SOME ASPECTS OF BILEVEL PROGRAMMING

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Submitted by

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DEDICATEDTO

GOD, MY PARENTS AND MY SUPERVISOR

DECLARATION

I hereby declare that the work which is being presented here in the dissertation entitled "SOME ASPECTS OF BILEVEL PROGRAMMING" in partial fulfilment of the requirement for the award of degree of **Master of Science** in **Mathematics and Computing** submitted in School of Mathematics and Computer Applications, Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of **Dr. Vikas Sharma**, Lecturer, SMCA and refers other researcher's work which are duly listed in the reference section.

The matter presented in this dissertation has not been submitted in any other University/Institute for the award of my degree.

Dated: 15/July/13

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It is certified that the above statement made by the student is correct to the best of my knowledge and belief.

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ABSTRACT

Bilevel programming involves two optimization problems where the constraint region of the first level problem is implicitly determined by another optimization problem. It has been applied to decentralized planning problems involving a decision process with a hierarchical structure. In this thesis we have discussed three problems in the field of bilevel optimization. In the first problem a bilevel linear/linear fractional programming problem has been discussed, where the objective function for the first level is linear, objective function for second level problem is linear fractional and the common constrained region is polyhedral. It has been discussed that an optimal solution can be found which is an extreme point of the polyhedron. Moreover, by taking into account the relationship between feasible solutions to the problem and bases of the technological coefficient submatrix associated to variables of the second level, an enumerative algorithm is proposed that finds a global optimum to the problem.

Second problem discussed in the thesis, consist of a linear fractional objective function at both the levels. It has been discussed that an optimal solution to this problem occurs at a boundary feasible extreme point. A Kth best algorithm has been proposed to solve this problems, further it has also been discussed this algorithm can also be extended to quasiconcave bilevel problems, provided that the first level objective function is explicitly quasimonotone.

The last problem we have considered is a nonlinear bilevel programming problem, where the objective function of first level is a indefinite quadratic function and the lower level objective function is linear. By making use of duality theory, bilevel program is transformed into an equivalent single level programming problem, which can further be converted into a programming problem without constraints. By using genetic algorithm and optimal solution of this problem is obtained.

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CHAPTER 1

INTRODUCTION

In this dissertation, we have studied theoretical properties of some hierarchical decision models, a class of decision problems with rich application potential. A hierarchical model, however, is comprised of several levels of DMs, whose decisions are made sequentially and may affect the options available to those lower in the hierarchy and the payoff of those higher in the hierarchy. A common example of such a model is that faced by the federal government. Policy decisions made at the federal level affect future decisions made by state and local governments, each of which acts in its own self-interest in reaction to federal directives. Decisions made by the state and local governments, in turn, affect the degree to which the federal government accomplishes its original objective. Thus, in order to perform an accurate analysis, the federal government must consider the reaction of the lower-level bodies, and make policy decisions accordingly. The same analysis applies in the corporate setting, where company policy is set at the highest level, interpreted and applied in smaller organizational units. Unlike other multiple-objective mathematical programming techniques, Multilevel Mathematical programming (MLMP) emphasizes the non cooperative character of the system.

Let x be the vector of decision variable controlled by the highest level, F the corresponding objective function, which is also the objective of the overall system, y_1 and f_1 the decision variable vector and the objective function, respectively, of the first lower level decision maker, y_2 and f_2 those of the second lower level,….., y_L and f_L those of lowest level, and G the vector of constraint functions. The general form of multilevel mathematical programming can be defined as:

$$
\begin{aligned}
&MAX_x\ F(x, y_1, y_2, ..., y_L) \\
&where\ y_1, y_2, ..., y_L\ solve: \\
&MAX_{y1}\ f_1(x, y_1, y_2, ..., y_L) \\
&where\ y_2, y_3, ..., y_L\ solve: \\
&MAX_{y2}\ f_2(x, y_1, y_2, ..., y_L) \\
&where\ y_3, y_4, ..., y_L\ solve: \\
&\quad\quad\quad . \\
&\quad\quad\quad . \\
&\quad\quad\quad . \\
&MAX_{y_{L-1}}\ f_{L-1}(x, y_1, y_2, ..., y_L) \\
&where\ y_L\ solve: \\
&MAX_{y_L}\ f_L(x, y_1, y_2, ..., y_L) \\
&\quad such\ that: \\
&G(x, y_1, y_2, ..., y_L) \leq 0
\end{aligned}
$$

Multilevel Mathematical Programming is an Operations Research technique dealing with the optimization of the objective of the highest level of a hierarchical organization, taking into account the tendency of the lower levels of the hierarchy to improve their own objectives. The decisions of the lower levels are not dictated by their superiors; however, their reactions to the upper levels' actions are perfectly known. Even though the headquarters neither does nor exercise direct control on the day to day operations of the subunits, he can influence those operations by setting transfer prices, allocating required resources, establishing product lines, or channeling capital investment. As shown in the above formulation, the hierarchical nature of the problem is reflected by the order imposed on the choice of decision first, followed by the next highest, until the lowest ones. At one level of the hierarchy, a decision maker has his own set of feasible solutions determined, in part, by the upper levels. Conversely, the decision instruments he controls may allow him to influence the operations of lower levels.

A large class of Multilevel Mathematical Programs involves only two levels and is called bilevel linear programming (BLP). The bilevel programming problem is an optimization problem whose constraints are (in part) determined by another optimization problem. In other words it is a hierarchical optimization problem consisting of two levels, the first of which (the leader's level) is dominant over the other (the follower's one).

In this dissertation we have discussed three categories of bilevel programming problem. We will consider the main results along with the most used algorithms for the following types of BLPP:

(i) Linear fractional bilevel program (both leader and follower objectives are fractional)
(ii) Bilevel linear/linear fractional programming (leader's objective is linear and follower's objective is fractional)
(iii) Bilevel non-linear programming problem

One of the most important and widely studied class of bilevel programming problems is bilevel linear programming problem. The Bilevel (or Two-Level) linear programming (BLP) is similar to a standard linear programming (LP), except that the constraint region is modified by including a defined linear objective function. But general linear programming techniques cannot be applied to solve this model, since the feasible region is non convex. The general form of BLP can be defined as

$$\max_x f_1(x,y) = c_1 x + d_1 y$$

where y solves:

$$\max f_2(x,y) = c_2 x + d_2 y \tag{1.1}$$

such that:

$$Ax + By \le b$$
$$x, y \ge 0$$

where:

$c_1, c_2 \in \Re^{n_1}$ *and* $d_1, d_2 \in \Re^{n_2}$ are constant vectors

$b \in \Re^m$ is a constant vector,

$A \in \Re^{m \times n_1}$ and $B \in \Re^{m \times n_2}$ are constant matrices,

x is a vector of the decision variables of the upper problem; its components are called upper variables,

y is a vector of the decision variables of the lower problem; its components are called lower variables,

f_1 is the objective function of the upper problem; it is called the upper objective, and,

f_2 is the objective function of the lower problem; it is called the lower objective.

The set of common constraint region is given by

$$S=\{(x,y):Ax+By\leq b, x,y\geq 0\}.$$

Definitions

a) Constraint region of the linear BLP:

$$S=\{(x,y):x\in X, y\in Y, A_1x+B_1y\leq b_1\ ,\ A_2x+B_2y\leq b_2\}.$$

b) Feasible set for the follower for each fixed $x\in X$:

$$S(x)=\{y\in Y, A_2x+B_2y\leq b_2\}$$

c) Projection of S onto the leader's decision space:

$$S(X)=\{x\in X:\exists y\in Y,\ A_1x+B_1y\leq b_1\ ,\ A_2x+B_2y\leq b_2\}.$$

d) Follower's rational reaction set for $x\in S(X)$:

$$P(x)=\{y\in Y: y\in \arg\min\{f(x,\hat{y}):\hat{y}\in S(x)\}\}$$

e) Inducible region:

$$IR=\{(x,y):(x,y)\in S, y\in P(x)\}$$

Another important class of bilevel programming problems is the class of bilevel fractional programming problem. It is worth mentioning that the objective functions, which are ratios frequently appear, for example, when an efficiency measure of system is to be optimized or in optimizing return on investment in resource allocation Fractional programming has received remarkable attention in the literature Schaible (1995) gives a survey on fractional programming which covers applications as well as major theoretical and algorithmic developments.

Bilevel Fractional programming (BFP), is a class of bilevel programming [Dempe (2003), Vicente and Calamai (1994)], has been proposed as a generalization of standard fractional programmig for dealing with hierarchical system with two decision levels. BFP problems assume that the objective function of both level are ratios of functions and the common constraint region to both levels is non empty and compact polyhedron. The bilevel fractional programming problem (BFPP), in which the leader's and follower's objective function is linear functional has been studied in this thesis. Mathematically a general bilevel fractional programming problem can be defined as:

$$(P) \qquad \min_{x_1, x_2} \; f_1(x_1, x_2) = \frac{h_1(x_1, x_2)}{g_1(x_1, x_2)}$$
$$\textit{where } x_2 \textit{ solve}:$$
$$\min_{x_2} \; f_2(x_1, x_2) = \frac{h_2(x_1, x_2)}{g_2(x_1, x_2)}$$
$$\textit{such that } (x_1, x_2) \in S$$

Where $x_1 \in \Re^{n_1}$ and $x_2 \in \Re^{n_2}$ are the variables controlled by the upper and the lower level decision maker, respectively; h_i and g_i are continuous functions, h_i are nonnegative and concave and g_i are positive and convex on S

$S = \{(x_1, x_2) : A_1 x_1 + A_2 x_2 \leq b, x_1 \geq 0, x_2 \geq 0\}$, which is assumed to be nonempty and bounded.

Let S_1 be the projection of S on $\Re^{n_1}$. For each $\tilde{x}_1 \in S_1$ provided by the upper level decision maker, the lower level one solves the fractional problem:

$$(P_1) \qquad \min_{x_2} \; f_2(\tilde{x}_1, x_2) = \frac{h_2(\tilde{x}_1, x_2)}{g_2(\tilde{x}_1, x_2)}$$
$$s.t \quad A_2 x_2 \le b - A_1 \tilde{x}_1$$
$$x_2 \ge 0$$

Let M($\tilde{x}_1$) denotes the set of optimal solutions to problem P_1 In order to ensure that the BFP problem is well posed it is also assumed that M ($\tilde{x}_1$) is a singleton for all $\tilde{x}_1 \in S_1$.

The feasible region of the upper level decision maker, also called the inducible region (IR), is implicitly defined by the lower level decision maker:

$$\text{IR} = \{(\tilde{x}_1, x_2) : \tilde{x}_1 \ge 0, \tilde{x}_2 = \arg\min \{ f_2(\tilde{x}_1, x_2) : A_1\tilde{x}_1 + A_2 x_2 \le b, x_2 \ge 0\}\}$$

Therefore, the FBP problem can also be stated as:

$$\min_{x_1, x_2} \; f_1(x_1, x_2) = \frac{h_1(x_1, x_2)}{g_1(x_1, x_2)}$$
$$s.t \qquad (x_1, x_2) \in IR$$

The BFP problem is a non convex optimization problem but, taking into account the quasiconcavity of f_2 and the properties of polyhedral, in Calvete and Gale (1998) it was proved that the inducible region is formed by the connected face.

While the bilevel linear and linear fractional programming problems have been extensively studied, the literature on nonlinear bilevel problem (NLBP) is rather poor. So far most research in this field is limited to convex or quadratic bilevel programming problems, and/or is mainly concerned with finding stationary points and local minima rather than global optimal solutions. For general nonlinear bilevel programs, most investigations have been concentrated on theoretical aspects. Very few exact methods exist, though some general properties of (NLBP), including its relation to multiobjective programming, have been discussed in Calamai and Vincete (1994). In Chapter 4 we have provided a globally convergent algorithm for a class of bilevel non linear programming where the upper level objective function is and indefinite quadratic function and the lower level objective function is linear. Mathematically this problem can be defined as follows:

$$\max_{x} F(x, y) = c_1^T x + d_1^T y + \frac{1}{2}(x^T . y^T) Q (x^T . y^T)$$

where y solves:

$$\max_{x} f(x, y) = c_2^T x + d_2^T y$$

$$s.t \quad Ax + By \le r,$$

$$x, y \ge 0$$

The genetic algorithm is one of the approaches to solve the bilevel non linear programming problem. In our discussion Genetic algorithm is used to solve the quadratic problem.

1.1 LITERATURE SURVEY

Hierarchical optimization was first defined by Bracken and McGill (1973) as a generalization of mathematical programming. In this context, the constraint region is implicitly determined by a series of optimization problems which must be solved in a predetermined sequence. The linear bilevel program was first shown to be NP–hard using satisfactory arguments common in computer science. The Bilevel programming problem (BLP) is a special case of the multilevel programming problems with two levels in a hierarchy, the upper level and lower level decision makers. The decision maker at the upper level, which is also termed as the Leader, makes the choice first to optimize his objective. Knowing the decision of the Leader, the Follower makes his response which in turn affects the Leader’s outcome. Since the formal formulation of the linear BLP proposed by Candler and Townsley (1982), many authors studied BLP intensively and contributed to this field. A lot of potential applications of BLP are presented by Dempe (2003), such as in the field of economics, engineering, ecology, transportation, game theory and so on. Due to the hierarchical structure, the BLP is generically non-convex and non-differentiable and intrinsically hard to solve, even if the objective functions of the both levels and the constraints are all linear. Bilevel programming has been proposed for dealing with decision processes involving two decision-makers with a hierarchical structure. The second level decision-maker optimizes his/her objective function under the given parameters from the first level decision-maker. This one, in return, with complete information on the possible reactions of the second level decision-maker, selects the parameters so as to optimize his/her own objective function. Hence, bilevel problems are characterized by the existence of two optimization problems in which the constraint region of the first level problem is implicitly determined by another optimization problem. In other words, bilevel problems have a subset of variables constrained to be an optimal solution of another problem parameterized by the remaining variables.

Bilevel programs were initially considered by Bracken and McGill (1973) in a series of papers that dealt with applications in the military field as well as in production and marketing decision making. By that time, such problems were called mathematical programs with optimization problems in the constraints, the terms bilevel and multilevel programming being introduced later by Candler and Norton [14]

This idea was further developed in Bard and Falk's 1989 paper where they generated the following mixed integer linear bi-level programming problem. Development continued and the more work performed, the more the conclusion that the solution to the BLPP was difficult and complex was reinforced. Some, like Bard and Moore [3], worked on algorithms for specific cases, specifically exploiting the follower's Kuhn-Tucker conditions. Hansen et.al [5] focused on determining necessary optimality conditions expressed in terms of the tightness of the follower's constraints and developing a penalty structure for the branch and bound method. Vincente et. al. [6] analyzed different discretizations of the set of variables. Studying the geometry of the feasible set and relating the classes of discrete linear problems to each other, they established equivalences. These equivalences were based on concave penalty functions and this would help to design penalty function methods for the solution of discrete linear programming problems. The bilevel programming problem is a nonconvex optimization problem that has received increasing attention in the literature Bard (1998), Dempe (2002) and Shimizu et al. (1997). A bibliography of references on bilevel and multilevel programming in both linear and nonlinear cases, which is up-dated biannually, can be found in Vicente and Calamai (1994). One of its main features is that, unlike general mathematical problems, the bilevel problem may not possess a solution even when f_1 and f_2 are continuous and S is compact.

1.2 SUMMARY OF THE THESIS

In this thesis, we have discussed three classes of bilevel programming problems. First problem discussed in Chapter 2 deals with linear/ linear fractional bilevel programming problem in which f_1 is linear and f_2 is linear fractional (LLFBP problem) originally studied by Calvete and Gale (1999). An enumerative algorithm has been proposed which finds a global optimum in a finite number of steps by examining implicitly only bases of the matrix. The advantage of this algorithm is that only linear problem needs to be solved. Mathematically, this problem can be stated as follows:

$$
\begin{aligned}
(P1): \quad & \min_{x_1} f_1 = k^1 x_1 + k^2 x_2, \\
& where\, x_2\, solves \\
& \min_{x_2} f_2 = \frac{\alpha + c^{11}x_1 + c^{12}x_2}{\beta + c^{21}x_1 + c^{22}x_2} \\
& s.t\ (x_1, x_2) \in S,
\end{aligned}
$$

where $x_1 \in \Re^{n_1}$ and $x_2 \in \Re^{n_2}$ are the variables controlled by the first level and the second level decision-maker, respectively; $k^1, k^2, c^{11}, c^{12}, c^{21}$ and c^{22} are vectors of conformal dimension; α and β are scalars; and the common constraint region to both levels is a polyhedron, i.e.

$$S = \{(x_1, x_2) : A^1 x_1 + A^2 x_2 = b, x_1 \geq 0, x_2 \geq 0\},$$

The algorithm discussed in this chapter is the first algorithm proposed for solving this particular kind of problem. An integer bilevel programming problem is considered in which the upper level objective function is linear and the lower level objective function is linear fractional. The variables at both the levels are related by the set of linear constraints. An algorithm is developed to find the in solution for the bilevel programming problem. Calvete and Gale (1994) developed this algorithm, which offers a global optimal solution to the bilevel linear/linear fractional programming problem.

Chapter 3 discussed another class of bilevel fractional programming problem, in which objectives at both the level are linear fractional. The Kth- best algorithm has been proposed to globally solve the FBP problem when both objective functions are linear fractional.

The problem discussed in this chapter has the following mathematical form:

$$\begin{aligned} &\min \quad f_1(x_1, x_2) = \frac{\alpha_1 + c_{11}x_1 + c_{12}x_2}{\beta_1 + d_{11}x_1 + d_{12}x_2}, \\ &\textit{where } x_2 \textit{ solves} \\ &\min \quad f_2(x_1, x_2) = \frac{\alpha_2 + c_{21}x_1 + c_{22}x_2}{\beta_2 + d_{21}x_1 + d_{22}x_2} \\ &s.t \quad (x_1, x_2) \in S, \end{aligned}$$

One of the main features of the BFP problem is that, even with the more complex objective functions they retain the most important property of the BLPP that is there exist an extreme point of S which solves the BFP problem

The Kth best algorithm has been proposed in Calvete and Gale (2004). It essentially asserts that the best of the extreme points of IR is an optimal solution to the problem. This property also applies to quasiconcave bilevel problems provided that the first level objective function is explicitly quasimonotonic.

Chapter 4 discusses about a special nonlinear bilevel programming problem (BLPP), where the upper level objective is a quadratic function and lower level objective is linear. By making use of duality theory for linear programming, this problem is transformed into an equivalent single-level programming. To solve the equivalent problem effectively, a genetic algorithm is discussed. Mathematically this problem can be stated as follows:

$$(BLP) \qquad \max_x F(x, y) = c_1^T x + d_1^T y + \frac{1}{2}(x^T . y^T) Q (x^T . y^T)$$

where y solves the following problem:

$$\begin{aligned} &\max_x \ f(x, y) = c_2^T x + d_2^T y \\ &s.t \quad Ax + By \le r, \\ &\qquad x, y \ge 0 \end{aligned}$$

CHAPTER -2

The bilevel linear/linear fractional programming problem

We consider a bilevel linear/linear fractional programming (BLLFP) problem, defined as:

$$
\begin{aligned}
(P1): \quad & \min_{x_1} f_1 = k^1 x_1 + k^2 x_2 \,, \\
& \textit{where}\; x_2 \; \textit{solves} \\
& \min_{x_2} f_2 = \frac{\alpha + c^{11} x_1 + c^{12} x_2}{\beta + c^{21} x_1 + c^{22} x_2} \\
& s.t \;\; (x_1, x_2) \in S,
\end{aligned}
$$

where $x_1 \in \Re^{n_1}$ and $x_2 \in \Re^{n_2}$ are the variables controlled by the first level and the second level decision-maker, respectively; $k^1, k^2, c^{11}, c^{12}, c^{21}$ and c^{22} are vectors of conformal dimension; α and β are scalars; and the common constraint region to both levels is a polyhedron, i.e.

$$S = \{(x_1, x_2) : A^1 x_1 + A^2 x_2 = b, x_1 \geq 0, x_2 \geq 0\},$$

where A^1 is an $m \times n_1$ matrix, A^2 is an $m \times n_2$ matrix and $b \in \Re^m$. Moreover, we assume that polyhedron S is nonempty and compact, matrix A^2 has full row rank and $m < n_2$ and $\beta + c^{21} x_1 + c^{22} x_2 > 0, \forall\, (x_1, x_2) \in S.$

2.1 Definitions and Assumptions

Let S_1 denotes the projection of S onto $\Re^{n_1}$, i.e. $S_1 = \{x_1 \in \Re^{n_1} : (x_1, x_2) \in S\}$ and by V_1, V_2 the sets of indices of first level and second level controlled variables, respectively. Notice that, for each $x_1 \in S_1$, the feasible reason of the second level decision maker

$S(x_1)=\{x_2 \in \Re^{n_2} : A^2 x_2 = b - A^1 x_1, x_2 \geq 0\}$ is also a nonempty compact polyhedron. Finally, the inducible region, or feasible region of the first-level decision-maker, will be denoted by

$$IR = \left\{ (x_1, x_2): x_1 \geq 0, x_2 = \arg\min \left\{ \frac{\alpha + c^{11} x_1 + c^{12} y_2}{\beta + c^{21} x_1 + c^{22} y_2} : A^1 x_1 + A^2 y_2 = b, y_2 \geq 0 \right\} \right\}.$$

Further it is assumed that for each value of the first-level variables $x_1 \in S_1$, there will be a unique solution to the second level problem.

2.2 Main theoretical results

It may be noted, if the objective functions of the first and second levels, f_1 and f_2, are quasiconcave and continuous functions and the common constraint region to both levels is a nonempty and compact polyhedron, the inducible region of the bilevel programming problem is comprised of the union of connected faces of the polyhedron and there is an extreme point of the polyhedron that solves the problem Calvete and Gale (1998). Hence we can conclude in the form of following remark.

Remark 1.

a) *The inducible region of P1 is formed by the union of connected faces of S.*
b) *An optimal solution to P1 occurs at an extreme point of polyhedron S.*

For each $x_1 \in S_1$ a feasible solution to P1, i.e. a point of IR, is obtained by solving the following linear fractional programming problem:

$$P(x_1): \quad \min \frac{\tilde{\alpha} + c^{12} x_2}{\tilde{\beta} + c^{22} x_2}$$
$$s.t. \quad x_2 \in S(x_1),$$

where $\tilde{\alpha} = \alpha + c^{11} x_1, \tilde{\beta} = \beta + c^{22} x_1$. Hence, x_2 which is an extreme point of the polyhedron $S(x_1)$ can be found which further solves the problem $P(x_1)$, and the point (x_1, x_2) so obtained belongs to the IR. Since a basis B of A^2 is associated to x_2, a basis of A^2 can be associated to each point of IR and so, we need only to consider these bases to find the points of inducible region IR. Therefore let us consider a basis B of A^2 and establish which conditions it should verify so as to be of interest.

To solve the problem $P(x_1)$, the parametric approach [Schaible (1995)] is considered. In this case, an optimal solution to the following linear parametric problem verifying $F(\lambda)=0$ is an optimal solution to $P(x_1)$:

$$LP(x_1): \qquad F(\lambda)=\min\left(\tilde{\alpha}+c^{12}x_2\right)-\lambda\left(\tilde{\beta}+c^{22}x_2\right)$$
$$s.t.\ x_2\in S(x_1).$$

Hence, in order to be able to obtain points of IR associated to a basis B, we have to test the following things:

- there exists $x_1\in S_1$ such that B is a feasible basis to $LP(x_1)$,
- that B verifies the optimality conditions of problem $LP(x_1)$ for some values of the parameter k and
- for one of these values $F(\lambda)=0$.

Regarding the optimality conditions of problem $LP(x_1)$, it suffices to check that the following reduced costs are greater than or equal to zero, regardless of the existence of x_1,

$$\left(c_j^{12}-\lambda c_j^{22}\right)-\left(c_B^{12}-\lambda c_B^{22}\right)B^{-1}A_j^2\geq 0 \quad \forall j\in V_2,$$

Where

c_j^{12} and c_j^{22} are the jth component of vectors c^{12} and c^{22}, respectively;

c_B^{12} and c_B^{22} are the m-row vectors of c^{12} and c^{22} associated to the basic variables of B; and

A_j^2 is the jth column of A^2.

Let $[\lambda^l,\lambda^u]$ be the interval of parameter λ computed by setting condition (1). If $\lambda^l=-\infty$ or $\lambda^u=\infty$ then the interval $[\lambda^l,\lambda^u]$ will be open in that extreme.

If there exists no value of λ such that condition 1 is verified by the basis B, then this basis is not of our interest as it become impossible to obtain a point of the inducible region corresponding to it. Therefore, in order to obtain points of IR, we must look for the existence of a $\lambda \in [\lambda^l, \lambda^u]$ such that $F(\lambda) = 0$ and a $x_1 \in S_1$ such that B is a feasible basis to $LP(x_1)$ .Thus, a subset of IR can be formed, which corresponds to each basis B from A^2 verifying condition (1):

$$\left\{ (x_1, x_2) : x_1 \geq 0, x_2 = \left(B^{-1}\left(b - A^1 x_1\right), 0\right), B^{-1}\left(b - A^1 x_1\right) \geq 0, \lambda^l \leq \frac{\alpha + c^{11} x_1 + c_B^{12} B^{-1}\left(b - A^1 x_1\right)}{\beta + c^{21} x_1 + c_B^{22} B^{-1}\left(b - A^1 x_1\right)} \leq \lambda^u \right\}$$

Therefore, if this set is nonempty, the best point of the inducible region corresponding to basis B is obtained by solving the following linear problem:

$$P(B): \quad \min \quad k^1 x_1 + k_B^2 x_{2B} \tag{2a}$$

$$s.t.$$

$$A^1 x_1 + B x_{2B} = b \tag{2b}$$

$$\left(\lambda^l c^{21} - c^{11}\right) x_1 + \left(\lambda^i c_B^{22} - c_B^{22}\right) x_{2B} \leq \alpha - \lambda^l \beta, \tag{2c}$$

$$\left(c^{11} - \lambda^u c^{21}\right) x_1 + \left(c_B^{22} - \lambda^u c_B^{22}\right) x_{2B} \leq \lambda^u \beta - \alpha, \tag{2d}$$

$$x_1, x_{2B} \geq 0, \tag{2e}$$

where x_{2B} stands for the variables of x_2 related to basis B and k_B^2 is the m-row vector of k^2 associated to these variables. Notice that, while basis B is being analyzed, the variables of the second level not associated to it remain equal to zero.

We also introduce the following relaxed problem, which does not take into account constraints (2c) and (2d):

$$P_R(B): \quad \min \quad k^1 x_1 + k_B^2 x_{2B}$$

$$s.t.$$

$$A^1 x_1 + B x_{2B} = b,$$

$$x_1, x_{2B} \geq 0,$$

Lemma 1. *If problem $P(B)$ is feasible then slack variables of constraints (2c) and (2d) are basic variables in the optimal solution.*

Proof. Let $(\tilde{x}_1, \tilde{x}_2)$ be an optimal solution to problem $P(B)$. To prove the lemma it suffices to note that $\tilde{x}_2$ is an optimal solution to the problem $LP(\tilde{x}_1)$ in the interval $[\lambda^i, \lambda^u]$, i.e. it verifies $B\tilde{x}_{2B} = b - A^1\tilde{x}_1$.

Remark 2. In most cases both constraints will be not binding. Indeed, assume for the time being that constraint (2c) is binding and let $(\tilde{x}_1, \tilde{x}_2)$ be an optimal solution to problem $P(B)$. Hence, $\tilde{x}_2$ is an optimal solution to problem $LP(\tilde{x}_1)$, B is an optimal basis in the interval $\left[\lambda^l, \lambda^u\right]$ and $F(\lambda)=0$ for $\lambda=\lambda^l$. Bearing in mind the properties of parametric linear programming problems, there is another optimal basis of $LP(\tilde{x}_1), \hat{B}$ associated to an interval $\left[\hat{\lambda}^l, \hat{\lambda}^u\right]$ such that $\hat{\lambda}^u = \lambda^l$, and $F\left(\hat{\lambda}^u\right)=0$. Therefore, for $\tilde{x}_1 \in S_1$ both bases have to represent the same extreme point of the second-level problem; otherwise it has alternative optimal solutions, which is a contradiction. The same argument applies if the binding constraint is Eq. (2d).

Lemma 2. *If problem $P(B)$ is feasible then reduced costs of problems $P(B)$ and $P_R(B)$ at the optimal solution are equal*.

Proof. Slack variables of constraints (2c) and (2d) are basic variables in the optimal solution of problem $P(B)$. Since slack variables have a zero cost coefficient, reduced cost coefficients are computed ignoring the elements of the corresponding rows in the optimal tableau. Hence, reduced costs in both problems, $P(B)$ and $P_R(B)$ are equal.

In fact, it is derived previously that an optimal solution to problem $P(B)$ is an optimal solution to problem $P_R(B)$. Moreover, if none of the optimal solutions to problem $P_R(B)$ verify constraints (2c) and (2d) then problem $P(B)$ is not feasible.

After examining a basis B verifying condition (1) to get the best point of IR associated to it, the next question to consider is what conditions must be satisfied for the bases that aim to provide a better point of IR. That is to say, we are looking for those vectors of A^2 that can improve the first-level objective function f_1. Notice that f_1 agrees with the objective function of $P(B)$ since, while basis B is being considered, the variables of the second level not associated to it are equal to zero. Let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ be the best point of IR associated to basis B, i.e. $\tilde{x}$ is an optimal solution of $P(B)$. Let T be the set of indices of variables associated to basis B and let R denote the set of indices of non basic variables corresponding to $\tilde{x}$ (notice that R contains some indices of first-level controlled variables, some indices of second-level controlled variables associated to B and all indices of second-level controlled variables not associated to B).

Lemma 3. *Any basis from A^2 capable of providing a point of IR better than $\tilde{x}$ must include at least one vector whose index belongs to the set*

$$C_1 = \{j \in V_2 - T : z_j < 0\},$$

where z_j denotes the jth reduced cost coefficient with respect to the optimal basis of $P(B)$.

Proof. Let $f_1(\tilde{x})$ denote the value of the first-level objective function at $\tilde{x}$. According to $\tilde{x}$ the matrix $[A^1, A^2]$ can be decomposed into $[Q, N]$ where Q is the $m \times m$ invertible matrix associated to basic variables of $\tilde{x}$. Hence for each $x \in IR$, we have:

$$f_1(x) = f_1(\tilde{x}) + \sum_{j \in \Re} z_j x_j ,$$

where $z_j = k_j - k_Q Q^{-1} A_j$, k_j is the jth cost coefficient in f_1, k_Q is the m-row vector of $k = [k^1, k^2]$ associated to basic variables of Q and A_j denotes the jth column vector of matrix $[A^1, A^2]$.

Therefore, in order to improve the first-level objective function we must consider variables with indices $j \in R$ such that $z_j < 0$. Since $\tilde{x}$ solves problem $P(B)$, $z_j \geq 0, \forall j \in V_1$ and $\forall j \in T$.

Further, this result also means that, if we have previously built sets $C_1^1, ..., C_1^i$, for a new basis $\hat{B}$ to be of interest (i.e. to be able to improve the current best point of the inducible region), it should include vectors of A^2 with at least one index from each of the sets $C_1^1,\dots\dots\dots,C_1^i$. Notice that if there is no basis with the above property then the current best point of IR is a global optimum to problem P1. Similarly, if in the course of the search $C_1=\phi$ then the current best point of IR is a global optimum to P1. We will denote by Ξ_1 the set of all sets C_1.

Consider now that problem $P_R(B)$ is not feasible, i.e. when solving this problem using the two-phase method, the optimal tableau of phase I contains at least one artificial variable with positive value. Then, when we construct new bases we should consider vectors that can contribute to avoiding this infeasibility. Hence we should consider those vectors of A^2 not in B whose associated variables can replace artificial variables by pivoting.

Lemma 4. *A necessary condition for the feasibility of any basis from A^2 is to include at least one vector whose index belongs to the set*

$$C_2=\{j\in V_2-T: z_j^I<0\}$$

where z_j^I denotes the jth reduced cost coefficient with respect to the objective function and the optimal basis of phase I of $P_R(B)$.

Proof. To reduce the infeasibility of the problem $P_R(B)$ we should drive out the artificial variables remaining in the optimal basis of phase I by placing variables $x_j,\ j\in V_1\bigcup V_2$, with $z_j^I>0$ into the basis. Since phase I for problem $P_R(B)$ has concluded, $z_j^I\geq 0, \forall j\in V_1$ and $j\in T$. Hence, only variables with indices in V_2-T have to be considered, according to the lemma.

As a result of lemma 4, if we have previously built sets $C_2^1,\dots\dots\dots C_2^i$, for a new basis $\hat{B}$ to be of interest, it should include vectors of A^2 with at least one index from each of the sets $C_2^1,\dots\dots\dots C_2^i$. It is worth mentioning that avoiding the infeasibility of problem $P_R(B)$ does not guarantee avoiding the infeasibility of problem $P(B)$. We will denote by Ξ_2 the set of all sets C_2.

Remark 3. In order not to return to a basis or a set of indices which are no longer of interest, we construct a set C_3 which includes all its indices. This is the case, for instance, when the selected basis B does not verify condition (1) or, if after solving problem $P_R(B)$ this problem is feasible but problem $P(B)$ is not. In this case, it is concluded that $F(\lambda)\neq 0, \forall \lambda \in [\lambda^l, \lambda^u]$, so this basis cannot be optimal for any fractional problem of the second level. Hence, since equations (2c) and (2d) will be different for a new basis, it does not make any sense to include vectors which avoid the infeasibility caused by these constraints in the new basis. Consequently, since the currently analyzed basis is no longer of interest, in order to avoid recovering it, we propose constructing set C_3 which contains the indices corresponding to all its vectors. Hence, if we have previously built sets $C_3^1,............,C_3^i$, for a new basis B to be of interest it should not include all vectors with indices in each of the sets $C_3^1,............,C_3^i$. We will denote by Ξ_3 the set of all sets C_3.

From the preceding lemmas and comments, we can conclude that after having previously analyzed bases $B_1,...........B_i$, sets Ξ_1, Ξ_2 and Ξ_3 will be constructed, whose elements are sets of indices. Then, in order to select a set of vectors of A^2 that can form a basis of interest, we will have to guarantee that the set of indices of these vectors includes at least one index from each element of Ξ_1, at least one index from each element of Ξ_2 and not all indices from each element of Ξ_3. Hence, to find this set of indices we suggest solving in w_j the following system:

$$P2: \sum_j w_j \delta_j \geq 1, \qquad \delta_j = \begin{cases} 1, & j \in C_1, \\ 0, & otherwise, \end{cases}$$

$$C_1 \in \Xi_1,$$

$$\sum_j w_j \delta_j \geq 1, \qquad \delta_j = \begin{cases} 1, & j \in C_2, \\ 0, & otherwise, \end{cases}$$

$$C_2 \in \Xi_2$$

$$\sum_j w_j \delta_j \leq \sum_j \delta_j - 1, \quad \delta_j = \begin{cases} 1, & j \in C_3, \\ 0, & otherwise, \end{cases}$$

$$C_3 \in \Xi_3$$

$$\sum_j w_j = m,$$

$$w_j \in \{0,1\}, \; j \in V_2$$

The required set will be constituted by vectors of A^2 with indices j such that the corresponding $w_j = 1$.

The following linear problem P_R provides a lower bound on the objective function of problem P1:

$$P_R: \qquad \min k^1 x_1 + k^2 x_2$$
$$s.t \;\; (x_1, x_2) \in S$$

Therefore, if in some step of the algorithm a point of the inducible region is found whose objective function value equals the optimum of P_R, then this point is a global optimum of P1.

2.3 Algorithm

The algorithm presented below is concerned with finding bases B, that can provide points of IR and examining them to obtain the best of these points associated to each of them. The algorithm begins with one of these B bases. In a typical iteration, it is determined if this basis can provide a better point of IR than the current best point. If this is so, this point becomes the new available best point of IR and a new basis which would be able to provide a better point of IR is considered. If it cannot, one of these new bases is directly constructed so as to continue the search for an optimum. This search is done in a way that prevents us from reconsidering any of the previously examined bases again which automatically leads us to possibly better solutions. So, since there are a finite number of these bases, the algorithm will find a global optimum to P1 in a finite number of steps. The following is a stepwise description of the algorithm.

Step 1.

- Solve problem P_R.
- If P_R is not feasible, neither is P1. Stop.
- Let $(\tilde{x}_1, \tilde{x}_2)$ be an optimal solution of P_R. Solve $P(\tilde{x}_1)$ using the parametric approach. Let $\tilde{x}_2$ be its optimal solution and B its optimal basis.
- If $\tilde{x}_2 = x_2$ stop, $(\tilde{x}_1, \tilde{x}_2)$ is a global optimum.
- $(\tilde{x}_1, \tilde{x}_2)$ is the current best point of IR. Set $\Xi_1 = \phi, \Xi_2 = \phi, \Xi_3 = \phi$.

Step 2.

- Given the basis B, solve $P_R(B)$ using the two-phase method.
- If $P_R(B)$ is feasible and any of its optimal solutions verify constraints (2c) and (2d), then go to Step 3.
- If $P_R(B)$ is feasible and none of the optimal solutions verify constraints (2c) and (2d), then go to Step 4.
- If $P_R(B)$ is not feasible, then go to Step 5

Step 3.

- Compare this optimal solution with the current best point of IR and update, if necessary, the latter.
- Compute C_1.
- If $C_1 = \phi$ stop, the current best point of IR is a global optimum to P1.
- Set $\Xi_1 = \Xi_1 \cup \{C_1\}$, then go to Step 6.

Step 4.

- Compute C_3. Set $\Xi_3 = \Xi_3 \cup \{C_3\}$, and then go to Step 6.

Step 5.

- Compute C_2. Set $\Xi_2 = \Xi_2 \cup C_2$.

Step 6.

- Solve P2.
- If P2 is not feasible, stop. The current best point of IR is a global optimum to P1.
- Let D be the constructed set of vectors. If $rank(D) = m$, then go to Step 7; otherwise go to Step 8.

Step 7.

- Set $B = D$. Compute $[\lambda^l, \lambda^u]$ by checking condition (1)
- If condition (1) is not verified, compute C_3, set $\Xi_3 = \Xi_3 \cup \{C_3\}$ and then go to step 6; otherwise go to step 2.

Step 8.

- Let $rank(D) = k$. Let $\hat{D}$ be the matrix of independent vectors of D. Set $D = \hat{D}$. Check the existence of a set G of $m-k$ vectors of A^2 so that $B = [D\ G]$ is a basis from A^2 verifying conditions given by sets Ξ_1, Ξ_2 and Ξ_3 and condition (1) for $\lambda \in [\lambda^l, \lambda^u]$.
- If it exists, then go to Step 1; otherwise compute C_3, set $\Xi_3 = \Xi_3 \cup \{C_3\}$ and go to Step 6.

Step 0 does the initialization. By solving $P(\tilde{x}_1)$ we get $(\hat{x}_1, \hat{x}_2) \in IR$ and a basis of A^2 associated to it. In order to solve $P(\tilde{x}_1)$ we use $LP(\tilde{x}_1)$ and determine the interval $[\lambda^l, \lambda^u]$ such that there exists λ belonging to it so that $F(\lambda) = 0$. Initial basis B is the basis associated to the optimal solution in this interval. The purpose of Steps 2-4 is either to provide the best point of IR associated to the B basis which is being analyzed, or to detect that there is none. Besides, whichever the case, conditions on the new bases to be considered are provided.

In Step 6 a linear program is solved so as to compute a set of vectors from A^2 verifying the current necessary conditions for obtaining a new basis which improves the current best point of IR. These conditions are given by sets Ξ_1, Ξ_2 and Ξ_3. Step 7 controls whether the obtained basis verifies condition (1) or not. In the first case the next iteration begins; otherwise we compute C_3 to avoid returning to this basis again in the future. In Step 7 we are given a submatrix D of A^2, with $rank(D) = k < m$. From now on, to simplify notations, we will denote D the m k-matrix of its independent vectors. In order to try to determine if there exists the set G, we suggest solving in u and λ the following linear system:

$$P3: \quad uD + \lambda c_D^{22} = c_D^{12}, \qquad (2a)$$

$$uH + \lambda c_H^{22} \le c_H^{12}, \qquad (2b)$$

where H is such that $A^2 = [D\ H]$, u is an m-row vector and $c_D^{12}, c_D^{22}, c_H^{12}, c_H^{22}$ are the row vector of c^{12} and c^{22} associated to columns of D and H. If problem P3 is feasible, we obtain a value $\lambda = \lambda_0$ such that the basis of A^2, provided by D and vectors of H for which the corresponding constraint (3b) id binding, will verify condition (1) at least for $\lambda = \lambda_0$, so it is a basis to be considered in order to obtain a better point of the inducible region Nevertheless, it is worth mentioning that, when solving problem P3, all vectors of A^2 are considered, hence it is possible to get a basis $B = [D\ G]$ including vectors which should be excluded according to conditions given by sets Ξ_1, Ξ_2 and Ξ_3. So it is necessary to check this possibility before going to Step 1. If P3 is not feasible then D cannot be completed to give a basis of interest, therefore it should not be included in any basis to be considered in future iterations of the algorithm, hence we compute set C_3 containing the indices corresponding to all vectors of D.

2.4 Illustrative example

Consider the following example:

$$P1: \quad \min_{(x_1,x_2)} f_1 = -8x_1 - 4x_2 + 4x_3 - 40x_4 - 4x_5,$$

$$where\,(x_3,......,x_8)\,solves$$

$$\min_{(x_3,...,x_8)} f_2 = \frac{1 + x_1 + x_2 + 2x_3 - x_4 + x_5}{6 + 2x_1 + x_3 + x_4 - 3x_5}$$

$$s.t.$$

$$-x_3 + x_4 + x_5 + x_6 = 1,$$

$$2x_1 - x_3 + 2x_4 - \frac{1}{2}x_5 + x_7 = 1,$$

$$2x_2 + 2x_3 - x_4 - \frac{1}{2}x_5 + x_8 = 1,$$

$$x_i \geq 0, i = 1,.....,8.$$

The Initialization Step and the First Iteration are shown in detail.

Step 1: The optimal solution to problem P_R is $(\tilde{x}_1, x_2) = (0,0,3/2,3/2,1,0,0,0)$. The optimal value of the objective function is -58, which constitutes a lower bound on the objective function of P1.

Optimal table for f_1

Table - 1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	sol
$z_j - c_j$	-28	-32	0	0	0	-22	-9	-34	-58
x_5	-2/3	2/3	0	0	1	1	-1/3	1/3	1
x_4	1	1	0	1	0	1/2	1/2	1/2	3/2
x_3	1/3	5/3	1	0	0	1/2	1/6	5/6	3/2

By fixing $\tilde{x}_1 = (0,0)$ and solving the fractional problem of the second level we get the optimal solution $\tilde{x}_2 = (0,1/2,0,1/2,0,3/2)$.

Table - 2

Optimal table for f_2

d_B	c_B	X_B	x_8	x_7	x_6	x_5	x_4	x_3	x_2	x_1
0	0	x_6=1/2	0	-1/2	1	5/4	0	-1/2	0	-1
1	-1	x_4=1/2	0	1/2	0	-1/4	1	-1/2	0	1
0	0	x_8=3/2	1	1/2	0	3/4	0	3/2	2	1
$z^2 = 13/2$	$z^1 = 1/2$	$f_2 = 1/13$								
		$z_j^1 - c_j$	0	-1/2	0	-3/4	0	-3/2	-1	-2
		$z_j^2 - d_j$	0	1/2	0	11/4	0	-3/2	0	-1
		Δ_j	0	7/2	0	89/8	0	9	13/2	11/4

Since $\tilde{x}_2 \neq x_2$ next we go to Step 2. The current best point of IR is $(0,0,0,1/2,0,1/2,0,3/2)$, $f_1 = -20$. Basis B_1 given by vectors with indices 4, 6 and 8 will be the first analysed basis. $[\lambda^l, \lambda^u] = [-3/11, 1]$.

First iteration: The corresponding $P_R(B_1)$ problem is:

$$\begin{aligned}
&\min \ f_1 = -8x_1 - 4x_2 - 40x_4, \\
&s.t. \\
&\quad x_4 + x_6 = 1, \\
&\quad 2x_1 + 2x_4 = 1, \\
&\quad 2x_2 - x_4 + x_8 = 1, \\
&x_1, x_2, x_4, x_6, x_8 \geq 0.
\end{aligned}$$

Whose optimal table is:

Iteration-1

Table - 3

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Sol.
$z_j - c_j$	34	0	-13	0	-31/2	0	21	2	-23
x_5	-1	0	-1/2	0	5/4	1	-1/2	0	½
x_4	1	0	-1/2	1	-1/4	0	½	0	½
x_2	1/2	1	¾	0	-3/8	0	¼	1/2	¾

Hence the optimal solution of this problem is $(0,3/4,0,1/2,0,1/2,0,0)$, $f_1 = -23$. Since, constraints (2c) and (2d)

$$-17x_1 - 11x_2 + 8x_4 \le 29$$
$$-x_1 + x_2 - 2x_4 \le 5.$$

are verified for this point and $-23 \le -20$, then we update the current best point of IR. The optimal tableau of $P_R(B_1)$ is shown in Table 3. Note that variables of the second level not associated to B_1, and therefore not to be considered when solving problem $P_R(B_1)$, are also included in order to construct the corresponding set C_1. Reduced cost of variables x_3 and x_5 are negative, so $C_1 = \{3,5\}, \Xi_1 = \{3,5\}$ and we go to Step 6. The corresponding P2 problem is

$$w_3 + w_5 \ge 1,$$
$$w_3 + w_4 + w_5 + w_6 + w_7 + w_8 = 3,$$
$$w \in \{0,1\}, i = 3,......,8.$$

There are several feasible solutions to this problem. We choose $w_3 = w_4 = w_5 = 1,$. Hence, D is the matrix formed by vectors associated to variables x_3, x_4 and x_5. Since $rank(D) = 3$ we check condition (1). There is no value of λ verifying this condition; hence $C_3 = \{3,4,5\}$, $\Xi_3 = \{3,4,5\}$ and we go to Step 5.

The corresponding P2 problem is the previous one with the added constraint $w_3 + w_4 + w_5 \le 2$. Choosing the feasible solution $w_3 = w_5 = w_6 = 1$, D is the matrix formed by vectors associated to variables x_3, x_5 and x_6. Note that $rank(D)=3$ and condition (1) is verified for $\lambda \in [0, \infty)$. Therefore, basis B_2 = D and a new iteration begin.

Iteration-2

Table - 4

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Sol.
$z_j - c_j$	0	4	0	-28	0	8	4	4	-16
x_5	0	4/3	0	2/3	1	4/3	0	2/3	2
x_1	1	1	0	1	0	1/2	½	1/2	3/2
x_3	0	4/3	1	-1/3	0	1/3	0	2/3	1

Constraint (2c) and (2d) are verified then we update the current best point of IR. The table 4 represents the optimal table of $P_R(B_2)$.

Iteration-3

Table - 5

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Sol.
$z_j - c_j$	128/3	52/3	0	-33	0	0	-23/3	44/3	-36
x_6	-2/3	2/3	0	2/3	0	1	1	4/3	7/8
x_5	4/3	2/3	0	1	1	0	½	2/3	7/8
x_3	2/3	4/3	1	-1/3	0	0	0	2/3	1

Table 5 gives the optimal table of $P_R(B_3)$. The optimal solution of $P_R(B_3)$ does not satisfy the conditions (2c) and (2d) hence, problem $P_R(B_3)$ is feasible but $P(B_3)$ is not.

Iteration-4

Table - 6

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Sol.
$z_j - c_j$	108/5	0	-96/5	0	0	62/5	74/5	2	-29.2
x_5	-4/5	0	-2/5	0	1	4/5	-2/5	0	2/5
x_4	4/5	0	-3/5	1	0	1/5	2/5	0	3/5
x_2	1/5	1	3/5	0	0	3/10	1/10	½	9/10

Table 6 gives the optimal table of $P_R(B_4)$. The optimal solution of $P_R(B_4)$ satisfy the conditions (2c) and (2d) hence we update the current best point of IR.

Iteration-5

Table - 7

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	R_1	R_2	R_3	Sol.
$z_j - c_J$	1	0	1/2	0	-5/4	-1	1/2	0	0	-1/2	-1	
R_1	-1	0	-1/2	0	5/4	1	-1/2	0	1	-1/2	0	1/2
x_4	1	0	-1/2	1	-1/4	0	1/2	0	0	½	0	1/2
x_2	1/2	1	3/4	0	-3/8	0	1/4	1/2	0	½	1/2	3/4

In the fifth iteration the corresponding $P_R(B_5)$ problem is not feasible, as artificial variables still remain in the optimal table of phase I as shown in the table. Also for this iteration, the P2 problem is not feasible, hence the current best point of the inducible region, $(x_1, x_2) = (0, 9/10, 0, 3/5, 2/5, 0, 0, 0)$ is a global optimum to problem P1. Its objective function value is $f_1 = -29.2$

Table 8 below shows the summary of the algorithm

Table - 8

	C_1	C_2	C_3	**D is formed by vectors with indices**	**Interval of λ**
Iteration 1	{3 ,5}			{3, 4, 5}	ϕ
			{3, 4, 5}	{3, 5, 6}	$[0,\infty)$
Iteration 2	{4}			{3, 4, 6}	$[1,\infty)$
Iteration 3			{3, 4, 6}	{4, 5, 6}	$[-3/11,\infty)$
Iteration 4	{3}			{3, 4, 7}	$[1/2,\infty)$
Iteration 5		{5, 6}		ϕ	

3.5 Summary and conclusions

In this chapter the linear/linear fractional bilevel problem has been discussed. In the discussed problem it has been considered that assumes that objective functions of both levels are linear and linear fractional respectively, and the feasible region is a polyhedron. For this problem we have shown that it is possible to extend the result concerning the linear bilevel problem which assures us that there is an extreme point of the feasible region that solves the problem.

The relationship between points of the inducible region and bases of the technological coefficient submatrix associated to variables of the second level, an algorithm is proposed which finds a global optimum to the BLLFP problem in a finite number of steps. This is the first algorithm proposed for solving this particular kind of problems. Furthermore, it is worth mentioning that one of the advantages of the procedure is that only linear problems need to be solved hence, the simplex algorithm or interior point methods for linear problems can be used.

CHAPTER – 3

Linear fractional bilevel programs

Bilevel programming involves two optimization problems where the constraint region of the first level problem is implicitly determined by another optimization problem. It has been applied to decentralized planning problems involving a decision process with a hierarchical structure. In terms of modeling, bilevel problems are programs which have a subset of their variables constrained to be an optimal solution of another problem parameterized by the remaining variables. The second level decision maker optimizes his objective function under the given parameters from the first level decision maker. This one, in return, with complete information on the possible reactions of the second level decision maker, selects the parameters so as to optimize his own objective function. Bilevel problems can be formulated as

$$\min_{(x_1,x_2)\in S} f_1(x_1,x_2) \, where \, x_2 \in \arg\min_{v\in S(x_1)} f_2(x_1,v) \tag{1}$$

where $x_1 \in \Re^{n_1}$ and $x_2 \in \Re^{n_2}$ are the variables controlled by the first level and the second level decision maker, respectively; $f_1, f_2 : \Re^n \to \Re, n = n_1 + n_2 ; S \subset \Re^n$ defines the common constraint region and $S(x_1) = \{x_2 \in \Re^{n_2} : (x_1, x_2) \in S\}$.

Let S_1 be the projection of S onto $\Re^{n_1}$. For each $x_1 \in S_1$, the second level decision maker solves problem (2)

$$\begin{aligned} &\min \quad f_2(x_1,x_2) \\ &s.t \quad x_2 \in S(x_1). \end{aligned} \tag{2}$$

The feasible region of the first level decision maker, called inducible region IR, is implicitly defined by the second level optimization problem

$$\text{IR} = \{(x_1, x_2^*) : x_1 \in S_1, x_2^* \in M(x_1)\},$$

where $M(x_1)$ denotes the set of optimal solutions to (2). We assume that S is not empty and that for all decisions taken by the first level decision maker, the second level decision maker has some room to respond, i.e. $M(x_1) \neq \phi$.

The bilevel programming problem (1) is a nonconvex optimization problem that has received increasing attention in the literature Bard (1998), Dempe (2002) and Shimizu et al. (1997). One of its main features is that, unlike general mathematical problems, the bilevel problem may not possess a solution even when f_1 and f_2 are continuous and S is compact. In particular $M(x_1)$, difficulties may arise when $M(x_1)$ is not single-valued for all permissible x_1 Bard (1998), Bialas and Karwan (1984), Dempe (2002) and Shimizu et al. (1997). Different approaches have been proposed in the literature to make sure that the bilevel problem is well posed. The most common one is to assume that, for each value of the first level variables x_1, there is a unique solution to the second level problem, i.e., the set $M(x_1)$ is a singleton for all $x_1 \in S_1$.

Other approaches focus on the way of selecting $x_2^* \in M(x_1)$, in order to evaluate $f_1(x_1, x_2)$, when $M(x_1)$,is not a singleton. Among the rules that have been proposed by Dempe (2002), it is worth mentioning the optimistic or weak approach and the pessimistic or strong approach. The first one assumes that the first level decision maker is able to influence the second level decision maker so that the latter always selects the variables x_2 to provide the best value of f_1. Thus, the first level decision maker has to solve the problem $\min_{x_1 \in S_1} \phi_0 \{x_1\}$ where $\phi_0(x_1) = \min_{x_2 \in M(x_1)} f(x_1, x_2)$. In the pessimistic approach, the first level decision maker behaves as though the second level decision maker always selected the optimal decision which gives the worst value of f_1. This leads to the problem $\min_{x_1 \in S_1} \phi_p \{x_1\}$ where $\phi_p(x_1) = \max_{x_2 \in M(x_1)} f(x_1, x_2)$. Finally, other approaches consider a local reduction of the problem by Falk and Liu (1995) and Stein and Still (2002).

In this chapter, a linear fractional bilevel programming (LFBP) problem is considered in which both objective functions are linear fractional and S is a polyhedron, which is assumed to be nonempty and bounded. Using the common notation in bilevel programming, the LFBP problem can be written as follows:

$$\begin{aligned} \min \quad & f_1(x_1, x_2) = \frac{\alpha_1 + c_{11}x_1 + c_{12}x_2}{\beta_1 + d_{11}x_1 + d_{12}x_2}, \\ & \textit{where } x_2 \textit{ solves} \\ \min \quad & f_2(x_1, x_2) = \frac{\alpha_2 + c_{21}x_1 + c_{22}x_2}{\beta_2 + d_{21}x_1 + d_{22}x_2} \\ s.t \quad & (x_1, x_2) \in S, \end{aligned} \tag{3}$$

where, for $i, j \in \{1, 2\}$, c_{ij}, d_{ij} are vectors of conformable dimensions, and α_i, β_i are scalars. We assume that $\beta_i + d_{i1}x_1 + d_{i2}x_2 > 0,\ i = 1,2, \forall\ (x_1, x_2) \in S$. Moreover, it is also assumed that $M(x_1)$ is a singleton for all $x_1 \in S_1$.

3.1 Theoretical properties

Before proving the main result on the optimal solution of problem (3) we list some preliminary definitions and results.

Definition 3.1.1 [Danao (1992)]. Let f be a real-valued function defined on a convex subset D of $\Re^n$,

- f is quasiconcave on D if and only if $d^1, d^2 \in D, \lambda \in [0, 1]$, and $f(d^1) \le f(d^2)$ imply $f(d^1) \le f[(1-\lambda)d^1 + \lambda d^2]$.

  The function f is quasiconvex if and only if $-f$ is quasiconcave.

- f is strongly quasiconcave on D iff $d^1, d^2 \in D, d^1 \ne d^2, \lambda \in (0, 1)$, and $f(d^1) < f(d^2)$ imply $f(d^1) < f[(1-\lambda)d^1 + \lambda d^2]$.

  The function f is strongly quasiconvex iff $-f$ is strongly quasiconcave.
- f is explicitly quasiconcave on D iff it is quasiconcave and strongly quasiconcave on D. The function f is explicitly quasiconvex iff $-f$ is explicitly quasiconcave.
- f is explicitly quasimonotonic on D iff it is explicitly quasiconcave and explicitly quasiconvex on D.

Note that the linear fractional functions

$$f_i(x_1, x_2) = \frac{\alpha_i + c_{i1}x_1 + c_{i2}x_2}{\beta_i + d_{i1}x_1\, d_{i2}x_2}, \qquad i = 1,2$$

are explicitly quasimonotonic on S if $\beta_i + d_{i1}x_1 + d_{i2} \neq 0$ in S ([Martos (1975) Theorem 3.53]).

On the other hand, since f_1 and f_2 are quasiconcave and S is a nonempty and compact polyhedron, the LFBP problem is a particular case of the quasiconcave bilevel problem [Calvete and Gale (1998)]. Hence:

1. The feasible region of the LFBP consists of the union of connected faces of the polyhedron *S*. As a consequence, in general IR is a nonconvex set.
2. There exists an extreme point of IR, thus an extreme point of the polyhedron *S*, which is an optimal solution of the LFBP problem

Definition 3.1.2 [Liu and Hart (1994)]. A point $(x_1, x_2) \in IR$ is a boundary feasible extreme point if there exists an edge E of S such that (x_1, x_2) is an extreme point of E, and the other extreme point of E is not an element of IR.

Consider the relaxed problem

$$(R.P) \qquad \min \quad f_1(x_1, x_2) = \frac{\alpha_1 + c_{11}x_1 + c_{12}x_2}{\beta_1 + d_{11}x_1 + d_{12}x_2}, \qquad (4)$$
$$s.t. \quad (x_1, x_2) \in S$$

Note that f_1 is a quasiconcave function and S is a nonempty and compact polyhedron, so that there is an extreme point of S which solves RP. If this extreme point is also a point of IR, then it is an optimal solution of the LFBP problem.

In general, an optimal feasible solution of relaxed problem may not be an optimal solution of problem P. In the next theorem it has been justified that an optimal solution is obtains at a boundary feasible extreme point.

Theorem 3.1.3. *If there exists an extreme point of S not in IR which is an optimal solution of the relaxed problem RP, then there exists a boundary feasible extreme point that solves the (LFBP) problem.*

Proof. Let $(\tilde{x}_1, \tilde{x}_2)$ be an optimal extreme point of S such that $(\tilde{x}_1, \tilde{x}_2) \in IR$. If it is a boundary feasible extreme point the proof is complete. If this is not so, every extreme point adjacent to $(\tilde{x}_1, \tilde{x}_2)$ is in IR and

$$f_1(\tilde{x}_1, \tilde{x}_2) \leq f_1(x_1, x_2) \tag{5}$$

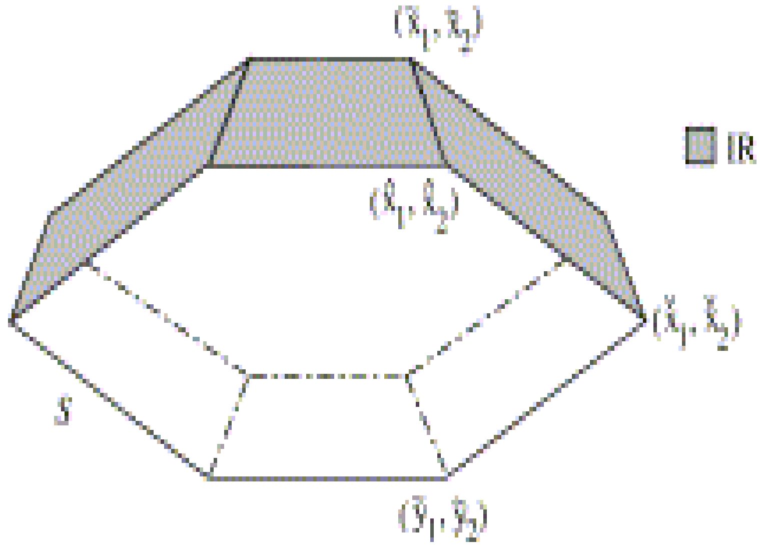
for all adjacent extreme point (x_1, x_2) of $(\tilde{x}_1, \tilde{x}_2)$. Firstly, we prove that there must be an extreme point $(\hat{x}_1, \hat{x}_2)$ adjacent to $(\tilde{x}_1, \tilde{x}_2)$ such that

$$f_1(\hat{x}_1, \hat{x}_2) = f_1(\tilde{x}_1, \tilde{x}_2) \tag{6}$$

For this purpose let us consider the relaxed problem (RP). Taking into account (5), $(\tilde{x}_1, \tilde{x}_2)$ is a local extreme-minimum point of f_1 in *S*. Since f_1 is quasiconcave and explicitly quasiconvex on *S*, we can conclude that $(\tilde{x}_1, \tilde{x}_2)$ is a global minimum of the relaxed problem (RP) ([Martos (1975), Theorem 5.13]), i.e.

$$f_1(\tilde{x}_1, \tilde{x}_2) \leq f_1(\hat{x}_1, \hat{x}_2) \quad \forall (x_1, x_2) \in S \tag{7}$$

By hypothesis, there exists an extreme point $(\tilde{y}_1, \tilde{y}_2) \in S$ not in IR which is an optimal solution of problem (RP). Thus $f_1(\tilde{x}_1, \tilde{x}_2) = f_1(\tilde{y}_1, \tilde{y}_2)$ Notice that $(\tilde{y}_1, \tilde{y}_2)$ cannot be adjacent to $(\tilde{x}_1, \tilde{x}_2)$ as $(\tilde{x}_1, \tilde{x}_2)$ is not a boundary feasible extreme point. Since f_1 is continuous, quasiconvex and explicitly quasiconcave on *S*, the optimum set of problem (RP) is the convex hull of some extreme points of *S* ([Martos (1975), Theorem 5.21), thus itself a polyhedron. Then, there exists an edge path in the optimum set of problem (RP) from $(\tilde{x}_1, \tilde{x}_2)$ to $(\tilde{y}_1, \tilde{y}_2)$. Hence, there must be an extreme point $(\hat{x}_1, \hat{x}_2)$ adjacent to $(\tilde{x}_1, \tilde{x}_2)$ pertaining to the optimum set of problem (RP), thus verifying (6) .If $(\hat{x}_1, \hat{x}_2)$ is a boundary feasible extreme point the proof is complete. If this is not so, we consider the extreme point $(\hat{x}_1, \hat{x}_2)$ instead of $(\tilde{x}_1, \tilde{x}_2)$ and repeat the same developments. Thus, we get an extreme point $(\breve{x}_1, \breve{x}_2)$ adjacent to $(\hat{x}_1, \hat{x}_2)$ verifying (6). This process is explained in fig.-1. If this new point is a boundary feasible extreme point the proof is complete. Otherwise, by repeating the process, because of the number of extreme points of *S* is finite, eventually a boundary feasible extreme point will be reached in a finite number of steps which solves the LFBP problem.



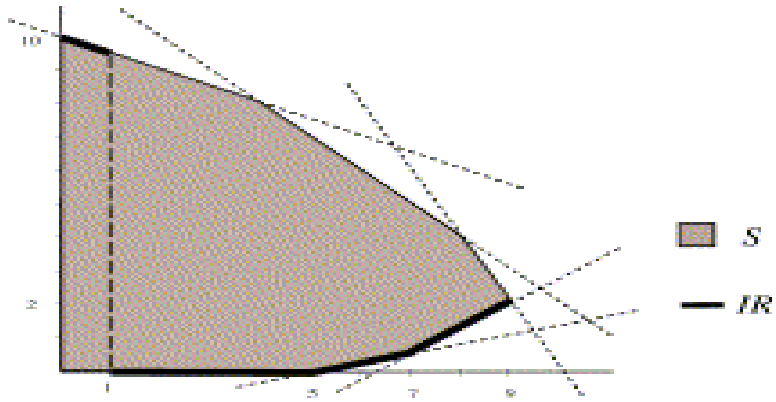
Remark 4. In the above discussion it has been assumed that $M(x_1)$ is a singleton set. If $M(x_1)$ not a singleton then feasible region will no longer be a union of connected faces as explained in the following example. Moreover, the first level decision maker could not reach his optimal decision without 'forcing' the decision of the second level decision maker.

Example: Consider the following LFBP problem

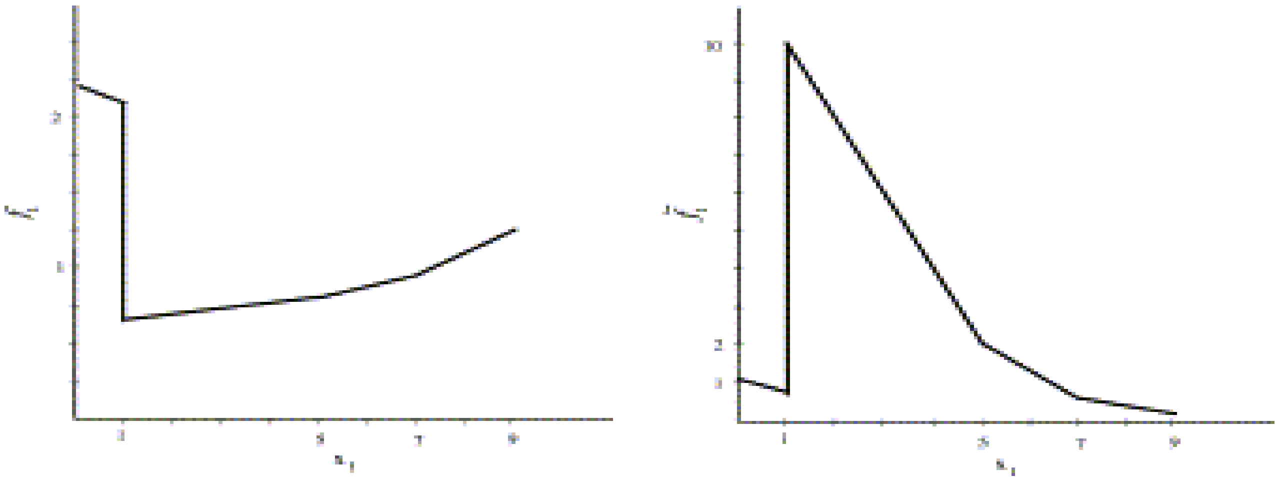
$$\begin{aligned}
&\min \quad \frac{x_1 + 3x_2 + 3}{x_1 + x_2 + 5}, \\
&\textit{where } x_2 \textit{ solves} \\
&\min \quad \frac{-x_1 + 2x_2 + 7}{x_1 + x_2 + 2} \\
&s.t \quad (x_1, x_2) \in S
\end{aligned} \tag{8}$$

$$\text{Where } S = \left\{ \begin{aligned}
&(x_1, x_2) \in \Re^2 : \\
&x_1 + 2x_2 \le 20; \\
&x_1 + x_2 \le 12; \\
&2x_1 + x_2 \le 20; \\
&3x_1 - 4x_2 \le 19; \\
&x_1 - 4x_2 \le 5; \\
&x_1, x_2 \ge 0
\end{aligned} \right\}$$

The common constraint region and the feasible region IR of the example are shown in Fig-2. Notice that for $x_1 = 1$ the second level problem has multiple optima, $M(1) = \left[0, \frac{19}{2}\right]$. This fact makes the inducible region not to consist of the union of faces of the polyhedron *S*.



Moreover, the optimization problem of the first level decision maker is not well defined. For completely evaluating $f_1(1, x_2)$ it is necessary to give a rule for selecting $x_2 \in M(1)$. The mapping of f_1 is plotted in Fig -3. Notice that the best value for the first objective function is $f_1 = \frac{2}{3}$ obtained when $x_1 = 1$ and $x_2 = 0$. However the first level decision maker cannot force this value because the second level decision maker is indifferent to each x_2 in the interval $\left[0, \frac{19}{2}\right]$. If the optimistic approach is taken the optimal solution to example (8) is therefore $x_1 = 1$ and $x_2 = 0$. Notice that this point is not an extreme point of the polyhedron *S*. However, if the pessimistic approach is used, then an optimal solution to the example does not exist.



On the other hand, if the first level objective function was

$$\tilde{f}_1 = \frac{-2x_1 - x_2 + 22}{x_1 + x_2 + 1},$$

then the first level decision maker could reach his minimum $\tilde{f}_1 = \frac{1}{6}$, obtained when $x_1 = 9$, since the second level problem given $x_1 = 9$ has a unique optimal solution $x_2 = 2$. Notice that in this case the optimal solution is a boundary feasible extreme point. The mapping of $\tilde{f}_1$ is also plotted in Fig - 3.

Remark 5. We can transform a (LFP) problem as a (LP) because Charnes and Cooper (C&C) transformation [Charnes and Cooper (1962)] allows us this reformulation. Hence, we wonder about the applicability of the C&C transformation to reformulate in a similar way the LFBP problem as a linear bilevel programming problem. Having this motivation in mind, consider that $S = \{(x_1, x_2): A_1x_1 + A_2x_2 \le b, x_1 \ge 0, x_2 \ge 0\}$ where b is a vector and A_1, A_2 are matrices of conformable dimensions.

For fixed $x_1 \in S_1$, let $z = \frac{1}{\beta_1 + d_{11}x_1 + d_{12}x_2}$ and $y_2 = zx_2$. Then the second level decision maker has to solve the following LP problem:

$$
\begin{aligned}
\min \quad & (\alpha_2 + c_{21}x_1)z + c_{22}y_2 \\
s.t. \quad & A_2 y_2 - (b - A_1 x_1)z \le 0, \\
& d_{22}y_2 + (\beta_2 + d_{21}x_1)z = 1, \\
& y_2 \ge 0, z \ge 0,
\end{aligned}
$$

By embedding this problem in the LFBP problem (3), we get:

$$
\begin{aligned}
\min \quad & \frac{\alpha_1 z + c_{11}x_1 z + c_{12}y_2}{\beta_1 z + d_{11}x_1 z + d_{12}y_2}, \\
& where\ y_2\ , z\ solve \\
\min \quad & (\alpha_2 + c_{21}x_1)z + c_{22}y_2 \\
s.t \quad & \\
& A_2 y_2 - (b - A_1 x_1)z \le 0, \\
& d_{22}y_2 + (\beta_2 + d_{21}x_1)z = 1, \\
& x_1 \ge 0\, , y_2 \ge 0, z \ge 0,
\end{aligned}
\tag{9}
$$

Notice that the first level objective function contains the nonlinear term $x_1 z$. In this case it definitely makes no sense to consider $y_1 = x_1 z$ as a single variable because x_1 is a variable controlled by the first level decision maker while z is controlled by the second level one. Since the reformulated problem is apparently more complicated to be solved than the original one, it does not seem very tempting to directly use the C&C transformation in the process of solving the LFBP problem. In the next section we will see that it can be used to solve LFP problems arising in successive iterations of the *K*-best algorithm.

3.2 The Kth – best algorithm

Since there is an extreme point of *S*, which solves the LFBP problem. Although an examination of all extreme points of the polyhedron *S* constitutes an algorithm that will find the solution of the LFBP problem in a finite number of steps, but this approach is not very efficient since the number of extreme points of S is, in general, very large. But in light of Theorem 3, we can propose the *K*th-best algorithm, a more successful enumeration scheme, for solving the LFBP problem. This algorithm was first proposed by Bialas and Karwan (1984) for solving the linear bilevel programming problem.

The Kth Best Algorithm

Step 1.

Let $\left(x_1^{[1]}, x_2^{[1]}\right)$ be an optimal solution to problem (4).

Let $W = \left\{\left(x_1^{[1]}, x_2^{[1]}\right)\right\}$ and $T = \phi$.

Set $i = 1$.

Go to step 2.

Step 2.

Set $x_1 = x_1^{[i]}$ and solve the second level problem.

Let x_2^* be its optimal solution.

If $x_2^* = x_2^{[i]}$, stop; $\left(x_1^{[i]}, x_2^{[i]}\right)$ is a global optimum to the LFBP problem.

Otherwise go to step 3.

Step 3.

Let $W^{[i]}$ denotes the set of adjacent extreme points of $\left(x_1^{[i]}, x_2^{[i]}\right)$.

Let $T = T \cup \left\{\left(x_1^{[i]}, x_2^{[i]}\right)\right\}$ and $W = \left(W \cup W^{[i]}\right) \setminus T$.

Go to step 4.

Step 4.

Set $i = i + 1$ and choose $\left(x_1^{[i]}, x_2^{[i]}\right)$ so that

$f_1\left(x_1^{[i]}, x_2^{[i]}\right) = \min\left\{f_1\left(x_1, x_2\right) : \left(x_1, x_2\right) \in W\right\}$.

Go to step 2.

According to the algorithm, which is described above, we find an optimal solution $\left(x_1^{[1]}, x_2^{[1]}\right)$ to the relaxed problem (RP). If this is a point of IR, then it is an optimal solution of the LFBP problem. If this is not so, the set of its adjacent extreme points $W^{[1]}$ is considered. Then, the extreme point in $W = W^{[1]}$ which provides the best value of f_1 is selected to test if it is a point of IR. If it is, the algorithm finishes. If this is not a point of IR then this point is eliminated from W and its adjacent extreme points with a worst value of f_1 are added to W. The algorithm continues by selecting the best extreme point in W with respect to f_1 and repeating the process.
In the following results the correctness of the algorithm is proved.

Let $\left(x_1^{[1]}, x_2^{[1]}\right)$, $\left(x_1^{[2]}, x_2^{[2]}\right)$,.….. $\left(x_1^{[m]}, x_2^{[m]}\right)$, denote the m ordered extreme point solutions to the relaxed problem (R.P), such that:

$$f_1\left(x_1^{[i]}, x_2^{[i]}\right) \le f_1\left(x_1^{[i+1]}, x_2^{[i+1]}\right), \ i = 1,......., m-1.$$

It will be justified that the (i+1) st best extreme point $\left(x_1^{[i+1]}, x_2^{[i+1]}\right)$ of S is adjacent to $\left(x_1^{[1]}, x_2^{[1]}\right)$, or $\left(x_1^{[2]}, x_2^{[2]}\right)$… or $\left(x_1^{[i]}, x_2^{[i]}\right)$. Hence, the algorithm successively computes the ordered sequence of extreme points, and it is obvious that $\left(x_1^{[k]}, x_2^{[k]}\right)$ is a global optimum to the LFBP problem if $k = \min_{i \in \{1,..m\}} \left\{i : \left(x_1^i, x_2^i\right) \in \Re\right\}$.

Theorem 3.2.1. *Let* $\left(\tilde{x}_1, \tilde{x}_2\right)$ *be an extreme point of S. There exists an edge path in S from* $\left(\tilde{x}_1, \tilde{x}_2\right)$ *to* $\left(x_1^{[1]}, x_1^{[1]}\right)$ *such that the value of is non increasing along it.*

Proof. Assume for the time being that every extreme point $\left(x_1, x_2\right)$ adjacent to $\left(\tilde{x}_1, \tilde{x}_2\right)$ verifies $f_1\left(x_1, x_2\right) \ge f_1\left(\tilde{x}_1, \tilde{x}_2\right)$.

Hence $\left(\tilde{x}_1, \tilde{x}_2\right)$ is an extreme point of S giving local minimum value of f_1. Since f_1 is quasiconcave and explicitly quasiconvex on S, then $\left(\tilde{x}_1, \tilde{x}_2\right)$ is a global minimum of the relaxed problem (RP), i.e.

$$f_1\left(\tilde{x}_1, \tilde{x}_2\right) = f_1\left(x_1^{[1]}, x_2^{[1]}\right),$$

Therefore $\left(x_1^{[1]}, x_2^{[1]}\right)$ and $\left(\tilde{x}_1, \tilde{x}_2\right)$ are extreme points of the optimum set of (RP). Since f_1 is continuous, quasiconvex and explicitly quasiconcave on S, this set is the convex hull of some extreme points of S. Then there exists an edge path in this polyhedron from $\left(\tilde{x}_1, \tilde{x}_2\right)$ to $\left(x_1^{[1]}, x_2^{[1]}\right)$. Since all the points of the edge path are from S and have the same value of f_1, this is the edge path we are looking for. Now consider that there exists at least an extreme point $\left(\hat{x}_1, \hat{x}_2\right)$ adjacent to $\left(\tilde{x}_1, \tilde{x}_2\right)$ such that

$$f_1\left(\hat{x}_1, \hat{x}_2\right) < f_1\left(\tilde{x}_1, \tilde{x}_2\right).$$

Let us now consider $\left(\hat{x}_1, \hat{x}_2\right)$ instead of $\left(\tilde{x}_1, \tilde{x}_2\right)$ and repeat the process explained before. Hence, either there exists an edge path P from $\left(\hat{x}_1, \hat{x}_2\right)$ to $\left(x_1^{[1]}, x_2^{[1]}\right)$ for which all points have the same value of f_1 and $(\tilde{x}_1, \tilde{x}_2) - (\hat{x}_1, \hat{x}_2) - P$ is the required edge path, or there exist an extreme point $(\breve{x}_1, \breve{x}_2)$ adjacent to $(\hat{x}_1, \hat{x}_2)$ such that

$$f_1(\breve{x}_1, \breve{x}_2) < f_1(\hat{x}_1, \hat{x}_2).$$

Next we consider $(\breve{x}_1, \breve{x}_2)$ and repeat the process. Since the number of extreme points of S is finite, eventually an edge path will be obtained along which the value of f_1 is non increasing.

Theorem 3.2.2. *The (k+1) st best extreme point of S,* $\left(x_1^{[k+1]}, x_2^{[k+1]}\right)$ *is adjacent to* $\left(x_1^{[1]}, x_2^{[1]}\right)$*, or* $\left(x_1^{[2]}, x_2^{[2]}\right)$*……, or* $\left(x_1^{[k]}, x_2^{[k]}\right)$*,* $k < m$*.*

Proof. Let $W^{[i]}$ denote the set of adjacent extreme points of $\left(x_1^{[i]}, x_2^{[i]}\right)$. Let $T = \left\{\left(x_1^{[1]}, x_2^{[1]}\right), \left(x_2^{[2]}, x_2^{[2]}\right), \ldots\ldots\ldots, \left(x_1^{[k]}, x_2^{[k]}\right)\right\}$ and $W = \left(W^{[1]} \cup W^{[2]} \cup \ldots\ldots \cup W^{[k]}\right)/T$.

Let $(y_1, y_2) \in W$ such that

$$f_1(y_1, y_2) = \min_{(w_1, w_2) \in W} \{f_1(w_1, w_2)\}.$$

Let $(\hat{x}_1, \hat{x}_2)$ be any extreme point of S such that $(\hat{x}_1, \hat{x}_2) \notin \bigcup_{i=1,\ldots k} W^{[i]}$. Taking into account that any edge path in S from $(\hat{x}_1, \hat{x}_2)$ to $\left(x_1^{[1]}, x_2^{[1]}\right)$ must contain at least a point of W as an intermediate point, and considering the edge path provided by Theorem 6, there exists $(\tilde{w}_1, \tilde{w}_2) \in W$ such that

$$f_1(\hat{x}_1, \hat{x}_2) \geq f_1(\tilde{w}_1, \tilde{w}_2) \geq f_1(y_1, y_2).$$

Since (y_1, y_2) minimizes the value of f_1 over the set of extreme points of S excluding T, then $(y_1, y_2) = \left(x_1^{[k+1]}, x_2^{[k+1]}\right).$

Theorem 3.1.4. *TheKth-best algorithm solves the LFBP problem.*

Proof. As a consequence of Theorem 3.2.2 the kth-best extreme point of the relaxed problem (4) is adjacent to either the 1st, 2nd, …, or $(k - 1)$th extreme point. Then, upon termination, the algorithm provides the best boundary feasible extreme point, i.e. the optimal solution to the LFBP problem.

As it was previously pointing out, it is worth noting that taking into account the C&C transformation only linear problems need to be solved when applying the Kth-best algorithm for solving the LFBP problem.

Example. The following linear fractional bilevel problem explains the process:

$$\min\ f_1 = \frac{1+y_1-y_2+2y_4}{8-y_1-2y_3+y_4+5y_5}$$

$$where\,(y_3,\dots\dots,y_8)\,solves$$

$$\min\ f_2 = \frac{1+y_1+y_2+2y_3-y_4+y_5}{6+2y_1+y_3+y_4-3y_5}$$

$$s.t$$

$$\begin{aligned} &-y_3+y_4+y_5+y_6=1,\\ &2y_1-y_3+2y_4-0.5y_5+y_7=1,\\ &2y_2+2y_3-y_4-0.5y_5+y_8=1,\\ &y_i\geq 0, i=3,\dots\dots,8. \end{aligned}$$

The optimal solution of the relaxed problem (RP) is $\left(x_1^{[1]},x_2^{[1]}\right)=(0,0.75,0,0,1,0,1.5,0)$. By fixing $x_1^{[1]}=(y_1,y_2)=(0,0.75),$, we get the following linear fractional problem corresponding to the second level:

$$\min\ \frac{1.75+2y_3-y_4+y_5}{6+y_3+y_4-3y_5}$$

$$s.t$$

$$\begin{aligned} &-y_3+y_4+y_5+y_6=1,\\ &-y_3+2y_4-0.5y_5+y_7=1,\\ &2y_3-y_4-0.5y_5+y_8=-0.5,\\ &y_i\geq 0, i=3,\dots\dots,8. \end{aligned}$$

Its optimal solution is $x_2^*=(y_3,\dots\dots,y_8)=(0,0.5,0,0.5,0,0)$. Hence $\left(x_1^{[1]},x_2^{[1]}\right)\notin IR$. Notice that $\left(x_1^{[1]},x_2^*\right)=(0,0.75,0,0.5,0,0.5,0,0)\notin$ IR, so that it provides an upper bound on the optimal value of f_1 for the example. The adjacent extreme points of $\left(x_1^{[1]},x_2^{[1]}\right)$ are given in Table. In this table are also shown the successive best extreme points computed and its adjacent extreme points. The optimal solution is reached at the fourth best extreme point. The following table gives the result of the Kth best algorithm for the example:

Iteration	$\left(x_1^{[i]}, x_2^{[i]}\right)$	$W^{[i]}$	
i=1	(0,0.75,0,0,1,0,1.5,0)∉IR	(0, 0, 1, 0, 2, 0, 3, 0)	f_1=0.0588
	f_1=0.0192	(0, 0, 0, 0, 1, 0, 1.5, 1.5)	f_1=0.0769
		(0.75, 0.75, 0, 0, 1, 0, 0, 0)	f_1=0.0816
		(0, 0.9, 0, 0.6, 0.4, 0, 0, 0)	f_1=0.1226
		(0, 0.5, 0, 0, 0, 1, 1, 0)	f_1=0.2
i=2	(0,0,1,0,2,0,3,0)∉IR	(0,0.75,0,0,1,0,1.5,0)	f_1=0.0192
	f_1=0.0588	(0, 0, 0, 0, 1, 0, 1.5, 1.5)	f_1=0.0769
		(0, 0, 0.5, 0, 0, 1.5, 1.5, 0)	f_1=0.125
		(1.5, 0, 1, 0, 2, 0, 0, 0)	f_1=0.1613
		(0, 0, 1.5, 1.5, 1, 0, 0, 0)	f_1=0.32
i=3	(0,0,0,0,1,0,1.5,1.5)∉IR	(0,0.75,0,0,1,0,1.5,0)	f_1=0.0192
	f_1=0.0769	(0, 0, 1, 0, 2, 0, 3, 0)	f_1=0.0588
		(0, 0, 0, 0, 0, 1, 1, 1)	f_1=0.125
		(0.75, 0, 0, 0, 1, 0, 0, 1.5)	f_1=0.1429
		(0, 0, 0, 0.6, 0.4, 0, 0, 1.8)	f_1=0.2075
i=4	(0.75,0.75,0,0,1,0,0,0)∈IR		
	f_1=0.0816		

Example 2 The example illustrated in chapter 2(solved by Kth best method)

$$
\begin{aligned}
&P1: \quad \min_{(x_1,x_2)} f_1 = -8x_1 - 4x_2 + 4x_3 - 40x_4 - 4x_5, \\
&where\,(x_3,......,x_8)\,solves \\
&\qquad \min_{(x_3,...,x_8)} f_2 = \frac{1+x_1+x_2+2x_3-x_4+x_5}{6+2x_1+x_3+x_4-3x_5} \\
&s.t. \\
&\qquad -x_3+x_4+x_5+x_6=1, \\
&\qquad 2x_1-x_3+2x_4-\frac{1}{2}x_5+x_7=1, \\
&\qquad 2x_2+2x_3-x_4-\frac{1}{2}x_5+x_8=1, \\
&x_i \ge 0, i=1,.....,8.
\end{aligned}
$$

Optimal table of leader :-

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Sol
$z_j - c_j$	-28	-32	0	0	0	-22	-9	-34	-58
x_5	-2/3	2/3	0	0	1	1	-1/3	1/3	1
x_4	1	1	0	1	0	½	1/2	1/2	3/2
x_3	1/3	5/3	1	0	0	½	1/6	5/6	3/2

Second best solution:-

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Sol.
$z_j - c_j$	128/3	52/3	0	-33	0	0	-23/3	44/3	-36
x_6	-2/3	2/3	0	2/3	0	1	1	4/3	7/8
x_5	4/3	2/3	0	1	1	0	½	2/3	7/8
x_3	2/3	4/3	1	-1/3	0	0	0	2/3	1

Third best solution :-

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Sol.
$z_j - c_j$	108/5	0	-96/5	0	0	62/5	74/5	2	-29.2
x_5	-4/5	0	-2/5	0	1	4/5	-2/5	0	2/5
x_4	4/5	0	-3/5	1	0	1/5	2/5	0	3/5
x_2	1/5	1	3/5	0	0	3/10	1/10	1/2	9/10

Hence, the Kth Best Algorithm can also solves the problem explained in chapter 2 and gives the result in less iterations as compare to the algorithm explained there.

2.4 The quasiconcave bilevel problem

Theorem3 is mainly based on the fact that the first level objective function is explicitly quasimonotonic. Hence, we can say that Theorem 3.1.1 is still valid for more general problems. So, consider the quasiconcave bilevel programming problem, in which f_1 and f_2 are continuous functions; f_1 is quasiconcave on S; f_2 is quasiconcave on $S(x_1)$, for all $x_1 \in S_1$; S is a polyhedron, which is assumed to be nonempty and bounded; and $M(x_1)$ is single valued for all $x_1 \in S_1$. As explained in the chapter that for this problem IR is formed by the union of connected faces of S [Calvete and Gale (1998)]. Hence, there exists an extreme point of the polyhedron S that solves it. Under the assumption that the first level objective function is explicitly quasimonotonic, the Theorem 3 can be replicated step by step to obtain that there exists a boundary feasible extreme point that solves the quasiconcave problem. Also no additional assumption for the second level objective function is required, so that this result is still valid for bilevel problems in which the first level objective function is linear or linear fractional and the second level objective function is linear, fractional or multiplicative.

The same can be said with regard to the *K*th-best algorithm. Under the mentioned assumptions, an optimal solution to the quasiconcave bilevel problem can be obtained by checking the best of the extreme points adjacent to all previously analyzed extreme points.

CHAPTER – 4

Non Linear Programming

The bilevel programming problem is a nested optimization problem with two levels in a hierarchy, the upper level and lower level decision-makers who have their own objective functions and constraint functions. The bilevel programming is neither continuous anywhere nor convex even if the objective functions of the upper level and lower level and the constraints are all linear because the objective function of the upper-level, which, generally speaking, is neither linear nor differentiable, is decided by the solution function of the lower-level problem. So, it is greatly difficult to solve the bilevel programming for its non-convexity and non-continuity, especially the bilevel nonlinear programming problem. Thus most researches on algorithms of bilevel programming are limited to the special structure of this problem or obtaining the locally optimum. So, we construct a solving method for a kind of bilevel nonlinear programming (BLP) with special structure.

4.1 Basic definitions of BLP

We consider the bilevel nonlinear programming (BLP) formulated as follows:

$$(BLP)\max_{x} F(x,y) = c_1^T x + d_1^T y + \frac{1}{2}\left(x^T . y^T\right) Q \left(x^T . y^T\right)^T \qquad (1)$$

where y solves the following problem:

$$\begin{aligned} &\max_{x} \; f(x,y) = c_2^T x + d_2^T y \\ &s.t \quad Ax + By \le r, \\ &\qquad\; x, y \ge 0 \end{aligned}$$

where $F(x, y)$, $f(x, y)$ is the upper-level's objective function and lower-level's objective function of BLP, respectively. $c_1, c_2 \in \Re^{n_1}, d_1, d_2 \in \Re^{n_2}, A \in \Re^{m\times n_1}, B \in \Re^{m\times n_2}, r \in \Re^{m}, Q \in \Re^{(n_1+n_2)\times(n_1+n_2)}$ is symmetric matrix. $x \in \Re^{n_1}, y \in \Re^{n_2}$ are the decision variables under the control of the upper level and lower level, respectively.

Next we give the following definitions of the BLP:

- The constraint region of BLP:

$$\Omega = \{(x, y) | Ax + By \leq r, x, y \geq 0\}.$$

- The constraint region of the lower programming for some fixed $x \geq 0$:

$$\Omega(x) = \{y | By \leq r - Ax, y \geq 0\}.$$

- The rational reaction set of the lower level programming for some fixed $x \geq 0$

$$M(x) = \{y | y \in \arg\max \{f(x, y), y \in \Omega(x)\}\}.$$

- The inducible region of BLP:

$$IR = \{(x, y) | (x, y) \in \Omega, y \in M(x)\}.$$

We consider Ω to be nonempty and bounded to ensure that there exists at least a solution to (BLP). So, feasible solution and optimal solution to BLP can be defined as follows:

Definition 4.1.1. A point (x, y) is called to be feasible to BLP if $(x, y) \in IR$.

Definition 4.1.2. A feasible point (x^*, y^*) is called to be optimal to BLP if $F(x^*, y^*) \geq F(x, y)$, $\forall (x, y) \in IR$. Now, we discuss the numerical algorithm to BLP under those above definitions.

4.2 The solution algorithm

For some fixed $x \geq 0$, the optimal solution to the lower programming can be obtained by solving the following linear programming:

$$\begin{aligned} &\max_{x} c_2^T x + d_2^T y \\ &s.t \quad By \leq r - Ax. \\ &\qquad y \geq 0. \end{aligned} \tag{2}$$

$c_2^T x$ is constant hence, we can assume $c_2 = 0$ to ignore this term without loss of generality when solving the lower programming. Thus, we can get the dual problem of problem (2) written as follows:

$$\min_{u} \ (r - Ax)^T u \qquad (3)$$
$$s.t \quad B^T u \geq d_2.$$
$$u \geq 0.$$

where $u \in \Re^m$ is the dual variable.

Theorem 4.2.1 (x^*, y^*) *is the optimal solution to the problem (1) if and only if there exists* u^* *such that* (x^*, y^*, u^*) *is the solution to the following programming:*

$$\max_{x,y,u} \ c_1^T x + d_1^T y + \frac{1}{2}(x^T, y^T) Q (x^T, y^T)^T \qquad (4)$$
$$s.t \quad Ax + By \leq r.$$
$$B^T u \geq d_2.$$
$$d_2^T y - (r - Ax)^T u = 0.$$
$$x, y, u \geq 0.$$

Proof :- According to the duality theorem of the linear programming, it is obvious that there exits x^*, y^*, u^* such that $d_2^T y^* - (r - Ax^*)^T u = 0$ if and only if y^* solve the following problem (2) for the fixed x^*.

By Theorem 4.2.1, the original bilevel problem (1) can be equally transformed into the traditional programming (4). Thus, we can get the solution of (1) by solving the problem (4). Note that the constraints except the nonlinear constraint $d_2^T y - (r - Ax)^T u = 0$ are all linear in problem (4), we can solve a series of nonlinear programming with only linear constraints by relaxing the nonlinear constraint to replace solving problem (4).

Let $U = \{u \mid B^T u \geq d_2, u \geq 0\}$ denote the feasible region to linear programming (3). Then the following conclusions are listed by the theory of the linear programming [Wan and Fei (2004)]:

Conclusion 1. The feasible region of the linear programming has at least one vertex and at most finite vertexes if it is not empty.

Conclusion 2. If there exist an optimal solution to the linear programming, it must be one vertex of the feasible region.

By the above conclusions we can say that there are finite vertexes in the feasible region U and u^* is one of them. Therefore, we can transform the problem (4) into a series of following nonlinear programming problems by obtaining all vertexes of U, denoted by $U^E = \{u^1, u^2, ...u^i\}$, according to the method in linear programming [Wei and Yan (2003)].

$$NP(u^i) \quad \max_{x,y} \; c_1^T x + d_1^T y + \frac{1}{2}(x^T, y^T)Q(x^T, y^T)^T \qquad (5)$$
$$s.t \quad Ax + By \le r.$$
$$d_2^T y - (r - Ax)^T u^i = 0.$$
$$x, y \ge 0$$

Solving the above problem is easier than solving the problem (4).

Either there exit an optimal solution or no feasible solution to the problem NP (u^i) for $i \in \{1, 2, \dots, t\}$ because Ω is compact and nonempty. Let $I \subseteq \{1, 2, \dots, t\}$ such that if $i \in I$, then there exist optimal solutions to the problem NP(u^i), otherwise the problem NP(u^i) has none feasible solution. There should exist an i such that the problem NP (u^i) has an optimal solution, hence $I \neq \phi$. For $j \in I$, let (x^j, y^j) be the optimal solution to the problem NP (u^i) and

$$F(x^k, y^k) = \max\{F(x^j, y^j) \mid j \in I\}.$$

With respect to the above we have the next theorem.

Theorem 4.2.2. (x^k, y^k) *is an optimal solution to the problem (1).*

Hence, we can obtain the optimal solution of BLP by solving a series of quadratic programming with linear constraints. Next, we discuss the algorithm to the problem (12).

Although quadratic programming is NP-Hard [Horst et al. (2000)], many researcher are devoted into this field and put forward various algorithms such as lagrangian method, active set method, labeling method, interior point method and so on, for its extensive applications [Horst et al. (2000)].

The genetic algorithm is one of the approach among the various approaches available, is extensively applied into solving the optimization problem because of its good characteristics such as few requirements for the differential of functions, globally convergence, robust, simplicity, and implicit parallelism and so on. In our method, the genetic algorithm is used to solve the quadratic programming problem (5), so following transformation are being made to the problem (5) so that the genetic algorithm can be used because it is difficult to deal with the constraints in genetic algorithm.

Hence, initially the problem (5) is transformed into the following formulation:

$$\max_{x,y}\ c_1^T x + d_1^T y + \frac{1}{2}\left(x^T, y^T\right) Q \left(x^T, y^T\right)^T \tag{6}$$

$$s.t \quad \begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \le r.$$

$$\begin{pmatrix} -u^{i^T} A & d_2^T \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \le r^T u^i.$$

$$\begin{pmatrix} u^{i^T} A & -d_2^T \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \le -r^T u^i.$$

$$\begin{pmatrix} -I_{n1} & 0_{n1\times n2} \\ 0_{n2\times n1} & -I_{n2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \le 0$$

$$\max_{x,y}\ c_1^T x + d_1^T y + \frac{1}{2}\left(x^T, y^T\right) Q \left(x^T, y^T\right)^T$$

$$s.t \quad \begin{pmatrix} A & B \\ -u^{i^T} & d_2^T \\ u^{i^T} & -d_2^T \\ -I_{n1} & 0_{n1\times n2} \\ 0_{n1\times n2} & -I_{n2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} r \\ r^T u^i \\ -r^T u^i \\ 0_{(n1+n2)\times 1} \end{pmatrix}. \tag{7}$$

By the duality theory of the nonlinear programming [Horst et al. (2000)], the quadratic programming problem (7) can be written as the following nonlinear programming without constraint:

$$\max_{\lambda\ge 0}\ -\frac{1}{2}\lambda^T M \lambda + d^T\lambda + \frac{1}{2}\left(c_1^T, d_1^T\right) Q^{-1}\left(c_1^T, d_1^T\right)^T \tag{8}$$

where

$$M = -\begin{pmatrix} A & B \\ -{u^i}^T & d_2^T \\ {u^i}^T & -d_2^T \\ -I_{n1} & 0_{n1\times n2} \\ 0_{n1\times n2} & -I_{n2} \end{pmatrix} Q^{-1} \begin{pmatrix} A & B \\ -{u^i}^T & d_2^T \\ {u^i}^T & -d_2^T \\ -I_{n1} & 0_{n1\times n2} \\ 0_{n1\times n2} & -I_{n2} \end{pmatrix}^T$$

$$d = -\left[\begin{pmatrix} r \\ r^T u^i \\ -r^T u^i \\ 0_{(n1+n2)\times 1} \end{pmatrix} - \begin{pmatrix} A & B \\ -{u^i}^T & d_2^T \\ {u^i}^T & -d_2^T \\ -I_{n1} & 0_{n1\times n2} \\ 0_{n1\times n2} & -I_{n2} \end{pmatrix} Q^{-1} \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \right]$$

Thus, the problem (5) is transformed into nonlinear programming without constraint so that the genetic algorithm is used to solve the problem. If λ^* solves the problem (8), then

$$Z^* = \begin{pmatrix} x^* \\ y^* \end{pmatrix} = Q^{-1} \left[-\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} A & B \\ -{u^i}^T & d_2^T \\ {u^i}^T & -d_2^T \\ -I_{n1} & 0_{n1\times n2} \\ 0_{n1\times n2} & -I_{n2} \end{pmatrix}^T \lambda^* \right]$$

solves the problem (5)

Next, the steps to solve the dual problem (8) by use of genetic algorithm are listed in details:

Step 1**:** Initialing. Set the population size $\mathrm{POP}_{\mathrm{Size}}$, probability of crossover P_{c}, probability of mutation P_{m}, the maximal generation of terminating the algorithm T and set $t = 0$.

Step 2**:** Generating initial population. The initialing population is obtained by randomly generating $\mathrm{POP}_{\mathrm{Size}}$ initial chromosomes.

Step 3**:** Calculating the values of the fitness functions. Calculating the fitness function value of each chromosome in current population.

Step 4**:** Generating the next population. Choosing the chromosomes by Roulette Wheel, then generate the new chromosomes by crossover and mutation operators to obtain the next population.

Step 5**:** Termination condition. If t is greater to T, then algorithm stops and output the optimal solution, otherwise set $t = t + 1$ and go to Step 3.

Above all we have done the following things

- we transform problem (4) into a series of problem (5) by the vertexes of *U*,
- then we can obtain the optimal solution to problem (1) by solving problem (5) in parallelism.
- At the same time, problem (5) can be transformed into problem (8) by the duality of the nonlinear programming to avoid dealing with the constraints in genetic algorithm.

The steps of the algorithm for solving (1) are listed as follows:

***Step 1*:** Generate all vertexes $U^E = \{u^1, u^2, ..., u^t\}$ of *U* by the method of linear programming.

***Step 2*:** Solve the problem $NP(u^k)(k=1,2,...,t)$ by genetic algorithm. If there is none feasible solution, then let (0, 0) denote the optimal solution and $F_k = -\infty$ denote the optimal value, otherwise let (x^k, y^k) denote the optimal solution and $F_k = F(x^k, y^k)$ denote the optimal value.

***Step 3*:** Compare F_k $(k=1,...,t)$, let $F^* = \max\{F_k, k=1,...,t\}$, and the corresponding (x^k, y^k) be the optimal solution (x^*, y^*).

***Step 4*:** If $F^* = -\infty$, then there is no feasible solution to problem (1) otherwise (x^*, y^*) is the solution to the problem (1) and F^* is the upper-level's optimal value of problem (1).

4.3 Numerical Experiment

In this section, the following example is solved by the proposed algorithm to demonstrate the feasibility and efficiency of our algorithm..

$$\begin{aligned} &\max_x F(x,y) = -x^2 - y^2 + 16x + 5xy \\ &s.t \quad 0 \le x \le 20, \\ &\max_y f(x,y) = y \\ &s.t \quad x + y - 20 \le 0, \\ &\qquad 0 \le y \le 10. \end{aligned} \tag{9}$$

For some fixed upper-level's decision variable *x*, the dual problem of the lower programming problem is written as follows:

$$\begin{aligned} &\min_u (20-x)u_1 + 10u_2 \\ &s.t \quad u_1 + u_2 \ge 1 \\ &\qquad u_1, u_2 \ge 0. \end{aligned} \tag{10}$$

By Theorem 4.3.1, it is well known that the optimal solution to the problem (9) can be obtained by solving the following problem:

$$\begin{aligned} \max_{x,y,u}\ & -x^2-y^2+16x+5xy \\ s.t\ \ & x\le 20 \\ & x+y-20\le 0 \\ & y\le 10 \\ & u_1+u_2\ge 1 \\ & y-(20-x)u_1-10u_2=0 \\ & x,y,u=(u_1,u_2)^T\ge 0 \end{aligned} \tag{11}$$

By use of the algorithm of linear programming, we obtain u^1 = (0, 1) and u^2 = (1, 0) which are the vertexes of the feasible region of the above problem. Hence, problem (11) can be transformed into the following nonlinear programming problems with only linear constraints:

$$\begin{aligned} \max_{x,y,u}\ & -x^2-y^2+16x+5xy \\ s.t\ \ & x\le 20 \\ & x+y-20\le 0 \\ & y\le 10 \\ & y-10=0 \\ & x,y,\ge 0 \end{aligned} \tag{12}$$

$$\begin{aligned} \max_{x,y,u}\ & -x^2-y^2+16x+5xy \\ s.t\ \ & x\le 20 \\ & x+y-20\le 0 \\ & y\le 10 \\ & y-(20-x)=0 \\ & x,y\ge 0 \end{aligned} \tag{13}$$

Then, the optimal solution to problem (9) is obtained by solving problems (12) and (13) to replace problem (11). However, it is difficult for genetic algorithm to deal with the constraints, so problems (12) and (13) can be transformed into the following nonlinear programming problems by duality theory of nonlinear programming to conveniently apply the genetic algorithm [6].

For any $i \in \{1, 2\}$, the reverent problem NP(u^i) is solved by genetic algorithm, and the results are listed in the following table:

i	u^i	(x^T, y^T)	F_i
1	(0, 1)	(10, 10)	460
2	(1, 0)	(11.14286, 8.85714)	469.14286

From the above table, we can see that the optimal solution is (11.14286, 8.85714) with the optimal value 469.14286. The optimal solution is (11.14, 8.86) and the optimal value is 469.14 in Zhong and Xu (1995). And the optimal solution is (11.1429, 8.8671) and the optimal value is 469.1429 in Li and Wang (2002).

4.4 Conclusion

A globally convergent algorithm is constructed for a special bilevel nonlinear programming. The bilevel nonlinear programming can be transformed into single level programming by the duality problem of the lower problem. And then the nonlinear constraints of the nonlinear programming are simplified to linear ones by use of the vertexes of the feasible region of the dual problem. Therefore, we can obtain the optimal solution to the bilevel nonlinear programming by solving a series of nonlinear programming problems with only linear constraints. To avoid the difficulty of dealing with the constraints in genetic algorithm, problem (5) is turned into the nonlinear programming (8) without constraint by the duality theory of the nonlinear programming. Thus, we can obtain the optimal solution to the BLP by solving a series of the nonlinear programming without constraint, which can easily be solved.

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