ALGORITHMS FOR SOLVING DECISION-MAKING PROBLEM UNDER TYPE-2 FUZZY SETS

A Thesis

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requirement for the award of the degree

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by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Algorithms For Solving Decision-Making Problem Under Type-2 Fuzzy Sets"** in partial fulfillment of the requirement for the award of degree of Degree of Philosophy and submitted in the School of Mathematics (SoM), Thapar Institute of Engineering & Technology, Patiala is an authentic record of my own carried out during a period from July, 2015 to June, 2020 under the supervision of Dr. Harish Garg, Assistant Professor, SoM, Thapar Institute of Engineering & Technology Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Date: June 2020

Abstract

Multiple attribute decision-making (MADM) is one of the hot topics in the field of the decision-making process to access the best alternative(s) from the feasible ones. In literature, many terms have been used for MADM such as multi-criteria decision analysis (MCDA), multi-objective decision-making (MODM), multi-criteria decision-making (MCDM), etc. and have been frequently used by the researchers to solve real-world decision-making problems. Generally, the MADM issue is explained in the two-stage process: (i) the aggregation of the estimations of criteria for every option (ii) the positioning or ranking between the options. The general process of MADM problem consists a set of alternatives $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ and attributes $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m\}$ such that it is divided into two mutually disjoint sets namely, F_1(the cost attribute) and F_2(the benefit attribute). The importance factor of these criteria is given in the form of weight vector $(w_1, w_2, \ldots, w_m)$ such that $w_j > 0$ and $\sum\limits_{j=1}^{m} w_j = 1$. The decision matrix corresponding to given alternatives is given as:

$$
\mathcal{M} = \begin{array}{c c}
 & \begin{array}{cccc} \mathcal{G}_1 & \mathcal{G}_2 & \ldots & \mathcal{G}_m \end{array} \\
\begin{array}{c} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_n \end{array} &
\begin{pmatrix}
\mathcal{A}_{11} & \mathcal{A}_{12} & \ldots & \mathcal{A}_{1m} \\
\mathcal{A}_{21} & \mathcal{A}_{22} & \ldots & \mathcal{A}_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{A}_{n1} & \mathcal{A}_{n2} & \ldots & \mathcal{A}_{nm}
\end{pmatrix}
\end{array}
$$

where $\mathcal{A}_{ij}$ represents the preference of decision maker for alternative $\mathcal{A}_i (i = 1, 2, \ldots, n)$ over the criteria $\mathcal{G}_j (j = 1, 2, \ldots, m)$.

In real decision-making, the decision makers (DMs) need to give their evaluation information of attributes by various types of the evaluation process, such as crisp numbers,

interval numbers, fuzzy numbers, and so. However, in many practical cases, because of the increasing uncertainty in the data and various cognition constraints of DMs, it is often difficult for DMs to use real values to express their preferences. To ease with it, a concept of fuzzy set (FS) is introduced by Zadeh in 1965 which adopts the membership degree (MD) to describe the information. After it, various extensions of FSs come into the existence such as intuitionistic fuzzy sets (IFSs), interval-valued IFSs (IVIFSs), Type-2 fuzzy set (T2FSs), Hesitant fuzzy sets (HFSs), and so on, to deal with the uncertain and imprecise information. In the theories of FSs and its extensions, a crisp membership function is assigned to its element. However, in many situations, uncertainty is not probabilities in nature but it is imprecise or vague in nature. To address it, the concept of type-2 fuzzy set (T2FS) was developed by Mendel in 2002, an extension of FS, in which membership values are type-1 FSs on [0,1] is developed. In T2FS, there is an additional membership function which provides an additional degree of freedom to the practices to model the uncertainties and each element is characterized by the degrees of the primary, secondary and a footprint of uncertainty (FOU).

After this pioneering work, researchers have been engaged in extensions and applications to different disciplines. However, the most important task for the decision-maker is to rank the objects so as to obtain the desired object(s). For this, researchers have made efforts to enrich the concept of information measures as well as aggregation operators in type-2 fuzzy environments. Among these, an aggregation operator is an important part of the decision-making which usually takes the form of mathematical function to aggregate all the input individual data into a single one. However, an information measure such as the distance and similarity measures, complementary to each other, are defined to differentiate between the two or more objects. Thus in order to handle the information in a more accurate and certain manner, there is a need to plan/adopt suitable methodologies to solve the decision-making problems. The aim of this work is to develop some novel techniques to access the best alternative(s) for the decision makers under the T2FSs and its extensions environment.

Keywords: Type-2 fuzzy and intuitionistic fuzzy sets; Aggregation operators; Triangular interval type-2 (TIT2) intuitionistic fuzzy sets; Symmetric TIT2 intuitionistic fuzzy sets.

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And above all, I thank and pay my regards to the Almighty for his love and blessings.

(Sukhveer Singh)

Patiala

June 8, 2020

List of Publications

Refereed Journals

(J1) Harish Garg, Sukhveer Singh, Algorithm for solving group decision making problems based on the similarity measures under type-2 intuitionistic fuzzy sets environment, *Soft Computing, Springer*, 24(10), 7361 - 7381, 2020, doi: 10.1007/s00500-019-04359-8 (**SCI: Impact Factor: 2.784**).

(J2) Harish Garg, Sukhveer Singh, A novel triangular interval type-2 intuitionistic fuzzy set and their aggregation operators, *Iranian Journal of Fuzzy Systems*, 15(5), 69 - 93, 2018, doi: 10.22111/IJFS.2018.4159 (**SCI: Impact Factor: 1.496**).

(J3) Sukhveer Singh, Harish Garg, Symmetric triangular interval type-2 intuitionistic fuzzy sets with their applications in multi criteria decision making, *Symmetry*, 10(9), 401; 2018, doi: 10.3390/sym10090401 (**SCI: Impact Factor: 2.143**).

(J4) Sukhveer Singh, Harish Garg, Distance measures between type-2 intuitionistic fuzzy sets and their application to multicriteria decision-making process, *Applied Intelligence, Springer* 46 (4), 788 - 799, 2017, doi: 10.1007/s10489-016-0869-9 (**SCI: Impact Factor: 2.882**).

List of Abbreviations

AT . . . . Archimedean t-norms.

AC . . . . Archimedean t-conorms.

DMs . . . . . . Decision Makers.

MCDM . . . . Multiple Criteria Decision Making.

MADM . . . . Multi-Attribute Decision Making.

MAGDM . . . Multi Attribute Group Decision Making.

LVs . . . . Linguistic Variables.

LMF . . . . . . Lower Membership Function.

UMF . . . . . . Upper Membership Function.

PMF . . . . . . Primary Membership Function.

SMF . . . . . . Secondary Membership Function.

PNMF . . . . . Primary Non-membership Function.

SNMF . . . . . Secondary Non-membership Function.

VMF . . . . . . Variance Margin Function.

NMD . . . . . Non-membership Degree.

FOU . . . . . . Footprint of Uncertainty.

FS . . . . Fuzzy Set.

HFS . . . . . . Hesitant Fuzzy Set.

IFS . . . . Intuitionistic Fuzzy Set.

IVIFS . . . . . Interval Valued Intuitionistic Fuzzy Set.

IT2FS . . . . . Interval Type-2 Fuzzy Set.

IT2IFS . . . . Interval Type-2 Intuitionistic Fuzzy Set.

T1FS . . . . . . Type-1 Fuzzy Set.

T2FS . . . . . . Type-2 Fuzzy Set.

T2IFS . . . . . Type-2 Intuitionistic Fuzzy Set.

TIT2FS . . . . Triangular Interval Type-2 Fuzzy Set.

TrIT2FS . . . Trapezoidal interval type-2 Fuzzy Set.

IFN . . . . . . Intuitionistic Fuzzy Number.

IT2FN . . . . . Interval Type-2 Fuzzy Number.

IVIFN . . . . . Interval Value Intuitionistic Fuzzy Number.

LIFNs . . . . . Linguistic Intuitionistic Fuzzy Number.

TIT2IFS . . . Triangular Interval Type-2 Intuitionistic Fuzzy Set.

STIT2FS . . . Symmetric Triangular Interval Type-2 Fuzzy Set.

STIT2IFS . . . Symmetric Triangular Interval Type-2 Intuitionistic Fuzzy Set.

T2FN . . . . . Type-2 Fuzzy Number.

T2IFN . . . . . Type-2 Intuitionistic Fuzzy Number.

TIT2FN . . . . Triangular Interval Type-2 Fuzzy Number.

TrIT2FN . . . Trapezoidal Interval Type-2 Fuzzy Number.

TIT2IFN . . . Triangular Interval Type-2 Intuitionistic Fuzzy Number.

AO . . . . Aggregation Operator.

HM . . . . Hamy Mean.

MM . . . . . . Muirhead Mean.

BM . . . . Benferroni Mean.

GBM . . . . . Generalized Benferroni Mean.

MSM . . . . . Maclaurin Symmetric Mean.

TOPSIS . . . . Technique for Order Preference by Similarity to Ideal Solution.

VIKOR . . . . VlseKriterijumska Optimizacijia I Kompromisno Resenje.

COPRAS . . . COmplex PRoportional Assessment.

AHP . . . . . . Analytic Hierarchy Process.

FAHP . . . . . Fuzzy Analytic Hierarchy Process.

ELECTRE . . Elimination Et Choice Translating REality.

ERP . . . . . . Enterprise Resource Planning.

EIS . . . . Enterprise Information System.

OWA . . . . . . Ordered Weighted Average.

QUALIFLEX QUALItative FLEXible.

IFWA . . . . . Intuitionistic Fuzzy Weighted Average.

IFWG . . . . . Intuitionistic Fuzzy Weighted Geometric.

IT2FWGBM . Interval Type-2 Fuzzy Weighted Geometric Bonferroni Mean.

STIT2HM . . Symmetric Triangular Interval Type-2 Fuzzy Hamy Mean.

STIT2FN . . . Symmetric Triangular Interval Type-2 Fuzzy Number.

STIT2IFN . . Symmetric Triangular Interval Type-2 Intuitionistic Fuzzy Number.

STIT2IHM . . Symmetric Triangular Interval Type-2 Intuitionistic Fuzzy Hamy Mean.

TIT2IFWA . . Triangular Interval Type-2 Intuitionistic Fuzzy Weighted Averaging.

TIT2IFOWA . Triangular Interval Type-2 Intuitionistic Fuzzy Ordered Weighted Averaging.

TIT2IFHA . . Triangular Interval Type-2 Intuitionistic Fuzzy Hybrid Averaging.

WSTIT2FHM Weighted Symmetric Triangular Interval Type-2 Fuzzy Hamy Mean.

WSTIT2IFHM Weighted Symmetric Triangular Interval Type-2 Intuitionistic Fuzzy Hamy Mean.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Multiple criteria decision-making (MCDM) is emerging area in the decision making process to find the finest alternative(s) from the given ones. In literature, many terms have been used for MCDM such as multi-criteria decision analysis (MCDA), multi-objective decision-making (MODM), multi-attribute decision-making (MADM), etc. and have been frequently used by the researchers to solve real-world decision-making problems. Generally, MCDM issue is explained in the two-stage process: (i) the aggregation of the estimations of criteria for every option (ii) the positioning or ranking between the options.

The general process of MCDM problem consists a set of alternatives $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ and criteria $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m\}$ such that it is divided into two mutually disjoint sets namely, F_1(the cost criteria) and F_2(the benefit criteria). The importance factor of these criteria is given in the form of weight vector $(w_1, w_2, \ldots, w_m)$ such that $w_j > 0$ and $\sum\limits_{j=1}^{m} w_j = 1$. The decision matrix corresponding to given alternatives is given as:

$$\begin{array}{c c}
 & \begin{array}{cccc} \mathcal{G}_1 & \mathcal{G}_2 & \ldots & \mathcal{G}_m \end{array} \\
\begin{array}{c} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_n \end{array} & \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \ldots & \mathcal{A}_{1m} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \ldots & \mathcal{A}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{n1} & \mathcal{A}_{n2} & \ldots & \mathcal{A}_{nm} \end{pmatrix}
\end{array}$$

where $\mathcal{A}_{ij}$ represents the preference of decision maker for alternative $\mathcal{A}_i (i = 1, 2, \ldots, n)$ over the criteria $\mathcal{G}_j (j = 1, 2, \ldots, m)$.

To evaluate the given alternatives, decision makers (DMs) may choose the crisp number to rate them. However, under the uncertainty environment, it is often difficult to access these accurately. Thus, a concept of fuzzy sets (FSs) [166] is utilized by assigning a membership degree (MD) to each element. After it, a concept of intuitionistic fuzzy sets (IFSs) [7] is given by adding the non-membership degree (NMD) along with MD into the analysis. Later on, several extension of the FSs such as Type-2 fuzzy set (T2FS) [98], interval Type-2 fuzzy set (IT2FS) [99], hesitant fuzzy set (HFS) [140], Neutrosophic set (NS) [132], the interval-valued IFS (IVIFS) [6] and so on appears simultaneously, to handle the uncertainties.

In the theories of FSs and its extensions, a crisp membership function is assigned to its element. However, in many situations, uncertainty is not probabilities in nature but it is imprecise or vague in nature. To address it, the concept of Type-2 fuzzy set (T2FS) [98], an extension of FS, in which membership values are type-1 FSs on [0,1] is developed. In T2FS, there is an additional membership function which provides an additional degree of freedom to the practices to model the uncertainties and each element is characterized by the degrees of the primary, secondary and a footprint of uncertainty (FOU). The aim of this work is to develop some novel techniques to access the best alternative(s) for the decision makers under the T2FSs and its extensions environment. Brief literature on various issues related to the MCDM process by using existing methods have been reviewed and are given section-wise hereafter.

1.1 Literature Review

In this section, a brief literature review with respect to various MCDM methods under different scenario is given.

1.1.1 Review of distance/similarity measures

Distance and similarity measures, complementary to each other, are defined to differentiate between the two or more objects. These two measures can be considered as two diverse perspectives of discrimination. The similarity measure utilized to show the proximity whereas the distance measure is utilized to show the contrast between the objects.

Since the distance and similarity measures play a significant role in real life, numerous analysts have been defined by various distance and similarity measures in different environments. For instance, Szmidt and Kacprzyk [137] defined the Hamming and Euclidean distance measures to distinguish between the two IFSs. Grzegorzewski [63] proposed some distance measures based on the Hausdorff metric. Dengfeng and Chuntian [32] introduced the concept of similarity measure to find the closeness degree between the two IFSs. Liang and Shi [91] presented an improved similarity measure for different IFSs. Hung and Yang [72] presented the similarity measures based on Hausdorff distance. Szmidt and Kacprzyk [139] proposed a similarity measure and discussed its importance in medical diagnosis. Xia and Xu [156] developed the aggregation operator (AO) for IFSs based on the similarity measures. Ye [164] highlighted the errors of existing similarity measure [90] and constructed the cosine similarity, weighted cosine similarity measure to solve pattern recognition and medical diagnosis problems. Besides that, Hwang et al. [74] introduced a new similarity measure induced by Sugeno integral. Based on the medians of intervals and the Hausdorff distance, Chen and Randyanto [16] discussed a similarity measure for IFSs. Later on, Boran and Akay [9] proposed a parametric similarity measure for IFSs and did a comparative analysis with some existing similarity measures[32, 91, 107, 164] in terms of counter-intuitive cases. Song et al. [133] discussed the similarity measure and applied it to the problem of pattern recognition. Chen et al. [13] presented a similarity measure based on the centroid points of transformed right-angled triangular to overcome the limitations of existing measures [9, 32, 72, 91]. Garg [42] characterized the distance and similarity measures for intuitionistic multiplicative preference relation and its application in DM issues. Later on, Song et al. [134] proposed a similarity measure to overcome the restrictions of existing measures [9, 32, 72, 74, 164]. Recently, Luo and Zhao [96] defined a distance measure based on a matrix norm and a binary function which is either strictly increasing or decreasing and discussed their application in DM issues. From the interval point of view, Ke et al. [83] described a new distance measure for IFSs. Jiang et al. [77] defined distance and similarity measure based on the transformed triangles and discussed their application in pattern recognition. Rani and Garg [125, 126] presented some series of distance measures and power aggregation operators for complex IFSs.

Figueroa-García et al. [36] presented a distance measures for IT2FNs. Hao and Mendel [65] presented a similarity measure for T2FSs based on α-plane representation. Heidarzade et al. [67] presented a hierarchical clustering method based on the distance measures for IT2FNs to solve a supplier selection problem. Hu et al. [69] presented a possibility degree measures for IT2FNs to solve the MCDM problems. Hung and Yang [71] presented the similarity measures between T2FSs. Hwang et al. [73] presented several information measures such as similarity, inclusion, entropy between the T2FSs based on Sugeno integral to solve the decision-making problems. Singh [131] presented a similarity measures for T2FSs. Singh [130] presented some series of distance measures for T2FSs and then solve the group decision-making problems. Apart from them, some other studies related to information measures under IFSs, T2FS or IT2FS environment are conducted which are summarized in [1, 3, 52, 60, 93, 142, 152–155, 163].

1.1.2 Review of Aggregation operators

The aggregation operators (AOs) is one of the most collective phase and process during ordering the alternatives. The basic principle of AOs is to aggregate the various values into the collective once. Since in MADM process, there always occur more than one attribute values towards a single alternative and hence the process of AOs play a significant role between them. In terms of IFS, Atanassov [8], De et al. [31] presented the basic operational laws such as addition, scalar multiplication, power for IFSs. Based on these operational laws, Xu [159], Xu and Yager [161] presented the weighted, ordered weighted, and the hybrid weighted averaging and geometric AOs for different pairs of intuitionistic fuzzy numbers (IFNs). Later on, Zhao et al. [174] extended these AOs into its generalized form. However, apart from them, some other AOs by using some other families of t-norms such as Einstein [146, 148, 176], Hamacher [45, 70] are proposed by the researchers for a different IFNs. Besides that, Wei [149], Xu and Xia [162] defined some induced AOs for aggregating the IFNs. Garg [39], Xia et al. [157] defined the weighted averaging and geometric AOs for the intuitionistic multiplicative environment instead of additive environment. Apart from them, some other AOs for solving the decision-making problems are presented under the IFS environment are summarized in [43, 48–51, 57, 94, 146, 159, 161].

However, under the T2FSs and its extensions, various authors have investigated the problems of the DM under the T2FSs environment by using different aggregation operators. For instance, [22] presented an ELECTRE-based method to solve the MCDM problem under IT2FSs. Qin [118] presented the concept of the symmetric triangular interval Type-2 fuzzy numbers (IT2FNs) and hence based on the Hamy mean, some AOs are developed to solve the MCDM problems. Wang, Ju, Liu, Ju and Liu [141] presented a trapezoidal interval Type-2 (TrIT2) fuzzy Heronian mean AOs to solve the multi attribute group decision making (MAGDM) problems. Zhang [172] presented the arithmetic operations between TrIT2 fuzzy sets and based on it they developed some AOs for aggregating trapezoidal IT2FSs. Zhang and Zhang [173] presented the notion of the trapezoidal interval Type-2 fuzzy soft sets. Qin and Liu [120] presented an AO for triangular IT2FS by using Frank norm operations and applied them to solve the decision-making problems. Chiclana and Zhou [29] gave a method which involves T1FS OWA to specify the centroid of the IT2FS. Qin and Liu [122] presented some triangle interval Type-2 fuzzy Frank (TIT2FF) aggregation operators namely the TIT2FF weighted averaging and geometric operator and engaged them in solving the MAGDM problems. Chen [24] used interval Type-2 trapezoidal and Generalized interval-valued Trapezoidal fuzzy numbers to develop weighted geometric AOs along with its applications in MCDM problems on the basis of trapezoidal IT2FNs. On the other hand, [145] used aggregation operators for solving MAGDM problems on the basis of IT2FNs and arithmetic operations. Zhang [172] developed trapezoidal interval Type-2 fuzzy aggregation operators for aggregating trapezoidal IT2FNs. Wang, Ju, Liu, Ju and Liu [141] extended Heronian mean to TIT2F environment Other than these, some others kinds of the methods for solving the decision making problems by using AOs are summarized in [28, 34, 46, 75, 88, 100–105, 114, 127, 128, 147, 175].

1.1.3 Review of TOPSIS and ranking approaches

Chen and Lee [14] developed an IT2FS-TOPSIS method and gave some examples of how to tackle fuzzy MAGDM problems on the basis of IT2FSs. Prior to this, Chen and Lee [15] established a new method in accordance with the arithmetic operations and the ranking values of IT2FSs to cope up with fuzzy MAGDM problems. Zhang and Zhang [173] developed a novel approach to MAGDM in context of IT2FSs environment. Abdullah and Otheman [2] explained IT2F-TOPSIS approach by attaching entropy weight for sub-criteria and illustrated it through an example of supplier selection. Chen [19] constructed decision-making model covering the closeness coefficient approach and signed distance-based operation to solve the MAGDM problems. On the basis of signed distances, Chen [20] proposed an IT2F linear assignment method for analyzing the MCDM problems. The feasibility of the defined method is illustrated through its application in the selection of landfill site. Moreover, an extended QUALIFLEX method suggested by Chen et al. [25] is apt at handling MCDM problems with respect to IT2FSs. Chen [22] developed an ELECTRE ("ELimination Et Choice Translating REality") based outranking method for MAGDM within IT2FSs framework. Further, Ghorabaee et al. [61] presented a ranking method together with the COPRAS method for solving fuzzy MAGDM supplier selection problem in the context of IT2FSs. Kahraman et al. [79] illustrated a supplier selection problem through developed IT2F AHP method along with a new ranking method for T2FSs. Wang and Chen [143] used a closeness coefficient based approach to developing an IT2F MCDM method in the context of IT2FSs along with its application in watershed site selection. Furthermore, Oztaysi [113] presented AHP and IT2FSs group decision-making approach on an ERP ("Enterprise Resource Planning") selection problem which has been a governing EIS ("Enterprise Information Systems") application with six criteria and four alternatives. Qin and Liu [122] investigated a method on the basis of combined ranking value to deal with MAGDM problems under IT2F environment. Çebi and Otay [12] introduced a TOPSIS method to solve the MADM problem with IT2FSs information. Chen [23] presented an approach to solving the decision-making problems under the interval Type-2 trapezoidal fuzzy environment. Erdogan and Kaya [35] presented a Fuzzy Analytic Hierarchy Process (FAHP) based on IT2FSs to obtain the weight of the criteria. Further, to rank the alternative, a TOPSIS method is presented. [123] extended the VIKOR ("VlseKriterijumska Optimizacijia I Kompromisno Resenje, in Serbian") method based on the prospect theory to accommodate interval Type-2 fuzzy numbers. Apart from these, some other studies under different environment are conducted which are summarized in Akram et al. [4], Ashraf et al. [5], Chen [21], Cheng et al. [27], Das et al. [30], Garg and Kumar [56, 58], Gong et al. [62], Jana et al. [76], Kar et al. [80], Kumar and Garg [85, 86], Kundu et al. [87], Lee and Chen [89], Liu and Wang [95], Manna et al. [97], Muhuri et al. [109], Nancy and Garg [111], Own [112], Panja and Mondal [115], Pedrycz and Song [116], Pramanik et al. [117], Wang et al. [144], Wu and Mendel [151], Zamri et al. [168].

1.2 Research Motivation

In the previous sections, we presented a concise literature review of measures and operators under IFS, IVFS, IVIFS, and T2FS environment. For solving the decision-making problems, the main important task for the decision-maker(s) is to choose the appropriate method according to the nature of the problem. Furthermore, in the case of group decision makers', an ambiguity may occur between them to choose the appropriate method and which can lead to extra expenditure of time, resources, and money. But the main important thing of a good DM procedure is to optimize the time and money. That's why there is a need to research the appropriate technique to handle such problems for obtaining a better result. From the existing work described in the above sections, we found they have some gaps in DM procedure given as:

1) As all the existing works have been examined under the T2FSs environment by considering only the degree of membership during an analysis. But, in the real-life situation, it is not possible to make a decision without considering the degree of non-membership (also called as a dissatisfactory degree), as it is difficult for the person to give their preferences towards an object in terms of a single or exact number. Thus, for handling it, there is a need for the degree of non-membership into the analysis.

2) All aforementioned AOs are usually based on the algebraic norm operations, which have the lack of flexibility and robustness. Thus, there is need to address such issue also.

3) In the comprehensive literature review, researchers light upon Bonferroni mean (BM) and generalized BM (GBM) based operators by adding features to the interconnection among the multi-input parameters. But, it has been noted that they considered only

two or three multi-parameters simultaneously. Therefore, these operators are incapable of analyzing the effect of the multi-input arguments into one analysis.

1.3 Objective of the thesis

The overall aim of this research is to develop the various MADM or MAGDM methods under the different and uncertain fuzzy environment under the Type-2 fuzzy sets and its extensions. For it, we develop several approaches based on the aggregation operators and the information measures to solve the decision-making problems. By motivating from the above literature and gaps, the main objectives of the work are summarized as:

(O1) To develop some information measures in Type-2 fuzzy sets.

(O2) To develop some decision-making aggregation operators under Type-2 fuzzy sets.

(O3) To test and validate the proposed technique on some decision making problems in some fields.

1.4 Structure of the thesis

The entire thesis has been organized into six chapters which are briefly summarized as follows:

A brief account of the related work of various authors in the evaluation of decision making approaches by using several approaches is presented in the first chapter.

In **Chapter 2**, the basics and preliminaries related to the intuitionistic fuzzy sets, information measures, aggregation operators, Type-2 fuzzy sets, etc., are given.

In **Chapter 3**, we presented the novel concept of the Type-2 intuitionistic fuzzy sets (T2IFSs) by embedding the features of the non-membership and the hesitancy degrees into the Type-2 fuzzy sets. Further, to explore the study of T2IFSs, we formulated a series of distance measures between two T2IFSs by using geometric and Hausdorff metrics. Also, several desirable relations between them are defined. Later, we present an efficient method based on the developed distance measures to solve the MADM problems under the T2IFSs environment where the information related to each object is taken in the form of Type-2 intuitionistic fuzzy numbers (T2IFNs). The method is explained with a practical example and compared their results with the existing studies.

Chapter 4 explore the theory of Type-2 intuitionistic fuzzy set (T2IFS) with similarity measures in which the membership degrees for each member of the object is itself an IFS. Keeping the features of T2IFS and the similarity measures, it is vital to amplify some new similarity measures to process the T2IFSs in MAGDM methods. Since in real life, most of the decision-making problems are always uncertain in nature and difficult to represent it in terms of crisp or precise numbers. Motivated from it, this chapter presents some new similarity measures based on the geometric model and some new extensions of it to compute the degree of similarity between the two or more T2IFSs. Further, to explore the MAGDM problems, a TOPSIS approach based on the similarity measure is addressed to find the finest alternative. The working of the proposed measure is demonstrated through an illustrative example.

In **Chapter 5**, we present a new idea about the triangular interval Type-2 (TIT2) intuitionistic fuzzy sets and studied their several properties. Some basic operational laws, as well as the relation between them by using Frank t-norm and t-conorm operations, are defined. The Frank t-norm operation has an additional parameter which can give a flexible environment to the decision makers to chose their decisions, according to their desired goals. Based on these Frank operations, some series of weighted averaging AOs are defined namely, TIT2 intuitionistic fuzzy weighted averaging, TIT2 intuitionistic fuzzy ordered weighted averaging and TIT2 intuitionistic fuzzy hybrid averaging. The characteristics of these operators and the influence of the Frank parameter have been discussed. Later, a novel model based on developed operators is presented to solve the MADM problems and explained them with the help of a numerical example. Finally, comparative studies with some of the existing methods are discussed.

Chapter 6 we developed the concept of the Symmetric triangular interval Type-2 intuitionistic fuzzy sets (STIT2IFSs) by taking the features of T2IFSs and the symmetric triangular number and studied their desired properties. In a practical decision-making process, there always occurs an inter-relationship among the multi-input arguments. To address it, Hamy means (HM) operator is a standout among the most critical operators that catches the inter-relationship together with the multi-input arguments. Motivated by these primary characteristics, it is interested to extend HM operator to the STIT2IFS and hence defined some new interval Type-2 (IT2) intuitionistic fuzzy aggregation operators, named as symmetric TIT2 intuitionistic fuzzy HM operator and weighted symmetric TIT2 intuitionistic fuzzy HM operator, which can consider the multi interaction between the input argument under a provision of Type-2 intuitionistic uncertain situation. Later, we develop a method to solve the decision-making problem and illustrate with a numerical number to exemplify the practicability of the proposed technique.

Chapter 2

Preliminaries

In this chapter, we present the basic concepts and the mathematical structure related to IFSs, T2FSs, aggregation operators, etc., over the universal set $\mathcal{X}$.

2.1 Intuitionistic fuzzy sets and its extensions

Definition 2.1.1. [7] An IFS $\mathcal{A}$ is given as

$$\mathcal{A} = \Big\{(x, u_{\mathcal{A}}(x), v_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\Big\}, \tag{2.1}$$

where, $u_{\mathcal{A}}(x), v_{\mathcal{A}}(x) \in [0,1]$, represents MD and NMD of x with $0 \leq u_{\mathcal{A}}(x) + v_{\mathcal{A}}(x) \leq 1$ holds for $\forall x$. We call, this pair $(u_{\mathcal{A}}, v_{\mathcal{A}})$ as an IFN [161].

Definition 2.1.2. [7] For any two IFNs $\mathcal{A} = (u_{\mathcal{A}}, v_{\mathcal{A}})$ and $\mathcal{B} = (u_{\mathcal{B}}, v_{\mathcal{B}})$, we have

(i) $\mathcal{A} \subseteq \mathcal{B}$ if $u_{\mathcal{A}} \leq u_{\mathcal{B}}$ and $v_{\mathcal{A}} \geq v_{\mathcal{B}}$.

(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \supseteq \mathcal{B}$.

(iii) $\mathcal{A}^c = (v_{\mathcal{A}}, u_{\mathcal{A}})$.

(iv) $\mathcal{A} \cup \mathcal{B} = (\max\{u_{\mathcal{A}}, u_{\mathcal{B}}\}, \min\{v_{\mathcal{A}}, v_{\mathcal{B}}\})$.

(v) $\mathcal{A} \cap \mathcal{B} = (\min\{u_{\mathcal{A}}, u_{\mathcal{B}}\}, \max\{v_{\mathcal{A}}, v_{\mathcal{B}}\})$.

Definition 2.1.3. [161] The score function for an IFN $\mathcal{A} = (u_{\mathcal{A}}, v_{\mathcal{A}})$ is defined as :

$$\mathcal{S}(\mathcal{A}) = u_{\mathcal{A}} - v_{\mathcal{A}}, \tag{2.2}$$

and accuracy function is defined as:

$$\mathcal{H}(\mathcal{A}) = u_{\mathcal{A}} + v_{\mathcal{A}} \tag{2.3}$$

Definition 2.1.4. [6] An IVIFS $\mathcal{A}$ in $\mathcal{X}$ is defined as

$$\mathcal{A} = \{(x, \widetilde{u}_{\mathcal{A}}(x), \widetilde{v}_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\}, \tag{2.4}$$

where $\widetilde{u}_{\mathcal{A}}(x) = [u^L_{\mathcal{A}}(x), u^U_{\mathcal{A}}(x)]$ and $\widetilde{v}_{\mathcal{A}}(x) = [v^L_{\mathcal{A}}(x), v^U_{\mathcal{A}}(x)]$ are all subsets of $[0,1]$, and represents the MDs and NMDs of x to $\mathcal{A}$ such that for any $x \in \mathcal{X}, 0 \leq u^U_{\mathcal{A}}(x) + v^U_{\mathcal{A}}(x) \leq 1$. The pair $\mathcal{A} = ([u^L, u^U], [v^L, v^U])$ is called as IVIFN.

Definition 2.1.5. Let $\mathcal{A} = ([u^L_{\mathcal{A}}, u^U_{\mathcal{A}}], [v^L_{\mathcal{A}}, v^U_{\mathcal{A}}])$ and $\mathcal{B} = ([u^L_{\mathcal{B}}, u^U_{\mathcal{B}}], [v^L_{\mathcal{B}}, v^U_{\mathcal{B}}])$ be any two IVIFNs, then

(i) $\mathcal{A} \subseteq \mathcal{B}$ if $u^L_{\mathcal{A}} \leq u^L_{\mathcal{B}}, u^U_{\mathcal{A}} \leq u^U_{\mathcal{B}}, v^L_{\mathcal{A}} \geq v^L_{\mathcal{B}}$ and $v^U_{\mathcal{A}} \geq v^U_{\mathcal{B}}$.

(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \supseteq \mathcal{B}$.

(iii) $\mathcal{A}^c = ([v^L_{\mathcal{A}}, v^U_{\mathcal{A}}], [u^L_{\mathcal{A}}, u^U_{\mathcal{A}}])$.

(iv) $\mathcal{A} \cup \mathcal{B} = ([\max\{u^L_{\mathcal{A}}, u^L_{\mathcal{B}}\}, \max\{u^U_{\mathcal{A}}, u^U_{\mathcal{B}}\}], [\min\{v^L_{\mathcal{A}}, v^L_{\mathcal{B}}\}, \min\{v^U_{\mathcal{A}}, v^U_{\mathcal{B}}\}])$;

(v) $\mathcal{A} \cap \mathcal{B} = ([\min\{u^L_{\mathcal{A}}, u^L_{\mathcal{B}}\}, \min\{u^U_{\mathcal{A}}, u^U_{\mathcal{B}}\}], [\max\{v^L_{\mathcal{A}}, v^L_{\mathcal{B}}\}, \max\{v^U_{\mathcal{A}}, v^U_{\mathcal{B}}\}])$.

Definition 2.1.6. [160] The score function for an IVIFN $\mathcal{A} = ([u^L_{\mathcal{A}}, u^U_{\mathcal{A}}], [v^L_{\mathcal{A}}, v^U_{\mathcal{A}}])$ is defined as:

$$\mathcal{S}(\mathcal{A}) = \frac{u^L_{\mathcal{A}} + u^U_{\mathcal{A}} - v^L_{\mathcal{A}} - v^U_{\mathcal{A}}}{2}, \tag{2.5}$$

and accuracy function is

$$\mathcal{H}(\mathcal{A}) = \frac{u^L_{\mathcal{A}} + u^U_{\mathcal{A}} + v^L_{\mathcal{A}} + v^U_{\mathcal{A}}}{2}. \tag{2.6}$$

To order the different IFNs and/or IVIFNs, a comparison law between them is defined as follow.

Definition 2.1.7. [159] Let $\mathcal{A}$ and $\mathcal{B}$ be either two IFNs and/or IVIFNs then based on the functions as defined in Definitions 2.1.3 and 2.1.6, an order relation $A \succeq B$, where "$\succeq$" refer "preferred to", occurs if anyone of the following condition met.

(i) If $\mathcal{S}(\mathcal{A}) \geq \mathcal{S}(\mathcal{B})$.

(ii) $\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{B})$ and $\mathcal{H}(\mathcal{A}) \geq \mathcal{H}(\mathcal{B})$.

2.2 Type-2 fuzzy sets

Definition 2.2.1. [100] A T2FS $\mathcal{A}$ in the universe of discourse $\mathcal{X}$, is characterized by a type-2 membership function $\mu_{\mathcal{A}}(x,u)$ and is defined as follows:

$$\mathcal{A} = \big\{((x,u), \mu_{\mathcal{A}}(x,u)) \mid x \in X, u \in j_x \subseteq [0,1]\big\}$$

in which $0 \le \mu_{\mathcal{A}}(x,u) \le 1$. Moreover, $\mathcal{A}$ can also be expressed as

$$\mathcal{A} = \int_{x\in X} \mu_{\mathcal{A}}(x)/x = \int_{x\in X} \left[\int_{u\in j_x} f_x(u)/u\right]/x,$$

where $\mu_{\mathcal{A}}(x) = \int_{x\in j_x} f_x(u)/u$ is the grade of the membership, f_x is named as a secondary membership function (SMF) and the value of $f_x(u)$ is named as secondary grade or secondary membership. In addition, u is an argument of the SMF and j_x is named as the primary membership function (PMF) of x.

Definition 2.2.2. (Interval type-2 fuzzy set) [18] An interval T2FS is one in which the membership grade of every domain point is a crisp set whose domain is some interval contained in [0,1].

Definition 2.2.3. (Footprint of uncertainty) [18] Uncertainty in the primary memberships of a T2FS consists of boundary region that we call the "footprint of the uncertainty" (FOU). Mathematically, it is the union of all primary membership functions, i.e. $FOU(\mathcal{A}) = \cup_{x\in X} j_x$.

Let $\mathcal{F}_2(\mathcal{X})$ be the class of all T2FSs in $\mathcal{X}$, $\mu_{\mathcal{A}}(x,u) : (x,u) \to [0,1]$ is the membership function of $\mathcal{A}$ in $\mathcal{F}_2(\mathcal{X})$, for all $x \in \mathcal{X}$, $u \in j_x \subseteq [0,1]$. For any $x \in \mathcal{X}$, $f_x(u)$ is defined as $\mu_{\mathcal{A}}(x,u)$, $\forall\ u \in j_x \subseteq [0,1]$ is called a secondary membership functions, where u is the primary membership function.

Definition 2.2.4. The variance margin function of a T2FS is defined as the difference between PMF and SMF. It is denoted by ξ. For a T2FS $\mathcal{A}$, $\xi_{\mathcal{A}} = \mid u_{\mathcal{A}}(x_i) - f_{x_i}(u_{\mathcal{A}}) \mid$ for all $x_i \in \mathcal{X}$.

2.3 Distance measures between T2FSs

A distance measure between T2FSs is defined by considering FOU of the primary membership and secondary membership function, as well as the variance margin function as below.

Definition 2.3.1. A real function $d : \mathcal{F}_2(\mathcal{X}) \times \mathcal{F}_2(\mathcal{X}) \to [0, 1]$ is called a distance measure, where d satisfies the following axioms for $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \in \mathcal{F}_2(\mathcal{X})$

(P1) $0 \leq d(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) If $\mathcal{A}_1 = \mathcal{A}_2$, then $d(\mathcal{A}_1, \mathcal{A}_2) = 0$.

(P3) $d(\mathcal{A}_1, \mathcal{A}_2) = d(\mathcal{A}_2, \mathcal{A}_1)$.

(P4) If $d(\mathcal{A}_1, \mathcal{A}_2) = 0,\ d(\mathcal{A}_1, \mathcal{A}_3) = 0$, then $d(\mathcal{A}_2, \mathcal{A}_3) = 0$.

To compute the distance measures between the T2FSs, Singh [130] presented some distance measures on it. To do so, for two T2FSs $\mathcal{A}$ and $\mathcal{B}$, the SMFs is denoted by $f_x(u_{\mathcal{A}}) = \mu_{\mathcal{A}}(x, u_{\mathcal{A}})$, $g_x(u_{\mathcal{B}}) = \mu_{\mathcal{B}}(x, u_{\mathcal{B}})$, $u_{\mathcal{A}}$, $u_{\mathcal{B}}$ denotes the PMF and FOU($\mathcal{A}$), FOU($\mathcal{B}$) represents the footprints of uncertainty for the T2FSs $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.3.2. [130] The distance measures between two T2FSs are defined based on the FOU of the PMFs and SMFs, as well as the variance margin function. The following distance measures between T2FSs $\mathcal{A}$ and $\mathcal{B}$ are defined as

i) The normalized distance,

$$h_2(\mathcal{A}, \mathcal{B}) = \frac{1}{2n} \sum_{i=1}^{n} \begin{pmatrix} |u_{\mathcal{A}}(x_i) - u_{\mathcal{B}}(x_i)| + |f_{x_i}(u_{\mathcal{A}}) - g_{x_i}(u_{\mathcal{B}})| \\ + |\xi_{\mathcal{A}}(x_i) - \xi_{\mathcal{B}}(x_i)| \end{pmatrix} \tag{2.7}$$

ii) The normalized weighted distance,

$$h_{2w}(\mathcal{A}, \mathcal{B}) = \frac{1}{2n} \sum_{i=1}^{n} w_i \begin{pmatrix} |u_{\mathcal{A}}(x_i) - u_{\mathcal{B}}(x_i)| + |f_{x_i}(u_{\mathcal{A}}) - g_{x_i}(u_{\mathcal{B}})| \\ + |\xi_{\mathcal{A}}(x_i) - \xi_{\mathcal{B}}(x_i)| \end{pmatrix} \tag{2.8}$$

iii) The normalized Euclidean distance

$$e_2(\mathcal{A},\mathcal{B}) = \left\{ \frac{1}{2n}\sum_{i=1}^{n} \left(\begin{array}{c} \mid u_{\mathcal{A}}(x_i) - u_{\mathcal{B}}(x_i) \mid^2 + \mid f_{x_i}(u_{\mathcal{A}}) - g_{x_i}(u_{\mathcal{B}}) \mid^2 \\ + \mid \xi_{\mathcal{A}}(x_i) - \xi_{\mathcal{B}}(x_i) \mid^2 \end{array} \right) \right\}^{\frac{1}{2}} \quad (2.9)$$

iv) The normalized weighted Euclidean distance

$$e_{2w}(\mathcal{A},\mathcal{B}) = \left\{ \frac{1}{2n}\sum_{i=1}^{n} w_i \left(\begin{array}{c} |u_{\mathcal{A}}(x_i) - u_{\mathcal{B}}(x_i)|^2 + |f_{x_i}(u_{\mathcal{A}}) - g_{x_i}(u_{\mathcal{B}})|^2 \\ + |\xi_{\mathcal{A}}(x_i) - \xi_{\mathcal{B}}(x_i)|^2 \end{array} \right) \right\}^{\frac{1}{2}} \quad (2.10)$$

v) Utmost normalized Hamming distance

$$h_{2U}(\mathcal{A},\mathcal{B}) = \frac{1}{2n}\sum_{i=1}^{n} \max \left(\begin{array}{c} |u_A(x_i) - u_B(x_i)|, |f_{x_i}(u_A) - g_{x_i}(u_B)|, \\ |\xi_A(x_i) - \xi_B(x_i)| \end{array} \right) \quad (2.11)$$

2.4 Triangular interval type-2 fuzzy set

The operations on the T2FSs are very complex and hence cannot be applied in a real-life situations. For it, the interval T2FSs [28] are defined as follows.

Definition 2.4.1. [28] A T2FS transform into interval T2FS (IT2FS) when the grades of all SMFs is equal to 1. Mathematically, an IT2FS $\mathcal{A}$, with a membership function $\mu_{\mathcal{A}}(x, u_{\mathcal{A}})$, may be expressed either as Eq. (2.12) or as Eq. (2.13):

$$\mathcal{A} = \{(x, u_{\mathcal{A}}), \mu_{\mathcal{A}}(x, u_{\mathcal{A}}) = 1 \mid \forall x \in \mathcal{X}, \forall u_{\mathcal{A}} \in j_x \subseteq [0,1]\} \quad (2.12)$$

$$\mathcal{A} = \int_{x\in\mathcal{X}} \int_{u_{\mathcal{A}}\in j_x} 1/(x, u_{\mathcal{A}}), j_x \subseteq [0,1] \quad (2.13)$$

Definition 2.4.2. [28] An IT2FS is normally described by a zone called as FOU, which is limited by two membership functions (MFs), known as lower MF (LMF) $\underline{\mu}_{\mathcal{A}}(x, u_{\mathcal{A}})$ and the upper MF (UMF) $\overline{\mu}_{\mathcal{A}}(x, u_{\mathcal{A}})$. That is FOU=$[\underline{\mu}_{\mathcal{A}}(x, u_{\mathcal{A}}), \overline{\mu}_{\mathcal{A}}(x, u_{\mathcal{A}})]$. Figure 2.1 shows the graphical representation of IT2 fuzzy number (IT2FN) with triangular MF shape.

Definition 2.4.3. [28] Let $\mathcal{A} = \langle [a, b], c, [d, e] \rangle$ be a triangular interval T2FS (TIT2FS) defined on $\mathcal{X}$ shown in Fig. 2.2, where a, b, c, d, e are reference points of the set, satisfying

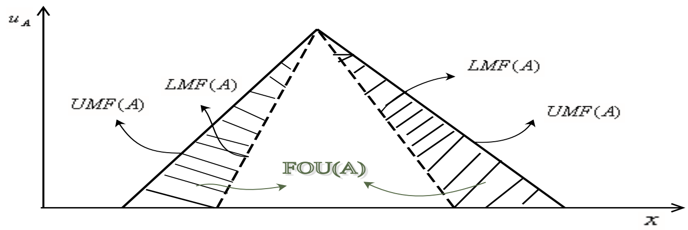

Figure 2.1: Representation of a IT2FS $\mathcal{A}$ with LMF (dashed), UMF (solid), FOU (shaded)

$0 \leq a \leq b \leq c \leq d \leq e \leq 1$. The UMF and LMF of $\mathcal{A}$ are defined as

$$\text{UMF}_{\mathcal{A}}(x) = \begin{cases} \frac{x-a}{c-a} & ; \quad a \leq x < c \\ 1 & ; \quad x = c \\ \frac{e-x}{e-c} & ; \quad c \leq x < e \end{cases} ; \qquad \text{LMF}_{\mathcal{A}}(x) = \begin{cases} \frac{x-b}{c-b} & ; \quad b \leq x < c \\ 1 & ; \quad x = c \\ \frac{d-x}{d-c} & ; \quad c \leq x < d \end{cases}$$

The FOU of $\mathcal{A}$ is depicted as a shaded portion in Fig. 2.2. If $\mathcal{X}$ is a set consists of all real numbers, then a TIT2FS in $\mathcal{X}$ is called TIT2FN ("triangular interval type-2 fuzzy number"). If $a = b, d = e$ then $\text{UMF}_{\mathcal{A}}(x) = \text{LMF}_{\mathcal{A}}(x)$ for all $x \in \mathcal{X}$, then the triangular

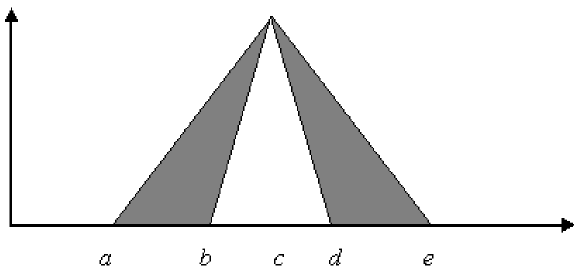

Figure 2.2: Representation of TIT2FS

IT2FS reduces to the triangular FS. If $\mathcal{X}$ is a set consists of all real numbers, then a TIT2FS in $\mathcal{X}$ is called triangular interval type-2 fuzzy number (TIT2FN).

2.5 Trapezoidal interval type-2 fuzzy sets

Since the operations on IT2FSs are very complex according to the decomposition theorems, so the concept of Trapezoidal interval T2FSs (TrIT2FSs) has been used to solve the decision making problems. The geometric representation of TrIT2FS is given in Fig. 2.3.

Definition 2.5.1. Let $\mathcal{A}^L$ and $\mathcal{A}^U$ be two TrIT2FSs whose heights are $h^L_{\mathcal{A}}$ and $h^U_{\mathcal{A}}$, respectively. Then, a TrIT2FS $\mathcal{A}$ in universe of discourse $\mathcal{X}$ is defined as

$$\mathcal{A} = [\mathcal{A}^L, \mathcal{A}^U] = \left[(a_1^L, a_2^L, a_3^L, a_4^L; h^L_{\mathcal{A}}), (a_1^U, a_2^U, a_3^U, a_4^U; h^U_{\mathcal{A}})\right] \tag{2.14}$$

where $a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, h^L_{\mathcal{A}}, h^U_{\mathcal{A}}$ are all real numbers and satisfy $a_1^L \leq a_2^L \leq a_3^L \leq a_4^L$, $a_1^U \leq a_2^U \leq a_3^U \leq a_4^U$, $0 \leq h^L_{\mathcal{A}} \leq h^U_{\mathcal{A}} \leq 1$. The upper and the lower membership function denoted by UMF and LMF respectively are defined as

$$\mathrm{UMF}_{\mathcal{A}}(x) = \begin{cases} \frac{h^U_{\mathcal{A}}(x-a_1^U)}{a_2^U - a_1^U} & ; a_1^U \leq x \leq a_2^U \\ h^U_{\mathcal{A}} & ; a_2^U \leq x \leq a_3^U \\ \frac{h^U_{\mathcal{A}}(a_4^U - x)}{(a_4^U - a_3^U)} & ; a_3^U \leq x \leq a_4^U \end{cases} \tag{2.15}$$

and

$$\mathrm{LMF}_{\mathcal{A}}(x) = \begin{cases} \frac{h^L_{\mathcal{A}}(x-a_1^L)}{a_2^L - a_1^L} & ; a_1^L \leq x \leq a_2^L \\ h^L_{\mathcal{A}} & ; a_2^L \leq x \leq a_3^L \\ \frac{h^L_{\mathcal{A}}(a_4^L - x)}{(a_4^L - a_3^L)} & ; a_3^L \leq x \leq a_4^L \end{cases} \tag{2.16}$$

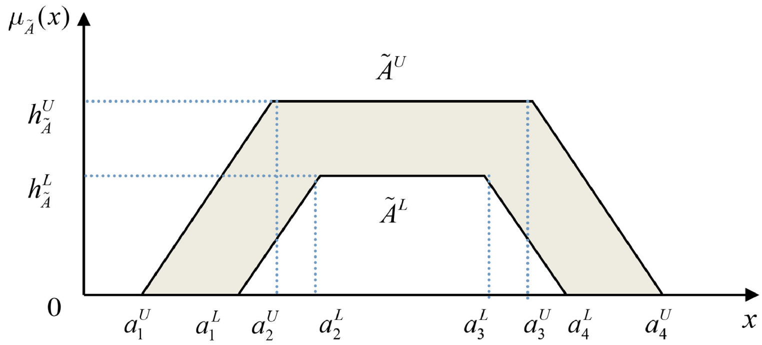

Figure 2.3: Representation of a Trapezoidal interval type-2 fuzzy set

Definition 2.5.2. Let $\mathcal{A}$ be IT2FS, then based on the definition of UMF and LMF, Karnik and Mendel [81] proposed a method to calculate the centroid of IT2FSs as

$$c(\mathcal{A}) = \frac{C_l(\mathcal{A}) + C_r(\mathcal{A})}{2} \tag{2.17}$$

where $C_l(\mathcal{A})$ and $C_r(\mathcal{A})$ represents the left and right endpoints of the centroid interval of IT2FSs and expressed mathematically as

$$C_l(\mathcal{A}) = \min_{\xi \in [a,b]} \frac{\int_a^{\xi} x\mathcal{A}^U(x)dx + \int_{\xi}^{b} x\mathcal{A}^L(x)dx}{\int_a^{\xi} \mathcal{A}^U(x)dx + \int_{\xi}^{b} \mathcal{A}^L(x)dx} \tag{2.18}$$

and

$$C_r(\mathcal{A}) = \max_{\xi\in[a,b]} \frac{\int_a^\xi x\mathcal{A}^L(x)dx + \int_\xi^b x\mathcal{A}^U(x)dx}{\int_a^\xi \mathcal{A}^L(x)dx + \int_\xi^b \mathcal{A}^U(x)dx} \tag{2.19}$$

Definition 2.5.3. Chen [19] presented the interval type-2 signed distance between two TrIT2FNs $\mathcal{A}$ and $\mathcal{B}$ by extending the concept of signed distance and the extension principle as

$$d(\mathcal{A},\mathcal{B}) = \frac{1}{8}\left|\begin{array}{r}(b_1^L - a_1^L + b_2^L - a_2^L + b_3^L - a_3^L + b_4^L - a_4^L) + 4(a_1^U - b_1^U)\\ +2(a_2^U - b_2^U) + 2(a_3^U - b_3^U) + 4(a_4^U - b_4^U)\\ +3(a_2^U + a_3^U - a_1^U - a_4^U)\dfrac{h_{\mathcal{A}}^L}{h_{\mathcal{A}}^U} - 3(b_2^U + b_3^U - b_1^U - b_4^U)\dfrac{h_{\mathcal{B}}^L}{h_{\mathcal{B}}^U}\end{array}\right| \tag{2.20}$$

2.6 Symmetric triangular interval type-2 fuzzy set

Qin [118] defined the symmetric triangular interval T2FS (STIT2FS) over the set $\mathcal{X}$ as follows:

Definition 2.6.1. [118] A STIT2FS $\mathcal{A}$ can be represented as

$$\mathcal{A} = \{(c_{\mathcal{A}}(x), \delta_{\mathcal{A}}(x), \underline{h}_{\mathcal{A}}(x), \bar{h}_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\}, \tag{2.21}$$

where $c_{\mathcal{A}}(x)$, $\delta_{\mathcal{A}}(x)$, $\underline{h}_{\mathcal{A}}(x)$, $\bar{h}_{\mathcal{A}}(x)$ are the reference points of the STT2FS at point x, satisfying the inequalities $\delta_{\mathcal{A}}(x) \leq c_{\mathcal{A}}(x)$, $0 \leq \underline{h}_{\mathcal{A}}(x) \leq \bar{h}_{\mathcal{A}}(x) \leq 1$. The pair $\mathcal{A} = (c_{\mathcal{A}}(x), \delta_{\mathcal{A}}(x), \underline{h}_{\mathcal{A}}, \bar{h}_{\mathcal{A}})$ as a symmetric TIT2FN (STIT2FN) (shown in Fig. 2.4) with UMF and the LMF are

$$\mathrm{UMF}_{\mathcal{A}}(x) = \begin{cases} \frac{\bar{h}_{\mathcal{A}}}{\delta_{\mathcal{A}}}(x - c_{\mathcal{A}} + \delta_{\mathcal{A}}) & ; c_{\mathcal{A}} - \delta_{\mathcal{A}} \leq x < c_{\mathcal{A}} \\ \bar{h}_{\mathcal{A}} & ; x = c_{\mathcal{A}} \\ \frac{\bar{h}_{\mathcal{A}}}{\delta_{\mathcal{A}}}(c_{\mathcal{A}} + \delta_{\mathcal{A}} - x) & ; c_{\mathcal{A}} \leq x < c_{\mathcal{A}} + \delta_{\mathcal{A}} \end{cases} \tag{2.22}$$

$$\mathrm{LMF}_{\mathcal{A}}(x) = \begin{cases} \frac{\underline{h}_{\mathcal{A}}}{\delta_{\mathcal{A}}}(x - c_{\mathcal{A}} + \delta_{\mathcal{A}}) & ; c_{\mathcal{A}} - \delta_{\mathcal{A}} \leq x < c_{\mathcal{A}} \\ \underline{h}_{\mathcal{A}} & ; x = c_{\mathcal{A}} \\ \frac{\underline{h}_{\mathcal{A}}}{\delta_{\mathcal{A}}}(c_{\mathcal{A}} + \delta_{\mathcal{A}} - x) & ; c_{\mathcal{A}} \leq x < c_{\mathcal{A}} + \delta_{\mathcal{A}} \end{cases} \tag{2.23}$$

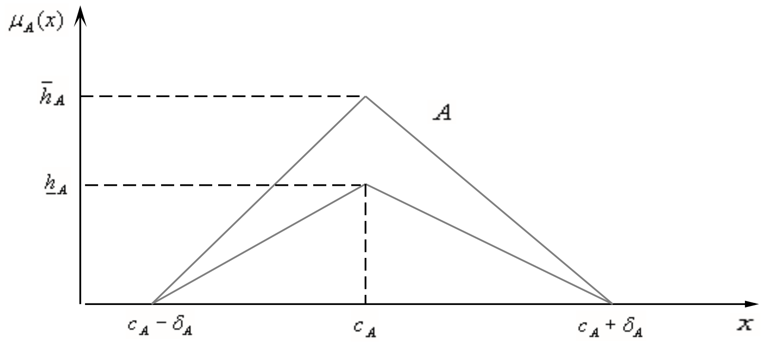

Figure 2.4: Representation of a Symmetric triangular interval type-2 fuzzy number

Definition 2.6.2. The score function of STIT2FN $\mathcal{A} = \left(c_{\mathcal{A}}, \delta_{\mathcal{A}}, \underline{h}_{\mathcal{A}}, \bar{h}_{\mathcal{A}}\right)$ is defined as

$$\begin{aligned} s(\mathcal{A}) &= (s_x(\mathcal{A}), s_y(\mathcal{A})) \\ &= \left(c_{\mathcal{A}} \frac{2\underline{h}_{\mathcal{A}}\bar{h}_{\mathcal{A}}}{\underline{h}_{\mathcal{A}}+\bar{h}_{\mathcal{A}}}, \frac{\underline{h}_{\mathcal{A}}+\bar{h}_{\mathcal{A}}}{2}\right) \end{aligned} \tag{2.24}$$

Definition 2.6.3. For two STIT2FNs $\mathcal{A}$ and $\mathcal{B}$, an order relation "(>)" to compare them is defined as

(i) If $s_x(\mathcal{A}) > s_x(\mathcal{B})$, then $\mathcal{A} > \mathcal{B}$;

(ii) If $s_x(\mathcal{A}) = s_x(\mathcal{B})$, then $\begin{cases} s_y(\mathcal{A}) > s_y(\mathcal{B}) & \Rightarrow \mathcal{A} > \mathcal{B}; \\ s_y(\mathcal{A}) = s_y(\mathcal{B}) & \Rightarrow \mathcal{A} = \mathcal{B}; \end{cases}$

Definition 2.6.4. [66] For non-negative real numbers $x_i (i = 1, 2, \ldots, n)$, the Hamy mean (HM) is given as

$$\mathrm{HM}^{(k)}(x_1, x_2, \cdots, x_n) = \frac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(\prod\limits_{j=1}^{k} x_{i_j}\right)^{\frac{1}{k}}}{\binom{n}{k}} \tag{2.25}$$

where k is the parameter, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $(i_1, i_2, \ldots, i_k)$ crosses all the $k-$tuple mix of $(1, 2, \ldots, n)$.

From Eq. (2.25), it is clear that the HM satisfies the following properties

1) $\mathrm{HM}^{(k)}(0, 0, \ldots, 0) = 0$.

2) $\mathrm{HM}^{(k)}(x, x, \ldots, x) = x$.

3) $\mathrm{HM}^{(k)}(x_1, x_2, \ldots, x_n) \leq \mathrm{HM}^{(k)}(y_1, y_2, \ldots, y_n)$, if $x_i \leq y_i$ for all i; and

4) $\min_i \{x_i\} \leq \mathrm{HM}^{(k)}(x_1, x_2, \ldots, x_n) \leq \max_i \{x_i\}$.

Definition 2.6.5. For a collection of STIT2FNs $\mathcal{A}_i = (c_{\mathcal{A}_i}, \delta_{\mathcal{A}_i}, \underline{h}_{\mathcal{A}_i}, \bar{h}_{\mathcal{A}_i})$ and parameter $k = 1, 2, \ldots, n$. If

$$\mathrm{STIT2HM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \frac{\bigoplus\limits_{\substack{1 \leq i_1 < \\ \ldots < i_k \leq n}} \left(\bigotimes\limits_{j=1}^{k} \mathcal{A}_{i_j} \right)^{\frac{1}{k}}}{\binom{n}{k}} \tag{2.26}$$

$$= \left(\frac{\sum\limits_{\substack{1 \leq i_1 < \\ \ldots < i_k \leq n}} \left(\prod\limits_{j=1}^{k} c_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum\limits_{\substack{1 \leq i_1 < \\ \ldots < i_k \leq n}} \left(\prod\limits_{j=1}^{k} \delta_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \left(\prod_{\substack{1 \leq i_1 < \\ \ldots < i_k \leq n}} \left(1 - \left(\prod_{j=1}^{k} (1 - \underline{h}_{\mathcal{A}_{i_j}}) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}, \right.$$
$$\left. 1 - \left(\prod_{\substack{1 \leq i_1 < \\ \ldots < i_k \leq n}} \left(1 - \left(\prod_{j=1}^{k} \bar{h}_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}} \right)$$

then $\mathrm{STIT2HM}^{(k)}$ is called the symmetric triangular interval type-2 fuzzy Hamy mean operator and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ represent the binomial coefficient.

2.7 Archimedean t-norms and Archimedean t-conorms

A t-norm (fuzzy intersection)[84, 108] '$\mathcal{T}$' is a binary operation on $[0, 1]$ i.e.

$$\mathcal{T} : [0, 1] \times [0, 1] \to [0, 1] \tag{2.27}$$

defined by

$$(\mathcal{B}_1 \cap \mathcal{B}_2)(x) = \mathcal{T}(\mathcal{B}_1(x), \mathcal{B}_2(x)) \qquad \forall x \in [0, 1] \tag{2.28}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are arbitrary fuzzy sets. Further, the mapping $\mathcal{T}$ satisfies the following axioms for all $a, b, d \in [0, 1]$

Axiom 1: $\mathcal{T}(a, 1) = a$ (Boundary Condition)

Axiom 2: If $b \leq d$ then $\mathcal{T}(a, b) \leq \mathcal{T}(a, d)$ (Monotonicity)

Axiom 3: $\mathcal{T}(a, b) = \mathcal{T}(b, a)$ (Commutativity)

Axiom 4: $\mathcal{T}(a, \mathcal{T}(b, d)) = \mathcal{T}(\mathcal{T}(a, b), d)$ (Associativity)

Alternatively, a T-conorm (fuzzy union)[84, 108] 'S' is also a binary operation on $[0, 1]$ given by

$$S : [0, 1] \times [0, 1] \to [0, 1] \tag{2.29}$$

defined by

$$(\mathcal{B}_1 \cup \mathcal{B}_2)(x) = S\left(\mathcal{B}_1(x), \mathcal{B}_2(x)\right) \qquad \forall x \in [0, 1] \tag{2.30}$$

Also, the mapping 'S' further satisfies the boundary, monotonicity, commutativity and associativity conditions.

The relation between 'S' and '$\mathcal{T}$' norms is given as

$$S(a, b) = 1 - \mathcal{T}(1 - a, 1 - b) \quad \forall a, b \in [0, 1] \tag{2.31}$$

A class of fuzzy intersection (t-norm) is obtained if t-norm also satisfies the additional axioms [84, 108], i.e.,

Axiom 5: $\mathcal{T}$ is continuous function (Continuity)

Axiom 6: $\mathcal{T}(a, a) < a$ (Subidempotency)

Axiom 7: If $a_1 < a_2$ and $b_1 < b_2$ implies $\mathcal{T}(a_1, b_1) < \mathcal{T}(a_2, b_2)$ (Strict monotonicity)

Similarly, for t-conorm, Axiom 6 is replaced by $S(a, a) > a$ and is called superidempotency.

A continuous t-norm that satisfy the subidempotency i.e., $\mathcal{T}(a, a) < a$ is called an Archimedean t-norm(AT)[108]. If it also satisfies the strict monotonicity then it is called strict Archimedean t-norm. On the other hand, a continuous t-conorm that satisfy the superidempotency i.e. $S(a, a) > a$ is called an Archimedean t-conorm(AC)[84, 108]. If it also satisfies the strict monotonicity then it is called strict Archimedean t-conorm.

Furthermore, strict AT and AC can be expressed in the form of continuous function $y : [0, 1) \to [0, 1]$ and $z : (0, 1] \to [0, 1]$ respectively for $a, b \in [0, 1]$ as

$$T(a, b) = z^{-1}(z(a) + z(b)) \quad \text{and} \quad S(a, b) = y^{-1}(y(a) + y(b))$$

where z(or y) is a decreasing(or increasing) function with $z(1) = 0$, $y(0) = 0$ and $z(a) = y(1 - a)$. However, some standard union and intersection form for $a, b \in [0, 1]$ are defined as [84, 108]:

(i) Standard intersection and union

$$\mathcal{T}(a,b) = \min(a,b) \quad ; \quad S(a,b) = \max(a,b)$$

(ii) Algebraic product and algebraic sum

$$\mathcal{T}(a,b) = ab \quad ; \quad S(a,b) = a + b - ab$$

(iii) Bounded Difference and Sum

$$\mathcal{T}(a,b) = \max(0, a+b-1) \quad ; \quad S(a,b) = \min(1, a+b)$$

(iv) Drastic intersection and union

$$\mathcal{T}(a,b) = \begin{cases} a & ;\text{when } b = 1 \\ b & ;\text{when } a = 1 \\ 0 & ;\text{otherwise} \end{cases} \quad ; \quad S(a,b) = \begin{cases} a & ;\text{when } b = 0 \\ b & ;\text{when } a = 0 \\ 1 & ;\text{otherwise} \end{cases}$$

(v) Yagar class of t-norm and t-conorm

$$\mathcal{T}(a,b) = 1 - \min\left(1, [(1-a)^p + (1-b)^p]^{1/p}\right);$$
$$S(a,b) = \min\left[1, (a^p + b^p)^{1/p}\right]$$

where $p > 0$.

Apart from them, some other well known ATs and ACs with their generator function are summarized in Table 2.1 [84, 108].

Table 2.1: Some AT and AC with their relative additive generators

Name	T-norm	Additive generator	S-norm	Additive generator
	$\mathcal{T}(a,b)$	$z(t)$	$S(a,b)$	$y(t)$
Algebraic	ab	$-\log(t)$	$a+b-ab$	$-\log(1-t)$
Einstein	$\frac{ab}{1+(1-a)(1-b)}$	$\log\left(\frac{2-t}{t}\right)$	$\frac{a+b}{1+ab}$	$\log\left(\frac{1+t}{1-t}\right)$
Hamacher ($\gamma > 0$)	$\frac{ab}{\gamma+(1-\gamma)(a+b-ab)}$	$\log\left(\frac{\gamma+(1-\gamma)t}{t}\right)$	$\frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab}$	$\log\left(\frac{\gamma+(1-\gamma)(1-t)}{1-t}\right)$
Frank($\lambda > 1$)	$\log_\lambda\left(1+\frac{(\lambda^a-1)(\lambda^b-1)}{\lambda-1}\right)$	$-\log\left(\frac{\lambda-1}{\lambda^t-1}\right)$	$\log_\lambda\left(1+\frac{(\lambda^{1-a}-1)(\lambda^{1-b}-1)}{\lambda-1}\right)$	$-\log\left(\frac{\lambda-1}{\lambda^{1-t}-1}\right)$

Chapter 3

Distance measure between type-2 intuitionistic fuzzy sets and its applications to decision making process[1]

In this chapter, we presented the notion of the type-2 intuitionistic fuzzy sets (T2IFSs) and formulated a series of distance measures between two T2IFSs by using geometric and Hausdorff metrics. Also, several desirable relations between them are defined. Later, we present an efficient method based on the developed distance measures to solve the MADM problems under the T2IFSs environment where the information related to each object is taken in the form of Type-2 intuitionistic fuzzy numbers (T2IFNs). The method is explained with a practical example and compared their results with the existing studies.

3.1 Introduction

In real life, most of the mathematical problems do not contain complete or exact information about the given problem and hence there is a big task for the decision maker to handle it before analyzing the problem. These inexact information has been handled by a theory of fuzzy set (FS) [166] and their corresponding extensions such as an intuitionistic fuzzy set (IFS) [7], type-2 fuzzy set (T2FS) [167] and so on. In the literature, various

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researchers [40, 44, 55, 159, 161] have analyzed the decision-making problem under FS as well as IFS environment either by using AOs or information measures. Among them, the distance measure is the function that gives the degree of discrimination among the two objects. In the literature, several distance measures are proposed by the researchers to solve the decision-making problems by utilizing the uncertain and fuzzy information. For instance, Szmidt [136] presented distance and similarity measures for IFSs and applied them to various decision making problems. Shen et al. [129] proposed a novel distance measure and discussed its various properties and further developed a TOPSIS ("technique for order preference by similarity to ideal solution") based on the proposed measure. Ye [165] put forward two similarity measures based on cosine functions. Song et al. [134] proposed a measure of similarity for IFSs and applied it to medical diagnosis problem. Garg and Kumar [58] presented the TOPSIS method based on exponential distance measures for solving MAGDM problems.

Since in the above-mentioned work, researchers have considered as a crisp membership function to its element. However, in many situations, uncertainty is not probabilities in nature but it is imprecise or vague in nature. To address it, the concept of T2FS [98], an extension of FS, in which membership values are type-1 FSs on [0,1] is developed. In T2FS, there is an additional membership function which provides an additional degree of freedom to the practices to model the uncertainties. In T2FS, each element is characterized by the degrees of the primary, secondary and a footprint of uncertainty (FOU). It has been observed from the above analysis that they have conducted an analysis by considering the degree of acceptance of an element only. But, in the real world, it is not possible for a decision-maker to give their preferences toward the object under the different parameters in terms of only the acceptance region (membership degree). Thus, for handling this, there is a need for the degree of non-membership degree (rejection degree) such that the sum of its membership and non-membership degree is less than or equal to one. Therefore to overcome it, a degree of membership, non-membership and their corresponding FOU have been considered during the present analysis and called a theory as a type-2 intuitionistic fuzzy set (T2IFS).

Considering the fact the T2IFS has a great powerful ability to model the imprecise and ambiguous information in real-world applications and the distance measure is one of the information measures to measure the degree of discrimination between the two sets. Therefore, by keeping the features of T2IFSs and the distance measures, it is vital to amplify some new distance measures to process the T2IFSs in MADM methods. In this light, the main objectives of this chapter are listed as follows:

1) to introduce some new distance measures under T2IFSs environment;

2) to create an algorithm to illuminate group decision-making issues with proposed measures;

3) to exhibit an illustration where the significance of preferences based on T2IFS decision problems has been clarified.

3.2 Type-2 Intuitionistic fuzzy set

Definition 3.2.1. A type-2 intuitionistic fuzzy set $\mathcal{A}$ in the universe of discourse $\mathcal{X}$ is set of pairs $\{x, \mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\}$ where x is the element of T2IFS, $\mu_{\mathcal{A}}(x)$ and $\nu_{\mathcal{A}}(x)$ are the grade of the membership and non-membership respectively which are defined in the interval $[0,1]$ as

$$\mu_{\mathcal{A}}(x) = \int\limits_{x \in j_x^1} f_x(u_{\mathcal{A}})/u_{\mathcal{A}} \quad ; \quad \nu_{\mathcal{A}}(x) = \int\limits_{x \in j_x^2} t_x(v_{\mathcal{A}})/v_{\mathcal{A}}$$

where $f_x(u_{\mathcal{A}})$ and $t_x(u_{\mathcal{A}})$ are named as secondary membership function (SMF) and secondary non-membership functions (SNMF). In addition, $u_{\mathcal{A}}, v_{\mathcal{A}}$ denotes the primary membership function (PMF) and primary non-membership functions (PNMF) and j_x^1, j_x^2 are named as the PMF and PNMF of x, respectively.

In other words, T2IFS $\mathcal{A}$ in the universe of discourse is defined as

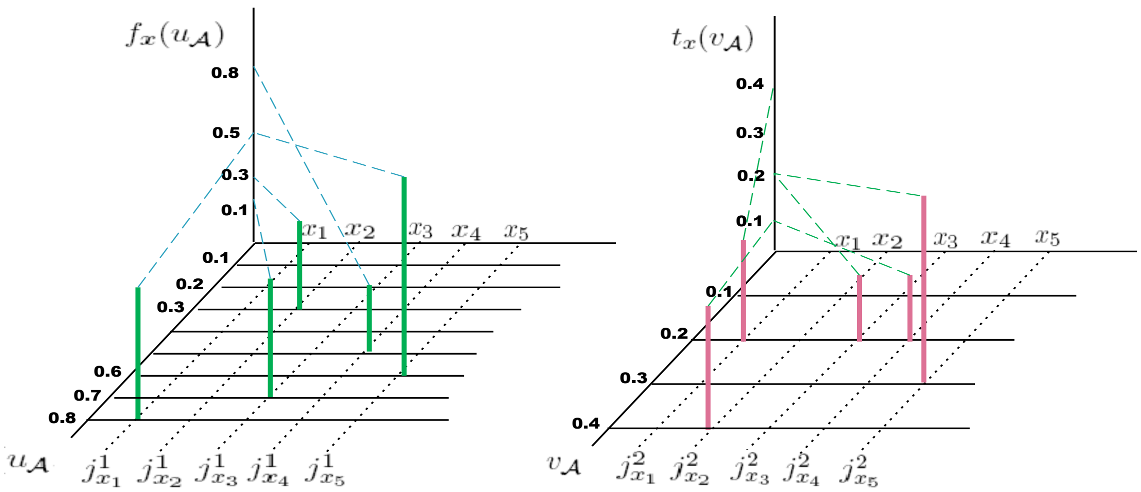
$$\mathcal{A} = \left\{ \left\langle (x, u_{\mathcal{A}}, v_{\mathcal{A}}), f_x(u_{\mathcal{A}}), t_x(v_{\mathcal{A}}) \right\rangle \mid x \in \mathcal{X}, u_{\mathcal{A}} \in j_x^1, v_{\mathcal{A}} \in j_x^2 \right\}$$

where the element of the domain $(x, u_{\mathcal{A}}, v_{\mathcal{A}})$ called as PMF $(u_{\mathcal{A}})$ and PNMF $(v_{\mathcal{A}})$ of $x \in \mathcal{X}$ while $f_x(u_{\mathcal{A}})$ and $t_x(v_{\mathcal{A}})$ be the memberships of the PMF and PNMF called as the SMF and SNMF respectively where $u_{\mathcal{A}} \in j_x^1 \subseteq [0,1]$, $v_{\mathcal{A}} \in j_x^2 \subseteq [0,1]$. For convenience, we denote this pair to be $\mathcal{A} = \langle x, (u_{\mathcal{A}}, f_x(u_{\mathcal{A}}), v_{\mathcal{A}}, t_x(v_{\mathcal{A}}))\rangle$ and called as type-2 intuitionistic fuzzy number (T2IFN).

Example 3.2.1. Consider a set of object $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$ and variable "beautiful". Then, a T2IFS $\mathcal{A}$ is given as follows:

$$\mathcal{A} = \begin{Bmatrix} ((x_1, 0.8, 0.2), (0.5, 0.4)), ((x_2, 0.3, 0.4), (0.3, 0.1)), ((x_3, 0.7, 0.2), (0.2, 0.2)), \\ ((x_4, 0.5, 0.2), (0.8, 0.1)), ((x_5, 0.6, 0.3), (0.5, 0.2)) \end{Bmatrix}$$

The diagrammatic representation of the proposed T2IFS corresponding to $\mathcal{A}$ is shown in Figure 3.1.



(a) Membership function (b) Non-Membership function

Figure 3.1: Diagrammatic representation of the T2IFS

Definition 3.2.2. Let $\mathcal{A}_1, \mathcal{A}_2$ be two T2IFSs then $\mathcal{A}_1 \subseteq \mathcal{A}_2$ if and only if $0 \leq f_x(u_{\mathcal{A}_1}) \leq f_x(u_{\mathcal{A}_2}) \leq 1, \forall u_{\mathcal{A}_1}, u_{\mathcal{A}_2} \in j^1_x \subseteq [0, 1]$ and $0 \leq t_x(u_{\mathcal{A}_1}) \leq t_x(u_{\mathcal{A}_2}) \leq 1, \forall v_{\mathcal{A}_1}, v_{\mathcal{A}_2} \in j^2_x \subseteq [0, 1]$.

Definition 3.2.3. The variance margin function (VMF) of T2IFS is defined as the difference between PMF and PNMF, and SMF and SNMF. It is denoted by ξ and η i.e., for T2IFS $\mathcal{A}$, variance margin functions are $\xi_{\mathcal{A}} =\mid u_{\mathcal{A}}(x_i) - f_{x_i}(u_{\mathcal{A}}) \mid$ and $\eta_{\mathcal{A}} =\mid v_{\mathcal{A}}(x_i) - t_{x_i}(v_{\mathcal{A}}) \mid \forall i$.

3.3 Distance measures between T2IFS

For a universal set $\mathcal{X}$, denote $F_2(\mathcal{X})$ be the set of all T2IFSs. Then, we define the concept of the distance measure of T2IFSs as follows.

Definition 3.3.1. Let $\mathcal{D} : F_2(\mathcal{X}) \times F_2(\mathcal{X}) \to [0,1]$ be a mapping, and let $\mathcal{A}_i \in F_2(\mathcal{X})$ $(i = 1,2,3)$. Then $\mathcal{D}(\mathcal{A}_1, \mathcal{A}_2)$ is called the distance degree, if following conditions are satisfied:

(P1) $0 \leq \mathcal{D}(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) If $\mathcal{A}_1 = \mathcal{A}_2$, then $\mathcal{D}(\mathcal{A}_1, \mathcal{A}_2) = 0$.

(P3) $\mathcal{D}(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{D}(\mathcal{A}_2, \mathcal{A}_1)$.

(P4) If $\mathcal{D}(\mathcal{A}_1, \mathcal{A}_2) = 0,\ \mathcal{D}(\mathcal{A}_1, \mathcal{A}_3) = 0$, then $\mathcal{D}(\mathcal{A}_2, \mathcal{A}_3) = 0$.

To compute the distance measures between the T2IFSs, we consider the FOU of the PMF, PNMF, SMF, SNMF as well as variance margin between the functions. For convenience, two T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$ defined in $\mathcal{X}$ are denoted by $\mathcal{A}_1 = \langle x, (u_{\mathcal{A}_1}, f_x(u_{\mathcal{A}_1}), v_{\mathcal{A}_1}, t_x(v_{\mathcal{A}_1})) \mid x \in \mathcal{X}\rangle$ and $\mathcal{A}_2 = \langle x, (u_{\mathcal{A}_2}, f_x(u_{\mathcal{A}_2}), v_{\mathcal{A}_2}, t_x(v_{\mathcal{A}_2})) \mid x \in \mathcal{X}\rangle$. Then, measures between $\mathcal{A}_1$ and $\mathcal{A}_2$ based on the geometric distance model is given as.

(i) The Hamming distance:

$$\mathcal{D}_1(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{4}\sum_{i=1}^{n} \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)| + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})| \\ + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)| + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)| \\ + |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix} \tag{3.1}$$

(ii) The normalized Hamming distance:

$$\mathcal{D}_2(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{4n}\sum_{i=1}^{n} \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)| + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})| \\ + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)| + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)| \\ + |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix} \tag{3.2}$$

(iii) The Euclidean distance:

$$\mathcal{D}_3(\mathcal{A}_1,\mathcal{A}_2)=\left[\frac{1}{4}\sum_{i=1}^{n}\left(\begin{array}{l}|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^2+|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^2\\ +|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|^2+|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|^2\\ +|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|^2+|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|^2\end{array}\right)\right]^{\frac{1}{2}} \tag{3.3}$$

(iv) The normalized Euclidean distance:

$$\mathcal{D}_4(\mathcal{A}_1,\mathcal{A}_2)=\left[\frac{1}{4n}\sum_{i=1}^{n}\left(\begin{array}{l}|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^2+|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^2\\ +|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|^2+|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|^2\\ +|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|^2+|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|^2\end{array}\right)\right]^{\frac{1}{2}} \tag{3.4}$$

Next, we show that the above defined measure $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)$ for $k=1,2,3,4$ satisfies the certain properties which are stated as below.

Theorem 3.3.1. The distances $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2),(k=2,4)$ between two T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfies the following properties:

(P1) $0\le\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)\le 1$.

(P2) If $\mathcal{A}_1=\mathcal{A}_2$, then $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)=0$.

(P3) $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)=\mathcal{D}_k(\mathcal{A}_2,\mathcal{A}_1)$.

(P4) If $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)=0$, $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_3)=0$, for $\mathcal{A}_3\in F_2(\mathcal{X})$ then $\mathcal{D}_k(\mathcal{A}_2,\mathcal{A}_3)=0$.

Proof. For $p=1,2$, we have

(P1) Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are T2IFSs, we have, $|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^p\ge 0$,$|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^p\ge 0$,$|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|^p\ge 0$, $|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|^p\ge 0$,$|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|^p\ge 0$, and $|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|^p\ge 0$, Thus $|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^p+|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^p+|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|^p+|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|^p+|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|^p+|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|^p\ge 0$, which implies that $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)\ge 0$. Further, $|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^p\le 1$,$|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^p\le 1$,$|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|^p\le 1$, $|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|^p\le 1$,$|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|^p\le 1$, and $|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|^p\le 1$. Therefore, $\sum\limits_{i=1}^{n}|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^p+|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^p+|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|^p+|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|^p+|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|^p+|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|^p\le 4$, $\Rightarrow\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)\le 1$. Thus $0\le\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)\le 1$.

(P2) $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)=0,\Leftrightarrow |u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^p+|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^p+|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|^p+|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|^p+|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|^p+|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|^p=0\Leftrightarrow u_{\mathcal{A}_1}(x_i)=u_{\mathcal{A}_2}(x_i),\ f_{x_i}(u_{\mathcal{A}_1})=f_{x_i}(u_{\mathcal{A}_2}),\ \xi_{\mathcal{A}_1}(x_i)=\xi_{\mathcal{A}_2}(x_i),\ v_{\mathcal{A}_1}(x_i)=v_{\mathcal{A}_2}(x_i),\ t_{x_i}(v_{\mathcal{A}_1})=t_{x_i}(v_{\mathcal{A}_2}),\ \eta_{\mathcal{A}_1}(x_i)=\eta_{\mathcal{A}_2}(x_i)$. Therefore, $\mathcal{A}_1=\mathcal{A}_2$.

(P3) $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)=\frac{1}{4n}\sum_{i=1}^{n}\Big(|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^p+|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^p+|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|^p+|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|^p+|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|^p+|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|^p\Big)$
$=\frac{1}{4n}\sum_{i=1}^{n}\Big(|u_{\mathcal{A}_2}(x_i)-u_{\mathcal{A}_1}(x_i)|^p+|f_{x_i}(u_{\mathcal{A}_2})-f_{x_i}(u_{\mathcal{A}_1})|^p+|\xi_{\mathcal{A}_2}(x_i)-\xi_{\mathcal{A}_1}(x_i)|^p+|v_{\mathcal{A}_2}(x_i)-v_{\mathcal{A}_1}(x_i)|^p+|t_{x_i}(v_{\mathcal{A}_2})-t_{x_i}(v_{\mathcal{A}_1})|^p+|\eta_{\mathcal{A}_2}(x_i)-\eta_{\mathcal{A}_1}(x_i)|^p\Big)=\mathcal{D}_k(\mathcal{A}_2,\mathcal{A}_1)$

(P4) Now $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2)=0\Rightarrow u_{\mathcal{A}_1}(x_i)=u_{\mathcal{A}_2}(x_i),\ f_{x_i}(u_{\mathcal{A}_1})=f_{x_i}(u_{\mathcal{A}_2}),\ \xi_{\mathcal{A}_1}(x_i)=\xi_{\mathcal{A}_2}(x_i),\ v_{\mathcal{A}_1}(x_i)=v_{\mathcal{A}_2}(x_i),\ t_{x_i}(v_{\mathcal{A}_1})=t_{x_i}(v_{\mathcal{A}_2}),\ \eta_{\mathcal{A}_1}(x_i)=\eta_{\mathcal{A}_2}(x_i)$ and $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_3)=0\Rightarrow u_{\mathcal{A}_1}(x_i)=u_{\mathcal{A}_3}(x_i),\ f_{x_i}(u_{\mathcal{A}_1})=f_{x_i}(u_{\mathcal{A}_3}),\ \xi_{\mathcal{A}_1}(x_i)=\xi_{\mathcal{A}_3}(x_i),\ v_{\mathcal{A}_1}(x_i)=v_{\mathcal{A}_3}(x_i),\ t_{x_i}(v_{\mathcal{A}_1})=t_{x_i}(v_{\mathcal{A}_3}),\ \eta_{\mathcal{A}_1}(x_i)=\eta_{\mathcal{A}_3}(x_i)$ therefore, $u_{\mathcal{A}_2}(x_i)=u_{\mathcal{A}_3}(x_i),\ f_{x_i}(u_{\mathcal{A}_2})=f_{x_i}(u_{\mathcal{A}_3}),\ \xi_{\mathcal{A}_2}(x_i)=\xi_{\mathcal{A}_3}(x_i),\ v_{\mathcal{A}_2}(x_i)=v_{\mathcal{A}_3}(x_i),\ t_{x_i}(v_{\mathcal{A}_2})=t_{x_i}(v_{\mathcal{A}_3}),\ \eta_{\mathcal{A}_2}(x_i)=\eta_{\mathcal{A}_3}(x_i)\Rightarrow\mathcal{D}_k(\mathcal{A}_2,\mathcal{A}_3)=0$.

Hence, $\mathcal{D}_k(\mathcal{A}_1,\mathcal{A}_2),(k=2,4)$ are valid distance measures for T2IFSs. □

Theorem 3.3.2. Measures $\mathcal{D}_1$ and $\mathcal{D}_3$ satisfies the following properties:

1) $0\le\mathcal{D}_1\le n$

2) $0\le\mathcal{D}_3\le n^{\frac{1}{2}}$

Proof. It can be easily obtain that $\mathcal{D}_1(\mathcal{A}_1,\mathcal{A}_2)=n\mathcal{D}_2(\mathcal{A}_1,\mathcal{A}_2)$ and thus by Theorem 3.3.1, we get $0\le\mathcal{D}_1(\mathcal{A}_1,\mathcal{A}_2)\le n$. Similarly, we can obtain $0\le\mathcal{D}_3(\mathcal{A}_1,\mathcal{A}_2)\le n^{\frac{1}{2}}$. □

However, in many practical situations, the different set may have taken different weights and thus, weight $w_i(i=1,2,\cdots,n)$ with $w_i>0$, $\sum_{i=1}^{n}w_i=1$, of the element $x_i\in\mathcal{X}$ should be taken into account. In the following, we develop a normalized weighted Hamming and the normalized weighted Euclidean distance measure between T2IFSs.

(i) The normalized weighted Hamming distance:

$$\mathcal{D}_5(\mathcal{A}_1,\mathcal{A}_2)=\frac{1}{4n}\sum_{i=1}^{n}w_i\begin{pmatrix}|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|+|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|\\ +|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|+|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|\\ +|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|+|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|\end{pmatrix} \tag{3.5}$$

(ii) The normalized weighted Euclidean distance:

$$\mathcal{D}_6(\mathcal{A}_1,\mathcal{A}_2)=\left[\frac{1}{4n}\sum_{i=1}^{n}w_i\begin{pmatrix}|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^2+|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^2\\ +|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|^2+|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|^2\\ +|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|^2+|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|^2\end{pmatrix}\right]^{\frac{1}{2}} \tag{3.6}$$

Theorem 3.3.3. The measures $\mathcal{D}_2$ and $\mathcal{D}_5$ satisfy the inequality $\mathcal{D}_5\leq\mathcal{D}_2$ for all T2IFSs.

Proof. For any two T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$ and $w_i>0$ be the weight vector of $x_i\in\mathcal{X}$ such that $\sum\limits_{i=1}^{n}w_i=1$ then by Definition of $\mathcal{D}_5$, we have

$$\begin{aligned}\mathcal{D}_5(\mathcal{A}_1,\mathcal{A}_2) &= \frac{1}{4n}\sum_{i=1}^{n}w_i\begin{pmatrix}|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|+|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|\\ +|\xi_{\mathcal{A}_1}(x_i)-\xi_{\mathcal{A}_2}(x_i)|+|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|\\ +|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|+|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|\end{pmatrix}\\
&= \frac{1}{4n}w_1\begin{pmatrix}|u_{\mathcal{A}_1}(x_1)-u_{\mathcal{A}_2}(x_1)|+|f_{x_1}(u_{\mathcal{A}_1})-f_{x_1}(u_{\mathcal{A}_2})|\\ +|\xi_{\mathcal{A}_1}(x_1)-\xi_{\mathcal{A}_2}(x_1)|+|v_{\mathcal{A}_1}(x_1)-v_{\mathcal{A}_2}(x_1)|\\ +|t_{x_1}(v_{\mathcal{A}_1})-t_{x_1}(v_{\mathcal{A}_2})|+|\eta_{\mathcal{A}_1}(x_1)-\eta_{\mathcal{A}_2}(x_1)|\end{pmatrix}\\
&+ \frac{1}{4n}w_2\begin{pmatrix}|u_{\mathcal{A}_1}(x_2)-u_{\mathcal{A}_2}(x_2)|+|f_{x_2}(u_{\mathcal{A}_1})-f_{x_2}(u_{\mathcal{A}_2})|\\ +|\xi_{\mathcal{A}_1}(x_2)-\xi_{\mathcal{A}_2}(x_2)|+|v_{\mathcal{A}_1}(x_2)-v_{\mathcal{A}_2}(x_2)|\\ +|t_{x_2}(v_{\mathcal{A}_1})-t_{x_2}(v_{\mathcal{A}_2})|+|\eta_{\mathcal{A}_1}(x_2)-\eta_{\mathcal{A}_2}(x_2)|\end{pmatrix}\\
&\vdots\\
&+ \frac{1}{4n}w_n\begin{pmatrix}|u_{\mathcal{A}_1}(x_n)-u_{\mathcal{A}_2}(x_n)|+|f_{x_n}(u_{\mathcal{A}_1})-f_{x_n}(u_{\mathcal{A}_2})|\\ +|\xi_{\mathcal{A}_1}(x_n)-\xi_{\mathcal{A}_2}(x_n)|+|v_{\mathcal{A}_1}(x_n)-v_{\mathcal{A}_2}(x_n)|\\ +|t_{x_n}(v_{\mathcal{A}_1})-t_{x_n}(v_{\mathcal{A}_2})|+|\eta_{\mathcal{A}_1}(x_n)-\eta_{\mathcal{A}_2}(x_n)|\end{pmatrix}\end{aligned}$$

Since $w_i \in [0,1]$, thus for each i, we have

$$w_i \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)| + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})| \\ +|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)| + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)| \\ +|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix}$$

$$\leq \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)| + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})| \\ +|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)| + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)| \\ +|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix}$$

Therefore, we have

$$\begin{aligned}
\mathcal{D}_5(\mathcal{A}_1, \mathcal{A}_2) \quad &\leq \quad \frac{1}{4n} \begin{pmatrix} |u_{\mathcal{A}_1}(x_1) - u_{\mathcal{A}_2}(x_1)| + |f_{x_1}(u_{\mathcal{A}_1}) - f_{x_1}(u_{\mathcal{A}_2})| \\ +|\xi_{\mathcal{A}_1}(x_1) - \xi_{\mathcal{A}_2}(x_1)| + |v_{\mathcal{A}_1}(x_1) - v_{\mathcal{A}_2}(x_1)| \\ +|t_{x_1}(v_{\mathcal{A}_1}) - t_{x_1}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_1) - \eta_{\mathcal{A}_2}(x_1)| \end{pmatrix} \\
&+ \quad \frac{1}{4n} \begin{pmatrix} |u_{\mathcal{A}_1}(x_2) - u_{\mathcal{A}_2}(x_2)| + |f_{x_2}(u_{\mathcal{A}_1}) - f_{x_2}(u_{\mathcal{A}_2})| \\ +|\xi_{\mathcal{A}_1}(x_2) - \xi_{\mathcal{A}_2}(x_2)| + |v_{\mathcal{A}_1}(x_2) - v_{\mathcal{A}_2}(x_2)| \\ +|t_{x_2}(v_{\mathcal{A}_1}) - t_{x_2}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_2) - \eta_{\mathcal{A}_2}(x_2)| \end{pmatrix} \\
&\vdots \\
&+ \quad \frac{1}{4n} \begin{pmatrix} |u_{\mathcal{A}_1}(x_n) - u_{\mathcal{A}_2}(x_n)| + |f_{x_n}(u_{\mathcal{A}_1}) - f_{x_n}(u_{\mathcal{A}_2})| \\ +|\xi_{\mathcal{A}_1}(x_n) - \xi_{\mathcal{A}_2}(x_n)| + |v_{\mathcal{A}_1}(x_n) - v_{\mathcal{A}_2}(x_n)| \\ +|t_{x_n}(v_{\mathcal{A}_1}) - t_{x_n}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_n) - \eta_{\mathcal{A}_2}(x_n)| \end{pmatrix} \\
&\leq \quad \frac{1}{4n} \sum_{i=1}^{n} \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)| + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})| \\ +|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)| + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)| \\ +|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix} \\
&= \quad \mathcal{D}_2(\mathcal{A}_1, \mathcal{A}_2)
\end{aligned}$$

Hence $\mathcal{D}_5(\mathcal{A}_1, \mathcal{A}_2) \leq \mathcal{D}_2(\mathcal{A}_1, \mathcal{A}_2)$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are arbitrary T2IFSs and therefore $\mathcal{D}_5 \leq \mathcal{D}_2$ for all T2IFSs. □

Theorem 3.3.4. The measures $\mathcal{D}_4$ and $\mathcal{D}_6$ satisfy the inequality $\mathcal{D}_6 \leq \mathcal{D}_4$ for all T2IFSs.

Proof. Similar to Theorem 3.3.3, so we omit here. □

Theorem 3.3.5. For three T2IFSs $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ and $w_i > 0$ be the weight vector of the element $x_i \in \mathcal{X}$, then the weighted distance $\mathcal{D}_k(\mathcal{A}_1, \mathcal{A}_2), (k = 5, 6)$ satisfies the following properties:

(P1) $0 \leq \mathcal{D}_k(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) If $\mathcal{A}_1 = \mathcal{A}_2$, then $\mathcal{D}_k(\mathcal{A}_1, \mathcal{A}_2) = 0$.

(P3) $\mathcal{D}_k(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{D}_k(\mathcal{A}_2, \mathcal{A}_1)$.

(P4) If $\mathcal{D}_k(\mathcal{A}_1, \mathcal{A}_2) = 0$, $\mathcal{D}_k(\mathcal{A}_1, \mathcal{A}_3) = 0$, then $\mathcal{D}_k(\mathcal{A}_2, \mathcal{A}_3) = 0$.

Proof. For any two T2IFSs $\mathcal{A}_1$, $\mathcal{A}_2$ and weight vector $w_i > 0$ with $\sum\limits_{i=1}^{n} w_i = 1$, we know that $\mathcal{D}_5(\mathcal{A}_1, \mathcal{A}_2) \leq \mathcal{D}_2(\mathcal{A}_1, \mathcal{A}_2)$ by Theorem 3.3.3. Further, it follows from Theorem 3.3.1 that $0 \leq \mathcal{D}_2(\mathcal{A}_1, \mathcal{A}_2) \leq 1$ and hence we get $0 \leq \mathcal{D}_5(\mathcal{A}_1, \mathcal{A}_2) \leq 1$ which implies that $\mathcal{D}_5$ satisfy (P1). Further, we can easily obtain that $\mathcal{D}_5(\mathcal{A}_1, \mathcal{A}_2)$ satisfy the properties (P2)-(P4). Hence, $\mathcal{D}_5$ defined in Eq. (3.5) is a valid distance measure for T2IFSs. Similarly, we can obtain that $\mathcal{D}_6$ defined in Eq. (3.6) is a valid distance measure. Hence, the result. $\square$

Theorem 3.3.6. The distance measures $\mathcal{D}_2$ and $\mathcal{D}_4$ satisfies the inequalities $\sqrt{\mathcal{D}_2} \geq \mathcal{D}_4$ for T2IFSs.

Proof. For any two T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$, we have $0 \leq u_{\mathcal{A}_1}, u_{\mathcal{A}_2} \leq 1$, $0 \leq v_{\mathcal{A}_1}, v_{\mathcal{A}_2} \leq 1$, $0 \leq f_{x_i}(u_{\mathcal{A}_1}), f_{x_i}(u_{\mathcal{A}_2}) \leq 1$, $0 \leq t_{x_i}(v_{\mathcal{A}_1}), t_{x_i}(v_{\mathcal{A}_2}) \leq 1$, $0 \leq \xi_{\mathcal{A}_1}, \xi_{\mathcal{A}_2} \leq 1$, $0 \leq \eta_{\mathcal{A}_1}, \eta_{\mathcal{A}_2} \leq 1$. Thus, we get $|u_{\mathcal{A}_1} - u_{\mathcal{A}_2}|^2 \leq |u_{\mathcal{A}_1} - u_{\mathcal{A}_2}|$, $|v_{\mathcal{A}_1} - v_{\mathcal{A}_2}|^2 \leq |v_{\mathcal{A}_1} - v_{\mathcal{A}_2}|$, $|f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^2 \leq |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|$, $|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^2 \leq |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|$, $|\xi_{\mathcal{A}_1} - \xi_{\mathcal{A}_2}|^2 \leq |\xi_{\mathcal{A}_1} - \xi_{\mathcal{A}_2}|$, $|\eta_{\mathcal{A}_1} - \eta_{\mathcal{A}_2}|^2 \leq |\eta_{\mathcal{A}_1} - \eta_{\mathcal{A}_2}|$. Hence, by definition of $\mathcal{D}_4$, we get

$$\begin{aligned}
\mathcal{D}_4(\mathcal{A}_1, \mathcal{A}_2) &= \left[\frac{1}{4n} \sum_{i=1}^{n} \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^2 + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^2 \\ +|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^2 + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^2 \\ +|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^2 + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^2 \end{pmatrix} \right]^{\frac{1}{2}} \\
&\leq \left[\frac{1}{4n} \sum_{i=1}^{n} \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)| + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})| \\ +|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)| + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)| \\ +|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix} \right]^{\frac{1}{2}} \\
&= \sqrt{\mathcal{D}_2(\mathcal{A}_1, \mathcal{A}_2)}
\end{aligned}$$

Since, $\mathcal{A}_1$ and $\mathcal{A}_2$ are arbitrary two T2IFSs and hence we get $\sqrt{\mathcal{D}_2} \geq \mathcal{D}_4$, which is the required result. $\square$

Theorem 3.3.7. The distance measures $\mathcal{D}_5$ and $\mathcal{D}_6$ satisfies the inequalities $\mathcal{D}_6 \leq \sqrt{\mathcal{D}_5}$ for T2IFSs.

Proof. Similar to Theorem 3.3.6, and hence we omit here. $\square$

Theorem 3.3.8. For T2IFSs, the distance measures $\mathcal{D}_2$ and $\mathcal{D}_6$ satisfies the inequalities $\mathcal{D}_6 \leq \sqrt{\mathcal{D}_2}$.

Proof. For any two T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$, $w_i > 0$ be the weight vector of $x_i \in \mathcal{X}$ with $\sum\limits_{i=1}^{n} w_i = 1$. Then, by the definition of $\mathcal{D}_6$, we have

$$\begin{aligned}
\mathcal{D}_6(\mathcal{A}_1, \mathcal{A}_2) &= \left[\frac{1}{4n}\sum_{i=1}^{n} w_i \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^2 + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^2 \\ +|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^2 + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^2 \\ +|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^2 + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^2 \end{pmatrix}\right]^{\frac{1}{2}} \\
&\leq \left[\frac{1}{4n}\sum_{i=1}^{n} \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^2 + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^2 \\ +|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^2 + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^2 \\ +|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^2 + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^2 \end{pmatrix}\right]^{\frac{1}{2}}
\end{aligned}$$

Further, T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$, we have $0 \leq u_{\mathcal{A}_1}, u_{\mathcal{A}_2} \leq 1$, $0 \leq v_{\mathcal{A}_1}, v_{\mathcal{A}_2} \leq 1$, $0 \leq f_{x_i}(u_{\mathcal{A}_1}), f_{x_i}(u_{\mathcal{A}_2}) \leq 1$, $0 \leq t_{x_i}(v_{\mathcal{A}_1}), t_{x_i}(v_{\mathcal{A}_2}) \leq 1$, $0 \leq \xi_{\mathcal{A}_1}, \xi_{\mathcal{A}_2} \leq 1$, $0 \leq \eta_{\mathcal{A}_1}, \eta_{\mathcal{A}_2} \leq 1$. Hence, by using the property that for any $a \in [0,1]$, we have $| a |^2 \leq | a |$. Thus,

$$\begin{aligned}
\mathcal{D}_6(\mathcal{A}_1, \mathcal{A}_2) &\leq \left[\frac{1}{4n}\sum_{i=1}^{n} \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)| + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})| \\ +|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)| + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)| \\ +|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix}\right]^{\frac{1}{2}} \\
&= \sqrt{\mathcal{D}_2(\mathcal{A}_1, \mathcal{A}_2)}
\end{aligned}$$

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are the arbitrary sets and hence we get $\mathcal{D}_6 \leq \sqrt{\mathcal{D}_2}$ holds for all T2IFSs. $\square$

Hung and Yang [72] proposed a Hausdorff metric for two intervals $\mathcal{A} = [a_1, a_2]$ and $\mathcal{B} = [b_1, b_2]$, as follows:

$$\mathcal{D}^U = \max\{|a_1 - a_2|, |b_1 - b_2|\} \tag{3.7}$$

Now, for any two T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathcal{X} = \{x_1, x_2, \cdots, x_n\}$, we propose the following utmost distance measures:

(i) utmost normalized Hamming distance

$$\mathcal{D}_1^U = \frac{1}{4n}\sum_{i=1}^{n} \max\begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|, |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|, \\ |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|, |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|, \\ |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|, |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix} \tag{3.8}$$

(ii) utmost normalized weighted Hamming distance

$$\mathcal{D}_2^U = \frac{1}{4n}\sum_{i=1}^{n} w_i \max\begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|, |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|, \\ |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|, |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|, \\ |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|, |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix} \tag{3.9}$$

(iii) utmost normalized Euclidean distance

$$\mathcal{D}_3^U = \left[\frac{1}{4n}\sum_{i=1}^{n} \max\begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^2, |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^2, \\ |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^2, |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^2, \\ |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^2, |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^2 \end{pmatrix}\right]^{\frac{1}{2}} \tag{3.10}$$

(iv) utmost normalized weighted Euclidean distance

$$\mathcal{D}_4^U = \left[\frac{1}{4n}\sum_{i=1}^{n} w_i \max\begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^2, |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^2, \\ |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^2, |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^2, \\ |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^2, |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^2 \end{pmatrix}\right]^{\frac{1}{2}} \tag{3.11}$$

Theorem 3.3.9. For $\mathcal{A}_1, \mathcal{A}_2 \in F_2(\mathcal{X})$, $\mathcal{D}_k^U$ $(k = 1, 3)$ are the valid distance measures.

Proof. For two T2IFSs $\mathcal{A}_1, \mathcal{A}_2 \in F_2(\mathcal{X})$ with n attributes. Then we have to prove that $\mathcal{D}_k^U, (k = 1, 3)$ satisfies the properties (P1)-(P4) as described in Definition 3.3.1. For it, let $p = 1, 2$ then we have

(P1) Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are T2IFSs, we have, $|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^p \geq 0$,$|f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^p \geq 0$,$|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^p \geq 0$, $|v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^p \geq 0$,$|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^p \geq 0$, and $|\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^p \geq 0$. Thus, we have

$$\mathcal{D}_k(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{4n}\sum_{i=1}^{n} \max \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^p, |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^p, \\ |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^p, |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^p, \\ |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^p, |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^p \end{pmatrix} \geq 0$$

Further for T2IFSs, we have $|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^p \leq 1$, $|f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^p \leq 1$, $|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^p \leq 1$, $|v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^p \leq 1$,$|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^p \leq 1$, and $|\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^p \leq 1$. Therefore, we have

$$\mathcal{D}_k(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{4n}\sum_{i=1}^{n} \max \begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^p, |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^p, \\ |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^p, |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^p, \\ |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^p, |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^p \end{pmatrix} \leq 1$$

Hence, $0 \leq \mathcal{D}_k^U(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) Let $\mathcal{D}_k^U(\mathcal{A}_1, \mathcal{A}_2) = 0, \Leftrightarrow \max\big\{|u_{\mathcal{A}_1}(x_i)-u_{\mathcal{A}_2}(x_i)|^p, |f_{x_i}(u_{\mathcal{A}_1})-f_{x_i}(u_{\mathcal{A}_2})|^p, |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^p$, $|v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^p$,$|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^p$, $|\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^p\big\} = 0 \Leftrightarrow u_{\mathcal{A}_1}(x_i) = u_{\mathcal{A}_2}(x_i)$, $f_{x_i}(u_{\mathcal{A}_1}) = f_{x_i}(u_{\mathcal{A}_2})$, $\xi_{\mathcal{A}_1}(x_i) = \xi_{\mathcal{A}_2}(x_i)$, $v_{\mathcal{A}_1}(x_i) = v_{\mathcal{A}_2}(x_i)$, $t_{x_i}(v_{\mathcal{A}_1}) = t_{x_i}(v_{\mathcal{A}_2})$, $\eta_{\mathcal{A}_1}(x_i) = \eta_{\mathcal{A}_2}(x_i)$. Therefore, $\mathcal{A}_1 = \mathcal{A}_2$.

(P3) $\mathcal{D}_k^U(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{4n}\sum_{i=1}^{n} \max\big\{|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^p$, $|f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^p$, $|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^p$, $|v_{\mathcal{A}_1}(x_i)-v_{\mathcal{A}_2}(x_i)|^p$, $|t_{x_i}(v_{\mathcal{A}_1})-t_{x_i}(v_{\mathcal{A}_2})|^p$, $|\eta_{\mathcal{A}_1}(x_i)-\eta_{\mathcal{A}_2}(x_i)|^p\big\} = \frac{1}{4n}\sum_{i=1}^{n} \max \big\{|u_{\mathcal{A}_2}(x_i)-u_{\mathcal{A}_1}(x_i)|^p$, $|f_{x_i}(u_{\mathcal{A}_2})-f_{x_i}(u_{\mathcal{A}_1})|^p$, $|\xi_{\mathcal{A}_2}(x_i)-\xi_{\mathcal{A}_1}(x_i)|^p$, $|v_{\mathcal{A}_2}(x_i)-v_{\mathcal{A}_1}(x_i)|^p$, $|t_{x_i}(v_{\mathcal{A}_2}) - t_{x_i}(v_{\mathcal{A}_1})|^p$, $|\eta_{\mathcal{A}_2}(x_i) - \eta_{\mathcal{A}_1}(x_i)|^p\big\} = \mathcal{D}_k^U(\mathcal{A}_2, \mathcal{A}_1)$.

(P4) When $\mathcal{D}_k^U(\mathcal{A}_1, \mathcal{A}_2) = 0$ which implies that $u_{\mathcal{A}_1}(x_i) = u_{\mathcal{A}_2}(x_i)$, $f_{x_i}(u_{\mathcal{A}_1}) = f_{x_i}(u_{\mathcal{A}_2})$, $\xi_{\mathcal{A}_1}(x_i) = \xi_{\mathcal{A}_2}(x_i)$, $v_{\mathcal{A}_1}(x_i) = v_{\mathcal{A}_2}(x_i)$, $t_{x_i}(v_{\mathcal{A}_1}) = t_{x_i}(v_{\mathcal{A}_2})$, $\eta_{\mathcal{A}_1}(x_i) = \eta_{\mathcal{A}_2}(x_i)$. Further, for $\mathcal{D}_k(\mathcal{A}_1, \mathcal{A}_3) = 0$ implies that $u_{\mathcal{A}_1}(x_i) = u_{\mathcal{A}_3}(x_i)$, $f_{x_i}(u_{\mathcal{A}_1}) = f_{x_i}(u_{\mathcal{A}_3})$, $\xi_{\mathcal{A}_1}(x_i) = \xi_{\mathcal{A}_3}(x_i)$, $v_{\mathcal{A}_1}(x_i) = v_{\mathcal{A}_3}(x_i)$, $t_{x_i}(v_{\mathcal{A}_1}) = t_{x_i}(v_{\mathcal{A}_3})$, $\eta_{\mathcal{A}_1}(x_i) = \eta_{\mathcal{A}_3}(x_i)$. Therefore, we get $u_{\mathcal{A}_2}(x_i) = u_{\mathcal{A}_3}(x_i)$, $f_{x_i}(u_{\mathcal{A}_2}) = f_{x_i}(u_{\mathcal{A}_3})$, $\xi_{\mathcal{A}_2}(x_i) = \xi_{\mathcal{A}_3}(x_i)$, $v_{\mathcal{A}_2}(x_i) = v_{\mathcal{A}_3}(x_i)$, $t_{x_i}(v_{\mathcal{A}_2}) = t_{x_i}(v_{\mathcal{A}_3})$, $\eta_{\mathcal{A}_2}(x_i) = \eta_{\mathcal{A}_3}(x_i)$ and hence by definition of $\mathcal{D}_k^U$ for $k = 1, 3$, we get $\mathcal{D}_k^U(\mathcal{A}_2, \mathcal{A}_3) = 0$.

Hence, the measures $\mathcal{D}_k^U(k = 1, 3)$ are the valid distance measures. □

Theorem 3.3.10. For $\mathcal{A}_1, \mathcal{A}_2 \in F_2(\mathcal{X})$, $\mathcal{D}_2^U$ and $\mathcal{D}_4^U$ are the distance measure.

Proof. Follows from Theorem 3.3.9. □

Theorem 3.3.11. For $\mathcal{A}_1, \mathcal{A}_2 \in F_2(\mathcal{X})$, $\mathcal{D}_2^U, \mathcal{D}_1^U$ and $\mathcal{D}_3^U, \mathcal{D}_4^U$ respectively satisfies following inequalities.

1) $\mathcal{D}_2^U \leq \mathcal{D}_1^U$.

2) $\mathcal{D}_4^U \leq \mathcal{D}_3^U$.

Proof. It can be easily follows from the definition of utmost measures as $w_i \in [0, 1]$ for all i. □

Theorem 3.3.12. Measures $\mathcal{D}_1^U$ and $\mathcal{D}_2$ have the following inequality $\mathcal{D}_2 \geq \mathcal{D}_1^U$ for T2IFSs.

Proof. Since for any positive numbers $a_i, i = 1, 2, \cdots, n$, $\sum\limits_{i=1}^{n} a_i \geq \max\limits_i\{a_i\}$. Thus, for any two T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$, we have

$$\begin{aligned}
\mathcal{D}_2(\mathcal{A}_1, \mathcal{A}_2) &= \frac{1}{4n}\sum_{i=1}^{n}\begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)| + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})| \\ +|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)| + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)| \\ +|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})| + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix} \\
&\geq \frac{1}{4n}\sum_{i=1}^{n}\max\begin{pmatrix} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|, |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|, \\ |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|, |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|, \\ |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|, |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)| \end{pmatrix} \\
&= \mathcal{D}_1^U(\mathcal{A}_1, \mathcal{A}_2)
\end{aligned}$$

As $\mathcal{A}_1$ and $\mathcal{A}_2$ are arbitrary T2IFSs, so we have $\mathcal{D}_2 \geq \mathcal{D}_1^U$ holds for all T2IFSs. □

Theorem 3.3.13. The measures $\mathcal{D}_3^U$ and $\mathcal{D}_4$ satisfies the following inequality $\mathcal{D}_4 \geq \mathcal{D}_3^U$ for T2IFSs.

Proof. Follows from Theorem 3.3.12 □

Theorem 3.3.14. Measures $\mathcal{D}_2, \mathcal{D}_5$ and $\mathcal{D}_1^U$ satisfies the following inequalities:

1) $\mathcal{D}_2 \geq \frac{\mathcal{D}_5+\mathcal{D}_1^U}{2}$.

2) $\mathcal{D}_2 \geq \sqrt{\mathcal{D}_2 \cdot \mathcal{D}_1^U}$.

Proof. For T2IFSs, we have $\mathcal{D}_2 \geq \mathcal{D}_5$ and $\mathcal{D}_2 \geq \mathcal{D}_1^U$. So by adding these inequalities, we get $\mathcal{D}_2 \geq \frac{\mathcal{D}_5+\mathcal{D}_1^U}{2}$, while by multiplying them, we get $\mathcal{D}_2 \geq \sqrt{\mathcal{D}_2 \cdot \mathcal{D}_1^U}$. □

3.4 Group decision making method with T2IFSs based on distance measures

In this section, we develop a method by using the proposed distance measures to solve the MADM problems under the T2IFS environment.

3.4.1 Distance measure based approach

Assume a decision making problems which consists of "t" different alternatives $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_t$, evaluated under the "n" attributes denoted by $\mathcal{G}_1$, $\mathcal{G}_2$, ..., $\mathcal{G}_n$. Let $w = (w_1, w_2, \ldots, w_n)^T$, satisfying $w_i > 0$ and $\sum\limits_{i=1}^{n} w_i = 1$ be the weight vector of the attributes $\mathcal{G}_i$. These given alternatives are evaluated by a set of the "l" decision makers $\mathcal{DM}_1$, $\mathcal{DM}_2$, ..., $\mathcal{DM}_l$ under the environment of T2IFS where characteristics of each alternative is accessed in terms of a PMF, SMF, PNMF and SNMF. To find the finest alternative(s) among the given ones, the proposed method has been described with the following steps:

Step 1: Arrange the decision makers' preferences in terms of PMF, SMF, PNMF and SNMF for each alternative $\mathcal{A}_i (i = 1, 2, \ldots, t)$ with respect to attributes $\mathcal{G}_j (j = 1, 2, \ldots, n)$.

Step 2: Compute the distance measure between the decision maker $\mathcal{DM}_k(k = 1, 2, \ldots, l)$ and the null decision $\mathcal{N}$, i.e. $\mathcal{D}(\mathcal{DM}_k, \mathcal{N})$, where $\mathcal{N}$ is a decision of the decision maker, having zero PMF and SMF while one PNMF and SNMF for each alternative $\mathcal{A}_i$ corresponding to each attribute $\mathcal{G}_j$.

Step 3: Find the maximum value of the distance measure corresponding to decision maker $\mathcal{DM}$ and hence construct the type-2 intuitionistic fuzzy alternative $\mathcal{A}_i, i = 1, 2, \ldots, t$ by corresponding maximum value of distance measure $\mathcal{D}$.

Step 4: Compute the distance measure between the alternative $\mathcal{A}_i$ and the null decision $\mathcal{N}$ by using the developed measures between the T2IFSs, i.e., $\mathcal{D}(\mathcal{A}_i, \mathcal{N})$.

Step 5: Rank the alternatives $\mathcal{A}_i(i = 1, 2, \ldots, t)$ based on the measurement values of the distances and select the most desired one(s).

3.4.2 Numerical example

The above developed method is explained with a numerical example as follows:

Consider a decision-making problem in which a person wants to invest some money into the market. For it, they bought a certain panel of experts ($\mathcal{DM}_1, \mathcal{DM}_2$ and $\mathcal{DM}_3$) having weight vector is $(0.40, 0.35, 0.25)^T$. The choice for a person to invest the money in one of the following possible five alternatives namely, $\mathcal{A}_1$ (" car company"), $\mathcal{A}_2$ ("food company"), $\mathcal{A}_3$ ("computer company"), $\mathcal{A}_4$ ("arms company") and $\mathcal{A}_5$ ("tire company"). The investor takes a decision under the different attributes namely, $\mathcal{G}_1$("the risk analysis"), $\mathcal{G}_2$ ("the growth analysis"), $\mathcal{G}_3$ ("the environmental impact analysis") and $\mathcal{G}_4$ ("the available space") whose weight vector is $(0.35, 0.30, 0.20, 0.15)^T$ under the T2IFS environment. To evaluate these alternatives, a group of the panel decides to rates them in terms of the linguistic grades of PMF, PNMF, and SMF, SNMF, which are summarized in Table 3.1. Then, the steps of the proposed method are utilized to access the best alternative(s) as follows:

Step 1: The rating values of each decision maker $\mathcal{DM}_k(k = 1, 2, 3)$ based on their knowledge and experience towards the evaluating of the given alternatives are summarized in Table 3.2.

Table 3.1: Linguistic grade and corresponding PMF,SMF, PNMF and SNMF values

Grades	PMF/SMF values	Grades	PNMF/SNMF values
Very Poor(VP)	0.0	Very Good (VG)	1
Poor (P)	0.2	Good (G)	0.7
Medium Poor (MP)	0.4	Medium Good (MG)	0.5
Fair(F)	0.5	Fair (F)	0.4
Medium Good (MG)	0.7	Medium Poor (MP)	0.2
Good (G)	0.9	Poor (P)	0.1
Very Good (VG)	1.0	Very Poor (VP)	0.0

Table 3.2: Graded values of the alternatives corresponding to each attribute

		$\mathcal{DM}_1$				$\mathcal{DM}_2$				$\mathcal{DM}_3$			
		PMF	SMF	PNMF	SNMF	PMF	SMF	PNMF	SNMF	PMF	SMF	PMNF	SNMF
	$\mathcal{A}_1$	VG	MG	VP	MP	G	VG	P	VP	G	MP	P	MG
	$\mathcal{A}_2$	MG	G	P	P	G	MG	VP	P	P	F	G	F
$\mathcal{G}_1$	$\mathcal{A}_3$	F	MP	F	MG	G	F	P	F	P	MP	G	MG
	$\mathcal{A}_4$	MG	F	MP	F	G	F	P	F	P	VP	G	VG
	$\mathcal{A}_5$	MG	F	MP	F	MG	MP	MP	MG	G	MG	VP	P
	$\mathcal{A}_1$	VG	G	VP	P	G	MP	P	MG	G	MG	P	MP
	$\mathcal{A}_2$	MG	MG	F	F	G	MG	P	MP	P	F	G	F
$\mathcal{G}_2$	$\mathcal{A}_3$	G	VG	P	VP	MG	MP	MP	MG	MP	MG	MG	MP
	$\mathcal{A}_4$	G	MG	P	MP	MG	MP	MP	MG	MP	F	MG	F
	$\mathcal{A}_5$	VG	G	VP	P	F	F	F	F	MP	MG	MG	MP
	$\mathcal{A}_1$	VG	MG	VP	MP	G	VG	P	VP	G	G	P	P
	$\mathcal{A}_2$	VG	MG	VP	MP	G	VG	P	VP	VG	G	VP	P
$\mathcal{G}_3$	$\mathcal{A}_3$	VG	G	VP	P	G	G	P	P	MP	MG	MG	MP
	$\mathcal{A}_4$	VG	G	VP	P	G	F	P	F	MP	MG	MG	MP
	$\mathcal{A}_5$	G	MP	P	MG	MG	MP	MP	MG	MP	MG	MG	MP
	$\mathcal{A}_1$	VG	G	VP	P	G	VG	P	VP	G	F	P	F
	$\mathcal{A}_2$	VG	G	VP	P	G	VG	P	VP	VG	MG	VP	MP
$\mathcal{G}_4$	$\mathcal{A}_3$	VG	MP	VP	MG	MG	G	MP	P	MP	MP	MG	MG
	$\mathcal{A}_4$	MG	F	MP	F	G	F	P	F	F	MG	F	MP
	$\mathcal{A}_5$	G	MG	P	MP	MG	MP	MP	MG	F	MG	F	MP

Step 2: Without loss of generality, we utilize $\mathcal{D}_2$ to compute the separation measure between the decision makers choices of each alternative $\mathcal{A}_i(i = 1, 2, \ldots, 5)$ from its

null decision $\mathcal{N}$, i.e., $\mathcal{D}_2(\mathcal{DM}_k, \mathcal{N})$. The results corresponding to them is represented in Table 3.3.

Table 3.3: Distance measures between $\mathcal{D}_2$ and $\mathcal{N}$

	$\mathcal{G}_1$			$\mathcal{G}_2$			$\mathcal{G}_3$			$\mathcal{G}_4$		
	$\mathcal{DM}_1$	$\mathcal{DM}_2$	$\mathcal{DM}_3$	$\mathcal{DM}_1$	$\mathcal{DM}_2$	$\mathcal{DM}_3$	$\mathcal{DM}_1$	$\mathcal{DM}_2$	$\mathcal{DM}_3$	$\mathcal{DM}_1$	$\mathcal{DM}_2$	$\mathcal{DM}_3$
$\mathcal{A}_1$	1.00	1.00	0.90	1.00	0.90	0.90	1.00	1.00	0.90	1.00	1.00	0.90
$\mathcal{A}_2$	0.80	0.95	0.55	0.65	0.90	0.55	1.00	0.90	1.00	1.00	1.00	1.00
$\mathcal{A}_3$	0.55	0.90	0.45	1.00	0.75	0.75	1.00	0.90	0.75	1.00	0.90	0.45
$\mathcal{A}_4$	0.75	0.90	0.15	0.90	0.75	0.55	1.00	0.90	0.75	0.75	0.90	0.75
$\mathcal{A}_5$	0.75	0.75	1.00	1.00	0.55	0.75	0.90	0.75	0.75	0.90	0.75	0.75

Step 3: Find the maximum value of $\mathcal{D}_2(\mathcal{DM}_k, \mathcal{N})$ from Table 3.3 for all alternatives $\mathcal{A}_i, (i = 1, 2, \ldots, 5)$ corresponding to each criteria $\mathcal{G}_j, (j = 1, 2, 3, 4)$ and hence construct the T2IFS alternative $\mathcal{A}_i = \{\mathcal{G}_j(u_{\mathcal{A}_i}, f_{\mathcal{G}_j}(\mathcal{A}_i), v_{\mathcal{A}_i}, t_{\mathcal{G}_j}(\mathcal{A}_i))\}$ as

$$\begin{aligned}
\mathcal{A}_1 &= \{\mathcal{G}_1(1, 0.7, 0, 0.2), \mathcal{G}_2(1, 0.9, 0, 0.1), \mathcal{G}_3(1, 0.7, 0, 0.2), \mathcal{G}_4(0.9, 1, 0.1, 0)\}; \\
\mathcal{A}_2 &= \{\mathcal{G}_1(0.9, 0.7, 0, 0.1), \mathcal{G}_2(0.9, 0.7, 0.1, 0.2), \mathcal{G}_3(1, 0.7, 0, 0.2), \mathcal{G}_4(0.9, 1, 0.1, 0)\}; \\
\mathcal{A}_3 &= \{\mathcal{G}_1(0.9, 0.5, 0.1, 0.4), \mathcal{G}_2(0.9, 1, 0.1, 0), \mathcal{G}_3(1.0, 0.9, 0, 0.1), \mathcal{G}_4(1, 0.4, 0, 0.5)\}; \\
\mathcal{A}_4 &= \{\mathcal{G}_1(0.9, 0.5, 0.1, 0.4), \mathcal{G}_2(0.9, 0.7, 0.1, 0.2), \mathcal{G}_3(1, 0.9, 0, 0.1), \mathcal{G}_4(0.9, 0.5, 0.1, 0.4)\}; \\
\mathcal{A}_5 &= \{\mathcal{G}_1(0.9, 0.7, 0, 0.1), \mathcal{G}_2(1, 0.9, 0, 0.1), \mathcal{G}_3(0.9, 0.4, 0.1, 0.5), \mathcal{G}_4(0.9, 0.7, 0.1, 0.2)\}
\end{aligned}$$

Step 4: Compute the measurement value of each alternative $\mathcal{A}_i$ from the null T2IFS $\mathcal{N}$ by using developed distance measure $\mathcal{D}_2$ and their corresponding measurement values are obtained as $\mathcal{D}_2(\mathcal{A}_1, \mathcal{N}) = 1.0000$, $\mathcal{D}_2(\mathcal{A}_2, \mathcal{N}) = 0.9625$, $\mathcal{D}_2(\mathcal{A}_3, \mathcal{N}) = 0.9750$, $\mathcal{D}_2(\mathcal{A}_4, \mathcal{N}) = 0.8875$ and $\mathcal{D}_2(\mathcal{A}_5, \mathcal{N}) = 0.9675$.

Step 5: Based on these measurement values, we conclude that the $\mathcal{A}_1$, i.e., the car company is the most suitable one to invest the money.

On the other hand, if we apply the other developed distance measures such as $\mathcal{D}_1, \mathcal{D}_3$, etc., for computing the best alternative(s), then Step 2 of the proposed approach has been executed for each distance measures and hence their corresponding T2IFS set has been constructed. Finally, based on these sets, the rating value of the distance measures corresponding to each alternative is computed and ranking has been done which are summarized in Table 3.4. From this table, it has been concluded that the best alternative is still $\mathcal{A}_1$ in all cases.

Table 3.4: Effect of the measures on to the final ranking order of the alternatives

Measures	Overall measurement values of $\mathcal{A}_i$ from $\mathcal{N}$					Ranking order
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	
$\mathcal{D}_1$	4.0000	3.8500	3.9000	3.5500	3.7500	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4$
$\mathcal{D}_2$	1.0000	0.9625	0.9750	0.8875	0.9675	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4$
$\mathcal{D}_3$	1.9097	1.8193	1.8432	1.6867	1.7349	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4$
$\mathcal{D}_4$	0.9539	0.9097	0.8639	0.8434	0.8675	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
$\mathcal{D}_5$	0.2500	0.2381	0.2412	0.2300	0.2194	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5$
$\mathcal{D}_6$	0.2144	0.1921	0.1919	0.1755	0.1648	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5$
$\mathcal{D}_1^U$	0.2500	0.2437	0.2437	0.2250	0.2375	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4$
$\mathcal{D}_2^U$	0.0625	0.0584	0.0603	0.0544	0.0559	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4$
$\mathcal{D}_3^U$	0.5000	0.4880	0.4880	0.4630	0.4757	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4$
$\mathcal{D}_4^U$	0.2500	0.2341	0.2415	0.2184	0.2247	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4$

3.4.3 Comparative study

Since interval-valued fuzzy sets, type-2 fuzzy sets are the special cases of the T2IFSs and hence in order to compare the performance of the developed methods with some existing methods, a comparative studies based on interval-valued and type-2 fuzzy set as proposed by the authors [10, 72, 130, 138, 150, 163, 169, 170] have been taken and their corresponding results are summarized in Table 3.5. From this table, it has been seen that the best company for investing the money is $\mathcal{A}_1$ than others and this result has been overlapped with the proposed results. Thus, the proposed technique can be suitably utilized to solve the problem of decision-making problem than the other existing measures. However, the considered theory under the T2IFS has different structure and generalization of these existing theory. Therefore, the developed methods contain wider information towards the decision-making process and hence handle the uncertainty in a more profitable way.

Table 3.5: Comparative analysis with the existing studies

Existing Techniques	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	Ranking
Burillo and Bustince [10]	0.8000	0.8000	0.7500	0.7750	0.7500	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5$
Szmidt and Kacprazyk [138]	0.8333	0.6042	0.7333	0.5063	0.5468	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4$
Hung and Yang [72]	0.6762	0.7278	0.3722	0.4771	0.4315	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Zeng and Li [169]	0.8000	0.7000	0.6500	0.5250	0.6000	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_4$
Zeng and Guo [170]	0.4000	0.4000	0.3750	0.3875	0.3750	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5$
Yang and Lin [163]	0.1814	0.1414	0.0903	0.1176	0.1252	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Wei and Wang [150]	0.7843	0.5556	0.4707	0.3520	0.5556	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Singh [130]	1.0000	0.9500	0.9750	0.8750	0.9250	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4$

3.5 Conclusion

In this chapter, a family of Hamming, Euclidean and utmost distance measures for T2IFSs has been proposed by considering the PMF, SMF, PNMF, SNMF, FOU, and VMF. Several desirable properties of these measures have been investigated in detail. Further, a ranking method based on these measures has been proposed for solving group decision-making problems and illustrated with a numerical example. The proposed method has more fuzziness and uncertainties due to the fact that it uses type-2 intuitionistic fuzzy sets rather than existed fuzzy sets. From the studies, it has been concluded that the proposed results coincide with the ones, shown in existing approaches and hence place an alternative way for solving the decision-making problems.

Chapter 4

Similarity measures between type-2 intuitionistic fuzzy sets and its applications to multiattribute group decision making process[1]

The objective of this chapter is to explore the theory of type-2 intuitionistic fuzzy set (T2IFS) in which the membership degrees for each member of the object is itself an IFS. In real life, most of the decision-making problems are always uncertain in nature and difficult to represent it in terms of crisp or precise numbers. For handling such types of information, T2IFS is one of the pioneer ways which provide an additional degree of freedom to the decision maker to reach the decision and is a generalization of several existing theories. Motivated from it, this chapter presents some families of the similarity measures to measure the degree of similarity between the two or more T2IFSs. Further, a group decision-making approach is explored based on proposed measures to rank the alternatives and demonstrate with an example.

4.1 Introduction

Multiple attribute group decision-making (MAGDM) is the important content of decision-making science, and its goal is to choose the best one from limited alternatives based on

[1]The content of this chapter is published as "Algorithm for solving group decision making problems based on the similarity measures under type-2 intuitionistic fuzzy sets environment", *Soft Computing, Springer*, 24(10), 7361 - 7381, 2020, doi: 10.1007/s00500-019-04359-8 (**SCI: Impact Factor: 2.784**).

decision-making attributes by multiple decision-makers (DMs). To access it, DMs can express their preferences either by fuzzy variables as they are more suitable for human cognitive activities. In recent years, some MAGDM methods have been proposed [38, 58, 129, 134, 136, 159, 165] based on the aggregation operators (AOs) and distance/similarity measures for IFSs. In [159], Xu presented averaging AOs for different IFNs. Garg [38] presented some generalized AOs for IFNs.

Similarity measure and distance measure are functions that give the degree of similarity and discrimination respectively among the two objects. Entropy measure quantifies the degree of fuzziness or uncertain information and the inclusion measure between two sets gives the extent to which a set is contained in another set. All these measures of information have been extensively explored by many researchers and scholars as vital topics. For instance, Szmidt [136] presented distance and similarity measures for IFSs and applied them to various DM problems. Shen et al. [129] proposed a novel distance measure and discussed its various properties and further developed a TOPSIS ("technique for order preference by similarity to ideal solution") based on the proposed measure. Ye [165] put forward two similarity measures based on cosine functions. Song et al. [134] proposed a measure of similarity for IFSs and applied it to medical diagnosis problem. Garg and Kumar [58] presented the TOPSIS method based on exponential distance measures for solving MAGDM problems.

Since in the above-mentioned work, researchers have considered as a crisp membership function to its element. However, in many situations, uncertainty is not probabilities in nature but it is imprecise or vague in nature. To address it, the concept of T2FS [98], an extension of FS, in which membership values are type-1 FSs on [0,1] is developed. In T2FS, there is an additional membership function which provides an additional degree of freedom to the practices to model the uncertainties. Some operations on T2FSs were studied in [15, 17]. Later on, Mendel and Wu [101, 102] enrich the study of T2FS into the different fields. Mitchell [106] rank the two different type-2 fuzzy numbers (T2FNs). Hung and Yang [71] presented the similarity measure between T2FSs. In [130], authors presented some similarity and distance measures for T2FSs. Apart from them, some other information measures for T2FSs are presented in [153, 154, 163]. Further, a concept of interval T2FSs is proposed by Mendel et al. [99] and hence it is widely. Under these environments, many methods have been developed to solve the decision-making problems by the scholars. For more details, we refers to read the articles [14, 15, 75, 89, 118, 120, 147]. In these existing studies, authors have considered only the membership degrees during the analysis. Later on, a concept of T2FS is extended to type-2 IFS (T2IFS) by embedding the features of the primary as well as secondary non-membership degree of an element. Further, under this T2IFS environment, authors have presented some series of distance measures. Considering the fact that the T2IFS is one of the pioneer ways to represent the uncertainty in the data with some more degree of freedom to the decision makers and the similarity measure is one of the information measures to measure the degree of similarity between the two sets. Therefore, by keeping the features of T2IFSs and the similarity measures, it is vital to amplify some new similarity measures to process the T2IFSs in MAGDM methods. These contemplations motivated us to think about the following fundamental targets for this chapter:

1) to introduce some new similarity measures under T2IFSs environment;

2) to create an algorithm to illuminate group decision-making issues with proposed measures;

3) to exhibit an illustration where the significance of preferences based on T2IFS decision problems has been clarified.

To achieve objective 1 is achieved by proposing some new similarity measures based on the geometric model and some new extensions of it to compute the degree of similarity between the two or more T2IFSs. The working of the proposed measure is demonstrated through an illustrative example. Further, we develop a multi-criteria group decision-making approach by using proposed measures between T2IFNs to complete the 2nd objective. Finally, the applicability of the proposed approach is demonstrated through a real-life numerical example and a details analysis of the impact of the decision maker is carried out to finalize the third objective.

4.2 Novel proposed Similarity measures

For a universal set $\mathcal{X}$, denote $F_2(\mathcal{X})$ be the set of all T2IFSs.

Definition 4.2.1. Let $S : F_2(\mathcal{X}) \times F_2(\mathcal{X}) \to [0,1]$ be a mapping, and let $\mathcal{A}_i \in F_2(\mathcal{X})$ $(i = 1,2,3)$. Then $\mathcal{S}$ is called the similarity degree, if following conditions are satisfied:

(P1) $0 \leq \mathcal{S}(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) If $\mathcal{A}_1 = \mathcal{A}_2$, then $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2) = 1$.

(P3) $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{S}(\mathcal{A}_2, \mathcal{A}_1)$.

(P4) If $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3$ then $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}(\mathcal{A}_1, \mathcal{A}_2)$ and $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}(\mathcal{A}_2, \mathcal{A}_3)$.

To compute the similarity measures between the T2IFSs, we consider the FOU of the PMF, PNMF, SMF, SNMF as well as variance margin between the functions. For convenience, two T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$ defined in $\mathcal{X}$ are denoted by $\mathcal{A}_1 = \langle x, (u_{\mathcal{A}_1}, f_x(u_{\mathcal{A}_1}), v_{\mathcal{A}_1}, t_x(v_{\mathcal{A}_1})) \mid x \in \mathcal{X} \rangle$ and $\mathcal{A}_2 = \langle x, (u_{\mathcal{A}_2}, f_x(u_{\mathcal{A}_2}), v_{\mathcal{A}_2}, t_x(v_{\mathcal{A}_2})) \mid x \in \mathcal{X} \rangle$. Then, measures between $\mathcal{A}_1$ and $\mathcal{A}_2$ based on the geometric distance model is given as .

Definition 4.2.2. If $\lambda \geq 1$ be a real number, then

$$\mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) = 1 - \left[\frac{1}{4} \sum_{i=1}^{n} w_i \begin{pmatrix} \mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid^{\lambda} + \mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid^{\lambda} \\ + \mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid^{\lambda} + \mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid^{\lambda} \\ + \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid^{\lambda} + \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid^{\lambda} \end{pmatrix} \right]^{\frac{1}{\lambda}} \tag{4.1}$$

where $w_i > 0$ and $\sum\limits_{i=1}^{n} w_i = 1$.

In particular, if $\lambda = 1$, then it becomes weighted Hamming similarity; if $\lambda = 2$ then it becomes weighted Euclidean similarity. If $w = (1/n, 1/n, \ldots, 1/n)^T$ then it reduces to normalized similarity.

Theorem 4.2.1. If $\lambda \to +\infty$, then similarity measure $\mathcal{S}_1$ reduces to

$$\lim_{\lambda \to +\infty} \mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) = 1 - \max_i \left\{ \begin{array}{l} \mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid, \mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid, \\ \mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid, \mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid, \\ \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid, \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid \end{array} \right\} \tag{4.2}$$

Proof. Let $l = \max_i \left\{ \begin{array}{l} \mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid, \mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid, \\ \mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid, \mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid, \\ \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid, \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid \end{array} \right\}$. Then, by definition of $\mathcal{S}_1$, we have

$$\begin{aligned}
&\lim_{\lambda \to +\infty} \mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) \\
= & \lim_{\lambda \to +\infty} \left[1 - \left[\frac{1}{4} \sum_{i=1}^{n} w_i \left(\begin{array}{l} \mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid^\lambda + \mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid^\lambda \\ + \mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid^\lambda + \mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid^\lambda \\ + \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid^\lambda + \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid^\lambda \end{array} \right) \right]^{\frac{1}{\lambda}} \right] \\
= & \lim_{\lambda \to +\infty} \left[1 - l \left[\tfrac{1}{4} \sum_{i=1}^{n} w_i \left(\begin{array}{l} \left(\frac{\mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid}{l} \right)^\lambda + \left(\frac{\mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid}{l} \right)^\lambda \\ + \left(\frac{\mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid}{l} \right)^\lambda + \left(\frac{\mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid}{l} \right)^\lambda \\ + \left(\frac{\mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid}{l} \right)^\lambda + \left(\frac{\mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid}{l} \right)^\lambda \end{array} \right) \right]^{\frac{1}{\lambda}} \right] \\
= & \; 1 - l \lim_{\lambda \to +\infty} \left(\tfrac{1}{4}\right)^{1/\lambda} \lim_{\lambda \to +\infty} \left[\sum_{i=1}^{n} w_i \left(\begin{array}{l} \left(\frac{\mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid}{l} \right)^\lambda + \left(\frac{\mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid}{l} \right)^\lambda \\ + \left(\frac{\mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid}{l} \right)^\lambda + \left(\frac{\mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid}{l} \right)^\lambda \\ + \left(\frac{\mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid}{l} \right)^\lambda + \left(\frac{\mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid}{l} \right)^\lambda \end{array} \right) \right]^{\frac{1}{\lambda}} \\
= & \; 1 - l
\end{aligned}$$

i.e., Eq. (4.2) holds. □

Proposition 4.2.1. The defined similarity $\mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2)$, between T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfies the following properties:

(P1) $0 \leq \mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) $\mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) = 1$, if $\mathcal{A}_1 = \mathcal{A}_2$.

(P3) $\mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{S}_1(\mathcal{A}_2, \mathcal{A}_1)$.

(P4) If $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3$ then $\mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2)$ and $\mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}_1(\mathcal{A}_2, \mathcal{A}_3)$.

Proof. For real number $\lambda \geq 1$, we have

(P1) Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are T2IFSs, we have, $|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda \geq 0$, $|f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda \geq 0$, $|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda \geq 0$, $|v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^\lambda \geq 0$, $|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^\lambda \geq 0$, and $|\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^\lambda \geq 0$. Thus, $|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^\lambda + |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^\lambda + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^\lambda \geq 0$ which implies that $\frac{1}{4}\sum_{i=1}^{n}(\omega_i(|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^\lambda + |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^\lambda + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^\lambda)) \geq 0$. Hence $\mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) \leq 1$. Further, $|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda \leq 1$, $|f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda \leq 1$, $|\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda \leq 1$, $|v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^\lambda \leq 1$, $|t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^\lambda \leq 1$ and $|\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^\lambda \leq 1$. Therefore, $\sum_{i=1}^{n}\omega_i(|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^\lambda + |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^\lambda + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^\lambda) \leq 4 \Rightarrow \mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) \geq 0$. Thus $0 \leq \mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) Let $\mathcal{A}_1 = \mathcal{A}_2$ which implies $u_{\mathcal{A}_1}(x_i) = u_{\mathcal{A}_2}(x_i)$, $f_{x_i}(u_{\mathcal{A}_1}) = f_{x_i}(u_{\mathcal{A}_2})$, $\xi_{\mathcal{A}_1}(x_i) = \xi_{\mathcal{A}_2}(x_i)$, $v_{\mathcal{A}_1}(x_i) = v_{\mathcal{A}_2}(x_i)$, $t_{x_i}(v_{\mathcal{A}_1}) = t_{x_i}(v_{\mathcal{A}_2})$, $\eta_{\mathcal{A}_1}(x_i) = \eta_{\mathcal{A}_2}(x_i)$. Therefore, $|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^\lambda + |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^\lambda + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^\lambda = 0$ and hence $\mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) = 1$.

(P3) $\mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) = 1 - (\frac{1}{4}\sum_{i=1}^{n}\omega_i\big[|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda + |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^\lambda + |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^\lambda + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^\lambda\big])^{1/\lambda}$
$= 1 - (\frac{1}{4}\sum_{i=1}^{n}\omega_i\big[|u_{\mathcal{A}_2}(x_i) - u_{\mathcal{A}_1}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_2}) - f_{x_i}(u_{\mathcal{A}_1})|^\lambda + |\xi_{\mathcal{A}_2}(x_i) - \xi_{\mathcal{A}_1}(x_i)|^\lambda + |v_{\mathcal{A}_2}(x_i) - v_{\mathcal{A}_1}(x_i)|^\lambda + |t_{x_i}(v_{\mathcal{A}_2}) - t_{x_i}(v_{\mathcal{A}_1})|^\lambda + |\eta_{\mathcal{A}_2}(x_i) - \eta_{\mathcal{A}_1}(x_i)|^\lambda\big])^{1/\lambda} = \mathcal{S}_1(\mathcal{A}_2, \mathcal{A}_1)$

(P4) As $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3$ which means that $u_{\mathcal{A}_1} \leq u_{\mathcal{A}_2} \leq u_{\mathcal{A}_3}$, $f_x(u_{\mathcal{A}_1}) \leq f_x(u_{\mathcal{A}_2}) \leq f_x(u_{\mathcal{A}_3})$ and $v_{\mathcal{A}_1} \geq v_{\mathcal{A}_2} \geq v_{\mathcal{A}_3}$, $t_x(u_{\mathcal{A}_1}) \leq t_x(u_{\mathcal{A}_2}) \leq t_x(u_{\mathcal{A}_3})$. Thus, $u_{\mathcal{A}_1} - u_{\mathcal{A}_2} \geq u_{\mathcal{A}_1} - u_{\mathcal{A}_3}$, $f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \geq f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_3})$ etc., and hence $\mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_2) \geq \mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_3)$. Similarly, we have $\mathcal{S}_1(\mathcal{A}_2, \mathcal{A}_3) \geq \mathcal{S}_1(\mathcal{A}_1, \mathcal{A}_3)$.

$\square$

Next, based on geometric model, we define similarity measures as:

Definition 4.2.3. Let $\lambda \geq 1$ be a real number, then the similarity measure between T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$ is defined as:

$$\mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2) = 1 - \left[\frac{1}{4} \sum_{i=1}^{n} w_i \left| \begin{array}{l} (u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)) + (f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})) \\ + (\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)) + (v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)) \\ + (t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})) + (\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)) \end{array} \right|^{\lambda} \right]^{\frac{1}{\lambda}} \quad (4.3)$$

Proposition 4.2.2. The defined measure $\mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2)$ between T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfies the following properties:

(P1) $0 \leq \mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) If $\mathcal{A}_1 = \mathcal{A}_2$, then $\mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2) = 1$.

(P3) $\mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{S}_2(\mathcal{A}_2, \mathcal{A}_1)$.

(P4) If $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3$ then $\mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2)$ and $\mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}_2(\mathcal{A}_2, \mathcal{A}_3)$.

Proof. For a real number $\lambda \geq 1$ and $\mathcal{A}_1$ and $\mathcal{A}_2$ are T2IFSs, we have

(P1) $u_{\mathcal{A}_1}(x_i) \geq 0$, $u_{\mathcal{A}_2}(x_i) \geq 0$, $f_{x_i}(u_{\mathcal{A}_1}) \geq 0$, $f_{x_i}(u_{\mathcal{A}_2}) \geq 0$, $\xi_{\mathcal{A}_1}(x_i) \geq 0$, $\xi_{\mathcal{A}_2}(x_i) \geq 0, v_{\mathcal{A}_1}(x_i) \geq 0$, $v_{\mathcal{A}_2}(x_i) \geq 0$, $t_{x_i}(v_{\mathcal{A}_1}) \geq 0$, $t_{x_i}(v_{\mathcal{A}_2}) \geq 0$, $\eta_{\mathcal{A}_1}(x_i) \geq 0$, $\eta_{\mathcal{A}_2}(x_i) \geq 0$ which give us that $\mid (u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)) + (f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})) + (\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)) + (v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)) + (t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})) + (\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)) \mid^{\lambda} \geq 0$. Thus, $\mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2) \leq 1$. Further, $u_{\mathcal{A}_1}(x_i) \leq 1$, $u_{\mathcal{A}_2}(x_i) \leq 1$, $f_{x_i}(u_{\mathcal{A}_1}) \leq 1$, $f_{x_i}(u_{\mathcal{A}_2}) \leq 1$, $\xi_{\mathcal{A}_1}(x_i) \leq 1$, $\xi_{\mathcal{A}_2}(x_i) \leq 1$, $v_{\mathcal{A}_1}(x_i) \leq 1$, $v_{\mathcal{A}_2}(x_i) \leq 1$, $t_{x_i}(v_{\mathcal{A}_1}) \leq 1$, $t_{x_i}(v_{\mathcal{A}_2}) \leq 1$, $\eta_{\mathcal{A}_1}(x_i) \leq 1$, $\eta_{\mathcal{A}_2}(x_i) \leq 1$ and $\omega_i > 0$ for all i. Thus, we have

$$\frac{1}{4} \sum_{i=1}^{n} w_i \left| \begin{array}{l} (u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)) + (f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})) \\ + (\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)) + (v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)) \\ + (t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})) + (\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)) \end{array} \right|^{\lambda} \leq 1$$

$$\Rightarrow \quad 1 - \left(\frac{1}{4} \sum_{i=1}^{n} w_i \left| \begin{array}{l} (u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)) + (f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})) \\ + (\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)) + (v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)) \\ + (t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})) + (\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)) \end{array} \right|^{\lambda} \right)^{1/\lambda} \geq 0$$

$$\Rightarrow \quad \mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2) \geq 0$$

Since, $\mathcal{A}_1$ and $\mathcal{A}_2$ are arbitrary T2IFSs so we have $0 \leq \mathcal{S}_2 \leq 1$.

(P2) Let $\mathcal{A}_1 = \mathcal{A}_2$ which implies $u_{\mathcal{A}_1}(x_i) = u_{\mathcal{A}_2}(x_i)$, $f_{x_i}(u_{\mathcal{A}_1}) = f_{x_i}(u_{\mathcal{A}_2})$, $\xi_{\mathcal{A}_1}(x_i) = \xi_{\mathcal{A}_2}(x_i)$, $v_{\mathcal{A}_1}(x_i) = v_{\mathcal{A}_2}(x_i)$, $t_{x_i}(v_{\mathcal{A}_1}) = t_{x_i}(v_{\mathcal{A}_2})$, $\eta_{\mathcal{A}_1}(x_i) = \eta_{\mathcal{A}_2}(x_i)$. Therefore, $\mid (u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)) + (f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})) + (\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)) + (v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)) + (t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})) + (\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)) \mid^{\lambda} = 0$ and hence $\mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2) = 1$.

(P3) For any two positive real numbers a and b, we know that $\mid a - b \mid = \mid b - a \mid$. Thus, by definition of $\mathcal{S}_2$, we have $\mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{S}_2(\mathcal{A}_2, \mathcal{A}_1)$.

(P4) Since $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3$ then we have $u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \geq u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_3}(x_i)$, $f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \geq f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_3})$ etc., which implies $\mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_2) \geq \mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_3)$ and $\mathcal{S}_2(\mathcal{A}_2, \mathcal{A}_3) \geq \mathcal{S}_2(\mathcal{A}_1, \mathcal{A}_3)$.

Hence, $\mathcal{S}_2$ is a valid similarity measure. □

Next, we propose some new similarity measure by considering the length of the subinterval, the median of the subinterval. For it, we modify the definition of similarity measures. Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two T2IFSs. Let $\varphi_1 = \varphi_{\mu_{\mathcal{A}_1\mathcal{A}_2}} + \varphi_{\nu_{\mathcal{A}_1\mathcal{A}_2}}$, $\varphi_2 = \mid \varphi_{\mathcal{A}_1}(x_i) - \varphi_{\mathcal{A}_2}(x_i) \mid$, where

$$\varphi_{\mathcal{A}_1}(x_i) = \frac{2 + u_{\mathcal{A}_1}(x_i) + f_{x_i}(u_{\mathcal{A}_1}) + \xi_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_1}(x_i) - t_{x_i}(v_{\mathcal{A}_1}) - \eta_{\mathcal{A}_1}(x_i)}{4},$$

$$\varphi_{\mathcal{A}_2}(x_i) = \frac{2 + u_{\mathcal{A}_2}(x_i) + f_{x_i}(u_{\mathcal{A}_2}) + \xi_{\mathcal{A}_2}(x_i) - v_{\mathcal{A}_2}(x_i) - t_{x_i}(v_{\mathcal{A}_2}) - \eta_{\mathcal{A}_2}(x_i)}{4},$$

$$\varphi_{\mu_{\mathcal{A}_1\mathcal{A}_2}} = \frac{1}{4} \left(\mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid + \mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid + \mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid\right),$$

$$\varphi_{\nu_{\mathcal{A}_1\mathcal{A}_2}} = \frac{1}{4} \left(\mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid + \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid + \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid\right),$$

$$l_{\mathcal{A}_1} = 1 - \frac{v_{\mathcal{A}_1}(x_i) - t_{x_i}(v_{\mathcal{A}_1}) - \eta_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_1}(x_i) - f_{x_i}(u_{\mathcal{A}_1}) - \xi_{\mathcal{A}_1}(x_i)}{4},$$

$$l_{\mathcal{A}_2} = 1 - \frac{v_{\mathcal{A}_2}(x_i) - t_{x_i}(v_{\mathcal{A}_2}) - \eta_{\mathcal{A}_2}(x_i) - u_{\mathcal{A}_2}(x_i) - f_{x_i}(u_{\mathcal{A}_2}) - \xi_{\mathcal{A}_2}(x_i)}{4},$$

φ_3 denotes degree of dissimilarity of the length and is defined as

$$\varphi_3 = \max(l_{\mathcal{A}_1}(x_i), l_{\mathcal{A}_2}(x_i)) - \min(l_{\mathcal{A}_1}(x_i), l_{\mathcal{A}_2}(x_i)).$$

Definition 4.2.4. Assume that φ_1, φ_2 and φ_3 are given. Then

$$\mathcal{S}_3(\mathcal{A}_1, \mathcal{A}_2) = 1 - \left[\sum_{i=1}^{n} w_i \left(\sum_{j=1}^{3} \beta_j \varphi_j(x_i) \right)^{\lambda} \right]^{\frac{1}{\lambda}} \tag{4.4}$$

where $\beta_j > 0$, $\sum\limits_{j=1}^{3} \beta_j = 1$.

It is clear from the definition that $\mathcal{S}_3(\mathcal{A}_1, \mathcal{A}_2)$ contains more information on T2IFSs than $\mathcal{S}_1$ and $\mathcal{S}_2$, so it measures the similarity between the set more correctly and accurately than $\mathcal{S}_1$ or $\mathcal{S}_2$.

Proposition 4.2.3. For all $\mathcal{A}_1, \mathcal{A}_2 \in F_2(\mathcal{X})$, $\mathcal{S}_3(\mathcal{A}_1, \mathcal{A}_2)$ is a valid similarity measure.

Proof. Let $\lambda \geq 1$ be a real number and $\mathcal{A}_1, \mathcal{A}_2 \in F_2(\mathcal{X})$. Then, we have

(P1) $\mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid \geq 0$, $\mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid \geq 0$, $\mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid \geq 0$, $\mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid \geq 0$, $\mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid \geq 0$ and $\mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid \geq 0$ which implies that $\mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid + \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid + \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid + \mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid + \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid + \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid \geq 0$. Also, $\mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid \leq 1, \mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid \leq 1, \mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid \leq 1$, $\mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid \leq 1, \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid \leq 1, \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid \leq 1$ which implies that $\mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid + \mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid + \mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid \leq 2$ and $\mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid + \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid + \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid \leq 2$. Hence, $\varphi_{\mu_{\mathcal{A}_1 \mathcal{A}_2}} \leq \frac{1}{2}$ and $\varphi_{\nu_{\mathcal{A}_1 \mathcal{A}_2}} \leq \frac{1}{2}$. Therefore, $\varphi_1(x_i) = \varphi_{\mu_{\mathcal{A}_1 \mathcal{A}_2}}(x_i) + \varphi_{\nu_{\mathcal{A}_1 \mathcal{A}_2}}(x_i) \leq 1$.

Further, $\mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid + \mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid + \mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid + \mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid + \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid + \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid \leq 4$ which gives us that

$$
\begin{aligned}
\varphi_2 &= \mid \varphi_{\mathcal{A}_1}(x_i) - \varphi_{\mathcal{A}_2}(x_i) \mid \\
&= \left| \begin{array}{l} \dfrac{2 + u_{\mathcal{A}_1}(x_i) + f_{x_i}(u_{\mathcal{A}_1}) + \xi_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_1}(x_i) - t_{x_i}(v_{\mathcal{A}_1}) - \eta_{\mathcal{A}_1}(x_i)}{4} \\ - \dfrac{2 + u_{\mathcal{A}_2}(x_i) + f_{x_i}(u_{\mathcal{A}_2}) + \xi_{\mathcal{A}_2}(x_i) - v_{\mathcal{A}_2}(x_i) - t_{x_i}(v_{\mathcal{A}_2}) - \eta_{\mathcal{A}_2}(x_i)}{4} \end{array} \right| \\
&\leq \frac{1}{4} \left(\begin{array}{l} \mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid + \mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid + \mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid \\ + \mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid + \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid + \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid \end{array} \right) \\
&\leq 1
\end{aligned}
$$

Thus, $0 \leq \varphi_2 \leq 1$. Similarly, we have $0 \leq \varphi_3 \leq 1$. Now, for any real number λ, $\sum_{j=1}^{3} \beta_j = 1$, we have $\sum_{j=1}^{3} \beta_j \varphi_j(x_i) \in [0,1]$ which further gives us that $\left[\sum_{i=1}^{n} w_i \left(\sum_{j=1}^{3} \beta_j \varphi_j(x_i) \right)^{\lambda} \right]^{\frac{1}{\lambda}} \in [0,1]$. Hence, $0 \leq \mathcal{S}_3(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) Let $\mathcal{A}_1 = \mathcal{A}_2$ which implies $u_{\mathcal{A}_1}(x_i) = u_{\mathcal{A}_2}(x_i)$, $f_{x_i}(u_{\mathcal{A}_1}) = f_{x_i}(u_{\mathcal{A}_2})$, $\xi_{\mathcal{A}_1}(x_i) = \xi_{\mathcal{A}_2}(x_i)$, $v_{\mathcal{A}_1}(x_i) = v_{\mathcal{A}_2}(x_i)$, $t_{x_i}(v_{\mathcal{A}_1}) = t_{x_i}(v_{\mathcal{A}_2})$, $\eta_{\mathcal{A}_1}(x_i) = \eta_{\mathcal{A}_2}(x_i)$. Then $\varphi_1 = 0, \varphi_2 = 0$ and $\varphi_3 = 0$. Therefore, by definition of $\mathcal{S}_3$, we have $\mathcal{S}_3(\mathcal{A}_1, \mathcal{A}_2) = 1$.

(P3) It is trivial.

(P4) Since $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3$ then we have $u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \geq u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_3}(x_i)$, $f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \geq f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_3})$ etc., which implies $\varphi_{\mu_{\mathcal{A}_1\mathcal{A}_2}}(x_i) \geq \varphi_{\mu_{\mathcal{A}_1\mathcal{A}_3}}(x_i)$ and $\varphi_{\nu_{\mathcal{A}_1\mathcal{A}_2}}(x_i) \geq \varphi_{\nu_{\mathcal{A}_1\mathcal{A}_3}}(x_i)$. Similarly, we can obtain for others. Hence, by definition of $\mathcal{S}_3$, we get $\mathcal{S}_3(\mathcal{A}_1, \mathcal{A}_2) \geq \mathcal{S}_3(\mathcal{A}_1, \mathcal{A}_3)$ and $\mathcal{S}_3(\mathcal{A}_2, \mathcal{A}_3) \geq \mathcal{S}_3(\mathcal{A}_1, \mathcal{A}_3)$.

Therefore, $\mathcal{S}_3$ is a valid similarity measure. □

Next, we present a new similarity measure by modifying the similarity measure $\mathcal{S}_1$ in order to adopt a statistical approach as follows:

$$\mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_2) = \frac{\gamma_\mu(\mathcal{A}_1, \mathcal{A}_2) + \gamma_\nu(\mathcal{A}_1, \mathcal{A}_2)}{2} \tag{4.5}$$

where $\lambda \geq 1$ and

$$\gamma_\mu(\mathcal{A}_1, \mathcal{A}_2) = 1 - \left[\frac{1}{2} \sum_{i=1}^{n} w_i \begin{pmatrix} \mid u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \mid^\lambda + \mid f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \mid^\lambda \\ + \mid \xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) \mid^\lambda \end{pmatrix} \right]^{\frac{1}{\lambda}}$$

$$\gamma_\nu(\mathcal{A}_1, \mathcal{A}_2) = 1 - \left[\frac{1}{2} \sum_{i=1}^{n} w_i \begin{pmatrix} \mid v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i) \mid^\lambda + \mid t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2}) \mid^\lambda \\ + \mid \eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i) \mid^\lambda \end{pmatrix} \right]^{\frac{1}{\lambda}}$$

denotes the similarity measures of "low" membership degrees $u_{\mathcal{A}_1}$ and $u_{\mathcal{A}_2}$, $f_{x_i}(u_{\mathcal{A}_1})$ and $f_{x_i}(u_{\mathcal{A}_2})$ and the "high" membership functions of $1 - v_{\mathcal{A}_1}$ and $1 - v_{\mathcal{A}_2}$ and so on.

Proposition 4.2.4. The measure $\mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_2)$, between the two T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfies the following properties:

(P1) $0 \leq \mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) If $\mathcal{A}_1 = \mathcal{A}_2$, then $\mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_2) = 1$.

(P3) $\mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{S}_4(\mathcal{A}_2, \mathcal{A}_1)$.

(P4) If $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3$ then $\mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_2)$ and $\mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}_4(\mathcal{A}_2, \mathcal{A}_3)$.

Proof. For two T2IFSs and a real number $\lambda \geq 1$, we have

(P1) $0 \leq |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda \leq 2$ which implies that

$$0 \leq \frac{1}{2} \sum_{i=1}^{n} w_i \left\{ |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda \right\} \leq 1.$$

Thus, $0 \leq \gamma_\mu(\mathcal{A}_1, \mathcal{A}_2) \leq 1$. Similarly, we have $0 \leq \gamma_\nu(\mathcal{A}_1, \mathcal{A}_2) \leq 1$. Thus, by definition of $\mathcal{S}_4$, we have $0 \leq \dfrac{\gamma_\nu(\mathcal{A}_1, \mathcal{A}_2) + \gamma_\nu(\mathcal{A}_1, \mathcal{A}_2)}{2} \leq 1$ and hence $0 \leq \mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) Let $\mathcal{A}_1 = \mathcal{A}_2$ which gives that $\gamma_\mu(\mathcal{A}_1, \mathcal{A}_2) = 1$ and $\gamma_\nu(\mathcal{A}_1, \mathcal{A}_2) = 1$. Therefore, $\mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_2) = 1$.

(P3) It is trivial, so we omit here.

(P4) Since $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3$ then we have $u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) \geq u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_3}(x_i)$, $f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) \geq f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_3})$ etc., which implies $|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda \leq |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_3}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_3})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_3}(x_i)|^\lambda$ and $|v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^\lambda + |t_{x_i}(u_{\mathcal{A}_1}) - t_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^\lambda \leq |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_3}(x_i)|^\lambda + |t_{x_i}(u_{\mathcal{A}_1}) - t_{x_i}(u_{\mathcal{A}_3})|^\lambda + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_3}(x_i)|^\lambda$. Thus, we have $\mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_2)$. Similarly, we have $\mathcal{S}_4(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}_3(\mathcal{A}_2, \mathcal{A}_3)$.

Hence $\mathcal{S}_4$ is a valid similarity measure. □

Next, we have proposed some new similarity measure by combining the normalized Hamming distance and Hausdorff distance.

Definition 4.2.5. For two T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$, the measure defined as below is a valid similarity measure.

$$\mathcal{S}_5(\mathcal{A}_1, \mathcal{A}_2) = 1 - \left[\frac{1}{3} \sum_{i=1}^{n} w_i \left(\begin{array}{l} \frac{|u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda}{2} \\ + \frac{|v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^\lambda + |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^\lambda + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^\lambda}{2} \\ + \frac{1}{2} \max \left(\begin{array}{c} |u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)|^\lambda + |f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})|^\lambda + |\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)|^\lambda, \\ |v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)|^\lambda + |t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})|^\lambda + |\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)|^\lambda \end{array} \right) \end{array} \right) \right]^{\frac{1}{\lambda}} \tag{4.6}$$

Definition 4.2.6. For two T2IFSs, we define

$$\mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_2) = 1 - \frac{1}{2} \sum_{i=1}^{n} w_i \widetilde{d}(I_{\mathcal{A}_1}(x_i), I_{\mathcal{A}_2}(x_i)) \tag{4.7}$$

where

$$\begin{aligned} I_{\mathcal{A}_1}(x_i) &= \big(u_{\mathcal{A}_1}(x_i) + f_{x_i}(u_{\mathcal{A}_1}) + \xi_{\mathcal{A}_1}(x_i), v_{\mathcal{A}_1}(x_i) + t_{x_i}(v_{\mathcal{A}_1}) + \eta_{\mathcal{A}_1}(x_i)\big) \\ I_{\mathcal{A}_2}(x_i) &= \big(u_{\mathcal{A}_2}(x_i) + f_{x_i}(u_{\mathcal{A}_2}) + \xi_{\mathcal{A}_2}(x_i), v_{\mathcal{A}_2}(x_i) + t_{x_i}(v_{\mathcal{A}_1}) + \eta_{\mathcal{A}_2}(x_i)\big) \\ \widetilde{d}(a,b) &= \max\big\{|\, a^L - b^L \,|, |\, a^U - b^U \,|\big\}\,; \text{where } a = (a^L, a^U), b = (b^L, b^U) \end{aligned}$$

Proposition 4.2.5. The measure $\mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_2)$ between T2IFSs $\mathcal{A}_1$ and $\mathcal{A}_2$ have the properties:

(P1) $0 \leq \mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) If $\mathcal{A}_1 = \mathcal{A}_2$, then $\mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_2) = 1$.

(P3) $\mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{S}_6(\mathcal{A}_2, \mathcal{A}_1)$.

(P4) If $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3$ then $\mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_2)$ and $\mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_3) \leq \mathcal{S}_6(\mathcal{A}_2, \mathcal{A}_3)$.

and hence it is a valid similarity measure.

Proof. In order to proof $\mathcal{S}_6$ is a valid similarity measures, we shall prove the parts (P1) and (P2) while others can be obtained similarly.

(P1) Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are T2IFSs, we have

$$\Rightarrow \quad u_{\mathcal{A}_1}(x_i) \geq 0, u_{\mathcal{A}_2}(x_i) \geq 0, f_{x_i}(u_{\mathcal{A}_1}) \geq 0, f_{x_i}(u_{\mathcal{A}_2}) \geq 0, \xi_{\mathcal{A}_1}(x_i) \geq 0, \xi_{\mathcal{A}_2}(x_i) \geq 0,$$
$$v_{\mathcal{A}_1}(x_i) \geq 0, v_{\mathcal{A}_2}(x_i) \geq 0, t_{x_i}(v_{\mathcal{A}_1}) \geq 0, t_{x_i}(v_{\mathcal{A}_2}) \geq 0, \eta_{\mathcal{A}_1}(x_i) \geq 0, \eta_{\mathcal{A}_2}(x_i) \geq 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} w_i \max \begin{pmatrix} \mid (u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)) + (f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})) \\ + (\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)) \mid, \mid (v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)) \\ + (t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})) + (\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)) \mid \end{pmatrix} \geq 0$$

$$\Rightarrow \quad 1 - \frac{1}{2}\sum_{i=1}^{n} w_i \max \begin{pmatrix} \mid (u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)) + (f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})) \\ + (\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)) \mid, \mid (v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)) \\ + (t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})) + (\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)) \mid \end{pmatrix} \leq 1$$

which implies $\mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

Further, $u_{\mathcal{A}_1}(x_i) \leq 1, u_{\mathcal{A}_2}(x_i) \leq 1, f_{x_i}(u_{\mathcal{A}_1}) \leq 1, f_{x_i}(u_{\mathcal{A}_2}) \leq 1, \xi_{\mathcal{A}_1}(x_i) \leq 1, \xi_{\mathcal{A}_2}(x_i) \leq 1$, $v_{\mathcal{A}_1}(x_i) \leq 1, v_{\mathcal{A}_2}(x_i) \leq 1, t_{x_i}(v_{\mathcal{A}_1}) \leq 1, t_{x_i}(v_{\mathcal{A}_2}) \leq 1, \eta_{\mathcal{A}_1}(x_i) \leq 1, \eta_{\mathcal{A}_2}(x_i) \leq 1$ which implies that

$$\Rightarrow \quad \sum_{i=1}^{n} w_i \max \begin{pmatrix} \mid (u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)) + (f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})) \\ + (\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)) \mid, \mid (v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)) \\ + (t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})) + (\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)) \mid \end{pmatrix} \leq 2$$

$$\Rightarrow \quad 1 - \frac{1}{2}\sum_{i=1}^{n} w_i \max \begin{pmatrix} \mid (u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i)) + (f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2})) \\ + (\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i)) \mid, \mid (v_{\mathcal{A}_1}(x_i) - v_{\mathcal{A}_2}(x_i)) \\ + (t_{x_i}(v_{\mathcal{A}_1}) - t_{x_i}(v_{\mathcal{A}_2})) + (\eta_{\mathcal{A}_1}(x_i) - \eta_{\mathcal{A}_2}(x_i)) \mid \end{pmatrix} \geq 0$$

Hence, $0 \leq \mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) Let $\mathcal{A}_1 = \mathcal{A}_2$ which gives that $u_{\mathcal{A}_1}(x_i) - u_{\mathcal{A}_2}(x_i) = 0$, $f_{x_i}(u_{\mathcal{A}_1}) - f_{x_i}(u_{\mathcal{A}_2}) = 0$, $\xi_{\mathcal{A}_1}(x_i) - \xi_{\mathcal{A}_2}(x_i) = 0$ and so on. Thus, by definition of $\mathcal{S}_6$, we have $\mathcal{S}_6(\mathcal{A}_1, \mathcal{A}_2) = 1$.

$\square$

In order to demonstrate the above-proposed similarity measures, we take a numerical example which is presented below

Example 4.2.1. Consider the following three alternatives $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ with four attributes $x_i, i = 1, 2, 3, 4$ whose weight vectors are $\omega = (0.3, 0.1, 0.2, 0.4)$ and are represented in the form of the following T2IFS:

$$\begin{aligned}
\mathcal{A}_1 &= \{x_1(0.3, 1.0, 0.5, 0.0), x_2(0.7, 0.5, 0.2, 0.4), x_3(1.0, 0.3, 0.0, 0.5), x_4(0.6, 0.7, 0.3, 0.1)\}, \\
\mathcal{A}_2 &= \{x_1(0.5, 0.6, 0.4, 0.2), x_2(0.7, 0.2, 0.1, 0.7), x_3(0.7, 0.3, 0.1, 0.6), x_4(0.3, 0.4, 0.5, 0.6)\}, \\
\mathcal{A}_3 &= \{x_1(0.2, 0.8, 0.5, 0.1), x_2(0.1, 0.3, 0.7, 0.4), x_3(0.8, 0.3, 0.1, 0.5), x_4(0.1, 0.7, 0.6, 0.2)\}.
\end{aligned}$$

Let **0** is a null T2IFS with zero PMF and SMF and one PNMF, SNMF values and this set is denoted by $\mathcal{N} = (0, 0, 1, 1)$. Then, by using the proposed similarity measures corresponding to $\lambda = 2$, we get

$$\begin{array}{lll}
\mathcal{S}_1(\mathcal{A}_1, \mathcal{N}) = 0.1935; & \mathcal{S}_1(\mathcal{A}_2, \mathcal{N}) = 0.4156; & \mathcal{S}_1(\mathcal{A}_3, \mathcal{N}) = 0.3083 \\
\mathcal{S}_2(\mathcal{A}_1, \mathcal{N}) = 0.8696; & \mathcal{S}_2(\mathcal{A}_2, \mathcal{N}) = 0.8239; & \mathcal{S}_2(\mathcal{A}_3, \mathcal{N}) = 0.7373 \\
\mathcal{S}_3(\mathcal{A}_1, \mathcal{N}) = 0.3730; & \mathcal{S}_3(\mathcal{A}_2, \mathcal{N}) = 0.5038; & \mathcal{S}_3(\mathcal{A}_3, \mathcal{N}) = 0.5431 \\
\mathcal{S}_4(\mathcal{A}_1, \mathcal{N}) = 0.1939; & \mathcal{S}_4(\mathcal{A}_2, \mathcal{N}) = 0.4201; & \mathcal{S}_4(\mathcal{A}_3, \mathcal{N}) = 0.3090 \\
\mathcal{S}_5(\mathcal{A}_1, \mathcal{N}) = 0.1802; & \mathcal{S}_5(\mathcal{A}_2, \mathcal{N}) = 0.3923; & \mathcal{S}_5(\mathcal{A}_3, \mathcal{N}) = 0.2974
\end{array}$$

If instead of null T2IFSs, if consider $\mathcal{B} \in \text{T2IFS}(\mathcal{X})$ which will be recognized, where

$$\mathcal{B} = \{x_1(0.5, 0.3, 0.4, 0.5), x_2(0.4, 0.2, 0.5, 0.6), x_3(0.8, 0.1, 0.1, 0.6), x_4(0.3, 0.2, 0.6, 0.7)\},$$

then we want to classify $\mathcal{B}$ with one of the classes $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$. Assuming that the weight vector of $x_i, i = 1, 2, 3, 4$ is $\omega = (0.3, 0.1, 0.2, 0.4)$ and hence we get the measurement values are

$$\begin{array}{lll}
\mathcal{S}_1(\mathcal{A}_1, \mathcal{B}) = 0.4980; & \mathcal{S}_1(\mathcal{A}_2, \mathcal{B}) = 0.7571; & \mathcal{S}_1(\mathcal{A}_3, \mathcal{B}) = 0.4995 \\
\mathcal{S}_2(\mathcal{A}_1, \mathcal{B}) = 0.9293; & \mathcal{S}_2(\mathcal{A}_2, \mathcal{B}) = 0.9163; & \mathcal{S}_2(\mathcal{A}_3, \mathcal{B}) = 0.8297
\end{array}$$

$$\mathcal{S}_3(\mathcal{A}_1,\mathcal{B}) = 0.7041; \quad \mathcal{S}_3(\mathcal{A}_2,\mathcal{B}) = 0.8768; \quad \mathcal{S}_3(\mathcal{A}_3,\mathcal{B}) = 0.7366$$

$$\mathcal{S}_4(\mathcal{A}_1,\mathcal{B}) = 0.4998; \quad \mathcal{S}_4(\mathcal{A}_2,\mathcal{B}) = 0.7575; \quad \mathcal{S}_4(\mathcal{A}_3,\mathcal{B}) = 0.5050$$

$$\mathcal{S}_5(\mathcal{A}_1,\mathcal{B}) = 0.4740; \quad \mathcal{S}_5(\mathcal{A}_2,\mathcal{B}) = 0.7444; \quad \mathcal{S}_5(\mathcal{A}_3,\mathcal{B}) = 0.4756$$

$$\mathcal{S}_6(\mathcal{A}_1,\mathcal{B}) = 0.6500; \quad \mathcal{S}_6(\mathcal{A}_2,\mathcal{B}) = 0.6800; \quad \mathcal{S}_6(\mathcal{A}_3,\mathcal{B}) = 0.6200$$

4.3 Group decision making approach based on the similarity measures between T2IFNs

In this section, we propose a new MAGDM method based on the similarity measure between T2IFSs. For it, let $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ be the set of universe and $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m\}$ be the finite set of the alternatives. Let $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$ be the finite set of attributes which are divided into two disjoint sets $\mathcal{F}_1$ and $\mathcal{F}_2$, where $\mathcal{F}_1$ represents the cost type while $\mathcal{F}_2$ represents the benefit type attributes such that $\mathcal{F}_1 \subseteq \mathcal{C}, \mathcal{F}_2 \subseteq \mathcal{C}$ and $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$. Let $\mathcal{DM} = \{\mathcal{DM}_1,\, \mathcal{DM}_2,\, \ldots,\, \mathcal{DM}_p\}$ be the finite set of the decision makers' who have the responsibility to evaluate the given alternatives. Let $D_k = (l_{ij}^k)_{m\times n} = \left((u_i^k(\mathcal{G}_j), f_{\mathcal{G}_j}(u_i^k), v_i^k(\mathcal{G}_j), t_{\mathcal{G}_j}(v_i^k))\right)_{m\times n}; k = 1, 2, \ldots, p$ be the decision matrices given by the k decision makers, where $l_{ij}^k = \left(u_i^k(\mathcal{G}_j), f_{\mathcal{G}_j}(u_i^k), v_i^k(\mathcal{G}_j), t_{\mathcal{G}_j}(v_i^k)\right)$ is the evaluating T2IFN of alternative $\mathcal{A}_i$ with respect to criterion $\mathcal{G}_j$ given by the $\mathcal{DM}_k$ decision maker, $u_i^k(\mathcal{G}_j) \in [0,1], f_{\mathcal{G}_j}(u_i^k) \in [0,1],\ v_i^k(\mathcal{G}_j) \in [0,1], t_{\mathcal{G}_j}(u_i^k) \in [0,1]$, $0 \le u_i^k(\mathcal{G}_j) + v_i^k(\mathcal{G}_j) \le 1$ and $f_{\mathcal{G}_j}(u_i^k) + t_{\mathcal{G}_j}(u_i^k) \le 1$. Here m represent the number of alternatives, n be number of criteria and p be the number of decision makers. Assume that each decision maker gives their rating values in term of the linguistic information according to linguistic term set $S = \{$"very low (VL)", "low", "medium low (ML)", "Medium (M)", "High (H)", "very High (VH)"$\}$. The proposed MAGDM method based on the similarity measure between T2IFNs is now presented as follows:

Step 1: Arrange the collective information of each decision maker, in terms of linguistic variable of T2IFNs and represented as a decision matrices $D_k = \left(l_{ij}^k\right)_{m\times n} (k = 1, 2, \ldots, p)$. The representation of D_k is given as below:

$$D_k = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{G}_1 & \mathcal{G}_2 & \dots & \mathcal{G}_n \end{array} \\ \begin{pmatrix} l_{11}^k & l_{12}^k & \dots & l_{1n}^k \\ l_{21}^k & l_{22}^k & \dots & l_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}^k & l_{m2}^k & \dots & l_{mn}^k \end{pmatrix} \end{array}$$

where $l_{ij}^k = \big(u_i^k(\mathcal{G}_j), f_{\mathcal{G}_j}(u_i^k), v_i^k(\mathcal{G}_j), t_{\mathcal{G}_j}(v_i^k)\big)$.

Step 2: Normalize each criterion values by using the following equation.

$$D_k = \begin{cases} (l_{ij}^k)^c & ; \text{ for } \mathcal{F}_1 \\ l_{ij}^k & ; \text{ for } \mathcal{F}_2 \end{cases} \tag{4.8}$$

where the complement of l_{ij}^k is taken as per the information given in Table 4.1.

Table 4.1: Linguistic grades and Compliments

LT	VL	L	ML	M	MH	H	VH
Compliment	VH	H	MH	M	ML	L	VL

Step 3: Take the positive ideal reference set, denoted by $\mathcal{N}$ of the criteria in terms of linguistic T2IFS whose rating value is represented as $(n_{ij}^k)^+$ and the negative ideal reference set, denoted by $\bar{\mathcal{N}}$ whose rating value is represented as $(n_{ij}^k)^-$.

Step 4: According to the proposed measures, calculate $(f_{ij}^k)^+$ between the evaluating T2IFN l_{ij}^k of alternative $\mathcal{A}_i$ with respect to criteria $\mathcal{G}_j$ and the obtained reference set $(n_{ij}^k)^+$ of criterion $\mathcal{G}_j$ to construct the positive similarity matrix $(G^k)^+ = ((f_{ij}^k)^+)_{m\times n}$, where $(f_{ij}^k)^+ = \mathcal{S}(l_{ij}^k, (n_j^k)^+)$ such that $(f_{ij}^k)^+ \in [0,1]$.

Step 5: According to the proposed measures, calculate $(f_{ij}^k)^-$ between the evaluating T2IFN l_{ij}^k of alternative $\mathcal{A}_i$ with respect to criteria $\mathcal{G}_j$ and the obtained reference set $(n_{ij}^k)^-$ of criterion $\mathcal{G}_j$ to construct the negative similarity matrix $(G^k)^- = ((f_{ij}^k)^-)_{m\times n}$, where $(f_{ij}^k)^- = \mathcal{S}(l_{ij}^k, (n_j^k)^-)$ such that $(f_{ij}^k)^- \in [0,1]$.

Step 6: Compute the weighted positive score $(\mathcal{S}_i^k)^+$ of each alternative $\mathcal{A}_i$ with respect to each decision maker $\mathcal{DM}_k$, where $(\mathcal{S}_i^k)^+ = \sum_{j=1}^{n} w_j (f_{ij}^k)^+$ such that $(\mathcal{S}_i^k)^+ \in [0,1]$, w_j is the weight of criterion $\mathcal{G}_j$, $w_j \in (0,1]$ and $\sum_{j=1}^{n} w_j = 1$. Compute the weighted negative score $(\mathcal{S}_i^k)^-$ of each alternative $\mathcal{A}_i$ with respect to each decision maker $\mathcal{DM}_k$, where $(\mathcal{S}_i^k)^- = \sum_{j=1}^{n} w_j (f_{ij}^k)^-$ such that $(\mathcal{S}_i^k)^- \in [0,1]$.

Step 7: Compute the relative degree of closeness $\mathfrak{C}_i^k$ for each expert by using Eq. (4.9) as

$$\mathfrak{C}_i^k = \frac{(\mathcal{S}_i^k)^-}{(\mathcal{S}_i^k)^+ + (\mathcal{S}_i^k)^-} \quad , \tag{4.9}$$

where $(\mathcal{S}_i^k)^+ \neq 0$, $\mathfrak{C}_i^k \in [0,1]$. The larger the value of $\mathfrak{C}_i^k$, the better the performance of the candidate in terms of decision maker $\mathcal{DM}_k$. The prominent characteristics of Eq. (4.9) is that they not only take into account the distance between T2IFS but also examine similar or dissimilar between the sets so as to avoid to draw the conclusion based on the small distance or large similarity.

Step 8: Since each expert may obtain the different ranking of the alternatives which leads to difficult to trace the best alternative(s). To overcome this, we compute the aggregative values of expert, by assigning the priority value, $w = (w^1, w^2, \ldots, w^p)^T$ such that $w^k > 0$ and $\sum_{k=1}^{p} w^k = 1$, to each expert, as

$$\mathcal{S}_i^+ = \sum_{k=1}^{p} w^k \ (\mathcal{S}_i^k)^+ \ ; \ \mathcal{S}_i^- = \sum_{k=1}^{p} w^k \ (\mathcal{S}_i^k)^- \tag{4.10}$$

Step 9: The closeness coefficient is determined for an alternative $\mathcal{A}_i (i = 1, 2, \ldots, m)$ as

$$\mathfrak{C}_i = \frac{\mathcal{S}_i^-}{\mathcal{S}_i^+ + \mathcal{S}_i^-}; \qquad \text{provided } \mathcal{S}_i^+ \neq 0 \tag{4.11}$$

and hence rank them according to the descending values of $\mathfrak{C}_i$'s.

4.4 Illustrative example

To demonstrate the above-mentioned approach, we present a numerical example regarding finding the best university in a country so that students can choose their studies accordingly. In order to demonstrate it, we consider a decision-making problem which involves the evaluations of five different universities $\mathcal{A}_i (i = 1, 2, \ldots, 5)$ of the country. To evaluate these different universities, a ministry of education appoint three senior educationalist named as $\mathcal{DM}_1$, $\mathcal{DM}_2$ and $\mathcal{DM}_3$, who have the responsibility to evaluate such university under the five criteria namely "Research potential" ($\mathcal{G}_1$), "Teaching records" ($\mathcal{G}_2$), "employability potential" ($\mathcal{G}_3$) and "placement records and international standard collaboration" ($\mathcal{G}_4$). Assume that the importance of these different criteria are taken as $\omega = (0.35, 0.30, 0.20, 0.15)$ and for the decision makers are $w = (0.40, 0.35, 0.25)$. The three experts give the judgments by using the linguistic term set of T2IFNs, where the linguistic grade of the PMF, PNMF, SMF, and SNMF are represented in Table 4.2.

Table 4.2: Linguistic grade and corresponding PMF, SMF, PNMF, SNMF values

Linguistic Terms	PMF/SMF value	Linguistic Terms	PNMF/SNMF value
VL	0.0	VH	1
L	0.2	H	0.7
ML	0.4	MH	0.5
M	0.5	M	0.4
MH	0.7	ML	0.2
H	0.9	L	0.1
VH	1.0	VL	0.0

By using the proposed approach, the steps are executed as below:

Step 1: The rating values of each decision maker towards the assessment of the alternatives are represented in Table 4.3.

Step 2: Since all the criteria are of same type, so there is no need of normalizing the rating values.

Step 3: W.l.o.g., we consider the positive reference set as $\mathcal{N} = (H, MH, VL, L)$ such that $(\eta_{ij}^k)^+ = (0.9, 0.7, 0.0, 0.1)_{5\times 4}$ for each decision maker. However, the consequently the negative reference set is given as $\bar{\mathcal{N}} = (L, ML, MH, MH)$ and hence $(\eta_{ij}^k)^- = (0.2, 0.4, 0.5, 0.5)_{5\times 4}$ for $k = 1, 2, 3$. Further, we take $\lambda = 2$ for the sake of calculation.

Table 4.3: Graded values of the alternative corresponding to each attribute (criteria)

		$\mathcal{DM}_1$				$\mathcal{DM}_2$				$\mathcal{DM}_3$			
		PMF	SMF	PNMF	SNMF	PMF	SMF	PNMF	SNMF	PMF	SMF	PMNF	SNMF
	$\mathcal{A}_1$	H	MH	L	L	M	VL	ML	M	MH	H	L	VL
	$\mathcal{A}_2$	MH	M	L	L	H	MH	L	ML	ML	MH	MH	ML
$\mathcal{G}_1$	$\mathcal{A}_3$	L	M	H	M	H	H	VL	L	VL	M	MH	VL
	$\mathcal{A}_4$	VH	H	VL	L	MH	L	ML	MH	H	MH	L	L
	$\mathcal{A}_5$	M	ML	M	MH	L	M	H	M	L	MH	ML	ML
	$\mathcal{A}_1$	M	MH	ML	ML	L	ML	MH	MH	H	MH	VL	L
	$\mathcal{A}_2$	ML	M	ML	ML	M	MH	ML	ML	VH	MH	VL	ML
$\mathcal{G}_2$	$\mathcal{A}_3$	H	MH	L	L	MH	ML	ML	L	ML	M	MH	ML
	$\mathcal{A}_4$	L	H	MH	VL	H	ML	VL	MH	ML	M	M	ML
	$\mathcal{A}_5$	VH	H	VL	VL	VL	H	H	VL	H	MH	VL	ML
	$\mathcal{A}_1$	L	M	H	ML	H	VH	VL	VL	H	MH	VL	ML
	$\mathcal{A}_2$	MH	ML	ML	MH	MH	H	ML	L	MH	H	ML	L
$\mathcal{G}_3$	$\mathcal{A}_3$	H	MH	VL	L	VL	M	H	M	ML	L	ML	L
	$\mathcal{A}_4$	M	L	M	H	M	ML	M	MH	M	ML	L	MH
	$\mathcal{A}_5$	MH	VL	L	H	M	M	ML	M	ML	H	MH	L
	$\mathcal{A}_1$	H	H	VL	L	L	VH	H	VL	H	M	VL	M
	$\mathcal{A}_2$	VL	H	M	L	MH	VH	ML	VL	ML	MH	M	ML
$\mathcal{G}_4$	$\mathcal{A}_3$	M	ML	ML	MH	H	H	L	L	L	ML	H	MH
	$\mathcal{A}_4$	MH	M	ML	M	MH	M	ML	M	M	MH	MH	ML
	$\mathcal{A}_5$	H	MH	VL	ML	VL	ML	MH	MH	H	MH	VL	ML

Step 4: For the computational process, here we take the similarity measure $\mathcal{S}_1$ as given in Eq. (4.1). Now, by applying Eq. (4.1), the positive degree of similarity between the alternatives and N is given as $(f_{11}^1)^+ = 0.9293$, $(f_{11}^2)^+ = 0.5310$, $(f_{11}^3)^+ = 0.7261$, $(f_{12}^1)^+ = 0.6918$, $(f_{12}^2)^+ = 0.4615$, $(f_{12}^3)^+ = 1.0000$, $(f_{13}^1)^+ = 0.3597$, $(f_{13}^2)^+ = 0.7764$, $(f_{13}^3)^+ = 0.9293$ and so on. The complete positive similarity matrix for each decision maker is represented in Table 4.4.

Step 5: By applying the Eq. (4.1), the negative degree of similarity between the alternatives and $\bar{N}$ is given as $(f_{11}^1)^- = 0.4299$, $(f_{11}^2)^- = 0.5204$, $(f_{11}^3)^- = 0.4432$, $(f_{12}^1)^- = 0.6258$, $(f_{12}^2)^- = 0.8586$, $(f_{12}^3)^- = 0.4169$, $(f_{13}^1)^- = 0.5528$, $(f_{13}^2)^- = 0.3597$, $(f_{13}^3)^- = 0.4432$ and so on. The complete negative similarity matrix for

Table 4.4: Positive Similarity matrix

	$(\mathcal{S}_{ij}^1)^-$				$(\mathcal{S}_{ij}^2)^-$				$(\mathcal{S}_{ij}^3)^-$			
	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	0.9293	0.6918	0.3597	0.8586	0.5310	0.4615	0.7764	0.1754	0.7261	1.0000	0.9293	0.7450
$\mathcal{A}_2$	0.8419	0.6683	0.6918	0.2286	0.9293	0.6918	0.7172	0.6394	0.5204	0.9000	0.7172	0.5641
$\mathcal{A}_3$	0.3836	0.9293	1.0000	0.6464	0.8586	0.7655	0.2789	0.8419	0.3000	0.5528	0.6192	0.3917
$\mathcal{A}_4$	0.8775	0.3000	0.5050	0.7655	0.6063	0.6464	0.6192	0.7655	0.9293	0.6000	0.6394	0.5699
$\mathcal{A}_5$	0.6192	0.8586	0.4084	0.9293	0.3836	0.1056	0.6918	0.3519	0.4901	0.9293	0.4343	0.9293

each decision maker is represented in Table 4.5.

Table 4.5: Negative Similarity matrix

	$(\mathcal{S}_{ij}^1)^-$				$(\mathcal{S}_{ij}^2)^-$				$(\mathcal{S}_{ij}^3)^-$			
	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	0.4299	0.6258	0.5528	0.4084	0.5204	0.8586	0.3597	0.2824	0.4432	0.4169	0.4432	0.4432
$\mathcal{A}_2$	0.5050	0.6683	0.6000	0.4084	0.4615	0.6258	0.4901	0.4169	0.6000	0.3917	0.4901	0.6258
$\mathcal{A}_3$	0.6838	0.4299	0.4169	0.7172	0.4084	0.4901	0.6394	0.4212	0.4708	0.6258	0.5584	0.7551
$\mathcal{A}_4$	0.3675	0.3917	0.6838	0.6127	0.5204	0.4169	0.7551	0.6127	0.4299	0.6536	0.6760	0.5877
$\mathcal{A}_5$	0.7551	0.3367	0.3795	0.4432	0.6838	0.2720	0.7172	0.8000	0.6258	0.4432	0.4756	0.4432

Step 6: Computing the weighted positive score $(\mathcal{S}_i^k)^+$ and negative scores $(\mathcal{S}_i^k)^-$ for each alternative $\mathcal{A}_i$ and each expert $\mathcal{DM}_k$, where $1 \leq i \leq 5, 1 \leq k \leq 3$, we can get $(\mathcal{S}_1^1)^+ = 0.6605$, $(\mathcal{S}_2^1)^+ = 0.6127$, $(\mathcal{S}_3^1)^+ = 0.6085$ and so on. Similarly for other decision makers we can compute and their results are summarized in Table 4.6.

Table 4.6: Separation measures from ideal solutions corresponding to each decision maker

	$\mathcal{DM}_1$			$\mathcal{DM}_2$			$\mathcal{DM}_3$		
	$(\mathcal{S}_i^1)^-$	$(\mathcal{S}_i^1)^+$	$\mathfrak{C}_i^1$	$(\mathcal{S}_i^2)^-$	$(\mathcal{S}_i^2)^-$	$\mathfrak{C}_i^2$	$(\mathcal{S}_i^3)^-$	$(\mathcal{S}_i^3)^-$	$\mathfrak{C}_i^3$
$\mathcal{A}_1$	0.5018	0.6605	0.4317	0.5043	0.4746	0.5151	0.4352	0.8076	0.3502
$\mathcal{A}_2$	0.5492	0.6127	0.4727	0.5035	0.7436	0.4037	0.5101	0.6422	0.4427
$\mathcal{A}_3$	0.5390	0.6085	0.4697	0.4739	0.6377	0.4263	0.5664	0.4379	0.5640
$\mathcal{A}_4$	0.4582	0.5423	0.4580	0.5342	0.6408	0.4546	0.5570	0.6783	0.4509
$\mathcal{A}_5$	0.4742	0.6429	0.4245	0.5352	0.3255	0.6219	0.5065	0.6035	0.4563

Step 7: Utilize Eq. (4.9) to compute the closeness degrees $\mathfrak{C}_i^k$ of each alternative $\mathcal{A}_i$, we

can summarized their results in the third column of each expert in Table 4.6. It is clearly seen from this table that the best alternative for the 1st decision maker is $\mathcal{A}_2$ while $\mathcal{A}_5$ and $\mathcal{A}_3$ respectively for the other decision makers. Therefore, sometimes, it is difficult to choose the final best one among them. To reduce the uncertainties, the final aggregation values are aggregated by taking the weight vector of the DMs.

Step 8: Use expert weight $w = (0.40, 0.35, 0.25)^T$ and Eq. (4.10) to aggregate the preferences of each expert. The measurement values of each alternative are found as $\mathcal{S}_1^+ = 0.6322$, $\mathcal{S}_2^+ = 0.6659$, $\mathcal{S}_3^+ = 0.5761$, $\mathcal{S}_4^+ = 0.6108$, $\mathcal{S}_5^+ = 0.5219$, $\mathcal{S}_1^- = 0.4860$, $\mathcal{S}_2^- = 0.5234$, $\mathcal{S}_3^- = 0.5231$, $\mathcal{S}_4^- = 0.5095$ and $\mathcal{S}_5^- = 0.5036$.

Step 9: Compute $\mathfrak{C}_i (i = 1, 2, 3, 4, 5)$ for each alternative by Eq. (4.11) and we get $\mathfrak{C}_1 = 0.4346$, $\mathfrak{C}_2 = 0.4401$, $\mathfrak{C}_3 = 0.4759$, $\mathfrak{C}_4 = 0.4548$ and $\mathfrak{C}_5 = 0.4911$. Thus, the ranking order of the alternatives is $\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$ and hence conclude that $\mathcal{A}_5$ is the best university in the country.

However, apart from the similarity measure $\mathcal{S}_1$, if we considered other proposed similarity measures $\mathcal{S}_2 - \mathcal{S}_6$ on to the considered data then the final closeness degree of the alternatives with respect to each decision maker and the overall rating are obtained corresponding to each similarity measures. The results corresponding to them are summarized in Table 4.7. From this table, it is clearly seen that the best alternative after incorporating all the expert suggestion is $\mathcal{A}_5$ which is conservative in nature with all the proposed similarity measures.

Furthermore, in order to measure the influence of the parameter λ on to the similarity measures indices and hence on to the final ranking order of the alternatives, we perform an experiment where we vary the value of λ from 1 to 20. The final ranking order along with their closeness degree of the alternatives are summarized in Table 4.8. However, the complete variation of the alternative behavior with λ is given in Fig. 4.1. From this figure, it is seen that for larger values of λ, the alternative $\mathcal{A}_3$ gives the best from $\mathcal{S}_1$, $\mathcal{S}_3, \mathcal{S}_4$ and $\mathcal{S}_5$ similarity measures. However, for smaller values of λ, the ranking order for all the alternatives changes which helps the decision maker to choose the desired one

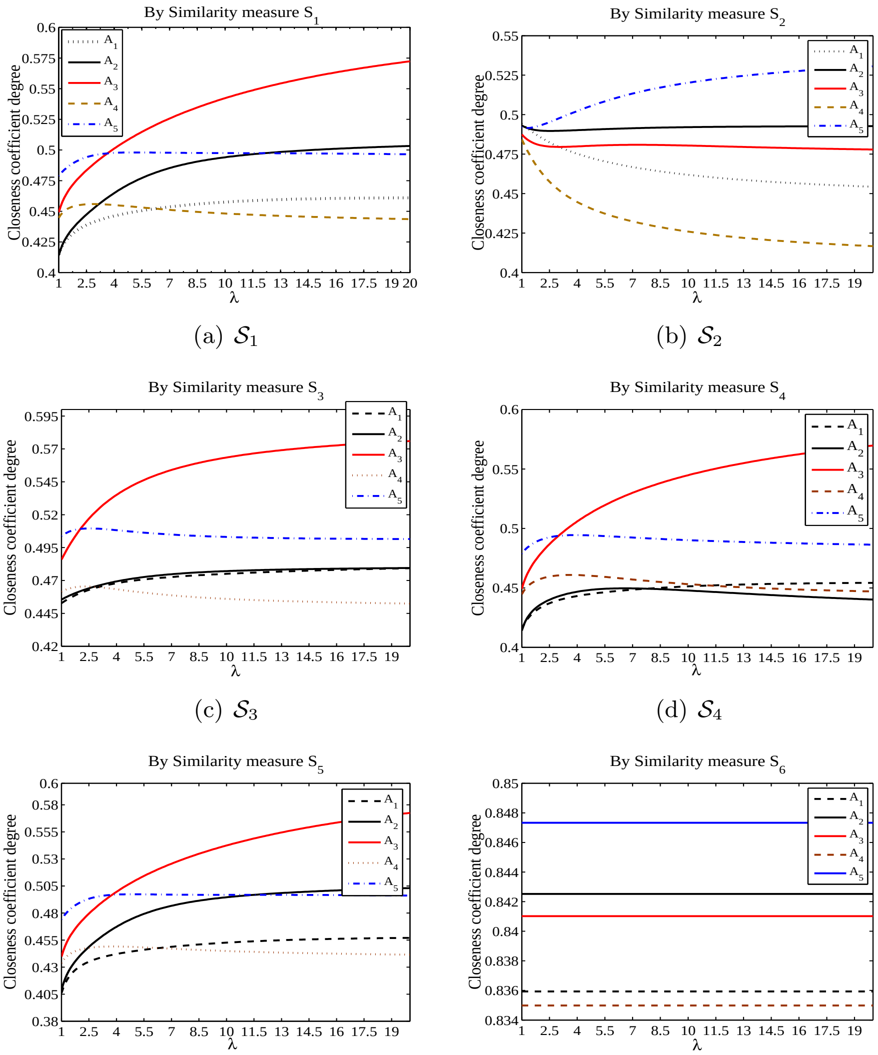
Table 4.7: Optimal results by using different measures corresponding to $\lambda = 2$

	By $\mathcal{S}_1$				By $\mathcal{S}_2$				By $\mathcal{S}_3$			
	$\mathfrak{C}_i^1$	$\mathfrak{C}_i^2$	$\mathfrak{C}_i^3$	$\mathfrak{C}_i$	$\mathfrak{C}_i^1$	$\mathfrak{C}_i^2$	$\mathfrak{C}_i^3$	$\mathfrak{C}_i$	$\mathfrak{C}_i^1$	$\mathfrak{C}_i^2$	$\mathfrak{C}_i^3$	$\mathfrak{C}_i$
$\mathcal{A}_1$	0.4317	0.5151	0.3502	0.4346	0.4733	0.5125	0.4682	0.4858	0.4585	0.5400	0.3631	0.4603
$\mathcal{A}_2$	0.4727	0.4037	0.4427	0.4401	0.5297	0.4733	0.4539	0.4899	0.4946	0.4210	0.4683	0.4617
$\mathcal{A}_3$	0.4697	0.4263	0.5640	0.4759	0.4624	0.4703	0.5284	0.4806	0.4896	0.4504	0.6170	0.5085
$\mathcal{A}_4$	0.4580	0.4546	0.4509	0.4548	0.4733	0.4444	0.4807	0.4644	0.4859	0.4393	0.4720	0.4653
$\mathcal{A}_5$	0.4245	0.6219	0.4563	0.4911	0.4559	0.5207	0.5191	0.4931	0.4208	0.6496	0.4763	0.5091
Best	$\mathcal{A}_2$	$\mathcal{A}_5$	$\mathcal{A}_3$	$\mathcal{A}_5$	$\mathcal{A}_2$	$\mathcal{A}_5$	$\mathcal{A}_3$	$\mathcal{A}_5$	$\mathcal{A}_2$	$\mathcal{A}_5$	$\mathcal{A}_3$	$\mathcal{A}_5$
	By $\mathcal{S}_4$				By $\mathcal{S}_5$				By $\mathcal{S}_6$			
	$\mathfrak{C}_i^1$	$\mathfrak{C}_i^2$	$\mathfrak{C}_i^3$	$\mathfrak{C}_i$	$\mathfrak{C}_i^1$	$\mathfrak{C}_i^2$	$\mathfrak{C}_i^3$	$\mathfrak{C}_i$	$\mathfrak{C}_i^1$	$\mathfrak{C}_i^2$	$\mathfrak{C}_i^3$	$\mathfrak{C}_i$
$\mathcal{A}_1$	0.4319	0.5106	0.3497	0.4333	0.4246	0.5184	0.3457	0.4305	0.8333	0.8611	0.7872	0.8359
$\mathcal{A}_2$	0.4649	0.4006	0.4421	0.4359	0.4797	0.4036	0.4338	0.4403	0.8551	0.8305	0.8361	0.8425
$\mathcal{A}_3$	0.4698	0.4244	0.5747	0.4784	0.4620	0.4279	0.5570	0.4707	0.8333	0.8148	0.8750	0.8410
$\mathcal{A}_4$	0.4567	0.4630	0.4513	0.4576	0.4535	0.4404	0.4440	0.4461	0.8319	0.8276	0.8485	0.8350
$\mathcal{A}_5$	0.4273	0.6216	0.4486	0.4903	0.4145	0.6299	0.4627	0.4896	0.8182	0.8750	0.8387	0.8473
Best	$\mathcal{A}_3$	$\mathcal{A}_5$	$\mathcal{A}_3$	$\mathcal{A}_5$	$\mathcal{A}_2$	$\mathcal{A}_5$	$\mathcal{A}_3$	$\mathcal{A}_5$	$\mathcal{A}_2$	$\mathcal{A}_5$	$\mathcal{A}_3$	$\mathcal{A}_5$

according to their needs. Here, the parameter λ represents the attitude characteristic of the decision makers towards the decision process. It is also noted from the figure that there is no change in the variation in the last graph when computed from $\mathcal{S}_6$ measure. This is because the measure $\mathcal{S}_6$ is independent of the parameter λ. From the graph and their respective degree values summarized in Table 4.8, a decision maker can analyze the behavior and choose the best one among their desired goals. Thus, it gives various choices to the decision makers to select their goals.

4.5 Conclusion

In this chapter, we present some series of similarity measures by considering the PMF, SMF, PNMF, SNMF, FOU, and VMF to accommodate the T2IFS information. Some desirable properties of its are investigates in details. Further, to explore the structure of T2IFSs and by similarity measure, we present a TOPSIS method based on the similarity measures to access the finest alternatives. Later, a group decision-making approach is presented based on the proposed method between T2IFSs. A numerical example is taken to explain the method. Also, from the study, it is concluded that some of the existing measures under T2FSs, IFSs can be easily deduce from the present study. Thus, the presented measure is one of the generalizations of the existing ones and provides us with a useful way to deal MAGDM in T2IFS environments.



(a) $\mathcal{S}_1$ (b) $\mathcal{S}_2$

(c) $\mathcal{S}_3$ (d) $\mathcal{S}_4$

(e) $\mathcal{S}_5$ (f) $\mathcal{S}_6$

Figure 4.1: Variation in ranking values with respect to λ

Table 4.8: Variation of closeness degree with respect to λ

λ	Measure	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	Ranking order
1	$\mathcal{S}_1$	0.4149	0.4141	0.4493	0.4447	0.4789	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_2$
	$\mathcal{S}_2$	0.4942	0.4932	0.4875	0.4843	0.4922	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
	$\mathcal{S}_3$	0.4528	0.4555	0.4858	0.4616	0.5038	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\mathcal{S}_4$	0.4149	0.4141	0.4493	0.4447	0.4789	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_2$
	$\mathcal{S}_5$	0.4060	0.4089	0.4397	0.4338	0.4741	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
1.5	$\mathcal{S}_1$	0.4281	0.4309	0.4662	0.4522	0.4863	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\mathcal{S}_2$	0.4897	0.4908	0.4828	0.4731	0.4917	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
	$\mathcal{S}_3$	0.4569	0.4589	0.4982	0.4642	0.5075	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\mathcal{S}_4$	0.4273	0.4291	0.4675	0.4537	0.4863	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\mathcal{S}_5$	0.4229	0.4293	0.4593	0.4426	0.4838	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
2.5	$\mathcal{S}_1$	0.4388	0.4473	0.4834	0.4557	0.4941	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\mathcal{S}_2$	0.4825	0.4897	0.4798	0.4576	0.4952	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
	$\mathcal{S}_3$	0.4629	0.4641	0.5170	0.4654	0.5096	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\mathcal{S}_4$	0.4371	0.4401	0.4868	0.4596	0.4926	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\mathcal{S}_5$	0.4350	0.4487	0.4795	0.4478	0.4932	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
5	$\mathcal{S}_1$	0.4493	0.4736	0.5106	0.4538	0.4980	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
	$\mathcal{S}_2$	0.4718	0.4906	0.4804	0.4394	0.5065	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2$
	$\mathcal{S}_3$	0.4700	0.4717	0.5431	0.4616	0.5070	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
	$\mathcal{S}_4$	0.4455	0.4488	0.5155	0.4600	0.4940	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\mathcal{S}_5$	0.4449	0.4758	0.5098	0.4486	0.4973	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
7.5	$\mathcal{S}_1$	0.4545	0.4873	0.5289	0.4506	0.4977	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
	$\mathcal{S}_2$	0.4658	0.4915	0.4809	0.4310	0.5147	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
	$\mathcal{S}_3$	0.4731	0.4753	0.5560	0.4582	0.5045	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
	$\mathcal{S}_4$	0.4491	0.4495	0.5329	0.4564	0.4920	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\mathcal{S}_5$	0.4496	0.4885	0.5289	0.4466	0.4969	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
10	$\mathcal{S}_1$	0.4575	0.4941	0.5423	0.4482	0.4975	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
	$\mathcal{S}_2$	0.4619	0.4921	0.4804	0.4259	0.5202	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
	$\mathcal{S}_3$	0.4751	0.4771	0.5635	0.4560	0.5031	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
	$\mathcal{S}_4$	0.4513	0.4479	0.5448	0.4532	0.4902	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_2$
	$\mathcal{S}_5$	0.4526	0.4946	0.5424	0.4449	0.4968	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$

Chapter 5

A novel triangular interval type-2 intuitionistic fuzzy sets and their aggregation operators[1]

In this chapter, we have presented a new idea about the triangular interval type-2 (TIT2) intuitionistic fuzzy sets and studied their several properties. Some basic operational laws, as well as the relation between them by using Frank Archimedean t-norm (AT) and Archimedean t-conorm (AC) operations, are defined. Based on these operations, some series of weighted averaging AOs are defined namely, TIT2 intuitionistic fuzzy weighted averaging, TIT2 intuitionistic fuzzy ordered weighted averaging and TIT2 intuitionistic fuzzy hybrid averaging. The characteristics of these operators and the influence of the Frank AT and AC parameter have been discussed. Later, a novel model based on developed operators is presented to solve the MADM problems and explained them with the help of a numerical example. Finally, comparative studies with some of the existing methods are discussed.

5.1 Introduction

As reviewed from the Section 1.1.2 of Chapter 1 that AOs play a significant role in any decision making the process to aggregate the different values into a single one. Also, it

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has been noticed that the most of the researchers studied the theories under the ordinary fuzzy set (henceforth called as a type-1 fuzzy set (T1FS)) environment, in which they have been assumed that the membership function corresponding to their element is exact. In order to overcome it, a T2FS [167], as an extension of the T1FS, is widely used. Further, due to the high complexities of T2FSs, it is difficult to apply in the real situation. For this, an interval type-2 fuzzy sets (IT2FSs) has been considered [99] which contain membership values from zero to one. Castillo et al. [11] discussed the short remarks on FS, IT2FS, general T2FS, and IFS. Chen and Lee [15], Chen, Yang, Lee and Yang [17] presented a fuzzy decision-making method based on the ranking values as well as the arithmetic operations of IT2FSs. Lee and Chen [89] presented a decision-making method under the IT2FS by using the concept of TOPSIS ("Technique for Order of Preference by Similarity to Ideal Solution").

For a decision-making problem, various authors have investigated the problems under the T2FSs environment by using different AOs as well as information measures [28, 75, 89, 100–102, 120, 147, 175]. All aforementioned AOs are usually based on the algebraic norm and probabilistic sum [33] operations, which have the lack of flexibility and robustness. Thus, there is need to address such issue also. Apart from them, some of the existing AOs are derived by either the fuzzy extension principle or based on the triangular norms [38, 41, 146, 159]. Frank triangle norms [37], is one of the most important norm operations and the generalizations of probabilistic and product t-norm and t-conorm. Further, frank triangle norms are the only triangle norm which satisfying the compatibility property. Also, it consists of additional parameter which provide more flexible to model the process. To the best of our knowledge, very fewer investigations on AOs based on Frank t-norms have been done by the authors. For instance, Qin and Liu [120] have presented an AO for TIT2FS. Nancy and Garg [110] presented AOs under single-valued neutrosophic environment. Qin et al. [124] presented a hesitant fuzzy AO based on the Frank t-norm operations.

As IT2FS considered only the degree of membership during analysis and hence there is a need to extend it by considering the degree of nonmembership also into the study. To handle the decision preference and to provide more degree of freedom to the expert during evaluating the objects, we presented a new theory named as an interval type-2 intuitionistic fuzzy sets (IT2IFSs) by considering a degree of membership, non-membership and their corresponding FOU. To enrich the study of the IT2IFSs, an AO is a crucial factor so it is meaningful to study AOs based on the triangular norm operations.

Thus, keeping inspiration from the fact that IT2IFSs have the great powerful ability to model the information in real-world applications, the present chapter has presented the various AOs for the triangular IT2IFSs based on the Frank t-norms. For it, firstly the basic operational laws based on the Frank t-norms have been defined on the triangular interval type-2 intuitionistic fuzzy numbers (TIT2IFNs). Based on these laws, some AOs, namely TIT2 intuitionistic fuzzy (TIT2IF) weighted averaging operators and their corresponding properties are presented in details. Later, a new method based on these operators is presented to solve the MADM problems and explained them with the help of a numerical example. Finally, comparative studies with some of the existing methods are discussed.

5.2 Interval type-2 Intuitionistic fuzzy set

In this section, we have defined the concept of the triangular interval type-2 Intuitionistic fuzzy sets (TIT2IFSs) by considering the upper and lower membership and non-membership functions by the triangular fuzzy numbers.

5.2.1 Triangular interval type-2 Intuitionistic fuzzy set

Let $\mathcal{A} = ([a, b], c, [d, e]; [A, B], C, [D, E])$ be a triangular IT2IFS (TIT2IFS) defined on $\mathcal{X}$, shown in Fig. 5.1, where $0 \leq a \leq b \leq c \leq d \leq e \leq 1$ and $0 \leq E \leq D \leq C \leq B \leq A \leq 1$ such that $e + E \leq 1$, $a + A \leq 1$ is characterized by a linear upper and lower membership and non-membership functions, which are defined as follows:

$$\mathrm{UMF}_{\mu}(x) = \begin{cases} \frac{x-a}{c-a} & ; \quad a \leq x < c \\ 1 & ; \quad x = c \\ \frac{e-x}{e-c} & ; \quad c \leq x < e \end{cases} \quad ; \quad \mathrm{LMF}_{\mu}(x) = \begin{cases} \frac{x-b}{c-b} & ; \quad b \leq x < c \\ 1 & ; \quad x = c \\ \frac{d-x}{d-c} & ; \quad c \leq x < d \end{cases}$$

and

$$\mathrm{UMF}_\nu(x)=\begin{cases}\frac{C-x}{C-E} & ; \quad E\le x<C\\ 0 & ; \quad x=C\\ \frac{x-C}{A-C} & ; \quad C\le x<A\end{cases} \quad ; \quad \mathrm{LMF}_\nu(x)=\begin{cases}\frac{C-x}{C-D} & ; \quad D\le x<C\\ 0 & ; \quad x=C\\ \frac{x-C}{B-C} & ; \quad C\le x<B\end{cases}$$

The FOU of the membership and non-membership functions are depicted as a shaded portion in Fig. 5.1. If X is a set consists of all real numbers, then a TIT2IFS in X is called triangular interval type-2 intuitionistic fuzzy number (TIT2IFN).

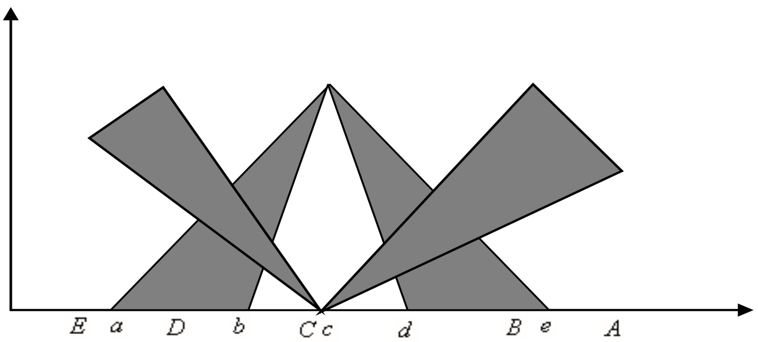

Figure 5.1: Representation of TIT2IFS

A compliment of TIT2IFN $\mathcal{A}=([a,b],c,[d,e];[A,B],\ C,\ [D,E])$ is given by

$$\mathcal{A}^c=([A,B],C,[D,E];[a,b],c,[d,e]) \tag{5.1}$$

To rank the different TIT2IFNs $\mathcal{A}=([a,b],c,[d,e];[A,B],\ C,\ [D,E])$, we define rank as

$$\Re(\mathcal{A})=\left(\frac{(a-A)+(e-E)}{2}+1\right)\left(\frac{(a-A)+(b-B)+4(c-C)+(d-D)+(e-E)}{8}\right) \tag{5.2}$$

The larger the Rank($\mathcal{A}$), the greater the TIT2IFN. Furthermore, to compare two or more different TIT2IFNs, we define an ordering relation between two TIT2IFNs $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}\prec\mathcal{B}$ if Rank($\mathcal{A}$) $<$ Rank($\mathcal{B}$) and $\mathcal{A}=\mathcal{B}$ if Rank($\mathcal{A}$) $=$ Rank($\mathcal{B}$)

To illustrate the working of Eq. (5.2), consider $\mathcal{A}=([0.3,0.4],\ 0.5,\ [0.6,0.7];\ [0.6,0.5],$ $0.3,\ [0.2,0.1])$ and $\mathcal{B}=([0.4,0.45],\ 0.50,\ [0.60,\ 0.70];\ [0.6,\ 0.4],\ 0.35,\ [0.25,\ 0.15])$ be two TIT2IFNs. Then, by using Eq. (5.2), we get $\Re(\mathcal{A})=0.1725$ and $\Re(\mathcal{B})=0.1982$. Since $\Re(\mathcal{A})<\Re(\mathcal{B})$ and thus, we have $\mathcal{B}\succ\mathcal{A}$.

Next, we define the partial order for two TIT2IFNs $\mathcal{A}_1 = ([a_1, b_1], c_1, [d_1, e_1]; [A_1, B_1], C_1, [D_1, E_1])$ and $\mathcal{A}_2 = ([a_2, b_2], c_2, [d_2, e_2]; [A_2, B_2], C_2, [D_2, E_2])$ denoted by $\mathcal{A}_1 \succeq_P \mathcal{A}_2$ if and only if $a_1 \geq a_2, b_1 \geq b_2, \ldots, e_1 \geq e_2$ and $A_1 \leq A_2, B_1 \leq B_2, \ldots, E_1 \leq E_2$. Especially, $\mathcal{A}_1 = \mathcal{A}_2$ if and only if $a_1 = a_2, b_2 = b_2, \ldots, e_1 = e_2$ and $A_1 = A_2, \ldots, E_1 = E_2$.

From the above, it has been observed that if $\mathcal{A}_1 \succeq_P \mathcal{A}_2$ which indicate that $\Re(\mathcal{A}_1) \geq \Re(\mathcal{A}_2)$. If $\Re(\mathcal{A}_1) > \Re(\mathcal{A}_2)$ then $\mathcal{A}_1 \succ \mathcal{A}_2$; if Rank$(\mathcal{A}_1) = \Re(\mathcal{A}_2)$ and since $a_1 \geq a_2, \ldots, e_1 \geq e_2$ and $A_1 \leq A_2, \ldots, E_1 \leq E_2$ then $a_1 = a_2, \ldots, e_1 = e_2, A_1 = A_2, \ldots, E_1 = E_2$, which indicates that $\mathcal{A}_1 = \mathcal{A}_2$. Thus, we can say if $\mathcal{A}_1 \succeq_P \mathcal{A}_2$ then, we have $\mathcal{A}_1 \succeq \mathcal{A}_2$.

5.2.2 Frank t-norms Operational laws of TIT2IFNs

The set-theoretical operators had an important role since at the beginning of the fuzzy set. By using the concept of the AT and AC, many AOs are defined in the literature. Among the various existing t-norm and t-conorms, Frank norm operations [37] is a general and flexible family of continuous triangular norms. Frank operations include the Frank product($\otimes_F$) and Frank sum ($\oplus_F$) which are defined in the following ways:

$$\begin{aligned} x \oplus_F y &= 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-x} - 1)(\lambda^{1-y} - 1)}{\lambda - 1}\right), \lambda > 1 \quad \forall (x, y) \in [0, 1]^2 \\ x \otimes_F y &= \log_\lambda \left(1 + \frac{(\lambda^{x} - 1)(\lambda^{y} - 1)}{\lambda - 1}\right), \lambda > 1 \quad \forall (x, y) \in [0, 1]^2 \end{aligned}$$

Further, it can be seen that

- $\big(x \oplus_F y\big) + \big(x \otimes_F y\big) = x + y$
- $\frac{\partial(x \oplus_F y)}{\partial x} + \frac{\partial(x \otimes_F y)}{\partial x} = \frac{\partial(x \oplus_F y)}{\partial y} + \frac{\partial(x \otimes_F y)}{\partial y} = 1$

In addition, Frank AT and AC have also two special cases given as follows.

(i) When $\lambda \to 1$, we have $x \oplus_F y = x + y - xy$ and $x \otimes_F y = xy$ which are the algebraic AT and AC respectively, defined in Table 2.1 of Chapter 2.

(ii) When $\lambda \to +\infty$, we have $x \oplus_F y = \min(x + y, 1)$, $x \otimes_F y = \max(0, x + y - 1)$, which are the Lukasiewicz AT anc AC respectively.

By keeping the features of Frank AT and AC, we define the basic operational laws for the different TIT2IFNs as follows:

Let $\mathcal{A} = ([a, b], c, [d, e]; [A, B], C, [D, E])$, $\mathcal{A}_1 = ([a_1, b_1], c_1, [d_1, e_1]; [A_1, B_1], C_1, [D_1, E_1])$ and $\mathcal{A}_2 = ([a_2, b_2], c_2, [d_2, e_2]; [A_2, B_2], C_2, [D_2, E_2])$ be three TIT2IFNs and $k > 0$, $\lambda > 1$ be two real numbers. Then, we have

(i) Addition operations:

$$\mathcal{A}_1 \oplus_F \mathcal{A}_2 = \begin{pmatrix} [1 - f(a_1, a_2), 1 - f(b_1, b_2)], 1 - f(c_1, c_2), [1 - f(d_1, d_2), \\ 1 - f(e_1, e_2)]; [f(1 - A_1, 1 - A_2), f(1 - B_1, 1 - B_2)], \\ f(1 - C_1, 1 - C_2), [f(1 - D_1, 1 - D_2), f(1 - E_1, 1 - E_2)] \end{pmatrix} \tag{5.3}$$

(ii) Multiplication operations:

$$\mathcal{A}_1 \otimes_F \mathcal{A}_2 = \begin{pmatrix} [f(1 - a_1, 1 - a_2), f(1 - b_1, 1 - b_2)], f(1 - c_1, 1 - c_2), \\ [f(1 - d_1, 1 - d_2), f(1 - e_1, 1 - e_2)]; [1 - f(A_1, A_2), \\ 1 - f(B_1, B_2)], 1 - f(C_1, C_2), [1 - f(D_1, D_2), 1 - f(E_1, E_2)] \end{pmatrix} \tag{5.4}$$

(iii) Multiplication by ordinary number $k > 0$:

$$k \cdot_F \mathcal{A} = \begin{pmatrix} [1 - g_k(a), 1 - g_k(b)], 1 - g_k(c), [1 - g_k(d), 1 - g_k(e)]; \\ [g_k(1 - A), g_k(1 - B)], g_k(1 - C), [g_k(1 - D), g_k(1 - E)] \end{pmatrix} \tag{5.5}$$

(iv) Power operation by number $k > 0$:

$$\mathcal{A}^k = \begin{pmatrix} [g_k(1 - a), g_k(1 - b)], g_k(1 - c), [g_k(1 - d), g_k(1 - e)]; \\ [1 - g_k(A), 1 - g_k(B)], 1 - g_k(C), [1 - g_k(D), 1 - g_k(E)] \end{pmatrix} \tag{5.6}$$

where the functions $f(\cdot)$ and $g(\cdot)$ are defined as

$$f(x, y) = \log_\lambda \left(1 + \frac{(\lambda^{1-x} - 1)(\lambda^{1-y} - 1)}{\lambda - 1} \right) \tag{5.7}$$

$$\text{and} \qquad g_k(x) = \log_\lambda \left(1 + \frac{(\lambda^{1-x} - 1)^k}{(\lambda - 1)^{k-1}} \right) \tag{5.8}$$

respectively, for $x, y \in [0, 1]$.

Theorem 5.2.1. If $\mathcal{A}, \mathcal{A}_1$ and $\mathcal{A}_2$ be three TIT2IFNs; then, $\mathcal{A}_3 = \mathcal{A}_1 \oplus_F \mathcal{A}_2$, $\mathcal{A}_4 = \mathcal{A}_1 \otimes_F \mathcal{A}_2$, $\mathcal{A}_5 = k \cdot_F \mathcal{A}$ and $\mathcal{A}_6 = \mathcal{A}^k$, $k > 0$ are also TIT2IFNs.

Proof. For any two TIT2IFNs $\mathcal{A}_1 = ([a_1, b_1], c_1, [d_1, e_1]; [A_1, B_1], C_1, [D_1, E_1])$ and $\mathcal{A}_2 = ([a_2, b_2], c_2, [d_2, e_2]; [A_2, B_2], C_2, [D_2, E_2])$. Consider $\mathcal{A}_3 = \mathcal{A}_1 \oplus \mathcal{A}_2 = ([a_3, b_3], c_3, [d_3, e_3]; [A_3, B_3], C_3, [D_3, E_3])$ where by the definition of addition operations of TIT2IFNs and by Eqs. (5.7)-(5.8), we have

$$\begin{aligned}
a_3 &= 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a_1} - 1)(\lambda^{1-a_2} - 1)}{\lambda - 1}\right) \quad ; \quad A_3 = \log_\lambda \left(1 + \frac{(\lambda^{A_1} - 1)(\lambda^{A_2} - 1)}{\lambda - 1}\right) \\
b_3 &= 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-b_1} - 1)(\lambda^{1-b_2} - 1)}{\lambda - 1}\right) \quad ; \quad B_3 = \log_\lambda \left(1 + \frac{(\lambda^{B_1} - 1)(\lambda^{B_2} - 1)}{\lambda - 1}\right) \\
c_3 &= 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-c_1} - 1)(\lambda^{1-c_2} - 1)}{\lambda - 1}\right) \quad ; \quad C_3 = \log_\lambda \left(1 + \frac{(\lambda^{C_1} - 1)(\lambda^{C_2} - 1)}{\lambda - 1}\right) \\
d_3 &= 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-d_1} - 1)(\lambda^{1-d_2} - 1)}{\lambda - 1}\right) \quad ; \quad D_3 = \log_\lambda \left(1 + \frac{(\lambda^{D_1} - 1)(\lambda^{D_2} - 1)}{\lambda - 1}\right) \\
e_3 &= 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-e_1} - 1)(\lambda^{1-e_2} - 1)}{\lambda - 1}\right) \quad ; \quad E_3 = \log_\lambda \left(1 + \frac{(\lambda^{E_1} - 1)(\lambda^{E_2} - 1)}{\lambda - 1}\right)
\end{aligned}$$

Then, in order to show $\mathcal{A}_3$ be TIT2IFN we have to show that $0 \leq a_3 \leq b_3 \leq c_3 \leq d_3 \leq e_3 \leq 1$, $1 \geq A_3 \geq B_3 \geq C_3 \geq D_3 \geq E_3 \geq 0$ and $e_3 + E_3 \leq 1$, $a_3 + A_3 \leq 1$. Since $\mathcal{A}_j$, $(j = 1, 2)$ be TIT2IFNs which implies that $0 \leq a_j \leq b_j \leq c_j \leq d_j \leq e_j \leq 1$; $1 \geq A_j \geq B_j \geq C_j \geq D_j \geq E_j \geq 0$, $a_j + A_j \leq 1$ and $e_j + E_j \leq 1$ for $j = 1, 2$, then, for $\lambda > 1$ be a real number, we have

$$\begin{aligned}
& \frac{(\lambda^{1-e_1} - 1)(\lambda^{1-e_2} - 1)}{\lambda - 1} \leq \frac{(\lambda^{1-d_1} - 1)(\lambda^{1-d_2} - 1)}{\lambda - 1} \leq \frac{(\lambda^{1-c_1} - 1)(\lambda^{1-c_2} - 1)}{\lambda - 1} \\
& \qquad \leq \frac{(\lambda^{1-b_1} - 1)(\lambda^{1-b_2} - 1)}{\lambda - 1} \leq \frac{(\lambda^{1-a_1} - 1)(\lambda^{1-a_2} - 1)}{\lambda - 1} \\
\Leftrightarrow \quad & 0 \leq \log_\lambda \left(1 + \frac{(\lambda^{1-e_1} - 1)(\lambda^{1-e_2} - 1)}{\lambda - 1}\right) \leq \log_\lambda \left(1 + \frac{(\lambda^{1-d_1} - 1)(\lambda^{1-d_2} - 1)}{\lambda - 1}\right) \\
& \qquad \leq \log_\lambda \left(1 + \frac{(\lambda^{1-c_1} - 1)(\lambda^{1-c_2} - 1)}{\lambda - 1}\right) \leq \log_\lambda \left(1 + \frac{(\lambda^{1-b_1} - 1)(\lambda^{1-b_2} - 1)}{\lambda - 1}\right) \\
& \qquad \leq \log_\lambda \left(1 + \frac{(\lambda^{1-a_1} - 1)(\lambda^{1-a_2} - 1)}{\lambda - 1}\right) \leq 1 \\
\Leftrightarrow \quad & 0 \leq a_3 \leq b_3 \leq c_3 \leq d_3 \leq e_3 \leq 1
\end{aligned}$$

and

$$\frac{(\lambda^{A_1}-1)(\lambda^{A_2}-1)}{\lambda-1} \geq \frac{(\lambda^{B_1}-1)(\lambda^{B_2}-1)}{\lambda-1} \geq \frac{(\lambda^{C_1}-1)(\lambda^{C_2}-1)}{\lambda-1}$$
$$\geq \frac{(\lambda^{D_1}-1)(\lambda^{D_2}-1)}{\lambda-1} \geq \frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}$$

$$\begin{aligned}\Leftrightarrow \quad & 1 \geq \log_\lambda\left(1+\frac{(\lambda^{A_1}-1)(\lambda^{A_2}-1)}{\lambda-1}\right) \geq \log_\lambda\left(1+\frac{(\lambda^{B_1}-1)(\lambda^{B_2}-1)}{\lambda-1}\right)\\ & \geq \log_\lambda\left(1+\frac{(\lambda^{C_1}-1)(\lambda^{C_2}-1)}{\lambda-1}\right) \geq \log_\lambda\left(1+\frac{(\lambda^{D_1}-1)(\lambda^{D_2}-1)}{\lambda-1}\right)\\ & \geq \log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right) \geq 0\\ \Leftrightarrow \quad & 1 \geq A_3 \geq B_3 \geq C_3 \geq D_3 \geq E_3 \geq 0\end{aligned}$$

Finally, $e_3 + E_3 = 1 - \log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)(\lambda^{1-e_2}-1)}{\lambda-1}\right) + \log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right) \leq 1-\log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right)+\log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right) \leq 1$. Similarly, $a_3+A_3 \leq 1$ which indicates that $\mathcal{A}_3 = \mathcal{A}_1 \oplus_F \mathcal{A}_2$ is TIT2IFN. Similarly, we can prove that $\mathcal{A}_4 = \mathcal{A}_1 \otimes_F \mathcal{A}_2$, $\mathcal{A}_5 = k \cdot_F \mathcal{A}$ and $\mathcal{A}_6 = \mathcal{A}^k$ are also TIT2IFNs. □

Theorem 5.2.2. Let $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ be three TIT2IFNs and k, k_1, k_2 be three positive real numbers then, we have

(i) $\mathcal{A}_1 \oplus_F \mathcal{A}_2 = \mathcal{A}_2 \oplus_F \mathcal{A}_1$

(ii) $\mathcal{A}_1 \otimes_F \mathcal{A}_2 = \mathcal{A}_2 \otimes_F \mathcal{A}_1$

(iii) $k \cdot_F (\mathcal{A}_1 \oplus_F \mathcal{A}_2) = k \cdot_F \mathcal{A}_1 \oplus_F k \cdot_F \mathcal{A}_2$

(iv) $(\mathcal{A}_1 \otimes_F \mathcal{A}_2)^k = \mathcal{A}_1^k \otimes_F \mathcal{A}_2^k$

(v) $(k_1 \cdot_F \mathcal{A}) \oplus_F (k_2 \cdot_F \mathcal{A}) = (k_1 + k_2) \cdot_F \mathcal{A}$

(vi) $\mathcal{A}^{k_1} \otimes_F \mathcal{A}^{k_2} = \mathcal{A}^{k_1+k_2}$

Proof. We prove the parts (i), (iii) and (v) and hence similar for others.

(i) By the definition of addition operations, we can easily obtain it.

(iii) Since $\mathcal{A}_1$ and $\mathcal{A}_2$ be two TIT2IFNs and then by addition operations, we get

$$
\begin{aligned}
& k \cdot_F (\mathcal{A}_1 \oplus_F \mathcal{A}_2) \\
& = \left(\begin{array}{l}
\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)^k(\lambda^{1-a_2}-1)^k}{(\lambda-1)^{2k-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)^k(\lambda^{1-b_2}-1)^k}{(\lambda-1)^{2k-1}}\right)\right], \\
1-\log_\lambda\left(1+\frac{(\lambda^{1-c_1}-1)^k(\lambda^{1-c_2}-1)^k}{(\lambda-1)^{2k-1}}\right), \left[1-\log_\lambda\left(1+\frac{(\lambda^{1-d_1}-1)^k(\lambda^{1-d_2}-1)^k}{(\lambda-1)^{2k-1}}\right), \right. \\
\left. 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)^k(\lambda^{1-e_2}-1)^k}{(\lambda-1)^{2k-1}}\right)\right]; \left[\log_\lambda\left(1+\frac{(\lambda^{A_1}-1)^k(\lambda^{A_2}-1)^k}{(\lambda-1)^{2k-1}}\right), \right. \\
\left. \log_\lambda\left(1+\frac{(\lambda^{B_1}-1)^k(\lambda^{B_2}-1)^k}{(\lambda-1)^{2k-1}}\right)\right], \log_\lambda\left(1+\frac{(\lambda^{C_1}-1)^k(\lambda^{C_2}-1)^k}{(\lambda-1)^{2k-1}}\right), \\
\left[\log_\lambda\left(1+\frac{(\lambda^{D_1}-1)^k(\lambda^{D_2}-1)^k}{(\lambda-1)^{2k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{E_1}-1)^k(\lambda^{E_2}-1)^k}{(\lambda-1)^{2k-1}}\right)\right]
\end{array} \right) \\
& = \left(\begin{array}{l}
\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)^k}{(\lambda-1)^{k-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)^k}{(\lambda-1)^{k-1}}\right)\right], \\
1-\log_\lambda\left(1+\frac{(\lambda^{1-c_1}-1)^k}{(\lambda-1)^{k-1}}\right), \left[1-\log_\lambda\left(1+\frac{(\lambda^{1-d_1}-1)^k}{(\lambda-1)^{k-1}}\right), \right. \\
\left. 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)^k}{(\lambda-1)^{k-1}}\right)\right]; \left[\log_\lambda\left(1+\frac{(\lambda^{A_1}-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{B_1}-1)^k}{(\lambda-1)^{k-1}}\right)\right], \\
\log_\lambda\left(1+\frac{(\lambda^{C_1}-1)^k}{(\lambda-1)^{k-1}}\right), \left[\log_\lambda\left(1+\frac{(\lambda^{D_1}-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{E_1}-1)^k}{(\lambda-1)^{k-1}}\right)\right]
\end{array} \right) \\
& \oplus_F \left(\begin{array}{l}
\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_2}-1)^k}{(\lambda-1)^{k-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_2}-1)^k}{(\lambda-1)^{k-1}}\right)\right], \\
1-\log_\lambda\left(1+\frac{(\lambda^{1-c_2}-1)^k}{(\lambda-1)^{k-1}}\right), \left[1-\log_\lambda\left(1+\frac{(\lambda^{1-d_2}-1)^k}{(\lambda-1)^{k-1}}\right), \right. \\
\left. 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_2}-1)^k}{(\lambda-1)^{k-1}}\right)\right]; \left[\log_\lambda\left(1+\frac{(\lambda^{A_2}-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{B_2}-1)^k}{(\lambda-1)^{k-1}}\right)\right], \\
\log_\lambda\left(1+\frac{(\lambda^{C_2}-1)^k}{(\lambda-1)^{k-1}}\right), \left[\log_\lambda\left(1+\frac{(\lambda^{D_2}-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{E_2}-1)^k}{(\lambda-1)^{k-1}}\right)\right]
\end{array} \right) \\
& = k \cdot_F \mathcal{A}_1 \oplus_F k \cdot_F \mathcal{A}_2
\end{aligned}
$$

Hence, $k \cdot_F (\mathcal{A}_1 \oplus_F \mathcal{A}_2) = k \cdot_F \mathcal{A}_1 \oplus_F k \cdot_F \mathcal{A}_2$.

(v) For $k_i > 0$, $i = 1, 2$, we have

$$k_i \cdot_F \mathcal{A} = \left(\begin{array}{l} \left[1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a} - 1)^{k_i}}{(\lambda - 1)^{k_i - 1}}\right), 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-b} - 1)^{k_i}}{(\lambda - 1)^{k_i - 1}}\right)\right], \\ 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-c} - 1)^{k_i}}{(\lambda - 1)^{k_i - 1}}\right), \left[1 - \log_\lambda \left(1 + \frac{(\lambda^{1-d} - 1)^{k_i}}{(\lambda - 1)^{k_i - 1}}\right), \right. \\ \left. 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-e} - 1)^{k_i}}{(\lambda - 1)^{k_i - 1}}\right)\right]; \left[\log_\lambda \left(1 + \frac{(\lambda^{A} - 1)^{k_i}}{(\lambda - 1)^{k_i - 1}}\right), \right. \\ \left. \log_\lambda \left(1 + \frac{(\lambda^{B} - 1)^{k_i}}{(\lambda - 1)^{k_i - 1}}\right)\right], \log_\lambda \left(1 + \frac{(\lambda^{C} - 1)^{k_i}}{(\lambda - 1)^{k_i - 1}}\right), \\ \left[\log_\lambda \left(1 + \frac{(\lambda^{D} - 1)^{k_i}}{(\lambda - 1)^{k_i - 1}}\right), \log_\lambda \left(1 + \frac{(\lambda^{E} - 1)^{k_i}}{(\lambda - 1)^{k_i - 1}}\right)\right] \end{array} \right)$$

Thus,

$$\begin{aligned} & k_1 \cdot_F \mathcal{A} \oplus_F k_2 \cdot_F \mathcal{A} \\ = & \left(\begin{array}{l} \left[1 - \log_\lambda \left(1 + \frac{\frac{(\lambda^{1-a}-1)^{k_1}}{(\lambda-1)^{k_1-1}} \frac{(\lambda^{1-a}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{(\lambda - 1)}\right), 1 - \log_\lambda \left(1 + \frac{\frac{(\lambda^{1-b}-1)^{k_1}}{(\lambda-1)^{k_1-1}} \frac{(\lambda^{1-b}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{(\lambda - 1)}\right)\right], \\ 1 - \log_\lambda \left(1 + \frac{\frac{(\lambda^{1-c}-1)^{k_1}}{(\lambda-1)^{k_1-1}} \frac{(\lambda^{1-c}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{(\lambda - 1)}\right), \left[1 - \log_\lambda \left(1 + \frac{\frac{(\lambda^{1-d}-1)^{k_1}}{(\lambda-1)^{k_1-1}} \frac{(\lambda^{1-d}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{(\lambda - 1)}\right), \right. \\ \left. 1 - \log_\lambda \left(1 + \frac{\frac{(\lambda^{1-e}-1)^{k_1}}{(\lambda-1)^{k_1-1}} \frac{(\lambda^{1-e}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{(\lambda - 1)}\right)\right]; \left[\log_\lambda \left(1 + \frac{\frac{(\lambda^{A}-1)^{k_1}}{(\lambda-1)^{k_1-1}} \frac{(\lambda^{A}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{\lambda - 1}\right), \right. \\ \left. \log_\lambda \left(1 + \frac{\frac{(\lambda^{B}-1)^{k_1}}{(\lambda-1)^{k_1-1}} \frac{(\lambda^{B}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{\lambda - 1}\right)\right], \log_\lambda \left(1 + \frac{\frac{(\lambda^{C}-1)^{k_1}}{(\lambda-1)^{k_1-1}} \frac{(\lambda^{C}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{\lambda - 1}\right), \\ \left[\log_\lambda \left(1 + \frac{\frac{(\lambda^{D}-1)^{k_1}}{(\lambda-1)^{k_1-1}} \frac{(\lambda^{D}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{\lambda - 1}\right), \log_\lambda \left(1 + \frac{\frac{(\lambda^{E}-1)^{k_1}}{(\lambda-1)^{k_1-1}} \frac{(\lambda^{E}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{\lambda - 1}\right)\right] \end{array} \right) \\ = & \left(\begin{array}{l} \left[1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a} - 1)^{k_1+k_2}}{(\lambda - 1)^{k_1+k_2-1}}\right), 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-b} - 1)^{k_1+k_2}}{(\lambda - 1)^{k_1+k_2-1}}\right)\right], \\ 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-c} - 1)^{k_1+k_2}}{(\lambda - 1)^{k_1+k_2-1}}\right), \left[1 - \log_\lambda \left(1 + \frac{(\lambda^{1-d} - 1)^{k_1+k_2}}{(\lambda - 1)^{k_1+k_2-1}}\right), \right. \\ \left. 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-e} - 1)^{k_1+k_2}}{(\lambda - 1)^{k_1+k_2-1}}\right)\right]; \left[\log_\lambda \left(1 + \frac{(\lambda^{A} - 1)^{k_1+k_2}}{(\lambda - 1)^{k_1+k_2-1}}\right), \right. \\ \left. \log_\lambda \left(1 + \frac{(\lambda^{B} - 1)^{k_1+k_2}}{(\lambda - 1)^{k_1+k_2-1}}\right)\right], \log_\lambda \left(1 + \frac{(\lambda^{C} - 1)^{k_1+k_2}}{(\lambda - 1)^{k_1+k_2-1}}\right), \\ \left[\log_\lambda \left(1 + \frac{(\lambda^{D} - 1)^{k_1+k_2}}{(\lambda - 1)^{k_1+k_2-1}}\right), \log_\lambda \left(1 + \frac{(\lambda^{E} - 1)^{k_1+k_2}}{(\lambda - 1)^{k_1+k_2-1}}\right)\right] \end{array} \right) \\ = & (k_1 + k_2) \cdot_F \mathcal{A} \end{aligned}$$

$\square$

5.3 Aggregation Operators for TIT2IFNs

In this section, some series of weighted AOs for TIT2IFNs have been proposed based on above defined Frank t-norm operations.

5.3.1 Weighted averaging Operator

Definition 5.3.1. Let $\mathcal{A}_j, j = 1, 2, \ldots, n$ be a collection of TIT2IFNs and if

$$\text{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1 \cdot_F \mathcal{A}_1 \oplus_F \omega_2 \cdot_F \mathcal{A}_2 \oplus_F \ldots \oplus_F \omega_n \cdot_F \mathcal{A}_n \tag{5.9}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\mathcal{A}_j$ such that $\omega_j > 0$, $\sum\limits_{j=1}^{n} \omega_j = 1$ then, TIT2IFWA is called a TIT2IF weighted averaging operator.

Theorem 5.3.1. The aggregated value by using TIT2IFWA operator for a collection of TIT2IFNs $\mathcal{A}_j = ([a_j, b_j], c_j, [d_j, e_j]; [A_j, B_j], C_j, [D_j, E_j]), j = 1, 2, \ldots, n$ is still TIT2IFN and is given by

$$\begin{aligned} &\text{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ &= \left(\begin{array}{l} \left[1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-a_j} - 1)^{\omega_j} \right), 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-b_j} - 1)^{\omega_j} \right) \right], \\ 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-c_j} - 1)^{\omega_j} \right), \left[1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-d_j} - 1)^{\omega_j} \right), \right. \\ \left. 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-e_j} - 1)^{\omega_j} \right) \right]; \left[\log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{A_j} - 1)^{\omega_j} \right), \right. \\ \left. \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{B_j} - 1)^{\omega_j} \right) \right], \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{C_j} - 1)^{\omega_j} \right), \\ \left[\log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{D_j} - 1)^{\omega_j} \right), \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{E_j} - 1)^{\omega_j} \right) \right] \end{array} \right) \end{aligned} \tag{5.10}$$

Proof. We will prove the Eq. (5.10) by mathematical induction on n. Since for each j, $\mathcal{A}_j$ is a TIT2IFN. Then, the following steps of the mathematical induction have been followed:

Step 1: For $n = 2$, we have TIT2IFWA$(\mathcal{A}_1, \mathcal{A}_2) = \omega_1 \cdot_F \mathcal{A}_1 \oplus_F \omega_2 \cdot_F \mathcal{A}_2$ with $\omega_1 + \omega_2 = 1$. Thus, by the operation of TIT2IFNs for $i = 1, 2$, we get

$$\omega_j \cdot_F \mathcal{A}_j = \begin{pmatrix} [1 - g_{\omega_j}(a_j), 1 - g_{\omega_j}(b_j)], 1 - g_{\omega_j}(c_j), [1 - g_{\omega_j}(d_j), 1 - g_{\omega_j}(e_j)]; \\ [g_{\omega_j}(1 - A_j), g_{\omega_j}(1 - B_j)], g_{\omega_j}(1 - C_j), [g_{\omega_j}(1 - D_j), g_{\omega_j}(1 - E_j)] \end{pmatrix}$$

where $g_{\omega_j}(\cdot)$ is defined in Eq. (5.8). Then, by the addition operational laws of TIT2IFNs, we get

$$\begin{aligned} & \omega_1 \cdot_F \mathcal{A}_1 \oplus_F \omega_2 \cdot_F \mathcal{A}_2 \\ = & \begin{pmatrix} [1 - f\big(1 - g_{\omega_1}(a_1), 1 - g_{\omega_2}(a_2)\big), 1 - f\big(1 - g_{\omega_1}(b_1), 1 - g_{\omega_2}(b_2)\big)], \\ 1 - f\big(1 - g_{\omega_1}(c_1), 1 - g_{\omega_2}(c_2)\big), [1 - f\big(1 - g_{\omega_1}(d_1), 1 - g_{\omega_2}(d_2)\big), \\ 1 - f\big(1 - g_{\omega_1}(e_1), 1 - g_{\omega_2}(e_2)\big)]; [f\big(1 - g_{\omega_1}(1 - A_1), 1 - g_{\omega_2}(1 - A_2)\big), \\ f\big(1 - g_{\omega_1}(1 - B_1), 1 - g_{\omega_2}(1 - B_2)\big)], f\big(1 - g_{\omega_1}(1 - C_1), 1 - g_{\omega_2}(1 - C_2)\big), \\ [f\big(1 - g_{\omega_1}(1 - D_1), 1 - g_{\omega_2}(1 - D_2)\big), f\big(1 - g_{\omega_1}(1 - E_1), 1 - g_{\omega_2}(1 - E_2)\big)] \end{pmatrix} \end{aligned} \tag{5.11}$$

where $f(\cdot)$ is defined according to Eq. (5.7). Now, by Eq. (5.8), we have $g_{\omega_j}(a_j) = \log_\lambda\left(1 + \frac{(\lambda^{1-a_j} - 1)^{\omega_j}}{(\lambda - 1)^{\omega_j - 1}}\right)$ for $i = 1, 2$. Thus by Eq. (5.7), we get

$$\begin{aligned} & 1 - f\left(1 - g_{\omega_1}(a_1), 1 - g_{\omega_2}(a_2)\right) \\ = \;& 1 - \log_\lambda\left(1 + \frac{(\lambda^{g_{\omega_1}(a_1)} - 1)(\lambda^{g_{\omega_2}(a_2)} - 1)}{\lambda - 1}\right) \\ = \;& 1 - \log_\lambda\left(1 + \frac{\left(\frac{(\lambda^{1-a_1} - 1)^{\omega_1}}{(\lambda - 1)^{\omega_1 - 1}}\right)\left(\frac{(\lambda^{1-a_2} - 1)^{\omega_2}}{(\lambda - 1)^{\omega_2 - 1}}\right)}{\lambda - 1}\right) \\ = \;& 1 - \log_\lambda\left(1 + \prod_{j=1}^{2}\left(\lambda^{1-a_j} - 1\right)^{\omega_j}\right) \end{aligned}$$

Similarly, we can get

$$f(1 - g_{\omega_1}(1 - A_1), 1 - g_{\omega_2}(1 - A_2)) = \log_\lambda\left(1 + \prod_{i=1}^{2}\left(\lambda^{A_j} - 1\right)^{\omega_j}\right)$$

Therefore, by Eq. (5.11), result holds for $n = 2$.

Step 2: If Eq. (5.10) holds for $n = k$, then for $n = k+1$, we have

$$
\begin{aligned}
&\mathrm{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{k+1}) \\
&= \mathrm{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k) \oplus_F \omega_{k+1} \cdot_F \mathcal{A}_{k+1} \\
&= \left(\begin{array}{l}
\left[1-\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{1-a_j}-1)^{\omega_j}\right), 1-\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{1-b_j}-1)^{\omega_j}\right)\right], \\
1-\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{1-c_j}-1)^{\omega_j}\right), \left[1-\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{1-d_j}-1)^{\omega_j}\right), \right. \\
\left. 1-\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{1-e_j}-1)^{\omega_j}\right)\right]; \left[\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{A_j}-1)^{\omega_j}\right), \right. \\
\left. \log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{B_j}-1)^{\omega_j}\right)\right], \log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{C_j}-1)^{\omega_j}\right), \\
\left[\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{D_j}-1)^{\omega_j}\right), \log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{E_j}-1)^{\omega_j}\right)\right]
\end{array}\right) \\
&\bigoplus_F \left(\begin{array}{l}
\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_{k+1}}-1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_{k+1}}-1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right)\right], \\
1-\log_\lambda\left(1+\frac{(\lambda^{1-c_{k+1}}-1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right), \left[1-\log_\lambda\left(1+\frac{(\lambda^{1-d_{k+1}}-1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right), \right. \\
\left. 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_{k+1}}-1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right)\right]; \left[\log_\lambda\left(1+\frac{(\lambda^{A_{k+1}}-1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right), \right. \\
\left. \log_\lambda\left(1+\frac{(\lambda^{B_{k+1}}-1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right)\right], \log_\lambda\left(1+\frac{(\lambda^{C_{k+1}}-1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right), \\
\left[\log_\lambda\left(1+\frac{(\lambda^{D_{k+1}}-1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{E_{k+1}}-1)^{\omega_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right)\right]
\end{array}\right)
\end{aligned}
$$

$$
= \left(\begin{array}{l}
\left[1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{1-a_j}-1)^{\omega_j}}{(\lambda-1)^{\sum_{j=1}^{k+1}\omega_j-1}}\right), 1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{1-b_j}-1)^{\omega_j}}{(\lambda-1)^{\sum_{j=1}^{k+1}\omega_j-1}}\right)\right], \\
1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{1-c_j}-1)^{\omega_j}}{(\lambda-1)^{\sum_{j=1}^{k+1}\omega_j-1}}\right), \left[1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{1-d_j}-1)^{\omega_j}}{(\lambda-1)^{\sum_{j=1}^{k+1}\omega_j-1}}\right), \right. \\
\left. 1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{1-e_j}-1)^{\omega_j}}{(\lambda-1)^{\sum_{j=1}^{k+1}\omega_j-1}}\right)\right]; \left[\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{A_j}-1)^{\omega_j}}{(\lambda-1)^{\sum_{j=1}^{k+1}\omega_j-1}}\right), \right. \\
\left. \log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{B_j}-1)^{\omega_j}}{(\lambda-1)^{\sum_{j=1}^{k+1}\omega_j-1}}\right)\right], \log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{C_j}-1)^{\omega_j}}{(\lambda-1)^{\sum_{j=1}^{k+1}\omega_j-1}}\right), \\
\left[\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{D_j}-1)^{\omega_j}}{(\lambda-1)^{\sum_{j=1}^{k+1}\omega_j-1}}\right), \log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{E_j}-1)^{\omega_j}}{(\lambda-1)^{\sum_{j=1}^{k+1}\omega_j-1}}\right)\right]
\end{array}\right)
$$

$$
= \left(\begin{array}{l}
\left[1-\log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{1-a_j}-1)^{\omega_j}\right), 1-\log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{1-b_j}-1)^{\omega_j}\right)\right], \\
1-\log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{1-c_j}-1)^{\omega_j}\right), \left[1-\log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{1-d_j}-1)^{\omega_j}\right), \right. \\
\left. 1-\log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{1-e_j}-1)^{\omega_j}\right)\right]; \left[\log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{A_j}-1)^{\omega_j}\right), \right. \\
\left. \log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{B_j}-1)^{\omega_j}\right)\right], \log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{C_j}-1)^{\omega_j}\right), \\
\left[\log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{D_j}-1)^{\omega_j}\right), \log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{E_j}-1)^{\omega_j}\right)\right]
\end{array}\right)
$$

Thus, results holds for $n = k + 1$ and hence, by the principle of mathematical induction, result given in Eq. (5.10) holds for all $n \in \mathbb{Z}^+$. $\square$

It has been observed from the TIT2IFWA operator that it satisfies certain properties such as boundedness, idempotent, and monotonicity, invariance, etc., which can be stated as follows:

Theorem 5.3.2. (Idempotency:) If $\mathcal{A}_j = \mathcal{A}$ for all j, then

$$\text{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$$

Proof. Since for all j, $\mathcal{A}_j = \mathcal{A} = ([a,b], c, [d,e];\ [A,B],\ C,\ [D,E])$, and $\sum_{j=1}^{n} \omega_j = 1$ so by Theorem 5.3.1, we have

$$\begin{aligned}
&\text{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\\
&= \left(\begin{array}{l}
\left[1 - \log_\lambda\left(1 + \prod_{j=1}^{n} (\lambda^{1-a} - 1)^{\omega_j}\right), 1 - \log_\lambda\left(1 + \prod_{j=1}^{n} (\lambda^{1-b} - 1)^{\omega_j}\right)\right],\\
1 - \log_\lambda\left(1 + \prod_{j=1}^{n} (\lambda^{1-c} - 1)^{\omega_j}\right), \left[1 - \log_\lambda\left(1 + \prod_{j=1}^{n} (\lambda^{1-d} - 1)^{\omega_j}\right),\right.\\
\left.1 - \log_\lambda\left(1 + \prod_{j=1}^{n} (\lambda^{1-e} - 1)^{\omega_j}\right)\right]; \left[\log_\lambda\left(1 + \prod_{j=1}^{n} (\lambda^{A} - 1)^{\omega_j}\right),\right.\\
\left.\log_\lambda\left(1 + \prod_{j=1}^{n} (\lambda^{B} - 1)^{\omega_j}\right)\right], \log_\lambda\left(1 + \prod_{j=1}^{n} (\lambda^{C} - 1)^{\omega_j}\right),\\
\left[\log_\lambda\left(1 + \prod_{j=1}^{n} (\lambda^{D} - 1)^{\omega_j}\right), \log_\lambda\left(1 + \prod_{j=1}^{n} (\lambda^{E} - 1)^{\omega_j}\right)\right]
\end{array} \right)\\
&= \left(\begin{array}{l}
\left[1 - \log_\lambda \lambda^{1-a}, 1 - \log_\lambda \lambda^{1-b}\right], 1 - \log_\lambda \lambda^{1-c}, \left[1 - \log_\lambda \lambda^{1-d},\right.\\
\left.1 - \log_\lambda \lambda^{1-e}\right]; \left[\log_\lambda \lambda^{A}, \log_\lambda \lambda^{B}\right], \log_\lambda \lambda^{C}, \left[\log_\lambda \lambda^{D}, \log_\lambda \lambda^{E}\right]
\end{array} \right)\\
&= ([a,b], c, [d,e]; [A,B], C, [D,E])\\
&= \mathcal{A}
\end{aligned}$$

Thus, proof is completed. □

Theorem 5.3.3. (Boundedness:) Let $\mathcal{A}^- = \big([\min_j\{a_j\}, \min_j\{b_j\}], \min_j\{c_j\}, [\min_j\{d_j\}, \min_j\{e_j\}];$ $[\max_j\{A_j\},\ \max_j\{B_j\}],\ \max_j\{C_j\},\ [\max_j\{D_j\}, \max_j\{E_j\}]\big)$ and $\mathcal{A}^+ = \big([\max_j\ \{a_j\},\ \max_j\{b_j\}],$ $\max_j\{c_j\},\ [\max_j\{d_j\},\ \max_j\{e_j\}];\ [\min_j\{A_j\},\ \min_j\{B_j\}],\ \min_j\{C_j\},\ [\min_j\{D_j\}, \min_j\{E_j\}]\big)$ then

$$\mathcal{A}^- \leq \text{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$$

Proof. As for all j, we have $\min_j\{a_j\} \leq a_j \leq \max_j\{a_j\}$, this implies that $1 - \max_j\{a_j\} \leq 1 - a_j \leq 1 - \min_j\{a_j\}$. Hence, for $\lambda > 1$, we have $\lambda^{1-\max_j\{a_j\}} - 1 \leq \lambda^{1-a_j} - 1 \leq \lambda^{1-\min_j\{a_j\}} - 1$ $\Leftrightarrow \lambda^{1-\max_j\{a_j\}} - 1 \leq \prod_{j=1}^{n}(\lambda^{1-a_j} - 1)^{\omega_j} \leq \lambda^{1-\min_j\{a_j\}} - 1 \Leftrightarrow 1 - \max_j\{a_j\} \leq \log_\lambda \big(1 + \prod_{j=1}^{n}(\lambda^{1-a_j} - 1)^{\omega_j}\big) \leq 1 - \min_j\{a_j\}$. Therefore,

$$\min_j\{a_j\} \leq 1 - \log_\lambda \left(1 + \prod_{j=1}^{n}(\lambda^{1-a_j} - 1)^{\omega_j}\right) \leq \max_j\{a_j\} \tag{5.12}$$

Furthermore, for all j, we have $\min_j\{A_j\} \leq A_j \leq \max_j\{A_j\}$, this implies $\lambda^{\min_j\{A_j\}} - 1 \leq \lambda^{A_j} - 1 \leq \lambda^{\max_j\{A_j\}} - 1 \Leftrightarrow \lambda^{\min_j\{A_j\}} - 1 \leq \prod_{j=1}^{n}(\lambda^{A_j} - 1)^{\omega_j} \leq \lambda^{\max_j\{A_j\}} - 1$. Thus,

$$\min_j\{A_j\} \leq \log_\lambda \left(1 + \prod_{j=1}^{n}(\lambda^{A_j} - 1)^{\omega_j}\right) \leq \max_j\{A_j\} \tag{5.13}$$

Let $\text{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = ([a, b], c, [d, e]; [A, B], C, [D, E])$. Then from Eq. (5.12) and (5.13) we have, $\min_j\{a_j\} \leq a \leq \max_j\{a_j\}$ and $\min_j\{A_j\} \leq A \leq \max_j\{A_j\}$. Similarly, we have $\min_j\{b_j\} \leq b \leq \max_j\{b_j\}, \min_j\{c_j\} \leq c \leq \max_j\{c_j\}, \min_j\{d_j\} \leq d \leq \max_j\{d_j\}$, $\min_j\{e_j\} \leq e \leq \max_j\{e_j\}, \min_j\{B_j\} \leq B \leq \max_j\{B_j\}, \min_j\{C_j\} \leq C \leq \max_j\{C_j\}$, $\min_j\{D_j\} \leq D \leq \max_j\{D_j\}, \min_j\{E_j\} \leq E \leq \max_j\{E_j\}$. Hence, by using Eq. (5.2), we have

$$\begin{aligned}
\mathfrak{R}(\mathcal{A}) &= \left(\frac{(a - A) + (e - E)}{2} + 1\right)\left(\frac{a - A + b - B + 4(c - C) + d - D + e - E}{8}\right) \\
&\leq \left(\frac{\max_j\{a_j\} - \min_j\{A_j\} + \max_j\{e_j\} - \min_j\{E_j\}}{2} + 1\right) \times \\
&\times \left(\frac{\begin{array}{c}\max_j\{a_j\} - \min_j\{A_j\} + \max_j\{b_j\} - \min_j\{B_j\} + 4\max_j\{c_j\} \\ - 4\min_j\{C_j\} + \max_j\{d_j\} - \min_j\{D_j\} + \max_j\{e_j\} - \min_j\{E_j\}\end{array}}{8}\right) \\
&= \mathfrak{R}(\mathcal{A}^+)
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{R}(\mathcal{A}) &= \left(\frac{(a-A)+(e-E)}{2}+1\right)\left(\frac{a-A+b-B+4(c-C)+d-D+e-E}{8}\right)\\
&\geq \left(\frac{\min\limits_j\{a_j\}-\max\limits_j\{A_j\}+\min\limits_j\{e_j\}-\max\limits_j\{E_j\}}{2}+1\right)\times\\
&\times \left(\frac{\begin{array}{c}\min\limits_j\{a_j\}-\max\limits_j\{A_j\}+\min\limits_j\{b_j\}-\max\limits_j\{B_j\}+4\min\limits_j\{c_j\}\\ -4\max\limits_j\{C_j\}+\min\limits_j\{d_j\}-\max\limits_j\{D_j\}+\min\limits_j\{e_j\}-\max\limits_j\{E_j\}\end{array}}{8}\right)\\
&= \mathfrak{R}(\mathcal{A}^-)
\end{aligned}$$

Therefore, $\mathcal{A}^- \leq \text{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+.$ □

Theorem 5.3.4. (Monotonicity:) If $\mathcal{A}_j$ and $\mathcal{B}_j$ be collections of two TIT2IFNs such that $\mathcal{A}_j \leq \mathcal{B}_j$ then

$$\text{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{TIT2IFWA}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n)$$

Theorem 5.3.5. (Shift-invariance:) If $\mathcal{B}$ be another TIT2IFN, then

$$\text{TIT2IFWA}(\mathcal{A}_1 \oplus_F \mathcal{B}, \mathcal{A}_2 \oplus_F \mathcal{B}, \ldots, \mathcal{A}_n \oplus_F \mathcal{B}) = \text{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2 \ldots, \mathcal{A}_n) \oplus_F \mathcal{B}$$

Theorem 5.3.6. (Homogeneity:) If $\mathcal{B} > 0$ be a real number, then

$$\text{TIT2IFWA}(\mathcal{B} \cdot_F \mathcal{A}_1, \mathcal{B} \cdot_F \mathcal{A}_2, \ldots, \mathcal{B} \cdot_F \mathcal{A}_n) = \mathcal{B} \cdot_F \text{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2 \ldots, \mathcal{A}_n)$$

Proof. The proof of these Theorems can be easily derived from the Frank operational laws of TIT2IFNs; thus, it is omitted here. □

5.3.2 Ordered weighted averaging Operator

Definition 5.3.2. Suppose Ω be a family of TIT2IFNs $\mathcal{A}_j$ for $j = 1, 2, \ldots, n$ and TIT2IFOWA : $\Omega^n \longrightarrow \Omega$, if

$$\text{TIT2IFOWA}(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \omega_1 \cdot_F \mathcal{A}_{\delta(1)} \oplus_F \omega_2 \cdot_F \mathcal{A}_{\delta(2)} \oplus_F \ldots \oplus_F \omega_n \cdot_F \mathcal{A}_{\delta(n)}$$

where $\omega = (\omega_1, \omega_2 \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_j$, $(\delta(1), \delta(2), \ldots, \delta(n))$ is a permutation of $(1, 2, 3, \ldots, n)$ such that $\mathcal{A}_{\delta(j-1)} \geq \mathcal{A}_{\delta(j)}$ for $j = 2, 3, \ldots, n$ then, TIT2IFOWA is called triangular interval type-2 intuitionistic fuzzy ordered weighted averaging operator

Theorem 5.3.7. The aggregated value by using TIT2IFOWA operator for a collection of TIT2IFNs $\mathcal{A}_j$, $(j = 1, 2, \ldots n)$ is again TIT2IFN, and is given by

$$\mathrm{TIT2IFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$
$$= \left(\begin{array}{c} \left[1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-a_{\delta(j)}} - 1)^{\omega_j} \right), 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-b_{\delta(j)}} - 1)^{\omega_j} \right) \right], \\ 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-c_{\delta(j)}} - 1)^{\omega_j} \right), \left[1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-d_{\delta(j)}} - 1)^{\omega_j} \right), \right. \\ \left. 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-e_{\delta(j)}} - 1)^{\omega_j} \right) \right]; \left[\log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{A_{\delta(j)}} - 1)^{\omega_j} \right), \right. \\ \left. \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{B_{\delta(j)}} - 1)^{\omega_j} \right) \right], \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{C_{\delta(j)}} - 1)^{\omega_j} \right), \\ \left[\log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{D_{\delta(j)}} - 1)^{\omega_j} \right), \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{E_{\delta(j)}} - 1)^{\omega_j} \right) \right] \end{array} \right)$$

Proof of this result is similar to Theorem 5.3.1, so we omit here.

Example 5.3.1. Let $\mathcal{A}_1 = ([0.4, 0.5], 0.5, [0.6, 0.7]; [0.5, 0.4], 0.3, [0.2, 0.1])$, $\mathcal{A}_2 = ([0.2, 0.3], 0.4, [0.4, 0.5]; [0.7, 0.6], 0.5, [0.3, 0.2])$, $\mathcal{A}_3 = ([0.3, 0.5], 0.5, [0.6, 0.7]; [0.6, 0.4], 0.4, [0.3, 0.2])$ be three TIT2IFNs and $\omega = (0.3, 0.4, 0.3)^T$ be their corresponding weight vector. Now, based on the ranking formula, we have $\mathcal{A}_1 \geq \mathcal{A}_3 \geq \mathcal{A}_2$ thus, we have $\mathcal{A}_{\delta(1)} = \mathcal{A}_1$, $\mathcal{A}_{\delta(2)} = \mathcal{A}_3$ and $\mathcal{A}_{\delta(3)} = \mathcal{A}_2$. If we take $\lambda = 2$ for simplification, then we have, $1 - \log_\lambda \left(1 + \prod_{j=1}^{3} (\lambda^{1-a_{\delta(j)}} - 1)^{\omega_j} \right) = 0.3034$, $\log_\lambda \left(1 + \prod_{j=1}^{3} (\lambda^{A_{\delta(j)}} - 1)^{\omega_j} \right) = 0.5959$, $1 - \log_\lambda \left(1 + \prod_{j=1}^{3} (\lambda^{1-b_{\delta(j)}} - 1)^{\omega_j} \right) = 0.4456$, $1 - \log_\lambda \left(1 + \prod_{j=1}^{3} (\lambda^{1-c_{\delta(j)}} - 1)^{\omega_j} \right) = 0.4716$, $1 - \log_\lambda \left(1 + \prod_{j=1}^{3} (\lambda^{1-d_{\delta(j)}} - 1)^{\omega_j} \right) = 0.5470$, $1 - \log_\lambda \left(1 + \prod_{j=1}^{3} (\lambda^{1-e_{\delta(j)}} - 1)^{\omega_j} \right) = 0.6491$, $\log_\lambda \left(1 + \prod_{j=1}^{3} (\lambda^{B_{\delta(j)}} - 1)^{\omega_j} \right) = 0.4530$, $\log_\lambda \left(1 + \prod_{j=1}^{3} (\lambda^{C_{\delta(j)}} - 1)^{\omega_j} \right) = 0.3933$, $\log_\lambda \left(1 + \prod_{j=1}^{3} (\lambda^{D_{\delta(j)}} - 1)^{\omega_j} \right) = 0.2660$, $\log_\lambda \left(1 + \prod_{j=1}^{3} (\lambda^{E_{\delta(j)}} - 1)^{\omega_j} \right) = 0.1629$. Thus, $\mathrm{TIT2IFOWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = \big([0.3034, 0.4456], 0.4716, [0.5470, 0.6491]; [0.5959, 0.4530], 0.3933, [0.2660, 0.1629]\big)$.

As similar to those of TIT2IFWA operator, the TIT2IFOWA operator also follows the boundedness, idempotency, and monotonicity properties. Besides the aforementioned properties, the TIT2IFOWA operator has the following desirable results.

Theorem 5.3.8. For a collection of TIT2IFNs $\mathcal{A}_j (j = 1, 2, \ldots, n)$, we have the following:

(i) If $\omega = (1, 0, \ldots, 0)^T$ then TIT2IFOWA$(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \max\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$

(ii) If $\omega = (0, 0, \ldots, 1)^T$ then TIT2IFOWA$(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \min\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$

(iii) If $\omega_j = 1$ and $\omega_i = 0 (i \neq j)$, then TIT2IFOWA$(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}_{\delta(j)}$ where $\mathcal{A}_{\delta(j)}$ is the j^{th} largest of $\mathcal{A}_j$, $(j = 1, 2, \ldots, n)$.

5.3.3 Hybrid Averaging Operator

Since TIT2IFWA operator weights only the TIT2IFNs while TIT2IFOWA operator weights the ordered positions of TIT2IFNs. However, in order to combine these two aspects in one, we now introduce a hybrid AO, which weight both the given TIT2IFNs and their ordered positions.

Definition 5.3.3. A triangular interval type-2 intuitionistic fuzzy hybrid averaging (TIT2IFHA) operator is a mapping TIT2IFHA : $\Omega^n \longrightarrow \Omega$, such that

$$\text{TIT2IFHA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \phi_1 \cdot_F \dot{\mathcal{A}}_{\delta(1)} \oplus_F \phi_2 \cdot_F \dot{\mathcal{A}}_{\delta(2)} \oplus_F \ldots \oplus_F \phi_n \cdot_F \dot{\mathcal{A}}_{\delta(n)}$$

where Ω is the set of all TIT2IFNs, and $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ is the weighted vector associated with TIT2IFHA, such that $\phi_j > 0$ and $\sum_{j=1}^{n} \phi_j = 1$; $\dot{\mathcal{A}}_j = (n\omega_j) \cdot_F \mathcal{A}_j, j = 1, 2, \ldots, n$, $\dot{\mathcal{A}}_{\delta(j)}$ is the j^{th} largest of the weighted TIT2IFNs $\dot{\mathcal{A}}_j$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\mathcal{A}_j$ with $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 5.3.9. For a collection of TIT2IFNs, $\mathcal{A}_j (j = 1, 2, \ldots, n)$, the aggregated value based on the TIT2IFHA operator is also TIT2IFN and can be expressed as

$$\mathrm{TIT2IFHA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$
$$= \left(\begin{array}{l} \left[1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-a_{\delta(j)}} - 1)^{\phi_j} \right), 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-b_{\delta(j)}} - 1)^{\phi_j} \right) \right], \\ 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-c_{\delta(j)}} - 1)^{\phi_j} \right), \left[1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-d_{\delta(j)}} - 1)^{\phi_j} \right), \right. \\ \left. 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-e_{\delta(j)}} - 1)^{\phi_j} \right) \right]; \left[\log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{A_{\delta(j)}} - 1)^{\phi_j} \right), \right. \\ \left. \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{B_{\delta(j)}} - 1)^{\phi_j} \right) \right], \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{C_{\delta(j)}} - 1)^{\phi_j} \right), \\ \left[\log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{D_{\delta(j)}} - 1)^{\phi_j} \right), \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{E_{\delta(j)}} - 1)^{\phi_j} \right) \right] \end{array} \right)$$

The proof is similar to Theorem 5.3.1, so it is omitted here.

Similar to those of TIT2IFWA and TIT2IFOWA operators, the TIT2IFHA operator has also follows the same properties.

Theorem 5.3.10. The TIT2IFWA operator as defined in Theorem 5.3.1 is a special case of the TIT2IFHA operator.

Proof. Let $\phi = (1/n, 1/n, \ldots, 1/n)^T$ then, $\mathrm{TIT2IFHA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \frac{1}{n}(\dot{\mathcal{A}}_{\delta(1)} \oplus_F \dot{\mathcal{A}}_{\delta(2)} \oplus_F \ldots \oplus_F \dot{\mathcal{A}}_{\delta(n)}) = \omega_1 \cdot_F \mathcal{A}_1 \oplus_F \ldots \oplus_F \omega_n \cdot_F \mathcal{A}_n = \mathrm{TIT2IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$. □

Theorem 5.3.11. The TIT2IFOWA operator as defined in Theorem 5.3.7 is a special case of the TIT2IFHA operator.

Proof. Let $\omega = (1/n, 1/n, \ldots, 1/n)^T$ then, $\dot{\mathcal{A}}_j = (n\omega_j) \cdot_F \mathcal{A}_j = \mathcal{A}_j$, for all j. Thus, $\mathrm{TIT2IFHA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \phi_1 \cdot_F \dot{\mathcal{A}}_{\delta(1)} \oplus_F \phi_2 \cdot_F \dot{\mathcal{A}}_{\delta(2)} \oplus_F \ldots \oplus_F \phi_n \cdot_F \dot{\mathcal{A}}_{\delta(n)} = \phi_1 \cdot_F \mathcal{A}_{\delta(1)} \oplus_F \phi_2 \cdot_F \mathcal{A}_{\delta(2)} \oplus_F \ldots \oplus_F \phi_n \cdot_F \mathcal{A}_{\delta(n)} = \mathrm{TIT2IFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$. □

5.4 Decision-making approach based on proposed operators

In this section, a decision-making method by using the above-defined AOs has been presented followed by an illustrative example for demonstrating the approach.

5.4.1 Decision-making approach

Let $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m\}$ be the set of "m"alternatives and $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$ be a set of "n" criteria with the associated weight $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ satisfying $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. The characteristics of each alternative $\mathcal{A}_i (i = 1, 2, \ldots, m)$ with respect to each criteria $\mathcal{G}_j (j = 1, 2, \ldots, n)$ is assessed in terms of TIT2IFNs $\mathcal{A}_{ij} = \big([a_{ij}, b_{ij}], c_{ij}, [d_{ij}, e_{ij}];\ [A_{ij}, B_{ij}], C_{ij}, [D_{ij}, E_{ij}]\big)$ where $0 \leq a_{ij} \leq b_{ij} \leq c_{ij} \leq d_{ij} \leq e_{ij} \leq 1$ and $1 \geq A_{ij} \geq B_{ij} \geq C_{ij} \geq D_{ij} \geq E_{ij} \geq 0$ such that $e_{ij} + E_{ij} \leq 1$ and $a_{ij} + A_{ij} \leq 1$. Then, in the following, we develop an approach based on the proposed operator to solve the decision-making problems with TIT2IF information, which involves the following steps.

Step 1: Collect the information as decision matrix $D = (\mathcal{A}_{ij})_{m \times n}$.

Step 2: Normalize the data $\mathcal{A}_{ij}$ into r_{ij}, if required, by converting the cost type criteria $(\mathcal{F}_1)$ into the benefit type $(\mathcal{F}_2)$ using Eq. (5.14) as

$$r_{ij} = \begin{cases} \mathcal{A}_{ij} & ; \quad j \in \mathcal{F}_2 \\ \mathcal{A}_{ij}^c & ; \quad j \in \mathcal{F}_1 \end{cases} \tag{5.14}$$

where $\mathcal{A}_{ij}^c$ is the complement of $\mathcal{A}_{ij}$ and obtain the normalized TIT2IFN decision matrix $\mathcal{R} = (r_{ij})_{m \times n}$.

Step 3: Aggregate the TIT2IFNs $r_{ij} (j = 1, 2, \ldots, n)$ for each alternative $\mathcal{A}_i (i = 1, 2, \ldots, m)$ into the overall preference value r_i either by using the proposed TIT2IFWA, TIT2IFOWA or TIT2IFHA operators.

Step 4: Determine the ranking value of each aggregated value $r_i, (i = 1, 2, \ldots, m)$ by using Eq. (5.2) and select the best one(s).

Step 5: Perform the sensitivity analysis on the parameter λ according to decision makers' preferences.

5.4.2 Numerical Example

The above decision-making procedure has been illustrated with the case study that a person wants to invest a money in the market. For this, they have chosen the four multi-national companies namely $\mathcal{A}_1$: Infosys, $\mathcal{A}_2$: Wipro, $\mathcal{A}_3$: Dell and $\mathcal{A}_4$: Apple. In order to assess these alternatives, the investors have brought a panel with three experts e_1, e_2 and e_3 whose weight vector is 0.35, 0.35 and 0.30. These three experts have evaluated the each alternative $\mathcal{A}_i$, $i = 1, 2, 3, 4$ with respect to the four attributes namely $\mathcal{G}_1$ ("the growth analysis"), $\mathcal{G}_2$ ("the development of society"), $\mathcal{G}_3$ (" the technical support") and $\mathcal{G}_4$ (" the quality") whose weight vector is $\omega = (0.3, 0.2, 0.1, 0.4)^T$. Then, the following steps have been performed based on the proposed decision-making approach to find the most desirable alternative(s).

Step 1: The three experts $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ have evaluated the given alternatives $\mathcal{A}_i$, $i = 1, 2, 3, 4$ with respect to attributes $\mathcal{G}_j$, $j = 1, 2, 3, 4$ in the form of TIT2IFNs which are represented in Tables 5.1-5.3 respectively.

Step 2: Since all the attributes are of same types, so there is no need of the normalization.

Step 3: Let $\phi = (0.35, 0.25, 0.25, 0.15)^T$ be the weight vector associated with TIT2IFHA operator. Without loss of generality, here we utilize TIT2IFHA and TIT2IFWA operators to aggregate the given data by taking $\lambda = 2$. For this, firstly we compute $\dot{\mathcal{A}}_{ij}^k = 4\omega_j \mathcal{A}_{ij}^k$; $k = 1, 2, 3$ and hence the corresponding values for each decision makers $\mathcal{E}_k$, $k = 1, 2, 3$ are summarized in Tables 5.4-5.6 respectively. By taking weight vector 0.35, 0.35, 0.30 of the three experts and utilizing TIT2IFWA operator to aggregate these expert preferences into the collective TIT2IF matrix $R = (r_{ij})_{4\times 4}$. The resultant matrix is given in Table 5.7. Later, by taking TIT2IFWA operator, corresponding to the weight vector $\phi = (0.35, 0.25, 0.25, 0.15)^T$ of the criteria $\mathcal{G}_j$, to aggregate all the performance values $r_{ij}, (j = 1, 2, 3, 4)$ of the i^{th} alternative and get the overall performance value r_i corresponding to alternative $\mathcal{A}_i (i = 1, 2, 3, 4)$ are r_1 = ([0.4726, 0.5792], 0.6807, [0.7398, 0.8464]; [0.4221, 0.2996], 0.2559, [0.1802, 0.0910]), r_2 = ([0.4857, 0.6062], 0.7059, [0.7393, 0.8466]; [0.4326, 0.2918], 0.2596, [0.2066, 0.1074]), r_3 = ([0.3910, 0.4960], 0.5939, [0.6794, 0.7817]; [0.4984, 0.3807], 0.3080, [0.2239, 0.1287]), and r_4 = ([0.4357, 0.5387], 0.6259, [0.7059, 0.8016]; [0.4649, 0.3497], 0.2766, [0.2193, 0.1225]).

Step 4: By Eq. (5.2), we compute the ranking values of $r_i (i = 1, 2, 3, 4)$ as $\mathfrak{R}(r_1) = 0.5864$, $\mathfrak{R}(r_2) = 0.5977$, $\mathfrak{R}(r_3) = 0.3595$ and $\mathfrak{R}(r_4) = 0.4509$.

Step 5: Since $\mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_4 > \mathfrak{R}_3$ and thus, the ordering of the given alternatives is $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$. Therefore, $\mathcal{A}_2$ is the most desirable one while $\mathcal{A}_3$ is the least one.

5.4.3 Effect of Frank norm parameter λ on the ranking

To analyze the effect of the parameter λ on the most desirable alternatives of the given attributes, we use the different values of λ in the proposed approach for describing the changing trend of ranking order as well as the measuring value corresponding to each alternative. The complete variation of the ranking value of each alternative with respect to parameter λ is summarized in Fig. 5.2, while the ranking values for some parametric values of $\lambda \to 1, 2, 2.5, 5, 7.5, 10$ are summarized in Table 5.8. From this table and figure, it has been seen that with the increase in the value of λ, the ranking value corresponding to each alternative also increases but the ranking order of these alternatives remain same i.e., $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$ and hence the best alternative is Wipro ($\mathcal{A}_2$) for investing a money in the market.

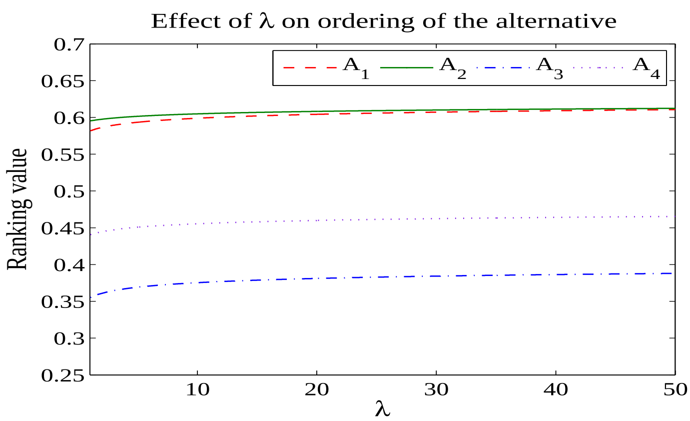

Figure 5.2: Effect of Frank norm parameter λ on the ranking order of the alternatives

5.4.4 Comparative studies

In order to compare the performance of the proposed methods with some existing methods, a comparative study has been taken in which the existing operators based on algebraic averaging operator [159], geometric operator [161], Einstein based t-norm [146], Hamacher based t-norm operator [92], etc., have been considered. The results corresponding to it has been shown in Table 5.9. From this table, it has been analyzed that the best company for investing the money in the market is Wipro ($\mathcal{A}_2$) than others and this result has been overlapped with the existing studies results which validate the stability of the approach. Furthermore, the proposed operator involves a certain parameter λ, which makes them more flexible in the process of information fusion and is more adequate to model practical decision-making problems. Thus, the proposed technique can be suitably utilized to solve the problem of decision-making problem than the other existing measures. According to the above comparison analysis, the proposed method for addressing the decision-making problems has the following merits with respect to the existing ones.

(i) Compared with IFWA operator (or IFWG operator) proposed by Xu [159] (or Xu and Yager [161]), the IFWA operator (or IFWG operator) is only a special case of our proposed operators when parameter $\lambda \to 1$. So, our methods are more general. Furthermore, the proposed operators based on Frank t-norms, are more robust and can capture the relationship between the arguments. Moreover, when $\lambda \to \infty$, the proposed operators are reduced to the operators based on Lukasiewicz product and Lukasiewicz sum. Therefore, the Frank AOs can contain almost all of the arithmetic AOs and geometric AOs for TIT2IFNs according to different values of parameter λ.

(ii) The proposed methods include a parameter, which can adjust the aggregate value based on the real decision needs, and capture many existing hesitant fuzzy AOs. Therefore, the benefit is that the proposed operators come with their higher generality and flexibility. In other words, the decision maker can use the appropriate parameter value based on their risk preference and actual needs.

5.5 Conclusion

The objective of this chapter is to present triangular interval type 2 intuitionistic fuzzy AOs by considering the Frank operational laws. For this, some operational laws based on Frank t-norm and conorm has been presented under the triangular type 2 intuitionistic fuzzy environment and then based on its some series of weighted averaging operators such as TIT2IFWA, TIT2IFOWA and TIT2IFHA have been proposed. Various desirable properties of its have also been stated and discussed in details. An illustrative example related to the decision-making process has been taken for demonstrating the approach. A comparative study with some existing operators shows that the proposed operators and their corresponding techniques provide an alternative way to solve the MCDM problem in a more effective manner. A sensitivity analysis has also been conducted for showing the impact of the decision parameters on to the ranking of the alternatives. In addition, the proposed results corresponding to different values of λ will offer the various choices for the decision maker for assessing the decisions.

Table 5.1: TIT2 Intuitionistic fuzzy decision matrix given by Expert $\mathcal{E}_1$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$([0.40, 0.50], 0.60, [0.70, 0.80]; [0.50, 0.40], 0.30, [0.20, 0.10])$	$([0.50, 0.60], 0.70, [0.80, 0.90]; [0.40, 0.30], 0.30, [0.20, 0.10])$	$([0.60, 0.70], 0.80, [0.90, 0.95]; [0.30, 0.25], 0.20, [0.10, 0.05])$	$([0.60, 0.70], 0.80, [0.80, 0.90]; [0.30, 0.25], 0.20, [0.20, 0.10])$
$\mathcal{A}_2$	$([0.50, 0.60], 0.70, [0.80, 0.90]; [0.40, 0.30], 0.30, [0.20, 0.10])$	$([0.50, 0.60], 0.70, [0.80, 0.90]; [0.40, 0.30], 0.30, [0.20, 0.10])$	$([0.40, 0.50], 0.60, [0.70, 0.80]; [0.50, 0.40], 0.40, [0.30, 0.20])$	$([0.60, 0.70], 0.80, [0.80, 0.90]; [0.40, 0.20], 0.20, [0.20, 0.10])$
$\mathcal{A}_3$	$([0.30, 0.40], 0.50, [0.60, 0.70]; [0.60, 0.50], 0.40, [0.30, 0.20])$	$([0.40, 0.50], 0.60, [0.70, 0.80]; [0.45, 0.40], 0.30, [0.20, 0.10])$	$([0.20, 0.30], 0.40, [0.50, 0.60]; [0.60, 0.50], 0.40, [0.30, 0.20])$	$([0.30, 0.40], 0.50, [0.60, 0.70]; [0.70, 0.50], 0.40, [0.30, 0.20])$
$\mathcal{A}_4$	$([0.20, 0.30], 0.40, [0.50, 0.60]; [0.70, 0.60], 0.50, [0.40, 0.30])$	$([0.60, 0.70], 0.80, [0.85, 0.90]; [0.40, 0.20], 0.20, [0.15, 0.05])$	$([0.30, 0.40], 0.50, [0.60, 0.70]; [0.60, 0.50], 0.40, [0.30, 0.20])$	$([0.40, 0.50], 0.60, [0.70, 0.80]; [0.50, 0.40], 0.30, [0.20, 0.10])$

Table 5.2: TIT2 Intuitionistic fuzzy decision matrix given by Expert $\mathcal{E}_2$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$([0.20, 0.30], 0.35, [0.40, 0.50]; [0.70, 0.60], 0.50, [0.40, 0.30])$	$([0.30, 0.35], 0.40, [0.40, 0.50]; [0.60, 0.50], 0.50, [0.40, 0.30])$	$([0.30, 0.50], 0.60, [0.60, 0.70]; [0.60, 0.40], 0.40, [0.30, 0.20])$	$([0.30, 0.40], 0.50, [0.50, 0.60]; [0.60, 0.30], 0.30, [0.20, 0.10])$
$\mathcal{A}_2$	$([0.25, 0.30], 0.35, [0.35, 0.40]; [0.65, 0.55], 0.45, [0.35, 0.25])$	$([0.30, 0.40], 0.40, [0.50, 0.60]; [0.60, 0.50], 0.40, [0.40, 0.30])$	$([0.30, 0.50], 0.55, [0.60, 0.70]; [0.70, 0.50], 0.40, [0.40, 0.20])$	$([0.20, 0.40], 0.45, [0.50, 0.60]; [0.70, 0.50], 0.45, [0.40, 0.30])$
$\mathcal{A}_3$	$([0.45, 0.50], 0.50, [0.60, 0.70]; [0.40, 0.30], 0.30, [0.20, 0.10])$	$([0.50, 0.60], 0.60, [0.70, 0.80]; [0.50, 0.40], 0.30, [0.20, 0.10])$	$([0.40, 0.60], 0.70, [0.80, 0.90]; [0.50, 0.40], 0.30, [0.20, 0.10])$	$([0.30, 0.40], 0.55, [0.60, 0.70]; [0.60, 0.50], 0.40, [0.30, 0.20])$
$\mathcal{A}_4$	$([0.50, 0.55], 0.55, [0.60, 0.75]; [0.40, 0.30], 0.20, [0.20, 0.10])$	$([0.60, 0.70], 0.80, [0.85, 0.90]; [0.40, 0.30], 0.20, [0.10, 0.05])$	$([0.40, 0.50], 0.60, [0.70, 0.80]; [0.50, 0.40], 0.30, [0.20, 0.10])$	$([0.50, 0.60], 0.65, [0.70, 0.80]; [0.40, 0.30], 0.30, [0.30, 0.20])$

Table 5.3: TIT2 Intuitionistic fuzzy decision matrix given by Expert $\mathcal{E}_3$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$([0.40, 0.50], 0.60, [0.70, 0.80]; [0.50, 0.40], 0.30, [0.20, 0.10])$	$([0.50, 0.60], 0.70, [0.80, 0.90]; [0.450, 0.40], 0.25, [0.15, 0.10])$	$([0.40, 0.50], 0.60, [0.70, 0.80]; [0.45, 0.35], 0.30, [0.30, 0.20])$	$([0.50, 0.60], 0.70, [0.80, 0.90]; [0.40, 0.30], 0.30, [0.20, 0.10])$
$\mathcal{A}_2$	$([0.30, 0.40], 0.50, [0.60, 0.70]; [0.60, 0.50], 0.40, [0.30, 0.20])$	$([0.60, 0.70], 0.80, [0.80, 0.90]; [0.35, 0.30], 0.20, [0.15, 0.10])$	$([0.40, 0.50], 0.60, [0.70, 0.80]; [0.45, 0.40], 0.30, [0.20, 0.10])$	$([0.60, 0.70], 0.80, [0.80, 0.90]; [0.30, 0.20], 0.20, [0.15, 0.05])$
$\mathcal{A}_3$	$([0.10, 0.20], 0.30, [0.40, 0.50]; [0.60, 0.50], 0.40, [0.30, 0.20])$	$([0.60, 0.70], 0.80, [0.80, 0.90]; [0.30, 0.20], 0.20, [0.20, 0.10])$	$([0.30, 0.40], 0.50, [0.60, 0.70]; [0.55, 0.45], 0.40, [0.30, 0.20])$	$([0.40, 0.50], 0.60, [0.70, 0.80]; [0.50, 0.40], 0.35, [0.30, 0.20])$
$\mathcal{A}_4$	$([0.20, 0.30], 0.40, [0.50, 0.60]; [0.70, 0.60], 0.50, [0.40, 0.30])$	$([0.20, 0.30], 0.40, [0.50, 0.60]; [0.60, 0.50], 0.40, [0.30, 0.20])$	$([0.10, 0.20], 0.30, [0.40, 0.50]; [0.50, 0.40], 0.30, [0.20, 0.10])$	$([0.30, 0.40], 0.50, [0.60, 0.70]; [0.60, 0.50], 0.40, [0.40, 0.30])$

Table 5.4: Decision maker preference corresponding to Expert $\mathcal{E}_1$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	([0.7845, 0.8678], 0.9332, [0.9332, 0.9788]; [0.1332, 0.0972], 0.0668, [0.0668, 0.0212])	([0.4622, 0.5700], 0.6731, [0.7705, 0.8604]; [0.4300, 0.3269], 0.2295, [0.1396, 0.0599])	([0.4208, 0.5131], 0.6106, [0.7158, 0.8345]; [0.4869, 0.3894], 0.3894, [0.2842, 0.1655])	([0.2920, 0.3616], 0.4475, [0.5685, 0.6638]; [0.6384, 0.5982], 0.5525, [0.4315, 0.3362])
$\mathcal{A}_2$	([0.7845, 0.8678], 0.9332, [0.9332, 0.9788]; [0.2155, 0.0668], 0.0668, [0.0668, 0.0212])	([0.5700, 0.6731], 0.7705, [0.8604, 0.9401]; [0.3269, 0.2295], 0.2295, [0.1396, 0.0599])	([0.4208, 0.5131], 0.6106, [0.7158, 0.8345]; [0.4869, 0.3894], 0.3894, [0.2842, 0.1655])	([0.1785, 0.2320], 0.2920, [0.3616, 0.4475]; [0.7680, 0.7080], 0.7080, [0.6384, 0.5525])
$\mathcal{A}_3$	([0.4434, 0.5707], 0.6849, [0.7845, 0.8678]; [0.5566, 0.3151], 0.2155, [0.1322, 0.0668])	([0.3507, 0.4622], 0.5700, [0.6731, 0.7705]; [0.5378, 0.4300], 0.3269, [0.2295, 0.1396])	([0.3321, 0.4208], 0.5131, [0.6106, 0.7158]; [0.5336, 0.4869], 0.3894, [0.2842, 0.1655])	([0.0839, 0.1295], 0.1785, [0.2320, 0.2920]; [0.8215, 0.7680], 0.7080, [0.6384, 0.5525])
$\mathcal{A}_4$	([0.5707, 0.6849], 0.7845, [0.8678, 0.9332]; [0.3151, 0.2155], 0.1322, [0.0668, 0.0212])	([0.2362, 0.3507], 0.4622, [0.5700, 0.6731]; [0.6493, 0.5378], 0.4300, [0.3269, 0.2295])	([0.5131, 0.6106], 0.7158, [0.7729, 0.8345]; [0.4869, 0.2842], 0.2842, [0.2271, 0.0961])	([0.1295, 0.1785], 0.2320, [0.2920, 0.3616]; [0.8215, 0.7680], 0.7080, [0.6384, 0.5525])

Table 5.5: Decision maker preference corresponding to Expert $\mathcal{E}_2$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	([0.4434, 0.5707], 0.6849, [0.6849, 0.7845]; [0.4293, 0.1322], 0.1322, [0.0668, 0.0212])	([0.2362, 0.3507], 0.4069, [0.4622, 0.5700]; [0.6493, 0.5378], 0.4300, [0.3269, 0.2295])	([0.2463, 0.2889], 0.3321, [0.3321, 0.4208]; [0.6679, 0.5792], 0.5792, [0.4869, 0.3894])	([0.1295, 0.2320], 0.2920, [0.2920, 0.3616]; [0.8215, 0.7080], 0.7080, [0.6384, 0.5525])
$\mathcal{A}_2$	([0.3048, 0.5707], 0.6295, [0.6849, 0.7845]; [0.5566, 0.3151], 0.2634, [0.2155, 0.1322])	([0.2938, 0.3507], 0.5166, [0.4069, 0.4622]; [0.5931, 0.4834], 0.3778, [0.2774, 0.1835])	([0.2463, 0.3321], 0.3321, [0.4208, 0.5131]; [0.6679, 0.5792], 0.4869, [0.4869, 0.3894])	([0.1295, 0.2320], 0.2610, [0.2920, 0.3616]; [0.8705, 0.7680], 0.7080, [0.7080, 0.5525])
$\mathcal{A}_3$	([0.4434, 0.5707], 0.7366, [0.7845, 0.8678]; [0.4293, 0.3151], 0.2155, [0.1322, 0.0668])	([0.5166, 0.5700], 0.5700, [0.6731, 0.7705]; [0.3269, 0.2295], 0.2295, [0.1396, 0.0599])	([0.4208, 0.5131], 0.5131, [0.6106, 0.7158]; [0.5792, 0.4869], 0.3894, [0.2842, 0.1655])	([0.1785, 0.2920], 0.3616, [0.4475, 0.5685]; [0.7680, 0.7080], 0.6384, [0.5525, 0.4315])
$\mathcal{A}_4$	([0.6849, 0.7845], 0.8283, [0.8678, 0.9322]; [0.2155, 0.1322], 0.1322, [0.1322, 0.0668])	([0.5700, 0.6222], 0.6222, [0.6731, 0.8165]; [0.3269, 0.2295], 0.1396, [0.1396, 0.0599])	([0.5131, 0.6106], 0.7158, [0.7729, 0.8345]; [0.4869, 0.3894], 0.2842, [0.1655, 0.0961])	([0.1785, 0.2320], 0.2920, [0.3616, 0.4475]; [0.7680, 0.7080], 0.6384, [0.5525, 0.4315])

Table 5.6: Decision maker preference corresponding to Expert $\mathcal{E}_3$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	([0.6849, 0.7845], 0.8678, [0.9332, 0.9788]; [0.2155, 0.1322], 0.1322, [0.0668, 0.0212])	([0.4622, 0.5700], 0.6731, [0.7705, 0.8604]; [0.4300, 0.3269], 0.2295, [0.1396, 0.0599])	([0.4208, 0.5131], 0.6106, [0.7158, 0.8345]; [0.5336, 0.4869], 0.3380, [0.2271, 0.1655])	([0.1785, 0.2320], 0.2920, [0.3616, 0.4475]; [0.7390, 0.6747], 0.6384, [0.6384, 0.5525])
$\mathcal{A}_2$	([0.7845, 0.8678], 0.9332, [0.9332, 0.9788]; [0.1322, 0.0668], 0.0668, [0.0413, 0.0068])	([0.3507, 0.4622], 0.5700, [0.6731, 0.7705]; [0.5378, 0.4300], 0.3269, [0.2295, 0.1396])	([0.5131, 0.6106], 0.7158, [0.7158, 0.8345]; [0.4389, 0.3894], 0.2842, [0.2271, 0.1622])	([0.1785, 0.2320], 0.2920, [0.3616, 0.4475]; [0.7390, 0.7080], 0.6384, [0.5525, 0.4315])
$\mathcal{A}_3$	([0.5707, 0.6849], 0.7845, [0.8678, 0.9332]; [0.3151, 0.2155], 0.1717, [0.1322, 0.0668])	([0.1191, 0.2362], 0.3507, [0.4622, 0.5700]; [0.5378, 0.4300], 0.3269, [0.2295, 0.1396])	([0.5131, 0.6106], 0.7158, [0.7158, 0.8345]; [0.3984, 0.2842], 0.2842, [0.2842, 0.1655])	([0.1295, 0.1785], 0.2320, [0.2920, 0.3616]; [0.7954, 0.7390], 0.7080, [0.6384, 0.5525])
$\mathcal{A}_4$	([0.4434, 0.5707], 0.6849, [0.7845, 0.8678]; [0.4293, 0.3151], 0.2155, [0.2155, 0.1322])	([0.2362, 0.3507], 0.4622, [0.5700, 0.6731]; [0.6493, 0.5378], 0.4300, [0.3269, 0.2295])	([0.1626, 0.2463], 0.3321, [0.4208, 0.5131]; [0.6679, 0.5792], 0.4869, [0.3894, 0.2842])	([0.0409, 0.0839], 0.1295, [0.1785, 0.2320]; [0.7680, 0.7080], 0.6384, [0.5525, 0.4315])

Table 5.7: Collective information by the decision maker

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$([0.6601, 0.7664], 0.8573, [0.8835, 0.9516];$ $[0.2336, 0.1188], 0.1043, [0.0668, 0.0212])$	$([0.3902, 0.5016], 0.5950, [0.6878, 0.7906];$ $[0.4984, 0.3906], 0.2873, [0.1891, 0.0966])$	$([0.3638, 0.4424], 0.5271, [0.6122, 0.7387];$ $[0.5599, 0.4795], 0.4305, [0.3227, 0.2248])$	$([0.2034, 0.2795], 0.3499, [0.4204, 0.5087];$ $[0.7295, 0.6582], 0.6300, [0.5583, 0.4778])$
$\mathcal{A}_2$	$([0.6688, 0.7978], 0.8762, [0.8835, 0.9516];$ $[0.2639, 0.1165], 0.1090, [0.0879, 0.0289])$	$([0.4186, 0.5142], 0.6381, [0.6954, 0.8011];$ $[0.4700, 0.3619], 0.3045, [0.2067, 0.1147])$	$([0.3952, 0.4894], 0.5682, [0.6326, 0.7555];$ $[0.5287, 0.4487], 0.3842, [0.3227, 0.2248])$	$([0.1615, 0.2320], 0.2813, [0.3379, 0.4186];$ $[0.7937, 0.7286], 0.6865, [0.6345, 0.5135])$
$\mathcal{A}_3$	$([0.4845, 0.6082], 0.7357, [0.8136, 0.8921];$ $[0.4301, 0.2815], 0.2014, [0.1322, 0.0668])$	$([0.3544, 0.4447], 0.5119, [0.6191, 0.7217];$ $[0.4536, 0.3469], 0.2892, [0.1932, 0.1041])$	$([0.4212, 0.5152], 0.5841, [0.6453, 0.7578];$ $[0.5007, 0.4159], 0.3547, [0.2842, 0.1655])$	$([0.1313, 0.2034], 0.2619, [0.3307, 0.4204];$ $[0.7947, 0.7380], 0.6830, [0.6072, 0.5073])$
$\mathcal{A}_4$	$([0.5820, 0.6961], 0.7764, [0.8467, 0.9179];$ $[0.3039, 0.2045], 0.1533, [0.1211, 0.0551])$	$([0.3708, 0.4598], 0.5237, [0.6089, 0.7321];$ $[0.5149, 0.4033], 0.2938, [0.2442, 0.1447])$	$([0.4235, 0.5215], 0.6288, [0.6959, 0.7686];$ $[0.5363, 0.3950], 0.3352, [0.2402, 0.1369])$	$([0.1213, 0.1704], 0.2243, [0.2852, 0.3573];$ $[0.7864, 0.7286], 0.6621, [0.5814, 0.4710])$

Table 5.8: Ranking values for the different values of λ for each alternative

Alternative	$\lambda \to 1$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 5$	$\lambda = 7.5$	$\lambda = 10$
$\mathcal{A}_1$	0.6662	0.5864	0.5434	0.4740	0.4605	0.4557
$\mathcal{A}_2$	0.6779	0.5977	0.5541	0.4649	0.4525	0.4436
$\mathcal{A}_3$	0.4426	0.3595	0.3162	0.2784	0.2911	0.3005
$\mathcal{A}_4$	0.5337	0.4509	0.4069	0.3496	0.3559	0.3590

Table 5.9: Comparative study with existing methods

Method	Parameter	Order of alternatives
Xu and Yager [161]	None	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Xu [159]	None	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Wang and Liu [146]	None	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Liu [92]	$\gamma = 2$	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Proposed operator	$\lambda = 2$	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$

Chapter 6

A novel symmetric triangular interval type-2 intuitionistic fuzzy sets and their aggregation operators[1]

In this chapter, we developed the concept of the Symmetric triangular interval T2IFSs (STIT2IFSs) by taking the features of T2IFSs and the symmetric triangular number and studied their desired properties. In a practical decision-making process, there always occurs an inter-relationship among the multi-input arguments. To address it, Hamy means (HM) operator is a standout among the most critical operators that catches the inter-relationship together with the multi-input arguments. Motivated by these primary characteristics, it is interested to extend HM operator to the STIT2IFS and hence defined some new triangular interval type-2 (TIT2) intuitionistic fuzzy aggregation operators, named as symmetric TIT2 intuitionistic fuzzy HM operator which can consider the multi interaction between the input argument under a provision of type-2 intuitionistic uncertain situation. Later, we develop a method to solve the decision-making problem and illustrate with a numerical number to exemplify the practicability of the proposed technique.

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6.1 Introduction

In these existing works, authors have investigated the problem by taking a quantitative environment to access the alternatives. However, not all alternatives are accessed in terms of quantitative. For this, there exists the concept of qualitative assessment in terms of linguistic variables/terms (LVs/LTs) [68, 158]. By taking the advantages of LTs, Zhang [171] presented the linguistic intuitionistic fuzzy (LIF) AOs to aggregate the LIF numbers. Chen et al. [26] presented an approach to solving the MCDM problem under LIFS environment. Garg and Kumar [54] presented AOs for LIF numbers (LIFNs) by using set pair analysis theory. Garg and Kumar [53] presented new possibility degree measure for LIFNs and an AO to aggregate the different LIFNs to solve MCDM problems. In many practical problems, it is not easy for any decision maker (DM) to discover an exact membership function of an FS corresponding to its element. To overthrow this limitation, type-2 fuzzy set (T2FS), an extension of T1FS, is applied to the model and is characterized by two functions: PMF and SMF. Unfortunately, T2FSs are highly complex, it is troublesome for the DMs to implement it in the real situation; hence, their use is not yet widespread. To reduce the computational complexity, Interval type-2 fuzzy (IT2F) sets (IT2FSs) [99] is the most widely used in T2FSs. In past decades, many methods have been developed to extend the theory of MCDM under IT2FS environment. Chen et al. [25] built up an expanded QUALIFLEX strategy for taking care of DM issues in view of IT2FSs and gave a contextual analysis of medicinal basic leadership. Chen [22] built up an ELECTRE-base outranking strategy for decision-making problems using IT2FSs. Wu and Mendel [151] proposed a linguistic weighted average AOs to deal with analytical hierarchical process (AHP) process under IT2F environment. Qin and Liu [120] investigated a family of type-2 fuzzy AOs in light of Frank triangular norm and built up another way to deal with MCDM problems under the IT2FSs setting. Gong et al. [62] extended the generalized Bonferroni mean (GBM) operator to the trapezoidal IT2F environment. Apart from these, some other studies under T2FS environment are conducted which are summarized in [28, 62, 75, 89, 95, 100–102, 116, 118, 120, 147, 175].

In all these above AOs, researchers have described the information by considering the independent of argument assumptions during the aggregation. However, the interaction between the multi-input parameters have commonly occurred and thus, it is necessary to add their features into the process. In that direction, Bonferroni mean (BM) and generalized BM (GBM)-based operators are proposed by the researchers [47, 82]. But from them, it has been observed that they have considered only two or three multi-parameter at a single time. However, they are unable to analyze the effect of the multi-input argument into one analysis. Furthermore, in BM and GBM, there is a need for two and three parameters from the irrational set during the process which increases the computational complexity. An alternative to BM operators, Hamy mean (HM) [66] or Maclaurin symmetric mean (MSM) or Muirhead mean (MM) operator has advantages of capturing the inter-relationship among the multiple input arguments. Qin [118] make a correlation between the HM and the MSM and conclude that the MSM is an instance of HM [119, 121]. Garg and Nancy [59] develop MCDM method by prioritized MM aggregation operators. Additionally, the HM operator involves the parameter, which can provide more flexibility and robustness during the aggregation operator. The existing - arithmetic and geometric mean- operators can be easily deduced from the HM by setting a particular value to its parameter. Be that as it may, the HM just accomplished a couple of research results on the hypothesis and application of inequality [64, 78]. Therefore, it is a means to study the AOs using the HM operator.

Thus, keeping in mind the advantages of T2IFS and the multiple input interaction between the argument of HM operator, this chapter has presented the concept of the symmetric TIT2IFS and their desired properties. These considerations have led us to consider the main objectives of this chapter:

1) to propose the concept of the symmetric TIT2IFS (STIT2IFSs);

2) to propose some new AOs for STIT2IFSs under the linguistic intuitionistic features;

3) to develop an algorithm to solve the decision-making problems based on proposed operators;

4) to present some example to validate and compare the results.

6.2 Symmetric Triangular Interval T2IFS

In this section, we developed a concept of a symmetric triangular IT2IFS and characterize their fundamental operational laws.

Definition 6.2.1. Let $\mathcal{X}$ be the universal set. A symmetric triangular interval type-2 IFS (STIT2IFS) can be represented as follows:

$$\mathcal{A} = \{(\zeta_{\mathcal{A}}(x), \varrho_{\mathcal{A}}(x), \varphi_{\mathcal{A}}(x), \varphi^*_{\mathcal{A}}(x), \vartheta_{\mathcal{A}}(x), \vartheta^*_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\} \tag{6.1}$$

where $\zeta_{\mathcal{A}}(x), \varrho_{\mathcal{A}}(x), \varphi_{\mathcal{A}}(x), \varphi^*_{\mathcal{A}}(x),\ \vartheta_{\mathcal{A}}(x), \vartheta^*_{\mathcal{A}}(x)$ are the real numbers satisfying the inequalities, $\zeta_{\mathcal{A}}(x) \geq \varrho_{\mathcal{A}}(x)$, $0 \leq \varphi_{\mathcal{A}}(x) \leq \varphi^*_{\mathcal{A}}(x) \leq 1$, $0 \leq \vartheta^*_{\mathcal{A}}(x) \leq \vartheta_{\mathcal{A}}(x) \leq 1$ such that $\varphi_{\mathcal{A}}(x) + \vartheta_{\mathcal{A}}(x) \leq 1$ and $\varphi^*_{\mathcal{A}}(x) + \vartheta^*_{\mathcal{A}}(x) \leq 1$.

For convenience, we represent this pair as $\mathcal{A} = (\zeta_{\mathcal{A}}, \varrho_{\mathcal{A}}, \varphi_{\mathcal{A}}, \varphi^*_{\mathcal{A}}, \vartheta_{\mathcal{A}}, \vartheta^*_{\mathcal{A}})$ and called as symmetric triangular IT2 intuitionistic fuzzy (IT2IF) number (STIT2IFN) where $\zeta_{\mathcal{A}} \geq \varrho_{\mathcal{A}}$, $\varphi_{\mathcal{A}} + \vartheta_{\mathcal{A}} \leq 1$, $\varphi^*_{\mathcal{A}} + \vartheta^*_{\mathcal{A}} \leq 1$ and $\varphi_{\mathcal{A}} \leq \varphi^*_{\mathcal{A}}$, $\vartheta_{\mathcal{A}} \geq \vartheta^*_{\mathcal{A}}$. The graphical representation of STIT2IFN is given in Figure 6.1.

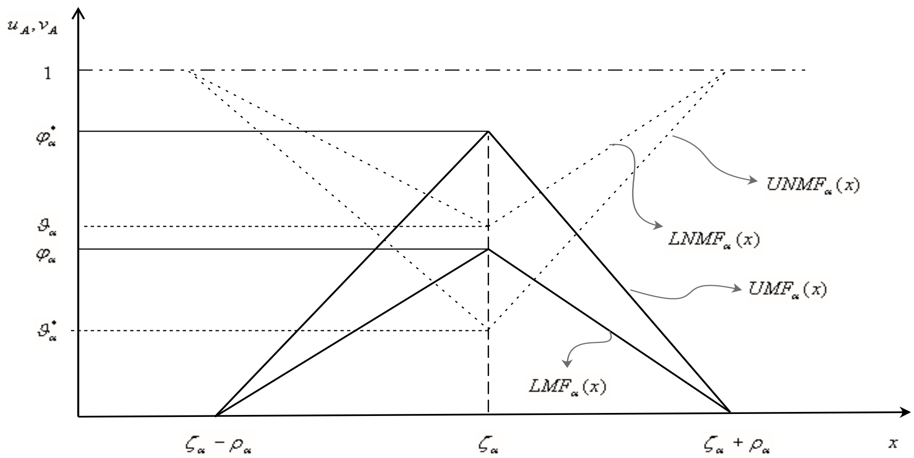

Figure 6.1: Representation of Symmetric Triangular Interval T2IFN $\mathcal{A}$

Definition 6.2.2. For a STIT2IFN $\mathcal{A} = (\zeta_{\mathcal{A}}, \varrho_{\mathcal{A}}, \varphi_{\mathcal{A}}, \varphi^*_{\mathcal{A}}, \vartheta_{\mathcal{A}}, \vartheta^*_{\mathcal{A}})$, the lower and upper membership and non-membership functions denoted by LMF, UMF, LNMF and UNMF are defined as

$$\mathrm{UMF}_{\mathcal{A}}(x) = \begin{cases} \dfrac{\varphi^*_{\mathcal{A}}}{\varrho_{\mathcal{A}}}(x-\zeta_{\mathcal{A}}+\varrho_{\mathcal{A}}) & ;\zeta_{\mathcal{A}}-\varrho_{\mathcal{A}} \leq x < \zeta_{\mathcal{A}} \\ \varphi^*_{\mathcal{A}} & ;x=\zeta_{\mathcal{A}} \\ \dfrac{\varphi^*_{\mathcal{A}}}{\varrho_{\mathcal{A}}}(\zeta_{\mathcal{A}}+\varrho_{\mathcal{A}}-x) & ;\zeta_{\mathcal{A}} < x \leq \varrho_{\mathcal{A}}+\zeta_{\mathcal{A}} \end{cases} ;$$

$$\mathrm{UNMF}_{\mathcal{A}}(x) = \begin{cases} \dfrac{(\vartheta^*_{\mathcal{A}}-1)(x-\zeta_{\mathcal{A}}+\varrho_{\mathcal{A}})+\varrho_{\mathcal{A}}}{\varrho_{\mathcal{A}}} & ;\zeta_{\mathcal{A}}-\varrho_{\mathcal{A}} \leq x < \zeta_{\mathcal{A}} \\ \vartheta^*_{\mathcal{A}} & ;x=\zeta_{\mathcal{A}} \\ \dfrac{(1-\vartheta^*_{\mathcal{A}})(x-\zeta_{\mathcal{A}})+\vartheta^*_{\mathcal{A}}\varrho_{\mathcal{A}}}{\varrho_{\mathcal{A}}} & ;\zeta_{\mathcal{A}} < x \leq \varrho_{\mathcal{A}}+\zeta_{\mathcal{A}} \end{cases}$$

$$\mathrm{LMF}_{\mathcal{A}}(x) = \begin{cases} \dfrac{\varphi_{\mathcal{A}}}{\varrho_{\mathcal{A}}}(x-\zeta_{\mathcal{A}}+\varrho_{\mathcal{A}}) & ;\zeta_{\mathcal{A}}-\varrho_{\mathcal{A}} \leq x < \zeta_{\mathcal{A}} \\ \varphi_{\mathcal{A}} & ;x=\zeta_{\mathcal{A}} \\ \dfrac{\varphi_{\mathcal{A}}}{\varrho_{\mathcal{A}}}(\zeta_{\mathcal{A}}+\varrho_{\mathcal{A}}-x) & ;\zeta_{\mathcal{A}} < x \leq \varrho_{\mathcal{A}}+\zeta_{\mathcal{A}} \end{cases} ;$$

$$\mathrm{LNMF}_{\mathcal{A}}(x) = \begin{cases} \dfrac{(\vartheta_{\mathcal{A}}-1)(x-\zeta_{\mathcal{A}}+\varrho_{\mathcal{A}})+\varrho_{\mathcal{A}}}{\varrho_{\mathcal{A}}} & ;\zeta_{\mathcal{A}}-\varrho_{\mathcal{A}} \leq x < \zeta_{\mathcal{A}} \\ \vartheta_{\mathcal{A}} & ;x=\zeta_{\mathcal{A}} \\ \dfrac{(1-\vartheta_{\mathcal{A}})(x-\zeta_{\mathcal{A}})+\vartheta_{\mathcal{A}}\varrho_{\mathcal{A}}}{\varrho_{\mathcal{A}}} & ;\zeta_{\mathcal{A}} < x \leq \varrho_{\mathcal{A}}+\zeta_{\mathcal{A}} \end{cases}$$

Definition 6.2.3. The score function of STIT2IFN $\mathcal{A} = (\zeta_{\mathcal{A}}, \varrho_{\mathcal{A}}, \varphi_{\mathcal{A}}, \varphi^*_{\mathcal{A}}, \vartheta_{\mathcal{A}}, \vartheta^*_{\mathcal{A}})$ is defined as

$$\begin{aligned} s(\mathcal{A}) &= (s_x(\mathcal{A}), s_y(\mathcal{A})) \\ &= \left(\zeta_{\mathcal{A}}\frac{2\varphi_{\mathcal{A}}\varphi^*_{\mathcal{A}}}{\varphi_{\mathcal{A}}+\varphi^*_{\mathcal{A}}} - \zeta_{\mathcal{A}}\frac{2\vartheta_{\mathcal{A}}\vartheta^*_{\mathcal{A}}}{\vartheta_{\mathcal{A}}+\vartheta^*_{\mathcal{A}}}, \frac{\vartheta_{\mathcal{A}}+\varphi^*_{\mathcal{A}}}{2} - \frac{\varphi_{\mathcal{A}}+\vartheta^*_{\mathcal{A}}}{2}\right) \end{aligned} \tag{6.2}$$

Definition 6.2.4. For two STIT2IFNs $\mathcal{A}$ and $\mathcal{B}$, an order relation "$(>)$" to compare them is defined as

(i) If $s_x(\mathcal{A}) > s_x(\mathcal{B})$, then $\mathcal{A} > \mathcal{B}$;

(ii) If $s_x(\mathcal{A}) = s_x(\mathcal{B})$, then $\begin{cases} s_y(\mathcal{A}) > s_y(\mathcal{B}) & \Rightarrow \mathcal{A} > \mathcal{B}; \\ s_y(\mathcal{A}) = s_y(\mathcal{B}) & \Rightarrow \mathcal{A} = \mathcal{B}; \end{cases}$

Definition 6.2.5. For two STIT2IFNs $\mathcal{A} = \big(\zeta_{\mathcal{A}},\ \varrho_{\mathcal{A}},\ \varphi_{\mathcal{A}},\ \varphi^*_{\mathcal{A}},\ \vartheta_{\mathcal{A}},\ \vartheta^*_{\mathcal{A}}\big)$ and $\mathcal{B} = \big(\zeta_{\mathcal{B}}, \varrho_{\mathcal{B}},$ $\varphi_{\mathcal{B}},\ \varphi^*_{\mathcal{B}},\ \vartheta_{\mathcal{B}},\ \vartheta^*_{\mathcal{B}}\big)$, $\lambda > 0$, then the operational laws of it are shown as follows:

(i) $\mathcal{A} \oplus \mathcal{B} = \Big(\zeta_{\mathcal{A}} + \zeta_{\mathcal{B}}, \varrho_{\mathcal{A}} + \varrho_{\mathcal{B}}, \varphi_{\mathcal{A}}\varphi_{\mathcal{B}}, \varphi^*_{\mathcal{A}} + \varphi^*_{\mathcal{B}} - \varphi^*_{\mathcal{A}}\varphi^*_{\mathcal{B}}, \vartheta_{\mathcal{A}} + \vartheta_{\mathcal{B}} - \vartheta_{\mathcal{A}}\vartheta_{\mathcal{B}}, \vartheta^*_{\mathcal{A}}\vartheta^*_{\mathcal{B}}\Big)$;

(ii) $\mathcal{A} \otimes \mathcal{B} = \Big(\zeta_{\mathcal{A}}\zeta_{\mathcal{B}}, \varrho_{\mathcal{A}}\varrho_{\mathcal{B}}, \varphi_{\mathcal{A}} + \varphi_{\mathcal{B}} - \varphi_{\mathcal{A}}\varphi_{\mathcal{B}}, \varphi^*_{\mathcal{A}}\varphi^*_{\mathcal{B}}, \vartheta_{\mathcal{A}}\vartheta_{\mathcal{B}}, \vartheta^*_{\mathcal{A}} + \vartheta^*_{\mathcal{B}} - \vartheta^*_{\mathcal{A}}\ \vartheta^*_{\mathcal{B}}\Big)$;

(iii) $\lambda\mathcal{A} = \Big(\lambda\zeta_{\mathcal{A}}, \lambda\varrho_{\mathcal{A}}, (\varphi_{\mathcal{A}})^{\lambda}, 1 - (1 - \varphi^*_{\mathcal{A}})^{\lambda}, 1 - (1 - \vartheta_{\mathcal{A}})^{\lambda}, (\vartheta^*_{\mathcal{A}})^{\lambda}\Big)$;

(iv) $\mathcal{A}^{\lambda} = \Big(\zeta^{\lambda}_{\mathcal{A}}, \varrho^{\lambda}_{\mathcal{A}}, 1 - (1 - \varphi_{\mathcal{A}})^{\lambda}, (\varphi^*_{\mathcal{A}})^{\lambda}, (\vartheta_{\mathcal{A}})^{\lambda}, 1 - (1 - \vartheta^*_{\mathcal{A}})^{\lambda}\Big)$

Theorem 6.2.1. For STIT2IFNs $\mathcal{A}$ and $\mathcal{B}$, the operations defined in Definition 6.2.5 are again STIT2IFNs.

Proof. Consider two STIT2IFNs $\mathcal{A} = (\zeta_{\mathcal{A}}, \varrho_{\mathcal{A}}, \varphi_{\mathcal{A}}, \varphi^*_{\mathcal{A}}, \vartheta_{\mathcal{A}}, \vartheta^*_{\mathcal{A}})$ and $\mathcal{B} = (\zeta_{\mathcal{B}}, \varrho_{\mathcal{B}}, \varphi_{\mathcal{B}}, \varphi^*_{\mathcal{B}}, \vartheta_{\mathcal{B}}, \vartheta^*_{\mathcal{B}})$. So by Definition 6.2.1, we have $\zeta_{\mathcal{A}} \geq \varrho_{\mathcal{A}}$, $\varphi_{\mathcal{A}} \leq \varphi^*_{\mathcal{A}}$, $\vartheta_{\mathcal{A}} \geq \vartheta^*_{\mathcal{A}}$, $\varphi_{\mathcal{A}} + \vartheta_{\mathcal{A}} \leq 1$, $\varphi^*_{\mathcal{A}} + \vartheta^*_{\mathcal{A}} \leq 1$, $\zeta_{\mathcal{B}} \geq \varrho_{\mathcal{B}}$, $\varphi_{\mathcal{B}} \leq \varphi^*_{\mathcal{B}}$, $\vartheta_{\mathcal{B}} \geq \vartheta^*_{\mathcal{B}}$ $\varphi_{\mathcal{B}} + \vartheta_{\mathcal{B}} \leq 1$, $\varphi^*_{\mathcal{B}} + \vartheta^*_{\mathcal{B}} \leq 1$. Let $\mathcal{A} \oplus \mathcal{B} = \gamma = \big(\zeta_{\gamma}, \varrho_{\gamma}, \varphi_{\gamma}, \varphi^*_{\gamma}, \vartheta_{\gamma}, \vartheta^*_{\gamma}\big)$ and thus by Definition 6.2.5, we get $\zeta_{\gamma} = \zeta_{\mathcal{A}} + \zeta_{\mathcal{B}}$, $\varrho_{\gamma} = \varrho_{\mathcal{A}} + \varrho_{\mathcal{B}}$, $\varphi_{\gamma} = \varphi_{\mathcal{A}}\varphi_{\mathcal{B}}$, $\varphi^*_{\gamma} = \varphi^*_{\mathcal{A}} +^*_{\mathcal{B}} - \varphi^*_{\mathcal{A}}\varphi^*_{\mathcal{B}}$, $\vartheta_{\gamma} = \vartheta_{\mathcal{A}} + \vartheta_{\mathcal{B}} - \vartheta_{\mathcal{A}}\vartheta_{\mathcal{B}}$, $\vartheta^*_{\gamma} = \vartheta^*_{\mathcal{A}}\vartheta^*_{\mathcal{B}}$. Now, to show $\mathcal{A} \oplus \mathcal{B}$ is again an STIT2IFN, we need to prove that $\zeta_{\gamma} \geq \varrho_{\gamma}$, $\varphi_{\gamma} \leq \varphi^*_{\gamma}$, $\vartheta_{\gamma} \geq \vartheta^*_{\gamma}$, $\varphi_{\gamma} + \vartheta_{\gamma} \leq 1$, $\varphi^*_{\gamma} + \vartheta^*_{\gamma} \leq 1$.

As $\zeta_{\mathcal{A}} \geq \varrho_{\mathcal{A}}$ and $\zeta_{\mathcal{B}} \geq \varrho_{\mathcal{B}}$ which implies that $\zeta_{\gamma} \geq \varrho_{\gamma}$. Further $\varphi_{\mathcal{A}} \leq \varphi^*_{\mathcal{A}}$, $\varphi_{\mathcal{B}} \leq \varphi^*_{\mathcal{B}}$, $\vartheta_{\mathcal{A}} \geq \vartheta^*_{\mathcal{A}}$, $\vartheta_{\mathcal{B}} \geq \vartheta^*_{\mathcal{B}}$, $\varphi_{\mathcal{A}} + \vartheta_{\mathcal{A}} \leq 1$, $\varphi^*_{\mathcal{A}} + \vartheta^*_{\mathcal{A}} \leq 1$ which gives that

$$
\begin{aligned}
\varphi_{\gamma} + \vartheta_{\gamma} &= \varphi_{\mathcal{A}}\varphi_{\mathcal{B}} + (\vartheta_{\mathcal{A}} + \vartheta_{\mathcal{B}} - \vartheta_{\mathcal{A}}\vartheta_{\mathcal{B}}) \\
&= \varphi_{\mathcal{A}}\varphi_{\mathcal{B}} + 1 - (1 - \vartheta_{\mathcal{A}})(1 - \vartheta_{\mathcal{B}}) \\
&\leq \varphi_{\mathcal{A}}\varphi_{\mathcal{B}} + 1 - \varphi_{\mathcal{A}}\varphi_{\mathcal{B}} \\
&\leq 1
\end{aligned}
$$

and

$$\begin{aligned}
\varphi^*_\gamma + \vartheta^*_\gamma &= \varphi^*_\mathcal{A}\varphi^*_\mathcal{B} - \varphi^*_\mathcal{A}\varphi^*_\mathcal{B} + \vartheta^*_\mathcal{A}\vartheta^*_\mathcal{B} \\
&= 1 - (1 - \varphi^*_\mathcal{A})(1 - \varphi^*_\mathcal{B}) + \vartheta^*_\mathcal{A}\vartheta^*_\mathcal{B} \\
&\leq 1 - \vartheta^*_\mathcal{A}\vartheta^*_\mathcal{B} + \vartheta^*_\mathcal{A}\vartheta^*_\mathcal{B} \\
&\leq 1
\end{aligned}$$

Finally, $\varphi_\gamma = \varphi_\mathcal{A}\varphi_\mathcal{B} \leq \varphi^*_\mathcal{A}\varphi^*_\mathcal{B} = \varphi^*_\gamma$ and $\vartheta_\gamma = \vartheta_\mathcal{A} + \vartheta_\mathcal{B} - \vartheta_\mathcal{A}\vartheta_\mathcal{B} = 1 - (1 - \vartheta_\mathcal{A})(1 - \vartheta_\mathcal{B}) \geq 1 - (1 - \vartheta^*_\mathcal{A})(1 - \vartheta^*_\mathcal{B}) = \vartheta^*_\gamma$. Therefore, we conclude that $\mathcal{A} \oplus \mathcal{B}$ becomes STIT2IFN. Similarly, we can prove that $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A}^\lambda$ and $\lambda\mathcal{A}$ are also STIT2IFNs. □

6.3 STIT2IF Hamy Mean Aggregation Operators

Let Ω be the gathering of all non-empty STIT2IFNs $\mathcal{A}_i = (\zeta_i, \varrho_i, \varphi_i, \varphi^*_i, \vartheta_i, \vartheta^*_i), (i = 1, 2, \ldots, n)$. Here, we present HM-based AOs for STIT2IFNs.

6.3.1 STIT2IFHM Operator

Definition 6.3.1. A STIT2IFHM is a mapping STIT2IFHM : $\Omega^n \to \Omega$ defined as

$$\text{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \frac{\bigoplus\limits_{\substack{1 \leq i_1 < \\ \ldots < i_k \leq n}} \left(\bigotimes\limits_{j=1}^{k} \mathcal{A}_{i_j} \right)^{\frac{1}{k}}}{\binom{n}{k}} \tag{6.3}$$

then STIT2IHM$^{(k)}$ is called the symmetric triangular IT2IF Hamy mean operator, where $k = 1, 2, \ldots, n$ is the parameter and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ represent the binomial coefficient.

Theorem 6.3.1. The aggregated value for n STIT2IFNs $\mathcal{A}_i = (\zeta_i, \varrho_i, \varphi_i, \varphi^*_i, \vartheta_i, \vartheta^*_i)$ by using Definition 6.3.1 is again STIT2IFN which is given as

$$\begin{aligned}
&\text{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\
= &\left(\begin{array}{l}
\dfrac{\sum\limits_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \dfrac{\sum\limits_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \left(\prod\limits_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(1 - \left(\prod\limits_{j=1}^{k} (1 - \varphi_{\mathcal{A}_{i_j}}) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}, \\
1 - \left(\prod\limits_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(1 - \left(\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod\limits_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(1 - \left(\prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}, \\
\left(\prod\limits_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(1 - \left(\prod\limits_{j=1}^{k} (1 - \vartheta^*_{\mathcal{A}_{i_j}}) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}
\end{array} \right)
\end{aligned} \tag{6.4}$$

Proof. The first part of the result can be easily obtained from Theorem 6.2.1. So, there is a need to prove only that Eq. (6.4) is kept.

According to the operational laws of STIT2IFNs, we get

$$\bigotimes_{j=1}^{k} \mathcal{A}_{i_j} = \left(\begin{array}{l}
\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}, \prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}}, 1 - \prod\limits_{j=1}^{k} (1 - \varphi_{\mathcal{A}_{i_j}}), \\
\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}}, \prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}_{i_j}}, 1 - \prod\limits_{j=1}^{k} (1 - \vartheta^*_{\mathcal{A}_{i_j}})
\end{array} \right)$$

and

$$\begin{aligned}
&\left(\bigotimes_{j=1}^{n} \mathcal{A}_{i_j} \right)^{\dfrac{1}{k}} \\
= &\left(\begin{array}{l}
\left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}} \right)^{\dfrac{1}{k}}, \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}} \right)^{\dfrac{1}{k}}, 1 - \left(\prod\limits_{j=1}^{k} (1 - \varphi_{\mathcal{A}_{i_j}}) \right)^{\dfrac{1}{k}}, \\
\left(\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}} \right)^{\dfrac{1}{k}}, \left(\prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}_{i_j}} \right)^{\dfrac{1}{k}}, 1 - \left(\prod\limits_{j=1}^{k} (1 - \vartheta^*_{\mathcal{A}_{i_j}}) \right)^{\dfrac{1}{k}}
\end{array} \right)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\bigoplus_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(\bigotimes_{j=1}^{n} \mathcal{A}_{i_j}\right)^{\dfrac{1}{k}} \\
&= \left(\begin{array}{l}
\sum\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}, \sum\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}, \prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k}(1-\varphi_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}}\right), \\
1-\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k} \varphi^{*}_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right), 1-\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right), \\
\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k}(1-\vartheta^{*}_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}}\right)
\end{array}\right)
\end{aligned}
$$

Subsequently, we have

$$
\begin{aligned}
&\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\
&= \frac{\bigoplus\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(\bigotimes\limits_{j=1}^{k} \mathcal{A}_{i_j}\right)^{\dfrac{1}{k}}}{\binom{n}{k}} \\
&= \left(\begin{array}{l}
\dfrac{\sum\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}}, \dfrac{\sum\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}}, \left(\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k}(1-\varphi_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, \\
1-\left(\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k} \varphi^{*}_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, 1-\left(\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, \\
\left(\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k}(1-\vartheta^{*}_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}
\end{array}\right)
\end{aligned}
$$

□

In what follows, we investigate the certain property of STIT2IFHM operator.

Theorem 6.3.2. (Idempotency) If $\mathcal{A}_i = \mathcal{A} = (\zeta_{\mathcal{A}}, \varrho_{\mathcal{A}}, \varphi_{\mathcal{A}}, \varphi^{*}_{\mathcal{A}}, \vartheta_{\mathcal{A}}, \vartheta^{*}_{\mathcal{A}})$ for all i, then

$$\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}.$$

Proof. Since $\mathcal{A}_i = \mathcal{A} = (\zeta_{\mathcal{A}}, \varrho_{\mathcal{A}}, \varphi_{\mathcal{A}}, \varphi^*_{\mathcal{A}}, \vartheta_{\mathcal{A}}, \vartheta^*_{\mathcal{A}})$ for all i then based on Theorem 6.3.1, we have

$$
\begin{aligned}
&\text{STIT2IFHM}^{(k)}(\mathcal{A}, \mathcal{A}, \ldots, \mathcal{A}) \\
&= \left(\begin{array}{l}
\dfrac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \dfrac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} (1 - \varphi_{\mathcal{A}}) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}, \\
1 - \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}} \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}} \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}, \\
\left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} (1 - \vartheta^*_{\mathcal{A}}) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}
\end{array} \right) \\
&= \left(\begin{array}{l}
\dfrac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} (\zeta^k_{\mathcal{A}})^{\frac{1}{k}}}{\binom{n}{k}}, \dfrac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} (\varrho^k_{\mathcal{A}})}{\binom{n}{k}}, \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} (1 - (1 - \varphi_{\mathcal{A}})) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} (1 - \varphi^*_{\mathcal{A}}) \right)^{\frac{1}{\binom{n}{k}}}, \\
1 - \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} (1 - \vartheta_{\mathcal{A}}) \right)^{\frac{1}{\binom{n}{k}}}, \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} (1 - (1 - \vartheta^*_{\mathcal{A}})) \right)^{\frac{1}{\binom{n}{k}}}
\end{array} \right) \\
&= \left(\begin{array}{l}
\dfrac{\binom{n}{k} (\zeta^k_{\mathcal{A}})^{\frac{1}{k}}}{\binom{n}{k}}, \dfrac{\binom{n}{k} (\varrho^k_{\mathcal{A}})^{\frac{1}{k}}}{\binom{n}{k}}, 1 - (1 - \varphi_{\mathcal{A}})^{\frac{\binom{n}{k}}{\binom{n}{k}}}, 1 - (1 - \varphi^*_{\mathcal{A}})^{\frac{\binom{n}{k}}{\binom{n}{k}}}, \\
1 - (1 - \vartheta_{\mathcal{A}})^{\frac{\binom{n}{k}}{\binom{n}{k}}}, (1 - (1 - \vartheta^*_{\mathcal{A}}))^{\frac{\binom{n}{k}}{\binom{n}{k}}}
\end{array} \right) \\
&= (\zeta_{\mathcal{A}}, \varrho_{\mathcal{A}}, \varphi_{\mathcal{A}}, \varphi^*_{\mathcal{A}}, \vartheta_{\mathcal{A}}, \vartheta^*_{\mathcal{A}}) \\
&= \mathcal{A}
\end{aligned}
$$

□

Theorem 6.3.3. (Commutativity) Let $\mathcal{A}_i (i = 1, 2, \ldots, n)$ be a collection of STIT2IFNs, and $\overline{\mathcal{A}}_i$ be any permutation of $\mathcal{A}_i$. Then

$$\text{STIT2IFHM}^{(k)}(\overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2, \ldots, \overline{\mathcal{A}}_n) = \text{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

Proof. Based on the Definition 6.3.1, we have

$$\begin{aligned}
\mathrm{STIT2IFHM}^{(k)}(\overline{\mathcal{A}}_1,\overline{\mathcal{A}}_2,\ldots,\overline{\mathcal{A}}_n) &= \frac{\bigoplus\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(\bigotimes\limits_{j=1}^{k}\tilde{\mathcal{A}}_{i_j}\right)^{\frac{1}{k}}}{\binom{n}{k}}\\
&= \frac{\bigoplus\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(\bigotimes\limits_{j=1}^{k}\mathcal{A}_{i_j}\right)^{\frac{1}{k}}}{\binom{n}{k}}\\
&= \mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)
\end{aligned}$$

□

Theorem 6.3.4. (Monotonicity) For two different STIT2IFNs $\mathcal{A}_i=\big(\zeta_{\mathcal{A}_i},\,\varrho_{\mathcal{A}_i},\,\varphi_{\mathcal{A}_i},\,\varphi^*_{\mathcal{A}_i},\,\vartheta_{\mathcal{A}_i},\,\vartheta^*_{\mathcal{A}_i}\big)$, and $\mathcal{B}_i=\big(\zeta_{\mathcal{B}_i},\,\varrho_{\mathcal{B}_i},\,\varphi_{\mathcal{B}_i},\,\varphi^*_{\mathcal{B}_i},\,\vartheta_{\mathcal{B}_i},\,\vartheta^*_{\mathcal{B}_i}\big)$, $(i=1,2,\ldots,n)$. If $\zeta_{\mathcal{A}_i}\le\zeta_{\mathcal{B}_i}$, $\varrho_{\mathcal{A}_i}\ge\varrho_{\mathcal{B}_i}$, $\varphi_{\mathcal{A}_i}\ge\varphi_{\mathcal{B}_i}$, $\varphi^*_{\mathcal{A}_i}\le\varphi^*_{\mathcal{B}_i}$, $\vartheta_{\mathcal{A}_i}\le\vartheta_{\mathcal{B}_i}$ and $\vartheta^*_{\mathcal{A}_i}\ge\vartheta^*_{\mathcal{B}_i}$ for all i, then

$$\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\le\mathrm{STIT2IFHM}^{(k)}(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_n).$$

Proof. Let $\mathcal{A}=\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)$ and $\mathcal{B}=\mathrm{STIT2IFHM}^{(k)}(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_n)$. Then according to Theorem 6.3.1, we get

$$\begin{aligned}
\mathcal{A} &= \mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\
&= \left(\begin{array}{l}
\dfrac{\sum\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(\prod\limits_{j=1}^{k}\zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}},\dfrac{\sum\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(\prod\limits_{j=1}^{k}\varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}},\left(\prod\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}(1-\varphi_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}},\\
1-\left(\prod\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}\varphi^*_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}},1-\left(\prod\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}\vartheta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}},\\
\left(\prod\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}(1-\vartheta^*_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}
\end{array}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B} &= \text{STIT2IFHM}^{(k)}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n) \\
&= \left(\begin{array}{l}
\frac{\sum\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{B}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{B}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}}, \left(\prod\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k}(1-\varphi_{\mathcal{B}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, \\
1-\left(\prod\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k}\varphi^*_{\mathcal{B}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, 1-\left(\prod\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k}\vartheta_{\mathcal{B}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, \\
\left(\prod\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(1-\left(\prod\limits_{j=1}^{k}(1-\vartheta^*_{\mathcal{B}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}
\end{array} \right)
\end{aligned}$$

Since $\zeta_{\mathcal{A}_i} \le \zeta_{\mathcal{B}_i}$ which implies that

$$\frac{\sum\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}} \le \frac{\sum\limits_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{B}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}}$$

Also, $\varphi_{\mathcal{A}_i} \ge \varphi_{\mathcal{B}_i}$ implies that $\prod\limits_{j=1}^{k}(1-\varphi_{\mathcal{A}_{i_j}}) \le \prod\limits_{j=1}^{k}(1-\varphi_{\mathcal{B}_{i_j}})$. Thus,

$$\left(\prod_{j=1}^{k}(1-\varphi_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}} \le \left(\prod_{j=1}^{k}(1-\varphi_{\mathcal{B}_{i_j}})\right)^{\frac{1}{k}}$$

which give us that

$$\left(\prod_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(1-\left(\prod_{j=1}^{k}(1-\varphi_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}} \ge \left(\prod_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(1-\left(\prod_{j=1}^{k}(1-\varphi_{\mathcal{B}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}$$

Similarly for $\varphi^*_{\mathcal{A}_i} \le \varphi^*_{\mathcal{B}_i}$, $\vartheta_{\mathcal{A}_i} \le \vartheta_{\mathcal{A}_i}$ and $\vartheta^*_{\mathcal{A}_i} \ge \vartheta^*_{\mathcal{B}_i}$ for all i, we have

$$1-\left(\prod_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(1-\left(\prod_{j=1}^{k}\vartheta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}} \le 1-\left(\prod_{\substack{1\le i_1<\\ \ldots<i_k\le n}} \left(1-\left(\prod_{j=1}^{k}\vartheta_{\mathcal{B}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}};$$

$$\left(\prod_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(1-\left(\prod_{j=1}^{k}(1-\vartheta^*_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\dfrac{1}{\binom{n}{k}}}\ge\left(\prod_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(1-\left(\prod_{j=1}^{k}(1-\vartheta^*_{\mathcal{B}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\dfrac{1}{\binom{n}{k}}};$$

and
$$1-\left(\prod_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(1-\left(\prod_{j=1}^{k}\varphi^*_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\dfrac{1}{\binom{n}{k}}}\le 1-\left(\prod_{\substack{1\le i_1<\\ \ldots<i_k\le n}}\left(1-\left(\prod_{j=1}^{k}\varphi^*_{\mathcal{B}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\dfrac{1}{\binom{n}{k}}}$$

Therefore, by using these inequalities and Definition 6.2.4, we get

$$\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\le\mathrm{STIT2IFHM}^{(k)}(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_n)$$

□

Theorem 6.3.5. (Boundedness) For n STIT2IFNs $\mathcal{A}_i$, $\mathcal{A}^-=\big(\min_i\{\zeta_i\},\max_i\{\varrho_i\},\min_i\{\varphi_i\},\max_i\{\varphi^*_i\},\max_i\{\vartheta_i\},\min_i\{\vartheta^*_i\}\big)$, and $\mathcal{A}^+=\big(\max_i\{\zeta_i\},\min_i\{\varrho_i\},\max_i\{\varphi_i\},\min_i\{\varphi^*_i\},\min_i\{\vartheta_i\},\max_i\{\vartheta^*_i\}\big)$, we have

$$\mathcal{A}^-\le\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\le\mathcal{A}^+$$

Proof. Clearly, we get $\mathcal{A}^-\le\mathcal{A}_i\le\mathcal{A}^+$. Thus, based on Theorems 6.3.3 and 6.3.4, we have

$$\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\ge\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}^-,\mathcal{A}^-,\ldots,\mathcal{A}^-)=\mathcal{A}^-$$
$$\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\le\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}^+,\mathcal{A}^+,\ldots,\mathcal{A}^+)=\mathcal{A}^+$$

□

Lemma 6.3.1. [66] For n non-negative real numbers x_i, we have

$$HM^{(1)}(x_1,x_2,\ldots,x_n)\ge HM^{(2)}(x_1,x_2,\ldots,x_n)\ge\ldots\ge HM^{(n)}(x_1,x_2,\ldots,x_n)\quad(6.5)$$

with equality holding iff $x_1=x_2=\ldots=x_n$.

Lemma 6.3.2. [78] Let $x_i\ge0$, $y_i>0$ and $\sum_{i=1}^{n}y_i=1$. Then

$$\prod_{i=1}^{n}x_i^{y_i}\le\sum_{i=1}^{n}x_iy_i\quad(6.6)$$

with equality holding if and only if $x_1=x_2=\ldots=x_n$.

Theorem 6.3.6. For given STIT2IFNs $\mathcal{A}_i$, the operator STIT2IFHM is monotonically decreasing with parameter k.

Proof. For STIT2IFNs $\mathcal{A}_i$ and $k = 1, 2, \ldots, n$, we denote

$$C(k) = \frac{\sum\limits_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}}, \quad \Delta(k) = \frac{\sum\limits_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}},$$

$$T(k) = \left(\prod_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(1 - \left(\prod_{j=1}^{k}(1-\varphi_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}},$$

$$S(k) = 1 - \left(\prod_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(1 - \left(\prod_{j=1}^{k}\varphi^*_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}},$$

$$T^*(k) = 1 - \left(\prod_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(1 - \left(\prod_{j=1}^{k}\vartheta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}},$$

$$S^*(k) = \left(\prod_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(1 - \left(\prod_{j=1}^{k}(1-\vartheta^*_{\mathcal{A}_{i_j}})\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}$$

Based on Theorem 6.3.1, we have

$$\begin{aligned} \text{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= (C(k), \Delta(k), T(k), S(k), T^*(k), S^*(k)) \\ \text{and } \text{STIT2IFHM}^{(k+1)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \begin{pmatrix} C(k+1), \Delta(k+1), T(k+1), \\ S(k+1), T^*(k+1), S^*(k+1) \end{pmatrix} \end{aligned}$$

Following Definition 6.2.3 and Lemma 6.3.1, we obtained

$$\begin{aligned} s_x(\text{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)) &\geq \frac{\sum\limits_{\substack{1\leq i_1< \\ \ldots<i_k\leq n}} \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}} \\ &\geq \frac{\sum\limits_{\substack{1\leq i_1< \\ \ldots<i_{k+1}\leq n}} \left(\prod\limits_{j=1}^{k+1} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k+1}}}{\binom{n}{k+1}} \\ &\geq s_x(\text{STIT2IFHM}^{(k+1)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)) \end{aligned}$$

Then, two cases are arisen:

Case 1 If $s_x\big(\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\big) > s_x\big(\mathrm{STIT2IFHM}^{(k+1)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\big)$, following the Definition 6.2.4 we get

$$\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) > \mathrm{STIT2IFHM}^{(k+1)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

Case 2 If $s_x\big(\mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\big) = s_x\big(\mathrm{STIT2IFHM}^{(k+1)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\big)$. Then, by Lemmas 6.3.1 and 6.3.2, we get

$$\begin{aligned}
S(k) &= 1 - \left(\prod_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}} \\
&\geq 1 - \frac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}} \right)}{\binom{n}{k}} \\
&= \sum_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \frac{\left(\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}
\end{aligned}$$

To check the monotonic behavior of $S(k)$, we assume that it is increasing with k, i.e.,

$$S(n) > S(n-1) > \ldots > S(1) \tag{6.7}$$

Also since

$$S(1) \geq 1 - \sum_{1 \le i_1 \le n} \frac{\prod\limits_{j=1}^{1} \left(1 - \varphi^*_{\mathcal{A}_{i_j}} \right)}{\binom{n}{1}} = 1 - \frac{n - \sum\limits_{i=1}^{n} (\varphi^*_{\mathcal{A}_i})}{n} = \frac{\sum\limits_{i=1}^{n} \varphi^*_{\mathcal{A}_i}}{n} \tag{6.8}$$

which implies that

$$\begin{aligned}
S(n) > S(1) &= \frac{\sum_{i=1}^{n} \varphi^*_{\mathcal{A}_i}}{n} \\
\Rightarrow \quad \left(\prod_{i=1}^{n} \varphi^*_{\mathcal{A}_i} \right)^{\frac{1}{n}} &> \frac{\sum_{i=1}^{n} \varphi^*_{\mathcal{A}_i}}{n}
\end{aligned}$$

which contradict the Lemma 6.3.2. Hence with parameter k, $S(k)$ is monotonically decreasing. Similarly, we can get $T^*(k)$ is also monotonically decreasing with parameter

k. Also, the functions $T(k)$ and $S^*(k)$ are monotonically increasing with parameter k. Therefore,

$$\begin{aligned}
& s_y\big(\text{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\big) \\
= \; & \frac{S(k) + T^*(k)}{2} - \frac{T(k) + S^*(k)}{2} \\
> \; & \frac{S(k+1) + T^*(k+1)}{2} - \frac{T(k+1) + S^*(k+1)}{2} \\
= \; & s_y\big(\text{STIT2IFHM}^{(k+1)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\big)
\end{aligned}$$

Thus, by both the cases, we get $\text{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \geq \text{STIT2IFHM}^{(k+1)}$ $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$. $\square$

Furthermore, we will talk about a few special cases of the STIT2IFHM operator concerning the parameter the k.

1) When $k = 1$, Eq. (6.4) reduces to the triangular IT2IF averaging operator.

$$\begin{aligned}
& \text{STIT2IFHM}^{(1)}(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_m) \\
= & \left(\begin{array}{l}
\dfrac{\sum\limits_{1 \leq i_1 \leq n} \left(\prod\limits_{j=1}^{1} \zeta_{\mathcal{A}_{i_j}} \right)^{\frac{1}{1}}}{\binom{n}{1}}, \dfrac{\sum\limits_{1 \leq i_1 \leq n} \left(\prod\limits_{j=1}^{1} \varrho_{\mathcal{A}_{i_j}} \right)^{\frac{1}{1}}}{\binom{m}{1}}, \left(\prod\limits_{1 \leq i_1 \leq n} \left(1 - \left(\prod\limits_{j=1}^{1} \left(1 - \varphi_{\mathcal{A}_{i_j}}\right) \right)^{\frac{1}{1}} \right) \right)^{\frac{1}{\binom{n}{1}}}, \\
1 - \left(\prod\limits_{1 \leq i_1 \leq n} \left(1 - \left(\prod\limits_{j=1}^{1} \varphi^*_{\mathcal{A}_{i_j}} \right)^{\frac{1}{1}} \right) \right)^{\frac{1}{\binom{n}{1}}}, 1 - \left(\prod\limits_{1 \leq i_1 \leq n} \left(1 - \left(\prod\limits_{j=1}^{1} \vartheta_{\mathcal{A}_{i_j}} \right)^{\frac{1}{1}} \right) \right)^{\frac{1}{\binom{n}{1}}}, \\
\left(\prod\limits_{1 \leq i_1 \leq n} \left(1 - \left(\prod\limits_{j=1}^{1} \left(1 - \vartheta^*_{\mathcal{A}_{i_j}}\right) \right)^{\frac{1}{1}} \right) \right)^{\frac{1}{\binom{n}{1}}}
\end{array} \right) \\
= & \left(\begin{array}{l}
\dfrac{\sum\limits_{i=1}^{n} \zeta_{\mathcal{A}_i}}{n}, \dfrac{\sum\limits_{i=1}^{n} \varrho_{\mathcal{A}_i}}{n}, \left(\prod\limits_{i=1}^{n} \left(1 - \left(1 - \varphi_{\mathcal{A}_{i_j}}\right)\right) \right)^{\frac{1}{n}}, 1 - \left(\prod\limits_{i=1}^{n} \left(1 - \varphi^*_{\mathcal{A}_{i_j}}\right) \right)^{\frac{1}{n}}, \\
1 - \left(\prod\limits_{i=1}^{n} \left(1 - \vartheta_{\mathcal{A}_{i_j}}\right) \right)^{\frac{1}{n}}, \left(\prod\limits_{i=1}^{n} \left(1 - \left(1 - \vartheta^*_{\mathcal{A}_{i_j}}\right)\right) \right)^{\frac{1}{n}}
\end{array} \right) \\
= & \left(\begin{array}{l}
\dfrac{\sum\limits_{i=1}^{n} \zeta_{\mathcal{A}_i}}{n}, \dfrac{\sum\limits_{i=1}^{n} \varrho_{\mathcal{A}_i}}{n}, \left(\prod\limits_{i=1}^{n} \varphi_{\mathcal{A}_{i_j}} \right)^{\frac{1}{n}}, 1 - \left(\prod\limits_{i=1}^{n} \left(1 - \varphi^*_{\mathcal{A}_{i_j}}\right) \right)^{\frac{1}{n}}, \\
1 - \left(\prod\limits_{i=1}^{n} \left(1 - \vartheta_{\mathcal{A}_{i_j}}\right) \right)^{\frac{1}{n}}, \left(\prod\limits_{i=1}^{n} \vartheta^*_{\mathcal{A}_{i_j}} \right)^{\frac{1}{n}}
\end{array} \right)
\end{aligned}$$

2) When $k = n$, Eq. (6.4) will reduce to triangular IT2IF geometric operator.

$$\begin{aligned}
&\mathrm{STIT2IFHM}^{(m)}(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n) \\
&= \left(\begin{array}{l}
\dfrac{\sum\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}}{\binom{n}{n}}, \dfrac{\sum\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(\prod\limits_{j=1}^{k} \varrho_{\gamma_{i_j}}\right)^{\frac{1}{n}}}{\binom{n}{n}}, \left(\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \varphi_{\gamma_{i_j}}\right)\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{\binom{n}{n}}}, \\
1 - \left(\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{\binom{n}{n}}}, 1 - \left(\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{\binom{n}{n}}}, \\
\left(\prod\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \vartheta^*_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}\right)\right)\right)^{\frac{1}{\binom{n}{n}}}
\end{array} \right) \\
&= \left(\begin{array}{l}
\left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}, \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}, \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \varphi_{\mathcal{A}_{i_j}}\right)\right)^{\frac{1}{n}}\right), 1 - \left(1 - \left(\prod\limits_{j=1}^{k} \varphi^*_{\gamma_{i_j}}\right)^{\frac{1}{n}}\right), \\
1 - \left(1 - \left(\prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}\right), \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \vartheta^*_{\mathcal{A}_{i_j}}\right)\right)^{\frac{1}{n}}\right)
\end{array} \right) \\
&= \left(\begin{array}{l}
\left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}, \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}, \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \varphi_{\mathcal{A}_{i_j}}\right)\right)^{\frac{1}{n}}\right), \\
\left(\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}, \left(\prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}, \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \vartheta^*_{\mathcal{A}_{i_j}}\right)^{\frac{1}{n}}\right)\right)
\end{array} \right)
\end{aligned}$$

6.3.2 WSTIT2IFHM Operator

Definition 6.3.2. For a collection of n STIT2IFNs, $\mathcal{A}_i$, $w = (w_1, w_2, \cdots, w_n)^T$ is weight vector of $\mathcal{A}_i$, where $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$, we define WSTIT2IFHM operator as

$$\begin{aligned}
&\mathrm{WSTIT2IFHM}_w^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_m) \\
&= \begin{cases}
\dfrac{\bigoplus\limits_{\substack{1\le i_1< \\ \ldots<i_k\le n}} \left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right) \left(\bigotimes\limits_{j=1}^{k} \mathcal{A}_{i_j}\right)^{\frac{1}{k}}}{\binom{n-1}{k}} & ; 1 \le k < n \\
\bigotimes\limits_{j=1}^{k} \mathcal{A}_j^{\frac{1-w_j}{n-1}} & ; k = n
\end{cases}
\end{aligned} \tag{6.9}$$

then $\mathrm{WSTIT2IFHM}_w^{(k)}$ is stated as weighted symmetric triangular IT2IF Hamy mean operator.

Theorem 6.3.7. For n STIT2IFNs $\mathcal{A}_i = (\zeta_i, \varrho_i, \varphi_i, \varphi_i^*, \vartheta_i, \vartheta_i^*)$ $(i = 1, 2, \ldots, n)$, the value obtained through Eq. (6.9) is also STIT2IFN, and is given as

$$\mathrm{WSTIT2IFHM}_w^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_m)$$
$$= \left(\begin{array}{l} \dfrac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right) \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \dfrac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right) \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \\ \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \varphi_{\mathcal{A}_{i_j}}\right) \right)^{\frac{1}{k}} \right)^{\left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}, 1 - \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}} \right)^{\left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}, \\ 1 - \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}} \right)^{\left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}, \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \vartheta^*_{\mathcal{A}_{i_j}}\right) \right)^{\frac{1}{k}} \right)^{\left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right)} \right)^{\frac{1}{\binom{n-1}{k}}} \end{array} \right) \tag{6.10}$$

for $1 \le k < n$ and if $k = n$, then

$$\mathrm{WSTIT2IFHM}_w^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n)$$
$$= \left(\begin{array}{l} \prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_j}^{\frac{1-w_j}{n-1}}, \prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_j}^{\frac{1-w_j}{n-1}}, 1 - \prod\limits_{j=1}^{k} \left(1 - \varphi_{\mathcal{A}_j}\right)^{\frac{1-w_j}{n-1}}, \\ \prod\limits_{j=1}^{k} \left(\varphi^*_{\mathcal{A}_j}\right)^{\frac{1-w_j}{n-1}}, \prod\limits_{j=1}^{k} \left(\vartheta_{\mathcal{A}_j}\right)^{\frac{1-w_j}{n-1}}, 1 - \prod\limits_{j=1}^{k} \left(1 - \vartheta^*_{\mathcal{A}_j}\right)^{\frac{1-w_j}{n-1}} \end{array} \right) \tag{6.11}$$

Proof. Similar to the proof of Theorem 6.3.1. □

Theorem 6.3.8. The operator STIT2IFHM is a special case of the WSTIT2IFHM operator.

Proof. Assume that $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$, then by Theorem 6.3.4, for $1 \le k < n$, we have

$$\mathrm{WSTIT2IFHM}_w^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n)$$
$$= \left(\begin{array}{l} \dfrac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \frac{k}{n}\right) \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \dfrac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \frac{k}{n}\right) \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \\ \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \varphi_{\mathcal{A}_{i_j}}\right) \right)^{\frac{1}{k}} \right)^{\left(1 - \frac{k}{n}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}, 1 - \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}} \right)^{\left(1 - \frac{k}{n}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}, \\ 1 - \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \vartheta_{\mathcal{A}_{i_j}} \right)^{\frac{1}{k}} \right)^{\left(1 - \frac{k}{n}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}, \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \vartheta^*_{\mathcal{A}_{i_j}}\right) \right)^{\frac{1}{k}} \right)^{\left(1 - \frac{k}{n}\right)} \right)^{\frac{1}{\binom{n-1}{k}}} \end{array} \right)$$

$$= \left(\begin{array}{l} \frac{\sum\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}} \left(1-\frac{k}{n}\right)\left(\prod\limits_{j=1}^{k}\zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}\frac{n-k}{n}}, \frac{\sum\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}} \left(1-\frac{k}{n}\right)\left(\prod\limits_{j=1}^{k}\varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}\frac{n-k}{n}}, \\ \left(\prod\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}\left(1-\varphi_{\mathcal{A}_{i_j}}\right)\right)^{\frac{1}{k}}\right)^{\left(1-\frac{k}{n}\right)}\right)^{\frac{1}{\binom{n}{k}\frac{n-k}{n}}}, 1-\left(\prod\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}\varphi^*_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)^{\left(1-\frac{k}{n}\right)}\right)^{\frac{1}{\binom{n}{k}\frac{n-k}{n}}}, \\ 1-\left(\prod\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}\vartheta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)^{\left(1-\frac{k}{n}\right)}\right)^{\frac{1}{\binom{n}{k}\frac{n-k}{n}}}, \left(\prod\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}\left(1-\vartheta^*_{\mathcal{A}_{i_j}}\right)\right)^{\frac{1}{k}}\right)^{1-\frac{k}{n}}\right)^{\frac{1}{\binom{n}{k}\frac{n-k}{n}}} \end{array} \right)$$

$$= \left(\begin{array}{l} \frac{\sum\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}} \left(\prod\limits_{j=1}^{k}\zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}} \left(\prod\limits_{j=1}^{k}\varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k}}, \left(\prod\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}\left(1-\varphi_{\mathcal{A}_{i_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, \\ 1-\left(\prod\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}\varphi^*_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, 1-\left(\prod\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}\vartheta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, \\ \left(\prod\limits_{\substack{1\le i_1< \\ \cdots<i_k\le n}}\left(1-\left(\prod\limits_{j=1}^{k}\left(1-\vartheta^*_{\mathcal{A}_{i_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}} \end{array} \right)$$

$$= \mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n)$$

On the other hand, for $k = n$, we have

$$\begin{aligned} & \mathrm{WSTIT2IFHM}_w^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n) \\ = & \left(\begin{array}{l} \prod\limits_{j=1}^{k}\zeta_{\mathcal{A}_j}^{\frac{1-\frac{1}{n}}{n-1}}, \prod\limits_{j=1}^{k}\varrho_{\mathcal{A}_j}^{\frac{1-\frac{1}{n}}{n-1}}, 1-\prod\limits_{j=1}^{k}\left(1-\varphi_{\mathcal{A}_j}\right)^{\frac{1-\frac{1}{n}}{n-1}}, \\ \prod\limits_{j=1}^{k}\left(\varphi^*_{\mathcal{A}_j}\right)^{\frac{1-\frac{1}{n}}{n-1}}, \prod\limits_{j=1}^{k}\left(\vartheta_{\mathcal{A}_j}\right)^{\frac{1-\frac{1}{n}}{n-1}}, 1-\prod\limits_{j=1}^{k}\left(1-\vartheta^*_{\mathcal{A}_j}\right)^{\frac{1-\frac{1}{n}}{n-1}} \end{array} \right) \\ = & \left(\begin{array}{l} \prod\limits_{j=1}^{k}\zeta_{\mathcal{A}_j}^{\frac{1}{n}}, \prod\limits_{j=1}^{k}\varrho_{\mathcal{A}_j}^{\frac{1}{n}}, 1-\prod\limits_{j=1}^{k}(1-\varphi_{\mathcal{A}_j})^{\frac{1}{n}}, \\ \prod\limits_{j=1}^{k}(\varphi^*_{\mathcal{A}_j})^{\frac{1}{n}}, \prod\limits_{j=1}^{k}(\vartheta_{\mathcal{A}_j})^{\frac{1}{n}}, 1-\prod\limits_{j=1}^{k}(1-\vartheta^*_{\mathcal{A}_j})^{\frac{1}{n}} \end{array} \right) \\ = & \mathrm{STIT2IFHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n) \end{aligned}$$

$\square$

6.4 An Approach to MCDM based on the proposed operator

In this section, an MCDM approach is developed under the triangular IT2IF (TIT2IF) environment. The description of the problem, as well as the procedure steps, are explained as below.

6.4.1 Proposed approach

Assume an MCDM problem which consists of 'n' different alternatives $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and a set of 'm' attributes $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ whose weight vector is $w = (w_1, w_2, \cdots, w_m)^T$, satisfying $w_j > 0$ and $\sum_{j=1}^{m} w_j = 1$. An expert has evaluated these given alternatives and rate them under TIT2IF environment denoted by $l_{pj}(p = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ where l_{pj} represent the linguistic information about the alternatives. Furthermore, the importance of attributes plays a dominant role during the decision-making process. During handling the MCDM problems, if the sum of the relative coefficient w.r.t. each criterion is small, it relates that such criteria demonstrate a major impact on the overall values of the alternative. Similarly, if the relative coefficient sum is large then it shows such criterion plays a less significant role. Hence, the relative coefficient of the alternative under the certain criteria is inversely proportional to the corresponding weights of criteria. Therefore, the weight of the criteria is determined by using the Spearman method [135] which main steps are summarized in Algorithm 1.

By using this weight vector, we summarized the following steps based on the proposed AO to rank the alternatives under STIT2IFS environment.

Step 1: Arrange the information of each alternative in decision matrix $\overline{L}$ as

$$\overline{L} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{G}_1 & \mathcal{G}_2 & \ldots & \mathcal{G}_n \end{array} \\ \begin{pmatrix} \bar{l}_{11} & \bar{l}_{12} & \ldots & \bar{l}_{1n} \\ \bar{l}_{21} & \bar{l}_{22} & \ldots & \bar{l}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{l}_{m1} & \bar{l}_{m2} & \ldots & \bar{l}_{mn} \end{pmatrix} \end{array} \tag{6.16}$$

Algorithm 1 Weight determination using Spearman coefficient Method

1: Take two criteria $\mathcal{G}_k$ and $\mathcal{G}_j$ and then compute their relative coefficients as

$$\Delta_{kj} = 1 - \frac{6\sum_{p=1}^{n}(l_{pk} - l_{pj})^2}{m(m-1)} \tag{6.12}$$

and hence construct the matrix $\Delta_{m\times m} = (\Delta_{kj})_{m\times m}$ as

$$\Delta_{m\times m} = \begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1m} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2m} \\ \dots & \dots & \ddots & \dots \\ \Delta_{m1} & \Delta_{m2} & \cdots & \Delta_{mm} \end{pmatrix} \tag{6.13}$$

2: Compute the relative coefficient sum of each criteria by using Eq. (6.14).

$$\Delta_j = \sum_{\substack{k=1 \\ k\neq j}}^{m} \Delta_{jk} \tag{6.14}$$

3: Compute the weight of each criteria as

$$w_j = \frac{\sigma_j}{\sum_{j=1}^{m}\sigma_j} \tag{6.15}$$

where $\sigma_j = \dfrac{1}{\Delta_j}$ represent the contribution index of the criteria.

---

where $\bar{l}_{pj} = \left(\bar{\zeta}_{pj}, \bar{\varrho}_{pj}, \overline{\varphi}_{pj}, \overline{\varphi}^*_{pj}, \overline{\vartheta}_{pj}, \overline{\vartheta}^*_{pj}\right)$ be the STIT2IFNs provided by an expert.

Step 2: Compute the normalized decision matrix L from $\overline{L}$ by using the normalized formula

$$l_{pj} = \begin{cases} \left(\bar{\zeta}_{pj}, \bar{\varrho}_{pj}, \overline{\varphi}_{pj}, \overline{\varphi}^*_{pj}, \overline{\vartheta}_{pj}, \overline{\vartheta}^*_{pj}\right) & ;\text{for the benefit type criteria} \\ \left(\bar{\zeta}_{pj}, \bar{\varrho}_{pj}, \overline{\vartheta}_{pj}, \overline{\vartheta}^*_{pj}, \overline{\varphi}_{pj}, \overline{\varphi}^*_{pj}\right) & ;\text{for the cost type criteria} \end{cases} \tag{6.17}$$

Step 3: Compute the weight vector to each criteria by using Algorithm 1.

Step 4: Combine the different values of STIT2IFNs $l_{pj}(j = 1, 2, \dots, m)$ into the single one l_p of each alternative $\mathcal{A}_p(p = 1, 2, \dots, n)$ by using WSTIT2IFHM operator as follows:

$$l_p = \text{WSTIT2IFHM}_w^{(k)}(l_{p1}, l_{p2}, \cdots, l_{pn})$$

$$
= \left(\begin{array}{l}
\dfrac{\sum\limits_{\substack{1 \le p_1 < \\ \ldots < p_k \le n}} \left(1 - \sum\limits_{j=1}^{k} w_{p_j}\right) \left(\prod\limits_{j=1}^{k} \zeta_{p_j}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \dfrac{\sum\limits_{\substack{1 \le p_1 < \\ \ldots < p_k \le n}} \left(1 - \sum\limits_{j=1}^{k} w_{p_j}\right) \left(\prod\limits_{j=1}^{k} \varrho_{p_j}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \\
\left(\prod\limits_{\substack{1 \le p_1 < \\ \ldots < p_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} (1 - \varphi_{p_j})\right)^{\frac{1}{k}}\right)^{\left(1 - \sum\limits_{j=1}^{k} w_{p_j}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}, 1 - \left(\prod\limits_{\substack{1 \le p_1 < \\ \ldots < p_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \varphi^*_{p_j}\right)^{\frac{1}{k}}\right)^{\left(1 - \sum\limits_{j=1}^{k} w_{p_j}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}, \\
1 - \left(\prod\limits_{\substack{1 \le p_1 < \\ \ldots < p_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \vartheta_{p_j}\right)^{\frac{1}{k}}\right)^{\left(1 - \sum\limits_{j=1}^{k} w_{p_j}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}, \left(\prod\limits_{\substack{1 \le p_1 < \\ \ldots < p_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \vartheta^*_{p_j}\right)\right)^{\frac{1}{k}}\right)^{\left(1 - \sum\limits_{j=1}^{k} w_{p_j}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}
\end{array} \right)
$$

Step 5: Compute the score value of the l_p by using Eq. (6.2).

Step 6: Rank all the alternatives by using an order relation defined in Definition 6.2.4 and hence select the most feasible alternative(s).

6.4.2 A Case Study

Jharkhand is the eastern state of the India, which has the 40 percent mineral resources of the country and second leading state of the mineral wealth after Chhattisgarh state. It is also known for its vast forest resources. Jamshedpur, Bokaro and Dhanbad cities of the Jharkhand are famous for industries in all over the world. After that, it is the widespread poverty state of the India because it is the primarily a rural state as 76 percent of the population live in the villages which depend on the agriculture and wages. Only 30 percent villages are connected by roads while only 55 percent villages have accessed to electricity and other facilities. But in the today's life, everyone is changing fast to himself for a better life, therefore, everyone moves to the urban cities for a better job. To stop this emigration, Jharkhand government wants to set up the industries based on the agriculture in the rural areas. For this, the government has been organized MOMENTUM JHARKHAND global investor submit 2017 in Ranchi to invite the companies for investment in the rural areas. Government announced the various facilities for setup the five food processing plants in the rural areas and consider the six attributes required for company selection to setup them, namely, project cost ($\mathcal{G}_1$), completion time ($\mathcal{G}_2$), technical capability ($\mathcal{G}_3$), financial status ($\mathcal{G}_4$), company background ($\mathcal{G}_5$), reference from previous project ($\mathcal{G}_6$) and assign the weights of relative importance of each attributes. The six companies taken as in the form of the alternatives, namely, Surya Food and Agro Pvt. Ltd. ($\mathcal{A}_1$), Mother Dairy Fruit and Vegetable Pvt. Ltd. ($\mathcal{A}_2$), Parle Products Ltd. ($\mathcal{A}_3$), Heritage Food Ltd. ($\mathcal{A}_4$), Verka Pvt. Ltd. ($\mathcal{A}_5$) and Reliance Pvt. Ltd. ($\mathcal{A}_6$) interested for these projects. Then the main object of the government is to choose the best company among them for the task. In order to find the best feasible alternative(s) for the required task, the authority called an expert to evaluate these alternatives and rate their preferences in terms of linguistic terms (LTs). The standardized LTs such as "Very High" (VH), "High"(H), "Medium"(M), "Medium Low"(ML), "Low"(L), "Very Low"(VL) are defined in terms of STIT2IFNs given in Table 6.1.

Table 6.1: Linguistic grade and corresponding values of Symmetric TIT2IFNs

LTs	Symmetric Triangular IT2IFNs
VL	(0.20,0.10,0.60,0.65,0.35,0.30)
L	(0.30,0.10,0.65,0.70,0.30,0.25)
ML	(0.40,0.20,0.70,0.75,0.20,0.18)
M	(0.50,0.20,0.75,0.80,0.16,0.15)
MH	(0.60,0.30,0.80,0.85,0.13,0.12)
H	(0.70,0.30,0.85,0.90,0.10,0.08)
VH	(0.80,0.40,0.90,0.95,0.07,0.03)

Furthermore, the complementary relation corresponding to LTs is presented in Table 6.2.

Table 6.2: Linguistic grades and compliments.

LT	VL	L	ML	M	MH	H	VH
Complemented LT	VH	H	MH	M	ML	L	VL

The above mentioned steps are executed to locate the best alternative(s).

Step 1: An expert has evaluated each alternative and present their rating values in terms of LTs which are summarized as

$$\overline{L} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \\ \mathcal{A}_4 \\ \mathcal{A}_5 \\ \mathcal{A}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 & \mathcal{G}_5 & \mathcal{G}_6 & \mathcal{G}_7 \end{array} \\ \begin{pmatrix} \text{VH} & \text{H} & \text{M} & \text{MH} & \text{H} & \text{VH} & \text{H} \\ \text{M} & \text{ML} & \text{H} & \text{VH} & \text{H} & \text{VH} & \text{VH} \\ \text{H} & \text{VH} & \text{VH} & \text{M} & \text{MH} & \text{L} & \text{VL} \\ \text{MH} & \text{VL} & \text{MH} & \text{H} & \text{VL} & \text{MH} & \text{H} \\ \text{VH} & \text{H} & \text{VL} & \text{H} & \text{M} & \text{VL} & \text{L} \\ \text{ML} & \text{VL} & \text{VH} & \text{M} & \text{VL} & \text{L} & \text{H} \end{pmatrix} \end{array} \tag{6.18}$$

Step 2: As the criteria $\mathcal{G}_1$ and $\mathcal{G}_2$ are the cost type, so we normalize their rating values by using Table 6.2 and Eq. (6.17), we get

$$L = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \\ \mathcal{A}_4 \\ \mathcal{A}_5 \\ \mathcal{A}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 & \mathcal{G}_5 & \mathcal{G}_6 & \mathcal{G}_7 \end{array} \\ \begin{pmatrix} \text{VL} & \text{L} & \text{M} & \text{MH} & \text{H} & \text{VH} & \text{H} \\ \text{M} & \text{MH} & \text{H} & \text{VH} & \text{H} & \text{VH} & \text{VH} \\ \text{L} & \text{VL} & \text{VH} & \text{M} & \text{MH} & \text{L} & \text{VL} \\ \text{ML} & \text{VH} & \text{MH} & \text{H} & \text{VL} & \text{MH} & \text{H} \\ \text{VL} & \text{L} & \text{VL} & \text{H} & \text{M} & \text{VL} & \text{L} \\ \text{MH} & \text{VH} & \text{VH} & \text{M} & \text{VL} & \text{L} & \text{H} \end{pmatrix} \end{array} \tag{6.19}$$

Step 3: Apply the Algorithm 1 to compute the weight vector to each criteria. For it, we follows the steps of the algorithm and summarized as below

1. By using Eq. (6.12), construct the relative coefficient matrix Δ for each criteria as

$$\Delta = \begin{array}{c} \\ \mathcal{G}_1 \\ \mathcal{G}_2 \\ \mathcal{G}_3 \\ \mathcal{G}_4 \\ \mathcal{G}_5 \\ \mathcal{G}_6 \\ \mathcal{G}_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 & \mathcal{G}_5 & \mathcal{G}_6 & \mathcal{G}_7 \end{array} \\ \begin{pmatrix} 1 & 0.9666 & 0.9344 & 0.9094 & 0.9044 & 0.9174 & 0.9344 \\ 0.9666 & 1 & 0.9344 & 0.9311 & 0.8444 & 0.9144 & 0.9694 \\ 0.9344 & 0.9344 & 1 & 0.9344 & 0.9014 & 0.9144 & 0.9374 \\ 0.9094 & 0.9314 & 0.9344 & 1 & 0.9414 & 0.9464 & 0.9574 \\ 0.9044 & 0.8444 & 0.9014 & 0.9414 & 1 & 0.9504 & 0.9004 \\ 0.9174 & 0.9144 & 0.9144 & 0.9464 & 0.9504 & 1 & 0.9714 \\ 0.9344 & 0.9694 & 0.9374 & 0.9574 & 0.9004 & 0.9714 & 1 \end{pmatrix} \end{array}$$

2. The relative coefficient sum of each criteria is computed by using Eq. (6.14) and get

$$\Delta_1 = 5.564,\ \Delta_2 = 5.558,\ \Delta_3 = 5.554,\ \Delta_4 = 5.618,$$
$$\Delta_5 = 5.440,\ \Delta_6 = 5.612,\ \Delta_7 = 5.668.$$

3. By using Eq. (6.15), the weight vector of each criteria is obtained as

$$w_1 = 0.1431,\ w_2 = 0.1432,\ w_3 = 0.1433,\ w_4 = 0.1417,$$
$$w_5 = 0.1463,\ w_6 = 0.1419,\ w_7 = 0.1405.$$

Step 4: Aggregate all the values by using WSTIT2IFHM operator into a collective one $l_p (p = 1, 2, \ldots, 6)$. Here, without loss of generality, we take $k = 2$ and the obtained results are

$$\begin{aligned}
l_1 &= \mathrm{WSTIT2IFHM}_w^{(2)}(l_{11}, l_{12}, \cdots, l_{17}) \\
&= \left(\begin{array}{l}
\dfrac{\sum\limits_{1 \le p_1 < p_2 \le 7} \left(1 - \sum\limits_{j=1}^{2} w_{p_j}\right) \left(\prod\limits_{j=1}^{2} \zeta_{p_j}\right)^{\frac{1}{2}}}{\binom{6}{2}}, \dfrac{\sum\limits_{1 \le p_1 < p_2 \le 7} \left(1 - \sum\limits_{j=1}^{2} w_{p_j}\right) \left(\prod\limits_{j=1}^{2} \varrho_{p_j}\right)^{\frac{1}{2}}}{\binom{6}{2}}, \\
\left(\prod\limits_{1 \le p_1 < p_2 \le 7} \left(1 - \left(\prod\limits_{j=1}^{2} (1 - \varphi_{p_j})\right)^{\frac{1}{2}}\right)^{\left(1 - \sum\limits_{j=1}^{2} w_{p_j}\right)} \right)^{\frac{1}{\binom{6}{2}}}, 1 - \left(\prod\limits_{1 \le p_1 < p_2 \le 7} \left(1 - \left(\prod\limits_{j=1}^{2} \varphi^*_{p_j}\right)^{\frac{1}{2}}\right)^{\left(1 - \sum\limits_{j=1}^{2} w_{p_j}\right)} \right)^{\frac{1}{\binom{6}{2}}}, \\
1 - \left(\prod\limits_{1 \le p_1 < p_2 \le 7} \left(1 - \left(\prod\limits_{j=1}^{2} \vartheta_{p_j}\right)^{\frac{1}{2}}\right)^{\left(1 - \sum\limits_{j=1}^{2} w_{p_j}\right)} \right)^{\frac{1}{\binom{6}{2}}}, \left(\prod\limits_{1 \le p_1 < p_2 \le 7} \left(1 - \left(\prod\limits_{j=1}^{2} \left(1 - \vartheta^*_{p_j}\right)\right)^{\frac{1}{2}}\right)^{\left(1 - \sum\limits_{j=1}^{2} w_{p_j}\right)} \right)^{\frac{1}{\binom{6}{2}}}
\end{array} \right) \\
&= (0.5154, 0.2276, 0.7820, 0.8314, 0.1596, 0.1339)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
l_2 &= (0.6950, 0.3239, 0.8546, 0.9053, 0.0974, 0.0687); \\
l_3 &= (0.3846, 0.1681, 0.7166, 0.7633, 0.2243, 0.1927); \\
l_4 &= (0.5481, 0.2612, 0.7952, 0.8449, 0.1436, 0.1210); \\
l_5 &= (0.3201, 0.1342, 0.6769, 0.7244, 0.2642, 0.2292); \\
l_6 &= (0.5272, 0.2390, 0.7914, 0.8414, 0.1536, 0.1232)
\end{aligned}$$

Step 5: The score values of $l_p (p = 1, 2, \ldots, 6)$ are computed by Eq. (6.2) and get

$$\begin{aligned} s(l_1) &= (0.3404, 0.0375); s(l_2) = (0.5550, 0.0396); s(l_3) = (0.2046, 0.0392) \\ s(l_4) &= (0.3770, 0.0362); s(l_5) = (0.1455, 0.0412); s(l_6) = (0.3579, 0.0402) \end{aligned}$$

Step 6: Since $s_x(l_2) > s_x(l_4) > s_x(l_6) > s_x(l_1) > s_x(l_3) > s_x(l_5)$ and thus by Definition 6.2.4, we get the ranking order of the alternatives as $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_6 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_5$. Here "$\succ$" means "preferred to". Therefore, $\mathcal{A}_2$ is the best alternative.

6.4.3 Influence of the parameter k on Alternatives

Keeping in mind the end goal to investigate the impact of the parameter k on to the final positioning order of the alternatives, we use an alternate estimation of k in our test. Here n is 7 in our case, so we shift k from 1 to 7 and their outcomes relating to the proposed technique have been outlined in Table 6.3. From this table, it is seen that with the expansion of the interaction of the multi-input options, the general score estimations of it diminishes which recommend that the proposed operator reflect the risk preferences to the decision makers. This examination will propose distinctive decisions to the analyst as indicated by his/her decision. For example, in the event that he will cover the risk parameters during the aggregation then they will allocate a little incentive to the parameter k with the goal that score esteems increments while, if the analyst is pessimistic in nature towards the choice then the bigger estimation of k can be allocated during the procedure.

Table 6.3: Effect of the parameter k on to ranking of alternatives.

Value of	Score Values (s_x, s_y) of the Alternatives						Ranking
k	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	Order
1	(0.3615, 0.0762)	(0.5627, 0.0523)	(0.2268, 0.0872)	(0.3953, 0.0677)	(0.1577, 0.0702)	(0.3836, 0.0856)	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_6 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_5$
2	(0.3404, 0.0375)	(0.5550, 0.0396)	(0.2046, 0.0392)	(0.3770, 0.0362)	(0.1455, 0.0412)	(0.3579, 0.0402)	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_6 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_5$
3	(0.3324, 0.0241)	(0.5526, 0.0840)	(0.1997, 0.0250)	(0.3702, 0.0268)	(0.1427, 0.0321)	(0.3484, 0.0240)	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_6 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_5$
4	(0.3285, 0.0177)	(0.5507, 0.0329)	(0.1976, 0.0181)	(0.3656, 0.0203)	(0.1415, 0.0275)	(0.3437, 0.0161)	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_6 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_5$
5	(0.3260, 0.0138)	(0.5498, 0.0314)	(0.1964, 0.0141)	(0.3631, 0.0170)	(0.1409, 0.0247)	(0.3408, 0.0115)	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_6 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_5$
6	(0.3244, 0.0113)	(0.5492, 0.0304)	(0.1957, 0.0114)	(0.3613, 0.0148)	(0.1405, 0.0228)	(0.3389, 0.0086)	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_6 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_5$
7	(0.3232, 0.0095)	(0.5488, 0.0298)	(0.1952, 0.0094)	(0.3601, 0.0131)	(0.1402, 0.0215)	(0.3376, 0.0064)	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_6 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_5$

Furthermore, in some other existing Bonferroni mean (BM) and generalized Bonferroni mean (GBM) operators, the information takes only two or three arguments during an aggregation. Also, in BM operator there is need of two additional parameters (p, q) while the three parameters (p, q, r) for GBM from an infinite rational set. Thus, the computational complexity is too high in such cases. On the other hand, in the proposed operator, there is only one parameter k from a finite integer set and hence the computational complexity is low and easier to understand. Finally, the several operators such as averaging, BM and geometric for the T2IFNs can be deduced from the proposed ones by setting $k = 1$, $k = 2$ and $k = n$ respectively. Subsequently, our proposed operator and the strategy are more summed up and adaptable to tackle the decision-making problems.

6.4.4 Comparative Study

In this section, we perform some comparative analysis of the proposed method result with some of the existing approaches result in [62, 95, 116, 118] under the uncertain environment. The results computed from them on to the considered problem are summarized as below:

1) In Gong et al. [62], authors proposed the weighted geometric Bonferroni mean operator under the type-2 fuzzy environment, denoted by IT2FWGBM, which is defined as

$$\begin{aligned} d_k &= \text{IT2FWGBM}_w^{p,q}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m) \\ &= \frac{1}{p+q}\left(\bigotimes_{\substack{i,j=1 \\ i\neq j}}^{m} (p(\mathcal{A}_i)^{w_i} \oplus q(\mathcal{A}_j)^{w_j})\right)^{1/m(m-1)} \end{aligned} \tag{6.20}$$

By applying Eq. (6.20) on to the considered data, we get the aggregated value corresponding to each alternative as

$$\begin{aligned} d_1 &= \text{IT2FWGBM}_w^{1,1}(\mathcal{A}_{11}, \mathcal{A}_{12}, \mathcal{A}_{13}, \mathcal{A}_{14}, \mathcal{A}_{15}, \mathcal{A}_{16}, \mathcal{A}_{17}) \\ &= (0.8321, 0.9050, 0.9050, 0.9534, 0.6065) \\ d_2 &= \text{IT2FWGBM}_w^{1,1}(\mathcal{A}_{21}, \mathcal{A}_{22}, \mathcal{A}_{23}, \mathcal{A}_{24}, \mathcal{A}_{25}, \mathcal{A}_{26}, \mathcal{A}_{27}) \\ &= (0.8671, 0.9486, 0.9486, 1.0000, 0.7500) \\ d_3 &= \text{IT2FWGBM}_w^{1,1}(\mathcal{A}_{31}, \mathcal{A}_{32}, \mathcal{A}_{33}, \mathcal{A}_{34}, \mathcal{A}_{35}, \mathcal{A}_{36}, \mathcal{A}_{37}) \\ &= (0.7980, 0.8676, 0.8676, 0.9137, 0.6015) \end{aligned}$$

$$\begin{aligned} d_4 &= \text{IT2FWGBM}_w^{1,1}(\mathcal{A}_{41}, \mathcal{A}_{42}, \mathcal{A}_{43}, \mathcal{A}_{44}, \mathcal{A}_{45}, \mathcal{A}_{46}, \mathcal{A}_{47}) \\ &= (0.8317, 0.9131, 0.9131, 0.9656, 0.6080) \\ d_5 &= \text{IT2FWGBM}_w^{1,1}(\mathcal{A}_{51}, \mathcal{A}_{52}, \mathcal{A}_{53}, \mathcal{A}_{54}, \mathcal{A}_{55}, \mathcal{A}_{56}, \mathcal{A}_{57}) \\ &= (0.7802, 0.8456, 0.8456, 0.8895, 0.6085) \\ d_6 &= \text{IT2FWGBM}_w^{1,1}(\mathcal{A}_{61}, \mathcal{A}_{62}, \mathcal{A}_{63}, \mathcal{A}_{64}, \mathcal{A}_{65}, \mathcal{A}_{66}, \mathcal{A}_{67}) \\ &= (0.8318, 0.9073, 0.9073, 0.9569, 0.6000) \end{aligned}$$

Therefore, the score values of these aggregated numbers are $s(d_1) = 0.5405$, $s(d_2) = 0.7079$, $s(d_3) = 0.5182$, $s(l_4) = 0.5450$, $s(l_5) = 0.5052$, and $s(l_6) = 0.5418$ and hence the final ranking of all alternatives $\mathcal{A}_k (k = 1, 2, \ldots, 6)$ is found as $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_6 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_5$.

2) If we use the existing WSTIT2FHM operator as proposed by Qin [118] under the T2FS environment

$$\begin{aligned} l_p &= \text{WSTIT2FHM}^{(k)}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ &= \left(\begin{array}{c} \dfrac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right) \left(\prod\limits_{j=1}^{k} \zeta_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \dfrac{\sum\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right) \left(\prod\limits_{j=1}^{k} \varrho_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \\ \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \left(1 - \varphi_{\mathcal{A}_{i_j}}\right)\right)^{\frac{1}{k}}\right)^{\left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right)} \right)^{\frac{1}{\binom{n-1}{k}}}, \\ 1 - \left(\prod\limits_{\substack{1 \le i_1 < \\ \ldots < i_k \le n}} \left(1 - \left(\prod\limits_{j=1}^{k} \varphi^*_{\mathcal{A}_{i_j}}\right)^{\frac{1}{k}}\right)^{\left(1 - \sum\limits_{j=1}^{k} w_{i_j}\right)} \right)^{\frac{1}{\binom{n-1}{k}}} \end{array} \right), \end{aligned} \tag{6.21}$$

then, the aggregated values corresponding to each alternative (by taking $k = 2$) are obtained as $l_1 = (0.5154,\ 0.2276,\ 0.7820,\ 0.8314)$, $l_2 = (0.6950,\ 0.3239,\ 0.8546,\ 0.9054)$, $l_3 = (0.3846,\ 0.1681,\ 0.7166,\ 0.7633)$, $l_4 = (0.5481,\ 0.2612,\ 0.7951,\ 0.8449)$, $l_5 = (0.3201,\ 0.1342,\ 0.6769,\ 0.7244)$, and $l_6 = (0.5272,\ 0.2390,\ 0.7914,\ 0.8414)$. Thus, the score values are

$$\begin{aligned} s(l_1) &= (0.2077, 0.8067); s(l_2) = (0.3055, 0.8799); s(l_3) = (0.1422, 0.7400) \\ s(l_4) &= (0.2245, 0.8200); s(l_5) = (0.1120, 0.7006); s(l_6) = (0.2150, 0.8164) \end{aligned}$$

and hence ordering is $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_6 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_5$

From the above examinations, it is revealed that the ranking order of the alternatives stays same yet the computational procedure is altogether unique. For instance, in [62, 118] authors have introduced AOs under TIT2FNs by considering just the degree of membership during an examination. But it is quite recognizable that the level of non-membership likewise assumes a predominant part during the aggregation process. Thus, the outcomes processes by these methodologies [62, 118] might be unreasonable under some specific constraints where the degree of non-membership pays more significance than the degree of agreement.

However, apart from these, we give some characteristics comparison of our proposed method and the aforementioned methods, which are listed in Table 6.4.

Table 6.4: The characteristic comparisons of different methods.

Methods	Whether Captures Interrelationship of Two Aggregated Arguments	Whether Captures Interrelationship of Multiple Aggregated Arguments	Whether It Makes the Method Flexible by the Parameter Vector	Whether Criteria Weights Are Depends on the Collective Information	Whether Describe Information Using Linguistic Features	Whether Flexible to Express a Wider Range of Information
Gong et al. [62]	✓	×	×	×	×	×
Liu and Wang [95]	×	×	×	×	×	×
Pedrycz and Song [116]	×	×	×	✓	✓	×
Qin [118]	✓	✓	✓	×	✓	×
The proposed method	✓	✓	✓	✓	✓	✓

In [95], authors presented an analytical method for solving the problems by using the fuzzy weighted average. In [62], the authors have presented the BM by considering simultaneously the values of UMF and LMF to aggregate IT2FS information. On the other hand, the present study is based on the HM operator which is more adaptable and robustness in the process of information fusion than others such as BM, GBM. The outstanding characteristic of the HM operator is to catch the inter-relationship between more than two input arguments with a parameter k from the finite integer set. Furthermore, in [118], the author developed HM operator by taking into account the membership degree only but in practical problems, it is sometimes not possible for DM to give their preferences in terms of acceptance degree only. Therefore, the non-membership degree is required for handling the problems in which rejection degree is not equal to one minus acceptance degree. Also by comparing with the AHP-based method [116], the proposed method does not require any software package to compute the results while the technique proposed in [116] requires it. Thus, the computation complexity of the proposed technique is comparatively easy. Furthermore, the AHP-based technique is usually dependent on various parameters and thus the final ranking may some time suffers from inconsistency, in the case of inappropriate parameter selection. On the other hand, the proposed method draws up a more authentic ranking result as it can terminate the difference, draws up for the flaws of already existing aggregation methods that do not capture experts utility or decision preference and achieves more stationary and commendable interrelationships result with less information loss. The proposed method takes into consideration the uniformity of the alternatives as well as highlights the significance and interactions in association with any solutions to alternatives. On the other hand, the AHP-based technique is good at calculating only the optimal ranking values of the alternatives beyond inter-relationships.

6.5 Conclusions

In this chapter, an endeavor has been made to exhibit some new AOs to accommodate the IT2IF conditions. IT2IFS is one of the augmentations of the conventional FS, IFS by considering grades of the PMFs also. On the other hand, in practical application problems, the criteria interrelationship phenomenon occurs frequently. To address it, Hamy means (HM) operator is a standout among the most critical operators that catches the inter-relationship together with the multi-input arguments. Furthermore, to diminish the computational complexity of the IT2IFS, we introduce symmetric IT2IFS and characterize some operation laws. Then, keeping the advantages of STIT2IFS and HM operators, we exhibit the STIT2IFHM and WSTIT2IFHM operators under a provision of a type-2 intuitionistic uncertain situation. Various beneficial characteristics of these operators have endorsed. Furthermore, in light of these operators, a decision-making approach is introduced to solve the MCDM problems. The presented approach has been tried and clarified with a numerical illustration and registered that it can efficiently deal with the available information by eliminating more amount of fuzziness as compared to the existing approaches. The major advantages of the proposed operator with respect to the existing ones are that it need only one parameter k from a finite integer set while other needs more than one from an infinite rational set such as BM and GBM etc., and hence the computational complexity is low and easier to understand. Additionally, a portion of the existing studies can be effectively concluded from the proposed operators by setting $k = 1$, $k = 2$ and $k = n$. Thus, it expresses a better technique for taking care of decision-making problems with additional benefits.

Chapter 7

Summary and Future Scope

The chapter presents a comprehensive summary of the research contributions made during the period of this thesis. It also outlines the managerial implications for the implementation of recommendations. Finally the scope for future work has been outlined.

7.1 Summary of the work

In this thesis, we discuss in detail the concept of fuzzy, intuitionistic fuzzy sets and type-2 intuitionistic fuzzy, which mathematically represent vague phenomena. In the Chapters 3 and 4, we have extensively searched and have introduced a number of new information theoretic measures associated with vagueness. In Chapters 5 and 6, we have extended the area of aggregation operators once again on type-2 fuzzy and type-2 intuitionistic fuzzy sets. The conclusion made from the work presented in this thesis are summarized below:

1) The research work presented in this thesis is an attempt to give an alternative approach for addressing the decision-making problems under the type-2 intuitionistic fuzzy set environment. Here, in the literature, there are two well-known forms as type-1 fuzzy sets (T1FS) and intuitionistic fuzzy sets. One is based on the consideration of the membership functions and another one uses the membership and non-membership functions such that their sum at each element is less than or equal to one. In these study, researchers have evaluated the objects in terms of a crisp membership function. Apart from these, in many practical situations, uncertainty is not probabilities in nature but it is imprecise or vague. To address this, the concept of T2FS, an extension of T1FS,

in which membership values are type 1 FSs on [0,1] is developed. In T2FS, there is an additional membership function which provides an additional degree of freedom to the practices to model the uncertainties.

2) In the literature, authors have developed the several measures and their respective theories by considering only the membership degrees during the analysis. However, the other component such as non-membership degrees is treated as a complement to them. To address this issue and to incorporate more degrees of freedom to the decision makers during the analysis, in this thesis, a concept of T2FS has been extended to type 2 IFS (T2IFS) in which each element of the object has been characterized with a pair of the primary as well as secondary non-membership degrees also along with the primary and secondary membership degrees. From the study, we can conclude that the several existing sets such as T1FS, IFSs, T2FSs can be considered as a special cases of the T2IFS. Thus, the applicability range of the T2IFS is much wider than the existing sets.

3) The limitation and shortcomings of the existing methods are discussed in earlier chapters. In the real-life situation, it is not possible to make decision without considering the degree of non-membership, as it is difficult for the person to their preferences towards an object in terms of single or exact value. All existing aggregation operators are usually based on the algebraic norm operations, which have lack of flexibility and robustness. BM and GBM based operators considered only two or three parameters simultaneously. Therefore, these operators are incapable of analyzing the effect of multi-input arguments in to one analysis.

4) The new measures proposed, briefly mentioned below, have enriched the study in wider way and have provided tools and methods for multiple criteria decision making problems, which are greatly needed in the contemporary society looking for quantitative as well as qualitative and scientific temper.

    a) Developed Hamming, Euclidean and utmost distance measures for type-2 intuitionistic fuzzy environment by using linguistic variables.

b) Developed series of similarity measures for type-2 intuitionistic fuzzy environment by using linguistic variables.

c) Presented a TOPSIS method based on similarity measure to access finest alternative.

The proposed measures have several elegant properties which enhance the employability of these measures. Hence, the presented measure are one of the generalizations of existing ones and offers an effective way to handle MAGDM in T2IFSs information followed with illustrated examples.

5) Another allied theme, addressed in Chapters 5 and 6, is that of aggregation which is a lately emerged area and compassing the ever usefully and widely employed concept of averages. Aggregation employs not only the values but also their various types of interaction in arriving at a representative value. In Chapters 5 and 6, we have proposed some new weighted aggregation operators for aggregating the triangular interval type-2 intuitionistic fuzzy (TIT2IF) information and symmetric triangular interval type-2 intuitionistic fuzzy information:

a) TIT2IF averaging operator.

b) TIT2IF ordered weighted averaging (TIT2IFOWA) operator.

c) TIT2IF hybrid averaging (TIT2IFHA) operator.

d) Symmetric TIT2IF Hamy mean (STIT2IFHM) operator.

e) Weighted symmetric TIT2IF Hamy mean (WSTIT2IFHM) operator.

A distinguishing characteristic of these operators is that they take into account aggregation among the aggregated arguments. These operators satisfy a number of interesting properties that outlines a wide ground for applications in different areas. To enhance its practicability, real life examples have also been included. Moreover, sensitivity analysis has been performed in order to show the influence of the decision parameters on to the ranking of the alternatives. In addition, the proposed results in correspondence with the different values of λ provides a number of choices to the decision maker for evaluating the decision.

7.2 Future scope of the work

There are several new directions which can be looked forward in the future with the work studied in the thesis. The future scope of the study can be suggested as follows:

1) The presented methodology will be further extended to prioritized average and geometric aggregation operators with type-2 intuitionistic fuzzy linguistic information.

2) The different preferences of the expert can be aggregated by using generalized power aggregation operator based on t-norm and co-norm in terms of type-2 intuitionistic fuzzy information.

3) The presented methodology will be further extended and improved by using some more generalized information measures such as correlation coefficients, divergence measures and belief functions with type-2 intuitionistic fuzzy environment.

4) The study based on the proposed algorithm may be extended for the applications part in different real life problems, including medical diagnosis, pattern recognition, human resource management, location selection etc.

5) In our study, we have taken the single data without any hesitancy. In the future, we may try to extend for the information under the hesitant fuzzy set environment and hence develop their corresponding MAGDM algorithm.

6) The study, based on these arithmetic operations may be extended for the applications part in reliability optimization, resource allocation, facility planning and management, inventory control, network analysis and job shop scheduling.

Bibliography

[1] Abdullah, L. and Najib, L.: 2014, A new type-2 fuzzy set of linguistic variables for the fuzzy analytic hierarchy process, *Expert Systems with Applications* **41**(7), 3297–3305.

[2] Abdullah, L. and Otheman, A.: 2013, A new entropy weight for sub-criteria in interval type-2 fuzzy TOPSIS and its application, *International Journal of Intelligent Systems and Applications* **5**(2), 25.

[3] Abdullah, L. and Zulkifli, N.: 2018, A new DEMATEL method based on interval type-2 fuzzy sets for developing causal relationship of knowledge management criteria, *Neural Computing and Applications* pp. 1–17.
URL: *10.1007/s00521-017-3304-1*

[4] Akram, M., Ilyas, F. and Garg, H.: 2020, Multi-criteria group decision making based on ELECTRE I method in Pythagorean fuzzy information, *Soft Computing* **24**(5), 3425 – 3453.

[5] Ashraf, Z., Roy, M. L., Muhuri, P. K. and Lohani, Q. M. D.: 2018, A novel image steganography approach based on interval type-2 fuzzy similarity, *2018 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pp. 1 – 8.

[6] Atanassov, K. and Gargov, G.: 1989, Interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **31**, 343 – 349.

[7] Atanassov, K. T.: 1986, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20**(1), 87 – 96.

[8] Atanassov, K. T.: 1994, Operators over interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **64**, 159 – 174.

[9] Boran, F. E. and Akay, D.: 2014, A biparametric similarity measure on intuitionistic fuzzy sets with applications to pattern recognition, *Information Sciences* **255**, 45 – 57.

[10] Burillo, P. and Bustince, H.: 1996, Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets, *Fuzzy Sets and Systems* **78**(3), 305 – 316.

[11] Castillo, O., Melin, P., Tsvetkov, R. and Atanassov, K.: 2014, Short remark on interval type-2 fuzzy sets and intuitionistic fuzzy sets, *Notes on Intuitionistic Fuzzy Sets* **20**(2), 1 – 5.

[12] Çebi, F. and Otay, İ.: 2015, Multi-criteria and multi-stage facility location selection under interval type-2 fuzzy environment: a case study for a cement factory, *International Journal of Computational Intelligence Systems* **8**(2), 330–344.

[13] Chen, S. M., Cheng, S. H. and Lan, T. C.: 2016, A novel similarity measure between intuitionistic fuzzy sets based on the centroid points of transformed fuzzy numbers with applications to pattern recognition, *Information Sciences* **343 - 344**, 15– 40.

[14] Chen, S. M. and Lee, L. W.: 2010a, Fuzzy multiple attributes group decision-making based on the interval type-2 TOPSIS method, *Expert Systems with Applications* **37**(4), 2790–2798.

[15] Chen, S. M. and Lee, L. W.: 2010b, Fuzzy multiple attributes group decision-making based on the ranking values and the arithmetic operations of interval type - 2 fuzzy sets, *Expert Systems with Applications* **37**(1), 824 – 833.

[16] Chen, S. M. and Randyanto, Y.: 2013, A novel similarity measure between intuitionistic fuzzy sets and its applications, *International Journal of Pattern Recognition and Artificial Intelligence* **27**(7), 1350021[34pages].

[17] Chen, S. M., Yang, M. W., Lee, L. W. and Yang, S. W.: 2012, Fuzzy multiple attributes group decision-making based on ranking interval type-2 fuzzy sets, *Expert Systems with Applications* **39**(5), 5295–5308.

[18] Chen, S. M., Yang, M. W., Yang, S. W., Sheu, T. W. and Liau, C. J.: 2012, Multicriteria fuzzy decision making based on interval-valued intuitionistic fuzzy sets, *Expert Systems with Applications* **39**, 12085 – 12091.

[19] Chen, T. Y.: 2013a, An interactive method for multiple criteria group decision analysis based on interval type-2 fuzzy sets and its application to medical decision making, *Fuzzy Optimization and Decision Making* **12**(3), 323–356.

[20] Chen, T. Y.: 2013b, A linear assignment method for multiple-criteria decision analysis with interval type-2 fuzzy sets, *Applied Soft Computing* **13**(5), 2735–2748.

[21] Chen, T. Y.: 2013c, A signed-distance-based approach to importance assessment and multi-criteria group decision analysis based on interval type-2 fuzzy set, *Knowledge and Information Systems* **35**(1), 193–231.

[22] Chen, T. Y.: 2014, An ELECTRE-based outranking method for multiple criteria group decision making using interval type-2 fuzzy sets, *Information Sciences* **263**, 1 – 21.

[23] Chen, T. Y.: 2015a, An interval type-2 fuzzy LINMAP method with approximate ideal solutions for multiple criteria decision analysis, *Information Sciences* **297**, 50–79.

[24] Chen, T. Y.: 2015b, Likelihoods of interval type-2 trapezoidal fuzzy preference relations and their application to multiple criteria decision analysis, *Information Sciences* **295**, 303–322.

[25] Chen, T. Y., Chang, C. H. and Lu, J. F. R.: 2013, The extended QUALIFLEX method for multiple criteria decision analysis based on interval type-2 fuzzy sets and applications to medical decision making, *European Journal of Operational Research* **226**(3), 615–625.

[26] Chen, Z., Liu, P. and Pei, Z.: 2015, An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers, *Journal of Computational Intelligence Systems* **8**(4), 747 – 760.

[27] Cheng, S. H., Chen, S. M. and Huang, Z. C.: 2016, Autocratic decision making using group recommendations based on ranking interval type-2 fuzzy sets, *Information Sciences* **361**, 135–161.

[28] Chiao, K. P.: 2011, Multiple criteria group decision making with triangular interval type-2 fuzzy sets, *in Proceedings of the IEEE International Conference on Fuzzy Systems (FUZZ 11)*, pp. 2572 – 2582.

[29] Chiclana, F. and Zhou, S.-M.: 2013, Type-reduction of general type-2 fuzzy sets: the type-1 OWA approach, *International Journal of Intelligent Systems* **28**(5), 505–522.

[30] Das, P., Chatterjee, K. and Kar, S.: 2013, Supplier selection in risk environment, fuzzy interval type-2 TOPSIS method, *International Conference on Advances in Communication and Computing Technologies*, pp. 16 – 21.

[31] De, S. K., Biswas, R. and Roy, A. R.: 2000, Some operations on intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **117**, 477 – 484.

[32] Dengfeng, L. and Chuntian, C.: 2002, New similarity measure of intuitionistic fuzzy sets and application to pattern recognitions, *Pattern Recognition Letters* **23**, 221 – 225.

[33] Deschrijver, G. and Kerre, E.: 2002, A generalization of operators on intuitionistic fuzzy sets using triangular fuzzy t - norms and t - conorms, *IEEE Transactions on Fuzzy Systems* **8**(1), 19 –27.

[34] Dutta, A. and Jana, D. K.: 2017, Expectations of the reductions for type-2 trapezoidal fuzzy variables and its application to a multi-objective solid transportation problem via goal programming technique, *Journal of Uncertainty Analysis and Applications* **5**(1), 1 – 21.

[35] Erdogan, M. and Kaya, I.: 2015, An integrated multi-criteria decision-making methodology based on type-2 fuzzy sets for selection among energy alternatives in Turkey, *Iranian Journal of Fuzzy Systems* **12**(1), 1–25.

[36] Figueroa-García, J. C., Chalco-Cano, Y. and Román-Flores, H.: 2015, Distance measures for interval type-2 fuzzy numbers, *Discrete Applied Mathematics* **197**, 93–102.

[37] Frank, M. J.: 1979, On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$, *Aequationes Mathematicae* **19**(1), 194 – 226.

[38] Garg, H.: 2016a, Generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision making, *Computers and Industrial Engineering* **101**, 53 – 69.

[39] Garg, H.: 2016b, Generalized intuitionistic fuzzy multiplicative interactive geometric operators and their application to multiple criteria decision making, *International Journal of Machine Learning and Cybernetics* **7**(6), 1075 – 1092.

[40] Garg, H.: 2016c, A new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems, *Applied Soft Computing* **38**, 988 – 999.

[41] Garg, H.: 2016d, Some series of intuitionistic fuzzy interactive averaging aggregation operators, *SpringerPlus* **5**(1), 999.
URL: *10.1186/s40064-016-2591-9*

[42] Garg, H.: 2017, Distance and similarity measure for intuitionistic multiplicative preference relation and its application, *International Journal for Uncertainty Quantification* **7**(2), 117 – 133.

[43] Garg, H.: 2018a, Novel correlation coefficients under the intuitionistic multiplicative environment and their applications to decision - making process, *Journal of Industrial and Management Optimization* **14**(4), 1501 – 1519.

[44] Garg, H.: 2018b, Some arithmetic operations on the generalized sigmoidal fuzzy numbers and its application, *Granular Computing* **3**(1), 9 – 25.

[45] Garg, H.: 2019, Intuitionistic fuzzy hamacher aggregation operators with entropy weight and their applications to multi-criteria decision-making problems, *Iranian Journal of Science and Technology, Transactions of Electrical Engineering* **43**(3), 597 – 613.

[46] Garg, H.: 2020, A novel trigonometric operation-based q-rung orthopair fuzzy aggregation operators and its fundamental properties, *Neural Computing and Applications* pp. 1–23.
URL: *10.1007/s00521-020-04859-x*

[47] Garg, H. and Arora, R.: 2018, Bonferroni mean aggregation operators under intuitionistic fuzzy soft set environment and their applications to decision-making, *Journal of the Operational Research Society* **69**(11), 1711 – 1724.

[48] Garg, H. and Arora, R.: 2019, Generalized intuitionistic fuzzy soft power aggregation operator based on t-norm and their application in multi criteria decision-making, *International Journal of Intelligent Systems* **34**(2), 215–246.

[49] Garg, H. and Arora, R.: 2020, Maclaurin symmetric mean aggregation operators based on t-norm operationsfor the dual hesitant fuzzy soft set, *Journal of Ambient Intelligence and Humanized Computing* **11**(1), 375 – 410.
URL: *10.1007/s12652-019-01238-w*

[50] Garg, H. and Kaur, G.: 2020, Quantifying gesture information in brain hemorrhage patients using probabilistic dual hesitant fuzzy sets with unknown probability information, *Computers and Industrial Engineering* **140**, 106211.
URL: *10.1016/j.cie.2019.106211*

[51] Garg, H. and Kumar, K.: 2018a, An advanced study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making, *Soft Computing* **22**(15), 4959 – 4970.

[52] Garg, H. and Kumar, K.: 2018b, Distance measures for connection number sets based on set pair analysis and its applications to decision making process, *Applied Intelligence* **48**(10), 3346 –3359.

[53] Garg, H. and Kumar, K.: 2018c, Group decision making approach based on possibility degree measures and the linguistic intuitionistic fuzzy aggregation operators using Einstein norm operations, *Journal of Multiple-Valued Logic & Soft Computing* **31**(1/2), 175–209.

[54] Garg, H. and Kumar, K.: 2018d, Some aggregation operators for linguistic intuitionistic fuzzy set and its application to group decision-making process using the set pair analysis, *Arabian Journal for Science and Engineering* **43**(6), 3213 – 3227.

[55] Garg, H. and Kumar, K.: 2019a, An advanced study on operations of connection number based on set pair analysis, *National Academy Science Letters* **42**(4), 351 – 354.

[56] Garg, H. and Kumar, K.: 2019b, Improved possibility degree method for ranking intuitionistic fuzzy numbers and their application in multiattribute decision-making, *Granular Computing* **4**(2), 237 – 247.

[57] Garg, H. and Kumar, K.: 2019c, Linguistic interval-valued Atanassov intuitionistic fuzzy sets and their applications to group decision-making problems, *IEEE Transactions on Fuzzy Systems* **27**(12), 2302 – 2311.

[58] Garg, H. and Kumar, K.: 2020, A novel exponential distance and its based TOPSIS method for interval-valued intuitionistic fuzzy sets using connection number of SPA theory, *Artificial Intelligence Review* **53**(1), 595 – 624.

[59] Garg, H. and Nancy: 2018, Multi-criteria decision-making method based on prioritized muirhead mean aggregation operator under neutrosophic set environment, *Symmetry* **10**(7), 280.
URL: *10.3390/sym10070280*

[60] Garg, H. and Nancy: 2019, Algorithms for possibility linguistic single-valued neutrosophic decision-making based on COPRAS and aggregation operators with new information measures, *Measurement* **138**, 278 – 290.

[61] Ghorabaee, M. K., Amiri, M., Sadaghiani, J. S. and Goodarzi, G. H.: 2014, Multiple criteria group decision-making for supplier selection based on COPRAS method with interval type-2 fuzzy sets, *The International Journal of Advanced Manufacturing Technology* **75**(5-8), 1115–1130.

[62] Gong, Y., Hu, N., Zhang, J., Liu, G. and Deng, J.: 2015, Multi-attribute group decision making method based on geometric Bonferroni mean operator of trapezoidal interval type-2 fuzzy numbers, *Computers and Industrial Engineering* **81**, 167 – 176.

[63] Grzegorzewski, P.: 2004, Distance between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the hausdorff metric, *Fuzzy Sets and Systems* **148**(2), 319 – 328.

[64] Guan, K.: 2006, The hamy symmetric function and its generalization, *Mathematical Inequalities & Applications* **9**(4), 797 – 805.

[65] Hao, M. and Mendel, J. M.: 2014, Similarity measures for general type-2 fuzzy sets based on the α-plane representation, *Information Sciences* **277**, 197–215.

[66] Hara, T., Uchiyama, M. and Takahasi, S.-E.: 1998, A refinement of various mean inequalities, *Journal of Inequalities and Applications* **1998**(4), 932025.

[67] Heidarzade, A., Mahdavi, I. and Mahdavi-Amiri, N.: 2016, Supplier selection using a clustering method based on a new distance for interval type-2 fuzzy sets: A case study, *Applied Soft Computing* **38**, 213–231.

[68] Herrera, F. and Martínez, L.: 2001, A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making, *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)* **31**(2), 227 – 234.

[69] Hu, J., Zhang, Y., Chen, X. and Liu, Y.: 2013, Multi-criteria decision making method based on possibility degree of interval type-2 fuzzy number, *Knowledge-Based Systems* **43**, 21–29.

[70] Huang, J. Y.: 2014, Intuitionistic fuzzy Hamacher aggregation operator and their application to multiple attribute decision making, *Journal of Intelligent and Fuzzy Systems* **27**, 505 – 513.

[71] Hung, W. L. and Yang, M. S.: 2004a, Similarity measures between type-2 fuzzy sets, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **12**(6), 827–841.

[72] Hung, W. L. and Yang, M. S.: 2004b, Similarity measures of intuitionistic fuzzy sets based on hausdorff distance, *Pattern Recognition Letters* **25**, 1603 – 1611.

[73] Hwang, C. M., Yang, M. S., Hung, W. L. and Lee, E. S.: 2011, Similarity, inclusion and entropy measures between type-2 fuzzy sets based on the Sugeno integral, *Mathematical and Computer Modelling* **53**(9-10), 1788–1797.

[74] Hwang, C. M., Yang, M. S., Hung, W. L. and Lee, M. G.: 2012, A similarity measure of intuitionistic fuzzy sets based on the Sugeno integral with its application to pattern recognition, *Information Sciences* **189**, 93–109.

[75] Jana, D. K.: 2016, Novel arithmetic operations on type-2 intuitionistic fuzzy and its applications to transportation problem, *Pacific Science Review A: Natural Science and Engineering* **18**(3), 178 – 189.

[76] Jana, D. K., Pramanik, S. and Maiti, M.: 2017, Mean and CV reduction methods on Gaussian type-2 fuzzy set and its application to a multilevel profit transportation problem in a two-stage supply chain network, *Neural Computing and Applications* **28**(9), 2703 – 2726.

[77] Jiang, Q., Jin, X., Lee, S. J. and Yao, S.: 2019, A new similarity/distance measure between intuitionistic fuzzy sets based on the transformed isosceles triangles and its applications to pattern recognition, *Expert Systems with Applications* **116**, 439–453.

[78] Jiang, W. D.: 2007, Some properties of dual form of the hamy's symmetric function., *Journal of Mathematical Inequalities* **1**(1), 117 – 125.

[79] Kahraman, C., Öztayşi, B., Sarı, İ. U. and Turanoğlu, E.: 2014, Fuzzy analytic hierarchy process with interval type-2 fuzzy sets, *Knowledge-Based Systems* **59**, 48–57.

[80] Kar, M. B., Roy, B., Kar, S., Majumder, S. and Pamucar, D.: 2019, Type-2 multi-fuzzy sets and their applications in decision making, *Symmetry* **11**(2), 170.
URL: *10.3390/sym11020170*

[81] Karnik, N. N. and Mendel, J. M.: 2001, Centroid of a type-2 fuzzy set, *Information Sciences* **132**(1-4), 195–220.

[82] Kaur, G. and Garg, H.: 2018, Multi - attribute decision - making based on Bonferroni mean operators under cubic intuitionistic fuzzy set environment, *Entropy* **20**(1), 65.
URL: *10.3390/e20010065*

[83] Ke, D., Song, Y. and Quan, W.: 2018, New distance measure for Atanassov's intuitionistic fuzzy sets and its application in decision making, *Symmetry* **10**(10), 429.

[84] Klir, G. J. and Yuan, B.: 2005, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall of India Private Limited, New Delhi.

[85] Kumar, K. and Garg, H.: 2018a, Connection number of set pair analysis based TOPSIS method on intuitionistic fuzzy sets and their application to decision making, *Applied Intelligence* **48**(8), 2112 – 2119.

[86] Kumar, K. and Garg, H.: 2018b, TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment, *Computational and Applied Mathematics* **37**(2), 1319 – 1329.

[87] Kundu, P., Kar, S. and Maiti, M.: 2017, A fuzzy multi-criteria group decision making based on ranking interval type-2 fuzzy variables and an application to transportation mode selection problem, *Soft Computing* **21**(11), 3051–3062.

[88] Kundu, P., Majumder, S., Kar, S. and Maiti, M.: 2019, A method to solve linear programming problem with interval type-2 fuzzy parameters, *Fuzzy Optimization and Decision Making* **18**(1), 103–130.

[89] Lee, L. W. and Chen, S. M.: 2008, Fuzzy multiple attributes group decision-making based on the extension of TOPSIS method and interval type - 2 fuzzy sets, *Proceedings of 2008 International Conference on Machine Learning and Cybernetics, IEEE*, Vol. 1-7, pp. 3260 – 3265.

[90] Li, Y. H., Olson, D. L. and Qin, Z.: 2007, Similarity measures between intuitionistic fuzzy (vague) sets: a comparative analysis, *Pattern Recognition Letters* **28**, 278 – 285.

[91] Liang, Z. and Shi, P.: 2003, Similarity measures on intuitionistic fuzzy sets, *Pattern Recognition Letters* **24**, 2687 – 2693.

[92] Liu, P.: 2014, Some hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making, *IEEE Transactions on Fuzzy Systems* **22**(1), 83 – 97.

[93] Liu, P., Gao, H. and Ma, J.: 2019, Novel green supplier selection method by combining quality function deployment with partitioned Bonferroni mean operator in interval type-2 fuzzy environment, *Information Sciences* **490**, 292 – 316.

[94] Liu, P. and Wang, Y.: 2019, Intuitionistic fuzzy interaction Hamy mean operators and their application to multi-attribute group decision making, *Group Decision and Negotiation* **28**(1), 197–232.

[95] Liu, X. and Wang, Y.-M.: 2013, An analytical solution method for the generalized fuzzy weighted average problem, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **21**(03), 455–480.

[96] Luo, M. and Zhao, R.: 2018, A distance measure between intuitionistic fuzzy sets and its application in medical diagnosis, *Artificial intelligence in medicine* **89**, 34–39.

[97] Manna, S., Basu, T. M. and Mondal, S. K.: 2019, Trapezoidal interval type-2 fuzzy soft stochastic set and its application in stochastic multi-criteria decision-making, *Granular Computing* pp. 1 – 15.
URL: *10.1007/s41066-018-0119-0*

[98] Mendel, J. M. and John, R. B.: 2002, Type-2 fuzzy sets made simple, *IEEE Transactions on Fuzzy Systems* **10**(2), 117–127.

[99] Mendel, J. M., John, R. I. and LIu, F.: 2006, Interval type-2 fuzzy logic systems made simple, *IEEE Transactions on Fuzzy Systems* **14**(6), 808 – 821.

[100] Mendel, J. M. and Mouzouris, G. C.: 1999, Type-2 fuzzy logic systems, *IEEE Transactions on Fuzzy Systems* **7**, 643 – 658.

[101] Mendel, J. M. and Wu, H.: 2006, Type-2 fuzzistics for symmetric interval type-2 fuzzy sets: Part 1, forward problems, *Fuzzy Systems, IEEE Transactions on* **14**(6), 781–792.

[102] Mendel, J. M. and Wu, H.: 2007, Type-2 fuzzistics for symmetric interval type-2 fuzzy sets: Part 2, inverse problems, *IEEE Transactions on Fuzzy Systems* **15**(2), 301–308.

[103] Merigo, J. M.: 2009, Probabilistic decision making with the OWA operator and its application in investment management, *Proceeding of the IFSA-EUSFLAT International Conference, Lisbon, Portugal*, pp. 1364 – 1369.

[104] Merigo, J. M.: 2010, Fuzzy decision making with immediate probabilities, *Computers & Industrial Engineering* **58**(4), 651 – 657.

[105] Merigo, J. M.: 2012, The probabilistic weighted average and its application in multiperson decision making, *International Journal of Intelligent Systems* **27**(5), 457 – 476.

[106] Mitchell, H.: 2006, Ranking type-2 fuzzy numbers, *IEEE Transactions on Fuzzy Systems* **14**(2), 287–294.

[107] Mitchell, H. B.: 2003, On the dengfeng chuntian similarity measure and its application to pattern recognition, *Pattern Recognition Letters* **24**, 3101 – 3104.

[108] Mizumoto, M.: 1989, Pictorial representations of fuzzy connectives, part I: cases of t-norms, t-conorms and averaging operators, *Fuzzy sets and systems* **31**(2), 217–242.

[109] Muhuri, P. K., Ashraf, Z. and Lohani, Q. M. D.: 2018, Multi-objective reliability-redundancy allocation problem with interval Type-2 fuzzy uncertainty, *IEEE Transaction on Fuzzy Systems* **26**(3), 1339 –1355.

[110] Nancy and Garg, H.: 2016, Novel single-valued neutrosophic decision making operators under Frank norm operations and its application, *International Journal for Uncertainty Quantification* **6**(4), 361 – 375.

[111] Nancy and Garg, H.: 2019, A novel divergence measure and its based TOPSIS method for multi criteria decision - making under single - valued neutrosophic environment, *Journal of Intelligent & Fuzzy Systems* **36**(1), 101 – 115.

[112] Own, C. M.: 2009, Switching between type-2 fuzzy sets and intuitionistic fuzzy sets: an application in medical diagnosis, *Applied Intelligence* **31**(3), 283–291.

[113] Oztaysi, B.: 2015, A group decision making approach using interval type-2 fuzzy ahp for enterprise information systems project selection., *Journal of Multiple-Valued Logic & Soft Computing* **24**(5).

[114] Pal, S. S. and Kar, S.: 2019, A hybridized forecasting method based on weight adjustment of neural network using Generalized Type-2 fuzzy set, *International Journal of Fuzzy Systems* **21**(1), 308 – 320.

[115] Panja, S. and Mondal, S. K.: 2019, Analyzing a four-layer green supply chain imperfect production inventory model for green products under type-2 fuzzy credit period, *Computers & Industrial Engineering* **129**, 435 – 453.

[116] Pedrycz, W. and Song, M.: 2014, A granulation of linguistic information in AHP decision-making problems, *Information Fusion* **17**, 93–101.

[117] Pramanik, S., Jana, D. K., Mondal, S. K. and Maiti, M.: 2015, A fixed-charge transportation problem in two-stage supply chain network in Gaussian type-2 fuzzy environments, *Information Sciences* **325**, 190–214.

[118] Qin, J.: 2017, Interval type-2 fuzzy hamy mean operators and their application in multiple criteria decision making, *Granular Computing* **2**(4), 249–269.

[119] Qin, J. and Liu, X.: 2014a, An approach to intuitionistic fuzzy multiple attribute decision making based on Maclaurin symmetric mean operators, *Journal of Intelligent & Fuzzy Systems* **27**(5), 2177–2190.

[120] Qin, J. and Liu, X.: 2014b, Frank aggregation operators for triangular interval type-2 fuzzy set and its application in multiple attribute group decision making, *Journal of Applied Mathematics* **2014**, Article ID 923213 24 pages.

[121] Qin, J. and Liu, X.: 2015a, Approaches to uncertain linguistic multiple attribute decision making based on dual Maclaurin symmetric mean, *Journal of Intelligent & Fuzzy Systems* **29**(1), 171–186.

[122] Qin, J. and Liu, X.: 2015b, Multi-attribute group decision making using combined ranking value under interval type-2 fuzzy environment, *Information Sciences* **297**, 293–315.

[123] Qin, J., Liu, X. and Pedrycz, W.: 2015, Hesistant fuzzy Maclaurin symmetric mean operators and its application to multiple-attribute decision-making, *International Journal of Fuzzy Systems* **17**(4), 509 – 520.

[124] Qin, J., Liu, X. and Pedrycz, W.: 2016, Frank aggregation operators and their application to hesitant fuzzy multiple attribute decision making, *Applied Soft Computing* **41**, 428 – 452.

[125] Rani, D. and Garg, H.: 2017, Distance measures between the complex intuitionistic fuzzy sets and its applications to the decision - making process, *International Journal for Uncertainty Quantification* **7**(5), 423 – 439.

[126] Rani, D. and Garg, H.: 2018, Complex intuitionistic fuzzy power aggregation operators and their applications in multi-criteria decision-making, *Expert Systems.* **35**(6), e12325.

[127] Samanta, S. and Jana, D. K.: 2019, A multi-item transportation problem with mode of transportation preference by MCDM method in interval type-2 fuzzy environment, *Neural Computing and Applications* **31**(2), 605 – 617.

[128] Seth, T., Gupta, P. K. and Muhuri, P. K.: 2018, Hesitant fuzzy linguistic term sets for group decision making in supplier performance evaluation, *2018 IEEE International Conference on Fuzzy Systems*, pp. 1 – 8.

[129] Shen, F., Ma, X., Li, Z., Xu, Z. S. and Cai, D.: 2018, An extended intuitionistic fuzzy TOPSIS method based on a new distance measure with an application to credit risk evaluation, *Information Sciences* **428**, 105 – 119.

[130] Singh, P.: 2014, Some new distance measures for type-2 fuzzy sets and distance measure based ranking for group decision making problems, *Frontiers of Computer Science* **8**(5), 741 – 752.

[131] Singh, P.: 2015, Similarity measure for type-2 fuzzy sets with an application to students' evaluation, *Computer Applications in Engineering Education* **23**(5), 694–702.

[132] Smarandache, F.: 1998, *Neutrosophy. Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning*, Ann Arbor, Michigan, USA.

[133] Song, Y., Wang, X., Lei, L. and Xue, A.: 2015, A novel similarity measure on intuitionistic fuzzy sets with its applications, *Applied Intelligence* **42**(2), 252–261.

[134] Song, Y., Wang, X., Quan, W. and Huang, W.: 2017, A new approach to construct similarity measure for intuitionistic fuzzy sets, *Soft computing* **23**(6), 1985–1998.

[135] Spearman, C.: 1904, The proof and measurement of association between two things, *The American journal of psychology* **15**(1), 72–101.

[136] Szmidt, E.: 2014, *Distances and similarities in intuitionistic fuzzy sets*, Springer.

[137] Szmidt, E. and Kacprzyk, J.: 2000, Distances between intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **114**, 505 – 518.

[138] Szmidt, E. and Kacprzyk, J.: 2001, Entropy for intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **118**(3), 467 – 477.

[139] Szmidt, E. and Kacprzyk, J.: 2004, A similarity measure for intuitionistic fuzzy sets and its application in supporting medical diagnostic reasoning, *Lecture Notes in Computer Science* **3070**, 388 – 393.

[140] Torra, V.: 2010, Hesitant fuzzy sets, *International Journal of Intelligent Systems* **25**(6), 529 – 539.

[141] Wang, H., Ju, Y., Liu, P., Ju, D. and Liu, Z.: 2018, Some trapezoidal interval type-2 fuzzy heronian mean operators and their application in multiple attribute group decision making, *Journal of Intelligent & Fuzzy Systems* **32**(2), 2323–2337.

[142] Wang, H., Liu, P. and Liu, Z.: 2018, Trapezoidal interval type-2 fuzzy maclaurin symmetric mean operators and their applications to multiple attribute group decision making, *International Journal for Uncertainty Quantification* **8**(4), 343 – 360.

[143] Wang, J. C. and Chen, T. Y.: 2014, A closeness coefficient-based multiple criteria decision-making method using interval type-2 fuzzy sets and its application to watershed site selection, *Journal of Industrial and Production Engineering* **31**(1), 1–16.

[144] Wang, J., Chen, Q. H., Zhang, H. Y., Chen, X. H. and Wang, J. Q.: 2017, Multi-criteria decision-making method based on type-2 fuzzy sets, *Filomat* **31**(2), 431–450.

[145] Wang, J. Q., Yu, S. M., Wang, J., Chen, Q. H., Zhang, H. Y. and Chen, X. H.: 2015, An interval type-2 fuzzy number based approach for multi-criteria group decision-making problems, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **23**(4), 565–588.

[146] Wang, W. and Liu, X.: 2012, Intuitionistic fuzzy information aggregation using Einstein operations, *IEEE Transactions on Fuzzy Systems* **20**(5), 923 – 938.

[147] Wang, W., Liu, X. and Qin, Y.: 2012, Multi-attribute group decision making models under interval type-2 fuzzy environment, *Knowledge-Based Systems* **30**, 121–128.

[148] Wang, W. Z. and Liu, X. W.: 2011, Intuitionistic fuzzy geometric aggregation operators based on Einstein operations, *Intenational Journal of Intelligent Systems* **26**, 1049 – 1075.

[149] Wei, G. W.: 2010, Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making, *Applied Soft Computing* **10**, 423 – 431.

[150] Wei, G. W., Wang, H. J. and Lin, R.: 2011, Application of correlation coefficient to interval-valued intuitionistic fuzzy multiple attribute decision-making with incomplete weight information, *Knowledge and Information Systems* **26**(2), 337 – 349.

[151] Wu, D. and Mendel, J. M.: 2007a, Aggregation using the linguistic weighted average and interval type-2 fuzzy sets, *IEEE Transactions on Fuzzy Systems* **15**(6), 1145 – 1161.

[152] Wu, D. and Mendel, J. M.: 2007b, Uncertainty measures for interval type-2 fuzzy sets, *Information sciences* **177**(23), 5378–5393.

[153] Wu, D. and Mendel, J. M.: 2008, A vector similarity measure for linguistic approximation: Interval type-2 and type-1 fuzzy sets, *Information Sciences* **178**(2), 381–402.

[154] Wu, D. and Mendel, J. M.: 2009, A comparative study of ranking methods, similarity measures and uncertainty measures for interval type-2 fuzzy sets, *Information Sciences* **179**(8), 1169–1192.

[155] Wu, D. and Mendel, J. M.: 2019, Similarity measures for closed general type-2 fuzzy sets: Overview, comparisons, and a geometric approach, *IEEE Transactions on Fuzzy Systems* **27**(3), 515–526.

[156] Xia, M. M. and Xu, Z. S.: 2010, Some new similarity measures for intuitionistic fuzzy values and their application in group decision making, *Journal of System Science and System Engineering* **19**, 430 – 452.

[157] Xia, M., Xu, Z. and Liao, H.: 2013, Preference relations based on intuitionistic multiplicative information, *IEEE Transactions on Fuzzy Systems* **21**(1), 113 – 132.

[158] Xu, Z.: 2005, An approach based on similarity measure to multiple attribute decision making with trapezoid fuzzy linguistic variables, *International Conference on Fuzzy Systems and Knowledge Discovery*, pp. 110 – 117.

[159] Xu, Z. S.: 2007a, Intuitionistic fuzzy aggregation operators, *IEEE Transactions on Fuzzy Systems* **15**, 1179 – 1187.

[160] Xu, Z. S.: 2007b, Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making, *Control and Decision* **22**(2), 215 – 219.

[161] Xu, Z. S. and Yager, R. R.: 2006, Some geometric aggregation operators based on intuitionistic fuzzy sets, *International Journal of General Systems* **35**, 417 – 433.

[162] Xu, Z. and Xia, M.: 2011, Induced generalized intuitionistic fuzzy operators, *Knowledge-Based Systems* **24**(2), 197–209.

[163] Yang, M. S. and Lin, D. C.: 2009, On similarity and inclusion measures between type-2 fuzzy sets with an application to clustering, *Computers & Mathematics with Applications* **57**(6), 896 – 907.

[164] Ye, J.: 2011, Cosine similarity measures for intuitionistic fuzzy sets and their applications, *Mathematical and Computer Modelling* **53**(1-2), 91 – 97.

[165] Ye, J.: 2016, Similarity measures of intuitionistic fuzzy sets based on cosine function for the decision making of mechanical design schemes, *Journal of Intelligent & Fuzzy Systems* **30**, 151 – 158.

[166] Zadeh, L. A.: 1965, Fuzzy sets, *Information and Control* **8**, 338–353.

[167] Zadeh, L. A.: 1975, The concept of a linguistic variable and its application to approximate reasoning: Part-1, *Information Science* **8**, 199–251.

[168] Zamri, N., Abdullah, L., Hitam, M. S., Noor, M. and Jusoh, A.: 2013, A novel hybrid fuzzy weighted average for MCDM with interval triangular type-2 fuzzy sets, *WSEAS Transactions on Systems* **12**(4), 212–228.

[169] Zeng, W. and Li, H.: 2006, Relationship between similarity measure and entropy of interval-valued fuzzy sets, *Fuzzy Sets and Systems* **157**(11), 1477 – 1484.

[170] Zeng, W. Y. and Guo, P.: 2008, Normalized distance, similarity measure, inclusion measure and entropy of interval-valued fuzzy sets and their relationship, *Information Sciences* **178**, 1334 – 1342.

[171] Zhang, H.: 2014, Linguistic intuitionistic fuzzy sets and application in MAGDM, *Journal of Applied Mathematics* **2014**, Article ID 432092 11 pages.
URL: *http://dx.doi.org/10.1155/2014/432092*

[172] Zhang, Z.: 2018, Trapezoidal interval type-2 fuzzy aggregation operators and their application to multiple attribute group decision making, *Neural Computing and Applications* **29**(4), 1039–1054.

[173] Zhang, Z. and Zhang, S.: 2013, A novel approach to multi attribute group decision making based on trapezoidal interval type-2 fuzzy soft sets, *Applied Mathematical Modelling* **37**(7), 4948–4971.

[174] Zhao, H., Xu, Z., Ni, M. and Liu, S.: 2010, Generalized aggregation operators for intuitionistic fuzzy sets, *International Journal of Intelligent Systems* **25**(1), 1–30.

[175] Zhao, T. and Xia, J.: 2012, Type-2 intuitionistic fuzzy sets, *Control Theory & Applications* **29**(9), 1215 – 1222.

[176] Zhao, X. and Wei, G. W.: 2013, Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making, *Knowledge Based Systems* **37**, 472 – 479.