SYMMETRIES AND EXACT SOLUTIONS OF SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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DECLARATION

It is certified that the thesis is entirely my own and that the ideas and references cited herein have been duly acknowledged.

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CERTIFICATE

This is to certify that the thesis entitled “ **Symmetries and Exact Solutions of Some Nonlinear Partial Differential Equations** ” submitted by Mr. Sachin Kumar in fulfilment of the requirements for the award of degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala is a record of candidate's own work carried out by him under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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Dedicated to

My Father
and
My Wife

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Abstract

The thesis entitled SYMMETRIES AND EXACT SOLUTIONS OF SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS is devoted to find symmetries and exact solutions of some nonlinear partial differential equations (PDEs) which represent some physically relevant systems. The thesis comprises seven chapters.

Chapter 1 is introductory and consists of prerequisites of the present work. It presents primarily the methodologies utilized in the thesis and a brief account of the related studies made by various authors in the field.

In **Chapter 2**, we have investigated the symmetries and invariant solutions of b-family equation and modified b-family equation. Firstly, the Lie group method is utilized for the purpose of obtaining the group infinitesimals of b-family equation

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx},\ t > 0,\ x \in R,$$

where b is a positive integer. The basic fields of the optimal system lead to reductions that are inequivalent with respect to the symmetry transformations. Secondly, we used Direct method introduced by Clarkson and Krusksal to find symmetries of b-family equation. We presented some exact solutions of b-family equation.

In this chapter, We have also investigated the symmetries of modified b-family equation

$$u_t - u_{xxt} + (b+1)u^2u_x = bu_xu_{xx} + uu_{xxx},$$

which describe the balance between the convection and the stretching for small viscosity in the dynamics of $1D$ nonlinear waves in fluids. We have shown that only non constant similarity reduction obtainable either by Lie classical method or Direct method, is travelling wave solution of b-family equation.

In **Chapter 3**, coupled Higgs field equation and Hamiltonian amplitude equation are studied using the Lie classical method. Symmetry reductions and exact solutions are reported for Higgs equations and Hamiltonian amplitude equation. We also establish the travelling wave solutions involving parameters of the coupled Higgs equations and Hamiltonian amplitude equation by using $(\frac{G'}{G})$-expansion method. The travelling waves solutions expressed by hyperbolic, trigonometric and the rational functions are obtained.

In **Chapter 4**, we considered the variable coefficient form of (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation (vcZKMEW) that is given as

$$u_t + 3\alpha(t)u^2u_x + \beta(t)u_{xxt} + \delta(t)u_{xyy} = 0,$$

where $\alpha(t)$, $\beta(t)$ and $\delta(t)$ are arbitrary functions of t. By using Lie group analysis, symmetries for the equation are obtained. Using symmetries, the vcZKMEW equation is reduced to two dimensional partial differential equation. Exact solutions of reduced two dimensional PDE are obtained and corresponding exact solutions of vcZKMEW equation are shown.

In **Chapter 5**, the variable coefficients version of the Benjamin-Bona-Mahony (BBM) equation

$$u_t + \alpha(t)u_x + \beta(t)uu_x - \delta(t)u_{xxt} = 0,$$

where $\alpha(t), \beta(t)$ and $\delta(t)$ are arbitrary functions of t, has been investigated for symmetries and some interesting exact solutions have been derived. The Painlevé analysis of an ODE has also been performed which shows that it is not integrable.

In **Chapter 6**, the Painlevé analysis of (2+1)- dimensional variable coefficients Broer-Kaup (VCBK) equation is performed by the Weiss-Kruskal approach to check the Painlevé property. Similarity reductions of the VCBK equation to one-dimensional partial differential equations including Burgers equation are investigated. The Lie group formalism is applied again on one of the investigated partial differential equation to derive symmetries, and the ordinary differential equations deduced from the optimal system of subalgebras are further studied and some exact solutions are obtained.

In **Chapter 7**, the variable-coefficients Gardner (vc-Gardner) equation is considered. By using the Painlevé analysis and Lie group analysis, the Painlevé properties and symmetries for the equation are obtained. The exact solutions generated from the symmetries and Painlevé analysis are presented.

List of Research Papers

1. "Benjamin-Bona-Mahony (BBM) Equation with Variable Coefficients: Similarity Reductions and Painlev$\acute{e}$ Analysis ", Applied Mathematics and Computation, 217 (2011) 7021-7027. **(Impact Factor 1.317) (SCI)**

2. "Exact Solutions of b-family Equation: Classical Lie Approach and Direct Method ", International Journal of Nonlinear Science 11 (1) (2011) 59-67.

3. "Painleve Analysis, Lie Symmetries and Exact Solutions for (2+1)-Dimensional Variable Coefficients Broer-Kaup Equations", Communications in Nonlinear Science and Numerical Simulations, 17 (2012) 1529-1541. **(Impact Factor 2.806) (SCI)**

4. "Symmetry Reductions and Exact Solutions of Modified b-family Equation", Indian Journal of Industrial and Applied Mathematics (Accepted).

5. "Coupled Higgs Field Equation and Hamiltonian Amplitude Equation: Lie Classical Approach and $(\frac{G'}{G})$-Expansion Method", Parmana-Journal of Physics 79 (1) (2012) 41-60. (**Impact Factor .575) (SCI)**

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Chapter 1

Introduction

A single or a system of differential equations are at the core of exact sciences, those disciplines readily quantified and modeled mathematically. In many situations, problems of physical interest are often translated in terms of differential equations that may be ordinary, partial, linear or nonlinear in nature. Exact solutions of these resulting equations are of much interest, both from mathematical and application points of view. On account of their applications in such conventional areas as physics and engineering and most recent applications in the disciplines of biology, chemistry, ecology and economics, differential equations have retained there central role. However, Physical examples of linear systems are relatively rare. With advent of high speed computing facilities study of nonlinear partial differential equations is of current interest and emphasis is shifted from the classical study of linear systems to the fascinating problems encountered in the study of nonlinear systems.

In study of nonlinear partial differential equations (PDEs), finding explicit solutions has great theoretical and practical importance. Nonlinear equations are difficult to solve and their study is regarded as a confusing endeavor. But the strong desire of exact and more general solutions to nonlinear PDEs governing nonlinear phenomenon in technological enhancement and for research purpose made tremendous growth in research interest in this field. There is much current interest in obtaining exact solutions of nonlinear PDEs; these solutions provide information about nonlinear phenomena and describe various aspects of the physical phenomena.

In order to unify and extend various specialized solution methods for ordinary differential equations (ODEs), Lie (1881) introduced the notion of continuous groups now known as Lie groups. Lie's method of infinitesimal transformation groups which essentially reduces the number of independent variables in PDE and reduces the order of ODE has been widely used in equations of mathematical physics. Lie's work has also been responsible for systematically relating a large number of topics and methods in ordinary differential equations. We now turn our attention to one of the most important group in relation to the differential equation that is, symmetry group. A symmetry group of a system of differential equations is a group of transformations which maps any solution to another solution of the system. In Lie's framework such a group depends on continuous parameters and consists of either the point transformations or more generally, contact transformations.

The symmetry group method pioneered by Lie [65] is an original and powerful method for finding symmetry reductions of nonlinear partial differential equations. Though the method is entirely algorithmic, it often involves a large amount of tedious algebra and auxiliary calculations which are virtually very difficult to manage manually. Many symbolic manipulation programs [42] have been developed in order to facilitate the determination of the associated similarity reductions. Ovsiannikov [88] and his coworkers began a systematic programme of successfully applying the Lie continuous group of transformations method to wide range of problems. Bluman and Cole [12] proposed a generalization of Lie's method which they called the "nonclassical method of group-invariant solutions," which itself have been generalized by Olver and Rosenau [87]. All these methods determine Lie point transformations of a given partial differential equation. i.e., transformations depending only on the independent and dependent variables. Noether [82] recognized that Lie's method could be generalized by allowing the transformation to depend upon the derivatives of the dependent variables as well as the independent and dependent variables. The associated symmetries, called Lie-Bäcklund symmetries, can also be determined by an algorithmic method [3, 86]. Bluman et al. [14] introduce an algorithmic method which yields new classes of symmetries of a given partial differential equation that are neither Lie point nor Lie-B$\ddot{a}$cklund symmetries. This method was further generalized by Olver and Rosenau [87] and the concept of generalized conditional symmetry was proposed by Fokas and Liu [33]. A common characteristic of all these methods for finding symmetries and associated similarity reductions of a given partial differential equation is the use of group theory. Clarkson [20] developed a direct, algorithmic method for finding similarity reductions of PDEs, and the novel features of this method are entirely straightforward without group analysis. Since the symmetry group method can be used to obtain symmetry reductions of nonlinear PDEs with arbitrary functions [12, 86], an open problems was then proposed by Clarkson [20, 23] to develop the direct method for seeking symmetry reductions of nonlinear PDEs with arbitrary functions.

The work carried out in this thesis is devoted to find the symmetries and exact solutions of nonlinear PDEs. The prime objective and motivation in carrying out the proposed study is to demonstrate the importance and efficacy of group theoretic methods to find the symmetries and exact solutions of some important physical systems.

The present chapter is divided into two sections. The first section gives important preliminaries. The second section contains methodology, more specifically, a brief resume of the 'Lie classical method (1891) [65]', 'Direct method' due to Clarkson and Kruskal (1989) [22] and $(\frac{G'}{G})$-expansion method put forward by Wang et al. (2008) [105], has been given. Also included in this section are necessary mathematical tools for establishing the integrability of a nonlinear differential equation, namely the 'Painlev$\acute{e}$ analysis' as suggested by Wiess, Tabor and Carnevale (1983) [112].

Sections, subsections, theorems, remarks, equations etc., are numbered consecutively along with the chapter number. For example, Section 2.1 means Section 1 of Chapter 2, Subsection 6.4.1 means Subsection 1 of Section 4 in Chapter 6 and Theorem 6.1 means Theorem 1 in Chapter 6.

1.1 Preliminaries

In this section, some basic definition and fundamentals of present work are given.

1.1.1 One-Parameter Lie Group of Transformations and Infinitesimal Transformations

Definition 1.1.1 A *group* G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

(i) **Closure property:** For any elements a and b of G, $\phi(a,b)$ is an element of G.

(ii) **Associative property:** For any elements a, b, c of G:

$\phi(a, \phi(b,c)) = \phi(\phi(a,b),c)$.

(iii) **Identity element:** There exists a unique identity element e of G such that for any element a of G:

$\phi(a,e) = \phi(e,a) = a$

(iv) **Inverse element:** For any element a of G, there exists a unique inverse element a^{-1} in G such that

$\phi(a,a^{-1}) = \phi(a^{-1},a) = e$.

Definition 1.1.2 [12] Let $\bar{x} = (x_1, x_2, x_3, ..., x_n)$ lie in region $D \subset \Re^n$. The set of transformations

$$\bar{x}^* = \bar{X}(\bar{x};\epsilon),$$

defined for each $\bar{x}$ in D and parameter ϵ in set $T \subset \Re$, with $\phi(\epsilon,\delta)$ defining a law of composition of parameters ϵ and δ in T, forms a *one-parameter group of transformations* on D, if the following hold:

(i) For each ϵ in T, the transformations are one-to-one onto D. [Hence, $\bar{x}^*$ lies in D.]

(ii) T with the law of composition ϕ forms a group G.

(iii) For each $\bar{x}$ in D, $\bar{x}^* = \bar{x}$ when $\epsilon = \epsilon_0$ corresponds to the identity e, i.e.,

$$\bar{X}(\bar{x};\epsilon_0) = \bar{x}.$$

(iv) If $\bar{x}^* = \bar{X}(\bar{x};\epsilon),\ \bar{x}^{**} = \bar{X}(\bar{x}^*;\delta)$ then

$$\bar{x}^{**} = \bar{X}(\bar{x};\phi(\epsilon,\delta)).$$

Definition 1.1.3 [12] A one-parameter group of transformations defines a *one-parameter Lie group of transformations* if, in addition to satisfying axioms (i)-(iv) of definition (1.1.2) the following hold:

(v) ϵ is a continuous parameter, i.e., T is an interval in $\Re$. Without loss of generality, $\epsilon = 0$ corresponds to the identity element e.

(vi) $\bar{X}$ is infinitely differentiable with respect to $\bar{x}$ in D and an analytic function of ϵ in T.

(vii) $\phi(\epsilon,\delta)$ is an analytical function of ϵ and δ, for $\epsilon,\delta \in T$.

Definition 1.1.4 [12] Consider a one-parameter Lie group of transformations

$$\bar{x}^* = X(\bar{x},\epsilon) \tag{1.1.1}$$

with the identity e and law of composition ϕ. Expanding (1.1.1) about $\epsilon = 0$, we get

$$\begin{aligned}\bar{x}^* &= \bar{x} + \epsilon\left(\frac{\partial \bar{X}(\bar{x};\epsilon)}{\partial \epsilon}\,|_{\epsilon=0}\right) + \frac{1}{2}\epsilon^2\left(\frac{\partial^2 \bar{X}(\bar{x};\epsilon)}{\partial \epsilon^2}\,|_{\epsilon=0}\right) + ...\\ &= \bar{x} + \epsilon\left(\frac{\partial \bar{X}(\bar{x};\epsilon)}{\partial \epsilon}\,|_{\epsilon=0}\right) + O(\epsilon^2)\end{aligned}$$

Let $\bar{\xi}(\bar{x}) = \frac{\partial \bar{X}(\bar{x};\epsilon)}{\partial \epsilon}\,|_{\epsilon=0}$, then the transformation $\bar{x} + \epsilon\bar{\xi}(\bar{x})$ is called the *infinitesimal transformation* of the Lie group of transformations (1.1.1). The components of $\bar{\xi}(\bar{x})$ are called the infinitesimals of (1.1.1).

Theorem 1.1.1 [10] (First Fundamental Theorem of Lie). There exists a parametrization $\tau(\epsilon)$ such that the Lie group of transformations (1.1.1) is equivalent to the solution of an initial value problem for a system of first-order ODEs given by

$$\frac{dx^*}{d\tau} = \xi(x^*), \tag{1.1.2}$$

with $x^* = x$ when $\tau = 0$.

In particular,

$$\tau(\epsilon) = \int_0^{\epsilon} \Gamma(\epsilon')d\epsilon', \tag{1.1.3}$$

where $\Gamma(\epsilon) = \frac{\partial \phi(a,b)}{\partial b}|_{(a,b)=(\epsilon^{-1},\epsilon)}$ and $\Gamma(0) = 1$.
[ϵ^{-1} denotes the inverse element of ϵ].

Definition 1.1.5 [12] The *infinitesimal generator* of the one-parameter Lie group of transformations (1.1.1) is the operator

$$X(\bar{x}) = \bar{\xi}(\bar{x}).\nabla = \sum_{i=1}^{n} \xi_i(\bar{x}) \frac{\partial}{\partial x_i},$$

where ∇ is the gradient operator $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n} \right)$.

1.1.2 Invariance of Partial Differential Equations

Definition 1.1.6 [12] An infinitely differentiable function $F(\bar{x})$ is *invariant function* of the Lie group of transformations (1.1.1) if and only if, for any group transformation (1.1.1),

$$F(\bar{x}^*) = F(\bar{x}).$$

If $F(\bar{x})$ is an invariant function of (1.1.1), then $F(\bar{x})$ is said to be invariant under (1.1.1).

Theorem 1.1.2 [10] $F(\bar{x})$ is invariant under a Lie group of transformations (1.1.1) if and only if

$$XF(\bar{x}) = 0.$$

Definition 1.1.7 (Invariance for a System of PDEs) [10] Consider a system of N PDEs with m dependent variables $u = (u^1, u^2, u^3, ..., u^m)$ and n independent variables $x = (x_1, x_2, x_3, ..., x_n)$ given by

$$F^{\mu}(x, u, \partial u, \partial^2 u, ...\partial^k u) = 0, \quad \mu = 1, 2, 3, ..., N. \tag{1.1.4}$$

The one-parameter Lie group of point transformations

$$\begin{aligned} x^* &= X(x, u; \epsilon) \\ u^* &= U(x, u; \epsilon) \end{aligned} \tag{1.1.5}$$

leaves invariant the system of PDEs (1.1.4), if and only if its kth extension, leaves invariant the N surfaces in $(x, u, \partial u, \partial^2 u, ...\partial^k u)$-space, defined by (1.1.4).

Theorem 1.1.3 [10] (Infinitesimal Criterion for the Invariance of a System of PDEs). Let

$$X = \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta^\mu(x, u)\frac{\partial}{\partial u^\mu} \tag{1.1.6}$$

be the infinitesimal generator of the Lie group of transformations (1.1.5). Let

$$\begin{aligned} X^{(k)} = \xi_i(x, u)\tfrac{\partial}{\partial x_i} + \eta^\mu(x, u)\tfrac{\partial}{\partial u^\mu} + \eta^{(1)\mu}(x, u, \partial u)\tfrac{\partial}{\partial u_i^\mu} + ... \\ \eta^{(k)\mu}_{i_1, i_2, ..., i_k}(x, u, \partial u, ..., \partial^\mu u)\tfrac{\partial}{\partial u^\mu_{i_1, i_2, ..., i_k}} \end{aligned} \tag{1.1.7}$$

be the kth-extended infinitesimal generator of (1.1.6) [10], where $\mu = 1, 2, ..., m$ and $i_j = 1, 2, ..., n$ for $j = 1, 2, ..., k$, in terms of

$$\xi(x, u) = (\xi_1(x, u), \xi_2(x, u), ..., \xi_n(x, u)), \;\; \eta(x, u) = (\eta^1(x, u), \eta^2(x, u), ..., \eta^m(x, u)).$$

Then the one-parameter Lie group of point transformations (1.1.5) is admitted by the system of PDEs (1.1.4) if and only if

$$X^{(k)}F^\sigma(x, u, \partial u, \partial^2 u, ...\partial^k u) = 0, \tag{1.1.8}$$

when u satisfies (1.1.4) for each $\sigma = 1, 2, ..., N$.

Definition 1.1.8 (Invariant Solutions) $u = \theta(x)$, with components $u^\nu = \theta^\nu(x), \;\nu = 1, 2, ..., m$ is an invariant solution of the system of PDEs (1.1.4) resulting from admitted point symmetry with infinitesimal generator (1.1.6) if and only if:

(i) $u^\nu = \theta^\nu(x)$ is an invariant surface of (1.1.8) for each $\nu = 1, 2, ..., m$.

(ii) $u = \theta(x)$ satisfies (1.1.4).

It follows that $u = \theta(x)$ is an invariant of the system of PDEs (1.1.4), resulting from its invariance under the Lie group of point transformations (1.1.5), if and only if $u = \theta(x)$ satisfies

$$\begin{aligned} &(i)\; X(u^\nu - \theta^\nu(x)) \text{ when } u = \theta(x), \;\nu = 1, 2, ..., m, \text{ i.e.,} \\ &\quad \xi_i(x, \theta(x))\tfrac{\partial \theta(x)}{\partial x_i} = \eta^\nu(x, \theta(x)), \;\nu = 1, 2, ..., m. \end{aligned} \tag{1.1.9}$$

$$\begin{aligned}&(ii)\ F^{\mu}(x,u,\partial u,\partial^2 u,...\partial^k u)=0,\ \text{when}\ u=\theta(x),\ \mu=1,2,3,...,N,\ \text{i.e.,}\\ &\quad F^{\mu}(x,u,\partial\theta(x),\partial^2\theta(x),...\partial^k\theta(x))=0,\ \mu=1,2,3,...,N,\end{aligned} \tag{1.1.10}$$

Equations (1.1.9) are the invariant surface conditions for the invariant solutions of the system of PDEs (1.1.4) resulting from its invariance under the point symmetry (1.1.5). As is the situation for the scalar PDE, invariant solutions can be determined by the following procedure:

Invariant Form Method: Here, we solve the invariant surface conditions (1.1.9) by explicitly solving the corresponding characteristics equations for $u=\theta(x)$ given by

$$\frac{dx_1}{\xi_1(x,u)}=\frac{dx_2}{\xi_2(x,u)}=...=\frac{dx_n}{\xi_n(x,u)}=\frac{du^1}{\eta^1(x,u)}=\frac{du^2}{\eta^2(x,u)}=...=\frac{du^m}{\eta^m(x,u)}. \tag{1.1.11}$$

If $y_1(x,u),y_2(x,u),...,y_{n-1}(x,u),\rho^1(x,u),\rho^2(x,u),...,\rho^m(x,u)$, are $n+m-1$ functionally independent constants that arise from solving the system of $n+m-1$ first order ODEs (1.1.11) with the Jacobian $\frac{\partial(\nu^1,\nu^2,...,\nu^m)}{\partial(u^1,u^2,...,u^m)}\neq 0$, then the general solution $u=\theta(x)$ of the system of PDEs (1.1.9) is given, implicitly, by the invariant form

$$\nu^{\nu}(x,u)=\phi^{\nu}(y_1(x,u),y_2(x,u),...,y_{n-1}(x,u)), \tag{1.1.12}$$

where ϕ^{ν} is differentiable function of $y_1(x,u),y_2(x,u),...,y_{n-1}(x,u)$, for $\nu=1,2,...,m$. Note that $y_1(x,u),y_2(x,u),...,y_{n-1}(x,u),\nu^1(x,u),\nu^2(x,u),...,\nu^m(x,u)$ are $n+m-1$ functionally independent group invariants and hence are $n+m-1$ canonical coordinates for the Lie group of points transformations (1.1.5). Let $y_n(x,u)$ be the $(n+m)th$ canonical coordinate satisfying

$$Xy_n=1.$$

If the system of PDEs (1.1.4) is transformed into a system of PDEs in terms of independent variables $y_1,y_2,...,y_n$ and dependent variables $\nu^1,\nu^2,...,\nu^m$, then the transformed system of PDEs admits the one-parameter Lie group of transformations.

$$\begin{aligned}&y_i^*=y_i,\ i=1,2,..,n-1,\\ &y_n^*=y_n+\epsilon,\\ &\nu^{*\nu}=\nu^{\nu},\ \nu=1,2,...,m.\end{aligned} \tag{1.1.13}$$

Thus, the variable y_n does not appear explicitly in the transformed system of ODEs and, hence, the transformed system of PDEs has solutions of the form (1.1.12) consequently, the system of PDEs (1.1.4) has invariant solutions given implicitly by the invariant form (1.1.12). Such solutions are found by solving a reduced system of differential equations with $n-1$ independent variables $y_1, y_2, ..., y_{n-1}$ and m dependent variables $\nu^1, \nu^2, ..., \nu^m$. The variables $y_1, y_2, ..., y_{n-1}$ are commonly called similarity variables. The reduced system of differential equations is found by substituting the invariant form (1.1.12) into the given system of PDEs (1.1.4). We assume that this substitution does not lead to a singular differential equation. Note that if $\frac{\partial \xi}{\partial u} = 0$ as is typically the case, then $y_i = y_i(x),\ i = 1, 2, .., n-1$. If $n = 2$, then the reduced system of differential equations is a system ODEs.

1.1.3 Lie Algebra

Definition 1.1.9 [10] For an r-parameter Lie group of transformations with infinitesimal generators $X_\alpha,\ \alpha = 1, 2, ..., r$, *the commutator* (Lie Bracket) of X_α and X_β is a first-order operator defined by

$$[X_\alpha,\ X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha. \tag{1.1.14}$$

It immediately follows that

$$[X_\alpha,\ X_\beta] = -[X_\beta,\ X_\alpha]. \tag{1.1.15}$$

Theorem 1.1.4 [10] (Second Fundamental Theorem of Lie) The commutator of any two infinitesimal generators of an r-parameter Lie group of transformations is also an infinitesimal generator. In particular,

$$[X_\alpha,\ X_\beta] = \sum_{\gamma=1}^{r} C^\gamma_{\alpha\beta} X_\gamma, \tag{1.1.16}$$

where the coefficients $C^\gamma_{\alpha\beta}$ are constants called *structure constants*, $\alpha,\ \beta,\ \gamma = 1, 2, ..., r$.

Definition 1.1.10 Equations (1.1.16) are called the *commutation relations* of the r-parameter Lie group of transformations.

For any three infinitesimal generators X_α, X_β and X_γ, by direct computation one can show that *Jacobi's identity* holds:

$$[X_\alpha,[X_\beta,X_\gamma]]+[X_\beta,[X_\gamma,X_\alpha]]+[X_\gamma,[X_\alpha,X_\beta]]=0. \tag{1.1.17}$$

Theorem 1.1.5 [10] (Third Fundamental Theorem of Lie) The structure constants defined by the commutation relations (1.1.16) satisfy the relations

$$\begin{aligned} &C^\gamma_{\alpha\beta}=-C^\gamma_{\beta\alpha},\\ &\textstyle\sum_{\rho=1}^{r}[C^\rho_{\alpha\beta}C^\delta_{\rho\gamma}+C^\rho_{\beta\gamma}C^\delta_{\rho\alpha}+C^\rho_{\gamma\alpha}C^\delta_{\rho\beta}]=0. \end{aligned} \tag{1.1.18}$$

In particular, these relations are equivalent to the commutator anti-symmetry property (1.1.15) and Jacobi's identity (1.1.17), respectively.

Definition 1.1.11 A *Lie algebra* **L** is a vector space over **R** or **C** with a bilinear bracket operation (the commutator) satisfying the properties (1.1.15), (1.1.17) and, most important, (1.1.16). In particular, the set of infinitesimal generators $\{X_\alpha\}$, $\alpha=1,2,...r$, of an r-parameter Lie group of transformations forms an r-dimensional Lie algebra over **R**.

Proposition. [86] Let G be a Lie group with Lie algebra **L**. For each vector $v\in\mathbf{L}$, *the adjoint vector Ad v* at $w\in\mathbf{L}$ is

$$ad\ v|_w=[w,v]=-[v,w]. \tag{1.1.19}$$

The adjoint representation *Ad G* of the underlying Lie group can be reconstructed by summing the Lie series [86]

$$\begin{aligned} Ad(\exp\epsilon v)w_0&=\textstyle\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}(ad\ v)^n(w_0)\\ &=w_0-\epsilon[v,w_0]+\frac{\epsilon^2}{2}[v,[v,w_0]]-... \end{aligned} \tag{1.1.20}$$

1.1.4 Classification of Subgroup and Sub Algebras

Let G be a Lie group. An optimal system of s-parameter subgroups is a list of conjugacy inequivalent s-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s-parameter sub algebras forms an optimal system if every s-parameter sub algebra of the Lie algebra **L** is equivalent to a unique member of the list under some element of the adjoint representation.

The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of sub algebras. For one dimensional sub algebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by a nonzero vector in **L**. This problem is attacked by the naïve approach of taking a general element in the Lie algebra **L** and subjecting it to its various adjoint transformations so as to “simplify” it as much as possible. (Refer to [86]).

1.2 Methodology

In this thesis, we mainly deal with the methods of group invariant solutions, based on the theory of continuous group of transformations, better known as ‘Lie groups’, acting on the space of independent and dependent variables of the system. We also applied Direct method by Clarkson [22] and $\frac{G'}{G}$-expansion method [105, 125] on some nonlinear partial differential equations.

We now provide the brief outlines of the methods mentioned above. More emphasis has been laid on the implementation than on the mathematical intricacies of the techniques, thereby making the methods algorithmic in nature and thus easy to apply.

1.2.1 Classical Lie Method

The classical method [10, 12, 65, 88] essentially consists of finding symmetry reductions of PDEs with the help of determining equations obtained under the condition of invariance (1.1.8) of the system of PDEs. More specifically, when a given system of PDEs (1.1.4) is subjected to invariance under one-parameter Lie group of transformations (1.1.5), one arrives at an over determined linear system of differential equations for the group infinitesimals. These infinitesimals of the transformations help us obtain

the reductions of the system. The stepwise procedure is as follows:
Consider a system of N PDEs with m dependent variables $u = (u^1, u^2, ..., u^m)$ and n independent variables $x = (x_1, x_2, ..., x_n)$ given by

$$F^{\mu}(x, u, \partial u, \partial^2 u, ...\partial^k u) = 0, \quad \mu = 1, 2, 3, ..., N. \tag{1.2.1}$$

1. Let the one-parameter Lie group of point transformations (1.1.5) leaves invariant the system of PDEs (1.2.1).

2. Apply the prolonged operator $X^{(k)}$ given by (1.1.7) to each equation of the system (1.2.1) and require that

$$X^{(k)} F^{\mu}|_{F^{\nu}=0} = 0, \qquad \mu, \nu = 1, 2, ..., N \tag{1.2.2}$$

   The meaning of the condition (1.2.2) is that $X^{(k)}$ vanishes on the solution set of the originally given system (1.2.1). Precisely, this condition assures that $u(x)$ is solution of (1.2.1) whenever $u^*(x^*)$ is one.

3. From the invariance condition, a system of linear PDEs for ξ and η that constitutes a set of determining equations for the infinitesimal generator X admitted by the given system of PDEs (1.2.1) is obtained.

4. The solutions of the determining equations will lead to the explicit forms of ξ and η.

5. Construct the corresponding characteristics equations (1.1.11) and obtain u in terms of $n - 1$ new independent variables.

6. Rewrite the system (1.2.1) in these new coordinates to get the reduced form of the system.

1.2.2 Direct Method

Clarkson and Kruskal (1989) [22] developed an algorithmic method for finding similarity reductions (known as direct method) and using it Clarkson obtained new similarity reductions of the Boussinesq equation [22]. The novel characteristic of this direct method in comparison to the methods mentioned here, is that it involves no use of group theory. Further, for many equations the method appears to be simple to implement than either to classical [10, 86] or non-classical methods [11]. Also, some times direct methods in addition to classical symmetries, gives symmetries of PDEs which are different from classical symmetries (As in [20]).

The basic idea is to seek reduction of a given partial differential equation (with two independent variables x, t and one dependent variable q) in the form

$$q(x,t) = F(x,t,\eta), \tag{1.2.3}$$

with $\eta = \eta(z)$ and $z = z(x,t)$, which is the most general form for a similarity reduction (Bluman and Cole (1974) [12]). Substitution of this assumed form into the partial differential equation and demanding that the result be an ordinary differential equation for $\eta(z)$ imposes conditions upon $F(x,t,\eta)$, $z(x,t)$ and their derivatives in the form of an over determined system of equations, whose solution yields the similarity reductions.

It may be remarked here that the said approach has certain resemblances to the so-called method of free parameter analysis (Hansen (1984) [41]), wherein the boundary conditions are crucially used in the determination of the similarity reduction.

In a number of situations, it is convenient to seek solutions in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)\eta(z(x,t)), \tag{1.2.4}$$

rather than in the more general form

$$u(x,t) = F(x,t,\eta(z)), \tag{1.2.5}$$

where $\alpha(x,t)$, $\beta(x,t)$ and $z(x,t)$ are assumed to be sufficiently differentiable functions and $\eta(z)$ is n-times differentiable (n being the order of PDE). On substituting equation (1.2.4) into the PDE under consideration and collecting the coefficients of like derivatives and powers of $\eta(z)$, we shall arrive at a PDE.

In order that resulting equation may become an ordinary differential equation for $\eta(z)$, we require that the ratios of different derivatives and powers of $\eta(z)$ to be functions of z alone. This given an overdetermined system of equations for $\alpha(x,t),\ \beta(x,t),\ z(x,t)$ whose solutions yield the desired similarity solutions.
In order to put the above procedure into practice, we have to follow the remarks recorded below (for details see Clarkson [22]).

Remark 1.2.2.1 We substitute (1.2.4) into the partial differential equation and then require that the resulting equation is an ordinary differential equation for $\eta(z)$, so it is necessary that the ratios of different derivatives and powers of $\eta(z)$ be function of z only. This gives a set of conditions for $\alpha(x,t), \beta(x,t), z(x,t)$ in the form of an overdetermined system of equations, any solution of which yield a similarity reduction. (These conditions are both necessary and sufficient for (1.2.4) to reduce the partial differential equation for $u(x,t)$ to an ordinary differential equation for $\eta(z)$).

Remark 1.2.2.2 We use the coefficient of highest derivatives of $\eta(z)$ as normalizing coefficient and therefore require that the other coefficients be of the form of the normalising coefficient multiplied by $\Gamma(z)$, where Γ is a function of z to be determined.

Remark 1.2.2.3 We reserve uppercase greek letters for undetermined functions of z so that after performing operations the result can be denoted by the same letter [e.g, the derivative of $\Gamma(z)$ will be called $\Gamma(z)$].

Remark 1.2.2.4 There are three freedoms in the determination of α, β, z and η we can exploit, without loss of generality, that are valuable in keeping the method manageable: (i) if $\alpha(x,t)$ has the form $\alpha = \alpha_0(x,t) + \beta(x,t)\Omega(z)$, then we can choose $\Omega \equiv 0$ [by substituting $\eta(z) \to \eta(z) - \Omega(z)$] (ii) if $\beta(x,t)$ has the form $\beta = \beta_0(x,t)\Omega(z)$, then we can take $\Omega \equiv 1$ [by substituting $\eta(z) \to \eta(z)/\Omega(z)$]; and (iii) if $z(x,t)$ is determined by the equation of the form $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ [by substituting $z \to \Omega^{-1}(z)$].

1.2.3 Painlev*é* Analysis

Previous sections give us an idea of the considerable progress made in familiarizing one with nonlinear PDEs in terms symmetries. Another important aspect in relation to the said dynamical system that deserves special attention is to trace out the progress made in developing an approach that helps in deciding whether it is integrable or not. In the case of ordinary differential equations; the singularity structure analysis (also called Painlev*é* test) of the solution in the complex plane has played an important role in deciding between integrable and non-integrable dynamical systems. More specifically, one could classify an ODE or a system of ODEs in the complex domain to be of Painlev*é* type or has Painlev*é* property (PP) if the only movable singularities of all its solutions are poles. Fundamental contribution connecting Painlev*é* property and integrability in the case of ODE has been made by Kovalevskaya (1889) [60], Yoshida (1983) [121], Erconlani and Siggia (1986) [32].

Weiss, Tabor and Carnevale (WTC (1983)) [112] have introduced the Painlev*é* test for PDEs and have shown that there exists a close relationship between PP and integrability. This has been successfully carried over to KdV, KP and Boussinesq equation and led to what is termed as B*ä*cklund transformation.

In this section, the description of so called WTC technique has been given, with special reference to the main steps for its application to PDEs and different stages as 'leading order', 'resonance analysis' and 'compatibility condition'.

Painlev*é* Analysis for Partial Differential Equations: Integrability

While extending the idea of connection between PP and its integrability in the case of ODE(s) or PDE(s), Weiss et al. (1983) [112] have required that the solutions be single-valued around movable singularity manifolds. Further, they have pointed out that the singularity of PDEs are 'in general' not isolated as the solutions are functions of several complex variable $(z_1, z_2, ..., z_n)$, but rather lie on manifolds determined by the condition

$$\phi(z_1, z_2, ..., z_n) = 0. \tag{1.2.6}$$

Consider the evolution equation

$$\frac{\partial u}{\partial t} = A(u), \tag{1.2.7}$$

where A is polynomial in u and its spatial derivatives.

Thus, if $u = u(z_1, z_2, ..., z_n)$ is a solution of the PDE (1.2.7) then we require that in the neighbourhood of the manifold, equation (1.2.6) can be expanded into

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j, \tag{1.2.8}$$

where $u_0 \neq 0,\ u_j = u_j(z_1, z_2, ..., z_n)$ and $\phi = \phi(z_1, z_2, ..., z_n)$ are analytic functions of z_js in a neighbourhood of the manifold (1.2.6) and that α is negative integer.

Implementation of this procedure is direct and follows algorithmically in a manner similar to that of the ODE(s).

There are essentially four steps involved in the Painlev*é* analysis of PDE(s) (see Lakshmanan and Tamizhmani (1988)).

1. Determine of the leading order behaviors.

2. Identification of the powers at which arbitrary functions can enter into the Laurent series called resonances.

3. Verifying that at the resonance values sufficient number of arbitrary functions exist without the introduction of movable critical manifolds.

4. Establishing connections with the solutions and other integrability properties.

The remarkable feature of the Painlev*é* analysis, particularly for soliton equations, is that a natural connection exists between the P-property and the linearization property, Lax pairs, B*ä*cklund transformations, integrability, etc.

In the following we outline briefly each of the stages.

1. Leading Order Analysis

   As pointed out earlier α occurring in the expansion (1.2.8) has to be so determined that α is negative integer so that no movable critical manifolds enter. Consequently, we start with the determination of all possible value(s) of α and u_0 in the expansion (1.2.8). For each value of α, the homogeneous terms with the highest degree may balance each other. These terms are called leading terms (or dominant terms). The values for u_0 can be determined by equating the coefficients of the dominant terms to zero and solving the resulting algebraic equation for u_0.

2. Resonance Analysis

   Next, one has to find the "resonance" values, j, that is the power(s) at which the coefficient u_j of the term $\phi^{j+\alpha}$ in the expansion (1.2.8) is arbitrary.

   To find these, we substitute (1.2.8) into the equation (1.2.7) and obtain appropriate recursion relation for u_j and extract the coefficient $\tilde{Q}(j) = Q(j)u_j$ of the term $\phi^{j+\alpha-N}$, where N is the order of the PDE. Then $Q(j) = 0$ is called the resonance equation, for which -1 is always a root, which corresponds to the arbitrary nature of ϕ. In order to avoid any movable critical singular manifold, we require that these remaining roots are non-negative integers.

3. Arbitrary Functions

   Let j_s be the highest of the allowed resonance values. On substituting

$$u = \sum_{j=0}^{j_s} u_j \phi^{j+\alpha} \tag{1.2.9}$$

   into equation (1.2.7) and collecting the coefficient of $\phi^{j+\alpha-N}$, we get

$$Q(j)u_j + R_j = 0, \tag{1.2.10}$$

   where R_j is a polynomial in the partial derivatives of ϕ and u_k, $(k = 0, 1, ..., j-1)$. Since $Q(j) = 0$, for any resonance value j, R_j should identically vanish. In this case u_j is arbitrary. In case it is not so, we have to introduce logarithmic term of the form $a_j + b_j \log \phi$ in the series. But due to this addition, logarithmic singularities will appear in the solution manifold. Thus, $R_j = 0$ is a condition to ensure that the solution is free from movable critical manifold at a particular resonance value j. In this way we can check that the general solution is free from movable critical manifolds.

1.2.4 Travelling Wave Solutions to Nonlinear Partial Differential Equations

1.2.4.1 $\frac{G'}{G}$-Expansion Method

The main idea of this method is that the travelling wave solutions of nonlinear equations can be expressed by a polynomial in $(\frac{G'}{G})$, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x + ct$, where λ, μ and c are arbitrary constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear term appearing in the given nonlinear equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method.
Suppose that we have a complex nonlinear PDE in the following form:

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0, \tag{1.2.11}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. Now we are giving the main steps [105] for solving equation (1.2.11) using the $(\frac{G'}{G})-$expansion method:

Step (i) Seek travelling wave solutions of (1.2.11) by taking $u(x,t) = \phi(\xi)e^{\iota(px+rt)}$, $\xi = x + ct$, and transform (1.2.11) to the ordinary differential equation (ODE)

$$Q(\phi, \phi', \phi'', ...) = 0, \tag{1.2.12}$$

where prime ($'$) denotes the derivative with respect to ξ.

Step (ii) If possible, integrate (1.2.12) term by term as many times as possible yields constant(s) of integration. For simplicity the integration constant(s) can be set to zero.

Step (iii) Suppose that the solution $u(\xi)$ of ODE (1.2.12) can be expressed as a finite series in the form

$$u(\xi) = a_m(\frac{G'}{G})^m + a_{m-1}(\frac{G'}{G})^{m-1} + ... + a_0, \quad a_m \neq 0 \tag{1.2.13}$$

where $G = G(\xi)$ satisfies the second order Linear ODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{1.2.14}$$

$a_0, a_1, ..., a_m, \lambda$ and μ are constants to be determined later. m is a positive integer, which is determined by the homogeneous balancing method.

Step (iv) Substituting (1.2.13) together with (1.2.14) into (1.2.12) yields an algebraic equation involving powers of $(\frac{G'}{G})$. Equating the coefficients of each power of $(\frac{G'}{G})$ to zero, to obtain a system of algebraic equations for a_i, λ, μ and c. Then, to determine these constants we solve the system with the aid of softwares, such as Maple. Since the solutions of (1.2.14) have been well known for us depending on the sign of the discriminant $\triangle = \lambda^2 - 4\mu$, the exact solutions of given (1.2.11) can be obtained.

1.2.4.2 Tanh-Sech Rational Function Method

The tanh-sech rational function method [96] is one of the most effective straightforward method to construct exact solutions. The method generates solutions in the form of the rational function of tanh and sech functions. Here, the method is described for a system of two partial differential equations and it can be easily extended to a system with more number of differential equations. The method mainly consists following steps:

Step (i) Transform the PDEs into ODEs:

For a given system of partial differential equations, say, in two independent variables,

$$\begin{aligned} P(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{xx}, ...) = 0 \\ Q(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{xx}, ...) = 0, \end{aligned} \tag{1.2.15}$$

where $u(x,t)$ and $v(x,t)$ are functions of variables x and t.

P and Q are polynomials about u, v and their derivatives. To begin with, by using the simple transformation: $\xi = x - ct$, where ξ is the traveling wave variable, c is the velocity of the traveling wave, the system (1.2.15) becomes a system of ordinary differential equations,

$$\begin{aligned} P(u, v, u', v', u'', v'', ...) = 0 \\ Q(u, v, u', v', u'', v'', ...) = 0, \end{aligned} \tag{1.2.16}$$

where $(')$ denotes derivative with respect to ξ.

Let us assume that the system (1.2.16) admits a solution in the form

$$\begin{aligned} F(\xi) = \frac{a_0 \tanh(\xi) + a_1 + a_2 \sec h(\xi)}{b_0 \tanh(\xi) + b_1 + b_2 \sec h(\xi)} \\ G(\xi) = \frac{c_0 \tanh(\xi) + c_1 + c_2 \sec h(\xi)}{d_0 \tanh(\xi) + d_1 + d_2 \sec h(\xi)}, \end{aligned} \tag{1.2.17}$$

where $a_i,\ b_i,\ c_i$ and d_i's are constants to be determined later.

Step (ii) Derive the algebraic system for the coefficients:

The substitution of the form of $F(\xi)$ and $G(\xi)$ in equation (1.2.16) brings forth different possibilities for $a_i,\ b_i,\ c_i$ and d_i's. We substitute (1.2.17) in (1.2.16), we can express all higher derivatives of tanh and $\sec h$ as polynomials in tanh and $\sec h$. Because the coefficients of tanh and $\sec h$ have to vanish, we get two sets of algebraic equations comprising the nonlinear equations in $a_i,\ b_i,\ c_i$ and d_i's, involving c. The solutions to the algebraic equations give us various relations among the physical parameters and the undetermined constants in the form (1.2.17).

Step (iii) Solve the nonlinear parameterized algebraic system:

Most of these algebraic systems can be directly solved by using Ritt-Wu method, which is an algorithmic method and has been implemented and contributed by Dongming Wang to the Maple share library with the package Charsets. The function Csolve involved in Charsets can be directly used to solve system of algebra equations.

However for a few of algebraic systems which are too complicated to be solved automatically by using the Wu method, in this case, we can reduce the algebraic system to several simpler algebraic systems, and then solve them respectively. Ritt-Wu method guarantees that all solutions of nonlinear algebraic systems can be obtained.

Step (iv) Build and test the solutions:

The outputs of solving the algebraic system comprise a list of the form $a_i,\ b_i,\ c_i$ and d_i's, with c. Using these constants in (1.2.17), solutions of the system (1.2.15) can be obtained.

Chapter 2

b-Family And Modified b-Family Equations [1]

2.1 Introduction

Holm and Staley [47] studied a one-dimensional version of active fluid transport that is described by the following family of (1+1) evolutionary equations

$$m_t + \underbrace{um_x} + \underbrace{bu_x m} = \underbrace{\epsilon m}, \;\; u = g * m \tag{2.1.1}$$

where the fluid velocity $u(t,x)$ is defined on the real line vanishing at spatial infinity and $u = g * m$ denote the convolution (or filtering)

$$u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy,$$

which relates velocity u to momentum density m by integration against kernel $g(x)$ over the real line. Here the kernel g is chosen to be the Green's function for the Helmholtz operator on the line, that is, $g(x) = \frac{1}{2}e^{-|x|}$. This means $m = u - u_{xx}$. The family of equations (2.1.1) is characterized by the kernel g and the real dimensionless constant b, which is the ratio of stretching to convective transport. The parameter b is also the number of covariant dimensions associated with the momentum density m. The function $g(x)$ will determine the traveling wave shape and length scale for

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equation (2.1.1), while the constant b will provide a balance or bifurcation parameter for the nonlinear solution behavior. The quadratic term in equation (2.1.1) represent the competition, or balance, in fluid convection between nonlinear transport and amplification due to b-dimensional stretching. On the other hand, in a recent study of soliton equations, it is found that equation (2.1.1) for any $b \neq -1$ is included in the family of shallow water equations at quadratic order accuracy that are asymptotically equivalent under Kodama transformations [29].
Degasperis and Procesi [27] found, using the method of asymptotic integrability, that only three equations from the following six-parameter family

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \partial_x (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}), \tag{2.1.2}$$

where $c_0, c_1, ...\alpha, \gamma$ are real, were integrable up to third order: the KdV equation ($\alpha = c_2 = c_3 = 0$), the Camassa-Holm equation ($c_1 = \frac{-3c_3}{2\alpha^2}, c_2 = \frac{c_3}{2}$), and one new equation ($c_1 = \frac{-2c_3}{\alpha^2}, c_2 = c_3$), which on proper scaling, shifting the dependent variable, and finally applying Galilean boost reads as [25, 26]

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{2.1.3}$$

KdV type of equations has been an important and well studied class of nonlinear evolution equations with numerous applications in physical sciences, engineering fields and arises in various physical contexts.

The Camassa-Holm equation was first introduced by Camassa and Holm as a shallow water equation [19]. Like KdV equation it admits solitary waves that are solitons. In addition to that, the Camassa -Holm equation models wave breaking which the KdV does not. It models the propagation of unidirectional shallow water waves on a flat bottom, and then represents the fluid velocity at time t in the horizontal direction x [19, 55]. The Camassa-Holm equation is a water wave equation at quadratic order in an asymptotic expansion for unidirectional shallow water waves described by the incompressible Euler equations, while the KdV equation appears at first order in this expansion [27, 55]. For further details on Camassa-Holm equation one may refer to a paper by Holm and Staley [47].

As mentioned above, the Degasperis-Procesi equation (2.1.3) was first introduced in [27] by an asymptotic integrability test within a family of third order dispersive equations. Then Degasperis et al. [26] proved the exact integrability of (2.1.3) by constructing a Lax pair. The n-peakon solutions of equation (2.1.3) are derived by Lundmark and Szmigielski [74] using inverse scattering approach. Mustafa [80] proved that smooth solutions to (2.1.3) have infinite speed of propagation, that is, they lose instantly the property of having compact support. Well-posedness (in terms of existence, uniqueness, and stability of solutions) of the Cauchy problem for the Degasperis-Procesi equation (2.1.3) was studied by Yin in a series of papers [117, 118, 119, 120] and by Coclite and Karlsen [24].

In this chapter, we investigate the symmetries of the following one parameter family of non-evolution equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \tag{2.1.4}$$

where the parameter b is real, which includes both the Camassa-Holm equation for $b = 2$ and the Degasperis- Procesi equation for $b = 3$ as special cases. Since it arises from (2.1.1) when the peakon kernel $g(x) = \frac{1}{2}e^{-|x|}$ is chosen, we refer to (2.1.4) as the peakon b-family of equations..The equation is introduced by D. D. Holm and M. F. Struley [47], which describes the balance between the convection and the stretching for small viscosity in the dynamics of $1D$ nonlinear waves in fluids. In the particular case $b = 2$ equation (2.1.4) becomes the dispersionless version of the integrable Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \tag{2.1.5}$$

It may be noted that for $b = 3$, equation (2.1.4) takes the form of Degasperis-Procesi equation (2.1.3).

We also consider a family of important physically equation which is called modified b-equation [69, 109, 111] as expressed in following form

$$u_t - u_{xxt} + (b+1)u^2 u_x = b u_x u_{xx} + u u_{xxx}, \tag{2.1.6}$$

where b is a positive integer. When $b = 2, b = 3$, (2.1.6) is called modified Camassa-Holm (mCH) equation and modified Degasperis-Procesi (mDP) equation, respectively. Wazwaz [109] uses sine-cosine, tanh-function methods to obtain their solitary wave solutions. Ben-gong Zhang et al. [126] employed HPM for finding the solution of modified CamassaHolm and DegasperisProcesi equations. We have applied the direct method to find symmetry reductions of a modified b-family equation.

Exact solutions play a vital role in the study of nonlinear phenomena as these solutions provide much information on various aspects of the physical phenomena. Since equations (2.1.4) and (2.1.6), represents an important class of nonlinear partial differential equations including two physically relevant systems (Camassa-Holm and Degasperis-Procesi), its exact solutions are desirable. The present work, which is purely due to the intrinsic theoretical interest in the nonlinear systems (2.1.4) and (2.1.6), is devoted to extract exact solutions of these systems. We present some explicit exact solutions to equations (2.1.4) and (2.1.6). From these solutions; one can easily derive the solutions of Camassa-Holm equation and Degasperis-Procesi equations as particular case.

The outline of this chapter is as follows: In section 2.2, the classical Lie method and Direct method is utilized to obtain symmetries and exact solutions of (2.1.4) and In section 2.3, we used the Lie classical method and Direct method to obtain similarity reductions of modified b-family equation. In this section we also have given travelling wave solutions of the equation. In section 2.4 we have drawn some conclusion.

2.2 b-Family Equation

In this section, we apply Lie classical method and Direct method to obtain symmetries and exact solutions of b-family equation (2.1.4).

2.2.1 Classical Lie Symmetry Analysis

In this section, we will apply Lie's method [86, 95] of infinitesimal transformation groups on b-family equations. As mentioned in [86], We let $\tau(x,t,u),\ \xi(x,t,u),\ \eta(x,t,u)$ be infinitesimals corresponding to $t,\ x,\ u$, respectively and impose the condition of invariance on (2.1.4). The invariance under infinitesimal transformations means that if u is solution of equation (2.1.4), then u^* is also a solution of it.

Herein, on invoking the invariance criterion as mentioned in [86], the following relation from the coefficients of the first order of ϵ is deduced:

$$-\eta^t + \eta^{txx} - (b+1)[\eta u_x + u\eta^x] + b[\eta^x u_{xx} + \eta^{xx} u_x] + u\eta^{xxx} + \eta u_{xxx} = 0 \qquad (2.2.1)$$

where $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ and η^{txx} are extended (prolonged) infinitesimals corresponding to $u_t, u_x, u_{xx}, u_{xxx}$ and u_{txx} respectively. The set of determining equations for the group infinitesimals ξ, τ and η, which is obtained from (2.2.1), after equating the coefficients of various derivative terms to zero, is as follows:

$$\begin{aligned}
&\xi_u = 0\\
&\tau_x = \tau_u = 0\\
&\eta_{uu} = 0\\
&2\eta_{xu} - \xi_{xx} = 0\\
&\eta_{xxu} - 2\xi_x = 0\\
&u\tau_t - \xi_t + \eta - u\xi_x = 0\\
&-\tau_t + \xi_x - \eta_u = 0\\
&\eta_{tu} - 2\xi_{tx} + b\eta_x - 3u\eta_{xu} = 0\\
&\eta_t - \eta_{txx} + (b+1)u\eta_x - u\eta_{xxx} = 0u\eta_u - bu\eta_{xxu} + \xi_t - (b+1)\eta + b\eta_{xx} = 0.
\end{aligned} \qquad (2.2.2)$$

The set of equations (2.2.2) helps us to obtain the infinitesimals ξ, τ and η, as follows:

$$\begin{aligned}\xi &= a_1\\ \tau &= -a_3 t + a_2\\ \eta &= a_3 u,\end{aligned} \tag{2.2.3}$$

where a_1, a_2 and a_3 are arbitrary constants. The Lie algebra associated with equation (2.1.4) consists of the following three vector fields

$$V_1 = \frac{\partial}{\partial x},\ V_2 = \frac{\partial}{\partial t} \text{ and } V_3 = u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t}.$$

The similarity variable and form can be obtained by solving the characteristic equations

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta}. \tag{2.2.4}$$

The general solution of these equations involves two constants; one becomes the new independent variable ξ and the other, say F, plays the role of new dependent variable. On substituting these solutions of (2.2.4) in equation (2.1.4), one gets the reduced ordinary differential equation (ODE).

As mentioned in Olver [86], the commutator Table-2.1 and the adjoint Table-2.2 for above Lie algebra can be easily constructed as follows:

Table 2.1: Commutator Table

	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	$-V_2$
V_3	0	V_2	0

We need only present one solution from each equivalence class, as the rest may be found by applying appropriate group symmetries; a complete set of such solutions

Table 2.2: Adjoint Table

	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3+\epsilon V_3$
V_3	V_1	$V_2 e^{-\epsilon}$	V_3

is referred to as an "optimal system" of group invariant solutions. The problem of deriving an optimal system of group invariant solutions is equivalent to finding an optimal system of Lie symmetries (or subalgebras spanned by these operators). The method used here is given by Olver [86].

For brevity we omit the details, and just state the result that an optimal system comprises of two vector fields viz. (i)$V_1 + \mu V_3$ and (ii)V_3. Now we primary focus on the reductions associated with these vector fields and attempt to find some exact solutions.

Vector field $V_1 + \mu V_3$

For this vector field, on using the characteristic equations (2.2.4), the similarity variable and the form of the similarity solution are as follows:

$$\xi(t,x) = te^{\mu x}, \qquad u(t,x) = \frac{1}{t}F(\xi).$$

On using these in equation (2.1.4), the reduced ODE is given by

$$\begin{aligned}&-\xi F'(\xi) + F + \mu^2\xi^3 F'''(\xi) + 2\mu^2\xi^2 F'' - \mu(b+1)\xi F F' + b\mu^3\xi^3 F'F'' + b\mu^3\xi^2(F')^2\\ &+\mu^3\xi^3 F F''' + 3\mu^3\xi^2 F F'' + \mu^3\xi F F' = 0,\end{aligned} \tag{2.2.5}$$

where $'$ prime denotes the differentiation with respect to the variable ξ. On transforming the independent variable by the relation , $\xi = exp(\zeta)$, the ODE (2.2.5) becomes

$$-\dot{F} + F + \mu^2\dddot{F} - \mu^2\ddot{F} - \mu(b+1)F\dot{F} + b\mu^3\dot{F}\ddot{F} + \mu^3 F\dddot{F} = 0. \tag{2.2.6}$$

Solution of the equation (2.2.6) has been presented in section 2.2.3.

Vector field V_3

In this case, the form of the similarity variable and similarity solution is as follows:

$$\xi = x, \qquad u(t,x) = \frac{1}{t}F(\xi).$$

The reduced ODE in this case is as follows:

$$FF''' + bF''F'' - (b+1)FF' - F'' + F = 0. \tag{2.2.7}$$

Some interesting solution of the equation (2.2.7) has been presented in section 2.2.3.

2.2.2 Similarity Reductions by Direct Method

In this section, we use direct method introduced by Clarkson and Kruksal [22] to obtain similarity reductions of the equation (2.1.4). The novel features of it are entirely straightforward without group analysis.The direct method to find similarity reductions is a very simple method that does not use group theory. The main idea is to seek a reduction of a given PDE in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)), \tag{2.2.8}$$

where $\alpha(x,t), \beta(x,t)$ and $z(x,t)$ are to be determined.

Substituting (2.2.8) into (2.1.4) and collecting monomials of w and its derivatives yields

$$\begin{aligned}
&-\beta^2 z_x^3 ww''' - (\beta z_t z_x^2 + \alpha\beta z_x^3)w''' - (3\beta^2 z_x z_{xx} + 3\beta_x\beta z_x^2 + b\beta\beta_x z_x^2)ww'' \\
&-b\beta^2 z_x^3 w'w'' + (-2\beta_x z_x z_t - 3\alpha\beta_x z_x^2 - \beta_t z_x^2 - 3\alpha\beta z_x z_{xx} - 2\beta z_x z_{xt} \\
&-\beta z_t z_{xx} - b\alpha_x\beta z_x^2)w'' + (-2b\beta_x^2 z_x - b\beta\beta_x z_{xx} - \beta^2 z_{xxx} - 3\beta\beta_x z_{xx} - 3\beta\beta_{xx} z_x \\
&+(b+1)\beta^2 z_x - b\beta\beta_{xx} z_x)ww' + (-2\beta_x z_{xt} - b\alpha_x\beta z_{xx} + b\alpha\beta z_x - 2\beta_{xt} z_x \\
&+\alpha\beta z_x - 3\alpha\beta_{xx} z_x - \beta_{xx} z_t + \beta z_t - \alpha\beta z_{xxx} - b\alpha_{xx}\beta z_x - \beta z_{xxt} - 3\alpha\beta_x z_{xx} \\
&-2b\alpha_x\beta_x z_x)w' + ((b+1)\beta\beta_x - \beta\beta_{xxx} - b\beta_x\beta_{xx})w^2 + (-b\beta^2 z_x z_{xx} - 2b\beta\beta_x z_x^2)w'^2 \\
&+(-b\alpha_x\beta_{xx} - b\alpha_{xx}\beta_x + \alpha\beta_x + \alpha_x\beta - \alpha\beta_{xxx} - \alpha_{xxx}\beta + b\alpha\beta_x + b\alpha_x\beta + \beta_t)w \\
&+(b+1)\alpha\alpha_x - \alpha\alpha_{xxx} + \alpha_t - \alpha_{xxt} - b\alpha_x\alpha_{xx} = 0.
\end{aligned} \tag{2.2.9}$$

Demand that result be a ODE, impose conditions upon w and z and their derivatives that enable one to solve for α, β and z.

We shall now proceed to find general symmetry reduction of b-family equation using this method. Use the coefficient of $w'''w$ as normalizing coefficient and using the freedoms as mentioned in remarks (1-4) as explained in chapter 1 section 1.2.2, we find that

$$\alpha = 0,\ \beta = \tfrac{\sigma'(t)}{\theta'(x)} \text{ and } z = \theta(x) + \sigma(t).$$

Put these values in (2.2.8), on simplification we get

$$\begin{aligned}&-\sigma'^2\theta'(ww''' + w''' + bw'w'') - (b+6)\sigma'^2\tfrac{\theta''}{\theta'}ww'' + (\sigma'^2\tfrac{\theta''}{\theta'} - \tfrac{\sigma''}{\theta'})w'' \\ &+(-3(b+1)\sigma'^2\tfrac{\theta''^2}{\theta'^3} + (b+1)\tfrac{\sigma'^2}{\theta'} - (b+4)\sigma'^2\tfrac{\theta'''}{\theta'^2})ww' + (\sigma''\tfrac{\theta''}{\theta'} + \sigma'^2\tfrac{\theta''}{\theta'^2} + \tfrac{\sigma'^2}{\theta'})w' + \\ &(-(b+1)\sigma'^2\tfrac{\theta''}{\theta'^3} + (2b+6)\sigma'^2\tfrac{\theta''^3}{\theta'^5} - (b+6)\sigma'^2\tfrac{\theta''\theta'''}{\theta'^4} + \sigma'^2\tfrac{\theta''''}{\theta'^3})w^2 + b\sigma'^2\tfrac{\theta''}{\theta'}w'^2 \\ &+\tfrac{\sigma''}{\theta'}w = 0.\end{aligned} \tag{2.2.10}$$

We continue to make this an ordinary differential equation for $w(z)$. Then the remaining coefficients yield

$$-\sigma'^2\theta'\Gamma_1(z) = -\sigma'^2\frac{\theta''}{\theta'} \tag{2.2.11}$$

$$-\sigma'^2\theta'\Gamma_2(z) = -\sigma'^2\frac{\theta''}{\theta'} - \frac{\sigma''}{\theta'} \tag{2.2.12}$$

$$-\sigma'^2\theta'\Gamma_3(z) = -3(b+1)\sigma'^2\frac{\theta''^2}{\theta'^3} + (b+1)\frac{\sigma'^2}{\theta'} - (b+4)\sigma'^2\frac{\theta'''}{\theta'^2} \tag{2.2.13}$$

$$-\sigma'^2\theta'\Gamma_4(z) = \sigma''\frac{\theta''}{\theta'} - 2\sigma'^2\frac{\theta''^2}{\theta'^3} + \sigma'^2\frac{\theta'''}{\theta'^2} + \frac{\sigma'^2}{\theta'} \tag{2.2.14}$$

$$-\sigma'^2\theta'\Gamma_5(z) = -(b+1)\sigma'^2\frac{\theta''}{\theta'^3} + (6+2b)\sigma'^2\frac{\theta''^3}{\theta'^5} - (b+6)\sigma'^2\frac{\theta'\theta'''}{\theta'^4} + \sigma'^2\frac{\theta'\theta''''}{\theta'^3} \tag{2.2.15}$$

$$-\sigma'^2\theta'\Gamma_6(z) = \sigma'^2\frac{\theta''}{\theta'} \tag{2.2.16}$$

$$-\sigma'^2\theta'\Gamma_7(z) = \frac{\sigma''}{\theta'}, \tag{2.2.17}$$

where $\Gamma_1(z), \Gamma_2(z), \Gamma_3(z), \Gamma_4(z), \Gamma_5(z), \Gamma_6(z)$ and $\Gamma_7(z)$ has to be determined.

First consider (2.2.11), equation (2.2.10) will be ordinary differential equation if $\theta'' = A\theta'$, where A is a arbitrary constant.

Now from equation (2.2.17), equation (2.2.10) will reduce to ODE only if

$$\theta' = 1. \tag{2.2.18}$$

So from equation (2.2.12),

$$\sigma'' = B\sigma' \tag{2.2.19}$$

where B is a arbitrary constant. Similarly other equations will be satisfied by taking

$$\Gamma_2 = B, \Gamma_7 = -B \text{ and } \Gamma_3(z) = \Gamma_4(z) = C,$$

where C is a arbitrary constant. From equation (24), we get

$$\sigma = -\frac{\ln(Bt + D)}{B} + E, \tag{2.2.20}$$

where D and E are arbitrary constants.

Using equation (2.2.18) and (2.2.20), we get

$$z = x - \tfrac{\ln(Bt+D)}{B} + C_1 \text{ and } \beta = -\tfrac{1}{Bt+D},$$

where C_1 is constant.

So b-family equation possesses similarity reduction of the form

$$u(x,t) = -\tfrac{1}{Bt+D} w(z) \text{ and } z = x - \tfrac{\ln(Bt+D)}{B} + C_1,$$

where $w(z)$ satisfies

$$ww''' + w''' + bw'w'' + Bw'' - (b+1)ww' - w' - Bw = 0. \tag{2.2.21}$$

Now we seek solutions of the b-family equation (2.1.4) in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)(y(x)). \tag{2.2.22}$$

Substituting this into (2.1.4) yields

$$
\begin{aligned}
&-\beta^2 yy''' - \alpha\beta y''' - b\beta^2 y'y'' - (b+3)\beta\beta_x yy'' + (-\beta_t - 3\alpha\beta_x - b\alpha_x\beta)y'' \\
&-2b\beta\beta_x y'^2 + (-(b+3)\beta\beta_{xx} + (b+1)\beta - 2b{\beta_x}^2)yy' + ((b+1)\alpha\beta - b\alpha_{xx}\beta \\
&-3\alpha\beta_{xx} - 2\beta_{xt} - 2b\alpha_x\beta_x)y' + ((b+1)\beta\beta_x - b\beta_x\beta_{xx} - \beta\beta_{xxx})y^2 + ((b+1)\alpha_x\beta \\
&+(b+1)\alpha\beta_x - \alpha\beta_{xxx} - \beta_{xxt} - b\beta_x\alpha_{xx} - \alpha_{xxx}\beta - b\alpha_x\beta_{xx})y + (b+1)\alpha\alpha_x + \alpha_t \\
&-\alpha\alpha_{xxx} - b\alpha_x\alpha_{xx} - \alpha_{xxt} = 0.
\end{aligned}
\tag{2.2.23}
$$

This is an ordinary differential equation for $y(x)$ if the ratios of coefficients of different powers and derivatives of y are functions of x only. There are three cases to consider (since the calculations are similar to those done in the more general case above, details are omitted).

case (i) $\beta_t = 0$

In this case, we get solution in the form of $u = e^x y(x)$, where $y(x)$ is given be

$$yy''' + (b+3)yy'' + 2(b+1)yy' + byy'' + 2by'^2 = 0. \tag{2.2.24}$$

case (ii) $\beta_x = 0,\ \beta_t \neq 0$

In this case, we get solution in the form of $u = -\frac{1}{t+C_1}y(x)$, where C_1 is constant and $y(x)$ is given by

$$yy''' + byy'' + y'' - (b+1)yy' - y = 0. \tag{2.2.25}$$

case (iii) $\beta_x \neq 0,\ \beta_t \neq 0$

In this case, we get $\beta = \frac{1}{t+C_2}$,where C_2 is constant, which contradicts the initial assumption that $\beta_x \neq 0$. Therefore there are no special solutions of the b-family equation in this case.

Now we seek solutions of the b-family equation (2.1.4) in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)(y(t)).$$

In this case, we will get solution of equation (2.1.4) as

$$u(x,t) = (\theta_1(t)e^{x} + \theta_2(t)e^{-x})y(t)$$

or

$$u(x,t) = (\theta_1(t)\sinh(x) + \theta_2(t)\cosh(x))y(t), \tag{2.2.26}$$

where $\theta_1(t)$ and $\theta_2(t)$ are arbitrary functions of t.

2.2.3 Some Exact Solutions of b-Family Equation

In above section, we have given various reductions of b-family equation using Lie classical method and Direct method. In this section we have given some exact solution corresponding to the ODEs that are obtained by the reduction of b-family equation. For equation (2.2.6), let us assume a special solution of the form

$$F(\zeta) = \frac{k\tanh(\zeta) + l + m\sec h(\zeta)}{tanh(\zeta) + p + q\sec h(\zeta)},$$

where k, l, m, p and q are constants to be found out.

The substitution of the form of $F(\zeta)$ in equation (2.2.6) brings forth the following four possibilities:

(i) $p = -1, k = l = -qm, \mu = 1$

(ii) $p = -1, k = l = -qm, \mu = -1$

(iii) $p = -1, k = -qm, l = mq, \mu = 1$

(iv) $p = -1, k = -qm, l = mq, \mu = -1$

The solution to equation (2.1.4) for the above cases can be obtained, respectively, in the following forms

$$\begin{aligned}
&(i)\ u(x,t) = \tfrac{m(-q\tanh(x+\ln(t))-q+\sec h(x+\ln(t))}{t(\tanh(x+\ln(t))-1+\sec h(x+\ln(t))}\\
&(ii)\ u(x,t) = \tfrac{m(-q\tanh(-x+\ln(t))-q+\sec h(-x+\ln(t))}{t(\tanh(-x+\ln(t))-1+\sec h(-x+\ln(t))}\\
&(iii)\ u(x,t) = \tfrac{m(-q\tanh(x+\ln(t))+q+\sec h(x+\ln(t))}{t(\tanh(x+\ln(t))+1+\sec h(x+\ln(t))}\\
&(iv)\ u(x,t) = \tfrac{m(-q\tanh(-x+\ln(t))+q+\sec h(-x+\ln(t))}{t(\tanh(-x+\ln(t))+1+\sec h(-x+\ln(t))}.
\end{aligned} \tag{2.2.27}$$

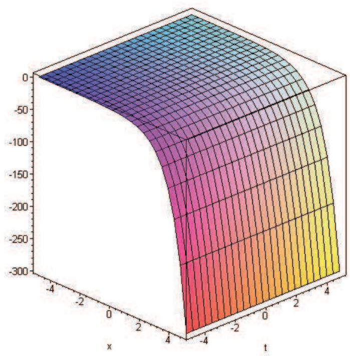

Figure 2.1: Kink wave solution (2.2.27)(i) for $q = 1$ and $m = 2$

Proceeding in a similar manner as in the previous case, and assuming a solution of the reduced ODE (2.2.7) in the form

$$F(\xi) = \frac{k\tanh(\xi) + l + m\sec h(\xi)}{tanh(\xi) + p + q\sec h(\xi)},$$

where k, l, m, p and q are constants to be found.

In this case, following two possibilities arise:

$(i)\quad p = 1, k = -qm, l = mq$

$(ii)\quad p = -1, k = -qm, l = -mq$

and the final solution to equation (2.1.4) can be expressed, in respective order of cases, as follows:

$$\begin{aligned} &(i)\;\; u(x,t) = \tfrac{m(-q\tanh(x)+q+\sec h(x))}{t(\tanh(x)+1+q\sec h(x))} \\ &(ii)\;\; u(x,t) = \tfrac{m(-q\tanh(x)-q+\sec h(x))}{t(\tanh(x)-1+q\sec h(x))}. \end{aligned} \qquad (2.2.28)$$

For equation (2.2.21), let us assume a special solution of the form

$$w(z) = \frac{k\tanh(z) + l + m\sec h(z)}{tanh(z) + p + q\sec h(z)},$$

where k, l, m, p and q are constants to be found out.

The substitution of the form of $w(z)$ in equation (2.2.21) brings forth the following four possibilities:

$(i)\quad p=-1, k=l=-qm,$

$(ii)\quad p=-1, k=-qm, l=mq.$

The solution to equation (2.1.4) for the above cases can be obtained, respectively, in the following forms

$$\begin{array}{l} (i)\ \ u(x,t)=-\frac{1}{Bt+D}\frac{m(-q\tanh(x-\frac{\ln(Bt+D)}{B}+C_1)-q+\sec h(x-\frac{\ln(Bt+D)}{B}+C_1)}{(\tanh(x-\frac{\ln(Bt+D)}{B}+C_1)-1+\sec h(x-\frac{\ln(Bt+D)}{B}+C_1)}\\ (ii)\ \ u(x,t)=-\frac{1}{Bt+D}\frac{m(-q\tanh(x-\frac{\ln(Bt+D)}{B}+C_1)+q+\sec h(x-\frac{\ln(Bt+D)}{B}+C_1)}{t(\tanh(x-\frac{\ln(Bt+D)}{B}+C_1)+1+\sec h(x-\frac{\ln(Bt+D)}{B}+C_1)}, \end{array} \tag{2.2.29}$$

where B, D and C_1 are arbitrary constants.

Solution of ODE (2.2.24) are

$$\begin{array}{l} (i)\ \ y\,(x)=C_1e^{\left(-2\,b-3/2-1/2\,\sqrt{16\,b^2+16\,b+1}\right)x}\\ (ii)\ \ y\,(x)=C_1\,e^{\left(-2\,b-3/2+1/2\,\sqrt{16\,b^2+16\,b+1}\right)x}, \end{array} \tag{2.2.30}$$

where C_1 is a arbitrary constants.

In this case, we get stationary solution of equation (2.1.4) as

$$\begin{array}{l} (i)\ \ u(x)=e^xC_1\,e^{\left(-2\,b-3/2-1/2\,\sqrt{16\,b^2+16\,b+1}\right)x}\\ (i)\ \ u(x)=e^xC_1\,e^{\left(-2\,b-3/2+1/2\,\sqrt{16\,b^2+16\,b+1}\right)x}. \end{array} \tag{2.2.31}$$

Now consider the ODE (2.2.25). Solutions of ODE (2.2.25) are given as

$$\begin{array}{l} (i)\ \ y\,(x)=C_2\,e^x\\ (ii)\ \ y\,(x)=C_3-\frac{C_1+C_2\,x}{(b+1)C_2}, \end{array} \tag{2.2.32}$$

where C_2 and C_3 are arbitrary constants and final solution of equation (2.1.4) can be expressed as

$$\begin{array}{l} (i)\ \ u(x,t)=-\frac{1}{t+C_1}C_2\,e^x\\ (ii)\ \ u(x,t)=-\frac{1}{t+C_1}(C_3-\frac{C_4+C_2\,x}{(b+1)C_2}). \end{array} \tag{2.2.33}$$

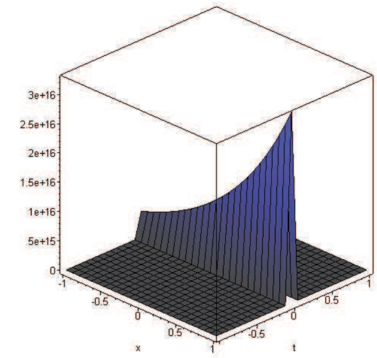

Figure 2.2: Singularity solution (2.2.33)(i) for $C_1 = 0$ and $C_2 = 1$

Let us assuming a solution of the reduced ODE (2.2.25) in the form

$$y(x) = \frac{k\tanh(x) + l + m\sec h(x)}{tanh(x) + p + q\sec h(x)},$$

where k, l, m, p and q are constants to be found.

In this case, following two possibilities arise:

$(i)\quad p = 1, k = -qm, l = mq$

$(ii)\quad p = -1, k = -qm, l = -mq$

and the final solution to equation (2.1.4) can be expressed, in respective order of cases, as follows:

$$\begin{array}{l} (i)\ \ u(x,t) = -\frac{1}{t+C_1}\frac{m(-q\tanh(x)+q+\sec h(x))}{(\tanh(x)+1+q\sec h(x))} \\ (ii)\ \ u(x,t) = -\frac{1}{t+C_1}\frac{m(-q\tanh(x)-q+\sec h(x))}{t(\tanh(x)-1+q\sec h(x))}, \end{array} \tag{2.2.34}$$

where C_1 is arbitrary constant.

From (2.2.26), We get the solution of (2.1.4) as follows

$$u(x,t) = (\theta_1(t)e^x + \theta_2(t)e^{-x})y(t)$$

or

$$u(x,t) = (\theta_1(t)\sinh(x) + \theta_2(t)\cosh(x))y(t),$$

where $\theta_1(t)$ and $\theta_2(t)$ are arbitrary functions of t.

2.3 Modified b-Family Equation

In this section, we find symmetries and exact solutions of modified b-family equation.

2.3.1 Symmetries by Lie Classical Method

In this section, we apply Lie classical method [86, 95] to obtain symmetries of modified b-family equation (2.1.6). Proceeding in similar manner as mentioned earlier, to find the Lie symmetries of equation (2.1.6), we obtain following invariance condition form the coefficients of the first order of ϵ:

$$\eta^t - \eta^{xxt} + (b+1)[2uu_x\eta + u^2\eta^x] - b[\eta^x u_{xx} + \eta^{xx}u_x] - u\eta^{xxx} - \eta u_{xxx} = 0, \quad (2.3.1)$$

where $\eta^x,\ \eta^t,\ \eta^{xxx}$ and η^{xxt} are extended (prolonged) infinitesimals acting on an enlarged space (jet space) that includes all derivatives of the dependent variables $u_x,\ u_t,\ u_{xxx}$ and u_{xxt}.

Set of determining equations for the group infinitesimals $\xi,\ \tau$ and η, which we get from (2.3.1) after equating the coefficients of various derivative terms to zero, is as follows:

$$\begin{aligned}
&\tau_u = \tau_x = 0\\
&\xi_u = \eta_{uu} = 0\\
&\eta_t - \eta_{xxt} + (b+1)\eta_x u^2 - u\eta_{xxx} = 0\\
&\eta_u - \tau_t - \eta_{xxu} - (\eta_u - 2\xi_x - \tau_t) = 0\\
&-\eta_u - \tau_t - \eta_{xxu} + 3\xi_x = 0\\
&(\eta_u - \tau_t)u - \eta_{xxu}u + \xi_t - \eta - (\eta_u - 3\xi_x)u = 0\\
&2\eta_{xu} - \xi_{xx} = 0\\
&-(\eta_{tu} - 2\xi_{xt}) - b\eta_x - 3(\eta_{xu} - \xi_{xx})u = 0\\
&-(b+1)(\eta_u - \tau_t)u^2 + (b+1)\eta_{xxu}u^2 - \xi_t - (2\eta_{xtu} - \xi_{xxt}) + 2(b+1)\eta u\\
&+(b+1)(\eta_u - \xi_x)u^2 - b\eta_{xx} - (3\eta_{xxu} - \xi_{xxx})u = 0.
\end{aligned} \quad (2.3.2)$$

Solving determining equations (2.3.2), we have following infinitesimals

$$\begin{aligned}\xi &= a\\ \tau &= b\\ \eta &= 0,\end{aligned} \tag{2.3.3}$$

where $a,\ b$ are arbitrary constants.

So using Lie classical method, we obtained trivial symmetries of modified b-family equation (2.1.6). Using these symmetries, we will get travelling wave solutions the equation (2.1.6) that has been presented in section 2.3.3.

2.3.2 Similarity Reductions by Direct Method

In this section, we use direct method introduced by Clarkson and Kruksal [22] to obtain similarity reductions of the equation (2.1.6). The novel features of it are entirely straightforward without group analysis. The direct method to find similarity reductions is a very simple method that does not use group theory. The main idea is to seek a reduction of a given PDE in the form:

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)), \tag{2.3.4}$$

where $\alpha(x,t), \beta(x,t)$ and $z(x,t)$ are to be determined. Substituting (2.3.4) into (2.1.6) and collecting monomials of w and its derivatives yields:

$$\begin{aligned}&-\beta^2 z_x^3 ww''' - (\alpha\beta z_x^3 + \beta z_x z_t)w''' - b\beta^2 z_x^3 w'w'' - (b\beta\beta_x z_x^2\\
&+3\beta\beta_x z_x^2 + 3\beta^2 z_x z_{xx})ww'' - (2\beta_x z_t z_x + 3\alpha\beta z_x z_{xx} + 3\alpha\beta_x z_x^2\\
&+\beta z_t z_{xx} + 2\beta z_x z_{xt} + \beta_t z_x^2 + b\alpha_x\beta z_x^2)w'' + (2(b+1)\alpha\beta^2 z_x\\
&-(b+3)\beta\beta_x z_{xx} - (b+1)\beta\beta_{xx} z_x - \beta^2 z_{xxx} - 2b\beta_x^2 z_x)ww' - (2b\beta\beta_x z_x^2\\
&+b\beta^2 z_x z_{xx})w'^2 + (b+1)\beta^2 z_x^3 w'w^2 + (\beta z_t - \beta_t z_{xx} - 3\alpha\beta_{xx} z_x\\
&+(1+b)\alpha^2\beta z_x - \alpha\beta z_{xxx} - 3\alpha\beta_x z_{xx} - 2\beta_x z_{xt} - \beta_{xx} z_t - \beta z_{xxt}\\
&-2\beta_{xt} z_x - b\beta z_x\alpha_{xx} - b\alpha_x\beta z_{xx} - 2b\alpha_x\beta_x z_x)w' + (b+1)\beta^2\beta_x w^3\\
&+(-\beta\beta_{xxx} + 2(b+1)\alpha\beta\beta_x - b\beta_x\beta_{xx} + (b+1)\beta^2\alpha_x)w^2 + (-\beta\alpha_{xxx}\\
&-\beta_{xxx}\alpha - b\beta_x\alpha_{xx} + 2(b+1)\alpha\beta\alpha_x - b\alpha_x\beta_{xx} + \beta_t - \beta_{xxt} + (b+1)\alpha^2\beta_x)w\\
&+(b+1)\alpha^2\alpha_x - \alpha_{xxt} - \alpha\alpha_{xxx} - b\alpha_x\alpha_{xx} = 0.\end{aligned} \tag{2.3.5}$$

Demand that result be a ODE, impose conditions upon w and z and their derivatives that enable one to solve for α, β and z. However, before doing this we make some remarks about this direct method of seeking similarity reductions (using the simplified ansatz (2.1)).

Remark 2.1. We substitute (2.3.4) into the partial differential equation and then require that the resulting equation is an ordinary differential equation for $w(z)$, so it is necessary that the ratios of different derivatives and powers of $w(z)$ be function of z only. This gives a set of conditions for $\alpha(x,t), \beta(x,t), z(x,t)$ in the form of an overdetermined system of equations, any solution of which yield a similarity reduction. (These conditions are both necessary and sufficient for (2.3.4) to reduce the partial differential equation for $u(x,t)$ to an ordinary differential equation for $w(z)$).

Remark 2.2. We use the coefficient of $w'''w$ as normalizing coefficient and therefore require that the other coefficients be of the form $\beta^2 z_x^3 \Gamma(z)$, where Γ is function to be determined.

Remark 2.3. We reserve uppercase greek letters for undetermined functions of z so that after performing operations the result can be denoted by the same letter [e.g, the derivative of $\Gamma(z)$ will be called $\Gamma(z)$].

*Remark*2.4. There are three freedoms in the determination of α, β, z and w we can exploit, without loss of generality, that there are valuable in keeping the method manageable:

1. If $\alpha(x,t)$ has the form $\alpha = \alpha_0(x,t) + \beta(x,t)\Omega(z)$, then we can choose $\Omega \equiv 0$ [by substituting $w(z) \to w(z) - \Omega(z)$]

2. If $\beta(x,t)$ has the form $\beta = \beta_0(x,t)\Omega(z)$, then we can take $\Omega \equiv 1$ [by substituting $w(z) \to w(z)/\Omega(z)$]

3. If $z(x,t)$ is determined by the equation of the form $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ [by substituting $z \to \Omega^{-1}(z)$].

We shall now proceed to find general symmetry reduction of b-family equation using this method. The coefficient of ww''' and $w'w^2$ yields common constraint

$$\beta^2 z_x^3 \Gamma(z) = \beta^3 z_x, \tag{2.3.6}$$

where $\Gamma(z)$ is a function to be determined. Hence using the freedom mentioned in Remark 2.4 (2) above, we choose

$$\beta = z_x^2 \tag{2.3.7}$$

The coefficient of w^3 yields

$$\beta^2 z_x^3 \Gamma(z) = \beta^2 \beta_x, \tag{2.3.8}$$

where $\Gamma(z)$ another function to be determined. Hence using (2.3.7), we get

$$z_x \Gamma(z) = 2\frac{z_{xx}}{z_x}. \tag{2.3.9}$$

Now rescaling $\Gamma(z)$ and integrating, we get

$$\Gamma(z) + \log z_x = \Theta(t). \tag{2.3.10}$$

(Recall Remark 2.3) Integrating again, we get

$$\Gamma(z) = x\Theta(t) + \Sigma(t), \tag{2.3.11}$$

with $\sum(t)$ is another function of integration. By Remark 2.4 (3), we have

$$z = x\theta(t) + \sigma(t), \tag{2.3.12}$$

where $\theta(t)$ and $\sigma(t)$ are to be determined. From equation (2.3.7) and (2.3.12)

$$\beta = \theta(t)^2. \tag{2.3.13}$$

The coefficient of w''' yields

$$\beta^2 z_x^3 \Gamma(z) = \alpha\beta z_x^3 + \beta z_x^2 z_t, \tag{2.3.14}$$

where $\Gamma(z)$ has to be determined. Using (2.3.12) and (2.3.13), this simplifies to

$$\theta(t)^3 \Gamma(z) = \alpha\theta + (x\frac{d\theta}{dt} + \frac{d\sigma}{dt}). \tag{2.3.15}$$

Hence by Remark 2.4 (1)above

$$\alpha = -\frac{1}{\theta}(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}). \tag{2.3.16}$$

Let us see how the equation (2.3.5) look with simplifications as determined so far, viz. (2.3.12)-(2.3.16):

$$\begin{aligned} &-\theta^7(ww''' + b^2w'w'' - (b+1)w'w^2) + (4-b)\tfrac{d\theta}{dt}\theta^3 w'' - 2(b+1)\theta^4(\tfrac{d\theta}{dt}x \\ &+\tfrac{d\sigma}{dt})ww' + (\theta^2(\tfrac{d\theta}{dt}x + \tfrac{d\sigma}{dt}) + (1+b)(\theta(\tfrac{d\theta}{dt}x + \tfrac{d\sigma}{dt})^2)w'(b+1)\theta^3\tfrac{d\theta}{dt}w^2 \\ &+(2(b+1)\tfrac{d\theta}{dt}(\tfrac{d\theta}{dt}x + \tfrac{d\sigma}{dt}) + 2\theta\tfrac{d\theta}{dt})w + \tfrac{1}{\theta^2}\tfrac{d\theta}{dt}(\tfrac{d\theta}{dt}x \\ &+\tfrac{d\sigma}{dt}) - \tfrac{1}{\theta}(\tfrac{d^2\theta}{dt^2}x + \tfrac{d^2\sigma}{dt^2})\tfrac{(b+1)}{\theta^3}\tfrac{d\theta}{dt}(\tfrac{d\theta}{dt}x + \tfrac{d\sigma}{dt})^2 = 0. \end{aligned} \tag{2.3.17}$$

We continue to make this an ordinary differential equation for $w(z)$. Then the remaining coefficients yield

$$\theta^7\gamma_1(z) = \theta^3\frac{d\theta}{dt} \tag{2.3.18}$$

$$\theta^7\gamma_2(z) = \theta^4(\frac{d\theta}{dt}x + \frac{d\sigma}{dt}) \tag{2.3.19}$$

$$\theta^7\gamma_3(z) = \theta^2(\frac{d\theta}{dt}x + \frac{d\sigma}{dt}) + (1+b)\theta(\frac{d\theta}{dt}x + \frac{d\sigma}{dt})^2 \tag{2.3.20}$$

$$\theta^7\gamma_4(z) = \theta^3\frac{d\theta}{dt} \tag{2.3.21}$$

$$\theta^7\gamma_5(z) = (b+1)\frac{d\theta}{dt}(\frac{d\theta}{dt}x + \frac{d\sigma}{dt}) + \theta\frac{d\theta}{dt} \tag{2.3.22}$$

$$\theta^7\gamma_6(z) = \tfrac{1}{\theta^2}\tfrac{d\theta}{dt}(\tfrac{d\theta}{dt}x + \tfrac{d\sigma}{dt}) - \tfrac{1}{\theta}(\tfrac{d^2\theta}{dt^2}x + \tfrac{d^2\sigma}{dt^2}) - \tfrac{(b+1)}{\theta^3}\tfrac{d\theta}{dt}(\tfrac{d\theta}{dt}x + \tfrac{d\sigma}{dt})^2 \tag{2.3.23}$$

with $\gamma_1(z), \gamma_2(z), \gamma_3(z), \gamma_4(z), \gamma_5(z)$ and $\gamma_6(z)$ are to be determined.

First consider (2.3.18), since $z = x\theta(t) + \sigma(t)$ [from equation (2.3.12)], it is necessary that $\gamma_1(z) = A$, where A is constant.

Hence, we get

$$\frac{d\theta}{dt} = A\theta^4. \tag{2.3.24}$$

It can then easily seen that equation (2.3.21) is satisfied identically. In equation (2.3.19), right hand side is linear in x. so taking $\gamma_2(z) = Bz + C$. Now using (2.3.12) and equating coefficients of powers of x, we get $B = A$ and

$$\frac{d\sigma}{dt} = \theta^3(A\sigma + C). \tag{2.3.25}$$

Clearly from equation (2.3.22), equation (2.3.17) reduces to ODE only if $\theta = 1$. So from equations (2.3.12), (2.3.24) and (2.3.25), we get $\beta = 1$ and $z = x + Ct + C_0$

Hence condition for modified b-family equation to have similarity reduction in the form $u(x,t) = U(x,t,w(z))$, for some U and z, is that $u(x,t) = w(z)$ with $z = x+Ct+C_0$, where C, C_0 are arbitrary constants, which is just travelling wave solution. Now let us consider the solutions in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)y(x). \tag{2.3.26}$$

Substituting this in equation (2.1.6) yields

$$\begin{aligned}
&-\beta^2 yy''' - \alpha\beta y''' - b\beta^2 y'y'' - (b+3)\beta\beta_x yy'' \\
&-(b\alpha_x\beta + 3\alpha\beta_x + \beta_t)y'' \\
&-2b\beta\beta_x y'^2 + (b+1)\beta^3 y^2 y' + (-2b\beta_x^2 + 2(b+1)\alpha\beta^2 - (b+3)\beta\beta_{xx})yy' \\
&+(-2b\alpha_x\beta_x - 2\beta_{xt} + (b+1)\alpha^2\beta - b\beta\alpha_{xx} - 3\alpha\beta_{xx})y' \\
&+(b+1)\beta^2\beta_x y^3 + (-b\beta_x\beta_{xx} + (b+1)\beta^2\alpha_x + 2(b+1)\alpha\beta\beta_x - \beta\beta_{xxx})y^2 \\
&+(\beta_t - b\alpha_x\beta_{xx} + (b+1)\alpha^2\beta_x + 2(b+1)\alpha\beta\alpha_x - \beta\alpha_{xxx} - \beta_{xxt} \\
&\alpha\beta_{xxx} - b\beta_x\alpha_{xx})y - b\alpha_x\alpha_{xx} + (b+1)\alpha^2\alpha_x - \alpha\alpha_{xxx} + \alpha_t - \alpha_{xxt} = 0.
\end{aligned} \tag{2.3.27}$$

It can be easily shown that equation (2.3.27) will be ODE only if $\alpha = 0$ and $\beta = 1$. Using this Equation, (2.3.27) reduces to

$$(b+1)\left(y\left(x\right)\right)^2 \tfrac{d}{dx}y\left(x\right) - b\left(\tfrac{d}{dx}y\left(x\right)\right)\tfrac{d^2}{dx^2}y\left(x\right) - y\left(x\right)\tfrac{d^3}{dx^3}y\left(x\right) = 0 \tag{2.3.28}$$

This is an ordinary differential equation, so there are no special solution of equation (2.1.6) in this case. We will get only stationary solution of equation.

Therefore we conclude that the only non-constant similarity reduction of the modified b-family equation obtainable either using classical Lie method or direct method due to Clarkson and Kruskal (1989), is the travelling wave solution.

2.3.3 Some Exact Solutions of Modified b-Family Equation

To find the explicit exact solutions of the modified b-family equation, we make the transformation

$$\begin{aligned} &u(x,t) = w(z) \\ &z = x - ct \end{aligned} \tag{2.3.29}$$

where $w(z)$ satisfies

$$\begin{aligned} &(c - w(z)) \tfrac{d^3}{dz^3} w(z) + (b+1)(w(z))^2 \tfrac{d}{dz} w(z) \\ &-b \left(\tfrac{d}{dz} w(z)\right) \tfrac{d^2}{dz^2} w(z) - c \tfrac{d}{dz} w(z) = 0. \end{aligned} \tag{2.3.30}$$

We derived some traveling wave solutions of equation (2.1.6) with aid of Maple as follows:

1. **Solutions in terms of** tan() **function**

$$\begin{aligned} &(i)\ \ u(x,t) = \tfrac{1}{b+1}(-c + 2\,b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} + 4\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}) \\ &+\tfrac{3}{b+1}\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\tan(C_1 + 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct)))^2 \\ &(ii)\ \ u(x,t) = \tfrac{1}{b+1}(-c - 2\,b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} - 4\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}) \\ &-\tfrac{3}{b+1}\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\tan(C_1 - 1/2\,i\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct)))^2 \end{aligned} \tag{2.3.31}$$

where C_1 is arbitrary constant.

2. **Solutions in terms of** tanh() **function**

$$\begin{aligned} &(i)\ \ u(x,t) = -\tfrac{1}{b+1}(c + 2\,b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} + 4\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}) \\ &+\tfrac{3}{b+1}\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\tanh(C_1 - 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct)))^2 \\ &(ii)\ \ u(x,t) = -\tfrac{1}{b+1}(c + 2\,b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} + 4\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}) \\ &+\tfrac{3}{b+1}\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\tanh(C_1 + 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct))^2 \end{aligned} \tag{2.3.32}$$

where C_1 is arbitrary constant.

3. **Solutions in terms of** cot() **function**

$$\begin{aligned} &(i)\ \ u(x,t) = \tfrac{1}{b+1}(-c + 2\,b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} + 4\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}) \\ &+\tfrac{3}{b+1}\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\cot(C_1 - 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct)))^2 \\ &(ii)\ \ u(x,t) = \tfrac{1}{b+1}(-c + 2\,b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} + 4\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}) \\ &+\tfrac{3}{b+1}\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\cot(C_1 + 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct)))^2 \end{aligned} \tag{2.3.33}$$

where C_1 is arbitrary constant.

4. **Solutions in terms of** coth() **function**

$$\begin{aligned}&(i)\;\; u(x,t) = -\tfrac{1}{b+1}(c + 2\,b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} + 4\sqrt{\tfrac{c(b+1-c)}{b(b+2)}})\\&+\tfrac{3}{b+1}\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\coth(C_1 - 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct)))^2\\&(ii)\;\; u(x,t) = -\tfrac{1}{b+1}(c + 2\,b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} + 4\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}})\\&+\tfrac{3}{b+1}\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\coth(C_1 + 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct)))^2\end{aligned} \tag{2.3.34}$$

where C_1 is arbitrary constant.

5. **Solutions in terms of** sec() **function**

$$\begin{aligned}&(i)\;\; u(x,t) = -\tfrac{1}{b+1}(2\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} + c + b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}})\\&+\tfrac{3}{b+1}\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\sec(C_1 - 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct)))^2\\&(ii)\;\; u(x,t) = -\tfrac{1}{b+1}(2\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} + c + b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}})\\&+\tfrac{3}{b+1}\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\sec(C_1 + 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c}{b(b+2)}}(x-ct)))^2\end{aligned} \tag{2.3.35}$$

where C_1 is arbitrary constant.

6. **Solutions in terms of sech() function**

$$\begin{aligned}&(i)\;\; u(x,t) = \tfrac{1}{b+1}(2\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} - c + b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}})\\&-\tfrac{3}{b+1}\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\mathrm{sech}(C_1 - 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct)))^2\\&(ii)\;\; u(x,t) = \tfrac{1}{b+1}(2\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}} - c + b\sqrt{\tfrac{c(b+1-c)}{b(b+2)}})\\&-\tfrac{3}{b+1}\,\sqrt{\tfrac{c(b+1-c)}{b(b+2)}}(b+2)(\mathrm{sech}(C_1 + 1/2\,\sqrt{2}\sqrt[4]{\tfrac{c(b+1-c)}{b(b+2)}}(x-ct)))^2\end{aligned} \tag{2.3.36}$$

where C_1 is arbitrary constant.

2.4 Discussion

In this chapter, we have investigated the symmetries and invariant solutions of b-family equation and modified b-family equation. Firstly, the Lie group method is utilized for the purpose of obtaining the group infinitesimals of b-family equation (2.1.4). The basic fields of the optimal system lead to reductions that are inequivalent with respect to the symmetry transformations. Secondly, we used direct method introduced by Clarkson and Krusksal to find symmetries of b-family equation. We obtain the exact solutions of b-family equation corresponding to reduced ODEs, which have been verified by putting them back into the original equation using Maple. One can easily derive the exact solutions of Camassa-Holm equation and Degasperis-Procesi equations as particular case.

We have also investigated the symmetries of modified b-family equation (2.1.6), which describe the balance between the convection and the stretching for small viscosity in the dynamics of $1D$ nonlinear waves in fluids. we performed Direct method introduced by Clarkson and Kruksal (1989) for symmetries of the modified b-family equation. We have shown that only non constant similarity reduction obtainable either by Lie classical method or Direct method, is travelling wave solution of equation (2.1.6). We have derived some travelling wave solution. The authenticity of solutions have been checked with aid of software Maple.

Chapter 3

Coupled Higgs Field Equation And Hamiltonian Amplitude Equation [1]

3.1 Introduction

Nonlinear evolution equations (NEEs) are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma wave and chemical physics. Various methods have been utilized to explore different kinds of solutions of physical models described by nonlinear partial differential equations (PDEs). One of the basic physical problems for those models is to obtain their exact solutions. In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors who are interested in nonlinear physical phenomena. Various methods for obtaining exact travelling wave solutions to nonlinear equations have been presented, such as the Homogeneous balance method [102], the Tanh function method [76, 1], the Jacobi elliptic function method [73, 114], the F-expansion method [103, 104]. Among various methods, the Lie symmetry method, also called as Lie group method, is one of the most powerful methods to determine exact solutions of nonlinear PDEs. It is based upon the study of the invariance under one-parameter Lie group of point transformations [86, 15, 44]. In the recent past, there have been considerable developments in symmetry methods

[1] A part of this chapter has been appeared in *Parmana-Journal of Physics* 79 (1) (2012) 41-60

for differential equations as is evident by the number of research papers, books and new symbolic software devoted to the subject. Some recent and important contributions are in [39, 97].

The Higgs equation [99]

$$\begin{aligned} &u_{tt} - u_{xx} - \alpha u + \beta |u|^2 u - 2uv = 0 \\ &v_{tt} + v_{xx} - \beta(|u|^2)_{xx} = 0, \end{aligned} \tag{3.1.1}$$

describes a system of conserved scalar nucleons interacting with neutral scalar mesons. The equation (3.1.1) is the coupled nonlinear Klein-Gorden equation for $\alpha < 0,\ \beta < 0$ and the Coupled Higgs field equation for $\alpha > 0, \beta > 0$. Tajiri obtained N-soliton solutions to the system (3.1.1) in [99]. Zhao constructed more general travelling wave solutions of system (3.1.1) in [129].

A new Hamiltonian amplitude equation

$$\iota u_x + u_{tt} - 2\eta |u|^2 u - \beta u_{xt} = 0, \tag{3.1.2}$$

where $\eta = \pm 1, \beta << 1$, was introduced by Wadati et al. [101]. This equation governs certain instabilities of modulated wave trains; the addition of the term $-\beta u_{xt}$ overcomes the ill-posedness of the unstable nonlinear Schródinger equation. The equation is apparently not integrable, but a Hamiltonian analogue of the Kuramoto-Sivashinsky equation, which arises in dissipative system. Yan found solitary wave solutions for a Hamiltonian amplitude equation by using a simple transformation in [115].

The chapter has been structured as follows. In subsection 3.2.1 and 3.2.2, we applied Lie classical method to the system (3.1.1) and equation (3.1.2) respectively, to reduce them to ordinary differential equations (ODEs) and some exact solution are derived. In subsection 3.3.1, we applied $(\frac{G'}{G})-$ expansion method for finding exact travelling wave solutions of Higgs field equation. Subsection 3.3.2 is devoted to find travelling wave solutions of Hamiltonian amplitude equation. In Section 4, some conclusions are given.

3.2 Lie Symmetry Analysis

Lie's method [86, 15, 44] is an effective and simplest method among group theoretic techniques and a large number of equations are solved with the aid of this method. The Lie group method, is also called symmetry analysis sometimes. Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. Once one has determined the symmetry group of a system of differential equations, a number of applications become available. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones.

3.2.1 Higgs Field Equation

In this subsection, we will perform Lie symmetry analysis for the Higgs field equation. As u is a complex variable, so to sperate the real and imaginary parts of u, we will consider two cases.

Case (i) Here, on setting

$$u(x,t) = \psi_1(x,t) + \iota\psi_2(x,t), \tag{3.2.1}$$

system (3.1.1) decomposes into following system of equations

$$\begin{aligned} &\psi_{1tt} - \psi_{1xx} + \alpha\psi_1 + \beta(\psi_1^2 + \psi_2^2)\psi_1 - 2\psi_1 v = 0 \\ &\psi_{2tt} - \psi_{2xx} + \alpha\psi_2 + \beta(\psi_1^2 + \psi_2^2)\psi_2 - 2\psi_2 v = 0 \\ &v_{tt} + v_{xx} - \beta(\psi_1^2 + \psi_2^2)_{xx} = 0. \end{aligned} \tag{3.2.2}$$

Let us consider the Lie group of point transformations

$$\begin{aligned} t^* &= t + \epsilon\tau(x,t,\psi_1,\psi_2,v) + O(\epsilon^2) \\ x^* &= x + \epsilon\xi(x,t,\psi_1,\psi_2,v) + O(\epsilon^2) \\ \psi_1^* &= \psi_1 + \epsilon\eta(x,t,\psi_1,\psi_2,v) + O(\epsilon^2) \\ \psi_2^* &= \psi_2 + \epsilon\phi(x,t,\psi_1,\psi_2,v) + O(\epsilon^2) \\ v^* &= v + \epsilon\zeta(x,t,\psi_1,\psi_2,v) + O(\epsilon^2), \end{aligned} \tag{3.2.3}$$

with small parameter $\epsilon \ll 1$. The method for determining the symmetry group of (3.2.2) consists of finding the infinitesimals ξ, τ, η, ϕ and ζ, which are functions of x, t, ψ_1, ψ_2 and v.

Assuming that equations (3.2.2) are invariant under the transformations (3.2.3), the infinitesimals ξ, τ, η, ϕ and ζ must satisfy the symmetry conditions

$$\begin{aligned}
&\eta^{tt} - \eta^{xx} + \alpha\eta + 3\beta\psi_1^2\eta + 2\beta\phi\psi_1\psi_2 - 2\psi_1\zeta + \beta\psi_2^2\eta - 2\eta v = 0 \\
&\phi^{tt} - \phi^{xx} + \alpha\phi + 3\beta\psi_2^2\phi + 2\beta\eta\psi_1\psi_2 - 2\psi_2\zeta + \beta\psi_1^2\phi - 2\phi v = 0 \\
&\zeta^{tt} + \zeta^{xx} - 4\beta\psi_{1x}\eta^x - 4\beta\psi_{2x}\phi^x - 2\beta\psi_1\eta^{xx} - 2\beta\eta\psi_{1xx} - 2\beta\psi_2\phi^{xx} - 2\beta\phi\psi_{2xx} = 0,
\end{aligned} \tag{3.2.4}$$

where $\eta^x, \eta^{xx}, \eta^{tt}, \phi^x, \phi^{xx}, \phi^{tt}, \zeta^{xx}$ and ζ^{tt} are extended (prolonged) infinitesimals acting on an enlarged space (jet space) that includes all derivatives of the dependent variables (for more details the readers can refer to [15]). Substituting value of $\eta^x, \eta^{xx}, \eta^{tt}, \phi^x, \phi^{xx}, \phi^{tt}, \zeta^{xx}$ and ζ^{tt} into (3.2.4), then equating the coefficients of the various monomials in the first, second and the other order partial derivatives of ψ_1, ψ_2, v and their powers, we can find the determining equations for the symmetry group of the Higgs field equation. Solving these equations, we get the following forms of the coefficient functions

$$\xi = a_1, \ \tau = a_2, \ \eta = -\psi_2 a_3, \ \phi = \psi_1 a_3, \ \zeta = 0, \tag{3.2.5}$$

where a_1, a_2 and a_3 are arbitrary constants.

Now we can reduce the system (3.1.1) to system of ODEs using characteristic equation:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{d\psi_1}{\eta} = \frac{d\psi_2}{\phi} = \frac{dv}{\zeta}. \tag{3.2.6}$$

Solving characteristic equation and using (3.2.1) we have the following similarity variables for system (3.1.1)

$$\rho = x - b_0 t, u = P(\rho)e^{\iota a_0 t}e^{\iota Q(\rho)}, v = R(\rho), \tag{3.2.7}$$

where $a_0 = \frac{a_3}{a_2}, b_0 = \frac{a_1}{a_2}$. Here $\rho(x,t)$ is new independent variable and $P(\rho), Q(\rho), R(\rho)$ are new dependent variables.

Using (3.2.7) in (3.1.1) and separating real and imaginary parts, we have

$$(b_0^2-1)P''+\beta P^3+(1-b_0^2)PQ'^2+2a_0b_0PQ'-2PR-(a_0^2+\alpha)P=0 \qquad (3.2.8)$$

$$2(b_0^2-1)P'Q'-2a_0b_0P'+(b_0^2-1)PQ''=0 \qquad (3.2.9)$$

$$(b_0^2+1)R''-2\beta PP''-2\beta P'^2=0, \qquad (3.2.10)$$

where ($'$) denotes derivative with respect to ρ.

Integrating (3.2.10) twice we get

$$R=\frac{\beta}{b_0^2+1}P^2+\frac{C_1}{b_0^2+1}\rho+\frac{C_2}{b_0^2+1}, \qquad (3.2.11)$$

where C_1 and C_2 are arbitrary constants.

Let

$$Q'=Z(P) \qquad (3.2.12)$$

so that

$$Q''=P'\frac{dZ}{dP}. \qquad (3.2.13)$$

Using (3.2.12) and (3.2.13) in (3.2.9) and integrating we have

$$Q'=Z=\frac{a_0b_0}{b_0^2-1}+\frac{C_0}{P^2}, \qquad (3.2.14)$$

where C_0 is arbitrary constant.

Now using (3.2.11), (3.2.14) in (3.2.8) with $C_1=0$ and integrating once we get

$$P'^2=-\frac{\beta}{2(b_0^2+1)}P^4+\frac{1}{b_0^2-1}(\alpha+a_0^2+\frac{2C_2}{b_0^2+1}-\frac{a_0^2b_0^2}{b_0^2-1})P^2-\frac{C_0}{P^2}+C_3, \qquad (3.2.15)$$

where C_3 is arbitrary constant.

Subcase(i) If $C_0=0,\ C_3\neq 0$

Integrating (3.2.15) we get

$$P(\rho)=\frac{sn\left(\frac{1}{2}\sqrt{-2\,b_2+2\,\sqrt{{b_2}^2+4\,b_1C_3}}\rho+C_4,m\right)C_3\sqrt{2}}{\sqrt{C_3\left(-b_2+\sqrt{{b_2}^2+4\,b_1C_3}\right)}}, \qquad (3.2.16)$$

where

$$m = \frac{\sqrt{\left(-2\,b_1C_3-{b_2}^2+b_2\sqrt{{b_2}^2+4\,b_1b_3}\right)2b_1C_3}}{-2\,b_1C_3-{b_2}^2+b_2\sqrt{{b_2}^2+4\,b_1C_3}},\ b_1 = \frac{\beta}{2(b_0^2+1)}, b_2 = \frac{1}{b_0^2-1}(\alpha + a_0^2 + \frac{2C_2}{b_0^2+1} - \frac{a_0^2b_0^2}{b_0^2-1}) \tag{3.2.17}$$

and sn is Jacobi elliptic sine function.

Using (3.2.16) in (3.2.11), we have

$$R(\rho) = \frac{\beta}{b_0^2+1}\left(\frac{sn\left(\frac{1}{2}\sqrt{-2\,b_2+2\,\sqrt{{b_2}^2+4\,b_1C_3}}\rho+C_4,m\right)C_3\sqrt{2}}{\sqrt{C_3\left(-b_2+\sqrt{{b_2}^2+4\,b_1C_3}\right)}}\right)^2 + \frac{C_2}{b_0^2+1}, \tag{3.2.18}$$

where $m,\ b_1$ and b_2 are given by equation (3.2.17).

Integrating (3.2.14) we have

$$Q(\rho) = \frac{a_0b_0}{b_0^2-1}\rho + C_5, \tag{3.2.19}$$

where C_5 is arbitrary constant.

Using (3.2.16), (3.2.18) and (3.2.19) in (3.2.7), solution of the main system (3.1.1) is given as

$$\begin{aligned} u(x,t) &= \left(\frac{sn\left(\frac{1}{2}\sqrt{-2\,b_2+2\,\sqrt{{b_2}^2+4\,b_1C_3}}\rho+C_4,m\right)C_3\sqrt{2}}{\sqrt{C_3\left(-b_2+\sqrt{{b_2}^2+4\,b_1C_3}\right)}}\right)e^{\iota a_0t}e^{\iota(\frac{a_0b_0}{b_0^2-1}\rho+C_5)} \\ v(x,t) &= \frac{\beta}{b_0^2+1}\left(\frac{sn\left(\frac{1}{2}\sqrt{-2\,b_2+2\,\sqrt{{b_2}^2+4\,b_1C_3}}\rho+C_4,m\right)C_3\sqrt{2}}{\sqrt{C_3\left(-b_2+\sqrt{{b_2}^2+4\,b_1C_3}\right)}}\right)^2 + \frac{C_2}{b_0^2+1}, \end{aligned} \tag{3.2.20}$$

where $m,\ b_1,\ b_2$ are given by equation (3.2.17) and $\rho = x - b_0t$.

Subcase(ii) If $C_0 = 0,\ C_3 = 0$

In this case (3.2.15) becomes

$$P'^2 = -b_1P^4 + b_2P^2, \tag{3.2.21}$$

where b_1 and b_2 are given by equation (3.2.17).

Using transformation

$$X(\rho) = P(\rho)^2 \tag{3.2.22}$$

in (3.2.21), we have

$$X'^2 = -4b_1X^3 + 4b_2X^2. \tag{3.2.23}$$

Equation (3.2.23) admit the following solutions

$$X = P^2 = \begin{cases} \dfrac{b_2}{b_1}\text{sech}\left(\sqrt{b_2}\,(\pm\rho\ + C_4\,)\right)^2, & \text{when } b_2 > 0 \\ \dfrac{b_2}{b_1}\sec\left(\sqrt{-b_2}\,(\pm\rho\ + C_4\,)\right)^2, & \text{when } b_2 < 0 \end{cases} \tag{3.2.24}$$

where C_4 is arbitrary constant.

Corresponding solutions of the main system (3.1.1) are given as:

When $b_2 > 0$

$$\begin{aligned} u(x,t) &= \sqrt{\frac{b_2}{b_1}}\text{sech}\left(\sqrt{b_2}\,(\pm\rho\ + C_4\,)\right)e^{\iota a_0}e^{\iota(\frac{a_0b_0}{b_0^2-1}\rho+C_5)} \\ v(x,t) &= \frac{b_2\beta}{b_1(b_0^2+1)}\text{sech}\left(\sqrt{b_2}\,(\pm\rho\ + C_4\,)\right)^2 + \frac{C_2}{b_0^2+1}, \end{aligned} \tag{3.2.25}$$

where $b_1,\ b_2$ are given by equation (3.2.17) and $\rho = x - b_0t$.

When $b_2 < 0$

$$\begin{aligned} u(x,t) &= \sqrt{\frac{b_2}{b_1}}\sec\left(\sqrt{-b_2}\,(\pm\rho\ + C_4\,)\right)e^{\iota a_0}e^{\iota(\frac{a_0b_0}{b_0^2-1}\rho+C_5)} \\ v(x,t) &= \frac{b_2\beta}{b_1(b_0^2+1)}\sec\left(\sqrt{-b_2}\,(\pm\rho\ + C_4\,)\right)^2 + \frac{C_2}{b_0^2+1}, \end{aligned} \tag{3.2.26}$$

where $b_1,\ b_2$ are given by equation (3.2.17) and $\rho = x - b_0t$.

In this case, we get trivial Lie symmetries of system (3.1.1).

Case(ii) Now on setting

$$u(x,t) = \psi_1(x,t)e^{\iota\psi_2(x,t)}, \tag{3.2.27}$$

system (3.1.1) decomposes to following system

$$\begin{aligned} &\psi_{1tt} - \psi_1\psi_{2t}^2 - \psi_{1xx} + \psi_1\psi_{2x}^2 - \alpha\psi_1 + \beta\psi_1^3 - 2\psi_1v = 0 \\ &2\psi_{1t}\psi_{2t} + \psi_1\psi_{2tt} - 2\psi_{1x}\psi_{2x} - \psi_1\psi_{2xx} = 0 \\ &v_{tt} + v_{xx} - \beta(\psi_1)_{xx}^2 = 0. \end{aligned} \tag{3.2.28}$$

Apply the Lie classical method on system (3.2.28) as mentioned in case (i), we get the following symmetries

$$\eta = a_1\psi_1,\ \phi = a_2,\ \zeta = a_1(2v+\alpha),\ \tau = -a_1t + a_4,\ \xi = -a_1x + a_3, \tag{3.2.29}$$

where $a_1,\ a_2,\ a_3$ and a_4 are arbitrary constants.

Now we will give symmetry reductions and corresponding exact solutions of system (3.1.1).

Subcase (i) When $a_1 \neq 0$ and $a_2 = a_3 = a_4 = 0$.

Solving the characteristic equation (3.2.6) and using (3.2.27), we have following similarity variables

$$\rho = \frac{x}{t},\ u = \frac{F(\rho)}{t}e^{G(\rho)},\ v = \frac{1}{2}\left(\frac{H(\rho)}{t^2} - \alpha\right). \tag{3.2.30}$$

Here $\rho(x,t)$ is new independent variable and $F(\rho),\ G(\rho),\ H(\rho)$ are new dependent variables.

Using (3.2.30) in 3.1.1, we have

$$(\rho^2-1)F'' - (\rho^2-1)FG'^2 + 4\rho F' + \beta F^3 - FH + 2F = 0 \tag{3.2.31}$$

$$(1-\rho^2)FG'' + 2(1-\rho^2)F'G' - 4\rho FG' = 0 \tag{3.2.32}$$

$$(1+\rho^2)H'' + 6\rho H' + 6H - 2\beta(F^2)'' = 0, \tag{3.2.33}$$

where ($'$) denotes derivative with respect to ρ.

From equation (3.2.32), we have

$$G = \int \frac{C_1}{F^2(1-\rho^2)^2}d\rho. \tag{3.2.34}$$

Now using equation (3.2.34) in (3.2.31), we have

$$H = \frac{((\rho^2-1)F)''}{F} + \frac{1}{(1-\rho^2)^3F^4} + \beta F^2. \tag{3.2.35}$$

Substituting value of H from (3.2.35) in (3.2.33), we have

$$\begin{aligned}&(1+\rho^2)(\tfrac{((\rho^2-1)F)''}{F} + \tfrac{1}{(1-\rho^2)^3F^4} + \beta F^2)'' + 6\rho(\tfrac{((\rho^2-1)F)''}{F} + \tfrac{1}{(1-\rho^2)^3F^4} + \beta F^2)' \\ &+6(\tfrac{((\rho^2-1)F)''}{F} + \tfrac{1}{(1-\rho^2)^3F^4} + \beta F^2) - 2\beta(F^2)'' = 0.\end{aligned} \tag{3.2.36}$$

Solution of main system (3.1.1) is given as

$$u = \frac{F}{t}e^{\int \frac{C_1}{F^2(1-\rho^2)^2}d\rho},\ v = \frac{1}{2}\left(\frac{\frac{((\rho^2-1)F)''}{F} + \frac{1}{(1-\rho^2)^3F^4} + \beta F^2}{t^2} - \alpha\right), \tag{3.2.37}$$

where F is given by the equation (3.2.36) and ρ is given by (3.2.30).

Subcase (ii) When $a_1 = a_3 = 0, a_2 \neq 0$ and $a_4 \neq 0$

For simplicity let $a_2 = a_3 = 1$ and solving characteristic equation, we get following similarity variables

$$\rho = x, \; u = F(\rho)e^{t+G(\rho)}, w = H(\rho). \tag{3.2.38}$$

Here $\rho(x,t)$ is new independent variable and $F(\rho), \; G(\rho), \; H(\rho)$ are new dependent variables.

Using (3.2.38) in 3.1.1, we have

$$H'' - 2\,\beta\,F'^2 - 2\,\beta\,F F'' = 0 \tag{3.2.39}$$

$$-F - F'' + FG'^2 - \alpha\,F + \beta\,F^3 - 2\,F H = 0 \tag{3.2.40}$$

$$-2\,F'\,G' - FG'' = 0, \tag{3.2.41}$$

where $(')$ denotes derivative with respect to ρ.

From (3.2.39), we have

$$H = \beta F^2 + C_1\rho + C_2, \tag{3.2.42}$$

where $C_1, \; C_2$ are arbitrary constants.

Now integrating (3.2.41), we have

$$G' = \frac{C_3}{F^2}, \tag{3.2.43}$$

where C_3 is an arbitrary constant.

Using (3.2.42) and (3.2.43) in (3.2.40), we have

$$F'' - \frac{C_3^2}{F^3} + (1+\alpha)F - \beta\,F^3 + 2\,F(\beta F^2 + C_1\rho + C_2) = 0. \tag{3.2.44}$$

Here choosing $C_3 = C_1 = 0$ and using (3.2.38), corresponding to ODE (3.2.44), solutions of main system (3.1.1) can be given as

$$
\begin{aligned}
&(i)\ u = -\frac{\sqrt{-\beta\,(1+\alpha+2\,C_2)}\tanh\left(C_5-1/2\,\sqrt{2+2\,\alpha+4\,C_2}x\right)\mathrm{e}^{\iota(t+C_4)}}{\beta},\\
&\quad v = -\,(1+\alpha+2\,C_2)\tanh\left(C_5-1/2\,\sqrt{2+2\,\alpha+4\,C_2}x\right)^2 + C_2,\\
&(ii)\ u = \frac{1}{\beta}\sqrt{-\beta\,\left(2\,C_2+\alpha+1-{C_6}^2\right)}\mathrm{cn}\left(C_5+C_6\,x, 1/2\,\frac{\sqrt{-4\,C_2-2\,\alpha-2+2\,{C_6}^2}}{C_6}\right)\mathrm{e}^{\iota(t+C_4)}\\
&\quad v = -\left(2\,C_2+\alpha+1-{C_6}^2\right)\mathrm{cn}\left(C_5+C_6\,x, 1/2\,\frac{\sqrt{-4\,C_2-2\,\alpha-2+2\,{C_6}^2}}{C_6}\right)^2 + C_2\\
&(iii)\ u = \frac{1}{\sqrt{\beta}}\sqrt{2}C_6\,\mathrm{dn}\left(C_5+C_6\,x, \frac{\sqrt{2\,{C_6}^2+1+\alpha+2\,C_2}}{C_6}\right)\mathrm{e}^{\iota(t+C_4)}\\
&\quad v = 2\,{C_6}^2\mathrm{dn}\left(C_5+C_6\,x, \frac{\sqrt{2\,{C_6}^2+1+\alpha+2\,C_2}}{C_6}\right)^2 + C_2
\end{aligned}
\tag{3.2.45}
$$

where $C_4,\ C_5$ are arbitrary constants.

3.2.2 Hamiltonian Amplitude Equation

In this subsection, we will find the symmetries and exact solution of Hamiltonian amplitude equation (3.1.2). Here too, we will consider two cases.

Case (i) On setting

$$u = \psi_1 + \iota\psi_2 \tag{3.2.46}$$

equation (3.1.2) decomposes into following system

$$
\begin{aligned}
&-\psi_{2x} + \psi_{1tt} + 2\eta(\psi_1^2+\psi_2^2)\psi_1 - \beta\psi_{1xt} = 0\\
&\psi_{1x} + \psi_{2tt} + 2\eta(\psi_1^2+\psi_2^2)\psi_2 - \beta\psi_{2xt} = 0.
\end{aligned}
\tag{3.2.47}
$$

Proceeding in similar manner as mentioned earlier, to find the Lie symmetries of system (3.2.47), we get

$$\xi = a_1,\ \tau = a_2,\ \phi_1 = a_3 v,\ \phi_2 = -a_3 u, \tag{3.2.48}$$

where ξ, τ, ϕ_1 and ϕ_2 are infinitesimals corresponding to x, t, ψ_1 and ψ_2, respectively. Solving the characteristic equation we get the following similarity variables of the system (3.1.2)

$$\rho = x - b_0 t,\ u = P(\rho)e^{\iota a_0 t}e^{\iota Q(\rho)}, \tag{3.2.49}$$

where $a_0 = \frac{a_3}{a_2}$ and $b_0 = \frac{a_1}{a_2}$.

Using (3.2.49) in (3.1.2) we have

$$b_0(b_0+\beta)P'' + 2\eta P^3 - Pa_0^2 - b_0(b_0+\beta)PQ'^2 + (\beta a_0 + 2a_0b_0 - 1)PQ' = 0 \quad (3.2.50)$$

$$b_0(b_0+\beta)PQ'' + (1-\beta a_0 - 2a_0b_0)P' + 2b_0(b_0+\beta)P'Q' = 0, \quad (3.2.51)$$

where $(')$ denotes derivative with respect to ρ.

Using

$$Q' = Z(P) \quad (3.2.52)$$

in (3.2.51) and integrating we get

$$Q' = Z = \frac{2a_0b_0 + \beta a_0 - 1}{2b_0(\beta+b_0)} + \frac{C_0}{P^2}, \quad (3.2.53)$$

where C_0 is arbitrary constant.

Using (3.2.53) in (3.2.50) and integrating, we have

$$P'^2 = -\frac{\eta}{b_0(\beta+b_0)}P^4 - \frac{1}{b_0(\beta+b_0)}\left(\frac{(\beta a_0 + 2a_0b_0 - 1)^2}{4b_0(\beta+b_0)} - a_0^2\right)P^2 - \frac{C_0}{P^2} + C_3, \quad (3.2.54)$$

where C_3 is arbitrary constant.

Here we will consider two cases as follows:

Subcase(i) when $C_0 = 0, C_3 \neq 0$

Integrating (3.2.54) we get solution in form of Jacobi elliptical sine function

$$P(\rho) = \frac{\sqrt{2C_3}\,sn\left(\frac{1}{2}\sqrt{2\,b_2 - 2\,\sqrt{{b_2}^2 - 4\,b_1C_3}}\rho + C_4, m\right)}{\left(-b_2 + \sqrt{{b_2}^2 - 4\,b_1C_3}\right)}, \quad (3.2.55)$$

where

$$m = \frac{\sqrt{-2\left(2\,b_1C_3 - {b_2}^2 + b_2\sqrt{{b_2}^2 - 4\,b_1C_3}\right)b_1C_3}}{2\,b_1C_3 - {b_2}^2 + b_2\sqrt{{b_2}^2 - 4\,b_1C_3}},\ b_1 = \frac{\eta}{b_0(\beta+b_0)},\ b_2 = \frac{1}{b_0(\beta+b_0)}\left(\frac{(\beta a_0 + 2a_0b_0 - 1)^2}{4b_0(\beta+b_0)} - a_0^2\right) \quad (3.2.56)$$

and $C_3,\ C_4$ are arbitrary constants.

Using (3.2.53) and (3.2.55) in (3.2.49), solution of Hamiltonian amplitude equation

(3.1.2) is given as

$$u(x,t)=\frac{\sqrt{2C_3}sn\left(\frac{1}{2}\sqrt{2\,b_2-2\,\sqrt{{b_2}^2-4\,b_1C_3}}(x-b_0t)+C_4,m\right)}{\left(-b_2+\sqrt{{b_2}^2-4\,b_1C_3}\right)}e^{\iota a_0}e^{\iota(\frac{2a_0b_0+\beta a_0-1}{2b_0(\beta+b_0)}(x-b_0t)+C_5)}, \tag{3.2.57}$$

where $m,\ b_1,\ b_2$ are given by (3.2.56) and $C_4,\ C_5$ are arbitrary constants.

Subcase(ii) when $C_0=0,\ C_3=0$

In this case equation (3.2.54) reduces to

$$P'^2=-b_1P^4-b_2P^2, \tag{3.2.58}$$

where b_1 and b_2 are given by (3.2.56).

Using the transformation (3.2.22), equation (3.2.58) reduces to

$$X'^2=-4b_1X^3-4b_2X^2. \tag{3.2.59}$$

Equation (3.2.59) admits the following solutions

$$X=P^2=\begin{cases}-\dfrac{b_2}{b_1}\text{sech}\left(\sqrt{-b_2}\,(\pm\rho\ +C_4\,)\right)^2, & \text{when } b_2<0\\ -\dfrac{b_2}{b_1}\sec\left(\sqrt{b_2}\,(\pm\rho\ +C_4\,)\right)^2, & \text{when } b_2>0\end{cases} \tag{3.2.60}$$

Corresponding solutions of the main equation (3.1.2) are given as

$$u(x,t)=\begin{cases}\sqrt{-\dfrac{b_2}{b_1}}\text{sech}\left(\sqrt{-b_2}\,(\pm(x-b_0t)\ +C_4\,)\right)e^{\iota a_0}e^{\iota(\frac{2a_0b_0+\beta a_0-1}{2b_0(\beta+b_0)}(x-b_0t)+C_5)},\\ \qquad\qquad \text{when } b_2<0,\ b_1>0\\ \sqrt{-\dfrac{b_2}{b_1}}\sec\left(\sqrt{b_2}\,(\pm(x-b_0t)\ +C_4\,)\right)e^{\iota a_0}e^{\iota(\frac{2a_0b_0+\beta a_0-1}{2b_0(\beta+b_0)}(x-b_0t)+C_5)},\\ \qquad\qquad \text{when } b_2>0,\ b_1<0\end{cases} \tag{3.2.61}$$

where b_1 and b_2 are given by (3.2.56) and $C_4,\ C_5$ are arbitrary constants.

Case(ii) Now on setting

$$u(x,t)=\psi_1(x,t)e^{\iota\psi_2(x,t)}, \tag{3.2.62}$$

system (3.1.2) decomposes to following system

$$\begin{aligned}&\psi_{1tt} - \psi_1\psi_{2x} - \psi_1\psi_{2t}^2 - \beta\psi_{xt} + \beta\psi_1\psi_{2x}\psi_{2t} - 2\eta\psi_1^3 = 0\\&\psi_1\psi_{2tt} + \psi_{1x} + 2\psi_{1t}\psi_{2t} - \beta(\psi_{1t}\psi_{2x} + \psi_{1x}\psi_{2t}) - \beta\psi_1\psi_{2xt} = 0.\end{aligned} \tag{3.2.63}$$

Now applying Lie classical method on (3.2.63), we get following symmetries

$$\xi = a_1x + a_2,\ \tau = -a_1\left(t + \frac{2x}{\beta}\right) + a_3,\ \phi_1 = 0,\ \phi_2 = -a_1\left(\frac{t}{\beta}\right) + a_4, \tag{3.2.64}$$

where $\xi, \tau, \phi_1, \phi_2$ are infinitesimals corresponding to x, t, ψ_1, ψ_2, respectively and $a_1,\ a_2,$ $a_3,\ a_4$ are arbitrary constants.

Now we will give similarity reduction and corresponding exact solutions of equation (3.1.2).

Subcase (i) $a_1 \neq 0$ and $a_2 = a_3 = a_4 = 0$

Solving characteristic equation, we have following similarity variables

$$\rho = tx + \frac{x^2}{\beta},\ \psi_1 = F(\rho),\ \psi_2 = \frac{2x}{\beta^2} + \frac{t}{\beta} + G(\rho), \tag{3.2.65}$$

where ρ is new independent variable and $F(\rho),\ G(\rho)$ are new dependent variables.

Using (3.2.65) in (3.2.63), we have

$$\beta^3\rho F'' - \beta^3\rho F G'^2 + \beta^3 F' + 2\eta\beta^2 F^3 + F = 0 \tag{3.2.66}$$

$$\rho F G'' + 2\rho F' G' + F G' = 0, \tag{3.2.67}$$

where $(')$ denotes derivative with respect to ρ.

Integrating (3.2.67) once, we have

$$G' = \frac{C_1}{\rho F^2}, \tag{3.2.68}$$

where C_1 is arbitrary constant.

Substituting (3.2.68) in (3.2.66), we have

$$\beta^3\rho^2F^3F'' + \beta^3\rho F^3F' + 2\eta\beta^2\rho F^6 + \rho F^4 - \beta^3C_1^2 = 0. \tag{3.2.69}$$

Using (3.2.65 and (3.2.68) in (3.2.62), solution of main equation (3.1.2) can be given as

$$u = F(\rho)e^{\iota(\frac{2x}{\beta^2}+\frac{t}{\beta}+\int\frac{C_1}{\rho F^2}d\rho+C_2)}, \tag{3.2.70}$$

where ρ is given by (3.2.65) and F is given by (3.2.69).

Subcase (ii) $a_2 \neq 0,\ a_4 \neq 0$ and $a_1 = a_3 = 0$

Let $a_2 = a_4 = 1$ and solving characteristic equation, we have following similarity variables

$$\rho = t,\ \psi_1 = F(\rho),\ \psi_2 = x + G(\rho), \tag{3.2.71}$$

where ρ is new independent variable and F, G are new dependent variables.

Using (3.2.71) into system (3.2.63), we have

$$F'' - FG'^2 - 2\eta F^3 + \beta FG' - F = 0 \tag{3.2.72}$$

$$FG'' + 2F'G' - \beta F' = 0, \tag{3.2.73}$$

where $(')$ denotes derivative with respect to ρ.

Corresponding to ODEs (3.2.72)-(3.2.73), solutions of Hamiltonian equation (3.1.2) are given as

$$\begin{aligned}
&(i)\ u = -\frac{\sqrt{2\eta\,(\beta^2-4)}\tanh\left(C_2 - 1/4\sqrt{2\,\beta^2-8}t\right)}{4\eta}\mathrm{e}^{i\left(x+\frac{\beta t}{2}+C_1\right)}\\
&(ii)\ u = \frac{1}{4\eta}\sqrt{2\eta\,\left(\beta^2-4\,{C_3}^2-4\right)}\mathrm{cn}\left(C_2+C_3\,t,\ \frac{\sqrt{-2\,\beta^2+8\,{C_3}^2+8}}{4C_3}\right)\mathrm{e}^{i\left(x+\frac{\beta t}{2}+C_1\right)}\\
&(iii)\ u = \frac{1}{2\eta}\sqrt{\eta\,\left(4\,{C_3}^2-4+\beta^2\right)}\mathrm{nd}\left(C_2+C_3\,t,\ \frac{\sqrt{8\,{C_3}^2+\beta^2-4}}{2C_3}\right)\mathrm{e}^{i\left(x+\frac{\beta t}{2}+C_1\right)},
\end{aligned} \tag{3.2.74}$$

where $C_1,\ C_2,\ C_3$ are arbitrary functions.

3.3 $(\frac{G'}{G})-$Expansion Method and Travelling Wave Solutions

In this section, we shall use $(\frac{G'}{G})-$ expansion method [105, 125] to obtain the some new exact solutions of (3.1.1) and (3.1.2) equations. The description of method has been given in chapter 1, section 1.2.4.1.

3.3.1 Coupled Higgs Equation

To find the explicit exact solutions of the coupled Higgs equation (3.1.1), we make the transformation

$$u = e^{\iota\theta}U(\xi), \quad v = V(\xi), \tag{3.3.1}$$

where $\theta = px + rt$, $\xi = x + ct$, we have a relation $p = rc$ and system (3.1.1) reduces to the following system of ODEs

$$\begin{aligned}&(c^2-1)U'' + [r^2(c^2-1)-\alpha]U - 2UV + \beta U^3 = 0,\\ &(c^2+1)V'' - \beta(U^2)'' = 0.\end{aligned} \tag{3.3.2}$$

Integrating the second equation in the system (3.3.2), we find

$$(c^2+1)V' - 2\beta UU' + C = 0, \tag{3.3.3}$$

where C is constant of integration.
Suppose that solution of system can be expressed by polynomials in $(\frac{G'}{G})$ as follows:

$$\begin{aligned}U(\xi) &= a_m(\frac{G'}{G})^m + ...,\\ V(\xi) &= b_n(\frac{G'}{G})^n + ...,\end{aligned} \tag{3.3.4}$$

where $G = G(\xi)$ satisfies the second order Linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{3.3.5}$$

where λ, μ and c are arbitrary constants to be determined later. Balancing the highest order derivatives in linear terms with the nonlinear terms, we get $m = 1$ and $n = 2$. We can suppose that the solution of system (3.3.2)-(3.3.3) is of the form

$$U(\xi) = a_1(\frac{G'}{G}) + a_0, \quad a_1 \neq 0, \tag{3.3.6}$$

$$V(\xi) = b_2(\frac{G'}{G})^2 + b_1(\frac{G'}{G}) + b_0, \quad b_2 \neq 0. \tag{3.3.7}$$

Substituting (3.3.6)-(3.3.7) together with (3.3.5) into first equation of (3.3.2) and in equation (3.3.3), collecting all terms with equal power of $(\frac{G'}{G})$ and setting each coefficient to zero, we obtain the following set of algebraic equations:

$$\begin{aligned}
&\beta\,{a_1}^3-2\,a_1-2\,a_1b_2+2\,c^2a_1=0\\
&3\,\beta\,{a_1}^2a_0-2\,a_0b_2-2\,a_1b_1+3\,c^2\lambda\,a_1-3\,\lambda\,a_1=0\\
&c^2\lambda^2a_1+2\,c^2a_1\mu+r^2c^2a_1-\alpha\,a_1+3\,\beta\,a_1{a_0}^2-2\,a_1b_0-\lambda^2a_1-2\,a_1\mu-r^2a_1-2\,a_0b_1=0\\
&-2\,a_0b_0+r^2c^2a_0-r^2a_0+c^2\lambda\,\mu\,a_1+\beta\,{a_0}^3-\lambda\,\mu\,a_1-\alpha\,a_0=0\\
&-2\,b_2+2\,\beta\,{a_1}^2-2\,c^2b_2=0\\
&-2\,c^2\lambda\,b_2-2\,\lambda\,b_2-b_1+2\,\beta\,a_1a_0-c^2b_1+2\,\beta\,{a_1}^2\lambda=0\\
&2\,\beta\,{a_1}^2\mu+2\,\beta\,a_1a_0\lambda-c^2b_1\lambda-2\,b_2\mu-b_1\lambda-2\,c^2b_2\mu=0\\
&-c^2b_1\mu+2\,\beta\,a_1a_0\mu-b_1\mu=0.
\end{aligned}\tag{3.3.8}$$

Solving the algebraic equations above, yields two different cases as follows

Case (i)

$$\{a_0=a_0,a_1=a_1,b_0=-\tfrac{\alpha}{2}+\tfrac{{a_0}^2\beta}{2},b_1=\beta\,a_1a_0,b_2=\tfrac{\beta\,{a_1}^2}{2},c=\pm 1\}\tag{3.3.9}$$

Case (ii)

$$\begin{aligned}
&\{a_0=\{\pm\tfrac{\lambda}{2}\sqrt{-\tfrac{2\,c^2+2}{\beta}},a_1=\pm\sqrt{-\tfrac{2\,c^2+2}{\beta}},\\
&b_0=-\tfrac{c^2\lambda^2}{4}+c^2\mu+\tfrac{r^2c^2}{2}-\tfrac{\alpha}{2}-\tfrac{\lambda^2}{4}-\mu-\tfrac{r^2}{2},b_1=-2\,\lambda,b_2=-2,c=c\}
\end{aligned}\tag{3.3.10}$$

For **case (i)** substituting (3.3.9) into (3.3.6)-(3.3.7), we have

$$\begin{aligned}
&U(\xi)=a_1(\tfrac{G'}{G})+a_0,\\
&V(\xi)=\tfrac{\beta\,{a_1}^2}{2}(\tfrac{G'}{G})^2+\beta\,a_1a_0(\tfrac{G'}{G})-\tfrac{\alpha}{2}+\tfrac{{a_0}^2\beta}{2},\\
&\xi=x\pm t.
\end{aligned}\tag{3.3.11}$$

Substituting the general solution of (3.3.5), we have three types of travelling wave solutions of the coupled Higgs equation (3.1.1) as follows:

Subcase (i) When $\lambda^2-4\mu>0$

$$\begin{aligned}
&u=\left(a_1\left(-\tfrac{1}{2}\,\lambda+\tfrac{1}{2}\,\frac{\left(C_1\,\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)\right)\sqrt{\lambda^2-4\,\mu}}{C_1\,\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)}\right)+a_0\right)e^{\iota\theta},\\
&v=\frac{\beta\,{a_1}^2}{2}\left(-\tfrac{1}{2}\,\lambda+\tfrac{1}{2}\,\frac{\left(C_1\,\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)\right)\sqrt{\lambda^2-4\,\mu}}{C_1\,\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)}\right)^2\\
&+\beta\,a_1a_0\left(-\tfrac{1}{2}\,\lambda+\tfrac{1}{2}\,\frac{\left(C_1\,\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)\right)\sqrt{\lambda^2-4\,\mu}}{C_1\,\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}\xi\right)}\right)-\frac{\alpha}{2}+\frac{{a_0}^2\beta}{2},
\end{aligned}\tag{3.3.12}$$

where $\xi = x \pm t$, and $\theta = px + rt$,

C_1 and C_2 are two arbitrary constants.

Subcase (ii) When $\lambda^2 - 4\mu < 0$

$$\begin{aligned}
u &= \left(a_1 \left(-\frac{1}{2}\lambda + \frac{1}{2} \frac{\left(-C_1 \sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right) + C_2 \cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)\right)\sqrt{-\lambda^2+4\mu}}{C_1 \cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right) + C_2 \sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)} \right) + a_0 \right) e^{\iota\theta}, \\
v &= \frac{\beta a_1{}^2}{2} \left(-\frac{1}{2}\lambda + \frac{1}{2} \frac{\left(-C_1 \sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right) + C_2 \cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)\right)\sqrt{-\lambda^2+4\mu}}{C_1 \cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right) + C_2 \sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)} \right)^2 \\
&+ \beta a_1 a_0 \left(-\frac{1}{2}\lambda + \frac{1}{2} \frac{\left(-C_1 \sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right) + C_2 \cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)\right)\sqrt{-\lambda^2+4\mu}}{C_1 \cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right) + C_2 \sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)} \right) - \frac{\alpha}{2} + \frac{a_0{}^2\beta}{2},
\end{aligned} \tag{3.3.13}$$

where $\xi = x \pm t$, and $\theta = px + rt$,

C_1 and C_2 are two arbitrary constants.

Subcase (iii) When $\lambda^2 - 4\mu = 0$

$$\begin{aligned}
u &= \left(a_1 \left(\frac{2C_2 - C_1\lambda - C_2\lambda\xi}{2(C_1 + C_2\xi)}\right) + a_0 \right) e^{\iota\theta}, \\
v &= \frac{\beta a_1{}^2}{2} \left(\frac{2C_2 - C_1\lambda - C_2\lambda\xi}{2(C_1 + C_2\xi)}\right)^2 + \beta a_1 a_0 \left(\frac{2C_2 - C_1\lambda - C_2\lambda\xi}{2(C_1 + C_2\xi)}\right) - \frac{\alpha}{2} + \frac{a_0{}^2\beta}{2},
\end{aligned} \tag{3.3.14}$$

where $\xi = x \pm t$, and $\theta = px + rt$,

C_1 and C_2 are two arbitrary constants.

For **case (ii)** substituting (3.3.10) into (3.3.6)-(3.3.7), we have

$$\begin{aligned}
U(\xi) &= \pm\sqrt{-\frac{2c^2+2}{\beta}}\left(\frac{G'}{G}\right) \pm \frac{\lambda}{2}\sqrt{-\frac{2c^2+2}{\beta}}, \\
V(\xi) &= -2\left(\frac{G'}{G}\right)^2 - 2\lambda\left(\frac{G'}{G}\right) - \frac{c^2\lambda^2}{4} + c^2\mu + \frac{r^2c^2}{2} - \frac{\alpha}{2} - \frac{\lambda^2}{4} - \mu - \frac{r^2}{2}, \\
\xi &= x + ct.
\end{aligned} \tag{3.3.15}$$

Substituting general solution of (3.3.5), again we have three type solutions as given below

Subcase (i) When $\lambda^2 - 4\mu > 0$

$$u(x,t) = \pm\frac{1}{2}\sqrt{-\frac{2\,c^2+2}{\beta}}\left(\frac{\left(C_1\,\sinh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\cosh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)\right)\sqrt{\lambda^2-4\,\mu}}{C_1\,\cosh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\sinh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)}\right)e^{\iota\theta}$$
$$v(x,t) = -2\left(-\frac{1}{2}\,\lambda+\frac{1}{2}\,\frac{\left(C_1\,\sinh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\cosh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)\right)\sqrt{\lambda^2-4\,\mu}}{C_1\,\cosh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\sinh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)}\right)^2$$
$$-2\,\lambda\left(-\frac{1}{2}\,\lambda+\frac{1}{2}\,\frac{\left(C_1\,\sinh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\cosh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)\right)\sqrt{\lambda^2-4\,\mu}}{C_1\,\cosh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)+C_2\,\sinh\left(\frac{1}{2}\,\sqrt{\lambda^2-4\,\mu}\xi\right)}\right)-\frac{c^2\lambda^2}{4}+c^2\mu$$
$$+\frac{r^2c^2}{2}-\frac{\alpha}{2}-\frac{\lambda^2}{4}-\mu-\frac{r^2}{2}, \tag{3.3.16}$$

where $\xi = x + ct$, and $\theta = px + rt$,

C_1 and C_2 are two arbitrary constants.

Subcase (ii) When $\lambda^2 - 4\mu < 0$

$$u(x,t) = \pm\frac{1}{2}\sqrt{-\frac{2\,c^2+2}{\beta}}\left(\frac{\left(-C_1\,\sin\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)+C_2\,\cos\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)\right)\sqrt{-\lambda^2+4\,\mu}}{C_1\,\cos\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)+C_2\,\sin\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)}\right)e^{\iota\theta}$$
$$v(x,t) = -2\left(-\frac{1}{2}\,\lambda+\frac{1}{2}\,\frac{\left(-C_1\,\sin\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)+C_2\,\cos\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)\right)\sqrt{-\lambda^2+4\,\mu}}{C_1\,\cos\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)+C_2\,\sin\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)}\right)^2$$
$$-2\,\lambda\left(-\frac{1}{2}\,\lambda+\frac{1}{2}\,\frac{\left(-C_1\,\sin\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)+C_2\,\cos\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)\right)\sqrt{-\lambda^2+4\,\mu}}{C_1\,\cos\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)+C_2\,\sin\left(\frac{1}{2}\,\sqrt{-\lambda^2+4\,\mu}\xi\right)}\right)-\frac{c^2\lambda^2}{4}+c^2\mu$$
$$+\frac{r^2c^2}{2}-\frac{\alpha}{2}-\frac{\lambda^2}{4}-\mu-\frac{r^2}{2}, \tag{3.3.17}$$

where $\xi = x + ct$, and $\theta = px + rt$,

C_1 and C_2 are two arbitrary constants.

Subcase (iii) When $\lambda^2 - 4\mu = 0$

$$u(x,t) = \pm\sqrt{-\frac{2\,c^2+2}{\beta}}\left(\frac{2C_2-C_1\lambda-C_2\lambda\xi}{2(C_1+C_2\xi)}\right)\pm\frac{\lambda}{2}\sqrt{-\frac{2\,c^2+2}{\beta}},$$
$$v(x,t) = -2\left(\frac{2C_2-C_1\lambda-C_2\lambda\xi}{2(C_1+C_2\xi)}\right)^2-2\,\lambda\left(\frac{2C_2-C_1\lambda-C_2\lambda\xi}{2(C_1+C_2\xi)}\right)-\frac{c^2\lambda^2}{4}+c^2\mu \tag{3.3.18}$$
$$+\frac{r^2c^2}{2}-\frac{\alpha}{2}-\frac{\lambda^2}{4}-\mu-\frac{r^2}{2},$$

where $\xi = x + ct$, and $\theta = px + rt$,

C_1 and C_2 are two arbitrary constants.

3.3.2 Hamiltonian Amplitude Equation

To find explicit exact travelling wave solutions of Hamiltonian amplitude equation (3.1.2), let

$$u = e^{\iota\theta}\phi(\xi), \quad \theta = px + rt, \quad \xi = x + ct. \tag{3.3.19}$$

Substituting (3.3.19) into (3.1.2), we have

$$(c^2-\beta c)\phi''(\xi)+\iota(1+2rc-\beta cp-\beta r)\phi'(\xi)-(p+r^2-\beta pr)\phi(\xi)+2\eta\phi^3(\xi) = 0. \tag{3.3.20}$$

Employing the condition

$$1 + 2rc - \beta cp - \beta r = 0, \tag{3.3.21}$$

(3.3.20) transformed into following equation:

$$(c^2 - \beta c)\phi''(\xi) - (p + r^2 - \beta pr)\phi(\xi) + 2\eta\phi^3(\xi) = 0. \tag{3.3.22}$$

By balancing the highest-order derivative term ϕ'' with the nonlinear term ϕ^3, we get $m = 1$. Therefore we suppose that (3.3.22) has the following formal solution with $a_1 \neq 0$,

$$\phi(\xi) = a_1(\frac{G'}{G}) + a_0, \tag{3.3.23}$$

where a_0 and a_1 are constants to be determined later.

Substituting (3.3.23) together with (3.3.5) into (3.3.22), collecting all terms with equal power of $(\frac{G'}{G})$ and setting each coefficient to zero, we obtain the following set of algebraic equations:

$$\begin{aligned}
&2\,(c^2 - \beta\, c)\, a_1 + 2\,\eta\, {a_1}^3 = 0\\
&3\,(c^2 - \beta\, c)\,\lambda\, a_1 + 6\,\eta\, a_0{a_1}^2 = 0\\
&(c^2 - \beta\, c)\,(\lambda^2 a_1 + 2\, a_1\mu) + 6\,\eta\, {a_0}^2 a_1 - (p + r^2 - \beta\, pr)\, a_1 = 0\\
&(c^2 - \beta\, c)\,\lambda\,\mu\, a_1 + 2\,\eta\, {a_0}^3 - (p + r^2 - \beta\, pr)\, a_0 = 0.
\end{aligned} \tag{3.3.24}$$

Solving the algebraic equations above, yields

$$\begin{aligned}
&a_0 = \pm\frac{\lambda\left(-p-r^2+\beta\, pr\right)}{\sqrt{\frac{2(-p-r^2+\beta\, pr)}{(-\lambda^2+4\,\mu)\eta}}(-\lambda^2+4\,\mu)\eta}, a_1 = \pm\sqrt{\frac{2(-p-r^2+\beta\, p\, r)}{\eta(-\lambda^2+4\,\mu)}},\\
&c = \frac{-\beta\,\lambda^2+4\,\beta\,\mu\pm\sqrt{(-\lambda^2+4\,\mu)(4\,\beta^2\mu+8\,p-\lambda^2\beta^2+8\,r^2-8\,\beta\, p\, r)}}{2(-\lambda^2+4\,\mu)}
\end{aligned} \tag{3.3.25}$$

Substituting (3.3.25) into (3.3.23), we have

$$\phi(\xi) = \pm \frac{\lambda\,(-p - r^2 + \beta\, pr)}{\sqrt{\frac{2(-p-r^2+\beta\, pr)}{(-\lambda^2+4\,\mu)\eta}}\,(-\lambda^2 + 4\,\mu)\,\eta}\left(\frac{G'(\xi)}{G(\xi)}\right) \pm \sqrt{\frac{2(-p - r^2 + \beta\, p\, r)}{\eta(-\lambda^2 + 4\,\mu)}}. \quad (3.3.26)$$

Substituting the general solution of (3.3.5) into (3.3.26),we have solution of equation (3.1.2) with relation (3.3.21) as:

Case(i) when $\lambda^2 - 4\mu > 0$

$$u = \pm \frac{(-p - r^2 + \beta\, pr)\left(C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\,\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\,\mu}}{2}\xi\right)\right)\sqrt{2}}{2\left(C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\,\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\,\mu}}{2}\xi\right)\right)\eta\sqrt{-\frac{-p-r^2+\beta\, pr}{\eta}}}e^{\iota\theta}, \quad (3.3.27)$$

where $\xi = x + (\frac{-\beta\,\lambda^2+4\,\beta\,\mu\pm\sqrt{(-\lambda^2+4\,\mu)(4\,\beta^2\mu+8\,p-\lambda^2\beta^2+8\,r^2-8\,\beta\,p\,r)}}{2(-\lambda^2+4\,\mu)})t$ and $\theta = px + rt$.
C_1 and C_2 are arbitrary constants.

Case(ii) when $\lambda^2 - 4\mu < 0$

$$u = \pm \frac{(-p - r^2 + \beta\, pr)\left(C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) - C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\right)\sqrt{2}}{2\left(C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\right)\eta\sqrt{\frac{-p-r^2+\beta\, pr}{\eta}}}e^{\iota\theta}, \quad (3.3.28)$$

where $\xi = x + (\frac{-\beta\,\lambda^2+4\,\beta\,\mu\pm\sqrt{(-\lambda^2+4\,\mu)(4\,\beta^2\mu+8\,p-\lambda^2\beta^2+8\,r^2-8\,\beta\,p\,r)}}{2(-\lambda^2+4\,\mu)})t$ and $\theta = px + rt$.
C_1 and C_2 are arbitrary constants.

3.4 Concluding Remarks

In this chapter, We have investigated the symmetries and invariant solutions of Higgs field equation and Hamiltonian amplitude equation. Using the symmetries, we have reduced the equations (3.1.1) and (3.1.2) into system of ODEs and then certain exact solutions of Higgs field and Hamiltonian amplitude equations are obtained. Along with the Lie Symmetry method, the travelling wave solutions of the coupled Higgs equation and Hamiltonian amplitude equation are successfully found by using the $(\frac{G'}{G})$-expansion method, which include hyperbolic function solutions, trigonometry function solutions and rational solutions.

Chapter 4

Variable Coefficients Zakharov-Kuznetsov Modified Equal Width Equation

4.1 Introduction

The Zakharov-Kuznetsov (ZK) equation [123]

$$u_t + auu_x + (u_{xx} + ru_{yy})_x = 0, \tag{4.1.1}$$

is two dimensional generalization of KdV equations.

The modified equal-width (MEW) equation

$$u_t + 3u^2u_x - \alpha u_{xxt} = 0, \tag{4.1.2}$$

has been discussed in [37] and [124].

Wazwaz [108] has introduced a nonlinear (2+1) dimensional problem

$$u_t + 3au^2u_x + (bu_{xxt} + ru_{xyy}) = 0, \tag{4.1.3}$$

where a, b and r are real arbitrary constants, as an extension of MEW equation (4.1.2) established in ZK (4.1.1) sense, known as Zakharov-Kuznetsov modified equal width (ZK-MEW) equation. The first term represents the evolution term, the coefficient of a is the nonlinear term while the third term and fourth term together, in parentheses, are the dispersion terms. The solitons are a result of a delicate balance between dispersion and nonlinearity. The solutions of (4.1.3) have been studied in various aspects. See for example the papers [7, 79, 110] and references therein.

We will consider the variable coefficient form of (4.1.3) that is given as

$$u_t + 3\alpha(t)u^2u_x + \beta(t)u_{xxt} + \delta(t)u_{xyy} = 0, \tag{4.1.4}$$

where $\alpha(t),\ \beta(t)$ and $\delta(t)$ are arbitrary functions of t. By taking $\alpha(t) = a,\ \beta(t) = b$ and $\delta(t) = r$ (4.1.4) reduces to (4.1.3). For $\alpha(t) = 1,\ \beta(t) = -\alpha$ and $\delta(t) = 0$ it takes the form (4.1.2).

The chapter has been organized as follows. Section 4.2 is devoted to application of Lie classical method to generate various symmetries of the ZK-MEW equation. Section 4.3 contains the reduced partial differential equation (PDE) and the exact solutions of (4.1.4). Some conclusions are drawn in section 4.4.

4.2 Invariance Analysis

A Lie point symmetry [65] of a partial differential equation (PDE) is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. The symmetry group [10, 86, 89, 40] of ZK-MEW equation (4.1.4) will be generated by the vector field of the form

$$V = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \xi^3\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial \psi}, \tag{4.2.1}$$

where $\xi^i,\ i = 1, 2, 3$ and η depend on x, y, t and ψ. Apply the third prolongation $pr^{(3)}V$ to (4.1.4), we obtain the over determined system of linear PDEs. The solution of this large system helps us to obtain the infinitesimals $\xi^i,\ i = 1, 2$ and η, as follows:

$$\begin{aligned} \xi^1 &= C_1x + C_2 \\ \xi^2 &= C_3y + C_4 \\ \eta &= C_5\psi, \end{aligned} \tag{4.2.2}$$

where C_1, C_2, C_3, C_4 and C_5 are arbitrary constants. Infinitesimal ξ^3 and the admissible forms of various coefficients in equation are govern by the following conditions:

$$\begin{aligned} &\xi^3\beta' - C_1\beta = 0 \\ &\xi^3_t\alpha + \xi^3\alpha' = C_1 - 2\alpha C_5 \\ &\xi^3_t\delta + \xi^3\delta' = C_1 + 2\delta C_3, \end{aligned} \tag{4.2.3}$$

where $\xi^3_t = \frac{d\xi^3}{dt}$ and ($'$) denotes derivative with respect to t.
Corresponding infinitesimal generators are given as

$$\begin{aligned} &V_1 = x\frac{\partial}{\partial x} + \frac{2\beta}{\beta'}\frac{\partial}{\partial t} \\ &V_2 = \frac{\partial}{\partial x} \\ &V_3 = y\frac{\partial}{\partial y} \\ &V_4 = \frac{\partial}{\partial y} \\ &V_5 = \psi\frac{\partial}{\partial \psi}. \end{aligned} \tag{4.2.4}$$

The commutation relations between these vector fields is given by Table 1, the entry in row i and column j representing $[V_i, V_j]$:

Table 4.1: Commutator Table of the Lie algebra of Eq. (4.1.4)

	V_1	V_2	V_3	V_4	V_5
V_1	0	-V_2	0	0	0
V_2	V_2	0	0	0	0
V_3	0	0	0	-V_4	0
V_4	0	0	V_4	0	0
V_5	0	0	0	0	0

4.3 Reduction and Exact Solutions

In this section, we will drive some exact solutions of variable coefficients ZK-MEW equation . One way to obtain exact solutions of (4.1.4) is by reducing it to ordinary differential equations. This can be achieved with the use of Lie point symmetries admitted by (4.1.4). It is well known that the reduction of a partial differential

equation with respect to r-dimensional (solvable) subalgebra of its Lie symmetry algebra leads to reducing the number of independent variables by r.
First we use the vector field $V_2 + V_4$ and corresponding similarity variables are

$$\rho = ax - \int \alpha(t)dt,\ \ \tau = ay - \int \alpha(t)dt, \psi = F(\rho, \tau), \tag{4.3.1}$$

and admissible coefficients obtained from (4.2.3) with $C_1 = C_3 = C_5 = 0,\ C_2 = C_4 = 1$ are as follows

$$\beta = \text{arbitrary constant},\ \ \alpha(t) = \delta(t) = \text{arbitrary functions of } t. \tag{4.3.2}$$

Here a is arbitrary constant. $\rho,\ \tau$ are new independent variables and F is new dependent variable.
Using (4.3.1) and (4.3.2) in main equation (4.1.4), we have

$$a^3 F_{\rho\tau\tau} - \beta a^2 F_{\rho\rho\tau} - \beta a^2 F_{\rho\rho\rho} + 3aF^2 F_\rho - F_\rho - F_\tau = 0. \tag{4.3.3}$$

Solutions of (4.3.3) are given as

$$\begin{aligned}
&(i)\ F(\rho,\tau) = \frac{2\,k_2\,aA_1}{\tanh\left(\frac{4\,k_1\,a^3k_2+4\,{k_2}^2\rho\,a^3-\tau+2\,\tau\,\beta\,a^2{k_2}^2+\tau\,\sqrt{4\,\beta^2a^4{k_2}^4-4\,\beta\,a^2{k_2}^2+1-8\,a^3{k_2}^2+16\,a^5{k_2}^4\beta}}{4a^3k_2}\right)}\\
&(ii)\ F(\rho,\tau) = A_2\text{sn}\left(k_2 + k_3\,\rho + k_4\,\tau,\ \ \text{m}_1\right)\\
&(iii)\ F(\rho,\tau) = A_3\tan\left(k_1 + k_2\,\rho + \ \tfrac{A_4}{4a^3k_2}\tau\right),
\end{aligned} \tag{4.3.4}$$

where $k_1,\ k_2,\ k_3,\ k_4$ are arbitrary constants and

$$\begin{aligned}
A_1 &= \frac{1}{\sqrt{-\sqrt{4\,\beta^2a^4{k_2}^4-4\,\beta\,a^2{k_2}^2+1-8\,a^3{k_2}^2+16\,a^5{k_2}^4\beta}+4\,a^3{k_2}^2-1+2\,\beta\,a^2{k_2}^2}}\\
A_2 &= \frac{\sqrt{2ak_3\left(a^3{k_4}^2k_3+k_3+k_4-\beta\,a^2{k_3}^3-\beta\,a^2k_4\,{k_3}^2\right)}}{ak_3}\\
\text{m}_1 &= \frac{\sqrt{-k_3\left(-\beta\,{k_3}^2-\beta\,k_4\,k_3+a{k_4}^2\right)\left(a^3{k_4}^2k_3+k_3+k_4-\beta\,a^2{k_3}^3-\beta\,a^2k_4\,{k_3}^2\right)}}{k_3\left(-\beta\,{k_3}^2-\beta\,k_4\,k_3+a{k_4}^2\right)a}\\
A_3 &= -\frac{\sqrt{-\sqrt{4\,\beta^2a^4{k_2}^4+4\,\beta\,a^2{k_2}^2+1+8\,a^3{k_2}^2+16\,a^5{k_2}^4\beta}-4\,a^3{k_2}^2-1-2\,\beta\,a^2{k_2}^2}}{2a^2k_2}\\
A_4 &= \left(1+2\,\beta\,a^2{k_2}^2+\sqrt{4\,\beta^2a^4{k_2}^4+4\,\beta\,a^2{k_2}^2+1+8\,a^3{k_2}^2+16\,a^5{k_2}^4\beta}\right)\ .
\end{aligned} \tag{4.3.5}$$

Consequently, Using (4.3.1) with admissible coefficients given by (4.3.2) solutions of main equation (4.1.4) are given as

$$
\begin{aligned}
&(i)\ \psi = \frac{2\,k_2\,aA_1}{\tanh\left(\frac{4\,k_1\,a^3k_2+4\,k_2{}^2a^3(ax-\int\alpha(t)dt)-(1-2\,\beta\,a^2k_2{}^2-\sqrt{4\,\beta^2a^4k_2{}^4-4\,\beta\,a^2k_2{}^2+1-8\,a^3k_2{}^2+16\,a^5k_2{}^4\beta})(ay-\int\alpha(t)dt)}{4a^3k_2}\right)}\\
&(ii)\ \psi = A_2\ \mathrm{sn}\left(k_2+k_3\left(ax-\int\alpha(t)dt\right)+k_4\left(ay-\int\alpha(t)dt\right),\ \mathrm{m}_1\right)\\
&(iii)\ \psi = A_3\tan\left(k_1+k_2\left(ax-\int\alpha(t)dt\right)+\frac{A_4}{4a^3k_2}\left(ay-\int\alpha(t)dt\right)\right),
\end{aligned}
\tag{4.3.6}
$$

where $A_1,\ A_2,\ A_3,\ A_4,\ \mathrm{m}_1$ are given by (4.3.5).

By taking $\alpha(t)=\beta=k_3=k_4=1, a=.1$ and $t=C_2=0$ in (4.3.6)(ii), we have following soliton solution. (see Fig 4.1)

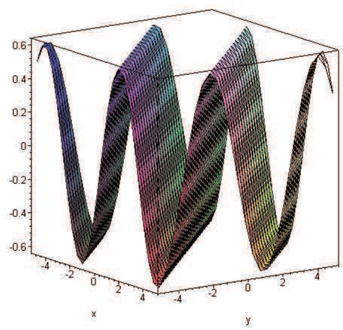

Figure 4.1: Soliton solution (4.3.6)(ii)

By taking $\alpha(t)=k_2=1,\ \beta=a=-1$ and $k_1=t=0$ in (4.3.6)(iii), we have following profile of solution(see Fig 4.2)

4.4 Conclusion

We have investigated the symmetries and invariant solutions of a (2+1) dimensional variable coefficient ZK-MEW equation (4.1.4). The Lie group method is utilized for

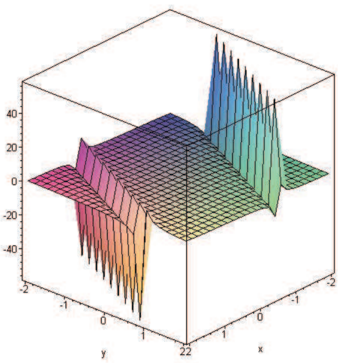

Figure 4.2: Profile of solution (4.3.6)(iii)

the purpose of obtaining the symmetries. Using symmetries, (2+1) dimensional PDE is reduced to two dimensional PDE. Exact solutions of reduced PDE are obtained and corresponding to which solutions of main equation (4.1.4) are shown.

Chapter 5

Benjamin-Bona-Mohany (BBM) Equation With Variable Coefficients [1]

5.1 Introduction

The regularized long-wave (RLW) equation, also known as the Benjamin-Bona-Mahony (BBM) equation,

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{5.1.1}$$

investigated, for the first time, by Benjamin, Bona and Mahony [6] as a regularized version of the Korteweg-de-Vries (KdV) equation [59]

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{5.1.2}$$

for shallow water waves, was originally derived as approximation for surface water waves in a uniform channel [6, 59]. In certain theoretical investigations, the BBM equation is superior as a model for long waves, and the word "regularized" refers to the fact that, from the standpoint of existence, uniqueness and stability, the BBM equation offers considerable technical advantages over the KdV equation, as demonstrated in [6].

In addition to shallow water waves, the BBM equation is applicable to the

[1] A part of this chapter has appeared in *Applied Mathematics and Computation* 217 (2011) 7021-7027

study of drift waves in plasma or the Rossby waves in rotating fluids ([77] and references therein). Under certain conditions, it also provides a model of one-dimensional transmitted waves. These find applications in semiconductor devices, optical devices, etc. ([113] and references therein). Other interesting BBM-related issues developed can be seen elsewhere, such as in Refs. [8, 9, 19, 30, 78, 127]. The wide applicability of these equations is the main reason why, during the last decades, they have attracted so much attention from mathematicians.

In contrast to the KdV case, the BBM equation is not an evolution equation in the strict sense due to the apppearance of u_{xxt} (as shown in Eq. (5.1.1)), but it is still considered to be a nonlinear partial differential equation (PDE) of evolution type. The main mathematical difference between KdV and BBM models can be most readily appreciated by comparing the dispersion relation for the respective linearized equations. It can be easily seen that these relations are comparable only for small wave numbers (i.e. long waves) and they generate drastically different responses to short waves (which are irrelevant to its role as a physical model). This is one of the reasons why, whereas existence and regularity theory for the KdV equation is difficult, the theory of the BBM equation is comparatively simple. The computing is also much easier for (5.1.1) than for (5.1.2).

During the last two decades, various versions of the BBM equation in the literature have occurred. General form of the BBM equation is as follows:

$$u_t + \alpha u_x + \beta u u_x - \delta u_{xxt} = 0,$$

where α, β and δ are constants with the nonlinear and dispersion coefficients $\beta \neq 0$ and $\delta > 0$. This equation covers the following types of the BBM equation as seen in the literature:

1. $\alpha = 1, \beta = 1, \delta = 1$: [6, 85]
2. $\alpha = 1, \beta = -1, \delta = 1$: [91]
3. $\alpha = 1, \beta = 2, \delta = 1$: [17]
4. $\alpha = 1, \beta = \pm 6, \delta = 1$: [58]

5. $\alpha = 1, \beta = 12, \delta = 1$: [78]
6. $\alpha = 1, \beta$ arbitrary, $\delta = 1$: [43]
7. $\alpha = 0, \beta = 1, \delta > 0$: [127]
8. $\alpha = 0, \beta = 1, \delta = 1$: [113]
9. $\alpha = 0, \beta = 6, \delta = 1$: [30]
10. α arbitrary, $\beta = 1, \delta = 1$: [113]
11. $\alpha = v_d, \beta = -v_d, \delta = \rho_s^2$: [77] with definitions of v_d and ρ_s as therein,
12. $\alpha = 2\kappa, \beta = 3, \delta = 1$: [19], where κ is a constant related to the critical shallow water wave speed.

The physical situations in which nonlinear equations arise tend to be highly idealized due to assumption of constant coefficients. Due to this, much attention has been paid on study of nonlinear equations with variable coefficients [31, 94, 95, 96, 106, 107]. Often, it is very difficult to solve explicitly these nonlinear equations for exact solutions. Consequently perturbation, asymptotic and numerical methods are applied to obtain approximate solutions of these equations. However, there is much current interest in obtaining exact solutions of these equations. These solutions provide much information about nonlinear phenomena and well described various aspects of the physical phenomena. These solutions are useful to discuss and examine the sensitivity of physical phenomena with several important parameters described by variable coefficients. The exact solutions are also helpful in designing and testing of numerical algorithm. Lie's method [15, 44, 70, 86] is an effective and simplest method among group theoretic techniques and a large number of equations are solved with the aid of this method. Some recent contributions can be seen in the articles [4, 28, 35, 51, 83, 92, 122]. In this chapter, we study the variable coefficients version of the BBM equation

$$u_t + \alpha(t)u_x + \beta(t)uu_x - \delta(t)u_{xxt} = 0, \qquad (5.1.3)$$

for exact solutions with the help of Lie's classical method [86].

The chapter has been organized as follows. Section 5.2 is devoted to the outline of Lie classical method to generate various symmetries of the BBM equation and an optimal system comprising basic vector fields is identified. Section 5.3 contains the reduced ordinary differential equations (ODEs) and their exact solutions. In Section 5.4, we give the Painlevé analysis for an ordinary differential equation (ODE). Some conclusions are drawn in Section 5.5.

5.2 Symmetry Group

In this section, by using Lie symmetry analysis method, we will obtain the symmetries of variable coefficients BBM equation (5.1.3). The technique has earlier been used to obtain the exact solutions of various nonlinear partial differential equations [31, 94] and also we have given the brief description in first chapter, hence we avoid to discuss the method in detail. In this section, we will obtain the symmetry groups using the Lie's classical method. The method mainly consists of following steps:
(i) Let us consider the Lie group of point transformations

$$\begin{aligned} t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2) \\ x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2) \\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \end{aligned} \tag{5.2.1}$$

which leaves the equation (5.1.3) invariant. In other words, the transformations are such that if u is solution of equation (5.1.3), then u^* is also a solution. The method for determining the symmetry group of (5.1.3) consists of finding the infinitesimals τ, ξ and η, which are functions of x, t, u.

(ii) Assuming that equation (5.1.3) is invariant under the transformations (5.2.1), we get the following relations from the coefficients of the first order of ϵ :

$$\eta^t + \alpha'(t)\tau u_x + \alpha(t)\eta^x + \beta'(t)\tau u u_x + \beta(t)\eta u_x + \beta(t)u\eta^x - \delta'(t)\tau u_{xxt} - \delta(t)\eta^{xxt} = 0, \tag{5.2.2}$$

where η^t, η^x and η^{xxt} are extended (prolonged) infinitesimals acting on an enlarged space (jet space) that includes all derivatives of the dependent variables u_t, u_x, u_{xx} and

u_{xxt} (for more details the readers can refer to [15]). The infinitesimals are determined from invariance condition (5.2.2), by setting the coefficients of different differentials equal to zero. We obtain a large number of PDEs in ξ, τ and η as follows

$$\begin{aligned}
&\tau_x = \tau_u = 0\\
&\xi_t = \xi_u = 0\\
&\eta_{uu} = \eta_{tu} = 0\\
&\xi_{xx} - 2\eta_{xu} = 0\\
&\eta_t = \delta\eta_{xxt} + \beta\eta_x u + \alpha\eta_x = 0\\
&2\delta\xi_x - \delta'\tau - \delta^2\eta_{xxu} = 0\\
&\beta\tau_t u + \beta\eta + \alpha'\tau + \alpha\delta\eta_{xxu} - \beta\xi_x u + \beta\delta\eta_{xxu}u - \alpha\xi_x + \beta'\tau u + \alpha\tau_t = 0,
\end{aligned} \tag{5.2.3}$$

where $(')$ denotes derivative with respect to t.

The general solution of system (5.2.3) provides following forms for the infinitesimal elements ξ, τ, η and admissible forms of various coefficients in the equation (5.1.3).

$$\begin{aligned}
&\xi = ax + b\\
&\eta = cu + d\\
&\tau\delta'(t) - 2\delta(t)a = 0,
\end{aligned} \tag{5.2.4}$$

where a, b, c and d are arbitrary constants. The functions $\alpha(t), \beta(t)$ and $\delta(t)$ are governed by the following conditions:

$$\begin{aligned}
&\beta(t)\tau_t - \beta(t)a + \beta'(t)\tau + \beta(t)c = 0\\
&\alpha(t)\tau_t - \alpha(t)a + \alpha'(t)\tau + \delta(t)d = 0.
\end{aligned} \tag{5.2.5}$$

(iii)The infinitesimal generators of the corresponding Lie algebra are given by

$$\begin{aligned}
&V_1 = x\tfrac{\partial}{\partial x} + \tfrac{2\delta(t)}{\delta'(t)}\tfrac{\partial}{\partial t}\\
&V_2 = \tfrac{\partial}{\partial x}\\
&V_3 = u\tfrac{\partial}{\partial u}\\
&V_4 = \tfrac{\partial}{\partial u}.
\end{aligned} \tag{5.2.6}$$

In general, there are infinite number of subalgebras of this Lie algebra formed from any linear combination of generators $V_j; j = 1, 2, 3, 4$ and to each subalgebra one can get the reduction using characteristic equations:

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta}. \tag{5.2.7}$$

However, this problem becomes manageable by recognizing that if two algebras are similar, i.e. they are connected to each other by a transformation from the symmetry group, then their corresponding invariant solutions are connected to each other by the same transformation. Therefore, it is sufficient to put all similar subalgebras in to one class and select a representative from each class. The set of all these representatives is called an optimal system [86]. The optimal system for equation (5.1.3) consists of following vector fields.

$$\begin{aligned}
&(i) V_1 + \mu V_3 \\
&(ii) V_2 + \gamma V_3 \\
&(iii) V_3 \\
&(iv) V_4,
\end{aligned} \tag{5.2.8}$$

where μ and γ are arbitrary constants.

The similarity variable and form can be obtained by solving characteristic equations (5.2.7). The general solution of these equations involves two constants, one becomes new independent variable ξ and other play the role of new dependent variable. On substituting solution of (5.2.7) in equation (5.1.3), we get the reduced ODE.

5.3 Reduced ODEs and Exact Solutions

1. Vector field $V_1 + \mu V_3$

For this vector field, on using the characteristic equations (5.2.7), the similarity variable and the form of the similarity solution are as follows:

$$\begin{aligned} u(t,x) &= x^{\mu}F(\zeta) \\ \zeta(t,x) &= \tfrac{x^2}{\delta(t)} \\ \alpha(t) &= \tfrac{\delta(t)^{\frac{-1}{2}}\delta'(t)}{2} \\ \beta(t) &= \tfrac{\delta(t)^{\frac{-1-\mu}{2}}\delta'(t)}{2}. \end{aligned} \tag{5.3.1}$$

On using these in equation (5.1.3), the reduced ODE is given by

$$\begin{aligned} &(-2\zeta^2 + 2\zeta^{\frac{3}{2}} + 2\mu^2\zeta + 4\zeta + 6\mu\zeta)F' + \mu\zeta^{\frac{1}{2}}F + \mu\zeta^{\frac{\mu+1}{2}}F^2 + 2\zeta^{\frac{\mu+3}{2}}FF' \\ &+(2\mu+5)4\zeta^2F'' + 8\zeta^3F''' = 0, \end{aligned} \tag{5.3.2}$$

where prime ($'$) denotes the differentiation with respect to the variable ζ. In this case because of the complexity of the reduced system, the following two particular cases have been worked out.

case (i) $\mu = 0$

In this case, form of the similarity variable, similarity solution and coefficients functions are as follows:

$$\begin{aligned} u(t,x) &= F(\zeta) \\ \zeta(t,x) &= \tfrac{x^2}{\delta(t)} \\ \alpha(t) &= \tfrac{\delta(t)^{\frac{-1}{2}}\delta'(t)}{2} \\ \beta(t) &= \tfrac{\delta(t)^{\frac{-1}{2}}\delta'(t)}{2}. \end{aligned} \tag{5.3.3}$$

The reduced ODE is

$$(2\zeta + \zeta^{\frac{3}{2}} - \zeta^2)F' + \zeta^{\frac{3}{2}}FF' + 4\zeta^3F''' + 10\zeta^2F'' = 0. \tag{5.3.4}$$

To get a solution for equation (5.2.5), let us assume that the (5.2.5) admits a solution in the form

$$F(\zeta) = a_0 + a_1\zeta^{\frac{1}{2}}.$$

Substituting this in equation (5.2.5), we get $a_0 = -1$ and $a_1 = 1$.

We find the solution of equation (5.1.3) as follows

$$u(x,t) = -1 + \frac{x}{\delta(t)^{\frac{1}{2}}}.$$

case(ii) $\mu = 2$

In this case, form of the similarity variable, similarity solution and coefficients functions are as follows:

$$\begin{aligned} u(t,x) &= x^2F(\zeta) \\ \zeta(t,x) &= \tfrac{x^2}{\delta(t)} \\ \alpha(t) &= \tfrac{\delta(t)^{\frac{-1}{2}}\delta'(t)}{2} \\ \beta(t) &= \tfrac{\delta(t)^{\frac{-3}{2}}\delta'(t)}{2}. \end{aligned} \tag{5.3.5}$$

The reduced ODE is

$$(-\zeta^2 + \zeta^{\frac{3}{2}} + 12\zeta)F' + \zeta^{\frac{3}{2}}F^2 + \zeta^{\frac{1}{2}}F + \zeta^{\frac{5}{2}}FF' + 18\zeta^2F'' + 4\zeta^3F''' = 0. \tag{5.3.6}$$

To get a solution for (5.3.6), let us assume that the ODE (5.3.6) admits a solution in the form

$$F(\zeta) = a_0 + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^{\frac{1}{2}}}.$$

In this case we get $a_0 = 0, a_1 = 1$ and $a_2 = -1$.

We find the solution of equation(1.3) as follows

$$u(x,t) = \delta(t) - x\delta(t)^{\frac{1}{2}}.$$

2. Vector field $V_2 + \gamma V_3$

From equation (5.2.4) following two cases arise:

Either $\tau = 0$, $\delta(t)$ is arbitrary or $\delta(t) =$ constant, τ is arbitrary.

case(i) $\tau = 0$, $\delta(t)$ is arbitrary

Using characteristic equations (5.2.7) the similarity variable and the form of the similarity solution are as follows:

$$\begin{aligned} u(t,x) &= F(\zeta) \\ \zeta &= x \\ \alpha(t) &= \text{arbitrary} \\ \beta(t) &= \text{arbitrary}. \end{aligned} \tag{5.3.7}$$

On using these in equation (5.1.3), the reduced ODE is given by

$$F'(1+F) = 0, \tag{5.3.8}$$

and gives the constant solution of (5.1.3)

case(ii) $\delta(t)$ =constant, τ is arbitrary.

On using the characteristic equations (5.2.7), the similarity variable and the form of the similarity solution are as follows:

$$\begin{aligned} u(t,x) &= \exp(\gamma \textstyle\int \alpha(t)dt)F(\zeta) \\ \zeta(t,x) &= x - \textstyle\int \alpha(t)dt \\ \alpha(t) &= \text{arbitrary} \\ \beta(t) &= \alpha(t)\exp(-\gamma \textstyle\int \alpha(t)dt). \end{aligned} \tag{5.3.9}$$

On using these in equation (5.1.3), the reduced ODE is given by

$$-\gamma F - FF' + \gamma\delta F'' - \delta F''' = 0. \tag{5.3.10}$$

In next section, we perform Painlevé analysis of this ODE.

3. Vector field V_3 and V_4

Corresponding to these vector fields we get only constant solutions of the equation (5.1.3).

5.4 Painlevé Analysis for ODE

According to the Ablowitz, Ramani and Segur (ARS) conjecture [2], *every ordinary differential equation obtained by an exact reduction of a nonlinear partial differential equation solvable by Inverse Scattering Transform (IST) method, has the Painlevé property.* This conjecture therefore provides a necessary condition for the integrability of a given partial differential equation.

The Painlevé property for ODEs is defined as follows. The solutions of a system of ODEs are regarded as (analytic) functions of a complex variable. The "movable" singularities of the solution are the singularities of the solution (as a function of ζ)

whose location depends on the initial conditions and are, hence, movable. (Fixed singularities occur at points where the coefficients of the equation are singular). The ODEs system is said to possess the Painlevé property when all the movable singularities are single-valued (simple poles) [52].

The ARS algorithm was developed in order to determine whether or not a non-linear ODE (or a system of ODEs) admits movable branch points, either algebraic or logarithmic. It is important to keep in mind that the occurrence of movable essential singularities can not be detected by this procedure.

The ARS algorithm proceeds in three steps, dealing with the dominant behaviors, the resonances and to find the constants of integration and check compatibility conditions respectively [2, 61].

To apply the Painlevé analysis on equation (5.3.10), we look for a solution of it in the form

$$F = F_0(\zeta - \zeta_0)^p + O((\zeta - \zeta_0)^{p-1}, \tag{5.4.1}$$

where ζ_0 is arbitrary. Substituting (5.4.1) in to (5.3.10) shows that for certain values of p, two or more terms in the equation may balance (depending on F_0), and the rest can be ignored as $\zeta \to \zeta_0$. For each such choice of p, the terms which can be balanced are called the leading terms. Requiring that the leading terms do balance (usually) determines F_0.

Here for $\gamma \neq 0$, we find $p = -2$ and $F_0 = -12\delta$. Corresponding resonances are $r_1 = -1, r_2 = 4$ and $r_3 = 6$. For $r_2 = 4$ compatibility condition cannot be satisfied. so for $\gamma \neq 0$ equation does not pass Painlevé test.

For $\gamma = 0$, integrating equation (5.3.10) twice with respect to ζ, we obtain .

$$\delta F'^2 = -\frac{F^3}{3} + c_1 F + c_2, \tag{5.4.2}$$

where c_1 and c_2 denote arbitrary constants. Corresponding to ODE (5.4.2), choosing $c_2 = 0$ and using (5.3.9), some other solutions of equation (5.1.3) can be given as:

$$u\left(x,t\right) = \sqrt{3}\sqrt{c_1}\left(cn\left(C_2 - 1/6\,\sqrt{6}\sqrt{\frac{\sqrt{3}\sqrt{c_1}}{\delta}}\left(x - \int \alpha\left(t\right)dt\right), 1/2\,\sqrt{2}\right)\right)^2 \tag{5.4.3}$$

$$u(x,t) = \sqrt{3}\sqrt{c_1}\left(cn\left(C_2 + 1/6\,\sqrt{6}\sqrt{\frac{\sqrt{3}\sqrt{c_1}}{\delta}}\left(x - \int \alpha(t)\,dt\right), 1/2\,\sqrt{2}\right)\right)^2 \quad (5.4.4)$$

$$u(x,t) = -\sqrt{3}\sqrt{c_1}\left(cn\left(C_2 - 1/6\,i\sqrt{6}\sqrt{\frac{\sqrt{3}\sqrt{c_1}}{\delta}}\left(x - \int \alpha(t)\,dt\right), 1/2\,\sqrt{2}\right)\right)^2 \quad (5.4.5)$$

$$u(x,t) = -\sqrt{3}\sqrt{c_1}\left(cn\left(C_2 + 1/6\,i\sqrt{6}\sqrt{\frac{\sqrt{3}\sqrt{c_1}}{\delta}}\left(x - \int \alpha(t)\,dt\right), 1/2\,\sqrt{2}\right)\right)^2 \quad (5.4.6)$$

$$u(x,t) = \sqrt{3}\sqrt{c_1}\left(dn\left(C_2 - 1/6\,\sqrt{3}\sqrt{\frac{\sqrt{3}\sqrt{c_1}}{\delta}}\left(x - \int \alpha(t)\,dt\right), \sqrt{2}\right)\right)^2 \quad (5.4.7)$$

$$u(x,t) = \sqrt{3}\sqrt{c_1}\left(dn\left(C_2 + 1/6\,\sqrt{3}\sqrt{\frac{\sqrt{3}\sqrt{c_1}}{\delta}}\left(x - \int \alpha(t)\,dt\right), \sqrt{2}\right)\right)^2 \quad (5.4.8)$$

$$u(x,t) = -\sqrt{3}\sqrt{c_1}\left(nc\left(C_2 + 1/6\,\sqrt{6}\sqrt{\frac{\sqrt{3}\sqrt{c_1}}{\delta}}\left(x - \int \alpha(t)\,dt\right), 1/2\,\sqrt{2}\right)\right)^2 \quad (5.4.9)$$

$$u(x,t) = -\sqrt{3}\sqrt{c_1}\left(nd\left(C_2 - 1/6\,\sqrt{3}\sqrt{\frac{\sqrt{3}\sqrt{c_1}}{\delta}}\left(x - \int \alpha(t)\,dt\right), \sqrt{2}\right)\right)^2 \quad (5.4.10)$$

$$u(x,t) = -\sqrt{3}\sqrt{c_1}\left(nd\left(C_2 + 1/6\,\sqrt{3}\sqrt{\frac{\sqrt{3}\sqrt{c_1}}{\delta}}\left(x - \int \alpha(t)\,dt\right), \sqrt{2}\right)\right)^2 \quad (5.4.11)$$

$$u(x,t) = -12\,\delta\,{C_4}^2\wp\left(C_3 + C_4\,\left(x - \int \alpha(t)\,dt\right), 1/12\,\frac{c_1}{{C_4}^4\delta^2}, 0\right), \quad (5.4.12)$$

where C_2, C_3, C_4 are arbitrary constants. cn, dn, nd, nc and $\wp$ denotes jacobi elliptic functions and weierstrassP function, respectively.

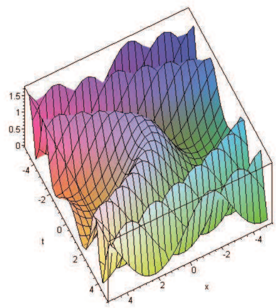

Figure 5.1: Soliton solution (5.4.3) for $C_1 = \delta = 1, \alpha(t) = t$ and $C_2 = 0$,

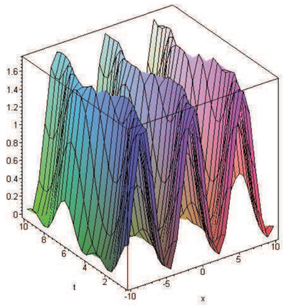

Figure 5.2: Periodic Solution (5.4.8) for $C_1 = \delta = 1, \alpha(t) = \sin(t)$ and $C_2 = 0$,

5.5 Discussion

We have investigated the symmetries and invariant solutions of a variable coefficient BBM equation. The Lie group method is utilized for the purpose of obtaining the group infinitesimals. The basic fields of the optimal system lead to reductions that are inequivalent with respect to the symmetry transformations. We obtain the exact solutions of BBM equation corresponding to reduced ODEs, which have been verified by putting them back into the original equation using Maple. In one of the cases of reduced ODEs, we showed that it does not pass the Painlevé test. According to the ARS conjecture BBM equation is not solvable by IST. Though we keep from discussing the physical implications of the solutions reported in this chapter, yet we feel worth mentioning that the solutions obtained are such that one can choose the arbitrary function $\delta(t)$, along with various other arbitrary parameters, in a suitable manner, to simulate physical situations governed by the equation (5.1.3) to obtain particular solutions having desired features.

Chapter 6

(2+1)-Dimensional Variable Coefficients Broer-Kaup System [1]

6.1 Introduction

A large variety of physical, chemical, and biological phenomena is governed by nonlinear evolution equations. Many completely integrable models were presented during the course of studying shallow water waves, for example, KdV-type equations, the WBK equation, the integrable long wave equation, the Boussinesq equation, etc. Finding exact solutions of these nonlinear evolution equations plays an important role for these equations which are drawn from diverse interesting nonlinear phenomena. As a result, the research on exact solutions of nonlinear evolution equations has become more and more important.

A (2 + 1)-dimensional Broer-Kaup-Kupershmidt (BKK) system [72, 116]

$$\begin{aligned} H_{ty} &= H_{xxy} - 2(HH_x)_y - 2G_{xx} \\ G_t &= -G_{xx} - 2(GH)_x, \end{aligned} \tag{6.1.1}$$

can be obtained from the symmetry reduction of KP equation [71]. It has been widely applied in many branches of physics like plasma physics, fluid dynamics, nonlinear fiber optic communication, etc. The BKK system was used to model nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow water

[1] A part of this chapter has appeared in *Communications in Nonlinear Science and Numerical Simulation* 17 (2012) 1529-1541

of uniform depth, and can be derived from KP equation.

We study the variable coefficient version of the Broer-Kaup (VCBK) system [75, 128]

$$\begin{aligned} H_{ty} &= B(t)[H_{xxy} - 2(HH_x)_y - 2G_{xx}] \\ G_t &= B(t)[-G_{xx} - 2(GH)_x], \end{aligned} \tag{6.1.2}$$

where $B(t)$ is an arbitrary function of time variable t, and $B \neq 0$. When $B(t) = 1$, the VCBK system (6.1.2) reduces to the celebrated (2+1)-dimensional Broer-Kaup system (6.1.1). The VCBK system (6.1.2) is an important mathematical model in nonlinear optics, plasma physics and statistical physics [5, 50, 75, 128].
By letting $G = H_y$, VCBK system (6.1.2) can be converted into a new equation[5]

$$H_{ty} + B(t)[H_{xxy} + 2(HH_x)_y] = 0. \tag{6.1.3}$$

The study of nonlinear models for integrable properties is one of the important works in nonlinear science. Therefor, in this chapter, we check the Painlevé property of the VCBK system(6.1.2) firstly, and then the symmetries of equation (6.1.3) are considered. The exact solutions generated from the symmetries are presented.
The chapter has been organized as follows. In section 6.2, we perform Painlevé analysis for VCBK system. Section 6.3 is devoted to outline of Lie classical method to generate various symmetries of the VCBK system. Section 6.4 contains the reduced partial differential equations and corresponding ordinary differential equations (ODEs) and their exact solutions. In section 6.5, we have given some more exact solutions. Some conclusions are drawn in Section 6.6.

6.2 Painlevé Analysis for VCBK System

We check the integrability of VCBK system (6.1.2) by means of the Painlevé analysis (as mentioned in chapter 1 section 1.2.3). Using the standard Weiss-Kruskal approach [112, 54], the expansion about the singular manifold has the form

$$H = \phi^{\alpha} \sum h_j \phi^j, \qquad G = \phi^{\beta} \sum g_j \phi^j, \tag{6.2.1}$$

where $h_j = h_j(x,y,t),\ g_j = g_j(x,y,t),\ \phi = \phi(x,y,t)$ are analytic functions of (x,y,t). Substituting equation (6.2.1) into equation (6.1.2) leads to

$$\alpha = -1,\ \beta = -2,\ h_0 = \phi_x,\ g_0 = -\phi_x\phi_y, \tag{6.2.2}$$

and the recursion relations have the forms

$$\begin{aligned}&2(j-2)(j-3)^2\phi_x^2\phi_y B(t)h_j - 2(j-2)(j-3)\phi_x^2 B(t)g_j = z_1(h_i,g_i)\quad i \le j-1,\\ &2(j-3)\phi_x^2\phi_y B(t)h_j + j(3-j)\phi_x^2 B(t)g_j = z_2(h_i,g_i)\quad i \le j-1,\end{aligned} \tag{6.2.3}$$

where z_1 and z_2 are complicated functions of $h_i, g_i (i \le j-1)$ and the derivatives of the singularity ϕ. From equation (6.2.3), it is found that resonances occur at $j = -1, 2, 3, 3, 4$. The resonance at $j = -1$ corresponds to the arbitrary function ϕ defining the singularity manifold for the (2+1)-dimensional VCBK system. After detailed calculation, we find that compatibility conditions at $j = 2, 3, 3, 4$ are satisfied identically. According to Weiss-Kruskal approach, system (6.1.2) possesses Painlevé property.

6.3 Classical Symmetry Analysis

In this section, we shall obtain the symmetries of equation (6.1.3) using Lie classical method. We let the group of infinitesimal transformations be defined as

$$\begin{aligned}&t^* \to t + \epsilon T(x,t,y,H) + O(\epsilon^2)\\ &x^* \to x + \epsilon X(x,t,y,H) + O(\epsilon^2)\\ &y^* \to y + \epsilon Y(x,t,y,H) + O(\epsilon^2)\\ &H^* \to H + \epsilon\eta(x,t,y,H) + O(\epsilon^2),\end{aligned} \tag{6.3.1}$$

and impose the condition of invariance on (6.1.3).

On invoking the invariance criterion, the following relation from the coefficients of the first order of ϵ is deduced:

$$\begin{aligned}&\eta^{ty} + \dot{B}(t)TH_{xxy} + B(t)\eta^{xxy} + 2\dot{B}(t)TH_xH_y + 2B(t)\eta^xH_y + 2B(t)H_x\eta^y\\ &+2T\dot{B}(t)HH_{xy} + 2B(t)\eta H_{xy} + 2B(t)H\eta^{xy} = 0\end{aligned} \tag{6.3.2}$$

where $(\dot{})$ denotes the derivative with respect to t and $\eta^x, \eta^y, \eta^{ty}, \eta^{xy}$ and η^{xxy} are extended (prolonged) infinitesimals acting on an enlarged space corresponding to $H_x, H_y, H_{ty}, H_{xy}, H_{xxy}$ respectively. The method for determining the symmetry group of (6.1.3) mainly consists of finding the infinitesimals X, Y, T and η, which are functions of x, y, t and H. The general solution of equation (6.3.2) provides the infinitesimal elements X, Y, T and η, for which the equation (6.1.3) possesses Lie symmetry. We need to use the expressions for $\eta^x, \eta^y, \eta^{ty}, \eta^{xy}$ and η^{xxy} in equation (6.3.2) and replace H_{ty} by equation (6.1.3). On equating the coefficients of different differentials to zero, we get a number of PDEs in X, Y, T and η, as follows

$$\begin{aligned}
&\eta_y = \eta_{HH} = 0\\
&Y_H = Y_t = Y_x = 0\\
&T_x = T_y = T_H = 0\\
&X_y = X_H\\
&BT_t + B'T - 2BX_x = 0\\
&2B\eta_x + B\eta_{xxH} + 2BH\eta_{xH} + \eta_{tH} = 0\\
&B\tau_t + B'T - BX_x + B\eta_H = 0\\
&-BX_xH - BX_{xx} + 2B\eta + 2BT_tH - X_t + 2B\eta_{xH} + 2B'TH = 0,
\end{aligned} \tag{6.3.3}$$

where $(')$ denotes derivative with respect to t.

The special solution of this large system (6.3.3) helps us to obtain the infinitesimals X, Y, T and η, as follows:

$$\begin{aligned}
X &= Q(t)\\
T &= \tfrac{a_1}{B(t)}\\
Y &= R(y)\\
\eta &= \tfrac{\dot{Q}(t)}{2B(t)},
\end{aligned} \tag{6.3.4}$$

where a_1 is arbitrary constants. $Q(t), R(y)$ are arbitrary functions of t and y, respectively and $(\dot{})$ denotes t-derivative. Then the vector field V can be written as

$$V = V_1 + V_2 + V_3, \tag{6.3.5}$$

where

$$\begin{aligned} V_1 &= \tfrac{1}{B(t)}\tfrac{\partial}{\partial t}, \\ V_2 &= R(y)\tfrac{\partial}{\partial y}, \\ V_3 &= \tfrac{\dot{Q}(t)}{2B(t)}\tfrac{\partial}{\partial H} + Q(t)\tfrac{\partial}{\partial x}. \end{aligned} \tag{6.3.6}$$

6.4 Reduction and Invariant Solutions of the VCBK Equations

To obtain the symmetry reductions of equation (6.1.3), we have to solve the characteristic equation,

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dt}{T} = \frac{dH}{\eta}, \tag{6.4.1}$$

where X, Y, T and η are given by equation (6.3.4). To solve equation (6.4.1) two cases should be considered: (i) $R(y) \neq 0$ and (ii) $R(y) = 0, Q(t) \neq 0$

Case (i) $R(y) \neq 0$

On solving equations (6.4.1) we have,

$$\begin{aligned} \rho &= \int \tfrac{1}{R(y)} dy - \int B(t)dt \\ \sigma &= x - \int B(t)Q(t)dt \end{aligned} \tag{6.4.2}$$

$$H = \frac{Q(t)}{2} + F(\rho, \sigma), \tag{6.4.3}$$

where ρ, σ are new independent variables and F is new dependent variable. Substituting equation (6.4.3) along with equations (6.4.2) into equation (6.1.3), we obtain the reduced equation which reads

$$-F_{\rho\rho} + F_{\rho\sigma\sigma} + 2FF_{\rho\sigma} + 2F_\rho F_\sigma = 0. \tag{6.4.4}$$

Integrating equation (6.4.4) with respect to ρ, we get

$$-F_\rho + F_{\sigma\sigma} + 2FF_\sigma = K(\sigma), \tag{6.4.5}$$

where $K(\sigma)$ is arbitrary function of σ. Equation (6.4.5) is a nonhomogeneous Burger's equation [100, 93]. For $K(\sigma) = 0$ it is Burger's equation in (1+1)-dimensions [18].

Burger's equation is the simplest and best known equation in nonlinear wave theory. Although it is a nonlinear equation, it is rather simple because it can be mapped into the linear heat equation through the Hopf-Cole transformation [49]. Many authors [34, 53, 81] have found different solutions of Burgers equation. We can find variety of solutions of VCBK equation from well known solutions of Burgers equation.

Case (ii) $R(y) = 0,\ Q(t) \neq 0$

Following the same way as in case (i), we get the following invariants

$$\begin{aligned} \rho &= y \\ \sigma &= x - \textstyle\int B(t)Q(t)dt \end{aligned} \tag{6.4.6}$$

$$H = \frac{Q(t)}{2} + F(\rho, \sigma), \tag{6.4.7}$$

where F being the reduction field with respect to ρ and σ. Substituting equation (6.4.7) along with equations (6.4.6) into equation (6.1.3) yields the second type of similarity reduction

$$F_{\rho\sigma\sigma} + 2F_\rho F_\sigma + 2FF_{\rho\sigma} = 0. \tag{6.4.8}$$

Apply Lie classical method on equation (6.4.8) as already mentioned in chapter 1 sec 1.2.1. In this case we get symmetries as follows:

$$\begin{aligned} \tau_1 &= -a_1\sigma + a_2 \\ \tau_2 &= f(\rho) \\ \phi &= a_1 F, \end{aligned} \tag{6.4.9}$$

where τ_1, τ_2 and ϕ are infinitesimals corresponding to ρ, σ and F, respectively, a_1, a_2 are arbitrary constants and $f(\rho)$ is a arbitrary function of ρ. Lie algebra admitted by (6.4.8) is

$$L_3 = \{V_1 = F\frac{\partial}{\partial F} - \sigma\frac{\partial}{\partial \sigma},\ V_2 = \frac{\partial}{\partial \sigma},\ V_3 = f(\rho)\frac{\partial}{\partial \rho}\}. \tag{6.4.10}$$

We define a relation between two invariant solutions to hold true if the first one can be mapped to the other by applying a transformation group generated by a linear combinations of the operators in (6.4.10). Since these mappings are reflexive, symmetric and transitive, the relation is an equivalence relation, which induces a natural partition on the set of all group invariant solutions into equivalence classes.

An optimal system of generators corresponding to Lie algebra L_3 (6.4.10) is

$$\begin{aligned} &(i)\ V_1 + \mu V_3 \\ &(ii)\ V_2 + \gamma V_3 \\ &(iii)\ V_3, \end{aligned} \tag{6.4.11}$$

where μ and γ are arbitrary real constants.
We use the method of characteristics to determine the invariants and the reduced ODEs corresponding to each subalgebra given in (6.4.11). Symmetry variables and the invariants of the subalgebras of the Lie algebra L_3 are given in Table 1.

Table 6.1: Similarity variables of equation (6.4.8)

Subalgebra	Symmetry variable	Function $F(\rho,\sigma)$
$V_1 + \mu V_3$	$\xi = \sigma \exp(\frac{1}{\mu}\int \frac{1}{f(\rho)} d\rho)$	$F = \frac{1}{\sigma} J(\xi)$
$V_2 + \gamma V_3$	$\xi = \sigma - \frac{1}{\gamma}\int \frac{1}{f(\rho)} d\rho$	$F = J(\xi)$
V_3	$\xi = \sigma$	$F = J(\xi)$

Now reduction of variable is performed to obtain ODE. Exact solutions of ODE are discussed in each case.

Vector field $V_1 + \mu V_3$

The reduced ODE is

$$\xi^3 J''' + 2\xi^2 J'' J + 2\xi^2 J'^2 - 2\xi J' J = 0, \tag{6.4.12}$$

where 'prime' denotes ξ-derivative. By the transformation $\zeta = \log \xi$, equation (6.4.12) can be rewritten as

$$\dddot{J} - 3\ddot{J} + 2J\ddot{J} - 4J\dot{J} + 2\dot{J}^2 + 2\dot{J} = 0, \tag{6.4.13}$$

where dots denote ζ-derivatives.

Integrating equation (6.4.13) with respect to ζ once we get

$$\dddot{J} - 3\dot{J} + 2J\dot{J} - 2J^2 + 2J = a_1, \tag{6.4.14}$$

where a_1 is an arbitrary constant.

For $a_1 = 0$, Solutions of equation (6.4.14) are as follows:

$$\begin{aligned} &(i)\ J(\zeta) = \tfrac{1}{2} - \tfrac{1}{2}\tanh\left(C_1 - \tfrac{1}{2}\zeta\right)\\ &(ii)\ J(\zeta) = \tfrac{1}{2} + \tfrac{1}{2}\coth\left(C_1 + \tfrac{1}{2}\zeta\right)\\ &(iii)\ J(\zeta) = \tfrac{1}{2} - \tfrac{1}{2}\iota\tan\left(C_1 + \tfrac{1}{2}\iota\zeta\right)\\ &(iv)\ J(\zeta) = \tfrac{1}{C_1}\tanh\left(\tfrac{e^{\zeta} - C_2}{C_1}\right)e^{\zeta}, \end{aligned} \tag{6.4.15}$$

where C_1 and C_2 denote arbitrary constants.

Consequently, the solutions of the main system (6.1.2) are given by

$$\begin{aligned} &(i)\ H(x,y,t) = \frac{Q(t)}{2} + \frac{\left(\frac{1}{2}+\frac{1}{2}\tanh\left(C_1+\frac{1}{2}\ln\left(\left(x-\int B(t)Q(t)dt\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}}\right)\right)\right)}{\left(x-\int B(t)Q(t)dt\right)}\\ &G(x,y,t) = \frac{1-\left(\tanh\left(C_1+\frac{1}{2}\ln\left(\left(x-\int B(t)Q(t)dt\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}}\right)\right)\right)^2}{4\left(x-\int B(t)Q(t)dt\right)\mu(f(y))} \end{aligned} \tag{6.4.16}$$

$$\begin{aligned} &(ii) H(x,y,t) = \frac{Q(t)}{2} + \frac{\left(\frac{1}{2}+\frac{1}{2}\coth\left(C_1+\frac{1}{2}\ln\left(\left(x-\int B(t)Q(t)dt\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}}\right)\right)\right)}{\left(x-\int B(t)Q(t)dt\right)}\\ &G(x,y,t) = \frac{1-\left(\coth\left(C_1+\frac{1}{2}\ln\left(\left(x-\int B(t)Q(t)dt\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}}\right)\right)\right)^2}{4\left(x-\int B(t)Q(t)dt\right)\mu(f(y))} \end{aligned} \tag{6.4.17}$$

$$\begin{aligned} &(iii)\ H(x,y,t) = \frac{Q(t)}{2} + \frac{\left(\frac{1}{2}-\frac{1}{2}\,i\tan\left(C_1+\frac{1}{2}\,i\ln\left(\left(x-\int B(t)Q(t)dt\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}}\right)\right)\right)}{\left(x-\int B(t)Q(t)dt\right)}\\ &G(x,y,t) = \frac{1+\left(\tan\left(C_1+\frac{1}{2}\,i\ln\left(\left(x-\int B(t)Q(t)dt\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}}\right)\right)\right)^2}{4\left(x-\int B(t)Q(t)dt\right)\mu(f(y))} \end{aligned} \tag{6.4.18}$$

$$\begin{aligned} &(iv)\ H(x,y,t) = \frac{Q(t)}{2} + \frac{\tanh\left(\left(\left(x-\int B(t)Q(t)dt\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}} - C_2\right){C_1}^{-1}\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}}}{C_1}\\ &G(x,y,t) = \frac{\left(1-\left(\tanh\left(\left(\left(x-\int B(t)Q(t)dt\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}} - C_2\right){C_1}^{-1}\right)\right)^2\right)\left(x-\int B(t)Q(t)dt\right)\left(e^{\frac{\int(f(y))^{-1}dy}{\mu}}\right)^2}{\mu(f(y)){C_1}^2}\\ &+\frac{\tanh\left(\left(\left(x-\int B(t)Q(t)dt\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}} - C_2\right){C_1}^{-1}\right)e^{\frac{\int(f(y))^{-1}dy}{\mu}}}{\mu(f(y))C_1}. \end{aligned} \tag{6.4.19}$$

Vector field $V_2 + \gamma V_3$

In this case, the reduced ODE is

$$J''' + 2J''J + 2J'^2 = 0. \tag{6.4.20}$$

Integrating equation (6.4.20) with respect to ξ twice we get,

$$J' + J^2 = a_2\xi + a_3, \tag{6.4.21}$$

where a_2 and a_3 are arbitrary constants.

Using

$$u' = Ju, \tag{6.4.22}$$

where u is function of ξ and prime ($\prime$) denotes ξ-derivative. Equation (6.4.21) becomes

$$u'' = a_2\xi u + a_3 u, \tag{6.4.23}$$

which is general Airy differential equation.

Thus, solutions of equation (6.4.20) are:

$$\begin{aligned} &(i)\ J(\xi) = C_2\ \tanh(C_1 + C_2\,\xi)\\ &(ii)\ J(\xi) = C_2\ \coth(C_1 + C_2\,\xi)\\ &(iii)\ J(\xi) = -C_2\ \tan(C_1 + C_2\,\xi)\\ &(iv)\ J(\xi) = C_2\ \cot(C_1 + C_2\,\xi)\,, \end{aligned} \tag{6.4.24}$$

where C_1 and C_2 denote arbitrary constants.

The general solution of equation (6.4.23) is given by:

$$u(\xi) = C_1\,Ai(\frac{a_2\xi + a_3}{(-a_2)^{2/3}}) + C_2\,Bi(\frac{a_2\xi + a_3}{(-a_2)^{2/3}}), \tag{6.4.25}$$

where Ai and Bi are the Airy Ai and Bi wave functions, respectively.

Solution of equation (6.4.21) for $a_2 = a_3 = 0$ is given by:

$$J(\xi) = \frac{1}{(\xi + C_1)}. \tag{6.4.26}$$

As a result the solutions of the main system (6.1.2) corresponding to solutions (6.4.24) are as

$$\begin{aligned}(i)\ H(x,y,t) &= \frac{Q(t)}{2} + C_2 \tanh\left(C_1 + C_2\left(x - \int B(t)\,Q(t)\,dt - \frac{\int (f(y))^{-1}dy}{\gamma}\right)\right)\\ G(x,y,t) &= \frac{-C_2{}^2\left(1-\left(\tanh\left(C_1+C_2\left(x-\int B(t)Q(t)dt-\frac{\int (f(y))^{-1}dy}{\gamma}\right)\right)\right)^2\right)}{\gamma(f(y))}\end{aligned} \tag{6.4.27}$$

$$\begin{aligned}(ii)\ H(x,y,t) &= \frac{Q(t)}{2} + C_2 \coth\left(C_1 + C_2\left(x - \int B(t)\,Q(t)\,dt - \frac{\int (f(y))^{-1}dy}{\gamma}\right)\right)\\ G(x,y,t) &= \frac{-C_2{}^2\left(1-\left(\coth\left(C_1+C_2\left(x-\int B(t)Q(t)dt-\frac{\int (f(y))^{-1}dy}{\gamma}\right)\right)\right)^2\right)}{\gamma(f(y))}\end{aligned} \tag{6.4.28}$$

$$\begin{aligned}(iii)\ H(x,y,t) &= \frac{Q(t)}{2} - C_2 \tan\left(C_1 + C_2\left(x - \int B(t)\,Q(t)\,dt - \frac{\int (f(y))^{-1}dy}{\gamma}\right)\right)\\ G(x,y,t) &= \frac{C_2{}^2\left(1+\left(\tan\left(C_1+C_2\left(x-\int B(t)Q(t)dt-\frac{\int (f(y))^{-1}dy}{\gamma}\right)\right)\right)^2\right)}{\gamma(f(y))}\end{aligned} \tag{6.4.29}$$

$$\begin{aligned}(iv)\ H(x,y,t) &= \frac{Q(t)}{2} + C_2 \cot\left(C_1 + C_2\left(x - \int B(t)\,Q(t)\,dt - \frac{\int (f(y))^{-1}dy}{\gamma}\right)\right)\\ G(x,y,t) &= \frac{-C_2{}^2\left(-1-\left(\cot\left(C_1+C_2\left(x-\int B(t)Q(t)dt-\frac{\int (f(y))^{-1}dy}{\gamma}\right)\right)\right)^2\right)}{\gamma(f(y))}.\end{aligned} \tag{6.4.30}$$

Using equation (6.4.22) and equation (6.4.25), the solution of the main system (6.1.2) is given as:

$$\begin{aligned}&H(x,y,t)\\ &= \frac{Q(t)}{2} + \frac{\left(\frac{C_1\,a_2 Ai\left(1,\frac{\left(a_2\left(x-\int B(t)Q(t)dt-\frac{\int (f(y))^{-1}dy}{\gamma}\right)+a_3\right)}{(-a_2)^{2/3}}\right)}{(-a_2)^{2/3}} + \frac{C_2\,a_2 Bi\left(1,\frac{\left(a_2\left(x-\int B(t)Q(t)dt-\frac{\int (f(y))^{-1}dy}{\gamma}\right)+a_3\right)}{(-a_2)^{2/3}}\right)}{(-a_2)^{2/3}}\right)}{\left(C_1\,Ai\left(\frac{\left(a_2\left(x-\int B(t)Q(t)dt-\frac{\int (f(y))^{-1}dy}{\gamma}\right)+a_3\right)}{(-a_2)^{2/3}}\right)+C_2\,Bi\left(\frac{\left(a_2\left(x-\int B(t)Q(t)dt-\frac{\int (f(y))^{-1}dy}{\gamma}\right)+a_3\right)}{(-a_2)^{2/3}}\right)\right)},\end{aligned} \tag{6.4.31}$$

where $G(x,y,t)$ can be easily found by differentiating $H(x,y,t)$ with respect to y. Due to lengthy expression we are not mentioning it here.

The solution of system (6.1.2) corresponding to solution (6.4.26) is given by:

$$\begin{aligned} H\left(x,y,t\right) &= \tfrac{Q(t)}{2} + \frac{1}{\left(x-\int B(t)Q(t)dt-\frac{\int (f(y))^{-1}dy}{\gamma}+C_1\right)} \\ G\left(x,y,t\right) &= \frac{1}{\left(x-\int B(t)Q(t)dt-\frac{\int (f(y))^{-1}dy}{\gamma}+C_1\right)^2 \gamma(f(y))}. \end{aligned} \tag{6.4.32}$$

Vector field V_3

For this case the partial differential equation (6.4.8) is identically satisfied. Accordingly, the solution of the main system (6.1.2) is given as

$$\begin{aligned} H\left(x,y,t\right) &= \tfrac{Q(t)}{2} + J\left(\sigma\right) \\ G\left(x,y,t\right) &= 0, \end{aligned} \tag{6.4.33}$$

where $\sigma = x - \int B\left(t\right) Q\left(t\right) dt$.

6.5 Some More Exact Solutions

We can find more solutions of the main system (6.1.2) corresponding to the solutions of PDE (6.4.8). Traveling wave solutions of PDE (6.4.8) are as follows:

$$\begin{aligned} &(i)\;\; F\left(\rho,\sigma\right) = C_3\,\tanh\left(C_1 + C_2\,\rho + C_3\,\sigma\right) \\ &(ii)\;\; F\left(\rho,\sigma\right) = C_3\,\coth\left(C_1 + C_2\,\rho + C_3\,\sigma\right) \\ &(iii)\;\; F\left(\rho,\sigma\right) = -C_3\,\tan\left(C_1 + C_2\,\rho + C_3\,\sigma\right) \\ &(iv)\;\; F\left(\rho,\sigma\right) = C_3\,\cot\left(C_1 + C_2\,\rho + C_3\,\sigma\right), \end{aligned} \tag{6.5.1}$$

where C_1, C_2 and C_3 are arbitrary constants.

Now integrating PDE (6.4.8) with respect to ρ and σ, we get

$$F_\sigma + F^2 = \int g_1(\sigma)d\sigma + g_2(\rho), \tag{6.5.2}$$

where g_1 is arbitrary function of σ and g_2 is arbitrary function of ρ.

To solve equation (6.5.2), we consider two cases:

Case(i) $g_1 = 0$

In this case, solution of equation (6.5.2) can be given as

$$F(\rho,\sigma) = \sqrt{g_2(\rho)}\,\tanh(g_2(\rho)(\sigma + h(\rho))), \tag{6.5.3}$$

where h is arbitrary function of ρ.

Case(ii) $g_1 = g_2 = 0$

Solutions of equation (5.2) can be given as

$$F(\rho,\sigma) = \frac{1}{\sigma + g_3(\rho)}, \tag{6.5.4}$$

where g_3 is arbitrary function of ρ.

The solutions of the main system (6.1.2) corresponding to solutions (6.5.1) are:

$$\begin{aligned}(i)H(x,y,t) &= \tfrac{Q(t)}{2} + C_3 \tanh\left(C_1 + C_2\, y + C_3\, \left(x - \int B\,(t)\, Q\,(t)\, dt\right)\right)\\ G(x,y,t) &= C_3C_2\, \left(1 - \left(\tanh\left(C_1 + C_2\, y + C_3\, \left(x - \int B\,(t)\, Q\,(t)\, dt\right)\right)\right)^2\right)\end{aligned} \tag{6.5.5}$$

$$\begin{aligned}(ii)H(x,y,t) &= \tfrac{Q(t)}{2} + C_3 \coth\left(C_1 + C_2\, y + C_3\, \left(x - \int B\,(t)\, Q\,(t)\, dt\right)\right)\\ G(x,y,t) &= C_3C_2\, \left(1 - \left(\coth\left(C_1 + C_2\, y + C_3\, \left(x - \int B\,(t)\, Q\,(t)\, dt\right)\right)\right)^2\right)\end{aligned} \tag{6.5.6}$$

$$\begin{aligned}(iii)H(x,y,t) &= \tfrac{Q(t)}{2} - C_3 \tan\left(C_1 + C_2\, y + C_3\, \left(x - \int B\,(t)\, Q\,(t)\, dt\right)\right)\\ G(x,y,t) &= -C_3C_2\, \left(1 + \left(\tan\left(C_1 + C_2\, y + C_3\, \left(x - \int B\,(t)\, Q\,(t)\, dt\right)\right)\right)^2\right)\end{aligned} \tag{6.5.7}$$

$$\begin{aligned}(iv)H(x,y,t) &= \tfrac{Q(t)}{2} + C_3 \cot\left(C_1 + C_2\, y + C_3\, \left(x - \int B\,(t)\, Q\,(t)\, dt\right)\right)\\ G(x,y,t) &= C_3C_2\, \left(-1 - \left(\cot\left(C_1 + C_2\, y + C_3\, \left(x - \int B\,(t)\, Q\,(t)\, dt\right)\right)\right)^2\right).\end{aligned} \tag{6.5.8}$$

The solution of the main system (6.1.2) corresponding to solution (6.5.3) is

$$H(x,y,t) = \frac{Q(t)}{2} + \sqrt{g_2\,(y)}\tanh\left(\left(x - \int B\,(t)\, Q\,(t)\, dt\right)\sqrt{g_2\,(y)} + h\,(y)\,\sqrt{g_2\,(y)}\right), \tag{6.5.9}$$

where $G(x,y,t)$ can be easily found by differentiating $H(x,y,t)$ with respect to y. A variety of solutions of VCBK equations can be found by taking $g_2(y)$ and $h(y)$ as any functions of y as $sn, cn, \sinh, \tanh, \wp$, etc., where sn, cn are Jacobi elliptic functions and $\wp$ is WeierstrassP function, respectively.

The solution of the main system corresponding to solution (6.5.4) is given by:

$$\begin{aligned}H(x,y,t) &= \tfrac{Q(t)}{2} + \frac{1}{\left(x - \int B(t)Q(t)dt + g_3(y)\right)}\\ G(x,y,t) &= -\frac{\frac{d}{dy}g_3(y)}{\left(x - \int B(t)Q(t)dt + g_3(y)\right)^2}.\end{aligned} \tag{6.5.10}$$

Here, too, we can take g_3 as any function of y to find variety of solutions of VCBK.

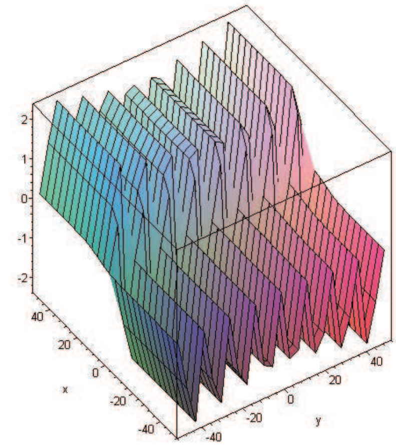

Figure 6.1: Periodic solution $H(x,y,t)$ (6.4.19) for $Q(t)=0, C_1=C_2=\mu\ =0$ and $f(y)=\sin(y)$

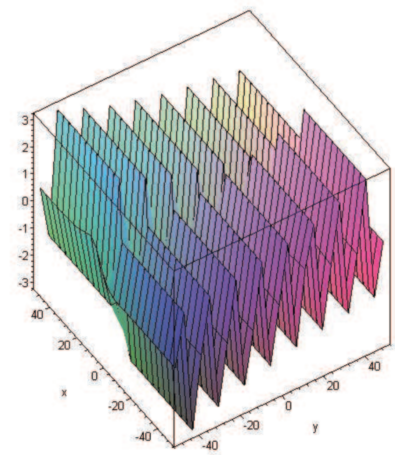

Figure 6.2: Periodic solution $G(x,y,t)$ (6.4.19) for $Q(t)=0, C_1=C_2=\mu\ =0$ and $f(y)=\sin(y)$

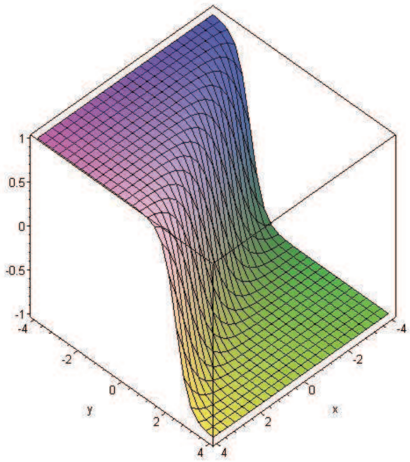

Figure 6.3: Kink wave solution $H(x, y, t)$ (6.4.27) for $Q(t) = C_1 = 0$ and $C_2 = f(y) = 1$

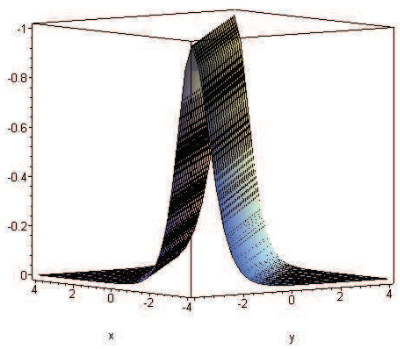

Figure 6.4: A single soliton solution $G(x, y, t)$ (6.4.27) for $Q(t) = C_1 = 0$ and $C_2 = f(y) = 1$

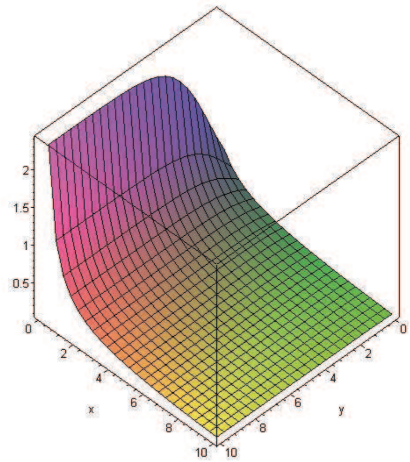

Figure 6.5: Kink wave solution $H(x, y, t)$ (6.4.16) for $Q(t) = C_1 = 0$ and $C_2 = f(y) = \mu = 1$

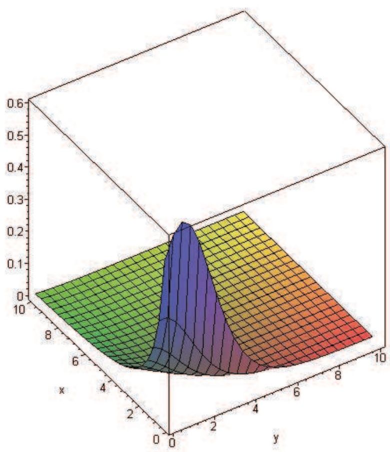

Figure 6.6: Singularity at $x = 0$ of solution $G(x, y, t)$ (6.4.16) for $Q(t) = C_1 = 0$ and $C_2 = f(y) = \mu = 1$

6.6 Conclusion and Remark

In this chapter, we have considered the Painlevé property and the symmetries for the (2+1)-dimensional VCBK system. Using Weiss-Kruskal approach, we check the Painlevé property of (2+1)-dimensional VCBK system and prove that the system (6.1.2) is integrable. The Lie algebra of the system involves arbitrary functions. Using the subalgebras, we have reduced equation (6.1.3) to PDEs in two independent variables. One of these PDEs is well known Burger's equation. The other PDE is investigated again by Lie symmetry method and reduction to ODEs and certain exact solutions are obtained for each element in the optimal system. We have derived some more exact solutions of system (6.1.2) corresponding to the PDE (6.4.8). The solution (6.4.19) for $Q(t) = 0,\ C_1 = C_2 = \mu = 1$ and $f(y) = \sin(y)$ gives the periodic solutions as shown in Figure 6.1 ($H(x, y, t)$) and Figure 6.2 ($G(x, y, t)$). For $Q(t) = C_1 = 0,\ C_2 = f(y) = 1$ solution (6.4.27) furnishes a kink wave and a single soliton solution as in Figure 6.3 ($H(x, y, t)$) and Figure 6.4 ($G(x, y, t)$), respectively. In Figure 6.5 ($H(x, y, t)$) for $Q(t) = C_1 = 0,\ C_2 = f(y) = \mu = 1$ we have kink wave solution (6.4.16). There is a singularity at $x = 0$ for solution (6.4.16) as depicted in Figure 6.6 ($G(x, y, t)$).

Remark 6.6.1 When $B(t) = 1$, the VCBK system (6.1.2) reduces to the celebrated (2+1)-dimensional Broer-Kaup system (6.1.1). Though we keep from discussing the physical implications of the solutions reported in this chapter, yet we feel worth mentioning that the solutions obtained are such that one can choose the arbitrary function $B(t)$ in a suitable manner to simulate physical situations governed by the equation (6.1.2) to obtain particular solutions having desired features.

Chapter 7

Variable Coefficients Gardner Equation [1]

7.1 Introduction

A large variety of physical, chemical and biological phenomena is governed by nonlinear partial differential equations (PDEs). The nonlinear PDEs, which exhibit a rich variety of nonlinear phenomena, arise in many physical fields like the condense matter physics, fluid mechanics, plasma physics and optics, etc. The physical situations in which nonlinear equations arise tend to be highly idealized due to assumption of constant coefficients. When the inhomogeneities of media and non-uniformity of boundaries are taken into account in various real physical situations, the variable-coefficient PDEs often can provide more powerful and realistic models than their constant-coefficient counterparts in describing a large variety of real phenomena. Due to this, much attention has been paid on study of nonlinear equations with variable coefficients [31, 67, 97, 62]. Finding exact solutions of such nonlinear PDEs plays an important role for these equations which are drawn from diverse interesting nonlinear phenomena. As a result, the research on exact solutions of nonlinear evolution equations has become more and more important. In the past decades, a wealth of methods have been developed to deal with these exact solutions of PDEs though it

[1] A part of this chapter has been communicated in *Parmana Journal of Physics*

is rather difficult. Some of the most important methods are the inverse scattering transformation (IST) [36], Darboux and Bäcklund transformations [64], Hirotas direct method [64, 46, 66], Lie symmetry analysis [97, 86, 39], CK method [22, 20], etc. As is well known, the Lie group method is a powerful and direct approach to construct exact solutions of nonlinear differential equations. Furthermore, based on the Lie group method, many types of exact solutions of PDEs can be considered, such as the traveling wave solutions, similarity solutions, soliton wave solutions, fundamental solutions, and so on. Some recent contributions are in [67, 97, 62, 67].

Recently, Liu et al [68] considered the Gardner equation for time dependent coefficients in following form

$$u_t + \alpha t^m u u_x + \beta t^n u^2 u_x + \gamma t^p u_{xxx} + \delta t^q u_x + \mu t^r u = 0, \tag{7.1.1}$$

where $u = u(x,t)$ denotes the amplitude of the relevant wave model, such as for the internal waves in a stratified ocean, x is the horizontal coordinate, and t is the time. While $\alpha, \beta, \gamma, \delta$ and μ are arbitrary constant parameters, m, n, p, q and r are given real numbers. Liu et al [68] have done the Painlevé analysis and applied Lie classical method to find some exact solutions of equation (7.1.1) for $\mu \neq 0$ and $\mu = 0$.

In the present Chapter, we will consider the Gardner equation with variable coefficients of the form

$$u_t + \alpha(t) u u_x + \beta(t) u^2 u_x + \gamma(t) u_{xxx} + \delta(t) u_x + \mu(t) u = 0, \tag{7.1.2}$$

where $\alpha(t), \beta(t), \gamma(t), \delta(t)$ and $\mu(t)$ are arbitrary functions of t. For $\alpha(t) = \alpha t^m, \beta(t) = \beta t^n, \gamma(t) = \gamma t^p, \delta(t) = \delta t^q$ and $\mu(t) = \mu t^r$ equation reduces to equation (7.1.1). Equation (7.1.2) provide more powerful model then the equation (7.1.1) as one can choose the arbitrary functions $\alpha(t), \beta(t), \gamma(t), \delta(t), \mu(t)$ in suitable manner to simulate physical situation govern by the equation. The equation (7.1.2) also covers Gardner's equation of the form of (7.1.1). Equation (7.1.2) can be used to model such physical situations as the dust-acoustic solitary waves in dusty plasmas, internal solitary waves in stable, stratified shear flows in ocean and atmosphere, ion acoustic waves in plasmas with a negative ion, interfacial solitary waves over slowly varying topographies, and wave motion in a nonlinear elastic structural element with large deflection, etc.

Variable-coefficient NLEEs are not completely integrable unless the variable coefficients satisfy some specific constraint conditions. Thereby, we will find the conditions for the equation (7.1.2) to pass the Painlevé test firstly, then the symmetries and exact solutions are considered.

The rest of this chapter is organized as follows. In Section 7.2, Painlevé analysis of equation (7.1.2) with damping term (so, $\mu(t) \neq 0$) and without damping term (so, $\mu(t) = 0$) is performed. In Section 7.3, the symmetries of the equation for $\mu(t) \neq 0$ and $\mu(t) = 0$ are obtained by the Lie group analysis method. In Sect. 7.4, we investigate the symmetry reductions and exact explicit solutions for the vc-Gardner equation. In Sect. 7.5, we conclude and make some remarks.

7.2 Painlevé Analysis

Firstly, we check the integrability of equation (7.1.2) by means of the Painlevé analysis. Using the standard Kruskal's simplified method, the expansion about the singular manifold has the form

$$u = \phi^{-p} \sum_{j=0}^{\infty} u_j \phi^j, \tag{7.2.1}$$

where $\phi = x + \psi(t)$, $u_j(t)$ are analytic functions in a neighborhood of the non characteristic singular manifold, $u_0 \neq 0$ and p is a positive integer.

Through the leading order analysis, it is readily found that $p = 1$ and $u_0 = \sqrt{\frac{-6\gamma(t)}{\beta(t)}}$. Then substituting (7.2.1) into (7.1.2), we have

$$j = 0, \quad u_0 = \sqrt{\frac{-6\gamma(t)}{\beta(t)}}, \tag{7.2.2}$$

$$j = 1, \quad u_1 = -\frac{\alpha(t)}{2\beta(t)}, \tag{7.2.3}$$

$$j = 2, \quad u_2 = -\frac{{u_1}^2}{u_0} - \frac{\psi_t}{\beta(t)u_0} - \frac{\delta(t)}{\beta(t)u_0} - \frac{\alpha(t)u_1}{\beta(t)u_0}, \tag{7.2.4}$$

$$j = 3, \quad \mu(t)\, u_0 + u_{0t} = 0, \tag{7.2.5}$$

$$j = 4, \quad u_2\psi_t + u_{1t} + \delta(t)\, u_2 + \mu(t)\, u_1 + \alpha(t)\, u_1 u_2 + \beta(t)\, u_1^2 u_2 + \alpha(t)\, u_0 u_3 + \beta(t)\, u_0 u_2^2 + 2\beta(t)\, u_1 u_0 u_3 = 0. \tag{7.2.6}$$

From above system we can get u_0, u_1, and u_2 in unique manner but we cannot get u_3, u_4, so $j = -1, 3, 4$ are resonances where $j = -1$ corresponds to the arbitrariness of the singular manifold. The compatibility conditions at $j = 3, 4$ are satisfied identically for arbitrary chosen u_3 and u_4. Therefore, (7.1.2) possesses the Painlevé property under the conditions (7.2.5) and (7.2.6). Specializing (7.2.1) by setting the resonance functions $u_3 = u_4 = 0$ and also by requiring $u_2 = 0$, it can be easily shown that $u_j = 0$, for all $j \geq 2$.

Then the condition (7.2.6) reduces to

$$u_{1t} + \mu(t)\, u_1 = 0. \tag{7.2.7}$$

Solving (7.2.5) and (7.2.7), we get

$$\beta(t) = k_1\gamma(t) \exp\left(\int 2\mu(t)dt\right), \quad \beta(t) = k_2\alpha(t) \exp\left(\int \mu(t)dt\right) \tag{7.2.8}$$

where k_1, k_2 are arbitrary constant. Therefore, under the condition (7.2.8), we can say that (7.1.2) possesses the Painlevé property and becomes integrable. For $\mu(t) = 0$ condition (7.2.8) reduces to

$$\beta(t) = k_1\gamma(t), \quad \beta(t) = k_2\alpha(t). \tag{7.2.9}$$

So equation (7.1.2) for $\mu(t) = 0$ possesses Painlevé property under the condition (7.2.9).

Remark 2.1 If we take $\alpha(t) = \alpha t^m, \beta(t) = \beta t^n, \gamma(t) = \gamma t^p, \mu(t) = \mu t^r$ in equations (7.2.8) and (7.2.9), we have integrability condition of equation (7.1.1) as $t^n = k_1 t^p \exp(\frac{2\mu}{r+1} t^{r+1}), t^n = k_2 t^m \exp(\frac{\mu}{r+1} t^{r+1})$ for $\mu \neq 0$ while we have integrability condition as $t^n = k_1 t^p, t^n = k_2 t^m$ for $\mu = 0$. Liu et al [68] obtained these conditions for equation (7.1.1).

7.3 Symmetries of the vc-Gardner Equation

In this section, we will investigate the symmetries of equation (7.1.2) by using Lie classical method. Firstly, let us consider a one-parameter Lie group of infinitesimal transformation

$$\begin{aligned} x &\to x + \epsilon\xi(x,t,u) \\ t &\to t + \epsilon\tau(x,t,u) \\ u &\to u + \epsilon\eta(x,t,u), \end{aligned} \tag{7.3.1}$$

with a small parameter $\epsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{7.3.2}$$

The symmetry group of equation (7.1.2) will be generated by the vector field of the form (7.3.2). Applying the third prolongation $\mathrm{pr}^{(3)}V$ of V to equation (7.1.2), we find that the coefficient functions ξ, τ and η must satisfy the symmetry condition

$$\begin{aligned} &\eta^t + \alpha'\tau u u_x + \alpha\eta u_x + \alpha u\eta^x + \beta'\tau u^2 u_x + 2\beta\eta u u_x + \beta u^2\eta^x + \gamma'\tau u_{xxx} + \gamma\eta^{xxx} + \delta'\tau u_x \\ &+\delta\eta^x + \mu'\tau u + \mu\eta = 0, \end{aligned} \tag{7.3.3}$$

where $(')$ denotes t-derivative and $\eta^t, \eta^x, \eta^{xxx}$ are coefficients of $\mathrm{pr}^{(3)}V$. Using the expressions for η^t, η^x and η^{xxx} in equation (7.3.3) and u_t must be replaced by equation (7.1.2). On substituting the coefficients of different differentials equal to zero lead to the system of determining equations.

Solving this system of determining equations provides following forms for the infinitesimal elements ξ, τ, η and admissible forms of various coefficients in the equation (7.1.2)

$$\xi = C_1 x + f(t), \tau = \tfrac{C_2 - g(t)}{\mu}, \eta = g(t)u, \tag{7.3.4}$$

where C_1, C_2 are arbitrary constants and $f(t), g(t)$ are arbitrary functions of t. The functions $\alpha(t), \beta(t), \gamma(t)$ and $\delta(t)$ are governed by the following conditions:

$$\begin{aligned} &2\beta g(t) - \beta C_1 + \tau\beta' + \beta\tau_t = 0 \\ &-\alpha C_1 + \alpha g(t) + \tau\alpha' + \alpha\tau_t = 0 \\ &-f'(t) - \delta C_1 + \tau\delta' + \delta\tau_t = 0 \\ &-3\gamma C_1 + \tau\gamma' + \gamma\tau_t = 0. \end{aligned} \tag{7.3.5}$$

The infinitesimal generators of the corresponding Lie algebra are given by

$$\begin{aligned} &V_1 = x\tfrac{\partial}{\partial x} \\ &V_2 = \tfrac{1}{\mu(t)}\tfrac{\partial}{\partial t} \\ &X(f) = f(t)\tfrac{\partial}{\partial x} \\ &Y(g) = -\tfrac{g(t)}{\mu(t)}\tfrac{\partial}{\partial t} + g(t)u\tfrac{\partial}{\partial u}. \end{aligned} \tag{7.3.6}$$

It should be noted that generators V_1, V_2 commute. The commutation relations for generators $X(f), Y(g)$ are easy to obtain and are

$$\begin{aligned} &[X(f_1), X(f_2)] = 0 \\ &[Y(g_1), Y(g_2)] = Y(\tfrac{g_2\dot{g}_1 - g_1\dot{g}_2}{\mu(t)}) \\ &[Y(g), X(f)] = X(\tfrac{g\dot{f}}{\mu(t)}). \end{aligned} \tag{7.3.7}$$

$\{X(f), Y(g)\}$ construct a Kac-Moody algebra.

Now as mentioned above again solving the system of determining equations, obtained from invariance condition (7.3.3), for $\mu(t) = 0$ provides following infinitesimal elements ξ, τ, η and admissible forms of various coefficients

$$\begin{aligned} &\xi = C_1 x + f(t) \\ &\eta = C_2 u + C_3 \\ &\tau = \tfrac{3C_1}{\gamma(t)}\textstyle\int \gamma dt + \tfrac{C_4}{\gamma(t)}, \end{aligned} \tag{7.3.8}$$

where C_1, C_2, C_3, C_4 are arbitrary constants and $f(t)$ is arbitrary function of t. The coefficients $\alpha(t), \beta(t)$ and $\delta(t)$ are governed by the following conditions:

$$\begin{aligned} &2\beta C_2 - \beta C_1 + \tau\beta' + \beta\tau_t = 0 \\ &2\beta C_3 - \alpha C_1 + \alpha C_2 + \tau\alpha' + \alpha\tau_t = 0 \\ &-f'(t) - \delta C_1 + \alpha C_3 + \tau\delta' + \delta\tau_t = 0 \\ &-3\gamma C_1 + \tau\gamma' + \gamma\tau_t = 0. \end{aligned} \tag{7.3.9}$$

Corresponding infinitesimal generators are given by

$$\begin{aligned} &V_1 = x\tfrac{\partial}{\partial x} + \tfrac{3}{\gamma(t)}\int \gamma(t)dt\tfrac{\partial}{\partial t} \\ &V_2 = u\tfrac{\partial}{\partial u} \\ &V_3 = \tfrac{\partial}{\partial u} \\ &V_4 = \tfrac{1}{\gamma(t)}\tfrac{\partial}{\partial t} \\ &X(f) = f(t)\tfrac{\partial}{\partial x}. \end{aligned} \tag{7.3.10}$$

It should be noted that generators V_1, V_2, V_3 and V_4 commute.

7.4 Symmetry Reductions and Exact Solutions

In this section we will consider reductions of equation (7.1.2) for $\mu(t) \neq 0$ and $\mu(t) = 0$. Firstly we will find similarity reductions and exact solutions of equation (7.1.2) for $\mu(t) \neq 0$.

(i) For the generator $aV_1 + bV_2 + X(f) + Y(g)$ with $g(t) \neq b$, we have following similarity variables:

$$\begin{aligned} &\zeta = x\mathrm{e}^{a\int \frac{\mu(t)}{-b+g(t)}dt} + \int \frac{\mu(t)\, f(t)}{-b+g(t)}\mathrm{e}^{a\int \frac{\mu(t)}{-b+g(t)}dt}dt \\ &u(x,t) = \mathrm{e}^{\int \frac{g(t)\mu(t)}{b-g(t)}dt}F(\zeta) \end{aligned} \tag{7.4.1}$$

and admissible coefficients are given as

$$\begin{aligned} &\alpha(t) = k_1\tfrac{\mu(t)}{(b-g(t))}\mathrm{e}^{\int -\frac{\mu(t)(-a+g(t))}{(b-g(t))}dt} \\ &\beta(t) = k_2\tfrac{\mu(t)}{(b-g(t))}\mathrm{e}^{\int -\frac{\mu(t)(-a+2g(t))}{(b-g(t))}dt} \\ &\gamma(t) = k_3\tfrac{\mu(t)}{(b-g(t))}\mathrm{e}^{\int \frac{3a\mu(t)g(t)}{(b-g(t))}dt} \\ &\delta(t) = \tfrac{\mu(t)}{(b-g(t))}\mathrm{e}^{\int \frac{a\mu(t)g(t)}{(b-g(t))}dt}\left(k_4 + f'(t)\mathrm{e}^{\int -\frac{a\mu(t)g(t)}{(b-g(t))}dt}\right) \end{aligned} \tag{7.4.2}$$

where k_1, k_2, k_3, k_4 are arbitrary constants and $(')$ denotes t-derivative. Substituting (7.4.1) with (7.4.2) into (7.1.2), we get

$$k_3F''' + k_2F^2F' + k_1FF' + a\zeta F' + k_4F' - bF = 0, \tag{7.4.3}$$

where $(')$ denote ζ-derivatives.

For solutions of ordinary differential equation (ODE)(7.4.3), we will consider the following cases:

Case(i) $a \neq 0, b \neq 0, k_4 \neq 0$

For this case we get trivial solution.

Case(ii) $a \neq 0, b = 0, k_4 \neq 0$

For this case we get constant solution of ODE (7.4.3) as

$$F = C_1,$$

where C_1 is arbitrary constant. Corresponding solution of main equation (7.1.2) is

$$u(x,t) = C_1 \mathrm{e}^{-\int \mu(t)dt}. \tag{7.4.4}$$

Case(iii) $a = 0, b = 0, k_4 \neq 0$

For this case we get following solutions of ODE (7.4.3)

$$
\begin{aligned}
&(i)\ F(\zeta) = \\
&\quad -\tfrac{1}{2k_2}\left(k_1 \pm \sqrt{12\,{C_3}^2 k_2 k_3 - 12\,k_2 k_4 + 3\,{k_1}^2}\,\mathrm{cn}\left(C_2 + C_3\zeta,\ \frac{\sqrt{2k_2k_3\left(4\,{C_3}^2k_2k_3 - 4\,k_2k_4 + {k_1}^2\right)}}{4k_2k_3C_3}\right)\right)\\
&(ii)\ F(\zeta) = \\
&\quad -\tfrac{1}{2k_2}\left(k_1 \pm \sqrt{24\,{C_3}^2 k_2 k_3 - 24\,k_2 k_4 + 6\,{k_1}^2}\ \mathrm{sn}\left(C_2 + C_3\zeta,\ \frac{\sqrt{-k_2k_3\left(4\,{C_3}^2k_2k_3 - 4\,k_2k_4 + {k_1}^2\right)}}{2k_2k_3C_3}\right)\right)\\
&(iii)\ F(\zeta) = -\tfrac{1}{2k_2}\left(k_1 \pm \sqrt{12\,k_2k_4 - 3\,{k_1}^2}\cot\left(\frac{-4\,C_1\,k_2k_3 + \sqrt{2k_2k_3\left(-4\,k_2k_4 + {k_1}^2\right)}\zeta}{4k_2k_3}\right)\right)\\
&(iv)\ F(\zeta) = -\tfrac{1}{2k_2}\left(k_1 \pm \sqrt{-12\,k_2k_4 + 3\,{k_1}^2}\tanh\left(\frac{-4\,C_1\,k_2k_3 + \sqrt{-2k_2k_3\left(-4\,k_2k_4 + {k_1}^2\right)}\zeta}{4k_2k_3}\right)\right)\\
&(v)\ F(\zeta) = -\tfrac{k_1}{2k_2} \pm \tfrac{1}{\zeta}\sqrt{-6\,\tfrac{k_3}{k_2}},\quad \text{with condition}\quad k_4 = \tfrac{{k_1}^2}{4k_2}
\end{aligned}
\tag{7.4.5}
$$

where C_1, C_2, C_3 are arbitrary constants and sn, cn denote Jacobi elliptic functions. Consequently, the solutions of main equation (7.1.2) are given as

$$(i)\ u(x,t) = -\frac{1}{2k_2}\left(k_1 \pm \sqrt{12\,{C_3}^2 k_2 k_3 - 12\,k_2 k_4 + 3\,{k_1}^2}\,\mathrm{cn}\left(C_2 + C_3\zeta,\ \frac{\sqrt{2k_2 k_3\left(4\,{C_3}^2 k_2 k_3 - 4\,k_2 k_4 + {k_1}^2\right)}}{4k_2 k_3 C_3}\right)\right)\mathrm{e}^{\int -\mu(t)dt}$$

$$(ii)\ u(x,t) = -\frac{1}{2k_2}\left(k_1 \pm \sqrt{24\,{C_3}^2 k_2 k_3 - 24\,k_2 k_4 + 6\,{k_1}^2}\,\mathrm{sn}\left(C_2 + C_3\zeta,\ \frac{\sqrt{-k_2 k_3\left(4\,{C_3}^2 k_2 k_3 - 4\,k_2 k_4 + {k_1}^2\right)}}{2k_2 k_3 C_3}\right)\right)\mathrm{e}^{\int -\mu(t)dt}$$

$$(iii)\ u(x,t) = -\frac{1}{2k_2}\left(k_1 \pm \sqrt{12\,k_2 k_4 - 3\,{k_1}^2}\cot\left(\frac{-4\,C_1\,k_2 k_3 + \sqrt{2k_2 k_3\left(-4\,k_2 k_4 + {k_1}^2\right)}\zeta}{4k_2 k_3}\right)\right)\mathrm{e}^{\int -\mu(t)dt}$$

$$(iv)\ u(x,t) = -\frac{1}{2k_2}\left(k_1 \pm \sqrt{-12\,k_2 k_4 + 3\,{k_1}^2}\tanh\left(\frac{-4\,C_1\,k_2 k_3 + \sqrt{-2k_2 k_3\left(-4\,k_2 k_4 + {k_1}^2\right)}\zeta}{4k_2 k_3}\right)\right)\mathrm{e}^{\int -\mu(t)dt}$$

$$(v)\ u(x,t) = \left(-\frac{k_1}{2k_2} \pm \frac{1}{\zeta}\sqrt{-6\,\frac{k_3}{k_2}}\right)\mathrm{e}^{\int -\mu(t)dt}, \quad \text{with condition } k_4 = \frac{{k_1}^2}{4k_2}, \tag{7.4.6}$$

where $\zeta(x,t) = x + \int \frac{\mu(t)f(t)}{g(t)}dt$.

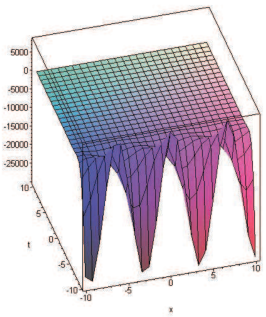

Figure 7.1: Periodic solution (7.4.6) (i) for $k_1 = k_2 = k_3 = k_4 = C_3 = 1, C_2 = 0$ and $f(t) = g(t) = t, \mu(t) = 1$

Case(iv) $a = 0, b = 0, k_4 = 0$

For this case solution of ODE (7.4.3) is given as

$$F(\zeta) = -\frac{12k_3 k_1}{\zeta^2 {k_1}^2 - 2\,C_1\,\zeta\,{k_1}^2 + {C_1}^2{k_1}^2 + 6\,k_3 k_2}. \tag{7.4.7}$$

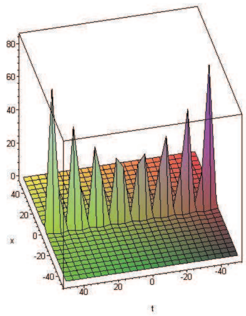

Figure 7.2: Periodic solution (7.4.6) (v) for $k_1 = k_2 = k_3 = 1$ and $f(t) = g(t) = t, \mu(t) = 1$

Corresponding solution of main equation (7.1.2) is given as

$$u(x,t) = -\frac{12k_3k_1}{\zeta^2{k_1}^2 - 2\,C_1\,\zeta\,{k_1}^2 + {C_1}^2{k_1}^2 + 6\,k_3k_2}\mathrm{e}^{\int -\mu(t)dt}, \tag{7.4.8}$$

where $\zeta(x,t) = x + \int \frac{\mu(t)f(t)}{g(t)}dt$.

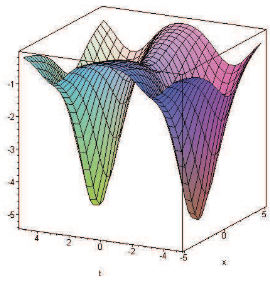

Figure 7.3: Two dromion solution (7.4.8) for $k_1 = k_2 = C_1 = 1, k_3 = -1,$ and $f(t) = \cos(t), \mu(t) = -sin(t), g(t) = 1$

(ii) For the generator $bV_2 + X(f) + Y(g)$ with $g(t) = b$, solving (7.3.4) and (7.3.5) by taking $C_1 = 0$ and $g(t) = C_2 = b$, we have following infinitesimals

$$\xi = f(t), \tau = \frac{C_3}{\gamma(t)}, \eta = bu, \tag{7.4.9}$$

where C_3 is arbitrary constant. Corresponding similarity variables are

$$\zeta(x,t) = x - \tfrac{1}{c_3}\int f\,(t)\,\gamma\,(t)\,dt, \quad u(x,t) = \mathrm{e}^{\frac{b}{c_3}\int\gamma(t)dt}F(\zeta) \tag{7.4.10}$$

and admissible coefficients are given by

$$\begin{aligned}
\alpha(t) &= k_1\gamma(t)\mathrm{e}^{-\frac{b}{c_3}\int\gamma(t)dt}\\
\beta(t) &= k_2\gamma(t)\mathrm{e}^{-\frac{2b}{c_3}\int\gamma(t)dt}\\
\delta(t) &= \tfrac{f(t)\gamma(t)}{c_3} + \tfrac{\gamma(t)k_3}{c_3}\\
\mu(t) &= k_4\gamma(t),
\end{aligned} \tag{7.4.11}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants. Substituting (7.4.10) with (7.4.11) into main equation (7.1.2), we have following ODE

$$C_3F''' + k_2C_3F^2F' + k_1C_3FF' + k_3F' + k_4C_3F + bF = 0, \tag{7.4.12}$$

where ($'$) denotes derivative with respect to ζ.

Employing conditions $b = -k_4C_3$ and putting $\frac{k_3}{C_3} = k_5$, ODE (7.4.12) reduces to

$$F''' + k_2F^2F' + k_1FF' + k_5F' = 0. \tag{7.4.13}$$

Solutions of (7.4.13) are given as

$$\begin{aligned}
&(i)\ F\,(\zeta) = -\tfrac{1}{2k_2}\left(k_1 \pm \sqrt{12\,{C_4}^2k_2 - 12\,k_2k_5 + 3\,{k_1}^2}\,\mathrm{cn}\left(C_2 + C_4\,\zeta,\ \frac{\sqrt{2k_2\left(4\,{C_4}^2k_2 - 4\,k_2k_5 + {k_1}^2\right)}}{4k_2C_4}\right)\right)\\
&(ii)\ F\,(\zeta) = -\tfrac{k_1}{2k_2} \pm \tfrac{1}{\sqrt{k_2}}\sqrt{6}C_4\,\mathrm{dn}\left(C_2 + C_4\,\zeta,\ \frac{\sqrt{k_2\left(8\,{C_4}^2k_2 + 4\,k_2k_5 - {k_1}^2\right)}}{2k_2C_4}\right)\\
&(iii)\ F\,(\zeta) = -\tfrac{1}{2k_2}\left(k_1 \pm \sqrt{-12\,k_2k_5 + 3\,{k_1}^2}\tanh\left(\frac{-4\,C_1\,k_2 + \sqrt{2}\sqrt{-k_2\left(-4\,k_2k_5 + {k_1}^2\right)}\zeta}{4k_2}\right)\right)\\
&(iv)\ F\,(\zeta) = -\tfrac{1}{2k_2}\left(k_1 \pm \sqrt{-24\,k_2k_5 + 6\,{k_1}^2}\csc\left(C_1 - \frac{\sqrt{-k_2\left(-4\,k_2k_5 + {k_1}^2\right)}\zeta}{2k_2}\right)\right)\\
&(v)\ F\,(\zeta) = -\frac{12k_1}{\zeta^2{k_1}^2 - 2\,C_1\,\zeta{k_1}^2 + {C_1}^2{k_1}^2 + 6\,k_2}\ \text{for } k_5 = 0,
\end{aligned} \tag{7.4.14}$$

where C_1, C_2 and C_4 are arbitrary constants.

Consequently, solutions of the main equation (7.1.2) are given as

$$
\begin{aligned}
&(i)\ u(x,t) = \\
&-\tfrac{1}{2k_2}\left(k_1 \pm \sqrt{12\,{C_4}^2k_2 - 12\,k_2k_5 + 3\,{k_1}^2}\,\mathrm{cn}\left(C_2 + C_4\,\zeta,\ \tfrac{\sqrt{2k_2\left(4\,{C_4}^2k_2 - 4\,k_2k_5 + {k_1}^2\right)}}{4k_2C_4}\right)\right)\mathrm{e}^{\frac{b}{c_3}\int\gamma(t)dt}\\
&(ii)\ u(x,t) = -\tfrac{k_1}{2k_2} \pm \tfrac{1}{\sqrt{k_2}}\sqrt{6}C_4\,\mathrm{dn}\left(C_2 + C_4\,\zeta,\ \tfrac{\sqrt{k_2\left(8\,{C_4}^2k_2 + 4\,k_2k_5 - {k_1}^2\right)}}{2k_2C_4}\right)\mathrm{e}^{\frac{b}{c_3}\int\gamma(t)dt}\\
&(iii)\ u(x,t) = -\tfrac{1}{2k_2}\left(k_1 \pm \sqrt{-12\,k_2k_5 + 3\,{k_1}^2}\tanh\left(\tfrac{-4\,C_1\,k_2 + \sqrt{2}\sqrt{-k_2\left(-4\,k_2k_5 + {k_1}^2\right)}\zeta}{4k_2}\right)\right)\mathrm{e}^{\frac{b}{c_3}\int\gamma(t)dt}\\
&(iv)\ u(x,t) = -\tfrac{1}{2k_2}\left(k_1 \pm \sqrt{-24\,k_2k_5 + 6\,{k_1}^2}\csc\left(C_1 - \tfrac{\sqrt{-k_2\left(-4\,k_2k_5 + {k_1}^2\right)}\zeta}{2k_2}\right)\right)\mathrm{e}^{\frac{b}{c_3}\int\gamma(t)dt}\\
&(v)\ u(x,t) = -\frac{12k_1}{\zeta^2{k_1}^2 - 2\,C_1\,\zeta{k_1}^2 + {C_1}^2{k_1}^2 + 6\,k_2}\mathrm{e}^{\frac{b}{c_3}\int\gamma(t)dt}\ \text{for } k_5 = 0,
\end{aligned}
\tag{7.4.15}
$$

where $\zeta(x,t) = x - \frac{1}{c_3}\int f(t)\,\gamma(t)\,dt$ with $b = -k_4C_3$ and $\frac{k_3}{C_3} = k_5$.

Now we will find the reductions of equation (7.1.2) for $\mu(t) = 0$.

For the generator $bV_4 + X(f)$, we have following similarity variables

$$\zeta = bx - \int\lambda(t)\,f(t)\,dt, \quad u(x,t) = F(\zeta), \tag{7.4.16}$$

and admissible coefficients are given by

$$
\begin{aligned}
\alpha(t) &= k_1\lambda(t)\\
\beta(t) &= k_2\lambda(t)\\
\delta(t) &= \tfrac{\lambda(t)}{b}(k_3 + f(t)),
\end{aligned}
\tag{7.4.17}
$$

where k_1, k_2, k_3 are arbitrary constants.

Substituting (7.4.16) with (7.4.17) in equation (7.1.2) for $\mu(t) = 0$, we get following ODE

$$b^3F''' + k_2bF^2F' + k_1bFF' + k_3F' = 0, \tag{7.4.18}$$

where $(')$ denotes ζ-derivative.

Solutions of ODE (7.4.18) are given as

$$(i)\ F(\zeta)$$
$$= -\tfrac{1}{2bk_2}\left(k_1 b \pm \sqrt{3b\left(4\,{C_3}^2 k_2 b^3 - 4\,k_2 k_3 + {k_1}^2 b\right)}\mathrm{cn}\left(C_2 + C_3\,\zeta,\ \tfrac{\sqrt{2k_2 b\left(4\,{C_3}^2 k_2 b^3 - 4\,k_2 k_3 + {k_1}^2 b\right)}}{4k_2 b^2 C_3}\right)\right)$$
$$(ii) F(\zeta) = -\tfrac{1}{2k_2^{\frac{3}{2}}}\left(k_1\sqrt{k_2} \pm 2\sqrt{6}bC_3\,\mathrm{dn}\left(C_2 + C_3\,\zeta,\ \tfrac{\sqrt{k_2 b\left(8\,{C_3}^2 k_2 b^3 + 4\,k_2 k_3 - {k_1}^2 b\right)}}{2k_2 b^2 C_3}\right) k_2\right)$$
$$(iii) F(\zeta) = -\tfrac{k_1}{2k_2} \pm \tfrac{\sqrt{6b\left(-4\,k_2 k_3 + {k_1}^2 b\right)}}{2k_2 b}\,\csc\left(C_1 + \tfrac{\sqrt{-k_2 b\left(-4\,k_2 k_3 + {k_1}^2 b\right)}\zeta}{2k_2 b^2}\right)$$
$$(iv)\ F(\zeta) = -\tfrac{1}{2bk_2}\left(k_1 b \pm \sqrt{3}\sqrt{b\left(-4\,k_2 k_3 + {k_1}^2 b\right)}\tanh\left(\tfrac{4\,C_1\,k_2 b^2 + \sqrt{2}\sqrt{-k_2 b\left(-4\,k_2 k_3 + {k_1}^2 b\right)}\zeta}{4k_2 b^2}\right)\right)$$
$$(v) F(\zeta) = -\frac{12 k_1 b^2}{\zeta^2 {k_1}^2 - 2\,C_1\,\zeta\,{k_1}^2 + {C_1}^2 {k_1}^2 + 6\,b^2 k_2}\ \text{for}\ k_3 = 0, \tag{7.4.19}$$

where C_1, C_2 and C_3 are arbitrary constants.

Consequently, solutions of the main equation (7.1.2) for $\mu(t) = 0$ are given as

$$(i)\ u(x,t) =$$
$$-\tfrac{1}{2bk_2}\left(k_1 b \pm \sqrt{3b\left(4\,{C_3}^2 k_2 b^3 - 4\,k_2 k_3 + {k_1}^2 b\right)}\mathrm{cn}\left(C_2 + C_3\,\zeta,\ \tfrac{\sqrt{2k_2 b\left(4\,{C_3}^2 k_2 b^3 - 4\,k_2 k_3 + {k_1}^2 b\right)}}{4k_2 b^2 C_3}\right)\right)$$
$$(ii)\ u(x,t) = -\tfrac{1}{2k_2^{\frac{3}{2}}}\left(k_1\sqrt{k_2} \pm 2\sqrt{6}bC_3\,\mathrm{dn}\left(C_2 + C_3\,\zeta,\ \tfrac{\sqrt{k_2 b\left(8\,{C_3}^2 k_2 b^3 + 4\,k_2 k_3 - {k_1}^2 b\right)}}{2k_2 b^2 C_3}\right) k_2\right)$$
$$(iii)\ u(x,t) = -\tfrac{k_1}{2k_2} \pm \tfrac{\sqrt{6b\left(-4\,k_2 k_3 + {k_1}^2 b\right)}}{2k_2 b}\,\csc\left(C_1 + \tfrac{\sqrt{-k_2 b\left(-4\,k_2 k_3 + {k_1}^2 b\right)}\zeta}{2k_2 b^2}\right)$$
$$(iv)\ u(x,t) = -\tfrac{1}{2bk_2}\left(k_1 b \pm \sqrt{3}\sqrt{b\left(-4\,k_2 k_3 + {k_1}^2 b\right)}\tanh\left(\tfrac{4\,C_1\,k_2 b^2 + \sqrt{2}\sqrt{-k_2 b\left(-4\,k_2 k_3 + {k_1}^2 b\right)}\zeta}{4k_2 b^2}\right)\right)$$
$$(v)\ u(x,t) = -\frac{12 k_1 b^2}{\zeta^2 {k_1}^2 - 2\,C_1\,\zeta\,{k_1}^2 + {C_1}^2 {k_1}^2 + 6\,b^2 k_2}\ \text{for}\ k_3 = 0, \tag{7.4.20}$$

where $\zeta(x,t) = bx - \int \lambda(t)\, f(t)\, dt$.

Now we find solution of equation (7.1.2) for $\mu(t) = 0$ based on Painlevé analysis as mentioned in sec-2. Truncating the Painlevé series, we have

$$u(x,t) = \tfrac{u_0}{\phi(x,t)} - \tfrac{\alpha(t)}{2\beta(t)}, \tag{7.4.21}$$

where u_0 is given by (7.2.2). From (7.2.4), we have

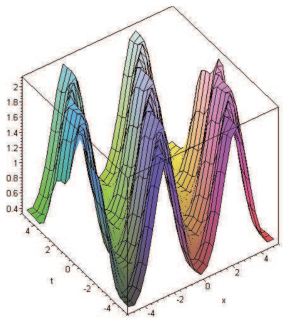

Figure 7.4: Periodic solution (7.4.20)(i) for $k_1 = k_2 = C_3 = b = 1, C_2 = 0, f(t) = \lambda(t) = \cos(t)$ and $k_3 = -1$,

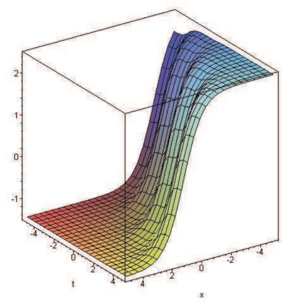

Figure 7.5: Kink wave Solution (7.4.20)(iv) for $k_1 = k_3 = C_1 = b = 1, f(t) = \lambda(t) = \cos(t)$ and $k_2 = -1$,

$$\phi(x,t) = x + \int \frac{\alpha(t)^2}{4\beta(t)} - \delta(t)\,dt + C_1, \tag{7.4.22}$$

where C_1 is arbitrary constant.

Under the integrability condition (7.2.9), equation (7.1.2) for $\mu(t) = 0$ has the solution

$$u(x,t) = \frac{\sqrt{-6\,(\beta(t))^2\,k_1'}}{\beta(t)\left(x + \int -\delta(t) + \frac{{k_2'}^2\beta(t)}{4}dt + C_1\right)} - \frac{k_2'}{2}, \tag{7.4.23}$$

where $k_1' = \frac{1}{k_1}$ and $k_2' = \frac{1}{k_2}$.

7.5 Conclusion and Remarks

In this chapter, we have considered the Painlevé property and the symmetries for the vc-Gardner equation (7.1.2) for $\mu(t) \neq 0$ and $\mu(t) = 0$. Using the standard Kruskal's simplified method, we have deduced the integrability conditions for equation (7.1.2) for $\mu(t) \neq 0$ and $\mu(t) = 0$. The symmetries of the equations are obtained by using Lie group analysis method. It is shown that vc-Gardner equation have infinite dimensional symmetry group, the Lie algebra of which involve arbitrary functions. The reduced ODEs obtained from subalgebras are presented, and the exact explicit solutions are investigated. Exact solution of equation (7.1.2) for $\mu(t) = 0$ generated from Painlevé analysis is also presented.

Remark 7.5.1 Particularly on taking $\mu(t) = \mu t^r$ in (7.4.4), we have the solution of equation (7.1.1) as $u(x,t) = C_1 \mathrm{e}^{-\mu\frac{t^{r+1}}{r+1}}$. This was the only solution derived by Liu et al [13] for equation (7.1.1) using Lie symmetries with condition $r \neq -1$. Using (7.4.4), we can also find the solution of (7.1.1) for $r = -1$ that is $u(x,t) = \frac{1}{t^\mu}$.

Remark 7.5.2 Though we keep from discussing the physical implications of the solutions reported in this chapter, yet we feel worth mentioning that the solutions obtained are such that one can choose the arbitrary functions $\alpha(t), \beta(t), \gamma(t), \lambda(t)$ and $\mu(t)$ in a suitable manner to simulate physical situations governed by the equation (7.1.2) for $\mu(t) \neq 0$ and $\mu(t) = 0$ to obtain particular solutions having desired features.

Summary

The nonlinear phenomena are encountered in a variety of situations in physics as well as in other natural and applied sciences. Most of these phenomena are governed by nonlinear PDE(s). Finding solutions of such equations is an arduous task and only in certain special cases one can write down the solutions explicitly. However, exact solutions to nonlinear PDE(s) play an important role for understanding of qualitative as well as quantitative features of many phenomena and processes. Exact solutions visually demonstrate and make it possible to understand the mechanism of complex nonlinear effects. The thesis entitled "SYMMETRIES AND EXACT SOLUTIONS OF SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS" is an attempt to obtain symmetries and the exact solutions of some nonlinear PDE(s) representing some interesting physical systems. To determine the admissible symmetries of nonlinear PDE(s), the methods- the classical Lie approach and the direct method due to Clarkson, have been utilized. Both of these methods are systematic and are easy to apply.

In general determining all the symmetries of a partial differential equation is a formidable task. However, Sophus Lie observed that if we restrict ourself to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries. In chapters 2, 3, 4, 5, 6 and 7, Lie classical method has been applied on a system under investigation and after that a formal approach of identifying an optimal system of Lie sub algebras has been adopted with help of the adjoint action of the Lie algebra.

The basic generators contained in the optimal system have then been exploited to achieve the desired reduction of PDE(s) to ODE(s). Then, the solutions of systems are obtained from reduced ODE(s) via some techniques based on special functions such as Jacobi elliptic, hyperbolic functions, etc.

The direct method to find similarity reductions is a very simple method that does not use group theory. The novel features of it are entirely straightforward without group analysis. In chapter 2, Direct method has been used to find the symmetries of modified b-family equations and b-family equations. Exact solutions are obtained from corresponding reductions.

Another important aspect that deserves special attention is to trace out the progress made in developing an approach that helps in deciding whether system under study is integrable or not. one could classify an ODE or a system of ODEs in the complex domain to be of Painlev*é* type or has Painlev*é* property (PP) if the only movable singularities of all its solutions are poles. In chapter 5, 6 and 7, We performed Painlev*é* analysis of ODE(s) or PDE(s).

We also establish the travelling wave solutions involving parameters of the coupled Higgs equations and Hamiltonian amplitude equation by using $(\frac{G'}{G})$-expansion method put forward by Wang et al. (2008). The travelling waves solutions expressed by hyperbolic, trigonometric and the rational functions are obtained. The $\tanh - sech$ rational function method is one of the most effective straightforward method to construct exact solutions. The method generates solutions in the form of the rational function of tanh and sech functions. We have find some exact solutions of b-family equations using $\tanh - sech$ rational function method.

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