THE NOTION OF EXHAUSTIVENESS AND GENERALIZED ASCOLI THEOREM

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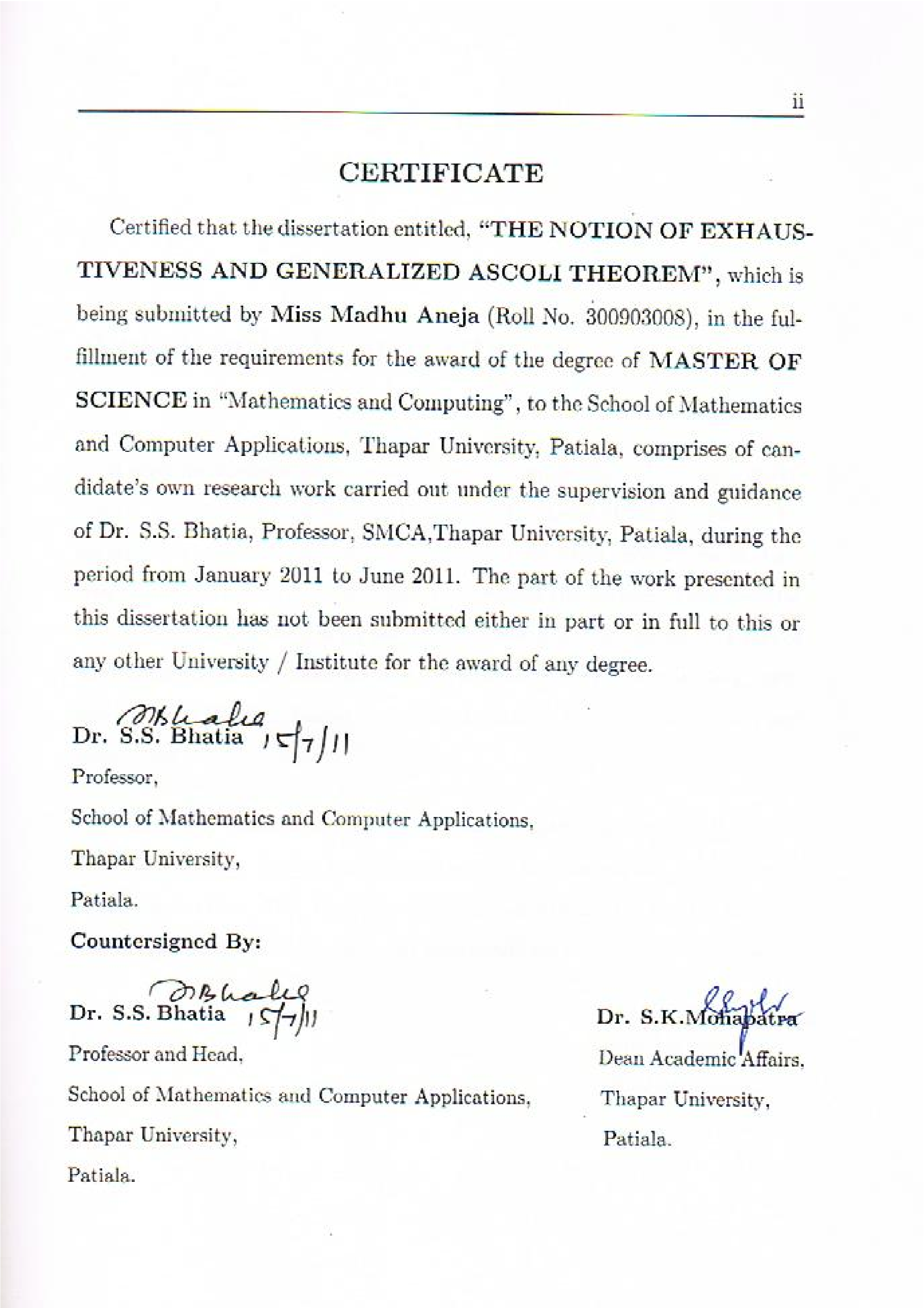
INDIA

DEDICATED

TO

MY PARENTS AND GOD



CERTIFICATE

Certified that the dissertation entitled, **"THE NOTION OF EXHAUSTIVENESS AND GENERALIZED ASCOLI THEOREM"**, which is being submitted by **Miss Madhu Aneja** (Roll No. 300903008), in the fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications, Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of Dr. S.S. Bhatia, Professor, SMCA,Thapar University, Patiala, during the period from January 2011 to June 2011. The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University / Institute for the award of any degree.

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Abstract

The present dissertation entitled "**THE NOTION OF EXHAUSTIVENESS AND GENERALIZED ASCOLI THEOREM**" embodies a study carried out by me under the supervision of Dr. S.S. Bhatia, Professor and Head, School of Mathematics and Computer Applications, Thapar University, Patiala.

In this dissertation, we study the notion of exhaustiveness which applies for both families and sequence of functions. This new notion is close to equicontinuity and describes the relation between pointwise convergence for functions and α-convergence (continuous convergence). Using these results we have studied the Generalized Ascoli theorem dealing with exhuastiveness instead of equicontinuity.

The work presented in this dissertation is divided into four chapters. Chapter I is introductory. In this chapter, we have represented some basic notations and definitions used in the sequel.

In Chapter II we have studied the notion of exhaustiveness introduced by V. Gregoriades and N. Papanastassiou [5] which is close to equicontinuity. The notion of exhaustiveness enables us to view the convergence of a sequence of functions in terms of properties of the sequence and not of properties of functions as single members.

In chapter III we have studied the notion of α-convergence (known as continuous convergence) introduced by R. Das [4] which turned out to be useful for characterizing compactness in metric spaces. The notion of α-convergence is stronger than pointwise convergence. On the other hand, if the limit function f is continuous, then this convergence is weaker than uniform convergence.

Chapter IV is devoted to the study of the Generalization of Ascoli Theorem using the notion of exhaustiveness.

Towards the end, references of various publications cited in the present dissertation have been reported.

List of Notations

Symbol	**Meaning**
$\forall$	for all
$\in$	belongs to
$\subseteq$	subset
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}(X, \mathbb{R})$	the set of all continuous real-valued functions defined on topological space X.
$\cup$, $\cap$	union, intersection
$\emptyset$	empty set
$\square$	end of a proof
$X \setminus S$	complement of S in X
$\neq$	not equal to

Contents

Chapter I

Introduction

Let $C(X)$ be set of all continuous real-valued functions defined on topological space X, and let S be a subset of $C(X)$. A set S is said to be equicontinuous at some $x \in X$ if for each $\varepsilon > 0$ there exists a neighborhood V of x such that $y \in V$ implies $|f(y) - f(x)| < \varepsilon$ for every $f \in S$. If S is equicontinuous at every point of X, then S is called equicontinuous set.

We know that in a metric space a closed and bounded set need not be compact. However, if X is a compact topological space, then a closed and bounded (with respect to uniform metric) subset of $C(X)$ is compact if and only if it is equicontinuous. This result is known ias Arzela- Ascoli Theorem.

In Mathematics, the Arzela-Ascoli theorem gives a necessary and sufficient condition to decide whether every subsequence of given sequence of real-valued continuous functions defined on a closed and bounded interval has a uniformly convergent subsequence. The main condition in this theorem is the equicontinuity of the sequence of functions. The theorem is a fundamental result in mathematics. This theorem has many application. In particular, it forms the basis for the proof of "Peano Existence theorem" in the theory of ordinary differential equations and "Montel's theorem" in complex analysis. It also plays a decisive role in the proof of "Peter-Weyl theorem".

The notion of equicontinuity was introduced at around the same time by Ascoli (1883-1884). A weak form of the theorem was proven by Ascoli (1882-1883), who established the sufficient condition for compactness, and by, Arzela (1895), who established the necessary condition for compactness and gave the the first clear presentation of the result. In its simplest form, the theorem of Ascoli with which we are concerned is an extension of the Bolzano-Weierstrass theorem.

The notion of α-convergence (otherwise continuous convergence or "Stetige Konvergenz") has been known by the the beginning of the 20th century (see [3, 6]). Around 1950s Stoilov [8] and Arens [2] came up with some results which characterize α-convergence and very helpful for this dissertation. We study the basic facts about α-convergence in chapter III.

V.Gregoriades and N.Papanastassiou [5] have introduced the notion of exhaustiveness which applies for both families and sequences of functions. This new notion is close to equicontinuity and describes the relation pointwise convergence for functions and α-convergence (continuous convergence). The notion of exhaustiveness enables us to view the convergence of a sequence of functions in terms of properties of the sequence and not of properties of functions as single members. An example of this is given in Chapter III (Theorem 3.2) which measures the step from pointwise convergence to α-convergence using the notion of exhaustiveness.

In this dissertation we have studied the Generalization of Ascoli Theorem using the notion of exhaustiveness.

Basic Notations and Definitions

Definition 1.1 A metric (or a distance) d on a nonempty set X is a function $d : X \times X \to \mathbb{R}$ satisfying the three properties:

a. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y)= 0$ if and only if $x = y$;

b. $d(x, y) = d(y, x)$ for all $x, y \in X$;

c. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality).

The pair (X, d) is called metric space.

We here give some notations. With X and Y we mean metric spaces, unless stated otherwise. If it is not mentioned explicitly the symbol d stands for the metric on X and the symbol p for the metric on Y.

If x is a member of X and δ is a positive number, with $S(x, \delta)$ we mean the (open) ball of radius δ , that is $S(x, \delta) = \{y \in X / d(y, x) < \delta\}$. Also if X and Y are metric spaces we denote with $C(X, Y)$ the set of all continuous functions from X to Y.

Definition 1.2 A metric space $(X,\, d)$ is called bounded if there exists a number $M > 0$ such that $d(x,\, y) \leq M$ for all $x,\, y \in X$. The diameter of a subset A of a metric space $(X,\, d)$ is defined by

$$d(A) = sup\{d(x,\, y) : x,\, y \in A\}.$$

Thus, $(X,\, d)$ is bounded if and only if the diameter of X is finite.

Definition 1.3 Let $(X,\, d)$ be a metric space. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to be covergent if there is a point $x \in X$ such that for each $\varepsilon > 0$, there exists a positive integer n_0 such that $d(x_n,\, x) < \varepsilon$ for all $n \geq n_0$.

Definition 1.4 A sequence $(x_n)_{n \in \mathbb{N}}$ of a metric space $(X,\, d)$ is called a cauchy sequence if for every $\varepsilon > 0$, there exists n_0 (depending upon ε) such that $d(x_n,\, x_m) < \varepsilon$ for all $n,\, m > n_0$.

Clearly, every convergent sequence is a cauchy sequence. However, in general, the converse is not true. As an example, let $X = (0,\, \infty)$ with distance $d(x,\, y)$ $= |x - y|$, and $x_n = \dfrac{1}{n}$ for each n. Then $(x_n)_{n \in \mathbb{N}}$ is a cauchy sequence that does not converge in X.

If a metric space has the property that all of its Cauchy sequences converge(in the space), then the metric space is called complete metric space.

Definition 1.5 Let X be a nonempty set. A collection τ of subsets of X is said to be a topology on X if τ satisfies the following properties:

a. $X \in \tau$ and $\phi \in \tau$.

b. If U and V belong to τ, then $U \cap V \in \tau$.

c. If $\{V_i\}_{i \in I}$ is a family of members of τ, then $\bigcup_{i \in I} V_i \in \tau$.

If τ is a topology on a set X, then the pair (X, τ) is called a topological space.

Definition 1.6 Consider a sequence $(f_n)_{n \in \mathbb{N}}$ of real-valued functions defined on a set X such that $\lim\limits_{n \to \infty} f_n(x)$ exists in $\mathbb{R}$ for each $x \in X$. Then a new function f can be defined by $f(x) = \lim\limits_{n \to \infty} f_n(x)$ for each $x \in X$. If this happens, then the sequence $(f_n)_{n \in \mathbb{N}}$ is said to converge pointwise to f (or that f is the pointwise limit of $(f_n)_{n \in \mathbb{N}}$) and is written symbolically as $f_n \underset{\longrightarrow}{pw} f$. In other words, $f_n \to f$ if for each $\varepsilon > 0$ and each $x \in X$ there exists some n_0 (depending upon both ε and x) such that $|f_n(x) - f(x)| < \varepsilon \ \forall\ n \geq n_0$.

Definition 1.7 A sequence $(f_n)_{n \in \mathbb{N}}$ of real-valued functions is said to converge uniformly on X to a function f if for each $\varepsilon > 0$ there exists some n_0 (depending upon ε) such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq n_0$ and all $x \in X$.

We shall write $f_n \underset{\longrightarrow}{pw} f$ to denote that $(f_n)_{n \in \mathbb{N}}$ pointwise converges to f. Also we will keep the analogous notation about uniform convergence, that is, we will denote $f_n \underset{\longrightarrow}{u} f$.

Note that the pointwise limit of a sequence of continuous functions need not to be a continuous function. For an example take $X = [0,\ 1]$, and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions defined by $f_n(x) = x^n$ for each $x \in [0, 1]$. Then each f_n is a continuous function, and $f_n \underset{\longrightarrow}{pw} f$ holds for the function f defined by $f(x) = 0$ if $x \in [0,\ 1)$ and $f(1) = 1$. Clearly, f is not continuous. Also, it is easy to see that the convergence is not uniform.

Definition 1.8 A set S is said to be compact set if every open cover of S admits finite subcovers of S.

Chapter II

Exhuastiveness

V. Gregoriades and N. Papanastassiou [5] have introduced a new notion, called exhaustiveness, which is close to the notion of equicontinuity introduced by Ascoli in (1883-1884) in the following way:

Definition 2.1 Let $(X,\, d)$, $(Y,\, p)$ be metric spaces, $x \in X$, F be a family of functions from X to Y and $f_n : X \to Y, n \in \mathbb{N}$.

(a) If F is infinite, we call the family F exhaustive at x iff for every $\varepsilon > 0$ there exists $\delta > 0$ and A a finite subset of F such that: for every $y \in S(x, \delta)$ and for every $f \in F \setminus A$ we have that $p(f(y),\ f(x)) < \varepsilon$.

(b) In case where F is finite we define F to be exhaustive at x iff each member of F is continuous function at X.

(c) F is exhaustive iff F is exhaustive at every x.

(d) The sequence $(f_n)_{n\,\in\,\mathbb{N}}$ is called exhaustive at x iff for all $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $y \in S(x,\ \delta)$ and all $n\ \geq\ n_0$ we have that $p(f_n(y),\ f_n(x)) < \varepsilon$.

(e) The sequence $(f_n)_{n\,\in\,\mathbb{N}}$ is called exhaustive iff it is exhaustive at every $x \in X$.

Notice that in the most interesting case where $(f_n)_{n\,\in\,\mathbb{N}}$ is a sequence of functions for which $f_n \neq f_m$ for $n \neq m$, then the family $F = \{f_n / n \in \mathbb{N}\}$ is exhaustive at some $x_o \in X$ if and only if the sequence $(f_n)_{n\,\in\,\mathbb{N}}$ is exhaustive at x_o.

Remarks 2.1

(a) An equicontinuous family is an exhaustive family such that for every $\varepsilon > 0$ the finite set A in Definition 2.1(a) can be taken to be empty set. So equicontinuity implies exhaustiveness.

(b) Saying that F is exhaustive does not imply that there exists a finite subset of F (call it A) such that $F \setminus A$ is equicontinuous. (That is because the set A in the definition depends on $\varepsilon > 0$.)

Proposition 2.1 Let $(X,\ d)$, $(Y,\ p)$ be metric spaces, $x \in X$, F a family of functions from X to Y and $f_n : X \to Y,\ n \in \mathbb{N}$.

(a) F is equicontinuous at x if and only if F is exhaustive at x and for each $f \in F$, f is continuous at x.

(b) The family $\{f_n / n \in \mathbb{N}\}$ is equicontinuous at x if and only if the sequence $(f_n)_{n\,\in\,\mathbb{N}}$ is exhaustive at x and each f_n is continuous at x.

Proof: (a) Assume that F is infinite. If F is equicontinuous at x then F is exhaustive at x. For the inverse direction : Let $\varepsilon > 0$, there exist $\delta_1 > 0$ and A finite subset of F such that for every $y \in S(x,\, \delta_1)$ and for every $f \in F \backslash A$ we have $p(f(y),\, f(x)) < \varepsilon$. Since each f is continuous at x there exists $\delta_f > 0$ such that for every $y \in S(x,\, \delta_f)$ we have $p(f(y),\, f(x)) < \varepsilon$. Put $\delta = \min \{\delta_1,\, \delta_f / f \in A\} > 0$. One can check that for every $y \in S(x,\, \delta)$ and for every $f \in F$ we have that

$$p(f(y),\, f(x)) < \varepsilon. \qquad \square$$

(b) Assume that F is infinite. If F is equicontinuous at x then F is exhaustive at x. For the inverse direction : Let $\varepsilon > 0$, there exist $\delta_1 > 0$ and $n_0 \in \mathbb{N}$ such that for every $y \in S(x,\, \delta_1)$ and all $n \geq n_o$ we have $p(f_n(y),\, f_n(x)) < \varepsilon$. Since each f_n is continuous at x there exists $\delta_{f_n} > 0$ such that for every $y \in S(x,\, \delta_{f_n})$ we have $p(f_n(y),\, f_n(x)) < \varepsilon$. Put $\delta = \min \{\delta_1,\, \delta_{f_n} / n < n_0\} > 0$. One can check that for every $y \in S(x,\, \delta)$ and for every $f_n \in F$ we have that

$$p(f_n(y),\, f_n(x)) < \varepsilon. \qquad \square$$

The preceding proposition suggests that there exists an exhaustive sequence (similarly family) which contains no continuous functions. This can be seen in the following example:

Example 2.1 For $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) = \dfrac{1}{n}$, for $x \leq 0$ and $f_n(x) = \dfrac{1}{2n}$, for $x > 0$. Show that sequence $(f_n)_{n \in \mathbb{N}}$ is exhaustive at 0

Proof: Given $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) = \dfrac{1}{n}$, for $x \leq 0$ and $f_n(x) = \dfrac{1}{2n}$, for $x > 0$. Clearly, no f_n is continuous at 0. We claim that the sequence $(f_n)_{n \in \mathbb{N}}$ is exhaustive at 0. Let $\varepsilon > 0$, then there exists an integer $n_o > \dfrac{1}{2\varepsilon}$ such that for $\delta = 1$, for all $y \in (-1,\, 1)$ and for all $n \geq n_o$ we have that $|f_n(y) - f_n(x)| \leq \dfrac{1}{2n} < \varepsilon$. □

The preceding example gives the following picture for an exhaustive family of functions F : the family F is equicontinuous "as a whole" with the continuity of each member of F erased.

Some of the results of equicontinuity apply for exhaustiveness. For example, we know that the pointwise limit of an equicontinuous sequence of functions is a continuous function. It can be seen from the following proposition that the same holds if we replace equicontinuity with exhaustiveness.

Proposition 2.2 Let $(X,\, d)$, $(Y,\, p)$ be metric spaces and $f,\ f_n,\ n = 1, 2, ...,$ Y - valued functions defined on X. If the sequence $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f and $(f_n)_{n \in \mathbb{N}}$ is exhaustive at $x \in X$ then f is continuous at x.

Proof: Since $(f_n)_{n \in \mathbb{N}}$ is exhaustive at x. Therefore, there exists $\delta > 0$ and there exists $n_0 \in \mathbb{N}$ such that for all $y \in S(x,\, \delta)$ and all $n \geq n_o$ we have that

$$p(f_n(y),\, f_n(x)) < \frac{\varepsilon}{3}.$$

Let $y \in S(x,\, \delta)$. Since $f_n \underset{\longrightarrow}{pw} f$ there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ it holds that

$$p(f_n(y),\, f(y)) < \frac{\varepsilon}{3} \ \text{ and } \ p(f_n(x),\, f(x)) < \frac{\varepsilon}{3}.$$

Put $n_2 = \max\{n_0, n_1\}$. Then

$p(f(y), f(x)) \leq p(f(y), f_{n_2}(y)) + p(f_{n_2}(y), f_{n_2}(x)) + p(f_{n_2}(x), f(x)) < \varepsilon$.

Hence f is continuous. $\square$

If in the previous proposition we replace the condition "$(f_n)_{n \in \mathbb{N}}$ is exhaustive at x" with "$\{f_n / n \in \mathbb{N}\}$ is exhaustive at x" then the same conclusion still holds. To see that, let us take an interesting case where the set $\{f_n / n \in \mathbb{N}\}$ is infinite. Choose naturals $k_1 < k_2 < k_3 < ... < k_n < ...$ such that $f_{k_n} \neq f_{k_m}$ for $n \neq m$. The sequence $(f_{k_n})_{n \in \mathbb{N}}$ is exhaustive at x and then by following the same steps as in the proof of Proposition 2.2, we shall get the result.

Chapter III

α-convergence

In this chapter, we study the basic facts about α-convergence introduced by R. Das [4].

Definition 3.1 Let $f,\ f_n,\ n \in \mathbb{N}$ be functions from X to Y. The sequence $(f_n)_{n \in \mathbb{N}}$ α-converges to f iff for every $x \in X$ and for every sequence $(x_n)_{n \in \mathbb{N}}$ of points of X converging to x, the sequence $(f_n(x_n))_{n \in \mathbb{N}}$ converges to $f(x)$.

We shall write $f_n \ \underset{\to}{\alpha}\ f$ to denote that $(f_n)_{n \in \mathbb{N}}$ α-converges to f.

Remarks 3.1

(a) It is obvious that α-convergence is stronger than pointwise convergence.

(b) The usual convergences such as pointwise and uniform do not require a topology for the domain space. However a topology is needed for α-convergence.

(c) Take $f : \mathbb{R} \to \mathbb{R}$ any non-continuous function and $x_n \to x$ such that the sequence $(f(x_n))_{n \in \mathbb{N}}$ does not converge to $f(x)$. If we put $f_n \equiv f$ for all $n \in \mathbb{N}$, we see that $(f_n)_{n \in \mathbb{N}}$ does not α-converge to f although the sequence $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f.

(d) For all $n \in \mathbb{N}$ define $f_n : (0, 1] \to \mathbb{R}$ such that $f_n(x) = 1 - nx$, for $x \leq \dfrac{1}{n}$ and $f_n(x) = 0$, for $x > \dfrac{1}{n}$. Then we can see that the sequence (f_n) α-converges to zero function but does not converge uniformly.

Proof: The proofs of (a) and (b) are quite obvious.

(c) Given $f : \mathbb{R} \to \mathbb{R}$ is a non continuous function.

$\Rightarrow$ There exist a sequence $(x_n)_{n \,\epsilon\, \mathbb{N}}$ in $\mathbb{R}$ such that

$$x_n \to x \quad \text{in} \quad \mathbb{R} \quad \text{and} \quad f(x_n) \nrightarrow f(x) \quad \text{in} \quad \mathbb{R}.$$

Also, given $f_n \equiv f \quad \forall \quad n \in \mathbb{N}$.

$$\Rightarrow \; f_n(x) = f(x) \;\; \forall \;\; n \in \mathbb{N} \;\; \text{and} \;\; \forall \;\; x \in \mathbb{R}. \tag{3.1.1}$$

Since, $f(x_n) \nrightarrow f(x)$ implies that $f_n(x_n) \nrightarrow f(x)$

Therefore, the sequence $(f_n)_{n \in \mathbb{N}}$ does not α-converge to f.

Also, from (3.1.1) $(f_n)_{n \in N}$ is a constant sequence of functions.

Therefore, the sequence $(f_n)_{n \in N}$ converges uniformly to f.

(d) Given $f_n : (0, 1] \to \mathbb{R}$ such that $f_n(x) = 1 - nx$, for $x \leq \dfrac{1}{n}$ and $f_n(x) = 0$, for $x > \dfrac{1}{n}$.

Now, we have to show that $(f_n)_{n \in \mathbb{N}}$ α-converges to f.

Let $x_n \to x$ for all $n \in \mathbb{N}$. Therefore, for $\varepsilon = \dfrac{1}{4}$ there exist a positive integer $n_0 \in \mathbb{N}$ such that

$$d(x_n,\, x) < \varepsilon \;\; \forall \;\; n \geq n_0.$$

In particular, $d(x_{n_0},\, x) < \varepsilon$.
$\Rightarrow \; \dfrac{-1}{4} \; < \; x_{n_0} - x \; < \; \dfrac{1}{4}$.
$\Rightarrow \; x - \dfrac{1}{4} \; < \; x_{n_o} \; < \; x + \dfrac{1}{4}$.
Choose n_1 such that $\dfrac{1}{n_1} \; < \; x - \dfrac{1}{4}$. Take $n_2 = \max\{n_0,\, n_1\}$, we have then

$$f_n(x_n) = 0 \;\; \forall \;\; n \geq n_2.$$

Hence the sequence $(f_n)_{n \,\in\, \mathbb{N}}$ α-converges to 0.

Now, we show that $(f_n)_{n \,\in \mathbb{N}}$ does not converge uniformly to f.
Suppose that $(f_n)_{n \,\in \mathbb{N}}$ converges uniformly to 0. Therefore, for $\varepsilon = \dfrac{1}{2}$ there exist some $n_o \in \mathbb{N}$ such that

$$|f_n(x) - 0| \; < \; \frac{1}{2} \;\; \forall\, x \quad \text{and} \quad n \geq n_0.$$

In particular, $|f_{n_0}(x) - 0| \; < \; \dfrac{1}{2} \;\; \forall\, x$.

$$\Rightarrow |f_{n_0}(x)| \; < \; \frac{1}{2} \;\; \forall\, x. \tag{3.1.2}$$

Take $x = \dfrac{1}{2n_0} \; < \; \dfrac{1}{n}$. So, $f_{n_0}(x) = 1 - n_0\, \dfrac{1}{2n_0} = \dfrac{1}{2}$, which is a contradiction to (3.1.2). Therefore $(f_n)_{n \,\in \mathbb{N}}$ does not converges uniformly to 0. □

Proposition 3.1 For all functions $f,\; f_n,\; n \in \mathbb{N}$, from X to Y, if $(f_n)_{n \,\in\, \mathbb{N}}$ α-converges to f then each subsequence $(f_{k_n})_{n \,\in\, \mathbb{N}}$ also α-converges to f.

Proof: Let $x \in X$ and $x_n \to x$. Define

$$\begin{aligned} y_n &= x_i, \text{ if } n = k_i \text{ for some } i \in \mathbb{N} \\ &= x, \text{ otherwise.} \end{aligned}$$

We have that $y_{k_n} = x_n$ for each $n \in \mathbb{N}$ and also $y_n \to x$.

Since,

$$f_n \underset{\to}{\alpha} f$$

we obtain that $f_n(y_n) \to f(x)$.

Therefore, $f_{k_n}(y_{k_n}) \to f(x)$. Since $y_{k_n} = x_n$, so we have that $f_{k_n}(x_n) \to f(x)$.

□

Notice that in fact we have proved the following: if for all $(x_n)_{n \in \mathbb{N}}$ in X with $x_n \to x$ we have that $f_n(x_n) \to f(x)$, then for all $(x_n)_{n \in \mathbb{N}}$ in X with $x_n \to x$ we have that $f_{k_n}(x_n) \to f(x)$.

The connection between the notion of exhaustiveness with the notion of α-convergence is shown in the following theorem.

Theorem 3.2 Let (X, d), (Y, p) be metric spaces and functions $f_n, f : X \to Y$, $n \in \mathbb{N}$. Then the following are equivalent:

- (a) The sequence $(f_n)_{n \in \mathbb{N}}$ α-converges to f.
- (b) The sequence $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f and $(f_n)_{n \in \mathbb{N}}$ is exhaustive.

Proof: We first prove that $(a) \Rightarrow (b)$

Let the sequence $(f_n)_{n \in \mathbb{N}}$ α-converges to f. Therefore, the sequence $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f.

Let us assume that $(f_n)_{n \in \mathbb{N}}$ is not exhaustive at some point x. This means that there exist some $\varepsilon > 0$ such that for every $\delta > 0$ and for every $n \in \mathbb{N}$ there exist $y_n \in S(x,\, \delta)$ and $n_k \geq n$ such that

$$p(f_{n_k}(y_n),\, f_{n_k}(x)) \geq \varepsilon$$

Take $\delta = 1$ and $n = 1$. Then there exist some $n_1 > 1$ and $y_1 = x_{n_1} \in S(x,\, 1)$ such that

$$p(f_{n_1}(y_1),\, f_{n_1}(x)) \geq \varepsilon.$$

Take $\delta = \frac{1}{2}$ and $n = n_1$. Then there exist some $n_2 > n_1$ and $y_2 = x_{n_2} \in S(x,\, \frac{1}{2})$ such that

$$p(f_{n_2}(y_2),\, f_{n_2}(x)) \geq \varepsilon.$$

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On continuing like this, take $\delta = \frac{1}{n}$ and $n = n_{k-1}$. Then there exist some $n_k > n_{k-1}$ and $y_n = x_{n_k} \in S(x,\, \frac{1}{n})$ such that

$$p(f_{n_k}(y_n),\, f_{n_k}(x)) \geq \varepsilon.$$

In this way, we get a sequence (y_n) and set of natural numbers $(1 < n_1 < n_2 < ... < n_k < ...)$ such that

$$d(y_n,\, x) < \frac{1}{n} \quad \text{and} \quad p(f_{n_k}(y_n),\, f_{n_k}(x)) \geq \varepsilon \text{ for each } n \in \mathbb{N}$$

(3.2.1)

Since $f_n \underset{\rightarrow}{\alpha} f$ and $(f_{n_k})_{n \in \mathbb{N}}$ is a subsequence of $(f_n)_{n \in \mathbb{N}}$, therefore from Proposition 3.1 we obtain that

$$f_{n_k} \underset{\rightarrow}{\alpha} f.$$

Moreover, since $y_n \to x$, so $f_{n_k}(y_n) \to f(x)$.

Also, $f_{n_k}(x) \underset{\longrightarrow}{pw} f(x)$ because α-convergence is stronger than pointwise convergence.

Now, $p(f_{n_k}(y_n),\ f_{n_k}(x)) \ \le\ p(f_{n_k}(y_n),\ f(x))\ +\ p(f_{n_k}(x),\ f(x))$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall\ \ n \ge n_k.$$

$< \varepsilon\ \ \forall\ \ n \ge n_k$, which is contradiction to (3.2.1).

Hence, $(f_n)_{n\,\in\,\mathbb{N}}$ is exhaustive at x.

Now we prove that $(b) \Rightarrow (a)$

Let $x \in X$ and $x_n \to x$. We need to prove that $f_n(x_n) \to f(x)$.

Assume $\varepsilon > 0$, since $f_n \underset{\longrightarrow}{pw} f$ there exists $n_1\ \in \mathbb{N}$ such that for all $n \ge n_1$ we have that

$$p(f_n(x),\ f(x)) < \frac{\varepsilon}{2}.$$

Also $(f_n)_{n\,\in\,\mathbb{N}}$ is exhaustive and so there exist $\delta > 0$ and $n_2 \in \mathbb{N}$ such that for all $y \in S(x,\ \delta)$ and for all $n \ge n_2$ we have that

$$p(f_n(y),\ f_n(x)) < \frac{\varepsilon}{2}.$$

Since $x_n \to x$, for $\delta > 0$ there exist $n_3 \in \mathbb{N}$ such that for all $n \ge n_3$ we have that $d(x_n,\ x) < \delta$. Therefore if $n \ge \max\{n_2,\ n_3\}$ from the previous two statements we have that

$$p(f_n(x_n),\ f_n(x)) < \frac{\varepsilon}{2}.$$

Put $n_o = \max\{n_1,\ n_2,\ n_3\}$ and let $n \ge n_0$, then

$$\begin{aligned} p(f_n(x_n),\ f(x)) \ &\le\ p(f_n(x_n),\ f_n(x))\ +\ p(f_n(x),\ f(x)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$ □

Remark 3.2 A careful look on the proof of the preceding theorem reveals that we have actually proved the following:

Let $(X,\, d)$, $(Y,\, p)$ be metric spaces, $x \in X$ and functions $f_n,\ f : X \to Y$, $n \in \mathbb{N}$. The following are equivalent:

(a) For all sequences $(x_n)_{n\,\in\,\mathbb{N}}$ in X with $x_n \to x$ we have that $f_n(x_n) \to f(x)$.

(b) The sequence $(f_n(x))_{n\,\in\,\mathbb{N}}$ converges to $f(x)$ and the sequence $(f_n)_{n\,\in\,\mathbb{N}}$ is exhaustive at x.

The next proposition is due to Stoilov [8] describes some interesting results about α-convergence.

Proposition 3.3 Let $(X,\, d)$, $(Y,\, p)$ be metric spaces and functions $f_n,\ f : X \to Y$, $n \in \mathbb{N}$. Then

(a) If the sequence $(f_n)_{n\,\in\,\mathbb{N}}$ α-converges to f, then f is continuous.

(b) The sequence $(f_n)_{n\,\in\,\mathbb{N}}$ α-converges to f if and only if f is continuous and $(f_n)_{n\,\in\,\mathbb{N}}$ converges to f uniformly on compact space X.

In particular:

(c) If $(f_n)_{n\,\in\,\mathbb{N}}$ converges to f uniformly and f is continuous, then $(f_n)_{n\,\in\,\mathbb{N}}$ α-converges to f.

And also:

(d) If A is compact subset of X and $(f_n)_{n\,\in\,\mathbb{N}}$ α-converges to f, then $(f_n)_{n\,\in\,\mathbb{N}}$ converges to f uniformly.

Proof: (a) Given the sequence $(f_n)_{n \in N}$ α-converges to f. From Theorem 3.2 and Proposition 2.2 we have that exhuastiveness and α-convergence are equivalent and limit of exhuastive sequence of functions is continuous. $\square$

(b) Firstly Suppose that the sequence $(f_n)_{n \in \mathbb{N}}$ α-converges to f. Then by proposition 2.3(a) f is continuous.

Let $\varepsilon > 0$ be arbitrary. Since f is continuous on a compact space X, f is uniformly continuous so there exists some $\delta > 0$ such that whenever

$$d(x,\, y) < \delta, \text{ we have } \ p(f(x),\, f(y)) < \varepsilon.$$

Since every α-convergent sequence is exhaustive, so for each $x \in X$, there exists some $\delta_x > 0$ and $n_x \in \mathbb{N}$, such that for all $y \in S(x,\, \delta_x)$ and for all $n \geq n_x$ we have

$$p(f_n(y),\, f_n(x)) < \varepsilon$$

without loss of generality we can assume δ_x to be less than δ.

Since $X \subseteq \bigcup\limits_{x \in X} S(x,\, \delta_x)$ and X is compact, there exists some $x_1, x_2, ..., x_k \in X$ such that $X = \bigcup\limits_{i=1}^{k} S(x_i, \delta_{x_i})$.

Now, given that sequence $(f_n)_{n \in \mathbb{N}}$ α-converges to f. So the sequence $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f. Therefore for each i, there exists some $m_i \in \mathbb{N}$ such that

$$p(f_n(x_i), f(x_i)) < \varepsilon \quad \text{whenever} \quad n \geq m_i.$$

Take $n_0 = \max\{n_{x_1},\, n_{x_2},\, ...,\, n_{x_k},\, m_1, ...,\, m_k\}$.

Let $y \in X$ be arbitrary. So $y \in S(x_i,\, \delta_{x_i})$ for some i.

Then,

$$d(x_i,\, y) < \delta_{x_i} < \delta.$$

So,

$$p(f(x_i), f(y)) < \varepsilon \text{ and } p(f_n(y), f_n(x_i)) < \varepsilon.$$

Therefore,

$$\begin{aligned} p(f_n(y),\, f(y)) &\le p(f_n(y),\, f_n(x_i)) + p(f_n(x_i),\, f(x_i)) + p(f(x_i), f(y)) \\ &< \quad \varepsilon \ + \ \varepsilon \ + \ \varepsilon \\ &= 3\varepsilon \ \ \forall\, n \ge n_0. \end{aligned}$$

Hence, the sequence $(f_n)_{n\,\in\,\mathbb{N}}$ converges uniformly to f.

Conversly, Suppose that $f_n \overset{u}{\longrightarrow} f$ and f is continuous.

Let $x_n \to x$ and $\varepsilon > 0$ be arbitrary. So there exists some $n_0 \in \mathbb{N}$ such that

$$p(f_n(x),\, f(x)) < \varepsilon \ \ \forall \ \ n \ge n_0 \text{ and } \forall \ \ x \in X.$$

In particular,

$$p(f_n(x_n), f(x_n)) < \varepsilon \ \ \forall \ \ n \ge n_o \quad .$$

Since f is continuous on X, we have $f(x_n) \to f(x)$. So there exist $n_1 \in \mathbb{N}$ such that

$$p(f(x_n),\, f(x)) < \varepsilon \ \ \forall \ \ n \ge n_1.$$

Let $n_2 = \max\{n_0,\, n_1\}$ then we have,

$$\begin{aligned} p(f_n(x_n),\, f(x)) &\le p(f_n(x_n),\, f(x_n)) \ + \ p(f(x_n),\, f(x)). \\ &< \ \varepsilon + \varepsilon \\ &= 2\varepsilon \ \ \forall \ \ n \ge n_2. \end{aligned}$$

Hence, the sequence $(f_n)_{n\,\in\,\mathbb{N}}$ α-converges to f. □

(c) Suppose that $f_n \overset{u}{\longrightarrow} f$ and f is continuous.

Let $x_n \to x$ and $\varepsilon > 0$ be arbitrary. So there exists some $n_0 \in \mathbb{N}$ such that

$$p(f_n(x),\, f(x)) < \varepsilon \;\; \forall \;\; n \geq n_0 \text{ and } \forall \;\; x \in X.$$

In particular,

$$p(f_n(x_n), f(x_n)) < \varepsilon \;\; \forall \;\; n \geq n_o \quad .$$

Since f is continuous on X, we have $f(x_n) \to f(x)$. So there exist $n_1 \in \mathbb{N}$ such that

$$p(f(x_n),\, f(x)) < \varepsilon \;\; \forall \;\; n \geq n_1.$$

Let $n_2 = \max\{n_0,\, n_1\}$ then we have,

$$\begin{aligned} p(f_n(x_n),\, f(x)) \;&\leq\; p(f_n(x_n),\, f(x_n)) \;+\; p(f(x_n),\, f(x)). \\ &<\; \varepsilon \;+\; \varepsilon \\ &= 2\varepsilon \;\; \forall \;\; n \geq n_2. \end{aligned}$$

Hence, the sequence $(f_n)_{n \in \mathbb{N}}$ α-converges to f. □

(d) Suppose that the sequence $(f_n)_{n \in \mathbb{N}}$ α-converges to f. Then by proposition 2.3(a) f is continuous.

Let $\varepsilon > 0$ be arbitrary. Since, f is continuous on a compact subset A of X, f is uniformly continuous so there exists some $\delta > 0$ such that whenever

$$d(x,\, y) < \delta, \text{ we have } \; p(f(x),\, f(y)) < \varepsilon.$$

Since every α-convergent sequence is exhaustive, so for each $x \in X$, there exists some $\delta_x > 0$ and $n_x \in \mathbb{N}$ such that for all $y \in S(x,\, \delta_x)$ and for all $n \geq n_x$, we have

$$p(f_n(y),\, f_n(x)) < \varepsilon$$

without loss of generality we can assume δ_x to be less than δ.

Since, A is a subset of X. Then $A \subseteq \bigcup\limits_{x \,\in\, A} S(x,\, \delta_x)$ and A is compact, there exists some $x_1, x_2, ..., x_k \in A$ such that $A = \bigcup_{i=1}^{k} S(x_i, \delta_{x_i})$.

Now, given that sequence $(f_n)_{n \in \mathbb{N}}$ α-converges to f. So the sequence $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f. Therefore for each i, there exists some $m_i \in \mathbb{N}$ such that

$$p(f_n(x_i), f(x_i)) < \varepsilon \quad \text{whenever} \quad n \geq m_i.$$

Take $n_0 = \max\{n_{x_1},\, n_{x_2},\, ...,\, n_{x_k},\, m_1, ...,\, m_k\}$.

Let $y \in A$ be arbitrary. So $y \in S(x_i,\, \delta_{x_i})$ for some i.

Then

$$d(x_i,\, y) < \delta_{x_i} < \delta.$$

So,

$$p(f(x_i), f(y)) < \varepsilon \text{ and } p(f_n(y), f_n(x_i)) < \varepsilon.$$

Therefore,

$$\begin{aligned} p(f_n(y),\, f(y)) &\leq p(f_n(y),\, f_n(x_i)) + p(f_n(x_i),\, f(x_i)) + p(f(x_i), f(y)) \\ &< \varepsilon + \varepsilon + \varepsilon \\ &= 3\varepsilon \ \ \forall\, n \geq n_0. \end{aligned}$$

Hence, the sequence $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f. □

Chapter IV

Generalized Ascoli Theorem

In this chapter we have studied the Generalization of Ascoli Theorem using the notion of exhaustiveness.

Connections with general topology

4.1 A generalization of the classical Ascoli theorem

Let X be a metric space. We define $Bd(X)$, the space of bounded functions on X as,

$$Bd(X) = \{f : X \to \mathbb{R} \ / \ sup_{x \in X}|f(x)| < \infty\} \ .$$

The supremum norm on $Bd(X)$ is defined by

$$\|f\| = sup_{x \in X}|f(x)|.$$

We shall denote the corresponding metric space with $(Bd(X), \ \| \, . \, \| \ )$. The topology induced from this norm is the topology of uniform convergence.

The sequence $(f_n)_{n \in \mathbb{N}}$ converges to f in $(Bd(X), \ \| \ . \ \|)$ if and only if $(f_n)_{n \in \mathbb{N}}$ converges to f uniformly. If X is compact metric space then every real-valued function is bounded. So if X is compact then the set $C(X, \, R)$ is a subset of $Bd(X)$. Consequently, $C(X, \mathbb{R})$ is a metric space equipped with the metric generated by the norm $\| \; . \; \|$.

The proof of main theorem is based on the following theorems.

Theorem 4.1 For any space X, $(Bd(X), \| \; . \; \|)$ is a complete metric space.

Proof: Let $(f_n)_{n \in \mathbb{N}}$ be a cauchy sequence in $Bd(X)$.

Therefore, for given $\varepsilon > 0$ there exist some $n_o \in \mathbb{N}$ such that

$$\|f_n - f_m\| \; < \; \varepsilon \quad \forall \quad n, \; m \geq n_0$$

$$\Rightarrow \; Sup\{|f_n(x) - f_m(x)| \; : \; x \in X\} \; < \; \varepsilon \; \forall \; n, \; m \geq n_0.$$

$\Rightarrow \quad |f_n(x) - f_m(x)| < \varepsilon \quad \forall \quad x \in X \quad$ and $\quad \forall \; n, \; m \geq n_0$.

$\Rightarrow \quad (f_n(x))_{n \in \mathbb{N}}$ is a cauchy sequence of real numbers for each $x \in X$. Since $\mathbb{R}$ is complete. So every cauchy sequence in $\mathbb{R}$ is convergent.

Let $f(x) = \lim\limits_{n \to \infty} f_n(x)$ for each $x \in X$.

Now, we show that $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f.

For this, let $\varepsilon > 0$. Choose n_o such that

$$\|f_n - f_m\| < \frac{\varepsilon}{2} \text{ for } n, \; m \geq n_o.$$

As we know that

$$|f_n(x) - f_m(x)| < \|f_n - f_m\| \text{ for all } x \in X.$$

Therefore, $|f_n(x) - f_m(x)| < \dfrac{\varepsilon}{2}$ holds for all $n, \; m \geq n_0$ and all $x \in X$. Take limit m tends to ∞, then we have

$$|f_n(x) - f(x)| \; \leq \; \frac{\varepsilon}{2} \; < \; \varepsilon \text{ holds for all } n \; \geq \; n_0 \text{ and all}$$

$x \in X$.

Therefore, $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f.

Claim : $f \in Bd(X)$.

As $\lim\limits_{n\to\infty} f_n(x) = f(x)$, therefore, for given $\varepsilon > 0$ there exist a positive integer n_0 such that

$$|f_n(x) - f(x)| \ < \ \varepsilon \quad \forall \quad x \in X \quad \text{and} \quad \forall \ \ n \geq n_o.$$

Taking $\varepsilon = 1$, we get

$$|f_n(x) - f(x)| < 1 \quad \forall \ \ x \in X \quad \text{and} \quad \forall \ \ n \geq n_0.$$

In particular,

$$|f_{n_0}(x) - f(x)| < 1 \quad \forall \ \ x \in X.$$

$$\Rightarrow \quad -1 \ < \ f_{n_0}(x) - f(x) \ < \ 1.$$

$$\Rightarrow \quad -1 + f_{n_0}(x) < f(x) < 1 + f_{n_0}(x) \qquad (4.1.1)$$

But $f_{n_0}(x) \in Bd(X)$. Therefore, there exist a positive integer $M > 0$ such that $|f_{n_0}(x)| < M \quad \forall \ \ x \in X$. Hence from (4.1.1), we have

$$|f(x)| < 1 + M \quad \forall \ \ x \in X.$$

This implies that $f \in Bd(X)$. Therefore, $(Bd(X), \| \, . \, \|)$ is a complete metric space.

Theorem 4.2 Let X be a topological space and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of $C(X)$. If $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f on X, then f is a continuous function.

Proof: We need to show that f is continuous at every point of X.

Let $a \in X$ and $\varepsilon > 0$.

Since $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f on X, so there exists some $m \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall \ n \geq m.$$

In particular,

$$|f_m(x) - f(x)| < \varepsilon.$$

On the other hand, since f_m is a continuous function, therefore there exists a neighborhood V of a such that

$$|f_m(x) - f_m(a)| < \varepsilon \quad \forall\, x \in V.$$

Now note that if $x \in V$, then

$$\begin{aligned} |f(x) - f(a)| &\leq |f(x) - f_m(x)| \;+\; |f_m(x) - f_m(a)| \;+\; |f_m(a) - f(a)| \\ &< \varepsilon \;+\; \varepsilon \;+\; \varepsilon \\ &= 3\varepsilon, \text{ this shows that } f \text{ is continuous at } a. \end{aligned}$$

□

Theorem 4.3 For a subset A of a metric space $(X,\, d)$ the following statements are equivalent :

- (a) A is a compact set.
- (b) Every infinite subset of A has an accumulation point in A.
- (c) Every sequence in A has a subsequence which converges to a point of A.

The proof of this theorem is given in C. Aliprantis and O. Burkinshaw [1] on page 49.

Main Result: The main result of this chapter is the following theorem:

Theorem 4.4 (Generalized Ascoli theorem). Let X be a compact metric space and let F be an infinite subset of $(Bd(X),\, \|\ .\ \|)$. Then (*a*) If F is closed, bounded and exhaustive, then F is compact.

(*b*) Moreover, if every cluster point of F is a continuous function, then the converse of (*a*) is also true.

Proof: (*a*) The main frame is the same with the classical proof. It is enough to prove that F is sequentially compact.

Since X is compact metrizable space and every compact metrizable space is separable. So, let $(x_n)_{n \in \mathbb{N}}$ a dense subset of X and $(f_n)_{n \in \mathbb{N}} \subseteq F$. Now given that F is bounded. Therefore,

$$Dia(F) = Sup\{\ d(f,\, g) : f,\, g \in F\ \} = m.$$

$$\Rightarrow d(f,\, g) \leq m \ \ \forall \ \ f,\, g \in F.$$

$$\Rightarrow |f(x) - g(x)| \leq m \ \ \forall \ \ f,\, g \in F \ \text{ and } \ \forall \ \ x \in X.$$

$$\Rightarrow |f(x)| - |g(x)| \leq m \ \ \forall \ \ f,\, g \in F \quad \text{and} \quad \forall \ \ x \in X.$$

$$\Rightarrow |f(x)| \leq m \ + \ |g(x)| \ \ \forall \ \ f,\, g \in F \quad \text{and} \quad \forall \ \ x \in X.$$

(4.4.1)

Now, $g \in F$ and let it be fixed. Since X is compact and g is bounded. Therefore, there exist some positive integer n such that

$$|g(x)| \leq n \ \ \forall \ \ x \in X.$$

From (4.4.1) we have $|f(x)| \leq m + n = M \ \ \forall \ \ x \in X \quad \text{and} \quad \forall \ \ f \in F$.
Since $|f_n(x_1)| \leq M$ as each $f_n \in F$. Thus $(f_n(x_1))_{n \in \mathbb{N}}$ is a bounded sequence in $\mathbb{R}$.

Therefore, by Bolzano Weierstrass Theorem it has a convergent subsequence. Let $(f_{k_n^1})_{n \in \mathbb{N}}$ be a subsequence of $(f_n)_{n \in \mathbb{N}}$ such that lim $f_{k_n^1}(x_1)$ exists in $\mathbb{R}$.

Now, again the sequence $(f_{k_n^1}(x_2))_{n\,\in\,\mathbb{N}}$ is bounded. Therefore, there exist a convergent subsequence $(f_{k_n^2})_{n\,\in\,\mathbb{N}}$ of $(f_{k_n^1})_{n\,\in\,\mathbb{N}}$ such that lim $(f_{k_n^2}(x_2))_{n\,\in\,\mathbb{N}}$ exists in $\mathbb{R}$.

Inductively we obtain sequences of naturals ... $\subseteq$ (k_n^{j+1}) $\subseteq$ (k_n^j) $\subseteq$ ... (k_n^1) such that for each $j \in \mathbb{N}$ the sequence $(f_{k_n^j}(x_j))_{n\,\in\,\mathbb{N}}$ is convergent. It can easily be checked that for each $j \in \mathbb{N}$ the diagonal sequence $(f_{k_n^n}(x_j))_{n\,\in\,\mathbb{N}}$ is also convergent. Therefore the sequence $(f_{k_n^n}(x_j))_{n\,\in\,\mathbb{N}}$ is cauchy sequence in $\mathbb{R}$.

Using the fact that F is exhaustive, we obtain for each $x \in X$ the sequence $(f_{k_n^n}(x))_{n\,\in\,\mathbb{N}}$ is a Cauchy sequence (the method for this, is very much the same with the classical one).
Put $f(x) = \lim_{n\,\in\,\mathbb{N}} f_{k_n^n}(x)$.
Using the exhaustiveness and the compactness of X from Theorem 3.2 and Proposition 3.3(b) it follows that $f_{k_n^n} \overset{u}{\longrightarrow} f$.
Since F is closed, so we have that $f \in F$ and hence F is compact.

(b) Assume that F is compact but not exhaustive at some point x. Then by definition there exists $\varepsilon > 0$ such that for every $\delta > 0$ and for every finite subset A of F, there exists $x_{\delta,\,A} \in S(x,\,\delta)$ and $f_{\delta,\,A} \in F\setminus A$ such that

$$|f_{\delta,\,A}(x_{\delta,\,A}) \;-\; f_{\delta,\,A}(x)| \;\geq\; \varepsilon$$

Take $\delta = 1$. Let $x_1 \in S(x,\,1)$ and $A = \phi$. Then

$$|f_1(x_1) - f_1(x)| \geq \varepsilon \quad \text{whenever} \quad d(x,\,x_1) < 1.$$

Take $\delta = \dfrac{1}{2}$. Let $x_2 \in S(x,\,\dfrac{1}{2})$ and $A = \{f_1\}$. Then

$$|f_2(x_2) - f_2(x)| \geq \varepsilon \quad \text{whenever} \quad d(x,\, x_2) < \frac{1}{2}.$$

Take $\delta = \dfrac{1}{3}$. Let $x_3 \in S(x,\, \dfrac{1}{3})$ and $A = \{f_1,\ f_2\}$.

$$|f_3(x_3) - f_3(x)| \geq \varepsilon \quad \text{whenever} \quad d(x,\, x_3) < \frac{1}{3}.$$

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On continuing like this take $\delta = \dfrac{1}{n}$. Let $x_n \in S(x,\, \dfrac{1}{n})$ and $A = \{f_1,\ f_2,\ f_3,\ \ldots,\, f_{n-1}\}$. Then

$$|f_n(x_n) - f_n(x)| \geq \varepsilon \quad \text{whenever} \quad d(x,\, x_n) < \frac{1}{n}.$$

Let $\varepsilon > 0$ be any real number and choose a positive integer $m \in \mathbb{N}$ so large such that

$$\frac{1}{m} \; < \; \varepsilon.$$

Let $n \geq m$ be any natural number then $\dfrac{1}{n} \leq \dfrac{1}{m} < \varepsilon$.

$\Rightarrow \dfrac{1}{n} < \varepsilon \ \ \forall \ \ n \geq m.$

Therefore, from above $d(x_n,\, x) < \varepsilon \ \ \forall \ \ n \geq m.$

This implies that $x_n \to x$. Thus, by induction we define two sequences $(x_n)_{n \,\in\, \mathbb{N}} \subseteq X$ and $(f_n)_{n \,\in\, \mathbb{N}} \subseteq F$ such that

$f_n \neq f_m$ for $n \neq m$, $x_n \to x$ and $|f_n(x_n) - f_n(x)| \geq \varepsilon$ for each $n \in \mathbb{N}$. (4.4.2)

Since F is compact, so there exists a subsequence $(f_{k_n})_{n \,\in\, \mathbb{N}}$ and $f \in F$ such that $f_{k_n} \xrightarrow{u} f$.

From the hypothesis, the function f is continuous and so $f(x_{k_n}) \to f(x)$.

For $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ such that for all $n \geq n_o$, we have that

$$|f(x_{k_n}) - f(x)| < \frac{\varepsilon}{3}.$$

Also, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$

$$\|f_{k_n} - f\| < \frac{\varepsilon}{3} \text{ holds.}$$

Let $n = \max\{n_o,\ n_1\}$, then

$$|f_{k_n}(x_{k_n}) - f_{k_n}(x)| < |f_{k_n}(x_{k_n}) - f(x_{k_n})| + |f(x_{k_n}) - f(x)| + |f(x) - f_{k_n}(x)|.$$
$$\leq\ 2.\|f_{k_n} - f\|\ +\ |f(x_{k_n}) - f(x)|\ <\ \varepsilon,$$

which contradicts (4.4.2). □

Note that F is not necessarily a subset of $C(X,\, R)$. Using the fact that an exhaustive family F which consists of continuous functions is equicontinuous (Proposition 2.1(a)) it is clear that this theorem is indeed a generalization of classical Ascoli theorem.

Corollary 4.5 (Classical Ascoli - Arzela Theorem). Let X be a compact topological space, and let A be a subset of $C(X)$. Then the following statements are equivalent :

- (a) A is compact subset of the metric space $C(X)$ (equipped with the uniform metric).
- (b) A is closed, bounded, and equicontinuous.

Proof: We first prove that $(a) \Rightarrow (b)$

Given A is a compact subset of a metric space $C(X)$. We have to show that A is closed, bounded and equicontinuous.

We know that comapact set is closed and bounded. What remains to be shown is that A is equicontinuous.

To this end, let $\varepsilon > 0$ and $\{S(f,\, \varepsilon) : f \in A\}$ be an open cover A.

Since, A is compact. This open covering is reducible to finite subcovering. Let $f_1,\, f_2,\, f_3,\, ...,\, f_n \in A$ such that $A \subseteq \bigcup\limits_{i=1}^{n} S(f_i,\, \varepsilon)$.

If $x \in X$, then pick a neighborhood $V_{x,\, i}$ of x such that

$|f_i(y) - f_i(x)| < \varepsilon$ holds for all $y \in V_{x,\, i}$ and all $i = 1,\, 2, ...,\, n$.

Now, let $V_x = \bigcap\limits_{i=1}^{n} V_{x,\, i}$. Take $y \in V_x$ and $f \in A$. Then on choosing some i such that $f \in S(f_i,\, \varepsilon)$, we note that

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_i(y)| \ + \ |f_i(y) - f_i(x)| \ + \ |f_i(x) - f(x)| \\ &< \varepsilon \ + \ \varepsilon \ + \ \varepsilon \\ &= 3\varepsilon. \end{aligned}$$

This shows that A is equicontinuous at x, and since x was arbitrary, A is an set of functions.

We now prove that $(b) \Rightarrow (a)$

It's given that A is closed, bounded and equicontinuous, and we have to show that A is compact. In order to prove this, it is sufficient to show that every sequence in A has a convergent subsequence. For this let $(f_n)_{n \,\in\, \mathbb{N}}$ be a sequence in A. Our claim is to show that $(f_n)_{n \,\in\, \mathbb{N}}$ has a convergent subsequence.

Given that A is bounded. So, choose $m > 0$ satisfying

$$Dia(A) = Sup\{\ d(f,\, g) : f,\, g \in A\ \} = m.$$

$$\Rightarrow d(f,\, g) \leq m \;\; \forall \;\; f,\, g \in A.$$

$$\Rightarrow |f(x) - g(x)| \leq m \;\; \forall \;\; f,\, g \in A \;\text{ and }\; \forall \;\; x \in X.$$

$$\Rightarrow |f(x)| - |g(x)| \leq m \;\; \forall \;\; f,\, g \in A \quad \text{and} \quad \forall \;\; x \in X.$$

$$\Rightarrow |f(x)| \leq m \; + \; |g(x)| \;\; \forall \;\; f,\, g \in A \quad \text{and} \quad \forall \;\; x \in X$$

(4.5.1).

Now $g \in A$ and let it be fixed. Since X is compact and g is a continuous function defined on X. Therefore, g is bounded. Then there exist some positive integer n such that

$$|g(x)| \leq n \;\; \forall \;\; x \in X.$$

From (4.5.1) we have $|f(x)| \leq m + n = M \;\; \forall \;\; x \in X \quad \text{and} \quad \forall \;\; f \in A.$

Let k be a natural number. Since A is equicontinuous, Therefore, for every $y \in X$, there exists an open set V_y such that

$$|f(x) - f(y)| < \frac{1}{k} \text{ whenever } x \in V_y \text{ and } f \in S.$$

Note that $\{V_y : y \in X\}$ is an open cover for X. Since, X is compact, this open cover has a finite subcover.

Let $F_k = \{y_1,\, y_2,\, ...,\, y_n\}$ such that $X = \bigcup_{y \in F_k} V_y$ and let $F = \bigcup_{i=1}^{\infty} F_i$. Since each F_i is finite, F is atmost countable. Let $F = \{x_1,\, x_2,\, ...\}$ be an enumeration of F. Since, $|f_n(x_1)| \leq M$ as $f_n \in A$ and this holds for all n. Therefore, $\{f_n(x_1)\}$ is bounded sequence in $\mathbb{R}$ and so it has a convergent subsequence. So, let $(g_n^1)_{n \in \mathbb{N}}$ be a subsequence of $(f_n)_{n \in \mathbb{N}}$ such that limit $g_n^1(x_1)$ exists in $\mathbb{R}$.

Similarly, there exists a subsequence $(g_n^2)_{n \in \mathbb{N}}$ of $(g_n^1)_{n \in \mathbb{N}}$ such that limit $g_n^2(x_2)$ exists in $\mathbb{R}$.

Continuing this way,

we can choose (inductively) sequences $(g_n^i)_{n \in \mathbb{N}}$ $(i = 1, 2, ...)$ such that

a. $(g_n^1)_{n \in \mathbb{N}}$ is a subsequence of $(f_n)_{n \in \mathbb{N}}$.

b. $(g_n^{i+1})_{n \in \mathbb{N}}$ is a subsequence of $(g_n^i)_{n \in \mathbb{N}}$ for each $i = 1, 2, ...$, and

c. $\lim_{n \to \infty} g_n^i(x_i)$ exists in $\mathbb{R}$ for each $i = 1, 2, ...$

Now, consider the diagonal sequence $h_n = g_n^n$ and note that $(h_n)_{n \in \mathbb{N}}$ is a subsequence of $(f_n)_{n \in \mathbb{N}}$ such that limit $h_n(x_i)$ exists in $\mathbb{R}$ for each i. Now, we claim that $(h_n)_{n \in \mathbb{N}}$ is a cauchy sequence in $C(X)$.

Let k be natural number. Since, limit $h_n(y)$ exists for each $y \in F_k$ and $(h_n)_{n \in \mathbb{N}}$ is a cauchy sequence. So there exists some n_0 such that

$$|h_n(y) - h_m(y)| < \frac{1}{k} \text{ holds for all } n,\, m \geq n_0 \text{ and all } y \in F_k.$$

If $x \in X$, then take some $y \in F_k$ such that $x \in V_y$ and that

$$\begin{aligned} |h_n(x) - h_m(x)| &\leq |h_n(x) - h_n(y)| \;+\; |h_n(y) - h_m(y)| \;+\; |h_m(y) - h_m(x)| \\ &< \frac{1}{k} + \frac{1}{k} + \frac{1}{k} \\ &= \frac{3}{k} \text{ holds for all } n,\, m \geq n_0. \end{aligned}$$

Moreover,

$$\begin{aligned} \|h_n - h_m\| &= sup\{|h_n(x) - h_m(x)| : x \in X\} \\ &= \frac{3}{k}. \end{aligned}$$

holds for all n, $m \geq n_0$ and thus $(h_n)_{n \in \mathbb{N}}$ is a cauchy sequence in $C(X)$. Since, $C(X)$ is complete. So, $(h_n)_{n \in \mathbb{N}}$ converges to some $h \in C(X)$. Finally since, A is closed and thus $h \in A$, therefore the sequence $(f_n)_{n \in \mathbb{N}}$ of A has a convergent subsequence. Consequently, A is compact. □

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