On n–Color Partitions and Combinatorics

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Submitted by

Priyanka
Roll no. 301403012

Under the Supervision of

Dr. Meenakshi Rana
Assistant Professor
School of Mathematics

Thapar University
Patiala-147004(PUNJAB)
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Certificate

This is to certify that the thesis entitled **"On n–Color Partitions and Combinatorics"**, being submitted by Ms. Priyanka for the award of the degree of Master of Science in the School of Mathematics, Thapar University, Patiala, is a bonafide survey work carried out under the supervision of **Dr. Meenakshi Rana** and that no part of this thesis has been submitted for the award of any other degree.

Priyanka

Priyanka

Roll N0. 301403012

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Meenakshi

Dr. Meenakshi Rana
Assistant Professor
School of Mathematics,
Thapar University, Patiala

Dr. A.K. Lal
Associate Professor & Head, SOM
Thapar University,
Patiala

Dr. S.S. Bhatia
Dean of Academic Affairs
Thapar University,
Patiala

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Abstract

The present dissertation contains a detail study of investigations carried out by various authors on enumerative combinatorics using n–color partitions of certain q–series. The whole work is divided into three chapters.

Chapter 1 is introductory including elementary definitions, notations and generating functions which will be required for later chapters. This chapter includes some celebrated identities such as Rogers–Ramanujan Identities and Göllnitz–Gordan Identities.

In Chapter 2, we have discussed n–color partitions introduced by Agarwal and Andrews [“N copies of N”, J. Combin. Theory Ser. A 45, (1987), 40-49]. This chapter is further devoted to the study of $(n + t)$–color partitions.

In Chapter 3, we have done the survey of further advances in n–color partitions. It particularly include the combinatorial interpretations of q–series using n–color partitions, further Rogers–Ramanujan type Identities for n–color partitions are also explored.

Contents

Chapter 1

Introduction

1.1 Introduction

Partition theory is an area of the additive number theory, a subject concerning the representation of integers as the some of other integers. The concept of partition of non-negative integers also belongs to combinatorics. The works of Srinivasa Ramanujan - the legendary Indian mathematician of the twentieth century, made a profound impact on many areas of modern number theoretic research.

The real development started with Euler (1674). He was first who discovered the important properties of the partition function and presented them in his book "Introduction in Analys in Infinitorum". The theory has been further developed by many of the other great mathematicians - prominent among them are Gauss, Jacobi, Cayley, Sylvester, Hardy, Ramanujan, Schur, MacMahon, Gupta, Gordon, Andrews and Stanley. The celebrated joint work of Ramanujan with Hardy indeed revolutionized the study of partitions. For example, partitions, continued fractions, definite integrals and mock theta functions. The theory of partition has found many applications in different areas like probability, statistical mechanism and particle physics. In our thesis we will discuss about partitions, Generating functions of ordinary partitions, Combinatorial interpretations of q–series and Rogers–Ramanujan identities and a detailed work on n–color partitions.

1.1.1 Basic Definitions

Definition 1.1.1 *A partition of a positive integer n is a finite non-increasing sequence of a positive integer $a_1 \geq a_2 \geq a_3 \geq \cdots\cdots \geq a_n$, whose sum is n. where a_i are called part of the partition and $p(n)$ denote the number of partition of n.*

Example. $p(5) = 7$, the relevant partition of 5 are

5

4+1

3+2

3+1+1

2+2+1

2+1+1+1

1+1+1+1+1

Remark 1.1 We take $p(n)$=0 for all $n < 0$ and $p(0) = 1$.

1.1.2 Notations

Rising factorial. Let 'a' be a complex number, then define

$(a)_0 = 1$

$(a)_n = a(a+1)(a+2)(a+3)\cdots(a+n-1)$

Taking a=1 , $(1)_n = 1(2)(3)(4)\cdots(1+n-1) = n!$

q–rising factorial. Let 'a' be non-negative integer, then define

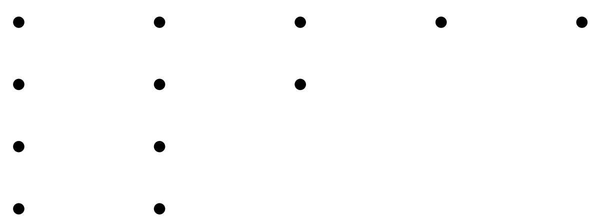
$(a, q)_0 = 1$

$(a, q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$, where $n > 0$

1.2 Graphical Representation of Partition

Ferrer's Graph. The Ferrer's Graph named after NORMAN MACLEOD FERRERS is a partition $\pi = (t_1, t_2, t_3, \cdots, t_i)$ of n is a set of i –rows of equispaced dots alligned on left, where jth-rows has t_j dots.

Example. The Ferrer's Graph of a partition 5+3+2+2 of 12 as following,

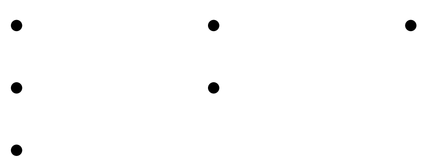


If we read above graph vertically (i.e column wise, then we will get a **conjugate partition** of π, which is denoted by π^c.

$$\pi^c = 4 + 4 + 2 + 1 + 1$$

Self – Conjugate Partition. A partition π which is identical with its conjugate π^c is called self–conjugate partition.

Example. 3+2+1 is self–conjugate partition of 6.



1.3 Generating Function

Table 1.1: Different Type of Generating Functions

Generating Function		q–series	
Partitions	: $\sum_{n=0}^{\infty} p(n)q^n$	$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$	$= \frac{1}{(q;q)_\infty}$
Partitions of n into atmost m parts	: $\sum_{n=0}^{m} p_m(n)q^n$	$\prod_{n=1}^{m} \frac{1}{(1-q^n)}$	$= \frac{1}{(q;q)_m}$
Distinct parts	: $\sum_{n=0}^{\infty} D(n)q^n$	$\prod_{n=1}^{\infty} (1+q^n)$	$=(-q;q)_\infty$
Odd parts	: $\sum_{n=0}^{\infty} O(n)q^n$	$\prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})}$	$= \frac{1}{(q;q^2)}_\infty$
Distinct-odd parts	: $\sum_{n=0}^{\infty} d(n)q^n$	$\prod_{n=1}^{\infty} (1+q^{2n-1})$	$= (-q;q^2)_\infty$

1.4 Rogers–Ramanujan Identities

The following two " sum-product " identities are known as Rogers–Ramanujan Identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} (1-q^{5n-1})^{-1}(1-q^{5n-4})^{-1} \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} (1-q^{5n-2})^{-1}(1-q^{5n-3})^{-1} \tag{1.2}$$

They were first discovered and proved by Leonard James Rogers in (1894) [15] and were rediscovered by Srinivasa Ramanujan in (1913) [17]. Ramanujan had no proof, but rediscovered Roger's paper in 1917 and published a paper in 1919 which contins two proofs (one by Ramanujan and the other by Rogers) and after the publication of this paper these identities are known as Rogers–Ramanujan identities(RRI). MacMahon [14] gave the following combinatorial interpretations of equation (1.1) and equation (1.2) respectively:

Theorem 1.4.1 *The number of partitions of n into parts with the minimal difference 2 equals the number of partitions of n into parts which are congruent to* $\pm 1(mod\ 5)$.

Theorem 1.4.2 *The number of partitions of n into parts with the minimal part 2 and minimal difference 2 equals the number of partitions of n into parts which are congruent to* $\pm 2(mod\ 5)$.

1.5 Göllnitz–Gordan Identities

The Göllnitz–Gordan Identities, given below, are due to Göllnitz [10] and were included in his 1916 unpublished honors thesis. However, essentially no one know about the results untill Gordan (1965) [11] independently rediscovered them.

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{8n+1})(1-q^{8n+4})(1-q^{8n+7})} \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{8n+3})(1-q^{8n+4})(1-q^{8n+5})} \tag{1.4}$$

Theorem 1.5.1 ***First Göllnitz–Gordan Identity:*** *The number of partitions of n into summands differing by at least 2, among which no two consecutive even summands are appear equals the number of partition of n into summands* $\equiv 1,\ 4$ *or* $7\ (mod\ 8)$.

Theorem 1.5.2 ***Second Göllnitz–Gordan Identity:*** *The number of partitions of n into summands differing by at least 2, among which no two con-*

secutive even summands are appear and with each summands ≥ 3 *equals the number of partition of* n *into summands* $\equiv 3,\ 4$ *or* $5\ (mod\ 8)$.

Some other mathematicians have interpreted q–series using ordinary partitions, see for instance [9, 20]. The next chapter discuss the combinatorial interpretations using n–color partitions.

Chapter 2

Colored Partitions

In this chapter we study Rogers–Ramanujan type identites and q–series using n–color partitions.

2.1 n–Color Partitions

The n–color partitions were defined by Agarwal and Andrews [2] in 1987 which extends the ordinary partitions and which are further extended to $(n+t)$–color partitions.

Generating Function. If $P_M(v)$ denote the number of partitions of v with n copies of n, then

$$\sum_{v=0}^{\infty} P_M(v)q^v = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \tag{2.1}$$

It was pointed out in [4] that since the right hand side of equation (2.1) is also a generating function for the MacMahon's plane partition function so the number of n–color partitions of v equals to the number of plane partitions of v.

Definition 2.1.1 *An n–**Color partition** of a positive integer v is a parti-*

tion in which a summands of size n, can come in n different colors denoted by subscripts: $n_1, n_2, n_3, \cdots\cdots, n_n$ and summands satisfy the order

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < 5_1 < \cdots\cdots$$

Table 2.1: Partitions of some integers

Integer	Colored Partition	Ordinary Partitions
1	1_1	1
2	$2_1, 2_2, 1_1$	2, 1+1
3	$3_1, 3_2, 3_3, 2_1 1_1, 2_2 1_1, 1_1 1_1 1_1$	3, 2+1, 1+1+1

Definition 2.1.2 *The **Weighted Difference** of two parts m_i, n_j, $m \geq n$ is defined by $m - n - i - j$ and denoted by $((m_i - n_j))$.*

2.2 Rogers–Ramanujan Type Identities with n-color partitions

Analogues to RRI, many q–series identities have been given by several mathematicians. See, for instance, Gordan [11], Subbarao [21]. In all these results only ordinary partitions were used. Analogous to MacMahon's combinatorial interpretations (Theorem 1.4.1 and Theorem 1.4.2) of RRI (1.1) and (1.2) respectively, Agarwal and Andrews [4] using n–color partitions proved the Theorems (2.1.1)–(2.1.2), given below;

Theorem 2.2.1 *The partitions of v with n copies of n wherein each pair of parts has positive weighted difference are equinumerous with the ordinary partitions of v into parts $\not\equiv 0, \pm 4 (mod\ 10)$.*

Ordinary Partitions with parts $\not\equiv 0, \pm 4 (mod\ 10)$	Colored partitions with Positive Weighted Difference
5	5_1
3+2	5_2
3+1+1	5_3
2+2+1	5_4
2+1+1+1	5_5
1+1+1+1+1	$4_1 + 1_1$

Theorem 2.2.2 *The partitions of v with n copies of n wherein each pair of parts has nonnegative weighted difference are equinumerous with the ordinary partitions of v into parts $\not\equiv 0, \pm 6(mod\ 14)$.*

Ordinary Partitions with parts $\not\equiv 0, \pm 6 (mod\ 14)$	Colored partitions with Nonnegative Weighted Difference
5	5_1
4+1	5_2
3+2	5_3
3+1+1	5_4
2+2+1	5_5
2+1+1+1	$4_1 + 1_1$
1+1+1+1+1	$4_2 + 1_1$

In [1], Agarwal first prove the following theorem using n–color partitions.

Theorem 2.2.3 *For $k \geq -3$, let $C_k(n)$ denote the number of partitions with "N Copies of N" of n such that each pair of parts m_i, r_j satisfies $|m - r| > i + j + k$. Then*

$$\sum_{n=0}^{\infty} C_k(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n[1+\frac{(k+3)(n-1)}{2}]}}{(q;q)_n (q;q^2)_n}$$

Proof 2.2.1 Let $C_k(m, n)$ denote the number of partitions enumerated by $C_k(n)$ with the added restriction that there be exactly m parts. We shall first prove that

$$C_k(m,n) = C_k(m,n-m) + C_k(m-1,n-km-3m+k+2)$$
$$+C_k(m,n-2m+1) - C_k(m,n-3m+1). \qquad (2.2)$$

We split the partitions enumerated by $C_k(m,n)$ into three classes:

(1) that don't contain k_k as a part,

(2) that contain 1_1 as a part ,

(3) that contain $k_k (k > 1)$ as a part.

Now, we transform the partitions in class(1) by deleting 1 from each parts ignoring the subscripts. Then this transformation will not disturb the inequalities between the parts and so the transformed partitions will be of type enumerated by $C_k(m,n-m)$.

In class(2) we transform the partitions by deleting the parts 1_1 and then subscripts k+3 from all the remaining parts ignoring the subscripts. The transformed partitions will be of the type enumerated by $C_k(m-1,n-km-3m+k+2)$. Here note that k can't be lass than -3.

Finally, we transform the partitions in class(3) by replacing k_k by $k-1_{k-1}$ and then subtracting 2 from all the remaining parts. This will produce a partitions of $n-1-2(m-1) = n-2m+1$ into m parts. It is important to note here that by this transformations we get only those partitions of $n-2m+1$ into m-parts which contain $k-1_{k-1}$ as a part. Therefore the actual number of partitions which belongs to class(3) is $C_k(m,n-2m+1) - C_k(m,n-3m+1)$, where $C_k(m,n-3m+1)$ is number of partitions of $n-2m+1$ into m parts which are free from the parts like k_k.

The above transformations clearly establish a bijection between the partitions enumerated by $C_k(m,n)$ and those enumerated by $C_k(m,n-m) + C_k(m-1,n-km-3m+k+2) + C_k(m,n-2m+1) - C_k(m,n-3m+1)$. Thus identity (2.2) is established.

Let

$$f_k(z,q) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_k(m,n) z^m q^n \tag{2.3}$$

Then equation (2.2) implies that

$$\begin{aligned}
f_k(z,q) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_k(m,n) z^m q^n = C_k(m,n-m) + C_k(m-1,n-km-3m+k+2) \\
&+ C_k(m,n-2m+1) - C_k(m,n-3m+1) z^m q^n \qquad (2.4) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_k(m,n-m)(zq)^m q^{n-m} \\
&+ zq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_k(m-1,n-km-3m+k+2)(zq^{k+3})^{m-1} q^{n-m(k+3)+k+2} \\
&+ \frac{1}{q}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_k(m,n-2m+1)(zq^2)^m q^{n-2m+1} \\
&- \frac{1}{q}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_k(m,n-3m+1)(zq^3)^m q^{n-3m+1}
\end{aligned}$$

$$= f_k(zq,q) + zqf_k(zq^{k+3},q) + \frac{1}{q} f_k(zq^2,q) - \frac{1}{q} f_k(zq^3,q) \tag{2.5}$$

Put $f_k(z,q) = \sum_{n=0}^{\infty} \lambda_{k,n}(q) z^n$, and then comparing coefficients of z^n on both side of equation (2.4),we see that

$$\lambda_{k,n}(q) = \frac{\lambda_{k,n-1}(q) q^{(n-1)(k+3)+1}}{(1-q^n)(1-q^{2n-1})} \tag{2.6}$$

Iterating Equation (2.6) n times and observing that $\lambda_{k,\ 0}(q) = 1$, we find that

$$\lambda_{k,n}(q) = \frac{q^n[1+\frac{(k+3)(n-1)}{2}]}{(q;q)_n (q;q^2)_n}, \tag{2.7}$$

therefore

$$f_k(z,q) = \frac{q^n[1+\frac{(k+3)(n-1)}{2}]z^n}{(q;q)_n(q;q^2)_n} \tag{2.8}$$

$$\sum_{n=0}^{\infty} C_k(n)q^n = \sum_{n=0}^{\infty}[\sum_{m=0}^{\infty} C_k(m,n)]q^n = f_k(1,q)$$
$$= \frac{q^n[1+\frac{(k+3)(n-1)}{2}]}{(q;q)_n(q;q^2)_n}$$

This completes the proof of the theorem.

2.2.1 Particular Cases

If $k=0$, then Theorem 2.2.3, in conjunction with identity [19, I(46), p.156]

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q;q)_n(q;q^2)_n} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty}(1-q^{10})(1-q^{10n-6})(1-q^{10n-4})$$

reduces to Theorem 2.2.1.

If $k=-1$, then Theorem 2.2.3, in conjunction with identity [19, I(61), p.158]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n(q;q^2)_n} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty}(1-q^{14n})(1-q^{14n-6})(1-q^{14n-8})$$

reduces to Theorem 2.2.2.

2.3 $(n+t)$–color Partitions

In [4], Agarwal and Andrews extended n–color partitions to $(n+t)$–color partitions which further helped to interpret other q–series combinatorially which were not possible to interpret by n–color partitions. Before starting some of the results let us state the definitions of $(n+t)$–color partitions.

Definition 2.3.1 *A partition with "$(n+t)$ copies of (n)", $t \geq 0$, is a partition in which a summands of $n, (n \geq 0)$, can come in $(n+t)$ different colors , denoted by the subscripts, $n_1, n_2, n_3, ..., n_{n+t}$.*

Example. Partitions of 2 with "$(n+1)$ copies of n" are,

$$\begin{aligned} &2_1, \quad 2_1 0_1, \quad 1_1 1_1, \quad 1_1 1_1 0_1, \\ &2_2, \quad 2_2 0_1, \quad 1_2 1_1, \quad 1_2 1_1 0_1, \\ &2_3, \quad 2_3 0_1, \quad 1_2 1_2, \quad 1_2 1_2 0_1. \end{aligned}$$

Note that zeros are permitted if and only if $t \geq 1$.

2.4 q–series identities using n–color Partitions

In [2], Agarwal proved some generalized theorems using $(n+1)$–color partition and $(n+2)$–color partitions respectively, all these q–series also appeared in [19] derived using Bailey transformation.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-2})(1-q^{10n-8})(1-q^{20n-14})(1-q^{20n-6})(1-q^{10n}) \tag{2.9}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-3})(1-q^{10n-7})(1-q^{20n-16})(1-q^{20n-4})(1-q^{10n}) \tag{2.10}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-4})(1-q^{10n-6})(1-q^{20n-18})(1-q^{20n-2})(1-q^{10n}) \tag{2.11}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1}})) \tag{2.12}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-1})(1-q^{10n-9})(1-q^{20n-8})(1-q^{20n-12})(1-q^{10n}) \tag{2.13}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{8n-1})(1-q^{8n-7})(1-q^{16n-10})(1-q^{16n-6})(1-q^{8n}) \tag{2.14}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{8n-3})(1-q^{8n-5})(1-q^{16n-14})(1-q^{16n-2})(1-q^{8n}) \tag{2.15}$$

We shall now discuss the combinatorial counterparts of (2.9)–(2.15) in the following Theorems (2.4.1)–(2.4.7), respectively.

Theorem 2.4.1 *Let $A_1(v)$ denote the number of partitions of v with "n copies of n" where each pair of parts has nonnegative weighted difference and even parts appear with even subscripts and odd with odd subscripts. Let $B_1(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0, \pm 2, \pm 6$, ± 8,± 10(mod 20). Then $A_1(v) = B_1(v)$.*

Proof 2.4.1 We split partitions enumerated by $A_1(m, v)$ into three classes:
(1) that do not contain k_k as a summand,
(2) that contain 1_1 as a summand, and
(3) that contain $k_k(k > 1)$ as a summand.
Following the method of proof of Theorem 2.2.3 it can be proved that the partitions in class (1) are counted by $A_1(m, v - 2m)$, in class(2) by $A_1(m - 1, v - 2m + 1)$ and
in class(3) by $A_1(m, v - 2m + 1)$-$A_1(m, v - 4m + 1)$, and so

$$\begin{aligned} A_1(m, v) = A_1(m, v - 2m) + A_1(m - 1, v - 2m + 1) + A_1(m, v - 2m + 1) \\ - A_1(m, v - 4m + 1) \end{aligned}$$

$$\begin{aligned} \sum_{v=0}^{\infty}\sum_{m=0}^{\infty} A_1(m, v)z^m q^v &= \sum_{v=0}^{\infty}\sum_{m=0}^{\infty} A_1(m, v - 2m)z^m q^v \\ &+ \sum_{v=0}^{\infty}\sum_{m=0}^{\infty} A_1(m - 1, v - 2m + 1)z^m q^v \\ &+ \sum_{v=0}^{\infty}\sum_{m=0}^{\infty} A_1(m, v - 2m + 1)z^m q^v \\ &+ \sum_{v=0}^{\infty}\sum_{m=0}^{\infty} A_1(m, v - 4m + 1)z^m q^v \qquad (2.16) \end{aligned}$$

Let

$$h(z,q)=\sum_{v=0}^{\infty}\sum_{m=0}^{\infty}A_1(m,v)z^m q^v \tag{2.17}$$

Substituting for $A_1(m,v)$ from (2.16) in (2.17) and then simplifying we get

$$h(z,q)=h(zq^2,q)+zqh(zq^2,q)+\frac{1}{q}h(zq^2,q)-\frac{1}{q}h(zq^4,q) \tag{2.18}$$

Setting $h(z,q)=\sum_{n=0}^{\infty}\alpha_n(q)z^n$, and then comparing the cofficients of z^n on each side of (2.18), we see that

$$\alpha_n(q)=\frac{q^{2n-1}}{(1-q^{2n})(1-q^{2n-1})}\alpha_{n-1}(q) \tag{2.19}$$

Iterating (2.19) n times and observing that $\alpha_0(q)=1$, we find that

$$\alpha_n(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}$$

$$h(z,q)=\sum_{n=0}^{\infty}\frac{q^{n^2}z^n}{(q;q)_{2n}}=f_1(z,q)$$

Now

$$\begin{aligned}\sum_{n=0}^{\infty}A_1(v)q^v&=\sum_{v=0}^{\infty}(\sum_{m=0}^{\infty}A_1(m,v))q^v\\&=h(1,q)\\&=f_1(1,q)\\&=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}\\&=\frac{1}{(q;q)_\infty}\prod_{n=1}^{\infty}(1-q^{10n-2})(1-q^{10n-8})(1-q^{20n-14})(1-q^{20n-6})(1-q^{10n})\\&=\sum_{v=0}^{\infty}B_1(v)q^v\end{aligned}$$

Theorem 2.4.2 *Let $A_2(v)$ denote the number of partitions of v with "n+1 copies of n" in which for some i, i_{i+1} is a part, the parts are nonnegative, each pair of parts has nonnegative weighted difference, and even part appear with odd subscripts and odd with even. Let $B_2(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0, \pm 3, \pm 4, \pm 7, \pm 10 (mod\ 20)$.*
Then $A_2(v) = B_2(v)$.

Proof 2.4.2 The proof is similar to that of Theorem 2.4.1, hence we omit the details and give only q–functional equation used in this case.

$$f_1(z, q) - f_1(zq^2, q) = zqf_2(zq, q)$$

Using this q-functional equation, we get

$$f_2(z, q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}z^n}{(q;q)_{2n+1}}$$

Theorem 2.4.3 *Let $A_3(v)$ denote the number of partitions of v with "n+2 copies of n" in which for some i, i_{i+2} is a part, the parts are nonnegative, each pair of parts has nonnegative weighted difference, and even part appear with even subscripts and odd with odd. Let $B_3(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0, \pm 2, \pm 4, \pm 6, \pm 10 (mod\ 20)$. Then $A_3(v) = B_3(v)$.*

Proof 2.4.3 The proof is similar to that of Theorem 2.4.1, hence we omit the details and give only q–functional equation used in this case.

$$f_1(z, q) - f_1(zq^2, q) = zqf_3(z, q)$$

using this q-functional equation, we get

$$f_3(z, q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}z^n}{(q;q)_{2n+1}}$$

Theorem 2.4.4 *Let $A_4(v)$ denote the number of partitions of v with "n copies of n" wherein each pair of parts has weighted difference > 1 and even part appear with even subscripts and odd with odd. Let $B_4(v)$ denote the number of ordinary partitions of v into distinct parts. Then $A_4(v) = B_4(v)$.*

Proof 2.4.4 The proof is similar to that of Theorem 2.4.1, hence we omit the details and give only q–functional equation used in this case.

$$f_4(z,q) = f_4(zq^2,q) + zqf_4(zq^4,q) + q^{-1}f_4(zq^2,q) - q^{-1}f_4(zq^4,q)$$

using this q-functional equation, we get

$$f_4(z,q) = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}z^n}{(q;q)_{2n+1}}$$

Theorem 2.4.5 *Let $A_5(v)$ denote the number of partitions of v with "n copies of n" wherein each pair of parts has weighted difference which is either nonnegative or equal to -2 and even part appear with even subscripts and odd parts with odd subscripts greater than 1. Let $B_5(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0, \pm 1, \pm 8, \pm 9, \pm 10 (mod\ 20)$. Then $A_5(v) = B_5(v)$.*

Proof 2.4.5 The proof is similar to that of Theorem 2.4.1, hence we omit the details and give only q–functional equation used in this case.

$$f_5(z,q) = f_5(zq^2,q) + zq^2f_5(zq^2,q) + q^{-1}f_5(zq^2,q) - q^{-1}f_5(zq^4,q)$$

using this q-functional equation, we get

$$f_5(z,q) = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}z^n}{(q;q)_{2n}}$$

Theorem 2.4.6 *Let $A_6(v)$ denote the number of partitions of v with "n copies of n" wherein each pair of parts has weighted difference which is* *either ≥ 2 or equal to 0 and even part appear with even subscripts and odd parts with odd subscripts greater than 1. Let $B_6(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0, \pm 1, \pm 6, \pm 7, \pm 8 (mod\ 16)$. Then $A_6(v) = B_6(v)$.*

Proof 2.4.6 The proof is similar to that of Theorem 2.4.1 , hence we omit the details and give only q–functional equation used in this case.

$$f_6(z,q) = f_6(zq^2,q) + zq^2 f_6(zq^4,q) + q^{-1} f_6(zq^2,q) - q^{-1} f_6(zq^4,q)$$

using this q-functional equation, we get

$$f_6(z,q) = \sum_{n=0}^{\infty} \frac{q^{2n^2} z^n}{(q;q)_{2n}}$$

Theorem 2.4.7 *Let $A_7(v)$ denote the number of partitions of v with "n+2 copies of n" wherein for some i, i_{i+2} is a part, the parts are nonnegative, each pair of parts has weighted difference which is either ≥ 2 or equal to 0 and even part appear with even subscripts and odd parts with odd subscripts greater than 1. Let $B_7(v)$ denote the number of ordinary partitions of v into parts $\not\equiv 0, \pm 2, \pm 3, \pm 5, \pm 8 (mod\ 16)$. Then $A_7(v) = B_7(v)$.*

Proof 2.4.7 The proof is similar to that of Theorem 2.4.1, hence we omit the details and give only q–functional equation used in this case.

$$f_6(z,q) - f_6(zq^2,q) = zq^2 f_7(zq^2,q),$$

following the usual method, we get

$$f_7(z,q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)} z^n}{(q;q)_{2n+1}}$$

Chapter 3

Some more advances in n–Color Patitions

This chapter is devoted to some more advances in colored partitions since last 20 years. Result discussed are all combinatorial and follow similar techniques as discussed in Chapter 2.

3.1 Mock-theta functions using n–color partitions

In his last letter to Hardy, Ramanujan listed 17 functions which he called mock theta functions. He separated these 17 functions into three classes. First containing functions of order 3, second containing 10 functions of order 5 and third containing 3 functions of order 7. Waston [22] found three more functions of order 3 and two more of order 5 appear in the lost notebook [16]. In [3] the number theoretic interpretations of the following mock theta functions are given.

$$\Psi(q) = \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q;q^2)_m} \tag{3.1}$$

$$F_0(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q;q^2)_m} \tag{3.2}$$

$$\Phi_0(q) = \sum_{m=0}^{\infty} q^{m^2}(-q;q^2)_m \tag{3.3}$$

$$\Phi_1(q) = \sum_{m=0}^{\infty} q^{(m+1)^2}(-q;q^2)_m \tag{3.4}$$

we remark here that $\Psi(q)$ is of order 3 while the remaining three are of order 5.

In 2004, using n–color partitions Agarwal in [3] gave the number theoretic interpretations of the mock theta functions (3.1)–(3.4) as follows;
Following are combinatorial counterpart of above respectively.

Theorem 3.1.1 *For $v \geq 1$, let $A_1(v)$ denote the number of n–color partitions of v such that even parts appear with even subscripts and odd with odd, for some k, k_k is a part, and weighted difference of any two consecutive parts is 0. Then,*

$$\sum_{v=1}^{\infty} A_1(v)q^v = \Psi(q) \tag{3.5}$$

Proof 3.1.1 We split the partitions mentioned by $A_1(m, v)$ into two classes:
(1) that contain 1_1 as a part, and
(2) that contain $k_k, (k > 1)$ as a part.
By applying the method of [1] it can be easily proved that the partitions in class (1) are mentioned by $A_1(m-1, v-2m+1)$ and in class (2) by $A_1(m, v-2m+1)$
therefore,

$$A_1(m, v) = A_1(m-1, v-2m+1) + A_1(m, v-2m+1) \tag{3.6}$$

By using theorem (2.2.3) of Chapter 2, we get the q–functional equation,

$$f_1(z,q) = zqf_1(zq^2,q) + q^{-1}f_1(zq^2,q) \tag{3.7}$$

From [1] we obtained the following expression.

$$f_1(z,q) = \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q;q^2)_n} \tag{3.8}$$

Now,

$$\begin{aligned}
\sum_{v=0}^{\infty} A_1(v)q^v &= \sum_{v=0}^{\infty}(\sum_{n=0}^{\infty} A_1(m,v)q^v \\
&= f_1(1,q) \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} \\
&= \Psi(q)
\end{aligned}$$

Theorem 3.1.2 *For $v \geq 0$, let $A_2(v)$ denote the number of n–color partitions of v such that even parts appear with even subscripts and odd with odd > 1, for some k, k_k is a part, and weighted difference of any two consecutive parts is 0. Then,*

$$\sum_{v=1}^{\infty} A_2(v)q^v = F_0(q) \tag{3.9}$$

Proof 3.1.2 We split the partitions mentioned by $A_2(m,v)$ into two classes:
(1) that contain 2_2 as a part, and
(2) that contain $k_k, (k > 2)$ as a part.
By applying the method of [1] it can be easily proved that the partitions in class (1) are mentioned by $A_2(m-1, v-4m+2)$ and in class (2) by $A_2(m, v-2m+1)$

therefore,

$$A_2(m,v) = A_2(m-1, v-4m+2) + A_2(m, v-2m+1) \tag{3.10}$$

The q-functional equation using in this case.

$$f_2(z,q) = zq^2 f_2(zq^4, q) + q^{-1} f_2(zq^2, q) \tag{3.11}$$

Proceeding in usual manner, we get

$$f_2(z,q) = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q;q^2)_m} \tag{3.12}$$

Theorem 3.1.3 *For $v \geq 0$, let $A_3(v)$ denote the number of n–color partitions of v such that only the first copy of odd parts and second copy of even parts are used, that is , the parts are of the type $(2k-1)$ or $(2k)_2$, the minimum part is 1_1 or 2_2, and weighted difference of any two consecutive parts is 0. Then,*

$$\sum_{v=1}^{\infty} A_3(v) q^v = \Phi_0(q) \tag{3.13}$$

Proof 3.1.3 We split the partitions mentioned by $A_3(m,v)$ into two classes: (1) that contain 1_1 as a part, and (2) that contain 2_2 as a part. By applying the method of [1], we see that partitions in class (1) are counted by $A_3(m-1, v-2m+1)$ and in class (2) by $A_3(m-1, v-4m+2)$.
therefore,

$$A_3(m,v) = A_3(m-1, v-2m+1) + A_3(m-1, v-4m+2) \tag{3.14}$$

This gives the q-functional equation,

$$f_3(z,q) = zqf_3(zq^2,q) + zq^2 f_3(zq^4,q) \tag{3.15}$$

using this q-functional equation in [1], we obtained the following expression.

$$f_3(z,q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n z^n \tag{3.16}$$

$$\begin{aligned}\sum_{v=0}^{\infty} A_3(v)q^v &= \sum_{v=0}^{\infty}(\sum_{n=0}^{\infty} A_3(m,v)q^v \\ &= f_3(1,q) \\ &= \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n \\ &= \Phi_0(q)\end{aligned}$$

Theorem 3.1.4 *For $v \geq 1$, let $A_4(v)$ denote the number of n–color partitions of v such that only the first copy of odd parts and second copy of even parts are used, the minimum part is 1_1 or 2_2, and weighted difference of any two consecutive parts is 0. Then,*

$$\sum_{v=1}^{\infty} A_4(v)q^v = \Phi_1(q) \tag{3.17}$$

Proof 3.1.4 The partitions mentioned by $A_4(m,v)$ are mainly those partitions which belong to class (1) of Theorem 3.1.3

$$A_4(z,v) = A_3(m-1, v-2m+1) \tag{3.18}$$

using equations (3.14) and (3.18), we can easily obtain the q-functional equation:

$$f_4(z,q) = f_3(z,q) - zq^2 f_3(z,q) \tag{3.19}$$

From [1], we obtain the following expression

$$f_4(z,q) = \sum_{n=1}^{\infty} q^{n^2}(-q;q^2)_{n-1} z^n \tag{3.20}$$

$$\begin{aligned}
\sum_{v=0}^{\infty} A_4(v)q^v &= \sum_{v=0}^{\infty}(\sum_{n=0}^{\infty} A_4(m,v)q^v \\
&= f_4(1,q) \\
&= \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_{n-1} \\
&= \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n \\
&= \Phi_1(q)
\end{aligned}$$

Later, in 2009, Agarwal and Rana [6] gave the combinatorial interpretations of following fifth order mock theta function using "$n+2$ copies of n"

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} \tag{3.21}$$

Theorem 3.1.5 *For $v \geq 0$, let $B(v)$ denote the number of partitions of v with "$n+2$ copies of n" in which even parts appear with even subscripts and odd with odd greater than 1. For some i, i_{i+2} is a part and weighted difference of any two consecutive parts is zero.*

$$\sum_{v=0}^{\infty} B(v)q^v = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} = F_1(q) \tag{3.22}$$

Proof 3.1.5 To prove this theorem, we recall (3.12), given below

$$f(z,q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} z^n \tag{3.23}$$

Now suppose that $B(m,v)(m > 0)$ denotes the number of partitions mentioned by $B(v)$ into m parts. Clearly if we subtract 2 from each nonempty part of a partition mentioned by $A_2(m,v)$ we get a partition mentioned by $B(m,v-2m)$ and since the transformation is reversible, we see that

$$A_2(m,v) = B(m,v-2m), m > 0, v \geq 2m \tag{3.24}$$

Now for $|q| < 1$ and $|z| < |q^{-1}|$, let $\sum\limits_{v=0}^{\infty}\sum\limits_{m=1}^{\infty} B(m,v)z^m q^v = g(z,q) = \sum\limits_{n=1}^{\infty} \beta_(q)z^n$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \alpha_n(q)z^n &= \sum_{v=2}^{\infty}\sum_{m=1}^{\infty} A_2(m,v)z^m q^v \\
&= \sum_{v\geq 2m}^{\infty}\sum_{m=1}^{\infty} B(m,v-2m)z^m q^v \\
&= g(zq^2,q) \\
&= \sum_{n=1}^{\infty} \beta_n(q)(zq^2)^n
\end{aligned} \tag{3.25}$$

where

$$\alpha_n(q) = \frac{q^{2n^2}}{(q;q^2)_n}; \quad \alpha_0(q) = 1$$

On comparing the coefficients of $z^n(n > 0)$ in the extremes of (3.25), we get

$$\beta_n(q) = \frac{q^{2n^2-2n}}{(q:q^2)_n}, n > 0$$

therefore,

$$g(z,q)=\sum_{n=1}^{\infty}\frac{q^{2n^2-2n}}{(q:q^2)_n}z^n$$

Now,

$$\begin{aligned}\sum_{v=0}^{\infty}B(v)q^v &= \sum_{v=0}^{\infty}(\sum_{m=1}^{\infty}B(m,v))q^v\\ &= g(1,q)\\ &= \sum_{n=1}^{\infty}\frac{q^{2n^2-2n}}{(q:q^2)_n}\\ &= \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q:q^2)_{n+1}}=F_1(q)\end{aligned}$$

3.2 New combinatorial version of Göllnitz–Gordan Identities

In 2009, Agarwal and Rana [7] establish an infinite family of 3-way combinatorial identities using n–color partitions and lattice paths. We discuss some particular cases which provide serval new combinatorial versions of the "Göllnitz–Gordan Identities".

Theorem 3.2.1 *Given a positive integer k, let $A_k(v)$ denote the number of partitions of v in which each part $\geq k$, minimal difference ≥ 2 between the parts, consecutive odd integers are not allowed if k is even and consecutive even integers are not allowed if k is odd.*
Let $B_k(v)$ denote the number of n–color partitions of v such that the parts $\geq k$, parts used are of the type
$(2l-1)_1$ and $(2l)_2$ if k is odd,
$(2l-1)_2$ and $(2l)_1$ if k is even.
The weighted difference between any two parts is nonnegative and even.

Then

$$\sum_{v=0}^{\infty} A_k(v)q^v = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n+k-1} = \sum_{v=0}^{\infty} B_k(v)q^v \tag{3.26}$$

Proof 3.2.1 Let $B_k(m, v)$ denote the number of partitions mentioned by $B_k(v)$ into exactly m-parts. If k is odd, then we split the partitions mentioned by $B_k(v)$ into three classes: (1) those that have least part equal to k_1, (2) those that have least part equal to $(k+1)_2$, and (3) those that have least part greater than or equal to $(k+2)_1$. By using these classes and applying transformation on these classes we find the recurrence relations:

$$\begin{aligned} B_k(m, v) &= B_k(m-1, v-k-2m+2) + B_k(m-1, v-k-4m+3) \\ &+ B_k(m, v-2m) \end{aligned} \tag{3.27}$$

Let

$$g_k(z;q) = \sum_{v=0}^{\infty}\sum_{n=0}^{\infty} B_k(m, v)z^m q^v \tag{3.28}$$

using (3.27) in (3.28) , we get the q-functional equation:

$$g_k(z;q) = zq^k g_k(zq^2;q) + zq^{k+1} g_k(zq^4;q) + g_k(zq^2;q) \tag{3.29}$$

Putting

$$g_k(z,q) = \sum_{n=0}^{\infty} \beta_k(n;q)z^n \tag{3.30}$$

then comparing coefficient of z^n on each side of (3.30), we get

$$\beta_k(n;q) = \frac{(1+q^{2n-1})q^{2n-2+k}}{(1-q^{2n})} \beta(n-1;q) \tag{3.31}$$

Iterating (3.31) n–times and observing that $\beta_k(0;q) = 1$, we see that

$$\beta_k(n;q) = \frac{(-q;q^2)_n q^{n(n+k-1)}}{(q^2;q^2)_n}$$

Therefore,

$$g_k(z;q) = \sum_{n=0}^{\infty} \beta_k(n;q) z^n = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+k-1)}}{(q^2;q^2)_n} z^n$$

Now,

$$\sum_{v=0}^{\infty} B_k(v) q^v = \sum_{v=0}^{\infty} [\sum_{m=0}^{\infty} B_k(m,v)] q^v$$

$$= g_k(1;q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+k-1)}}{(q^2;q^2)_n}$$

.

3.2.1 Particular Cases.

If $k = 1$ then equation (3.26), in conjunction with [19, I(36), p.155] reduces to **First Göllintz–Gordan Identity** given by (1.3)

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{8n+1})(1-q^{8n+4})(1-q^{8n+7})} \tag{3.32}$$

If $k = 3$ then equation (3.26), in conjunction with [19, I(34), p.155] reduces to **Second Göllintz–Gordan Identity** given by (1.4)

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{8n+3})(1-q^{8n+4})(1-q^{8n+5})} \tag{3.33}$$

In 2010, Goyal and Agarwal [13] further interpret the similar basic q–series as combinatorially with n–color partitions, given below as:

$$\sum_{v=0}^{\infty} A_k(v)q^v = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n}$$

Theorem 3.2.2 *For a positive integer k, let $A_k(v)$ denote the number of n–color partitions of v such that*

(1) parts are of the form $(2l-1)_1$ or $(2l)_2$, if k is odd and of form $(2l-1)_2$ or $(2l)_1$, if k is even,

(2) if m_i is the smallest or only part in the partition, then $m \equiv i+k-1(mod\ 4)$ and

(3) the weighted difference between any two consecutive parts is non-negative and is $\equiv 0(mod\ 4)$.

Proof 3.2.2 Let $A_k(m,v)$ denote the number of partitions mentioned by $A_k(v)$ into m parts. We split the partitions mentioned by $A_k(m,v)$ into three classes: (1) that have least part equal to k_1,

(2) that have least part equal to $(k+1)_2$, and

(3) that have least part greater than or equal to $(k+2)_1$,

using these three classes we find the recurrence relation.

$$\begin{aligned} A_k(m,v) &= A_k(m-1, v-k-2m+2) + A_k(m-1, v-k-4m+3) \\ &+ A_k(m, v-4m) \end{aligned} \tag{3.34}$$

using this recurrence relation q–functional equation

$$f_k(z;q) = zq^k f_k(zq^2;q) + zq^{k+1} f_k(zq^4;q) + f_k(zq^4;q) \tag{3.35}$$

since $f_k(0;q) = 1$, then

$$f_k(z;q) = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n z^n}{(q^4;q^4)_n} \tag{3.36}$$

Now

$$\begin{aligned}\sum_{v=0}^{\infty} A_k(v)q^v &= \sum_{v=0}^{\infty}(\sum_{n=0}^{\infty} A_k(m,v))q^v \\ &= f_k(1;q) \\ &= \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n}\end{aligned}$$

3.3 Further Rogers–Ramanujan Identities for n–color partitions.

In [12], we find the various q–identities, combinatorially interpreted using colored partitions. These identities had been introduced by Rogers [15].

$$\sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q;q^2)_n(q^4;q^4)_n} = \frac{(-q^3,-q^5,-q^7;q^{10})_\infty}{(q^4,q^6;q^{10})_\infty} \tag{3.37}$$

$$\sum_{n=0}^{\infty} \frac{q^{3n^2-2n}}{(q;q^2)_n(q^4;q^4)_n} = \frac{(-q,-q^5,-q^9;q^{10})_\infty}{(q^2,q^8;q^{10})_\infty} \tag{3.38}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n(q^4;q^4)_n} = \frac{(-q^3,-q^7,-q^{11};q^{14})_\infty}{(q^2,q^6,q^8,q^{12};q^{14})_\infty} \tag{3.39}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_n(q^4;q^4)_n} = \frac{(-q^5,-q^7,-q^9;q^{14})_\infty}{(q^4,q^6,q^8,q^{10};q^{14})_\infty} \tag{3.40}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}(q^4;q^4)_n} = \frac{(-q,-q^7,-q^{13};q^{14})_\infty}{(q^2,q^4,q^{10},q^{12};q^{14})_\infty} \tag{3.41}$$

We observe that identities (3.37) and (3.38) were also obtained by Bailey [8] and appear in [19], Identities (3.39)–(3.41) are also mentioned as Rogers–Selberg identities [15,18,19].

The q–identities (3.37)–(3.41) have their combinatorial counterparts in the following theorems, respectively.

where $f_i(z, q)$ will denote the 2-variable generating function

$$f_i(z;q) = \sum_{v=0}^{\infty}\sum_{m=0}^{\infty} A_i(m,v) z^m q^v \tag{3.42}$$

where $|q| < 1$ and $|z| < |q|^{-1}$.

Theorem 3.3.1 *Let $A_1(v)$ denote the number of n–color partitions of v such that even parts appear with even subscripts and odd with odd, all subscripts are > 2, if m_i is smallest or only part in the partition, then $m \equiv i(mod\ 4)$ and the weighted difference of any two consecutive parts is non-negative and is $\equiv 0(mod\ 4)$. Let*

$$B_1(v) = \sum_{k=0}^{v} C_1(v-k) D_1(k)$$

where $C_1(v)$ is the number of partitions of v into parts $\equiv \pm 4(mod\ 10)$ and $D_1(v)$ denotes the number of partitions of v into distinct parts $\equiv \pm 3, 5(mod\ 10)$. Then

$$A_1(v) = B_1(v), \forall v$$

Proof 3.3.1 We split the partitions mentioned by $A_1(m, v)$ into three class:

(1) that do not contain k_k as a part,

(2) that contain 3_3 as a part, and

(3) that contain $k_k, k > 3$ as a part.

Using these three classes we find the recurrence relations:

$$\begin{aligned} A_1(m,v) &= A_1(m,v-4m) + A_1(m-1,v-6m+3) \\ &+ A_1(m,v-2m+1) - A_1(m,v-6m+1) \end{aligned} \tag{3.43}$$

Substituting $A_1(m,v)$ from (3.43) into equation (3.42) and then simplifying, we get

$$f_1(z;q) = f_1(zq^4;q) + zq^3 f_1(zq^6;q) + q^{-1} f_1(zq^2;q) - q^{-1} f_1(zq^6;q) \tag{3.44}$$

Since $f_1(0;q) = 1$, we may easily check by coefficient in equation (3.44) that

$$f_1(z;q) = \sum_{n=0}^{\infty} \frac{q^{3n^2} z^n}{(q;q^2)_n (q^4;q^4)_n} \tag{3.45}$$

Now

$$\begin{aligned} \sum_{v=0}^{\infty} A_1(v) q^v &= \sum_{v=0}^{\infty} \Big(\sum_{m=0}^{\infty} A_1(m,v)\Big) q^v \\ &= f_1(1,q) \\ &= \sum_{n=0}^{\infty} \frac{q^{3n^2} z^n}{(q;q^2)_n (q^4;q^4)_n} \\ &= \frac{(-q^3,-q^5,-q^7;q^{10})_\infty}{(q^4,q^6;q^{10})_\infty} \\ &= \sum_{v=0}^{\infty} B_1(v) q^v \end{aligned}$$

Theorem 3.3.2 *Let $A_2(v)$ denote the number of n–color partitions of v such that even parts appear with even subscripts and odd with odd, if m_i is smallest or only part in the partition, then $m \equiv i (mod\ 4)$ and the weighted difference of any two consecutive parts ≥ 4 and is $\equiv 0 (mod\ 4)$. Let*

$$B_2(v) = \sum_{k=0}^{v} C_2(v-k)D_2(k)$$

where $C_2(v)$ *is the number of partitions of* v *into parts* $\equiv \pm 2(mod\ 10)$ *and* $D_2(v)$ *denotes the number of partitions of* v *into distinct parts* $\equiv \pm 1, 5(mod\ 10)$. *Then*

$$A_2(v) = B_2(v), \forall\ v$$

Proof 3.3.2 We split the partitions mentioned by $A_2(m, v)$ into three class:
(1) that do not contain k_k as a part,
(2) that contain 1_1 as a part, and
(3) that contain $k_k, k > 1$ as a part.
using these three classes we find the recurrence relations:

$$\begin{aligned} A_2(m, v) &= A_2(m, v-4m) + A_2(m-1, v-6m+5) \\ &+ A_2(m, v-2m+1) - A_2(m, v-6m+1) \end{aligned} \quad (3.46)$$

Substituting $A_2(m, v)$ from (3.46) into equation (3.42) and then simplifying, we get

$$f_2(z;q) = f_2(zq^4;q) + zqf_2(zq^6;q) + q^{-1}f_2(zq^2;q) - q^{-1}f_2(zq^6;q) \quad (3.47)$$

using this q-functional equation in [1] ,we obtain the following expression:

$$\begin{aligned} \sum_{v=0}^{\infty} A_2(v)q^v &= \sum_{v=0}^{\infty}(\sum_{m=0}^{\infty} A_2(m, v))q^v \\ &= f_2(1, q) \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{q^{3n^2-2n} z^n}{(q;q^2)_n (q^4;q^4)_n}$$
$$= \frac{(-q, -q^5, -q^9; q^{10})_\infty}{(q^2, q^8; q^{10})_\infty}$$
$$= \sum_{v=0}^{\infty} B_2(v) q^v$$

Theorem 3.3.3 *Let $A_3(v)$ denote the number of n–color partitions of v such that even parts appear with even subscripts and odd with odd, all subscripts are > 3, if m_i is smallest or only part in the partition, then $m \equiv i(mod\ 4)$ and the weighted difference of any two consecutive parts is nonnegative and is $\equiv 0(mod\ 4)$. Let*

$$B_3(v) = \sum_{k=0}^{v} C_3(v-k) D_3(k)$$

where $C_3(v)$ is the number of partitions of v into parts $\equiv \pm 2, \pm 6(mod\ 14)$ and $D_3(v)$ denotes the number of partitions of v into distinct parts $\equiv \pm 3, 7(mod\ 14)$. Then

$$A_3(v) = B_3(v), \forall\ v$$

Proof 3.3.3 We split the partitions mentioned by $A_3(m, v)$ into three class:
(1) that do not contain k_k as a part,
(2) that contain 2_2 as a part, and
(3) that contain $k_k, k > 2$ as a part.
using these three classes we find the recurrence relations:

$$\begin{aligned} A_3(m, v) &= A_3(m, v-4m) + A_3(m-1, v-4m+2) \\ &+ A_3(m, v-2m+1) - A_3(m, v-6m+1) \qquad (3.48) \end{aligned}$$

Substituting $A_3(m, v)$ from (3.48) into equation (3.42) and then simplifying, we get

$$f_3(z;q) = f_3(zq^4;q) + zq^2 f_3(zq^4;q) + q^{-1} f_3(zq^2;q) - q^{-1} f_3(zq^6;q) \quad (3.49)$$

using this q-functional equation in [1] ,we obtain the following expression:

$$\begin{aligned}\sum_{v=0}^{\infty} A_3(v)q^v &= \sum_{v=0}^{\infty}(\sum_{m=0}^{\infty} A_3(m,v))q^v \\ &= f_3(1,q) \\ &= \sum_{n=0}^{\infty} \frac{q^{2n^2} z^n}{(q;q^2)_n (q^4;q^4)_n} \\ &= \frac{(-q^3,-q^7,-q^{11};q^{14})_\infty}{(q^2,q^6,q^8,q^{12};q^{14})_\infty} \\ &= \sum_{v=0}^{\infty} B_3(v)q^v\end{aligned}$$

Theorem 3.3.4 *Let $A_4(v)$ denote the number of n–color partitions of v such that even parts appear with even subscripts and odd with odd, all subscripts are > 3, if m_i is smallest or only part in the partition, then $m \equiv i (mod\ 4)$ and the weighted difference of any two consecutive parts is ≥ -4 and is $\equiv 0 (mod\ 4)$. Let*

$$B_4(v) = \sum_{k=0}^{v} C_4(v-k) D_4(k)$$

where $C_4(v)$ is the number of partitions of v into parts $\equiv \pm 4, \pm 6 (mod\ 14)$ and $D_4(v)$ denotes the number of partitions of v into distinct parts $\equiv \pm 5, 7 (mod\ 14)$. Then

$$A_4(v) = B_4(v), \forall\ v$$

Proof 3.3.4 We split the partitions mentioned by $A_4(m,v)$ into three class: (1) that do not contain k_k as a part,

(2) that contain 4_4 as a part, and

(3) that contain $k_k, k > 4$ as a part.

Using these three classes we find the recurrence relations:

$$\begin{aligned} A_4(m,v) &= A_4(m, v-4m) + A_4(m-1, v-4m) \\ &+ A_4(m, v-2m+1) - A_4(m, v-6m+1) \qquad (3.50) \end{aligned}$$

Substituting $A_4(m,v)$ from (3.50) into equation (3.42) and then simplifying, we get

$$f_4(z;q) = f_4(zq^4;q) + zq^4 f_4(zq^4;q) + q^{-1} f_4(zq^2;q) - q^{-1} f_4(zq^6;q) \quad (3.51)$$

Using this q-functional equation in [1] ,we obtain the following expression:

$$\begin{aligned} \sum_{v=0}^{\infty} A_4(v) q^v &= \sum_{v=0}^{\infty} (\sum_{m=0}^{\infty} A_4(m,v)) q^v \\ &= f_4(1,q) \\ &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)} z^n}{(q;q^2)_n (q^4;q^4)_n} \\ &= \frac{(-q^5, -q^7, -q^9; q^{14})_\infty}{(q^4, q^6, q^8, q^{10}; q^{14})_\infty} \\ &= \sum_{v=0}^{\infty} B_4(v) q^v \end{aligned}$$

Theorem 3.3.5 *Let $A_5(v)$ denote the number of n–color partitions of v with "$n+2$ copies of n" such that even parts appear with even subscripts and odd with odd, all subscripts are > 1, for some $i, i+2$ is a part and the weighted difference of any two consecutive parts is non-negative and is $\equiv 0 (mod\ 4)$. Let*

$$B_5(v) = \sum_{k=0}^{v} C_5(v-k) D_5(k)$$

where $C_5(v)$ *is the number of partitions of* v *into parts* $\equiv \pm 2, \pm 4 (mod\ 14)$ *and* $D_5(v)$ *denotes the number of partitions of* v *into distinct parts* $\equiv \pm 1, 7 (mod\ 14)$. *Then*

$$A_5(v) = B_5(v), \forall\ v$$

Proof 3.3.5 The proof is similar to that of Theorem 3.3.4 , hence we omit the details and give only recurrence relation used in this case.

$$A_5(m-1, v-2m) = A_3(m, v) - A_3(m, v-4m) \tag{3.52}$$

Using this recurrence relations, we find q-functional equations:

$$zq^2 f_5(zq^2, q) = f_3(zq, q) - f_3(zq^4, q)$$

Using this q-functional equation in [1] ,we obtain the following expression:

$$\begin{aligned}
\sum_{v=0}^{\infty} A_5(v) q^v &= \sum_{v=0}^{\infty} (\sum_{m=0}^{\infty} A_5(m, v)) q^v \\
&= f_5(1, q) \\
&= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)} z^n}{(q; q^2)_{n+1} (q^4; q^4)_n} \\
&= \frac{(-q, -q^7, -q^{13}; q^{14})_\infty}{(q^2, q^4, q^{10}, q^{12}; q^{14})_\infty} \\
&= \sum_{v=0}^{\infty} B_5(v) q^v
\end{aligned}$$

3.4 n–color partitions with weighted difference equal to -2

In 1997, Agarwal and Balasubrananian [5] discuss the study on n–color partitions with weighted difference equal to -2. It is shown in [5] that these partitions give rise to an explicit expression for the sum of the divisors of odd integers. These partitions are also linked with conjugate and self-conjugate n–color partitions.

Definition 3.4.1 *Let* $\Pi = (a_1)_{b_1} + (a_2)_{b_2} + (a_3)_{b_3} + \cdots + (a_r)_{b_r}$ *be an* n*–color partitions of* v*. Then we call* $(a_i)_{a_i - b_i + 1}$ *the conjugate of* $(a_i)_{b_i}$ *and conjugate of* Π *denoted by* Π^c*.*

An n–color partitions is said to be **Self-Conjugate** if it is identical with its conjugate. Thus $5_3 + 3_2 + 1_1$ is a self-conjugate partitions of 9.

3.4.1 Some intersecting identity on n–color partitions

Theorem 3.4.1 *Let* $A(v)$ *denote the number of* n*–color partitions of positive integer* v *with weighted difference of each pair of part is -2. Let* $B(v)$ *denote the number of* n*–color partitions of* v *such that in each pair of parts* $m_i, n_j (m > n)$ n *is the arithmetic mean of the subscripts* i *and* j*. Then*

$$A(v) = B(v), \qquad \forall\ v$$

Example 3.4.1 $A(5) = B(5) = 11.$
where the relevant partitions for $A(5)$ *are* $5_1, 5_2, 5_3, 5_4, 5_5,$
$4_4 1_1, 3_1 2_2, 3_2 2_1, 3_3 1_1 1_1, 2_2 1_1 1_1 1_1, 1_1 1_1 1_1 1_1 1_1.$
and the relevant partitions for $B(5)$ *are* $5_1, 5_2, 5_3, 5_4, 5_5,$
$4_1 1_1, 3_3 2_1, 3_2 2_2, 3_1 1_1 1_1, 2_1 1_1 1_1 1_1, 1_1 1_1 1_1 1_1 1_1.$

Theorem 3.4.2 *Let* $A(m, v)$ *denote the number of* n*–color partitions of* v *enumerated by* $A(v)$ *into exactly* m*-parts and the weighted difference of each*

pair of parts is -2.
Let $C(m,v)$ denote the number of ordinary partitions of all numbers $\leq v$ with minimum part m and the difference between parts 0 and $m-1$. Then

$$A(m,v) = C(m,v)$$

Theorem 3.4.3 *Let $D(m,v)$ denote the number of n–color partitions of v enumerated by $D(v)$ into exactly m-parts and the weighted difference of each pair of parts is -2.*
Let $F(m,v)$ denote the number of ordinary partitions of all numbers $\leq v$ into parts where the lowest part is m which does not repeat and differences between parts are 0 and $m-1$. Then

$$D(m,v) = F(m,v)$$

.

Example 3.4.2 *Consider the case when m=2 and v=11.*
$D(2,11) = F(2,11) = 4$,
since the relevant partitions for $D(2,11)$ are $10_{10}1_1, 9_72_2, 8_43_3, 7_14_4$.
also, relevant partitions for $F(2,11)$ are $2, 2+3, 2+3+3, 2+3+3+3$.

Theorem 3.4.4 *Let $F(v)$ denote the number of self-conjugate n–color partitions of v and $G(v)$ denote those n–color partitions of v where each pair of parts has weighted difference greater than 1 and even parts appear with even subscripts and odd with odd. Then*

$$F(v) = G(v) \qquad \forall\ v$$

Proof 3.4.1 We observe that if an n–order partitions is self conjugate then in each part m_i, m must be odd. Because $m_i = m_{m-i+1} \Rightarrow m = 2i+1$. Thus if we ignore the subscripts of all parts in a self-conjugate n–color partition of v, we get a unique ordinary partitions of v into odd parts.

Conversely, if we consider an ordinary partitions of v into odd parts and replace each part by $2a-1$ by $(2a-1)_a$. We get a unique self-conjugate n–color partitions of v. This bijections shows that the number of self-conjugate n–color partitions of v equals the number of ordinary partitions of v into odd parts. That is,

$$1 + \sum_{v=1}^{\infty} F(v)q^v = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})}$$

Example 3.4.3 *$F(7) = G(7) = 5$,*
the relevant partitions for $F(7)$ are $7_4, 5_3 1_1 1_1, 3_2 3_2 1_1, 3_2 1_1 1_1 1_1 1_1, 1_1 1_1 1_1 1_1 1_1 1_1 1_1$.
and the relevant partitions for $G(7)$ are $7_1, 7_3, 7_5, 7_7, 6_2 1_1$.

3.5 Conclusion

The colored partitions have proved to be a successful tools in enumerating the q–series and are closely related with other combinatorial tools such as lattice paths, F–partitions and plane partitions.

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