DISCONTINUOUS GALERKIN METHOD FOR LWR TRAFFIC FLOW MODEL

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Submitted by

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Certificate

This is to certify that the thesis "Discontinuous Galerkin Method For LWR Traffic Flow Model." submitted by Daljeet Kaur of M.Sc (Mathematics), Thapar University, Patiala, was carried out by me under supervision of Dr. Sapna Sharma. She has not submitted this material for credit towards any other degree at Thapar University Patiala or any other University.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

In the present work we discussed the Discontinuous Galerkin method for LWR model for Traffic flow modeling. The basics of traffic flow theory are studied briefly. Traffic flow modeling is classified and some earlier important work in the field is discussed concisely. The 1st chapter of the thesis introduce the traffic flow modeling and various models discribing the traffic flow phenomenon. The LWR model is presented in details.

Chapter two deals with the classification of differential equations, some important class of methods are disscussed to find the solution of the complex partial differential equations.

Hyperbolic PDEs are used to model traffic flow phenomena. There are very few methods available to find the solution of the complex PDEs. The Discontinuous Galerkin method is one of the higher order method used to simulate the traffic flow models. The discription of the method, numerical experiments and graphs are given in chapter 3.

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Chapter 1

Traffic Flow

Transportation problem have been topic of concern for man long before the automobiles invention. However, in recent years, the scientific analysis of traffic problems has become an attentive topic of deliberation and research for scientist and engineers.Traffic problem include: installation of traffic light or stop signs; cycle timing of traffic lights; where to construct flyovers, whether to change a two way street to a one way street, number of lanes on a road; where to construct overpasses, exits and entrances. In particular, the main aim is to study the traffic phenomena with an objective of eventually making decisions which may palliate congestion, reduce accidents, maximize traffic flow, minimize automobile exhaust pollution etc.

1.1 Introduction

Traffic flow is the scientific study of movement of individual drivers, vehicles, infrastructure (highways, traffic signage and traffic control devices) and interaction they make with one another. The main purpose of the study is to develop an optimal road network with maximum flow of traffic and minimum traffic congestion.
The first beginnings for traffic flow descriptions on a highway are derived from observations by Greenshields, firstly shown to the public exactly 75 years ago. He carried out tests to measure traffic flow, traffic density and speed using photographic measurement methods for the first time.

1.2 Mathematical Modeling of Traffic Flow

Mathematical Modeling of traffic flow has been a key tool to simulate a behavior of transportation system. There are mainly three traffic flow models, which can be classified in 3 categories: microscopic car following model, macroscopic continuum model and mesoscopic model.

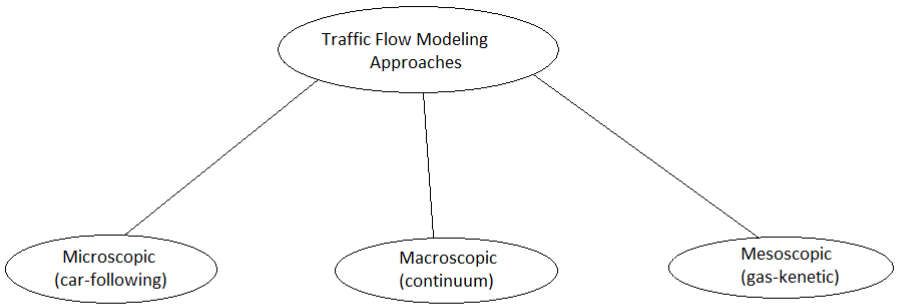


Figure 1.1: General Classification of Traffic Flow Modeling Approaches

1.2.1 Microscopic Approach

Microscopic model describes both the space-time behavior of the system entities as well their interaction at a high level of detail. For example: Car following model, Cellular Automata model etc.

1.2.2 Mesoscopic Approach

Simulate individual vehicles, but describes their activities and interactions based on aggregate (macroscopic) relationships.

1.2.3 Macroscopic Approach

Instead of describing the individual behavior of each vehicles, in this model we look at the traffic flow from a global perspective. For example: Hydrodynamic model, Gas

Boltzmann model etc.

Different Approaches are useful at different level details. Our research will focus on macroscopic modeling approaches so we will discuss some literature work on this modeling approach.

1.3 Types of Traffic Flow

Traffic flow can be classified into two primary types. Understanding what type of flow is happening in a given situation, it will help you decide which analysis methods and descriptions are the most applicable.
Uninterrupted flow is the first type of traffic flow. It is flow described by vehicle-vehicle interactions and interactions between vehicles and road. For example, vehicles traveling on highway are performing in uninterrupted flow.
Interrupted flow is the second type of traffic flow and it is flow determined by an external means, such as a traffic signal. Under the conditions of interrupted flow, vehicle-vehicle interactions and vehicle-road interactions play a secondary role in defining the traffic flow.

1.4 Fundamental variables of Traffic Flow

Though traffic flow is a complex phenomenon of many particle systems yet it is represent with the help of a common set of terms. There are three main parameters to understand the traffic stream: Speed(ν), Density(ρ) and Flow(q).

1.4.1 Speed(ν)

The speed of a vehicle in traffic flow is defined as the distance covered per unit time. It is not possible to record the speed of each individual vehicles on road. Thus average speed is based on taking the all vehicles over a period of time or space. If time is kept as reference then it is called time mean speed and if speed is measured by space reference, then it is called space mean speed.
Time Mean Speed: It is defined as the average speed of all vehicles passing reference point on a highway over some specified time period. With the help of Loop detectors,

we measure it. When spread over a reference area, then loop detectors can record the signature of vehicles and can track the speed of each vehicle. But average speed measurements obtained from this method are not accurate, because when vehicles are traveling at distinct speeds over the same distance then instantaneous speed average among individual vehicles does not account for the difference in travel time for the vehicle

$$\nu_t \;=\; (1/m)\sum_{i=1}^{m} v_i$$

where m represent the number of vehicles which are passing through the fixed point.
Space Mean Speed: It is defined as the average speed of all the vehicles occupying a given section of a highway over some specified time period. Successive photographs and video of a road segment record the speed of each individual vehicle, and then calculate the average speed. Space mean speed is more accurate than the time mean speed. The info for space calculating space mean speed may be taken from a camera, satellite photographs, or both.

$$\nu_s \;=\; n\Big(\sum_{i=1}^{m} \frac{1}{\nu_i}\Big)^{-1}$$

where n represent the number of vehicles which are passing the given road segment.

1.4.2 Density(ρ)

Density refers to the number of vehicles per unit length of road. Normally, it is expressed as vehicles per mile or vehicles per kilometer. High densities imply that individual vehicles are very close with each other, while low densities indicate greater distances between vehicles. Spacing, gap, headway and clearance are various measures for describing the space between vehicles. Let n represent the number of vehicles which are occupied in 1 length of the road section. Then

$$\rho = \frac{n}{l}$$

1.4.3 Flow(q)

Flow is one of the most common parameter in traffic flow. It refers to the average number of cars passing per unit time through given point on the roadway. Normally, it

is given in terms of vehicles per hour. All the fundamental parameters discussed above depends on position of vehicles and time.

1.5 Fundamental Law

In the last section, we have discussed the fundamental traffic parameters: velocity, density and flow. They are all related to each other. Under the conditions of uninterrupted flow, there is a close relationship among all three parameters which is as:

$$q = \rho\nu$$

where
q = flow (which is expressed in vehicles per hour)
ν = speed (which is expressed in miles per hour or kilometer per hour)
ρ = density (which is expressed in vehicles per miles or vehicles per kilometer)
This relationship is known as fundamental law of traffic flow.

1.6 Conservation Law of Cars

We will see that on a unidirectional (simplex) road with no entrance and no exit in this section. The number of vehicles between two points are conserved. Assume on some section of road, between $x = \alpha$ and $x = \beta$, the total number of cars are N at time t within the interval $[\alpha, \beta]$, as shown in Fig. 1.2,

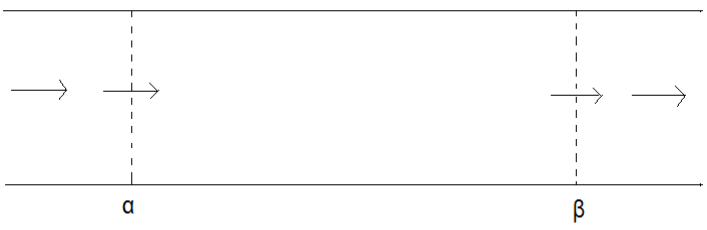


Figure 1.2: Cars entering and leaving the road segment

Then,

$$N = \int_{\alpha}^{\beta} \rho(x,t)dx \tag{1.1}$$

If there are no entrances and exits on the road. The N increases due to cars entering from α and N decreases due to cars leaving from β. Then the rate of change of the number of cars is, equals the number of cars crossing $x = \alpha$ per unit time $q(\alpha, t)$ minus number of cars leaving $x = \beta$ per unit time $q(\beta, t)$, or

$$\frac{dN}{dT} \;=\; q(\alpha, t) \;-\; q(\beta, t) \tag{1.2}$$

So plugging equations (1.1) and (1.2), we get results

$$\frac{d}{dt}\int_{\alpha}^{\beta} \rho(x, t)dx \;=\; q(\alpha, t) \;-\; q(\beta, t) \tag{1.3}$$

Now,

$$\begin{aligned} Flow &= [number\ of\ cars\ crossing\ the\ point\ x\ per\ unit] \\ &= (density)(velocity) \\ &= \rho(x, t)\nu(x, t) \end{aligned}$$

or $\qquad q(x, t) = \rho(x, t) * \nu(x, t)$

Assume $\nu = \nu(\rho)$ i.e. the car velocity ν is dependent only on the density ρ of vehicles on road.
So, the rate of change of the number of cars is:

$$\frac{d}{dt}\int_{\alpha}^{\beta} \rho(x, t)\ dx = \rho(\alpha, t)\nu(\alpha, t) - \rho(\beta, t)\nu(\beta, t) \tag{1.4}$$

This is one integral form of conservation law. Another form is obtained by integrating this in time t_1 to t_2 giving an expression for the mass in $[\alpha, \beta]$ at time $t_2 > t_1$ in terms of the mass at time t_1 and total (integrated) flux at each boundary during this period time:

$$\int_{\alpha}^{\beta} \rho(x, t_2)\ dx - \int_{\alpha}^{\beta} \rho(x, t_1)\ dx = \int_{t_1}^{t_2} \rho(\alpha, t)\nu(\alpha, t)\ dt - \int_{t_1}^{t_2} \rho(\beta, t)\nu(\beta, t)\ dt \tag{1.5}$$

To derive the differential equation of the conservation law, we must now assume that

$\rho(x,t)\nu(x,t)$ are differentiable function. Then using:

$$\rho(x,t_2) - \rho(x,t_1) = \int_{t_1}^{t_2} \frac{\partial}{\partial t}\rho(x,t)\ dt$$

and

$$\rho(\alpha,t)\nu(\alpha,t) - \rho(\beta,t)\nu(\beta,t) = \int_{\alpha}^{\beta} \frac{\partial}{\partial x}\rho(x,t)\nu(x,t)$$

put these values in (1.5) and we get:

$$\int_{t_1}^{t_2}\int_{\alpha}^{\beta} \left\{\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\rho(x,t)\nu(x,t)\right\}\ dx\ dt = 0 \tag{1.6}$$

Since this must hold for any section $[\alpha,\beta]$ and over any time interval $[t_1,t_2]$. We conclude that in fact the integrand in (1.6) must be indentically zero i.e.,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\nu)}{\partial x} = 0$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1.7}$$

Equation (1.7) is represent the conservation of cars in mathematical form.

1.7 Fundamental Diagram

Beside the fundamental relation, there are also experimental relations between the traffic flow parameters.

These relationship are called fundamental diagrams. They are obtained from

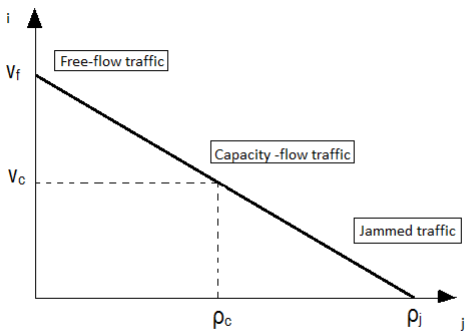


Figure 1.3: The fundamental v-ρ diagram according to Greenshields

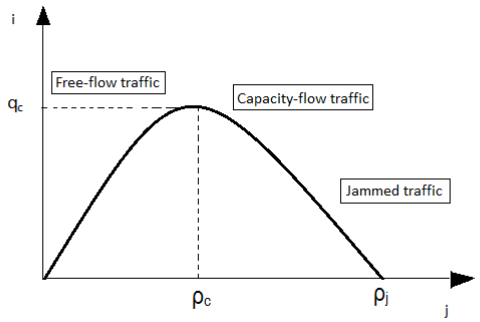


Figure 1.4: The fundamental q- ρ diagram according to Greenshield

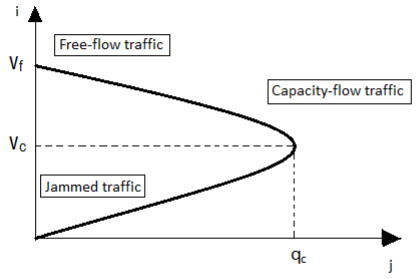


Figure 1.5: The fundamental v-q diagram according to Greenshield

measurements. These fundamental diagram are:

- Speed-density relationship:
$$\nu = \nu_f.(1 - \frac{\rho}{\rho_j})$$
- Speed-flow relationship:
$$q = \nu_f.\rho.(1 - \frac{\rho}{\rho_j})$$
- Flow-density relationship:
$$q = \rho_j.\nu.(1 - \frac{\nu}{\nu_f})$$

Fig 1.3, 1.4 and 1.5 give graphical overviews of the fundamental diagrams according to Greenshields[6]. Greenshield made the assumption that, under uninterrupted flow conditions, speed and density are linearly related. The following can be derived from Greenshields model (characterized by variables related to the traffic state):

- This is the mean speed that vehicles will travel on a roadway when traffic density of vehicles is very low. Under low-density conditions, drivers can obtain free flow speed without any under stress caused by other vehicles on the roadways. In other words, Free-flow traffic is characterized by a low density where speed is very high, which results in a free-flow *speed* ν_f. Mostly ν_f is the maximum allowed speed. When the density is zero, the flow is zero because there are no vehicles on the roadway.

- Capacity-flow traffic is characterized by a maximum flow which is called the capacity $flow$ q_c. As the density increases, the flow also increases to some maximum flow conditions.

- Jammed traffic is characterized by a maximum density where speed is very low called the jam *density* ρ_j. Extremely high densities can bring traffic on a roadway to a complete stop. When the density reaches a maximum, the flow must be zero because the vehicles tend to line up end to end (parking lot conditions).

Assume a road is homogeneous such that the car velocity is dependent only on the density of vehicles on road. Since the traffic flow is equals to density time velocity, so the flow only depends upon the density of vehicles along the road.

$$q = \rho\nu(\rho) \tag{1.8}$$

This traffic flow has some certain properties. Flow is the product of speed and density, then traffic flow may be zero in two ways:

1. if there is no traffic on the road that means $\rho = 0$.
2. if the velocity is zero it means no vehicle is moving that is the case of $\rho = \rho_{max}$.

So for all other values of density between 0 and ρ_{max} traffic flow is positive. The maximum traffic flow is called the capacity of the road . It is assumed that the flow-density relationship is concave downwards, $\frac{d^2q}{dq^2} < 0$. So as ρ increases $\frac{dq}{d\rho}$ decreases.

1.8 Earlier Important Work in Continuum Modeling Approach

In continuum model traffic flow is described in terms of average density, average velocity and average flow.

1.8.1 LWR Model

The research on traffic flow was started in $1950's$. To describe the dynamical properties on a homogeneous and unidirectional highway, Lighthill(1955), Whitham(1955) and Richards(1956) independently proposed a continuum model, which is known as LWR model. In LWR model, densities, speed values and flows were defined as continuous variable in each point in time and space (continuum, macroscopic model). The first order partial differential equation (PDE) from this model is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1.9}$$

Crucial to the approach of Lighthill, Whitham and Richards was the fundamental hypothesis, i.e. flow is a function of density and speed:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \nu)}{\partial x} = 0 \tag{1.10}$$

where $\rho(x,t)$ be the density and $\nu(x,t)$ the velocity of the traffic flow. Lighthill and Whitham assumed that the fundamental hypothesis holds at all traffic densities, not just for light-density traffic but also for congested traffic conditions. Using the fundamental diagrams to relate the two dependent variables in the left-hand side of (1.10) (density ρ and flow q) to one another, it is possible to solve the partial differential equation by given initial and boundary conditions.
Above equation is not consistent in self but needs an additional relation which is supplemented by the equation of traffic flow

$$q = \rho \nu \tag{1.11}$$

and the relation ship between the mean velocity and the traffic density under steady-state uniform flow.

$$\nu = \nu_e(\rho) \tag{1.12}$$

So, (1.10) becomes:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \nu_e(\rho))}{\partial x} = 0$$

Therefore we can write above equation in the form:

$$\rho_t + (\rho \nu_e(\rho))_x = 0, \tag{1.13}$$

Where $\nu_e(\rho)$ is the equilibrium velocity; x and t represent the space and time respectively. Using LWR modeling approach exhibits a wide range of phenomena such as shock formation and rarefaction waves.

1.8.2 Payne - Whitham Model

The simplest macro model is the LWR (Lighthill WithamRichards) model, which can be expressed by the following equation:

$$\rho_t + (\rho \nu_e(\rho))_x = 0, \tag{1.13}$$

where $\rho,\ \nu_e(\rho)$ are respectively the density and equilibrium speed. Equation (1.14) can reproduce the formation and propagation of shock wave, but it cannot be used to explore the non-equilibrium properties of traffic flow since the speed in the model cannot deviate from $\nu_e(\rho)$. To conquer this drawback, some density-gradient (DG) models and speed-gradient (SG) models were developed. The first non equilibrium DG model was given by Payne Whitham known as, PW model. Which was

$$\partial_t \rho + \partial_x(\rho \nu) = 0 \tag{1.15}$$

$$\partial_t + \nu \partial_x \nu + \frac{c^2(\rho)}{\rho} \partial_x \rho = \frac{V_*(\rho) - \nu}{\tau} \tag{1.16}$$

Where V_* is the equilibrium velocity, τ is the relaxation time and c is the sound speed. The relaxation time is the time taken by driver to adjust it's velocity due to front stimuli.

This model is able to explain the cluster formation.

1.8.3 Zhang Model

The wrong way problem was the major flaw in the PW model, that is why new higher model was developed by Zhang.

$$\partial_t \rho + \partial_x(\rho\nu) = 0 \tag{1.17}$$

$$\partial_t + \nu\partial_x\nu + 2\beta c(\rho)\partial_x\nu + \frac{c^2(\rho)}{\rho}\partial_x\rho = \frac{V_*(\rho) - \nu}{\tau} + \mu(\rho)\partial_{xx}\nu \tag{1.18}$$

where

$$\mu(\rho) = 2\beta\tau c^2(\rho) \tag{1.18}$$

$$c(\rho) = \rho V_*{}'(\rho) \tag{1.18}$$

μ is the viscosity coefficient and β is the dimensionless parameter. Different models are used for the equilibrium velocity.
Traffic flow is a complex phenomenon and it is modeled by hyperbolic equations. It is not easy to find its analytical solution, so numerical techniques are used to find the solution of complex differential equations. So the next chapter is based on the numerical methods of partial differential equations.

Chapter 2

Numerical Techniques

2.1 Differential equation

A differential equation is a mathematical equation that relates some function with its derivatives. In real life applications, the functions usually represent physical quantities, the derivatives represent their rate of change and the equation defines a relationship between the two. Differential equation play a prominent role in many disciplines including engineering, physics, economics, biology etc.

There are two types of differential equations:

1. Ordinary differential equation
2. Partial differential equation

2.1.1 Ordinary differential equation

Ordinary differential equation is a differential equation which contains derivatives of a dependent variable with respect to only single independent variable.

$$f(x, y, y', y", ...y^n) = 0$$

Above equation is n^{th} order differential equation. Where x is the independent variable and y is a dependent variable which is function of x. Here $y', y'', ...y^{(n)}$ are the first, second and n^{th} derivatives of y with respect to x respectively.
There are two types of ordinary differential equation.

Linear ordinary Differential equation:

A differential equation is said to be linear if the unknown function and all of its derivatives occurring only in the first degree and they are not multiplied with each other.

$$a_n(x)y^{(n)} + a_{(n-1)}(x)y^{(n-1)} + ... + a_1(x)y^{'} + a_0(x)y = Q(x).$$

If Q(x)=0, then it is said to be homogeneous.

Non- Linear ordinary Differential equation

A differential equations is said to be nonlinear if it is not a linear. Problems involving nonlinear differential equations are extremely diverse, and methods of solution or analysis are problem dependent.

$$f(x, y, y', y", ...y^n) = 0,$$

where the equation is non-linear in $y, y', y", ...y^n$.

2.1.2 Partial differential equation

A partial differential equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables. PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics. The general form of PDE:

$$f(x, y; u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}.....) \ = \ 0$$

Where f is some function x,y,... are independent variables and u(x,y,....) is called a dependent variable.

Classification of 1^{st} Order Partial Differential Equation

Partial Differential Equations can be classified as:

Linear PDE: This type of PDEs, the dependent variable and its derivatives enter the equation linearly, i.e, there is no product of the dependent variable or its derivatives.

$$a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y} = c(x,y)z + d(x,y)$$

where a, b and c are functions of independent variables or constant.

Semi linear PDE: An equation is called semilinear if it can be written in the form

$$a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y} = c(x,y,z)$$

It is linear in $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where a, b are functions of independent variables x and y or constant.

Quasi linear PDE: A first order partial differential equation is called quasi-linear if it can be written in the form

$$a(x,y,z)\frac{\partial z}{\partial x} + b(x,y,z)\frac{\partial z}{\partial y} = c(x,y,z)$$

where derivatives are linear with the coefficients a, b and c which are function of x, y and z. Linear and semi-linear equations are special cases of quasi-linear equations. But, a quasi linear PDE needs not be linear.

Non-linear PDE: A PDE which is not linear, is called non-linear PDEs. Two solutions to a nonlinear equation cannot be added to produce a third solution that also satisfies the original equation. For example

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z$$

Classification of 2^{nd} Order Partial Differential Equation

Consider the general second order Linear partial differential equation of form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \tag{2.1}$$

where

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}, \quad p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y}$$

where R, S and T are continuous functions of x and y and possessing partial derivatives defined in some domain D on the xy-plane.
Equation (2.1) is called

- Hyperbolic at a point (x,y) in D, if

  $$S^2 - 4RT > 0$$

  Then equation (2.1) has two distinct roots. Hence we get two distinct family of characteristics.

- Elliptic at a point (x,y) in D, if

  $$S^2 - 4RT < 0$$

  Then (2.1) has complex roots and we get two family of complex characteristics.

- Parabolic at a point (x,y) in D, if

  $$S^2 - 4RT = 0$$

  Then we get only one root and also get only one family of characteristic.

2.2 Initial value problem(IVP)

An **IVP** is an ordinary differential equation together with a specified value, called the initial condition, of the unknown function at a given point in the domain of the solution.

It is also known as the Cauchy problem. For example,

$$\frac{d^2y}{dx^2} + y = 0, \quad for\ 0 < x \leq 1$$

$$y(0) = y_0 \quad \left(\frac{dy}{dx}\right)_{x=0} = y_0$$

2.3 Boundary value problem(BVP)

A Boundary value problem is a differential equations together with a set of additional constraints, called the boundary conditions. For example,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0, \quad for\ a < x < b$$

$$y(a) = y_0, \quad y'(b) = y_0$$

Four types of Boundary Conditions are define as follows:

Dirichlet conditions

Dirichlet boundary condition specifies the value of the function on the boundary. For example,

$$\frac{d^2y}{dx^2} + cy = x^2, \qquad 0 < x < 1$$
$$y(0) = 0, \qquad y(1) = 1$$

Neumann condition

A boundary condition which specifies the value of the normal derivative of the function on a boundary. For example,

$$\frac{d^2y}{dx^2} + cy = x^2, \qquad 0 < x < 1$$
$$y'(0) = 0, \qquad y'(1) = \frac{4}{3}$$

Mixed conditions

In a mixed boundary value problem, the solution is required to satisfy a Dirichlet or a Neumann boundary condition in a mutually exclusive way on disjoint parts of the boundary. For example,

$$\frac{d^2y}{dx^2} + cy = x^2, \qquad 0 < x < 1$$
$$y(0) = 0, \qquad y'(1) = 0$$

Robin boundary conditions

It is a weighted combination of Dirichlet boundary conditions and Neumann boundary conditions. For example,

$$\frac{d^2y}{dx^2} = xy, \qquad 0 < x < 1$$
$$y(0) + y'(0) = 1, \qquad y(1) = 1$$

2.4 Methodology

There are two types of methods for solving differential equations:

2.4.1 Analytical methods

It is a analytical technique which use to find the exact solution of given differential equation. There are various analytical method which are following:

1. Variable separable method
2. Homogeneous differential equation
3. Exact differential equation
4. Linear equation with constant coefficient
5. Method of variation of parameters

6. Lagrange's method

7. Charpit's method

8. Method of characteristics

2.4.2 Numerical methods:

Many differential equations cannot be solved using symbolic computation ("analysis"). Numerical methods used to find numerical approximations to the solutions of differential equations. In real life problems, such as in engineering, physics, biology and chemistry etc, modeled equations are very complex and not easily solvable. So, it is worth to discuss the numerical techniques those may work well for these equations.
Some of the numerical methods are follows:

1. Finite Difference Method

2. Weighted Residual Methods

3. Finite Element Method

Finite difference schemes

The finite difference method is one of the several techniques for obtaining numerical solution to differential equations. In all numerical solutions the continuous partial differential equation (PDE) is replaced with a discrete approximation. In this context the word discrete means that the numerical solution is known only at a finite number of points in the physical domain. In general, increasing the number of points not only increases the resolution (i.e., detail), but also the accuracy of the numerical solution. The mesh is the set of locations where the discrete solution is computed. These points are called nodes. The finite difference solution of a boundary value problem is obtained by replacing the differential equation at each nodal point by using the Taylor's series expansion. Thus we get a system of equations in the unknown function values at the nodal points. Simplifying the system of equations, we get the solution values at the nodal point of mesh. Depending on the Taylor's series approximation, we get various difference schemes.

Finite difference formule for first order derivatives term:
Forward Difference:

$$y'(x_i) = \frac{y_{i+1} - y_i}{h} + O(h)$$

Backward Difference:

$$y'(x_i) = \frac{y_i - y_{i-1}}{h} + O(h)$$

Central Difference:

$$y'(x_i) = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2)$$

Finite difference formule for second order derivatives term:
Forward Difference:

$$y''(x_i) = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2} + O(h)$$

Backward Difference:

$$y''(x_i) = \frac{y_i - 2y_{i-1} + y_{i-2}}{h^2} + O(h)$$

Central Difference:

$$y''(x_i) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O(h^2)$$

Advantages of Finite Difference Method

- Simple to implement
- High-order is feasible
- Explicit in time
- Direction can be exploited - upwind
- Strong theory
- it gives very efficient and high quality results.

Disadvantages of Finite Difference Method

- Geometric flexibility are not agreeable
- It is more difficult to use for handling material discontinuities.

Weighted residual method

In general, a solution to a PDE can be described as a superposition of a base set of functions:

$$\bar{u} = \sum c_i \phi_i \tag{2.2}$$

Where the coefficients $c_i's$ are unknown parameters and are to be determined by a chosen method. The method tries to minimize the error, for instance, finite differences seek to minimize the error specifically at the chosen grid points. WRM's represent a particular group of methods where an integral error is minimized. This error is minimized in a certain way and so that defining the specific method.
Now consider a boundary value problem:

$$\mathbf{Lu=g} \tag{2.3}$$

Where L is differential operator, u is dependent variable which is exact solution and g is known function.
However, the numerical solution is approximate solution, i.e.,$\bar{u} \neq u$. Now substitute the approximate solution in (2.3) and it produce a residual i.e.,

$$Re = L\bar{u} - g \tag{2.4}$$

The goal of WRM's is to choose the coefficients $c_i's$ such that residual Re becomes minimize over a chosen domain. For this, we multiply the residual by weight function w and then integrate over the chosen domain,

$$\int_a^b w \ Re \ dx = 0 \tag{2.5}$$

If it satisfy for any w, then 'Re' approaches to zero and the numerical solution will approach the exact solution.
Now, there are various methods, which differ from each other in the way the integral is

minimized by choosing weight function to determine the parameters.

There are different types of methods:

1. **Petro-galerkin method**
2. **Galerkin method**
3. **Least square method**
4. **Collocation method**

Example: Consider the differential equation:

$$\frac{d^2y}{dx^2} + y = x^2, \quad 0 < x < 1 \tag{2.6}$$

$$y(0) = 0, \qquad y'(1) = 1$$

Solution: Now we assume the approximation solution is:

$$\bar{u} = \sum_{i=1}^{N} c_i\phi_i + \phi_0$$

where $c_i's$ are unknown coefficient to be determined and ϕ_0 is pre-selected function which satisfy initial and boundary conditions.
Let us consider, possible test or trial functions are
$\phi_0 = x, \;\; \phi_1 = -x(2-x), \quad \phi_2 = x^2(1-\frac{2}{3}x)$

$$\begin{aligned} Re &= -(\sum_{i=1}^{N} c_i\frac{d^2\phi}{dx^2} + 0) - (\phi_0 + \sum_{i=1}^{N} c_i\phi_i) + x^2 \\ &= -(c_1\frac{d^2\phi}{dx^2} + c_2\frac{d^2\phi}{dx^2} + 0) - (x - c_1\phi_1 + c_2\phi_2) + x^2 \\ &= (2 - 2x + x^2)c_1 + (-2 + 4x - x^2 + \left(\frac{2}{3}\right)x^2)c_2 - x + x^2 \end{aligned}$$

Now we consider various methods:

Petro Galerkin method:

Let the weight function be $w_1 = x, \quad w_2 = x^2$, we get,

$$\int_0^1 xRe\ dx = 0 \quad and \qquad \int_0^1 x^2 Re\ dx = 0$$

or

$$and \qquad \begin{aligned} \frac{7}{12}c_1 + \frac{13}{60}c_2 - \frac{1}{12} &= 0 \\ \frac{11}{30}c_1 + \frac{11}{45}c_2 - \frac{1}{20} &= 20 \end{aligned}$$

by solving for c, we get:

$$c_1 = \tfrac{103}{682}, \qquad c_2 = -\tfrac{15}{682}$$

so, solution is:

$$u_{PG} = 1.302053x - 0.173021x^2 - 0.04663x^3$$

The Galerkin method:

In this method, the weight functions are chosen to be identical to the trial functions. Here, we take weight function $w_i = \phi_i$

$\int_0^1 x(2-x)Re\ dx = 0, \qquad \int_0^1 x^2(1-\frac{2}{3})Re\ dx = 0$

or

$$and \qquad \begin{aligned} \frac{4}{5}c_1 + \frac{28}{45}c_2 - \frac{7}{60} &= 0 \\ \frac{17}{90}c_1 + \frac{29}{315}c_2 - \frac{1}{36} &= 0 \end{aligned}$$

by solving these equations for c,

$$c_1 = \tfrac{623}{4306}, \qquad c_2 = \tfrac{21}{4306}$$

Hence the solution is:

$$u_G = 1.2894x - 0.1398x^2 - 0.00325x^3$$

The least square method:

In this method, the weight function is the derivatives of the residual itself. Therefore, taking weight function $w_i = \frac{\partial Re}{\partial c_i}$, we get

$\int_0^1 (2 - 2x + x^2)Re\ dx = 0, \qquad \int (2 - 4x + x^2 - \frac{2}{3}x^3)Re\ dx = 0$

or

$$\frac{28}{15}c_1 - \frac{47}{90}c_2 - \frac{13}{60} = 0$$

$$and \qquad -\frac{47}{90}c_1 + \frac{253}{315}c_2 + \frac{1}{36} = 0$$

by solving these equations, we get

$c_1 = \frac{1292}{9935}, \qquad c_2 = \frac{991}{19870}$

Hence the solution is:

$$u_{L.S} = 1.2601x - 0.08017x^2 - 0.03325x^3$$

Collocation method:

In this method, the weight functions are selected to be Dirac delta functions such that the error is zero at the selected nodes x_m

$$w(x_m) = \delta(x - x_m)$$

In given domain, choosing the points $x = \frac{1}{3}\ and\ x = \frac{2}{3}$ as the collocation points, we evaluate the residual at these points and then set them equal to zero.

$Re(\frac{1}{3}) = 0 \qquad \Rightarrow 117c_1 - 61c_2 = 18$

$Re(\frac{2}{3}) = 0 \qquad \Rightarrow 90c_1 + 34c_2 = 18$

solving these equations, we get,

$c_1 = \frac{1710}{9468}, \qquad c_2 = \frac{486}{9468}$

Hence the solution is:

$$U_c = 1.3612x - 0.12927x^2 - 0.03422x^3$$

Finite Element Method

Finite element method (FEM) is a numerical method for finding approximate solution for differential equations. It is also defined as finite element analysis (FEA). FEM subdivides a large problem into smaller, simpler parts, which are called finite elements. The method essentially consists of considering the piecewise function for the solution. Then obtaining the parameters of the functions in a manner that reduces the error in

the solution. In most applications, the Galerkin formulation for the weighted residuals use in finite element method. The approximate functions are simple polynomials defined in local domain:

$$\bar{u} = \sum_{j=1}^{n} u_j \phi_j(x)$$

where for a suitable set of functions the local domain can be any shape. Since the finite element method is used with local coordinates. The domain can be subdivided uniformly to increase the resolution, where it is desired. The interpolating functions are called shape functions. A step by step procedure of the finite element method of the given problem is:

- Discretization of the domain
- Weak formulation over the element
- A set of algebraic equations

Example: Consider a second order linear differential equation for finding the finite element solution,

$$\frac{d^2u}{dx^2} - u + x = 0 \qquad\qquad 0 \leq x \leq 1.0$$

with boundary conditions;

$$u(0) = 1 \quad \textit{and} \quad \left(\frac{du}{dx}\right)\Big|_{x=1} = 1$$

solution: We take the nodes as $x = 0, 0.2, 0.5, 0.8, 1.0$.
Therefore, we have the following elements:

$$[0, 0.2], \quad [0.2, 0.5], \quad [0.5, 0.8], \quad [0.8, 1.0]$$

.
Now, at each node x_i, we define the basis functions $\phi_i(x)$ as

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in [x_i, x_{i+1}] \end{cases}$$

Let the finite element approximate solution be

$$\bar{u} = \sum_{j=1}^{5} \Phi_j(x) u_j. \tag{2.7}$$

Now, the weak formulation is given by

$$\int_0^1 \left(\frac{d^2\bar{u}}{dx^2} - \bar{u} + x \right) \Phi_i(x) dx = 0, \quad i = 1(1)5.$$

Integrating by parts and using $Q = -du/dx$, we get

$$\int_0^1 \frac{d\Phi_i}{dx} \frac{d\bar{u}}{dx} dx + \int_0^1 \Phi_i(x)\bar{u} dx = \int_0^1 x\Phi_i(x) dx - \Phi_i(x_5)Q_5 + \Phi_i(x_1)Q_1. \tag{2.8}$$

Substituting the value of $\bar{u}$ from (2.7), then equation (2.8) becomes

$$\sum_{j=1}^{5} u_j \int_0^1 \frac{d\Phi_i}{dx} \frac{d\Phi_j}{dx} dx + \sum_{j=1}^{5} u_j \int_0^1 \Phi_i(x)\Phi_j(x) dx$$
$$= \int_0^1 x\Phi_i(x) dx - [\Phi_i(x_5)Q_5 - \Phi_i(x_1)Q_1], i = 1(1)5.$$

Rewriting the above integral element wise, we get

$$\sum_{r=1}^{4} \sum_{j=1}^{5} u_j \int_{e_r} \frac{d\Phi_i}{dx} \frac{d\Phi_j}{dx} dx + \sum_{r=1}^{4} \sum_{j=1}^{5} u_j \int_{e_r} \Phi_i(x)\Phi_j(x) dx = R - T, \quad i = 1(1)5 \tag{2.9}$$

where

$$R = \sum_{r=1}^{4} \int_{e_r} x\Phi_i(x) dx \tag{2.10}$$

and

$$T = [\Phi_i(x_5)Q_5 - \Phi_i(x_1)Q_1]. \tag{2.11}$$

Writing the first term on the left hand side of equation (2.9) in matrix form, we get

$$AU = \sum_{r=1}^{4}\sum_{j=1}^{5} u_j \int_{e_r} \frac{d\Phi_i}{dx}\frac{d\Phi_j}{dx}dx, \quad i = 1(1)5$$

Evaluating the integral over an element e having nodes 1 and 2, we get

$$A^e = \begin{bmatrix} 1/L & -1/L \\ -1/L & 1/L \end{bmatrix} \tag{2.12}$$

where L is defined as distance between the nodes.
Consider the second term on the left hand side of equation (2.9), and denote it in matrix form as

$$BU = \sum_{r=1}^{4}\sum_{j=1}^{5} u_j \int_{e_r} \Phi_i(x)\Phi_j(x)dx, \quad i = 1(1)5$$

Again, evaluating the integral over element e, we get

$$B^e = \begin{bmatrix} L/3 & L/6 \\ L/6 & L/3 \end{bmatrix} \tag{2.13}$$

Adding (2.12) and (2.13), we get

$$P^e = \begin{bmatrix} 3 + L^2/3L & 6 - L^2/6L \\ -(6 - L^2)/6L & 3 + L^2/3L \end{bmatrix}$$

Substituting the values $L_1 = 0.2, L_2 = 0.3, L_3 = 0.3, L_4 = 0.2$, we get

$$P_1 = \begin{bmatrix} 15.2/3 & -14.9/3 \\ -14.5/3 & 15.2/3 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 15.2/3 & -14.9/3 \\ -14.5/3 & 15.2/3 \end{bmatrix}$$

Therefore, we have $P_3 = P_2$ and $P_4 = P_1$ since $L_3 = L_2$ and $L_4 = L_1$.
These values of P_1, P_2, P_3, P_4 will be inserted in appropriate rows and columns of the global matrix.

The $i-$th term of the column vector R of equation (2.10) is given by

$$R_i = \int_0^1 x\phi_i(x)dx, \quad i = 1(1)5.$$
$$R_i^e = \sum_{r=1}^{4} \int_{e_r} x\Phi_i(x)dx, \quad i = 1(1)5.$$

Evaluating the above integral over the element e having nodes 1 and 2.

$$R_1^e = \frac{1}{2}x_1L + \frac{L^2}{6}$$
$$R_2^e = \frac{1}{2}x_1L - \frac{L^2}{6}$$

Similarly, we can get the expression for R over other elements.

$$R = \begin{bmatrix} \frac{1}{2}x_1L_1 + \frac{L_1^2}{6} \\ \frac{1}{2}x_2L_1 + \frac{L_1^2}{6} + \frac{1}{2}x_2L_2 - \frac{L_2^2}{6} \\ \frac{1}{2}x_3L_2 + \frac{L_2^2}{6} + \frac{1}{2}x_3L_3 - \frac{L_3^2}{6} \\ \frac{1}{2}x_4L_3 + \frac{L_3^2}{6} + \frac{1}{2}x_4L_4 - \frac{L_4^2}{6} \\ \frac{1}{2}x_5L_4 + \frac{L_4^2}{6} \end{bmatrix}$$

$$R = \begin{bmatrix} 0.00667 \\ 0.05833 \\ 0.15 \\ 0.19167 \\ 0.09333 \end{bmatrix}$$

Given that, $Q_5 = -1$,
Using nodal values for $x_1 = 0, x_2 = 0.2, x_3 = 0.5, x_4 = 0.8, x_5 = 1.0$, in $T_i = [-\Phi_i(x_5)Q_5 + \Phi_i(x_1)Q_1]$, we get

$$T = \begin{bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Using the values of $P_1, P_2, P_3, P_4, P_5, R$ and T, the system (2.9) becomes

$$\begin{aligned}
5.0667u_1 - 4.9667u_2 &= 0.00667 - Q_1 \\
-4.9667u_1 + 8.5u_2 - 3.2833u_3 &= 0.05833 \\
-3.2833u_2 + 6.8667u_3 - 3.28337u_4 &= 0.15 \\
-3.2833u_3 + 8.5u_4 - 4.9669u_5 &= 0.191667 \\
-4.9667u_4 + 5.0667u_5 &= 1.0933.
\end{aligned}$$

Since Q_1 is not known, therefore we neglect first equation and consider the above remaining equations. Since $u_1 = 1$ is given, solving these linear equations, we get

$$\begin{aligned}
u_2 &= 1.0667 \quad & u_3 &= 1.2307 \\
u_4 &= 1.4610 \quad & u_5 &= 1.6480
\end{aligned}$$

Advantages of Finite Element Method

- High-order accuracy
- complex geometries

Disadvantages of Finite Element Method

- Not well suited for problems with direction

Chapter 3

Discontinuous Galerkin(DG) method

3.1 Introduction

Discontinuous Galerkin (DG) Methods are a extension of finite-element methods. These methods using totally discontinuous basis functions, which are usually taken as piecewise polynomials. Now, the basis functions can be completely discontinuous, these methods have the flexibility which is not used by typical finite-element methods, such as rather easy implementation with hanging nodes, complete freedom in changing the polynomial degrees in individual element independent of that in the neighbors (p-adaptivity), small size of elements can be taken (h-adaptivity). A good example to clarify the capacity of the discontinuous Galerkin method in h-p adaptivity, efficiency in parallel dynamic load balancing, and excellent resolution properties is the successful reproduction of the Rayleigh- Taylor flow instabilities.

The methods are popularly emerging for the numerical approximation the solutions of partial differential equations. The first discontinuous galerkin method was introduced in 1973 by Reed and Hill for solving a time independent hyperbolic equations of the steady state neutron transport, namely,

$$\sigma u + \bigtriangledown.(au) = f, \qquad in \;\; \Omega$$

The above linear problem has been studied by several authors. A major development of the DG method is carried out by Cockburn et al, called Local discontinuous method (LDG), in which they have established a framework to easily solve non-linear time-dependent problems, such as the Euler equations of gas dynamics, using explicit, non

linearity stable high-order Runge-Kutta time discretizations. The first error analysis of the DG method for this problem was carried out in 1974 by Lesaint and Raviart [11]. Later in 1986, Johnson and Pitkarnta [10] showed that the method convergences with order $k+\frac{1}{2}$ for general triangulations and polynomials of degree k. In 1991, Peterson [12] numerically confirmed this rate to be optimal. In 1988, Richter [13] obtained the optimal order of convergence of k+1 for some structured two-dimensional non-Cartesian grids and polynomials of degree k. In 1996, Lin et al. [14] showed first order convergence for the DG method using piecewise-constant approximations. A review of several techniques of analysis for finite element methods for hyperbolic problems including the DG method and the continuous Galerkin method can be found in the paper by Falk [15].

Some little literature is available concerned with macroscopic traffic flow modeling using the Galerkin finite element method (FEM). In [1], [2], and [3] a Galerkin FEM type is used to solve the macroscopic Lighthill Witham and Richards (LWR) [4], [5] traffic flow model in conjunction with Greenshields flow-density relationship[6]. A wavelet-Galerkin FEM is used in [7] to solve the macroscopic non-constant speed LWR traffic model. A discontinuous Galerkin FEM is used in [8] for solving red-and-green light models for the traffic flow.
In the present work a numerical solution of the LWR traffic flow model with constant speed. The validation of results is done by using the analytical method of characterstics [9].

3.2 Discontinuous Galerkin(DG) method for LWR Model

The Discontinuous Galerkin method was first designed as an effective numerical method for solving hyperbolic conservation laws, which may have discontinuous solutions.

Let one-dimensional conservation law from equation(1.8), we get

$$u_t \; + \; f(u)_x \; = \; 0, \qquad u(x,0) = u_0(x), \qquad \forall x \in (0,1) \tag{3.1}$$

Where $f(u) = \rho\nu_e(\rho)$ which represent the density. The main difficulty of a numerical solution to (3.1) is the occurrence of shocks even if the initial condition $u_0(x)$ is smooth. Discontinuous galerkin method must satisfy the following properties:
1. It is locally conservative.
2. It is high-order accurate in smooth regions of the solution.
3. It has sharp and monotone (non-oscillatory) shock transitions.

Let us consider the following mesh to cover the computational domain [0,1] consisting of cells $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, for $1 \leq i \leq N$, where the nodes are

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < .... < x_{N+\frac{1}{2}} = 1.$$

Let the length between two nodes is,

$$\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad 1 \leq i \leq N; \quad h = \max_{1 \leq i \leq N} \Delta x_i$$

Now consider the mesh is regular, namely there is a constant $c < 1$ do not dependent of h such that

$$\Delta x_i \geq ch, \qquad 1 \leq i \leq N$$

We define a finite-element space containing of piecewise polynomials

$$V_h^k = \{v : v \mid_{I_i} \in P^k(I_i); \quad 1 \leq i \leq N\} \tag{3.2}$$

where $P^k(I_i)$ represent the set of polynomials up to k degree defined on the cell I_i. The semi-discrete DG method for solving (3.1) is refered as follows: find the unique function $u_h = u_h(t) \in V_h^k$ such that, for all test functions $v_h \in V_h^k$ and for all $1 \leq i \leq N$.

3.2.1 Weak Formulation

Firstly multiply the equation with a test function v_h then integrate over a cell $I_i = [x_{i+\frac{1}{2}}, x_{i-\frac{1}{2}}]$, we get a following equation by using integrate by parts:

$$\int_{I_i} (u_h)_t v_h \, dx + \int_{I_i} f(u_h)_t v_h dx = 0$$

$$\int_{I_i} (u_h)_t v_h dx + \left| v_h f(u_h) \right|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} - \int_{I_i} v_h f(u_h) dx = 0$$

we have

$$\int_{I_i} (u_h)_t (v_h) dx - \int_{I_i} f(u_h)(v_h)_x dx + f(u_{i+\frac{1}{2}}) v_h(x_{i+\frac{1}{2}}) - f(u_{i-\frac{1}{2}}) v_h(x_{i-\frac{1}{2}}) = 0 \tag{3.3}$$

Values from inside I_i for the test function v_h:

$$v^-_{j+\frac{1}{2}},\ v^+_{j-\frac{1}{2}}$$

Hence the DG scheme is:

$$\int_{I_i}(u_h)_t(v_h)dx - \int_{I_i} f(u_h)(v_h)_x dx + \widehat{f}_{i+\frac{1}{2}}v_h^-(x_{i+\frac{1}{2}}) - \widehat{f}_{i-\frac{1}{2}}v_h^+(x_{i-\frac{1}{2}}) = 0 \tag{3.4}$$

Here, $\widehat{f}_{i+\frac{1}{2}}$ is a single-valued function defined at the cell interfaces which is again the numerical flux. In general, it depends on the values of the numerical solution u_h from both sides of the interface.

$$\widehat{f}_{i+\frac{1}{2}} = \widehat{f}(u_h(x^-_{i+\frac{1}{2}}, t), u_h(x^+_{i+\frac{1}{2}}, t))$$

We use the so-called monotone fluxes from finite-difference and finite-volume schemes for solving hyperbolic conservation laws, In order to obtain stable and convergent scheme numerical flux satisfy the following conditions:

- Consistency: $\widehat{f}(u, u) = f(u)$.
- Continuity: $\widehat{f}(u^-, u^+)$ is at least Lipschitz continuous with respect to both arguments u^- and u^+
- Monotonicity: $\widehat{f}(u^-, u^+)$ is a non-decreasing function of its first argument u^- and a non-increasing function of its other argument u^+. Symbolically, it is represent as $\widehat{f}(\uparrow, \downarrow)$.

There are different types of existing flux:

Lax-Friedrichs flux

$$\widehat{f}(u^-, u^+) = \frac{1}{2}(f(u^-) + f(u^+) - \alpha(u^+ - u^-)),$$

$$\alpha = \max_u \mid f^{'}(u) \mid$$

The Godunov flux

$$\widehat{f}(u^-,u^+) = \begin{cases} min_{u^-\le u\le u^+} f(u), & if\ u^- < u^+, \\ max_{u^+\le u\le u^-} f(u), & if\ u^- \ge u^+, \end{cases}$$

The Engquist-Osher flux

$$\widehat{f} = \int_0^{u^-} max(f'(u),0)du + \int_0^{u^+} min(f'(u),0)du + f(0).$$

This completes the definition of the DG space approximation.
Several comments about this DG space discretization are :

- The class of monotone schemes is one of the great achievements of the development of numerical schemes for non-linear scalar hyperbolic conservation laws. The stability and convergence properties of these schemes are corner stones for the construction of high-resolution finite volume and finite difference schemes. The same thing can be said about DG space discretizations which, as we have seen, try to capture those properties by incorporating their numerical fluxes.
- In the case of zero degree polynomial for k=0, set $v_h\Big|_{I_i}$=1, then the method is the same as the finite volume method,

$$\frac{d}{dt}\int_{I_i} u_h\ dx + \hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}} = 0$$

$$\frac{du_i}{dt} + \frac{1}{\Delta x}(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}) = 0$$

- In the linear case f(u)=au, all the above numerical fluxes coincide with the so-called upwind numerical flux, namely,

$$\widehat{f}(u^-,u^+) = \begin{cases} au^- & if\ \ a \ge 0 \\ au^+ & if\ \ a < 0 \end{cases}$$

3.2.2 Time Discretization

Here hyperbolic problem is dependent on spatial variables and time. Therefore, for spatial variables, we are used the discontinuous galerkin method and the time discretiztaion is achieved by explicit Runga-Kutta methods. This is approach of the so-called Runga-Kutta discontinuous galerkin method [19,18,22,21]

$$\frac{du}{dt} = L(u) \tag{3.5}$$

where L(u) is a discretization of the spatial operator, the third order Runge-Kutta is simply:

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}) \end{aligned}$$

If the PDEs involve high-order spatial derivatives with coefficients not very small, then explicit time marching methods such as the Runge-Kutta methods described above suffer from severe time-step restrictions. It is an important research subject to study efficient time discretization for such situations.

3.3 Cell Entropy Inequality and L^2 - Stability

Proposition 3.3.1. *: The solution u_h to the semi-discrete DG scheme (3.4) satisfies the following cell entropy inequality*

$$\frac{d}{dt}\int_{I_i} U(u_h)\, dx + \widehat{F}_{i+\frac{1}{2}} - \widehat{F}_{i-\frac{1}{2}} \leq 0 \tag{3.6}$$

for the square entropy $U(u) = \frac{u^2}{2}$, for some consistent entropy flux

$$\widehat{F}_{i+\frac{1}{2}} = \widehat{F}(u_h(x^-_{i+\frac{1}{2}}, t), u_h(x^+_{i+\frac{1}{2}}, t))$$

satisfying $\widehat{F}(u, u) = F(u)$

Proof: We introduce a short-hand notation

$$B_i(u_h; v_h) = \int_{I_i} (u_h)_t(v_h)\ dx - \int_{I_i} f(u_h)(v_h)_x\ dx + \widehat{f}_{i+\frac{1}{2}} v_h^-(x_{i+\frac{1}{2}}) - \widehat{f}_{i-\frac{1}{2}} v_h^+(x_{i-\frac{1}{2}}) \quad (3.7)$$

where $f_{j+i} = \widehat{f}(u_h(x^-_{i+\frac{1}{2}}, t), u_h(x^+_{i+\frac{1}{2}}, t))$ is a Lipschitz continuous monotone flux (i.e., f is nondecreasing in the first argument u^- and nonincreasing in the second argument u^+). If we take $v_h = u_h$ in the (3.4), we get

$$B_i(u_h; u_h) = \int_{I_i} (u_h)_t(u_h)\ dx - \int_{I_i} f(u_h)(u_h)_x\ dx + \widehat{f}_{i+\frac{1}{2}} u_h^-(x_{i+\frac{1}{2}}) - \widehat{f}_{i-\frac{1}{2}} u_h^+(x_{i-\frac{1}{2}}) = 0 \quad (3.8)$$

If we define $\widetilde{F} = \int^u f(u)du$, then (3.8) rewrite as

$$B_i(u_h, u_h) = \int_{I_i} U(u_h)_t\ dx - \widetilde{F}(u_h^-(x_{i+\frac{1}{2}})) + \widetilde{F}(u_h^+(x_{i-\frac{1}{2}})) + \widehat{f}_{i+\frac{1}{2}} u_h^-(x_{i+\frac{1}{2}}) - \widehat{f}_{i-\frac{1}{2}} u_h^+(x_{i-\frac{1}{2}}) = 0 \quad (3.9)$$

Now adding and subtract some terms, we get,

$$B_i(u_h, u_h) = \int_{I_i} U(u_h)_t\ dx - \widetilde{F}(u_h^-(x_{i+\frac{1}{2}})) + \widetilde{F}(u_h^+(x_{i-\frac{1}{2}})) + \widehat{f}_{i+\frac{1}{2}} u_h^-(x_{i+\frac{1}{2}}) - \widehat{f}_{i-\frac{1}{2}} u_h^+(x_{i-\frac{1}{2}}) + \widetilde{F}(u_h^-(x_{i-\frac{1}{2}})) + \widehat{f}_{i-\frac{1}{2}} u_h^-(x_{i-\frac{1}{2}}) - \widetilde{F}(u_h^-(x_{i-\frac{1}{2}})) - \widehat{f}_{i-\frac{1}{2}} u_h^-(x_{i-\frac{1}{2}}) = 0 \quad (3.3.1)$$

or

$$B_i(u_h, u_h) = \int_{I_i} U(u_h)_t\ dx + \widehat{F}_{i+\frac{1}{2}} - \widehat{F}_{i-\frac{1}{2}} + \Theta_{i-\frac{1}{2}} = 0 \quad (3.10)$$

where

$$\begin{aligned} \widehat{F}_{i+\frac{1}{2}} &= -\widetilde{F}(u_h^-(x_{i+\frac{1}{2}})) + \widehat{f}_{i+\frac{1}{2}} u_h^-(x_{i+\frac{1}{2}}), \\ \widehat{F}_{i-\frac{1}{2}} &= -\widetilde{F}(u_h^-(x_{i-\frac{1}{2}})) + \widehat{f}_{i-\frac{1}{2}} u_h^-(x_{i-\frac{1}{2}}), \end{aligned} \quad (3.11)$$

and

$$\Theta_{i-\frac{1}{2}} = -\widetilde{F}(u_h^-(x_{i-\frac{1}{2}})) + \widehat{f}_{i-\frac{1}{2}} u_h^-(x_{i-\frac{1}{2}}) + \widetilde{F}(u_h^+(x_{i-\frac{1}{2}})) - \widehat{f}_{i-\frac{1}{2}} u_h^+(x_{i-\frac{1}{2}}). \quad (3.12)$$

It is easy to simplify that the numerical entropy flux $\widetilde{F}$ defined by (3.11) is consistent with the entropy flux $F(u) = \int^u U^{'}(u) f^{'}(u)\, du$, for $U(u) = \frac{u^2}{2}$. It is also easy to simplify

$$\Theta = -\widetilde{F}(u_h^-) + \widehat{f} u_h^- + \widetilde{F}(u_h^+) - \widehat{f} u_h^+ = (u_h^+ - u_h^-)(\widetilde{F}^{'}(\xi) - \widehat{f}) \geq 0,$$

here we have dropped the subscript $i-\frac{1}{2}$, since there all quantities are evaluated in $\Theta_{i+\frac{1}{2}}$. We have used the mean value theorem and ξ is a value between u^- *and* u^+. We have used the fact $\widetilde{F}^{'}(\xi) = f(\xi)$ and the last inequality is due to the property of monotonicity of the flux function $\widehat{f}$. This complete the proof of the cell entropy inequality (3.6).

we note that the proof does not depend on the accuracy of the scheme, namely it holds for any k degree piecewise polynomial space (3.2). Also , the same entropy inequality can be given for the multi-dimensional DG scheme on any triangulation.

The cell entropy inequality trivially implies an L^2- stability of the approximate solution.

Proposition 3.3.2. *For periodic or compactly supported boundary conditions, the solution u_h to the semi-discrete DG scheme (3.4) satisfies the following L^2 -stability*

$$\frac{d}{dt} \int_0^1 (u_h)^2 \, dx \leq 0 \tag{3.13}$$

Proof: We introduce a short-hand notation

$$B_i(u_h; v_h) = \int_{I_i} (u_h)_t (v_h) \, dx - \int_{I_i} f(u_h)(v_h)_x \, dx + \widehat{f}_{i+\frac{1}{2}} v_h^-(x_{i+\frac{1}{2}}) - \widehat{f}_{i-\frac{1}{2}} v_h^+(x_{i-\frac{1}{2}}) \tag{3.14}$$

where $f_{j+i} = \widehat{f}(u_h(x_{i+\frac{1}{2}}^-, t), u_h(x_{i+\frac{1}{2}}^+, t))$ is a Lipschitz continuous monotone flux (i.e., f is nondecreasing in the first argument u^- and nonincreasing in the second argument u^+). If we take $v_h = u_h$ in the (3.4), we get

$$B_i(u_h; u_h) = \int_{I_i} (u_h)_t (u_h) \, dx - \int_{I_i} f(u_h)(u_h)_x \, dx + \widehat{f}_{i+\frac{1}{2}} u_h^-(x_{i+\frac{1}{2}}) - \widehat{f}_{i-\frac{1}{2}} u_h^+(x_{i-\frac{1}{2}}) = 0 \tag{3.15}$$

If we define

$$\widetilde{F} = \int^u f(u) du,$$

then (3.15) rewrite as

$$B_i(u_h, u_h) = \frac{d}{dt}\int_{I_i} (u_h)^2 \, dx - \widetilde{F}(u_h^-(x_{i+\frac{1}{2}})) + \widetilde{F}(u_h^+(x_{i-\frac{1}{2}})) + (\widehat{f}_{i+\frac{1}{2}} u_h^-(x_{i+\frac{1}{2}}) - \widehat{f}_{i-\frac{1}{2}} u_h^+(x_{i-\frac{1}{2}}) = 0 \quad (3.16)$$

or

$$B_i(u_h, u_h) = \frac{d}{dt}\int_{I_i} (u_h)^2 dx + \widehat{F}_{i+\frac{1}{2}} - \widehat{F}_{i-\frac{1}{2}} + \Theta_{i-\frac{1}{2}} = 0 \quad (3.17)$$

where

$$\begin{aligned} \widehat{F}_{i+\frac{1}{2}} &= -\widetilde{F}(u_h^-(x_{i+\frac{1}{2}})) + \widehat{f}_{i+\frac{1}{2}} u_h^-(x_{i+\frac{1}{2}}), \\ \widehat{F}_{i-\frac{1}{2}} &= -\widetilde{F}(u_h^-(x_{i-\frac{1}{2}})) + \widehat{f}_{i-\frac{1}{2}} u_h^-(x_{i-\frac{1}{2}}), \end{aligned} \quad (3.18)$$

and

$$\Theta_{i-\frac{1}{2}} = -\widetilde{F}(u_h^-(x_{i-\frac{1}{2}})) + \widehat{f}_{i-\frac{1}{2}} u_h^-(x_{i-\frac{1}{2}}) + \widetilde{F}(u_h^+(x_{i-\frac{1}{2}})) - \widehat{f}_{i-\frac{1}{2}} u_h^+(x_{i-\frac{1}{2}}). \quad (3.19)$$

It is easy to simplify that the numerical entropy flux $\widetilde{F}$ defined by (3.18) is consistent with the entropy flux $F(u) = \int^u U'(u) f'(u)\, du$, for $U(u) = \frac{u^2}{2}$. It is also easy to calculuate

$$\Theta = -\widetilde{F}(u_h^-) + \widehat{f} u_h^- + \widetilde{F}(u_h^+) - \widehat{f} u_h^+ = (u_h^+ - u_h^-)(\widetilde{F}'(\xi) - \widehat{f}) \geq 0,$$

Therefore, by (3.17), we get the result,

$$\frac{d}{dt}\int_{I_i} (u_h)^2 \, dx + \widehat{F}_{i+\frac{1}{2}} - \widehat{F}_{i-\frac{1}{2}} \leq 0 \quad (3.20)$$

Now numerical flux $\widehat{F}_{i+\frac{1}{2}}$ *and* $\widehat{F}_{i-\frac{1}{2}} \geq 0$ because it is non-decreasing functions on argument u^-. Therefore we get,

$$\frac{d}{dt}\int_0^1 (u_h)^2 \, dx \leq 0$$

3.4 Error Estimates for Smooth Solutions

Now consider the exact solution of (3.4) is smooth, we can obtain optimal L^2 - error estimates. Such error estimates can be obtained for the general nonlinear conservation

law (3.1) and for fully discretized RKDG methods, see [17]. However, for simplicity we will give the proof here only for the semi-discrete DG scheme and the linear version of (3.1):

$$u_t + u_x = 0 \tag{3.21}$$

for which the monotone flux is taken as the simple upwind flux $\widehat{f}(u^-, u^+) = u^-$. Clearly the proof is the same for $u_t + au_x = 0$ with any constant a.

Proposition 3.4.1. *The solution u_h of the DG scheme (3.4) for the PDE (3.21) with a smooth solution u satisfies the error estimate*

$$\|u - u_h\| \leq Ch^{k+1} \tag{3.22}$$

where C depends on u and its derivatives but it is independent of h.

Proof: The DG scheme (3.4) can be written as, when we using the notation in (3.7),

$$B_i(u_h; v_h) = 0 \tag{3.23}$$

for all $v_h \in V_h$ and for all i. It is easy to verify that the exact solution of the (3.21) also satisfies

$$B_i(u; v_h) = 0 \tag{3.24}$$

for all $v_h \in V_h$ and for all i. Subtracting (3.24) from (3.25) and using the linearity B_i with respect to its first argument u^-, we get the error equation

$$B_i(u - u_h; v_h) = 0 \tag{3.25}$$

for all $v_h \in V_h$ and for all i.

Now we define a special projection P into V_h. For a given smooth function w, the projection Pw is the unique function in v_h which satisfies, for each i

$$\int_{I_i} (Pw(x) - w(x))v_h(x)dx = 0 \qquad \forall v_h \in P^{k-1}(I_i); \qquad Pw(x^-_{i+\frac{1}{2}}) = w(x_{i+\frac{1}{2}}). \tag{3.26}$$

Standard approximation theory [16] implies,

$$\|Pw(x) - w(x)\| \leq Ch^{k+1} \tag{3.27}$$

where here and below C is a generic constant depending on w and its derivatives but it

is independent of h (which may not have the same value in different places). In (3.26), in particular $C = \widetilde{C}\|w\|_H^{k+1}$ where $\widetilde{C}$ is a constant which is independent of w and $\|w\|_H^{k+1}$ is the standard Sobolev (k+1) norm.

Now we take:

$$v_h = Pu - u_h \tag{3.28}$$

then (3.24) becomes and it is denote as

$$\varepsilon_h = u - Pu, \qquad e_h = Pu - u_h \tag{3.29}$$

We get

$$B_i(e_h + \varepsilon_h, e_h) = 0$$

$$B_i(e_h; e_h) + B_i(\varepsilon_h; e_h) = 0$$

$$B_i(e_h; e_h) = -B_i(\varepsilon_h; e_h) \tag{3.30}$$

now by using the cell entropy inequality (3.6) for the left side of (3.30), to obtain

$$B_i(e_h; e_h) = \frac{1}{2}\frac{d}{dt}\int_{I_i}(e_h)^2 dx + \widehat{F}_{i+\frac{1}{2}} - \widehat{F}_{i-\frac{1}{2}} + \Theta_{i-\frac{1}{2}}, \tag{3.31}$$

where $\Theta_{i-\frac{1}{2}} \geq 0$. Firstly we write all the terms of the right -hand side of (3.30)
$-B_i(\varepsilon_h; e_h) = -\int_{I_i}(\varepsilon_h)_t e_h dx + \int_{I_i}\varepsilon_h(e_h)_x dx - (\varepsilon_h)^-_{i+\frac{1}{2}}(e_h)^-_{i+\frac{1}{2}} + (\varepsilon_h)^-_{i-\frac{1}{2}}(e_h)^+_{i+\frac{1}{2}}$
Noticing the properties (3.25) of the Projection P,

$$\int_{I_i}\varepsilon_h(e_h)_x dx = 0$$

because $(e_h)_x$ is of at most (k-1) degree polynomial,and for all i

$$(\varepsilon_h)^-_{i+\frac{1}{2}} = u_{i+\frac{1}{2}} - (Pu)^-_{i+\frac{1}{2}} = 0, \quad \forall i$$

Therefore, we get the right-side of (3.30)

$$-B_i(\varepsilon_h; e_h) = -\int_{I_i}(\varepsilon_h)_t e_h dx \leq \frac{1}{2}\Big(\int_{I_i}((\varepsilon_h)_t)^2 dx + \int_{I_i}(e_h)^2 dx\Big). \tag{3.33}$$

Plugging (3.31) and (3.32) into the equality (3.30), take summing up over i, by using

the approximation solution (3.27), we get

$$\frac{d}{dt}\int_0^1 (e_h)^2 dx \leq \int_0^1 (e_h)^2 dx + Ch^{2k+2}.$$

A Gronwall's inequality, the fact that the initial error

$$\|u(.,0) - u_h(.,0)\| \leq Ch^{k+1}$$

(usually the initial condition $u_h(.,0)$ is taken as the L^2- projection of the analytical initial u(.,0), and finally the approximation solution give us the error estimate (3.22).

3.5 Numerical Results

Example: Consider the transport equation

$$\rho_t + (\rho\nu_e(\rho))_x = 0 \qquad in \;\; (0,x)\times(0,T)$$

where ρ represent the density and $\nu_e(\rho)$ is constant velocity or

$$u_t + u_x = 0 \qquad 0 < x < 1$$

where $u = \rho$ and take $\nu_e(\rho)$=1.
with initial conditions,

$$u(x,0) = u_0(x)$$

and periodic boundary conditions, with the DG approximation obtained with polynomial approximations of degree k=0,1,2. To march in time, a Runge-Kutta method of order k +1 was used.
Solution: Firstly, to discretize the transport equation in space by using a DG scheme. Then we seek a discontinuous approximation solution u_h, which are chosen as piecewise polynomial. There is no restriction in how to chose the space $v^k(p)$, although in space a typical choice of the polynomial of degree at most k. To solve these type of problem, we take a orthogonal polynomials. By taking k=0, 1, 2 and get following cases,
Case 1: Let the degree of polynomial is zero i.e., k=0, then by taking approximation solution is:

$$u(x,t) = u_1$$

By using this approximation solution in given equation, by solving for k=0, we get it in the form of:

$$AU = B$$

where

$$A = \begin{bmatrix} h \end{bmatrix},\ U = \begin{bmatrix} \dot{u_1} \end{bmatrix}\ and\ B = \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} \end{bmatrix} \begin{bmatrix} u_{j-1}^1 \\ u_j^1 \end{bmatrix}$$

Case 2: Let the degree of polynomial is one i.e., k=1, then taking approximation solution is:

$$u(x,t) = u_1 + \frac{(x - x_j)}{h} u_2$$

by using these approximation solution in given equation, by evaluating, we get,

$$AU = B$$

where,

$$A = \begin{bmatrix} h & 0 \\ 0 & \frac{h}{12} \end{bmatrix}, \qquad U = \begin{bmatrix} \dot{u_1} \\ \dot{u_2} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} & \frac{1}{2h} & -\frac{1}{h} \\ -\frac{6}{h} & -\frac{6}{h} & -\frac{3}{h} & -\frac{3}{h} \\ \frac{30}{h} & -\frac{30}{h} & \frac{15}{h} & \frac{15}{h} \end{bmatrix} \begin{bmatrix} u_{j-1}^1 \\ u_j^1 \\ u_{j-1}^2 \\ u_j^2 \end{bmatrix}$$

Case 3: Let the degree of polynomial is two i.e., k=2, let approximation solution is:

$$u(x,t) = u_1 + \frac{(x - x_j)}{h} u_2 + \Big(\Big(\frac{x - x_j}{h} \Big)^2 - \frac{1}{12} \Big) u_3$$

by using these approximation solution in given equation, by evaluating, we get,

$$AU = B$$

where,

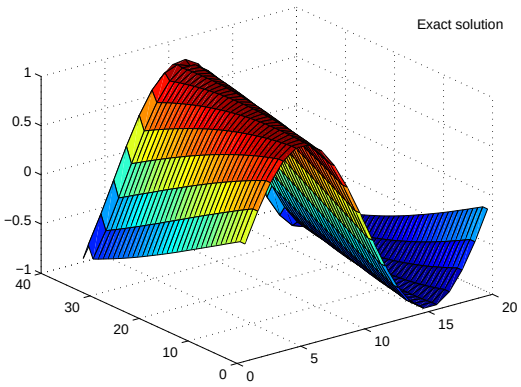
$$A = \begin{bmatrix} h & 0 & 0 \\ 0 & \frac{h}{12} & 0 \\ 0 & 0 & \frac{h}{180} \end{bmatrix}, \qquad U = \begin{bmatrix} ui_1 \\ ui_2 \\ ui_3 \end{bmatrix},$$

and

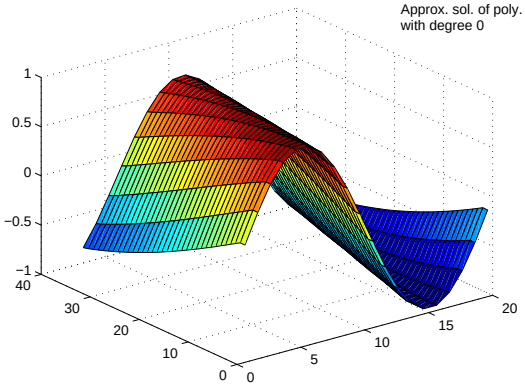
$$B = \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} & \frac{1}{2h} & -\frac{1}{h} & \frac{1}{6h} & -\frac{1}{6h} \\ -\frac{6}{h} & -\frac{6}{h} & -\frac{3}{h} & -\frac{3}{h} & -\frac{1}{h} & -\frac{1}{h} \\ \frac{30}{h} & -\frac{30}{h} & \frac{15}{h} & \frac{15}{h} & \frac{15}{h} & -\frac{15}{h} \end{bmatrix} \begin{bmatrix} u_{j-1}^1 \\ u_j^1 \\ u_{j-1}^2 \\ u_j^2 \\ u_{j-1}^3 \\ u_j^3 \end{bmatrix}$$

3.5.1 Example 1:

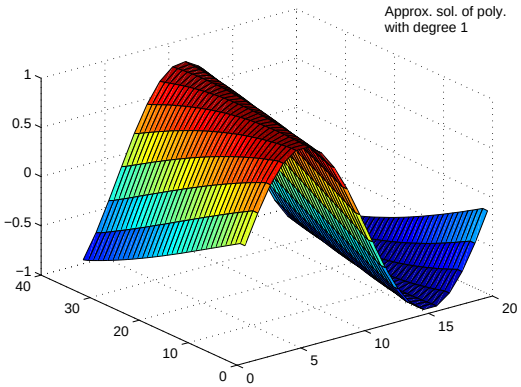
When initial condition is given by: $u(x,0) = sin(x), \qquad 0 < x < 2\pi$



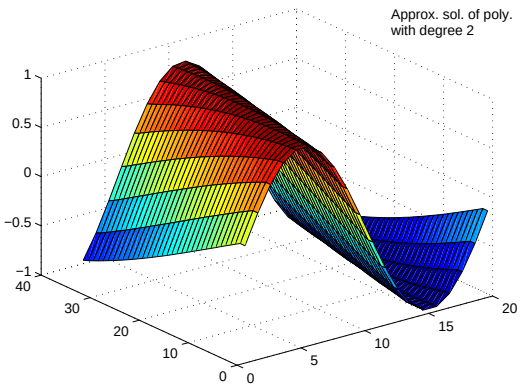
(a) Exact solution



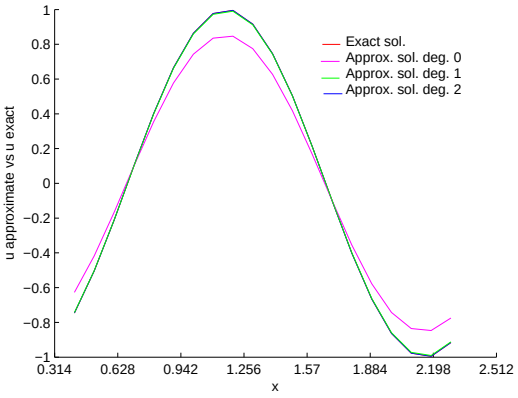
(b) Approx sol. for 0 degree poly.



(c) Approx sol. for 1 degree poly.



(d) Approx sol. for 2 degree poly.

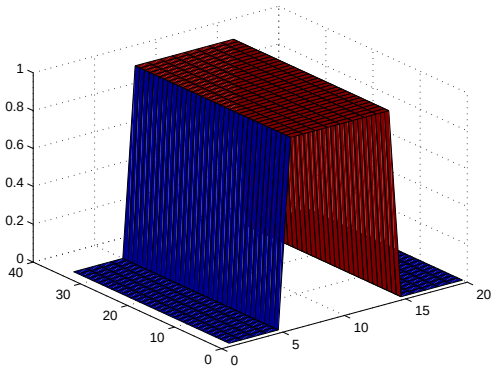


(e) Approx sol vs exact sol

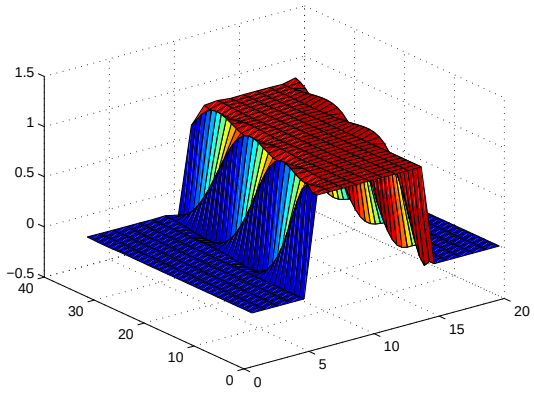
3.5.2 Example 2:

When initial condition is given by:

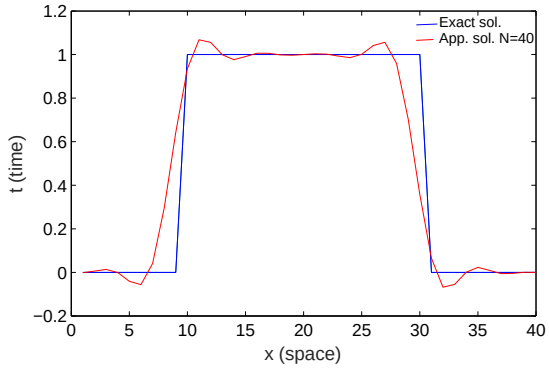
$$u(x,0) = \begin{cases} 1, & if \quad x \in (\frac{\pi}{2}, \frac{3\pi}{2}) \\ 0, & otherwise \end{cases} \qquad 0 < x < 2\pi$$



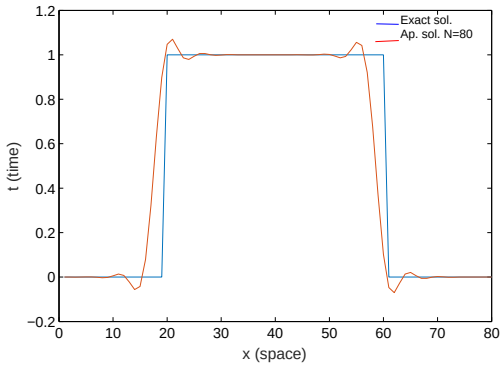
(a) Exact solution



(b) Approx. sol. for 2 degree polynomial.



(c) Approx. solution vs exact solution for N=40



(d) Approx. solution vs exact sol. for N=80

Conclusion

Discontinuous Galerkin(DG) method can be used to solve the first-order macroscopic LWR traffic flow model with constant speed. The results of the Discontinuous Galerkin method analysis are compared with that of the analytical method of characteristics. Approximate solution of polynomial of degree 0, 1 and 2 are taken into consideration and found that polynomial of 2 degree as approximate solution is much and more closure to exact results.

Future thesis will concentrate on the application of the Discontinuous Galerkin method to the LWR model will non-constant speed.

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