6 are obtained without any contribution of these indices.

In addition to these, some of the prevailing AOs can be obtained from the proposed AOs by taking $\delta_j = \delta'_j \ \forall \ j$; phase terms corresponding to each CIFN equal to zero and different forms of additive generator t, which are summarized as follows:

Additive generator	Reduction of operator			
$t(a) =$	GCIFCWA to	GCIFCOWA to	GCIFCWG to	GCIFCOWG to
$-\log(a)$	IFWA_ω[179]	IFOWA_ω[179]	IFWG_ω[185]	IFOWG_ω[185]
$\log\left(\frac{2-a}{a}\right)$	$\text{IFWA}^{\varepsilon}_\omega$[156]	$\text{IFOWA}^{\varepsilon}_\omega$[156]	$\text{IFWG}^{\varepsilon}_\omega$[161]	$\text{IFOWG}^{\varepsilon}_\omega$[161]
$\log\left(\frac{\gamma+(1-\gamma)a}{a}\right)$	IFHWA_ω[83]	IFHOWA_ω[83]	-	-

In the above tabular representation $0 < a \leq 1$ and $\gamma \in (0, \infty)$ are real numbers. The above tabulated representation depicts that the AOs presented in the present chapter are more generalized than the existing ones. Moreover, our proposed approach can handle DM problems under IFS environment also by setting phase terms equal to zero. Therefore, proposed MCDM method is more generalized as it can handle DM problems under IFS as well as CIFS theories.

8.5.3 Further Discussion

In addition to the above comparative studies, we give some characteristic comparison of our proposed MCDM approach and the DM methods proposed in [27, 44, 46, 47, 59, 72, 83, 130, 156, 169, 201] which is tabulated in Table 8.7. In this table, the symbol '✓' describes that the associated DM approach uses generalized operators based on t-norm and co-norm, handles group DM problems, determines the criteria weights objectively, handles optimistic as well as pessimistic behavior of decision-maker, tackles with time-periodic problems and can represent two-dimensional information simultaneously whereas the symbol '×' means that the corresponding method fails. The values, tabulated in Table 8.7 depict that the operators presented in [59, 169] and our proposed AOs are based on t-norms and co-norms. Also, the operators proposed in [83, 156, 161, 179, 185] are special cases of proposed operators and therefore, our presented work is more generalized and can be utilized to solve

DM problems under FS, IFS and CFS environment also. The decision-maker may choose the desired norm during aggregation process in accordance with his/her attitude and situation. Further, the MCDM approaches proposed in [130] and the presented method can handle group DM problems. Moreover, by taking $k = 1$, our developed approach can handle single decision-maker problems as well. Also, the MCDM method presented in [44, 72] and our proposed approach determines the weights corresponding to criteria objectively. Since the criteria weights play an important part in the process of making decisions therefore, the method of determining and choosing weights affects the ranking results directly. The random choice of criteria weights may lead us to wrong decision results. In the DM methods proposed in [27, 46, 47, 59, 83, 130, 156, 169, 201] weights are chosen subjectively. Therefore, our presented approach is more reliable as compared to the approaches of [27, 46, 47, 59, 83, 130, 156, 169, 201]. In addition to these, the DM approaches proposed in [27, 44, 47, 72] are based on the geometric operators only whereas the methods defined in [59, 83, 156, 169] are based on averaging operators only. But, our proposed method and the approach given in [130] provide the choice to decision-maker to utilize averaging or geometric operator in accordance with the DM problem and their optimistic or pessimistic attitude towards the problem. Besides this, the MCDM methods proposed in [27, 44, 46, 47, 72, 83, 156, 169, 201] deal with real membership and non-membership degrees, which fail to handle time-periodic problems and cannot represent more than one-dimensional information in one set. On the other hand, the proposed method can handle complex problems which involve periodicity and can aggregate two dimensional data together in one set. This discussion leads to the conclusion that the presented approach can handle time-periodic complex problems more efficiently which are either difficult or impossible to be solved using existing theories [27, 44, 46, 47, 72, 83, 156, 169, 201].

8.5.4 The influences of exponential and logarithmic operations and selection of δ in practice

In this section, we discuss the influence of exponential and logarithmic operations on CIFNs and the choice of δ practically. As we have already pointed out that for any CIFN

$\mathcal{C}$, $\delta^{\mathcal{C}}$ and $\log_{\delta}(\mathcal{C})$ are also CIFNs such that $\delta^{\mathcal{C}} = \mathcal{C}$ and $\log_{\delta}(\mathcal{C}) = \mathcal{C}$ do not hold always. This gives that there is difference between $\delta^{\mathcal{C}}$ and $\mathcal{C}$ and similarly between $\log_{\delta}(\mathcal{C})$ and $\mathcal{C}$. Obviously, this difference varies in accordance with variation in the value of δ. Therefore, it is essential to choose the δ wisely while applying these operations to practical problems.

Let $\mathcal{C} = \Big((\zeta, w_{\zeta}), (\vartheta, w_{\vartheta})\Big)$ be a CIFN and $\delta \in (0,1)$ be a real number. Then, we have

$$\delta^{\mathcal{C}} = \left(\begin{pmatrix} t^{-1}\left((1-\zeta)t(\delta)\right), \\ t^{-1}\left((1-w_{\zeta})\,t(\delta)\right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\vartheta t(\delta)\right), \\ s^{-1}\left(w_{\vartheta} t(\delta)\right) \end{pmatrix} \right)$$

$$\text{and} \quad \log_{\delta}(\mathcal{C}) = \left(\begin{pmatrix} 1 - \dfrac{t(\zeta)}{t(\delta)}, \\ 1 - \dfrac{t\left(w_{\zeta}\right)}{t(\delta)} \end{pmatrix}, \begin{pmatrix} \dfrac{s(\vartheta)}{t(\delta)}, \\ \dfrac{s\left(w_{\vartheta}\right)}{t(\delta)} \end{pmatrix} \right).$$

From the above definings of $\delta^{\mathcal{C}}$ and $\log_{\delta}(\mathcal{C})$, we observe the following points:

(P1) There exists a real number $\delta_1 = t^{-1}\left(\frac{t(\zeta)}{1-\zeta}\right)$ such that

(a) If $\delta = \delta_1$ then, $t^{-1}\left((1-\zeta)t(\delta)\right) = \zeta$ and $1 - \frac{t(\zeta)}{t(\delta)} = \zeta$;

(b) If $\delta < \delta_1$ then, $t^{-1}\left((1-\zeta)t(\delta)\right) < \zeta$ and $1 - \frac{t(\zeta)}{t(\delta)} > \zeta$;

(c) If $\delta > \delta_1$ then, $t^{-1}\left((1-\zeta)t(\delta)\right) > \zeta$ and $1 - \frac{t(\zeta)}{t(\delta)} < \zeta$.

(P2) There exists a real number $\delta_2 = t^{-1}\left(\frac{t\left(w_{\zeta}\right)}{1-w_{\zeta}}\right)$ such that

(a) If $\delta = \delta_2$ then, $t^{-1}\left((1-w_{\zeta})\,t(\delta)\right) = w_{\zeta}$ and $1 - \frac{t\left(w_{\zeta}\right)}{t(\delta)} = w_{\zeta}$;

(b) If $\delta < \delta_2$ then, $t^{-1}\left((1-w_{\zeta})\,t(\delta)\right) < w_{\zeta}$ and $1 - \frac{t\left(w_{\zeta}\right)}{t(\delta)} > w_{\zeta}$;

(c) If $\delta > \delta_2$ then, $t^{-1}\left((1-w_{\zeta})\,t(\delta)\right) > w_{\zeta}$ and $1 - \frac{t\left(w_{\zeta}\right)}{t(\delta)} < w_{\zeta}$.

(P3) There exists a real number $\delta_3 = t^{-1}\left(\frac{s(\vartheta)}{\vartheta}\right)$ such that

(a) If $\delta = \delta_3$ then, $s^{-1}\left(\vartheta t(\delta)\right) = \vartheta$ and $\frac{s(\vartheta)}{t(\delta)} = \vartheta$;

(b) If $\delta < \delta_3$ then, $s^{-1}\left(\vartheta t(\delta)\right) > \vartheta$ and $\frac{s(\vartheta)}{t(\delta)} < \vartheta$;

(c) If $\delta > \delta_3$ then, $s^{-1}\left(\vartheta t(\delta)\right) < \vartheta$ and $\frac{s(\vartheta)}{t(\delta)} > \vartheta$.

(P4) There exists a real number $\delta_4 = t^{-1}\left(\frac{s(w_{\vartheta})}{w_{\vartheta}}\right)$ such that

(a) If $\delta = \delta_4$ then, $s^{-1}(w_\vartheta t(\delta)) = w_\vartheta$ and $\frac{s(w_\vartheta)}{t(\delta)} = w_\vartheta$;

(b) If $\delta < \delta_4$ then, $s^{-1}(w_\vartheta t(\delta)) > w_\vartheta$ and $\frac{s(w_\vartheta)}{t(\delta)} < w_\vartheta$;

(c) If $\delta > \delta_4$ then, $s^{-1}(w_\vartheta t(\delta)) < w_\vartheta$ and $\frac{s(w_\vartheta)}{t(\delta)} > w_\vartheta$.

(P5) If $\delta < \delta_1 < \delta_2 < \delta_3 < \delta_4$ then, $\delta^{\mathcal{C}} \subset \mathcal{C} \subset \log_\delta(\mathcal{C})$. It gives that the value of CIFN $\mathcal{C}$ will decrease (increase) after applying exponential (logarithmic) operation on it for such δ.

(P6) If $\delta_1 < \delta < \delta_2 < \delta_3 < \delta_4$ then, $t^{-1}((1-\zeta)t(\delta)) > \zeta$, $s^{-1}(\vartheta t(\delta)) > \vartheta$, $1 - \frac{t(\zeta)}{t(\delta)} < \zeta$, $\frac{s(\vartheta)}{t(\delta)} < \vartheta$, $t^{-1}((1-w_\zeta)\,t(\delta)) < w_\zeta$, $s^{-1}(w_\vartheta t(\delta)) > w_\vartheta$, $1 - \frac{t(w_\zeta)}{t(\delta)} > w_\zeta$ and $\frac{s(w_\vartheta)}{t(\delta)} < w_\vartheta$. It implies that the membership values of $\mathcal{C}$ associated with amplitude and phase terms will increase (decrease) and decrease (increase) respectively whereas, the non-membership values associated with amplitude and phase terms will increase (decrease) after applying exponential (logarithmic) operation on it for such δ.

(P7) If $\delta_1 < \delta_2 < \delta < \delta_3 < \delta_4$ then, $t^{-1}((1-\zeta)t(\delta)) > \zeta$, $s^{-1}(\vartheta t(\delta)) > \vartheta$, $1 - \frac{t(\zeta)}{t(\delta)} < \zeta$, $\frac{s(\vartheta)}{t(\delta)} < \vartheta$, $t^{-1}((1-w_\zeta)\,t(\delta)) > w_\zeta$, $s^{-1}(w_\vartheta t(\delta)) > w_\vartheta$, $1 - \frac{t(w_\zeta)}{t(\delta)} < w_\zeta$ and $\frac{s(w_\vartheta)}{t(\delta)} < w_\vartheta$. This gives that the MDs and NMDs of $\mathcal{C}$ associated with amplitude and phase terms will increase (decrease) after applying exponential (logarithmic) operation on it for such δ.

(P8) If $\delta_1 < \delta_2 < \delta_3 < \delta < \delta_4$ then, $t^{-1}((1-\zeta)t(\delta)) > \zeta$, $s^{-1}(\vartheta t(\delta)) < \vartheta$, $1 - \frac{t(\zeta)}{t(\delta)} < \zeta$, $\frac{s(\vartheta)}{t(\delta)} > \vartheta$, $t^{-1}((1-w_\zeta)\,t(\delta)) > w_\zeta$ and $s^{-1}(w_\vartheta t(\delta)) > w_\vartheta$, $1 - \frac{t(w_\zeta)}{t(\delta)} < w_\zeta$ and $\frac{s(w_\vartheta)}{t(\delta)} < w_\vartheta$. It implies that the membership values of $\mathcal{C}$ associated with amplitude and phase terms will increase (decrease) whereas, the non-membership values associated with amplitude and phase terms will decrease (increase) and increase (decrease) respectively after applying exponential (logarithmic) operation on it for such δ.

(P9) If $\delta_1 < \delta_2 < \delta_3 < \delta_4 < \delta$ then, $\log_\delta(\mathcal{C}) \subset \mathcal{C} \subset \delta^{\mathcal{C}}$. It gives that the value of CIFN $\mathcal{C}$ will increase (decrease) after applying exponential (logarithmic) operation on it for such δ.

8.6 Conclusion

The key contribution of this chapter is described as follows:

1) In this chapter, we presented some exponential, logarithmic and compensative exponential of logarithmic operational laws under CIF environment and investigated their properties in detail.

2) Based on the compensative operational laws, some AOs namely GCIFCWA, GCIFCOWA, GCIFCWG and GCIFCOWG are proposed which are more generalized and reduce to existing operators [59, 83, 156, 161, 179, 185] by giving different forms to additive generator t.

3) A group MCDM approach has been presented for solving DM problems in a more efficient way under the CIF environment in which the criteria weights are determined objectively. An example is illustrated in order to justify the application of the presented work in real life. Also, the proposed approach has been validated by comparing the results of the example with existing studies.

4) The impact of the parameters used in exponential and logarithmic operations on CIFNs is discussed in detail. The proposed MCDM method can be efficiently used for solving complex time-periodic DM problems and it can be applied on IFS data as well by setting phase terms equal to zero.

Table 8.1: Preferences given by expert $\mathcal{E}^{(1)}$

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$((0.6, 0.3), (0.1, 0.2))$	$((0.7, 0.5), (0.2, 0.2))$	$((0.3, 0.4), (0.5, 0.3))$	$((0.9, 0.8), (0.1, 0.1))$
$\mathcal{V}_2$	$((0.5, 0.4), (0.3, 0.2))$	$((0.7, 0.3), (0.1, 0.3))$	$((0.5, 0.4), (0.2, 0.2))$	$((0.4, 0.4), (0.2, 0.1))$
$\mathcal{V}_3$	$((0.8, 0.4), (0.2, 0.3))$	$((0.6, 0.3), (0.4, 0.2))$	$((0.9, 0.3), (0.1, 0.1))$	$((0.8, 0.3), (0.2, 0.2))$
$\mathcal{V}_4$	$((0.8, 0.7), (0.1, 0.1))$	$((0.7, 0.8), (0.2, 0.2))$	$((0.9, 0.7), (0.1, 0.3))$	$((0.8, 0.7), (0.1, 0.2))$
$\mathcal{V}_5$	$((0.7, 0.6), (0.2, 0.3))$	$((0.8, 0.7), (0.1, 0.1))$	$((0.5, 0.4), (0.3, 0.3))$	$((0.4, 0.5), (0.3, 0.2))$

Table 8.2: Preferences given by expert $\mathcal{E}^{(2)}$

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$((0.7, 0.8), (0.1, 0.2))$	$((0.4, 0.4), (0.3, 0.4))$	$((0.5, 0.4), (0.2, 0.2))$	$((0.8, 0.7), (0.1, 0.1))$
$\mathcal{V}_2$	$((0.7, 0.6), (0.3, 0.3))$	$((0.5, 0.3), (0.4, 0.4))$	$((0.6, 0.3), (0.1, 0.2))$	$((0.5, 0.6), (0.2, 0.3))$
$\mathcal{V}_3$	$((0.6, 0.4), (0.1, 0.1))$	$((0.3, 0.3), (0.3, 0.5))$	$((0.7, 0.6), (0.1, 0.3))$	$((0.4, 0.3), (0.2, 0.4))$
$\mathcal{V}_4$	$((0.7, 0.8), (0.1, 0.2))$	$((0.9, 0.8), (0.1, 0.1))$	$((0.9, 0.8), (0.1, 0.2))$	$((0.7, 0.6), (0.2, 0.4))$
$\mathcal{V}_5$	$((0.7, 0.5), (0.3, 0.3))$	$((0.6, 0.4), (0.4, 0.4))$	$((0.3, 0.1), (0.4, 0.5))$	$((0.7, 0.6), (0.1, 0.2))$

Table 8.3: Preferences given by expert $\mathcal{E}^{(3)}$

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$((0.5, 0.6), (0.2, 0.4))$	$((0.3, 0.3), (0.2, 0.6))$	$((0.7, 0.4), (0.2, 0.1))$	$((0.5, 0.4), (0.3, 0.4))$
$\mathcal{V}_2$	$((0.3, 0.4), (0.1, 0.3))$	$((0.5, 0.4), (0.4, 0.3))$	$((0.8, 0.7), (0.1, 0.1))$	$((0.6, 0.3), (0.3, 0.2))$
$\mathcal{V}_3$	$((0.7, 0.6), (0.1, 0.2))$	$((0.4, 0.3), (0.4, 0.4))$	$((0.5, 0.3), (0.3, 0.2))$	$((0.4, 0.4), (0.4, 0.2))$
$\mathcal{V}_4$	$((0.9, 0.9), (0.1, 0.1))$	$((0.9, 0.6), (0.1, 0.2))$	$((0.8, 0.8), (0.2, 0.1))$	$((0.9, 0.8), (0.1, 0.1))$
$\mathcal{V}_5$	$((0.6, 0.5), (0.2, 0.4))$	$((0.6, 0.4), (0.2, 0.4))$	$((0.7, 0.4), (0.1, 0.4))$	$((0.4, 0.5), (0.5, 0.4))$

Table 8.4: Aggregated values of experts by using GCIFCOWA operator

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.7239, 0.7490), (0.0863, 0.1775)\big)$	$\big((0.7075, 0.6309), (0.1210, 0.2025)\big)$	$\big((0.1772, 0.0557), (0.6151, 0.4430)\big)$	$\big((0.8350, 0.7389), (0.1143, 0.1271)\big)$
$\mathcal{V}_2$	$\big((0.5415, 0.4898), (0.2042, 0.2711)\big)$	$\big((0.5044, 0.2303), (0.2872, 0.4109)\big)$	$\big((0.6157, 0.4599), (0.1541, 0.1827)\big)$	$\big((0.0338, 0.0282), (0.7505, 0.6724)\big)$
$\mathcal{V}_3$	$\big((0.8939, 0.7826), (0.0454, 0.0711)\big)$	$\big((0.7306, 0.6193), (0.1639, 0.1485)\big)$	$\big((0.2400, 0.0065), (0.6455, 0.7046)\big)$	$\big((0.7713, 0.5613), (0.1311, 0.1414)\big)$
$\mathcal{V}_4$	$\big((0.8253, 0.8323), (0.1000, 0.1275)\big)$	$\big((0.7591, 0.6230), (0.2123, 0.2498)\big)$	$\big((0.9600, 0.9189), (0.0400, 0.0675)\big)$	$\big((0.8527, 0.7596), (0.1068, 0.1636)\big)$
$\mathcal{V}_5$	$\big((0.9546, 0.9319), (0.0284, 0.0454)\big)$	$\big((0.9157, 0.8606), (0.0515, 0.0628)\big)$	$\big((0.6762, 0.4694), (0.1413, 0.2762)\big)$	$\big((0.0428, 0.0217), (0.7313, 0.8279)\big)$

Table 8.5: Influence of the operators used in Step 2 and Step 5 on ranking order of alternatives

Operator utilized in Step 2	Operator utilized in Step 5	Score values of					Ranking order
		$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	
GCIFCOWA	GCIFCWA	1.0694	0.9529	1.0798	1.5833	1.3085	$\mathcal{V}_4 \succ \mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2$
GCIFCOWA	GCIFCWG	-0.7570	0.6238	0.0735	1.0126	-1.2759	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5$
GCIFCOWG	GCIFCWA	0.9231	0.8381	1.0143	1.5387	1.2652	$\mathcal{V}_4 \succ \mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_2$
GCIFCOWG	GCIFCWG	-0.9008	0.5467	-0.0931	0.9888	-1.3232	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5$

Table 8.6: Comparative study $\left(\gamma = 3 \text{ in } [46, 83] \text{ and } \lambda = 0.5 \text{ in } [201]\right)$

Comparison with	Score values					Ranking
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	
Wang and Liu [156]	0.5590	-0.0032	0.5836	0.7608	0.5318	$\mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_2$
He, Chen, Zhau, Liu and Tao [72]	0.4203	-0.2362	0.4013	0.7631	0.0458	$\mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_2$
Huang [83]	0.5505	-0.0293	0.5758	0.7600	0.4969	$\mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_2$
Garg [44]	0.4548	-0.1970	0.4404	0.7659	0.1232	$\mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_2$
Chen and Chang [27]	0.5572	0.0063	0.5881	0.7314	0.5831	$\mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2$
Garg [46]	0.5299	-0.0576	0.5645	0.7275	0.4741	$\mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_2$
Ye [201]	0.4619	-0.1691	0.4873	0.7439	0.1283	$\mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_2$
Garg [47]	0.5397	-0.0315	0.5726	0.7285	0.5161	$\mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_2$
Goyal et al. [67]	0.5745	-0.1681	0.6032	0.8871	-0.1253	$\mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_2$

Table 8.7: The characteristic comparison of different approaches

Method	Generalized operators based on t-norm and co-norm	Handles group decision-making problems	Determination of criteria weights objectively	Handles optimistic as well pessimistic behavior	Ability to handle time-periodic problems	Ability to handle two-dimensional information
Wang and Liu [156]	×	×	×	×	×	×
He et al.[72]	×	×	✓	×	×	×
Huang [83]	×	×	×	×	×	×
Garg [44]	×	×	✓	×	×	×
Chen and Chang [27]	×	×	×	×	×	×
Xia et al. [169]	✓	×	×	×	×	×
Garg [47]	×	×	×	×	×	×
Ye [201]	×	×	×	✓	×	×
Garg [46]	×	×	×	×	×	×
Rani and Garg [130]	×	✓	×	✓	✓	✓
Garg and Rani [59]	✓	×	×	×	✓	✓
The proposed approach	✓	✓	✓	✓	✓	✓

Chapter 9

Complex intuitionistic fuzzy power aggregation operators and their applications in multi-criteria decision-making[1]

In this chapter, we developed some new power AOs namely CIF power averaging (CIFPA), CIF weighted power averaging (CIFWPA), CIF ordered weighted power averaging (CIFOWPA), CIF power geometric (CIFPG), CIF weighted power geometric (CIFWPG) and CIF ordered weighted power geometric (CIFOWPG). Some desirable properties of these operators are investigated. Based on the proposed operators, a MCDM approach is presented under the CIFS environment. An illustrative example related to the selection of the best alternative(s) is considered to demonstrate the efficiency of the proposed approach. The reliability of the presented method is explored by comparing its results with several existing studies.

9.1 Introduction

The AOs presented in the previous three chapters aggregate independent arguments. These operators do not consider any sort of interrelationship among arguments during aggregation process and fuse them by considering them to be independent of one another. However, this may not be the case always. There may exist some kind of dependency in

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the form of supportive correlation, interrelationship and prioritization relationship among the arguments to be aggregated. For handling such situations, Yager [190] originated the idea of PA operator which takes into account the correlation among the arguments and during aggregation process by PA operator, the arguments reinforce each other. Xu and Yager [187] proposed PG operator, ordered PG operator using geometric mean and PA operator [190]. Further, Xu [175] proposed PA operator under IFS theory and applied the proposed operator in developing MCGDM approach and extended the proposed operator and DM technique for IVIFSs. The detailed literature review on PA operator is done in Section 1.1.4 of Chapter 1.

As per our knowledge, the existing studies under CIFS environment do not consider interdependence among CIFNs during aggregation process. Therefore, in order to consider the dependency among CIFNs during their fusion, we develop power AOs in this chapter. For this, we first define some basic algebraic operational laws between the pairs of the CIFSs which involve both uncertainty and periodicity semantics and studied their properties. Then, based on these operations, we propose some power AOs named as CIF power averaging, CIF power geometric, CIF weighted power averaging and geometric as well as their corresponding ordered weighted operators to aggregate the different CIFNs. The various properties of these operators are investigated in details. Furthermore, we propose an MCDM approach based on the proposed operators for CIFSs. The presented MCDM method is delineated through a real life example. The results of the proposed method are compared with several existing studies under CIFS and IFS environment.

9.2 Operational laws and Power AOs of CIFNs

In this section, we introduce some basic operational laws for the collection of CIFNs, denoted by Ω, and their corresponding power AOs.

9.2.1 Operational laws of CIFNs

Definition 9.2.1. For two CIFNs $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$, $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$ and a positive real number ρ, the basic operational laws for CIFNs are defined as follows:

(i) $\mathcal{C}_1 \oplus \mathcal{C}_2 = \left(\left(1 - \prod_{j=1}^{2} (1-\zeta_j), 1 - \prod_{j=1}^{2} (1 - w_{\zeta_j}) \right), \left(\prod_{j=1}^{2} \vartheta_j, \prod_{j=1}^{2} w_{\vartheta_j} \right) \right)$;

(ii) $\mathcal{C}_1 \otimes \mathcal{C}_2 = \left(\left(\prod_{j=1}^{2} \zeta_j, \prod_{j=1}^{2} w_{\zeta_j} \right), \left(1 - \prod_{j=1}^{2} (1-\vartheta_j), 1 - \prod_{j=1}^{2} (1 - w_{\vartheta_j}) \right) \right)$;

(iii) $\rho\mathcal{C}_1 = \left(\left(1 - (1-\zeta_1)^{\rho}, 1 - (1 - w_{\zeta_1})^{\rho} \right), \left(\vartheta_1^{\rho}, (w_{\vartheta_1})^{\rho} \right) \right)$;

(iv) $\mathcal{C}_1^{\rho} = \left(\left(\zeta_1^{\rho}, (w_{\zeta_1})^{\rho} \right), \left(1 - (1-\vartheta_1)^{\rho}, 1 - (1 - w_{\vartheta_1})^{\rho} \right) \right)$.

Theorem 9.2.1. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two CIFNs and $\rho > 0$ is a real number, then $\mathcal{C}_1 \oplus \mathcal{C}_2$, $\rho\mathcal{C}_1$, $\mathcal{C}_1 \otimes \mathcal{C}_2$ and $\mathcal{C}_1^{\rho}$ are also CIFNs.

Proof. Let $\mathcal{C}_1 = \left((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}) \right)$, $\mathcal{C}_2 = \left((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}) \right)$ be two CIFNs such that $0 \leq \zeta_j, \vartheta_j \leq 1$; $0 \leq \zeta_j + \vartheta_j \leq 1$ and $0 \leq w_{\zeta_j}, w_{\vartheta_j} \leq 1$; $0 \leq w_{\zeta_j} + w_{\vartheta_j} \leq 1$ for $j = 1, 2$. Take, $\mathcal{C}_3 = \mathcal{C}_1 \oplus \mathcal{C}_2 = \left((\zeta_3, w_{\zeta_3}), (\vartheta_3, w_{\vartheta_3}) \right)$ where $\zeta_3 = 1 - \prod_{j=1}^{2} (1 - \zeta_j)$, $\vartheta_3 = \prod_{j=1}^{2} \vartheta_j$, $w_{\zeta_3} = 1 - \prod_{j=1}^{2} (1 - w_{\zeta_j})$ and $w_{\vartheta_3} = \prod_{j=1}^{2} w_{\vartheta_j}$.

Now, since $0 \leq \zeta_1, \zeta_2 \leq 1$ which implies that $0 \leq 1 - \zeta_j \leq 1$ and therefore $0 \leq 1 - \prod_{j=1}^{2} (1 - \zeta_j) \leq 1$. Hence, $0 \leq \zeta_3 \leq 1$. On the other hand, $0 \leq \vartheta_1, \vartheta_2 \leq 1$ which implies that $0 \leq \prod_{j=1}^{2} \vartheta_j \leq 1$ and hence $0 \leq \vartheta_3 \leq 1$. Further, $\zeta_j + \vartheta_j \leq 1$ for $j = 1, 2$ which implies that $\prod_{j=1}^{2} \vartheta_j \leq \prod_{j=1}^{2} (1 - \zeta_j)$ and hence $\zeta_3 + \vartheta_3 = 1 - \prod_{j=1}^{2} (1 - \zeta_j) + \prod_{j=1}^{2} \vartheta_j \leq 1 - \prod_{j=1}^{2} (1 - \zeta_j) + \prod_{j=1}^{2} (1 - \zeta_j) = 1$. Also $\zeta_3 + \vartheta_3 \geq 0$ as $\zeta_3 \geq 0$ and $\vartheta_3 \geq 0$. Hence, $0 \leq \zeta_3 + \vartheta_3 \leq 1$. Similarly, we can obtain that $0 \leq w_{\zeta_3}, w_{\vartheta_3} \leq 1$ such that $0 \leq w_{\zeta_3} + w_{\vartheta_3} \leq 1$. Thus, we get $\mathcal{C}_1 \oplus \mathcal{C}_2$ is a CIFN. Similarly, it can be proved that $\mathcal{C}_1 \otimes \mathcal{C}_2$, $\mathcal{C}_1^{\rho}$, $\rho\mathcal{C}_1$ are also CIFNs. □

Theorem 9.2.2. For three CIFNs $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ we have:

(i) $\mathcal{C}_1 \oplus \mathcal{C}_2 = \mathcal{C}_2 \oplus \mathcal{C}_1$.

(ii) $\mathcal{C}_1 \otimes \mathcal{C}_2 = \mathcal{C}_2 \otimes \mathcal{C}_1$.

(iii) $(\mathcal{C}_1 \oplus \mathcal{C}_2) \oplus \mathcal{C}_3 = \mathcal{C}_1 \oplus (\mathcal{C}_2 \oplus \mathcal{C}_3)$.

(iv) $\big(\mathcal{C}_1 \otimes \mathcal{C}_2\big) \otimes \mathcal{C}_3 = \mathcal{C}_1 \otimes \big(\mathcal{C}_2 \otimes \mathcal{C}_3\big)$.

Proof. Here we prove the parts (i) and (iii) only, as the others are similar.

(i) Since $\mathcal{C}_1 = \Big((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1})\Big)$ and $\mathcal{C}_2 = \Big((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2})\Big)$ are two CIFNs, then we have

$$
\begin{aligned}
\mathcal{C}_1 \oplus \mathcal{C}_2 &= \left(\left(1 - \prod_{j=1}^{2}(1-\zeta_j), 1 - \prod_{j=1}^{2}(1 - w_{\zeta_j})\right), \left(\prod_{j=1}^{2}\vartheta_j, \prod_{j=1}^{2} w_{\vartheta_j}\right)\right) \\
&= \Big(\Big(\zeta_1 + \zeta_2 - \zeta_1\zeta_2, w_{\zeta_1} + w_{\zeta_2} - w_{\zeta_1}w_{\zeta_2}\Big), \Big(\vartheta_1\vartheta_2, w_{\vartheta_1}w_{\vartheta_2}\Big)\Big) \\
&= \Big(\Big(\zeta_2 + \zeta_1 - \zeta_2\zeta_1, w_{\zeta_2} + w_{\zeta_1} - w_{\zeta_2}w_{\zeta_1}\Big), \Big(\vartheta_2\vartheta_1, w_{\vartheta_2}w_{\vartheta_1}\Big)\Big) \\
&= \mathcal{C}_2 \oplus \mathcal{C}_1
\end{aligned}
$$

(iii) Since $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$, $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$ and $\mathcal{C}_3 = ((\zeta_3, w_{\zeta_3}), (\vartheta_3, w_{\vartheta_3}))$ are three CIFNs, then

$$
\begin{aligned}
\big(\mathcal{C}_1 \oplus \mathcal{C}_2\big) \oplus \mathcal{C}_3 &= \left(\begin{pmatrix} 1 - \prod_{j=1}^{2}(1-\zeta_j), \\ 1 - \prod_{j=1}^{2}(1 - w_{\zeta_j}) \end{pmatrix}, \begin{pmatrix} \prod_{j=1}^{2}\vartheta_j, \\ \prod_{j=1}^{2} w_{\vartheta_j} \end{pmatrix}\right) \oplus \Big((\zeta_3, w_{\zeta_3}), (\vartheta_3, w_{\vartheta_3})\Big) \\
&= \left(\begin{pmatrix} 1 - \prod_{j=1}^{3}(1-\zeta_j), \\ 1 - \prod_{j=1}^{3}(1 - w_{\zeta_j}) \end{pmatrix}, \begin{pmatrix} \prod_{j=1}^{3}\vartheta_j, \\ \prod_{j=1}^{3} w_{\vartheta_j} \end{pmatrix}\right) \\
&= \Big((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1})\Big) \oplus \left(\begin{pmatrix} 1 - \prod_{j=2}^{3}(1-\zeta_j), \\ 1 - \prod_{j=2}^{3}(1 - w_{\zeta_j}) \end{pmatrix}, \begin{pmatrix} \prod_{j=2}^{3}\vartheta_j, \\ \prod_{j=2}^{3} w_{\vartheta_j} \end{pmatrix}\right) \\
&= \mathcal{C}_1 \oplus \big(\mathcal{C}_2 \oplus \mathcal{C}_3\big)
\end{aligned}
$$

□

Theorem 9.2.3. For two CIFNs $\mathcal{C}_1$ and $\mathcal{C}_2$ and positive real numbers ρ, ρ_1, ρ_2, we have

(i) $\rho\big(\mathcal{C}_1 \oplus \mathcal{C}_2\big) = \rho\mathcal{C}_1 \oplus \rho\mathcal{C}_2$;

(ii) $\left(\mathcal{C}_1 \otimes \mathcal{C}_2\right)^\rho = \mathcal{C}_1^\rho \otimes \mathcal{C}_2^\rho$;

(iii) $(\rho_1 + \rho_2)\mathcal{C}_1 = \rho_1\mathcal{C}_1 \oplus \rho_2\mathcal{C}_1$;

(iv) $\mathcal{C}_1^{\rho_1+\rho_2} = \mathcal{C}_1^{\rho_1} \otimes \mathcal{C}_1^{\rho_2}$.

Proof. Here we prove the parts (i) and (iii) only, while others are similar.

(i) Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are CIFNs.

$$\begin{aligned}
\rho\left(\mathcal{C}_1 \oplus \mathcal{C}_2\right) &= \rho\left(\left(1-\prod_{j=1}^{2}(1-\zeta_j), 1-\prod_{j=1}^{2}(1-w_{\zeta_j})\right), \left(\prod_{j=1}^{2}\vartheta_j, \prod_{j=1}^{2}w_{\vartheta_j}\right)\right) \\
&= \left(\left(1-\prod_{j=1}^{2}(1-\zeta_j)^\rho,\;\; 1-\prod_{j=1}^{2}\left(1-w_{\zeta_j}\right)^\rho\right), \left(\prod_{j=1}^{2}\vartheta_j^\rho,\;\; \prod_{j=1}^{2}w_{\vartheta_j}^\rho\right)\right) \\
&= \left(\left(1-(1-\zeta_1)^\rho,\;\; 1-(1-w_{\zeta_1})^\rho\right), \left(\vartheta_1^\rho,\;\; (w_{\vartheta_1})^\rho\right)\right) \\
&\oplus \left(\left(1-(1-\zeta_2)^\rho,\;\; 1-(1-w_{\zeta_2})^\rho\right), \left(\vartheta_2^\rho,\;\; (w_{\vartheta_2})^\rho\right)\right) \\
&= \rho\mathcal{C}_1 \oplus \rho\mathcal{C}_2
\end{aligned}$$

Hence, $\rho\left(\mathcal{C}_1 \oplus \mathcal{C}_2\right) = \rho\mathcal{C}_1 \oplus \rho\mathcal{C}_2$

(iii) Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are CIFNs.

$$\begin{aligned}
(\rho_1+\rho_2)\mathcal{C}_1 &= \left(\left(1-(1-\zeta_1)^{\rho_1+\rho_2},\;\; 1-(1-w_{\zeta_1})^{\rho_1+\rho_2}\right), \left(\vartheta_1^{\rho_1+\rho_2},\;\; (w_{\vartheta_1})^{\rho_1+\rho_2}\right)\right) \\
&= \left(\left(1-(1-\zeta_1)^{\rho_1}, 1-(1-w_{\zeta_1})^{\rho_1}\right), \left(\vartheta_1^{\rho_1}, (w_{\vartheta_1})^{\rho_1}\right)\right) \\
&\oplus \left(\left(1-(1-\zeta_1)^{\rho_2}, 1-(1-w_{\zeta_1})^{\rho_2}\right), \left(\vartheta_1^{\rho_2}, (w_{\vartheta_1})^{\rho_2}\right)\right) \\
&= \rho_1\mathcal{C}_1 \oplus \rho_2\mathcal{C}_1
\end{aligned}$$

Hence, $(\rho_1 + \rho_2)\mathcal{C}_1 = \rho_1\mathcal{C}_1 \oplus \rho_2\mathcal{C}_1$.

□

9.2.2 Power Averaging Aggregation Operators

In this section, based on the proposed operational laws of CIFNs, some power AOs are presented under CIF environment to aggregate the collection of the different CIFNs.

CIF power averaging operator

Definition 9.2.2. For a collection of CIFNs $\mathcal{C}_j (j = 1, 2, \ldots, n)$, a CIF power averaging (CIFPA) AO is a mapping CIFPA : $\Omega^n \to \Omega$ defined by:

$$\text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \sigma_1\mathcal{C}_1 \oplus \sigma_2\mathcal{C}_2 \oplus \ldots \oplus \sigma_n\mathcal{C}_n \tag{9.1}$$

where $\sigma_j = \frac{1+T(\mathcal{C}_j)}{\sum_{j=1}^{n}(1+T(\mathcal{C}_j))}$ and $T(\mathcal{C}_j) = \sum_{\substack{l=1\\ l\neq j}}^{n} \left(\text{Sup}(\mathcal{C}_j, \mathcal{C}_l)\right)$ $(j = 1, 2, \ldots, n)$. Here $\text{Sup}(\mathcal{C}_j, \mathcal{C}_l)$ is the support of $\mathcal{C}_j$ from $\mathcal{C}_l$ satisfying the properties mentioned in Definition 2.4.5 and $\text{Sup}(\mathcal{C}_j, \mathcal{C}_l) = 1 - \mathcal{D}(\mathcal{C}_j, \mathcal{C}_l)$ where $\mathcal{D}$ is the hamming distance measure as given in Eq. (3.1).

Theorem 9.2.4. For a collection of CIFNs $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right) (j = 1, 2, \ldots, n)$, the aggregated value obtained by using CIFPA operator is again a CIFN and is given by

$$\text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} 1 - \prod_{j=1}^{n}\left(1-\zeta_j\right)^{\sigma_j}, \\ 1 - \prod_{j=1}^{n}\left(1-w_{\zeta_j}\right)^{\sigma_j} \end{pmatrix}, \begin{pmatrix} \prod_{j=1}^{n}\vartheta_j^{\sigma_j}, \\ \prod_{j=1}^{n}\left(w_{\vartheta_j}\right)^{\sigma_j} \end{pmatrix}\right) \tag{9.2}$$

Proof. The fact that, the value obtained after applying CIFPA operator is still CIFN, follows from the Theorem 9.2.1. Now, by making use of mathematical induction, we will show that Eq. (9.2) holds.

Since for each j, $\mathcal{C}_j$ is a CIFN and $\sigma_j > 0$ therefore, we have $\sigma_j\mathcal{C}_j$ is also CIFN by using Theorem 9.2.1. Then, by utilizing the steps of the principal of mathematical induction on n, we have:

Step 1: For $n = 2$, we have $\mathcal{C}_1 = ((\zeta_1, w_{\zeta_1}), (\vartheta_1, w_{\vartheta_1}))$ and $\mathcal{C}_2 = ((\zeta_2, w_{\zeta_2}), (\vartheta_2, w_{\vartheta_2}))$. Thus, by the operation of CIFNs, we get

$$\begin{aligned} \sigma_1\mathcal{C}_1 &= \left(\left(1-\left(1-\zeta_1\right)^{\sigma_1},\ \ 1-\left(1-w_{\zeta_1}\right)^{\sigma_1}\right), \left(\vartheta_1^{\sigma_1},\ \ w_{\vartheta_1}^{\sigma_1}\right)\right) \\ \text{and}\quad \sigma_2\mathcal{C}_2 &= \left(\left(1-\left(1-\zeta_2\right)^{\sigma_2},\ \ 1-\left(1-w_{\zeta_2}\right)^{\sigma_2}\right), \left(\vartheta_2^{\sigma_2},\ \ w_{\vartheta_2}^{\sigma_2}\right)\right) \end{aligned}$$

Hence, by addition law of CIFNs, we get

$$\text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2) = \left(\begin{pmatrix} 1 - \prod_{j=1}^{2}\left(1-\zeta_j\right)^{\sigma_j}, \\ 1 - \prod_{j=1}^{2}\left(1-w_{\zeta_j}\right)^{\sigma_j} \end{pmatrix}, \begin{pmatrix} \prod_{j=1}^{2}\vartheta_j^{\sigma_j}, \\ \prod_{j=1}^{2}\left(w_{\vartheta_j}\right)^{\sigma_j} \end{pmatrix}\right)$$

Thus, the result is valid when $n = 2$.

Step 2: Assume that Eq. (9.2) holds for $n = m$, where m is any positive natural number i.e.,

$$\text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m) = \left(\begin{pmatrix} 1 - \prod_{j=1}^{m} \left(1 - \zeta_j\right)^{\sigma_j}, \\ 1 - \prod_{j=1}^{m} \left(1 - w_{\zeta_j}\right)^{\sigma_j} \end{pmatrix}, \begin{pmatrix} \prod_{j=1}^{m} \vartheta_j^{\sigma_j}, \\ \prod_{j=1}^{m} \left(w_{\vartheta_j}\right)^{\sigma_j} \end{pmatrix} \right)$$

then for $n = m + 1$, we have

$$\begin{aligned} \text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{m+1}) &= \text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m) \oplus \text{CIFPA}(\mathcal{C}_{m+1}) \\ &= \left(\begin{pmatrix} 1 - \prod_{j=1}^{m} \left(1 - \zeta_j\right)^{\sigma_j}, \\ 1 - \prod_{j=1}^{m} \left(1 - w_{\zeta_j}\right)^{\sigma_j} \end{pmatrix}, \begin{pmatrix} \prod_{j=1}^{m} \vartheta_j^{\sigma_j}, \\ \prod_{j=1}^{m} \left(w_{\vartheta_j}\right)^{\sigma_j} \end{pmatrix} \right) \\ &\oplus \left(\begin{pmatrix} 1 - \left(1 - \zeta_{m+1}\right)^{\sigma_{m+1}}, \\ 1 - \left(1 - w_{\zeta_{m+1}}\right)^{\sigma_{m+1}} \end{pmatrix}, \begin{pmatrix} \vartheta_{m+1}^{\sigma_{m+1}}, \\ w_{\vartheta_{m+1}}^{\sigma_{m+1}} \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} 1 - \prod_{j=1}^{m+1} \left(1 - \zeta_j\right)^{\sigma_j}, \\ 1 - \prod_{j=1}^{m+1} \left(1 - w_{\zeta_j}\right)^{\sigma_j} \end{pmatrix}, \begin{pmatrix} \prod_{j=1}^{m+1} \vartheta_j^{\sigma_j}, \\ \prod_{j=1}^{m+1} \left(w_{\vartheta_j}\right)^{\sigma_j} \end{pmatrix} \right) \end{aligned}$$

Thus, the result is true for $n = m + 1$ and hence, the Eq. (9.2) holds for all natural numbers n. □

The working of CIFPA operator is demonstrated with a numerical example as follows:

Example 9.2.1. Let $\mathcal{C}_1 = ((0.6, 0.8), (0.2, 0.1))$, $\mathcal{C}_2 = ((0.8, 0.7), (0.2, 0.1))$, $\mathcal{C}_3 = ((0.5, 0.6), (0.3, 0.4))$, $\mathcal{C}_4 = ((0.6, 0.7), (0.3, 0.2))$ be four CIFNs. Firstly, we calculate $\mathcal{D}(\mathcal{C}_j, \mathcal{C}_l)$ $\forall$ $j, l \in \{1, 2, 3, 4\}$, $j \neq l$ using Eq. (3.1) as given below.

$$\begin{aligned}
\mathcal{D}(\mathcal{C}_1, \mathcal{C}_2) &= \frac{1}{4}\left[|\zeta_1 - \zeta_2| + |\vartheta_1 - \vartheta_2| + |w_{\zeta_1} - w_{\zeta_2}| + |w_{\vartheta_1} - w_{\vartheta_2}|\right] \\
&= \frac{1}{4}\left[|0.6 - 0.8| + |0.2 - 0.2| + |0.8 - 0.1| + |0.1 - 0.1|\right] \\
&= 0.075
\end{aligned}$$

which gives that $\mathrm{Sup}(\mathcal{C}_1, \mathcal{C}_2) = 1 - \mathcal{D}(\mathcal{C}_1, \mathcal{C}_2) = 0.925$. Similarly, we can obtain, $\mathrm{Sup}(\mathcal{C}_1, \mathcal{C}_3) = 0.825$, $\mathrm{Sup}(\mathcal{C}_1, \mathcal{C}_4) = 0.925$, $\mathrm{Sup}(\mathcal{C}_2, \mathcal{C}_3) = 0.8$, $\mathrm{Sup}(\mathcal{C}_2, \mathcal{C}_4) = 0.9$ and $\mathrm{Sup}(\mathcal{C}_3, \mathcal{C}_4) = 0.9$. Therefore,

$$\begin{aligned}
T(\mathcal{C}_1) &= \sum_{\substack{l=1 \\ l\neq 1}}^{4} \big(\mathrm{Sup}(\mathcal{C}_1, \mathcal{C}_l)\big) \\
&= \big(\mathrm{Sup}(\mathcal{C}_1, \mathcal{C}_2)\big) + \big(\mathrm{Sup}(\mathcal{C}_1, \mathcal{C}_3)\big) + \big(\mathrm{Sup}(\mathcal{C}_1, \mathcal{C}_4)\big) \\
&= 0.925 + 0.825 + 0.925 \\
&= 2.6750
\end{aligned}$$

In the similar manner, we can obtain $T(\mathcal{C}_2) = 2.6250$, $T(\mathcal{C}_3) = 2.5250$ and $T(\mathcal{C}_4) = 2.7250$, which gives that

$$\begin{aligned}
\sum_{j=1}^{4}\Big(1 + T(\mathcal{C}_j)\Big) &= \Big(1 + T(\mathcal{C}_1)\Big) + \Big(1 + T(\mathcal{C}_2)\Big) + \Big(1 + T(\mathcal{C}_3)\Big) + \Big(1 + T(\mathcal{C}_4)\Big) \\
&= \big(1 + 2.6750\big) + \big(1 + 2.6250\big) + \big(1 + 2.5250\big) + \big(1 + 2.7250\big) \\
&= 14.55
\end{aligned}$$

Hence, $\sigma_1 = \frac{1+T(\mathcal{C}_1)}{\sum_{j=1}^{4}\Big(1+T(\mathcal{C}_j)\Big)} = \frac{1+2.6750}{14.55} = 0.2526$. Similarly, we can obtain $\sigma_2 = 0.2491$, $\sigma_3 = 0.2423$ and $\sigma_4 = 0.2560$. Further,

$$\begin{aligned}
\prod_{j=1}^{4}\big(1 - \zeta_j\big)^{\sigma_j} &= \big(1 - \zeta_1\big)^{\sigma_1}\big(1 - \zeta_2\big)^{\sigma_2}\big(1 - \zeta_3\big)^{\sigma_3}\big(1 - \zeta_4\big)^{\sigma_4} \\
&= (1 - 0.6)^{0.2526} \times (1 - 0.8)^{0.2491} \times (1 - 0.5)^{0.2423} \times (1 - 0.6)^{0.2560} \\
&= 0.3553 \\
\prod_{j=1}^{4}\big(1 - w_{\zeta_j}\big)^{\sigma_j} &= (1 - 0.8)^{0.2526} \times (1 - 0.7)^{0.2491} \times (1 - 0.6)^{0.2423} \times (1 - 0.7)^{0.2560} \\
&= 0.2903
\end{aligned}$$

$$\begin{aligned}
\prod_{j=1}^{4}(\vartheta_j)^{\sigma_j} &= (0.2)^{0.2526}\times(0.2)^{0.2491}\times(0.3)^{0.2423}\times(0.3)^{0.2560}\\
&= 0.2448\\
\prod_{j=1}^{4}\left(w_{\vartheta_j}\right)^{\sigma_j} &= (0.1)^{0.2526}\times(0.1)^{0.2491}\times(0.4)^{0.2423}\times(0.2)^{0.2560}\\
&= 0.1671
\end{aligned}$$

Now, by using Eq. (9.2), we get

$$\begin{aligned}
\text{CIFPA}(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4) &= \Big(\Big(1-0.3553, 1-0.2903\Big), \Big(0.2448, 0.1671\Big)\Big)\\
&= \Big(\Big(0.6447, 0.7097\Big), \Big(0.2448, 0.1671\Big)\Big)
\end{aligned}$$

It can be clearly seen that the aggregated value is again CIFN.

Based on the Theorem 9.2.4, it is observed that the CIFPA operator satisfies some properties which are stated as below.

Property 9.2.1. Let $\mathcal{C}_0$ be CIFN and if $\mathcal{C}_j=\mathcal{C}_0$ for all $j=1,2,\ldots,n$, then

$$\text{CIFPA}(\mathcal{C}_1,\mathcal{C}_2\ldots\mathcal{C}_n)=\mathcal{C}_0$$

This property is called Idempotency.

Proof. Let $\mathcal{C}_0=((\zeta_0,w_{\zeta_0}),(\vartheta_0,w_{\vartheta_0}))$ and $\mathcal{C}_j=((\zeta_j,w_{\zeta_j}),(\vartheta_j,w_{\vartheta_j}))$ be CIFNs such that $\mathcal{C}_j=\mathcal{C}_0$ for all j which implies that $\zeta_j=\zeta_0$, $\vartheta_j=\vartheta_0$, $w_{\zeta_j}=w_{\zeta_0}$ and $w_{\vartheta_j}=w_{\vartheta_0}$ for all j. Then, by using the defining of σ_j as defined in Definition 9.2.2, we have $\sum\limits_{j=1}^{n}\sigma_j=1$. So, by Theorem 9.2.4, we have

$$\begin{aligned}
\text{CIFPA}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n) &= \left(\begin{pmatrix} 1-\prod\limits_{j=1}^{n}(1-\zeta_0)^{\sigma_j}, \\ 1-\prod\limits_{j=1}^{n}(1-w_{\zeta_0})^{\sigma_j} \end{pmatrix}, \begin{pmatrix} \prod\limits_{j=1}^{n}\vartheta_0^{\sigma_j}, \\ \prod\limits_{j=1}^{n}(w_{\vartheta_0})^{\sigma_j} \end{pmatrix}\right)\\
&= \left(\begin{pmatrix} 1-\Big(1-\zeta_0\Big)^{\sum\limits_{j=1}^{n}\sigma_j}, \\ 1-(1-w_{\zeta_0})^{\sum\limits_{j=1}^{n}\sigma_j} \end{pmatrix}, \begin{pmatrix} \Big(\vartheta_0\Big)^{\sum\limits_{j=1}^{n}\sigma_j}, \\ (w_{\vartheta_0})^{\sum\limits_{j=1}^{n}\sigma_j} \end{pmatrix}\right)\\
&= \Big(\Big(\zeta_0, w_{\zeta_0}\Big), \Big(\vartheta_0, w_{\vartheta_0}\Big)\Big)\\
&= \mathcal{C}_0
\end{aligned}$$

Hence, $\text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2 \ldots \mathcal{C}_n) = \mathcal{C}_0$. $\square$

Property 9.2.2. For a collection of CIFNs $\mathcal{C}_j = \big((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\big)$, let $\mathcal{C}^- = \big(\big(\min\limits_j\{\zeta_j\}, \min\limits_j\{w_{\zeta_j}\}\big), \big(\max\limits_j\{\vartheta_j\}, \max\limits_j\{w_{\vartheta_j}\}\big)\big)$ and $\mathcal{C}^+ = \big(\big(\max\limits_j\{\zeta_j\}, \max\limits_j\{w_{\zeta_j}\}\big), \big(\min\limits_j\{\vartheta_j\}, \min\limits_j\{w_{\vartheta_j}\}\big)\big)$. Then,

$$\mathcal{C}^- \subseteq \text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2 \ldots \mathcal{C}_n) \subseteq \mathcal{C}^+$$

This property is called as Boundedness.

Proof. Take $\mathcal{C} = \text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)$ and hence by Theorem 9.2.4 we get $\mathcal{C} = \Big((\zeta_{\mathcal{C}}, w_{\zeta_{\mathcal{C}}}), (\vartheta_{\mathcal{C}}, w_{\vartheta_{\mathcal{C}}})\Big)$. For a CIFN $\mathcal{C}_j$, we have $\min\limits_j\{\zeta_j\} \le \zeta_j \le \max\limits_j\{\zeta_j\}$. $\Rightarrow 1 - \max\limits_j\{\zeta_j\} \le 1 - \zeta_j \le 1 - \min\limits_j\{\zeta_j\} \Rightarrow \left(1 - \max\limits_j\{\zeta_j\}\right)^{\sigma_j} \le (1 - \zeta_j)^{\sigma_j} \le \left(1 - \min\limits_j\{\zeta_j\}\right)^{\sigma_j} \Rightarrow 1 - \max\limits_j\{\zeta_j\} \le \prod\limits_{j=1}^{n} (1 - \zeta_j)^{\sigma_j} \le 1 - \min\limits_j\{\zeta_j\} \Rightarrow \min\limits_j\{\zeta_j\} \le 1 - \prod\limits_{j=1}^{n} (1 - \zeta_j)^{\sigma_j} \le \max\limits_j\{\zeta_j\} \Rightarrow \min\limits_j\{\zeta_j\} \le \zeta_{\mathcal{C}} \le \max\limits_j\{\zeta_j\}$. Also, $\min\limits_j\{\vartheta_j\} \le \vartheta_j \le \max\limits_j\{\zeta_j\} \Rightarrow \left(\min\limits_j\{\vartheta_j\}\right)^{\sigma_j} \le \left(\vartheta_j\right)^{\sigma_j} \le \left(\max\limits_j\{\vartheta_j\}\right)^{\sigma_j} \Rightarrow \prod\limits_{j=1}^{n} \left(\min\limits_j\{\vartheta_j\}\right)^{\sigma_j} \le \prod\limits_{j=1}^{n} \left(\vartheta_j\right)^{\sigma_j} \le \prod\limits_{j=1}^{n} \left(\max\limits_j\{\vartheta_j\}\right)^{\sigma_j} \Rightarrow \min\limits_j\{\vartheta_j\} \le \prod\limits_{j=1}^{n} \left(\vartheta_j\right)^{\sigma_j} \le \max\limits_j\{\vartheta_j\}$ $\Rightarrow \min\limits_j\{\vartheta_j\} \le \vartheta_{\mathcal{C}} \le \max\limits_j\{\vartheta_j\}$. Similarly, we can obtain that $\min\limits_j\{w_{\zeta_j}\} \le w_{\zeta_{\mathcal{C}}} \le \max\limits_j\{w_{\zeta_j}\}$ and $\min\limits_j\{w_{\vartheta_j}\} \le w_{\vartheta_{\mathcal{C}}} \le \max\limits_j\{w_{\vartheta_j}\}$. Thus, we get $\mathcal{C}^- \subseteq \text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \mathcal{C}^+$. $\square$

Property 9.2.3. For a collection of CIFNs $\mathcal{C}_j (j = 1, 2, \ldots, n)$, if $\big(\dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2, \ldots, \dot{\mathcal{C}}_n\big)$ be the permutation of $\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big)$ then

$$\text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \text{CIFPA}(\dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2, \ldots, \dot{\mathcal{C}}_n)$$

This property is called Commutativity.

Proof. As $\big(\dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2, \ldots, \dot{\mathcal{C}}_n\big)$ is an arbitrary arrangement of $\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big)$. Therefore,

$$\text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \frac{\bigoplus\limits_{j=1}^{n} \Big(1+T(\mathcal{C}_j)\Big)\mathcal{C}_j}{\sum\limits_{j=1}^{n} \Big(1+T(\mathcal{C}_j)\Big)} = \frac{\bigoplus\limits_{j=1}^{n} \Big(1+T(\dot{\mathcal{C}}_j)\Big)\dot{\mathcal{C}}_j}{\sum\limits_{j=1}^{n} \Big(1+T(\dot{\mathcal{C}}_j)\Big)} = \text{CIFPA}(\dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2, \ldots, \dot{\mathcal{C}}_n)$$

Hence, $\text{CIFPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \text{CIFPA}(\dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2, \ldots, \dot{\mathcal{C}}_n)$. $\square$

Remark 9.2.1. From the proposed CIFPA operator, it is observed that CIFPA operator is not monotonic, i.e., there exist some collections of CIFNs $\mathcal{C}_j$ and $\mathcal{Z}_j$ where

$j = 1, 2, \ldots, n$ which satisfying the relation $\mathcal{C}_j \subseteq \mathcal{Z}_j$ for all j but CIFPA$(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subsetneq$ CIFPA$(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n)$.

In order to demonstrate the non-monotonic property, we give one counter example as below:

Example 9.2.2. Consider two collections $\mathcal{C}_j$ and $\mathcal{Z}_j$ $(j = 1, 2, 3)$ of CIFNs as $\mathcal{C}_1 = ((0.2, 0.2), (0.2, 0.2))$, $\mathcal{C}_2 = ((0.4, 0.4), (0.4, 0.4))$, $\mathcal{C}_3 = ((0.1, 0.1), (0.89, 0.89))$ and $\mathcal{Z}_1 = ((0.2, 0.2), (0.2, 0.2))$, $\mathcal{Z}_2 = ((0.4, 0.4), (0.4, 0.4))$, $\mathcal{Z}_3 = ((0.11, 0.11), (0.11, 0.11))$. Clearly $\mathcal{C}_j \subseteq \mathcal{Z}_j$ for all j. Now, based on Theorem 9.2.4, we get CIFPA$(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$=((0.2477,0.2477), (0.4058, 0.4058)) and CIFPA$(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3) = ((0.2439, 0.2439), (0.2045, 0.2045))$ which gives that CIFPA$(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \subsetneq$ CIFPA$(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3)$. Hence, the result.

9.2.3 CIF weighted power averaging operator

In this section, we take into the account the different weightage of CIFNs $\mathcal{C}_j (j = 1, 2, \ldots, n)$ during an aggregation process and hence propose a new CIF weighted power averaging (CIFWPA) AO as follows.

Definition 9.2.3. For a collection of CIFNs $\mathcal{C}_j (j = 1, 2, \ldots, n)$, a CIFWPA operator is a map CIFWPA : $\Omega^n \to \Omega$ defined by:

$$\text{CIFWPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \phi_1 \mathcal{C}_1 \oplus \phi_2 \mathcal{C}_2 \oplus \ldots \oplus \phi_n \mathcal{C}_n \tag{9.3}$$

where $\phi_j = \frac{\xi_j (1 + T'(\mathcal{C}_j))}{\sum\limits_{j=1}^{n} \xi_j (1 + T'(\mathcal{C}_j))}$, $T'(\mathcal{C}_j) = \sum\limits_{\substack{l=1 \\ l \neq j}}^{n} \xi_l \big(\text{Sup}(\mathcal{C}_j, \mathcal{C}_l)\big)$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ is the weight vector assigned to CIFNs $\mathcal{C}_j$ such that $\xi_j > 0$ and $\sum\limits_{j=1}^{n} \xi_j = 1$.

Theorem 9.2.5. For a collection of CIFNs $\mathcal{C}_j = \big((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\big)$ with corresponding weight vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$, such that, $\xi_j > 0$ and $\sum\limits_{j=1}^{n} \xi_j = 1$, the aggregated value obtained by using CIFWPA operator is also a CIFN and is given by

$$\text{CIFWPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} 1 - \prod\limits_{j=1}^{n} \big(1 - \zeta_j\big)^{\phi_j}, \\ 1 - \prod\limits_{j=1}^{n} \big(1 - w_{\zeta_j}\big)^{\phi_j} \end{pmatrix}, \begin{pmatrix} \prod\limits_{j=1}^{n} \vartheta_j^{\phi_j}, \\ \prod\limits_{j=1}^{n} \big(w_{\vartheta_j}\big)^{\phi_j} \end{pmatrix} \right) \tag{9.4}$$

Proof. The proof of the theorem can be obtained by the principal of mathematical induction as similar to the Theorem 9.2.4, so we omit the proof here. □

For a collection of CIFNs $\mathcal{C}_j (j = 1, 2, \ldots, n)$ and their weight vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ such that each $\xi_j > 0$ and $\sum\limits_{j=1}^{n} \xi_j = 1$, CIFWPA operator also satisfies the same properties as that of CIFPA operator which are listed, without proof, as below

(P1) (Idempotency) If $\mathcal{C}_0$ be CIFN such that $\mathcal{C}_j = \mathcal{C}_0$ for all j, then, we have

$$\mathrm{CIFWPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathcal{C}_0$$

(P2) (Boundedness) If $\mathcal{C}^-$ and $\mathcal{C}^+$ respectively, be the lower and upper bounds of the CIFNs $\mathcal{C}_j (j = 1, 2 \ldots, n)$ then we have

$$\mathcal{C}^- \subseteq \mathrm{CIFWPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \mathcal{C}^+$$

(P3) (Commutativity) For any permutation $\left(\dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2, \ldots, \mathcal{C}_n\right)$ of CIFNs $\left(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\right)$ and corresponding permutation $\left(\dot{\xi}_1, \dot{\xi}_2, \ldots, \dot{\xi}_n\right)$ of weight vector $\left(\xi_1, \xi_2, \ldots, \xi_n\right)$, we have

$$\mathrm{CIFWPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathrm{CIFWPA}(\dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2, \ldots, \dot{\mathcal{C}}_n)$$

(P4) (Non-monotonicity) For two CIFNs $\mathcal{C}_j$ and $\mathcal{Z}_j$ satisfying $\mathcal{C}_j \subseteq \mathcal{Z}_j$, it is not necessary that

$$\mathrm{CIFWPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \mathrm{CIFWPA}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n)$$

9.2.4 CIF ordered weighted power averaging operator

In this section, we have extended the above proposed AO to its ordered weighted.

Definition 9.2.4. Let $\mathcal{C}_j (j = 1, 2, \ldots, n)$ be the collection of CIFNs. A CIF ordered weighted power averaging operator is a map CIFOWPA : $\Omega^n \to \Omega$ defined by

$$\mathrm{CIFOWPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \quad = \quad \varphi_1 \mathcal{C}_{\tau(1)} \oplus \varphi_2 \mathcal{C}_{\tau(2)} \oplus \ldots \oplus \varphi_n \mathcal{C}_{\tau(n)} \tag{9.5}$$

where Ω is the set of CIFNs and $\left(\tau(1), \tau(2), \ldots, \tau(n)\right)$ is a permutation of $(1, 2, \ldots, n)$ satisfying $\mathcal{S}\left(\mathcal{C}_{\tau(j-1)}\right) \geq \mathcal{S}\left(\mathcal{C}_{\tau(j)}\right)$ for $j = 2, 3, \ldots, n$. Further, φ_j is defined by

$$\varphi_j \quad = \quad g\left(\frac{B_j}{TV}\right) - g\left(\frac{B_{j-1}}{TV}\right) \tag{9.6}$$

where $B_j = \sum\limits_{l=1}^{j} V_{\tau(l)}$, $V_{\tau(j)} = 1 + \sum\limits_{\substack{l=1 \\ l \neq j}}^{n} \big(\mathrm{Sup}(\mathcal{C}_j, \mathcal{C}_l)\big)$ and $TV = \sum\limits_{j=1}^{n} V_{\tau(j)}$ and the function g : $[0,1] \to [0,1]$ is a basic unit-interval monotonic (BUM) function satisfying three properties namely $g(0) = 0$, $g(1) = 1$ and if $x \leq y$ then $g(x) \leq g(y)$.

Theorem 9.2.6. For a collection of CIFNs $\mathcal{C}_j = ((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j}))$, the combined value obtained by using CIFOWPA operator is also a CIFN and is given by

$$\mathrm{CIFOWPA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} 1 - \prod\limits_{j=1}^{n} \left(1 - \zeta_{\tau(j)}\right)^{\varphi_j}, \\ 1 - \prod\limits_{j=1}^{n} \left(1 - w_{\zeta_{\tau(j)}}\right)^{\varphi_j} \end{pmatrix}, \begin{pmatrix} \prod\limits_{j=1}^{n} \vartheta_{\tau(j)}^{\varphi_j}, \\ \prod\limits_{j=1}^{n} \left(w_{\vartheta_{\tau(j)}}\right)^{\varphi_j} \end{pmatrix} \right) \tag{9.7}$$

where φ_j is defined in Eq. (9.6).

Proof. The proof of this theorem can be easily obtained by the induction principle, so we omit here. □

Example 9.2.3. Let $\mathcal{C}_1 = ((0.6, 0.8), (0.2, 0.1))$, $\mathcal{C}_2 = ((0.8, 0.7), (0.2, 0.1))$, $\mathcal{C}_3 = ((0.5, 0.6), (0.3, 0.4))$, $\mathcal{C}_4 = ((0.6, 0.7), (0.3, 0.2))$ be four CIFNs. The score values of these numbers are computed by using Eq. (6.1) of Chapter 6 and are obtained as $\mathcal{S}(\mathcal{C}_1) = 1.1$, $\mathcal{S}(\mathcal{C}_2) = 1.2$, $\mathcal{S}(\mathcal{C}_3) = 0.4$ and $\mathcal{S}(\mathcal{C}_4) = 0.8$. Since, $\mathcal{S}(\mathcal{C}_2) > \mathcal{S}(\mathcal{C}_1) > \mathcal{S}(\mathcal{C}_4) > \mathcal{S}(\mathcal{C}_3)$. Therefore, we get $\mathcal{C}_{\tau(1)} = ((0.8, 0.7), (0.2, 0.1))$, $\mathcal{C}_{\tau(2)} = ((0.6, 0.8), (0.2, 0.1))$, $\mathcal{C}_{\tau(3)} = ((0.6, 0.7), (0.3, 0.2))$ and $\mathcal{C}_{\tau(4)} = ((0.5, 0.6), (0.3, 0.4))$.

Now, the distance measurement value between the CIFNs is computed as $\mathcal{D}(\mathcal{C}_1, \mathcal{C}_2) = 0.0750$ and hence $\mathrm{Sup}(\mathcal{C}_1, \mathcal{C}_2) = 1 - 0.0750 = 0.9250$. Similarly, we can obtain that $\mathrm{Sup}(\mathcal{C}_1, \mathcal{C}_3) = \mathrm{Sup}(\mathcal{C}_3, \mathcal{C}_1) = 0.9000$, $\mathrm{Sup}(\mathcal{C}_1, \mathcal{C}_4) = \mathrm{Sup}(\mathcal{C}_4, \mathcal{C}_1) = 0.2000$, $\mathrm{Sup}(\mathcal{C}_2, \mathcal{C}_3) = \mathrm{Sup}(\mathcal{C}_3, \mathcal{C}_2) = 0.9250$, $\mathrm{Sup}(\mathcal{C}_2, \mathcal{C}_4) = \mathrm{Sup}(\mathcal{C}_4, \mathcal{C}_2) = 0.8250$ and $\mathrm{Sup}(\mathcal{C}_3, \mathcal{C}_4) = \mathrm{Sup}(\mathcal{C}_4, \mathcal{C}_3) = 0.9000$. Now, using the equation $V_{\tau(j)} = 1 + \sum\limits_{\substack{l=1 \\ l \neq j}}^{n} \big(\mathrm{Sup}(\mathcal{C}_j, \mathcal{C}_l)\big)$ we have $V_{\tau(1)} = 3.6250$, $V_{\tau(2)} = 3.6750$, $V_{\tau(3)} = 3.7250$ and $V_{\tau(4)} = 3.5250$. Thus, $B_1 = 3.6250$, $B_2 = 7.3000$, $B_3 = 11.0250$ and $B_4 = 14.5500$. Also, $TV = \sum\limits_{j=1}^{n} V_{\tau(j)} = 14.5500$. Now, taking the value of $g(x) = x^2$ and using the Eq. (9.6), we get

$$\varphi_1 = g\left(\frac{B_1}{TV}\right) = g\left(\frac{3.6250}{14.5500}\right) = g(0.2491) = (0.2491)^2 = 0.0621$$

$$\varphi_2 = g\left(\frac{B_2}{TV}\right) - g\left(\frac{B_1}{TV}\right) = g\left(\frac{7.3000}{14.5500}\right) - g\left(\frac{3.6250}{14.5500}\right) = 0.1896$$
$$\varphi_3 = g\left(\frac{B_3}{TV}\right) - g\left(\frac{B_2}{TV}\right) = g\left(\frac{11.0250}{14.5500}\right) - g\left(\frac{7.3000}{14.5500}\right) = 0.3224$$
$$\varphi_4 = g\left(\frac{B_4}{TV}\right) - g\left(\frac{B_3}{TV}\right) = g\left(\frac{14.5500}{14.5500}\right) - g\left(\frac{11.0250}{14.5500}\right) = 0.4258$$

Based on this information, we have

$$\prod_{j=1}^{4} \left(1 - \zeta_{\tau(j)}\right)^{\varphi_j} = 0.4214; \qquad \prod_{j=1}^{4} \vartheta_{\tau(j)}^{\varphi_j} = 0.2709$$
$$\prod_{j=1}^{4} \left(1 - w_{\zeta_{\tau(j)}}\right)^{\varphi_j} = 0.3140; \qquad \prod_{j=1}^{4} \left(w_{\vartheta_{\tau(j)}}\right)^{\varphi_j} = 0.2257$$

Thus, by using Theorem 9.2.6, we get

$$\begin{aligned} \text{CIFOWPA}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4) &= \left(\left(1 - 0.4214, 1 - 0.3140\right), \left(0.2709, 0.2257\right)\right) \\ &= \left(\left(0.5786, 0.6860\right), \left(0.2709, 0.2257\right)\right) \end{aligned}$$

Also, for a collection of CIFNs, it is observed that CIFOWPA operator also satisfies the properties namely, idempotency, commutativity and boundedness.

9.2.5 Power Geometric Aggregation operator

In this section, we have extended the above defined averaging AOs to the geometric AOs under the CIF environment which includes: CIF power geometric (CIFPG) operator, CIF weighted power geometric (CIFWPG) operator and CIF ordered weighted power geometric (CIFOWPG) operator.

Definition 9.2.5. A CIFPG operator, defined on a collection of CIFNs $\mathcal{C}_j (j = 1, 2, \ldots, n)$ is a mapping CIFPG : $\Omega^n \to \Omega$ defined as

$$\text{CIFPG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathcal{C}_1^{\sigma_1} \otimes \mathcal{C}_2^{\sigma_2} \otimes \ldots \otimes \mathcal{C}_n^{\sigma_n} \tag{9.8}$$

where $\sigma_j = \frac{1+T(\mathcal{C}_j)}{\sum_{j=1}^{n}(1+T(\mathcal{C}_j))}$ and $T(\mathcal{C}_j) = \sum_{\substack{l=1 \\ l \neq j}}^{n} \left(\text{Sup}(\mathcal{C}_j, \mathcal{C}_l)\right)$ $(j = 1, 2, \ldots, n)$. Here $\text{Sup}(\mathcal{C}_j, \mathcal{C}_l)$ is the support of $\mathcal{C}_j$ from $\mathcal{C}_l$ defined as $\text{Sup}(\mathcal{C}_j, \mathcal{C}_l) = 1 - \mathcal{D}(\mathcal{C}_j, \mathcal{C}_l)$ where $\mathcal{D}$ is the distance measure.

Theorem 9.2.7. For a collection of CIFNs $\mathcal{C}_j = \left(\left(\zeta_j, w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right)$, the combined value obtained by using CIFPG operator is also a CIFN and is given by

$$\mathrm{CIFPG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} \prod_{j=1}^{n} (\zeta_j)^{\sigma_j}, \\ \prod_{j=1}^{n} \left(w_{\zeta_j}\right)^{\sigma_j} \end{pmatrix}, \begin{pmatrix} 1 - \prod_{j=1}^{n} \left(1 - \vartheta_j\right)^{\sigma_j}, \\ 1 - \prod_{j=1}^{n} \left(1 - w_{\vartheta_j}\right)^{\sigma_j} \end{pmatrix} \right) \tag{9.9}$$

Proof. It can be proved in a similar manner as that of Theorem 9.2.4. □

Definition 9.2.6. Consider a collection of CIFNs $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ with corresponding weights $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ such that, $\xi_j > 0$ and $\sum_{j=1}^{n} \xi_j = 1$. A CIFWPG is a map CIFWPG : $\Omega^n \to \Omega$ defined by:

$$\mathrm{CIFWPG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathcal{C}_1^{\phi_1} \otimes \mathcal{C}_2^{\phi_2} \otimes \ldots \otimes \mathcal{C}_n^{\phi_n} \tag{9.10}$$

where $\phi_j = \frac{\xi_j(1+T'(\mathcal{C}_j))}{\sum_{j=1}^{n} \xi_j(1+T'(\mathcal{C}_j))}$ and $T'(\mathcal{C}_j) = \sum_{\substack{l=1 \\ l \neq j}}^{n} \xi_l \big(\mathrm{Sup}(\mathcal{C}_j, \mathcal{C}_l)\big)$.

Theorem 9.2.8. Let $\mathcal{C}_j = ((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j}))$ be a collection of CIFNs with associated weight vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$, such that, $\xi_j > 0$ and $\sum_{j=1}^{n} \xi_j = 1$. The aggregated value obtained by using CIFWPG operator is also CIFN and is given by

$$\mathrm{CIFWPG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} \prod_{j=1}^{n} (\zeta_j)^{\phi_j}, \\ \prod_{j=1}^{n} \left(w_{\zeta_j}\right)^{\phi_j} \end{pmatrix}, \begin{pmatrix} 1 - \prod_{j=1}^{n} \left(1 - \vartheta_j\right)^{\phi_j}, \\ 1 - \prod_{j=1}^{n} \left(1 - w_{\vartheta_j}\right)^{\phi_j} \end{pmatrix} \right) \tag{9.11}$$

Proof. It can be proved in a similar manner as that of Theorem 9.2.4. □

Definition 9.2.7. A CIFOWPG operator, defined on a collection of CIFNs $\mathcal{C}_j(j = 1, 2, \ldots, n)$ is a mapping CIFOWPG : $\Omega^n \to \Omega$ defined as

$$\mathrm{CIFOWPG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathcal{C}_{\tau(1)}^{\varphi_1} \otimes \mathcal{C}_{\tau(2)}^{\varphi_2} \otimes \ldots \otimes \mathcal{C}_{\tau(n)}^{\varphi_n} \tag{9.12}$$

where $\big(\tau(1), \tau(2), \ldots, \tau(n)\big)$ is an arrangement of $(1, 2, \ldots, n)$ satisfying $\mathcal{S}\left(\mathcal{C}_{\tau(j-1)}\right) \geq \mathcal{S}\left(\mathcal{C}_{\tau(j)}\right)$ for $j = 2, 3, \ldots, n$ and φ_j is given in Eq. (9.6).

Theorem 9.2.9. The aggregated value for a collection of CIFNs $\mathcal{C}_j = ((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j}))$ by using CIFOWPG operator is still CIFN and is given by

$$\mathrm{CIFOWPG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} \prod_{j=1}^{n} \left(\zeta_{\tau(j)}\right)^{\varphi_j}, \\ \prod_{j=1}^{n} \left(w_{\zeta_{\tau(j)}}\right)^{\varphi_j} \end{pmatrix}, \begin{pmatrix} 1 - \prod_{j=1}^{n} \left(1 - \vartheta_{\tau(j)}\right)^{\varphi_j}, \\ 1 - \prod_{j=1}^{n} \left(1 - w_{\vartheta_{\tau(j)}}\right)^{\varphi_j} \end{pmatrix} \right) \tag{9.13}$$

Proof. It can be proved in a similar manner as that of Theorem 9.2.4. □

Further, it can be easily obtained from the above defined geometric operators that they also satisfy the properties of idempotency, boundedness, commutativity and do not satisfy monotonicity.

9.3 MCDM approach using proposed power AOs

The general description of MCDM problem is given in Section 2.5 of Chapter 2. Then, to determine the most desirable alternative(s), the proposed operators are utilized to develop a MCDM method, which involves the following steps:

Step 1: Construct the decision matrix $\mathcal{M}^{(z)} = \left(\mathcal{C}_{uv}^{(z)}\right)_{m\times n}$ corresponding to the rating values of each alternative given by the expert $\mathcal{E}^{(z)}(z = 1, 2, \ldots, k)$.

Step 2: Aggregate the rating values of each expert $\mathcal{M}^{(z)}(z = 1, 2, \ldots, k)$ into the overall collective CIF decision matrix $\mathcal{M} = (\mathcal{C}_{uv})$ where $\mathcal{C}_{uv} = ((\zeta_{uv}, w_{\zeta_{uv}}), (\vartheta_{uv}, w_{\vartheta_{uv}}))$ is calculated either by using CIFOWPA operator as follows

$$\begin{aligned} \mathcal{C}_{uv} &= \mathrm{CIFOWPA}\left(\mathcal{C}_{uv}^{(1)}, \mathcal{C}_{uv}^{(2)}, \ldots, \mathcal{C}_{uv}^{(k)}\right) \\ &= \left(\begin{pmatrix} 1 - \prod_{z=1}^{k} \left(1 - \zeta_{uv}^{\left(\tau(z)\right)}\right)^{\varphi_{uv}^{(z)}}, \\ 1 - \prod_{z=1}^{k} \left(1 - w_{\zeta_{uv}}^{\left(\tau(z)\right)}\right)^{\varphi_{uv}^{(z)}} \end{pmatrix}, \begin{pmatrix} \prod_{z=1}^{k} \left(\vartheta_{uv}^{\left(\tau(z)\right)}\right)^{\varphi_{uv}^{(z)}}, \\ \prod_{z=1}^{k} \left(w_{\vartheta_{uv}}^{\left(\tau(z)\right)}\right)^{\varphi_{uv}^{(z)}} \end{pmatrix} \right) \end{aligned}$$

or by using CIFOWPG operator as follows

$$\begin{aligned}\mathcal{C}_{uv} &= \text{CIFOWPG}\big(\mathcal{C}_{uv}^{(1)}, \mathcal{C}_{uv}^{(2)}, \ldots, \mathcal{C}_{uv}^{(k)}\big) \\ &= \left(\begin{pmatrix} \prod\limits_{z=1}^{k} \left(\zeta_{uv}^{\left(\tau(z)\right)}\right)^{\varphi_{uv}^{(z)}}, \\ \prod\limits_{z=1}^{k} \left(w_{\zeta_{uv}}^{\left(\tau(z)\right)}\right)^{\varphi_{uv}^{(z)}} \end{pmatrix}, \begin{pmatrix} 1 - \prod\limits_{z=1}^{k} \left(1 - \vartheta_{uv}^{\left(\tau(z)\right)}\right)^{\varphi_{uv}^{(z)}}, \\ 1 - \prod\limits_{z=1}^{k} \left(1 - w_{\vartheta_{uv}}^{\left(\tau(z)\right)}\right)^{\varphi_{uv}^{(z)}} \end{pmatrix}\right)\end{aligned}$$

where $\varphi_{uv}^{(1)}, \varphi_{uv}^{(2)}, \ldots, \varphi_{uv}^{(k)}$ are the standardized weights determined by using Eq. (9.6) and τ is the permutation map from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, k\}$.

Step 3: Aggregate the collective rating values $\mathcal{M} = (\mathcal{C}_{uv})$ of the alternative $\mathcal{V}_u (u = 1, 2, \ldots, m)$ into the overall assessment value $\mathcal{C}_u = ((\zeta_u, w_{\zeta_u}), (\vartheta_u, w_{\vartheta_u}))$ based on the either power averaging or geometric AOs. For instance, if we utilize CIFPA operator to aggregate each rating value of the alternative $\mathcal{V}_u$, then we get the overall assessment value $\mathcal{C}_u (u = 1, 2, \ldots, m)$ as

$$\begin{aligned}\mathcal{C}_u &= \text{CIFPA}\big(\mathcal{C}_{u1}, \mathcal{C}_{u2}, \ldots, \mathcal{C}_{un}\big) \\ &= \left(\begin{pmatrix} 1 - \prod\limits_{v=1}^{n} (1 - \zeta_{uv})^{\sigma_v}, \\ 1 - \prod\limits_{v=1}^{n} (1 - w_{\zeta_{uv}})^{\sigma_v} \end{pmatrix}, \begin{pmatrix} \prod\limits_{v=1}^{n} (\vartheta_{uv})^{\sigma_v}, \\ \prod\limits_{v=1}^{n} \left(w_{\vartheta_{uv}}\right)^{\sigma_v} \end{pmatrix}\right)\end{aligned}$$

On the other hand, if we utilize the CIFWPG operator to aggregate the preference values, then we have

$$\begin{aligned}\mathcal{C}_u &= \text{CIFWPG}\big(\mathcal{C}_{u1}, \mathcal{C}_{u2}, \ldots, \mathcal{C}_{un}\big) \\ &= \left(\begin{pmatrix} \prod\limits_{v=1}^{n} (\zeta_{uv})^{\phi_v}, \\ \prod\limits_{v=1}^{n} \left(w_{\zeta_{uv}}\right)^{\phi_v} \end{pmatrix}, \begin{pmatrix} 1 - \prod\limits_{v=1}^{n} (1 - \vartheta_{uv})^{\phi_v}, \\ 1 - \prod\limits_{v=1}^{n} (1 - w_{\vartheta_{uv}})^{\phi_v} \end{pmatrix}\right)\end{aligned}$$

Step 4: Compute the score values of the overall aggregated values $\mathcal{C}_u = ((\zeta_u, w_{\zeta_u}), (\vartheta_u, w_{\vartheta_u}))(u = 1, 2, \ldots, m)$ by using equation

$$\mathcal{S}(\mathcal{C}_u) = \zeta_u - \vartheta_u + w_{\zeta_u} - w_{\vartheta_u}.$$

If there is no difference between two score values then calculate the accuracy values of the alternatives as

$$\mathcal{H}(\mathcal{C}_u) = \zeta_u + \vartheta_u + w_{\zeta_u} + w_{\vartheta_u}.$$

Step 5: Rank all the feasible alternatives $\mathcal{V}_u(u = 1, 2, \ldots, m)$ according to the Definition 6.2.1 of Chapter 6 and hence select the most desirable alternative(s).

9.4 Illustrative Example

To illustrate the proposed approach, we consider example related to the Biometric based attendance devices (BBAD) here.

9.4.1 Description of the problem

Increasing technological advancements have led to various improvements which are quite helpful in the growth and the development of the businesses. One type of such a technology, which is becoming most popular nowadays is biometric technology. The biometric technologies are used to identify human characteristics and verify identity. Biometric based attendance devices (BBAD) are becoming very famous as they read each employee's unique fingerprint, hand shape, face or iris shape and they ensure that the employees cannot clock in for one another and these devices stop employee time theft cases. Due to these characteristics of BBAD, Bharat Sanchar Nagar Limited (BSNL) company, headquartered in New Delhi India, decides to set up BBAD in all of its offices spread all over the country. For achieving this, BSNL authority make a group meeting which consists of General manager, Chief Executive, Chairman and Managing Director and they consult three experts $\mathcal{E}^{(1)}, \mathcal{E}^{(2)}, \mathcal{E}^{(3)}$ to select the best model out of four possible ones namely, $\mathcal{V}_1$: CP Plus, $\mathcal{V}_2$: ESSI, $\mathcal{V}_3$: ESSL X990 and $\mathcal{V}_4$: T 60 with different production dates. The experts evaluate the model of BBAD on the basis of four criteria namely: $\mathfrak{B}_1$: User friendly, $\mathfrak{B}_2$: Provision for data backup, $\mathfrak{B}_3$: Battery backup and $\mathfrak{B}_4$: Employee tracking via GPS. Obviously, these criteria would be affected with the changes in production date. The target of the BSNL company is to choose the most optimal model of BBAD and the production date simultaneously. Thus the problem is two dimensional namely, model of BBAD and production date of BBAD. Therefore, the three experts $(\mathcal{E}^{(z)})(z=1,2,3)$ give their individual preferences corresponding to each alternative in terms of CIFNs. Assume that the weight vectors corresponding to four preferences factors is $\xi=(0.4,0.3,0.15,0.15)^T$. In what follows, we utilize the MCDM method proposed in above section to determine the most desirable alternative(s) under CIF environment.

Step 1: The given three experts evaluate the alternatives under the CIFS environment and their corresponding rating values are summarized in the decision matrices represented in Tables 9.1, 9.2, 9.3 respectively. In these tables, for instance, the rating value corresponding to the expert $\mathcal{E}^{(1)}$ for an alternative $\mathcal{V}_1$ under $\mathfrak{B}_1$ criteria is given as $((0.5,0.4),(0.4,0.5))$ which describes that the first expert is agreed 50% with the suitability of the model $\mathcal{V}_1$ at $\mathfrak{B}_1$ while disagree with 40%. On the other hand, the same expert satisfied 40% with the suitability of production date of BBAD at $\mathfrak{B}_1$ and not satisfied with the 50%. In the similar manner, the other data values can be interpreted.

Step 2a: The different preferences of the experts $\mathcal{C}_{uv}^{(z)}(z=1,2,3)$ are aggregated into a collective one $\mathcal{C}_{uv}(u=1,2,3,4;v=1,2,3,4)$ by taking a function $g(x)=x^2$ and utilizing CIFOWPA AO. The values obtained by using this operator are summarized in Table 9.4.

Step 2b: If we take CIFOWPG AO to aggregate the different preferences $\mathcal{C}_{uv}^{(z)}(z=1,2,3)$ of each expert into the collective one $\mathcal{C}_{uv}(u=1,2,3,4;v=1,2,3,4)$ corresponding to the function $g(x)=x^2$, then the values corresponding to each alternative $\mathcal{V}_u(u=1,2,3,4)$ are summarized in Table 9.5.

Step 3a: Without loss of generality, we take CIFWPA operator to aggregate the different values $\mathcal{C}_{uv}(v=1,2,3,4)$, obtained from Step 2a, by taking weight vector as $\xi=(0.4,0.3,0.15,0.15)^T$. The collective values of each alternative $\mathcal{V}_u(u=1,2,3,4)$ are obtained as $\mathcal{C}_1=((0.5324,0.4393),(0.3407,0.4167)),\mathcal{C}_2=((0.4900,0.4866),(0.3016,0.3683)),\mathcal{C}_3=((0.4787,0.5077),(0.3097,0.3523))$ and $\mathcal{C}_4$=((0.4312, 0.4119), (0.3342, 0.3611)).

Step 3b: If we take CIFWPG operator instead of CIFWPA operator to aggregate the collective values of $\mathcal{C}_{uv}(v = 1, 2, 3, 4)$,obtained from Step 2b then, the overall collective values $\mathcal{C}_u$ of the alternatives $\mathcal{V}_u(u = 1, 2, 3, 4)$ are computed as $\mathcal{C}_1 = ((0.5000, 0.4284), (0.3528, 0.4336))$, $\mathcal{C}_2$=((0.4751,0.4515), (0.3200,0.4014)), $\mathcal{C}_3 = $ ((0.4633,0.4998), (0.3290, 0.3652)) and $\mathcal{C}_4$=((0.4134,0.4007), (0.3570,0.3859)).

Step 4: The score values of the alternative $\mathcal{V}_u(u = 1, 2, 3, 4)$ are obtained based on the overall assessment values $\mathcal{C}_u(u = 1, 2, 3, 4)$ as $\mathcal{S}(\mathcal{C}_1) = 0.2143$, $\mathcal{S}(\mathcal{C}_2) = 0.3066$, $\mathcal{S}(\mathcal{C}_3) = 0.3244$, $\mathcal{S}(\mathcal{C}_4) = 0.1478$. On the other hand, the score values of the alternative $\mathcal{V}_u(u = 1, 2, 3, 4)$ according to the aggregated values obtained from Step 3b are $\mathcal{S}(\mathcal{C}_1) = 0.1420$, $\mathcal{S}(\mathcal{C}_2) = 0.2051$, $\mathcal{S}(\mathcal{C}_3) = 0.2689$, $\mathcal{S}(\mathcal{C}_4) = 0.0712$.

Step 5: Since $\mathcal{S}(\mathcal{C}_3) > \mathcal{S}(\mathcal{C}_2) > \mathcal{S}(\mathcal{C}_1) > \mathcal{S}(\mathcal{C}_4)$ and hence based on it, the ranking of all the feasible alternatives $\mathcal{V}_u(u = 1, 2, 3, 4)$ is shown as follows

$$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4,$$

where the symbol "$\succ$" means "preferred to". Thus, we conclude that the best alternative is $\mathcal{V}_3$, i.e., $\mathcal{V}_3$ is the most optimal device.

Besides this, in order to analyze the influence of the AOs on to the ranking order by changing the operators to aggregate the different preferences of the experts as well as criteria in the Step 2 and Step 3 respectively, an analysis has been conducted whose significance effect and the complete ranking order of the alternatives is represented in the Table 9.6. In this table, the first column depicts the operator used for aggregating individual preferences of three experts into the collective one while the second column presents the operator used for aggregating the preferences obtained in Step 2, corresponding to each alternative for different attributes. In the first four rows of the Table 9.6, an optimistic approach towards the aggregation of the experts values is utilized by taking ordered weighted power averaging AOs in Step 2 while in the next four rows pessimistic approach is utilized by taking ordered weighted power geometric AOs. Further, from this table, it is concluded that final score values of the alternatives are less for pessimistic approach than the optimistic approach. It is also observed that from the final ranking order of the alternatives that if no information about the weights corresponding to criteria is known during the aggregation in Step 3 then the alternative $\mathcal{V}_2$ is more preferable over $\mathcal{V}_3$. However, if the information about the weight vector is already given then $\mathcal{V}_3$ comes out to be more preferred over $\mathcal{V}_2$. Also, the different weights of the criteria lead to different ranking of the alternatives. Therefore, weights corresponding to criteria should be chosen carefully.

9.4.2 Comparative study

In this section, we compare the performance of the proposed MCDM approach with some of the existing approaches [27, 44, 46, 47, 72, 83, 156, 175, 201] under an IFS theory. For it, firstly the considered preferences of the experts are converted into the IFNs by taking the phase terms corresponding to each CIFN to be zero. Then, based on this information, we applied the existing approaches and their corresponding results are discussed as follows.

(i) If we apply the intuitionistic fuzzy power weighted average operator, proposed by Xu [175] on to the considered data, then the overall score values of the alternatives $\mathcal{V}_u (u = 1, 2, 3, 4)$ are computed as $\mathcal{S}(\mathcal{V}_1) = 0.1917$, $\mathcal{S}(\mathcal{V}_2) = 0.1884$, $\mathcal{S}(\mathcal{V}_3) = 0.1690$ and $\mathcal{S}(\mathcal{V}_4) = 0.0970$. Hence, their corresponding ranking order becomes: $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$ which gives that $\mathcal{V}_3$ is the required choice.

(ii) If we utilize the intuitionistic fuzzy weighted geometric interaction averaging operator, proposed by He, Chen, Zhau, Liu and Tao [72], then we get the score values of the alternatives as: $\mathcal{S}(\mathcal{V}_1) = 0.1941$, $\mathcal{S}(\mathcal{V}_2) = 0.1880$, $\mathcal{S}(\mathcal{V}_3) = 0.1710$, and $\mathcal{S}(\mathcal{V}_4) = 0.0962$. Hence, the corresponding ranking order of the alternatives becomes: $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$ which gives that $\mathcal{V}_1$ is the optimal choice.

(iii) On applying the intuitionistic fuzzy Hamacher weighted averaging operator, proposed by Huang [83], by taking $\gamma = 3$, the score values of the alternatives are obtained as: $\mathcal{S}(\mathcal{V}_1) = 0.1908$, $\mathcal{S}(\mathcal{V}_2) = 0.1858$, $\mathcal{S}(\mathcal{V}_3) = 0.1713$ and $\mathcal{S}(\mathcal{V}_4) = 0.0940$ and based on these score values, the corresponding ranking order of the alternatives is: $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$ which gives that $\mathcal{V}_1$ is the optimal choice.

(iv) If we apply the weighted geometric averaging operator, introduced by Chen and Chang [27], then the score values of the alternatives are: $\mathcal{S}(\mathcal{V}_1) = 0.1891$, $\mathcal{S}(\mathcal{V}_2) = 0.1719$, $\mathcal{S}(\mathcal{V}_3) = 0.1720$ and $\mathcal{S}(\mathcal{V}_4) = 0.0887$. Based on it, the ranking order of the alternatives is: $\mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4$ which gives $\mathcal{V}_1$ is the the most favored device.

(v) On utilizing the intuitionistic fuzzy Einstein weighted averaging operator, introduced by Wang and Liu [156], we get the score values as: $\mathcal{S}(\mathcal{V}_1) = 0.1920$, $\mathcal{S}(\mathcal{V}_2) = 0.1862$, $\mathcal{S}(\mathcal{V}_3) = 0.1718$ and $\mathcal{S}(\mathcal{V}_4) = 0.0946$ and the corresponding ranking becomes: $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$ which gives that $\mathcal{V}_1$ is the required device.

(vi) If we utilize the Einstein weighted geometric interactive averaging operator, given by Garg [44], then score values of the alternatives become: $\mathcal{S}(\mathcal{V}_1) = 0.1950$, $\mathcal{S}(\mathcal{V}_2) = 0.1900$, $\mathcal{S}(\mathcal{V}_3) = 0.1714$ and $\mathcal{S}(\mathcal{V}_4) = 0.0972$. Based on these values, their corresponding ranking order becomes: $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$ which shows that $\mathcal{V}_1$ is the most favored device.

(vii) On applying the Hamacher interactive weighted averaging operator, defined by Garg [46], by taking $\gamma = 3$, we obtain the score values as: $\mathcal{S}(\mathcal{V}_1) = 0.1837$, $\mathcal{S}(\mathcal{V}_2) = 0.1717$, $\mathcal{S}(\mathcal{V}_3) = 0.1699$ and $\mathcal{S}(\mathcal{V}_4) = 0.0868$. Hence, the ranking order of the alternatives becomes: $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$ which shows that $\mathcal{V}_1$ is the optimal choice.

(viii) If we utilize the Einstein interactive weighted averaging operator, introduced by Garg [47], we obtain the score values as: $\mathcal{S}(\mathcal{V}_1) = 0.1856$, $\mathcal{S}(\mathcal{V}_2) = 0.1718$, $\mathcal{S}(\mathcal{V}_3) = 0.1706$ and $\mathcal{S}(\mathcal{V}_4) = 0.0874$. Based on these values, the ranking order of the alternatives becomes: $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$ which gives that $\mathcal{V}_1$ is the best choice.

(ix) On utilizing the intuitionistic fuzzy hybrid weighted arithmetic and geometric AO, introduced by Ye [201], by taking $\lambda = 0.5$, the score values of the alternatives are obtained as: $\mathcal{S}(\mathcal{V}_1) = 0.1857$, $\mathcal{S}(\mathcal{V}_2) = 0.1833$, $\mathcal{S}(\mathcal{V}_3) = 0.1695$ and $\mathcal{S}(\mathcal{V}_4) = 0.0919$ and based on these score values, the corresponding ranking order of the alternatives is: $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_4$ which shows that $\mathcal{V}_1$ is the optimal choice.

From the above study, it is observed that the best alternative is different from the proposed ranking order. This is due to the fact that an IFS contains the information with a real-valued membership and non-membership degrees and only considered amplitude term which causes loss of information during the execution. On the other hand, CIFS contains more information (both the membership and non-membership degrees are complex valued) with amplitude and phase terms than the CFS (contains only complex valued membership degree), IFS (with a real-valued membership and non-membership degrees and only considered amplitude term), FS (with only crisp membership degrees with amplitude term only). Thus, the proposed AOs under CIFSs environment are more generalized than the existing operators.

9.5 Conclusion

The main contribution of this chapter is summarized as follows:

1) New operational laws based on algebraic norm for CIFNs are developed. The properties of these operations are investigated in detail.

2) By considering the interdependence among the arguments to be aggregated, a series of the power averaging and geometric AOs named as CIFPA, CIFWPA, CIFOWPA, CIFPG, CIFWPG and CIFOWPG is presented and their properties are proved.

3) Based on the proposed operators, a decision-making approach is presented to find the best alternative in the CIFS environment. An illustrative example is taken for illustrating the developed approach and its results are compared with some of the existing approaches in order to validate it.

Table 9.1: Preferences of the alternatives given by expert $\mathcal{E}^{(1)}$

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.5,0.4),(0.4,0.5)\big)$	$\big((0.6,0.4),(0.3,0.5)\big)$	$\big((0.3,0.6),(0.5,0.3)\big)$	$\big((0.4,0.5),(0.3,0.5)\big)$
$\mathcal{V}_2$	$\big((0.7,0.6),(0.2,0.2)\big)$	$\big((0.4,0.6),(0.3,0.3)\big)$	$\big((0.6,0.7),(0.3,0.2)\big)$	$\big((0.4,0.5),(0.2,0.3)\big)$
$\mathcal{V}_3$	$\big((0.5,0.5),(0.2,0.3)\big)$	$\big((0.4,0.6),(0.2,0.3)\big)$	$\big((0.6,0.5),(0.3,0.3)\big)$	$\big((0.5,0.6),(0.4,0.2)\big)$
$\mathcal{V}_4$	$\big((0.4,0.3),(0.3,0.5)\big)$	$\big((0.7,0.5),(0.3,0.4)\big)$	$\big((0.5,0.4),(0.3,0.3)\big)$	$\big((0.6,0.3),(0.2,0.4)\big)$

Table 9.2: Preferences of the alternatives given by expert $\mathcal{E}^{(2)}$

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.6,0.5),(0.3,0.4)\big)$	$\big((0.7,0.5),(0.3,0.3)\big)$	$\big((0.4,0.4),(0.3,0.5)\big)$	$\big((0.5,0.4),(0.3,0.3)\big)$
$\mathcal{V}_2$	$\big((0.4,0.3),(0.4,0.6)\big)$	$\big((0.6,0.5),(0.2.(0.4)\big)$	$\big((0.5,0.6),(0.5,0.3)\big)$	$\big((0.5,0.5),(0.3,0.4)\big)$
$\mathcal{V}_3$	$\big((0.5,0.5),(0.4,0.4)\big)$	$\big((0.6,0.5),(0.2,0.3)\big)$	$\big((0.4,0.6)),(0.3,0.2)\big)$	$\big((0.5,0.5),(0.4,0.3)\big)$
$\mathcal{V}_4$	$\big((0.4,0.4),(0.5,0.2)\big)$	$\big((0.4,0.5),(0.2,0.5)\big)$	$\big((0.6,0.5),(0.3,0.4)\big)$	$\big((0.4,0.5),(0.4,0.4)\big)$

Table 9.3: Preferences of the alternatives given by expert $\mathcal{E}^{(3)}$

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.8,0.6),(0.2,0.3)\big)$	$\big((0.4,0.5),(0.4,0.3)\big)$	$\big((0.3,0.4),(0.5,0.3)\big)$	$\big((0.6,0.3),(0.3,0.5)\big)$
$\mathcal{V}_2$	$\big((0.5,0.3),(0.3,0.4)\big)$	$\big((0.6,0.5),(0.2,0.2)\big)$	$\big((0.5,0.6),(0.3,0.3)\big)$	$\big((0.6,0.5),(0.2,0.4)\big)$
$\mathcal{V}_3$	$\big((0.7,0.6),(0.2,0.4)\big)$	$\big((0.5,0.5),(0.4,0.4)\big)$	$\big((0.3,0.4),(0.4,0.5)\big)$	$\big((0.4,0.4),(0.3,0.4)\big)$
$\mathcal{V}_4$	$\big((0.5,0.5),(0.2,0.4)\big)$	$\big((0.3,0.4),(0.4,0.3)\big)$	$\big((0.5,0.5),(0.3,0.2)\big)$	$\big((0.4,0.4),(0.5,0.4)\big)$

Table 9.4: Aggregated values of experts by CIFOWPA operator

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.5800, 0.4602), (0.3366, 0.4387)\big)$	$\big((0.5569, 0.4470), (0.3302, 0.3979)\big)$	$\big((0.3340, 0.4266), (0.4239, 0.3539)\big)$	$\big((0.5306, 0.3849), (0.3000, 0.4724)\big)$
$\mathcal{V}_2$	$\big((0.4750, 0.3391), (0.3376, 0.4666)\big)$	$\big((0.5006, 0.5575), (0.2497, 0.3159)\big)$	$\big((0.5120, 0.6123), (0.3982, 0.2871)\big)$	$\big((0.4819, 0.5000), (0.2507, 0.3636)\big)$
$\mathcal{V}_3$	$\big((0.5271, 0.5120), (0.2939, 0.3632)\big)$	$\big((0.4817, 0.5360), (0.2931, 0.3516)\big)$	$\big((0.3765, 0.4878), (0.3512, 0.3457)\big)$	$\big((0.4475, 0.4610), (0.3417, 0.3357)\big)$
$\mathcal{V}_4$	$\big((0.4120, 0.3591), (0.3393, 0.3607)\big)$	$\big((0.3952, 0.4471), (0.3066, 0.3680)\big)$	$\big((0.5356, 0.4466), (0.3000, 0.3154)\big)$	$\big((0.4252, 0.4261), (0.4213, 0.4000)\big)$

Table 9.5: Aggregated values of experts by using CIFOWPG operator

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.5596, 0.4508), (0.3478, 0.4484)\big)$	$\big((0.5334, 0.4420), (0.3351, 0.4188)\big)$	$\big((0.3292, 0.4185), (0.4425, 0.3722)\big)$	$\big((0.5136, 0.3672), (0.3000, 0.4810)\big)$
$\mathcal{V}_2$	$\big((0.4572, 0.3221), (0.3486, 0.5066)\big)$	$\big((0.4805, 0.5525), (0.2564, 0.3256)\big)$	$\big((0.5100, 0.6101), (0.4191, 0.2898)\big)$	$\big((0.4739, 0.5000), (0.2573, 0.3686)\big)$
$\mathcal{V}_3$	$\big((0.5187, 0.5100), (0.3182, 0.3682)\big)$	$\big((0.4736, 0.5315), (0.3174, 0.3571)\big)$	$\big((0.3578, 0.4708), (0.3566, 0.3907)\big)$	$\big((0.4425, 0.4515), (0.3472, 0.3470)\big)$
$\mathcal{V}_4$	$\big((0.4100, 0.3491), (0.3640, 0.4044)\big)$	$\big((0.3632, 0.4421), (0.3274, 0.3857)\big)$	$\big((0.5311, 0.4416), (0.3000, 0.3249)\big)$	$\big((0.4175, 0.4180), (0.4416, 0.4000)\big)$

Table 9.6: Effect of the operators used in Step 2 and Step 3 on ranking order of alternatives

Operator used in Step 2	Operator used in Step 3	Overall score values of				Ranking order
		$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
CIFOWPA	CIFPA	0.1829	0.3516	0.2933	0.1669	$\mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4$
CIFOWPA	CIFPG	0.1532	0.3111	0.2837	0.1511	$\mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4$
CIFOWPA	CIFWPA	0.2143	0.3066	0.3244	0.1478	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4$
CIFOWPA	CIFWPG	0.1919	0.2623	0.3159	0.1349	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4$
CIFOWPG	CIFPA	0.1325	0.3128	0.2437	0.1150	$\mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4$
CIFOWPG	CIFPG	0.1043	0.2639	0.2334	0.0961	$\mathcal{V}_2 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4$
CIFOWPG	CIFWPA	0.1631	0.2591	0.2778	0.0870	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4$
CIFOWPG	CIFWPG	0.1420	0.2051	0.2689	0.0712	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4$

Chapter 10

New generalized Bonferroni mean aggregation operators of complex intuitionistic fuzzy information based on Archimedean t-norm and t-conorm[1]

This chapter presents AOs namely generalized CIF Bonferroni mean and generalized CIF weighted Bonferroni mean which encapsulate the interaction among the criteria and preferences under CIF conditions. Some properties related to proposed operators are investigated. Based on the developed operators, a decision-making method is put forward and is explained with the aid of an example. The reliability of the presented decision-making method is examined with the help of a validity test and by comparing the results of the example with several predominating studies.

10.1 Introduction

In the modern DM environment, parameters considered in DM process are usually dependent on each other. In other words, the interrelationship among the objects occurs

[1]The content of this chapter is published as "New generalized Bonferroni mean aggregation operators of complex intuitionistic fuzzy information based on Archimedean t-norm and t-conorm", *Journal of Experimental and Theoretical Artificial Intelligence*, 32(1), pp. 81-109, 2020 (**SCI: Impact Factor: 2.340**).

frequently such as cost, efficiency, reliability etc., which directly influence the decision outcome. Therefore, it is essential to consider such interrelationship into the analysis in order to reach at optimal conclusion. To address this, Bonferroni [17] suggested a mean-type operator named as BM, which encapsulates the interconnection among the criteria and preferences. Due to this characteristic of BM operator, researchers are paying more attention to it. For example, Yager [192] introduced the BM operator with its interpretation and suggested its generalization. Xu and Yager [188] pointed out that BM operator can be used for aggregating not only crisp numbers but can be used for fuzzy arguments also and introduced BM operator for IFS theory. Li et al. [95] generalized BM operator and geometric mean and hence, proposed generalized Bonferroni mean and investigated their desirable properties. The extensive literature review on AOs which aggregate dependent arguments is done in section 1.1.4 of Chapter 1. Furthermore, it is analyzed that the CIFSs can express the data with a wider range and handle the two dimensional information at the same time. Based on the analysis of CIFS theory and BM operator, we can derive the following results:

1) The CIFSs are the successful means to deal with the uncertain and periodic information than IFSs and their representation is more flexible and broader for solving DMPs.

2) The existing algebraic AOs consider the independent nature of arguments during aggregation process. On the other hand, BM operator has a prominent characteristic of aggregating the different values by considering the interrelationship among the pairs of arguments.

3) The structure of BM operator gives two flexible parameters p and q which reflect the attitude character towards the decision-making process.

Based on the aforementioned comprehensive analysis, it is important and useful to develop BM operator under CIFS environment. The aim of this chapter is as follows:

1) to represent the information of the expert in terms of CIFSs.

2) to propose some new AOs called as generalized CIF Bonferroni mean (GCIFBM) and generalized CIF weighted Bonferroni mean (GCIFWBM) to accumulate the rating

values of the decision-makers and further discuss some special cases of the proposed operators.

3) to develop an algorithm to solve the MCDM problems and delineate it through an example.

10.2 BM operator under CIFS environment

In this section, we propose GCIFBM and GCIFWBM operators under CIF environment and investigate their various properties. For this, throughout this section, we consider CIFNs $\mathcal{C}_j = \big((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j})\big)$ $(j = 1, 2, \ldots, n)$ and Ω as a set of all CIFNs.

10.2.1 GCIFBM operator

We define generalized BM operator, using t-norm and conorm, under CIF environment, as follows:

Definition 10.2.1. For positive real numbers p and q, we define a map $\mathrm{GCIFBM}^{p,q} : \Omega^n \to \Omega$ by

$$\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{j,l=1 \\ j \neq l}}^{n} \left(\mathcal{C}_j^p \otimes \mathcal{C}_l^q \right) \right) \right)^{\frac{1}{p+q}}. \tag{10.1}$$

Then, $\mathrm{GCIFBM}^{p,q}$ is called GCIFBM operator.

Theorem 10.2.1. The aggregated value acquired on applying GCIFBM operator on CIFNs $\mathcal{C}_j$ remains CIFN and is given as

$$\begin{aligned} &\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\ = &\left(\begin{pmatrix} t^{-1}\left(\dfrac{1}{p+q} t\left(\mathcal{A}(\zeta_j, \zeta_l)\right)\right), \\ t^{-1}\left(\dfrac{1}{p+q} t\left(\mathcal{A}(w_{\zeta_j}, w_{\zeta_l})\right)\right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\dfrac{1}{p+q} s\left(\mathcal{B}(\vartheta_j, \vartheta_l)\right)\right), \\ s^{-1}\left(\dfrac{1}{p+q} s\left(\mathcal{B}(w_{\vartheta_j}, w_{\vartheta_l})\right)\right) \end{pmatrix} \right) \end{aligned} \tag{10.2}$$

where the terms $\mathcal{A}$ and $\mathcal{B}$ are defined as

$$\mathcal{A}(x_j, x_l) = s^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} s\Big(t^{-1}\big(pt(x_j)+qt(x_l)\big)\Big)\right) \tag{10.3}$$

$$\text{and } \mathcal{B}(x_j, x_l) = t^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} t\Big(s^{-1}\big(ps(x_j)+qs(x_l)\big)\Big)\right) \tag{10.4}$$

Proof is given in Section 10.6.

Example 10.2.1. Let $\mathcal{C}_1 = \big((0.5, 0.6), (0.1, 0.2)\big)$, $\mathcal{C}_2 = \big((0.4, 0.3), (0.2, 0.5)\big)$, $\mathcal{C}_3 = \big((0.7, 0.4), (0.2, 0.3)\big)$ be three CIFNs and let $p, q = 1$. Consider the additive generator $t(a) = -\log a$ for $0 < a \leq 1$ with $t(0) = \infty$. Then, Eq. (10.2) reduces to the following Eq. (10.5) given as:

$$\begin{aligned}
&\mathrm{GCIFBM}^{1,1}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)\\
&= \left(\left(\begin{array}{c}\left(\prod\limits_{\substack{j,l=1\\ j\neq l}}^{3}(1-\zeta_j\zeta_l)^{\frac{1}{3(3-1)}}\right)^{\frac{1}{2}},\\ \left(\prod\limits_{\substack{j,l=1\\ j\neq l}}^{3}(1-w_{\zeta_j}w_{\zeta_l})^{\frac{1}{3(3-1)}}\right)^{\frac{1}{2}}\end{array}\right), \left(\begin{array}{c}1-\left(1-\prod\limits_{\substack{j,l=1\\ j\neq l}}^{3}\big(1-(1-\vartheta_j)(1-\vartheta_l)\big)^{\frac{1}{3(3-1)}}\right)^{\frac{1}{2}},\\ 1-\left(1-\prod\limits_{\substack{j,l=1\\ j\neq l}}^{3}\big(1-(1-w_{\vartheta_j})(1-w_{\vartheta_l})\big)^{\frac{1}{3(3-1)}}\right)^{\frac{1}{2}}\end{array}\right)\right)
\end{aligned} \tag{10.5}$$

Based on these information, we have

$$\begin{aligned}
\prod_{\substack{j,l=1\\ j\neq l}}^{3}(1-\zeta_j\zeta_l)^{\frac{1}{3(3-1)}} &= \big(1-0.5\times 0.4\big)^{\frac{1}{6}}\times\big(1-0.5\times 0.7\big)^{\frac{1}{6}}\times\big(1-0.4\times 0.5\big)^{\frac{1}{6}}\times\\
&\quad\times\big(1-0.4\times 0.7\big)^{\frac{1}{6}}\times\big(1-0.7\times 0.5\big)^{\frac{1}{6}}\times\big(1-0.7\times 0.4\big)^{\frac{1}{6}}\\
&= 0.7207
\end{aligned}$$

Similarly, we can obtain $\prod\limits_{\substack{j,l=1\\ j\neq l}}^{3}\Big(1-\big(1-\vartheta_j\big)\big(1-\vartheta_l\big)\Big)^{\frac{1}{3(3-1)}} = 0.3045$; $\prod\limits_{\substack{j,l=1\\ j\neq l}}^{3}\Big(1-w_{\zeta_j}w_{\zeta_l}\Big)^{\frac{1}{3(3-1)}} = 0.8185$ and $\prod\limits_{\substack{j,l=1\\ j\neq l}}^{3}\big(1-\big(1-w_{\vartheta_j}\big)\big(1-w_{\vartheta_l}\big)\big)^{\frac{1}{3(3-1)}} = 0.5557$. Thus, by using Eq. (10.5), we get $\mathrm{CIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \Big((0.5285, 0.4260), (0.1660, 0.3335)\Big)$.

Further, we analyze that certain properties hold under GCIFBM operator and these are explained as follows:

Property 10.2.1. (Idempotency) Let $\mathcal{C}_0$ be another CIFN such that $\mathcal{C}_j = \mathcal{C}_0$ for $j = 1, 2, \ldots, n$. Then, we have $\text{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2 \ldots \mathcal{C}_n) = \mathcal{C}_0$.

Property 10.2.2. (Monotonicity) For any two collections of CIFNs $\mathcal{C}_j = \big((\zeta_{\mathcal{C}_j}, w_{\zeta_{\mathcal{C}_j}}), (\vartheta_{\mathcal{C}_j}, w_{\vartheta_{\mathcal{C}_j}})\big)$ and $\mathcal{Z}_j = \big((\zeta_{\mathcal{Z}_j}, w_{\zeta_{\mathcal{Z}_j}}), (\vartheta_{\mathcal{Z}_j}, w_{\vartheta_{\mathcal{Z}_j}})\big)$ satisfying $\zeta_{\mathcal{C}_j} \leq \zeta_{\mathcal{Z}_j}$, $\vartheta_{\mathcal{C}_j} \geq \vartheta_{\mathcal{Z}_j}$, $w_{\zeta_{\mathcal{C}_j}} \leq w_{\zeta_{\mathcal{Z}_j}}$ and $w_{\vartheta_{\mathcal{C}_j}} \geq w_{\vartheta_{\mathcal{Z}_j}}$ $\forall\ j$, we have $\text{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \text{GCIFBM}^{p,q}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n)$.

Property 10.2.3. (Boundedness) For CIFNs $\mathcal{C}_j$, we have

$$\mathcal{C}^- \subseteq \text{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2 \ldots \mathcal{C}_n) \subseteq \mathcal{C}^+$$

where $\mathcal{C}^- = \big(\big(\min_j\{\zeta_j\}, \min_j\{w_{\zeta_j}\}\big), \big(\max_j\{\vartheta_j\}, \max_j\{w_{\vartheta_j}\}\big)\big)$ and $\mathcal{C}^+ = \big(\big(\max_j\{\zeta_j\}, \max_j\{w_{\zeta_j}\}\big), \big(\min_j\{\vartheta_j\}, \min_j\{w_{\vartheta_j}\}\big)\big)$.

Property 10.2.4. (Commutativity:) Let $\big(\dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2, \ldots, \dot{\mathcal{C}}_n\big)$ be an arbitrary arrangement of CIFNs $\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big)$. Then, we have

$$\text{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \text{GCIFBM}^{p,q}(\dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2, \ldots, \dot{\mathcal{C}}_n).$$

Proof of all these properties is given in Section 10.6.

Further, several particular cases of GCIFBM operator are discussed by taking different forms of the generator $t : [0, 1] \to [0, \infty)$ as follows:

(i) If $t(a) = -\log(a)$ then Eq. (10.2) reduces to CIF bonferroni mean:

$$\begin{aligned}
&\text{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
&= \left(\left(\begin{array}{c}
\left(1 - \prod\limits_{\substack{j,l=1\\ j\neq l}}^{n} \left(1 - \zeta_j^p \zeta_l^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, \\
\left(1 - \prod\limits_{\substack{j,l=1\\ j\neq l}}^{n} \left(1 - w_{\zeta_j}^p w_{\zeta_l}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}
\end{array}\right), \left(\begin{array}{c}
1 - \left(1 - \prod\limits_{\substack{j,l=1\\ j\neq l}}^{n} \left(1 - (1-\vartheta_j)^p (1-\vartheta_l)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, \\
1 - \left(1 - \prod\limits_{\substack{j,l=1\\ j\neq l}}^{n} \left(1 - (1-w_{\vartheta_j})^p (1-w_{\vartheta_l})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}
\end{array}\right)\right)
\end{aligned}$$

(ii) If $t(a) = \log\left(\frac{2-a}{a}\right)$ then Eq. (10.2) becomes CIF Einstein Bonferroni mean:

$$\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)$$
$$= \left(\left(\begin{array}{c}\dfrac{2\Big(P-Q\Big)^{\frac{1}{p+q}}}{\Big(P-Q\Big)^{\frac{1}{p+q}} + \Big(P+3Q\Big)^{\frac{1}{p+q}}}, \\ \dfrac{2\Big(P_1-Q_1\Big)^{\frac{1}{p+q}}}{\Big(P_1-Q_1\Big)^{\frac{1}{p+q}} + \Big(P_1+3Q_1\Big)^{\frac{1}{p+q}}}\end{array}\right), \left(\begin{array}{c}\dfrac{\Big(3M+N\Big)^{\frac{1}{p+q}} - \Big(N-M\Big)^{\frac{1}{p+q}}}{\Big(3M+N\Big)^{\frac{1}{p+q}} + \Big(N-M\Big)^{\frac{1}{p+q}}}, \\ \dfrac{\Big(3M_1+N_1\Big)^{\frac{1}{p+q}} - \Big(N_1-M_1\Big)^{\frac{1}{p+q}}}{\Big(3M_1+N_1\Big)^{\frac{1}{p+q}} + \Big(N_1-M_1\Big)^{\frac{1}{p+q}}}\end{array}\right)\right)$$

where $P = f(\zeta_j, \zeta_l)$, $Q = g(\zeta_j, \zeta_l)$, $P_1 = f(w_{\zeta_j}, w_{\zeta_l})$, $Q_1 = g(w_{\zeta_j}, w_{\zeta_l})$, $M = g(1-\vartheta_j, 1-\vartheta_l)$, $N = f(1-\vartheta_j, 1-\vartheta_l)$, $M_1 = g(1-w_{\vartheta_j}, 1-w_{\vartheta_l})$, $N_1 = f(1-w_{\vartheta_j}, 1-w_{\vartheta_l})$ and the functions f and g are defined as $f(x_j, x_l) = \prod\limits_{\substack{j,l=1 \\ j\neq l}}^{n} \Big((2-x_j)^p(2-x_l)^q + 3x_j^p x_l^q\Big)^{\frac{1}{n(n-1)}}$, $g(x_j, x_l) = \prod\limits_{\substack{j,l=1 \\ j\neq l}}^{n} \Big((2-x_j)^p(2-x_l)^q - x_j^p x_l^q\Big)^{\frac{1}{n(n-1)}}$.

(iii) If $t(a) = \log\left(\frac{\gamma+(1-\gamma)a}{a}\right)$, $\gamma \in (0, \infty)$ then, Eq. (10.2) reduces to CIF Hamacher Bonferroni mean:

$$\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n)$$
$$= \left(\left(\begin{array}{c}\dfrac{\gamma\Big(U-V\Big)^{\frac{1}{p+q}}}{\Big(U-(1-\gamma^2)V\Big)^{\frac{1}{p+q}} - (1-\gamma)\Big(U-V\Big)^{\frac{1}{p+q}}}, \\ \dfrac{\gamma\Big(U_1-V_1\Big)^{\frac{1}{p+q}}}{\Big(U_1-(1-\gamma^2)V_1\Big)^{\frac{1}{p+q}} - (1-\gamma)\Big(U_1-V_1\Big)^{\frac{1}{p+q}}}\end{array}\right), \left(\begin{array}{c}\dfrac{\Big(X-(1-\gamma^2)Y\Big)^{\frac{1}{p+q}} - \Big(X-Y\Big)^{\frac{1}{p+q}}}{\Big(X-(1-\gamma^2)Y\Big)^{\frac{1}{p+q}} - (1-\gamma)\Big(X-Y\Big)^{\frac{1}{p+q}}}, \\ \dfrac{\Big(X_1-(1-\gamma^2)Y_1\Big)^{\frac{1}{p+q}} - \Big(X_1-Y_1\Big)^{\frac{1}{p+q}}}{\Big(X_1-(1-\gamma^2)Y_1\Big)^{\frac{1}{p+q}} - (1-\gamma)\Big(X_1-Y_1\Big)^{\frac{1}{p+q}}},\end{array}\right)\right)$$

where $U = f_1(\zeta_j, \zeta_l)$, $V = g_1(\zeta_j, \zeta_l)$, $U_1 = f_1(w_{\zeta_j}, w_{\zeta_l})$, $V_1 = g(w_{\zeta_j}, w_{\zeta_l})$, $X = f_1(1-\vartheta_j, 1-\vartheta_l)$, $Y = g_1(1-\zeta_j, 1-\zeta_l)$, $X_1 = f_1(1-w_{\vartheta_j}, 1-w_{\vartheta_l})$, $Y_1 = g_1(1-w_{\zeta_j}, 1-w_{\zeta_l})$, and the functions f_1 and g_1 are defined as

$$f_1(x_j, x_l) = \prod_{\substack{j,l=1 \\ j\neq l}}^{n} \left(\Big(\gamma+(1-\gamma)x_j\Big)^p\Big(\gamma+(1-\gamma)x_l\Big)^q - (1-\gamma^2)x_j^p x_l^q\right)^{\frac{1}{n(n-1)}},$$

and $$g_1(x_j, x_l) = \prod_{\substack{j,l=1 \\ j\neq l}}^{n} \left(\Big(\gamma+(1-\gamma)x_j\Big)^p\Big(\gamma+(1-\gamma)x_l\Big)^q - x_j^p x_l^q\right)^{\frac{1}{n(n-1)}}.$$

Now, by varying the parameters p and q, numerous particular cases of GCIFBM aggregation operator are discussed as follows:

Case 1: If $q \to 0$ then, the terms $\mathcal{A}$ and $\mathcal{B}$ defined in Eq. (10.3) and (10.4) becomes $\mathcal{A}(x_j, x_l) = s^{-1}\Big(\frac{1}{n(n-1)}(n-1)\sum\limits_{j=1}^{n} s\big(t^{-1}\big(pt(x_j)\big)\big)\Big) = s^{-1}\Big(\frac{1}{n}\sum\limits_{j=1}^{n} s\big(t^{-1}\big(pt(x_j)\big)\big)\Big)$ and $\mathcal{B}(x_j, x_l) = t^{-1}\Big(\frac{1}{n}\sum\limits_{j=1}^{n} t\big(s^{-1}\big(ps(x_j)\big)\big)\Big)$. Therefore, Eq. (10.2) becomes

$$\begin{aligned}
&\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
= &\left(\begin{array}{c} \left(\begin{array}{l} t^{-1}\left(\frac{1}{p} t\left(s^{-1}\left(\frac{1}{n}\sum\limits_{j=1}^{n} s\Big(t^{-1}\Big(pt(\zeta_j)\Big)\Big)\right)\right)\right), \\ t^{-1}\left(\frac{1}{p} t\left(s^{-1}\left(\frac{1}{n}\sum\limits_{j=1}^{n} s\Big(t^{-1}\Big(pt(w_{\zeta_j})\Big)\Big)\right)\right)\right) \end{array} \right), \\ \left(\begin{array}{l} s^{-1}\left(\frac{1}{p} s\left(t^{-1}\left(\frac{1}{n}\sum\limits_{j=1}^{n} t\Big(s^{-1}\Big(ps(\vartheta_j)\Big)\Big)\right)\right)\right), \\ s^{-1}\left(\frac{1}{p} s\left(t^{-1}\left(\frac{1}{n}\sum\limits_{j=1}^{n} t\Big(s^{-1}\Big(ps(w_{\vartheta_j})\Big)\Big)\right)\right)\right) \end{array} \right) \end{array} \right) \\
= &\left(\frac{1}{n}\bigoplus_{j=1}^{n} \mathcal{C}_j^p\right)^{\frac{1}{p}}
\end{aligned}$$

Case 2: For $p = 2$ and $q \to 0$, Eq. (10.2) becomes generalized CIF square mean as:

$$\begin{aligned}
&\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
= &\left(\begin{array}{c} \left(\begin{array}{l} t^{-1}\left(\frac{1}{2} t\left(s^{-1}\left(\frac{1}{n}\sum\limits_{j=1}^{n} s\Big(t^{-1}\Big(2t(\zeta_j)\Big)\Big)\right)\right)\right), \\ t^{-1}\left(\frac{1}{2} t\left(s^{-1}\left(\frac{1}{n}\sum\limits_{j=1}^{n} s\Big(t^{-1}\Big(2t(w_{\zeta_j})\Big)\Big)\right)\right)\right) \end{array} \right), \\ \left(\begin{array}{l} s^{-1}\left(\frac{1}{2} s\left(t^{-1}\left(\frac{1}{n}\sum\limits_{j=1}^{n} t\Big(s^{-1}\Big(2s(\vartheta_j)\Big)\Big)\right)\right)\right), \\ s^{-1}\left(\frac{1}{2} s\left(t^{-1}\left(\frac{1}{n}\sum\limits_{j=1}^{n} t\Big(s^{-1}\Big(2s(w_{\vartheta_j})\Big)\Big)\right)\right)\right) \end{array} \right) \end{array} \right) \\
= &\left(\frac{1}{n}\bigoplus_{j=1}^{n} \mathcal{C}_j^2\right)^{\frac{1}{2}}
\end{aligned}$$

Case 3: For $p = 1$ and $q \to 0$ then, Eq. (10.2) is transformed to CIF average as:

$$\begin{aligned} \mathrm{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) &= \left(\begin{pmatrix} s^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} s(\zeta_j)\right), \\ s^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} s(w_{\zeta_j})\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} t(\vartheta_j)\right), \\ t^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} t(w_{\vartheta_j})\right) \end{pmatrix} \right) \\ &= \left(\frac{1}{n} \bigoplus_{j=1}^{n} \mathcal{C}_j \right) \end{aligned}$$

Case 4: For $p = q = 1$, Eq. (10.2) becomes generalized CIF interrelated square mean as:

$$\begin{aligned} &\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\ &= \left(\begin{array}{l} \begin{pmatrix} t^{-1}\left(\frac{1}{2}t\left(s^{-1}\left(\frac{1}{n(n-1)} \sum\limits_{\substack{j,l=1 \\ j \neq l}}^{n} s\Big(t^{-1}\big(t(\zeta_j) + t(\zeta_l)\big)\Big)\right)\right)\right), \\ t^{-1}\left(\frac{1}{2}t\left(s^{-1}\left(\frac{1}{n(n-1)} \sum\limits_{\substack{j,l=1 \\ j \neq l}}^{n} s\Big(t^{-1}\big(t(w_{\zeta_j}) + t(w_{\zeta_l})\big)\Big)\right)\right)\right) \end{pmatrix}, \\ \qquad \begin{pmatrix} s^{-1}\left(\frac{1}{2}s\left(t^{-1}\left(\frac{1}{n(n-1)} \sum\limits_{\substack{j,l=1 \\ j \neq l}}^{n} t\Big(s^{-1}\big(s(\vartheta_j) + s(\vartheta_l)\big)\Big)\right)\right)\right), \\ s^{-1}\left(\frac{1}{2}s\left(t^{-1}\left(\frac{1}{n(n-1)} \sum\limits_{\substack{j,l=1 \\ j \neq l}}^{n} t\Big(s^{-1}\big(s(w_{\vartheta_j}) + s(w_{\vartheta_l})\big)\Big)\right)\right)\right) \end{pmatrix} \end{array} \right) \\ &= \left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{j,l=1 \\ j \neq l}}^{n} \big(\mathcal{C}_j \otimes \mathcal{C}_l\big) \right) \right)^{\frac{1}{2}} \end{aligned}$$

10.2.2 GCIFWBM operator

In the above analysis, equal importance is given to all the data. But in real life situations, this may not be always possible. Some data is more important than the others. So we must take into account the proper weight given to the various CIFNs of data. Therefore, now we propose GCIFWBM operator.

Definition 10.2.2. For CIFNs $\mathcal{C}_j$ with corresponding weight vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ satisfying $\xi_j > 0$; $\sum_{j=1}^{n} \xi_j = 1$ and positive real numbers p and q, we define a map $\text{GCIFWBM}^{p,q} : \Omega^n \to \Omega$ by

$$\text{GCIFWBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{j,l=1 \\ j \neq l}}^{n} \Big(\big(\xi_j \mathcal{C}_j\big)^p \otimes \big(\xi_l \mathcal{C}_l\big)^q \Big) \right) \right)^{\frac{1}{p+q}}. \tag{10.6}$$

Then, $\text{GCIFWBM}^{p,q}$ is called GCIFWBM operator.

Theorem 10.2.2. For CIFNs $\mathcal{C}_j = \big((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j}) \big)$, $(j = 1, 2, \ldots, n)$ having associated weight vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ satisfying $\xi_j > 0$; $\sum_{j=1}^{n} \xi_j = 1$ and positive real numbers p and q, the aggregated value acquired after using GCIFWBM operator is also a CIFN and is given by

$$\text{GCIFWBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} t^{-1}\left(\dfrac{1}{p+q} t\left(\mathcal{A}_1(\zeta_j, \zeta_l)\right) \right), \\ t^{-1}\left(\dfrac{1}{p+q} t\left(\mathcal{A}_1(w_{\zeta_j}, w_{\zeta_l})\right) \right) \end{pmatrix}, \begin{pmatrix} s^{-1}\left(\dfrac{1}{p+q} s\left(\mathcal{B}_1(\vartheta_j, \vartheta_j)\right) \right), \\ s^{-1}\left(\dfrac{1}{p+q} s\left(\mathcal{B}_1(w_{\vartheta_j}, w_{\vartheta_j})\right) \right) \end{pmatrix} \right) \tag{10.7}$$

where the terms $\mathcal{A}_1$ and $\mathcal{B}_1$ are defined as

$$\mathcal{A}_1(x_j, x_l) = s^{-1}\left(\frac{1}{n(n-1)} \sum_{\substack{j,l=1 \\ j \neq l}}^{n} s\left(t^{-1}\left(pt\Big(s^{-1}(\xi_j s(x_j))\Big) + qt\Big(s^{-1}(\xi_l s(x_l))\Big) \right) \right) \right) \tag{10.8}$$

and

$$\mathcal{B}_1(x_j, x_l) = t^{-1}\left(\frac{1}{n(n-1)} \sum_{\substack{j,l=1 \\ j \neq l}}^{n} t\left(s^{-1}\left(ps\Big(t^{-1}(\xi_j t(x_j))\Big) + qs\Big(t^{-1}(\xi_l t(x_l))\Big) \right) \right) \right) \tag{10.9}$$

Proof. Same as the Theorem 10.2.1. □

10.3 Decision making approach based on proposed operators

This section presents an MCDM approach in order to evaluate the available alternatives characterized by different criteria under CIFS environment followed by an illustrative example.

10.3.1 Proposed operator based approach

The general description of MCDM problem is given in Section 2.5 of Chapter 2. Suppose that, an expert evaluated these different alternatives $\mathcal{V}_u$ $(u = 1, 2, \ldots, m)$ under the set of criteria $\mathfrak{B}_v$ $(v = 1, 2, \ldots, n)$ and gave their judgement values in terms of the CIFNs denoted by $\mathcal{C}_{uv} = ((\zeta_{uv}, w_{\zeta_{uv}}), (\vartheta_{uv}, w_{\vartheta_{uv}}))$ where $0 \leq \zeta_{uv}, \vartheta_{uv}, \zeta_{uv} + \vartheta_{uv} \leq 1$ and $0 \leq w_{\zeta_{uv}}, w_{\vartheta_{uv}}, w_{\zeta_{uv}} + w_{\vartheta_{uv}} \leq 1$. The target of the expert is to find out the most optimal alternative(s) among the available ones. For this, the following steps are summarized for solving the MCDM problems under CIFS environment by utilizing the proposed GCIFWBM operator.

Step 1: Collect the preferences given by expert as CIF matrix $\mathcal{M} = (\mathcal{C}_{uv})_{m \times n}$, as follows:

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \ldots & \mathcal{C}_{1n} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \ldots & \mathcal{C}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{m1} & \mathcal{C}_{m2} & \ldots & \mathcal{C}_{mn} \end{pmatrix} \end{array}$$

Step 2: If the criteria possess different types say benefit and cost then, convert cost criteria into benefit criteria by utilizing the following rule:

$$\mathcal{C}'_{uv} = \begin{cases} \big((\zeta_{uv}, w_{\zeta_{uv}}), (\vartheta_{uv}, w_{\vartheta_{uv}})\big) & ; \quad \text{for benefit criteria} \\ \big((\vartheta_{uv}, w_{\vartheta_{uv}}), (\zeta_{uv}, w_{\zeta_{uv}})\big) & ; \quad \text{for cost criteria} \end{cases}$$

and obtain the normalized CIF decision matrix $\mathscr{R} = (\mathcal{C}'_{uv})_{m \times n}$

Step 3: Aggregate the CIFNs $\mathcal{C}'_{uv}(v = 1, 2, \ldots, n)$ for each alternative $\mathcal{V}_u$ into collective one $\mathcal{C}_u = \big((\zeta_u, w_{\zeta_u}),\ (\vartheta_u, w_{\vartheta_u})\big)$ by utilizing the presented GCIFWBM operator given in Eq. (10.7) for real numbers p, q.

Step 4: Obtain the score values of aggregated CIFNs $\mathcal{C}_u = \big((\zeta_u, w_{\zeta_u}),\ (\vartheta_u, w_{\vartheta_u})\big)$ and hence of the alternatives $\mathcal{V}_u$ by using Eq. (10.10) stated as:

$$\mathcal{S}(\mathcal{V}_u) = \zeta_u - \vartheta_u + w_{\zeta_u} - w_{\vartheta_u}. \tag{10.10}$$

However, if any two of the score values of these aggregated numbers are equal then, compute their corresponding accuracy values using the Eq. (10.11) stated as:

$$\mathcal{H}(\mathcal{V}_u) = \zeta_u + \vartheta_u + w_{\zeta_u} + w_{\vartheta_u}. \tag{10.11}$$

Step 5: Obtain the ordering position of alternatives $\mathcal{V}_u$ using Definition 6.2.1.

10.3.2 Illustrative Example

The above MCDM method is illustrated via a case whose results are further compared with several prevailing studies. The description of the problem is as follows:

State Bank of India (SBI) is a prominent financial service providing government-owned Indian bank, having its head office in Mumbai, Maharashtra, with 14 regional hubs and 57 zonal offices which are situated at important cities throughout India. The 122nd Amendment Bill of the constitution of India introduced the Goods and Services Tax (GST) in India with effect from 1 July 2017. GST replaced several existing taxes which include: Services Tax, Central excise duty, State level value-added tax, Luxury tax etc. and therefore, affected all the areas of the business operations. Now, all the offices of the SBI have to comply in accordance with GST law as per the terms and conditions of the Government. Therefore, for the effective implementation of GST law, the higher authorities of the bank make a group meeting in which they decide to procure application software for full compliance under GST law. For this, they consult an information technology (IT) company which provides them information about five GST software namely: $\mathcal{V}_1$: SoftGST, $\mathcal{V}_2$: ClearTax, $\mathcal{V}_3$: Saral GST, $\mathcal{V}_4$: Marg ERP 9+ and $\mathcal{V}_5$: GSPINDIA with different software versions. The SBI authorities constitute a committee of experts which evaluates the GST software $\mathcal{V}_u$ on the basis of four criteria namely $\mathfrak{B}_1$: User-friendly interface, $\mathfrak{B}_2$: Data security, $\mathfrak{B}_3$: Cost and $\mathfrak{B}_4$: Intelligence. Obviously, the changes in the software version for a similar model of software will influence the criteria. The goal of SBI is to find out the most optimal software and its version simultaneously. Therefore, this problem has two dimensions which are: type of GST software and software version. Therefore, the committee of the experts provides their assessment values as CIFNs $\mathcal{C}_{uv}$ $(u = 1, 2, \ldots, 5; v = 1, 2, 3, 4)$

because the CIF model handles two-dimensional information simultaneously. The rating values of the committee for $\mathcal{V}_1$ at $\mathfrak{B}_1$ are given as $\mathcal{C}_{11} = \big((0.6, 0.8), (0.2, 0.2)\big)$ which describes that the committee of the experts is 60% agreed with the suitability of $\mathcal{V}_1$ at $\mathfrak{B}_1$ and 20% disagrees. The phase term that represents the version of the software is given as: the expert is 80% satisfied with the suitability of software version at $\mathfrak{B}_1$ and 20% is not satisfied. In a similar manner, all data of Table 10.1 can be interpreted. The weight vector associated with criteria $\mathfrak{B}_v$ is $\xi = (0.4, 0.3, 0.1, 0.2)^T$. The main procedure steps, for fulfilling the required purpose, by using the proposed method are

Step 1: The rating values of every alternative, given by the committee, are tabulated in Table 10.1.

Step 2: As the criteria $\mathfrak{B}_3$ is of cost type therefore, normalization is needed and the normalized data is given in the Table 10.2.

Step 3: Without loss of generality, by taking $p, q = 1$ and generator $t(a) = -\log(a)$ for $0 < a \leq 1$ with $t(0) = \infty$, the Eq. (10.7) reduces to the Eq. (10.12) given as:

$$\begin{aligned} &\mathrm{GCIFWBM}^{p,q}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \qquad (10.12) \\ &= \left(\left(\begin{array}{c} \left(1 - \prod_{\substack{j,l=1 \\ j \neq l}}^{n} \left(1 - \left(1 - (1-\zeta_j)^{\xi_j}\right)\left(1 - (1-\zeta_l)^{\xi_l}\right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}}, \\ \left(1 - \prod_{\substack{j,l=1 \\ j \neq l}}^{n} \left(1 - \left(1 - (1-w_{\zeta_j})^{\xi_j}\right)\left(1 - (1-w_{\zeta_l})^{\xi_l}\right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \end{array} \right), \left(\begin{array}{c} 1 - \left(1 - \prod_{\substack{j,l=1 \\ j \neq l}}^{n} \left(1 - (1-\vartheta_j)^{\xi_j}(1-\vartheta_l)^{\xi_l} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}}, \\ 1 - \left(1 - \prod_{\substack{j,l=1 \\ j \neq l}}^{n} \left(1 - (1-w_{\vartheta_j})^{\xi_j}(1-w_{\vartheta_l})^{\xi_l} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \end{array} \right) \right) \end{aligned}$$

Now, utilizing Eq. (10.12), in order to accumulate the CIFNs $\mathcal{C}_{uv}$ $(u = 1, 2, \ldots, 5; v = 1, 2, 3, 4)$ into collective one $\mathcal{C}_u$, the corresponding results are summarized as: $\mathcal{C}_1 = \big((0.2040, 0.1939), (0.7338, 0.7643)\big)$, $\mathcal{C}_2 = \big((0.1300, 0.2008), (0.7820, 0.7429)\big)$, $\mathcal{C}_3 = \big((0.1922, 0.2293), (0.7231, 0.7422)\big)$, $\mathcal{C}_4 = \big((0.1703, 0.1690), (0.7597, 0.7214)\big)$ and $\mathcal{C}_5 = \big((0.2425, 0.2015), (0.7160, 0.7169)\big)$.

Step 4: Using these aggregated CIFNs $\mathcal{C}_u$ the score values of alternatives $\mathcal{V}_u$ are obtained as $\mathcal{S}(\mathcal{V}_1) = -1.1002$, $\mathcal{S}(\mathcal{V}_2) = -1.1941$, $\mathcal{S}(\mathcal{V}_3) = -1.0439$, $\mathcal{S}(\mathcal{V}_4) = -1.1418$ and $\mathcal{S}(\mathcal{V}_5) = -0.9889$.

Step 5: From these score values, we conclude that the ordering position of $\mathcal{V}_u$ is $\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$ and hence, the most optimal choice is $\mathcal{V}_5$.

10.3.3 Influence of the parameters p and q on decision-making result

Further, we discuss the impact of parameters p and q on the decision-making results based on the GCIFWBM operator, the ranking results by varying p, q simultaneously are shown in Table 10.3 and the complete variation by fixing one parameter is shown in Figure 10.1. In these results, we arbitrarily choose the four values of q i.e., $q = 2, 5, 8, 10$ and discuss the impact on the score values of the alternatives by varying values of p from 1 to 10. Figure 10.1(a) depicts that for $p < 3.5326$ the ordering position of the alternatives becomes $\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$ and when $3.5326 < p < 8.2591$ the order of the ranking becomes $\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$ and for $p > 8.2591$ the ranking order of the alternatives is $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2$. However, for $p = 3.5326$, we obtain $\mathcal{S}(\mathcal{V}_3) = \mathcal{S}(\mathcal{V}_5) = -0.8593$ and therefore, we calculate the accuracy of the alternatives $\mathcal{V}_1$ and $\mathcal{V}_5$ and we get $\mathcal{H}(\mathcal{V}_3) = 1.8853$ and $\mathcal{H}(\mathcal{V}_5) = 1.8797$. Hence, by using the Definition 6.2.1 we get that $\mathcal{V}_3 \succ \mathcal{V}_5$ for $p = 3.5326$. Similarly, for $p = 8.5291$, we have $\mathcal{S}(\mathcal{V}_1) = \mathcal{S}(\mathcal{V}_5) = -0.6976$ and $\mathcal{H}(\mathcal{V}_1) = 1.8722$; $\mathcal{H}(\mathcal{V}_5) = 1.8931$ and hence, $\mathcal{V}_5 \succ \mathcal{V}_1$. Also, the Figure 10.1 show that the alternative $\mathcal{V}_4$ is always superior to $\mathcal{V}_2$ regardless of p, q values however, the ordering of alternatives $\mathcal{V}_1$, $\mathcal{V}_3$ and $\mathcal{V}_5$ changes by varying p, q values which is shown in Table 10.4.

In the above table p' denotes the value of p for which $\mathcal{S}(\mathcal{V}_1) = \mathcal{S}(\mathcal{V}_5)$. Also, the Figure 10.1 depict that, for each alternative $\mathcal{V}_u$, the maximum score value is obtained at $p = 10; q = 1$ and is $-0.5811, -0.8470, -0.5420, -0.6462$, -0.6171 and the minimum score value is obtained at $p = 1; q = 1$ and is $-1.1002, -1.1941, -1.0439$, $-1.1418, -0.9889$. From the above study, it has been concluded that the parameters p and q can be chosen in accordance with the degree of the positive or negative attitude of the decision maker. If the decision-maker is optimistic then, he/she may choose the large values of the parameters p and q while applying the BM operator. On the contrary, if the decision-maker is pessimistic then, he/she may choose the small values of the parameters p and q. Therefore, for the most pessimistic and most optimistic decision-maker, the parameters p, q can be assigned values 1 ; 1 and 1 ; 10 respectively. The choice of p and q values depends upon the preferences of decision-maker and the nature of the DM problem.

To further prove the effectiveness of the proposed method in this chapter, the impact of a change in p, q values on the decision taken by the decision-maker(s) based on

GCIFWBM operator is discussed. Figure 10.2 depict the influence on score values by varying p, q values from 1 to 10. From these figures, it is evident that the ranking order of the alternatives varies if we alter the values of p and q. However, the optimal alternative remains either $\mathcal{V}_3$ or $\mathcal{V}_5$ and the worst alternative is always $\mathcal{V}_2$. Besides this, the figures also depict that by interchanging the values of p and q, the corresponding score values of the alternatives remain unchanged and this property of interconvertability of p and q is also evident from Eq. (10.7). This impact of p and q esteems on the decision makes the presented MCDM method more adaptable as the decision-maker(s) can pick the parameters by their inclination and handy circumstances.

10.4 Comparative analysis

In order to show the superiority of proposed method, the results of the presented approach are compared with prevailing CIFS studies [6, 59, 129, 130] as well as IFS studies [156, 175, 188] and are elaborated as follows:

10.4.1 With CIFS studies

In order to prove the validity, superiority, and effectiveness of the proposed MCDM approach, we solve the presented illustrative example by using the existing CIF weighted power averaging (CIFWPA) operator [130], CIF weighted averaging (CIFWA) operator [59], distance measure [6] and weighted Euclidean distance measure [129] under CIF environment. For comparing the results with distance measures given in [6, 129], we take the the positive ideal alternative (PIA) ($\mathcal{V}^+$) as ideal alternative whose preferences are: $\mathcal{V}^+ = \{\mathcal{V}_1^+, \mathcal{V}_2^+, \ldots, \mathcal{V}_n^+\}$ where $\mathcal{V}_v^+ = \Big(\big(\max\limits_u\{\zeta_{uv}\}, \max\limits_u\{w_{\zeta_{uv}}\}\big), \big(\min\limits_u\{\vartheta_{uv}\}, \min\limits_u\{w_{\vartheta_{uv}}\}\big)\Big)$. Without loss of generality, we take $p = q = 1$ and additive generator $t(a) = -\log(a)$ for $0 < a \leq 1$ with $t(0) = \infty$ in the proposed GCIFWBM operator and then the results obtained by above studies are represented in Table 10.5. The results in Table 10.5 depict that the ranking order of alternatives obtained using the prevailing measures [6, 129] and the operators [59, 130] is identical with the one obtained on utilizing the proposed GCIFWBM operator for the cases $p = 1; q = 1$ and $p = 1; q = 0$ and is different when

$p = 1; q = 10$. Although the ordering position of alternatives remains same in some cases but the computational difference between the existing methods [6, 59, 129, 130] and the proposed approach is illustrated as follows:

(i) The CIFWPA operator [130] is based on the simple algebraic operations but the proposed GCIFWBM operator is based on generalized t-norm and co-norm operations. The algebraic operation is only the special case of the t-norm and co-norm operations which can be obtained by taking $t(a) = -\log(a)$ for $0 < a \leq 1$ with $t(0) = \infty$. Hence, the presented operators are more generalized. Moreover, the proposed operator captures the interrelationship among the arguments to be aggregated whereas the CIFWPA operator [130] do not have this property. Besides this, the presence of the parameters p and q in the proposed operator makes the proposed approach more flexible.

(ii) The CIFWA operator [59] is basic arithmetic weighted averaging operator which fuses the arguments by considering that the arguments to be aggregated are independent. In fact, by taking $p = 1$ and $q = 0$ the proposed GCIFWBM operator does not consider the dependency of the arguments and reduces to CIF simple arithmetic averaging operator. The ranking order of the alternatives obtained by using the proposed operator for the case $p = 1$ and $q = 0$ is given in the Table 10.5 and is identical with the one obtained on utilizing CIFWA operator which shows the validity and feasibility of the proposed operator. Thus the proposed operator can be utilized in both the cases whether the arguments are dependent or independent. If the arguments to be aggregated are dependent then the decision-maker may choose q to be any positive real number otherwise q can be taken as 0 for the case when there is no interconnection among the arguments. Hence, the proposed operator is more generalized than CIFWA operator.

(iii) The ranking order of the alternatives obtained using the distance measure [6] and weighted Euclidean distance [129] remains identical with the proposed GCIFWBM operator for some cases. But the results acquired by them do not take into account any interconnection among the arguments and cannot integrate information. Besides this, in [6, 129], the optimality of alternative(s) is assessed based on the distance between the alternative and PIA. However, taking into account the distance between alternative and PIA alone or negative ideal alternative (NIA) alone is not sufficient analysis to conclude the optimality of an alternative. The larger the distance between alternative and NIA and the smaller the distance between alternative and PIA, the better the alternative would be. Therefore, the distance of an alternative from PIA and NIA should be considered simultaneously. Moreover, the decision making methods given in [6, 129] neglect the decision maker's preferences and situations of the problem whereas the presence of parameters p and q in the proposed GCIFWBM operator allows the decision-maker to choose parameters in accordance with his/her attitude and preferences. Therefore, the proposed method is more effective and valid.

10.4.2 With IFS studies

In this section, we compare the proposed approach results with existing IFS studies [156, 175, 188]. For this, the phase terms corresponding to membership and non-membership degrees in each CIFN has been set to zero. Then, a comparative analysis is conducted with IFWBM operator [188], intuitionistic fuzzy power weighted averaging (IFPWA) operator [175] and intuitionistic fuzzy weighted einstein averaging (IFEWA) operator [156] and the corresponding results are tabulated in Table 10.6.

(i) The values depicts that on applying the IFWBM operator [188], the obtained score values are identical with the one obtained using proposed GCIFWBM operator. Thus, by setting the phase term of membership and non-membership degrees corresponding to each CIFN equal to zero and on taking the additive generator $t(a) = -\log(a)$ for $0 < a \leq 1$ with $t(0) = \infty$ in the proposed GCIFWBM operator, this operator reduces to the existing IFWBM operator [188]. This gives that the presented operator can handle CIFS as well as IFS data and the existing IFWBM operator is just one of its special cases. Moreover, the IFWBM operator is based on simple algebraic operations. On the other hand, the proposed GCIFBM operator is based on t-norm and co-norm operations and algebraic operations are just special cases of t-norm and co-norm operations.

(ii) The ranking order of alternatives obtained using the IFWPA operator [175] is identical with the proposed operator results. But there are computational differences between the proposed operators and the prevailing IFWPA operator. The IFWPA operator fuses one-dimensional data whereas the proposed GCIFWBM operator aggregates two-dimensional data. Besides this, IFWPA operator arguments reinforce and support each other. This operator does not consider the direct interrelationship among all pairs of arguments to be aggregated whereas the proposed operators have this property.

(iii) From the table, we can see that on using the existing IFEWA operator [156], the ordering position of the alternatives remains same with the one obtained on utilizing the proposed GCIFWBM operator. The IFEWA operator aggregates the argument by assuming that the arguments to be fused are independent. Also, by taking $q = 0$ in the proposed GCIFWBM operator, it does not consider the interconnection among arguments. Therefore, by taking $p = 1; q = 0$, the obtained corresponding results are tabulated in Table 10.6 which are identical with the existing IFEWA operator. This shows the reliability and validity of the proposed operator and also it can be used in both the situations whether the arguments are dependent or independent. Moreover, the prevailing IFEWA operator is based on Einstein operations which are just one of the special cases of t-norm and co-norm operations. Hence, the presented operator is more generalized and can be used in complex situations as well.

10.4.3 Further discussion

In addition to the above comparative studies, we give some characteristic comparison of our proposed MCDM approach and the DM methods proposed in [6, 59, 129, 130, 156, 175, 188] which are tabulated in Table 10.7.

In this table, the symbol '✓' describes that the associated DM approach uses generalized operators based on t-norm and co-norm, encapsulates the interrelationship among criteria, tackles with time-periodic problems, can represent two-dimensional information simultaneously, can fuse information and reflect decision-maker's preferences whereas the symbol '$\times$' means that the corresponding method fails. The values, tabulated in Table 10.7

depict that the operators presented in [59] and our proposed AOs are based on t-norms and co-norms. Also, the operators proposed in [188] are special cases of proposed operators and therefore, our presented work is more generalized and can be utilized to solve DM problems under FS, IFS and CFS environment also. The decision-maker may choose the desired norm during the aggregation process in accordance with his/her attitude and situation. Further, the AOs presented in [188] and our proposed operators take into account the interrelationship among the arguments during aggregation process whereas the operators are given in [59, 156] accumulate the numbers by considering that the arguments to be collected are independent of one another. Besides this, the MCDM methods proposed in [156, 175, 188] deal with real membership and non-membership degrees, which fail to handle time-periodic problems and cannot represent more than one-dimensional information in one set. On the other hand, the proposed method can handle complex problems which involve periodicity and can aggregate two-dimensional data together in one set. Moreover, the MCDM methods developed in [6, 129] do not have a function of fusing information whereas the approaches defined in [59, 130, 156, 175, 188] integrate the different input arguments into a single representative. Also, the presence of parameters in operators given in [188] and the presented operators allow the decision maker to choose the parameters in accordance with his/her attitude and the situation of the problem. This discussion leads to the conclusion that the presented approach can handle time-periodic complex problems more efficiently which are either difficult or impossible to be solved using existing theories [156, 175, 188].

10.4.4 Validity Test

To validate the efficiency of the proposed MCDM method, we perform certain test given by Wang and Triantaphyllou [162] on to considered problem.

Test by criterion 1: *The best alternative remains same by replacing a non-optimal alternative with another worse alternative:* We choose a non-optimal alternative $\mathcal{V}_2$ and rating value of it is replaced by an arbitrary worse alternative $\mathcal{V}_2'$ which are given as $\mathcal{V}_2' = \big\{\big((0.3, 0.2),\ (0.5, 0.6)\big),\ \big((0.1, 0.5),\ (0.4, 0.4)\big)\ ,\ \big((0.2, 0.1),\ (0.7, 0.8)\big),\ \big((0.1, 0.4),$ $(0.5, 0.4)\big)\big\}$. Now, the proposed approach is applied and hence, the final score values of each alternative $\mathcal{V}_u$ are obtained as: $\mathcal{S}(\mathcal{V}_1) = -1.1002$, $\mathcal{S}(\mathcal{V}_2') = -1.4163$, $\mathcal{S}(\mathcal{V}_3) = -1.0439$, $\mathcal{S}(\mathcal{V}_4) = -1.1419$ and $\mathcal{S}(\mathcal{V}_5) = -0.9889$. Hence, ordering position of alternatives becomes: $\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2'$ which gives that the best alternative is still $\mathcal{V}_5$ and therefore, the presented DM method is valid.

Test by criteria 2 & 3: *An MCDM method shows transitive property and ranking remains same even by decomposing the MCDM problem into smaller problems* Under it, we divide the given DM problem into four smaller problems as: $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$, $\{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5\}$, $\{\mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_1\}$ and $\{\mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_1, \mathcal{V}_2\}$. Now by following the steps of proposed MCDM approach, using GCIFWBM operator, the ranking order corresponding to each subproblem is $\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$, $\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2$, $\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4$ and $\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$ respectively and hence the overall combined ranking is $\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$. Thus, we observe that the combined ranking does not alter.

Hence, we conclude the proposed method fulfill these test criteria.

10.5 Conclusion

The key contribution of this chapter is outlined as follows:

1) New mean-type operators namely GCIFBM and GCIFWBM are presented which are more generalized and reduce to several existing operators by giving different forms to additive generator t and by taking different values of p and q.

2) The proposed operators are developed using generalized TC and TN operations and give the choice to decision-maker to choose appropriate additive generator in accordance with his/her preferences and the situation

3) An MCDM approach is presented for solving DM problems which involve dependent arguments. An example is illustrated in order to justify the application of the presented work in real life. Also, the proposed approach is validated by comparing the results of the example with existing studies and by using validity test criteria.

4) Furthermore, the impact of the parameters p and q used in the proposed AO on to the score values of alternatives is discussed in detail.

10.6 Appendix: Proof of the Results

Proof of the Theorem 10.2.1:

Proof. The first part of the result follows from the Theorem 6.2.5. Now, we will show that Eq. (10.2) holds. For positive real numbers p, q and CIFNs $\mathcal{C}_j$, we have:

$$\begin{aligned}
\mathcal{C}_j^p &= \left(\left(t^{-1}\Big(pt(\zeta_j)\Big), t^{-1}\Big(pt(w_{\zeta_j})\Big)\right), \left(s^{-1}\Big(ps(\vartheta_j)\Big), s^{-1}\Big(ps(w_{\vartheta_j})\Big)\right)\right) \\
\mathcal{C}_l^q &= \left(\left(t^{-1}\Big(qt(\zeta_l)\Big), t^{-1}\Big(qt(w_{\zeta_l})\Big)\right), \left(s^{-1}\Big(qs(\vartheta_l)\Big), s^{-1}\Big(qs(w_{\vartheta_l})\Big)\right)\right) \\
\text{and} \quad \left(\mathcal{C}_j^p \otimes \mathcal{C}_l^q\right) &= \left(\begin{pmatrix} t^{-1}\Big(pt(\zeta_j) + qt(\zeta_l)\Big), \\ t^{-1}\Big(pt(w_{\zeta_j}) + qt(w_{\zeta_l})\Big) \end{pmatrix}, \begin{pmatrix} s^{-1}\Big(ps(\vartheta_j) + qs(\vartheta_l)\Big), \\ s^{-1}\Big(ps(w_{\vartheta_j}) + qs(w_{\vartheta_l})\Big) \end{pmatrix}\right)
\end{aligned}$$

Now, in light to prove the truthfulness of Eq. (10.2), firstly we shall show that the following Eq. (10.13) holds.

$$\bigoplus_{\substack{j,l=1\\ j\neq l}}^{n} \left(\mathcal{C}_j^p \otimes \mathcal{C}_l^q\right) = \left(\begin{pmatrix} s^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{n} s\Big(t^{-1}\Big(pt(\zeta_j) + qt(\zeta_l)\Big)\Big)\right), \\ s^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{n} s\Big(t^{-1}\Big(pt(w_{\zeta_j}) + qt(w_{\zeta_l})\Big)\Big)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{n} t\Big(s^{-1}\Big(ps(\vartheta_j) + qs(\vartheta_l)\Big)\Big)\right), \\ t^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{n} t\Big(s^{-1}\Big(ps(w_{\vartheta_j}) + qs(w_{\vartheta_l})\Big)\Big)\right) \end{pmatrix}\right) \tag{10.13}$$

by utilizing mathematical induction on n.

Step 1: When $n = 2$, we acquire that

$$\begin{aligned}
&\bigoplus_{j,l=1;j\neq l}^{2} \left(\mathcal{C}_j^p \otimes \mathcal{C}_l^q\right) = \left(\mathcal{C}_1^p \otimes \mathcal{C}_2^q\right) \oplus \left(\mathcal{C}_2^p \otimes \mathcal{C}_1^q\right) \\
=& \left(\begin{pmatrix} t^{-1}\Big(pt(\zeta_1) + qt(\zeta_2)\Big), \\ t^{-1}\Big(pt(w_{\zeta_1}) + qt(w_{\zeta_2})\Big) \end{pmatrix}, \begin{pmatrix} s^{-1}\Big(ps(\vartheta_1) + qs(\vartheta_2)\Big), \\ s^{-1}\Big(ps(w_{\vartheta_1}) + qs(w_{\vartheta_2})\Big) \end{pmatrix}\right) \\
\oplus& \left(\begin{pmatrix} t^{-1}\Big(pt(\zeta_2) + qt(\zeta_1)\Big), \\ t^{-1}\Big(pt(w_{\zeta_2}) + qt(w_{\zeta_1})\Big) \end{pmatrix}, \begin{pmatrix} s^{-1}\Big(ps(\vartheta_2) + qs(\vartheta_1)\Big), \\ s^{-1}\Big(ps(w_{\vartheta_2}) + qs(w_{\vartheta_1})\Big) \end{pmatrix}\right) \\
=& \left(\begin{pmatrix} s^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{2} s\Big(t^{-1}\Big(pt(\zeta_j) + qt(\zeta_l)\Big)\Big)\right), \\ s^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{2} s\Big(t^{-1}\Big(pt(w_{\zeta_j}) + qt(w_{\zeta_l})\Big)\Big)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{2} t\Big(s^{-1}\Big(ps(\vartheta_j) + qs(\vartheta_l)\Big)\Big)\right), \\ t^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{2} t\Big(s^{-1}\Big(ps(w_{\vartheta_j}) + qs(w_{\vartheta_l})\Big)\Big)\right) \end{pmatrix}\right)
\end{aligned}$$

Thus, the Eq. (10.13) holds when $n = 2$.

Step 2: Consider that Eq. (10.13) is true when $n = m$, where $m \in \mathbb{Z}^+$. Then,

$$\bigoplus_{\substack{j,l=1\\ j\neq l}}^{m} \left(\mathcal{C}_j^p \otimes \mathcal{C}_l^q\right)$$

$$= \left(\begin{pmatrix} s^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{m} s\Big(t^{-1}\Big(pt(\zeta_j)+qt(\zeta_l)\Big)\Big)\right), \\ s^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{m} s\Big(t^{-1}\Big(pt(w_{\zeta_j})+qt(w_{\zeta_l})\Big)\Big)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{m} t\Big(s^{-1}\Big(ps(\vartheta_j)+qs(\vartheta_l)\Big)\Big)\right), \\ t^{-1}\left(\sum\limits_{\substack{j,l=1\\ j\neq l}}^{m} t\Big(s^{-1}\Big(ps(w_{\vartheta_j})+qs(w_{\vartheta_l})\Big)\Big)\right)\end{pmatrix}\right) \tag{10.14}$$

then for $n = m + 1$, we obtain

$$\bigoplus_{\substack{j,l=1\\ j\neq l}}^{m+1} \left(\mathcal{C}_j^p \otimes \mathcal{C}_l^q\right)$$

$$= \left(\bigoplus_{\substack{j,l=1\\ j\neq l}}^{m} \left(\mathcal{C}_j^p \otimes \mathcal{C}_l^q\right)\right) \oplus \left(\bigoplus_{j=1}^{m} \left(\mathcal{C}_j^p \otimes \mathcal{C}_{m+1}^q\right)\right) \oplus \left(\bigoplus_{l=1}^{m} \left(\mathcal{C}_{m+1}^p \otimes \mathcal{C}_l^q\right)\right) \tag{10.15}$$

Now, we shall prove the following Eq. (10.16) by the principle of induction on m

$$\bigoplus_{j=1}^{m} \left(\mathcal{C}_j^p \otimes \mathcal{C}_{m+1}^q\right)$$

$$= \left(\begin{pmatrix} s^{-1}\left(\sum\limits_{j=1}^{m} s\Big(t^{-1}\Big(pt(\zeta_j)+qt(\zeta_{m+1})\Big)\Big)\right), \\ s^{-1}\left(\sum\limits_{j=1}^{m} s\Big(t^{-1}\Big(pt(w_{\zeta_j})+qt(w_{\zeta_{m+1}})\Big)\Big)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum\limits_{j=1}^{m} t\Big(s^{-1}\Big(ps(\vartheta_j)+qs(\vartheta_{m+1})\Big)\Big)\right), \\ t^{-1}\left(\sum\limits_{j=1}^{m} t\Big(s^{-1}\Big(ps(w_{\vartheta_j})+qs(w_{\vartheta_{m+1}})\Big)\Big)\right)\end{pmatrix}\right) \tag{10.16}$$

Step 2a: For $m = 2$:

$$\bigoplus_{j=1}^{2} \left(\mathcal{C}_j^p \otimes \mathcal{C}_{m+1}^q\right) = \left(\mathcal{C}_1^p \otimes \mathcal{C}_{m+1}^q\right) \oplus \left(\mathcal{C}_2^p \otimes \mathcal{C}_{m+1}^q\right)$$

$$= \left(\begin{pmatrix} t^{-1}\Big(pt(\zeta_1)+qt(\zeta_{m+1})\Big), \\ t^{-1}\Big(pt(w_{\zeta_1})+qt(w_{\zeta_{m+1}})\Big)\end{pmatrix}, \begin{pmatrix} s^{-1}\Big(ps(\vartheta_1)+qs(\vartheta_{m+1})\Big), \\ s^{-1}\Big(ps(w_{\vartheta_1})+qs(w_{\vartheta_{m+1}})\Big)\end{pmatrix}\right)$$

$$\oplus \left(\begin{pmatrix} t^{-1}\Big(pt(\zeta_2)+qt(\zeta_{m+1})\Big), \\ t^{-1}\Big(pt(\zeta_2)+qt(\zeta_{m+1})\Big)\end{pmatrix}, \begin{pmatrix} s^{-1}\Big(ps(\vartheta_2)+qs(\vartheta_{m+1})\Big), \\ s^{-1}\Big(ps(w_{\vartheta_2})+qs(w_{\vartheta_{m+1}})\Big)\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{2} s\Big(t^{-1}\Big(pt(\zeta_j)+qt(\zeta_{m+1})\Big)\Big)\right), \\ s^{-1}\left(\sum_{j=1}^{2} s\Big(t^{-1}\Big(pt(w_{\zeta_j})+qt(w_{\zeta_{m+1}})\Big)\Big)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{2} t\Big(s^{-1}\Big(ps(\vartheta_j)+qs(\vartheta_{m+1})\Big)\Big)\right), \\ t^{-1}\left(\sum_{j=1}^{2} t\Big(s^{-1}\Big(ps(w_{\vartheta_j})+qs(w_{\vartheta_{m+1}})\Big)\Big)\right) \end{pmatrix} \right)$$

Thus, Eq. (10.16) holds when $m = 2$.

Step 2b: Consider that Eq. (10.16) is true for $m = m_0$, where m_0 is an arbitrary positive integer, i.e.,

$$\bigoplus_{j=1}^{m_0} \Big(\mathcal{C}_j^p \otimes \mathcal{C}_{m_0+1}^q\Big)$$

$$= \left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{m_0} s\Big(t^{-1}\Big(pt(\zeta_j)+qt(\zeta_{m_0+1})\Big)\Big)\right), \\ s^{-1}\left(\sum_{j=1}^{m_0} s\Big(t^{-1}\Big(pt(w_{\zeta_j})+qt(w_{\zeta_{m_0+1}})\Big)\Big)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{m_0} t\Big(s^{-1}\Big(ps(\vartheta_j)+qs(\vartheta_{m_0+1})\Big)\Big)\right), \\ t^{-1}\left(\sum_{j=1}^{m_0} t\Big(s^{-1}\Big(ps(w_{\vartheta_j})+qs(w_{\vartheta_{m_0+1}})\Big)\Big)\right) \end{pmatrix} \right)$$

then for $m = m_0 + 1$, we have

$$\bigoplus_{j=1}^{m_0+1} \Big(\mathcal{C}_j^p \otimes \mathcal{C}_{m_0+2}^q\Big) = \bigoplus_{j=1}^{m_0} \Big(\mathcal{C}_j^p \otimes \mathcal{C}_{m_0+2}^q\Big) \oplus \Big(\mathcal{C}_{m_0+1}^p \otimes \mathcal{C}_{m_0+2}^q\Big)$$

$$= \left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{m_0} s\Big(t^{-1}\Big(pt(\zeta_j)+qt(\zeta_{m_0+2})\Big)\Big)\right), \\ s^{-1}\left(\sum_{j=1}^{m_0} s\Big(t^{-1}\Big(pt(w_{\zeta_j})+qt(w_{\zeta_{m_0+2}})\Big)\Big)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{m_0} t\Big(s^{-1}\Big(ps(\vartheta_j)+qs(\vartheta_{m_0+2})\Big)\Big)\right), \\ t^{-1}\left(\sum_{j=1}^{m_0} t\Big(s^{-1}\Big(ps(w_{\vartheta_j})+qs(w_{\vartheta_{m_0+2}})\Big)\Big)\right) \end{pmatrix} \right)$$

$$\oplus \left(\begin{pmatrix} t^{-1}\Big(pt(\zeta_{m_0+1})+qt(\zeta_{m_0+2})\Big), \\ t^{-1}\Big(pt(w_{\zeta_{m_0+1}})+qt(w_{\zeta_{m_0+2}})\Big) \end{pmatrix}, \begin{pmatrix} s^{-1}\Big(ps(\vartheta_{m_0+1})+qs(\vartheta_{m_0+2})\Big), \\ s^{-1}\Big(ps(w_{\vartheta_{m_0+1}})+qs(w_{\vartheta_{m_0+2}})\Big) \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{m_0+1} s\Big(t^{-1}\Big(pt(\zeta_j)+qt(\zeta_{m_0+2})\Big)\Big)\right), \\ s^{-1}\left(\sum_{j=1}^{m_0+1} s\Big(t^{-1}\Big(pt(w_{\zeta_j})+qt(w_{\zeta_{m_0+2}})\Big)\Big)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{m_0+1} t\Big(s^{-1}\Big(ps(\vartheta_j)+qs(\vartheta_{m_0+2})\Big)\Big)\right), \\ t^{-1}\left(\sum_{j=1}^{m_0+1} t\Big(s^{-1}\Big(ps(w_{\vartheta_j})+qs(w_{\vartheta_{m_0+2}})\Big)\Big)\right) \end{pmatrix} \right)$$

Thus, Eq. (10.16) holds for $m = m_0 + 1$ and hence, Eq. (10.16) is true $\forall\ m$.

Similarly, we can prove that

$$\bigoplus_{l=1}^{m} \Big(\mathcal{C}_{m+1}^p \otimes \mathcal{C}_l^q\Big)$$

$$= \left(\begin{pmatrix} s^{-1}\left(\sum_{l=1}^{m} s\Big(t^{-1}\Big(pt(\zeta_{m+1})+qt(\zeta_l)\Big)\Big)\right), \\ s^{-1}\left(\sum_{l=1}^{m} s\Big(t^{-1}\Big(pt(w_{\zeta_{m+1}})+qt(w_{\zeta_l})\Big)\Big)\right) \end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{l=1}^{m} t\Big(s^{-1}\Big(ps(\vartheta_{m+1})+qs(\vartheta_l)\Big)\Big)\right), \\ t^{-1}\left(\sum_{l=1}^{m} t\Big(s^{-1}\Big(ps(w_{\vartheta_{m+1}})+qs(w_{\vartheta_l})\Big)\Big)\right) \end{pmatrix} \right) \tag{10.17}$$

Further, using the Eqs. (10.14), (10.16) and (10.17) in Eq. (10.15) we obtain:

$$\bigoplus_{\substack{j,l=1\\ j\neq l}}^{m+1}\left(\mathcal{C}_j^p\otimes\mathcal{C}_l^q\right)$$

$$=\left(\begin{pmatrix} s^{-1}\left(\sum_{\substack{j,l=1\\ j\neq l}}^{m} s\Big(t^{-1}\big(pt(\zeta_j)+qt(\zeta_l)\big)\Big)\right), \\ s^{-1}\left(\sum_{\substack{j,l=1\\ j\neq l}}^{m} s\Big(t^{-1}\big(pt(w_{\zeta_j})+qt(w_{\zeta_l})\big)\Big)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{\substack{j,l=1\\ j\neq l}}^{m} t\Big(s^{-1}\big(ps(\vartheta_j)+qs(\vartheta_l)\big)\Big)\right), \\ t^{-1}\left(\sum_{\substack{j,l=1\\ j\neq l}}^{m} t\Big(s^{-1}\big(ps(w_{\vartheta_j})+qs(w_{\vartheta_l})\big)\Big)\right)\end{pmatrix}\right)$$

$$\oplus\left(\begin{pmatrix} s^{-1}\left(\sum_{j=1}^{m} s\Big(t^{-1}\big(pt(\zeta_j)+qt(\zeta_{m+1})\big)\Big)\right), \\ s^{-1}\left(\sum_{j=1}^{m} s\Big(t^{-1}\big(pt(w_{\zeta_j})+qt(w_{\zeta_{m+1}})\big)\Big)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{j=1}^{m} t\Big(s^{-1}\big(ps(\vartheta_j)+qs(\vartheta_{m+1})\big)\Big)\right), \\ t^{-1}\left(\sum_{j=1}^{m} t\Big(s^{-1}\big(ps(w_{\vartheta_j})+qs(w_{\vartheta_{m+1}})\big)\Big)\right)\end{pmatrix}\right)$$

$$\oplus\left(\begin{pmatrix} s^{-1}\left(\sum_{l=1}^{m} s\Big(t^{-1}\big(pt(\zeta_{m+1})+qt(\zeta_l)\big)\Big)\right), \\ s^{-1}\left(\sum_{l=1}^{m} s\Big(t^{-1}\big(pt(w_{\zeta_{m+1}})+qt(w_{\zeta_l})\big)\Big)\right),\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{l=1}^{m} t\Big(s^{-1}\big(ps(\vartheta_{m+1})+qs(\vartheta_l)\big)\Big)\right), \\ t^{-1}\left(\sum_{l=1}^{m} t\Big(s^{-1}\big(ps(w_{\vartheta_{m+1}})+qs(w_{\vartheta_l})\big)\Big)\right)\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix} s^{-1}\left(\sum_{\substack{j,l=1\\ j\neq l}}^{m+1} s\Big(t^{-1}\big(pt(\zeta_j)+qt(\zeta_l)\big)\Big)\right), \\ s^{-1}\left(\sum_{\substack{j,l=1\\ j\neq l}}^{m+1} s\Big(t^{-1}\big(pt(w_{\zeta_j})+qt(w_{\zeta_l})\big)\Big)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\sum_{\substack{j,l=1\\ j\neq l}}^{m+1} t\Big(s^{-1}\big(ps(\vartheta_j)+qs(\vartheta_l)\big)\Big)\right), \\ t^{-1}\left(\sum_{\substack{j,l=1\\ j\neq l}}^{m+1} t\Big(s^{-1}\big(ps(w_{\vartheta_j})+qs(w_{\vartheta_l})\big)\Big)\right)\end{pmatrix}\right)$$

Thus, Eq. (10.13) is valid, when $n = m + 1$ and therefore, Eq. (10.13) is true for each positive integer n.

Now, by using operational law, as defined in Definition 6.2.2, we obtain:

$$\frac{1}{n(n-1)}\left(\bigoplus_{\substack{j,l=1\\ j\neq l}}^{n}\left(\mathcal{C}_j^p\otimes\mathcal{C}_l^q\right)\right)$$

$$=\left(\begin{pmatrix} s^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} s\Big(t^{-1}\big(pt(\zeta_j)+qt(\zeta_l)\big)\Big)\right), \\ s^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} s\Big(t^{-1}\big(pt(w_{\zeta_j})+qt(w_{\zeta_l})\big)\Big)\right)\end{pmatrix}, \begin{pmatrix} t^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} t\Big(s^{-1}\big(ps(\vartheta_j)+qs(\vartheta_l)\big)\Big)\right), \\ t^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} t\Big(s^{-1}\big(ps(w_{\vartheta_j})+qs(w_{\vartheta_l})\big)\Big)\right)\end{pmatrix}\right)$$

Further, by using operational law, as stated in Definition 6.2.2 and on utilizing Definition 10.2.1, we have:

$$\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n) = \left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{j,l=1\\ j\neq l}}^{n}\Big(\mathcal{C}_j^p\otimes\mathcal{C}_l^q\Big)\right)\right)^{\frac{1}{p+q}}$$

$$= \left(\begin{array}{c}\left(\begin{array}{c} t^{-1}\left(\frac{1}{p+q}t\left(s^{-1}\left(\frac{1}{n(n-1)}\sum\limits_{\substack{j,l=1\\ j\neq l}}^{n}s\Big(t^{-1}\Big(pt(\zeta_j)+qt(\zeta_l)\Big)\Big)\right)\right)\right), \\ t^{-1}\left(\frac{1}{p+q}t\left(s^{-1}\left(\frac{1}{n(n-1)}\sum\limits_{\substack{j,l=1\\ j\neq l}}^{n}s\Big(t^{-1}\Big(pt(w_{\zeta_j})+qt(w_{\zeta_l})\Big)\Big)\right)\right)\right)\end{array}\right), \\ \left(\begin{array}{c} s^{-1}\left(\frac{1}{p+q}s\left(t^{-1}\left(\frac{1}{n(n-1)}\sum\limits_{\substack{j,l=1\\ j\neq l}}^{n}t\Big(s^{-1}\Big(ps(\vartheta_j)+qs(\vartheta_l)\Big)\Big)\right)\right)\right), \\ s^{-1}\left(\frac{1}{p+q}s\left(t^{-1}\left(\frac{1}{n(n-1)}\sum\limits_{\substack{j,l=1\\ j\neq l}}^{n}t\Big(s^{-1}\Big(ps(w_{\vartheta_j})+qs(w_{\vartheta_l})\Big)\Big)\right)\right)\right)\end{array}\right)\end{array}\right)$$

which proves the result given in Eq. (10.2).

$\square$

Proof of the Property 10.2.1:

Proof. Let $\mathcal{C}_0 = \big((\zeta_0, w_{\zeta_0}), (\vartheta_0, w_{\vartheta_0})\big)$. Now, $\mathcal{C}_j = \mathcal{C}_0$ implies that $\zeta_j = \zeta_0$, $\vartheta_j = \vartheta_0$, $w_{\zeta_j} = w_{\zeta_0}$, $w_{\vartheta_j} = w_{\vartheta_0}$ for all j. Then, from Eqs. (10.3) and (10.4), we have

$$\begin{aligned}\mathcal{A}(\zeta_0,\zeta_0) &= s^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n}s\Big(t^{-1}\Big(pt(\zeta_0)+qt(\zeta_0)\Big)\Big)\right)\\ &= s^{-1}\left(\frac{1}{n(n-1)}n(n-1)s\Big(t^{-1}\Big(pt(\zeta_0)+qt(\zeta_0)\Big)\Big)\right)\\ &= t^{-1}\Big((p+q)t(\zeta_0)\Big)\end{aligned}$$

Similarly, we have $\mathcal{B}(\vartheta_0,\vartheta_0) = s^{-1}\Big((p+q)s(\vartheta_0)\Big)$.

Thus, by using Theorem 10.2.1, we get

$$
\begin{aligned}
&\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)\\
=&\left(\begin{pmatrix} t^{-1}\left(\frac{1}{p+q}t\left(t^{-1}\Big((p+q)t(\zeta_0)\Big)\right)\right),\\ t^{-1}\left(\frac{1}{p+q}t\left(t^{-1}\Big((p+q)t(w_{\zeta_0})\Big)\right)\right)\end{pmatrix},\begin{pmatrix} s^{-1}\left(\frac{1}{p+q}s\left(s^{-1}\Big((p+q)s(\vartheta_0)\Big)\right)\right),\\ s^{-1}\left(\frac{1}{p+q}s\left(s^{-1}\Big((p+q)s(w_{\vartheta_0})\Big)\right)\right)\end{pmatrix}\right)\\
=&\left(\begin{pmatrix} t^{-1}\left(\dfrac{1}{p+q}(p+q)t(\zeta_0)\right),\\ t^{-1}\left(\dfrac{1}{p+q}(p+q)t(w_{\zeta_0})\right)\end{pmatrix},\begin{pmatrix} s^{-1}\left(\dfrac{1}{p+q}(p+q)s(\vartheta_0)\right),\\ s^{-1}\left(\dfrac{1}{p+q}(p+q)s(w_{\vartheta_0})\right)\end{pmatrix}\right)\\
=&\ ((\zeta_0,w_{\zeta_0}),(\vartheta_0,w_{\vartheta_0}))
\end{aligned}
$$

Hence, $\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1,\mathcal{C}_2\ldots\mathcal{C}_n)=\mathcal{C}_0$. □

Proof of the Property 10.2.2:

Proof. Let $\mathcal{C}=\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)=\big((\zeta_{\mathcal{C}},w_{\zeta_{\mathcal{C}}}),(\vartheta_{\mathcal{C}},w_{\vartheta_{\mathcal{C}}})\big)$ and $\mathcal{Z}=\mathrm{GCIFBM}^{p,q}(\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\mathcal{Z}_n)=\big((\zeta_{\mathcal{Z}},w_{\zeta_{\mathcal{Z}}}),\ (\vartheta_{\mathcal{Z}},w_{\vartheta_{\mathcal{Z}}})\big)$ such that $\zeta_{\mathcal{C}_j}\le\zeta_{\mathcal{Z}_j}$, $\vartheta_{\mathcal{C}_j}\ge\vartheta_{\mathcal{Z}_j}$, $w_{\zeta_{\mathcal{C}_j}}\le w_{\zeta_{\mathcal{Z}_j}}$ and $w_{\vartheta_{\mathcal{C}_j}}\ge w_{\vartheta_{\mathcal{Z}_j}}$ $\forall\ j$. As, $\zeta_{\mathcal{C}_j}\le\zeta_{\mathcal{Z}_j}$ and t,s are decreasing and increasing functions respectively.

$$
\begin{aligned}
\Rightarrow\quad & pt(\zeta_{\mathcal{C}_j})+qt(\zeta_{\mathcal{C}_l})\ge pt(\zeta_{\mathcal{Z}_j})+qt(\zeta_{\mathcal{Z}_l})\\
\Rightarrow\quad & t^{-1}\left(pt(\zeta_{\mathcal{C}_j})+qt(\zeta_{\mathcal{C}_l})\right)\le t^{-1}\left(pt(\zeta_{\mathcal{Z}_j})+qt(\zeta_{\mathcal{Z}_l})\right)\\
\Rightarrow\quad & s\Big(t^{-1}\left(pt(\zeta_{\mathcal{C}_j})+qt(\zeta_{\mathcal{C}_l})\right)\Big)\le s\Big(t^{-1}\left(pt(\zeta_{\mathcal{Z}_j})+qt(\zeta_{\mathcal{Z}_l})\right)\Big)\\
\Rightarrow\quad & \frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} s\Big(t^{-1}\left(pt(\zeta_{\mathcal{C}_j})+qt(\zeta_{\mathcal{C}_l})\right)\Big)\le\frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} s\Big(t^{-1}\left(pt(\zeta_{\mathcal{Z}_j})+qt(\zeta_{\mathcal{Z}_l})\right)\Big)\\
\Rightarrow\quad & s^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} s\Big(t^{-1}\left(pt(\zeta_{\mathcal{C}_j})+qt(\zeta_{\mathcal{C}_l})\right)\Big)\right)\le s^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} s\Big(t^{-1}\left(pt(\zeta_{\mathcal{Z}_j})+qt(\zeta_{\mathcal{Z}_l})\right)\Big)\right)\\
\text{i.e. ,}\quad & \mathcal{A}(\zeta_{\mathcal{C}_j},\zeta_{\mathcal{C}_l})\le\mathcal{A}(\zeta_{\mathcal{Z}_j},\zeta_{\mathcal{Z}_l})\\
\Rightarrow\quad & t\left(\mathcal{A}(\zeta_{\mathcal{C}_j},\zeta_{\mathcal{C}_l})\right)\ge t\left(\mathcal{A}(\zeta_{\mathcal{Z}_j},\zeta_{\mathcal{Z}_l})\right)\\
\Rightarrow\quad & t^{-1}\left(\frac{1}{p+q}t\left(\mathcal{A}(\zeta_{\mathcal{C}_j},\zeta_{\mathcal{C}_l})\right)\right)\le t^{-1}\left(\frac{1}{p+q}t\left(\mathcal{A}(\zeta_{\mathcal{Z}_j},\zeta_{\mathcal{Z}_l})\right)\right)\\
\Rightarrow\quad & \zeta_{\mathcal{C}}\le\zeta_{\mathcal{Z}}
\end{aligned}
$$

Also, $\vartheta_{\mathcal{C}_j}\ge\vartheta_{\mathcal{Z}_j}$ which implies that $ps(\vartheta_{\mathcal{C}_j})+qs(\vartheta_{\mathcal{C}_l})\ge ps(\vartheta_{\mathcal{Z}_j})+ps(\vartheta_{\mathcal{Z}_l})$ and hence

$t\big(s^{-1}\left(ps(\vartheta_{\mathcal{C}_j})+qs(\vartheta_{\mathcal{C}_l})\right)\big)\le t\big(s^{-1}\left(ps(\vartheta_{\mathcal{Z}_j})+qs(\vartheta_{\mathcal{Z}_l})\right)\big)$. Therefore,

$$\Rightarrow\quad \frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} t\left(s^{-1}\left(ps(\vartheta_{\mathcal{C}_j})+qs(\vartheta_{\mathcal{C}_l})\right)\right)\le \frac{1}{n(n-1)}\sum_{\substack{j,l=1\\ j\neq l}}^{n} t\left(s^{-1}\left(ps(\vartheta_{\mathcal{Z}_j})+qs(\vartheta_{\mathcal{Z}_l})\right)\right)$$
$$\Rightarrow\quad \mathcal{B}(\vartheta_{\mathcal{C}_j},\vartheta_{\mathcal{C}_l})\ge \mathcal{B}(\vartheta_{\mathcal{Z}_j},\vartheta_{\mathcal{Z}_l})$$
$$\Rightarrow\quad s\left(\mathcal{B}(\vartheta_{\mathcal{C}_j},\vartheta_{\mathcal{C}_l})\right)\ge s\left(\mathcal{B}(\vartheta_{\mathcal{Z}_j},\vartheta_{\mathcal{Z}_l})\right)$$
$$\Rightarrow\quad s^{-1}\left(\frac{1}{p+q}s\left(\mathcal{B}(\vartheta_{\mathcal{C}_j},\vartheta_{\mathcal{C}_l})\right)\right)\ge s^{-1}\left(\frac{1}{p+q}s\left(\mathcal{B}(\vartheta_{\mathcal{Z}_j},\vartheta_{\mathcal{Z}_l})\right)\right)$$
$$\Rightarrow\quad \vartheta_{\mathcal{C}}\ge\vartheta_{\mathcal{Z}}$$

Similarly, we can prove that $w_{\zeta_{\mathcal{C}}}\le w_{\zeta_{\mathcal{Z}}}$ and $w_{\vartheta_{\mathcal{C}}}\ge w_{\vartheta_{\mathcal{Z}}}$. Hence, using the Definition 2.1.10, we obtain that GCIFBM$^{p,q}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)\subseteq$ GCIFBM$^{p,q}(\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\mathcal{Z}_n)$. □

Proof of the Property 10.2.3:

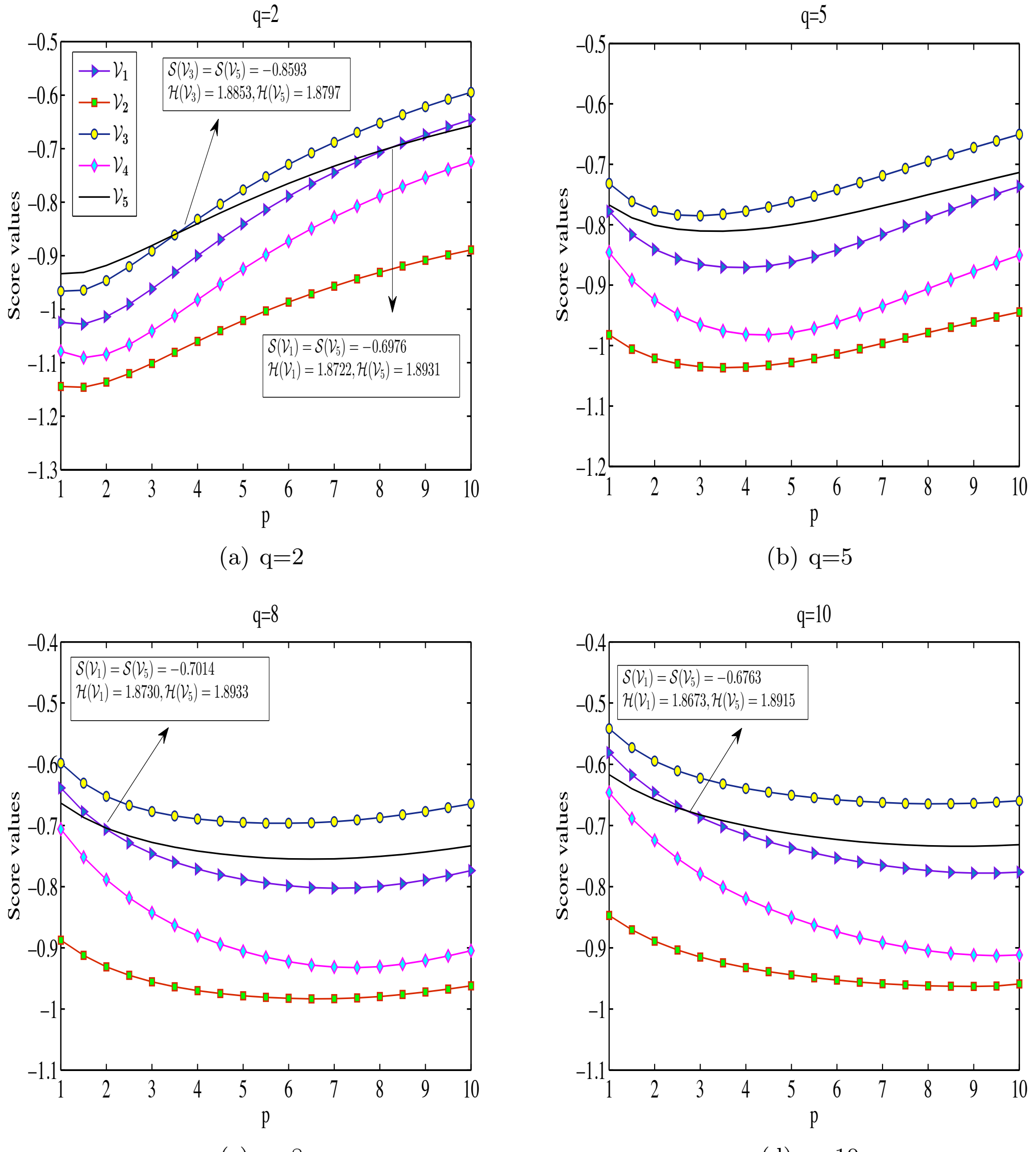
Proof. Since, $\mathcal{C}^-\subseteq\mathcal{C}_j\subseteq\mathcal{C}^+\ \forall\ j$. Then, by using Property 10.2.2, we have GCIFBM$^{p,q}(\mathcal{C}^-,\mathcal{C}^-,\ldots,\mathcal{C}^-)\subseteq$ GCIFBM$^{p,q}(\mathcal{C}_1,\mathcal{C}_2\ldots\mathcal{C}_n)\subseteq$ GCIFBM$^{p,q}(\mathcal{C}^+,\mathcal{C}^+,\ldots,\mathcal{C}^+)$. Further, by using Property 10.2.1, we obtain that $\mathcal{C}^-\subseteq$ GCIFBM$^{p,q}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)\subseteq\mathcal{C}^+$, which completes the proof. □

Proof of the Property 10.2.4:

Proof. As $\left(\dot{\mathcal{C}}_1,\dot{\mathcal{C}}_2,\ldots,\dot{\mathcal{C}}_n\right)$ is a permutation of $\left(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\right)$. Therefore,

$$\begin{aligned}\mathrm{GCIFBM}^{p,q}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)&=\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{j,l=1\\ j\neq l}}^{n}\left(\mathcal{C}_j^p\otimes\mathcal{C}_l^q\right)\right)\right)^{\frac{1}{p+q}}=\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{j,l=1\\ j\neq l}}^{n}\left(\dot{\mathcal{C}}_j^p\otimes\dot{\mathcal{C}}_l^q\right)\right)\right)^{\frac{1}{p+q}}\\&=\mathrm{GCIFBM}^{p,q}(\dot{\mathcal{C}}_1,\dot{\mathcal{C}}_2,\ldots,\dot{\mathcal{C}}_n)\end{aligned}$$

Hence, GCIFBM$^{p,q}(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n)=$ GCIFBM$^{p,q}(\dot{\mathcal{C}}_1,\dot{\mathcal{C}}_2,\ldots,\dot{\mathcal{C}}_n)$. □



(a) q=2 (b) q=5

(c) q=8 (d) q=10

Figure 10.1: Variation in parameter p by fixing q

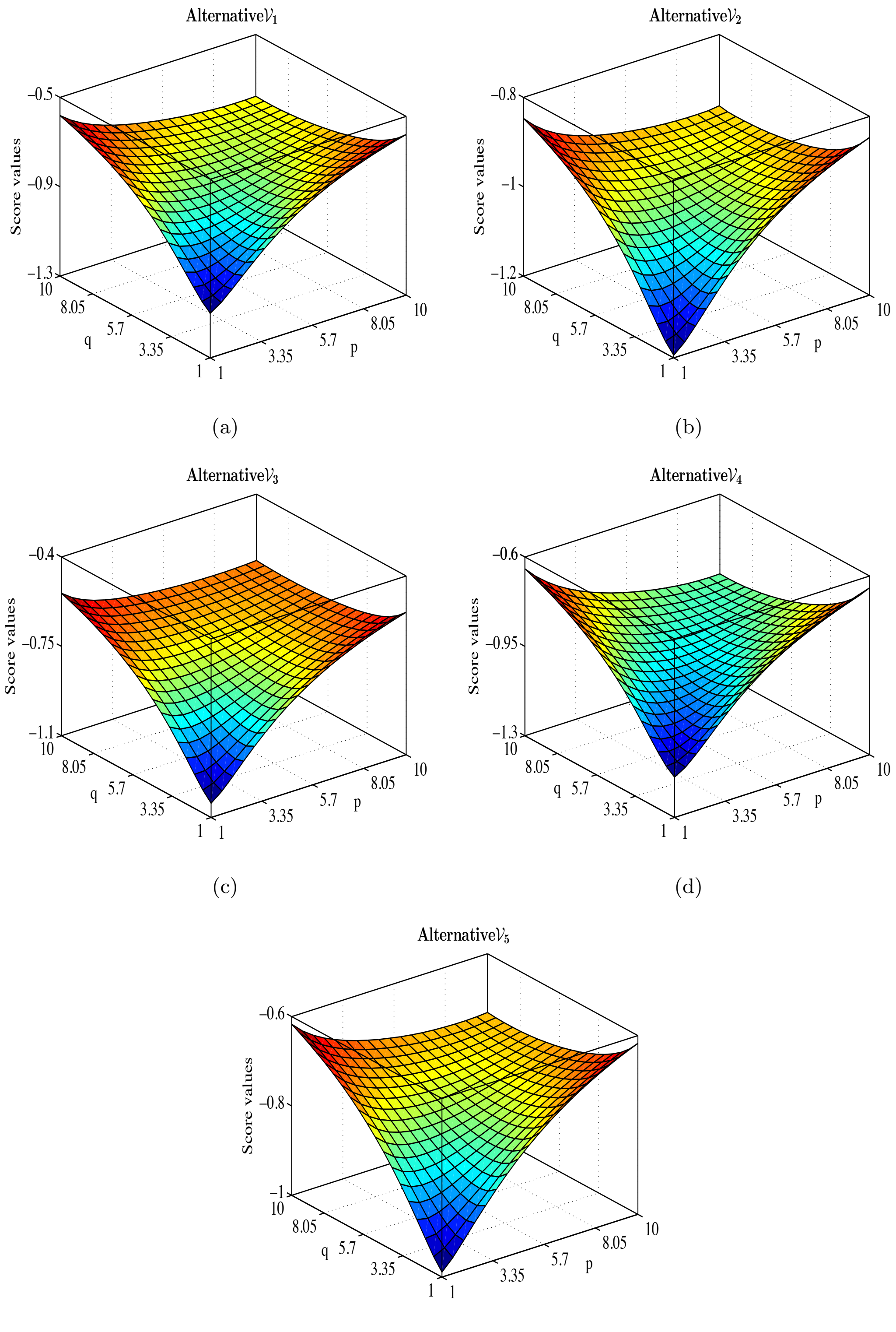


Figure 10.2: Score values of alternative $\mathcal{V}_u$ for different values of p, q

Table 10.1: Input values in terms of CIFNs

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.6,0.8),(0.2,0.2)\big)$	$\big((0.7,0.4),(0.2,0.5)\big)$	$\big((0.4,0.3),(0.5,0.5)\big)$	$\big((0.6,0.6),(0.4,0.3)\big)$
$\mathcal{V}_2$	$\big((0.6,0.5),(0.4,0.4)\big)$	$\big((0.3,0.8),(0.3,0.2)\big)$	$\big((0.3,0.4),(0.6,0.6)\big)$	$\big((0.3,0.5),(0.4,0.2)\big)$
$\mathcal{V}_3$	$\big((0.7,0.6),(0.3,0.3)\big)$	$\big((0.4,0.9),(0.2,0.1)\big)$	$\big((0.2,0.3),(0.7,0.7)\big)$	$\big((0.6,0.3),(0.3,0.6)\big)$
$\mathcal{V}_4$	$\big((0.4,0.8),(0.5,0.1)\big)$	$\big((0.7,0.3),(0.3,0.3)\big)$	$\big((0.1,0.3),(0.6,0.5)\big)$	$\big((0.5,0.5),(0.3,0.4)\big)$
$\mathcal{V}_5$	$\big((0.8,0.6),(0.2,0.3)\big)$	$\big((0.6,0.6),(0.3,0.2)\big)$	$\big((0.3,0.3),(0.6,0.5)\big)$	$\big((0.7,0.7),(0.2,0.2)\big)$

Table 10.2: Normalized rating values

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\big((0.6,0.8),(0.2,0.2)\big)$	$\big((0.7,0.4),(0.2,0.5)\big)$	$\big((0.5,0.5),(0.4,0.3)\big)$	$\big((0.6,0.6),(0.4,0.3)\big)$
$\mathcal{V}_2$	$\big((0.6,0.5),(0.4,0.4)\big)$	$\big((0.3,0.8),(0.3,0.2)\big)$	$\big((0.6,0.6),(0.3,0.4)\big)$	$\big((0.3,0.5),(0.4,0.2)\big)$
$\mathcal{V}_3$	$\big((0.7,0.6),(0.3,0.3)\big)$	$\big((0.4,0.9),(0.2,0.1)\big)$	$\big((0.7,0.7),(0.2,0.3)\big)$	$\big((0.6,0.3),(0.3,0.6)\big)$
$\mathcal{V}_4$	$\big((0.4,0.8),(0.5,0.1)\big)$	$\big((0.7,0.3),(0.3,0.3)\big)$	$\big((0.6,0.5),(0.1,0.3)\big)$	$\big((0.5,0.5),(0.3,0.4)\big)$
$\mathcal{V}_5$	$\big((0.8,0.6),(0.2,0.3)\big)$	$\big((0.6,0.6),(0.3,0.2)\big)$	$\big((0.6,0.5),(0.3,0.3)\big)$	$\big((0.7,0.7),(0.2,0.2)\big)$

Table 10.3: Effect of p and q on the ranking of alternatives

q	p	Score values of $\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	Ranking order
1	1	-1.1002	-1.1941	-1.0439	-1.1418	-0.9889	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	2	-1.0243	-1.1444	-0.9662	-1.0788	-0.9335	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	3	-0.9313	-1.0832	-0.8760	-0.9929	-0.8713	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	4	-0.8475	-1.0280	-0.7968	-0.9130	-0.8154	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
3	1	-0.9313	-1.0832	-0.8760	-0.9929	-0.8713	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	2	-0.9619	-1.1012	-0.8912	-1.0410	-0.8811	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	3	-0.9463	-1.0894	-0.8650	-1.0385	-0.8648	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	4	-0.9099	-1.0644	-0.8262	-1.0077	-0.8393	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
5	1	-0.7781	-0.9820	-0.7316	-0.8457	-0.7675	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	2	-0.8408	-1.0210	-0.7772	-0.9248	-0.8009	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	3	-0.8657	-1.0350	-0.7851	-0.9651	-0.8104	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	4	-0.8709	-1.0358	-0.7777	-0.9815	-0.8087	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
7	1	-0.6764	-0.9134	-0.6347	-0.7444	-0.6926	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	2	-0.7444	-0.9568	-0.6881	-0.8278	-0.7328	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	3	-0.7818	-0.9795	-0.7097	-0.8801	-0.7534	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	4	-0.8037	-0.9913	-0.7178	-0.9138	-0.7643	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
9	1	-0.6076	-0.8657	-0.5679	-0.6738	-0.6386	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	2	-0.6739	-0.9087	-0.6217	-0.7543	-0.6795	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	3	-0.7147	-0.9344	-0.6486	-0.8094	-0.7041	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
	4	-0.7420	-0.9505	-0.6635	-0.8436	-0.7205	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$

Table 10.4: Analysis of the Figures 10.1(b), 10.1(c) and 10.1(d)

Figure	Value of p'	Accuracy for $p = p'$	Ranking of the alternatives: When $p < p'$	When $p = p'$	When $p > p'$
10.1(b)	–	–	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$		
10.1(c)	1.9015	$\mathcal{H}(\mathcal{V}_1) = 1.8730$, $\mathcal{H}(\mathcal{V}_5) = 1.8933$	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2$	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
10.1(d)	2.7028	$\mathcal{H}(\mathcal{V}_1) = 1.8673$, $\mathcal{H}(\mathcal{V}_5) = 1.8915$	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2$	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$

Table 10.5: Comparative Analysis results with CIFS studies

Method used	Score values $\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	Ranking
Rani and Garg [130] method based on CIFWPA operator	0.7175	0.4597	0.8160	0.6263	0.8443	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
Garg and Rani [59] method based on CIFWA operator	0.7312	0.4600	0.8191	0.6309	0.8475	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
Alkouri and Salleh [6] method based on Distance measure			-			$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
Rani and Garg [129] method based on Euclidean distance measure			-			$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
The proposed method based on GCIFWPA operator ($p = 1; q = 1$)	-1.1002	-1.1941	-1.0439	-1.1418	-0.9889	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
The proposed method based on GCIFWPA operator ($p = 1; q = 10$)	-0.5811	-0.8470	-0.5420	-0.6462	-0.6171	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
The proposed method based on GCIFWPA operator ($p = 1; q = 0$)	-0.9947	-1.1356	-0.9389	-1.0369	-0.9196	$\mathcal{V}_5 \succ \mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$

Table 10.6: Comparative Analysis results with IFS studies

Method used	Score values $\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	Ranking
Xu and Yager [188] method based on IFBM operator ($p = 1; q = 1$)	-0.5298	-0.6521	-0.5309	-0.5894	-0.4735	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
Xu [175] method based on IFPWA operator	0.3750	0.1118	0.3556	0.2273	0.4735	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
Wang and Liu [156] method based on IFEWA operator	0.3754	0.1059	0.3482	0.2077	0.4756	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
The proposed based on GCIFWPA operator ($p = 1; q = 1$)	-0.5298	-0.6521	-0.5309	-0.5894	-0.4735	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
The proposed based on GCIFWPA operator ($p = 1; q = 0$)	-0.4871	-0.6256	-0.5015	-0.5774	-0.4278	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3 \succ \mathcal{V}_4 \succ \mathcal{V}_2$

Table 10.7: The characteristic comparison of different approaches

Method	Generalized operators based on t-norm and t-conorm	Captures interrelationship among arguments	Ability to capture information using complex numbers	Ability to handle two-dimensional information	Ability to integrate Information	Flexible according to decision-maker's preferences
Rani and Garg [130]	×	×	✓	✓	✓	×
Garg and Rani [59]	✓	×	✓	✓	✓	×
Alkouri and Salleh [6]	×	×	✓	✓	×	✓
Rani and Garg [129]	×	×	✓	✓	×	×
Xu and Yager [188]	×	✓	×	×	✓	✓
Xu [175]	×	×	×	×	✓	×
Wang and Liu [156]	×	×	×	×	✓	×
The proposed approach	✓	✓	✓	✓	✓	✓

Chapter 11

New prioritized aggregation operators with priority degrees among priority orders for complex intuitionistic fuzzy information[1]

The objective of this chapter is to present some new prioritized aggregation operators by considering priority degrees among priority orders for aggregating CIFNs. In it, we present prioritized averaging and geometric operators with and without priority degrees based on basic unit interval monotonic function. Some properties related to proposed operators and priority degrees are explored. Further, a group decision-making method is put forward and is illustrated with the aid of an example. The influence of the priority degrees on aggregation result is also discussed.

11.1 Introduction

In most of the practical multi-criteria decision-making (MCDM) problems, it is often required to accumulate some numerical values and this is when aggregation operators (AOs) play a fundamental role. However, we encounter many situations in which there exists strict prioritization relationship among the arguments to be aggregated. For instance, if

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we wish to purchase a plot in order to build home for residential purposes over the criteria of utility access ($\mathfrak{B}_1$), location ($\mathfrak{B}_2$) and cost ($\mathfrak{B}_3$) then, we do not want to give up utility access over location and location over cost. That is, in this situation there exists strict prioritization among criteria such that $\mathfrak{B}_1 \succ \mathfrak{B}_2 \succ \mathfrak{B}_3$ where '$\succ$' indicates preferred to. To deal with such prioritized MCDM issues, Yager [191], proposed a series of AOs such as prioritized scoring, prioritized averaging, prioritized "and" and prioritized "or" operators. Besides this, Yager [193] developed prioritized ordered weighted averaging operator based on the basic unit interval monotonic (BUM) function.

Inspired by the features of the CIFS model and importance of information aggregation, this chapter is focussed on exploring CIF prioritized operators by considering priority degrees among strict priority levels. In order to illustrate the concept of priority degrees, again consider the example of purchasing a plot as given above. Each priority level will be assigned a real non-negative number, called as priority degree. In the considered example, since $\mathfrak{B}_1 \succ \mathfrak{B}_2 \succ \mathfrak{B}_3$. The first priority order $\mathfrak{B}_1 \succ \mathfrak{B}_2$ is assigned a priority degree d_1 where $0 \leq d_1 < \infty$ and then, this prioritization relationship is expressed as $\mathfrak{B}_1 \succ_{d_1} \mathfrak{B}_2$. Similarly, the second priority order $\mathfrak{B}_2 \succ \mathfrak{B}_3$ is assigned a priority degree d_2 where $0 \leq d_2 < \infty$ and this prioritization relationship among $\mathfrak{B}_2$ and $\mathfrak{B}_3$ is expressed as $\mathfrak{B}_2 \succ_{d_2} \mathfrak{B}_3$. Thus, a two dimensional vector $d = (d_1, d_2)$ is assigned to prioritized criteria $\mathfrak{B}_1 \succ \mathfrak{B}_2 \succ \mathfrak{B}_3$ and hence, this relationship is expressed as $\mathfrak{B}_1 \succ_{d_1} \mathfrak{B}_2 \succ_{d_2} \mathfrak{B}_3$. Now, we illustrate the three special cases related to priority degrees as follows:

(1) If someone wants to pay extreme attention to the first criteria then, the first priority degree d_1 should be taken very large. Further, in this chapter, we will prove that when $d_1 \to \infty$, then the aggregated value depends on first criteria only and the other criteria values get ignored.

(2) If priority degree vector is considered as zero vector i.e., $d = (0, 0)$ then, we will prove that, all the criteria become equally important and no prioritization remains among the criteria.

(3) If each priority degree is equal to one i.e., $d = (1, 1)$ then, there exists normal prioritization among the criteria and we will prove that in this case proposed CIF prioritized

averaging operators with and without degrees become identical.

Thus, by considering the concept of priority degrees, we propose CIF prioritized averaging and geometric operators without priority degrees (CIFPrA,CIFPrG), CIF prioritized averaging and geometric operators with priority degrees ($\text{CIFPrA}_d, \text{CIFPrG}_d$), CIF prioritized ordered weighted averaging and geometric operators with priority degrees ($\text{CIFPrOWA}_d, \text{CIFPrOWG}_d$). Properties related to AOs and propositions by taking different priority vectors are proved. Furthermore, a group DM methodology is presented by considering the multi-dimensional complex data sets and is applied on a real life problem related to Indian stock exchange. The proposed DM method is validated by performing comparative studies with existing theories. Because of the discussed advantages of prioritized aggregation, priority degrees and CIFSs, it is essential to develop AOs under this environment for addressing those problems, which are either impossible or difficult to be handled with one dimensional grades of membership.

11.2 Proposed CIF prioritized operators with and without priority degrees

In this section, we propose some weighted averaging and geometric operators and discuss their various characteristics. For it, firstly we define the score and accuracy functions as follows:

Definition 11.2.1. For CIFN $\mathcal{C} = ((\zeta, w_\zeta), (\vartheta, w_\vartheta))$, the score S and accuracy H functions are

$$\mathcal{S}(\mathcal{C}) = \frac{2+\zeta-\vartheta+w_\zeta-w_\vartheta}{4}; \tag{11.1}$$

$$\text{and} \quad \mathcal{H}(\mathcal{C}) = \frac{2+\zeta+\vartheta+w_\zeta+w_\vartheta}{4} \tag{11.2}$$

such that $0 \leq \mathcal{S}(\mathcal{C}) \leq 1$ and $0 \leq \mathcal{H}(\mathcal{C}) \leq 1$. Based on these functions, if $\mathcal{S}(\mathcal{C}_1) > \mathcal{S}(\mathcal{C}_2)$ or $\mathcal{S}(\mathcal{C}_1) = \mathcal{S}(\mathcal{C}_2)$ and $\mathcal{H}(\mathcal{C}_1) > \mathcal{H}(\mathcal{C}_2)$ then, $\mathcal{C}_1 \succ \mathcal{C}_2$ which indicates that CIFN $\mathcal{C}_1$ is superior to $\mathcal{C}_2$.

11.2.1 CIF prioritized averaging operator

Let $\mathcal{C}_j = \left(\left(\zeta_j.w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right)$ $(j = 1, 2, \ldots, n)$ be a collection of "n" CIFNs having prioritization relationship among them such as $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \ldots \succ \mathcal{C}_n$ where $\mathcal{C}_j \succ \mathcal{C}_{j+1}$ indicates that the CIFN $\mathcal{C}_j$ has higher priority than $\mathcal{C}_{j+1}$ for each $j \in \{1, 2, \ldots, n-1\}$. We denote the collection of all such CIFNs by Θ.

Definition 11.2.2. A CIFPrA operator is a map CIFPrA : $\Theta^n \to \Theta$ defined as:

$$\text{CIFPrA}\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \quad = \quad \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \ldots \oplus \xi_n\mathcal{C}_n \tag{11.3}$$

where $\xi_j = \frac{T_j}{\sum\limits_{j=1}^{n} T_j}$, $T_j = \prod\limits_{l=1}^{j-1} \mathcal{S}(\mathcal{C}_l)$ for each $j \in \{2, 3, \ldots, n\}$ and $T_1 = 1$. Then CIFPrA operator is known as CIF prioritized averaging operator.

Theorem 11.2.1. The aggregated value of $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ obtained by using proposed CIFPrA operator is still CIFN and is given as

$$\text{CIFPrA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} 1 - \prod\limits_{j=1}^{n} \left(1 - \zeta_j\right)^{\xi_j}, \\ 1 - \prod\limits_{j=1}^{n} \left(1 - w_{\zeta_j}\right)^{\xi_j} \end{pmatrix}, \begin{pmatrix} \prod\limits_{j=1}^{n} \left(\vartheta_j\right)^{\xi_j}, \\ \prod\limits_{j=1}^{n} \left(w_{\vartheta_j}\right)^{\xi_j} \end{pmatrix}\right) \tag{11.4}$$

Proof. It can be obtained by implementing operations given in Definition 9.2.1 of Chapter 9 on the Eq. (11.3). □

Further, it is observed that the proposed CIFPrA operator satisfies the properties of idempotency and boundary and does not satisfy monotonicity. These can be demonstrated as follows:

Property 11.2.1. If $\mathcal{C}_0$ be another CIFN such that $\mathcal{C}_j = \mathcal{C}_0\ \forall\ j$ then, we have

$$\text{CIFPrA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathcal{C}_0$$

Proof. Let $\mathcal{C}_0 = ((\zeta_0, w_{\zeta_0}), (\vartheta_0, w_{\vartheta_0}))$ be such that $\mathcal{C}_j = \mathcal{C}_0$. It implies that $\zeta_j = \zeta_0$, $\vartheta_j = \vartheta_0$, $w_{\zeta_j} = w_{\zeta_0}$ and $w_{\vartheta_j} = w_{\vartheta_0}$ for all j. Also, by using Definition 11.2.2, we get that $\sum\limits_{j=1}^{n} \xi_j = 1$.

Hence, using Theorem 11.2.1, we have

$$
\begin{aligned}
& \operatorname{CIFPrA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \\
= & \left(\left(1 - \prod_{j=1}^{n} (1-\zeta_0)^{\xi_j}, 1 - \prod_{j=1}^{n} (1-w_{\zeta_0})^{\xi_j} \right), \left(\prod_{j=1}^{n} (\vartheta_0)^{\xi_j}, \prod_{j=1}^{n} (w_{\vartheta_0})^{\xi_j} \right) \right) \\
= & \left(\left(1 - (1-\zeta_0)^{\sum_{j=1}^{n} \xi_j}, 1 - (1-w_{\zeta_0})^{\sum_{j=1}^{n} \xi_j} \right), \left((\vartheta_0)^{\sum_{j=1}^{n} \xi_j}, (w_{\vartheta_0})^{\sum_{j=1}^{n} \xi_j} \right) \right) \\
= & \Big((\zeta_0, w_{\zeta_0}), (\vartheta_0, w_{\vartheta_0}) \Big) \\
= & \mathcal{C}_0
\end{aligned}
$$

Hence, $\operatorname{CIFPrA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \mathcal{C}_0$. □

Property 11.2.2. Let $\mathcal{C}^- = ((\min_j\{\zeta_j\}, \min_j\{w_{\zeta_j}\}), (\max_j\{\vartheta_j\}, \max_j\{w_{\vartheta_j}\}))$ and $\mathcal{C}^+ = ((\max_j\{\zeta_j\}, \max_j\{w_{\zeta_j}\}), (\min_j\{\vartheta_j\}, \min_j\{w_{\vartheta_j}\}))$ be the lower and upper bounds respectively of CIFNs $\mathcal{C}_j = ((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j}))$. Then,

$$\mathcal{C}^- \subseteq \operatorname{CIFPrA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \mathcal{C}^+$$

Proof. Let $\operatorname{CIFPrA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \Big((\zeta_{\mathcal{C}}, w_{\zeta_{\mathcal{C}}}), (\vartheta_{\mathcal{C}}, w_{\vartheta_{\mathcal{C}}}) \Big)$. For a CIFN $\mathcal{C}_j$, we have $\min\{\zeta_j\} \leq \zeta_j \leq \max\{\zeta_j\} \Rightarrow 1 - \max\{\zeta_j\} \leq 1 - \zeta_j \leq 1 - \min\{\zeta_j\} \Rightarrow \prod_{j=1}^{n} \Big(1 - \max\{\zeta_j\}\Big)^{\xi_j} \leq \prod_{j=1}^{n} \Big(1 - \zeta_j\Big)^{\xi_j} \leq \prod_{j=1}^{n} \Big(1 - \min\{\zeta_j\}\Big)^{\xi_j} \Rightarrow \Big(1 - \max\{\zeta_j\}\Big)^{\sum_{j=1}^{n} \xi_j} \leq \prod_{j=1}^{n} \Big(1 - \zeta_j\Big)^{\xi_j} \leq \Big(1 - \min\{\zeta_j\}\Big)^{\sum_{j=1}^{n} \xi_j} \Rightarrow 1 - \max\{\zeta_j\} \leq \prod_{j=1}^{n} \Big(1 - \zeta_j\Big)^{\xi_j} \leq 1 - \min\{\zeta_j\} \Rightarrow \min\{\zeta_j\} \leq 1 - \prod_{j=1}^{n} \Big(1 - \zeta_j\Big)^{\xi_j} \leq \max\{\zeta_j\}$. Hence, $\min\{\zeta_j\} \leq \zeta_{\mathcal{C}} \leq \max\{\zeta_j\}$. Also, $\min\{\vartheta_j\} \leq \vartheta_j \leq \max\{\vartheta_j\} \Rightarrow \prod_{j=1}^{n} \Big(\min\{\vartheta_j\}\Big)^{\xi_j} \leq \prod_{j=1}^{n} \vartheta_j^{\xi_j} \leq \prod_{j=1}^{n} \Big(\max\{\vartheta_j\}\Big)^{\xi_j} \Rightarrow \Big(\min\{\vartheta_j\}\Big)^{\sum_{j=1}^{n} \xi_j} \leq \prod_{j=1}^{n} \vartheta_j^{\xi_j} \leq \Big(\max\{\vartheta_j\}\Big)^{\sum_{j=1}^{n} \xi_j} \Rightarrow \min\{\vartheta_j\} \leq \vartheta_{\mathcal{C}} \leq \max\{\vartheta_j\}$. Similarly, we can obtain that $\min_j\{w_{\zeta_j}\} \leq w_{\zeta_{\mathcal{C}}} \leq \max_j\{w_{\zeta_j}\}$ and $\min_j\{w_{\vartheta_j}\} \leq w_{\vartheta_{\mathcal{C}}} \leq \max_j\{w_{\vartheta_j}\}$. Hence, by using Definition 2.1.10, we obtain that $\mathcal{C}^- \subseteq \operatorname{CIFPrA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subseteq \mathcal{C}^+$. □

Remark 11.2.1. Further, it is noticed that the proposed CIFPrA operator is not monotonic. That is, there exist collections of CIFNs $\mathcal{C}_j$ and $\mathcal{Z}_j$ where $j = 1, 2, \ldots, n$ such that $\mathcal{C}_j \subseteq \mathcal{Z}_j$ for each j but $\text{CIFPrA}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \subsetneq \text{CIFPrA}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n)$.

In order to prove that the proposed CIFPrA operator is not monotonic, we give an example as follows:

Example 11.2.1. Consider two collections of CIFNs $\mathcal{C}_j$ and $\mathcal{Z}_j$ $(j = 1, 2, 3)$ such that $\mathcal{C}_1 = \big((0.7, 0.7), (0.2, 0.2)\big)$, $\mathcal{C}_2 = \big((0.5, 0.5), (0.4, 0.4)\big)$, $\mathcal{C}_3 = \big((0.2, 0.2), (0.7, 0.7)\big)$ and $\mathcal{Z}_1 = \big((0.71, 0.71), (0.2, 0.2)\big)$, $\mathcal{Z}_2 = \big((0.6, 0.6), (0.4, 0.4)\big)$, $\mathcal{Z}_3 = \big((0.25, 0.25), (0.7, 0.7)\big)$. Evidently, $\mathcal{C}_j \subseteq \mathcal{Z}_j$ for each $j = 1, 2, 3$. Now, firstly we aggregate $\mathcal{C}_j$ $(j = 1, 2, 3)$ using proposed CIFPrA operator. Using the Eq. (11.1), we obtain that $\mathcal{S}(\mathcal{C}_1) = 0.7500$, $\mathcal{S}(\mathcal{C}_2) = 0.5500$ and $\mathcal{S}(\mathcal{C}_3) = 0.2500$. Now, using $T_j = \prod\limits_{l=1}^{j-1} \mathcal{S}(\mathcal{C}_l)$ for $j = 2, 3, \ldots, n$, we get that $T_2 = \mathcal{S}(\mathcal{C}_1) = 0.7500$ and $T_3 = \mathcal{S}(\mathcal{C}_1)\mathcal{S}(\mathcal{C}_2) = 0.7500 \times 0.5500 = 0.4125$. Also, we have defined $T_1 = 1$. It implies that $\sum\limits_{j=1}^{3} T_j = 1 + 0.7500 + 0.4125 = 2.1625$. Further, using $\xi_j = \frac{T_j}{\sum\limits_{j=1}^{n} T_j}$ we get that $\xi_1 = \frac{1}{2.1625} = 0.4624$; $\xi_2 = \frac{0.7500}{2.1625} = 0.3468$ and $\xi_3 = \frac{0.4125}{2.1625} = 0.1908$. Now, using the Theorem 11.2.1, we obtain that

$$\prod_{j=1}^{3} \left(1 - \zeta_j\right)^{\xi_j} = (1 - 0.7)^{0.4624} \times (1 - 0.5)^{0.3468} \times (1 - 0.2)^{0.1908} = 0.4318$$

Similarly, we can obtain that $\prod\limits_{j=1}^{3} \left(1 - w_{\zeta_j}\right)^{\xi_j} = 0.4318$, $\prod\limits_{j=1}^{3} \left(\vartheta_j\right)^{\xi_j} = 0.3230$ and $\prod\limits_{j=1}^{3} \left(w_{\vartheta_j}\right)^{\xi_j} = 0.3230$. Further, on utilizing Eq. (11.4), we get that

$$\begin{aligned} \text{CIFPrA}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) &= \big((1 - 0.4318, 1 - 0.4318), (0.3230, 0.3230)\big) \\ &= \big((0.5682, 0.5682), (0.3230, 0.3230)\big) \end{aligned}$$

Proceeding in the similar manner, we may obtain that $\text{CIFPrA}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3) = \big((0.6066, 0.6066), (0.3278, 0.3278)\big)$, which gives that $\text{CIFPrA}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \subsetneq \text{CIFPrA}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3)$.

11.2.2 CIFPrA operator with priority degrees

Here, we consider $\mathcal{C}_j = \left(\left(\zeta_j . w_{\zeta_j}\right), \left(\vartheta_j, w_{\vartheta_j}\right)\right)$ $(j = 1, 2, \ldots, n)$, a collection of "n" CIFNs having a prioritization among them expressed by the strict priority orders $\mathcal{C}_1 \succ_{d_1} \mathcal{C}_2 \succ_{d_2}$

$\ldots \succ_{d_{n-1}} \mathcal{C}_n$ where $\mathcal{C}_j \succ_{d_j} \mathcal{C}_{j+1}$ means that the CIFN $\mathcal{C}_j$ has a d_j higher priority than $\mathcal{C}_{j+1}$ and $d = (d_1, d_2, \ldots, d_{n-1})$ is the priority degree vector satisfying $0 \leq d_j < \infty$ for $j \in \{1, 2, \ldots n-1\}$. We denote the collection of such CIFNs having strict priority orders along with priority degrees by Θ_d.

Definition 11.2.3. A CIFPrA$_d$ operator is a map CIFPrA$_d : \Theta_d^n \to \Theta_d$ defined as:

$$\text{CIFPrA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \quad = \quad \xi_1^{(d)}\mathcal{C}_1 \oplus \xi_2^{(d)}\mathcal{C}_2 \oplus \ldots \oplus \xi_n^{(d)}\mathcal{C}_n \tag{11.5}$$

where $\xi_j^{(d)} = \frac{T_j^{(d)}}{\sum\limits_{j=1}^{n} T_j^{(d)}}$, $T_j^{(d)} = \prod\limits_{l=1}^{j-1} \big(\mathcal{S}(\mathcal{C}_l)\big)^{d_l}$ for each $j \in \{2, 3, \ldots, n\}$ and $T_1 = 1$. Then, CIFPrA$_d$ operator is known as CIF prioritized averaging operator along with priority degrees.

Theorem 11.2.2. The aggregated value of $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ obtained by using proposed CIFPrA$_d$ operator is still CIFN and is given as

$$\text{CIFPrA}_d(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} 1 - \prod\limits_{j=1}^{n} \big(1 - \zeta_j\big)^{\xi_j^{(d)}}, \\ 1 - \prod\limits_{j=1}^{n} \big(1 - w_{\zeta_j}\big)^{\xi_j^{(d)}} \end{pmatrix}, \begin{pmatrix} \prod\limits_{j=1}^{n} \big(\vartheta_j\big)^{\xi_j^{(d)}}, \\ \prod\limits_{j=1}^{n} \big(w_{\vartheta_j}\big)^{\xi_j^{(d)}} \end{pmatrix} \right) \tag{11.6}$$

Proof. It can be proved by utilizing Eq. (11.5) and the operations defined in Definition 9.2.1. $\square$

Also, it is analyzed that the proposed CIFPrA$_d$ operator is idempotent and bounded. Besides this, it also satisfies the following propositions.

Proposition 11.2.1. For CIFNs $\mathcal{C}_j$, we have

$$\lim_{\substack{(d_1, d_2, \ldots, d_{n-1}) \\ \to (1,1,\ldots,1)}} \text{CIFPrA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \quad = \quad \text{CIFPrA}\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \tag{11.7}$$

Proof. For $(d_1, d_2, \ldots, d_{n-1}) \to (1, 1, \ldots, 1)$ we obtain that $T_j^{(d)} = \prod\limits_{l=1}^{j-1} \big(\mathcal{S}(\mathcal{C}_l)\big)^{d_l} \to \prod\limits_{l=1}^{j-1} \mathcal{S}(\mathcal{C}_l) = T_j$, which gives that $\xi_j^{(d)} \to \xi_j$. Hence,

$$\begin{aligned} \lim_{\substack{(d_1, d_2, \ldots, d_{n-1}) \\ \to (1,1,\ldots,1)}} \text{CIFPrA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) &= \lim_{\substack{(d_1, d_2, \ldots, d_{n-1}) \\ \to (1,1,\ldots,1)}} \Big(\xi_1^{(d)}\mathcal{C}_1 \oplus \xi_2^{(d)}\mathcal{C}_2 \oplus \ldots, \oplus \xi_n^{(d)}\mathcal{C}_n\Big) \\ &= \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \ldots, \oplus \xi_n\mathcal{C}_n \\ &= \text{CIFPrA}\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \end{aligned}$$

$\square$

Remark 11.2.2. The Proposition 11.2.1 gives that the presented CIFPrA operator is a special case of the proposed CIFPrA$_d$ operator when $d_1 = d_2 = \ldots = d_{n-1} = 1$. Hence, CIFPrA$_d$ operator is more generalized than CIFPrA operator.

Proposition 11.2.2. For CIFNs $\mathcal{C}_j$, such that $\mathcal{S}(\mathcal{C}_j) \neq 0$ for each j, we have

$$\lim_{(d_1,d_2,\ldots,d_{n-1})\to(0,0,\ldots,0)} \text{CIFPrA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \quad = \quad \frac{1}{n}\big(\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \ldots \oplus \mathcal{C}_n\big). \tag{11.8}$$

Proof. For $(d_1, d_2, \ldots, d_{n-1}) \to (0, 0, \ldots, 0)$, we have $T_j^{(d)} = \prod\limits_{l=1}^{j-1} \big(\mathcal{S}(\mathcal{C}_l)\big)^{d_l} = 1$. It gives that $\xi_j^{(d)} = \frac{T_j^{(d)}}{\sum\limits_{j=1}^{n} T_j^{(d)}} = \frac{1}{n}$. Hence, $\lim_{(d_1,d_2,\ldots,d_{n-1})\to(0,0,\ldots,0)} \text{CIFPrA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) = \frac{1}{n}\big(\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \ldots \oplus \mathcal{C}_n\big)$. $\square$

Remark 11.2.3. The Proposition 11.2.2 gives that when $(d_1, d_2, \ldots, d_{n-1}) \to (0, 0, \ldots, 0)$, the presented CIFPrA$_d$ operator reduces to CIFWA operator [62] for the weight vector $\big(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\big)^T$.

Proposition 11.2.3. For CIFNs $\mathcal{C}_j$ such that $\mathcal{S}(\mathcal{C}_1) \neq 0, 1$, we have

$$\lim_{d_1\to+\infty} \text{CIFPrA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \quad = \quad \mathcal{C}_1. \tag{11.9}$$

Proof. Since, $d_1 \to +\infty$. Now, for each $j = 2, 3, \ldots, n$, we have $T_j^{(d)} = \prod\limits_{l=1}^{j-1} \big(\mathcal{S}(\mathcal{C}_l)\big)^{d_l} = \big(\mathcal{S}(\mathcal{C}_1)\big)^{+\infty}\big(\mathcal{S}(\mathcal{C}_2)\big)^{d_2} \ldots \big(\mathcal{S}(\mathcal{C}_{j-1})\big)^{d_{j-1}} = 0$ because $0 < \mathcal{S}(\mathcal{C}_1) < 1$. It gives that $\sum\limits_{j=1}^{n} T_j^{(d)} = T_1^{(d)} = 1$ which implies that $\xi_1^{(d)} = \frac{T_1^{(d)}}{\sum\limits_{j=1}^{n} T_1^{(d)}} = 1$ and $\xi_j^{(d)} = \frac{T_j^{(d)}}{\sum\limits_{j=1}^{n} T_j^{(d)}} = 0$ for each $j = 2, 3, \ldots, n$. Hence, $\lim_{d_1\to+\infty} \text{CIFPrA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) = \mathcal{C}_1$. $\square$

Remark 11.2.4. The Proposition 11.2.3 gives that when $d_1 \to +\infty$, then the priority degree d_1 of CIFN $\mathcal{C}_1$ is very high as compared to the priority degrees of other CIFNs. It gives that CIFN $\mathcal{C}_1$ is highly important. Therefore, in this case the aggregation result obtained on applying proposed operator CIFPrA$_d$ is uniquely determined by $\mathcal{C}_1$.

Example 11.2.2. Consider the four CIFNs $\mathcal{C}_1 = ((0.8, 0.3), (0.1, 0.2))$, $\mathcal{C}_2 = ((0.6, 0.5), (0.4,\ 0.3))$, $\mathcal{C}_3 = ((0.5, 0.4), (0.1,\ 0.3))$ and $\mathcal{C}_4 = ((0.3, 0.2), (0.2, 0.4))$ such that there is a strict prioritization among them expressed by strict priority orders as $\mathcal{C}_1 \succ_{d_1} \mathcal{C}_2 \succ_{d_2} \mathcal{C}_3 \succ_{d_3} \mathcal{C}_4$. Using Eq. (11.1), it can be easily computed that $\mathcal{S}(\mathcal{C}_1) = 0.7000$, $\mathcal{S}(\mathcal{C}_2) = 0.6000$, $\mathcal{S}(\mathcal{C}_3) = 0.6250$ and $\mathcal{S}(\mathcal{C}_4) = 0.4750$. In the following, we will aggregate the CIFNs $\mathcal{C}_j$, $(j = 1, 2, 3, 4)$ for four different priority vectors $d = (d_1, d_2, d_3)$ in which we will keep the values of priority degrees d_2, d_3 same and vary the value of d_1 and discuss its impact on the aggregated results.

Case 1: When $d = (1, 1, 1)$.

Then, $T_1^{(d)} = 1$, $T_2^{(d)} = \left(\mathcal{S}(\mathcal{C}_1)\right)^{d_1} = 0.7000$, $T_3^{(d)} = \left(\mathcal{S}(\mathcal{C}_1)\right)^{d_1}\left(\mathcal{S}(\mathcal{C}_2)\right)^{d_2} = 0.7000 \times 0.6000 = 0.4200$ and $T_4^{(d)} = \left(\mathcal{S}(\mathcal{C}_1)\right)^{d_1}\left(\mathcal{S}(\mathcal{C}_2)\right)^{d_2}\ \left(\mathcal{S}(\mathcal{C}_3)\right)^{d_3} = 0.7 \times 0.6 \times 0.625 = 0.2625$. It gives that $T^{(d)} = \sum\limits_{j=1}^{4} T_j^{(d)} = 2.3825$. Now, using $\xi_j^{(d)} = \frac{T_j^{(d)}}{\sum\limits_{j=1}^{4} T_j^{(d)}}$, we obtain that $\xi_1^{(d)} = 0.4197$, $\xi_2^{(d)} = 0.2938$, $\xi_3^{(d)} = 0.1763$ and $\xi_4^{(d)} = 0.1102$.

Now,

$$\begin{aligned}\prod_{j=1}^{4} (1 - \zeta_j)^{\xi_j^{(d)}} &= (1 - 0.8)^{0.4197} \times (1 - 0.6)^{0.2938} \times (1 - 0.5)^{0.1763} \times (1 - 0.3)^{0.1102} \\ &= 0.3308\end{aligned}$$

Similarly, we can obtain that $\prod\limits_{j=1}^{4}\left(1 - w_{\zeta_j}\right)^{\xi_j^{(d)}} = 0.6263$, $\prod\limits_{j=1}^{4}\left(\vartheta_j\right)^{\xi_j^{(d)}} = 0.1622$ and $\prod\limits_{j=1}^{4}\left(w_{\vartheta_j}\right)^{\xi_j^{(d)}} = 0.2612$. Further, on utilizing Eq. (11.6), we get that

$$\begin{aligned}\mathrm{CIFPrA}_d(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4) &= \left(\left(1 - 0.3308, 1 - 0.6263\right), (0.1622, 0.2612)\right) \\ &= \left(\left(0.6692, 0.3737\right), (0.1622, 0.2612)\right)\end{aligned}$$

Case 2: When $d = (4, 1, 1)$.

Continuing in the same way, as in Case 1, we obtain that

$$\mathrm{CIFPrA}_d(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4) = \big(\,(0.7357, 0.3419)\,, (0.1308, 0.2319)\,\big)$$

Case 3: When $d = (8, 1, 1)$, then $\mathrm{CIFPrA}_d(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4)$=((0.7815, 0.3136), (0.1089, 0.2096)).

Case 4: When $d = (12, 1, 1)$, then $\text{CIFPrA}_d(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4)$=((0.7953, 0.3036), (0.1022, 0.2025)).

The aggregated results obtained from the above four cases depict that as the priority degree d_1 corresponding to CIFN $\mathcal{C}_1$ increases, the aggregated value comes closer to the rating values of the CIFN $\mathcal{C}_1$.

Proposition 11.2.4. For CIFNs $\mathcal{C}_j$ such that $\mathcal{S}(\mathcal{C}_j) \neq 0$ for $j = 1, 2, \ldots, l+1$ and $\mathcal{S}(\mathcal{C}_{l+1}) \neq 1$, we have

$$\lim_{(d_1, d_2, \ldots, d_l, d_{l+1}) \to (0, 0 \ldots, 0, +\infty)} \text{CIFPrA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) = \frac{1}{l+1}\big(\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \ldots \oplus \mathcal{C}_{l+1}\big). \tag{11.10}$$

Proof. Since $(d_1, d_2, \ldots, d_l, d_{l+1}) \to (0, 0 \ldots, 0, +\infty)$ therefore,

$$\begin{aligned}
T_j^{(d)} = \prod_{l=1}^{j-1} \big(\mathcal{S}(\mathcal{C}_l)\big)^{d_l} &= \big(\mathcal{S}(\mathcal{C}_1)\big)^{d_1}\big(\mathcal{S}(\mathcal{C}_2)\big)^{d_2} \ldots \big(\mathcal{S}(\mathcal{C}_{j-1})\big)^{d_{j-1}} \\
&\to \big(\mathcal{S}(\mathcal{C}_1)\big)^{0}\big(\mathcal{S}(\mathcal{C}_2)\big)^{0} \ldots \big(\mathcal{S}(\mathcal{C}_{j-1})\big)^{0} = 1 \text{ for each } j = 2, 3, \ldots, l+1 \\
T_j^{(d)} = \prod_{l=1}^{j-1} \big(\mathcal{S}(\mathcal{C}_l)\big)^{d_l} &= \big(\mathcal{S}(\mathcal{C}_1)\big)^{d_1}\big(\mathcal{S}(\mathcal{C}_2)\big)^{d_2} \ldots \big(\mathcal{S}(\mathcal{C}_{j-1})\big)^{d_{j-1}} \\
&\to \big(\mathcal{S}(\mathcal{C}_1)\big)^{0}\big(\mathcal{S}(\mathcal{C}_2)\big)^{0} \ldots \big(\mathcal{S}(\mathcal{C}_l)\big)^{0}\big(\mathcal{S}(\mathcal{C}_{l+1})\big)^{+\infty} \ldots \big(\mathcal{S}(\mathcal{C}_{j-1})\big)^{d_{j-1}} \\
&= 0 \text{ for each } j = l+2, l+3, \ldots, n
\end{aligned}$$

Thus, $\sum_{j=1}^{n} T_j^{(d)} = l+1$ and $\xi_j^{(d)} = \frac{T_j^{(d)}}{\sum_{j=1}^{n} T_j^{(d)}} \to \frac{1}{l+1}$ for each $j = 1, 2, \ldots, l+1$ and $\xi_j^{(d)} = \frac{T_j^{(d)}}{\sum_{j=1}^{n} T_j^{(d)}} \to \frac{0}{l+1} = 0$ for each $j = l+2, l+3, \ldots, n$. Hence,

$$\lim_{(d_1, d_2, \ldots, d_l, d_{l+1}) \to (0, 0 \ldots, 0, +\infty)} \text{CIFPrA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) = \frac{1}{l+1}\big(\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \ldots \oplus \mathcal{C}_{l+1}\big).$$

□

Remark 11.2.5. The Proposition 11.2.4 gives that when $(d_1, d_2, \ldots, d_l, d_{l+1}) \to (0, 0 \ldots, 0, +\infty)$, it implies that there is no prioritization relationship among the CIFNs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}$ and all of these CIFNs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}$ have very high priority than the CIFNs $\mathcal{C}_{l+2}, \mathcal{C}_{l+3}, \ldots, \mathcal{C}_n$. Therefore, the aggregated value depends only on CIFNs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}$ and these CIFNs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}$ have equal weightage in aggregation process.

Proposition 11.2.5. For CIFNs $\mathcal{C}_j$ such that $\mathcal{S}(\mathcal{C}_{l+1}) \neq 0, 1$, we have

$$\lim_{\substack{(d_1,d_2,\ldots,d_l,d_{l+1})\\ \to(1,1\ldots,1,+\infty)}} \mathrm{CIFPrA}_d\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big) = \mathrm{CIFPrA}\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_{l+1}\big). \tag{11.11}$$

Proof. Since $(d_1, d_2, \ldots, d_l, d_{l+1}) \to (1, 1\ldots, 1, +\infty)$ therefore,

$$\begin{aligned}
T_j^{(d)} = \prod_{l=1}^{j-1}\big(\mathcal{S}(\mathcal{C}_l)\big)^{d_l} &= \big(\mathcal{S}(\mathcal{C}_1)\big)^{d_1}\big(\mathcal{S}(\mathcal{C}_2)\big)^{d_2}\ldots\big(\mathcal{S}(\mathcal{C}_{j-1})\big)^{d_{j-1}}\\
&\to \mathcal{S}(\mathcal{C}_1)\mathcal{S}(\mathcal{C}_2)\ldots\mathcal{S}(\mathcal{C}_{j-1}) = T_j \text{ for each } j = 2,3,\ldots,l+1\\
T_j^{(d)} = \prod_{l=1}^{j-1}\big(\mathcal{S}(\mathcal{C}_l)\big)^{d_l} &= \big(\mathcal{S}(\mathcal{C}_1)\big)^{d_1}\big(\mathcal{S}(\mathcal{C}_2)\big)^{d_2}\ldots\big(\mathcal{S}(\mathcal{C}_{j-1})\big)^{d_{j-1}}\\
&\to \mathcal{S}(\mathcal{C}_1)\mathcal{S}(\mathcal{C}_2)\ldots\mathcal{S}(\mathcal{C}_l)\big(\mathcal{S}(\mathcal{C}_{l+1})\big)^{+\infty}\ldots\big(\mathcal{S}(\mathcal{C}_{j-1})\big)^{d_{j-1}}\\
&= 0 \text{ for each } j = l+2, l+3,\ldots,n
\end{aligned}$$

Thus, $\sum\limits_{j=1}^{n} T_j^{(d)} \to \sum\limits_{j=1}^{l+1} T_j$ and $\xi_j^{(d)} = \frac{T_j^{(d)}}{\sum_{j=1}^{n} T_j^{(d)}} \to \frac{T_j}{\sum_{j=1}^{l+1} T_j}$ for each $j = 1, 2, \ldots, l+1$ and $\xi_j^{(d)} = \frac{T_j^{(d)}}{\sum_{j=1}^{n} T_j^{(d)}} \to \frac{0}{\sum_{j=1}^{l+1} T_j} = 0$ for each $j = l+2, l+3, \ldots, n$. Hence,

$$\lim_{(d_1,d_2,\ldots,d_l,d_{l+1})\to(0,0\ldots,0,+\infty)} \mathrm{CIFPrA}_d\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big) = \mathrm{CIFPrA}\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_{l+1}\big).$$

□

Remark 11.2.6. The Proposition 11.2.5 gives that when $(d_1, d_2, \ldots, d_l, d_{l+1}) \to (1, 1\ldots, 1, +\infty)$, it implies that there is normal prioritization relationship among the CIFNs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}$ and all of these CIFNs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}$ have very high priority than $\mathcal{C}_{l+2}, \mathcal{C}_{l+3}, \ldots, \mathcal{C}_n$. Therefore, the aggregated value depends only on CIFNs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}$.

Further, consider the case when there are two prioritization relationships among the CIFNs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ given as:

$$\begin{aligned}
&\mathcal{C}_1 \succ_{d_1} \mathcal{C}_2 \succ_{d_2} \ldots \succ_{d_{l-1}} \mathcal{C}_l \succ_{d_l} \mathcal{C}_{l+1} \succ_{d_{l+1}} \ldots \succ_{d_{n-1}} \mathcal{C}_n\\
&\mathcal{C}_1 \succ_{d_1} \mathcal{C}_2 \succ_{d_2} \ldots \succ_{d_{l-1}} \mathcal{C}_l \succ_{d'_l} \mathcal{C}_{l+1} \succ_{d_{l+1}} \ldots \succ_{d_{n-1}} \mathcal{C}_n
\end{aligned}$$

i.e. the $n-1$ dimensional priority degree vectors of CIFNs $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ are given as $d = (d_1, d_2, \ldots, d_{l-1}, d_l,\ d_{l+1}, \ldots, d_n)$ and $d' = (d_1, d_2, \ldots, d_{l-1}, d'_l, d_{l+1}, \ldots, d_n)$ respectively. Assume that the associated weight vectors for the priority degree vectors d and

d' are $\xi^{(d)} = \left(\xi_1^{(d)}, \xi_2^{(d)}, \ldots, \xi_n^{(d)}\right)$ and $\xi'^{(d')} = \left(\xi_1'^{(d')}, \xi_2'^{(d')}, \ldots, \xi_n'^{(d')}\right)$. Then, based on the above considerations, we have the following proposition:

Proposition 11.2.6. If $d_l < d_l'$ then

(a) $\xi_j^{(d)} \leq \xi_j'^{(d')}$ for $j \leq l$;

(b) $\xi_j^{(d)} \geq \xi_j'^{(d')}$ for $j > l$.

Proof. Since $d_l < d_l'$

(a) When $j \leq l$. For convenience, let $\mathcal{S}(\mathcal{C}_j) = \rho_j$. Now,

$$\begin{aligned}
\xi_j^{(d)} &= \frac{T_j^{(d)}}{\sum\limits_{j=1}^{n} T_j^{(d)}} \\
&= \frac{\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}}{\begin{pmatrix} 1+\rho_1^{d_1}+\rho_1^{d_1}\rho_2^{d_2}+\ldots+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}\rho_l^{d_l} \\ +\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}\rho_l^{d_l}\rho_{l+1}^{d_{l+1}}+\ldots+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{n-1}^{d_{n-1}} \end{pmatrix}} \\
&= \frac{\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}}{\begin{pmatrix} 1+\rho_1^{d_1}+\rho_1^{d_1}\rho_2^{d_2}+\ldots+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}+\rho_l^{d_l}\left(\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}\right. \\ \left.+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}\rho_{l+1}^{d_{l+1}}+\ldots+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}\rho_{l+1}^{d_{l+1}}\ldots\rho_{n-1}^{d_{n-1}}\right) \end{pmatrix}}
\end{aligned}$$

Similarly, we can obtain that

$$\xi_j'^{(d')} = \frac{\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}}{\begin{pmatrix} 1+\rho_1^{d_1}+\rho_1^{d_1}\rho_2^{d_2}+\ldots+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}+\rho_l^{d_l'}\left(\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}\right. \\ \left.+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}\rho_{l+1}^{d_{l+1}}+\ldots+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}\rho_{l+1}^{d_{l+1}}\ldots\rho_{n-1}^{d_{n-1}}\right) \end{pmatrix}}$$

For simplicity, let $\mathcal{A} = 1+\rho_1^{d_1}+\rho_1^{d_1}\rho_2^{d_2}+\ldots+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}$ and $\mathcal{B} = \rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}} + \rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}\rho_{l+1}^{d_{l+1}}+\ldots+\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{l-1}^{d_{l-1}}\rho_{l+1}^{d_{l+1}}\ldots\rho_{n-1}^{d_{n-1}}$. Then, we obtain that

$$\xi_j^{(d)} = \frac{\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}}{\mathcal{A}+\rho_l^{d_l}\mathcal{B}} \quad \text{and} \quad \xi_j^{(d')} = \frac{\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}}{\mathcal{A}+\rho_l^{d_l'}\mathcal{B}}$$

Now, since $d_l < d_l'$. It implies that $\rho_l^{d_l} \geq \rho_l^{d_l'}$ which leads to $\mathcal{A}+\rho_l^{d_l}\mathcal{B} \geq \mathcal{A}+\rho_l^{d_l'}\mathcal{B}$. It gives that $\frac{1}{\mathcal{A}+\rho_l^{d_l}\mathcal{B}} \leq \frac{1}{\mathcal{A}+\rho_l^{d_l'}\mathcal{B}}$ which implies that, $\frac{\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}}{\mathcal{A}+\rho_l^{d_l}\mathcal{B}} \leq \frac{\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}}{\mathcal{A}+\rho_l^{d_l'}\mathcal{B}}$. Hence, $\xi_j^{(d)} \leq \xi_j'^{(d')}$ for $j \leq l$.

(b) When $j > l$, we have

$$\xi_j^{(d)} = \frac{\rho_1^{d_1}\rho_2^{d_2}\dots\rho_{l-1}^{d_{l-1}}\rho_l^{d_l}\rho_{l+1}^{d_{l+1}}\dots\rho_{j-1}^{d_{j-1}}}{\mathcal{A}+\rho_l^{d_l}\mathcal{B}} \quad \text{and} \quad \xi_j^{(d')} = \frac{\rho_1^{d_1}\rho_2^{d_2}\dots\rho_{l-1}^{d_{l-1}}\rho_l^{d_l'}\rho_{l+1}^{d_{l+1}}\dots\rho_{j-1}^{d_{j-1}}}{\mathcal{A}+\rho_l^{d_l'}\mathcal{B}}$$

Since $d_l < d_l'$. It implies that $\rho_l^{d_l} \geq \rho_l^{d_l'}$ which leads to $\frac{\rho_l^{d_l'}}{\rho_l^{d_l}} \leq 1$. Now,

$$\begin{aligned}
\xi_j^{(d)} &= \frac{\rho_1^{d_1}\rho_2^{d_2}\dots\rho_{l-1}^{d_{l-1}}\rho_l^{d_l}\rho_{l+1}^{d_{l+1}}\dots\rho_{j-1}^{d_{j-1}}}{\mathcal{A}+\rho_l^{d_l}\mathcal{B}} \\
&= \frac{\left(\rho_1^{d_1}\rho_2^{d_2}\dots\rho_{l-1}^{d_{l-1}}\rho_l^{d_l}\rho_{l+1}^{d_{l+1}}\dots\rho_{j-1}^{d_{j-1}}\right)\left(\frac{\rho_l^{d_l'}}{\rho_l^{d_l}}\right)}{\left(\mathcal{A}+\rho_l^{d_l}\mathcal{B}\right)\left(\frac{\rho_l^{d_l'}}{\rho_l^{d_l}}\right)} \\
&= \frac{\rho_1^{d_1}\rho_2^{d_2}\dots\rho_{l-1}^{d_{l-1}}\rho_l^{d_l'}\rho_{l+1}^{d_{l+1}}\dots\rho_{j-1}^{d_{j-1}}}{\mathcal{A}\left(\frac{\rho_l^{d_l'}}{\rho_l^{d_l}}\right)+\rho_l^{d_l'}\mathcal{B}}
\end{aligned}$$

Further, since $\frac{\rho_l^{d_l'}}{\rho_l^{d_l}} \leq 1$. It gives that $\mathcal{A}\left(\frac{\rho_l^{d_l'}}{\rho_l^{d_l}}\right) \leq \mathcal{A}$ which leads to $\mathcal{A}\left(\frac{\rho_l^{d_l'}}{\rho_l^{d_l}}\right) + \rho_l^{d_l'}\mathcal{B} \leq \mathcal{A}+\rho_l^{d_l'}\mathcal{B}$. It implies that

$$\frac{1}{\mathcal{A}\left(\frac{\rho_l^{d_l'}}{\rho_l^{d_l}}\right)+\rho_l^{d_l'}\mathcal{B}} \geq \frac{1}{\mathcal{A}+\rho_l^{d_l'}\mathcal{B}}$$

and therefore,

$$\frac{\rho_1^{d_1}\rho_2^{d_2}\dots\rho_{l-1}^{d_{l-1}}\rho_l^{d_l'}\rho_{l+1}^{d_{l+1}}\dots\rho_{j-1}^{d_{j-1}}}{\mathcal{A}\left(\frac{\rho_l^{d_l'}}{\rho_l^{d_l}}\right)+\rho_l^{d_l'}\mathcal{B}} \geq \frac{\rho_1^{d_1}\rho_2^{d_2}\dots\rho_{l-1}^{d_{l-1}}\rho_l^{d_l'}\rho_{l+1}^{d_{l+1}}\dots\rho_{j-1}^{d_{j-1}}}{\mathcal{A}+\rho_l^{d_l'}\mathcal{B}}$$

Hence, $\xi_j^{(d)} \geq \xi_j'^{(d')}$ for $j > l$.

$\square$

11.2.3 CIF prioritized ordered weighted averaging operator with priority degrees

Definition 11.2.4. A CIFPrOWA$_d$ operator is a map CIFPrOWA$_d$: $\Theta_d^n \to \Theta_d$ defined as:

$$\text{CIFPrOWA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \quad = \quad \xi_1^{(d)}\mathcal{C}_{\tau(1)} \oplus \xi_2^{(d)}\mathcal{C}_{\tau(2)} \oplus \ldots \oplus \xi_n^{(d)}\mathcal{C}_{\tau(n)} \tag{11.12}$$

where $\big(\tau(1), \tau(2), \ldots, \tau(n)\big)$ is an arrangement of $(1, 2, \ldots, n)$ such that $\mathcal{S}(\mathcal{C}_{\tau(j-1)}) \geq \mathcal{S}(\mathcal{C}_{\tau(j)})$ for each $j = 2, 3, \ldots, n$. Further, $\xi_j^{(d)}$ is defined as:

$$\xi_j^{(d)} \quad = \quad g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_j\right\}} T_\mathfrak{q}^{(d)}}{T^{(d)}}\right) - g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_{j-1}\right\}} T_\mathfrak{q}^{(d)}}{T^{(d)}}\right) \tag{11.13}$$

where $g : [0, 1] \to [0, 1]$ is a BUM function satisfying $g(0) = 0$, $g(1) = 1$ and $g(x) \geq g(y)$ whenever $x \geq y$. Further $\mathcal{H}_j = \left\{\mathcal{C}_{\tau(l)} | l = 1, 2, \ldots, j\right\}$ for $j = 1, 2, \ldots, n$; $\mathcal{H}_0 = \phi$, $T_j^{(d)} = \prod\limits_{l=1}^{j-1}\big(\mathcal{S}(\mathcal{C}_l)\big)^{d_l}$ for each $j \in \{2, 3, \ldots, n\}$, $T_1 = 1$ and $T^{(d)} = \sum\limits_{j=1}^{n} T_j^{(d)}$. Then, CIFPrOWA$_d$ operator is known as CIF prioritized ordered weighted averaging operator along with priority degrees.

Theorem 11.2.3. The aggregated value of $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ obtained by using proposed CIFPrOWA$_d$ operator is still CIFN and is given as

$$\text{CIFPrOWA}_d(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} 1 - \prod\limits_{j=1}^{n}\big(1 - \zeta_{\tau(j)}\big)^{\xi_j^{(d)}}, \\ 1 - \prod\limits_{j=1}^{n}\Big(1 - w_{\zeta_{\tau(j)}}\Big)^{\xi_j^{(d)}} \end{pmatrix}, \begin{pmatrix} \prod\limits_{j=1}^{n}\big(\vartheta_{\tau(j)}\big)^{\xi_j^{(d)}}, \\ \prod\limits_{j=1}^{n}\big(w_{\vartheta_{\tau(j)}}\big)^{\xi_j^{(d)}} \end{pmatrix}\right) \tag{11.14}$$

Proof. It can by obtained by applying the operations given in Definition 9.2.1 of Chapter 9 on the Eq. (11.12). □

In order to illustrate the working of the proposed CIFPrOWA$_d$ operator, we give an example as follows:

Example 11.2.3. Consider the four CIFNs $\mathcal{C}_1 = ((0.2, 0.4), (0.3, 0.1))$, $\mathcal{C}_2$=((0.7,0.5), (0.2, 0.3)), $\mathcal{C}_3 = ((0.5, 0.4), (0.1,\ 0.3))$ and $\mathcal{C}_4 = ((0.3, 0.2), (0.2, 0.4))$ such that there is a strict prioritization among them expressed by strict priority orders as $\mathcal{C}_1 \succ_{d_1} \mathcal{C}_2 \succ_{d_2} \mathcal{C}_3 \succ_{d_3} \mathcal{C}_4$ where $d = (d_1, d_2, d_3) = (0, 0, 2)$. Using Eq. (11.1), it can be easily computed that $\mathcal{S}(\mathcal{C}_1) = 0.5500$, $\mathcal{S}(\mathcal{C}_2) = 0.6750$, $\mathcal{S}(\mathcal{C}_3) = 0.6250$ and $\mathcal{S}(\mathcal{C}_4) = 0.4750$. Thus, $\mathcal{S}(\mathcal{C}_2) > \mathcal{S}(\mathcal{C}_3) > \mathcal{S}(\mathcal{C}_1) > \mathcal{S}(\mathcal{C}_4)$ which implies that $\mathcal{C}_{\tau(1)} = \big((0.7, 0.5), (0.2, 0.3)\big)$; $\mathcal{C}_{\tau(2)} = \big((0.5, 0.4), (0.1, 0.3)\big)$; $\mathcal{C}_{\tau(3)} = \big((0.2, 0.4), (0.3, 0.1)\big)$ and $\mathcal{C}_{\tau(4)} = \big((0.3, 0.2), (0.2, 0.4)\big)$. Now, using $T_j^{(d)} = \prod\limits_{l=1}^{j-1} (\mathcal{S}(\mathcal{C}_l))^{d_l}$ for $j = 2, 3, 4$ and $T_1^{(d)} = 1$, we obtain that $T_1^{(d)} = T_2^{(d)} = T_3^{(d)} = 1$ and $T_4^{(d)} = 1 \times 1 \times (0.6250)^2 = 0.3906$. It gives that $T^{(d)} = \sum\limits_{j=1}^{4} T_j^{(d)} = 1 + 1 + 1 + 0.3906 = 3.3906$.

Further, since $\mathcal{H}_0 = \phi$ and $\mathcal{H}_j = \big\{\mathcal{C}_{\tau(l)} | l = 1, 2, \ldots, j\big\}$ for $j = 1, 2, \ldots, n$. It implies that $\mathcal{H}_1 = \{\mathcal{C}_{\tau(1)}\} = \{\mathcal{C}_2\}$; $\mathcal{H}_2 = \{\mathcal{C}_{\tau(1)}, \mathcal{C}_{\tau(2)}\} = \{\mathcal{C}_2, \mathcal{C}_3\}$; $\mathcal{H}_3 = \{\mathcal{C}_{\tau(1)}, \mathcal{C}_{\tau(2)}, \mathcal{C}_{\tau(3)}\} = \{\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_1\}$ and $\mathcal{H}_4 = \{\mathcal{C}_{\tau(1)}, \mathcal{C}_{\tau(2)}, \mathcal{C}_{\tau(3)}, \mathcal{C}_{\tau(4)}\} = \{\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_1, \mathcal{C}_4\}$. Now, taking $g(x) = x^2$ and using the Eq. (11.13), we obtain that

$$
\begin{aligned}
\xi_1^{(d)} &= g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_1\right\}} T_\mathfrak{q}^{(d)}}{T^{(d)}}\right) - g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_0\right\}} T_\mathfrak{q}^{(d)}}{T^{(d)}}\right) \\
&= g\left(\frac{T_2^{(d)}}{T^{(d)}}\right) - g(0) = g\left(\frac{1}{3.3906}\right) = 0.0870 \\
\xi_2^{(d)} &= g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_2\right\}} T_\mathfrak{q}^{(d)}}{T^{(d)}}\right) - g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_1\right\}} T_\mathfrak{q}^{(d)}}{T^{(d)}}\right) \\
&= g\left(\frac{T_2^{(d)} + T_3^{(d)}}{T^{(d)}}\right) - 0.0870 = g\left(\frac{2}{3.3906}\right) - 0.0870 = 0.2610 \\
\xi_3^{(d)} &= g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_3\right\}} T_\mathfrak{q}^{(d)}}{T^{(d)}}\right) - g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_2\right\}} T_\mathfrak{q}^{(d)}}{T^{(d)}}\right) \\
&= g\left(\frac{T_2^{(d)} + T_3^{(d)} + T_1^{(d)}}{T^{(d)}}\right) - 0.3480 = g\left(\frac{3}{3.3906}\right) - 0.3480 = 0.4349
\end{aligned}
$$

$$
\begin{aligned}
\xi_4^{(d)} &= g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathcal{H}_4\right\}} T_{\mathfrak{q}}^{(d)}}{T^{(d)}}\right) - g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathcal{H}_3\right\}} T_{\mathfrak{q}}^{(d)}}{T^{(d)}}\right) \\
&= g\left(\frac{T_2^{(d)}+T_3^{(d)}+T_1^{(d)}+T_4^{(d)}}{T^{(d)}}\right) - 0.7829 = g\left(\frac{3.3906}{3.3906}\right) - 0.7829 = 0.2171
\end{aligned}
$$

Now, using the above information, we have

$$
\begin{aligned}
\prod_{j=1}^{4}\left(1-\zeta_{\tau(j)}\right)^{\xi_j^{(d)}} &= \left(1-0.7\right)^{0.0870}\left(1-0.5\right)^{0.2610}\left(1-0.2\right)^{0.4349}\left(1-0.3\right)^{0.2171} = 0.6312 \\
\prod_{j=1}^{4}\left(1-w_{\zeta_{\tau(j)}}\right)^{\xi_j^{(d)}} &= \left(1-0.5\right)^{0.0870}\left(1-0.4\right)^{0.2610}\left(1-0.4\right)^{0.4349}\left(1-0.2\right)^{0.2171} = 0.6286 \\
\prod_{j=1}^{4}\left(\vartheta_j\right)^{\xi_j^{(d)}} &= \left(0.2\right)^{0.0870}\left(0.1\right)^{0.2610}\left(0.3\right)^{0.4349}\left(0.2\right)^{0.2171} = 0.1991 \\
\prod_{j=1}^{4}\left(w_{\vartheta_j}\right)^{\xi_j^{(d)}} &= \left(0.3\right)^{0.0870}\left(0.3\right)^{0.2610}\left(0.1\right)^{0.4349}\left(0.4\right)^{0.2171} = 0.1980
\end{aligned}
$$

Finally, using the Theorem 11.2.3, we obtain that

$$
\begin{aligned}
\mathrm{CIFPrOWA}_d(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4) &= \Big(\left(\left(1-0.6312\right),\left(1-0.6286\right)\right),\left(0.1991,0.1980\right)\Big) \\
&= \Big(\left(0.3688,0.3714\right),\left(0.1991,0.1980\right)\Big)
\end{aligned}
$$

Further, for convenience, in this section, we shall denote $\mathcal{S}(\mathcal{C}_j)$ by ρ_j for each $j = 1,2,\ldots,n$. Also, it is observed that, when $d_1 = d_2 = \ldots d_{n-1} = 1$ then, $T_j^{(d)} = \prod\limits_{l=1}^{j-1}\rho_l^{d_l} = \prod\limits_{l=1}^{j-1}\rho_l = \rho_1\rho_2\ldots\rho_{j-1}$ and $T^{(d)} = \sum\limits_{j=1}^{n}T_j^{(d)} = 1+\rho_1+\rho_1\rho_2+\ldots+\rho_1\rho_1\ldots\rho_{n-1}$. Then, we will call the operator $\mathrm{CIFPrOWA}_d$ defined in Definition 11.2.4 as CIFPrOWA operator for $d = (1,1,\ldots,1)$. Based on it, we give the following proposition:

Proposition 11.2.7. For CIFNs $\mathcal{C}_j$, we have

$$
\lim_{(d_1,d_2,\ldots,d_{n-1})\to(1,1,\ldots,1)} \mathrm{CIFPrOWA}_d\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big) = \mathrm{CIFPrOWA}\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big)
$$

Proposition 11.2.8. For CIFNs $\mathcal{C}_j$, such that $\mathcal{S}(\mathcal{C}_j)\neq 0$ for each j, we have

$$
\lim_{(d_1,d_2,\ldots,d_{n-1})\to(0,0,\ldots,0)} \mathrm{CIFPrOWA}_d\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big) = \mathrm{CIFOWA}_g\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big)
$$

where $\text{CIFOWA}_g\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big)=\bigoplus\limits_{j=1}^{n}\xi_j\mathcal{C}_{\tau(j)}$ is an n-dimensional ordered weighted averaging operator in which the weights ξ_j are determined using the continuous BUM function i.e, $\xi_j=g\left(\frac{j}{n}\right)-g\left(\frac{j-1}{n}\right)$ for $j=1,2,\ldots,n$.

Proof. Since, $(d_1,d_2,\ldots,d_{n-1})\to(0,0,\ldots,0)$. Therefore, for each $j=2,3,\ldots,n$, $T_j^{(d)}=\prod\limits_{l=1}^{j-1}\rho_l^{d_l}=\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}\to\rho_1^0\rho_2^0\ldots\rho_{j-1}^0=1$ and $T_1^{(d)}=1$. So, $T^{(d)}=\sum\limits_{j=1}^{n}T_j^{(d)}\to n$. It gives that

$$\frac{\sum\limits_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_j\}}T_\mathfrak{q}^{(d)}}{T^{(d)}}\to\frac{j}{n}\ \text{ and }\ \frac{\sum\limits_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_{j-1}\}}T_\mathfrak{q}^{(d)}}{T^{(d)}}\to\frac{j-1}{n}\quad\Rightarrow\ \xi_j^{(d)}\to g\left(\frac{j}{n}\right)-g\left(\frac{j-1}{n}\right)$$

Hence,

$$\begin{aligned}\lim_{\substack{(d_1,d_2,\ldots,d_{n-1})\\ \to(0,0,\ldots,0)}}\text{CIFPrOWA}_d\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big) &= \lim_{(d_1,d_2,\ldots,d_{n-1})\to(0,0,\ldots,0)}\bigoplus_{j=1}^{n}\xi_j^{(d)}\mathcal{C}_{\tau(j)}\\ &\to \bigoplus_{j=1}^{n}\left(g\left(\frac{j}{n}\right)-g\left(\frac{j-1}{n}\right)\right)\mathcal{C}_{\tau(j)}\\ &= \text{CIFOWA}_g\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big)\end{aligned}$$

□

Proposition 11.2.9. For CIFNs $\mathcal{C}_j$, such that $\mathcal{S}(\mathcal{C}_1)\neq0,1$, we have

$$\lim_{d_1\to+\infty}\text{CIFPrOWA}_d\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big)=\mathcal{C}_1$$

Proof. Since, $d_1\to+\infty$. Therefore, for each $j=2,3,\ldots,n$ we have, $T_j^{(d)}=\prod\limits_{l=1}^{j-1}\rho_l^{d_l}=\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}\to\rho_1^{\infty}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}=0$ and $T_1^{(d)}=1$. It implies that $T^{(d)}=\sum\limits_{j=1}^{n}T_j^{(d)}=1$. It leads to that

$$\text{if }\ \mathcal{C}_1\in\mathcal{H}_j\ \text{ then }\ g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_j\right\}}T_\mathfrak{q}^{(d)}}{T^{(d)}}\right)=g(1)=1$$

$$\text{and}\qquad\text{if }\ \mathcal{C}_1\notin\mathcal{H}_j\ \text{ then }\ g\left(\frac{\sum\limits_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_j\right\}}T_\mathfrak{q}^{(d)}}{T^{(d)}}\right)=g(0)=0.$$

Let $\mathcal{C}_1 = \mathcal{C}_{\tau(l)}$ for some $l = 1, 2, \ldots, n$. Then, using the Eq. (11.13), we obtain that $\xi_1^{(d)}, \xi_2^{(d)}, \ldots, \xi_{l-1}^{(d)} \to 0$, $\xi_l^{(d)} \to 1$, $\xi_{l+1}^{(d)}, \xi_{l+2}^{(d)}, \ldots, \xi_n^{(d)} \to 0$. Hence,

$$\begin{aligned}
\lim_{d_1 \to +\infty} \text{CIFPrOWA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) &= \lim_{d_1 \to +\infty} \bigoplus_{j=1}^{n} \Big(\xi_j^{(d)} \mathcal{C}_{\tau(j)}\Big) \\
&\to \xi_l^{(d)} \mathcal{C}_{\tau(l)} \\
&= \mathcal{C}_1
\end{aligned}$$

□

Proposition 11.2.10. For CIFNs $\mathcal{C}_j$ such that $\mathcal{S}(\mathcal{C}_j) \neq 0$ for $j = 1, 2, \ldots, l+1$ and $\mathcal{S}(\mathcal{C}_{l+1}) \neq 1$, we have

$$\lim_{(d_1, d_2, \ldots, d_l, d_{l+1}) \to (0, 0, \ldots, 0, +\infty)} \text{CIFPrOWA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) = \text{CIFOWA}_g\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}\big)$$

where $\text{CIFOWA}_g\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) = \bigoplus_{j=1}^{l+1} \big(\psi_j \mathcal{C}_{\varsigma(j)}\big)$ is the $l+1$-dimensional ordered weighted averaging operator in which the weights ψ_j are determined using the continuous BUM function g i.e, $\psi_j = g\left(\frac{j}{l+1}\right) - g\left(\frac{j-1}{l+1}\right)$ for $j = 1, 2, \ldots, l+1$ and $\big(\varsigma(1), \varsigma(2), \ldots, \varsigma(l+1)\big)$ is an arrangement of $(1, 2, \ldots, l+1)$ such that $\mathcal{S}(\mathcal{C}_{\varsigma(j-1)}) \geq \mathcal{S}(\mathcal{C}_{\varsigma(j)})$ for each $j = 2, 3, \ldots, l+1$.

Proof. Since, $(d_1, d_2, \ldots, d_l, d_{l+1}) \to (0, 0, \ldots, 0, +\infty)$. Therefore, for each $j = 2, 3, \ldots, l+1$, $T_j^{(d)} = \prod_{l=1}^{j-1} \rho_l^{d_l} = \rho_1^{d_1} \rho_2^{d_2} \ldots \rho_{j-1}^{d_{j-1}} \to \rho_1^0 \rho_2^0 \ldots \rho_{j-1}^0 = 1$ and when $j = l+2, l+3, \ldots, n$, $T_j^{(d)} = \prod_{l=1}^{j-1} \rho_l^{d_l} = \rho_1^{d_1} \rho_2^{d_2} \ldots \rho_l^{d_l} \rho_{l+1}^{d_{l+1}} \ldots \rho_{j-1}^{d_{j-1}} \to 0$. It gives that $T^{(d)} = \sum_{j=1}^{n} T_j^{(d)} \to k+1$.

Since, $\big(\varsigma(1), \varsigma(2), \ldots, \varsigma(l+1)\big)$ is an arrangement of $(1, 2, \ldots, l+1)$ such that $\mathcal{S}(\mathcal{C}_{\varsigma(j-1)}) \geq \mathcal{S}(\mathcal{C}_{\varsigma(j)})$ for each $j = 2, 3, \ldots, l+1$. Also, we have assumed that $\big(\tau(1), \tau(2), \ldots, \tau(n)\big)$ is an arrangement of $(1, 2, \ldots, n)$ such that $\mathcal{S}(\mathcal{C}_{\tau(j-1)}) \geq \mathcal{S}(\mathcal{C}_{\tau(j)})$ for each $j = 2, 3, \ldots, n$. As, $\big\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}\big\} \subseteq \big\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big\}$ therefore, there must exist natural numbers $1 \leq m_1, m_2, \ldots, m_{l+1} \leq n$ satisfying $\mathcal{C}_{\varsigma(1)} = \mathcal{C}_{\tau(m_1)}$, $\mathcal{C}_{\varsigma(2)} = \mathcal{C}_{\tau(m_2)}$, …, $\mathcal{C}_{\varsigma(l+1)} = \mathcal{C}_{\tau(m_{l+1})}$. Using the Eq. (11.13), we have

$$\begin{aligned}
&\xi_{m_1}^{(d)} \to g\left(\frac{1}{l+1}\right) - g\left(\frac{0}{l+1}\right) = \psi_1 \quad ; \quad \xi_{m_2}^{(d)} \to g\left(\frac{2}{l+1}\right) - g\left(\frac{1}{l+1}\right) = \psi_2 \\
&\ldots \\
&\xi_{m_{l+1}}^{(d)} \to g\left(\frac{l+1}{l+1}\right) - g\left(\frac{l}{l+1}\right) = \psi_{l+1}
\end{aligned}$$

and all other weights tend to 0. Hence,

$$\begin{aligned}
\lim_{\substack{(d_1,d_2,\ldots,d_l,d_{l+1})\\ \to(0,0,\ldots,0,+\infty)}} \mathrm{CIFPrOWA}_d\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big) &= \lim_{\substack{(d_1,d_2,\ldots,d_l,d_{l+1})\\ \to(0,0,\ldots,0,+\infty)}} \bigoplus_{j=1}^{n}\Big(\xi_j^{(d)}\mathcal{C}_{\tau(j)}\Big)\\
&\to \psi_1\mathcal{C}_{\tau(m_1)}\oplus\psi_2\mathcal{C}_{\tau(m_2)}\oplus\ldots\psi_{l+1}\mathcal{C}_{\tau(m_{l+1})}\\
&= \psi_1\mathcal{C}_{\varsigma(1)}\oplus\psi_2\mathcal{C}_{\varsigma(2)}\oplus\ldots\psi_{l+1}\mathcal{C}_{\varsigma(l+1)}\\
&= \mathrm{CIFOWA}_g(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_{l+1})
\end{aligned}$$

□

Proposition 11.2.11. For CIFNs $\mathcal{C}_j$, such that $\mathcal{S}(\mathcal{C}_{l+1})\neq 0,1$, we have

$$\lim_{(d_1,d_2,\ldots,d_l,d_{l+1})\to(1,1,\ldots,1,+\infty)} \mathrm{CIFPrOWA}_d\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big) = \mathrm{CIFPrOWA}\big(\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_{l+1}\big)$$

Proof. Since, $(d_1,d_2,\ldots,d_l,d_{l+1})\to(1,1,\ldots,1,+\infty)$. Let $\rho_1\rho_2\ldots\rho_{j-1}=\mathbb{T}_j$ for $j=2,3,\ldots,l+1$; $\mathbb{T}_1=1$ and $\mathbb{T}=\sum_{j=1}^{l+1}\mathbb{T}_j$. Therefore, for each $j=2,3,\ldots,l+1$, $T_j^{(d)}=\prod_{l=1}^{j-1}\rho_l^{d_l}=\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_{j-1}^{d_{j-1}}\to\rho_1^1\rho_2^1\ldots\rho_{j-1}^1=\mathbb{T}_j$. Further, for $j=l+2,l+3,\ldots,n$, $T_j^{(d)}=\prod_{l=1}^{j-1}\rho_l^{d_l}=\rho_1^{d_1}\rho_2^{d_2}\ldots\rho_l^{d_l}\rho_{l+1}^{d_{l+1}}\ldots\rho_{j-1}^{d_{j-1}}\to\rho_1^1\rho_2^1\ldots\rho_l^1\rho_{l+1}^{+\infty}\rho_{l+2}^{d_{l+2}}\ldots\rho_{j-1}^{d_{j-1}}=0$. It implies that $T^{(d)}=\sum_{j=1}^{n}T_j^{(d)}=T_1^{(d)}+T_2^{(d)}+\ldots+T_{l+1}^{(d)}=\mathbb{T}_1+\mathbb{T}_2+\mathbb{T}_3+\ldots+\mathbb{T}_{l+1}=\sum_{j=1}^{l+1}\mathbb{T}_j=\mathbb{T}$.

Now, assume that $\big(\varsigma(1),\varsigma(2),\ldots,\varsigma(l+1)\big)$ is an arrangement of $(1,2,\ldots,l+1)$ such that $\mathcal{S}(\mathcal{C}_{\varsigma(j-1)})\geq\mathcal{S}(\mathcal{C}_{\varsigma(j)})$ for each $j=2,3,\ldots,l+1$. Further, let $\mathbb{H}_0=\phi$ and $\mathbb{H}_j=\big\{\mathcal{C}_{\varsigma(z)}|z=1,2,\ldots,j\big\}$ for $j=1,2,\ldots,l+1$. Also, we have assumed that $\big(\tau(1),\tau(2),\ldots,\tau(n)\big)$ is an arrangement of $(1,2,\ldots,n)$ such that $\mathcal{S}(\mathcal{C}_{\tau(j-1)})\geq\mathcal{S}(\mathcal{C}_{\tau(j)})$ for each $j=2,3,\ldots,n$. As, $\big\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_{l+1}\big\}\subseteq\big\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\big\}$ therefore, there must exist natural numbers $1\leq m_1,m_2,\ldots,m_{l+1}\leq n$ satisfying $\mathcal{C}_{\varsigma(1)}=\mathcal{C}_{\tau(m_1)}$, $\mathcal{C}_{\varsigma(2)}=\mathcal{C}_{\tau(m_2)}$, …, $\mathcal{C}_{\varsigma(l+1)}=\mathcal{C}_{\tau(m_{l+1})}$. Further, we have

$$\begin{aligned}
\sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_m\}} T_\mathfrak{q}^{(d)} &\to \sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathbb{H}_0\}} \mathbb{T}_\mathfrak{q}=0 \quad\text{for}\quad m\in\{0,1,2,\ldots m_1-1\};\\
\sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_m\}} T_\mathfrak{q}^{(d)} &\to \sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathbb{H}_1\}} \mathbb{T}_\mathfrak{q} \quad\text{for}\quad m\in\{m_1,m_1+1,m_1+2,\ldots m_2-1\};\\
\sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathcal{H}_m\}} T_\mathfrak{q}^{(d)} &\to \sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_\mathfrak{q}\in\mathbb{H}_2\}} \mathbb{T}_\mathfrak{q} \quad\text{for}\quad m\in\{m_2,m_2+1,m_2+2,\ldots m_3-1\};
\end{aligned}$$

$$\ldots$$

$$\sum_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathcal{H}_m\right\}} T_{\mathfrak{q}}^{(d)} \to \sum_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_{l+1}\right\}} \mathbb{T}_{\mathfrak{q}} \quad \text{for} \quad m \in \{m_{l+1}, m_{l+1}+1, m_{l+1}+2, \ldots n\}.$$

Now, using the Eq. (11.13), we have

$$\xi_{m_1}^{(d)} \to g\left(\frac{\sum_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_1\right\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right) - g\left(\frac{\sum_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_0\right\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right)$$

$$\xi_{m_2}^{(d)} \to g\left(\frac{\sum_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_2\right\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right) - g\left(\frac{\sum_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_1\right\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right)$$

$$\ldots$$

$$\xi_{m_{l+1}}^{(d)} \to g\left(\frac{\sum_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_{l+1}\right\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right) - g\left(\frac{\sum_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_l\right\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right)$$

and all other weights tend to 0. Finally, using the Definition 11.2.4, we obtain that

$$\begin{aligned}
& \lim_{(d_1,d_2,\ldots,d_l,d_{l+1})\to(1,1,\ldots,1,+\infty)} \mathrm{CIFPrOWA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \\
= & \lim_{(d_1,d_2,\ldots,d_l,d_{l+1})\to(1,1,\ldots,1,+\infty)} \Big(\xi_1^{(d)}\mathcal{C}_{\tau(1)} \oplus \xi_2^{(d)}\mathcal{C}_{\tau(2)} \oplus \ldots, \oplus\xi_n^{(d)}\mathcal{C}_{\tau(n)}\Big) \\
= & \xi_{m_1}^{(d)}\mathcal{C}_{\tau(m_1)} \oplus \xi_{m_2}^{(d)}\mathcal{C}_{\tau(m_2)} \oplus \ldots \oplus \xi_{m_{l+1}}^{(d)}\mathcal{C}_{\tau(m_{l+1})} \\
\to & \left(g\left(\frac{\sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_1\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right) - g\left(\frac{\sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_0\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right)\right)\mathcal{C}_{\varsigma(1)} \\
& \oplus \left(g\left(\frac{\sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_2\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right) - g\left(\frac{\sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_1\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right)\right)\mathcal{C}_{\varsigma(2)} \\
& \oplus \ldots \oplus \left(g\left(\frac{\sum_{\mathfrak{q}\in\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_{l+1}\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right) - g\left(\frac{\sum_{\mathfrak{q}\in\left\{\mathfrak{q}|\mathcal{C}_{\mathfrak{q}}\in\mathbb{H}_l\right\}} \mathbb{T}_{\mathfrak{q}}}{\mathbb{T}}\right)\right)\mathcal{C}_{\varsigma(l+1)} \\
= & \mathrm{CIFPrOWA}\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}\big)
\end{aligned}$$

Hence, $\lim_{(d_1,d_2,\ldots,d_l,d_{l+1})\to(1,1,\ldots,1,+\infty)} \mathrm{CIFPrOWA}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) = \mathrm{CIFPrOWA}\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{l+1}\big)$.

$\square$

11.2.4 CIF prioritized geometric operators

In this section, we extend the prioritized averaging operators, defined in above sections, to prioritized geometric operators under CIF environment.

Definition 11.2.5. Consider $\mathcal{C}_j$ as a collection of CIFNs such that $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \ldots \succ \mathcal{C}_n$ where $\mathcal{C}_j \succ \mathcal{C}_{j+1}$ indicates that the CIFN $\mathcal{C}_j$ has higher priority than $\mathcal{C}_{j+1}$ for each $j \in \{1, 2, \ldots, n-1\}$. A CIFPrG operator is a map CIFPrG : $\Theta^n \to \Theta$ defined as:

$$\text{CIFPrG}\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \;=\; \mathcal{C}_1^{\xi_1} \otimes \mathcal{C}_2^{\xi_2} \otimes \ldots \otimes \mathcal{C}_n^{\xi_n} \tag{11.15}$$

where $\xi_j = \frac{T_j}{\sum\limits_{j=1}^{n} T_j}$, $T_j = \prod\limits_{l=1}^{j-1} \mathcal{S}(\mathcal{C}_l)$ for each $j \in \{2, 3, \ldots, n\}$; $T_1 = 1$ and Θ is a collection of CIFNs which have strict prioritization relationship among them. Then CIFPrG operator is known as CIF prioritized geometric operator.

Theorem 11.2.4. The aggregated value of $\mathcal{C}_j = \Big(\big(\zeta_j, w_{\zeta_j}\big), \big(\vartheta_j, w_{\vartheta_j}\big)\Big)$ $(j = 1, 2, \ldots, n)$ obtained by using proposed CIFPrG operator is still CIFN and is given as

$$\text{CIFPrG}(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} \prod\limits_{j=1}^{n} \left(\zeta_j\right)^{\xi_j}, \\ \prod\limits_{j=1}^{n} \left(w_{\zeta_j}\right)^{\xi_j} \end{pmatrix}, \begin{pmatrix} 1 - \prod\limits_{j=1}^{n} \left(1 - \vartheta_j\right)^{\xi_j}, \\ 1 - \prod\limits_{j=1}^{n} \left(1 - w_{\vartheta_j}\right)^{\xi_j} \end{pmatrix} \right) \tag{11.16}$$

Proof. It can be obtained by using operations defined in Definition 9.2.1 and the Eq. (11.15). □

Definition 11.2.6. Let $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$ be a collection of CIFNs such that there is a prioritization among them expressed by the strict priority orders $\mathcal{C}_1 \succ_{d_1} \mathcal{C}_2 \succ_{d_2} \ldots \succ_{d_{n-1}} \mathcal{C}_n$ where $\mathcal{C}_j \succ_{d_j} \mathcal{C}_{j+1}$ means that the CIFN $\mathcal{C}_j$ has a d_j higher priority than $\mathcal{C}_{j+1}$ for $j \in \{1, 2, \ldots n-1\}$. A CIFPrG$_d$ operator is a map CIFPrG$_d$: $\Theta_d^n \to \Theta_d$ defined as:

$$\text{CIFPrG}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \;=\; \mathcal{C}_1^{\xi_1^{(d)}} \otimes \mathcal{C}_2^{\xi_2^{(d)}} \otimes \ldots \otimes \mathcal{C}_n^{\xi_n^{(d)}} \tag{11.17}$$

where $\xi_j^{(d)} = \frac{T_j^{(d)}}{\sum\limits_{j=1}^{n} T_j^{(d)}}$; $T_j^{(d)} = \prod\limits_{l=1}^{j-1} \big(\mathcal{S}(\mathcal{C}_l)\big)^{d_l}$ for each $j \in \{2, 3, \ldots, n\}$; $T_1 = 1$ and Θ_d as a collection of CIFNs which have strict prioritization relationship among them along with priority degrees. Then, CIFPrG$_d$ operator is known as CIF prioritized geometric operator along with priority degrees.

Theorem 11.2.5. The aggregated value of $\mathcal{C}_j = \Big((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j}) \Big)$ $(j = 1, 2, \ldots, n)$ obtained by using proposed CIFPrG$_d$ operator is still CIFN and is given as

$$\text{CIFPrG}_d(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} \prod_{j=1}^{n} (\zeta_j)^{\xi_j^{(d)}}, \\ \prod_{j=1}^{n} (w_{\zeta_j})^{\xi_j^{(d)}} \end{pmatrix}, \begin{pmatrix} 1 - \prod_{j=1}^{n} (1 - \vartheta_j)^{\xi_j^{(d)}}, \\ 1 - \prod_{j=1}^{n} (1 - w_{\vartheta_j})^{\xi_j^{(d)}} \end{pmatrix} \right) \tag{11.18}$$

Proof. It can be proved by implementing operations defined in Definition 9.2.1 and the Eq. (11.17). □

Also, it has been observed that the presented operators CIFPrG and CIFPrG$_d$ are idempotent and monotonic.

Definition 11.2.7. A CIFPrOWG$_d$ operator is a map CIFPrOWG$_d : \Theta_d^n \rightarrow \Theta_d$ defined as:

$$\text{CIFPrOWG}_d\big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\big) \quad = \quad \mathcal{C}_{\tau(1)}^{\xi_1^{(d)}} \otimes \mathcal{C}_{\tau(2)}^{\xi_2^{(d)}} \otimes \ldots \otimes \mathcal{C}_{\tau(n)}^{\xi_n^{(d)}} \tag{11.19}$$

where $\big(\tau(1), \tau(2), \ldots, \tau(n)\big)$ is an arrangement of $(1, 2, \ldots, n)$ such that $\mathcal{S}(\mathcal{C}_{\tau(j-1)}) \geq \mathcal{S}(\mathcal{C}_{\tau(j)})$ for each $j = 2, 3, \ldots, n$ and $\xi_j^{(d)}$ is same as defined in Definition 11.2.4. Then. CIFPrOWG$_d$ operator is known as CIF prioritized ordered weighted geometric operator along with priority degrees.

Theorem 11.2.6. The aggregated value of $\mathcal{C}_j = ((\zeta_j, w_{\zeta_j}), (\vartheta_j, w_{\vartheta_j}))$ $(j = 1, 2, \ldots, n)$ obtained by using proposed CIFPrOWG$_d$ operator is still CIFN and is given as

$$\text{CIFPrOWG}_d(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) = \left(\begin{pmatrix} \prod_{j=1}^{n} (\zeta_{\tau(j)})^{\xi_j^{(d)}}, \\ \prod_{j=1}^{n} (w_{\zeta_{\tau(j)}})^{\xi_j^{(d)}} \end{pmatrix}, \begin{pmatrix} 1 - \prod_{j=1}^{n} (1 - \vartheta_{\tau(j)})^{\xi_j^{(d)}}, \\ 1 - \prod_{j=1}^{n} \Big(1 - w_{\vartheta_{\tau(j)}}\Big)^{\xi_j^{(d)}} \end{pmatrix} \right) \tag{11.20}$$

Proof. It can be proved by using operations given in Definition 9.2.1 and the Eq. (11.19). □

11.3 MCDM approach based on prioritized operators

Consider a set $\{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m\}$ of 'm' alternatives characterized by another collection $\{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\}$ of 'n' criteria. These criteria are such that there is prioritization relationship among them which is expressed by strict priority order as $\mathfrak{B}_1 \succ_{d_1} \mathfrak{B}_2 \succ_{d_2} \ldots \succ_{d_{n-1}} \mathfrak{B}_n$ where $\mathfrak{B}_v \succ_{d_v} \mathfrak{B}_{v+1}$ signifies that the criteria $\mathfrak{B}_v$ has a d_v higher priority than the criteria $\mathfrak{B}_{v+1}$ for $v \in \{1, 2, \ldots n-1\}$. Here $d = (d_1, d_2, \ldots, d_{n-1})$ is the priority degree vector corresponding to criteria. Further, the alternatives $\mathcal{V}_u$ are evaluated by 'k' experts $\left\{\mathcal{E}^{(1)}, \mathcal{E}^{(2)}, \ldots, \mathcal{E}^{(k)}\right\}$ who give their rating values in terms of CIFNs $\mathcal{C}_{uv}^{(z)} = \left(\left(\zeta_{uv}^{(z)}, w_{\zeta_{uv}}^{(z)}\right), \left(\vartheta_{uv}^{(z)}, w_{\vartheta_{uv}}^{(z)}\right)\right)$ where $0 \leq \zeta_{uv}^{(z)}, w_{\zeta_{uv}}^{(z)}, \vartheta_{uv}^{(z)}, w_{\vartheta_{uv}}^{(z)}, \zeta_{uv}^{(z)} + w_{\zeta_{uv}}^{(z)}, \vartheta_{uv}^{(z)} + w_{\vartheta_{uv}}^{(z)} \leq 1$ for each $z = 1, 2, \ldots, k$; $u = 1, 2, \ldots, m$ and $v = 1, 2, \ldots n$. Also, there exists prioritization among the experts such as $\mathcal{E}^{(1)} \succ_{d'_1} \mathcal{E}^{(2)} \succ_{d'_2} \ldots \succ_{d'_{k-1}} \mathcal{E}^{(k)}$ where $d' = (d'_1, d'_2, \ldots, d'_{k-1})$ is the priority degree vector corresponding to experts. The main objective of the problem is to determine the most favorable alternative out of the available ones. For this, we develop an MCDM method, which involves the following steps:

Step 1: Construct the CIF decision matrices $\mathcal{M}^{(z)} = \left(\mathcal{C}_{uv}^{(z)}\right)_{m\times n}$ for the rating values corresponding to all alternatives under different criteria, given by 'k' experts as:

$$\mathcal{M}^{(z)} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \mathcal{C}_{11}^{(z)} & \mathcal{C}_{12}^{(z)} & \ldots & \mathcal{C}_{1n}^{(z)} \\ \mathcal{C}_{21}^{(z)} & \mathcal{C}_{22}^{(z)} & \ldots & \mathcal{C}_{2n}^{(z)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{m1}^{(z)} & \mathcal{C}_{m2}^{(z)} & \ldots & \mathcal{C}_{mn}^{(z)} \end{pmatrix} \end{array} \tag{11.21}$$

Step 2: Aggregate the individual CIF decision matrices $\mathcal{M}^{(z)} = \left(\mathcal{C}_{uv}^{(z)}\right)_{m\times n}$, $(z = 1, 2, \ldots, l)$ into collective one $\mathcal{M} = (\mathcal{C}_{uv})_{m\times n}$ by utilizing either CIFPrOWA$_d$ operator i.e.,

$$\begin{aligned} \mathcal{C}_{uv} &= \left(\left(\zeta_{uv}, w_{\zeta_{uv}}\right), \left(\vartheta_{uv}, w_{\vartheta_{uv}}\right)\right) \\ &= \text{CIFPrOWA}_{d'}\left(\mathcal{C}_{uv}^{(1)}, \mathcal{C}_{uv}^{(2)}, \ldots, \mathcal{C}_{uv}^{(k)}\right) \end{aligned}$$

$$= \left(\begin{pmatrix} 1-\prod_{z=1}^{k}\left(1-\zeta_{uv}^{(\tau(z))}\right)^{\xi_{uv}^{(z)(d')}}, \\ 1-\prod_{z=1}^{k}\left(1-w_{\zeta_{uv}}^{(\tau(z))}\right)^{\xi_{uv}^{(z)(d')}} \end{pmatrix}, \begin{pmatrix} \prod_{z=1}^{k}\left(\vartheta_{uv}^{(\tau(z))}\right)^{\xi_{uv}^{(z)(d')}}, \\ \prod_{z=1}^{k}\left(w_{\vartheta_{uv}}^{(\tau(z))}\right)^{\xi_{uv}^{(z)(d')}} \end{pmatrix}\right) \quad (11.22)$$

or CIFPrOWG$_d$ operator i.e.,

$$\begin{aligned} \mathcal{C}_{uv} &= \left(\left(\zeta_{uv}, w_{\zeta_{uv}}\right), \left(\vartheta_{uv}, w_{\vartheta_{uv}}\right)\right) \\ &= \text{CIFPrOWG}_{d'}\left(\mathcal{C}_{uv}^{(1)}, \mathcal{C}_{uv}^{(2)}, \ldots, \mathcal{C}_{uv}^{(k)}\right) \\ &= \left(\begin{pmatrix} \prod_{z=1}^{k}\left(\zeta_{uv}^{(\tau(z))}\right)^{\xi_{uv}^{(z)(d')}}, \\ \prod_{z=1}^{k}\left(w_{\zeta_{uv}}^{(\tau(z))}\right)^{\xi_{uv}^{(z)(d')}} \end{pmatrix}, \begin{pmatrix} 1-\prod_{z=1}^{k}\left(1-\vartheta_{uv}^{(\tau(z))}\right)^{\xi_{uv}^{(z)(d')}}, \\ 1-\prod_{z=1}^{k}\left(1-w_{\vartheta_{uv}}^{(\tau(z))}\right)^{\xi_{uv}^{(z)(d')}} \end{pmatrix}\right) \end{aligned} \quad (11.23)$$

Step 3: Aggregate the values $\mathcal{C}_{uv}$ obtained in Step 2 into collective one $\mathcal{C}_u$ $(u = 1, 2, \ldots, m)$ by utilizing either CIFPrA$_d$ operator i.e.,

$$\begin{aligned} \mathcal{C}_{u} &= \left(\left(\zeta_{u}, w_{\zeta_{u}}\right), \left(\vartheta_{u}, w_{\vartheta_{u}}\right)\right) \\ &= \text{CIFPrA}_{d}\left(\mathcal{C}_{u1}, \mathcal{C}_{u2}, \ldots, \mathcal{C}_{un}\right) \\ &= \left(\begin{pmatrix} 1-\prod_{v=1}^{n}\left(1-\zeta_{uv}\right)^{\xi_{uv}^{(d)}}, \\ 1-\prod_{v=1}^{n}\left(1-w_{\zeta_{uv}}\right)^{\xi_{uv}^{(d)}} \end{pmatrix}, \begin{pmatrix} \prod_{v=1}^{n}\left(\vartheta_{uv}\right)^{\xi_{uv}^{(d)}}, \\ \prod_{v=1}^{n}\left(w_{\vartheta_{uv}}\right)^{\xi_{uv}^{(d)}} \end{pmatrix}\right) \end{aligned} \quad (11.24)$$

or CIFPrG$_d$ operator i.e.,

$$\begin{aligned} \mathcal{C}_{u} &= \left(\left(\zeta_{u}, w_{\zeta_{u}}\right), \left(\vartheta_{u}, w_{\vartheta_{u}}\right)\right) \\ &= \text{CIFPrG}_{d}\left(\mathcal{C}_{u1}, \mathcal{C}_{u2}, \ldots, \mathcal{C}_{un}\right) \\ &= \left(\begin{pmatrix} \prod_{v=1}^{n}\left(\zeta_{uv}\right)^{\xi_{uv}^{(d)}}, \\ \prod_{v=1}^{n}\left(w_{\zeta_{uv}}\right)^{\xi_{uv}^{(d)}} \end{pmatrix}, \begin{pmatrix} 1-\prod_{v=1}^{n}\left(1-\vartheta_{uv}\right)^{\xi_{uv}^{(d)}}, \\ 1-\prod_{v=1}^{n}\left(1-w_{\vartheta_{uv}}\right)^{\xi_{uv}^{(d)}} \end{pmatrix}\right) \end{aligned} \quad (11.25)$$

However, if there is prioritization relationship among the criteria without priority degrees then, utilize proposed CIFPrA or CIFPrG operator.

Step 4: Calculate the score of overall aggregated values $\mathcal{C}_u$, $(u = 1, 2, \ldots, m)$ using Eq. (11.1). However, if any two score values of different $\mathcal{C}_u$ are equal then, calculate accuracy using Eq. (11.2). Finally, rank the alternatives using Definition 11.2.1 and hence, obtain the most optimal one.

11.4 Illustrative Example

In this section, in order to illustrate the working of the proposed MCDM approach, we solve a problem to analyze the sector that affected, Indian Stock Exchange (ISE), the most during calender year (CY) 2019. Also, we compare the results of the proposed method with some of the prevailing CIFS and IFS studies in order to show the validity and superiority of the proposed work.

11.4.1 Description and solution of the problem

Consider a decision making problem of finding out the most important sector which affected ISE during CY 2019. We take into account five sectors, which affect ISE and these are $\mathcal{V}_1$: Consumer goods and fast moving consumer goods (FMCG), $\mathcal{V}_2$: Automobile sector, $\mathcal{V}_3$: Banking and Financial services, $\mathcal{V}_4$: Construction, $\mathcal{V}_5$: Pharmaceutical sector. The problem is to order these five sectors in descending order from the most important to least important that affected ISE during CY 2019. Further, we consider the four major factors $\mathfrak{B}_v$, $(v = 1, 2, 3, 4)$ which influenced the role of these sectors $\mathcal{V}_u$, $(u = 1, 2, \ldots, 5)$ in ISE and these are $\mathfrak{B}_1$: Crude oil price movement, $\mathfrak{B}_2$: Performance of the debt market, $\mathfrak{B}_3$: Budget 2019 and $\mathfrak{B}_4$: Reduction in repo rate by the Reserve bank of India (RBI). These factors $\mathfrak{B}_v$ are such that there is prioritization relationship among them with priority degree vector $d = (1, 0.5, 0.2)$ i.e., $\mathfrak{B}_1 \succ_1 \mathfrak{B}_2 \succ_{0.5} \mathfrak{B}_3 \succ_{0.2} \mathfrak{B}_4$. We consult three experts $\mathcal{E}^{(z)}$, $(z = 1, 2, 3)$ with $d' = (0.3, 0.2)$ i.e., $\mathcal{E}^{(1)} \succ_{0.3} \mathcal{E}^{(2)} \succ_{0.2} \mathcal{E}^{(3)}$ and these experts assess the performance of sectors $\mathcal{V}_u$ under factors $\mathfrak{B}_v$ and give their rating values in terms of CIFNs. The experts use CIFNs to evaluate sectors $\mathcal{V}_u$ because it is time-periodic problem. The influence of a particular sector on ISE does not always remain throughout the whole year. Some sectors affect stock market for few months only and not for the whole year. Since, CIFS environment is the most optimal environment to handle time-periodic problems therefore CIFNs are used by experts to express their rating values. In this problem, the amplitude terms measure the influence degree of sectors under the mentioned criteria on ISE while the phase term indicates the period of this influence.

The main procedure steps in order to obtain the most important sector, using proposed MCDM method described in the above section, are summarized as follows:

Step 1: The rating values of every alternative, given by three experts, are tabulated in Table 11.1. As from this Table, we have the rating value of $\mathcal{V}_1$ under factor $\mathfrak{B}_1$, given by $\mathcal{E}^{(1)}$, as $\left(\left(0.5, \frac{3}{12}\right), \left(0.1, \frac{4}{12}\right)\right)$. The membership term $\left(0.5, \frac{3}{12}\right)$ reveals that the expert $\mathcal{E}^{(1)}$ agrees that there is 50 percent influence of $\mathcal{V}_1$ on ISE during CY 2019 under criteria $\mathfrak{B}_1$ and the time span of this influence is of 3 months out of 12 months. Similarly, for non-membership value $\left(0.1, \frac{4}{12}\right)$, the expert $\mathcal{E}_1$ believes with a degree of 10 percent that there is no influence of $\mathcal{V}_1$ on ISE under $\mathfrak{B}_1$ and the duration with no influence is of 4 months. In the similar manner, the rest of the data can be interpreted.

Step 2: Without loss of generality (WLOG), by using Eq. (11.22) and taking $g(x) = x^2$, the rating values $\mathcal{C}_{uv}^{(z)}$, $(z = 1, 2, 3)$ given by three experts are aggregated into collective one $\mathcal{C}_{uv}$. The acquired values corresponding to each alternative $\mathcal{V}_u$ $(u = 1, 2, \ldots, 5)$ are depicted in Table 11.2.

Step 3: WLOG, by using Eq. (11.24), the values $\mathcal{C}_{uv}$ $(u = 1, 2, \ldots, 5; v = 1, 2, 3, 4)$ are aggregated into $\mathcal{C}_u$. The accumulated values associated with each alternative $\mathcal{V}_u$ are obtained as:

$$\begin{aligned}
\mathcal{C}_1 &= \left(\left(0.6325, \frac{6.1836}{12}\right), \left(0.1753, \frac{2.4732}{12}\right)\right); \\
\mathcal{C}_2 &= \left(\left(0.6259, \frac{7.0491}{12}\right), \left(0.2057, \frac{3.3599}{12}\right)\right); \\
\mathcal{C}_3 &= \left(\left(0.6947, \frac{6.7554}{12}\right), \left(0.1319, \frac{3.0332}{12}\right)\right); \\
\mathcal{C}_4 &= \left(\left(0.6195, \frac{6.4657}{12}\right), \left(0.2079, \frac{3.4520}{12}\right)\right); \\
\mathcal{C}_5 &= \left(\left(0.6644, \frac{7.2085}{12}\right), \left(0.2101, \frac{2.6017}{12}\right)\right).
\end{aligned}$$

Step 4: On using Eq. (11.1), the score value corresponding to each alternative $\mathcal{V}_u$ is obtained as $\mathcal{S}(\mathcal{C}_1) = 0.6916$, $\mathcal{S}(\mathcal{C}_2) = 0.6819$, $\mathcal{S}(\mathcal{C}_3) = 0.7183$, $\mathcal{S}(\mathcal{C}_4) = 0.6657$ and $\mathcal{S}(\mathcal{C}_5) = 0.7095$. It implies that, the ranking order of the alternatives is $\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$. Hence, the sector $\mathcal{V}_3$ has the most influence on ISE during CY 2019.

Further, the impact of different AOs utilized for aggregating the rating values corresponding to individual expert and criteria in Step 2 and 3 respectively, on the ordering position of alternatives, is analyzed and depicted in Table 11.3. The initial column of this table exhibits the AO utilized in order to accumulate the individual rating values of three experts into a single CIFN whereas, the second column depicts AO applied for combining the criteria values, acquired in Step 2, into a unique one. An optimistic approach is followed by taking prioritized ordered averaging operator for accumulation of the rating estimations of three experts in the first four rows of this Table. On the other hand, in the next four rows of the Table 11.3, the preferences of three experts are accumulated with pessimistic attitude by opting prioritized ordered weighted geometric operator. The tabulated values depict that the final score values of alternatives are greater when optimistic approach is followed in Step 2 than the values obtained using pessimistic approach. Further, it is analyzed that on using prioritized ordered weighted averaging operator in Step 2 for expert estimations aggregation, the ranking order of alternative remains same irrespective of the operator used in Step 3 except for the case when prioritized geometric operator without degrees is used in Step 3. Also, on utilizing prioritized ordered weighted geometric operator in Step 2, the ordering position of the alternatives remains identical for the cases when averaging operators are utilized in Step 3.

11.4.2 Comparative Analysis

With the objective of validating the proposed operators and MCDM method and showing its superiority, in this section, we compare the results of the proposed work with some of the existing CIFS [6, 58–62, 129, 130] and IFS [43, 90, 152, 203, 204] studies as follows:

With CIFS studies

Here, to show the validity, superiority and efficiency of the developed MCDM method, we solve the presented example using some of the prevailing CIFS studies. For this, we utilize CIF weighted power averaging (CIFWPA) [130], CIF weighted power geometric (CIFWPG) [130], CIF weighted averaging (CIFWA) [62], CIF weighted geometric (CIFWG) [62], CIF Einstein weighted averaging (CIFEWA) [59], CIF Einstein weighted geometric (CIFEWG) [60], CIF weighted averaging (CIFWA) [61], distance measure $\mathcal{D}$ [6], weighted euclidean distance measure $\mathcal{D}_6$ [129], correlation coefficient $\mathcal{K}_4$ [58]. We apply the above said operators and measures on the aggregated values of three experts acquired in Step 2 and take $(0.25, 0.25, 0.25, 0.25)^T$ as weight vector. For comparing the results with measures given in [6, 58, 129], we take the the PIA ($\mathcal{V}^+$) as ideal alternative whose preferences are: $\mathcal{V}^+ = \{\mathcal{V}_1^+, \mathcal{V}_2^+, \ldots, \mathcal{V}_n^+\}$ where $\mathcal{V}_v^+ = ((\max\limits_{1\leq u\leq m}\{\zeta_{uv}\}, \max\limits_{1\leq u\leq m}\{w_{\zeta_{uv}}\}), (\min\limits_{1\leq u\leq m}\{\vartheta_{uv}\}, \min\limits_{1\leq u\leq m}\{w_{\vartheta_{uv}}\}))$ for each $v = 1, 2, \ldots, n$. Then, the results obtained by applying measures defined in [6, 58, 129] and AOs presented in [59–62, 130] are tabulated in Table 11.4.

The results tabulated in the Table 11.4 depict that the best alternative acquired on applying the prevailing measures [6, 58, 129] and the AOs [59–62, 130] remains same with the one obtained on utilizing the proposed MCDM method. Although the best alternative remains same but the computational difference between the existing methods [6, 58–62, 129, 130] and the proposed approach is explained as given below:

1) In the AOs CIFWPA and CIFWPG [130] the weightage of each argument is calculated by taking into account the similarity of the argument with all other arguments which are to be aggregated whereas the proposed operators aggregate those arguments which have strict priority arrangement among them with or without degrees. Also, if we take, priority degree vector $(d_1, d_2, \ldots, d_{n-1}) \rightarrow (0, 0, \ldots, 0)$ in the proposed CIFPrA$_d$ and CIFPrG$_d$ AOs then, Proposition 11.2.2 implies that the arguments to be aggregated have no prioritization among them and hence, they become independent. It leads to the fact that the AOs developed in [130] can be utilized only for the case when there is some sort of dependency among the arguments whereas the proposed operators can be

used not only for aggregating those arguments among which there exist strict priority orders but also for fusing independent arguments.

2) The CIFWA and CIFWG [62] operators are based on simple algebraic operations and accumulate the arguments by assuming that the arguments to be aggregated are independent of one another. On the other hand, the proposed operators are also based on simple algebraic operations and fuse those CIFNs which have strict prioritization relationship among them and may have degree of priority. Besides this, by taking priority degree vector $(d_1, d_2, \ldots, d_{n-1}) \rightarrow (0, 0, \ldots, 0)$ the developed CIFPrA$_d$ and CIFPrG$_d$ AOs reduce to CIFWA and CIFWG [62] operators respectively for the weight vector $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$ as shown in Proposition 11.2.2. Hence, the proposed operators are more generalized.

3) Authors in [59, 60] developed generalized basic arithmetic weighted averaging and geometric operators respectively which are based on t-norm and co-norm operations. The CIFEWA and CIFEWG operators are special cases of the operators given in [59] and [60] which are obtained by taking $t(a) = \log\left(\frac{2-a}{a}\right)$ and are based on einstein operations. Again, these operators aggregate the independent arguments whereas the proposed operators have the ability to fuse those arguments which are independent as well as those also among which there is strict priority ordering with or without priority degrees.

4) The CIFWA [61] operator is based on novel operational laws and the approach proposed in [61] uses possibility degree in order to rank CIFNs. It may be the reason that the ranking order of the alternatives $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_4$ obtained on utilizing the approach developed in [61] differs from the proposed one. Also, the CIFWA [61] operator aggregates independent arguments only whereas the proposed operators are efficient enough to handle both independent and dependent arguments.

5) The ranking order of the alternatives obtained using the distance measure $\mathcal{D}$ [6], weighted euclidean distance $\mathcal{D}_6$ [129] and correlation coefficient $\mathcal{K}_4$ [58] remains identical with the proposed operators. But the results acquired utilizing existing distance measure $\mathcal{D}$ [6], weighted euclidean distance $\mathcal{D}_6$ [129] and correlation coefficient $\mathcal{K}_4$ [58]

cannot fuse or integrate information and cannot consider prioritization among criteria. On the other hand, in the proposed work, a single CIFN is obtained by applying the proposed operators which is representative of all aggregated arguments and it is obtained by considering prioritization with and without degrees among arguments.

With IFS studies

In light of comparing the results of the developed approach with prevailing IFS studies [43, 90, 152, 203, 204], we set the phase term associated with each CIFN equal to zero. Then, we utilize IF prioritized weighted averaging (IFPWA) [203], Generalized IF prioritized weighted geometric (GIFPWG) [204], IF einstein prioritized weighted averaging (IFEPWA) [152], IF einstein prioritized weighted geometric (IFEPWG) [152], Pythagorean fuzzy hamachar prioritized averaging (PFHPA) [43], Pythagorean fuzzy hamachar prioritized geometric (PFHPG) [43], Pythagorean fuzzy prioritized weighted averaging (PFPWA) [90] and Pythagorean fuzzy prioritized weighted geometric (PFPWG) [90] operators. We apply these prioritized operators [43, 90, 152, 203, 204] in Step 3 of the presented MCDM method and tabulate the acquired results in Table 11.5.

From the results obtained in Table 11.5, it is analyzed that on applying the IFPWA [203] and IFEPWA [152] AOs in Step 3 of the presented MCDM method, the ordering position of alternatives $\mathcal{V}_1$ and $\mathcal{V}_5$ changes and the ranking order of other alternatives remains same with the proposed approach results. On the other hand, when we utilize GIFPWG [204], IFEPWG [152] and PFHPG [43] AOs in Step 3, the ranking order of two alternatives $\mathcal{V}_2$ and $\mathcal{V}_3$ remains identical with the presented method outcomes whereas the position of other three alternatives changes. Also, on applying PFHPA [43] the order of all alternatives changes except the optimal one $\mathcal{V}_3$. On utilizing PFPWA [90] operator the ranking position of alternatives $\mathcal{V}_2$ and $\mathcal{V}_4$ changes whereas the order of other alternatives remains same with the proposed one. This change in the positioning of the alternatives is due to the fact that the AOs used in prevailing studies [43, 90, 152, 203, 204] aggregate real valued MDs and NMDs and tackle with only one dimensional problem. They do not consider phase terms corresponding to MD and NMD. On the other hand, the presented work fuses complex membership and non-membership values and handles more than one dimensional problems. Besides this, the proposed AOs consider the degree of priority among criteria as well along with prioritization relationship among them whereas the existing prioritized AOs [43, 90, 152, 203, 204] do not take into account priority degree vector in the process of aggregation.

Also, by setting phase terms corresponding to MD and NMD in the proposed work, the developed MCDM method can solve problems under IFS theory as well and the proposed operators CIFPrA and CIFPrG reduce to IFPWA [203] and GIFPWG [204] (By taking $t(a) = -\log(a), 0 < a \leq 1$) operators respectively . All these points lead to the conclusion that the proposed work is more generalized and can be applied on problems under CIFS as well as IFS theories.

11.4.3 Characteristic Comparison

In addition to the above comparative studies, we give some characteristic comparison of our proposed MCDM approach and the DM methods proposed in [43, 59–62, 90, 130, 152, 203, 204] which are tabulated in Table 11.6. In this Table, the symbol '✓' signifies that the associated MCDM method considers prioritization relationship among criteria, degree of priority among them, handles optimistic as well as pessimistic behavior of decision-maker, deals with group DM problems, tackles with time-periodic problems and can represent two-dimensional information simultaneously whereas the symbol '$\times$' means that the corresponding method fails. Based on it, we find that the proposed MCDM method has the following advantages:

1) The values, tabulated in Table 11.6 depict that the operators presented in [43, 90, 152, 203, 204] and our proposed AOs have the ability to consider strict prioritization relationship among the criteria. Also, the proposed prioritized operators consider the priority degree vector during aggregation process which indicates by how much degree a criteria is prior as comparison to others whereas the AOs proposed in [43, 59–62, 90, 130, 152, 203, 204] fail to do so.

2) The MCDM approaches presented in [60, 204] are based on the geometric operators only whereas the DM method defined in [59] is based on averaging operators only. But,

our proposed method and the approaches given in [43, 61, 62, 90, 130, 152, 203] provide the choice to decision-maker to utilize averaging or geometric operator in accordance with the DM problem and their optimistic or pessimistic attitude towards the problem.

3) The proposed method and the approaches proposed in [90, 130, 152, 203, 204] can handle group DM problems. Moreover, by taking $l = 1$, our developed approach can handle single decision-maker problems as well. Therefore, the presented MCDM method is more generalized as it can handle both single and group DM problems.

4) The MCDM methods proposed in [43, 90, 152, 203, 204] deal with real membership and non-membership degrees, which fail to handle time-periodic problems and cannot represent more than one-dimensional information in one set. On the other hand, the proposed method can handle complex problems which involve periodicity and can aggregate two dimensional data together in one set.

Besides this, the operators proposed in [203, 204] are special cases of proposed CIFPrA and CIFPrG operators and therefore, our presented work is more generalized and can be utilized to solve DM problems under FS, IFS and CFS environment also. This discussion leads to the conclusion that the presented approach can handle time-periodic complex problems involving prioritization among the considered factors more efficiently which are either difficult or impossible to be solved using existing theories [43, 59–62, 90, 130, 152, 203, 204].

11.4.4 The influence of priority degree vector "d" on aggregation results

In our day to day life, we come across a number of prioritized MCDM problems. In this chapter, a real non-negative number is assigned to each priority order and is called priority degree. These priority degrees play an important role in aggregation process and with the change in priority degree vector, the aggregation result also gets changed. Now, in this section, we solve an MCDM problem by considering different priority degree vectors to illustrate the influence of d on the aggregated results.

Example 11.4.1. Consider a MCDM problem characterized by five alternatives $\mathcal{V}_u$ $(u = 1, 2, \ldots, 5)$ under four criteria $\mathfrak{B}_v$ $(v = 1, 2, 3, 4)$. Suppose that these criteria have strict prioritization among them such that $\mathfrak{B}_1 \succ_{d_1} \mathfrak{B}_2 \succ_{d_2} \mathfrak{B}_3 \succ_{d_3} \mathfrak{B}_4$ where $d = (d_1, d_2, d_3)$ is the priority degree vector associated with criteria. The goal is to find the most optimal alternative among the available ones. The rating values corresponding to each alternative $\mathcal{V}_u$ under four different criteria $\mathfrak{B}_v$ are tabulated in Table 11.7.

We discuss two cases in which we apply proposed CIFPrA$_d$ and CIFPrOWA$_d$ operators by taking different priority degree vectors d and analyze its influence on the results obtained.

Case 1: For each alternative $\mathcal{V}_u$ $(u = 1, 2, \ldots, 5)$ we utilize proposed CIFPrA$_d$ operator to accumulate the criteria values $\mathfrak{B}_v$ $(v = 1, 2, 3, 4)$. The results obtained by taking different priority degree vectors $d = (d_1, d_2, d_3)$ are tabulated in Table 11.8. From the values, tabulated in Table 11.8, it is analyzed that the ranking position of alternatives changes with the change in the priority degree vector $d = (d_1, d_2, d_3)$. This influence of d vector is deeply illustrated as follows:

(1) When $d = (1, 1, 1)$, it implies that each criteria $\mathfrak{B}_v$ is prior to $\mathfrak{B}_{v+1}$ with the same degree for $v = 1, 2, 3$ and there is normal prioritization among criteria. In this case, CIFPrA$_d$ and CIFPrG$_d$ operators reduce to CIFPrA and CIFPrG operators respectively. Also, by taking phase terms corresponding to MD and NMD equal to zero, the proposed operators CIFPrA$_d$ and CIFPrG$_d$ operators reduce to IFPWA [203] and IFPWG [203] operators respectively.

(2) When $d = (10, 1, 1)$, it indicates that the criteria $\mathfrak{B}_1$ is very important and its priority degree is high as comparison to criteria $\mathfrak{B}_v$ $(v = 2, 3, 4)$ whereas the other criteria are prior to one another with the same degree. As the criteria $\mathfrak{B}_1$ is highly important in this case, the accumulated values corresponding to each alternative, obtained after applying CIFPrA$_d$ operator, are very close to the rating values of criteria $\mathfrak{B}_1$. It leads to the conclusion that if the first priority degree is very large as compared to other priority degrees then, the aggregation result is highly near to the first criteria value.

(3) When $d = (0, 0, 0)$, it signifies that no prioritization relationship exists among the criteria and hence, all the criteria are at the same importance level. In other words, in

this case, the criteria become independent and using the Proposition 11.2.2, we have

$$\mathrm{CIFPrA}_d\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big) = \frac{1}{4}\mathfrak{B}_1 \oplus \frac{1}{4}\mathfrak{B}_2 \oplus \frac{1}{4}\mathfrak{B}_3 \oplus \frac{1}{4}\mathfrak{B}_4$$

(4) When $d = (+\infty, 1, 1)$, it gives that the criteria $\mathfrak{B}_1$ is extremely important as its priority degree tends to $+\infty$. In this case, the aggregated value is uniquely determined by the criteria $\mathfrak{B}_1$ and all the other criteria value do not have any impact on the accumulated value. In nutshell, if the first priority degree in any MCDM tends to $+\infty$ then, the role of other criteria values become nil and the aggregated result is determined by first criteria only. In other words, in this case, using Proposition 11.2.3, we have

$$\mathrm{CIFPrA}_d\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big) = \mathfrak{B}_1$$

(5) When $d = (0, +\infty, 0)$, it indicates that the criteria $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are at the same priority level and are extremely important than the remaining criteria $\mathfrak{B}_3$ and $\mathfrak{B}_4$ because the second priority degree d_2 tends to ∞ here. In this case, the aggregation result is determined by criteria $\mathfrak{B}_1$ and $\mathfrak{B}_2$ only with equal weightage and the role of other criteria values vanish in aggregation process. In other words, in this case, by Proposition 11.2.4, we have

$$\mathrm{CIFPrA}_d\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big) = \frac{1}{2}\mathfrak{B}_1 \oplus \frac{1}{2}\mathfrak{B}_2$$

(6) When $d = (1, +\infty, 1)$, it signifies that there is normal prioritization relationship between the criteria $\mathfrak{B}_1$ and $\mathfrak{B}_2$ as considered in CIFPrA operator and these criteria $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have extremely high importance than the remaining criteria $\mathfrak{B}_3$ and $\mathfrak{B}_4$ because the second priority degree d_2 tends to ∞. In this case, the accumulated value is uniquely determined by criteria $\mathfrak{B}_1$ and $\mathfrak{B}_2$ only with normal prioritization considered among them and the influence of other criteria values becomes nil in aggregation process. In other words, in this case, by Proposition 11.2.5, we have

$$\mathrm{CIFPrA}_d\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big) = \mathrm{CIFPrA}\big(\mathfrak{B}_1, \mathfrak{B}_2\big)$$

Case 2: Here, we use proposed $\mathrm{CIFPrOWA}_d$ operator, by taking $g(x) = x^2$, to accumulate the criteria values $\mathfrak{B}_v$ ($v = 1, 2, 3, 4$) corresponding to each alternative $\mathcal{V}_u$

($u = 1, 2, \ldots, 5$) by taking different priority degree vectors $d = (d_1, d_2, d_3)$ and tabulate the acquired results in Table 11.9. The values, tabulated in Table 11.9, depict that the ordering position of alternatives gets affected as the priority degree vector $d = (d_1, d_2, d_3)$ changes. This influence of d vector is deeply illustrated as follows:

(1) When $d = (1, 1, 1)$, it implies that each criteria $\mathfrak{B}_v$ is prior to $\mathfrak{B}_{v+1}$ with the same degree for $v = 1, 2, 3$ i.e., there is normal prioritization relationship among criteria. In this case, there is no role of priority degrees on the aggregation results and using Proposition 11.2.7, we have

$$\mathrm{CIFPrOWA}_d\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big) = \mathrm{CIFPrOWA}\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big)$$

(2) When $d = (10, 1, 1)$, it indicates that the first priority degree is very high as comparison to other degrees. As the criteria $\mathfrak{B}_1$ is highly important in this case, the accumulated values corresponding to each alternative, obtained after applying $\mathrm{CIFPrOWA}_d$ operator, are very close to the rating values of criteria $\mathfrak{B}_1$. It leads to the conclusion that if the first priority degree is very large as compared to other priority degrees then, the aggregation result is highly near to the first criteria value.

(3) When $d = (0, 0, 0)$, it signifies that no prioritization relationship exists among the criteria and hence, all the criteria are at the same importance level. In other words, in this case, the criteria become independent and using Proposition 11.2.8, we have

$$\mathrm{CIFPrOWA}_d\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big) = \mathrm{CIFPrOWA}_g\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big)$$

(4) When $d = (+\infty, 1, 1)$, it gives that the criteria $\mathfrak{B}_1$ is extremely important as its priority degree tends to $+\infty$. In this case, the aggregated value is uniquely determined by the criteria $\mathfrak{B}_1$ and all the other criteria value do not have any impact on the accumulated value. In nutshell, if the first priority degree in any MCDM tends to $+\infty$ then, the role of other criteria values become nil and the aggregated result is determined by first criteria only. In other words, in this case, by Proposition 11.2.9, we have

$$\mathrm{CIFPrOWA}_d\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big) = \mathfrak{B}_1$$

(5) When $d = (0, +\infty, 0)$, it indicates that the criteria $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are at the same priority level and are extremely important than the remaining criteria $\mathfrak{B}_3$ and $\mathfrak{B}_4$ because the second priority degree d_2 tends to ∞ here. In this case, the aggregation result is determined by criteria $\mathfrak{B}_1$ and $\mathfrak{B}_2$ only and the role of other criteria values vanish in aggregation process. In other words, in this case, by Proposition 11.2.10, we have

$$\text{CIFPrOWA}_d\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big) = \text{CIFPrOWA}_g\big(\mathfrak{B}_1, \mathfrak{B}_2\big)$$

(6) When $d = (1, +\infty, 1)$, it signifies that there is normal prioritization relationship between the criteria $\mathfrak{B}_1$ and $\mathfrak{B}_2$ as considered in CIFPrA operator and these criteria $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have extremely high importance than the remaining criteria $\mathfrak{B}_3$ and $\mathfrak{B}_4$ because the second priority degree d_2 tends to ∞. In this case, the accumulated value is uniquely determined by criteria $\mathfrak{B}_1$ and $\mathfrak{B}_2$ only with normal prioritization considered among them and the influence of other criteria values becomes nil in aggregation process. In other words, in this case, by Proposition 11.2.11, we have

$$\text{CIFPrOWA}_d\big(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\big) = \text{CIFPrOWA}\big(\mathfrak{B}_1, \mathfrak{B}_2\big)$$

11.5 Conclusion

The main contributions of the proposed work are outlined as follows:

1) The present work employs CIFSs in order to handle uncertainty existing in the data using complex valued membership and non-membership degrees. CIF model is an extension of IF environment and can deal with two dimensional problems simultaneously and can handle time periodic problems in a better way.

2) A series of prioritized averaging and geometric AOs namely CIFPrA,CIFPrG, CIFPrA$_d$, CIFPrG$_d$, CIFPrOWA$_d$ and CIFPrOWG$_d$ by considering the strict priority orders among the arguments with and without degrees have been proposed. Some propositions related to priority degree have been investigated in detail which are helpful in fusing multiple CIF information.

3) A group MCDM approach based on proposed prioritized operators has been developed for solving DM problems under the CIF environment. Also, it has been analyzed that the proposed approach and AOs can be applied on IFS data as well and hence, the presented work is more generalized and effective.

4) The proposed MCDM method is illustrated via an example and the results of the presented methodology are compared with several existing techniques available under CIFS and IFS studies. Besides this, the influence of the priority degrees on the aggregated results is discussed in detail.

In the future, we will utilize the proposed AOs and MCDM approach in several other areas such as pattern recognition, medical diagnosis and image processing. Also, we will try to develop some methods for acquiring priority degree objectively in the future.

Table 11.1: Preferences given by experts $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$ and $\mathcal{E}^{(3)}$

Expert	Alternatives	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{E}^{(1)}$	$\mathcal{V}_1$	$\left(\left(0.5, \frac{4}{12}\right), \left(0.1, \frac{4}{12}\right)\right)$	$\left(\left(0.8, \frac{6}{12}\right), \left(0.1, \frac{2}{12}\right)\right)$	$\left(\left(0.7, \frac{4}{12}\right), \left(0.3, \frac{1}{12}\right)\right)$	$\left(\left(0.7, \frac{7}{12}\right), \left(0.1, \frac{1}{12}\right)\right)$
	$\mathcal{V}_2$	$\left(\left(0.3, \frac{9}{12}\right), \left(0.4, \frac{1}{12}\right)\right)$	$\left(\left(0.8, \frac{8}{12}\right), \left(0.1, \frac{3}{12}\right)\right)$	$\left(\left(0.8, \frac{9}{12}\right), \left(0.2, \frac{3}{12}\right)\right)$	$\left(\left(0.7, \frac{5}{12}\right), \left(0.2, \frac{6}{12}\right)\right)$
	$\mathcal{V}_3$	$\left(\left(0.7, \frac{3}{12}\right), \left(0.3, \frac{1}{12}\right)\right)$	$\left(\left(0.8, \frac{10}{12}\right), \left(0.1, \frac{1}{12}\right)\right)$	$\left(\left(0.5, \frac{4}{12}\right), \left(0.1, \frac{2}{12}\right)\right)$	$\left(\left(0.9, \frac{10}{12}\right), \left(0.1, \frac{2}{12}\right)\right)$
	$\mathcal{V}_4$	$\left(\left(0.6, \frac{6}{12}\right), \left(0.3, \frac{4}{12}\right)\right)$	$\left(\left(0.6, \frac{5}{12}\right), \left(0.3, \frac{2}{12}\right)\right)$	$\left(\left(0.5, \frac{8}{12}\right), \left(0.4, \frac{3}{12}\right)\right)$	$\left(\left(0.7, \frac{8}{12}\right), \left(0.1, \frac{3}{12}\right)\right)$
	$\mathcal{V}_5$	$\left(\left(0.7, \frac{10}{12}\right), \left(0.3, \frac{2}{12}\right)\right)$	$\left(\left(0.4, \frac{4}{12}\right), \left(0.4, \frac{4}{12}\right)\right)$	$\left(\left(0.9, \frac{10}{12}\right), \left(0.1, \frac{1}{12}\right)\right)$	$\left(\left(0.7, \frac{9}{12}\right), \left(0.1, \frac{1}{12}\right)\right)$
$\mathcal{E}^{(2)}$	$\mathcal{V}_1$	$\left(\left(0.6, \frac{8}{12}\right), \left(0.2, \frac{2}{12}\right)\right)$	$\left(\left(0.5, \frac{7}{12}\right), \left(0.4, \frac{2}{12}\right)\right)$	$\left(\left(0.8, \frac{9}{12}\right), \left(0.1, \frac{2}{12}\right)\right)$	$\left(\left(0.7, \frac{7}{12}\right), \left(0.2, \frac{3}{12}\right)\right)$
	$\mathcal{V}_2$	$\left(\left(0.9, \frac{9}{12}\right), \left(0.1, \frac{1}{12}\right)\right)$	$\left(\left(0.5, \frac{6}{12}\right), \left(0.4, \frac{5}{12}\right)\right)$	$\left(\left(0.5, \frac{8}{12}\right), \left(0.3, \frac{4}{12}\right)\right)$	$\left(\left(0.8, \frac{5}{12}\right), \left(0.1, \frac{5}{12}\right)\right)$
	$\mathcal{V}_3$	$\left(\left(0.5, \frac{5}{12}\right), \left(0.2, \frac{4}{12}\right)\right)$	$\left(\left(0.9, \frac{8}{12}\right), \left(0.1, \frac{3}{12}\right)\right)$	$\left(\left(0.8, \frac{9}{12}\right), \left(0.2, \frac{2}{12}\right)\right)$	$\left(\left(0.8, \frac{8}{12}\right), \left(0.1, \frac{3}{12}\right)\right)$
	$\mathcal{V}_4$	$\left(\left(0.7, \frac{9}{12}\right), \left(0.1, \frac{3}{12}\right)\right)$	$\left(\left(0.8, \frac{5}{12}\right), \left(0.1, \frac{6}{12}\right)\right)$	$\left(\left(0.5, \frac{5}{12}\right), \left(0.1, \frac{5}{12}\right)\right)$	$\left(\left(0.7, \frac{8}{12}\right), \left(0.1, \frac{1}{12}\right)\right)$
	$\mathcal{V}_5$	$\left(\left(0.7, \frac{9}{12}\right), \left(0.1, \frac{1}{12}\right)\right)$	$\left(\left(0.6, \frac{7}{12}\right), \left(0.4, \frac{3}{12}\right)\right)$	$\left(\left(0.4, \frac{5}{12}\right), \left(0.2, \frac{2}{12}\right)\right)$	$\left(\left(0.9, \frac{8}{12}\right), \left(0.1, \frac{2}{12}\right)\right)$
$\mathcal{E}^{(3)}$	$\mathcal{V}_1$	$\left(\left(0.9, \frac{9}{12}\right), \left(0.1, \frac{1}{12}\right)\right)$	$\left(\left(0.8, \frac{8}{12}\right), \left(0.2, \frac{2}{12}\right)\right)$	$\left(\left(0.4, \frac{5}{12}\right), \left(0.4, \frac{4}{12}\right)\right)$	$\left(\left(0.7, \frac{8}{12}\right), \left(0.1, \frac{3}{12}\right)\right)$
	$\mathcal{V}_2$	$\left(\left(0.5, \frac{5}{12}\right), \left(0.1, \frac{4}{12}\right)\right)$	$\left(\left(0.6, \frac{6}{12}\right), \left(0.3, \frac{5}{12}\right)\right)$	$\left(\left(0.9, \frac{9}{12}\right), \left(0.1, \frac{1}{12}\right)\right)$	$\left(\left(0.8, \frac{9}{12}\right), \left(0.1, \frac{3}{12}\right)\right)$
	$\mathcal{V}_3$	$\left(\left(0.4, \frac{5}{12}\right), \left(0.1, \frac{5}{12}\right)\right)$	$\left(\left(0.8, \frac{8}{12}\right), \left(0.2, \frac{4}{12}\right)\right)$	$\left(\left(0.8, \frac{8}{12}\right), \left(0.1, \frac{2}{12}\right)\right)$	$\left(\left(0.7, \frac{7}{12}\right), \left(0.1, \frac{4}{12}\right)\right)$
	$\mathcal{V}_4$	$\left(\left(0.9, \frac{8}{12}\right), \left(0.1, \frac{1}{12}\right)\right)$	$\left(\left(0.5, \frac{5}{12}\right), \left(0.4, \frac{4}{12}\right)\right)$	$\left(\left(0.7, \frac{7}{12}\right), \left(0.1, \frac{5}{12}\right)\right)$	$\left(\left(0.6, \frac{5}{12}\right), \left(0.3, \frac{5}{12}\right)\right)$
	$\mathcal{V}_5$	$\left(\left(0.5, \frac{4}{12}\right), \left(0.3, \frac{4}{12}\right)\right)$	$\left(\left(0.6, \frac{7}{12}\right), \left(0.2, \frac{3}{12}\right)\right)$	$\left(\left(0.8, \frac{8}{12}\right), \left(0.1, \frac{4}{12}\right)\right)$	$\left(\left(0.9, \frac{8}{12}\right), \left(0.1, \frac{3}{12}\right)\right)$

Table 11.2: Aggregated values of experts obtained by using CIFPrOWA$_d$ operator

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$\left(\left(0.5955, \frac{5.5921}{12}\right), \left(0.1228, \frac{2.8732}{12}\right)\right)$	$\left(\left(0.6672, \frac{6.7777}{12}\right), \left(0.2304, \frac{2.0000}{12}\right)\right)$	$\left(\left(0.5862, \frac{5.2685}{12}\right), \left(0.3111, \frac{2.2262}{12}\right)\right)$	$\left(\left(0.7000, \frac{7.3425}{12}\right), \left(0.1468, \frac{2.6054}{12}\right)\right)$
$\mathcal{V}_2$	$\left(\left(0.5203, \frac{7.3125}{12}\right), \left(0.1671, \frac{2.0755}{12}\right)\right)$	$\left(\left(0.5864, \frac{6.3132}{12}\right), \left(0.3049, \frac{4.6734}{12}\right)\right)$	$\left(\left(0.6877, \frac{8.4789}{12}\right), \left(0.2350, \frac{3.1790}{12}\right)\right)$	$\left(\left(0.7437, \frac{5.5159}{12}\right), \left(0.1528, \frac{5.3396}{12}\right)\right)$
$\mathcal{V}_3$	$\left(\left(0.4899, \frac{4.7478}{12}\right), \left(0.1492, \frac{3.6828}{12}\right)\right)$	$\left(\left(0.8423, \frac{8.3223}{12}\right), \left(0.1450, \frac{3.0638}{12}\right)\right)$	$\left(\left(0.6529, \frac{6.4378}{12}\right), \left(0.1234, \frac{2.0000}{12}\right)\right)$	$\left(\left(0.7717, \frac{7.8573}{12}\right), \left(0.1000, \frac{3.3304}{12}\right)\right)$
$\mathcal{V}_4$	$\left(\left(0.6778, \frac{7.3099}{12}\right), \left(0.1946, \frac{3.2223}{12}\right)\right)$	$\left(\left(0.5837, \frac{5.0000}{12}\right), \left(0.3090, \frac{3.2015}{12}\right)\right)$	$\left(\left(0.5219, \frac{6.4574}{12}\right), \left(0.1658, \frac{4.1505}{12}\right)\right)$	$\left(\left(0.6507, \frac{6.6212}{12}\right), \left(0.1789, \frac{3.4916}{12}\right)\right)$
$\mathcal{V}_5$	$\left(\left(0.6064, \frac{7.6343}{12}\right), \left(0.2664, \frac{2.6825}{12}\right)\right)$	$\left(\left(0.4836, \frac{5.2774}{12}\right), \left(0.3773, \frac{3.5960}{12}\right)\right)$	$\left(\left(0.6589, \frac{6.9670}{12}\right), \left(0.1478, \frac{2.2781}{12}\right)\right)$	$\left(\left(0.8522, \frac{8.3890}{12}\right), \left(0.1000, \frac{1.9417}{12}\right)\right)$

Table 11.3: Influence of the operators used in Step 2 and Step 3 on ranking order of alternatives

Operator utilized in Step 2	Operator utilized in Step 3	Overall values of the alternatives $\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	Ranking order of alternatives
CIFPrOWA$_d$	CIFPrA$_d$	0.6916	0.6819	0.7183	0.6657	0.7095	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFPrOWA$_d$	CIFPrG$_d$	0.6835	0.6639	0.6956	0.6583	0.6864	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFPrOWA$_d$	CIFPrA	0.6906	0.6802	0.7129	0.6652	0.6971	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFPrOWA$_d$	CIFPrG	0.6828	0.6632	0.6891	0.6575	0.6764	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFPrOWG$_d$	CIFPrA$_d$	0.6624	0.6475	0.6994	0.6397	0.6748	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFPrOWG$_d$	CIFPrG$_d$	0.6488	0.6299	0.6743	0.6341	0.6520	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
CIFPrOWG$_d$	CIFPrA	0.6604	0.6433	0.6934	0.6393	0.6605	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFPrOWG$_d$	CIFPrG	0.6471	0.6274	0.6672	0.6335	0.6412	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2$

Table 11.4: Comparative Analysis results with CIFS studies

Method used	Overall value of the alternatives $\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	Ranking order of alternatives
CIFWPA operator [130]	1.0515	1.0115	1.1900	0.9331	1.1443	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFWPG operator [130]	1.0184	0.9364	1.1065	0.9007	1.0441	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFWA operator [62]	1.0508	1.0118	1.1860	0.9321	1.1436	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFWG operator [62]	1.0176	0.9366	1.1012	0.8995	1.0414	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFEWA operator [59]	1.0476	1.0037	1.1773	0.9284	1.1339	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFEWG operator [60]	1.0221	0.9475	1.1146	0.9046	1.0547	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
CIFWA operator [61]	0.1525	0.2084	0.2458	0.1527	0.2406	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1$
Distance measure [6]			–			$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
Weighted Euclidean distance [129]			–			$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
Correlation coefficient K_4 [58]			–			$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4$
The proposed approach	0.6916	0.6819	0.7183	0.6657	0.7095	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$

Here: $t(a) = \log\left(\frac{2-a}{a}\right)$ in [59, 60] and $\alpha_1 = \beta_1 = \sigma_1 = \alpha_2 = \beta_2 = \sigma_2 = \frac{1}{3}$ in [6]

Table 11.5: Comparative Analysis results with IFS studies

Method used	Overall values of the alternatives					Ranking order of alternatives
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	
IFPWA operator [203]	0.7913	0.7758	0.8430	0.7745	0.7776	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
GIFPWG operator [204]	0.7811	0.7614	0.8159	0.7650	0.7455	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_5$
IFEPWA operator [152]	0.7906	0.7743	0.8404	0.7736	0.7743	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_4$
IFEPWG operator [152]	0.7823	0.7634	0.8200	0.7664	0.7492	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_5$
PFHPA operator [43]	0.7431	0.7262	0.7817	0.7317	0.7371	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
PFHPG operator [43]	0.7376	0.7165	0.7570	0.7253	0.7150	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_5$
PFPWA operator [90]	0.7440	0.7298	0.7918	0.7336	0.7446	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2$
PFPWG operator [90]	0.7366	0.7133	0.7459	0.7234	0.7091	$\mathcal{V}_3 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_5$
The proposed approach results	0.7614	0.7526	0.7829	0.7371	0.7777	$\mathcal{V}_3 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4$

Here: $g(a) = -\log(a), 0 < a \leq 1$ in [204], $\gamma = 3$ in [43]

Table 11.6: The characteristic comparison of different approaches

Method	Considers prioritization relationship among criteria	Considers priority degree among criteria	Handles optimistic and pessimistic behavior	Handles group decision-making problems	Ability to handle time periodic problems	Ability to handle two dimensional information
Yu [203]	✓	×	✓	✓	×	×
Yu [204]	✓	×	×	✓	×	×
Verma and Sharma [152]	✓	×	✓	✓	×	×
Gao [43]	✓	×	✓	×	×	×
Khan et al. [90]	✓	×	✓	✓	×	×
Rani and Garg [130]	×	×	✓	✓	✓	✓
Garg and Rani [62]	×	×	✓	×	✓	✓
Garg and Rani [59]	×	×	×	×	✓	✓
Garg and Rani [61]	×	×	✓	×	✓	✓
The proposed approach	✓	✓	✓	✓	✓	✓

Table 11.7: Rating values of alternatives $\mathcal{V}_u$ under criteria $\mathfrak{B}_v$

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$((0.5,0.4),(0.4,0.5))$	$((0.6,0.5),(0.3,0.4))$	$((0.8,0.6),(0.2,0.3))$	$((0.6,0.4),(0.3,0.5))$
$\mathcal{V}_2$	$((0.7,0.5),(0.3,0.3))$	$((0.4,0.5),(0.4,0.3))$	$((0.3,0.6),(0.5,0.3))$	$((0.4,0.4),(0.3,0.5))$
$\mathcal{V}_3$	$((0.3,0.4),(0.5,0.3))$	$((0.4,0.5),(0.3,0.5))$	$((0.5,0.4),(0.3,0.3))$	$((0.6,0.3),(0.3,0.5))$
$\mathcal{V}_4$	$((0.7,0.6),(0.2,0.2))$	$((0.4,0.3),(0.4,0.6))$	$((0.5,0.3),(0.3,0.4))$	$((0.4,0.6),(0.3,0.3))$
$\mathcal{V}_5$	$((0.6,0.5),(0.2,0.4))$	$((0.6,0.5),(0.2,0.2))$	$((0.5,0.7),(0.3,0.2))$	$((0.5,0.6),(0.5,0.3))$

Table 11.8: Impact of d on CIFPrA$_d$ operator results

Priority degree vector $d=(d_1,d_2,d_3)$	Alternatives	Aggregated values obtained using CIFPrA$_d$ operator	Score values	Ranking order of alternatives
$d=(1,1,1)$	$\mathcal{V}_1$	$((0.5970, 0.4601), (0.3257, 0.4385))$	0.5732	$\mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.5513, 0.5103), (0.3550, 0.3134))$	0.5983	
	$\mathcal{V}_3$	$((0.3835, 0.4204), (0.3944, 0.3553))$	0.5136	
	$\mathcal{V}_4$	$((0.5735, 0.4795), (0.2743, 0.3265))$	0.6131	
	$\mathcal{V}_5$	$((0.5720, 0.5564), (0.2407, 0.2829))$	0.6512	
$d=(10,1,1)$	$\mathcal{V}_1$	$((0.5004, 0.4002), (0.3997, 0.4997))$	0.5003	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.6946, 0.5005), (0.3022, 0.3006))$	0.6481	
	$\mathcal{V}_3$	$((0.3002, 0.4000), (0.4997, 0.3001))$	0.4751	
	$\mathcal{V}_4$	$((0.6877, 0.5878), (0.2073, 0.2115))$	0.7142	
	$\mathcal{V}_5$	$((0.5991, 0.5020), (0.2012, 0.3954))$	0.6261	
$d=(0,0,0)$	$\mathcal{V}_1$	$((0.6443, 0.4820), (0.2913, 0.4162))$	0.6047	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.4756, 0.5051), (0.3663, 0.3409))$	0.5684	
	$\mathcal{V}_3$	$((0.4616, 0.4042), (0.3409, 0.3873))$	0.5344	
	$\mathcal{V}_4$	$((0.5179, 0.4708), (0.2913, 0.3464))$	0.5878	
	$\mathcal{V}_5$	$((0.5528, 0.5838), (0.2783, 0.2632))$	0.6488	
$d=(+\infty,1,1)$	$\mathcal{V}_1$	$((0.5000, 0.4000), (0.4000, 0.5000))$	0.5000	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.7000, 0.5000), (0.3000, 0.3000))$	0.6500	
	$\mathcal{V}_3$	$((0.3000, 0.4000), (0.5000, 0.3000))$	0.4750	
	$\mathcal{V}_4$	$((0.7000, 0.6000), (0.2000, 0.2000))$	0.7250	
	$\mathcal{V}_5$	$((0.6000, 0.5000), (0.2000, 0.4000))$	0.6250	
$d=(0,+\infty,0)$	$\mathcal{V}_1$	$((0.5528, 0.4523), (0.3464, 0.4472))$	0.5529	$\mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.5757, 0.5000), (0.3464, 0.3000))$	0.6073	
	$\mathcal{V}_3$	$((0.3519, 0.4523), (0.3873, 0.3873))$	0.5074	
	$\mathcal{V}_4$	$((0.5757, 0.4708), (0.2828, 0.3464))$	0.6043	
	$\mathcal{V}_5$	$((0.6000, 0.5000), (0.2000, 0.2828))$	0.6543	
$d=(1,+\infty,1)$	$\mathcal{V}_1$	$((0.5358, 0.4354), (0.3634, 0.4642))$	0.5359	$\mathcal{V}_5 \succ \mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.6058, 0.5000), (0.3360, 0.3000))$	0.6175	
	$\mathcal{V}_3$	$((0.3339, 0.4342), (0.4242, 0.3536))$	0.4976	
	$\mathcal{V}_4$	$((0.5985, 0.4939), (0.2674, 0.3174))$	0.6269	
	$\mathcal{V}_5$	$((0.6000, 0.5000), (0.2000, 0.3064))$	0.6484	

Table 11.9: Impact of d on CIFPrOWA$_d$ operator results

Priority degree vector $d=(d_1,d_2,d_3)$	Alternatives	Aggregated values obtained using CIFPrOWA$_d$ operator	Score values	Ranking order of alternatives
$d=(1,1,1)$	$\mathcal{V}_1$	$((0.5348,0.4198),(0.3685,0.4797))$	0.5266	$\mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.4582,0.5151),(0.3819,0.3262))$	0.5563	
	$\mathcal{V}_3$	$((0.3428,0.4084),(0.4478,0.3319))$	0.4929	
	$\mathcal{V}_4$	$((0.4961,0.4016),(0.3232,0.4243))$	0.5376	
	$\mathcal{V}_5$	$((0.5759,0.5328),(0.2501,0.3256))$	0.6333	
$d=(10,1,1)$	$\mathcal{V}_1$	$((0.5000,0.4000),(0.4000,0.5000))$	0.5000	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.6892,0.5009),(0.3045,0.3012))$	0.6461	
	$\mathcal{V}_3$	$((0.3000,0.4000),(0.5000,0.3000))$	0.4750	
	$\mathcal{V}_4$	$((0.6757,0.5759),(0.2145,0.2230))$	0.7036	
	$\mathcal{V}_5$	$((0.5993,0.5009),(0.2015,0.3990))$	0.6249	
$d=(0,0,0)$	$\mathcal{V}_1$	$((0.5777,0.4347),(0.3317,0.4644))$	0.5540	$\mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.3971,0.4950),(0.3714,0.3751))$	0.5364	
	$\mathcal{V}_3$	$((0.4409,0.3915),(0.3751,0.3873))$	0.5175	
	$\mathcal{V}_4$	$((0.4573,0.3914),(0.3317,0.4334))$	0.5209	
	$\mathcal{V}_5$	$((0.5528,0.5608),(0.3063,0.2966))$	0.6277	
$d=(+\infty,1,1)$	$\mathcal{V}_1$	$((0.5000,0.4000),(0.4000,0.5000))$	0.5000	$\mathcal{V}_4 \succ \mathcal{V}_2 \succ \mathcal{V}_5 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.7000,0.5000),(0.3000,0.3000))$	0.6500	
	$\mathcal{V}_3$	$((0.3000,0.4000),(0.5000,0.3000))$	0.4750	
	$\mathcal{V}_4$	$((0.7000,0.6000),(0.2000,0.2000))$	0.7250	
	$\mathcal{V}_5$	$((0.6000,0.5000),(0.2000,0.4000))$	0.6250	
$d=(0,+\infty,0)$	$\mathcal{V}_1$	$((0.5271,0.4267),(0.3722,0.4729))$	0.5272	$\mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.4955,0.5000),(0.3722,0.3000))$	0.5808	
	$\mathcal{V}_3$	$((0.3265,0.4267),(0.4401,0.3409))$	0.4931	
	$\mathcal{V}_4$	$((0.4955,0.3914),(0.3364,0.4559))$	0.5236	
	$\mathcal{V}_5$	$((0.6000,0.5000),(0.2000,0.3364))$	0.6409	
$d=(1,+\infty,1)$	$\mathcal{V}_1$	$((0.5122,0.4120),(0.3874,0.4878))$	0.5123	$\mathcal{V}_5 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1 \succ \mathcal{V}_3$
	$\mathcal{V}_2$	$((0.5349,0.5000),(0.3599,0.3000))$	0.5937	
	$\mathcal{V}_3$	$((0.3111,0.4112),(0.4742,0.3163))$	0.4830	
	$\mathcal{V}_4$	$((0.5247,0.4200),(0.3169,0.4148))$	0.5533	
	$\mathcal{V}_5$	$((0.6000,0.5000),(0.2000,0.3610))$	0.6347	

Chapter 12

Summary and Future Scope

This chapter depicts the summary of the work that we have done in this thesis. Also, it outlines the scope of the future work.

12.1 Summary of the work

In this thesis, we have reviewed existing work on IFSs, IVIFSs and CFSs theories in Chapter 1 and have discussed the gaps in existing FS and IFS environments which are filled by the CIFS model. In Chapter 2, some basic concepts related to IFS and CIFS models are described. In Chapters 3–5, we have proposed a number of new information measures which process uncertain and periodic information simultaneously. In Chapters 6 and 7, we have introduced new generalized averaging and geometric AOs which aggregate independent arguments under CIF environment. In Chapter 8, we have presented novel generalized exponential and logarithm operations for CIFNs and AOs based on these operations. In Chapters 9 and 10, we have developed power and Bonferroni mean AOs respectively which consider interdependence among CIFNs during aggregation process. In Chapter 11, we have defined some new prioritized aggregation operators by considering priority degrees among priority orders for aggregating CIFNs. The key contribution of the work, presented in this thesis, is summarized as follows:

1) IFS and IVIFS theories capture uncertainty and fuzziness using real numbers lying between 0 and 1. Although, researchers have developed a number of MCDM techniques

under these theories but, these models can deal with only one dimensional DM problems. However, many real world complex problems involve two dimensional data i.e., information related with the attributes and periodicity of the parameters concerned with the problem. In order to portray such two dimensional information using these theories, the decision-maker will have to consider two or more FSs/IFSs/IVIFSs which may increase execution time and the number of computations required while solving the problem. But CFSs, CIFSs and Complex IVIFSs (CIVIFSs) have the ability of portraying two dimensional information together in one set. In a nutshell, the fundamental gap in IFS/IVIFS theories is that these sets can deal with only uncertainty whereas CIFS/CIVIFS models fill this gap by handling uncertainty and periodicity simultaneously.

2) The information measures presented in Chapters 3 - 5 have provided tools for clustering analysis, for solving pattern recognition problems, medical diagnosis problems and many other DM problems which involve uncertainty and periodicity factors simultaneously. The measures proposed in these chapters are:

   a) Hamming, Euclidean and Hausdorff distances among CIFSs.

   b) Correlation coefficient and weighted correlation coefficient for CIFSs.

   c) A series of similarity measures among CIFSs and then constructed distance, entropy and inclusion measures based on proposed similarity measures using the transformation relationship among various information measures.

   The proposed information measures have been applied in patten recognition problems, medical diagnosis problems and MCDM problems. Also, clustering algorithm has been developed based on the developed similarity measures in order to cluster CIFSs.

3) An AO is a mathematical function possessing the capability of reducing a set of numbers into a unique representative one. We have developed new operational laws for CIFNs based on ATT operations and a series of weighted averaging and geometric AOs in order to aggregate independent CIFNs in Chapters 6 and 7. Besides this, we have developed exponential operations in which bases are real numbers and the exponents

are CIFNs and have presented logarithmic and exponential of logarithmic operations in Chapter 8. A list of the proposed AOs in Chapters 6 - 8 is given as follows:

a) Generalized CIF weighted and ordered weightedaveraging/geometric operators.

b) Generalized CIF hybrid averaging/geometric operators.

c) Generalized CIF weighted and ordered weighted averaging/geometric operators based on exponential of logarithmic operations.

The above presented operators are based on ATT operations and the decision maker can choose any norm operation such as algebraic, einstein, hamachar in accordance with his/her desire or present situation by giving different forms to additive generator. The fundamental properties of these operators have been investigated in detail. Besides this, the proposed operators can be applied on IFS data as well by setting the phase terms equal to zero which shows that the presented work is more generalized.

4) In order to aggregate dependent CIFNs, we have proposed power, BM and prioritized operators in Chapters 9, 10 and 11 respectively. The AOs developed in these two chapters are listed as follows:

a) CIF power averaging/geometric operators.

b) CIF weighted and ordered weighted power averaging/geometric operators.

c) Generalized CIF weighted bonferroni mean operator.

d) CIF prioritized averaging/geometric operators with and without priority degrees.

e) CIF prioritized ordered weighted averaging/geometric operators with priority degrees.

The fundamental property of the power operators developed in Chapter 9 is that in these AOs the weightage of each argument is calculated by taking into account the similarity of the argument with all other arguments which are to be aggregated. On the other hand, in Chapter 10, generalized BM operators are developed which take into account the interrelationship among each pair of CIFNs. Various special cases of BM operators have been obtained and discussed. Also, it has been analyzed that the existing BM operator under IFS theory is one of the special cases of the proposed BM operator which proves that the presented work is more generalized and reliable. Also, the CIF prioritized AOs, presented in Chapter 11, address the situations in which there exists strict prioritization relationship among the arguments to be aggregated with or without priority degrees among strict priority levels.

12.2 Future scope of the work

The future scope of the work presented in the thesis is described as follows:

1) Various information measures such as distance, similarity, correlation, divergence and entropy can be developed under CIVIFS environment and based on them MCDM techniques can be established.

2) The presented work may be further extended to develop new AOs under CIFS and CIVIFS studies by constructing some novel additive generator such as trigonometric operations etc.

3) Based on the addition and multiplication operations of CIFNs and Complex IVIFNs (CIVIFNs), the subtraction and division operations can be developed under CIFS and CIVIFS theories.

4) The proposed BM operator can be further extended to Maclaurin Symmetric Mean (MSM) operator based on ATT operations for aggregating dependent CIFNs and CIVIFNs. Also, some hybrid AOs may be developed under CIFS and CIVIFS theories by combining the characteristic of the different operators.

5) The proposed MCDM and MCGDM methods can be integrated with other various existing MCDM techniques such as WASPAS, CODAS, TOPSIS, and MULTIMOORA.

6) The proposed AOs, information measures and MCDM approaches may be extended in the direction of solving environmental issues such as solid waste management, greenhouse gas emissions, health-care solid waste management and green supplier selection.

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