MATHEMATICAL PROPERTIES OF WAVELET FILTERS

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Mathematical Properties of Wavelet Filters

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "Mathematical Properties of Wavelet Filters" in partial fulfillment of the requirements for the award of degree of Master of Science , School of Mathematics and Computer Applications, Thapar Universtiy, Patiala is an authentic record of my own work carried out under the supervision of Mr. Singara Singh.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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Abstract

As wavelets are a mathematical tool they can be used to extract information from many different kinds of data, including - but certainly not limited to - audio signals and images. Sets of wavelets are generally needed to analyze data fully. A set of "complementary" wavelets will deconstruct data without gaps or overlap so that the deconstruction process is mathematically reversible. Thus, sets of complementary wavelets are useful in wavelet based compression/decompression algorithms where it is desirable to recover the original information with minimal loss.

More technically, a wavelet is a mathematical function used to divide a given function or continuous-time signal into different scale components. Usually one can assign a frequency range to each scale component. Each scale component can then be studied with a resolution that matches its scale. A **wavelet transform** is the representation of a function by wavelets. The wavelets are scaled and translated copies (known as "daughter wavelets") of a finite-length or fast-decaying oscillating waveform (known as the "mother wavelet"). Wavelet transforms have advantages over traditional Fourier transforms for representing functions that have discontinuities and sharp peaks, and for accurately deconstructing and reconstructing finite, non-periodic and/or non-stationary signals.

Wavelets have certain mathematical properties. In this thesis, we determine some mathematical properties like approximation order, Holder's regularity, Sobolov regularity and their support length. The wavelets studied are LeGall 5/3, Daubechies 9/7, DB2, DB4, DB6, DB8 and DB10.

Abbreviations

FIR	Finite Impulse Response
MRA	Multiresolution Analysis
STFT	Short Term Fourier Transform

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Chapter 1

Introduction

There are a number of ways to decompose a signal. Fourier transform is one of them. Fourier transform of a function f(t) is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt.$$

As integration is an averaging operation. Therefore Fourier transform is an "average" analysis, where the averaging interval is all of time. Thus by looking at a particular Fourier transform, we can say that, for example, that there is a large component of frequency 10 kHz in a signal, but we can't tell when in time this component occurred. Another way of saying this is that Fourier analysis provides excellent localization in frequency and none in time. The converse is true for the time function f(t), which provides exact information about the value of the function at each instant of time but does not directly provide spectral information. It should be noted that both f(t) and $F(\omega)$ represent the same function, and all the information is present in each representation. However, each representation makes different kinds of information easily accessible.

If we have a very nonstationary signal, we would like to know not only the frequency components but when in time the particular frequency components occurred. One way to obtain this information is via the short-term Fourier transform (STFT). With the STFT, we break the function f(t) into pieces of length T and apply Fourier analysis to each piece. This way, we can say, for example, that a component at 10kHz occurred in the third piece- that is, between time 2T and time 3T. Thus, we obtain an analysis that is a function of both time and frequency. If we simply chopped the function into pieces, we could get distortion in the form of boundary effects. In order to reduce the boundary effects, we window each piece before we take the Fourier transform. If the window shape is given by g(t),the STFT is formally given by

$$F(\omega,\tau) = \int_{-\infty}^{\infty} f(t) g*(t-\tau) e^{i\omega t} dt$$

The problem with the STFT is the fixed window size. If we have a set of functions in which the number of cycles is constant, but the size of window keeps changing. The number of cycles of the sinusoid in each window is the same, as the size of the window gets smaller, these cycles occur in a smaller time interval i.e. the frequency of the sinusoid increases. Further the lower frequency functions cover a longer time interval, while the higher frequency functions cover a shorter time interval, thus avoiding the problem that we had with the STFT. If we can write our function in terms of these functions and their translates, we have a representation that gives us time and frequency localization and can provide high frequency resolution at low frequencies (longer time window) and high time resolution at high frequencies (shorter time window). This is the basic idea behind wavelets. [1]

Figure 1.1 Three wavelet basis functions

1.1 Discrete-Time Bases

We would like to carry over the elements of linear algebra in finite dimensions to infinite-dimensional signals [2]. A basic question is to determine whether a sequence of signals $(\varphi^{(n)})_{n=-\infty}^{\infty}$ is a basis, that is, if all signals x can be written uniquely as

(1.1) $$x = \sum_{n} c_n \varphi^{(n)} ,$$

For some numbers ($c_{(n)}$). Here we encounter a problem not present in finite dimensions. The series expansion involves an infinite number of terms, and this series must converge. It turns out that we can not find a basis for all signals and we will here restrict ourselves to those with finite energy. Then, we can carry over all the concepts from finite-dimensional vector spaces in a fairly straightforward way.

1.1.1 Convergence and Completeness

To discuss convergence we need a measure of the size of a signal. The norm of a signal provides us with such a measure. Just as in finite dimensions there are different norms, but the one we will use is the energy norm,

$$\|x\| = (\sum_{k} |x_k|^2)^{1/2}$$

A vector space equipped with a norm is called a normed space. Now, a sequence of signals $(x^{(n)})_{n=1}^{\infty}$ is said to converge (in the energy norm) to x if

$$\left\| x^{(n)} - x \right\| \to 0, \quad \text{as } n \to \infty .$$

Accordingly, the equality in (1.1) means that the sequence of partial sums

$$s^{(N)} = \sum_{n=-N}^{N} c_n \varphi^{(n)} ,$$

is convergent with limit x, that is, $\left\| s^{(N)} - x \right\| \to 0$, as $N \to \infty$.

Remark . In the definition of convergence we assumed that we had a candidate for the limit of the sequence. It would be convenient to have a definition, or test, of convergence that does not involve that limit explicitly. A fundamental property of the real and complex number system is that a sequence of numbers is convergent if and only if it is a Cauchy sequence. In a general normed space, a sequence $(x^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence if for all $\in >0$ there is an N such that

$$\left\| x^{(n)} - x^{(m)} \right\| < \in, \quad \text{for all } n, m > N.$$

Loosely speaking, a Cauchy sequence is a sequence where the vectors are getting closer and closer together, or a sequence that is trying to converge. In a general normed space all Cauchy sequences does not have to converge. An example is Q , the set of all rational numbers. A normed space where all Cauchy sequence are convergent is called a Banach space, or a complete normed space. The finite dimensional vector space R^n and C^n are complete. The space of all signals with finite energy,

$$\ell^2 \ (z) = \{x : Z \to C | \ \|x\| < \infty \},$$

is also complete. In this chapter we will assume that all signals are contained in this space.

1.1.2 Hilbert Spaces and Orthogonal Bases

Just as in finite dimensions, it is convenient to work with orthogonal bases. To discuss orthogonality we need to impose further structure on our space ℓ^2(z), by defining an inner product between two vectors. The inner product between two signals x and y in ℓ^2 (z) is defined as

$$\langle x, y \rangle = \sum_k x_k \overline{y_k} .$$

Convergence of this infinite sum follows from the Cauchy-Schwarz inequality,

$$|\langle x, y \rangle| \leq \|x\| \; \|y\| .$$

Note that we now can write the norm of x $\in \ell^2$ (z) as

$$\|x^2\| = \langle x, x \rangle .$$

A complete space with an inner product is called a Hilbert space. We have only looked at the particular Hilbert space R^n, C^n, and ℓ^2 (z) so far.

We now have all the tools necessary to define orthonormal bases for our signals. First, recall definition (1.1) of a basis. A basis $(\varphi^{(n)})_{n=-\infty}^{\infty}$ is orthonormal if

$$\langle \varphi^{(j)}, \varphi^{(k)} \rangle = \begin{Bmatrix} 1, & j = k \\ 0, & j \neq k \end{Bmatrix}$$

For an orthonormal basis the coordinates of a signal x are given by

(1.2) $$c_n = \langle x, \varphi^{(n)} \rangle .$$

It follows that the signal can be written as

(1.3) $$x = \sum_n \langle x, \varphi^{(n)} \rangle \, \varphi^{(n)} .$$

Equation (1.2) is referred to as an analysis and equation (1.3) as a synthesis of the signal x, respectively.

1.2 The Discrete-Time Haar Basis

How do we find bases for our space ℓ^2 (z), and what properties do we want these bases to have? Usually we want the basis function to be well localized both in time and in frequency. The coordinates of a signal in the basis will then provide a measure of the strength of the signal for different time and frequency intervals. We will construct bases starting from two prototype basis functions and their even translates. These bases are further characterized by the fact that the coordinates in the new basis can be computed using a filter bank. This is another important property, since it means that the coordinate transformation can be computed efficiently.

1.2.1 The Basis Functions

The discrete- time Haar basis [2] is an example of the special class of orthogonal bases that are related to filter banks. These orthogonal bases are characterized by the fact that they are formed as the even translates of two prototype basis functions φ and ψ. For the Haar basis these two functions are given by,

$$\varphi(k) = \begin{cases} 1/\sqrt{2} & k = 0,1 \\ 0 & otherwise \end{cases}$$

$$\text{and} \qquad \psi(k) = \begin{cases} 1/\sqrt{2} & k = 0 \\ -1/\sqrt{2} & k = 1 \\ 0 & otherwise \end{cases}$$

The basis functions ($\varphi^{(n)}$) are now formed as the even translates of these two prototypes (see figure 1.2).

$$\varphi_k^{(2n)} = \varphi_{k-2n} \quad \text{and} \qquad \varphi_k^{(2n+1)} = \psi_{k-2n}.$$

The coordinates of a signal x in this new basis are consequently,

$$\text{(1.4)} \qquad y_n^{(0)} = c_{2n} = \left\langle x, \varphi^{(2n)} \right\rangle = \frac{1}{\sqrt{2}}(x_{2n} + x_{2n+1}),$$

$$y_n^{(1)} = c_{2n+1} = \left\langle x, \varphi^{(2n+1)} \right\rangle = \frac{1}{\sqrt{2}}(x_{2n} - x_{2n+1}),$$

in other words, weighted averages and difference of pairwise values of x.

Another way of interpreting this is to say that we take pairwise values of x, and that rotate the coordinate system in the plane (R^2) 45 degrees counter-clockwise. Here, we have also introduced the sequences $y^{(0)}$ and $y^{(1)}$ consisting of the even- and odd-indexed coordinates, respectively. The basis functions is an orthonormal basis of $\ell^2(Z)$ and we can therefore reconstruct

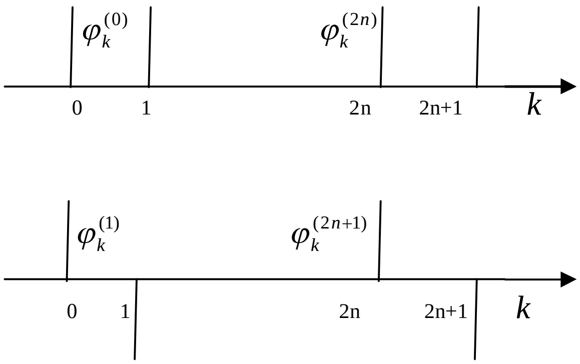

Figure 1.2: The discrete-time Haar basis.

the signal values as

$$
\begin{aligned}
x_k &= \sum_n c_n \varphi_k^{(n)} \\
&= \sum_n y_n^{(0)} \varphi_k^{(2n)} + \sum_n y_n^{(1)} \varphi_k^{(2n+1)} \qquad (1.5) \\
&= \sum_n y_n^{(0)} \varphi_{k-2n} + \sum_n y_n^{(1)} \psi_{k-2n}.
\end{aligned}
$$

1.2.2 Analysis

We will now show how we can use a filter bank to compute the coordinates in (1.4). If we define the impulse responses of two filters H and G as

$$h_k = \varphi_k \text{ and } g_k = \psi_k,$$

and if we let h* and g* denote the time-reverses of these filter, we can write the inner products in (1.4) as a convolution,

$$
\begin{aligned}
y_n^{(0)} &= \sum_k x_k \varphi_k^{(2n)} = \sum_k x_k h_{k-2n} = \sum_k x_k h^*_{2n-k} \\
&= (x * h^*)_{2n}.
\end{aligned}
$$

Similarly, we get $y_n^{(1)} = (x * g^*)_{2n}$.

The conclusion is that we can compute $y^{(0)}$ and $y^{(1)}$ by filtering x with $\tilde{H}$ and $\tilde{G}$, respectively, and then downsampling the output of these two filters (see figure 1.3). downsampling removes all odd-indexed values of a signal, and we define the downsampling operator $(\downarrow 2)$ as

$$(\downarrow 2)x = (\ldots, x_{-2}, x_0, x_2, \ldots).$$

Thus, we have $$y^{(0)} = (\downarrow 2)\tilde{H}x, \text{ and } y^{(1)} = (\downarrow 2)\tilde{G}x.$$

Here, $\tilde{H}$ and $\tilde{G}$ denotes the low and highpass. These two filters are non-causal since their impules responses are the time-reverses of causal filters. This is not necessarily a problem in applications, where the filters can always be made causal by delaying them a certain number of steps; the output is then delayed an equal number of steps.

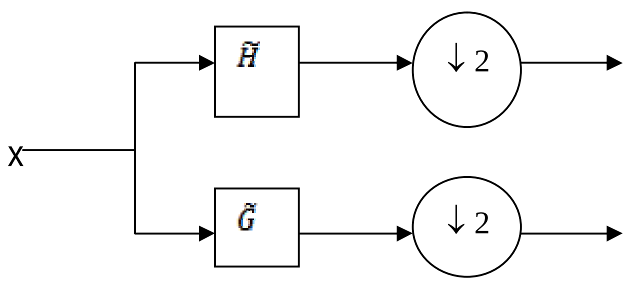

Figure 1.3: The analysis part of a filter bank.

1.2.3 Synthesis

So far we have seen how we can analyze a signal, or compute its coordinates, in the Haar basis using a filter bank. Let us now demonstrate how we can synthesize, or reconstruct, a signal from the knowledge of its coordinates. From the definition of the filters H and G, and from the reconstruction formula (1.5) we get

$$x_k = \sum_n y_n^{(0)} \varphi_{k-2n} + \sum_n y_n^{(1)} \psi_{k-2n}$$

$$= \sum_n y_n^{(0)} h_{k-2n} + \sum_n y_n^{(1)} g_{k-2n}$$

$$= (v^{(0)} * h)_k + (v^{(1)} * g)_k$$

where $v^{(0)} = (\uparrow 2)y^{(0)}$ and $v^{(1)} = (\uparrow 2)y^{(1)}$. Here the upsampling operator $(\uparrow 2)$ is defined as

$$(\uparrow 2)\text{y} = (\ldots, y_{-1}, 0, y_0, 0, y_{-1}, \ldots).$$

Hence, the signal x is the sum of to signals,

$$\text{X} = v^{(0)} * h + v^{(1)} * g$$

$$= \text{H}\,((\uparrow 2)\text{y}^{(0)}) + \text{G}((\uparrow 2)\text{y}^{(1)}) =: \text{x}^{(0)} + \text{x}^{(1)}.$$

The signals $x^{(0)}$ and $x^{(1)}$ are obtained by first upsampling $y^{(0)}$ and $y^{(1)}$, and then filtering the result with H and G, respectively; see figure 1.4. on the other hand, if we look back at the reconstruction formula (1.5), we can write x as

$$\text{x} = \text{x}^{(0)} + \text{x}^{(1)}$$

$$= \sum_n \left\langle x, \varphi^{(2n)} \right\rangle \varphi^{(2n)} + \sum_n \left\langle x, \varphi^{(2n+1)} \right\rangle \varphi^{(2n+1)}$$

which means that $x^{(0)}$ and $x^{(1)}$ are the orthogonal projection of x onto the subspaces spanned by the even and odd basis functions, respectively.

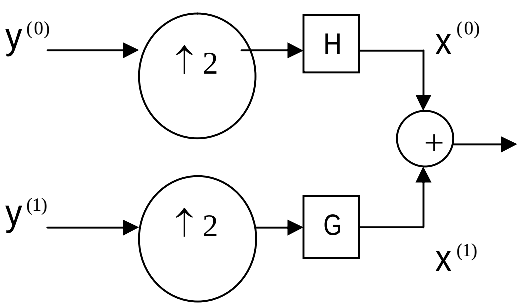

Figure 1.4: The synthesis part of a filter bank.

1.3 The Subsampling Operators

In this section we study the effect of down- and upsampling on a signal Specifically, we arrive at formulas for how the z- and fourier transform of a signal change, when the signal is subsampled.

1.3.1 Downsampling

The downsampling operator (↓ 2) removes all odd-indexed values of a signal and is consequently defined as

$$(\downarrow 2)\text{x} = (\ldots, x_{-2}, x_0, x_2, \ldots).$$

If we let y = (↓ 2)x we have $y_k = x_{2k}$, and in the z-domain we get

$$Y(z) = \frac{1}{2}[X(z^{1/2}) + X(-z^{1/2})].$$

The corresponding relation in the frequency domain is

$$Y(\omega) = \frac{1}{2}[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2}+\pi\right)]$$

From these relations we see that we get an alias component in the spectrum of y from the term $X(\omega/2+\pi)$. The filters before the upsampling operators will reduce this alias effect and if the filters are ideal low- and highpass filters, respectively, the alias component will be completely deleted.

1.3.2 Upsampling

The upsampling operator $(\uparrow 2)$ inserts a zero between every value of a signal,

$$(\uparrow 2)y = \left(...., y_{(-1)}, 0, y_0, 0, y_1, ...\right).$$

If we let $u = (\uparrow 2)y$ it is easy to verify that

(1.7) $$U(z) = Y(z^2),$$

and for the fourier transform we get

$$U(\omega) = Y(2\omega).$$

The spectrum $U(\omega)$ of u is thus a dilated version of $Y(\omega)$ by a factor two. This will result in the appearance of an image in $U(\omega)$. The low- and highpass filters after the upsampling reduce the effect of the image, and if the filters are ideal low- and highpass filters the image will be completely deleted.

1.3.3 Down- and Upsampling

By combining the results from the previous two sections, we obtain a relation between a signal x and the down- and upsampled signal

$$u = (\uparrow 2)(\downarrow 2)x = \left(...., x_{-2}, 0, x_0, 0, x_2, ...\right).$$

In the z-domain we have

$$U(z) = \frac{1}{2}[X(z)+X(-z)],$$

and in the fourier domain

$$U(\omega) = \frac{1}{2}[X(\omega)+X(\omega+\pi)].$$

1.4 Perfect Reconstruction

Above we studied the Haar basis which is an example of a special type of discrete-time bases. These bases were characterized by the fact that they were formed as the even translates of two prototype basis functions. We saw how low- and highpass filters, followed by downsampling, in the analysis part of a filter bank gave us the coordinates of the signal in the new basis. Similarly, upsampling followed by low- and highpass filters gave us the expansion of the signal in the new basis.

A filter bank consists of an analysis and a synthesis part as depicted in Figure 1.4. The goal is to find conditions on the low- and highpass filters so that the output $\hat{x}$ equals to the input x. This is called perfect reconstruction. For a general filter bank, the impulse responses of the low- and highpass filters in the synthesis part equals two prototype basis functions in a corresponding discrete-time basis. If the basis is orthogonal, the analysis filters are the time-reverses of the synthesis filters. More generally, we can have a biorthogonal basis, and then the analysis filters are denoted as $\tilde{H}$ and $\tilde{G}$.

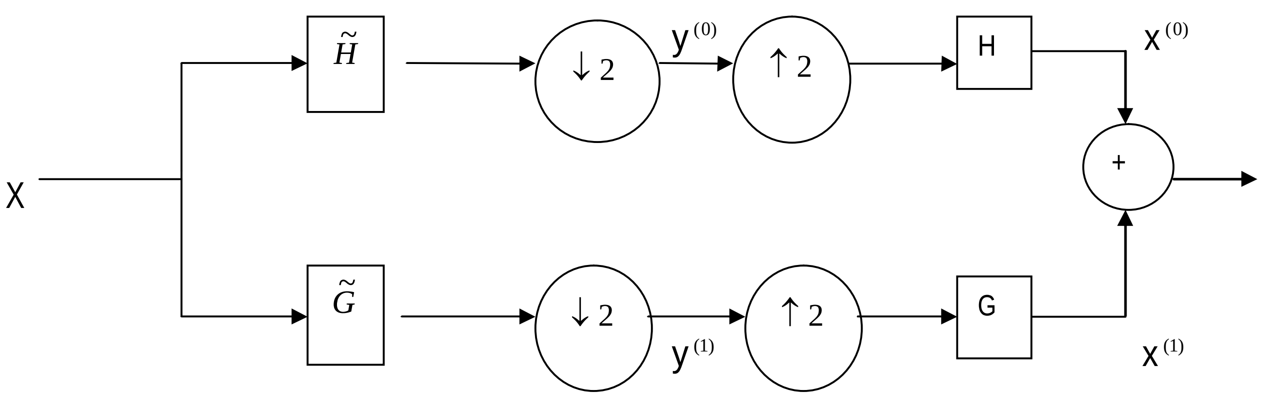

Figure 1.5: Filter bank.

1.4.1 Reconstruction Conditions

The results of the previous section for the down- and upsampling give us the following expressions for the z-transform of $x^{(0)}$ and $x^{(1)}$

$$X^{(0)}(z) = \frac{1}{2} H(z)\,[X(z)\tilde{H}(z) + X(-z)\,\tilde{H}(-z)]$$

$$X^{(1)}(z) = \frac{1}{2} G(z)\,[X(z)\tilde{G}(z) + X(-z)\,)\tilde{G}(-z)]$$

Adding these together, we obtain an expression for the z-transform of $\hat{x}$

$$\hat{X}(z) = \frac{1}{2}[H(z)\,\tilde{H}(z) + G(z)\tilde{G}(z)]\,X(z) + \frac{1}{2}[H(z)\,\tilde{H}(z) + G(z)\,\tilde{G}(-z)]\,X(-z),$$

where we have grouped the terms with the factor X(z) and X(-z) together respectively.

From this we see that we get perfect reconstruction, that is $x = \hat{x}$, if the factor in front of X(z) equals one, and the factor in front of X(-z) equals zero. We thus get the following two conditions on the filters.

(1.8) $$H(z)\tilde{H}(z) + G(z)\tilde{G}(z) = 2,$$

(1.9) $$H(z)\,\tilde{H}(-z) + G(z)\tilde{G}(-z) = 0.$$

The first condition ensures that there is no distortion of the signal, and the second condition that the alias component X(-z) is cancelled. These two conditions will appear again when we study wavelets. This is the key to the connection between filters banks and wavelets. Due to different normalizations, the right-hand side in the no distortion condition equals 1 for wavelets though.

1.4.2 Alias Cancellation and the Product Filter

At this stage we will define the highpass filters in terms of the lowpass filters, so that the alias cancellation condition (1.9) automatically is satisfied [2]. We let highpass filters equal

(1.10) $$G(z) = z^{-L}\,\tilde{H}(-z), \quad \text{and} \quad \tilde{G}(z) = z^{L}H(-z),$$

where L is an arbitrary odd integer.We will in most cases choose L = 1. Choosing the highpass filters in this way is a sufficient condition to ensure alias cancellation.

If we substitute the alias cancellation choice (1.10) into the no distortion condition (1.8), we get a single condition on the two lowpass filters for perfect reconstruction,

$$H(z)\,\tilde{H}(z) + H(-z)\,\tilde{H}(-z) = 2.$$

We now define the product filter $P(z) = H(z)\tilde{H}(z)$, and this finally reduces the perfect reconstruction condition to

(1.11) $$P(z) + P(-z) = 2.$$

The left hand side of this equation can be written as

$$P(z) + P(-z) = 2p_0 + 2\sum_n p_{2n} z^{-2n}.$$

From this we conclude that all even power in P(z) must be zero, except the constant term which should equal one. The odd powers all cancel and are the design variables in a filter bank.

The design of a perfect reconstruction filter bank is that equation of finding a product filter P(z) satisfying condition (1.11). Once such a product filter has been found, it is factored in some way as $P(z) = H(z)\,\tilde{H}(z)$. The highpass filters are then given by equation (1.10).

1.4.3 Orthogonal Filter Banks

In the first of this chapter, we saw that for orthogonal discrete-time bases the corresponding filters in the filter bank were related as

$$H(z) = \tilde{H}(z)\,, \quad \text{and} \quad G(z) = \tilde{G}(z)\,.$$

Such a filter bank is consequently called orthogonal. For orthogonal filter banks, we can write the perfect reconstruction condition solely in terms of the synthesis lowpass filter H(z). The relation between the analysis and synthesis filters implies that

$$P(z) = H(z)\tilde{H}^*(z) = H(z)H(z^{-1})\,,$$

that is, the sequence p is the autocorrelation of h . In the fourier domain we have

$$P(\omega) = H(\omega)\overline{H(\omega)} = |H(\omega)|^2 \geq 0\,.$$

This means that P(ω) is even and real-valued. Orthogonality thus implies that the coefficients in the product filter must be symmetric, that is, $p_n = p_{-n}$. In the fourier domain we can now write the perfect reconstruction condition (1.11) as

(1.12) $$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2\,.$$

This condition will appear again when we study orthogonal wavelet bases. As we mentioned earlier, the only difference is that for wavelets the left-hand side should equal 1.

1.4.4 Biorthogonal Bases

Recall that an orthogonal filter bank could be seen as a realization of the expansion of a signal into a special type of discrete-time basis. This basis was formed by the even translates of two basis function φ and ψ, where $\varphi_k = h_k$ and $\psi_k = g_k$. And we had

$$x = \sum_n \left\langle x, \varphi^{(2n)} \right\rangle \varphi^{(2n)} + \sum_n \left\langle x, \varphi^{(2n+1)} \right\rangle \varphi^{(2n+1)} \;,$$

where $\varphi_k^{(2n)} = \varphi_{k-2n}$ and $\varphi_k^{(2n+1)} = \psi_{k-2n}$.Now, a biorthogonal filterbank corresponds to the biorthogonal expansion

$$x = \sum_n \left\langle x, \varphi^{(2n)} \right\rangle \varphi^{(2n)} + \sum_n \left\langle x, \varphi^{(2n+1)} \right\rangle \varphi^{(2n+1)},$$

Here, $\varphi^{(2n)} = \varphi_{k-2n}$ and $\varphi_k^{(2n+1)} = \psi_{k-2n}; \varphi_k = h_k and \psi_k = g_k$.

1.4.5 Design of Filter Bank

The discussion in the previous section showed that the construction of a perfect reconstruction filter bank can be reduced to the following three steps [2]:

1. Find a product filter P(z) satisfying P(z)+P(-z) = 2.
2. Factor, in some way, the product filter into the two lowpass filters,
   $P(z) = H(z)\tilde{H}(z)$
3. Define the highpass filter as
   $G(z) = z^{-L}\tilde{H}(-z)$, and $\tilde{G}(z) = z^{L} H(-z)$,

   where L is an arbitrary odd integer.

The coefficient in the product filter satisfies $p_0 = 1$ and $p_{2n} = 0$. In an orthogonal filter bank we also have the symmetry condition $p_n = p_{-n}$. The simplest example of such a product filter came from the Haar basis, where

$$P(z) = \frac{1}{2}(z + 2 + z^{-1}).$$

The question is now to find product filters of higher orders. Such product filter would, in turn, give us low- and highpass filters of higher orders, since P(z) = H(z) $\tilde{H}^*(z)$. There are usually several different ways to factor P(z) into H(z) and $\tilde{H}^*(z)$. For an orthogonal filter bank we have H(z) = $\tilde{H}$ (z), and in the more general case we have a so-called biorthogonal filter bank.

In this section we will describe how to construct one family of product filters discovered by the mathematician Ingrid Daubechies. There are other types of product filters, (which define several families of wavelet bases).

1.4.6 The Daubechies Product Filter

In 1988, Daubechies [3] proposed a symmetric product filter of the following form

$$P(z) = \left(\frac{1+z}{2}\right)^N \left(\frac{1+z^{-1}}{2}\right)^N Q_N(z),$$

Where $Q_N(z)$ is a symmetric polynomial with 2N-1 powers in z,

$$Q_N(z) = a_{N-1}z^{N-1} + \ldots + a_1 z + a_0 + a_1 z^{-1} + \ldots + a_{N-1}z^{1-N}.$$

The polynomial Q_N (z) is chosen so that P(z) satisfies the perfect reconstruction condition, and it is unique.

So far, no conditions have actually stated that H and $\tilde{H}$ should be lowpass filters, or that G and $\tilde{G}$ should be highpass filters. But we see that the Daubechies product filter is chosen so that P(z) has a zero of order 2N for z = -1, that is, P(ω) has a zero of order 2N for ω = π. This means that P(z) is the product of two lowpass filters. As we will see when we present the wavelet theory, the number of zeros are related to the approximation properties of wavelet bases. This is where the theory of wavelets has influenced the design of filter banks. There is nothing in the discrete-time theory that suggest why there should be more than one zero at z = -1 for the lowpass filters.

Let us illustrate with two examples.

Example 1.1. For N = 1 we obtain the product filter for the Haar basis,

$$P(z) = \left(\frac{1+z}{2}\right)\left(\frac{1+z^{-1}}{2}\right)Q_1(z).$$

Here the condition $p_0 = 1$ implies $Q_1(z) = a_0 = 2$ and we have

$$P(z) = \frac{1}{2}(z + 2 + z^{-1}).$$

Example 1.2. For N = 2 we get the next higher-order product filter

$$P(z) = \left(\frac{1+z}{2}\right)^2\left(\frac{1+z^{-1}}{2}\right)^2 Q_2(z).$$

Here $Q_2(z) = a_1 z + a_0 + a_1 z^{-1}$, and if we substitute this expression into P(z) and simplify we get

$$P(z)=\frac{1}{16}\left(a_1 z^3 + (a_0 + 4a_1)z^2 + (4a_0 + 7a_1)z + (6a_0 + 8a_1) + (4a_0 + 7a_1)z^{-1} + (a_0 + 4a_1)z^{-2} + a_1 z^{-3}\right)$$

The perfect reconstruction condition $p_0 = 1$ and $p_2 = 0$ give the linear system

$$\begin{Bmatrix} 6a_0 + 8a_1 = 16 \\ a_0 + 4a_1 = 0 \end{Bmatrix}$$

With solution $a_0 = 4$ and $a_1 = -1$.We then have

$$P(z) = \frac{1}{16}\left(-z^3 + 9z + 16 + 9z^{-1} - z^{-3}\right).$$

1.4.7 Factorization

Let us first assume that we want to construct an orthogonal filter bank using the symmetric Daubechies product filter. Then, since P(z) = H(z) H(z^{-1}), we know that the zeros of P(z) always come in pairs as z_k and z_k^{-1} . when we factor P(z) we can, for each zero z_k, let either (z- z_k) or (z - z_k^{-1}) be a factor of H(z). If we always choose the zero that is inside or on the unit circle, $|z_k| \le 1$, then H(z) is called the minimum phase factor of P(z).

Now, suppose we also want the filter H(z) to be symmetric. Then the zeros of H(z) must come together as z_k and z_k^{-1}. But this contradicts the orthogonality condition except for the Haar basis, where both zeros are at z = -1. Thus orthogonal filter banks can not have symmetric filters.

In a biorthogonal basis, or filter bank, we factor the product filter as P(z) = H(z) $\tilde{H}$ (z). There are several ways of doing so, and we then obtain several different filter banks for a given product filter. In most cases, we want the filter H(z) and $\tilde{H}$ (z) to be symmetric, unless we are designing an orthogonal filter bank, that is.

Finally, since we want both H(z) and $\tilde{H}$ (z) to have real coefficients, we always let the complex conjugate zeros z_k and $\bar{z}_k$ belong to either H(z) or $\tilde{H}$ (z).

Chapter 2

Multiresolution Analysis

This chapter is devoted to the concept of Multi-Resolution Analysis(MRA) [6].As the name suggests, the basic idea is to analyze a function at different resolutions, or scales. Wavelets enter as a way to represent the difference between approximations at different scales.

We begin this chapter with the definition of scaling functions, multiresolution analysis and wavelets.Thereafter, we study orthogonal system.

2.1 Scaling Functions and Approximation

The central idea in a multiresolution analysis (MRA) is to approximate functions at different scales, or levels of resolution. These approximations are provided by the scaling function, which is sometimes also called the approximation function.

2.1.1 The Haar Scaling Function

The Haar scaling function is the simplest example; it is defined by

$$\varphi(t) = \begin{cases} 1 \ \ if \ 0 < t < 1, \\ 0 \ \ otherwise. \end{cases}$$

With this scaling function, we get piecewise constant approximations. For instance, we can approximate a function f with a function f_1 that is piecewise constant on the intervals (k/2,(k+1)/2); k$\in Z$. This function can be written as

(2.1) $$f_1(t) = \sum_k s_{1,k} \varphi(2t-k).$$

This should be obvious since $\varphi(2t-k)$ equals 1 on (k/2,(k+1)/2) and 0 otherwise. The subspace of functions in L^2 (R) of the form 2.1 is denoted V_1. The coefficients ($s_{1,k}$) may be chosen as the mean values of f over the intervals (k/2,(k+1)/2),

$$s_{1,k} = 2\int_{k/2}^{(k+1)/2} f(t)dt = 2\int_{-\infty}^{\infty} f(t)\varphi(2t-k)dt.$$

The approximation (2.1) is in fact the orthogonal projection of f onto V_1 having the ON basis $\{2^{1/2}\varphi(2t-k)\}$. For notational convenience, the factor $2^{1/2}$ is included in $s_{1,k}$.

The coefficients ($s_{1,k}$) could also be chosen as the sample values of f at t = k/2, $s_{1,k}$ = f(k/2).

We may approximate f on twice a coarser scale by a function f_0, that is piecewise constant on the intervals (k,k+1),

(2.2) $$f_0(t) = \sum_k s_{0,k}\varphi(t-k).$$

If the coefficients ($s_{0,k}$) are chosen as mean values over the intervals (k,k+1), it is easy to verify the relation, $s_{0,k} = \frac{1}{2}(s_{1,2k} + s_{1,2k+1}).$

The linear space of the functions in (2.2) is denoted V_0. More generally, we can get piecewise constant approximations on intervals $(2^{-j}k, 2^{-j}(k+1))$, $j \in Z$, with the function of the form

$$f_j(t) = \sum_k s_{j,k}\varphi(2^j t - k).$$

The corresponding linear space of functions is denoted V_j. The spaces V_j are referred to as approximation spaces at the scale 2^{-j}.

2.1.2 The Definition of Multiresolution Analysis

A disadvantage with the Haar scaling function is that it generates discontinuous approximations. Another scaling function is the hat function

$$\varphi(t) = \begin{cases} 1+t & if\ -1 \le t \le 0, \\ 1-t & if\ 0 \le t \le 1, \\ 0 & otherwie. \end{cases}$$

Taking linear combinations of $\varphi(t-k)$ as in (2.2) gives continuous, piecewise linear approximations.

The approximation spaces V_j in this case consist of continuous functions that are piecewise linear on the intervals $(2^{-j}k, 2^{-j}(k+1))$. This will often give better approximations compared to the Haar scaling function.

We now put forward a general framework for the construction of scaling functions and approximation spaces. This is the notion of a multiresolution analysis.

Definition 2.1. A multiresolution analysis (MRA) is a family of closed subspaces V_j of $L^2(R)$ with the following properties:

1. $V_j \subset V_{j+1}$ for all $j \in Z$,
2. $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$ for all $j \in Z$,
3. $\bigcup_j V_j$ is dense in $L^2(R)$,
4. $\bigcap_j V_j = \{0\}$,
5. There exists a scaling function $\varphi \in V_0$ such that $\{\varphi(t-k)\}$ is a Riesz basis for V_0.

The first condition just states that function in V_{j+1} contain more details than function in V_j: in a certain sense, we add information when we approximate a function at a finer scale. The second condition says that V_{j+1} approximates functions at twice a finer scale than V_j, and also gives a connection between the spaces V_j. The fifth condition requires the approximation spaces to be spanned by scaling functions. Let us introduce the dilated, translated, and normalized scaling functions $\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k)$.

After a scaling by 2^j, it is easy to see that for fixed j, the scaling functions constitute a Riesz basis for V_j. Thus, every $f_j \in V_j$ can be written as

(2.3) $$f_j(t) = \sum_k s_{j,k}\varphi_{j,k}(t).$$

The reason for the factors $2^{j/2}$ is that all scaling functions have equal norms, $\|\varphi_{j,k}\| = \|\varphi\|$. The remaining two conditions are of a more technical nature. They are needed to ensure that the wavelets, which we will introduce soon, give a Riesz basis for $L^2(R)$. The third basically says that any function can be approximated arbitrarily well with a function $f_j \in V_j$, if we just choose the scale fine enough. This is what is meant by density. Finally, the fourth condition says, loosely speaking, that the only function that can be approximated at an arbitrarily coarse scale is the zero function.

2.1.3 Properties of the Scaling Function

The definition of MRA imposes quite strong restrictions on the scaling function. In addition, there are some other properties we want the scaling function to have. It should be localized in time, which means a fast decay to zero as $|t| \to \infty$, preferably. It should have compact support, that is, be zero outside a bounded interval. This localization property ensures that the coefficients $(s_{j,k})$ in the approximations (2.3) contain local information about f. Note that the scaling functions we have seen so far, the Haar and hat scaling functions, both have compact support. We further want the scaling function to have integral one,

(2.4) $$\int_{-\infty}^{\infty} \varphi(t)dt = 1.$$

This condition has to do with the approximation properties of the scaling function.

Let us now return to the definition of MRA and see what it implies for the scaling function. Since the scaling function belongs to V_0, it also belongs to V_1, according to condition 1. Thus , it can be written as

(2.5) $$\varphi(t) = 2\sum_k h_k \varphi(2t-k),$$

for some coefficients (h_k). This is the scaling equation. Taking the Fourier transform of the scaling equation gives us

(2.6) $$\hat{\varphi}(w) = H(w/2)\hat{\varphi}(w/2)$$

where,

$$H(w) = \sum_k h_k e^{-ikw}.$$

Letting ω = 0 and using $\hat{\varphi}(0) = 1$, we get H(o) = $\sum h_k = 1$, and we see that the coefficients (h_k) may be interpreted as an averaging filter. In fact, as we will letter see, also H(π) = 0 holds, and thus H is a lowpass filter. It can be shown that the scaling function is uniquely defined by this filter together with the normalization (2.4). As a matter of fact, repeating (2.6) and again using $\hat{\varphi}(0) = 1$, yields (under certain conditions) the infinite product formula

(2.7) $$\hat{\varphi}(w) = \prod_{j>0} H(w/2^j).$$

The properties of the scaling function are reflected in the filter coefficients (h_k), and scaling functions are usually constructed by designing suitable filters.

Example 2.1. The B-spline of order N is defined by the convolution

$$S_N(t) = X(t) * \ldots * X(t), \qquad \text{(N factors)}$$

where X(t) denotes the Haar scaling function. The B-spline is a scaling function for each N. The translated functions S_N (t-k) gives N-2 times continuously differentiable, piecewise polynomial approximations of degree N-1. The cases N = 1 and N = 2 corresponds to the Haar and the hat scaling functions, respectively. When N grows larger, the scaling functions become more and more regular, but also more and more spread out.

Example 2.2. The sinc function

$$\varphi(t) = \frac{\sin \pi t}{\pi t},$$

is another scaling function. It does not have compact support, and the decay as $|t| \to \infty$ is very slow. Therefore, is it not used in practice. It has interesting theoretical properties though.

It is in a sense dual to the Haar scaling function, since its Fourier transform is given by the box function $$\hat{\varphi}(w) = \begin{Bmatrix} 1 \;\; if \; -\pi < w < \pi \\ 0 \;\; otherwise \end{Bmatrix}$$

It follows that every function in V_0 is band-limited with cut-off frequency π. In fact, the Sampling Theorem states that every such band-limited function f can be reconstructed from its sample values f (k) via $f(t) = \sum_k f(k) \sin c(t-k).$

Thus, V_0 equals the set of band-limited functions with cut-off frequency π. The V_j spaces, by scaling, become the set of functions band-limited to the frequency bands $(-2^j \pi, 2^j \pi)$.

2.2 Wavelets and Detail Spaces

We will now turn to a description of the difference between two successive approximation spaces in a multiresolution analysis: the wavelet or detail spaces.

2.2.1 The Haar Wavelet

A multiresolution analysis allows us to approximate functions at different levels of resolution. Let us look at the approximations of a function f at two consecutive scales, $f_0 \in V_0$ and $f_1 \in V_1$. The approximation f_1 contains more details than f_0 and the difference is the function

$$d_0 = f_1 - f_0.$$

We return again to the Haar system. Here, f_1 is piecewise constant on the interval (k/2, (k+1)/2) with values $s_{1,k}$, and f_0 is piecewise constant on the intervals (k,k+1) with values $s_{0,k}$ that are pairwise mean values of the $s_{1,k}$: s

(2.8) $$s_{0,k} = \frac{1}{2}(s_{1,2k} + s_{1,2k+1}).$$

We have plotted the function d_0 in the Haar system. It is piecewise constant on the intervals (k/2, (k+1)/2) so $d_0 \in V_1$. Let us consider the interval (k,k+1) .

Denote d_0: s value on the first half by $w_{0,k}$ .The value on the second half is then $-w_{0,k}$ and they both measure the deviation of f_1 from its mean value on the interval (k,k+1):

(2.9) $$w_{0,k} = \frac{1}{2}(s_{1,2k} - s_{1,2k+1}).$$

The Haar wavelet is defined by

$$\psi(t) = \begin{Bmatrix} 1 & if\, 0 < t < 1/2 \\ -1 & if\, 1/2 < t < 1 \\ 0 & otherwise \end{Bmatrix}$$

Using this wavelet we can write d_0 as

(2.10) $$d_0(t) = \sum_k w_{0,k}\psi(t-k),$$

with wavelet coefficients ($w_{0,k}$) at scale $1=2^0$. We want to generalize this to any MRA, and therefore make the following definition.

Definition 2.2. For a general MRA, a function ψ is said to be a wavelet if the detail space W_0 spanned by the functions ψ(t-k) complements V_0 $in V_1$. By this we mean that any $f_1 \in V_1$ can be uniquely written as $f_1 = f_0 + d_0$, where $f_0 \in V_0$ and $d_0 \in w_0$. We write this formally as $V_1 = V_0 \oplus W_0$. Finally, we require the wavelet ψ(t-k) to be a Riesz basis for W_0.

Note that the space W_0 need not be unique. However, the decomposition $f_1 = f_0 + d_0$ is unique, once the wavelet ψ (the space W_0) is chosen. However, if we require W_0 to be orthogonal to V_0, then W_0 is uniquely determined.

Example 2.3. When the scaling function is the hat function the wavelet can be chosen as the function in V_1 with values at the half-integers

f(0) = 3/2,

f(1/2) = f(-1/2) = -1/2,

f(1) = f(-1) = -1/4.

In this case, V_0 and W_0 are not orthogonal.

Example 2.4. For the sinc scaling function from Example 2.2, we choose the wavelet as

$$\hat{\psi}(w) = \begin{cases} 1 \;\; if \pi < |w| < 2\pi \\ 0 \;\; otherwise \end{cases}$$

It is easy to see that ψ (t) = sinc 2t-sinct. This is the sinc wavelet. The space W_0 will be the set of function that are band-limited to the frequency band $\pi < |w| < 2\pi$. More generally, the space W_j will contain all function that are band-limited to the frequency band $2^j \pi < |w| < 2^{j+1}\pi$.

2.2.2 Properties of the Wavelet

The function ψ is sometimes called the mother wavelet. As with the scaling function, we want it to be localized in time. We also want it to have integral zero $\int \psi(t)dt = 0,$

or $\hat{\psi}(0) = 0$, and it thus has to oscillate. This is also referred to as the cancellation property and it is connected to having wavelets represent differences. The function ψ will be a wave that quickly emerges and dies off, hence the term wavelet (small wave). Since $\psi \in V_1$ it can be written as

(2.11) $$\psi(t) = 2\sum_k g_k \varphi(2t-k),$$

for some coefficients (g_k). This is the wavelet equation. A Fourier transform gives

(2.12) $$\hat{\psi}(w) = G(w/2)\hat{\varphi}(w/2),$$

where

$$G(w) = \sum g_k e^{-ikw}.$$

using $\hat{\varphi}$ (0) = 1 and $\hat{\psi}$ (0) = 0, We get that G(0) = $\sum g_k$ = 0. Thus the coefficients (g_k) can be interpreted as a difference filter. Later we will see that also G(π) = 1 holds, and G is in fact a highpass filter. The wave and all its properties are determined by this filter, given the scaling function.

2.2.3 The Wavelet Decomposition

The dilated and translated wavelets $\psi_{j,k}$ are defined as $\psi_{j,k}(t) = 2^{j/2}\psi(2^j k - t)$.

The detail spaces W_j are defined as the set of functions of the form

(2.13) $$d_j(t) = \sum_k w_{j,k}\psi_{j,k}(t).$$

From definition 4.5 it follows that any $f_1 \in V_1$ can be decomposed as $f_1 = f_0 + d_0$, where $f_0 \in V_0$ is an approximation at twice are coarser scale, and $d_0 \in W_0$ contains the lost details. After a scaling with 2^j, we see that a function $f_{j+1} \in V_{j+1}$ can be decomposed as $f_{j+1} = f_j + d_j$, where $f_j \in V_j$ and $d_j \in W_j$, that is, $V_{j+1} = V_j \oplus W_j$.

Starting at a finest scale j, and repeating the decomposition $f_{j+1} = f_j + d_j$, until a certain level j_0, we can write any $f_j \in V_j$ as

$$f_j(t) = d_{j-1}(t) + d_{j-2}(t) + \ldots.d_{j0}(t) + f_{j0}(t)$$
$$= \sum_{j=j0}^{j-1}\sum_k w_{j,k}\psi_{j,k}(t) + \sum_k s_{j0,k}\varphi_{j0,k}(t).$$

We can express this in terms approximation and detail spaces as

$$V_j = W_{j-1} \oplus W_{j-2} \oplus \ldots.. \oplus W_{j0} \oplus V_{j0}.$$

Using the fourth condition in the definition of MRA one can show that f_{j0} goes to 0 in L^2 when $j_0 \to -\infty$. The third condition now implies that, choosing j larger and larger, we can approximate a function f with approximations f_j that become closer and closer to f. Letting $j \to \infty$ therefore gives us the wavelet decomposition of f

(2.14) $$f(t) = \sum_{j,k} w_{j,k} \varphi_{j,k}(t).$$

We have thus indicated how to prove that $\{\psi_{j,k}\}$ is a basis for L²(R). However, it still remains to construct the highpass filter G determining the mother wavelet ψ.

The decomposition $V_{j+1} = V_j \oplus W_j$ above is not unique. There are many ways to choose the wavelet ψ and the corresponding detail spaces W_j. Each such choice corresponds to a choice of the highpass filter G. There is a choice, which gives us an orthogonal system or orthogonal wavelet basis. This corresponds to choosing H and G as filters in an orthogonal filter bank.

We conclude this section by looking at the wavelet decomposition in the frequency domain. We saw earlier that for the sinc scaling function, the V_j spaces are the spaces of functions band-limited to the frequency bands $(0, 2^j \pi)$ (actually $(-2^j \pi, 2^j \pi)$, but we ignore negative frequencies to simplify the discussion). The detail spaces are the sets of band-limited functions in the frequency bands $(2^j \pi, 2^{j+1} \pi)$. The wavelet decomposition can in this case be seen as a decomposition of the frequency domain. For other wavelets, this frequency decomposition should be interpreted approximately, since the wavelet $\psi_{j,k}$ have frequency contents outside the band $(2^j \pi, 2^{j+1} \pi)$.

2.3 Orthogonal Systems

In this section, the wavelet space W_0 is to be orthogonal to the approximation space V_0. This means that we will get an orthogonal decomposition of V_1 into $V_1 = V_0 \oplus W_0$, and ultimately we will arrive at an orthogonal wavelet basis in L²(R).

2.3.1 Orthogonality Conditions

The first requirement is that the scaling functions $\varphi(t-k)$ constitute an orthogonal basis for V_0, that is,

$$\int_{-\infty}^{\infty} \varphi(t-k)\varphi(t-l)dt = \delta_{k,l}.$$

Using the scaling equation (2.5) we can transform this to a condition on the coefficients (h_k)

(2.15) $$\sum_l h_l h_{l+2k} = \delta_k / 2.$$

We also require the wavelets $\psi(t-k)$ to form an orthogonal basis for W_0

$$\int_{-\infty}^{\infty} \psi(t-k)\psi(t-l)dt = \delta_{k,l}.$$

Expressed in the coefficients (g_k) this becomes V_0 must be orthogonal to functions in

(2.16) $$\sum_l g_l g_{l+2k} = \delta_k / 2.$$

Finally, functions in W_0 which is formally written as $V_0 \perp W_0$. Then the scaling functions $\varphi(t-k)$ have to be orthogonal to each wavelet $\psi(t-l)$:

$$\int_{-\infty}^{\infty} \varphi(t-k)\psi(t-l)dt = 0, \text{ for all k and l.}$$

For the filter coefficients this means that

(2.17) $$\sum_m h_{m+2k} g_{m+2l} = 0.$$

All of the above orthogonality conditions can be transposed to an arbitrary scale. Using the scalar product notation we have

$$\langle \varphi_{j,k}, \varphi_{j,l} \rangle = \delta_{k,l},$$

$$\langle \psi_{j,k}, \psi_{j,l} \rangle = \delta_{k,l},$$

$$\langle \varphi_{j,k}, \psi_{j,l} \rangle = 0.$$

In other words, $\{\varphi_{j,k}\}$ is an orthonormal basis for $V_j, \{\psi_{j,k}\}$ for W_j, and $V_j \perp W_j$. The approximation $f_j \in V_j$ of a function f can chosen as the orthogonal projection onto V_j, which we denote by P_j. It can be computed as

$$f_j = p_j f = \sum_k \left\langle f, \varphi_{j,k} \right\rangle \varphi_{j,k}.$$

The detail d_j becomes the projection of f onto W_j,

$$d_j = Q_j f = \sum_k \left\langle f, \psi_{j,k} \right\rangle \psi_{j,k}.$$

In terms of the filter functions $H(\omega)$ and $G(\omega)$, (2.15) – (2.17) becomes

$$\begin{aligned} &|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1, \\ &|G(\omega)|^2 + |G(\omega+\pi)|^2 = 1, \\ &H(\omega)\overline{G(\omega)} + H(\omega+\pi)\overline{G(\omega+\pi)} = 0. \end{aligned} \tag{2.18}$$

Actually, these are the conditions on $\sqrt{2}$ H and $\sqrt{2}$ G to be low- and highpass filters in an orthogonal filter bank. This is a crucial observation, since construction of orthogonal scaling function and wavelet becomes equivalent with the construction of orthogonal filter banks.

2.3.2 Characterizations in the Orthonormal Case

As we now will see, orthogonal filter bank does not always lead to an orthonormal wavelet basis.

First of all, the infinite product (2.7) has to converge. If it does converge, it need not produce a function in L². Consider, for example, the most simple low-pass filter $h_k = \delta_k$, the identity filter. Together with the high-pass filter $g_k = \delta_{k-1}$, this gives an orthogonal filter bank. It is called the lazy filter bank, since it does not do anything except splitting the input signal into even- and odd-indexed samples. The filter function is $H(\omega) = 1$, and the infinite product formula implies that $\hat{\varphi}(\omega) = 1$. The scaling function is then the Dirac delta distribution, which is not even a function. Even if the product yields a function in L², we have to make sure that also conditions 3 and 4 in the definition of MRA are satisfied. An additional simple sufficient condition on the low-pass filter $H(\omega)$, determining the scaling function, to satisfied 3 and 4 is $H(\omega)>0$ for $|\omega| \le \pi/2$ (and $\hat{\varphi}(0) = 1$).

Now we are about to give precise characterizations on scaling function and wavelets in an orthogonal MRA. All equations are to hold almost everywhere (a.e.). We advice readers not familiar with this concept, to interpret almost everywhere as “everywhere except a finite number of points”.

Chapter 3

Mathematical Properties of Wavelets

Definition 1-Two-Scale Relation: The scaling function $\varphi(x)$ associated with the filter H(Z) is the L^2 -solution (if it exists) of the two-scaling relation [4]

$$\varphi(x) = \frac{2}{H(1)} \sum_{K \in Z} h_k \varphi(2x - k).$$

Definition 2-Approximation Order: The number L of factors $(1+z^{-1})$ that divide H(Z) is called approximation order [4].

Definition 3-Holder Regularity: The smallest real number r_c such that : 1) φ is N = $\lceil r \rceil - 1$ times continuously differentiable for $r < r_c$; and 2) [4]

$$\sup_{x,y \in R} \frac{\left|\varphi^{(N)}(y) + \varphi^{(N)}(x)\right|}{\left|y - x\right|^{r-N}} \text{ is finite for all } r < r_c$$

is called Holder critical regularity exponent.

Exact Computation of the Holder Exponent

In general, there is no systematic algorithm to compute the Holder regularity exponent for an arbitrary scaling function φ, and one usually has to turn to numerical methods to obtain estimates [7], [8], [9],[10]. Fortunately for us, an exact procedure is available when the filters are symmetric and positive on the unit circle, which is precisely the case for JPEG2000 filters. It is as follows.

- Eliminate all the order factors $(1+z^{-1})$ from H(z); i.e., find Q(z) such that H(z) = $(H(1)/2)z^{L/2}$ $(1+z^{-1})^{L}$ Q(z). Note that, because of the positivity on the unit circle, L is necessarily even.
- Build the matrix $[Q]_{k,l} = q_{2k-l}$ *where* $-L_Q \le k,l \le L_Q$ *and* $\{-L_Q......L_Q\}$ is the support of the filter Q.
- Find the largest eigenvalue, λ, of Q. Then r = $-\log_2|\lambda|$.

Definition 4-Sobolev Regularity: The smallest number s_c such that

$$\int |\varphi(\omega)|^2 (1+\omega^2)^8 d\omega \text{ is finite for all } s < s_c$$

is called Sobolev critical regularity exponent [4].

Exact Computation of the Sobolev Exponent

Computing the Sobolev exponent is an easier theoretical task; unlike the Holder exponent, it can always be achieved through the determination of the spectral radius of a reduced transition operator [9], [11], without any symmetry or positivity assumption on the filter. It turns out the algorithm is essentially the same as the one outlined above with the filter now being $H_2(z) = H(z)H(z^{-1})$. This is simply because the Sobolev exponent is one half the Holder exponent of the autocorrelation function $\varphi_2(x) = \int \varphi(\xi)\varphi(\xi + x)d\xi$ which is the scaling function corresponding to the refinement filter $H_2(z)$. Thus, the method consists in computing the Holder exponent r_2 *of* φ_2, which provides $s = (1/2)r_2$.

Both measures are linked through the following inequality: $r_c \le s_c \le r_c + 1/2$

Now we are calculating the important mathematical properties of some well known wavelets.

As the JPEG2000 LeGall 5/3 scaling filters are given by

$$\tilde{H}(\mathrm{z}) = \frac{1}{8} z(1+z^{-1})^2(-z-z^{-1}+4)$$

$$\mathrm{H(z)} = \frac{1}{2} z(1+z^{-1})^2$$

For analysis part of LeGall 5/3 scaling filter is

$$\tilde{H}(\mathrm{z}) = \frac{1}{8} z(1+z^{-1})^2(-z-z^{-1}+4)$$

So, its Approximation Order =2

To find Holders Regularity:

$$\tilde{H}(1) = 1$$

Equating equations , we get

$$\frac{1}{8}z(1+z^{-1})^2(-z-z^{-1}+4)=\frac{1}{2}z^{L/2}(1+z^{-1})^L Q(z)$$

Taking L=2 ,we get

$$\frac{1}{8}z(1+z^{-1})^2(-z-z^{-1}+4)=\frac{1}{2}z\ \ (1+z^{-1})^2 Q(z)$$

$$Q(z) = \frac{1}{4}(-z-z^{-1}+4)$$

$$= -\frac{1}{4}z^{-1}-\frac{1}{4}z+1$$

So,support of Q(z) is $\left\{\frac{-1}{4},1,\frac{-1}{4}\right\}$

$$Q_{k,l}=\begin{bmatrix}\frac{-1}{4} & 0 & 0\\ \frac{-1}{4} & 1 & \frac{-1}{4}\\ 0 & 0 & \frac{-1}{4}\end{bmatrix} \qquad \text{where } -1\le k,l\le 1$$

Eigen values are $\frac{-1}{4},\frac{-1}{4},1$

But largest value is 1.So r = 0

Hence, Holders Regularity for analysis part of LeGall 5/3 scaling filter is 0.

To find Sobolev Regularity:

$$\tilde{H}_2(z)=\tilde{H}(z)\tilde{H}(z^{-1})$$

$$=(\frac{1}{8}z(1+z^{-1})^2(-z-z^{-1}+4))(\frac{1}{8}z^{-1}(1+z)^2(-z-z^{-1}+4))$$

$$=\frac{1}{64}(1+z)^2(1+z^{-1})^2(-z-z^{-1}+4)^2$$

$$\tilde{H}_2(1)=1$$

Equating equations , we get

$$\frac{1}{64}(1+z)^2(1+z^{-1})^2(-z-z^{-1}+4)^2 = 2z^{L/2}(1+z^{-1})^L Q(z)$$

Taking L=2 ,we get

$$\frac{1}{64}(1+z)^2(1+z^{-1})^2(-z-z^{-1}+4)^2 = \frac{1}{2}z(1+z^{-1})^2 Q(z)$$

$$\text{Q(z)} = \frac{1}{32}z^{-1}(1+z)^2(-z-z+4)^2$$

So,support of Q(z) is $\left\{\frac{1}{32},\frac{-6}{32},\frac{3}{32},\frac{20}{32},\frac{3}{32},\frac{-6}{32},\frac{1}{32}\right\}$

Build matrix $Q_{k,l}$ (where $-3 \le k,l \le 3$)using the same method.

Eigen values are -0.2501, -0.2303, 0.1251, 0.2502, 0.5427, 0.0313, 0.0313

But largest value is 0.5427

s = 0.44

Hence, Sobolev Regularity for analysis part of LeGall 5/3 scaling filter is 0.44

Now for synthesis part of LeGall 5/3 scaling filter is

$$\text{H(z)} = \frac{1}{2}z(1+z^{-1})^2$$

Approximation Order = 2

To find Holders Regularity:

$$\text{H(1)} = 2$$

Equating equations , we get

$$\frac{1}{2}z(1+z^{-1})^2 = Z^{L/2}(1+z^{-1})^L Q(z)$$

Taking L=2,we get

$$\frac{1}{2}z(1+z^{-1})^2 = Z\ \ (1+z^{-1})^2 Q(z)$$

$$Q(z) = \frac{1}{2}$$

So,support of Q(z) is $\left\{\frac{1}{2}\right\}$

$Q_{k,l} = \left\{\frac{1}{2}\right\}$ where $1 \le k, l \le 1$

Eigen value is $\frac{1}{2}$

So, r = 1

Hence, Holders Regularity for synthesis part of LeGall 5/3 scaling filter is 1.

To find Sobolev Regularity:

$$H_2(z) = H(z)H(z^{-1})$$
$$= \frac{1}{4}(1+z)^2(1+z^{-1})^2$$
$$H_2(1) = 4$$

Equating equations , we get

$$\frac{1}{4}(1+z)^2(1+z^{-1})^2 = 2z^{L/2}(1+z^{-1})^L Q(z)$$

Taking L=2,we get

$$\frac{1}{4}(1+z)^2(1+z^{-1})^2 = 2z(1+z^{-1})^2 Q(z)$$

$$Q(z) = \frac{1}{8}(z^{-1}+2+z)$$

So,support of Q(z) is $\left\{\frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right\}$

$$Q_{k,l} = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{8} \end{bmatrix} \qquad where \ -1 \le k,l \le 1$$

Eigen values are 0.25, 0.125, 0.125

But largest value is 0.25

So r=2 and we know that s = r/2

s = 1

Hence, Sobolev Regularity for synthesis part of LeGall 5/3 scaling filter is 1

The JPEG2000 Daubechies 9/7 scaling filters are given by

$$\tilde{H}(z) = \frac{2^{-5}}{\frac{64}{5\rho} - 6 + \rho} z^2 (1 + z^{-1})^4 \times (z^2 + z^{-2} - (8-\rho)(z + z^{-1}) + \frac{128}{5\rho} + 2)$$

$$H(z) = \frac{2^{-3}}{\rho - 2} z^2 (1 + z^{-1})^4 (-z - z^{-1} + p)$$

For analysis part of Daubechies 9/7 scaling filter is

$$\tilde{H}(z) = \frac{2^{-5}}{\frac{64}{5\rho} - 6 + \rho} z^2 (1 + z^{-1})^4 \times (z^2 + z^{-2} - (8-\rho)(z + z^{-1}) + \frac{128}{5\rho} + 2)$$

Approximation Order = 4

To find Holders regularity:

H(1) = 1

Equating equations, we get

$$Q(z) = 0.0535(z^{-2} - 4.6305z^{-1} + 9.5976z^2 - 4.6305z + z^2)$$

So, support of Q(z) is {0.0535, -0.2477, 0.5135, -0.2477, 0.0535}

Build matrix $Q_{k,l}$ (where $-2 \le k,l \le 2$)using the same method.

Eigen values are 0.4769, -0.2111, -0.2477, 0.0535, 0.0535

Largest value is 0.4769

So, r = 1.07

Hence, Holders Regularity for analysis part of Daubechies scaling filter is 1.07

To find Sobolev Regularity:

$$\tilde{H}_2(z) = 0.0007(1+z)^4(1+z^{-1})^4(z^2+z^{-2}-4.6305z-4.6305z^{-1}+9.5975)^2$$

$$\tilde{H}_2(1) = 0.9783$$

Equating equations , we get

$$0.0007(1+z)^4(1+z^{-1})^4(z^2+z^{-2}-4.6305z-4.6305z^{-1}+9.5975)^2 = 0.48915z^{L/2}(1+z^{-1})^L Q(z)$$

Taking L = 4,we get

$$0.0007(1+z)^4(1+z^{-1})^4(z^2+z^{-2}-4.6305z-4.6305z^{-1}+9.5975)^2 = 0.48915z^2(1+z^{-1})^4 Q(z)$$

$$\begin{aligned} Q(z) &= 0.0014z^{-2}(1+z)^4(z^2+z^{-2}-4.6305z-4.6305z^{-1}+9.5975)^2 \\ &= 0.0014(z^{-6}-5.261z^{-5}+9.592z^{-4}+12.8346z^{-3}-47.8065z^{-2}+14.2596z^{-1}+118.0954+ \\ &\quad 14.2596z-47.8095z^2+12.8346z^3+9.592z^4-5.261z^5+z^6) \end{aligned}$$

So,support of Q(z) is

$$\begin{Bmatrix} 0.0014, -0.0074, 0.0134, 0.0179, -0.0669, 0.0199, 0.1653, 0.0199, -0.0669, 0.0179, 0.0134, \\ -0.0074, 0.0014 \end{Bmatrix}$$

Build matrix $Q_{k,l}$ (where $-6 \le k,l \le 6$) using the same method.

Eigen values are 0.1386, 0.1096, -0.0779, 0.0608, -0.0092, -0.0086, 0.0077, 0.0154, 0.0384, 0.0303, 0.0332, 0.0014, 0.0014

But maximum value is 0.1386

So r = 2.85 and we know that s = r/2.

s = 1.42

Hence, Sobolev Regularity for analysis part of Daubechies 9/7 scaling filter is 1.42

Now for synthesis part of Daubechies 9/7 scaling filter is

H(z) = $\frac{2^{-3}}{\rho-2} z^{2}(1+z^{-1})^{4}(-z-z^{-1}+p)$

Approximation Order = 4

To find Holders regularity:

H(1) = 2

Equating equations, we get

Q(z) = $0.0913(-z^{-1}+3.3695-z)$

So, support of Q(z) is {-0.0913, 0.3075, -0.0913}

Build matrix $Q_{k,l}$ (where $-1 \le k,l \le 1$)using the same method.

Eigen values are 0.3075, -0.0913, -0.0913

Largest value is 0.3075

So, r = 1.70

Hence, Holders Regularity for synthesis part of Daubechies 9/7 scaling filter is 1.70

To find Sobolev Regularity:

$$H_2(z) = 0.0083(1+z)^4(1+z^{-1})^4(-z-z^{-1}+3.3695)^2$$
$$H_2(1) = 4$$

Equating equations, we get

$$0.0083(1+z)^4(1+z^{-1})^4(-z-z^{-1}+3.3695)^2 = 2z^{L/2}(1+z^{-1})^L Q(z)$$

Taking L = 4, we get

$$0.0083(1+z)^4(1+z^{-1})^4(-z-z^{-1}+3.3695)^2 = 2z^2(1+z^{-1})^4 Q(z)$$

Q(z) = $0.0042\, z^{-2}(1+z)^4(-z-z^{-1}+3.3695)^2$

$=0.0042(z^{-4}-2.739z^{-3}-7.6025z^{-2}+10.241z^{-1}+28.209+10.241z-7.6025z^2-2.739z^3+z^4)$

So, support of Q(z) is

$\{0.0042, -0.0115, -0.0319, 0.0430, 0.1185, 0.0430, -0.0319, -0.0115, 0.0042\}$

$$Q_{k,l} = \begin{bmatrix} 0.0042 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0319 & -0.0115 & 0.0042 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1185 & 0.0430 & -0.0319 & -0.0115 & 0.0042 & 0 & 0 & 0 & 0 \\ -0.0319 & 0.0430 & 0.1185 & 0.0430 & -0.0319 & -0.0115 & 0.0042 & 0 & 0 \\ 0.0042 & -0.0115 & -0.0319 & 0.0430 & 0.1185 & 0.0430 & -0.0319 & -0.0115 & 0.0042 \\ 0 & 0 & 0.0042 & -0.0115 & -0.0319 & 0.0430 & 0.1185 & 0.0430 & -0.0319 \\ 0 & 0 & 0 & 0 & 0.0042 & -0.0115 & -0.0319 & 0.0430 & 0.1185 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0042 & -0.0115 & -0.0319 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0042 \end{bmatrix}$$

Where $-4 \le k,l \le 4$

Largest value is 0.0639

So, s = 1.98

Hence, Sobolev Regularity for synthesis part of Daubechies 9/7 scaling filter is 1.98

For db2 [5] scaling filter is

H(z) = $\frac{1}{2}(1+z^{-1})$

Approximation Order = 1

To find Holders regularity:

H(1) = 1

Equating equations, we get

$$\frac{1}{2}(1+z^{-1}) = \frac{1}{2} z^{L/2}(1+z^{-1})^L Q(z)$$

Taking L = 1,we get

$$\frac{1}{2}(1+z^{-1}) = \frac{1}{2}z^{1/2}(1+z^{-1})Q(z)$$

$$Q(z) = z^{-1/2}$$

So, support of Q(z) is {1}

$$Q_{k,l} = [1]$$

Eigen value is 1

So, r = 0

Hence, Holders Regularity for db2 is 0.

To find Sobolev Regularity:

$$H_2(z) = \frac{1}{4}(1+z)(1+z^{-1})$$

$$H_2(1) = 1$$

Equating equations, we get

$$\frac{1}{4}(1+z)(1+z^{-1}) = \frac{1}{2}z^{L/2}(1+z^{-1})^L Q(z)$$

Taking L = 1, we get

$$\frac{1}{4}(1+z)(1+z^{-1}) = \frac{1}{2}z^{1/2}(1+z^{-1})\ Q(z)$$

Q(z) = $z^{-1/2}(1+z)$

So, support of Q(z) is {1, 0, 1}

$$Q_{k,l} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textit{where } -1/2 \le k,l \le 1/2$$

Largest value is 1.

So, s = 0

Hence, Sobolev Regularity for db2 is 0.

For db4 [5] scaling filter is

H(z) = $\frac{1}{8}(1+z^{-1})^2((1+\sqrt{3})z^{-1}+1-\sqrt{3})$

Approximation Order = 2

To find Holders regularity:

H(1) = 1

Equating equations, we get

$$\frac{1}{8}(1+z^{-1})^2((1+\sqrt{3})z^{-1}+1-\sqrt{3}) = \frac{1}{2}z^{L/2}(1+z^{-1})^L Q(z)$$

Taking L = 2, we get

$$\frac{1}{8}(1+z^{-1})^2((1+\sqrt{3})z^{-1}+1-\sqrt{3}) = \frac{1}{2}z(1+z^{-1})^2 Q(z)$$

Q(z) = $\frac{1}{4}z^{-1}((1+\sqrt{3})z^{-1}+1-\sqrt{3})$

So, support of Q(z) is {0.6830, -0.1830}

$$Q_{k,l} = \begin{bmatrix} 0.683 & 0 & 0 & 0 & 0 \\ 0 & -0.1830 & 0.6830 & 0 & 0 \\ 0 & 0 & 0 & -0.1830 & 0.6830 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad where \ \ -2 \le k,l \le 2$$

Largest value is 0.6830

So, r = 0.55

Hence, Holders Regularity for db4 is 0.55

To find sobolev regularity:

$$H_2(z) = \frac{1}{64}(1+z)^2(1+z^{-1})^2(2.7321z^{-1} - 0.7321)(2.7321z - 0.7321)$$
$$H_2(1) = 1$$

Equating equations, we get

$$\frac{1}{64}(1+z)^2(1+z^{-1})^2(2.7321z^{-1} - 0.7321)(2.7321z - 0.7321) = \frac{1}{2}z^{L/2}(1+z^{-1})^L Q(z)$$

Taking L = 2, we get

$$\frac{1}{64}(1+z)^2(1+z^{-1})^2(2.7321z^{-1} - 0.7321)(2.7321z - 0.7321) = \frac{1}{2}z(1+z^{-1})Q(z)$$

$$Q(z) = \frac{1}{32}z^{-1}(1+z)^2(2.7321z^{-1} - 0.7321)(2.7321z - 0.7321)$$

$$= \frac{1}{32}(-2.0002z^{-2} + 3.9999z^{-1} + 12.0002 + 3.9999z - 2.0002z^2)$$

So, support of Q(z) is {-0.0625, 0.1249, 0.3750, 0.1249, -0.0625}

$$Q_{k,l} = \begin{bmatrix} -0.0625 & 0 & 0 & 0 & 0 \\ 0.3750 & 0.1249 & -0.0625 & 0 & 0 \\ -0.0625 & 0.1249 & 0.3750 & 0.1249 & -0.0625 \\ 0 & 0 & -0.0625 & 0.1249 & 0.3750 \\ 0 & 0 & 0 & 0 & -0.0625 \end{bmatrix} \quad where \ -2 \le k, l \le 2$$

Largest value is 0.2550

So, s = 0.9857

Hence, Sobolev regularity for db4 is 0.9857

For db6 scaling filter is

$H(z) = 0.3327z^{-2}(1+z^{-1})^3(z-(0.2873+0.1529i))(z-(0.2873-0.1529i))$

Approximation Order = 3

To find Holders regularity:

H(1) = 1.414

Equating equations, we get

Q(z) = $0.4706z^{-7/2}(z-(0.2873+0.1529i))(z-(0.2873-0.1529i))$

So, support of Q(z) is {0.0498, -0.2704, 0.4706}

Build matrix $Q_{k,l}$ (where $\frac{-7}{2} \leq k,l \leq \frac{7}{2}$)using the same method.

Largest value is 0.4706

So, r = 1.0874

Hence, Holders Regularity for db6 [5] is 1.0874

To find sobolev regularity:

$$H_2(z) = (0.3327)^2(1+z)^3(1+z^{-1})^3(z^2-0.5746z+0.1059)(z^{-2}-0.5746z^{-1}+0.1059)$$
$$H_2(1) = 1.999$$

Equating equations, we get

Q(z) $= 0.1107(1+z)^3 z^{-3/2}(z^2-0.5746z+0.1059)(z^{-2}-0.5746z^{-1}0.1059)$

So, support of Q(z) is {0.0117, -0.0352, -0.0273, 0.1758, 0.1758, -0.0273, -0.0352, 0.0117 }

Build the matrix $Q_{k,l}$ (where -7/2$\leq k,l \leq 7/2$)using the same procedure.

Largest value is 0.1405

So, s = 1.4157

Hence, Sobolev regularity for db6 is 1.4157

For db8 scaling filter is

H(z) = $0.1767z^{-3}(1+z^{-1})^4(z-0.5157)(z^2-0.025z+0.0579)$

Approximation Order = 4

To find Holders regularity:

H(1) = 1.414

Equating equations, we get

Q(z) = $0.2498z^{-5}(z-0.5157)(z^2-0.025z+0.0579)$

So, support of Q(z) is {-0.0074, 0.0177, -0.1351, 0.2498}

Build matrix $Q_{k,l}$ (where $-5 \le k,l \le 5$)using the same method.

Largest value is 0.2498

So, r = 2.001

Hence, Holders Regularity for db8 is 2.001

To find sobolev regularity:

$$H_2(z) = (0.0312)(1+z)^4(1+z^{-1})^4(z-0.5157)(z^2-0.025z+0.0579)(z^{-2}-0.025z^{-1}+0.0579)$$
$$H_2(1) = 1.9983$$

Equating equations, we get

Q(z) $= 0.0312(1+z)^4 z^{-2}(z^2-0.025z+0.0579)(z^{-2}-0.025z^{-1}0.0579)(z-0.5157)(z^{-1}-0.5157)$

So, support of Q(z) is {-0.0009, -0.001, -0.0128, -0.0195, 0.045, 0.1034, 0.045, -0.0195, -0.0128, 0.001, -0.0009 }

Build the matrix $Q_{k,l}$ (where $-5 \le k,l \le 5$)using the same procedure.

Eigen values are 0.0618, 0.0407, 0.0314, -0.0149, -0.0150, 0.0144,0.0074, 0.0006, 0.0005, -0.0009, -0.0009

Largest value is 0.0618

So, s = 2.008

Hence, Sobolev regularity for db8 is 2.008

For db10 scaling filter is

H(z) = $0.1601z^{-4}(1+z^{-1})^5(z^2-0.5542z+0.1707)(z^2-0.6744z+0.1220)$

Approximation Order = 5

To find Holders regularity:

H(1) = 1.414

Equating equations, we get

Q(z) = $0.2264z^{-13/2}(z^4 - 1.2286z^3 + 0.6665z^2 - 0.1828z + 0.0208)$

So, support of Q(z) is {0.0047, -0.0414, 0.1509, -0.2782, 0.2264}

Build matrix $Q_{k,l}$ (where $\frac{-13}{2} \leq k, l \leq \frac{13}{2}$)using the same method.

Eigen values are -0.2555, 0.1192, -0.0324,0.0047,0.2264, 0 ,0, 0, 0, 0, 0, 0, 0, 0

Largest value is 0.1192

So, r = 2.1

Hence, Holders Regularity for db10 is 2.1

To find sobolev regularity:

$$H_2(z) = (0.1601)^2(1+z)^5(1+z^{-1})^5(z^4 - 1.2286z^3 + 0.6665z^2 - 0.1828z + 0.0208)$$
$$H_2(1) = 1.9968$$

Equating equations, we get

Q(z) $= 0.0256(1+z)^5 z^{-5/2}(z^4 - 1.2286z^3 + 0.6665z^2 - 0.1828z + 0.0208)$

So, support of Q(z) is {0.0005, -0.0027, 0.0018, 0.0122, -0.0207, -0.0240, 0.0640, 0.0640, -0.0240, -0.0207, 0.0122, 0.0018, -0.0027, 0.0005 }

Build the matrix $Q_{k,l}$ (where -13/2$\leq k, l \leq 13/2$)using the same procedure.

Eigen values are -0.0443, 0.0544, -0.0311, 0.0311, -0.0039, 0.0037, 0.0019, 0.0037, 0.0079, 0.0147+0.0004i, 0.0147-0.0004i, 0.0157, 0.0005, 0.0005

Largest value is 0.0544

So, s = 2.1

Hence, Sobolev’s regularity for db10 is 2.1.

Now, in the following table, we are summarizing the properties of the discussed wavelet.

Name of The Wavelets	Approximation Order	Holders Regularity	Sobolev Regularity	Support Length
LeGall 5/3	2	1	1.5	5/3
Daubechies 9/7	4	1.70	2.12	9/7
db2	1	0	0	2
db4	2	0.55	0.9857	4
db6	3	1.0874	1.4157	6
db8	4	2.001	2.008	8
db10	5	2.1	2.1	10

Table 3.1 Comparison of Mathematical Properties of Wavelets

Chapter 4

Conclusion and Future Work

In this thesis, we discussed the mathematical properties of some of the wavelet filters which are being used in Image Compression algorithms, Edge detection techniques and Noise reduction techniques. The properties discussed in this thesis are- approximation order, vanishing moment, Holder's regularity, Sobolov's regularity, compact support, and orthogonality. There are some other wavelets which are being used in the image/signal processing applications. In our further work, we will include those wavelet families. There are some other important properties like approximation constant, wavelet constant, Riesz bounds and projection cosines, which we will study further.

References

[1] Khalid Sayood, "Introduction to data compression", Third Edition, Morgan Kaufmann Publishers.

[2] Jorgan Bergh, Fredrik Ekstedt and Martin Lindberg, "Wavelets", Overseas Press (India) Private Limited

[3] I. Daubecies, "Ten Lectures on Wavelets",SIAM 1992.

[4] Unser M., Blu T. "Mathematical Properties of JPEG2000 Wavelet Filters", IEEE Transaction on Image Processing, Vol. 12, No.9, pp. 1080-1090, Sept. 2003

[5] Rao R. M., Bopardikar A. S. "Wavelet Transforms- Introduction to Theory and Applications", Pearson Education, 2005.

[6] Mallat S., "A theory for multiresolution signal decomposition: The wavelet decomposition", IEEE Transaction on Pattern Analysis and Machine Intelligence, Vol.11, pp. 674-693, July 1983.

[7] L.Villemoes, "Wavelet analysis of refinement equations," SIAM Journal of Mathematical Analysis,vol.25, no.5, pp.1433-1460, Sept.1994.

[8] I.Daubechies and J. Largarias, "Two-scale difference equations I. Existence and global regularity of solutions," SIAM Journal of Mathematical Analysis, vol.22, no.5, pp. 1388-1410, Sept. 1991.

[9] O. Rioul, "Simple regularity criteria for subdivision schemes," SIAM Journal of Mathematical Analysis, Vol. 23, pp. 1544-1576, Nov. 1992.

[10] I. Daubechies and J. Lagarias, "Two scale difference equation II-Local regularity, infinite product matrices and fractals", SIAM Journal of Mathematical Analysis, Vol. 23, No. 4, pp. 1031-1079, July 1992.

[11] G. Strang, Eigen values of( ) H and convergence of the cascade algorithm", IEEE Transaction on Signal Processing, Vol. 44, No.2 , pp.233-238, 1996.