

FIXED POINT THEOREMS FOR DIFFERENT CONTRACTION MAPPINGS

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In

Mathematics and Computing

Submitted by

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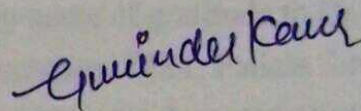


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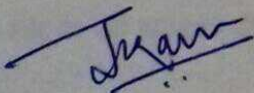
This is to certify that the thesis entitled, "**FIXED POINT THEOREMS FOR DIFFERENT CONTRACTION MAPPINGS** " in partial fulfillment of the requirements for the award of the degree of the Masters of Science in Mathematics and Computing to the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala is a record of my work studied under the supervision of Dr. Jatinderdeep Kaur. The matter embedded in this thesis has not been submitted by me for the award of any other degree of this or any other university/institute.



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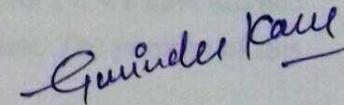
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ABSTRACT

The present dissertation entitled, "**FIXED POINT THEOREMS FOR DIFFERENT CONTRACTION MAPPINGS**", contains a study about Fixed point theory by me on existence of fixed points of self mappings in metric space under the supervision of **Dr. Jatinderdeep Kaur**, Associate Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

The aim of this work is to study and obtain some result on existence and uniqueness of fixed points. Fixed point theory has been revealed as a very powerful and important tool in the study of non-linear analysis. Various problems in physics, chemistry, biology, economics *etc.* can be solved by making use of fixed point theorems.

The work presented in this dissertation has been divided into four chapters. Chapter I is introduction which includes brief account of definitions and results which will be required for the later chapters. In Chapter II, we have studied Banach fixed point theorem which guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces. Also, we present some fixed point theorems in compact metric space.

The purpose of the Chapter III is to study fixed point theorems for generalized contraction mappings on a S -orbitally complete metric space and studied the existence and uniqueness of fixed points. In the Chapter IV, we have studied some fixed point theorems under quasi contraction condition. The purpose of this chapter is to extend the result presented in chapter III.

At the end of the present dissertation, we have added bibliography.

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List of symbols

- \mathbb{R} - Set of real numbers
- N - Set of natural numbers
- C - Set of complex numbers
- $\mathbb{R}^+ = [0, \infty)$
- \in - Belongs to
- \subset - Subset
- \forall - For all
- \implies - Implies
- $[a, b]$ - Closed interval
- (a, b) - Open interval
- R^n - Euclidean n-space

Chapter 1

Introduction

1.1 Origin of Fixed Point Theory

A characteristic approach to prove that an equation has a solution is to convert it as a fixed point problem, that is $g(t) = t$, here g is a function such that t is a solution of the equation if and only if $g(t) = t$. Then, we can show that g has a fixed point by using fixed point theorems. For example, let us take the simple quadratic equation $t^2 - 6t + 5 = 0$. Here $t = 1$ and $t = 5$ are the roots of this equation. Now, this equation can be written as:

$$t = \frac{t^2 + 5}{6}$$

Consider $g(t) = \frac{t^2 + 5}{6}$, we find that $t = 1$ and $t = 5$ are the two fixed points of g . So, in the end it can be concluded that the problem of finding the solution of an equation $G(t) = 0$ is the same as finding the fixed point of a function $g(t)$, where $g(t) = G(t) + t$.

The first fixed point theorem in topological fixed point theory was established by Brouwer[3] in 1912 which states that *a continuous mapping S of the closed unit ball in \mathbb{R}^n has at least one fixed point, that is a point t such that $S(t)=t$* . The Brouwer fixed point theorem forms the basis for several general fixed

point theorems and is an important fixed point theorem that applies to finite dimensional spaces.

In the present dissertation we will concentrate on metric fixed point theory under different contraction mappings. In the present chapter some basic concepts along with applicable results which will be often used in our forthcoming discussion have been studied.

1.2 Metric Fixed Point Theory

One of the most important and useful result in metric fixed point theory was established by Polish mathematician Stefan Banach in 1922, mainly known as Banach contraction principle. This principle states that a contraction mapping of a complete metric space into itself has a unique fixed point. It ensures the existence and uniqueness of fixed point under appropriate conditions. Till today, Banach contraction principle have been generalized and extended in various ways.

1.2.1 Basic Concepts and Definitions

We present a brief account of basic definitions, which will be used in the subsequent chapters for the sake of convenience.

Lipschitzian mapping.

Let (U, d) be a metric space. A mapping $S : U \rightarrow U$ is called a Lipschitzian mapping if there exists a constant $a \geq 0$ with

$$d(S(t_1), S(t_2)) \leq a d(t_1, t_2) \quad \forall t_1, t_2 \in U$$

It should be noted that a Lipschitzian map is necessarily continuous. Here a is called Lipschitz constant for S and is denoted by L . If $L < 1$, then S is said to be a contraction mapping whereas if $L = 1$, then S is said to be non-expansive mapping.

Metric space ([6], P.9).

Let U be non-empty set and $d : U \times U \rightarrow \mathbb{R}$ (set of reals) is said to be metric if d satisfies the following properties:

- (i) $d(t_1, t_2) \geq 0 \quad \forall t_1, t_2 \in U;$
- (ii) $d(t_1, t_2) = 0 \iff t_1 = t_2 \quad \forall t_1, t_2 \in U;$
- (iii) $d(t_1, t_2) = d(t_2, t_1) \quad \forall t_1, t_2 \in U; \quad (\text{symmetry})$
- (iv) $d(t_1, t_2) = d(t_1, t_3) + d(t_3, t_2) \quad \forall t_1, t_2, t_3 \in U; \quad (\text{triangle inequality})$

The ordered pair (U, d) is called a metric space.

Example. Let $U = \mathbb{R}$. For $t_1, t_2 \in U$, define

$$d(t_1, t_2) = |t_1 - t_2|$$

Then (U, d) is a metric space. This is called usual metric space and is denoted by \mathbb{R}_u .

Example. Let U be an arbitrary non-empty set. For $t_1, t_2 \in U$, define d by

$$d(t_1, t_2) = \begin{cases} 0 & : t_1 = t_2 \\ 1 & : t_1 \neq t_2 \end{cases}$$

Then (U, d) is a metric space. This is called discrete metric space and is denoted by U_d .

Convergent Sequence.

A sequence $\{t_n\}$ is said to be converge to the limit t if for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|t_n - t| < \varepsilon \quad \text{for } n > N$$

i.e.

$$\lim_{n \rightarrow \infty} t_n = t$$

Example. Let $t_n = \frac{1}{n}$ be a sequence. Let $\varepsilon > 0$, take $n \in \mathbb{N}$ such that the inequal-

ity $|t_n| = \frac{1}{n} < \varepsilon$ is true $\forall n > \frac{1}{\varepsilon}$ and hence $\forall n > N$ where N is any natural number such that $N > \frac{1}{\varepsilon}$.
 Then for any $\varepsilon > 0$, there is natural number N such that $|t_n| < \varepsilon$ for every $n \geq N$.

Cauchy Sequence.

Let (U, d) be a metric space and a sequence $\{t_n\}$ is said to be Cauchy sequence, if for every positive real number $\varepsilon > 0$, there is a positive integer N such that for all positive $m, n > N$ the distance $d(t_m, t_n) < \varepsilon$.

Example. $U = (0, 1]$ be the metric space with usual metric and $t_n = \frac{1}{n}$ be a sequence in U .

For each $\varepsilon > 0$ we have,

$$d(t_m, t_n) = \left| \frac{1}{m} - \frac{1}{n} \right| \leq \varepsilon \quad \forall m, n > \frac{1}{\varepsilon}$$

$$d(t_m, t_n) \rightarrow 0$$

therefore, $\{t_n\}$ is a Cauchy sequence.

Completeness.

A metric space (U, d) in which every Cauchy sequence converges to an element of U is called complete metric space.

Example. The real numbers are complete under the usual metric.

Contraction Mapping ([6], P.128).

Let (U, d) be a metric space. A mapping $S : U \rightarrow U$ is called a contraction on U if there exists a real number a with $0 \leq a < 1$ such that

$$d(S t_1, S t_2) \leq a d(t_1, t_2), \quad t_1 \neq t_2;$$

the number a is known as Lipschitz constant of S .

Example. Let $U = \mathbb{R}^+$ and mapping $S : U \rightarrow U$ defined by $S(t) = t^2$, $0 \leq t \leq \frac{1}{3}$

then, S is contraction mapping on $[0, \frac{1}{3}]$ with usual metric d . Let

$$\begin{aligned} d(St_1, St_2) &= d(t_1^2, t_2^2) = |t_1^2 - t_2^2| \\ &= |t_1 - t_2| |t_1 + t_2| \\ &\leq \frac{2}{3} |t_1 - t_2| \\ &\leq \frac{2}{3} d(t_1, t_2) \end{aligned}$$

Thus S is a contraction mapping.

Generalized contraction ([4], P.20).

Let $S : U \rightarrow U$ be a mapping and is said to be λ - *generalized contraction* if and only if for all $t_1, t_2 \in U$, there exists non-negative numbers a_1, a_2, a_3 and a_4 which may depend on both t_1 and t_2 , such that $\sup\{a_1 + a_2 + a_3 + 2a_4 ; t_1, t_2 \in U\} < 1$ and

$$d(St_1, St_2) \leq a_1.d(t_1, t_2) + a_2.d(t_1, St_1) + a_3.d(t_2, St_2) + a_4.[d(t_1, St_2) + d(t_2, St_1)].$$

Example. Let $U = [0, 10] \subset \mathbb{R}$ and mapping S is defined as $St = \frac{3t}{4}$ for all $t \in U$.

For example, $t_1 = 0, t_2 = 8$, then for $a_1 = \frac{3}{4}, a_2 = a_3 = a_4 = \frac{1}{20}$, S satisfies the generalized contraction condition.

Quasi-contraction ([5], P.268).

Let U be a metric space and $S : U \rightarrow U$ be a mapping, we say that S is a quasi contraction if and only if there exists a number $a, 0 \leq a < 1$ such that

$$d(St_1, St_2) \leq a \max\{d(t_1, t_2); d(t_1, St_1); d(t_2, St_2); d(t_1, St_2); d(t_2, St_1)\}$$

holds for every $t_1, t_2 \in U$

Example. (U, d) be a metric space *w.r.t.* usual metric space where $U = [0, \infty)$ and $S : [0, \infty) \rightarrow [0, \infty)$ defined by $S(t) = \frac{t}{2}$ is a quasi-contraction.

S-orbitally complete ([5], P.268).

Let $S : U \rightarrow U$; for $A \subset U$, let $\delta(A) = \sup\{d(t_1, t_2); t_1, t_2 \in A\}$ and for each $t \in U$, let

$$\begin{aligned}O(t, n) &= \{t, St, \dots, S^n t\} \quad n = 1, 2, 3, \dots \\O(t, \infty) &= \{t, St, \dots\}\end{aligned}$$

A space U is said to be S -orbitally complete iff every Cauchy sequence which is contained in $O(t, \infty)$ for some $t \in U$ converges in U .

Now, a brief chapter wise resume of the results contain in this dissertation has been given. The present dissertation consist of four chapters. Each chapter is divided into various sections.

In Chapter II, we have studied the existence and uniqueness of Banach fixed point theorem with some basic concepts.

The aim of the Chapter III is to study the fixed point theorem using complete metric space under the generalized contraction mapping.

In Chapter IV, we have studied some fixed point theorems under quasi contraction mapping.

Chapter 2

Banach fixed point theorem for contraction mapping

2.1 Introduction

Metric fixed point Theory has played a key role in the development of non-linear functional analysis. In 1922, Polish Mathematician Banach established a result which is considered as the base for the rest of Fixed Point Theory. He introduced the contraction mapping as follows.

Metric space ([6], P.9)

Let U be non-empty set and $d : U \times U \rightarrow \mathbb{R}$ (set of reals) is said to be metric if d satisfies the following properties:

- (i) $d(t_1, t_2) \geq 0 \quad \forall t_1, t_2 \in U;$
- (ii) $d(t_1, t_2) = 0 \iff t_1 = t_2 \quad \forall t_1, t_2 \in U;$
- (iii) $d(t_1, t_2) = d(t_2, t_1) \quad \forall t_1, t_2 \in U; \quad (\text{symmetry})$
- (iv) $d(t_1, t_2) \leq d(t_1, t_3) + d(t_3, t_2) \quad \forall t_1, t_2, t_3 \in U; \quad (\text{triangle inequality})$

The function d is also called distance function or simply distance. The ordered pair (U, d) is called a metric space.

Example. The most important examples of metric spaces are the euclidean space \mathbb{R}_u . Especially \mathbb{R}_1 (the real line) and \mathbb{R}_2 (the complex plane); the distance in \mathbb{R}_u is defined by $d(t_1, t_2) = |t_1 - t_2|$ ($t_1, t_2 \in \mathbb{R}_u$). Then d is a metric on \mathbb{R} and \mathbb{R}_u is known as usual metric space.

Complete metric space ([6], P.55).

A metric space (U, d) is said to be complete if every Cauchy sequence in U is convergent. In other words, (U, d) is a complete metric space if, whenever the sequence $\{t_n\}$ in U is such that $d(t_m, t_n) \rightarrow 0$ as $m, n \rightarrow \infty$, there exists an $t \in U$ with $d(t_n, t) \rightarrow 0$ as $n \rightarrow \infty$.

Fixed Point ([6], P.127)

Let (U, d) be a metric space and $S : U \rightarrow U$ be a mapping. The point $t \in U$ is called a fixed point of S if t is mapped onto itself; *i.e.* $St = t$.

Examples

1. A translation has no fixed point. $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(t) = t + 1$ then, it has no fixed point.
2. A rotation of the plan has the single fixed point. Indeed, the centre of rotation is the only fixed point.
 $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(t) = 2t$
3. $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(t) = t^2 - 3t + 4$, then 2 is fixed point of g .
4. The mapping $g : U \rightarrow U^2$ of \mathbb{R} into itself has two fixed points. Indeed, the points 0 and 1 are fixed points.
5. The projection $g : (t_1, t_2) \rightarrow t$ of \mathbb{R}^2 onto the t_1 -axis has infinitely many fixed points. In fact all points of the t_1 -axis are fixed points.

2.2 Contraction Mapping

Let (U, d) be a metric space. A mapping $S : U \rightarrow U$ is called a contraction on U if there exists a real number a with $0 \leq a < 1$ such that

$$d(S t_1, S t_2) \leq a d(t_1, t_2), \quad t_1 \neq t_2;$$

the number a is known as Lipschitz constant of S .

Example. Let $U = \mathbb{R} - \{0\}$ and mapping $S : U \rightarrow U$ defined by $S(t) = \frac{t}{2}$

$$\begin{aligned} d(S t_1, S t_2) &= \left| \frac{t_1}{2} - \frac{t_2}{2} \right| \\ &= \frac{1}{2} |t_1 - t_2| \\ &= \frac{1}{2} d(t_1, t_2) \end{aligned}$$

Thus S is a contraction mapping.

Theorem 2.2.1. A contraction mapping is uniformly continuous.

Proof. Let U be a metric space and $S : U \rightarrow U$ be a contraction mapping in a metric space with metric d . Thus for some $a \in [0, 1)$ we have for all $t_1, t_2 \in U$,

$$d(S t_1, S t_2) \leq a d(t_1, t_2)$$

we have to prove that S is uniformly continuous, let $\varepsilon > 0$ be given. There are two cases

(i) $a = 0$, our claim is trivial. Since then for all $t_1, t_2 \in U$

$$d(S t_1, S t_2) = 0 < \varepsilon$$

(ii) $a \in (0, 1)$, then for all $t_1, t_2 \in U$ with

$$d(t_1, t_2) < \frac{\varepsilon}{a}, \text{ we have}$$
$$d(St_1, St_2) \leq a d(t_1, t_2) < \varepsilon.$$

Therefore S is uniformly continuous.

□

2.3 Main Results

2.3.1 Banach's Fixed Point Theorem

Let (U, d) be a complete metric space and let $S : U \rightarrow U$ be a contraction on U . Then, S has a unique fixed point in U .

Proof. We will prove this theorem in two parts. In the first part we prove the existence of fixed point and in the second part we prove the uniqueness of fixed point.

Existence of fixed point Since S is a contraction therefore, let $a \in [0, 1[$ be the Lipschitz constant such that

$$d(St_1, St_2) \leq a d(t_1, t_2), \quad \forall t_1, t_2 \in U$$

(i) We construct a sequence $\{t_n\} \subset U$ as follows: Let $t_0 \in U$ be arbitrary and define a sequence inductively $\{t_n\}$ of points in U as:

$$t_1 = St_0$$
$$t_2 = St_1 = S^2t_0$$

$$\begin{aligned}
t_3 &= St_2 = S^3t_o \\
&\dots\dots\dots \\
&\dots\dots\dots \\
t_n &= St_{n-1} = S^nt_o
\end{aligned}$$

Clearly, $\{t_n\}$ is the sequence of images of t_o under repeated application of S .

(ii) In this step we will prove $\{t_n\}$ is a Cauchy sequence in U . Let $m, n \in N$ such that $m < n$. Then

$$\begin{aligned}
d(t_m, t_n) &= d(S^mt_o, S^nt_o) \\
&\leq a d(S^{m-1}t_o, S^{n-1}t_o) \\
&\dots\dots\dots \\
&\dots\dots\dots \\
&\leq a^m d(t_o, S^{n-m}t_o) \\
&\leq a^m [d(t_o, St_o) + d(St_o, S^2t_o) + \dots + d(S^{n-m-1}t_o, S^{n-m}t_o)] \\
&\quad \text{(By triangle inequality)} \\
&\leq a^m [d(t_o, St_o) + ad(t_o, St_o) + \dots + a^{n-m-1}d(t_o, St_o)] \\
&= a^m [1 + a + a^2 + \dots + a^{n-m-1}] d(t_o, St_o) \\
&\leq \frac{a^m}{1-a} d(t_o, St_o), (0 \leq a < 1) \\
&\longrightarrow 0
\end{aligned}$$

as $m, n \rightarrow \infty$, thus $d(t_m, t_n) < \epsilon$. Hence, $\{t_n\}$ is a Cauchy sequence.

(iii) In this step, we will show that t is a fixed point of S . Since U is complete, therefore the Cauchy sequence $\{t_n\}$ is convergent in U , let $\{t_n\}$ converges to t (where $t \in U$)

$$i.e. \lim_{n \rightarrow \infty} t_n = t$$

We have

$$\begin{aligned}
 d(t, St) &\leq d(t, t_n) + d(t_n, St) && \text{(by triangle inequality)} \\
 &\leq d(t, t_n) + a d(t_{n-1}, t) && \text{(S is a contraction)} \\
 &\longrightarrow 0, \text{ as } n \longrightarrow \infty \\
 \implies d(t, St) = 0 &\implies t = St
 \end{aligned}$$

Hence, t is a fixed point of S . Thus the existence of a fixed point is established.

Uniqueness. Let, if possible t_1 and t_2 be two fixed points of S in U .

Then $St_1 = t_1$ and $St_2 = t_2$ for some $t_1, t_2 \in U$

Consider,

$$\begin{aligned}
 d(t_1, t_2) &= d(St_1, St_2) \\
 &\leq a d(t_1, t_2)
 \end{aligned}$$

Since $0 \leq a < 1$. Therefore, $d(t_1, t_2)$ must be equal to zero which further implies that $t_1 = t_2$.

This proves the uniqueness of fixed point in U . □

Example. Let us consider the usual metric space \mathbb{R}_u . A mapping $g : \mathbb{R} \rightarrow \mathbb{R}$ is defined as $g(t) = \frac{t}{2}$. Here the metric space is complete and g is a contraction on \mathbb{R} and g has a fixed point 0 which is unique.

Remarks:

1. If U is not complete in above theorem, then S may not have a fixed point.

Example Consider $U =]0, \frac{1}{2}]$ and the mapping $S : U \rightarrow U$ is defined by $St = t^2$. Clearly U is not complete with the usual metric and also we see that S does not have any fixed point.

2. If S is not contraction in above theorem, then it may not have a fixed point.

Example Consider the complete metric space $U = [0, \infty[$ equipped with the metric of absolute value and consider the mapping $S : U \rightarrow U$ given by $St = \frac{1}{1+t^2}$.

Consider,

$$\begin{aligned} d(St_1, St_2) &= \left| \frac{1}{1+t_1^2} - \frac{1}{1+t_2^2} \right| \\ &= \left| \frac{t_2^2 - t_1^2}{(1+t_1^2)(1+t_2^2)} \right| \\ &< |t_1 - t_2| \end{aligned}$$

and hence S is a contractive map, while S is not a contraction and S has no fixed point.

2.3.2 Fixed point theorem in Compact metric space

Theorem 2.3.1. Let (U, d) be a compact metric space and $S : U \rightarrow U$ a contractive map (not necessarily contraction). Then, S has a unique fixed point in U .

Proof. Existence Define $g : U \rightarrow \mathbb{R}$ by $g(t) = d(t, St)$

We first establish that g is continuous. Let $\varepsilon > 0$ be given then,

$$\begin{aligned} |g(t_1) - g(t_2)| &= |d(t_1, St_1) - d(t_2, St_2)| \\ &\leq |d(t_1, St_1) - d(St_1, t_2)| + |d(St_1, t_2) - d(t_2, St_2)| \\ &\quad (\text{by triangle inequality}) \end{aligned}$$

Now, let (U, d) be a metric space then, for all $t_1, t_2, t_3 \in U$

$$\begin{aligned} d(t_1, t_3) &\leq d(t_1, t_2) + d(t_2, t_3) \\ d(t_3, t_2) &\leq d(t_1, t_2) + d(t_1, t_3) \\ d(t_1, t_3) - d(t_2, t_3) &\leq d(t_1, t_2) \end{aligned}$$

also, $-(-d(t_3, t_2) + d(t_1, t_3)) \leq d(t_1, t_2)$

So, $|d(t_1, t_3) - d(t_3, t_2)| \leq d(t_1, t_2)$

Therefore, $|g(t_1) - g(t_2)| \leq d(t_1, t_2) + d(St_1, St_2)$

As S is contractive. Therefore,

$$\begin{aligned} |g(t_1) - g(t_2)| &\leq 2 d(t_1, t_2) \\ &< \varepsilon, \text{ whenever } d(x, y) < \delta = \frac{\varepsilon}{2} \end{aligned}$$

Now, by using the result g is continuous and U is compact then $g(U)$ is compact as the continuous image of the compact space is compact.

Since g is continuous and U is compact, there exists $t_1 \in U$ such that $g(t_1) \leq g(t_2)$, for all $t_2 \in U$; *i.e.*, g attains its minimum at t_1 . We now, show that t_1 is a fixed point of S .

Let, if possible t_1 be not a fixed point of S . Then $St_1 \neq t_1$. Since $g(t_1) \leq g(t_2)$, for all $t_2 \in U$, taking $t_2 = St_1$, we have

$$\begin{aligned} g(t_1) &\leq g(St_1) \\ \implies d(t_1, St_1) &\leq d(St_1, S(St_1)) \\ &\leq d(t_1, St_1) \quad (\because S \text{ is contractive and } t_1 \neq St_1) \end{aligned}$$

which is a contradiction. Hence t_1 is fixed point of S .

Uniqueness. Let, if possible t_1 and t_2 be two fixed points of S in U . Then $St_1 = t_1$ and $St_2 = t_2$. Now, note that

$$\begin{aligned} d(t_1, t_2) &= d(St_1, St_2) \leq a d(t_1, t_2) \\ \implies d(t_1, t_2) &= 0 \\ \implies t_1 &= t_2. \end{aligned}$$

□

Example. Let us consider a compact metric space $[0,1]$ and a contractive map $g : [0, 1] \rightarrow [0, 1]$ which is defined as $g(t) = \frac{t}{3}$, it has a fixed point 0 which is unique.

The proof of the above theorem does not give an explicit procedure for

approximating the fixed point in the same way as being achieved in Banach fixed point theorem. In the next theorem the successive iterates St_o, S^2t_o, \dots of any point $t_o \in U$, can be converges to the unique fixed point t in U has been shown.

Theorem 2.3.2. Let (U, d) be a compact metric space, S a contractive map of U into itself. Then, for any $t_o \in U$, the successive iterates $St_o, S^2t_o, \dots, S^n t_o, \dots$ converge to the unique fixed point of S .

Proof. Let t_1 be the unique fixed point of S . Then,

$$\begin{aligned} d(S^{n+1}t_o, t_1) &= d(S(S^n t_o), St_1) \\ &\leq d(S^n t_o, t_1) \quad (\because S \text{ is contractive}) \end{aligned}$$

This is true for all $n \in \mathbb{N}$. Thus, $\{d(S^{n+1}t_o, t_1)\}$ is a decreasing sequence of non-negative real numbers and so converges to its infimum. Suppose

$$\lim_{n \rightarrow \infty} d(S^n t_o, t_1) = \lambda$$

Now, $\{S^n t_o\}$ being a sequence of points of a compact metric space U , there exists subsequence $(S^{n_k} t_o)_{k=1}^{\infty}$ which converges to same point $t_2 \in U$. Therefore,

$$d(t_2, t_1) = \lim_{k \rightarrow \infty} d(S^{n_k} t_o, t_1) = \lambda$$

If $\lambda \neq 0$, then $t_2 \neq t_1$ and so

$$\begin{aligned} \lambda &= d(t_2, t_1) > d(St_2, St_1) \\ &= d(St_2, t_1) \quad (\because St_1 = t_1) \\ &= \lim_{k \rightarrow \infty} d(S(S^{n_k} t_o), t_1) \\ &= \lim_{k \rightarrow \infty} d(S^{n_k+1} t_o, t_1) \\ &= \lambda \end{aligned}$$

This is a contradiction. Thus $\lambda = 0$. This verifies that

$$\lim_{n \rightarrow \infty} S^n t_0 = t_1$$

□

Theorem 2.3.3. Let (U, d) be a complete metric space and $S : U \rightarrow U$ be a mapping. If S^m is a contraction on U for some positive integer m , then S has a unique fixed point.

Proof. Write $K = S^m$. Under the hypothesis K is a contraction. By Theorem 2.3.1, the mapping K has a unique fixed point, say \tilde{t} . Thus

$$\begin{aligned} K\tilde{t} &= \tilde{t} \\ \implies K^n \tilde{t} &= \tilde{t} \end{aligned}$$

Again in view of Banach's fixed point theorem, we have

$$\lim_{n \rightarrow \infty} K^n t = \tilde{t}, \text{ for all } t \in U$$

In particular, taking $t = S\tilde{t}$, we get

$$\begin{aligned} \tilde{t} &= \lim_{n \rightarrow \infty} K^n t \\ &= \lim_{n \rightarrow \infty} K^n S\tilde{t} \\ &= \lim_{n \rightarrow \infty} S K^n \tilde{t} \\ &= \lim_{n \rightarrow \infty} S\tilde{t} \\ &= S\tilde{t} \end{aligned}$$

This shows that \tilde{t} is a fixed point of S . Also, since every fixed point of S is a fixed point of K , it follows that S cannot have more than one fixed point.3 □

Chapter 3

Fixed point theorems for generalized contraction

3.1 Introduction

In this chapter, we have defined generalized contraction and proved the fixed point theorem for generalized contraction mapping in S -orbitally complete metric space.

Let (U, d) be a metric space and $S : U \rightarrow U$ be a mapping. Mapping S is said to be *contraction* if there exists a number a , $0 \leq a < 1$, such that the condition

$$d(St_1, St_2) \leq a \cdot d(t_1, t_2) \tag{3.1}$$

holds for every $t_1, t_2 \in U$.

Kannan's mapping can be defined as, if U is a complete metric space and mapping $S : U \rightarrow U$ is such that the condition

$$d(St_1, St_2) \leq a' (d(t_1, St_1) + d(t_2, St_2)) ; \quad 0 < a' < \frac{1}{2} \tag{3.2}$$

holds for every $t_1, t_2 \in U$, then S leaves exactly one point of U fixed.

Remark: The condition (3.1) and (3.2) are independent.

Example 1: Let $U = [0, 1]$, $S(t) = \frac{t}{4}$ for $t \in [0, \frac{1}{2})$ and $S(t) = \frac{t}{5}$ for $t \in [\frac{1}{2}, 1]$ and d is the usual metric. Here S is discontinuous as $\frac{1}{2}$; consequently condition (3.1) is not satisfied but condition (3.2) is satisfied by taking $a' = \frac{4}{9}$.

Example 2: Let $U = [0, 1]$, $S(t) = \frac{t}{3}$ for $t \in [0, 1]$ and d is usual metric. Here condition (3.1) is satisfied but condition (3.2) is not satisfied if we take $t_1 = \frac{1}{3}$, $t_2 = 0$.

Generalized contraction([4], P.20).

Let $S : U \rightarrow U$ be a mapping and is said to be λ - *generalized contraction* if and only if for all $t_1, t_2 \in U$, there exists non-negative numbers a_1, a_2, a_3 and a_4 which may depend on both t_1 and t_2 , such that $\sup\{a_1 + a_2 + a_3 + 2a_4 ; t_1, t_2 \in U\} < 1$ and

$$d(St_1, St_2) \leq a_1.d(t_1, t_2) + a_2.d(t_1, St_1) + a_3.d(t_2, St_2) + a_4.[d(t_1, St_2) + d(t_2, St_1)]. \quad (3.3)$$

Remark 1: If S satisfies the condition (3.1), then S also satisfies the condition (3.3) for every $a_2(t_1, t_2), a_3(t_1, t_2), a_4(t_1, t_2)$ such that $0 \leq a_2(t_1, t_2) + a_3(t_1, t_2) + 2a_4(t_1, t_2) < 1 - a_1 - c$ ($0 < c \leq 1 - a_1$).

Remark 2: If S satisfies (3.2), then S also satisfies (3.3) for every $a_1(t_1, t_2)$ such that $0 \leq a_1(t_1, t_2) < 1 - 2a' - c$ ($0 < c \leq -2a'$).

Example: Let $U = [0, 10] \subset \mathbb{R}^1$ and let $S(t) = \frac{3t}{4}$ for all $t \in U$. If we take, $t_1 = 0$ and $t_2 = 8$, then S does not satisfy (3.2) for $a' < 3$. But S satisfies (3.3) on whole U with $a_1 = \frac{3}{4}$, $a_2(t_1, t_2) = a_3(t_1, t_2) = a_4(t_1, t_2) = \frac{1}{20}$.

S-orbitally complete ([5], P.268).

Let $S : U \rightarrow U$; for $A \subset U$, let $\delta(A) = \sup\{d(t_1, t_2); t_1, t_2 \in A\}$ and for each $t \in U$, let

$$O(t, n) = \{t, St, \dots, S^n t\}; \quad n = 1, 2, 3, \dots$$

$$O(t, \infty) = \{t, St, \dots\}$$

A space U is said to be S -orbitally complete iff every Cauchy sequence which is contained in $O(t, \infty)$ for some $t \in U$ converges in U .

Orbitally continuous ([4], P.21).

Let S be a mapping from $S : U \rightarrow U$ and S is said to be orbitally continuous if $u \in U$ and such that

$$u = \lim_{i \rightarrow \infty} S^{n_i} t$$

for some $t \in U$, then

$$Su = \lim_{i \rightarrow \infty} SS^{n_i} t.$$

3.2 Main Results

Theorem 3.2.1. Let S be a λ -generalized contraction of S -orbitally complete metric space U into itself. Then

- (i) There is in U a unique fixed-point u under S ,
- (ii) $S^n t \rightarrow u$ for every $t \in U$
- (iii) $d(S^n t, u) \leq \frac{\lambda^n}{1 - \lambda} \cdot d(t, St)$

Proof. Suppose $t \in U$ be an arbitrary point. Define a sequence.

$$t_0 = t, t_1 = St_0, \dots, t_n = St_{n-1} = S^n t_0, \dots \quad (3.4)$$

Since S is λ -generalized contraction, using (3.3) and definition of the sequence (3.4) it follows

$$\begin{aligned} d(t_n, t_{n+1}) &= d(St_{n-1}, St_n) \leq a_1(t_{n-1}, t_n) d(t_{n-1}, t_n) + a_2(t_{n-1}, t_n) d(t_{n-1}, St_{n-1}) \\ &\quad + a_3(t_{n-1}, t_n) d(t_n, St_n) + a_4(t_{n-1}, t_n) (d(t_{n-1}, St_n) + d(t_n, St_{n-1})) \\ &= a_1(t_{n-1}, t_n) d(t_{n-1}, t_n) + a_2(t_{n-1}, t_n) d(t_{n-1}, t_n) + a_3(t_{n-1}, t_n) \\ &\quad d(t_n, t_{n-1}) + a_4(t_{n-1}, t_n) d(t_{n-1}, t_{n+1}). \end{aligned}$$

Applying the triangle inequality we get

$$d(t_n, t_{n+1}) \leq (a_1(t_{n-1}, t_n) + a_2(t_{n-1}, t_n)) d(t_{n-1}, t_n) + a_3(t_{n-1}, t_n) d(t_n, t_{n+1}) + a_4(t_{n-1}, t_n) (d(t_{n-1}, t_n) + d(t_n, t_{n+1})).$$

Therefore

$$d(t_n, t_{n+1}) \leq \frac{a_1(t_{n-1}, t_n) + a_2(t_{n-1}, t_n) + a_4(t_{n-1}, t_n)}{1 - a_3(t_{n-1}, t_n) - a_4(t_{n-1}, t_n)} d(t_{n-1}, t_n) \quad (3.5)$$

As $\lambda < 1$, from

$$a_1(t_1, t_2) + a_2(t_1, t_2) + a_4(t_1, t_2) + \lambda a_3(t_1, t_2) + \lambda a_4(t_1, t_2) \leq \lambda$$

we get,

$$\frac{a_1(t_1, t_2) + a_2(t_1, t_2) + a_4(t_1, t_2)}{1 - a_3(t_1, t_2) - a_4(t_1, t_2)} \leq \lambda$$

satisfied for all $t_1, t_2 \in U$. Hence from (3.5) we get,

$$d(t_n, t_{n+1}) \leq d(t_{n-1}, t_n) \quad (3.6)$$

this implies that a generalized contraction is a contraction for certain pair of points. Repeating this process n -times, we attain

$$d(t_n, t_{n+1}) \leq \lambda d(t_{n-1}, t_n) \leq \dots \leq \lambda d(t, St).$$

Thus for any positive integer j , we have

$$d(t_n, t_{n+j}) \leq \sum_{i=1}^j d(t_{n+i-1}, t_{n+i}) \leq \sum_{i=1}^j \lambda^{n+i-1} d(t, St).$$

Hence

$$d(t_n, t_{n+j}) \leq \frac{\lambda^n}{1 - \lambda} d(t, St). \quad (3.7)$$

So, from (3.7) we conclude that sequence (3.4) is Cauchy because $\lambda < 1$, $\lambda^n \rightarrow 0$ when $n \rightarrow \infty$. As we know S is S -orbitally complete, there is a point $u \in U$ such

that

$$u = \lim_{n \rightarrow \infty} S^n t = \lim_{n \rightarrow \infty} t_n \quad (3.8)$$

Next we have to show that

$$Su = \lim_{n \rightarrow \infty} St_n = \lim_{n \rightarrow \infty} t_{n+1} = u \quad (3.9)$$

i.e. that u is a fixed point of S . Since S is the generalized contraction, by definition of generalized contraction and apply the triangle inequality we have,

$$\begin{aligned} d(Su, St_n) &\leq a_1(u, t_n)d(u, t_n) + a_2(u, t_n)d(u, Su) + a_3(u, t_n)d(t_n, St_n) + \\ &\quad a_4(u, t_n)[d(u, St_n) + d(Su, t_n)] \\ &\leq a_1(u, t_n)d(u, t_n) + a_2(u, t_n)(d(u, t_{n+1}) + d(t_{n+1}, Su)) + \\ &\quad a_3(u, t_n)d(t_n, t_{n+1}) + a_4(u, t_n)[d(u, St_n) + d(Su, t_n)] \\ &\leq \lambda d(u, t_n) + (a_2(u, t_n) + t(u, t_n))d(u, t_{n+1}) + a_2(u, t_n)d(St_n, Su) + \\ &\quad a_3(u, t_n)d(t_n, t_{n+1}) + a_4(u, t_n)[d(Su, St_n) + d(St_n, t_n)] \\ &\leq \lambda d(u, t_n) + \lambda d(u, t_{n+1}) + \\ &\quad [a_2(u, t_n) + a_4(u, t_n)] d(Su, St_n) + \lambda d(t_n, t_{n+1}) \\ &\leq \lambda (d(u, t_n) + d(u, t_{n+1}) + d(t_n, t_{n+1})) + \lambda d(Su, St_n). \end{aligned}$$

Therefore

$$d(Su, St_n) \leq \frac{\lambda}{1 - \lambda} (d(u, t_n) + d(u, t_{n+1}) + d(t_n, t_{n+1})). \quad (3.10)$$

Thus (3.9) follows from (3.8) and (3.10). Hence we show that S has at least one fixed point u in U .

Uniqueness: Let v be any other fixed point of S such that $Sv = v$, then by condition of generalized contraction we have

$$\begin{aligned} d(u, v) &= d(Su, Sv) \leq a(u, v).d(u, v) \\ &\implies (1 - a(u, v)).d(u, v) \leq 0 \\ &\implies d(u, v) = 0 \text{ i.e. } u = v \end{aligned}$$

Hence, we proved (i).

As t is arbitrary, (3.8) implies (ii). Similarly, when $j \rightarrow \infty$ (iii) follows from (3.7). □

Example. Let $U = [0, 10]$ and let $St = \frac{3t}{4}$ for all $t \in U$.

S is generalized contraction for $t_1 = 0, t_2 = 8$ with $a_1 = \frac{3}{4}, a_2 = a_3 = a_4 = \frac{1}{20}$ and S is S -orbitally complete. Firstly,

$$St = t \implies \frac{3t}{4} = t \implies t = 0.$$

0 is the only fixed point of S .

Now, let us suppose $St_o = t_1, S^2t_o = t_2$

As

$$\begin{aligned} t_2 &= St_1 = S^2t_o \\ t_3 &= St_2 = S^3t_o \\ t_1 &= St_o = \frac{3t_o}{4} \\ t_2 &= St_1 = S\left(\frac{3t_o}{4}\right) = \left(\frac{3}{4}\right)^2 t_o \\ t_3 &= St_2 = S\left[\left(\frac{3}{4}\right)^2 t_o\right] = \left(\frac{3}{4}\right)^3 t_o \\ &\dots\dots\dots \\ &\dots\dots\dots \\ t_n &= St_{n-1} = \left(\frac{3}{4}\right)^n t_o. \end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^n t_o \rightarrow 0$. Hence (ii) proved.

$$d(S^n t, u) \leq \frac{\lambda^n}{1 - \lambda} d(t, St); \text{ for } \lambda = \frac{1}{2}.$$

Thus, the result is true for this example.

Theorem 3.2.2. Let U be a metric space and S be a mapping from U to U . If there exists a positive integer l , such that $S^l = S S^{l-1}$ is λ -generalized contraction and U is S -orbitally complete, then
(i*) S has a unique fixed point $u \in U$ under S ,

(ii*) $S^n t \rightarrow u$ for every $t \in U$;

(iii*) $d(S^n t, u) \leq \lambda'^n \cdot \mu(t, St)$

where $\lambda' = \lambda^{\frac{1}{l}}$ and $\mu(t, St) = \max\{\lambda^{-1} d(S^r t, S^{r+l} t) : r = 0, 1, \dots, l-1\}$.

Proof. It can be easily see that from above Theorem 3.2.1 (i*) and (ii*) follows.

We need to show only (iii*). As S^l is a generalized contraction then for any positive integer n ; $n = ml + r$ with $m = \left[\frac{n}{l} \right]$ and $0 \leq r < l$.

Now, from (iii) we have

$$\begin{aligned} d(S^n t, u) &= d(S^{ml} S^r t, u) \leq \lambda^m d(S^l S^r t) \\ &= (\lambda^{\frac{1}{l}})^{ml+r-l} \cdot d(S^r t, S^{r+l} t) = (\lambda^{\frac{1}{l}})^n d(S^r t, S^{r+l} t). \end{aligned}$$

Therefore,

$$d(S^n t, u) \leq (\lambda^{\frac{1}{l}})^n \cdot \max\{\lambda^{-1} d(S^r t, S^{r+l} t) : r = 0, 1, \dots, l-1\}.$$

□

Chapter 4

Fixed point theorems for quasi-contractions

4.1 Introduction

In 1974, Ćirić [4] proved the fixed point theorem for quasi contractions in S orbitally complete metric space. In this chapter, we have defined quasi contraction and proved the fixed point theorem for quasi contraction mapping in S -orbitally complete metric space. According to the Banach's contraction mapping principle if $S : U \rightarrow U$ is a contraction on U i.e.

$$d(S t_1, S t_2) \leq a d(t_1, t_2)$$

for some $a < 1$ and all $t_1, t_2 \in U$ and U is complete, then

- (i) S has a unique fixed point u in U ,
- (ii) $\lim_{n \rightarrow \infty} S^n t = u$
- (iii) $d(S^n t, u) \leq a^{n(1-a)^{-1}} d(t, S t)$ for every $t \in U$.

Generalized contraction([4], P.20).

Let $S : U \rightarrow U$ be a mapping and is said to be λ - *generalized contraction* if and

only if for all $t_1, t_2 \in U$, there exists non-negative numbers a_1, a_2, a_3 and a_4 which may depend on both t_1 and t_2 , such that $\sup\{a_1 + a_2 + a_3 + 2a_4 ; t_1, t_2 \in U\} < 1$ and

$$d(St_1, St_2) \leq a_1.d(t_1, t_2) + a_2.d(t_1, St_1) + a_3.d(t_2, St_2) + a_4.[d(t_1, St_2) + d(t_2, St_1)].$$

S-orbitally complete ([5], P.268).

Let $S : U \rightarrow U$; for $A \subset U$, let $\delta(A) = \sup\{d(t_1, t_2); t_1, t_2 \in A\}$ and for each $t \in U$, let

$$O(t, n) = \{t, St, \dots, S^n t\} \quad n = 1, 2, 3, \dots$$

$$O(t, \infty) = \{t, St, \dots\}$$

A space U is said to be S -orbitally complete if and only if every Cauchy sequence which is contained in $O(t, \infty)$ for some $t \in U$ converges in U .

Quasi-contraction ([5], P.268).

Let U be a metric space and $S : U \rightarrow U$ be a mapping, we say that S is a quasi contraction if and only if there exists a number a , $0 \leq a < 1$ such that

$$d(St_1, St_2) \leq a \max\{d(t_1, t_2); d(t_1, St_1); d(t_2, St_2); d(t_1, St_2); d(t_2, St_1)\}$$

holds for every $t_1, t_2 \in U$

4.1.1 Lemmas

To prove the main results, the following lemmas are required.

Lemma 4.1.1. Let S be a quasi-contraction on U and let n be any positive integer. Then for each $t \in U$ and all positive integers i and j ; $i, j \in \{1, 2, 3, \dots, n\}$ implies $d(S^i t, S^j t) \leq a \delta [O(t, n)]$.

Proof. Let $t \in U$ be arbitrary, and n be any positive integer. Let i and j satisfy the

condition of Lemma 4.1.1 Then $S^{i-1}t, S^i t, S^{j-1}t, S^j t \in O(t, n)$

Since S is quasi contraction, we have

$$\begin{aligned} d(S^i t, S^j t) &= d(S S^{i-1} t, S S^{j-1} t) \\ &\leq a \max\{d(S^{i-1} t, S^{j-1} t); d(S^{i-1} t, S^i t); d(S^{j-1} t, S^j t); \\ &\quad d(S^{i-1} t, S^j t); d(S^i t, S^{j-1} t)\} \\ &\leq a\delta [O(t, n)] \end{aligned}$$

□

Remark: From above lemma, it follows that if S is a quasi contraction and $t \in U$, then for every positive integer n there exists a positive integer $k \leq n$, such that

$$d(t, S^k t) = \delta [O(t, n)].$$

Lemma 4.1.2. For any mapping the following holds:

if S is a quasi contraction on U , then

$$\delta [O(t, \infty)] \leq \frac{1}{1-a} d(t, St);$$

holds for all $t \in U$.

Proof. Let $t \in U$ be arbitrary. Since

$$\begin{aligned} \delta [O(t, 1)] &\leq \delta [O(t, 2)] \leq \dots \\ \implies \delta [O(t, \infty)] &= \sup\{\delta [O(t, n)]; n \in \mathbb{N}\} \end{aligned}$$

Now we show that

$$\delta [O(t, n)] \leq \frac{1}{1-a} d(t, St), \text{ for all } n \in \mathbb{N}$$

Suppose n be any positive integer. As we know from the previous remark, there

exists a $S^k x \in O(t, n)$ where $1 \leq k \leq n$ such that

$$d(t, S^k t) = \delta [O(t, n)]$$

Now, by using the triangle inequality and Lemma 4.1.1, we have

$$\begin{aligned} d(t, S^k t) &\leq d(t, St) + d(St, S^k t) \\ &\leq d(t, St) + a \delta [O(t, n)] \\ &= d(t, St) + a d(t, S^k t) \end{aligned}$$

Therefore,

$$\begin{aligned} d(t, S^k t)(1 - a) &= d(t, St) \\ \delta [O(t, n)] &= d(t, S^k t) \leq \frac{1}{1 - a} d(t, St) \end{aligned}$$

As n is arbitrary, the desired result is obtained. □

4.2 Main Results

The first main result of this chapter reads as follows:

Theorem 4.2.1. Let U be a S -orbitally complete metric space and S be a quasi contraction on it. Then

- (i) S has a unique fixed point u in U ;
- (ii) $\lim_{n \rightarrow \infty} S^n t = u$;
- (iii) $d(S^n t, u) \leq \frac{a^n}{1 - a} d(t, St)$, for every $t \in U$.

Proof. Let $t \in U$ be an arbitrary point. Define a sequence of iterates $\{S^n t\}$. We shall show that $\{S^n t\}$ is Cauchy sequence.

Let n and m be any positive integers such that $n < m$. From Lemma 4.1.1 it follows

that

$$\begin{aligned} d(S^n t, S^m t) &= d(SS^{n-1} t, S^{m-n+1} S^{n-1} t) \\ &\leq a \delta [O(S^{n-1} t, m-n+1)] \end{aligned}$$

Now, from remark to Lemma 4.1.1 , there exist $k_1, 1 \leq k_1 \leq m-n+1$ such that

$$\delta [O(S^{n-1} t, m-n+1)] = d(S^{n-1} t, S^{k_1} S^{n-1} t),$$

by again using the Lemma 4.1.1, we have

$$\begin{aligned} d(S^{n-1} t, S^{k_1} S^{n-1} t) &= d(SS^{n-2} t, S^{k_1+1} S^{n-2} t) \\ &\leq a \delta [O(S^{n-2} t, k_1+1)] \\ &\leq a \delta [O(S^{n-2} t, m-n+2)]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} d(S^n t, S^m t) &\leq a \delta [O(S^{n-1} t, S^{k_1} S^{n-1} t)] \\ &\leq a^2 \delta [O(S^{n-2} t, m-n+2)] \end{aligned}$$

Proceeding in this way, we obtain

$$d(S^n t, S^m t) \leq a \delta [O(S^{n-1} t, m-n+1)] \dots \dots \dots \leq a^n \delta [O(S t, m)].$$

As we know from Lemma 4.1.2

$$d(S^n t, S^m t) \leq \frac{a^n}{1-a} d(t, S t). \tag{4.1}$$

Since $\lim_{n \rightarrow \infty} a^n = 0$ where $0 \leq a < 1$

As $\{S^n t\}$ is a Cauchy Sequence.

Also, U is S -Orbitally complete . This implies $\{S^n t\}$ is converges in U to u .

Next, we show that $Su = u$.

Let us consider

$$\begin{aligned}
d(u, Su) &\leq d(u, S^{n+1}t) + d(SS^n t, Su) \\
&\leq d(u, S^{n+1}t) + a \max\{d(S^n t, u); d(S^n t, S^{n+1}t); d(u, Su); d(S^n t, Su); \\
&\quad d(S^n t, S^{n+1}t); d(u, Su); d(S^n t, Su); d(S^{n+1}t, u)\} \\
&\leq d(u, S^{n+1}t) + a.[d(S^n t, S^{n+1}t) + d(S^n t, u) + d(u, Su) + d(S^{n+1}t, u)] \\
d(u, Su)(1-a) &\leq d(u, S^{n+1}t) + a.d(S^n t, S^{n+1}t) + a.d(S^{n+1}t, u) + a.d(S^n t, u) \\
d(u.Su) &\leq \frac{1}{1-a} [(1+a)d(u, S^{n+1}t) + a.d(S^n t, S^{n+1}t) + a.d(u, S^n t)]
\end{aligned}$$

Since $\lim_{n \rightarrow \infty} S^n = u$. This implies $d(u, Su) = 0$. Therefore, u is a fixed point under S .

Uniqueness. Let u and v be two fixed points under S such that $Su = u$ and $Sv = v$ for $u \neq v$, then

$$\begin{aligned}
d(u, v) &\leq a \max\{d(u, v); d(u, Su); d(v, Sv); d(u, Sv); d(Su, v)\} \\
d(u, v) &\leq 0
\end{aligned}$$

which is contradiction, therefore $u = v$.

Let $m \rightarrow \infty$ in (4.1), we get

$$d(S^n t, u) \leq \frac{a^n}{1-a} d(t, St).$$

□

Example. Let (U, d) be a metric space w.r.t usual metric space where $U = [0, \infty]$ and $S : [0, \infty) \rightarrow [0, \infty)$ defined by $S(t) = \frac{t}{2}$

Proof. First, we, show that S is a quasi-contraction mapping.

Case(1) : $t_1 \geq t_2$

$$d(St_1, St_2) \leq a \max\{d(t_1, t_2); d(t_1, St_1); d(t_2, St_2); d(t_1, St_2); d(t_2, St_1)\}; \forall 0 \leq a < 1 \quad (4.2)$$

$$= a \max\{|t_1 - t_2|, \left|\frac{t_1}{2}\right|, \left|\frac{t_2}{2}\right|, \left|t_1 - \frac{t_2}{2}\right|, \left|t_2 - \frac{t_1}{2}\right|\} \quad (4.3)$$

As $t_1 > t_2$, therefore (4.2) implies

$$\frac{1}{2}|t_1 - t_2| \leq \frac{1}{2}\left|t_1 - \frac{t_2}{2}\right| \quad \text{for } a = \frac{1}{2}$$

This is true.

Case (2) : $t_1 < t_2$

Clearly $|t_1| < |t_2| \implies \left|\frac{t_1}{2}\right| < \left|\frac{t_2}{2}\right|$. Therefore, (4.2) becomes

$$\frac{1}{2}|t_1 - t_2| \leq \frac{1}{2}\left|t_2 - \frac{t_1}{2}\right| \quad \text{for } a = \frac{1}{2}.$$

This is true.

Therefore, S is a quasi contraction and S is S - orbitally complete also.

As S satisfies the requirements of Theorem 4.2.1, therefore $St_1 = t_1 \implies \frac{t_1}{2} = t_1 \implies t_1 = 0$

0 is the only fixed point. Therefore (i) proved.

Now, suppose $St_o = t_1$, $S^2t_o = t_2$

As

$$t_2 = St_1 = S^2t_o$$

$$t_3 = St_2 = S^3t_o$$

$$t_1 = St_o = \frac{t_o}{2}$$

$$t_2 = St_1 = S\left(\frac{t_o}{2}\right) = \frac{t_o}{2^2}$$

$$t_3 = St_2 = S\left(\frac{t_0}{2^2}\right) = \frac{t_0}{2^3}$$

.....

.....

$$t_n = St_{n-1} = \frac{t_0}{2^n}.$$

Therefore $\lim_{n \rightarrow \infty} \frac{t_0}{2^n} \rightarrow 0$. Hence (ii) proved.

$$d(S^n t, u) \leq \frac{a^n}{1-a} d(t, St); \text{ for } a = \frac{1}{2}.$$

Thus, the result is true for this example.

Theorem 4.2.2. Every contraction is quasi contraction but converse is not true.

Proof. Let S be a contraction mapping from U to U s.t.

$$d(St_1, St_2) \leq a d(t_1, t_2); 0 < a < 1$$

$$\begin{aligned} \text{As } d(St_1, St_2) &\leq a d(t_1, t_2) \\ &\leq a \max\{d(t_1, t_2); d(t_1, St_1); d(t_2, St_2); d(t_1, St_2); d(t_2, St_1)\}. \end{aligned}$$

This implies

$$d(St_1, St_2) \leq a \max\{d(t_1, t_2); d(t_1, St_1); d(t_2, St_2); d(t_1, St_2); d(t_2, St_1)\}$$

Therefore S is quasi contraction. □

Example: Let $U = [0, 2] \subset \mathbb{R}$ and let

$$St = \frac{t}{9}; 0 \leq t \leq 1$$

$$St = \frac{t}{10}; 1 \leq t \leq 2$$

S is not a contraction but is a quasi-contraction.

Case(1). $(t_1, t_2) \in [0, 1]$

$$St_1 = \frac{t_1}{9} \text{ and } St_2 = \frac{t_2}{9}$$

$$\left| \frac{t_1}{9} - \frac{t_2}{9} \right| < a|t_1 - t_2| \text{ where } 0 < a < 1$$

This is true.

Case(2). $(t_1, t_2) \in [1, 2]$

$$St_1 = \frac{t_1}{10} \text{ and } St_2 = \frac{t_2}{10}$$

$$\left| \frac{t_1}{10} - \frac{t_2}{10} \right| < a|t_1 - t_2| \text{ where } 0 < a < 1$$

This is true.

Case(3). $t_1 \in [0, 1]$ and $t_2 \in [1, 2]$

$$\left| \frac{t_1}{9} - \frac{t_2}{10} \right| < a|t_1 - t_2| \text{ where } 0 < a < 1$$

Take $t_1 = \frac{999}{1000}$ and $t_2 = \frac{1001}{1000}$ therefore,

$$\left| \frac{999}{9000} - \frac{1001}{10000} \right| = \left| \frac{109}{10,000} \right| < a|t_1 - t_2|$$

This is not hold for $0 < a < 1$

Hence, S is not a contraction.

Theorem 4.2.3. Every generalized contraction is quasi contraction but converse is not true.

Example. Let $U = \left\{ \frac{m}{n}; m = 0, 1, 3, 9, \dots; n = 1, 4, \dots, 3k + 1, \dots \right\}$,

$V = \left\{ \frac{m}{n}; m = 1, 3, 9, \dots; n = 2, 5, \dots, 3k + 2, \dots \right\}$,

and let $U_1 = U \cup V$ be a usual metric. Define $S : U_1 \rightarrow U_1$ by

$$St = \frac{3t}{5}, \text{ for } t \in U$$

$$St = \frac{t}{8}, \text{ for } t \in V$$

Case(1). either $t_1, t_2 \in U$ or $t_1, t_2 \in V$

For $a = \frac{3}{5}$

For $t_1, t_2 \in U$

$$\left| \frac{3t_1}{5} - \frac{3t_2}{5} \right| < a|t_1 - t_2|$$

This is true.

For $t_1, t_2 \in V$

$$\left| \frac{t_1}{8} - \frac{t_2}{8} \right| < a|t_1 - t_2|$$

This is true.

Case(2). Let $t_1 \in U$ and $t_2 \in V$

$$t_1 > \frac{5}{24}t_2 \implies d(St_1, St_2) = \left| \frac{3t_1}{5} - \frac{t_2}{8} \right| = \frac{3}{5} \left| t_1 - \frac{5}{24}t_2 \right|$$

As $t_1 > \frac{5}{24}t_2$

$$t_1 - \frac{1}{8}t_2 > \left(\frac{5}{24} - \frac{1}{8} \right) t_2$$

This holds. Now,

$$d(St_1, St_2) = \frac{3}{5} \left(t_1 - \frac{5}{24}t_2 \right) \leq \frac{3}{5} \left(t_1 - \frac{1}{8}t_2 \right) = \frac{3}{5}d(t_1, St_2)$$

$$t_1 < \frac{5}{24}t_2 \implies d(St_1, St_2) = \left| \frac{t_2}{8} - \frac{3t_1}{5} \right| = \frac{3}{5} \left(\frac{5}{24}t_2 - t_1 \right) \leq \frac{3}{5}(t_2 - t_1) = \frac{3}{5}d(t_1, t_2)$$

Therefore, quasi contraction conditions are hold.

$$d(St_1, St_2) \leq \frac{3}{5} \max\{d(t_1, t_2); d(t_1, St_2); d(t_2, St_1)\}$$

S is not generalized contraction on U_1 .

Let $t_1 = 1$ and $t_2 = \frac{1}{2}$. As we know from definition of generalized contraction

$$\begin{aligned} & a_1 \cdot d(t_1, t_2) + a_2 \cdot d(t_1, St_1) + a_3 \cdot d(t_2, St_2) + a_4 [d(t_1, St_2) + d(t_2, St_1)] \\ &= a_1 \cdot \frac{1}{2} + a_2 \cdot \frac{2}{5} + a_3 \cdot \frac{7}{16} + a_4 \cdot \frac{83}{80} \\ &< (a_1 + a_2 + a_3 + 2a_4) \cdot \frac{83}{160} < \frac{83}{160} < \frac{43}{80} = d(St_1, St_2) \end{aligned}$$

as $a_1 + a_2 + a_3 + 2a_4 < 1$,

therefore conditions of generalized contraction are not hold.

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