ON L^1 - CONVERGENCE OF CERTAIN TRIGONOMETRIC SUMS

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DEDICATED

TO

MY PARENTS AND GOD

CERTIFICATE

Certified that the dissertation entitled, " **ON L^1 CONVERGENCE OF CERTAIN TRIGONOMETRIC SUMS**", which is being submitted by **Miss Chinki Rani** (Roll No. 300903001), in the fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of Dr. Jatinderdeep Kaur and Dr. S.S. Bhatia during the period from January 2011 to June 2011. The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University / Institute for the award of any degree.

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ABSTRACT

The present dissertation entitled,**"On L^1-Convergence of Certain Trigonometric Sums"** contains a brief account of investigations carried out by various authors and by me on L^1-convergence of trigonometric series under the supervision of Dr. Jatinderdeep Kaur, Lecturer and Dr. S.S.Bhatia, Professor, School of Mathematics and Computer Applications, Thapar University, Patiala.

The work presented in this dissertation has been divided into four chapters . The first chapter is introductory. In this chapter, apart from setting up the notations and terminology to be used in sequel, we have presented some known results interrelated to our results along with a brief plan of our results presented in the subsequent chapters. The purpose of chapter II is to study the L^1-convergence of modified cosine sums introduced by C.S. Rees and Č.V. Stanajević in 1973 with class S of coefficient sequences. In chapter III, we have studied the L^1-convergence of the modified cosine sums given by Suresh Kumari and Babu Ram in 1987 with class S.

In chapter IV, we have obtained the L^1-convergence of the cosine sums given by Ram and Kumari with class S_r of Tomovski. We have also obtained the L^1-convergence of the r^th derivative of the cosine sums with S_r class.

Towards the end, references of various publications cited in the present dissertation have been reported.

CONTENTS

Chapter I

Introduction

1.1 The present dissertation comprises certain investigations carried out by the author "On L^1-Convergence of certain Trigonometric Sums". It is well known that if a trigonometric series converges in L^1 -metric to a function $f \in L^1(T)$, then it is the Fourier series of the function *f*. Riesz {[1], Vol.II, Ch.VIII§22} gave a counter example to show that in L^1-metric, the converse of the above said result does not hold good. This motivated various authors to study the L^1-convergence of trigonometric sums.

The work on this topic was initiated by Young W.H. [31] and Kolmogorov A.N. [13] by taking classes of convex sequence ($\Delta^2 a_n \geq 0$) and quasi-convex sequences ($\sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty$) respectively. Teljakovskǐi S.A [30] considered another class S introduced by Sidon [22] for L^1-convergence of trigonometric series. The results obtained by these authors were further generalized and extended by Hardy G.H. and Littlewood J.E. [8], Kano T. [10], Garrett J.W. and Stanojević Č.V. ([6], [7]), Ram B. ([15], [17]), Bojanic R. and Stanojević Č.V. [3], Bhatia S.S. and Ram B. [2], Tomovski Ž. ([25], [26], [27], [28]), and others by considering various generalizations of classes of sequences mentioned above for one-dimensional trigonometric series.

During their investigations, some authors introduced modified trigonometric sums, as these sums approximate their limits better than the classical trigonometric series in the sense that these sums converge in L^1-metric to the sum of trigometric series where as the classical series itself may not. In this concern, various authors like Rees, C.S. and Stanojević Č.V. [19], Kumari S. and Ram B. [14], Ram B. and Kumari S. [18], Ram B. and Bhatia S.S [9], Kaur K., Bhatia S.S. and Ram B. [12] have introduced various new modified trigonometric sums and have studied their L^1-convergence under various classes of coefficient sequences. Bhatia S.S. and Ram B. [2] generalized the results of Kumari S. and Ram B. [14] and obtained their results as corollary.

To provide sufficient background for later chapters, a summary of basic concepts, techniques and a brief chapter wise resume of the results contained in the dissertation has been given in this introductory chapter. However, some of the definitions and notations will be repeated occasionally in various chapters for the sake of convenience.

1.2 DEFINITIONS AND NOTATIONS

Let $\{a_n\}$ be a sequence. Then we write

$$\Delta a_n = a_n - a_{n+1}$$
$$\Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}$$

Abel's transformation ([1], Vol.I, p.1) If $a_0, a_1, a_2, ..., v_0, v_1, ..., v_n, ...$ are any real numbers, let us assume that

$$V_n = v_0 + v_1 + v_2 + ... + v_n.$$

Then for any values of m and n we find that

$$\sum_{k=m}^{n} a_k v_k = \sum_{k=m}^{n-1} \Delta a_k V_k + a_n V_n - a_m V_{m-1}$$

(under the condition that if $m = 0, V_{-1} = 0$).

Convex sequence. A sequence $\{a_n\}$ is said to be convex if $\Delta^2 a_n \geq 0$.

Quasi convex sequence ([1], Vol.II, p.202.) A sequence $\{a_n\}$ is said to quasi-convex if

$$\sum_{n=1}^{\infty} n|\triangle^2 a_n| < \infty.$$

Semi convex sequence [10]. A null sequence $\{a_n\}$ is said to be a semi-convex if

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \qquad (a_0 = 0)$$

It may be remarked here that every quasi-convex null sequence is semi-convex.

Generalized semi-convex sequence [11]. A null sequence $\{a_n\}$ is said to be generalized semi-convex if

$$\sum_{n=1}^{\infty} n^{\alpha}|\Delta^{\alpha+1}a_{n-1}+\Delta^{\alpha+1}a_n|<\infty, \qquad \text{for } \alpha>0, \qquad (a_0=0)$$

For $\alpha=1$. this generalized semi-convex sequence reduces to semi-convex sequence.

Quasi monotone sequence ([20], [24]). A sequence $\{a_n\}$ of non-negative numbers is said to be a quasi monotone if $a_{n+1}\le a_n(1+\frac{\alpha}{n})$ for some $\alpha>0$ and all $n>n_0(\alpha)$. An equivalent definition is that $n^{-\beta}a_n\downarrow 0$ for some $\beta>0$.

Monotone decreasing sequence. A sequence $\{a_n\}$ is said to be monotone decreasing sequence if $a_n\ge a_{n+1}$ for all values of n.

O- and o-relationships for series and integrals [1]. Using the notation now accepted in mathematical literature we write

$$u_n=o(v_n)$$

If $v_n>0$ $(n=0,1,2,..)$ and $\dfrac{u_n}{v_n}\to 0$ at $n\to\infty$.
If $\dfrac{u_n}{v_n}$ is bounded, we write

$$u_n=O(v_n)$$

If two positive constants A and B exists for which at sufficiently large n

$$A\le\frac{u_n}{v_n}\le B,$$

then we can write

$$u_n\sim v_n,$$

and finally

$$u_n\approx v_n,$$

will signify

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$$

(in this case it is usally said that u_n and v_n are asymptotically equal.)

Oliver's theorem. If $a_n \downarrow 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$ then n$a_n \to 0$ as $n \to \infty$

Fourier series. A trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called the Fourier series where the coefficients a_n and b_n can be determined by the Fourier formulas

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos nx\ dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\ dx \end{aligned}$$

derived from the integrable function $f(x)$. We can write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

And let the partial sum of this series be denoted by $S_n(x)$ such that

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

Dirichlet kernel ([1], Vol.I, p.85). Let

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + ... + \cos nx$$

then

$$\begin{aligned} 2\sin\frac{x}{2} D_n(x) &= \sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + ... + 2\sin\frac{x}{2}\cos nx \\ &= \sin\left(n + \frac{1}{2}\right)x \end{aligned}$$

hence $$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

This expression is known as Dirichlet's kernel. Moreover,

$$\begin{aligned}\tilde{D}_n(x) &= \sin x + \sin 2x + ... + \sin nx. \\ &= \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\end{aligned}$$

is called the kernel conjugate to the Dirichlet kernel.

Since the function $\frac{\sin x}{x}$ decreases in the interval $(0, \frac{\pi}{2}]$

Therefore,

$$\frac{\sin x}{x} \geq \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi} \quad \text{for} \quad 0 < x \leq \frac{\pi}{2} \qquad ......(1)$$

If $x \neq 0 \pmod{2\pi}$, then

$$|D_n(x)| \leq \frac{1}{2[\sin\frac{x}{2}]} \qquad ......(2)$$

and

$$|\tilde{D}_n(x)| \leq \frac{1}{[\sin\frac{x}{2}]} \qquad ......(3)$$

Using (1) and (2), we get

$$|D_n(x)| \leq \frac{\pi}{2x}, \quad for \quad 0 < |x| \leq \pi$$

and using (1) and (3), we get

$$|\tilde{D}_n(x)| \leq \frac{\pi}{x}, \quad for \quad 0 < |x| \leq \pi$$

Also, we shall make use of the uniform estimate

$$|D_n(x)| \leq n + \frac{1}{2}, \quad for \quad any \quad x$$

Fejer kernel ([1], [32]). The Fejer kernel $K_n(x)$ is defined as

$$\begin{aligned}K_n(x) &= \frac{1}{n+1}\sum_{j=0}^{n} D_j(x) \\ &= \frac{1}{n+1}\sum_{j=0}^{n} \frac{\sin(j+\frac{1}{2})x}{2\sin\frac{x}{2}}\end{aligned}$$

Using $|D_n(x)| \leq n+1$, it follows that $K_n(x) \leq n+1$

It has the properties that

$$(i) \quad K_n(x) \geq 0,$$

$$(ii) \quad \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1.$$

The conjugate Fejer kernel is defined as

$$\tilde{K}_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} \tilde{D}_j(x)$$

We have

$$\tilde{K}_n(x) > 0 \quad for \quad 0 < x < \pi, \quad n = 1, 2, 3, ..$$

and

$$|\tilde{K}_n(x)| < \frac{n}{2}$$

The class S ([22], [30]). A null sequence $\{a_k\}$ belongs to class S, if there exists a sequence $\{A_k\}$ such that

$$(i) \quad \{A_k\} \downarrow 0 \quad as \qquad k \to \infty.$$

$$(ii) \quad \sum_{k=0}^{\infty}\{A_k\} < \infty.$$

$$(iii) \quad |\Delta a_k| \leq A_k, \qquad \forall \ k$$

The class S is usually called as Sidon-Teljakovsk\u{i}i class. Since, firstly, Teljakovsk\u{i}i expressed Sidon's condition in a succinct equivalent form and secondly, he showed that the class S is also a class of L^1-convergence.

The class S' [23]. A sequence $\{a_n\}$ of numbers is said to belongs to the class S', if $a_n \to 0 \ \ as \ \ n \to \infty$ and there exists a sequence $\{A_n\}$ such that $\{A_n\}$ is quasi-monotone, $\sum_{n=0}^{\infty} A_n < \infty$ and $|\Delta a_n| \leq A_n$, for all n.

The class S_r [25]. A null sequence $\{a_k\}$ is said to belong to class $S_r,\ r = 0, 1, 2, 3, ...$ if there exists a sequence $\{A_k\}$ such that

$$
\begin{array}{ll}
(i) & \{A_k\} \downarrow 0. \quad as \quad k \to \infty. \\
(ii) & \sum_{k=1}^{\infty} k^r \{A_k\} < \infty. \\
(iii) & |\Delta a_k| \le A_k, \qquad \forall \ \ k
\end{array}
$$

When $r = 0$, we denote $S_r = S$. Clearly $S_{r+1} \subset S_r \ \ \forall \ \ r = 0, 1, 2, 3, ...$, but the converse of this inclusion is not true. (cf. Chapter IV)

The class $\mathcal{C}$ of Garrett and Stanojević [7]. A null sequence $\{a_n\}$ belongs to class $\mathcal{C}$ if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, independent of n, and such that

$$\int_0^\delta \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \epsilon, \qquad for \quad all \quad n \ge 0.$$

The class R [10]. A null sequence $\{a_n\}$ belongs to the class R, if

$$\sum_{k=1}^{\infty} k^2 \left| \Delta^2 \left(\frac{a_k}{k} \right) \right| < \infty.$$

The class $\mathbf{R}_r$ [2]. A null sequence $\{a_n\}$ belongs to the class $R_r \quad r = 0, 1, 2, .., $ if

$$\sum_{k=1}^{\infty} k^{r+2} \left| \Delta^2 \left(\frac{a_k}{k} \right) \right| < \infty.$$

Clearly, for $r = 0$ this class reduces to the class R.

(1.3) The following results about the behavior of cosine and sine series are known:

Theorem I ([1], [13], [31]). If $\{a_k\}$ is a quasi-convex null sequence, then

(1.3.1) $$f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L^1[0, \pi].$$

Theorem II ([1], [29]) . If $\{a_k\}$ is a quasi-convex null sequence, then

$$(1.3.2) \qquad \sum_{k=1}^{\infty} a_k \sin kx$$

is a Fourier series if and only if $\displaystyle\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty$.

In 1968, Kano [10] generalized Theorem I and Theorem II in the following form:

Theorem III. If $\{a_k\}$ is a null sequence such that

$$(1.3.3) \qquad \sum_{k=1}^{\infty} k^2 \left| \triangle^2 \left(\frac{a_k}{k} \right) \right| < \infty, \quad i.e. \ \ \{a_k\} \in class \ \ R,$$

then (1.3.1) and (1.3.2) are the Fourier series, or equivalently they represent integrable functions.

Concerning the integrability of trigonometric series belonging to the class S (introduced already in §1.2), Teljakovskiǐ [30] established the following theorems:

Theorem IV. Let the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

belongs to the class S. Then this is a Fourier series and the following relation holds:

$$\int_0^{\pi} \left| \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \right| dx \le C \sum_{k=0}^{\infty} A_k,$$

where C is an absolute constant.

Theorem V. Let the sine series

$$\sum_{k=1}^{\infty} a_k \sin kx$$

belongs to the class S. Then the following relation holds for $p = 1, 2, 3, ...$

$$\int_{\pi/(p+1)}^{\pi} |\sum_{k=1}^{\infty} a_k \sin kx| dx = \sum_{k=1}^{p} \frac{|a_k|}{k} + O(\sum_{k=1}^{\infty} A_k).$$

We observe that Theorem I and Theorem III provide just only the sufficient conditions for the integrability of cosine series.

Ram [16] showed that the condition S is sufficient for the integrability of Rees-Stanojević sums [19]

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_j)\cos kx \tag{1.3.4}$$

They proved the following theorem:

Theorem VI. Let the sequence $\{a_k\}$ satisfy the condition S. Then $g(x) = \lim_{n\to\infty} g_n(x)$ exists for $x \in (0, \pi]$, and

$$\int_0^{\pi} |g(x)|dx \leq C\sum_{k=0}^{\infty} A_k.$$

Consider the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.3.5}$$

Let the partial sums of (1.3.5) is denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$. Denote the class of sequences of Fourier coefficients $\{a_k\}$ by **F.** There are subclasses of **F** for which $a_n \log n = o(1), \quad n \to \infty$ is a necessary and sufficient condition for $||S_n - f||_{L^1} = o(1), \quad as \quad n \to \infty.$

A subclass **G** of **F** is called a class of L^1-convergence if $||S_n - f||_{L^1} = o(1), \quad n \to \infty$ if and only if $a_n \log n = o(1), \quad n \to \infty.$

There are three classical examples of classes of L^1-convergence. The first one is due to Young [31], where **G** is defined to be the class of all convex sequences $\{a_k\}$. The second one is the class of all quasi-convex sequences $\{a_k\}$, introduced by Kolmogorov [13]. The third example is class S due to Teljakovskiǐ [30]. These classical classes have been extended by various authors.

Concerning the L^1-convergence of the cosine series, we have the following classical result of Kolmogorov [13].

Theorem VII. If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex,then for the convergence of the series (1.3.5) in the metric space L, it is necessary and sufficient that $a_k \log k = o(1), \quad k \to \infty$.

The case in which the sequence $\{a_k\}$ is convex, of this theorem was established by Young [31].

Generalizing the above classical result, Teljakovskiǐ [30] proved the following result.

Theorem VIII. If the coefficient sequence $\{a_k\}$ of the cosine series (1.3.5) belongs to the class S, then a necessary and sufficient condition for L^1-convergence of (1.3.5) is $a_k \log k = o(1), \quad k \to \infty$.

Rees and Stanojević [19] introduced modified cosine sums (1.3.4) and obtained an analogue of Theorem XI for these sums. These modified cosine sums approximate their limits better than the classical cosine series as they converge in L^1-metric to the sum of the cosine series whereas the classical series itself may not. They proved the following result.

Theorem IX. Let f be the sum of the cosine series (1.3.5). Then $g_n(x)$ converges to f in L^1-metric if and only if $\{a_k\}$ belongs to the class $\mathcal{C}$.

Ram [15] proved the following result on L^1-convergence of Rees-Stanojević sums (1.3.4).

Theorem X. If (1.3.5) belongs to the class S. Then

$$||f - g_n||_{L^1} = o(1), \quad n \to \infty.$$

Theorem VIII of Teljakovskiǐ [30] follows as corollary of this theorem.

Later, Ram and Kumari ([14], [18]) introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \tag{1.3.6}$$

and

$$h_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \sin kx \tag{1.3.7}$$

and studied their L^1-convergence under the condition that the cosine series and sine series belongs to the class R and S. They also deduced the results about L^1-convergence of cosine and sine series. Their results are as below:

Theorem XI. Let $\{a_n\}$ belongs to the class S and $\lim_{n\to\infty} |a_{n+1}| \log n = 0$, then $||f - S_n||_{L^1} = o(1) \quad n \to \infty$.

Theorem XII. Let $\{a_n\}$ belongs to the class S. If $\lim_{n\to\infty} |a_{n+1}| \log n = 0$, then $||f - f_n||_{L^1} = o(1) \quad n \to \infty$.

Theorem XI of Ram [14] follows as corollary of this theorem.

Theorem XIII. Let $\{a_n\}$ belongs to the class R. If $t_n(x)$ represents $f_n(x)$ or $h_n(x)$, then $||f - t_n||_{L^1} = o(1) \quad n \to \infty$.

Further, Bhatia and Ram [2] studied the L^1-convergence of r-th differential of trigonometric sums introduced by Ram and Kumari [18] as:

Theorem XIV. Let $\{a_n\}$ belongs to the class R_r. If $t_n^r(x)$ represents $f_n^r(x)$ or $h_n^r(x)$, then $||f^r(x) - t_n^r(x)||_{L^1} = o(1) \quad n \to \infty$.

Hooda and Ram [9] further proved the following theorem in concern with Ram and Kumari sums (1.3.6)

Theorem XV. Let $\{a_n\}$ belong to the class S$'$. Then

$$||f - f_n||_{L^1} = o(1), \quad n \to \infty.$$

In chapter II, Theorem VIII, X have been studied for the cosine series using the modified cosine sum (1.3.4) of Rees-Stanojević [19].

Whereas, **in chapter III,** Theorem XI, XII have been studied for the cosine series using the modified cosine sum (1.3.6) of Ram and Kumari [18].

The objective of **chapter IV** is to generalize Theorem XII for the cosine series for an extended class $S_r,\ \ r = 0, 1, 2, ...$ of cofficient sequences and also obtain the L^1-convergence of the r^{th} derivative of the cosine series.

Chapter II

On L^1-Convergence of Certain Cosine Sums

2.1 Introduction

Consider the cosine trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{2.1.1}$$

Let the partial sum of (2.1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$.

Definition 1. A null sequence $\{a_k\}$ is said to belong to the class S, if there exists a sequence $\{A_k\}$ such that

$$\{A_k\} \downarrow 0 \quad as \quad k \to \infty, \tag{2.1.2}$$

$$\sum_{k=0}^{\infty} A_k < \infty, \tag{2.1.3}$$

$$|\Delta a_k| \leq A_k \quad \forall \ k. \tag{2.1.4}$$

This class S is introduced by Sidon in 1939 [22].

Garrett and Stanojević [7] proved that the partial Rees-Stanojević sums [19]

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx \tag{2.1.5}$$

converge in the L^1-metric to (2.1.1) if and only if given $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that

$$\int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \epsilon \tag{2.1.6}$$

for all $n \geq 0$.

It has been also shown that the classical Young-Kolmogorov-Stanojević sufficient condition for integrability of (2.1.1) imply (2.1.6).

Generalising a classical result [[1], p.204], Teljakovskiǐ [30] proved the following result:

Theorem A. If (2.1.1) belongs to the class S, then a necessary and sufficient condition for L^1 convergence of (2.1.1) is $a_n \log n = o(1),\ n \to \infty$.

The aim of this chapter is to study the L^1-convergence of Rees-Stanojević cosine sums (2.1.5) to a cosine trigonometric series belonging to the class S defined by Sidon.

(2.2) Lemmas. The proof of our results are based upon the following lemmas:

Lemma 1 [5]. If $|c_k| \leq 1$, then

$$\int_0^\pi \left| \sum_{k=0}^n c_k \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx \leq C\,(n+1)\,.$$

where C is a positive absolute constant.

Proof. Since

$$D_k(x) = \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

As, $D_k(x)$ is bounded.

So let,

$$\left| \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| \leq B$$

Therefore,

$$\begin{aligned} \int_0^\pi \left| \sum_{k=0}^n c_k \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx &\leq \int_0^\pi \left| B \sum_{k=0}^n 1 \right| dx \\ &\leq B(n+1)\pi = C(n+1) \end{aligned}$$

Hence

$$\int_0^\pi \left| \sum_{k=0}^n c_k \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx \leq C(n+1).$$

□

Lemma 2. If (2.1.1) belongs to the class S, then

$$g_n(x) = S_n(x) - a_{n+1}D_n(x).$$

where $D_n(x)$ denotes the Dirichlet kernel.

Proof. Since the modified cosine sum of Rees-Stanojević is

$$\begin{aligned}
g_n(x) &= \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta a_j \cos kx \\
&= \frac{1}{2}(a_0 - a_{n+1}) + \sum_{k=1}^{n}(a_k - a_{k+1} + a_{k+1} - ... + a_n - a_{n+1})\cos kx \\
&= \frac{a_0}{2} - \frac{a_{n+1}}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1}\left(D_n(x) - \frac{1}{2}\right) \\
&= S_n(x) - a_{n+1}D_n(x) + \frac{a_{n+1}}{2} - \frac{a_{n+1}}{2}
\end{aligned}$$

Hence,

$$g_n(x) = S_n(x) - a_{n+1}D_n(x). \qquad \square$$

2.3 Main Results: The main results of this chapter are given as under:

Theorem 1. If (2.1.1) belongs to the class S, then

$$||f - g_n||_{L^1} = o(1), \qquad n \to \infty.$$

Proof. Since

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta a_j \cos kx$$

By Lemma 2, we can write

$$\begin{aligned}
g_n(x) &= S_n(x) - a_{n+1}D_n(x). \\
f(x) - g_n(x) &= f(x) - (S_n(x) - a_{n+1}D_n(x)) \\
f(x) - g_n(x) &= \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx\right) - \left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1}D_n(x)\right) \\
&= \sum_{k=n+1}^{\infty} a_k \cos kx + a_{n+1}D_n(x)
\end{aligned}$$

On applying Abel's transformation, we get

$$\begin{aligned}
f(x) - g_n(x) &= \sum_{k=n+1}^{\infty}\Delta a_k D_k(x) - a_{n+1}D_n(x) + a_{n+1}D_n(x) \\
&= \sum_{k=n+1}^{\infty}\Delta a_k D_k(x)
\end{aligned}$$

Therefore,

$$\int_0^\pi |f(x) - g_n(x)| \ dx \ = \ \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| \ dx$$
$$= \ \int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| \ dx$$

Again applying Abel's transformation we get

$$= \ \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| \ dx$$
$$\leq \ \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| \ dx$$

By lemma 1,

$$\int_0^\pi |f(x) - g_n(x)| \ dx \ \leq \ C \ \sum_{k=n+1}^{\infty} (k+1) \ \Delta A_k$$

But,

$$\sum_{k=0}^{n} A_k = \sum_{k=0}^{n-1} (\Delta A_k) \ . \ (k+1) + (n+1) \ A_n$$

Since $A_n \downarrow 0$ as $n \to \infty$, and $\sum_{k=0}^{n} A_k < \infty$ as $n \to \infty$ then by Oliver's theorem $(n+1) \ A_n \to 0$ as $n \to \infty$. This implies

$$\sum_{k=0}^{n-1} (\Delta A_k) \ . \ (k+1) < \infty \qquad as \quad n \to \infty$$

Therefore,

$$\sum_{k=n+1}^{\infty} (\Delta A_k) \ . \ (k+1) = o(1)$$

Hence,

$$\int_0^\pi |f(x) - g_n(x)| \ dx \ = o(1)$$

i.e. $\qquad ||f - g_n||_{L^1} = o(1), \qquad n \to \infty.$

Hence proved. $\square$

Theorem 2. If (2.1.1) belongs to the class S, then a necessary and sufficient condition for L^1 convergence of (2.1.1) is $a_n \log n = o(1),\ n \to \infty$.

Proof. Since

$$\begin{aligned}
\int_0^\pi |f(x) - S_n(x)|dx &= \int_0^\pi |f(x) - g_n(x) + g_n(x) - S_n(x)|dx \\
&\le \int_0^\pi |f(x) - g_n(x)|dx + \int_0^\pi |g_n(x) - S_n(x)|dx \\
&\le \int_0^\pi |f(x) - g_n(x)|dx + \int_0^\pi |a_{n+1}D_n(x)|dx \\
&\le \int_0^\pi |f(x) - g_n(x)|dx + a_{n+1}\int_0^\pi |D_n(x)|dx
\end{aligned}$$

Since $\lim_{n\to\infty} \int_0^\pi |f(x) - g_n(x)|dx = 0$ by Theorem 1 and by the use of Zygmund's Theorem, we have

$$\begin{aligned}
\int_0^\pi |a_{n+1}D_n(x)|dx &\le a_{n+1}\int_0^\pi |D_n(x)|dx \\
&\le a_{n+1} \lim_{n\to\infty} \int_{\pi/n}^\pi \frac{1}{x} \\
&= a_{n+1} \log n \qquad as \quad n \to \infty.
\end{aligned}$$

Hence, $\qquad ||f - S_n|| = o(1), \quad n \to \infty$ □

Chapter III

On L^1-Convergence of a Modified Cosine Sums

3.1 Introduction

Consider the cosine trigonometric series

$$(3.1.1) \qquad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

Let the partial sum of (3.1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n \to \infty} S_n(x)$.

Definition 1. A null sequence $\{a_k\}$ is said to belong to the class S, if there exists a sequence $\{A_k\}$ such that

$$(3.1.2) \qquad \{A_k\} \downarrow 0 \;\; as \qquad k \to \infty,$$

$$(3.1.3) \qquad \sum_{k=0}^{\infty} A_k < \infty,$$

$$(3.1.4) \qquad |\Delta a_k| \leq A_k \qquad \forall \;\; k.$$

This class S is introduced by Sidon in 1939 [22].

Definition 2. Let

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + ... + \cos nx$$

be the Dirichlet kernel and

$$\tilde{D}_n(x) = \sin x + \sin 2x + \sin 3x + ... + \sin nx.$$

be the conjugate Dirichlet kernel.

The differentials $D_n'(x)$ and $\tilde{D}_n'(x)$ of $D_n(x)$ and $\tilde{D}_n(x)$ respectively can be written as

$$\begin{aligned} D_n'(x) &= -\sin x - 2\sin 2x - ... - n\sin nx \\ D_n'(x) &= -\sum_{k=1}^{n} k \sin kx \end{aligned}$$

and

$$\begin{aligned}\tilde{D}_n'(x) &= \cos x + 2\cos 2x + ... + n\cos nx\\ \tilde{D}_n'(x) &= \sum_{k=1}^{n} k\cos kx\end{aligned}$$

Concerning the L^1-convergence of Rees-Stanojević cosine sums [19]

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta a_k \cos kx$$

to a cosine trigonometric series, belonging to the class S, Ram [15] proved the following theorem:

Theorem A. If (3.1.1) belongs to the class S, then

$$||f - f_n||_{L^1} = o(1), \quad n \to \infty.$$

Later, in 1988 Ram and Kumari [14] introduced a new modified cosine sum as

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right) k\cos kx$$

The aim of this chapter is to study the L^1-convergence of this modified cosine sums under the class S of coefficient sequence.

(3.2) Lemmas. The following lemmas will be used in the proof of the theorem:

Lemma 1 [5]. If $|c_k| \le 1$, then

$$\int_0^{\pi}\left|\sum_{k=0}^{n} c_k \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}\right| dx \le C(n+1).$$

where C is a positive absolute constant.

Proof. Proof of this lemma has been given in chapter II.

Lemma 2 [21]. For the r^th derivatives of the Dirichlet's kernel $D_n(x)$ the following estimates hold

$$||\tilde{D}_n^{(r)}(x)||_{L^1} = O(n^r \log n), \qquad r = 0, 1, 2, ...$$

Proof. Let r be any non-negative integer.

Now with the Bernstein inquality in L^p-space, we have

$$\int_0^\pi |\tilde{D}_n^{(r)}(x)|dx \leq n^r \int_0^\pi |\tilde{D}_n(x)|dx.$$

On the other hand,

$$\begin{aligned}
\int_0^\pi |\tilde{D}_n(x)|dx &\leq \int_0^\pi \frac{\sin^2(\frac{nx}{2})}{x} dx + O(1) \\
&= \log(1 + n\pi) + O(1) = O(\log n).
\end{aligned}$$

Therefore

$$||\tilde{D}_n^{(r)}(x)||_{L^1} = O(n^r \log n)$$

In particular, for first derivative of $\tilde{D}_n(x)$ is

$$||\tilde{D}_n'(x)||_{L^1} = O(n \log n) \qquad \square$$

3.3 Main result: We prove the following results:

Theorem B. Let (3.1.1) belongs to the class S. If $\lim_{n\to\infty} |a_{n+1}| \log n = 0$, then

$||f - g_n|| = o(1), \quad n \to \infty.$

Proof: we have

$$\begin{aligned}
g_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + ... + \Delta\left(\frac{a_n}{n}\right)\right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1} + \frac{a_{k+1}}{k+1} - \frac{a_{k+2}}{k+2} + ... \frac{a_n}{n} - \frac{a_{n+1}}{n+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1} \right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx \\
&= S_n(x) - \frac{a_{n+1}}{n+1} \tilde{D}_n'(x)
\end{aligned}
$$

Therefore,

$$
f - g_n = \sum_{k=n+1}^{\infty} a_k \cos kx + \frac{a_{n+1}}{n+1} \tilde{D}_n'(x)
$$

Now, making use of Abel's transformation, we have

$$
f - g_n = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) + \frac{a_{n+1}}{n+1} \tilde{D}_n'(x)
$$

Now,

$$
\begin{aligned}
\int_0^{\pi} |f(x) - g_n(x)| dx &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx \\
&= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx
\end{aligned}
$$

Again using Abel's transformation, we get

$$
\begin{aligned}
\int_0^{\pi} |f(x) - g_n(x)| dx &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx \\
&\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx
\end{aligned}
$$

Using lemma 1, we get

$$
\leq C \sum_{k=n+1}^{\infty} (k+1) \Delta A_k + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx
$$

But,

$$
\sum_{k=0}^{n} A_k = \sum_{k=0}^{n-1} (\Delta A_k) \ . \ (k+1) + (n+1) \ A_n
$$

As $n \to \infty$, $(n+1) \ A_n \to 0$ by Oliver's theorem. Moreover $\sum_{k=0}^{n} A_k < \infty$ as $n \to \infty$ by (3.1.3) condition of class S. This implies

$$\sum_{k=0}^{n-1}(\Delta A_k)\ .\ (k+1) < \infty \qquad\qquad as \quad n \to \infty$$

Therefore,

$$\sum_{k=n+1}^{\infty}(\Delta A_k)\ .\ (k+1) = o(1)$$

Moreover, by Zygmund's Theorem ([1], p. 458) we have

$$\begin{aligned}\int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\right| dx &\leq \int_{-\pi}^{\pi} \left|\frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\right| dx \\ &= \frac{|a_{n+1}|}{n+1}\int_{-\pi}^{\pi}|\tilde{D}_n'(x)|dx\end{aligned}$$

Using lemma 2, we get

$$\begin{aligned}\int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\right| dx &= \frac{|a_{n+1}|}{n+1} \ \ n\log n \\ &= C \ \ |a_{n+1}|\log n \\ &\sim \ |a_{n+1}|\log n\end{aligned}$$

And $|a_{n+1}|\log n = 0 \ \ as \ \ n \to \infty$.

Therefore

$$\int_0^\pi |f(x) - g_n(x)|dx = o(1), \qquad\qquad n \to \infty.$$

Hence

$$||f - g_n|| = o(1), \quad n \to \infty.$$

Hence proved. □

Theorem C. Let (3.1.1) belongs to the class S and $\lim_{n\to\infty} |a_{n+1}|\log n = 0$, then $||f - S_n|| = o(1), \quad n \to \infty$.

Proof:- We notice that

$$\begin{aligned}\int_{-\pi}^{\pi}|f(x)-S_n(x)|dx &= \int_{-\pi}^{\pi}|f(x)-g_n(x)+g_n(x)-S_n(x)|dx \\ &\leq \int_{-\pi}^{\pi}|f(x)-g_n(x)|dx+\int_{-\pi}^{\pi}|g_n(x)-S_n(x)|dx\end{aligned}$$

Since

$$\begin{aligned}g_n(x) &= S_n(x)-\frac{a_{n+1}}{n+1}\tilde{D}_n'(x) \\ g_n(x)-S_n(x) &= \frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\end{aligned}$$

Therefore

$$\int_{-\pi}^{\pi}|f(x)-S_n(x)|dx \leq \int_{-\pi}^{\pi}|f(x)-g_n(x)|dx+\int_{-\pi}^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\right|dx$$

Since $\lim_{n\to\infty}\int_{-\pi}^{\pi}|f(x)-g_n(x)|dx = 0$ by our Theorem B and $\int_{-\pi}^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\right|dx$ behaves like $|a_{n+1}|\log n$ by Zygmund's Theorem, for large values of n.

Hence $||f-S_n|| = o(1), \quad n\to\infty.$ □

Chapter IV

On L^1-convergence of r-th Differential of Modified Trigonometric Sums

4.1 Introduction. Consider the cosine series

$$(4.1.1) \qquad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

Let the partial sum of (4.1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$. Further, let $f^r(x) = \lim_{n\to\infty} S_n^r(x), \quad r \in \{0,1,2,3....\}$ where $f^r(x)$ represents r^{th} derivative of $f(x)$ and $S_n^r(x)$ represents r^{th} derivative of $S_n(x)$.

Concerning the L^1- convergence of Ram and Kumari cosine sums [14]

$$(4.1.2) \qquad g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

Ram [14] proved the following result:

Theorem A. If (4.1.1) belongs to the class S and $\lim_{n\to\infty} |a_{n+1}| \log n = 0$, then

$$||f - g_n||_{L^1} = o(1), \quad n \to \infty.$$

Further, Tomovski [26] extended the Sidon-Teljakovskǐi class S to a new class $S_r, \quad r = 0, 1, 2, 3, ...,$ defined as follows:

Definition. A null sequence $\{a_n\}$ is said to belong to the class $S_r, \quad r = 0, 1, 2, 3, ...,$ if there exists a sequence $\{A_n\}$ such that

$$(4.1.3) \qquad A_n \downarrow 0, \quad n \to \infty,$$

$$(4.1.4) \qquad \sum_{n=0}^{\infty} n^r A_n < \infty,$$

$$(4.1.5) \qquad |\Delta a_n| \le A_n, \quad \forall \;\; n.$$

For r=0, this class reduces to class S. We note that, by $A_k \downarrow 0$ and $\sum_{k=1}^{\infty} k^r A_k < \infty$, it follows that $k^{r+1} A_k = o(1), \quad k \to \infty$.

It is obvious that $S_{r+1} \subset S_r, \qquad \forall \ \ r = 0, 1, 2, 3, ...$, but the converse of that inclusion is not true presented by the following examples:

Example 1. For $n = 1, 2, 3, ...$ define $a_n = \displaystyle\sum_{k=n+1}^{\infty} \frac{1}{k^2}$

Then $a_n \to 0 \ \ as \ \ n \to \infty$. And for $n = 0, 1, 2, 3, ..$

$$\begin{aligned}
\Delta a_n &= a_n - a_{n+1} = \sum_{k=n+1}^{\infty} \frac{1}{k^2} - \sum_{k=n+2}^{\infty} \frac{1}{k^2} \\
&= \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + ... \right] - \left[\frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + ... \right] \\
&= \frac{1}{(n+1)^2}
\end{aligned}$$

Firstly we shall show that $\{a_n\}_{n=1}^{\infty} \notin S_1$.

Let $\{A_n\}_{n=1}^{\infty}$ is an arbitrary positive sequence such that $A \downarrow 0$ and $\Delta a_n = |\Delta a_n| \le A_n$. However, $\displaystyle\sum_{n=1}^{\infty} n A_n \ge \sum_{n=1}^{\infty} \frac{n}{(n+1)^2}$ is divergent. i.e. $\{a_n\}_{n=1}^{\infty} \notin S_1$.

Now, for all $n = 0, 1, 2, ...$let $A_n = \dfrac{1}{n^2}$. Then $A_n \downarrow 0, \quad |\Delta a_n| \le A_n$

And $\displaystyle\sum_{n=0}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. i.e. $\{a_n\}_{n=0}^{\infty} \in S_0$.

We consider yet another example in support of Tomovski assertion as given below:

Example 2. For $n = 1, 2, 3, ..$ define $a_n = \displaystyle\sum_{k=n}^{\infty} \frac{1}{k^{r+2}}, \quad r = 0, 1, 2, 3, ...$

Then $a_n \to 0 \ \ as \ \ n \to \infty$. And for $n = 1, 2, 3, ..$

$$\begin{aligned}
\Delta a_n &= a_n - a_{n+1} = \sum_{k=n}^{\infty} \frac{1}{k^{r+2}} - \sum_{k=n+1}^{\infty} \frac{1}{k^{r+2}} \\
&= \left[\frac{1}{n^{r+2}} + \frac{1}{(n+1)^{r+2}} + ... \right] - \left[\frac{1}{(n+1)^{r+2}} + \frac{1}{(n+2)^{r+2}} + ... \right] \\
&= \frac{1}{n^{r+2}}
\end{aligned}$$

Let $\{A_n\}_{n=1}^{\infty}$ is an arbitrary positive sequence such that $A_n \downarrow 0$ and $|\Delta a_n| \le A_n$. However, $\displaystyle\sum_{n=1}^{\infty} n^{r+1} A_n \ge \sum_{n=1}^{\infty} n^{r+1} \frac{1}{n^{r+2}} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. i.e. $\{a_n\} \notin S_{r+1}$.

On the other hand, for all $n = 1, 2, 3, ...$ let $A_n = \dfrac{1}{n^{r+2}}$.

Then $A_n \downarrow 0, |\Delta a_n| \leq A_n$ and $\displaystyle\sum_{n=1}^{\infty} n^r A_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. i.e. $\{a_n\} \in S_r$.

The aim of this chapter is to generalize Theorem A for the cosine series with extended class S_r, $\quad r = 0, 1, 2, 3...$ of coefficient sequence and also to study the L^1-convergence of the r^{th} derivative of the cosine series.

4.2 Lemmas. The proofs of our results are based on the following lemmas:

Lemma 1 [21]. Let r be a nonnegative integer and $x \in [\frac{\pi}{n}, \pi]$, where $n \geq 1$. Then

$$D_n^{(r)}(x) = \sum_{k=0}^{r-1} \frac{\left(n+\frac{1}{2}\right)^k \ \sin\left[(n+\frac{1}{2})x + \frac{k\pi}{2}\right]}{\left(\sin(\frac{x}{2})\right)^{r+1-k}} \psi + \frac{\left(n+\frac{1}{2}\right)^r \sin\left[(n+\frac{1}{2})x + \frac{r\pi}{2}\right]}{2\sin\frac{x}{2}} \tag{4.2.1}$$

where the same ψ denotes various analytical functions of x independent of n. and $D_n(x)$ is the Dirichlet kernel.

Proof. The proof is straightforward in the case of $r = 0$. Assuming that Equation (4.2.1) holds, and taking the derivative of both sides in this equation, we have

$$\begin{aligned}
D_n^{(r+1)}(x) &= \sum_{k=0}^{r-1} \left(n+\frac{1}{2}\right)^{k+1} \sin\left[(n+\frac{1}{2})x + \frac{(k+1)\pi}{2}\right] \left(\sin(\frac{x}{2})\right)^{k-r-1} \psi \\
&+ \left(n+\frac{1}{2}\right)^{k} \sin\left[(n+\frac{1}{2})x + \frac{k\pi}{2}\right] \left(\sin(\frac{x}{2})\right)^{k-r-2} \psi \\
&+ \frac{\left(n+\frac{1}{2}\right)^{r+1} \sin\left[(n+\frac{1}{2})x + \frac{(r+1)\pi}{2}\right]}{2\sin\frac{x}{2}} \\
&+ \frac{\left(n+\frac{1}{2}\right)^{r} \ \sin\left[(n+\frac{1}{2})x + \frac{r\pi}{2}\right]}{\left(\sin^2(\frac{x}{2})\right)} \psi
\end{aligned}$$

Therefore,

$$\begin{aligned}
D_n^{(r+1)}(x) &= \sum_{k=0}^{r} \left(n+\frac{1}{2}\right)^{k} \sin\left[(n+\frac{1}{2})x + \frac{k\pi}{2}\right] \left(\sin(\frac{x}{2})\right)^{k-r-2} \psi \\
&+ \left(n+\frac{1}{2}\right)^{r+1} \sin\left[(n+\frac{1}{2})x + \frac{(r+1)\pi}{2}\right] 2\left(\sin(\frac{x}{2})\right)^{-1}.
\end{aligned}$$

The proof follows by induction.

Moreover, we can also write it as follow

$$D_n^{(r)}(x) = \sum_{k=0}^{r} \frac{\left(n+\frac{1}{2}\right)^k \sin\left[(n+\frac{1}{2})x + \frac{k\pi}{2}\right]}{\left(\sin(\frac{x}{2})\right)^{r+1-k}} \psi$$

□

Lemma 2 [26]. Let $\{\alpha_j\}_{j=0}^{k}$ be a sequence of real numbers. Then the following relation holds for $v = 0, 1, 2, ..., r$ and $r = 0, 1, 2, ...$

$$\begin{aligned} U_k &= \int_{\pi/(k+1)}^{\pi} \left| \sum_{j=0}^{k} \alpha_j \frac{(j+\frac{1}{2})^v \ \sin\left[(j+\frac{1}{2})x + \frac{v+3}{2}\pi\right]}{\left(\sin(\frac{x}{2})\right)^{r+1-v}} \right| dx \\ &= O\left((k+1)^{r-v+\frac{1}{2}} \ \left(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2v} \right)^{1/2} \right) \end{aligned}$$

Proof. Applying first Cauchy inequality, yields

$$\begin{aligned} U_k &\le \left[\int_{\pi/(k+1)}^{\pi} \frac{dx}{(\sin(\frac{x}{2}))^{2(r+1-v)}} \right]^{1/2} \\ &\times \left\{ \int_{\pi/(k+1)}^{\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right) x + \frac{(v+3)\pi}{2} \right] \right]^2 dx \right\}^{1/2} \end{aligned}$$

Since

$$\begin{aligned} \int_{\pi/(k+1)}^{\pi} \frac{dx}{\left(\sin(\frac{x}{2})\right)^{2(r+1-v)}} &\le \pi^{2(r+1-v)} \int_{\pi/(k+1)}^{\pi} \frac{dx}{x^{2(r+1-v)}} \\ &\le \frac{\pi(k+1)^{2(r+1-v)-1}}{2(r+1-v)-1} \\ &\le \pi(k+1)^{2(r+1-v)-1} \end{aligned}$$

we have

$$\begin{aligned} U_k &\le \left[\pi(k+1)^{2(r+1-v)-1} \right]^{1/2} \\ &\times \left\{ \int_{0}^{\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[\left(j+\frac{1}{2}\right) x + \frac{(v+3)\pi}{2} \right] \right]^2 dx \right\}^{1/2} \end{aligned}$$

$$\begin{aligned} U_k &\le \left[2\pi(k+1)^{2(r+1-v)-1} \right]^{1/2} \\ &\times \left\{ \int_{0}^{2\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^v \sin\left[(2j+1)t + \frac{(v+3)\pi}{2} \right] \right]^2 dt \right\}^{1/2} \end{aligned}$$

Then, applying Parseval's equality, we obtain:

$$U_k \leq \left[2\pi(k+1)^{2(r+1-v)-1}\right]^{1/2} \left[\sum_{j=0}^{k} |\alpha_j|^2 (j+1)^{2v}\right]^{1/2}$$

Finally,

$$U_k = O\left((k+1)^{r-v+\frac{1}{2}} \left(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2v}\right)^{1/2}\right)$$

□

Lemma 3 [26]. Let $\{a_k\}$ be a sequence of real numbers such that $|a_k| \leq 1, \quad \forall\ k$. Then there exists a constant $C > 0$ such that for any $n \geq 0$ and $r = 0, 1, 2, 3...$

$$\int_{\pi/(n+1)}^{\pi} \left|\sum_{k=0}^{n} a_k D_k^r(x)\right| dx \leq C(n+1)^{r+1}. \tag{4.2.2}$$

Proof. Since from lemma 1,

$$D_n^{(r)}(x) = \sum_{k=0}^{r} \frac{\left(n+\frac{1}{2}\right)^k \ \sin\left[(n+\frac{1}{2})x + \frac{k\pi}{2}\right]}{\left(\sin(\frac{x}{2})\right)^{r+1-k}} \psi_k(x)$$

where ψ_k denotes various entire $4\pi-$ periodic functions of x, independent of n.

Now, we have

$$\int_{\pi/(n+1)}^{\pi} \left|\sum_{k=0}^{n} a_k D_k^r(x)\right| dx \leq \int_{\pi/(n+1)}^{\pi} \left|\sum_{j=0}^{n} a_j \left(\sum_{v=0}^{r} \frac{\left(j+\frac{1}{2}\right)^v \ \sin\left[(j+\frac{1}{2})x + \frac{v+3}{2}\pi\right]}{\left(\sin(\frac{x}{2})\right)^{r+1-v}} \psi_v(x)\right)\right| dx$$

Since ψ_v are bounded, we have:

$$\int_{\pi/(n+1)}^{\pi} \left|\sum_{j=0}^{n} a_j \frac{\left(j+\frac{1}{2}\right)^v \ \sin\left[(j+\frac{1}{2})x + \frac{v+3}{2}\pi\right]}{\left(\sin(\frac{x}{2})\right)^{r+1-v}} \psi_v(x)\right| dx \leq k\ U_n$$

Where U_n is the integral as in lemma 2 and k=k(r) is a positive constant.

Applying lemma 2, to the last integral, we obtain:

$$\begin{aligned}
\int_{\pi/(n+1)}^{\pi} \left|\sum_{j=0}^{n} a_j \frac{\left(j+\frac{1}{2}\right)^v \ \sin\left[(j+\frac{1}{2})x + \frac{v+3}{2}\pi\right]}{\left(\sin(\frac{x}{2})\right)^{r+1-v}} \psi_v(x)\right| dx &= O\left((n+1)^{r-v+\frac{1}{2}} \left(\sum_{j=0}^{n} \alpha_j^2 (j+1)^{2v}\right)^{\frac{1}{2}}\right) \\
&= O\left((n+1)^{r-v+\frac{1}{2}}\ (n+1)^{v+\frac{1}{2}}\right) \\
&= O\left((n+1)^{r+1}\right)
\end{aligned}$$

Finally the inequality (4.2.2) is satisfied. □

Lemma 4 [5]. If $|a_k| \leq 1$, then

$$\int_0^\pi \left| \sum_{k=0}^{n} a_k D_k(x) \right| dx \leq C(n+1).$$

where C is positive absolute constant.

Lemma 5 [21]. $||D_n^r(x)||_{L^1} = O(n^r \log n), \quad r = 0, 1, 2, 3...$, where $D_n^r(x)$ represents the r^{th} derivative of Dirichlet kernel.

The proof of the lemma 4 and 5 have already been reported in chapter III.

4.3 Results. We prove the following two theorems:

Theorem 1. If (4.1.1) belongs to the class S_r and $\lim_{n\to\infty} |a_{n+1}| \log n = 0$, then

$$||f - g_n|| = o(1), \quad n \to \infty.$$

Proof. We have

$$\begin{aligned}
g_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{j} \right) k \cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\Delta \left(\frac{a_k}{k} \right) + \Delta \left(\frac{a_{k+1}}{k+1} \right) + ... + \Delta \left(\frac{a_n}{n} \right) \right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1} + \frac{a_{k+1}}{k+1} - \frac{a_{k+2}}{k+2} + ... \frac{a_n}{n} - \frac{a_{n+1}}{n+1} \right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1} \right)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
g_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx \\
&= S_n(x) - \frac{a_{n+1}}{n+1} \tilde{D}_n'(x)
\end{aligned}$$

Since $\{a_n\}$ is a null sequence, we have therefore

$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} S_n(x) = f(x)$ for $x \in (0,\pi]$.

Now, we consider

$$f(x) - g_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + \frac{a_{n+1}}{n+1}\tilde{D}'_n(x)$$

Making use of Abel's transformation and lemma 4, we have

$$\begin{aligned}
\int_0^{\pi} |f(x)-g_n(x)|dx &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1}D_n(x) + \frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx \\
&\le \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right| dx + \int_0^{\pi} \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx + |a_{n+1}| \int_0^{\pi} |D_n(x)|\, dx \\
&= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x)\right| dx + \int_0^{\pi} \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx + |a_{n+1}| \int_0^{\pi} |D_n(x)|\, dx
\end{aligned}$$

Again, an application of Abel's transformation, we have

$$\begin{aligned}
\int_0^{\pi} |f(x)-g_n(x)|dx &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x)\right| dx + \int_0^{\pi} \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx + |a_{n+1}| \int_0^{\pi} |D_n(x)|\, dx \\
&\le \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} k^r \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x)\right| dx + \int_0^{\pi} \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx + |a_{n+1}| \int_0^{\pi} |D_n(x)|\, dx \\
&\le C \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} k^r (k+1)\Delta A_k \right| + \int_0^{\pi} \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx + |a_{n+1}| \int_0^{\pi} |D_n(x)|\, dx
\end{aligned}$$

Since, by Zygmund's Theorem ([1], p. 458) we have

$$\begin{aligned}
\int_0^{\pi} \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx &\le \int_{-\pi}^{\pi} \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx \\
&= \frac{|a_{n+1}|}{n+1} \int_{-\pi}^{\pi} |\tilde{D}'_n(x)|dx
\end{aligned}$$

Using lemma 5, we get

$$\begin{aligned}
\int_0^{\pi} \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx &= \frac{|a_{n+1}|}{n+1} \quad n \log n \\
&= C \quad |a_{n+1}| \log n \\
&\sim |a_{n+1}| \log n
\end{aligned}$$

And, $|a_{n+1}| \displaystyle\int_0^{\pi} |D_n(x)|\, dx \approx |a_{n+1}| \log n$.

Therefore,

$$\int_0^\pi |f(x) - g_n(x)|dx = C\int_0^\pi \left|\sum_{k=n+1}^{\infty} k^r(k+1)\Delta A_k\right| + o(|a_{n+1}|\log n) + o(|a_{n+1}|\log n)$$

Moreover,

$$\sum_{k=0}^{n} k^r A_k = \sum_{k=0}^{n-1} \Delta A_k (k+1)k^r + n^{r+1}A_n$$

Now, $\sum_{k=0}^{n} k^r A_k < \infty$ as $n \to \infty$ by class S_r and by Oliver's theorem $n^{r+1}A_n = o(1)$ as $n \to \infty$.

Therefore,

$$\sum_{k=0}^{n-1} \Delta A_k (k+1)k^r < \infty \quad as \quad n \to \infty.$$

Hence, $||f - g_n|| = o(1), \quad n \to \infty$ if and only if $|a_{n+1}|\log n = o(1), \quad n \to \infty.$ □

Corollary 1. If (4.1.1) belongs to the class $S_r, \quad r = 0, 1, 2, 3, ...$ then

$||f - S_n||_{L^1} = o(1), \quad n \to \infty$ if and only if $|a_{n+1}|\log n = o(1), \quad n \to \infty.$

Proof. Consider,

$$\begin{aligned} ||f - S_n|| &= ||f - g_n + g_n - S_n|| \\ &\leq ||f - g_n|| + ||g_n - S_n|| \\ &= ||f - g_n|| + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx \end{aligned}$$

Further, $||f - g_n|| = o(1), \ n \to \infty$ (by theorem 1) and also we know that $\int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx$ behaves like $|a_{n+1}|\log n$ by Zygmund's Theorem ([1], Vol.II, p.458).

Thus,

$$||f - S_n||_{L^1} = o(1), \quad n \to \infty \text{ if and only if } |a_{n+1}|\log n = o(1), \quad n \to \infty, \qquad \square$$

Theorem 2. If (4.1.1) belongs to the class S_r and $n^r|a_{n+1}|\log n = o(1), \ n \to \infty$ then

$$||f^r - g_n^r|| = o(1), \quad n \to \infty.$$

Proof. Consider,

$$\begin{aligned}
g_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left(\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + ... + \Delta\left(\frac{a_n}{n}\right)\right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1} + \frac{a_{k+1}}{k+1} - \frac{a_{k+2}}{k+2} + ...\frac{a_n}{n} - \frac{a_{n+1}}{n+1}\right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n} k\cos kx \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n} k\cos kx \\
&= S_n(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n'(x)
\end{aligned}$$

We have then,

$$g_n^r(x) = S_n^r(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) \tag{4.3.1}$$

where $g_n^r(x)$ represents the r^{th} derivative of $g_n(x)$ and $\tilde{D}_n^r(x)$ represents the r^{th} derivative of conjugate Dirichlet kernel.

Since $\{a_n\} \in S_r$, $r \in \{0, 1, 2, 3....\}$, is a null sequence and $\tilde{D}_n^r(x)$ is bounded in $(0, \pi]$

Therefore,

$$\lim_{n\to\infty} g_n^r(x) = \lim_{n\to\infty} S_n^r(x) = f^r(x).$$

Now, it follows from (4.3.1) that

$$\begin{aligned}
f^r(x) - g_n^r(x) &= \sum_{k=1}^{\infty} a_k\ k^r\cos(kx + \frac{r\pi}{2}) - \sum_{k=1}^{n} a_k\ k^r\cos(kx + \frac{r\pi}{2}) + \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) \\
&= \sum_{k=n+1}^{\infty} a_k\ k^r\cos(kx + \frac{r\pi}{2}) + \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)
\end{aligned}$$

Making use of Abel's transformation, we have

$$f^r(x) - g_n^r(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) - a_{n+1}D_n^r(x) + \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)$$

where $D_n^r(x)$ denotes the r^{th} derivative of Dirichlet kernel.

We notice further that

$$\begin{aligned}
||f^r - g_n^r|| &= \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) - a_{n+1} D_n^r(x) + \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx \\
&\leq \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) \right| dx + \int_0^\pi |a_{n+1} D_n^r(x)| dx + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx \\
&= \int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k^r(x) \right| dx + \int_0^\pi |a_{n+1} D_n^r(x)| dx + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx
\end{aligned}$$

Applying Abel's transformation we have,

$$\begin{aligned}
||f^r - g_n^r|| &= \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i^r(x) \right| dx + \int_0^\pi |a_{n+1} D_n^r(x)| dx + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx \\
&\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i^r(x) \right| dx + \int_0^\pi |a_{n+1} D_n^r(x)| dx + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx
\end{aligned}$$

Moreover, by use of lemma 3 and lemma 5, we have

$$||f^r - g_n^r|| \leq C \int_0^\pi \left| \sum_{k=n+1}^{\infty} (k+1)^{r+1} \Delta A_k \right| dx + O(n^r |a_{n+1}| \log n) + O(n^r |a_{n+1}| \log n)$$

As we know that,

$$\sum_{k=0}^{n} k^r A_k = \sum_{k=0}^{n-1} \Delta A_k (k+1) k^r + n^{r+1} A_n$$

Now, by class S_r , $\sum_{k=0}^{n} k^r A_k < \infty$ as $n \to \infty$ and by Oliver's theorem $n^{r+1} A_n = o(1)$ as $n \to \infty$.

Therefore,

$$\sum_{k=0}^{n-1} \Delta A_k (k+1)^{r+1} = o(1) \quad as \quad n \to \infty.$$

Therefore, $$||f^r - g_n^r|| = o(1) + O(n^r |a_{n+1}| \log n)$$

Hence,

$$||f^r - g_n^r|| = o(1), \quad n \to \infty \text{ if and only if } n^r |a_{n+1}| \log n = o(1), \; n \to \infty. \qquad \square$$

Corollary 2. If (4.1.1) belongs to the class S_r, $\quad r = 0, 1, 2, 3....$ then

$||f^r - S_n^r||_{L^1} = o(1), \quad n \to \infty$ if and only if $|a_{n+1}|n^r \log n = o(1), \quad n \to \infty.$

Proof. Consider,

$$\begin{aligned} ||f^r - S_n^r|| &= ||f^r - g_n^r + g_n^r - S_n^r|| \\ &\leq ||f^r - g_n^r|| + ||g_n^r - S_n^r|| \\ &= ||f^r - g_n^r|| + \left|\left|\frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)\right|\right| \\ &= ||f^r - g_n^r|| + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)\right| dx \end{aligned}$$

Further, $||f^r - g_n^r||_{L^1} = o(1), \ \ n \to \infty$ (by theorem 2) and also we know that $\int_0^\pi \left|\frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)\right| dx$ behaves like $n^r|a_{n+1}|\log n$ for large value of n by lemma 5.

Thus,

$$||f^r - S_n^r||_{L^1} = o(1), \quad n \to \infty \text{ if and only if } n^r|a_{n+1}|\log n = o(1), \quad n \to \infty, \qquad \square$$

References

[1] Bary, N.K., A treatise on trigonometric series, Vol I and Vol II, **Pergamon Press**, London (1964).

[2] Bhatia, S.S. and Ram B., On L^1-convergence of certain modified trigonometric sums, **Indian J. Math.**, 35 (2)(1993), 171-176.

[3] Boljanic, R. and Stanojević, Č.V., A class of L^1-convergence, **Trans. Amer. Math. Soc.,** 269(2)(1982), 677-683.

[4] Bö; H., Convergence of certain cosine sums in the metric space L^1, **J. Sci. Eng.,** 2(1986), 123-128.

[5] Fomin, G.A., On Linear methods for summing Fourier series, **Mat. Sb.**, 66(107) (1964), 114-152.

[6] Garrett, J.W. and Stanojević, Č.V., On Integrability and L^1-convergence of certain cosine sums, **Notices, Amer. Math Soc.**, 22(1975), A-166.

[7] Garrett, J.W. and Stanojević, Č.V., On L^1-convergence of certain cosine sums, **Proc. Amer. Math. Soc.**, 54(1976), 101-105.

[8] Hardy, G.H. and Littlewood, J.E., Some new properties of Fourier coefficients, **J. London Math. Soc.**, 6 (1931), 3-9.

[9] Hooda, N. and Ram, B., Convergence of certain Modified Cosine Sum, **Indian J. Math.**, 1(2002), 41-46.

[10] Kano, T., Coefficients of some trigonometric series, **J. Fac. Sci. Shinshu Univ.**, 3(1968), 153-162.

[11] Kaur, K., Bhatia, S.S. and Ram, B., Integrability and L^1-convergence of Rees-Stanojević Sums with Generalized Semi-convex Coefficients, **International Journal of Mathematics & Mathematical Sciences**, 30(11)(2002), 645-650.

[12] Kaur, K., Bhatia, S.S. and Ram, B., L^1- convergence of certain trigonometric sums, **Georgian Math. J.**, 11(1)(2004), 99-104.

[13] Kolmogorov, A.N., Sur l'ordere de grandeur des coefficients de la series de Fourier - Lebesque, **Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys.**, (1923), 83-86.

[14] Kumari, S. and Ram, B., L^1-convergence of a modified cosine sum, **Indian J. Pure Appl. Math.**, 19(11)(1988), 1101-1104.

[15] Ram, B., Convergence of certain cosine sums in the metric space L, **Proc. Amer. Math. Soc.**, 66(2)(1977), 258-260.

[16] Ram, B., A sufficient condition for the integrability of Rees Stanojević sum, **Kyungpook Mathematical Journal**, 19(1979), 257-260.

[17] Ram, B., Integrability of Rees - Stanojević sums, **Acta Sci. Math.(Szeged)**, 42(1980), 153-155.

[18] Ram, B. and Kumari S., On L^1-convergence of certain Trigonometric Sums, **Indian J. Pure Appl. Math.**, 20(9)(1989), 908-914.

[19] Rees, C.S. and Stanojević, Č.V., Necessary and sufficient condition for integrability of certain cosine sums, **J. Math. Anal. Appl.**, 43(1973), 579-586.

[20] Shah, S.M., Trigonometric series with quasi-monotone coefficients, **Proc. Amer. Math. Soc.**, 13(1962), 266-273.

[21] Sheng, Shuyun, The extension of the theorems of Č.V.Stanojević and V.B.Stanojević, **Proc. Amer. Math. Soc.,** 110(1990), 895-904.

[22] Sidon, S., Hinreichende Bedingungen für den Fourier-Charakter einer trigonometrischen Reihe, **J. London Math. Soc.**, 14(1939), 158-160.

[23] Singh, N. and Sharma, K.M., Convergence of certain cosine sums in the metric space L, **Proc. Amer. Math. Soc.**, 72(1978), 117-120.

[24] Szász, O., Quasi-monotone series, **Amer. J. Math.**, 70(1948), 203-206.

[25] Tomovski, Ž, An Extension the Garrett-Stanojević class, **Approx. Theory and its Applications,** 16(1)(2000), 46-51.

[26] Tomovski, Ž, An Extension of The Sidon-Fomin Type Inequality and its Applications, **Math. Ineq. and Appl.,** 4(2)(2001),231-238.

[27] Tomovski, Ž, On the theorem of N.Singh and K.M. Sharma, **Math. Comm.,** 7(2002), 119-122.

[28] Tomovski, Ž, Remarks on some classes of Fourier coefficients, **Analysis Mathematica,** 29(2003), 165-170.

[29] Teljakovskiĭi, S.A., Some estimates for trigonometric series with quasi-convex coefficients, **Mat. Sb.**, 63(105)(1964), 426-444.

[30] Teljakovskiĭi, S.A., A sufficient condition of Sidon for the integrability of trigonometric series, **Mat. Zametki**, 14(3)(1973), 317-328.

[31] Young, W.H., On the Fourier series of bounded functions, **Proc. London Math. Soc.**, 12(2)(1913), 41-70.

[32] Zygmund, A., Trigonometric Series, Vol. I & Vol. II, **Cambridge Univ. Press**, 1959.