# A STUDY ON SOME INTUITIONISTIC/PYTHAGOREAN FUZZY AGGREGATION OPERATORS

Thesis submitted in partial fulfillment of the requirements for the

award of degree of

**Masters of Science**

**in**

**MATHEMATICS and COMPUTING**

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**July, 2017**

# CERTIFICATE

I hereby certify that the dissertation entitled, "**A STUDY ON SOME INTUITIONISTIC/ PYTHAGOREAN FUZZY AGGREGATION OPERATORS**", which is being submitted by **Miss. Diksha Garg** (Roll No. 301503006), in the partial fulfillment of the requirement for the award of the degree of Master of Science in the School of Mathematics, Thapar University, Patiala, comprises of candidate's own research work carried out under supervision and guidance of **Dr. Amit Kumar** during the period from January 2017 to June 2017.

The part of work presented in this disseration has not been submitted either in part or in full to this or any other University or Institute for the award of any degree.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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# ACKNOWLEDGEMENTS

Gurur–Brahma Gurur–Vishnur–Gururdevo Maheshvarah|
Gurure va Paraṁ Brahma Tasme Shrii – Gurve Namah||

First and foremost, I express my deep and sincere gratitude to my supervisor Dr. Amit Kumar, Associate Professor, School of Mathematics, Thapar University, Patiala for his expert guidance, valuable suggestions, support, advice and continuous encouragement throughout the period of my research work. The imperative comments, rendered by him during the discussions, are deeply appreciated.

I would like to acknowledge the adolescent inner blessings of Mehar. My supervisor, Dr. Amit Kumar, believes that Mata Vaishno Devi has appeared on the earth in the form of Mehar and without Mehar's blessings it would not have been possible to think the ideas presented in this thesis. Mehar is a lovely daughter of Dr. Parmpreet Kaur (Cousin sister of Dr. Amit Kumar).

My heartfelt thanks go to Dr. A. K. Lal (Head of School of Mathematics, Thapar University, Patiala), Prof. S. S. Bhatia (Dean of Academic Affairs, Thapar University, Patiala), and to all faculty members for their helpful and valuable suggestions.

I am forever grateful to my parents, who mean world to me. Thank you Mom and Dad for showing faith in me and giving me liberty to choose whait I desired.

I am ever grateful to God, the Creator and the Guardian and to whom I owe my very existence. Thank you God for the numerous blessings bestowed upon me in every aspect of my life.

Diksha

Patiala **Diksha Garg**

July, 2017

# ABSTRACT

In this thesis, need of intuitionistic/Pythagorean fuzzy aggregation operators is discussed. Furthermore, some existing properties of these aggregation operators as well as various extensions of these operators are studied.

**Contents**

# Chapter 1

# Introduction

In several real life problems, we need to aggregate the available information to make a decision e.g., to select the best student of class, the weighted arithmetic mean of the grades of different courses of each student of a class is evaluated. The student who secures maximum weighted arithmetic mean is considered as the best student. There are several operators which can be used to aggregate the available information e.g., Arithmetic mean , Geometric mean, Harmonic mean, Weighted arithmetic mean, Weighted geometric mean etc. However, in most of the real life problems Weighted arithmetic mean and Weighted geometric mean is used to aggregate the information. The weighted arithmetic mean and Weighted geometric mean can be defined as follows:

**Weighted arithmetic mean**: If $x_1, x_2$ ,..., $x_n$ are $n$ real numbers and $w = (w_1, w_1, \dots, w_1)^T$ , $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$ is the weight vector then the real number $w_1x_1 + w_2x_2 + \cdots + w_nx_n$ is known as weighted arithmetic mean of $x_1, x_2$ ,..., $x_n$.

**Weighted geometric mean**: If $x_1, x_2$,..., $x_n$ are $n$ real numbers and $w = (w_1, w_1, \dots, w_1)^T$ , $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$ is the weight vector then the real number ${x_1}^{w_1}, {x_2}^{w_2}$, ..., ${x_n}^{w_n}$ is known as weighted geometic mean of $x_1, x_2$ ,..., $x_n$.

It is well known fact that to find the weighted arithmetic mean and weighted geometric mean, the values of $x_1, x_2$,..., $x_n$ should be known in terms of real numbers. However, there exist several real life problems in which the obtained data/information cannot be represented as real numbers e.g. the following views of a teacher about a student cannot be represented as a real number.

1. The teacher is 80% satisfy with the statement that the student is good in Mathematics.

2. The teacher is 60% satisfy with the statement that the student is good in Physics.

It is pertinent to mention that the fuzzy set, introduced by Zadeh [6], can be used to represent such information. A fuzzy set , represented by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, is a generalization of crisp set $A = \{(x, X_A(x)) : x \in X\}$. The difference between crisp set $A$ and a fuzzy $\tilde{A}$ is that in the crisp set $A$, in the crisp set $A$, it is assumed that an element of Universal set $X$ will either belong to set $A$ or will not belong to set $A$. If the element $x$ belong to set $A$ then the value of $X_A(x)$ will be 1 and if the element $x$ does not belong to set $A$ then the value of $X_A(x)$ will be 0. While, in the fuzzy set $\tilde{A}$, it is assumed that an element of the universal set $X$ can partially/fully belong to set $\tilde{A}$. If the element $x$ of the universal set $X$ fully belong to $A$ then the value of $\mu_{\tilde{A}}(x)$ will be 1, if partially belong to $\tilde{A}$ then the value of $\mu_{\tilde{A}}(x)$ will be greater than 0 and less than 1. Otherwise the value of $\mu_{\tilde{A}}(x)$ will be 0.

In the literature, $w_1\tilde{x}_1 + w_2\tilde{x}_2 + \cdots + w_n\tilde{x}_n$ , $w_1, w_2, \dots, w_n \geq 0$, $\sum_{i=1}^{n} w_i = 1$ and $\tilde{x}_1{}^{w_1} \times \tilde{x}_2{}^{w_2} \times \dots \times \tilde{x}_n{}^{w_n}$, $w_i > 0$, $\sum_{i=1}^{n} w_i = 1$, where $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ are fuzzy numbers are named as weighted fuzzy arithmetic mean and weighted fuzzy geometric mean respectively. These expressions are obtained by replacing the real numbers $x_1, x_2$ ,..., $x_n$ with fuzzy numbers $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ in the expressions of crisp weighted arithmetic mean and crisp weighted geometric mean respectively.

Although, the fuzzy set is better representation of real life data as compared to crisp set. But, in the fuzzy set it is assumed that if $\mu_{\tilde{A}}(x)$ represent degree of membership then $1-\mu_{\tilde{A}}(x)$ represent degree of non-membership i.e., in the fuzzy set it is assumed that there is no hesitation. However, there exist several real life problems in which there also exist degree of hesitation with degree of membership and degree of non-membership e.g., if a person is 80% in favour to vote BJP then it does not imply that he/she 20% in appear to vote BJP. It may be that he/she is in confusion about his/her vote. In the literature, it is pointed out such information/data cannot be represented by a fuzzy set but can be represented by a intuitionistic fuzzy set [1].

An intuitionistic fuzzy set is generalization of fuzzy set. An intuitionistic fuzzy set is represented as $\tilde{A}=\{(x,\mu_{\tilde{A}}(x),\nu_{\tilde{A}}(x))\}$ where, $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ represented degree of membership and degree of non-membership of an element $x$ such that $0\leq\mu_{\tilde{A}}(x)\leq 1$ and $0\leq\nu_{\tilde{A}}(x)\leq 1$ is $0\leq\mu_{\tilde{A}}(x)+\nu_{\tilde{A}}(x)\leq 1$. The value $1-\left(\mu_{\tilde{A}}(x)+\nu_{\tilde{A}}(x)\right)$ is called degree of hesitation. Yager and Abbasov [5] pointed out that in some real life problems, for $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$, the condition $\mu_{\tilde{A}}(x)+\nu_{\tilde{A}}(x)\leq 1$ may not be satisfied but the condition $\mu_{\tilde{A}}^2(x)+\nu_{\tilde{A}}^2(x)\leq 1$ may be satisfied. Yager [5] named such type of intuitionistic fuzzy sets as Pythagorean fuzzy sets. In the literature, $w_1\tilde{x}_1+w_2\tilde{x}_2+\cdots+w_n\tilde{x}_n$ , $w_1,w_2,\ldots,w_n\geq 0$, $\sum_{i=1}^{n}w_i=1$ and $\tilde{x}_1{}^{w_1}\times\tilde{x}_2{}^{w_2}\times\ldots\times\tilde{x}_n{}^{w_n}$, $w_i>0$, $\sum_{i=1}^{n}w_i=1$, where $\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n$ are intuitionistic/pythagorean fuzzy sets are named as weighted intuitionistic/Pythagorean arithmetic mean and weighted intuitionistic/ Pythagorean geometric mean respectively.

It is pertinent to mention that the arithmetic operations of intuitionistic fuzzy sets /Pythagorean fuzzy sets are not unique and hence, on simplifying the general expression for evaluating the weighted arithmetic mean and weighted geometric mean i.e., $w_1\tilde{x}_1+w_2\tilde{x}_2+\cdots+$

$w_n\tilde{x}_n$ and $\tilde{x}_1^{w_1} \times \tilde{x}_2^{w_2} \times \ldots \times \tilde{x}_n^{w_n}$ respectively of intuitionistic/Pythagorean fuzzy set , different expressions are obtained.

Xu [4] proposed the intuitionistic fuzzy weighted averaging operator, the intuitionistic fuzzy ordered weighted averaging operator and the intuitionistic hybrid averaging operator.

Tan et al. [3] proposed Archimedean intuitionistic fuzzy weighted geometric operator, Archimedean intuitionistic fuzzy ordered weighted geometric operator and Archimedean intuitionistic fuzzy hybrid geometric operator. Ma and Xu [2] proposed the symmetric Pythagorean fuzzy weighted geometric/averaging operators.

In this thesis, the aggregation operatoras, proposed by Xu [4],Tan et al. [3] and Ma and Xu [4], are studied.

# Chapter 2

# Some properties of intuitionistic fuzzy aggregation operator and its extensions

In this chapter, the method for comparing intuitionistic fuzzy values, proposed by Xu [4], is discussed. Also, the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator and intuitionistic fuzzy hybrid averaging operator as well as various properties of these operators, proposed by Xu [4], are discussed.

## 2.1 Basic definitions

In this section, some basic definitions are reviewed [4].

**Definition 2.1** An aggregation function $f\colon [0,1]^n \rightarrow [0,1]$ is a function non-decreasing in each argument and satisfying $f(0,\ldots,0) = 0$ and $f(1,\ldots,1) = 1$.

**Definition 2.2** Let $WA\colon R^n \rightarrow R$, if $WA$

$$WA_\omega(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \omega_j\, a_j.$$

Then $WA$ is called a weighted averaging operator, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$, $R$ is the set of all real numbers.

**Definition 2.3** An ordered weighted averaging $(OWA)$ operator of dimension $n$ is a mapping

$$OWA : R^n \rightarrow R$$

that has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$ Furthermore,

$$OWA_\omega(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \omega_j\, b_j,$$

where $b_j$ is the $j$th largest of $a_i (i = 1, 2, \dots, n)$.

It is obvious from Definitions 2.2 and 2.3, the $WA$ operator first weights all the given arguments and then aggregates all these weighted arguments into a collective one. The fundamental aspect of the OWA operator is the reordering step; it first reorders all the given arguments in descending order and then weights these ordered arguments, and finally aggregates all these ordered weighted arguments into a collective one.

**Definition 2.3** Let $\widetilde{a_j} = \left[t_{\tilde{a}_j}, 1 - f_{\tilde{a}_j}\right]$ $(j = 1,2, \dots, n)$ be collection of intuitionistic fuzzy values. Then, intuitionistic fuzzy weighted averaging operator (IFWA) of dimension $n$ is a mapping, $IFWA: \Omega^{\mathrm{n}} \to \Omega$, where $\Omega$ is the set of all intuitionistic fuzzy values, defined by

$$IFWA_{\omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \omega_1 \tilde{a}_1 \oplus \omega_2 \tilde{a}_2 \oplus \dots . \oplus \ \omega_n \tilde{a}_n.$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $\widetilde{a_j}$ $(j = 1,2, \dots, n)$, with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Especially, if , $\omega = (1/n, 1/n, \dots, 1/n)^T$ then the IFWA operator is reduced to an intuitionistic fuzzy averaging (IFA) operator of dimension $n$, which is defined as follows:

$$IFA((\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = 1/n(\tilde{a}_1 \oplus \tilde{a}_2 \oplus \dots \oplus a_n).$$

**Definition 2.4** Let $\widetilde{a_j} = \left[t_{\tilde{a}_j}, 1 - f_{\tilde{a}_j}\right]$ $(j = 1,2, \dots, n)$ be a collection of intuitionistic fuzzy values. Then, an intuitionistic fuzzy $OWA$ (IFOWA) operator of dimension $n$ is a mapping IFOWA: $\Omega^n \to \Omega$, that has an associated vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Furthermore

$$IFOWA_{\omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \omega_1 \tilde{a}_{\sigma(1)} \oplus \omega_2 \tilde{a}_{\sigma(2)} \oplus \dots . \oplus \ \omega_n \tilde{a}_{\sigma(n)}.$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1,2, \dots, n)$ such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_j$, for all $j$.

Especially, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then the IFOWA operator is reduced to an IFA operator of dimension $n$

**Definition 2.5** An intuitionistic fuzzy hybrid aggregation operator of dimension $n$ is a mapping $IFHA\colon \Omega^n \to \Omega$ , which has an associated vector $w = (w_1, w_2, \dots, w_n)^T$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that

$$IFHA_{\omega,w}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = w_1\dot{\tilde{a}}_{\sigma(1)} \oplus w_2\dot{\tilde{a}}_{\sigma(2)} \oplus \dots \oplus w_n\dot{\tilde{a}}_{\sigma(n)},$$

where $\dot{\tilde{a}}_{\sigma(1)}$ is the $j^{th}$ largest of the weighted intuitionistic fuzzy values $\dot{\tilde{a}}_{\sigma(j)}\left(\dot{\tilde{a}}_{\sigma(j)} = n\omega_j\tilde{a}_j, j = 1,2,\dots,n\right)$ $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $\tilde{a}_j$ $(j = 1,2, \dots, n)$ , with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$, and $n$ is the balancing coefficient, which plays a role of balance (in such a case, if the vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ approaches $(1/n, 1/n, \dots, 1/n)^T$ , then the vector $(n\omega_1\tilde{a}_1, n\omega_2\tilde{a}_2, \dots, n\omega_n\tilde{a}_n)^T$ approaches $(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)^T$.

### 2.2 Arithmetic operations

In this section, arithmetic operations of intituonistic fuzzy values, proposed by Xu [4], are presented.

Let $\tilde{a} = [t_{\tilde{a}}, 1 - f_{\tilde{a}}]$ and $\tilde{b} = [t_{\tilde{b}}, 1 - f_{\tilde{b}}]$ be two intuitionistic fuzzy values; then

1) $\tilde{a} \oplus \tilde{b} = [t_{\tilde{a}} + t_{\tilde{b}} - t_{\tilde{a}}t_{\tilde{b}}, 1 - f_{\tilde{a}}f_{\tilde{b}}]$.

2) $\lambda\tilde{a} = \left[1 - (1 - t_{\tilde{a}})^{\lambda}, 1 - f_{\tilde{a}}^{\lambda}\right], \lambda > 0$.

### 2.3 Comparison of intutionistic fuzzy values

In this section, the method for comparing intutionistic fuzzy values, proposed by Xu [4] ,is presented.

Let $\tilde{a} = [t_{\tilde{a}}, 1 - f_{\tilde{a}}]$ and $\tilde{b} = [t_{\tilde{b}}, 1 - b_{\tilde{b}}]$ be two intuitionistic fuzzy values. Then, find scores $S(\tilde{a}) = t_{\tilde{a}} - f_{\tilde{a}}$ and $S(\tilde{b}) = t_{\tilde{b}} - f_{\tilde{b}}$ of $\tilde{a}$ and $\tilde{b}$ respectively and check that $S(\tilde{a}) > S(\tilde{b})$ or $S(\tilde{a}) < S(\tilde{b})$ or $S(\tilde{a}) = S(\tilde{b})$

**Case (1)** If $S(\tilde{a}) > S(\tilde{b})$ , then $\tilde{a} > \tilde{b}$.

**Case (2)** If $S(\tilde{a}) < S(\tilde{b})$ , then $\tilde{a} < \tilde{b}$.

**Case (3)** If $S(\tilde{a}) = S(\tilde{b})$ , then find accuracy degree $H(\tilde{a}) = t_{\tilde{a}} + f_{\tilde{a}}$ and $H(\tilde{b}) = t_{\tilde{b}} + f_{\tilde{b}}$ of $\tilde{a}$ and $\tilde{b}$ respectively

**Case (3a)** If $H(\tilde{a}) > H(\tilde{b})$ , then $\tilde{a} > \tilde{b}$.

**Case (3b)** If $H(\tilde{a}) < H(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$.

**Case (3c)** If $H(\tilde{a}) = H(\tilde{b})$, then $\tilde{a} = \tilde{b}$.

### 2.4 Some important results

In this section, some results, proposed by Xu [2], are presented.

**Theorem 2.1** Let $\tilde{a} = [t_{\tilde{a}}, 1 - f_{\tilde{a}}]$ and $\tilde{b} = [t_{\tilde{b}}, 1 - f_{\tilde{b}}]$ be two intuitionistic fuzzy values, $\lambda, \lambda_1$ ,$\lambda_2 > 0$ . Then,

1) $\tilde{a} \oplus \tilde{b} = \tilde{b} \oplus \tilde{a}$.

2) $\lambda(\tilde{a} \oplus \tilde{b}) = \lambda\tilde{a} \oplus \lambda\tilde{b}$.

3) $\lambda_1\tilde{a} \oplus \lambda_2\tilde{a} = (\lambda_1 + \lambda_2)\tilde{a}$.

Proof*:*

1) Using the operational law (1) of Section 2.2,

$$\begin{aligned} \tilde{a} \oplus \tilde{b} &= [t_{\tilde{a}} + t_{\tilde{b}} - t_{\tilde{a}}t_{\tilde{b}}, 1 - f_{\tilde{a}}f_{\tilde{b}}] \\ &= [t_{\tilde{b}} + t_{\tilde{a}} - t_{\tilde{b}}t_{\tilde{a}}, 1 - f_{\tilde{b}}f_{\tilde{a}}] \\ &= \tilde{b} \oplus \tilde{a}. \end{aligned}$$

2) Using the operational law (1) of Section 2.2,

$$\tilde{a} \oplus \tilde{b} = [t_{\tilde{a}} + t_{\tilde{b}} - t_{\tilde{a}}t_{\tilde{b}}, 1 - f_{\tilde{a}}f_{\tilde{b}}].$$

Using the operational law (2) of Section 2.2,

$$\lambda\left(\tilde{a} \oplus \tilde{b}\right) = [1 - (1 - (t_{\tilde{a}} + t_{\tilde{b}} - t_{\tilde{a}}t_{\tilde{b}}))^{\lambda}, 1 - (f_{\tilde{a}}f_{\tilde{b}})^{\lambda}]$$

$$= \left[1 - (1 - t_{\tilde{a}})^{\lambda}(1 - t_{\tilde{b}})^{\lambda}, 1 - (f_{\tilde{a}}f_{\tilde{b}})^{\lambda}\right]. \qquad (2.1)$$

Using operational law (2) of Section 2.2,

$$\lambda\,\tilde{a} = [1 - (1 - t_{\tilde{a}})^{\lambda}, 1 - f_{\tilde{a}}^{\lambda}],\ \lambda\tilde{b} = [1 - (1 - t_{\tilde{b}})^{\lambda}, 1 - f_{\tilde{b}}^{\lambda}]$$

Using operational law (1) of Section 2.2,

$$\lambda\tilde{a} \oplus \lambda\tilde{b} = \left[1 - (1 - t_{\tilde{a}})^{\lambda} + 1 - (1 - t_{\tilde{b}})^{\lambda} - \left(1 - (1 - t_{\tilde{a}})^{\lambda}\right) \times \left(1 - (1 - t_{\tilde{b}})^{\lambda}\right),\right.$$

$$\left. 1 - (f_{\tilde{a}}f_{\tilde{b}})^{\lambda}\right]$$

$$= [2 - (1 - t_{\tilde{a}})^{\lambda} - (1 - t_{\tilde{b}})^{\lambda} - (1 - (1 - t_{\tilde{a}})^{\lambda} - (1 - t_{\tilde{b}})^{\lambda} + (1 - t_{\tilde{a}})^{\lambda}$$

$$(1 - t_{\tilde{b}})^{\lambda}), 1 - (f_{\tilde{a}}f_{\tilde{b}})^{\lambda}]$$

$$= [1 - (1 - t_{\tilde{a}})^{\lambda}(1 - t_{\tilde{b}})^{\lambda}, 1 - (f_{\tilde{a}}f_{\tilde{b}})^{\lambda}]. \qquad (2.2)$$

Using (2.1) and (2.2),

$$\lambda(\tilde{a} \oplus \tilde{b}) = \lambda\tilde{a} \oplus \lambda\tilde{b}$$

3) using operational law (2) of Section 2.2,

$$\lambda_1\tilde{a} = \left[1 - (1 - t_{\tilde{a}})^{\lambda_1}, 1 - f_{\tilde{a}}^{\lambda_1}\right],$$

$$\lambda_2\tilde{a} = \left[1 - (1 - t_{\tilde{a}})^{\lambda_2}, 1 - f_{\tilde{a}}^{\lambda_2}\right].$$

Using operational law (1) of Section 2.2,

$$\lambda_1\tilde{a} \oplus \lambda_2\tilde{a} = [2-(1-t_{\tilde{a}})^{\lambda_1}-(1-t_{\tilde{a}})^{\lambda_2}-\left(1-(1-t_{\tilde{a}})^{\lambda_1}\right)\times\left(1-(1-t_{\tilde{a}})^{\lambda_2}\right)$$

$$,1-f_{\tilde{a}}^{\lambda_1}f_{\tilde{a}}^{\lambda_2}]$$

$$=[1-(1-t_{\tilde{a}})^{\lambda_1}(1-t_{\tilde{a}})^{\lambda_1},1-f_{\tilde{a}}^{\lambda_1}f_{\tilde{a}}^{\lambda_2}]$$

$$=[1-(1-t_{\tilde{a}})^{\lambda_1+\lambda_2},1-(f_{\tilde{a}})^{\lambda_1+\lambda_2}]$$

$$=(\lambda_1+\lambda_2)\tilde{a}.$$

**Theorem 2.2** Let $\tilde{a}_j = \left[t_{\tilde{a}_j}, 1-f_{\tilde{a}_j}\right]$ $(j=1,2,\dots,n)$ be a collection of intuitionistic fuzzy values; then their aggregated value by using the IFWA operator is

$$IFWA_{\omega}(\tilde{a}_1,\tilde{a}_2,\dots,\tilde{a}_n) = [1-\prod_{j=1}^{n}\left(1-t_{\tilde{a}_j}\right)^{\omega_j},1-\prod_{j=1}^{n}f_{\tilde{a}_j}^{\omega_j}], \qquad (2.3)$$

where $\omega=(\omega_1,\omega_2,\dots,\omega_n)^T$ is the weight vector of $\tilde{a}_j$ $(j=1,2,\dots,n)$, with $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1$. Especially, if $t_{\tilde{a}_j}=1-f_{\tilde{a}_j}$, for all , then (2.3) is reduced to the following form:

$$IFWA_{\omega}(\tilde{a}_1,\tilde{a}_2,\dots,\tilde{a}_n)=\left[1-\prod_{j=1}^{n}\left(1-t_{\tilde{a}_j}\right)^{\omega_j}\right].$$

**Proof** : This can be easily proved by mathematical induction as follows:

Firstly, prove that the result is true for $n=2$

Using operational law (2) of Section 2.2,

$$\omega_1\tilde{a}_1=\left[1-\left(1-t_{\tilde{a}_1}\right)^{\omega_1},1-f_{\tilde{a}_1}^{\omega_1}\right],$$

$$\omega_2\tilde{a}_2=\left[1-\left(1-t_{\tilde{a}_2}\right)^{\omega_2},1-f_{\tilde{a}_2}^{\omega_2}\right].$$

Using operational law (1) of Section 2.2,

$$IFWA_{\omega}(\tilde{a}_1,\tilde{a}_2)=\omega_1\tilde{a}_1\oplus\omega_2\tilde{a}_2$$

$$= [2 - (1 - t_{\tilde{a}_1})^{\omega_1} - (1 - t_{\tilde{a}_2})^{\omega_2} - (1 - (1 - t_{\tilde{a}_1})^{\omega_1}) \times (1 - (1 - t_{\tilde{a}_2})^{\omega_2}$$

$$, 1 - f_{\tilde{a}_1}^{\omega_1} f_{\tilde{a}_2}^{\omega_2}]$$

$$= \left[1 - (1 - t_{\tilde{a}_1})^{\omega_1} (1 - t_{\tilde{a}_2})^{\omega_2}, 1 - f_{\tilde{a}_1}^{\omega_1} f_{\tilde{a}_2}^{\omega_2}\right].$$

Therefore, the result is true for $n = 2$.

Now, let the result is true for $n = k$ i.e.,

$$IFWA_{\omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k) = [1 - \prod_{j=1}^{k} \left(1 - t_{\tilde{a}_j}\right)^{\omega_j}, 1 - \prod_{j=1}^{k} f_{\tilde{a}_j}^{\omega_j}]$$

and prove that it is also true for $n = k + 1$.

$$IFWA_{\omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_{k+1}) = [1 - \prod_{j=1}^{k} \left(1 - t_{\tilde{a}_j}\right)^{\omega_j} + (1 - (1 - t_{\tilde{a}_{k+1}})^{\omega_{k+1}}) - \Big(1 -$$

$$\prod_{j=1}^{k} \left(1 - t_{\tilde{a}_j}\right)^{\omega_j}\Big) \times (1 - (1 - t_{\tilde{a}_{k+1}})^{\omega_{k+1}}), 1 - \prod_{j=1}^{k+1} f_{\tilde{a}_j}^{\omega_j}]$$

$$= [1 - \prod_{j=1}^{k} \left(1 - t_{\tilde{a}_j}\right)^{\omega_j}, \ 1 - \prod_{j=1}^{k+1} f_{\tilde{a}_j}^{\omega_j}].$$

Since, the result is true for $n = k + 1$. Hence, the result is true for all values of $n$.

**Theorem 2.3** Let $\tilde{a}_j = \left[t_{\tilde{a}_j}, 1 - f_{\tilde{a}_j}\right]$ $(j = 1, 2, \dots, n)$ be a collection of intuitionistic fuzzy values; then their aggregated value by using the IFOWA operator is

$$IFOWA_{\omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = [1 - \prod_{j=1}^{n} \left(1 - t_{\tilde{a}_{\sigma(j)}}\right)^{\omega_j}, 1 - \prod_{j=1}^{n} f_{\tilde{a}_{\sigma(j)}}^{\omega_j}], \qquad (2.4)$$

where $\omega = (1/n, 1/n, \dots, 1/n)^T$ is the weighting vector of the IFOWA operator, with that $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Especially, if $t_{\tilde{a}_j} + f_{\tilde{a}_j} = 1$, for all $j = 1, 2, \dots, n$, i.e., all $\tilde{a}_j (j = 1, 2, \dots, n)$ are reduced to $t_{\tilde{a}_j} (j = 1, 2, \dots, n)$, respectively, then (2.4) is reduced to the following form:

$$IFOWA_{\omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = 1 - \prod_{j=1}^{n} \left(1 - t_{\tilde{a}_{\sigma(j)}}\right)^{\omega_j}.$$

**Proof** : This can be easily proved by mathematical induction as follows:

Firstly, prove that the result is true for $n = 2$

Using operational law (2) of Section 2.2,

$$\omega_1\tilde{a}_{\sigma(1)} = \left[1-\left(1-t_{\tilde{a}_{\sigma(1)}}\right)^{\omega_1}, 1-f_{\tilde{a}_{\sigma(1)}}^{\omega_1}\right],$$

$$\omega_2\tilde{a}_{\sigma(2)} = \left[1-\left(1-t_{\tilde{a}_{\sigma(2)}}\right)^{\omega_2}, 1-f_{\tilde{a}_{\sigma(2)}}^{\omega_2}\right].$$

Using operational law (1) of Section 2.2,

$$IFWOA_{\omega}(\tilde{a}_1, \tilde{a}_2) = \omega_1\tilde{a}_{\sigma(1)} \oplus \omega_2\tilde{a}_{\sigma(2)}$$

$$= [2-\left(1-t_{\tilde{a}_{\sigma(1)}}\right)^{\omega_1}-\left(1-t_{\tilde{a}_{\sigma(2)}}\right)^{\omega_2}-(1-\left(1-t_{\tilde{a}_{\sigma(1)}}\right)^{\omega_1})\times(1-$$

$$\left(1-t_{\tilde{a}_{\sigma(2)}}\right)^{\omega_2} \quad , 1-f_{\tilde{a}_{\sigma(1)}}^{\omega_1} f_{\tilde{a}_{\sigma(2)}}^{\omega_2}]$$

$$= \left[1-\left(1-t_{\tilde{a}_{\sigma(1)}}\right)^{\omega_1}\left(1-t_{\tilde{a}_{\sigma(2)}}\right)^{\omega_2}, 1-f_{\tilde{a}_{\sigma(1)}}^{\omega_1} f_{\tilde{a}_{\sigma(2)}}^{\omega_2}\right].$$

Therefore, the result is true for $n = 2$.

Now, let the result is true for $n = k$ i.e.,

$$IFWOA_{\omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k) = [1-\prod_{j=1}^{k}\left(1-t_{\tilde{a}_{\sigma(j)}}\right)^{\omega_j}, 1-\prod_{j=1}^{k} f_{\tilde{a}_{\sigma(j)}}^{\omega_j}]$$

and prove that it is also true for $n = k + 1$.

$$IFWOA_{\omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_{k+1}) = [1-\prod_{j=1}^{k}\left(1-t_{\tilde{a}_{\sigma(j)}}\right)^{\omega_j} +(1-\left(1-t_{\tilde{a}_{\sigma(k+1)}}\right)^{\omega_{k+1}})-$$

$$\left(1-\prod_{j=1}^{k}\left(1-\ t_{\tilde{a}_{\sigma(j)}}\right)^{\omega_j}\right)\times\left(1-\left(1-t_{\tilde{a}_{\sigma(k+1)}}\right)^{\omega_{k+1}}\right), 1-\prod_{j=1}^{k+1} f_{\tilde{a}_{\sigma(j)}}^{\omega_j}]$$

$$= [1-\prod_{j=1}^{k}\left(1-t_{\tilde{a}_{\sigma(j)}}\right)^{\omega_j}, \ 1-\prod_{j=1}^{k+1} f_{\tilde{a}_{\sigma(j)}}^{\omega_j}].$$

Since, the result is true for $n = k + 1$. Hence, the result is true for all values of $n$.

**Theorem 2.4**: The IFWA operator is a special case of the IFHA operator.

**Proof:** Let $\omega = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$; then

$$IFHA_{\omega,w}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = w_1 \dot{\tilde{a}}_{\sigma(1)} \oplus w_2 \dot{\tilde{a}}_{\sigma(2)} \oplus \dots \oplus w_n \dot{\tilde{a}}_{\sigma(n)}$$

$$= 1/n\left(\dot{\tilde{a}}_{\sigma(1)} \oplus \dot{\tilde{a}}_{\sigma(2)} \oplus \dots \oplus \dot{\tilde{a}}_{\sigma(n)}\right).$$

by Definition 2.3

$$= \omega_1 \tilde{a}_1 \oplus \omega_2 \tilde{a}_2 \oplus \dots \oplus \omega_n \tilde{a}_n$$

$$= IFWA_{\omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n).$$

**Theorem 2.5** The IFOWA operator is a special case of the IFHA operator.

**Proof:** Let $\omega = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$; then

$$IFHA_{\omega,w}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = w_1 \dot{\tilde{a}}_{\sigma(1)} \oplus w_2 \dot{\tilde{a}}_{\sigma(2)} \oplus \dots \oplus w_n \dot{\tilde{a}}_{\sigma(n)}$$

$$= 1/n\left(\dot{\tilde{a}}_{\sigma(1)} \oplus \dot{\tilde{a}}_{\sigma(2)} \oplus \dots \oplus \dot{\tilde{a}}_{\sigma(n)}\right)$$

by Definition 2.4

$$= w_1 \tilde{a}_{\sigma(1)} \oplus w_2 \tilde{a}_{\sigma(2)} \oplus \dots \oplus w_n \tilde{a}_{\sigma(n)}$$

$$= IFOWA_{\omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n).$$

# Chapter 3

# Generalized intuitionistic fuzzy geometric Aggregation operators and its properties

In this chapter, the generalized Archimedean intuitionistic fuzzy weighted geometric, generalized Archimedean intuitionistic fuzzy ordered weighted geometric and generalized Archimedean intuitionistic fuzzy hybrid geometric operators, proposed by Tan et al. [3], are studied.

## 3.1. Basic definitions

In this section, some basic definitions are reviewed [3]

**Definition 3. 1** A mapping $f\colon [0,1]^n \to [0,1]$ is called a generalized weighted geometric (GWG) operator of $n$-dimension if

$$f(x_1, x_2, \dots, x_n) = \frac{1}{\lambda}\left(\prod_{j=1}^{n}\left(\lambda x_j\right)^{w_j}\right),$$

where parameter $\lambda > 0, w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of the arguments $x_j (j = 1,2, \dots, n)$, satisfying $w_j \in [0,1]$ $(j = 1,2, \dots, n)$ and $\sum_{j=1}^{n} w_j = 1$.

**Definition 3.2** A mapping $f\colon [0,1]^n \to [0,1]$ is called a generalized ordered weighted geometric (GOWG) operator of $n$-dimension if

$$f(x_1, x_2, \dots, x_n) = \frac{1}{\lambda}\left(\prod_{j=1}^{n}\left(\lambda b_j\right)^{w_j}\right),$$

where parameter $\lambda > 0, w = (w_1, w_2, \dots, w_n)^T$ is the weight vector that is associated with, satisfying $w_j \in$ $[0,1]$ $(j = 1,2, \dots, n)$ and $\sum_{j=1}^{n} w_j = 1; b_j$ is the $jth$ largest of all numerical values $x_k (k = 1,2, \dots, n)$.

**Definition 3.3** A triangular norm (t-norm for short) $T$ is a bivariate aggregation function on the unit interval [0,1], that is, a function $T: [0,1] \times [0,1] \to [0,1]$, which is Commutative, Associative, Monotone in each component, and satisfies the boundary condition $T(x, 1) = x$.

There exist uncountable t-norms; the following four basic t-norms $T_M, T_P, T_L$ and $T_D$ are given by, respectively, $T_M(x, y) = \min(x, y)$, $T_P(x, y) = x.y$, $T_L(x, y) = (x + y - 1,0)$, and $T_D(x, y) = 0$,if $(x.y) \in [0,1)^2$ and otherwise $T_D(x, y) = \min(x, y)$.

**Definition 3.4** A triangular conorm (t-conorm for short) $S$ is a bivariate aggregation function on the unit interval [0,1], that is, a function $S: [0,1] \times [0,1] \to [0,1]$, which is Commutative, Associative, Monotone in each component, and satisfies the Boundary condition $S(x, 0) = x$.

**Definition 3.5** A t-norm is called Archimedean if for each $(x.y) \in (0,1)^2$ there is an integral $n \in \{1,2, \dots\}$ such that $T\frac{n-times}{(x,x,\dots,x)} < y$. A t-conorm is called Archimedean if for each $(x.y) \in (0,1)^2$ there is an integral $n \in \{1,2, \dots\}$ such that $S\frac{n-times}{(x,x,\dots,x)} > y$.

It is well known that a continuous Archimedean t-norm $T$ is expressed via its additive generator $g$ as $T(x,y) = g^{(-1)}(g(x) + g(y))$, where $g: [0,1] \to [0, \infty]$ with $g(1) = 0$ is a continuous strictly decreasing function, and $g^{(-1)}(t) = \max(0, g^{-1}(t))$ is the pseudo-inverse of $g$. The dual of any t-norm generated by $g$ is the corresponding t-conorm generated by $h$ with $h(t) = g(1 - t)$ and vice versa with respective t-norm and t-conorms satisfying $T(x, y) = 1 - S(1 - x, 1 - y)$. The corresponding t-conorms, using $h(t) = g(1 - t)$, is expressed as $S(x, y) = h^{(-1)}(h(x) + h(y))$, where $h^{(-1)}(t) = \min(1, h^{-1}(t))$ is the pseudo-inverse of $h$.

**Definition 3.6** Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set and $\alpha_j = \left(\mu_{\alpha_j}, \nu_{\alpha_j}\right)$ $(j = 1,2, \dots, n)$ be a collection of intutionistic fuzzy values on $X$. Based on Archimedean t-conorm and t-norm, the generalized Archimedean intuitionistic fuzzy weighted geometric (GAIFWG) operator of dimension $n$ is a mapping from $\Omega^n$ to $\Omega$ such that

$$GAIFWG_w(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{\lambda}\left((\lambda_{.\delta}\alpha_1)^{\wedge_\delta w_1} \otimes_\delta (\lambda_{.\delta}\alpha_2)^{\wedge_\delta w_2} \otimes_\delta \ldots (\lambda_{.\delta}\alpha_n)^{\wedge_\delta w_n}\right) =$$

$$\frac{1}{\lambda}\left(\otimes_{\delta}{}_{j=1}^{n}\left(\left(\lambda_{.\delta}\alpha_j\right)^{\wedge_\delta w_j}\right)\right)$$

where parameter $\lambda > 0, w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of the arguments $x_j (j = 1,2, \ldots, n)$, satisfying $w_j \in [0,1]$ $(j = 1,2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$.

**Definition 3.7** Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $\alpha_j = \left(\mu_{\alpha_j}, \nu_{\alpha_j}\right)$ $(j = 1,2, \ldots, n)$ be a collection of intuitionistic fuzzy values on $X$. Based on Archimedean t-conorm and t-norm, the generalized Archimedean intuitionistic fuzzy ordered weighted geometric (GAIFOWG) operator of dimension $n$ is a mapping from $\Omega^n$ to $\Omega$ that has an associated the weighting vector $w = (w_1, w_2, \ldots, w_n)$ with $w_j \in [0,1]$ $(j = 1,2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$,such that

$$GAIFOWG_w(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{\lambda}\left(\left(\lambda_{.\delta}\alpha_{(1)}\right)^{\wedge_\delta w_1} \otimes_\delta \left(\lambda_{.\delta}\alpha_{(2)}\right)^{\wedge_\delta w_2} \otimes_\delta \ldots \otimes_\delta \left(\lambda_{.\delta}\alpha_{(n)}\right)^{\wedge_\delta w_n}\right) =$$

$$\frac{1}{\lambda}\left(\otimes_{\delta}{}_{j=1}^{n}\left(\left(\lambda_{.\delta}\alpha_{(j)}\right)^{\wedge_\delta w_j}\right)\right),$$

where (.) indicates a permutation on $X$ such that $\alpha_{(1)} \geq \alpha_{(2)} \geq \cdots \geq \alpha_{(n)}, \lambda > 0$.

**Definition 3.8** Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $\alpha_j = \left(\mu_{\alpha_j}, \nu_{\alpha_j}\right)$ $(j = 1,2, \ldots, n)$ be a collection of intuitionistic fuzzy values on $X$. Based on Archimedean t-conorm and t-norm, the generalized Archimedean intuitionistic fuzzy hybrid geometric (GAIFHG) operator of dimension $n$ is a mapping from $\Omega^n$ to $\Omega$ that has an associated the weighting vector $w = (w_1, w_2, \ldots, w_n)$ with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, such that

$$GAIFHG_w(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{\lambda}\left(\left(\lambda_{.\delta}\alpha_{\sigma(1)}\right)^{\wedge_\delta w_1} \otimes_\delta \left(\lambda_{.\delta}\alpha_{\sigma(2)}\right)^{\wedge_\delta w_2} \otimes_\delta \ldots \otimes_\delta \left(\lambda_{.\delta}\alpha_{\sigma(n)}\right)^{\wedge_\delta w_n}\right) =$$

$$\frac{1}{\lambda}\left(\otimes_{\delta}{}_{j=1}^{n}\left(\left(\lambda_{.\delta}\alpha_{\sigma(j)}\right)^{\wedge_\delta w_j}\right)\right)$$

$$= \left(\left(1 - \left(1 - g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\tilde{\alpha}_{\sigma(j)}}\right)\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}, \left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\tilde{\alpha}_{\sigma(j)}}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}\right),$$

where $\lambda > 0, \tilde{\alpha}_{\sigma(j)} = \left(\mu_{\tilde{\alpha}_{\sigma(j)}}, \nu_{\tilde{\alpha}_{\sigma(j)}}\right)$ is the $jth$ largest of the weighted intuitionistic fuzzy values $\tilde{\alpha}_j \big(\tilde{\alpha}_j = n\omega_j \alpha_j, j = 1,2, \dots, n\big), \omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weight vector of $x_j (j = 1,2, \dots, n)$ witg $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$, and $n$ is the balancing coefficient; $g$ is an additive generator of a continuous Archimedean t-norm, and $h(t) = g(1 - t)$ is the generator of its dual t-conorm.

**Definition 3.9** Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy value in $X$ and $\lambda > 0$; then

(1) $A + B = (\mu_A + \mu_B - \mu_A \mu_B, \nu_A \nu_B)$.

(2) $A.B = (\nu_A + \nu_B - \nu_A \nu_B\ , \mu_A \mu_B)$.

(3)$\lambda A = \left(1 - (1 - \mu_A)^{\lambda}, \nu_A^{\lambda}\right)$ .

(4)$A^{\lambda}$ =($\mu_A^{\lambda}\ , 1 - (1 - \nu_A)^{\lambda}$).

**Definition 3.10** Let $\alpha_j = \left(\mu_{\alpha_j}, \nu_{\alpha_j}\right)$ $(j = 1,2)$ be two intuitonistic fuzzy values, then

(1) $\alpha_1 \oplus_{\delta} \alpha_2 = \left(h^{(-1)}\left(h\left(\mu_{\alpha_1}\right) + h\left(\mu_{\alpha_2}\right)\right), g^{(-1)}\left(g\left(\nu_{\alpha_1}\right) + g\left(\nu_{\alpha_2}\right)\right)\right)$.

(2) $\alpha_1 \otimes_{\delta} \alpha_2 = \left(g^{(-1)}\left(g\left(\mu_{\alpha_1}\right) + g\left(\mu_{\alpha_2}\right)\right), h^{(-1)}\left(h\left(\nu_{\alpha_1}\right) + h\left(\nu_{\alpha_2}\right)\right)\right)$.

(3) $\lambda_{.\delta} \alpha_i = \left(h^{-1}\left(\lambda h\left(\mu_{\alpha_i}\right)\right), g^{-1}\left(\lambda g\left(\nu_{\alpha_i}\right)\right)\right), \lambda > 0.$

(4) $(\alpha_i)^{\wedge\delta\lambda} = \left(g^{-1}\left(\lambda g\left(\mu_{\alpha_i}\right)\right), h^{-1}\left(\lambda h\left(\nu_{\alpha_i}\right)\right)\right), \lambda > 0.$

### 3.2 Some important results

In this section, some results, proposed by Tan et al.[3], are presented.

**Theorem 3.1** Le $\alpha_j = \left(\mu_{\alpha_i}, \nu_{\alpha_i}\right)$ $(i = 1,2)$ be two intuitionistic fuzzy values and let $a = \alpha_1 \oplus_{\delta} \alpha_2,$ for any $\lambda > 0$; then $a, b, c$ and $d$ are also intuitionistic fuzzy values.

**Proof:** Since $h(t) = g(1 - t)$, and $g: [0,1] \rightarrow [0, \infty]$ is a strictly decreasing function, $h(t)$ is a strictly increasing function, which indicates that

$$0 \leq g^{(-1)}\left(g\left(\mu_{\alpha_1}\right) + g\left(\mu_{\alpha_2}\right)\right) \leq 1, 0 \leq h^{(-1)}\left(h\left(\nu_{\alpha_1}\right) + h\left(\nu_{\alpha_2}\right)\right) \leq 1$$

and $g^{(-1)}\left(g\left(\mu_{\alpha_1}\right) + g\left(\mu_{\alpha_2}\right)\right) + h^{(-1)}\left(h\left(\nu_{\alpha_1}\right) + h\left(\nu_{\alpha_2}\right)\right)$

$$\leq h^{(-1)}\left(h\left(\nu_{\alpha_1}\right)+h\left(\nu_{\alpha_2}\right)\right)+g^{(-1)}\left(g\left(1-\nu_{\alpha_1}\right)+g\left(1-\nu_{\alpha_2}\right)\right)$$

$$\leq h^{(-1)}\left(h\left(\nu_{\alpha_1}\right)+h\left(\nu_{\alpha_2}\right)\right)+1-h^{(-1)}\left(h\left(\nu_{\alpha_1}\right)+h\left(\nu_{\alpha_2}\right)\right)=1.$$

Therefore, $a=\alpha_1 \oplus_\delta \alpha_2, and\ \ c=\alpha_1 \otimes_\delta \alpha_2$ are intuitionistic fuzzy values.

**Theorem 3.2** Let $X=\{x_1,x_2,\ldots,x_n\}$ be a finite set and $\alpha_j=\left(\mu_{\alpha_j},\nu_{\alpha_j}\right)$ $(j=1,2,\ldots,n)$ be a collection of IFVs on X. Then the aggregated value by using the $GAIFWG$ operator is also an intuitionistic fuzzy value. Furthermore,

$GAIFWG_w(\alpha_1,\alpha_2,\ldots,\alpha_n)=$

$$\left(\left(1-\left(1-g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right)\right)^{\frac{1}{\lambda}},\left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}\right),$$

where parameter $\lambda>0, w=(w_1,w_2,\ldots,w_n)^T$ is the weight vector of the arguments $x_j (j=1,2,\ldots,n)$, satisfying $w_j\in[0,1]$ $(j=1,2,\ldots,n)$ and $\sum_{j=1}^{n} w_j=1$, $g$ is an additive generator of a continuous Archimedean t-norm, and $h(t)=g(1-t)$ is the generator of its dual t-conorm.

**Proof:** This is easily proved by mathematical induction as follows:

Firstly, prove that the result is true for $n=2$

Since,

$$\lambda_{.\delta}\alpha_1=\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_1}\right)\right),g^{-1}\left(\lambda g\left(\nu_{\alpha_1}\right)\right)\right), \lambda_{.\delta}\alpha_2=\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_2}\right)\right),g^{-1}\left(\lambda g\left(\nu_{\alpha_2}\right)\right)\right),$$

then $(\lambda_{.\delta}\alpha_1)^{\wedge\delta w_1}=\left(g^{-1}\left(w_1 g\left(h^{-1}\left(\lambda h\left(\nu_{\alpha_1}\right)\right)\right)\right),h^{-1}\left(w_1 h\left(g^{-1}\left(\lambda g\left(\mu_{\alpha_1}\right)\right)\right)\right)\right)$

$$(\lambda_{.\delta}\alpha_2)^{\wedge\delta w_2}=\left(g^{-1}\left(w_2 g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_2}\right)\right)\right)\right),h^{-1}\left(w_2 h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_2}\right)\right)\right)\right)\right).$$

So,

$$(\lambda_{.\delta}\alpha_1)^{\wedge\delta w_1} \otimes_\delta (\lambda_{.\delta}\alpha_2)^{\wedge\delta w_2} = \Bigg(g^{-1}\Bigg(g\Bigg(g^{-1}\Bigg(w_1 g\Big(h^{-1}\Big(\lambda h\big(\mu_{\alpha_1}\big)\Big)\Big)\Bigg)\Bigg) +$$

$$g\Bigg(g^{-1}\Bigg(w_2 g\Big(h^{-1}\Big(\lambda h\big(\mu_{\alpha_2}\big)\Big)\Big)\Bigg)\Bigg)\Bigg),$$

$$\Bigg(h^{-1}\Bigg(h\Bigg(h^{-1}\Bigg(w_1 h\Big(g^{-1}\Big(\lambda g\big(\nu_{\alpha_1}\big)\Big)\Big)\Bigg)\Bigg) + h\Bigg(h^{-1}\Bigg(w_2 h\Big(g^{-1}\Big(\lambda g\big(\nu_{\alpha_2}\big)\Big)\Big)\Bigg)\Bigg)\Bigg)\Bigg)$$

$$=\Bigg(g^{-1}\Bigg(\textstyle\sum_{j=1}^{2} w_j g\Big(h^{-1}\Big(\lambda h\Big(\mu_{\alpha_j}\Big)\Big)\Big)\Bigg), h^{-1}\Bigg(\textstyle\sum_{j=1}^{2} w_j h\Big(g^{-1}\Big(\lambda g\Big(\nu_{\alpha_j}\Big)\Big)\Big)\Bigg)\Bigg).$$

Suppose result holds for $n = k$, that is

$$(\lambda_{.\delta}\alpha_1)^{\wedge\delta w_1} \otimes_\delta (\lambda_{.\delta}\alpha_2)^{\wedge\delta w_2} \otimes_\delta \ldots (\lambda_{.\delta}\alpha_n)^{\wedge\delta w_k} =$$

$$\Bigg(g^{-1}\Bigg(\textstyle\sum_{j=1}^{k} w_j g\Big(h^{-1}\Big(\lambda h\Big(\mu_{\alpha_j}\Big)\Big)\Big)\Bigg), h^{-1}\Bigg(\textstyle\sum_{j=1}^{k} w_j h\Big(g^{-1}\Big(\lambda g\Big(\nu_{\alpha_j}\Big)\Big)\Big)\Bigg)\Bigg).$$

Then, when $n = k + 1$,

$$(\lambda_{.\delta}\alpha_1)^{\wedge\delta w_1} \otimes_\delta (\lambda_{.\delta}\alpha_2)^{\wedge\delta w_2} \otimes_\delta \ldots (\lambda_{.\delta}\alpha_n)^{\wedge\delta w_{k+1}}$$

$$= \Bigg(g^{-1}\Bigg(\textstyle\sum_{j=1}^{k} w_j g\Big(h^{-1}\Big(\lambda h\Big(\mu_{\alpha_j}\Big)\Big)\Big)\Bigg), h^{-1}\Bigg(\textstyle\sum_{j=1}^{k} w_j h\Big(g^{-1}\Big(\lambda g\Big(\nu_{\alpha_j}\Big)\Big)\Big)\Bigg)\Bigg)$$

$$\otimes_\delta \Bigg(g^{-1}\Bigg(w_{k+1} g\Big(h^{-1}\Big(\lambda h\big(\nu_{\alpha_{k+1}}\big)\Big)\Big)\Bigg), h^{-1}\Bigg(w_{k+1} h\Big(g^{-1}\Big(\lambda g\big(\mu_{\alpha_{k+1}}\big)\Big)\Big)\Bigg)\Bigg)$$

$$=\Bigg(g^{-1}\Bigg(g\Bigg(g^{-1}\Bigg(\textstyle\sum_{j=1}^{k} w_j g\Big(h^{-1}\Big(\lambda h\Big(\mu_{\alpha_j}\Big)\Big)\Big)\Bigg)\Bigg) + g\Bigg(g^{-1}\Bigg(w_{k+1} g\Big(h^{-1}\Big(\lambda h\big(\mu_{\alpha_{k+1}}\big)\Big)\Big)\Bigg)\Bigg)\Bigg),$$

$$\Bigg(h^{-1}\Bigg(h\Bigg(h^{-1}\Bigg(\sum_{j=1}^{k} w_j h\Big(g^{-1}\Big(\lambda g\Big(\nu_{\alpha_j}\Big)\Big)\Big)\Bigg)\Bigg) + h\Bigg(h^{-1}\Bigg(w_{k+1} h\Big(g^{-1}\Big(\lambda g\big(\nu_{\alpha_{k+1}}\big)\Big)\Big)\Bigg)\Bigg)\Bigg)\Bigg)$$

$$=\Bigg(g^{-1}\Bigg(\textstyle\sum_{j=1}^{k+1} w_j g\Big(h^{-1}\Big(\lambda h\Big(\mu_{\alpha_j}\Big)\Big)\Big)\Bigg), h^{-1}\Bigg(\textstyle\sum_{j=1}^{k+1} w_j h\Big(g^{-1}\Big(\lambda g\Big(\nu_{\alpha_j}\Big)\Big)\Big)\Bigg)\Bigg).$$

That is, result holds for $n = k + 1$. Thus, result holds for all positive integral $n$.

Thus

$GAIFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) =$

$$=\frac{1}{\lambda}\left(\left(g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right), h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)$$

$$=\left(\left(1-\left(1-g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}, \left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}\right).$$

Since, $h(t) = g(1-t)$ and $g: [0,1] \to [0, \infty]$ is a strictly decreasing function, $h(t)$ is a strictly increasing function,which indicates that $0 \leq g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right) \leq 1$,

$$0 \leq h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right) \leq 1$$

and $\left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}} + \left(1-\left(1-g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}$

$$=\left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}} + \left(1-\left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(1-\mu_{\alpha_j}\right)\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}$$

$$\leq \left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(1-\mu_{\alpha_j}\right)\right)\right)\right)\right) + \left(1-\left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(1-\mu_{\alpha_j}\right)\right)\right)\right)\right)\right)^{\frac{1}{\lambda}} \leq 1.$$

That is to say, the aggregated value by using the generalized Archimedean intuitionistic fuzzy weighted geometric operator is also an intuitionistic fuzzy values.

**Theorem 3.3** The operation defined in Theorem 3.2 is consistent with the operator generalized weighted geometric expressed by

$$f(x_1, x_2, \dots, x_n) = \frac{1}{\lambda}\left(\prod_{j=1}^{n}\left(\lambda x_j\right)^{w_j}\right)$$

on ordinary fuzzy sets.

**Proof:** $GAIFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n)$

$$=\left(\left(1-\left(1-g^{-1}\left(\sum_{j=1}^n w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}, \left(h^{-1}\left(\sum_{j=1}^n w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}\right)$$

$$=\left(1-\left(1-g^{-1}\left(\sum_{j=1}^n w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}, \left(1-g^{-1}\left(\sum_{j=1}^n w_j g\left(1-g^{-1}\left(\lambda h\left(1-\mu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}$$

$$=\left(1-\left(1-g^{-1}\left(\sum_{j=1}^n w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}, \left(1-g^{-1}\left(\sum_{j=1}^n w_j g\left(\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}$$

which indicates that in this case $GAIFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n)$ is a fuzzy set, that is to say, the generalized Archimedean intuitionistic fuzzy weighted geometric operator is a natural extension of the generalized weighted geometric operator from fuzzy sets to intuitionistic fuzzy sets. Thus, the operation defined by Theorem 3.2 is consistent with the generalized weighted geometric operator expressed

$$f(x_1, x_2, \dots, x_n) = \frac{1}{\lambda}\left(\prod_{j=1}^{n}\left(\lambda x_j\right)^{w_j}\right),$$

on ordinary fuzzy sets.

**Theorem 3.4** Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set, $\alpha_j = \left(\mu_{\alpha_j}, \nu_{\alpha_j}\right)$ $(j = 1,2, \dots, n)$ be a collection of $IFVs$ on $X, \lambda > 0$, and $w = (w_1, w_2, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$ be the weight vector of the arguments $x_j (j = 1,2, \dots, n)$. If all $\alpha_j (j = 1,2, \dots, n)$ are equal, that is, $\alpha_j = \alpha = (\mu_\alpha, \nu_\alpha)$ for all $j$, then

$$GAIFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha.$$

**Proof:** Let $\alpha_j = \alpha = (\mu_\alpha, \nu_\alpha)$ for all $j$, then

$GAIFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n)$

$$=\left(\left(1-\left(1-g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right)\right)^{\frac{1}{\lambda}},\left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}\right)$$

$$=\left(1-\left(1-h^{-1}\left(\lambda h(\mu_\alpha)\right)\right)^{\frac{1}{\lambda}},\left(g^{-1}\left(\lambda g(\nu_\alpha)\right)\right)^{\frac{1}{\lambda}}\right)$$

$$=\left(h^{-1}\left(\lambda h(\mu_\alpha)\right),g^{-1}\left(\lambda g(\nu_\alpha)\right)\right)^{\frac{1}{\lambda}}$$

$$=\left((\mu_\alpha,\nu_\alpha)^{\lambda}\right)^{\frac{1}{\lambda}}=(\mu_\alpha,\nu_\alpha)=\alpha.$$

**Theorem 3.5** Let $X=\{x_1,x_2,\dots,x_n\}$ be a finite set, $\alpha_j=\left(\mu_{\alpha_j},\nu_{\alpha_j}\right)$ and $\beta_j=\left(\mu_{\beta_j},\nu_{\beta_j}\right)$ $(j=1,2,\dots,n)$ be two collection of $IFVs$ on $X,\lambda>0$, and $w=(w_1,w_2,\dots,w_n)$ with $w_j\in[0,1]$ and $\sum_{j=1}^{n}w_j=1$ be the weight vector of the arguments $x_j(j=1,2,\dots,n)$. If $\mu_{\alpha_j}\le\mu_{\beta_j}$ and $\nu_{\alpha_j}\ge\nu_{\beta_j}$ for all $j$, that is, $\alpha_j\le\beta_j$, then

$$GAIFWG_w(\alpha_1,\alpha_2,\dots,\alpha_n)\le GAIFWG_w(\beta_1,\beta_2,\dots,\beta_n).$$

**Proof:** Since, $h(t)=g(1-t)$ and $g:[0,1]\to[0,\infty]$ is a strictly decreasing function, therefore, $h(t)$ is a strictly increasing function. Let $\mu_{\alpha_j}\le\mu_{\beta_j}$ and $\nu_{\alpha_j}\ge\nu_{\beta_j}$ for all $j$, then

$$g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\ge g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\beta_j}\right)\right)\right)\right),$$

$$h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\le h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\beta_j}\right)\right)\right)\right)$$

$$\Rightarrow 1-\left(1-g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}\ge 1-\left(1-g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\beta_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}$$

$$\left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}\le\left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\beta_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}$$

$$\Rightarrow\qquad GAIFWG_w(\alpha_1,\alpha_2,\dots,\alpha_n)\le GAIFWG_w(\beta_1,\beta_2,\dots,\beta_n).$$

**Theorem 3.6** Let $\alpha_j = \left(\mu_{\alpha_j}, \nu_{\alpha_j}\right)$ $(j = 1,2,\dots,n)$ be a collection of $IFVs$ on $X, \lambda > 0$, and $w = (w_1, w_2, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ be the weight vector of the arguments $x_j(j = 1,2,\dots,n)$. If $\alpha^- = \left(\min_j \mu_{\alpha_j}, \max_j \nu_{\alpha_j}\right), \alpha^+ = \left(\max_j \mu_{\alpha_j}, \min_j \nu_{\alpha_j}\right)$, then

$$\alpha^- \leq GAIFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha^+$$

**Proof:** It is obvious that

$$\min_j \mu_{\alpha_j} \leq \mu_{\alpha_j} \leq \max_j \mu_{\alpha_j}, \min_j \nu_{\alpha_j} \leq \nu_{\alpha_j} \leq \max_j \nu_{\alpha_j}$$

Since, $g\colon [0,1] \to [0,\infty]$ is a strictly decreasing function, therefore, $h(t) = g(1-t)$ is a strictly increasing function.

$\Longrightarrow \quad g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\min_j \mu_{\alpha_j}\right)\right)\right)\right) \leq g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right) \leq$

$g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\max_j \mu_{\alpha_j}\right)\right)\right)\right)$,

$h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\min_j \nu_{\alpha_j}\right)\right)\right)\right) \leq h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right) \leq$

$h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\max_j \nu_{\alpha_j}\right)\right)\right)\right)$.

$$\Longrightarrow \quad h^{-1}\left(\lambda h\left(\min_j \mu_{\alpha_j}\right)\right) \leq g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right) \leq h^{-1}\left(\lambda h\left(\max_j \mu_{\alpha_j}\right)\right)$$

$$g^{-1}\left(\lambda g\left(\min_j \nu_{\alpha_j}\right)\right) \leq h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right) \leq g^{-1}\left(\lambda g\left(\max_j \nu_{\alpha_j}\right)\right)$$

$$\Longrightarrow \quad \left(h^{-1}\left(\lambda h\left(\min_j \mu_{\alpha_j}\right)\right), g^{-1}\left(\lambda g\left(\max_j \nu_{\alpha_j}\right)\right)\right)$$

$$\leq \left(g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right), h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)$$

$$\left(h^{-1}\left(\lambda h\left(\max_j \mu_{\alpha_j}\right)\right), g^{-1}\left(\lambda g\left(\min_j \nu_{\alpha_j}\right)\right)\right)$$

$$\geq \left(g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right), h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)$$

$$\Longrightarrow \lambda\left(\min_j \mu_{\alpha_j}, \max_j \nu_{\alpha_j}\right) \leq \left(g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right), h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)$$

$$\leq\ \lambda\left(\max_j \mu_{\alpha_j}, \min_j \nu_{\alpha_j}\right)$$

$$\Longrightarrow\ \ \alpha^- \leq \frac{1}{\lambda}\left(\left(g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right), h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)\right) \leq \alpha^+$$

$$\alpha^- \leq \left(1-\left(1-g^{-1}\left(\sum_{j=1}^{n} w_j g\left(h^{-1}\left(\lambda h\left(\mu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}, \left(h^{-1}\left(\sum_{j=1}^{n} w_j h\left(g^{-1}\left(\lambda g\left(\nu_{\alpha_j}\right)\right)\right)\right)\right)^{\frac{1}{\lambda}}\right)$$

$\Longrightarrow \alpha^- \leq GAIFWG_w(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+.$

### 3.3 Special Cases

(1) If parameter $\lambda$ and weight vector $w = (w_1, w_2, \ldots, w_n)$ are given by different values, and the additive generator $g$ is also assigned different forms, some specific intuitionistic fuzzy geometric aggregation operators can be obtained.

1. If $\lambda = 1$, then

$$ATS - IFWG_w(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(g^{-1}\left(\textstyle\sum_{j=1}^{n} w_j g\left(\mu_{\alpha_j}\right)\right), h^{-1}\left(\textstyle\sum_{j=1}^{n} w_j h\left(\nu_{\alpha_j}\right)\right)\right)$$

which is called the Archimedean t-conorm- and t-norm-based intuitionistic fuzzy weighted geometric (ATS-IFWG) operator. Furthermore, if $w = \left({}^{1}\!/_{n}, {}^{1}\!/_{n}, \ldots, {}^{1}\!/_{n}\right)$, then

$$AIFG_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \left( g^{-1}\left( \frac{1}{n} \sum_{j=1}^{n} g\left(\mu_{\alpha_j}\right) \right), h^{-1}\left( \frac{1}{n} \sum_{j=1}^{n} h\left(\nu_{\alpha_j}\right) \right) \right)$$

called the Archimedean intuitionistic fuzzy geometric (AIFG) operator.

2. If $g(t) = -\log t$, i.e. $h(t) = -\log(1-t)$, $g^{-1}(t) = e^{-t}$ ,$h^{-1}(t) = 1 - e^{-t}$, then

$$GIFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \left( 1 - \left( 1 - \prod_{j=1}^{n} \left( 1 - \left(1 - \mu_{\alpha_j}\right)^{\lambda} \right) \right)^{w_j} \right)^{\frac{1}{\lambda}}, \left( 1 - \prod_{j=1}^{n} \left( 1 - \nu_{\alpha_j}^{\lambda} \right)^{w_j} \right)^{\frac{1}{\lambda}}$$

which is called the generalized intuitionistic fuzzy weighted geometric (GIFWG) operator .

Furthermore, if $\lambda = 1$, then

$$IFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \left( \prod\nolimits_{j=1}^{n} \mu_{\alpha_j}^{w_j}, 1 - \prod\nolimits_{j=1}^{n} \left(1 - \nu_{\alpha_j}\right)^{w_j} \right)$$

which is called the intuitionistic fuzzy weighted geometric operator.

3. If $g(t) = log\left({(2-t)}/{t}\right)$, i.e. $h(t) = log\left({(2-(1-t))}/{(1-t)}\right)$, $g^{-1}(t) = {2}/{(e^t+1)}$

, $h^{-1}(t) = 1 - \left({2}/{(e^t+1)}\right)$

$$GEIFWG(\alpha_1, \alpha_2, \dots, \alpha_n) =$$

$$\left( 1 - \left( 1 - \frac{2 \prod_{j=1}^{n} \left( \frac{\left(1+\mu_{\alpha_j}\right)^{\lambda} - \left(1-\mu_{\alpha_j}\right)^{\lambda}}{\left(1+\mu_{\alpha_j}\right)^{\lambda} + \left(1-\mu_{\alpha_j}\right)^{\lambda}} \right)^{w_j}}{\prod_{j=1}^{n} \left( 2 - \frac{\left(1+\mu_{\alpha_j}\right)^{\lambda} - \left(1-\mu_{\alpha_j}\right)^{\lambda}}{\left(1+\mu_{\alpha_j}\right)^{\lambda} + \left(1-\mu_{\alpha_j}\right)^{\lambda}} \right)^{w_j} + \prod_{j=1}^{n} \left( \frac{\left(1+\mu_{\alpha_j}\right)^{\lambda} - \left(1-\mu_{\alpha_j}\right)^{\lambda}}{\left(1+\mu_{\alpha_j}\right)^{\lambda} + \left(1-\mu_{\alpha_j}\right)^{\lambda}} \right)^{w_j}} \right) \right)^{\frac{1}{\lambda}},$$

$$\left( \left( \frac{\prod_{j=1}^{n} \left( 1 + \frac{2\nu_{\alpha_j}^{\lambda}}{\left(2-\nu_{\alpha_j}\right)^{\lambda} + \nu_{\alpha_j}^{\lambda}} \right)^{w_j} - \prod_{j=1}^{n} \left( 1 - \frac{2\nu_{\alpha_j}^{\lambda}}{\left(2-\nu_{\alpha_j}\right)^{\lambda} + \nu_{\alpha_j}^{\lambda}} \right)^{w_j}}{\prod_{j=1}^{n} \left( 1 + \frac{2\nu_{\alpha_j}^{\lambda}}{\left(2-\nu_{\alpha_j}\right)^{\lambda} + \nu_{\alpha_j}^{\lambda}} \right)^{w_j} + \prod_{j=1}^{n} \left( 1 - \frac{2\nu_{\alpha_j}^{\lambda}}{\left(2-\nu_{\alpha_j}\right)^{\lambda} + \nu_{\alpha_j}^{\lambda}} \right)^{w_j}} \right)^{\frac{1}{\lambda}} \right)$$

called the generalized Einstein intuitionistic fuzzy weighted geometric (GEIFWG) operator. Furthermore, if $\lambda = 1$, then

$$EIFWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \left( \frac{2\prod_{j=1}^{n} \mu_{\alpha_j}^{w_j}}{\prod_{j=1}^{n} \left(2 - \mu_{\alpha_j}\right)^{w_j} + \prod_{j=1}^{n} \mu_{\alpha_j}^{w_j}}, \frac{\prod_{j=1}^{n} \left(1 - \nu_{\alpha_j}\right)^{w_j} - \prod_{j=1}^{n} \left(1 - \nu_{\alpha_j}\right)^{w_j}}{\prod_{j=1}^{n} \left(1 - \nu_{\alpha_j}\right)^{w_j} + \prod_{j=1}^{n} \left(1 - \nu_{\alpha_j}\right)^{w_j}} \right)$$

which is called the Einstein intuitionistic fuzzy weighted geometric (EIFWG) operator.

4. If $g(t) = log\left( {}^{(\gamma + (1-\gamma)t)}\big/_{t} \right), \lambda > 0$, i.e. $h(t) = log\left( {}^{(\gamma + (1-\gamma)(1-t))}\big/_{(1-t)} \right)$,

$g^{-1}(t) = {}^{\gamma}\big/_{(e^t + \gamma - 1)}, h^{-1} = 1 - \left( {}^{\gamma}\big/_{(e^t + \gamma - 1)} \right)$

$$GHIFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) =$$

$$\left( 1 - \left( 1 - \frac{\gamma \prod_{j=1}^{n} \left( \frac{\left(1+(\gamma-1)\mu_{\alpha_j}\right)^{\lambda} - \left(1-\mu_{\alpha_j}\right)^{\lambda}}{\left(1+(\gamma-1)\mu_{\alpha_j}\right)^{\lambda} + (1-\gamma)\left(1-\mu_{\alpha_j}\right)^{\lambda}} \right)^{w_j}}{\prod_{j=1}^{n} \left( 1+(\gamma-1) \left( \frac{\left(1+(\gamma-1)\mu_{\alpha_j}\right)^{\lambda} - \left(1-\mu_{\alpha_j}\right)^{\lambda}}{\left(1+(\gamma-1)\mu_{\alpha_j})\right)^{\lambda} + (1-\gamma)\left(1-\mu_{\alpha_j}\right)^{\lambda}} \right) \right)^{w_j} + (\gamma-1) \prod_{j=1}^{n} \left( \frac{\left(1+(\gamma-1)\mu_{\alpha_j}\right)^{\lambda} - \left(1-\mu_{\alpha_j}\right)^{\lambda}}{\left(1+(\gamma-1)\mu_{\alpha_j}\right)^{\lambda} + (\gamma-1)\left(1-\mu_{\alpha_j}\right)^{\lambda}} \right)^{w_j}} \right)^{\frac{1}{\lambda}}, \right.$$

$$\left. \left( \frac{\prod_{j=1}^{n} \left( 1+(\gamma-1) \left( \frac{\gamma \nu_{\alpha_j}}{\left(1+(\gamma-1)(1-\nu_{\alpha_j})\right)^{\lambda} + (1-\gamma)\nu_{\alpha_j}^{\lambda}} \right) \right)^{w_j} - \prod_{j=1}^{n} \left( 1 - \left( \frac{\gamma \nu_{\alpha_j}}{\left(1+(\gamma-1)(1-\nu_{\alpha_j})\right)^{\lambda} + (1-\gamma)\nu_{\alpha_j}^{\lambda}} \right) \right)^{w_j}}{\prod_{j=1}^{n} \left( 1+(\gamma-1) \left( \frac{\gamma \nu_{\alpha_j}}{\left(1+(\gamma-1)(1-\nu_{\alpha_j})\right)^{\lambda} + (1-\gamma)\nu_{\alpha_j}^{\lambda}} \right) \right)^{w_j} + (\gamma-1) \prod_{j=1}^{n} \left( 1 - \left( \frac{\gamma \nu_{\alpha_j}}{\left(1+(\gamma-1)(1-\nu_{\alpha_j})\right)^{\lambda} + (1-\gamma)\nu_{\alpha_j}^{\lambda}} \right) \right)^{w_j}} \right)^{\frac{1}{\lambda}} \right),$$

called the generalized Hamacher intuitionistic fuzzy weighted geometric (GHIFWG) operator. Especially, if $\gamma = 1$, then GIFWG operator; if $\gamma = 2$, then the GHIFWG operator is reduced to the generalized Einstien intuitionistic fuzzy weighted geometric operator.If $\gamma = 1$, then the GAIFWG operator is reduced to the following:

$$HIFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) =$$

$$\left(\frac{\gamma\prod_{j=1}^{n}\mu_{\alpha_j}^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(1-\mu_{\alpha_j}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{n}\mu_{\alpha_j}^{w_j}},\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)\nu_{\alpha_j}\right)^{w_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\nu_{\alpha_j}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{w_j}}\right)$$

which is called the Hamacher intuitionistic fuzzy weighted geometric (HIFWG) operator .

5. If $g(t)=\log\left({(\gamma-1)}/{(\gamma^{t}-1)}\right),\gamma>0$, i.e. $h(t)=\log\left({(\gamma-1)}/{(\gamma^{1-t}-1)}\right),g^{-1}(t)=$ $\log\left({\left({(\gamma-1+e^{t})}/{e^{t}}\right)}/{\log\gamma}\right),h^{-1}(t)=1-\log\left({\left({(\gamma-1+e^{t})}/{e^{t}}\right)}/{\log\gamma}\right)$,then

$$GFIFWG_{w}(\alpha_1,\alpha_2,\dots,\alpha_n)$$

$$=\left(\left(1-\left(1-\frac{\log 1+\prod_{j=1}^{n}\left(\gamma^{1-\frac{\log\left(1+(\gamma-1)\left(\frac{\gamma^{1-\mu_{\alpha_j}}}{\gamma-1}\right)^{\lambda}\right)}{\log\gamma}}-1\right)^{w_j}}{\log\gamma}\right)\right)^{\frac{1}{\lambda}},\left(1-\frac{\log 1+\prod_{j=1}^{n}\left(\gamma^{1-\frac{\log\left(1+(\gamma-1)\left(\frac{\gamma^{1-\mu_{\alpha_j}}}{\gamma-1}\right)^{\lambda}\right)}{\log\gamma}}-1\right)^{w_j}}{\log\gamma}\right)^{\frac{1}{\lambda}}\right)$$

called the generalized Frank intuitionistic fuzzy weighted geometric (GFIFWG) operator.

(2) In the following, we discuss the relations between the GAIFOWG operator and some existing intuitionistic fuzzy ordered weighted geometric aggregation operators.

If parameter $\lambda$ and weight vector $w=(w_1,w_2,\dots,w_n)$ are given by different values, and the additive generator $g$ is also assigned different forms, some specific intuitionistic fuzzy geometric aggregation operators can be obtained.

1. If $\lambda=1$, then

$$ATS-IFWG_{w}(\alpha_1,\alpha_2,\dots,\alpha_n)=\left(g^{-1}\left(\sum_{j=1}^{n}w_j g\left(\mu_{\alpha_{(j)}}\right)\right),h^{-1}\left(\sum_{j=1}^{n}w_j h\left(\nu_{\alpha_{(j)}}\right)\right)\right)$$

which is called the Archimedean t-conorm and t-norm based intuitionistic fuzzy weighted geometric (ATS-IFWG) operator. Furthermore, if $w=\left({1}/{n},{1}/{n},\dots,{1}/{n}\right)$, then

$$AIFG_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \left( g^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} g\left(\mu_{\alpha_{(j)}}\right)\right), h^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} h\left(\nu_{\alpha_{(j)}}\right)\right)\right)$$

called the Archimedean intuitionistic fuzzy geometric (AIFG) operator.

2. If $g(t) = -\log t$, i.e. $h(t) = -\log(1-t)$, $g^{-1}(t) = e^{-t}$ ,$h^{-1}(t) = 1 - e^{-t}$,

$$GIFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \mu_{\alpha_{(j)}}\right)^{\lambda}\right)\right)^{w_j}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{n}\left(1 - \nu_{\alpha_{(j)}}^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}}$$

which is called the generalized intuitionistic fuzzy weighted geometric (GIFWG) operator .

Furthermore, if $\lambda = 1$, then the generalized Archimedean intuitionistic fuzzy weighted geometric operator is reduced to the following:

$$IFWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\prod\nolimits_{j=1}^{n} \mu_{\alpha_{(j)}}^{w_j}, 1 - \prod\nolimits_{j=1}^{n}\left(1 - \nu_{\alpha_{(j)}}\right)^{w_j}\right)$$

which is called the intuitionistic fuzzy weighted geometric operator.

3. If $g(t) = log\left({(2-t)}/{t}\right)$, then $h(t) = log\left({(2-(1-t))}/{(1-t)}\right)$, $g^{-1}(t) = {2}/{(e^t + 1)}$ , $h^{-1}(t) = 1 - \left({2}/{(e^t+1)}\right)$ then

$$GEIFOWG(\alpha_1, \alpha_2, \dots, \alpha_n) =$$

$$\left(1-\left(1-\frac{2\prod_{j=1}^{n}\left(\frac{\left(1+\mu_{\alpha_{(j)}}\right)^{\lambda}-\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}{\left(1+\mu_{\alpha_{(j)}}\right)^{\lambda}+\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\frac{\left(1+\mu_{\alpha_{(j)}}\right)^{\lambda}-\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}{\left(1+\mu_{\alpha_{(j)}}\right)^{\lambda}+\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}\right)^{w_j}+\prod_{j=1}^{n}\left(\frac{\left(1+\mu_{\alpha_{(j)}}\right)^{\lambda}-\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}{\left(1+\mu_{\alpha_{(j)}}\right)^{\lambda}+\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}\right)^{w_j}}\right)^{\frac{1}{\lambda}}\right.,$$

$$\left.\left(\frac{\prod_{j=1}^{n}\left(1+\frac{2\nu_{\alpha_{(j)}}^{\lambda}}{\left(2-\nu_{\alpha_{(j)}}\right)^{\lambda}+\nu_{\alpha_{(j)}}^{\lambda}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\frac{2\nu_{\alpha_{(j)}}^{\lambda}}{\left(2-\nu_{\alpha_{(j)}}\right)^{\lambda}+\nu_{\alpha_{(j)}}^{\lambda}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\frac{2\nu_{\alpha_{(j)}}^{\lambda}}{\left(2-\nu_{\alpha_{(j)}}\right)^{\lambda}+\nu_{\alpha_{(j)}}^{\lambda}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\frac{2\nu_{\alpha_{(j)}}^{\lambda}}{\left(2-\nu_{\alpha_{(j)}}\right)^{\lambda}+\nu_{\alpha_{(j)}}^{\lambda}}\right)^{w_j}}\right)^{\frac{1}{\lambda}}\right)$$

called the generalized Einstein intuitionistic fuzzy ordered weighted geometric (GEIFOWG) operator.

Furthermore, if $\lambda = 1$, then

$$EIFOWG(\alpha_1, \alpha_2, \dots, \alpha_n)$$

$$= \left(\frac{2\prod_{j=1}^{n}\mu_{\alpha_{(j)}}^{w_j}}{\prod_{j=1}^{n}\left(2-\mu_{\alpha_{(j)}}\right)^{w_j}+\prod_{j=1}^{n}\mu_{\alpha_{(j)}}^{w_j}}, \frac{\prod_{j=1}^{n}\left(1-\nu_{\alpha_{(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\nu_{\alpha_{(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1-\nu_{\alpha_{(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\nu_{\alpha_{(j)}}\right)^{w_j}}\right)$$

which is called the Einstein intuitionistic fuzzy ordered weighted geometric (EIFWG) operator.

4. If $g(t) = log\left({(\gamma+(1-\gamma)t)}/{t}\right), \lambda > 0$, i.e. $h(t) = log\left({(\gamma+(1-\gamma)(1-t))}/{(1-t)}\right)$,

$g^{-1}(t) = {\gamma}/{(e^t+\gamma-1)}, h^{-1} = 1-\left({\gamma}/{(e^t+\gamma-1)}\right)$

$$GHIFOWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) =$$

$$\left(\begin{array}{c} 1-\left(1-\dfrac{\gamma\prod_{j=1}^{n}\left(\dfrac{\left(1+(\gamma-1)\mu_{\alpha_{(j)}}\right)^{\lambda}-\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}{\left(1+(\gamma-1)\mu_{\alpha_{(j)}}\right)^{\lambda}+(1-\gamma)\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(\dfrac{\left(1+(\gamma-1)\mu_{\alpha_{(j)}}\right)^{\lambda}-\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}{\left(1+(\gamma-1)\mu_{\alpha_{(j)}})\right)^{\lambda}+(1-\gamma)\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{n}\left(\dfrac{\left(1+(\gamma-1)\mu_{\alpha_{(j)}}\right)^{\lambda}-\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}{\left(1+(\gamma-1)\mu_{\alpha_{(j)}}\right)^{\lambda}+(\gamma-1)\left(1-\mu_{\alpha_{(j)}}\right)^{\lambda}}\right)^{w_j}}\right)^{\frac{1}{\lambda}}, \\ \left(\dfrac{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(\dfrac{\gamma\nu_{\alpha_{(j)}}}{\left(1+(\gamma-1)(1-\nu_{\alpha_{(j)}})\right)^{\lambda}+(1-\gamma)\nu_{\alpha_{(j)}}^{\lambda}}\right)\right)^{w_j}-\prod_{j=1}^{n}\left(1-\left(\dfrac{\gamma\nu_{\alpha_{(j)}}}{\left(1+(\gamma-1)(1-\nu_{\alpha_{(j)}})\right)^{\lambda}+(1-\gamma)\nu_{\alpha_{(j)}}^{\lambda}}\right)\right)^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(\dfrac{\gamma\nu_{\alpha_{(j)}}}{\left(1+(\gamma-1)(1-\nu_{\alpha_{(j)}})\right)^{\lambda}+(1-\gamma)\nu_{\alpha_{(j)}}^{\lambda}}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{n}\left(1-\left(\dfrac{\gamma\nu_{\alpha_{(j)}}}{\left(1+(\gamma-1)(1-\nu_{\alpha_{(j)}})\right)^{\lambda}+(1-\gamma)\nu_{\alpha_{(j)}}^{\lambda}}\right)\right)^{w_j}}\right)^{\frac{1}{\lambda}} \end{array}\right)$$

called the generalized Hamacher intuitionistic fuzzy ordeded weighted geometric (GHIFOWG) operator. Especially, if $\gamma = 1$, then the GHIFOWG operator is reduced to the GIFOWG operator; if $\gamma = 2$, then the GHIFOWG operator is reduced to the GEIFOWG operator.If $\gamma = 1$, then the GAIFOWG operator is reduced to the following:

$$HIFOWG_w(\alpha_1, \alpha_2, \dots, \alpha_n) =$$

$$\left(\frac{\gamma\prod_{j=1}^{n}\mu_{\alpha_{(j)}}^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(1-\mu_{\alpha_{(j)}}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{n}\mu_{\alpha_{(j)}}^{w_j}}, \frac{\prod_{j=1}^{n}\left(1+(\gamma-1)\nu_{\alpha_{(j)}}\right)^{w_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_{(j)}})^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\nu_{\alpha_{(j)}}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{n}(1-\nu_{\alpha_{(j)}})^{w_j}}\right)$$

which is called the Hamacher intuitionistic fuzzy ordered,weighted geometric (HIFOWG) operator .

5. If $g(t) = \log\left({}^{(\gamma-1)}\!\big/_{(\gamma^{t}-1)}\right), \gamma > 0$, i.e. $h(t) = \log\left({}^{(\gamma-1)}\!\big/_{(\gamma^{1-t}-1)}\right), g^{-1}(t) = \log\left({}^{\left({}^{(\gamma-1+e^{t})}\!/_{e^{t}}\right)}\!\Big/_{\log\gamma}\right), h^{-1}(t) = 1 - \log\left({}^{\left({}^{(\gamma-1+e^{t})}\!/_{e^{t}}\right)}\!\Big/_{\log\gamma}\right)$, then the GAIFOWG operator is reduced to the following:

$$GFIFOWG_w(\alpha_1, \alpha_2, \dots, \alpha_n)$$

$$=\left(\left(1-\left(1-\frac{\log 1+\prod_{j=1}^{n}\left(\gamma^{1-\frac{\log\left(1+(\gamma-1)\left(\frac{\gamma^{1-\mu_{\alpha(j)}}}{\gamma-1}\right)^{\lambda}\right)}{\log\gamma}}-1\right)^{w_j}}{\log\gamma}\right)\right)^{\frac{1}{\lambda}},\left(1-\frac{\log 1+\prod_{j=1}^{n}\left(\gamma^{1-\frac{\log\left(1+(\gamma-1)\left(\frac{\gamma^{1-\mu_{\alpha(j)}}}{\gamma-1}\right)^{\lambda}\right)}{\log\gamma}}-1\right)^{w_j}}{\log\gamma}\right)^{\frac{1}{\lambda}}\right)$$

called the generalized Frank intuitionistic fuzzy ordered weighted geometric (GFIFOWG) operator.

# Chapter 4

## Symmetric Pythagorean fuzzy weighted geometric/ averaging operators and their properties

In this chapter, the Pythagorean fuzzy weighted average/geometric operators and their power generalization, proposed by Ma & Zeshui [2], are studied.

### 4.1 Basic definitions

In this section, some basic definitions are reviewed [2]

**Definition 4.1** Let $X$ be a universe of discourse. A Pythagorean fuzzy set $A$ in $X$ is defined as follows:

$$A = \{x, \mu_A(x), \nu_A(x) | x \in X\},$$

where $\mu_A(x), \nu_A(x) \in [0,1]$ defines the membership and non-membership of $x$ to $A$, respectively and satisfy $\left(\mu_A(x)\right)^2 + \left(\upsilon_A(x)\right)^2 \leq 1$ . The expression pair $\pi_A^P(x) = \sqrt{1 - \mu_A^2(x) - \nu_A^2(x)}$ is called the degree of indeterminacy of $x$.

For a given $x \in X$, the pair $(\mu_A(x), \nu_A(x))$ is called as a Pythagorean fuzzy number (PFN), which is simply denoted as $\alpha = (\mu_\alpha, \nu_\alpha)$ where $\mu_\alpha, \nu_\alpha \in [0,1], \mu_\alpha^2 + \nu_\alpha^2 \leq 1$.

**Definition 4.2** Let $\alpha_j (j = 1,2, \ldots, n)$ be a collection of pythagorean fuzzy numbers, $PFWG_\omega$ and $PFWA_\omega \colon \mathcal{PFN}^n \to \mathcal{PFN}$,

(1) If

$$PFWG_\omega(\alpha) = \alpha_1^{\omega_1} \otimes \alpha_2^{\omega_2} \otimes \ldots \otimes \alpha_n^{\omega_n} = \left(\prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}, [1 - \prod_{j=1}^n (1 - \nu_{\alpha_j}^2)^{\omega_j}]^{\frac{1}{2}}\right)$$

then $PFWG_\omega$ is called a Pythagorean fuzzy weighted geometric (PFWG) operator of dimension $n$.

(2) If

$$PFWA_{\omega}(\alpha) = (\omega_1\alpha_1\ ) \oplus (\omega_2\alpha_2\ ) \oplus \ldots . \oplus (\omega_n\alpha_n\ )$$

$$= \left( [1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j}^2)^{\omega_j}]^{\frac{1}{2}}, \prod_{j=1}^{n} \nu_{\alpha_j}^{\omega_j} \right)$$

then $PFWA_{\omega}$ is called a Pythagorean fuzzy weighted averaging ($PFWA$) operator of dimension $n$,

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\alpha_j (j = 1,2, \ldots, n)$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

**Definition 4.3** Let $\alpha_j (j = 1,2, \ldots, n)$ be a collection of $PFNs$ and $SPFWG_{\omega}: \mathcal{PFN}^n \longrightarrow \mathcal{PFN}$

$SPFWG_{\omega}(\alpha) = (\omega_1 \boxdot \alpha_1) \boxplus (\omega_2 \boxdot \alpha_2) \boxplus \ldots \boxplus (\omega_n \boxdot \alpha_n)$. Then,

$$SPFWG_{\omega}(\alpha) = \left[ \frac{\prod_{j=1}^{n} \mu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^2\right)^{\omega_j} + \prod_{j=1}^{n} \mu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}}, \frac{\prod_{j=1}^{n} \nu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}^2\right)^{\omega_j} + \prod_{j=1}^{n} \nu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}} \right].$$

then $SPFWG_{\omega}$ is called a symmetric pythagorean fuzzy weighted geometric ($SPFWG$) operator of dimension $n$,

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\alpha_j (j = 1,2, \ldots, n)$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

**Definition 4.4** Let $\alpha_j (j = 1,2, \ldots, n)$ be a collection of $PFNs$ and $SPFWA_{\omega}: \mathcal{PFN}^n \longrightarrow \mathcal{PFN}$

$$SPFWA_{\omega}(\alpha) = (\omega_1 . \alpha_1) \boxplus (\omega_2 . \alpha_2) \boxplus \ldots \boxplus (\omega_n . \alpha_n)$$

$$= \left[ \left[ \frac{1 - \prod_{j=1}^{n} \left(1 - \mu_{\alpha_j}^2\right)^{\omega_j}}{2 - \prod_{j=1}^{n} \left(1 - \mu_{\alpha_j}^2\right)^{\omega_j} - \prod_{j=1}^{n} \mu_{\alpha_j}^{2\omega_j}} \right]^{\frac{1}{2}}, \left[ \frac{1 - \prod_{j=1}^{n} \left(1 - \nu\nu_{\alpha_j}^2\right)^{\omega_j}}{2 - \prod_{j=1}^{n} \left(1 - \nu_{\alpha_j}^2\right)^{\omega_j} - \prod_{j=1}^{n} \nu_{\alpha_j}^{2\omega_j}} \right]^{\frac{1}{2}} \right].$$

then $SPFWA_{\omega}$ is called a symmetric pythagorean fuzzy weighted averaging ($SPFWA$) operator of dimension $n$.

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $\alpha_j (j = 1,2, \dots, n)$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

**Definition 4.5** Let $\alpha_j (j = 1,2, \dots, n)$ be a collection of $PFNs$ , $PFWA_{\omega}^{\gamma}\, and\, PFWG_{\omega}^{\gamma}$ : $\mathcal{PFN}^n \longrightarrow \mathcal{PFN}$,

(1) If

$$PFWG_{\omega}^{\gamma}(\alpha) = \left(\prod_{j=1}^{n} \mu_{\alpha_j}^{\omega_j}, \prod_{j=1}^{n} \mu_{\alpha_j}^{\omega_j}\right)$$

then $PFWG_{\omega}^{\gamma}$ is called a Pythagorean fuzzy weighted geometric ($PFWG_{\omega}^{\gamma}$) operator of dimension $n$,

(2) If

$$PFWA_{\omega}^{\gamma}(\alpha) = (\sum_{j=1}^{n} \omega_j \mu_{\alpha_j}, \sum_{j=1}^{n} \omega_j \nu_{\alpha_j})$$

then $PFWA_{\omega}^{\gamma}$ is called a Pythagorean fuzzy weighted averaging ($PFWA_{\omega}^{\gamma}$) operator of dimension $n$.

**Definition 4.6** Let $\alpha = (\mu_\alpha, \nu_\alpha)$ and $\beta = (\mu_\beta, \nu_\beta)$ be two Pythagorean fuzzy numbers.Then,

(1) $\alpha \oplus \beta = (\mu_\alpha + \mu_\beta - \mu_\alpha \mu_\beta, \nu_\alpha \nu_\beta)$.

(2) $\alpha \otimes \beta = (\nu_\alpha + \nu_\beta - \nu_\alpha \nu_\beta, \mu_\alpha \mu_\beta)$.

(3)$\lambda\alpha = (1 - (1 - \mu_\alpha)^\lambda, \nu_\alpha^\lambda)$ , $\lambda > 0$.

(4)$\alpha^\lambda$ =$(\mu_\alpha^\lambda, 1 - (1 - \nu_\alpha)^\lambda), \lambda > 0$.

(5)$\alpha \boxplus \beta = \left(\frac{\mu_\alpha \mu_\beta}{[(1-\mu_\alpha^2)(1-\mu_\beta^2)+\mu_\alpha^2\mu_\beta^2]^{\frac{`}{2}}}, \frac{\nu_\alpha \nu_\beta}{[(1-\nu_\alpha^2)(1-\nu_\beta^2)+\nu_\alpha^2\nu_\beta^2]^{\frac{`}{2}}}\right)$.

(6)$\lambda \boxdot \alpha = \left(\frac{\mu_\alpha^\lambda}{[(1-\mu_\alpha^2)^\lambda+\mu_\alpha^{2\lambda}]^{\frac{1}{2}}}, \frac{\nu_\alpha^\lambda}{[(1-\nu_\alpha^2)^\lambda+\nu_\alpha^{2\lambda}]^{\frac{1}{2}}}\right)$.

### 4.2 Comparison of Pythagorean fuzzy numbers

In this section, the method for comparing two pythagorean fuzzy numbers, proposed by Ma and Xu [2] ,is presented.

Let $\alpha = [\mu_\alpha, 1 - \nu_\alpha]$ and $\beta = [\mu_\beta, 1 - \nu_\beta]$ be two pythagorean fuzzy numbers. Then, find scores $S(\alpha) = \sqrt{\mu_\alpha^2 - \nu_\alpha^2}$ and $S(\beta) = \sqrt{\mu_\beta^2 - \nu_\beta^2}$ be the scores of $\alpha$ and $\beta$ respectively and check that $S(\alpha) > S(\beta)$ or $S(\alpha) < S(\beta)$ or $S(\alpha) = S(\beta)$.

**Case (1)** If $S(\alpha) > S(\beta)$ , then $\alpha > \beta$.

**Case (2)** If $S(\alpha) < S(\beta)$ , then $\alpha < \beta$.

**Case (3)** If $S(\alpha) = S(\beta)$ , then find accuracy degree $H(\alpha) = \sqrt{\mu_\alpha^2 + \nu_\alpha^2}$ and $H(\beta) = \sqrt{\mu_\beta^2 + \nu_\beta^2}$ of $\tilde{a}$ and $\tilde{b}$ respectively.

**Case (3a)** If $H(\alpha) > H(\beta)$ , then $\alpha > \beta$.

**Case (3b)** If $H(\alpha) < H(\beta)$, then $\alpha < \beta$.

**Case (3c)** If $H(\alpha) = H(\beta)$, then $\alpha = \beta$.

### 4.3 Some important results

In this section, some results, proposed by Ma and Xu[2], are presented.

**Theorem 4.1** Let $\alpha, \beta$ be two pythagorean fuzzy numbers and $\lambda > 0$. Then $\alpha \boxplus \beta$ and $\lambda \boxdot \alpha$ are pythagorean fuzzy numbers.

**Proof:** Since, $\mu_\alpha^2 + \mu_\beta^2 - 2\mu_\alpha^2\mu_\beta^2 = \mu_\alpha^2(1 - \mu_\beta^2) + \mu_\beta^2(1 - \mu_\alpha^2) \geq 0$, then we have $\mu_\alpha^2\mu_\beta^2 \leq \mu_\alpha^2 + \mu_\beta^2 - \mu_\alpha^2\mu_\beta^2$. It follows that $\mu_\alpha^2\mu_\beta^2 \leq \frac{\mu_\alpha^2\mu_\beta^2}{1-\mu_\alpha^2-\mu_\beta^2+2\mu_\alpha^2\mu_\beta^2} = \frac{\mu_\alpha^2\mu_\beta^2}{(1-\mu_\alpha^2)(1-\mu_\beta^2)+\mu_\alpha^2\mu_\beta^2} \leq \mu_\alpha^2 + \mu_\beta^2 - \mu_\alpha^2\mu_\beta^2$ and hence $\mu_\alpha\mu_\beta \leq \frac{\mu_\alpha\mu_\beta}{\left[(1-\mu_\alpha^2)(1-\mu_\beta^2)+\mu_\alpha^2\mu_\beta^2\right]^{\frac{1}{2}}} \leq \left[\mu_\alpha^2 + \mu_\beta^2 - \mu_\alpha^2\mu_\beta^2\right]^{\frac{1}{2}}$. Thus $0 \leq \frac{\mu_\alpha\mu_\beta}{\left[(1-\mu_\alpha^2)(1-\mu_\beta^2)+\mu_\alpha^2\mu_\beta^2\right]^{\frac{1}{2}}} \leq 1.$

In a similar way, it holds that $0 \leq \frac{\nu_\alpha \nu_\beta}{\left[(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)+\nu_\alpha^2\nu_\beta^2\right]^{\frac{1}{2}}} \leq 1$.Since $\alpha, \beta$ are two pythagorean fuzzy numbers, therefore $\mu_\alpha^2 + \nu_\alpha^2 \leq 1$ and $\mu_\beta^2 + \nu_\beta^2 \leq 1$. Thus

$$\frac{\mu_\alpha^2\mu_\beta^2}{(1-\mu_\alpha^2)\left(1-\mu_\beta^2\right)+\mu_\alpha^2\mu_\beta^2} + \frac{\nu_\alpha^2\nu_\beta^2}{(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)+\nu_\alpha^2\nu_\beta^2}$$

$$= \frac{1}{\left(\frac{1}{\mu_\alpha^2}-1\right)\left(\frac{1}{\mu_\alpha^2}-1\right)+1} + \frac{\nu_\alpha^2\nu_\beta^2}{(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)+\nu_\alpha^2\nu_\beta^2}$$

$$\leq \frac{1}{\left(\frac{1}{1-\nu_\alpha^2}-1\right)\left(\frac{1}{1-\nu_\alpha^2}-1\right)+1} + \frac{\nu_\alpha^2\nu_\beta^2}{(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)+\nu_\alpha^2\nu_\beta^2}$$

$$= \frac{(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)}{(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)+\nu_\alpha^2\nu_\beta^2} + \frac{\nu_\alpha^2\nu_\beta^2}{(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)+\nu_\alpha^2\nu_\beta^2}$$

$$= 1$$

Thus $\alpha \boxplus \beta$ is a pythagorean fuzzy number.

If $0 < \lambda \leq 1$, it follows that $1-(1-\mu_\alpha^2)^\lambda \leq \frac{\mu_\alpha^{2\lambda}}{1+\mu_\alpha^{2\lambda}-\left[1-\left(1-\mu_\alpha^2\right)^\lambda\right]} = \frac{\mu_\alpha^{2\lambda}}{\mu_\alpha^{2\lambda}-\left(1-\mu_\alpha^2\right)^\lambda} \leq \mu_\alpha^{2\lambda}$, thus

$\left[1-(1-\mu_\alpha^2)^\lambda\right]^{\frac{1}{2}} \leq \frac{\mu_\alpha^\lambda}{\left[\mu_\alpha^{2\lambda}-\left(1-\mu_\alpha^2\right)^\lambda\right]^{\frac{1}{2}}} \leq \mu_\alpha^\lambda$.

If $\lambda > 1$, then similarly, $\mu_\alpha^\lambda \leq \frac{\mu_\alpha^\lambda}{\left[\mu_\alpha^{2\lambda}-\left(1-\mu_\alpha^2\right)^\lambda\right]^{\frac{1}{2}}} \leq \left[1-(1-\mu_\alpha^2)^\lambda\right]^{\frac{1}{2}}$. Thus $0 \leq \frac{\mu_\alpha^\lambda}{\left[\mu_\alpha^{2\lambda}-\left(1-\mu_\alpha^2\right)^\lambda\right]^{\frac{1}{2}}} \leq 1$ and $0 \leq \frac{\nu_\alpha^\lambda}{\left[\nu_\alpha^{2\lambda}-\left(1-\nu_\alpha^2\right)^\lambda\right]^{\frac{1}{2}}} \leq 1$ for all $\lambda > 0$.

Since $\alpha$ is a pythagorean fuzzy number, then it holds that $\mu_\alpha^2 + \mu_\alpha^2 \leq 1$. Thus for a $\lambda > 0$,

$$\frac{\mu_\alpha^{2\lambda}}{\left(1-\mu_\alpha^2\right)^\lambda+\mu_\alpha^{2\lambda}} + \frac{\nu_\alpha^{2\lambda}}{\left(1-\nu_\alpha^2\right)^\lambda+\nu_\alpha^{2\lambda}} = \frac{\nu_\alpha^{2\lambda}}{\left(\frac{1}{\mu_\alpha^2}-1\right)^\lambda+\nu_\alpha^{2\lambda}} + \frac{\nu_\alpha^{2\lambda}}{\left(1-\nu_\alpha^2\right)^\lambda+\nu_\alpha^{2\lambda}} \leq \frac{\nu_\alpha^{2\lambda}}{\left(\frac{1}{1-\nu_\alpha^2}-1\right)^\lambda+\nu_\alpha^{2\lambda}} + \frac{\nu_\alpha^{2\lambda}}{\left(1-\nu_\alpha^2\right)^\lambda+\nu_\alpha^{2\lambda}} \leq$$

$$\frac{\left(1-\nu_\alpha^2\right)^\lambda}{\left(1-\nu_\alpha^2\right)^\lambda+\nu_\alpha^{2\lambda}} + \frac{\nu_\alpha^{2\lambda}}{\left(1-\nu_\alpha^2\right)^\lambda+\nu_\alpha^{2\lambda}} = 1.$$

Thus $\lambda \boxdot \alpha$ is a $PNF$, which completes the proof of this theorem.

For given two PNFs $\alpha, \beta$ and $\lambda > 0$, from the proof of the above theorem that $\mu_\alpha \mu_\beta \leq$

$$\frac{\mu_\alpha \mu_\beta}{\left[(1-\mu_\alpha^2)\left(1-\mu_\beta^2\right)+\mu_\alpha^2 \mu_\beta^2\right]^{\frac{1}{2}}} \leq \left[\mu_\alpha^2 + \mu_\beta^2 - \mu_\alpha^2 \mu_\beta^2\right]^{\frac{1}{2}}. \nu_\alpha \nu_\beta \leq \frac{\nu_\alpha \nu_\beta}{\left[(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)+\nu_\alpha^2 \nu_\beta^2\right]^{\frac{1}{2}}} \leq \left[\nu_\alpha^2 + \nu_\beta^2 - \nu_\alpha^2 \nu_\beta^2\right]^{\frac{1}{2}}.$$

$$\left[1-(1-\mu_\alpha^2)^\lambda\right]^{\frac{1}{2}} \leq \frac{\mu_\alpha^\lambda}{\left[\mu_\alpha^{2\lambda}-(1-\mu_\alpha^2)^\lambda\right]^{\frac{1}{2}}} \leq \mu_\alpha^\lambda \quad , \left[1-(1-\nu_\alpha^2)^\lambda\right]^{\frac{1}{2}} \leq \frac{\nu_\alpha^\lambda}{\left[\nu_\alpha^{2\lambda}-(1-\nu_\alpha^2)^\lambda\right]^{\frac{1}{2}}} \leq \nu_\alpha^\lambda$$

For $0 < \lambda < 1$ and

$$\mu_\alpha^\lambda \leq \frac{\mu_\alpha^\lambda}{\left[\mu_\alpha^{2\lambda}-(1-\mu_\alpha^2)^\lambda\right]^{\frac{1}{2}}} \leq \left[1-(1-\mu_\alpha^2)^\lambda\right]^{\frac{1}{2}} \quad , \nu_\alpha^\lambda \leq \frac{\nu_\alpha^\lambda}{\left[\nu_\alpha^{2\lambda}-(1-\nu_\alpha^2)^\lambda\right]^{\frac{1}{2}}} \leq \left[1-(1-\nu_\alpha^2)^\lambda\right]^{\frac{1}{2}}$$

For $\lambda > 1$.

Thus, $\alpha\beta \leq \alpha \boxplus \beta \leq \alpha \oplus \beta$.

**Theorem 4.2** Let $\alpha, \beta$ be two $PNFs$ and $\lambda > 0$. Then,

(1) $\alpha \boxplus \beta = \beta \boxplus \alpha$,

(2) $\lambda \boxdot (\alpha \boxplus \beta) = (\lambda \boxdot \alpha) \boxplus (\lambda \boxdot \beta)$.

**Proof:** (1) $\alpha \boxplus \beta = \left(\frac{\mu_\alpha \mu_\beta}{[(1-\mu_\alpha^2)\left(1-\mu_\beta^2\right)+\mu_\alpha^2 \mu_\beta^2]^{\frac{1}{2}}}, \frac{\nu_\alpha \nu_\beta}{[(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)+\nu_\alpha^2 \nu_\beta^2]^{\frac{1}{2}}}\right)$

$$= \left(\frac{\nu_\alpha \nu_\beta}{[(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)+\nu_\alpha^2 \nu_\beta^2]^{\frac{1}{2}}}, \frac{\mu_\beta \mu_\alpha}{\left[\left(1-\mu_\beta^2\right)(1-\mu_\alpha^2)+\mu_\beta^2 \mu_\alpha^2\right]^{\frac{1}{2}}}\right) = \beta \boxplus \alpha.$$

(1) It follows that $\mu_{\lambda \boxdot (\alpha \boxplus \beta)} = \frac{\mu_{\alpha \boxplus \beta}^\lambda}{\left[\left(1-\mu_{\alpha \boxplus \beta}^2\right)^\lambda + \mu_{\alpha \boxplus \beta}^{2\lambda}\right]^{\frac{1}{2}}}$

$$= \frac{\left(\frac{\mu_\alpha\mu_\beta}{\left[(1-\mu_\alpha^2)\left(1-\mu_\beta^2\right)+\mu_\alpha^2\mu_\beta^2\right]^{\frac{1}{2}}}\right)^\lambda}{\left[\left(1-\left(\frac{\mu_\alpha\mu_\beta}{\left[(1-\mu_\alpha^2)\left(1-\mu_\beta^2\right)+\mu_\alpha^2\mu_\beta^2\right]^{\frac{1}{2}}}\right)^2\right)^\lambda+\left(\frac{\mu_\alpha\mu_\beta}{\left[(1-\mu_\alpha^2)\left(1-\mu_\beta^2\right)+\mu_\alpha^2\mu_\beta^2\right]^{\frac{1}{2}}}\right)^{2\lambda}\right]^{\frac{1}{2}}}$$

$$= \frac{\left(\mu_\alpha\mu_\beta\right)^\lambda}{\left[\left(1-\mu_\alpha^2\right)^\lambda\left(1-\mu_\beta^2\right)^\lambda+\left(\mu_\alpha\mu_\beta\right)^{2\lambda}\right]^{\frac{1}{2}}}$$

$$\mu_{(\lambda\boxdot\alpha)\boxplus(\lambda\boxdot\beta)} = \frac{\mu_{\lambda\boxdot\alpha}\mu_{\lambda\boxdot\beta}}{\left[\left(1-\mu_{\lambda\boxdot\alpha}^2\right)\left(1-\mu_{\lambda\boxdot\alpha}^2\right)+\mu_{\lambda\boxdot\alpha}^2\mu_{\lambda\boxdot\alpha}^2\right]^{\frac{1}{2}}}$$

$$= \frac{\frac{\mu_\alpha^\lambda}{\left[\left(1-\mu_\alpha^2\right)^\lambda+\mu_\alpha^{2\lambda}\right]^{\frac{1}{2}}}\,\frac{\mu_\beta^\lambda}{\left[\left(1-\mu_\beta^2\right)^\lambda+\mu_\beta^{2\lambda}\right]^{\frac{1}{2}}}}{\left[\left(1-\left(\frac{\mu_\alpha^\lambda}{\left[\left(1-\mu_\alpha^2\right)^\lambda+\mu_\alpha^{2\lambda}\right]^{\frac{1}{2}}}\right)^2\right)\left(1-\left(\frac{\mu_\alpha^\lambda}{\left[\left(1-\mu_\alpha^2\right)^\lambda+\mu_\alpha^{2\lambda}\right]^{\frac{1}{2}}}\right)^2\right)+\mu_{\lambda\boxdot\alpha}^2\mu_{\lambda\boxdot\alpha}^2\right]^{\frac{1}{2}}}$$

$$= \frac{\left(\mu_\alpha\mu_\beta\right)^\lambda}{\left[\left(1-\mu_\alpha^2\right)^\lambda\left(1-\mu_\beta^2\right)^\lambda+\left(\mu_\alpha\mu_\beta\right)^{2\lambda}\right]^{\frac{1}{2}}}.$$

and hence $\mu_{\lambda\boxdot(\alpha\boxplus\beta)} = \mu_{(\lambda\boxdot\alpha)\boxplus(\lambda\boxdot\beta)}$.

Similarly, $\nu_{\lambda\boxdot(\alpha\boxplus\beta)} = \nu_{(\lambda\boxdot\alpha)\boxplus(\lambda\boxdot\beta)}$.

Thus,

$$\lambda \boxdot (\alpha \boxplus \beta) = (\lambda \boxdot \alpha) \boxplus (\lambda \boxdot \beta).$$

**Theorem 4.3** Let $\alpha_j, \beta_j (j = 1,2, \ldots, n)$ be two collections of $PNFs$.

(1) If all $\alpha_j$ are equal,i.e., $\alpha_j = \delta = (\mu_\delta, \nu_\delta)$,for all $j$ ,then $SPFWG_\omega(\alpha) = \delta$;

(2) If $\alpha_j \le \beta_j$ for all $j$, then $SPFWG_\omega(\alpha) \le SPFWG_\omega(\beta)$;

(3) Let $\alpha^- = \left(min(\mu_\alpha), max(\nu_\alpha)\right)\ and\ \alpha^+ = \left(max(\mu_\alpha), min(\nu_\alpha)\right)$ ,then

$\alpha^- \leq SPFWG_\omega(\alpha) \leq \alpha^+$

**Proof:** (1) By Definition 4.3,

$$SPFWG_\omega(\alpha) = \left[\frac{\prod_{j=1}^{n} \mu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^2\right)^{\omega_j}+\prod_{j=1}^{n} \mu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}}, \frac{\prod_{j=1}^{n} \nu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}^2\right)^{\omega_j}+\prod_{j=1}^{n} \nu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}}\right] = (\mu_\delta, \nu_\delta) = \delta,$$

(2) Since $\alpha_j \leq \beta_j$ for all $j$,i.e., $\mu_{\alpha_j} \leq \mu_{\alpha_j}$ and $\nu_{\alpha_j} \leq \nu_{\alpha_j}$. Therefore,

$$\frac{\prod_{j=1}^{n} \mu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^2\right)^{\omega_j}+\prod_{j=1}^{n} \mu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}} = \frac{1}{\left[1+\prod_{j=1}^{n}\left(\frac{1}{\mu_{\alpha_j}^2}-1\right)^{\omega_j}\right]^{\frac{1}{2}}} \leq \frac{1}{\left[1+\prod_{j=1}^{n}\left(\frac{1}{\mu_{\alpha_j}^2}-1\right)^{\omega_j}\right]^{\frac{1}{2}}} = \frac{\prod_{j=1}^{n} \mu_{\beta_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\mu_{\beta_j}^2\right)^{\omega_j}+\prod_{j=1}^{n} \mu_{\beta_j}^{2\omega_j}\right]^{\frac{1}{2}}}$$

.

Similarly, it holds that $\frac{\prod_{j=1}^{n} \nu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}^2\right)^{\omega_j}+\prod_{j=1}^{n} \nu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}} \geq \frac{\prod_{j=1}^{n} \nu_{\beta_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\nu_{\beta_j}^2\right)^{\omega_j}+\prod_{j=1}^{n} \nu_{\beta_j}^{2\omega_j}\right]^{\frac{1}{2}}}$. Thus $SPFWG_\omega(\alpha) \leq SPFWG_\omega(\beta)$.

(4) Since $min_j\left\{\mu_{\alpha_j}\right\} \leq \mu_{\alpha_j} \leq max_j\left\{\mu_{\alpha_j}\right\}$ and $min_j\left\{\nu_{\alpha_j}\right\} \leq \nu_{\alpha_j} \leq max_j\left\{\nu_{\alpha_j}\right\}$, it follows that

$$\min_j\left\{\mu_{\alpha_j}\right\} \leq \frac{\prod_{j=1}^{n} \mu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^2\right)^{\omega_j}+\prod_{j=1}^{n} \mu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}} \leq \max_j\left\{\mu_{\alpha_j}\right\},$$

$$\min_j\left\{\nu_{\alpha_j}\right\} \leq \frac{\prod_{j=1}^{n} \nu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}^2\right)^{\omega_j}+\prod_{j=1}^{n} \nu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}} \leq \max_j\left\{\mu_{\alpha_j}\right\},$$

Thus $\alpha^- \leq SPFWG_\omega(\alpha) \leq \alpha^+$.

**Theorem 4.4** Let $\alpha, \beta$ be two $PFNs$ and $\lambda > 0$. Then $\alpha + \beta$ and $\lambda . \alpha$ are $PFNs$.

**Proof:** For given two PFNs $\alpha, \beta\ and\ \lambda > 0$,

$$\mu_\alpha\mu_\beta \leq \left[\frac{1-(1-\mu_\alpha^2)\left(1-\mu_\beta^2\right)}{2-(1-\mu_\alpha^2)\left(1-\mu_\beta^2\right)-\mu_\alpha^2\mu_\beta^2}\right]^{\frac{1}{2}} \leq \left[\mu_\alpha^2+\mu_\beta^2-\mu_\alpha^2\mu_\beta^2\right]^{\frac{1}{2}},$$

$$\nu_\alpha\nu_\beta \leq \left[\frac{1-(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)}{2-(1-\nu_\alpha^2)\left(1-\nu_\beta^2\right)-\nu_\alpha^2\nu_\beta^2}\right]^{\frac{1}{2}} \leq \left[\nu_\alpha^2+\nu_\beta^2-\nu_\alpha^2\nu_\beta^2\right]^{\frac{1}{2}},$$

$$\left[1-(1-\mu_\alpha^2)^\lambda\right]^{\frac{1}{2}} \leq \left[\frac{1-\left(1-\mu_\alpha^2\right)^\lambda}{2-\mu_\alpha^{2\lambda}-\left(1-\mu_\alpha^2\right)^\lambda}\right]^{\frac{1}{2}} \leq \mu_\alpha^\lambda,$$

.

$$\left[1-(1-\nu_\alpha^2)^\lambda\right]^{\frac{1}{2}} \leq \left[\frac{1-\left(1-\nu_\alpha^2\right)^\lambda}{2-\nu_\alpha^{2\lambda}-\left(1-\nu_\alpha^2\right)^\lambda}\right]^{\frac{1}{2}} \leq \nu_\alpha^\lambda.$$

For $0 < \lambda < 1$ and

$$\mu_\alpha^\lambda \leq \left[\frac{1-(1-\mu_\alpha^2)^\lambda}{2-\mu_\alpha^{2\lambda}-(1-\mu_\alpha^2)^\lambda}\right]^{\frac{1}{2}} \leq \left[1-(1-\mu_\alpha^2)^\lambda\right]^{\frac{1}{2}} \ ,$$

$$\nu_\alpha^\lambda \leq \left[\frac{1-(1-\nu_\alpha^2)^\lambda}{2-\nu_\alpha^{2\lambda}-(1-\nu_\alpha^2)^\lambda}\right]^{\frac{1}{2}} \leq \left[1-(1-\nu_\alpha^2)^\lambda\right]^{\frac{1}{2}},$$

For $\lambda > 1$.

Thus, $\alpha\beta \leq \alpha \boxplus \beta \leq \alpha \oplus \beta$.

**Theorem 4.5** Let $\alpha_j (j = 1,2, \ldots.., n)$ be a collections of $PNFs$. Then

(1) $PFWG_\omega(\alpha) \prec SPFWG_\omega(\alpha) \prec PFWA_\omega(\alpha)$.

(2) $PFWG_\omega(\alpha) \prec SPFWA_\omega(\alpha) \prec PFWA_\omega(\alpha)$.

**Proof:** Since, $\prod_{j=1}^{n} \mu_{\alpha_j}^{2\omega_j} \leq 1 - \prod_{j=1}^{n} \left(1-\mu_{\beta_j}^2\right)^{\omega_j}$, and the equality hold iff $\mu_{\alpha_1} = \mu_{\alpha_2} = \cdots = \mu_{\alpha_n}$, from Theorem 4.4 it follows that

$$\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j} \leq \frac{\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^2\right)^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}} \leq \left[1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^2\right)^{\omega_j}\right]^{\frac{1}{2}}$$

Similarly,

$$\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j} \leq \frac{\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}^2\right)^{\omega_j}+\prod_{j=1}^{n}\nu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}} \leq \left[1-\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}^2\right)^{\omega_j}\right]^{\frac{1}{2}}$$

and the equality hold iff $\nu_{\alpha_1}=\nu_{\alpha_2}=\cdots=\nu_{\alpha_n}$. Thus

$$\left(\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j},\left[1-\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}^2\right)^{\omega_j}\right]^{\frac{1}{2}}\right)$$

$$\leq\left(\frac{\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^2\right)^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}},\frac{\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j}}{\left[\prod_{j=1}^{n}\left(1-\nu_{\alpha_j}^2\right)^{\omega_j}+\prod_{j=1}^{n}\nu_{\alpha_j}^{2\omega_j}\right]^{\frac{1}{2}}}\right)$$

$$\leq\left(\left[1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^2\right)^{\omega_j}\right]^{\frac{1}{2}},\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j}\right)$$

and the equality hold iff $\alpha_1=\alpha_2=\cdots=\alpha_n$. Thus $PFWG_\omega(\alpha)\prec SPFWG_\omega(\alpha)\prec PFWA_\omega(\alpha)$.

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