On Generalized Convergence in Sequence Spaces

Thesis

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in

MATHEMATICS

by

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Certificate

This is to certify that the thesis entitled **"On Generalized Convergence in Sequence Spaces"**, submitted by **Sudhir Kumar (Reg. No: 951011006)**, in fulfillment of the requirement for the award of the degree of *Doctor of Philosophy* in School of Mathematics, Thapar University, Patiala is a record of candidate's own independent and original research work carried out by him under our supervision and guidance. The matter embodied in this thesis has not been submitted in part or full to any other University or Institute for the award of any degree.

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Declaration

It is certified that the thesis is entirely my own and that the ideas and references cited herein have been duly acknowledged.

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Dedicated

To

My Parents

To my supervisors ...

Guru Govind dou Khade, Kaake laagoon Paye
Balihari guru aapno, govind diyo milaye.

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December, 2017

(Sudhir Kumar)

Preface

The present thesis entitled **"On Generalized Convergence in Sequence Spaces"** comprises certain investigations carried out by me at the School of Mathematics (SOM), Thapar University, Patiala, under the supervision of Dr. S. S. Bhatia, Professor, School of Mathematics, Thapar University, Patiala and Dr. V. K. Kaushik, Professor, Panipat Institute of Engineering and Technology, Panipat, Haryana.

In modern analysis, the concept of convergence includes the study of analysis, operators and transformations theory, topology and functional analysis, which provides a general framework for finding solutions of different type of problems in pure and applied mathematics. Apart this, a study of convergence theory in itself presents a new insight and understanding into the processes and techniques of elementary analysis, which we accustomed to use in our daily calculations. Many generalizations of an ordinary convergence have appeared in literature so far, out of which a generalized summability method called as statistical convergence has become an active area of research in recent years.

Over the years, statistical convergence has been studied under the different names in the theory of analysis. In 1951, H. Fast [54] provided a different direction to ordinary convergence by introducing a regular summability method called as statistical convergence. Moreover, Fridy [58] accelerated the progress of statistical convergence by presenting statistical analogue of numerous concepts known in the history of ordinary convergence.

In recent years, much attention has been paid to study generalized convergence in some abstract spaces. In the present thesis, we have made efforts to develop some classical methods of summability and introduce some summability theories in different spaces like 2-normed Spaces, probabilistic normed spaces and double sequence spaces followed by characterization of certain properties of these generalized convergence methods in the above-mentioned sequence spaces.

The whole work in the thesis has been divided into five chapters. **Chapter – I** is an introductory one in which, the basic terminology of some generalized summability methods and abstract spaces under consideration in present thesis has been presented. We start with the definition of statistical convergence, which was initially introduced by H. Fast [54] and see how it has been extended in various abstract spaces to resolve many problems in the area of mathematical analysis. We also record some facts related to statistical convergence and its generalizations from the history, which will be needed in the sequel and form the base for the present thesis.

Chapter – II is devoted to the development of new sequence spaces $V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$, $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$ and $V_\lambda^f[\Delta_p^m, \mathcal{I}]_\infty$. These sequence spaces have been defined by the use of de la vallee poussin mean, modulus function f and the difference operator Δ. After making some early remarks on these spaces, some of their important properties and inclusion relations have been established in this chapter. Some concrete examples in support of our results have also been provided. Subsequently, the concept of $S_\lambda^{\Delta^m}(\mathcal{I})$-convergence has also been introduced and a condition is obtained under which $S_\lambda^{\Delta^m}(\mathcal{I})$-convergence coincides with above-mentioned sequence spaces.

In **Chapter – III**, we consider one of the generalized form of abstract spaces i.e. 2-norm space. The aim in this chapter is to extend the sequence spaces $V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$, $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$ and $V_\lambda^f[\Delta_p^m, \mathcal{I}]_\infty$ introduced in Chapter-II to the sequence spaces $V_\lambda^f[\|, \|, \Delta^m, \lambda, p]_0$, $V_\lambda^f[\|, \|, \Delta^m, \lambda, p]_L$ and $V_\lambda^f[\|, \|, \Delta^m, \lambda, p]_\infty$ in 2-norm spaces and have obtained some of their properties. A new class of generalized statistical convergent sequences has also been defined in this chapter with the help of an ideal and difference sequences. Further, we also establish a strong connection between this type of convergence and the aforesaid sequence space.

The aim of **Chapter – IV** is to introduce some new sequence spaces using ideals, modulus function and invariant means and examine some of their topological properties. We start with the introduction of the new sequence spaces $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$, $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_L$ and $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_\infty$ and prove that these are linear spaces. We also find some inclusion relations among these spaces by imposing some conditions on the modulus function f. Finally, a new kind of summability method called as $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})$-convergence for sequences of numbers has been presented in this chapter. Towards the end of this chapter, lacunary generalized convergence in probabilistic normed space has been developed.

The goal of **Chapter – V** is to introduce the concepts of $\mathcal{I}-$lacunary strongly Cesàro summability $(N_{\theta_{rs}}(\mathcal{I}_2))$ and $\mathcal{I}$-lacunary statistical convergence $(S_{\theta_{rs}}(\mathcal{I}_2))$ for double sequence of numbers with help of double lacunary sequence. Some important properties of these notions have been obtained in this chapter and it is proved that the set of all lacunary ideal statistically convergent bounded sequences is closed. Subsequently, a stronger form of $S_{\theta_{rs}}(\mathcal{I}_2)$-convergence has been defined with the help of ideals. Further, in this chapter some relevant connections between these notions have also been established.

Research Publications

Papers Published/Accepted in Referred Journals

1. Sudhir Kumar, V. Kumar and S.S. Bhatia "Generalized Sequence Spaces in 2-normed spaces defined by Ideal and a Modulus Function", **Analele Stiintifice ale Universitatii Alexandruioan Cuza din Iasi-Serie Noua-Mathematica DOI 10.2478/aicu-2014-0024, pp. 1-12. (SCI)**

2. Sudhir Kumar, V. Kumar and S.S. Bhatia "On Ideal Version of Lacunary Statistical Convergence of Double Sequences", **General Mathematics Notes, Vol. 17, No. 1, (2013) pp. 32-44.**

3. Sudhir Kumar, V. Kumar and S.S. Bhatia "Some New Difference Sequence Spaces of Invariant Means Defined by Ideal and Modulus Function", **International Journal of Analysis, Vol. 2014, Article Id. 631301, 7 pages, doi.org/10.1155/2014/631301.(HINDAWI)**

4. Sudhir Kumar, S.S. Bhatia and V. Kumar "On Generalized Sequence Spaces defined by Ideal and Modulus Function" **Atti Della Fondazione Giorgio Ronchi(Topology), Anno LXXI, No. 2, (2016), pp. 229-240.**

5. Sudhir Kumar, S.S. Bhatia and V. Kumar "On Statistically and Lacunary Statistically Ideal Convergence of Double Sequences", **Southeast Asian Bulletin of Mathematics, Vol. 41, (2017) pp. 421-430.**

Communicated Papers

1. Sudhir Kumar, S. S. Bhatia and V. Kumar " Generalized Statistical Convergence in Probabilistic Normed Space" **General Mathematics Notes.**

List of Symbols

1. $\mathbb{N}$ the set of positive integers.
2. $\mathbb{R}$ the set of real numbers.
3. $\mathbb{R}_0^+$ the set of positive real numbers.
4. χ_K the characteristic function of the set K.
5. $\delta(A)$ natural density of the set A.
6. $\overline{\delta}(A)$ upper natural density of the set A.
7. $\underline{\delta}(A)$ lower natural density of the set A.
8. $\delta_\theta(A)$ θ-density of the set A.
9. $\delta_\lambda(A)$ λ-density of the set A.
10. S the set of all statistically convergent sequences of numbers.
11. S_θ the set of all lacunary statistically convergent sequences of numbers.
12. S_λ the set of all λ-statistical convergent sequences of numbers.
13. ℓ_∞ the normed linear space of all bounded sequences of real numbers.
14. $P(X)$ the power set of any set X.
15. $\|.\|$ the norm of a normed linear space.
16. $\|.,\|$ the norm of a 2-normed space.
17. $|\sigma_1|$ the space of all strongly Casaro summable sequences of real numbers.
18. w_p the space of all strongly p-Casaro summable sequences of real numbers.

19. $[V_\sigma]$ the set of all σ-convergent sequences of real numbers.

20. f stands for modulus function.

21. Δ the difference operator.

22. $S_\lambda(\Delta^m)$ the set of all λ-statistical convergent generalized difference sequences $(\Delta^m x_k)$ of real numbers.

23. St_{θ_σ} the set of all strong σ-lacunary statistically convergent sequences of real number.

24. $S(\mathcal{I})$ the set of all $\mathcal{I}$-statistically convergent sequences of real numbers.

25. $N_\theta(\mathcal{I})$ the set of all $\mathcal{I}$-lacunary strong Casàro summable sequences of real numbers.

26. $S_\theta(\mathcal{I})$ the set of all $\mathcal{I}$-lacunary statistical convergent sequences of real numbers.

27. $S_\lambda(\mathcal{I})$ the set of all$\mathcal{I}$ -λ -statistical convergent sequences of real numbers.

28. $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})$ the set of all strong $\mathcal{I}$ - σ-statistically convergent generalized difference sequences $(\Delta^m x_k)$ of real numbers.

29. $S_\lambda^{\Delta^m}(\mathcal{I})$ the set of all $\mathcal{I}$-λ-statistically convergent generalized difference sequences $(\Delta^m x_k)$ of real numbers.

30. $(C, 1, 1)$ the set of all Casàro summable double sequences of real numbers.

31. $|\sigma_{11}|$ the space of all strongly Casàro summable double sequences of real numbers.

32. $\delta_2(A)$ double natural density of the set A.

33. S_2 the set of all statistical convergent double sequences of real numbers.

34.	ℓ_2^∞	the set of all bounded double sequences of real numbers.
35.	$S_2(\mathcal{I}_2)$	the set of all $\mathcal{I}_2$-statistically convergent double sequences of real numbers.
36.	$S_2(\mathcal{I}_2^*)$	the set of all $\mathcal{I}_2^*$-statistically convergent double sequences of real numbers.
37.	$S_{\theta_{rs}}$	the set of all lacunary statistically convergent double sequences of real numbers.
38.	$S_{\theta_{rs}}(\mathcal{I}_2)$	the set of all $\mathcal{I}_2$-lacunary statistically convergent double sequences of real numbers.
39.	$S_{\theta_{rs}}(\mathcal{I}_2^*)$	the set of all $\mathcal{I}_2^*$-lacunary statistically convergent double sequences of real numbers.
40.	D	the set of all distribution functions on $\mathbb{R}$.
41.	D^+	the set of all distance distribution functions on $\mathbb{R}_0^+$.
42.	$S^F(\mathcal{I})$	the set of all $\mathcal{I}$-statistically convergent sequences in probabilistic normed space $(X, F, *)$
43.	$S_\theta^F(\mathcal{I})$	the set of all $\mathcal{I}$-lacunary statistically convergent sequences in probabilistic normed space $(X, F, *)$
44.	$S_\lambda^{\Delta^m}(\|.,\|,\mathcal{I})$	the set of all $\mathcal{I}$-λ-statistically convergent generalized difference sequences $(\Delta^m x_k)$ of numbers in 2-normed space.
45.	$\mathbb{C}$	the set of real scalers.

Table of Contents

Chapter 1

Introduction

1.1 General Introduction

Mathematical analysis is that branch of mathematics, which includes calculus, theory of functions of real and complex variable, measure theory, differential equations, infinite series, numerical analysis, harmonic analysis etc. Among these, calculus was developed at the end of seventeenth century by Newton, Leibnitz and many others. It has been useful to resolve many problems in the area of Mathematics, Physics and Engineering sciences. Infinite series played important role in the development of calculus, but on the same side many mathematicians were facing problems in dealing with limits of infinite series and related theories, particularly with those series which were having divergent behavior. As a result, mathematicians tried to think, how a divergent series be summed up to finite limit, which today we call as "generalized limit" or "summability method". In this way, a new branch of mathematical analysis came into the existence with the aim to assign a limit(in some sense) to divergent sequences. This idea of research had grown very fast in the last century and many mathematicians played pioneer role in the development of many summability methods for divergent sequences and series. One among these was given by Cesàro as follows;

For any sequence $x = (x_k)$, if a new sequence $y = (y_n)$ is defined as:

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{k=1}^{n} x_k.$$

Then, the sequence $x = (x_k)$ is called Cesàro summable to a number ξ if $\lim_{n\to\infty} y_n = \xi$. Symbolically, it can be written as $x_k \to \xi\ (C, 1)$ or $\lim_{k\to\infty} x_k = \xi\ (C, 1)$.

The following example is a milestone in which the sequence is $(C, 1)$-summable, although it is divergent.

Example 1.1.1. *If* $x_k = (-1)^k (k \in \mathbb{N})$, *then*

$$y_n = \begin{cases} 0, & \text{for } n = 2k, \\ -\frac{1}{n}, & \text{for } n = 2k+1. \end{cases}$$

We see $\lim_{n\to\infty} y_n = 0$, which gives that $x = (x_k)$ *is* $(C, 1) -$ *summable to* 0. *That is*

$$\lim_{k\to\infty} x_k = 0\ (C, 1).$$

It is also noted that $(C, 1)$-summability preserve the usual convergence i.e. every convergent sequence is always $(C, 1)$-summable to their own limit. Thus, with the help of arithmetic means, we see a divergent sequence may be treated as a convergent sequence. Moreover, $(C, 1)$-summability can be extended to $(C, 2)$-summability and further upto (C, k)-summability. These generalized summability methods can also be expressed using infinte matrix transformation as follows:

For any sequence $x = (x_k)$, the A-transformation is given by $y = (y_n)$ where

$$y_n = Ax = \left(\sum_{k=1}^{\infty} a_{nk} x_k \right)_n , \qquad n = 1, 2, ... \tag{1.1.1}$$

The sequence $x = (x_k)$ is said to be A-convergent (or A-summable) to a number L, if the transformed sequence $y = (y_n)$ exists and is convergent to L. In this case, we write

$$A - \lim_{k \longrightarrow \infty} x_k = L.$$

Let $F(A)$ denotes the set of all A-summable sequences and is usually called the convergence field of the matrix transformation A.
If we take the matrix A as follows

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

then A-summability as defined in (1.1.1) coincides with the Casàro summability.

A matrix summability method is called regular, if $F(A)$ contains all convergent sequences such that $\lim_{k\to\infty} x_k = A - \lim_{k\to\infty} x_k$. Moreover, the matrix A is said to be regular matrix. Silverman and Toeplitz gave the following important theorem for a matrix to be regular:

Theorem 1.1.1. *An infinite matrix $A = (a_{nk})$ defines a regular summability method if and only if it satisfies the following properties;*

(i) there exists $c > 0$ such that for each $n = 1, 2, \cdots$ we have $\sum_{k=1}^{\infty} |a_{nk}| \le c$;

(ii) $\lim_{n\to\infty} a_{nk} = 0$ *holds for each fixed positive integer k.*

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1.$

Mathematicians like Cesàro, Hardy, Schur, Teoplitz have developed summabilty methods via infinite matrix transformation. H. Fast [54] in 1951 gave a non-matrix summability method with the help of natural density of subset of $\mathbb{N}$ and called it as statistical convergence. In the next section, we present a brief introduction and historical background of statistical convergence which was introduced about 80 years ago and has now become an active area of research and forms the basis of present study. Further in this introductory chapter, a summary of basic concepts, techniques and a brief chapter wise resume of the results contained in the thesis has been provided to give sufficient background for the later chapters. However, some of the definitions and notations will be repeated occasionally in various chapters for the sake of convenience.

1.2 Statistical Convergence

The idea of statistical convergence goes back to the first edition of most celebrated monograph of Zygmund [156] published in 1935, but it was formally introduced by Steinhaus [143] and Fast [54] in 1951 and was later reintroduced by Schoenberg [138] in 1959. The concept of statistical convergence is based on the notion of natural density of subsets of $\mathbb{N}$, the set of positive integers. Before proceeding further, we present the notion of natural density of subsets of $\mathbb{N}$.

For any set $K \subset \mathbb{N}$, the asymptotic (or natural) density of K, denoted as $\delta(K)$ and is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_K\,(k) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$$

provided the limit on right hand side exists. Here χ_K denotes the characteristic function of K and the sum on the right side of the above expression runs over the cardinality of the set $\{k \leq n : k \in K\}$. If $\delta(K) = 0$, then K is said to be a thin subset, otherwise K is called a non-thin subset of $\mathbb{N}$. Since $\delta(K)$ does not exist for all subsets of $\mathbb{N}$, so sometimes it is convenient to use the upper asymptotic (natural) density denoted by $\overline{\delta}(K)$, defined by

$$\overline{\delta}(K) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k) = \limsup_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|.$$

In the similar way, we can define the lower asymptotic (natural) density $\underline{\delta}(K)$ as

$$\underline{\delta}(K) = \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k) = \liminf_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|.$$

For sake of convenience, we recall some basic properties of δ, $\overline{\delta}$ and $\underline{\delta}$, which will be used frequently in the present thesis.

For arbitrary subsets A and B of $\mathbb{N}$.

(i) If $\delta(A)$ exists, then $\delta(A) = \underline{\delta}(A) = \overline{\delta}(A)$;

(ii) A is non-thin if and only if $\overline{\delta}(A) > 0$;

(iii) If $A \subset B$, then $\overline{\delta}(A) \leq \overline{\delta}(B)$;

(iv) If A is finite set, then $\delta(A) = 0$;

(v) If $A = a_1 < a_2 < \cdots < a_n < \cdots$ is an infinite set, then

$\delta(A) = \lim_{n\to\infty} \frac{n}{a_n}$, provided the limit exists;

(vi) $\delta(A^c) = 1 - \delta(A)$, where $A^c = \mathbb{N} - A$, provided $\delta(A)$ exists;

(vii) $\delta(\{2n : n \in \mathbb{N}\}) = \delta(\{2n-1 : n \in \mathbb{N}\}) = \frac{1}{2}$;

(viii) $\delta(\{n^2 : n \in \mathbb{N}\}) = 0$.

Now, we present the formal definition of statistical convergence introduced by Henry Fast [54] in 1951 by using the concept of natural(asymptotic) density.

Definition 1.2.1. [54] A number sequence $x = (x_k)$ is said to be statistical convergent to a number L provided that for each $\epsilon > 0$, we have

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, L is said to be the statistical limit of $x = (x_k)$ and is abbreviated as $S - \lim x_k = L$.

Let S denotes the set of all statistically convergent sequences of real numbers. This set S is a linear subspace of w, the space of all scalar valued sequences; and statistical limit acts as a linear functional on this space. Since every finite set has its natural density zero, hence it follows that an ordinary convergence implies statistical convergence, whereas the converse need not be true in general as evident from the following example:

Define a sequence $x = (x_k)$ as:

$$x_k = \begin{cases} 1, & \text{if } k = n^2, \\ 0, & \text{otherwise.} \end{cases}$$

For every $\epsilon > 0$,

$$\begin{aligned} \{k \leq n : |x_k - 0| \geq \epsilon\} &= \{k \leq n : x_k = 1\} \\ &= \{k \leq n : k \,\text{is a square}\} \\ &\subset \{1^2, 2^2, 3^2, ...\}. \end{aligned}$$

Since, the latter set has natural density zero, therefore it follows that

$$S - \lim_{k \longrightarrow \infty} x_k = 0,$$

whereas (x_k) is not convergent in usual sense.

The above example shows that statistical convergence is a generalized form of classical convergence. Further, we find number of properties in literature, which hold true for

ordinary convergence, but are not true for statistical convergence. For example:
(i) It is well-known that a subsequence of a convergent sequence is convergent, but this is not always true for statistical convergence.
(ii) Every convergent sequence must be bounded in usual sense, but it does not hold good in case of statistical convergence.
In 1959, I. J. Schoenberg [138] studied some basic properties of statistical convergence and established this concept as a summability method. Later, Salat [124] in 1980, gave following two important characterizations of statistical convergence.

Theorem 1.2.1. *[124] A sequence $x = (x_k)$ is statistically convergent to a number L, if and only if, there exists a set $K = \{k_1, k_2, k_3, ...\} \subset \mathbb{N}$, such that $\delta(K) = 1$ and $\lim_n x_{k_n} = L$.*

Theorem 1.2.2. *[124] The set of all bounded statistically convergent sequences of real numbers is a nowhere dense subset of the normed linear spaceℓ_∞, the set of all bounded sequences of real numbers.*

The notion of statistical convergence has been used by numerous authors in versatile areas of mathematics such as approximation theory ([1], [3], [18], [32], [42]), measure theory ([15], [93]), trigonometric series ([56], [129], [156]), summability theory ([15], [17], [22], [31], [32], [33], [36], [51], [56], [102] and [149]), number theory ([45], [71], [89]), Banach spaces ([32]), spectral theory([43], [122]), finitely additive set function ([33], ([34])), locally convex spaces ([88]), tauberian theory([20], [41], [49]) the fuzzy set theory ([7], [11], [12]).

Although, the concept of statistical convergence was established in 1951, but a major development towards this area started with the paper of Fridy [58] who gave the statistical analogue of Cauchy criterion of convergence as follows:

Definition 1.2.2. [58] A number sequence $x = (x_k)$ is said to be statistically Cauchy if for every $\epsilon > 0$, there exists a positive integer $m = m(\epsilon)$ such that $\delta(\{k \leq n : |x_k - x_m| \geq \epsilon\}) = 0$.

Theorem 1.2.3. *[58] A sequence $x = (x_k)$ is statistically convergent if and only if it is statistically Cauchy.*

In 1988, Connor [30] developed a natural relationship between statistical convergence and strong p-Cesàro summability. He has shown that if "A sequence $x = (x_k)$ is strong p-Cesàro summable for $0 < p < \infty$, then the sequence must be statistically convergent and a bounded statistically convergent sequence must be strong p-Cesàro summable". A sequence $x = (x_k)$ is said to be strongly Cesàro summable or strongly $(C, 1)$-summable to a number L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0.$$

We denote the space of all strongly Cesàro summable sequences by $|\sigma_1|$ i.e.

$$|\sigma_1| = \left\{ x : \text{for some L}, \lim_{n} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \right) \right\}.$$

A sequence $x = (x_k)$ is said to be strongly p-Cesàro summable to L (where p is a positive real number) if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L|^p = 0.$$

We denote the space of all strongly $p-$Cesàro summable sequences by w_p i.e.

$$w_p = \left\{ x : \text{for some L}, \lim_{n} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k - L|^p = 0 \right) \right\}.$$

Let $x = (x_k)$ be a sequence and $(x_{k_{(j)}})$ be a subsequence of x. If the asymptotic density $\delta(K) = 0$, where $K = \{k_j : j \in \mathbb{N}\}$, then $(x_{k_{(j)}})$ is called a thin subsequence.

On the other hand, $(x_{k_{(j)}})$ is said to be a non-thin subsequence of x, if $\delta(K) \neq 0$. In this case, either $\delta(K) > 0$ or K fails to have natural density.

Definition 1.2.3. [59] Let $x = (x_k)$ be a number sequence. A number ξ is said to be a statistical limit point of x if there is non-thin subsequence of x that converges to ξ.

Symbolically, Λ_x denotes the set of all statistical limit points of the sequence $x = (x_k)$.

Definition 1.2.4. [59] Let $x = (x_k)$ be a number sequence. A number ξ is said to be a statistical cluster point of x if for each $\epsilon > 0$,

$$\limsup_n \frac{1}{n}(\{k \in \mathbb{N} : |x_k - \xi| < \epsilon\}) > 0.$$

Here, Γ_x denotes the set of all statistical cluster points of the sequence x.
For a sequence $x = (x_k)$, define the sets

$$A_x = \{a \in \mathbb{R} : \delta(\{k \in \mathbb{N} : x_k < a\}) \neq 0\};$$

$$B_x = \{b \in \mathbb{R} : \delta(\{k \in \mathbb{N} : x_k > b\}) \neq 0\}.$$

The statistical limit inferior and superior of the sequence $x = (x_k)$ are defined as:

Definition 1.2.5. [62] Let $x = (x_k)$ be sequence of numbers, then the statistical limit inferior and statistical limit superior of x are respectively defined as:

$$s - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \phi, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$s - \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \phi, \\ -\infty, & \text{otherwise.} \end{cases}$$

Note that for any sequence $x = (x_k)$,

$$\liminf x \leq s - \liminf x \leq s - \limsup x \leq \limsup x.$$

Definition 1.2.6. [62] A sequence $x = (x_k)$ is called statistically bounded if there is a number B such that $\delta(\{k : |x_k| > B\}) = 0$.

In 1972, Schaefer[137] developed a summability method by using invariant means or σ-means, called as σ-convergence. Mursaleen[96] extended this idea by defining strongly $\sigma-$convergence and established the relations between σ-convergence and strongly σ-convergence. Later, in 2009, Mursaleen along with Edley[101] used the concept of statistical convergence to define statistically σ-convergence for number sequences. Before proceeding further, we recall the following definitions and important results from the papers of Schaefer[137], Mursaleen[96] and Mursaleen and Edley[101].

Let σ be the mapping of the set of all positive integers into itself. A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or σ-mean if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$,

(iii) $\varphi(x_{\sigma(n)}) = \varphi(x)$ for all $x \in \ell_\infty$.

Definition 1.2.7. [137] A bounded sequence $x = (x_k)$ is said to be σ-convergent to the number L, if and only if

$$\lim_{k\to\infty} t_{k_m} = L \qquad \text{uniformly in m},$$

where $t_{k_m} = \frac{x_m + x_{\sigma(m)} + \cdots + x_{\sigma^k(m)}}{k+1}$

We denote the set of all σ-convergent sequences by V_σ i.e.

$$V_\sigma = \left\{ x \in \ell_\infty : \text{for some L}, \lim_{k\to\infty} t_{k_m} = L \quad \text{uniformly in m} \right\}.$$

Definition 1.2.8. [96] A bounded sequence $x = (x_k)$ is said to be strongly σ-convergent to the limit L if

$$\lim_p \frac{1}{p}\sum_{j=1}^{p} |x_{\sigma^j(m)} - L| = 0 \qquad \text{uniformly in m.}$$

$[V_\sigma]$ denotes the set of all strongly σ-convergent sequences.

Definition 1.2.9. [101] A sequence $x = (x_k)$ is said to be statistically σ-convergent to L if for every $\epsilon > 0$

$$\lim_n \frac{1}{n}|\{k \leq n : |t_{k_m} - L| \geq \epsilon\}| = 0, \qquad \text{uniformly in m.}$$

In this case, we write $\delta(\sigma) - \lim x = L$.

Here, S_σ denotes the set of all statistically σ-convergent sequences.

Schaefer[137] and Mursaleen and Edley[101], proved the following results concerning $\sigma-$convergence and statistically $\sigma-$convergence.

Theorem 1.2.4. *[137] The class of σ-regular and σ-coercive matrices are disjoint.*

Theorem 1.2.5. *[101] If a sequence $x = (x_k)$ is bounded and σ-statistically convergent to L, then it is statistically σ-convergent to L.*

Theorem 1.2.6. *[101] A sequence $x = (x_k)$ is statistically σ-convergent to L if and only if there exists a set $K = \{k_1 < k_2 < \cdots < k_r < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\sigma - \lim x_{k_n} = L$.*

Theorem 1.2.7. *[101] A sequence $x = (x_k)$ is σ-statistically convergent to L if and only if there exists a set $K = \{k_1 < k_2 < \cdots < k_r < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\sigma(K) = 1$ and $\lim x_{k_n} = L$.*

In yet another direction, Çolak [25] in 2008, introduced an interesting version of statistical convergence with the help of the α-density and called it as statistical convergence of order α, where $\alpha \in (0,1]$.

For $K \subset \mathbb{N}$, the α-density of K is defined as

$$\delta_\alpha(K) = \lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \leq n : k \in K\}|,$$

provided this limit exists.

Remark 1.2.1. (i) α-density of any finite subset of $\mathbb{N}$ is always zero.

(ii) The equality $\delta_\alpha(K^c) = 1 - \delta_\alpha(K)$ holds good for $\alpha = 1$ but it does not hold for $0 < \alpha < 1$.

(iii)For $\alpha = 1$, the α-density coincides with the natural density. Further, for any set $K \subset \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$, $\delta_\beta(K) \leq \delta_\alpha(K)$.

Definition 1.2.10. [25] A sequence $x = (x_k)$ of numbers is said to be statistically convergent of order α to a number L, if for each $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $s^\alpha - \lim x_k = L$. The set of all statistically convergent sequences of order α is denoted by S^α.

For further work in this direction, we refer Cakan and Altay [19], Cakan et al. [20], Cakan et al. [21], Çolak et al. [28], Connor [30], Connor et al. [32], Conner and Kline [33], Edely and Mursaleen [43], etc.

1.2.1 Lacunary Statistical Convergence

In 1988, Connor [30] gave a natural relationship between statistical convergence and the space $|\sigma_1|$ of strongly Cesàro summable sequences. Analogously, Freedman et al.[56] introduced some Cesàro type summability using lacunary sequences and defined corresponding sequence space N_θ as follows:

By a lacunary sequence, we mean an increasing sequence of positive integers $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Definition 1.2.11. [56] A sequence $x = (x_k)$ of numbers is said to be lacunary strongly convergent or N_θ-summable to a number L, provided that

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0$$

It is denoted by $N_\theta - \lim x_k = L$ or $x_k \longrightarrow L(N_\theta)$.

Let N_θ denotes the space of all lacunary strongly convergent sequences as

$$N_\theta = \left\{ x = x_k : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0 \quad \text{for some } L \right\}.$$

The space N_θ is BK-space with the norm

$$\|(x_k)\|_\theta = \sup_r \frac{1}{h_r} \sum_{k \in I_r} |x_k|.$$

For a particular choice of $\theta = (2^r), N_\theta$ coincides with $|\sigma_1|$.

In the year 1993, an interesting generalization of statistical convergence, namely, lacunary statistical convergence was introduced by Fridy and Orhan [60] by using lacunary sequence. They did so by replacing the set $\{k : k \leq n\}$ in definition 1.2.1. by the set $\{k : k_{r-1} \leq k \leq k_r\}$.

Definition 1.2.12. [60] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ of numbers is said to be lacunary statistically convergent or S_θ-convergent to a number L, if for every $\epsilon > 0$

$$\lim_{r \to \infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0$$

In this case, we write $S_\theta - \lim x = L$. The set of all lacunary statistical convergent sequences is denoted by S_θ.

The limit in the above definition can also be obtained by using matrix transformation of the characteristic function χ_K of the set $K = K(x, L, \epsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ and is written as $\lim_n(C_{\theta_{\chi_K}}) = 0$, where C_θ is the matrix given as

$$C_\theta[n, k] = \begin{cases} \frac{1}{h_r}, & \text{if } k \in I_r, \\ 0, & \text{if } k \notin I_r. \end{cases}$$

A subset K of $\mathbb{N}$ has θ-density, if $\delta_\theta(K) = \lim_r \frac{|K \cap I_r|}{h_r}$ exists.

Later, this work was further carried out by Fridy and Orhan [61], where they introduced the S_θ-analogue of Cauchy sequence and presented Cauchy convergence criterion for lacunary statistical convergence.

Definition 1.2.13. [61] A sequence $x = (x_k)$ is said to be lacunary statistically Cauchy or $S_\theta-$Cauchy if there exists a subsequence $(x_{k'(r)})$ of (x_k) such that $k^{'}(r) \in I_r$, for each r, $\lim_{r\to\infty} x_{k'(r)} = L$ and for each $\epsilon > 0$,

$$\lim_r \frac{1}{h_r} \left|\left\{k \in I_r : |x_k - x_{k'(r)}| \geq \epsilon\right\}\right| = 0.$$

We now present some important properties of lacunary statistical convergent sequences proved by various authors.

Theorem 1.2.8. *[56] Let $\theta = (k_r)$ be a lacunary sequence, then $|\sigma_1| \subset N_\theta$ if and only if* $\inf_r q_r > 1$.

Theorem 1.2.9. *[56] Let $\theta = (k_r)$ be a lacunary sequence, then $N_\theta \subset |\sigma_1|$ if and only if* $\sup_r q_r < \infty$.

Theorem 1.2.10. *[56] Let* $\theta = (k_r)$ *be a lacunary sequence, then* $N_\theta = |\sigma_1|$ *it is necessary and sufficient that* $1 < \inf_r q_r \leq \sup_r q_r < \infty$.

Theorem 1.2.11. *[60] Let* $\theta = (k_r)$ *be a lacunary sequence, then*

(i) (a) $x_k \longrightarrow L(N_\theta)$ *implies* $x_k \longrightarrow L(S_\theta)$ *and*

(b) N_θ *is a proper subset of* S_θ*,*

(ii) $x \in \ell_\infty$ *and* $x_k \longrightarrow L(S_\theta)$ *imply* $x_k \longrightarrow L(N_\theta)$*,*

(iii) $S_\theta \cap \ell_\infty = N_\theta \cap \ell_\infty$*, where* ℓ_∞ *denotes the set of bounded sequences.*

Theorem 1.2.12. *[60] For any lacunary sequence* θ*,* $S - \lim x = L$ *implies* $S_\theta - \lim x = L$ *if and only if* $\lim\inf_r q_r > 1$*.*

If $\lim\inf_r q_r = 1$, then there exists a bounded S_θ-summable sequence that is not S-summable.

Theorem 1.2.13. *[60] For any lacunary sequence* θ*,* $S_\theta - \lim x = L$ *implies* $S - \lim x = L$ *if and only if* $\lim\sup_r q_r < \infty$*.*

If $\lim\sup_r q_r = \infty$, then there exists a bounded S-summable sequence that is not S_θ-summable (to any limit).

Theorem 1.2.14. *[60] For any lacunary sequence* θ*,* $S_\theta = S$ *if and only if*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty;$$

Using the Definition 1.2.13, Fridy and Orhan [61] stated the equivalent condition for S_θ-Cauchy sequence as follows:

Theorem 1.2.15. *[61] A sequence* $x = (x_k)$ *is* $S_\theta-$*convergent, if and only if it is a* $S_\theta-$*Cauchy sequence.*

The idea of lacunary statistical convergence was further explored by different authors in various directions. For instance : Jinlu Li [87] studied the inclusion properties of different lacunary sequences and solved the open problem of Fridy and Orhan [60], whereas K. Demirci [39] introduced and studied the concepts of lacunary statistical limit points and lacunary statistical cluster points for sequences of numbers. The idea of statistically σ-convergent was further extended by Karakaya[76] by the use of lacunary sequences and analogously obtained strong σ-lacunary statistically convergence.

Definition 1.2.14. [76] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be strong σ-lacunary statistically convergent to L, if for every $\epsilon > 0$

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |t_{k_m}(x - L)| \geq \epsilon\}| = 0, \quad \text{uniformly in}\, m.$$

The set of all strong σ-lacunary statistically convergent sequences is denoted by St_{θ_σ}.

Karakaya[76] proved the following important results related to strong σ-lacunary statistical convergence.

Theorem 1.2.16. *[76] Let $M = (M_k)$ be a sequence of Orlicz functions. Then $w_\sigma^0 \subset St_{\theta_\sigma}^0$ if and only if $\inf_k M_k(|t_{k_m}(x)|) > 0$ for all m.*

Theorem 1.2.17. *[76] $St_{\theta_\sigma}^0 \subset w_\sigma^0[M_k]_\theta$ if $\sup_k M_k(|t_{k_m}(x)|) < \infty$ for all m.*

Later in 1995, Cakalli [23] introduced lacunary statistical convergence in metrizable topological groups and proved some inclusion relations between the sets S and S_θ. Apart from this, many aspects of S_θ-convergence have also been discussed subsequently by various authors. For instance: Patterson and Savas[109] studied the concept of asymptotic equivalence of real sequences with the help of lacunary sequences, whereas Usulu and Nuray[154] extended this notion for the sequences of sets and discussed their topological properties. For some more work on S_θ-convergence, we refer (Gok et al.[70], Nuray[107], Pehlivan and Fisher[112], Savas[128], Savas and Nuray[134], Tripathy and Dutta[148]).

1.2.2 $\lambda-$Statistical Convergence

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 1.$$

The generalized de la Vallèe-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

Leindler [86] used generalized de la Vallèe-Poussin mean to define generalized summability called as V_λ-summability as follows.

Definition 1.2.15. [86] A sequence $x = (x_k)$ is said to be V_λ-summable to L if $t_n(x) \to L$ as $n \to \infty$.

The space V_λ is defined as

$$V_\lambda = \left\{ x = (x_k) : \exists\, L \in \mathbb{R}, \quad \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0 \right\}.$$

If $\lambda_n = n$, then V_λ-summablility reduces to $(C, 1)$-summability, where

$$(C, 1) = \left\{ x = (x_k) : \exists\, L \in \mathbb{R}, \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \right\}.$$

Mursaleen[97] used V_λ-summability to generalize the concept of statistical convergence with the help of λ-density and called it as λ-statistical convergence.

For $K \subseteq \mathbb{N}$, the set of positive integers, the $\lambda-$density of K is defined by

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} \left|\{n - \lambda_n + 1 \le k \le n : k \in K\}\right|.$$

In particular, if we take $\lambda_n = n$, then $\lambda-$density reduces to natural density.

Definition 1.2.16. [97] A number sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent or $S_\lambda-$convergent to a number L if for each $\epsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \left|\{k \in I_n : |x_k - L| \ge \epsilon\}\right| = 0.$$

In this case, we write $S_\lambda - \lim x_k = L$ or $x_k \to L(S_\lambda)$. The set of all $\lambda-$ statistically convergent sequences of numbers is denoted by S_λ.

Mursaleen [97] further observed that for the choice $\lambda_n = n$, $S_\lambda-$convergence coincides with statistical convergence and that $S_\lambda \subseteq S$ for all λ. However, converse of this i.e. $S \subseteq S_\lambda$ holds true if and only if

$$\liminf_n \frac{\lambda_n}{n} > 0.$$

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1$. For $\alpha \in (0, 1]$, the $\lambda_\alpha-$density of a subset $K \subseteq \mathbb{N}$ is defined as

$$\delta_\lambda^\alpha(K) = \lim_n \frac{1}{\lambda_n^\alpha} |\{k \in I_n : k \in K\}|,$$

if the limit exists.

Çolak and Bektas [27] introduced a stronger form of $\lambda-$statistical convergence called as "$\lambda-$statistical convergence of order α" where $\alpha \in (0, 1]$ in the following way:

Definition 1.2.17. [27] The sequence $x = (x_k)$ of numbers is called $\lambda-$statistically convergent of order α if there is a number L such that

$$\lim_n \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $S_\lambda^\alpha - \lim x_k = L$ or $x_k \to L(S_\lambda^\alpha)$. The set of all $\lambda-$statistically convergent sequences of order α is denoted by S_λ^α.

Çolak and Bektas [27] proved the following results concerning $S_\lambda^\alpha-$convergence:

Theorem 1.2.18. *Let $0 < \alpha \leq \beta \leq 1$. Then $S_\lambda^\alpha \subseteq S_\lambda^\beta$ and the inclusion is strict for some α and β such that $\alpha < \beta$.*

Theorem 1.2.19. *$S_\lambda^\alpha \subseteq S$ for all λ and each $\alpha \in (0,1)$.*

Theorem 1.2.20. *$S \subseteq S_\lambda^\alpha$ if and only if*

$$\liminf_{n \longrightarrow \infty} \frac{(\lambda_n)^\alpha}{n} > 0.$$

Theorem 1.2.21. *Let $0 < \alpha \leq \beta \leq 1$ and p be a positive real number. Then $[w_p^\alpha] \subseteq [w_p^\beta]$ and the inclusion is strict for some α and β such that $\alpha < \beta$.*

Remark 1.2.2. (i) If we take $\lambda_n = n$, $\lambda-$statistical convergence of order α coincides with statistical convergence of order α.

(ii) If we take $\alpha = 1$, $\lambda-$statistical convergence of order α coincides with $\lambda-$statistical convergence.

(iii) If we take $\lambda_n = n$ and $\alpha = 1$, $\lambda-$statistical convergence of order α coincides with statistical convergence.

The concepts of statistical convergence and λ-statistical convergence for sequence of real numbers have been further studied by Et and Nuray [52] and Et et al. [48] by the use of difference operator Δ.

For the set X of all real numbers, the operator $\Delta^m : X \to X$ is defined as

$$\begin{aligned} \Delta^0 x_k &= x_k, \\ \Delta^1 x_k &= x_k - x_{k+1}, \\ \vdots &= \vdots \\ \Delta^m x_k &= \Delta^1(\Delta^{m-1} x_k). \end{aligned}$$

Definition 1.2.18. [52] A sequence $x = (x_k)$ is said to be Δ^m-statistically convergent to a numbers L provided that for each $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \|\Delta^m x_k - L\| \ge \epsilon\}| = 0,$$

The set of all Δ^m_X-statistically convergent sequences is denoted by $S(\Delta^m)$.

Definition 1.2.19. [48] A sequence $x = (x_k)$ is said to be λ^m_X-statistically convergent to a number L if for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^m x_k - L\| \ge \epsilon\}| = 0.$$

In this case, we write $S_\lambda(\Delta^m, X)$- $\lim_k x_k = L$ or $x_k \longrightarrow L(S_\lambda(\Delta^m, X))$.

Et et al.[48] proved the following results for Δ^m_X-statistically convergence.

Theorem 1.2.22. *[48] If* $\liminf_n \frac{\lambda_n}{n} > 0$, *then* $S(\Delta^m, X) \subseteq S_\lambda(\Delta^m, X)$.

Theorem 1.2.23. *[48] Let f be a modulus function and* $\sup_k p_k = H$. *Then*

$$[V, \lambda, f, p](\Delta^m, X) \subset S_\lambda(\Delta^m, X).$$

Theorem 1.2.24. *[48] Let f be bounded and* $0 < \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$, *then*

$$S_\lambda(\Delta^m, X) \subseteq [V, \lambda, f, p](\Delta^m, X).$$

After the introduction of λ-statistical convergence and its various generalizations, many other concepts known for statistical convergence and lacunary statistical convergence have been developed and studied analogously for λ-statistical convergence. For more details in this direction, see (Alghamdi and Mursaleen [4], Altin et al. [7], Altin and Et [8], Altin et al. [9], Altinok et al. [10], Altinok and Mursaleen [11], Çinar, Karakuş and Et [24], Çolak et al. [26], Çolak et al. [28], Çolak [29], Et et al. [48], Et, Tripathy and Dutta [53], Ghosal [69], Savaş [126], Subramanian and Esi [144], Tuncer and Benli [153] etc).

1.2.3 $\mathcal{I}-$Convergence

In the earlier sections, we have presented various generalizations of statistical convergence. In this section, we present a review of $\mathcal{I}-$convergence introduced by Kostyrko, Macaj and Salàt [80] with the help of ideals of subsets of $\mathbb{N}$. It gives an unifying approach to different types of generalized convergence. We, now present some basic definitions, which will directly or indirectly be used in the subsequent chapters.
Let X be a non-empty set and $\mathcal{P}(X)$ be the power set of X.

Definition 1.2.20. [80] A family of sets $\mathcal{I} \subset \mathcal{P}(X)$ is called an ideal in X if and only if

(i) $\emptyset \in \mathcal{I}$;

(ii) For each $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$;

(iii) For each $A \in \mathcal{I}$ and $B \subseteq A$, we have $B \in \mathcal{I}$.

Definition 1.2.21. [80] A nonempty family of sets $\mathcal{F} \subset \mathcal{P}(X)$ is called a filter on X if and only if

(i) $\emptyset \notin \mathcal{F}$;

(ii) For each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$;

(iii) For $A \in \mathcal{F}$ and $B \supseteq A$ we have, $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$.

Definition 1.2.22. [80] If $\mathcal{I} \subset \mathcal{P}(X)$ is a non-trivial ideal in X, then the class $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \backslash A : A \in \mathcal{I}\}$ is a filter on X.

The filter $\mathcal{F} = \mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

Definition 1.2.23. [80] A non-trivial ideal $\mathcal{I} \subset \mathcal{P}(X)$ is called an admissible ideal in X if and only if it contains all singletons *i.e.*, if it contains $\{\{x\} : x \in X\}$.

Definition 1.2.24. [80] An admissible ideal $\mathcal{I} \subset \mathcal{P}(X)$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $A_1, A_2, \cdots$ belonging to $\mathcal{I}$, there exists a countable family $B_1, B_2, \cdots$ belonging to $\mathcal{I}$ such that $A_i \triangle B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i \in \mathcal{I}$.

By using the terminology of ideals, Kostyrko et al.[80] introduced the concept of $\mathcal{I}$-convergence of a sequence in a fixed metric space (X, d), but in order to make it more convenient in respect of the present work, we shall consider definitions in $\mathbb{R}$ with respect to usual metric in place of (X, d).

Definition 1.2.25. [80] Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}-$convergent to L if for every $\epsilon > 0$, the set

$$A(\epsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in \mathcal{I}.$$

The number L is called $\mathcal{I}-$limit of (x_k) and we write $\mathcal{I} - \lim x_k = L$.

Example 1.2.1. *(i) If we consider $\mathcal{I}_0 = \{\phi\}$, then clearly $\mathcal{I}_0$ is a non-trivial ideal. Moreover, a sequence is $\mathcal{I}_0$-convergent if and only if it is a constant sequence.*
(ii) If we take $\mathcal{I} = \mathcal{I}_f$ the class of all finite subsets of $\mathbb{N}$, then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence in $\mathbb{R}$.
(iii) If $\mathcal{I}_\delta = \{K \subseteq \mathbb{N} : \delta(K) = 0\}$, then $\mathcal{I}_\delta$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_\delta$-convergence coincides with the statistical convergence.

It is remarkable here that if $\mathcal{I}$ is an admissible ideal, then usual convergence implies $\mathcal{I}$-convergence of numbers. Thus, for an admissible ideal $\mathcal{I}$, $\mathcal{I}$-convergence is a more generalized form of ordinary convergence and depends upon the variations of the ideal $\mathcal{I}$.

Kostyrko et al.[80] introduced another type of convergence called as $\mathcal{I}^*-$convergence. This type of convergence is closely associated to $\mathcal{I}-$convergence and is defined as;

Definition 1.2.26. [80] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}^*-$convergent to $L \in X$ if there exists a set $M = \{m_1, m_2, \cdots\} \subset \mathbb{N}$ such that $M \in \mathcal{F}(\mathcal{I})$ and $\lim x_{m_k} = L$.

Kostyrko et al.[80] obtained following relations between $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence for a number sequence.

Theorem 1.2.25. *[80] Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{I}^* - \lim x_k = L$, then $\mathcal{I} - \lim x_k = L$.*

Theorem 1.2.26. *[80] $\mathcal{I} - \lim x_k = L$ implies $\mathcal{I}^* - \lim x_k = L$ if and only if $\mathcal{I}$ satisfies the condition (AP).*

Dems [40] defined $\mathcal{I}$-Cauchy sequences and proved that Cauchy type condition is necessary and sufficient for $\mathcal{I}$-convergence for a number sequence. Pehlivan et al.[113] on the other hand studied the notion of $\mathcal{I}^*-$Cauchy and its relationship with $\mathcal{I}-$convergence and $\mathcal{I}-$Cauchy sequences.

Definition 1.2.27. [40] For an admissible ideal $\mathcal{I}$, a sequence, $x = (x_k)$ of numbers is said to be $\mathcal{I}-$Cauchy sequence if for each $\epsilon > 0$, there exists a positive integer m such that the set $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in \mathcal{I}$.

Dems [40] proved the following results concerning $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy.

Theorem 1.2.27. *[40] If (X, ρ) is a complete space, then every $\mathcal{I}-$Cauchy sequence in X is $\mathcal{I}-$convergent.*

Theorem 1.2.28. *[40] If every $\mathcal{I}-$Cauchy sequence in X is $\mathcal{I}-$convergent in X, then X is complete.*

Theorem 1.2.29. *[40] If a sequence $x = (x_k)$ in X is $\mathcal{I}-$convergent, then it is $\mathcal{I}-$Cauchy.*

Kostyrko et al.[81] further developed the concepts of $\mathcal{I}-$limit points and $\mathcal{I}-$cluster points and studied their inclusion relations. Continuing on this way, Demirci [38] defined $\mathcal{I}-$limit inferior, $\mathcal{I}-$limit superior and obtained some interesting properties. Before proceeding further, we recall several definitions and important results from the papers of Kostyrko et al. [81] and Demirci [38].

Definition 1.2.28. [81] An element $\xi \in X$ is said to be an $\mathcal{I}-$limit point of the sequence $x = (x_k) \in X$ provided that there is a set $M = \{m_1, m_2, \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim x_{m_k} = \xi$.

The set $\Lambda_x(\mathcal{I})$ denotes the set of $\mathcal{I}-$limit points of the sequence $x = (x_k)$

Definition 1.2.29. [81] An element $\gamma \in X$ is said to be an $\mathcal{I}-$cluster point of the sequence $x = (x_k) \in X$ if and only if $\{n \in \mathbb{N} : |x_k - \gamma| < \epsilon\} \notin \mathcal{I}$, for each $\epsilon > 0$.

The set $\Gamma_x(\mathcal{I})$ denotes the set of $\mathcal{I}-$cluster points of the sequence $x = (x_k)$.

For a sequence $x = (x_k)$, define the sets

$$A_x = \{a \in \mathbb{R} : \{k \in \mathbb{N} : x_k < a\} \in \mathcal{I}\};$$
$$B_x = \{b \in \mathbb{R} : \{k \in \mathbb{N} : x_k > b\} \in \mathcal{I}\}.$$

Demirci [38] defined the concepts of $\mathcal{I}$-limit inferior and $\mathcal{I}$-limit superior of a sequence $x = (x_k)$ as:

Definition 1.2.30. [38] Let $\mathcal{I}$ be an admissible ideal and $x = (x_k) \in X$ be a number sequence. Then the $\mathcal{I}$-limit inferior and $\mathcal{I}$-limit superior of x are respectively defined as :

$$\mathcal{I} - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \phi, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\mathcal{I} - \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \phi, \\ -\infty, & \text{otherwise.} \end{cases}$$

Definition 1.2.31. [38] The real number sequence $x = (x_k)$ is called $\mathcal{I}$-bounded if there is a number B such that $\{k : |x_k| > B\} \in I$.

Demirci [38] proved the following important results concerning $\mathcal{I}-$limit points and $\mathcal{I}-$cluster points.

Theorem 1.2.30. *[38] The $\mathcal{I}$-bounded sequence $x = (x_k)$ is $\mathcal{I}$-convergent if and only if*

$$\mathcal{I} - \liminf x = \mathcal{I} - \limsup x.$$

Theorem 1.2.31. *[38] For any real number sequence $x = (x_k)$,*

$$\mathcal{I} - \liminf x \leq \mathcal{I} - \limsup x.$$

Theorem 1.2.32. *[38] Let $x = (x_k) \in \ell_\infty$ and $\mathcal{I}(\Gamma_x)$ be the set of all $\mathcal{I}-$cluster points of x. Then*

$$\mathcal{I} - \limsup x = \max \mathcal{I}(\Gamma_x).$$

Theorem 1.2.33. *[38] Let $x = (x_k)$ be a bounded sequence, then*

$$\mathcal{I} - \liminf x = \min \mathcal{I}(\Gamma_x).$$

Later in 2011, Savas and Das[127] unified the approaches of ideal convergence, statistical convergence and $\lambda-$statistical convergence to $\mathcal{I}-$statistical convergence and $\mathcal{I} - \lambda-$statistical convergence respectively, these two new concepts are defined as below:

Definition 1.2.32. [127] A sequence $x = (x_k) \in X$ is said to be $\mathcal{I}-$statistically convergent or $S(\mathcal{I})-$convergent to L, if for each $\epsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(S(\mathcal{I}))$ or $S(\mathcal{I}) - \lim_{k \to \infty} x_k = L$.

Definition 1.2.33. [127] A sequence $x = (x_k) \in X$ is said to be $\mathcal{I} - [V, \lambda]$ summable to L, if for any $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \|x_k - L\| \geq \delta \right\} \in \mathcal{I},$$

where $I_n = [n - \lambda_n + 1, n]$.

Definition 1.2.34. [127] A sequence $x = (x_k)$ is said to be $\mathcal{I} - \lambda-$statistically convergent or $\mathcal{I} - S_\lambda$ convergent to L, if for every $\epsilon > 0$ and $\delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(\mathcal{I} - S_\lambda)$ or $\mathcal{I} - S_\lambda - \lim_{k\to\infty} x_k = L$.

Savas and Das[127] proved the following important results concerning $\mathcal{I}-$statistical convergence and $\mathcal{I} - \lambda-$statistical convergence.

Theorem 1.2.34. *[127] $S(\mathcal{I}) \subset S_\lambda(\mathcal{I})$, if* $\liminf_{n\longrightarrow\infty} \frac{\lambda_n}{n} > n$.

Theorem 1.2.35. *[127] If $\lambda \in \Delta$ be such that* $\lim_n \frac{\lambda_n}{n} = 1$, *then $S_\lambda(\mathcal{I}) \subset S(\mathcal{I})$.*

Theorem 1.2.36. *[127] $S_\lambda(\mathcal{I}) \cap m(X)$ is a closed subset of $m(X)$, if X is a Banach space.*

Following the work of Savaş and Das [127], Das et al.[37] introduced the notion of $\mathcal{I}-$lacunary strong Cesàro summability and $\mathcal{I}-$lacunary statistical convergence and investigated their various topological properties.

Definition 1.2.35. [37] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}-$lacunay statistical convergent or $S_\theta(\mathcal{I})-$convergent to L if for every $\epsilon > 0$ and $\delta > 0$

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(S_\theta(\mathcal{I}))$ or $S_\theta(\mathcal{I}) - \lim_{k\to\infty} x_k = L$. The set of all $\mathcal{I}-$lacunary statistically convergent sequences will be denoted by $S_\theta(\mathcal{I})$.

Definition 1.2.36. [37] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ of numbers is said to be $N_\theta(\mathcal{I})-$convergent to L, if for every $\epsilon > 0$ we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \geq \epsilon \right\} \in \mathcal{I}.$$

Symbolically, It is written as $x_k \to L(N_\theta(\mathcal{I}))$.

Das et al.[37] obtained the following results concerning $\mathcal{I}-$lacunary strong Ces$\grave{a}$ro summability and $\mathcal{I}-$lacunary statistical convergence.

Theorem 1.2.37. *[37] For any lacunary sequence $\theta = (k_r)$, $\mathcal{I}-$statistical convergence implies $\mathcal{I}-$lacunary statistical convergence if* $\liminf_r q_r > 1$. *If* $\liminf_r q_r = 1$, *then there exists bounded sequence (x_n), which is $\mathcal{I}-$statistically convergent but not $\mathcal{I}-$lacunary statistically convergent.*

Theorem 1.2.38. *[37] Let $\mathcal{I}$ be an admissible ideal satisfying condition(AP), and let $\theta \in F(\mathcal{I})$. If $x \in S(\mathcal{I}) \cap S_\theta(\mathcal{I})$, then $S(\mathcal{I}) - \lim x = S_\theta(\mathcal{I}) - \lim x$.*

Theorem 1.2.39. *[37] $S_\theta(\mathcal{I}) \cap \ell_\infty$ is a closed subset of ℓ_∞.*

The ideas of $\mathcal{I}$ and $\mathcal{I}^*$-convergence attracted many mathematicians to work towards these directions. For extensive view in this regard, we refer(Kostyrko et al. [82], Kumar ([83], [85]), Mursaleen and Alotaibi [98], Mursaleen and Mohiuddine [104], Mursaleen et al. [105], Savaş [125], Tripathy and Tripathy [147] etc.)

1.3 Sequence Spaces defined by Modulus Function

In this section some known sequence spaces alongwith some examples have been presented with the help of modulus function and difference sequences.

A sequence space is a linear space whose elements are sequences chosen from another linear space. The concept of a linear space involves an algebric structure given by the definition of two operations, namely the sum of any two of its vectors and the product of any scalar with any vector. Sequence spaces have various applications in several branches of functional analysis like the theory of functions, the theory of locally convex spaces, matrix transformations as well as summability theory. Now, we recall some of the familiar sequence spaces.

Let X denotes the space of all sequences $x = (x_k)$ of real numbers. The space of all null, convergent and bounded sequences are defined as

$$c_0 = \left\{ x = (x_k) \in X : \lim_{k \longrightarrow \infty} x_k \longrightarrow 0 \right\};$$

$$c = \left\{ x = (x_k) \in X : \exists\ L \in \mathbb{C} \text{ such that } \lim_{k \longrightarrow \infty} |x_k - L| \longrightarrow 0 \right\}$$

and

$$\ell_\infty = \left\{ x = (x_k) \in X : \sup_k |x_k| < \infty \right\};$$

respectively. It is well known that the spaces c_0 c and ℓ_∞ are Banach spaces with respect to the norm

$$\|x\|_\infty = \sup_k |x_k|.$$

The space of all bounded and convergent series is defined by

$$b_s = \left\{ x = (x_k) \in X : \sup_n \sum_{k=1}^{n} |x_k| < \infty \right\};$$

and

$$c_s = \left\{ x = (x_k) \in X : \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} |x_k| \text{ exists} \right\}.$$

These spaces are Banach spaces with respect to the norm

$$\|x\|_{bs} = \sup_n \sum_{k=1}^{n} |x_k|.$$

The space of all absolutely convergent series is defined as

$$\ell_1 = \left\{ x = (x_k) \in X : \sum_k |x_k| < \infty \right\};$$

and is a Banach space with respect to the norm

$$\|x\|_1 = \sum_k |x_k|.$$

For $1 \leq p < \infty$, the space ℓ_p denotes the space of all sequences such that

$$\sum_k |x_k|^p < \infty.$$

The space $\ell_p (1 \leq p < \infty)$ is also a Banach space with respect to the norm

$$\|x\| = \left(\sum_k |x_k|^p \right)^{\frac{1}{p}}.$$

For $0 < p < 1, \ell_p$ is a complete p-normed space with respect to the norm

$$\|x\| = \sum_k |x_k|^p.$$

In the present thesis, X, c_0, c and ℓ_∞ denote the space of all, null, convergent and bounded sequences of real and complex numbers.

We now turn towards the definition of modulus function, which was initially given by Nakano[106] in 1953 as follows:

Definition 1.3.1. [106] A modulus function f is a function $f : [0, \infty) \longrightarrow [0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from right at 0.

A modulus function may be bounded or unbounded. For example, $f(x) = x^p$ for $0 < p \leq 1$ is unbounded, but $f(x) = \frac{x}{1+x}$ is bounded.

Ruckle[120] studied the concept of modulus function f from the point of view of the sequence space and constructed a class of FK-spaces as follows:

$$L(f) = \left\{ x = (x_k) : \sum f(|x_k|) < \infty \right\}.$$

Note that, the space $L(f)$ is closely related with ℓ_1 space, $L(f)$ space reduces to ℓ_1 space particularly for $f(x) = x$ for all real $x \geq 0$. Towards this direction, Ruckle[119] and [121]) proved that for any modulus function f, $L(f) \subset \ell_1$ and $[L(f)]^\alpha = \ell_\infty$, where $[L(f)]^\alpha$ denotes the space

$$[L(f)]^\alpha = \left\{ y = (y_k) : \sum_{k=1} f(|x_k y_k|) < \infty \right\}.$$

Pehlivan and Fisher[111] generalized, the notion of lacunary strong convergence by using modulus function f as:

$$N_\theta(f) = \left\{ x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} f(|x_k - L|) = 0 \text{ for some } L \right\}.$$

$$N_\theta^0(f) = \left\{ x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} f(|x_k|) = 0 \right\}.$$

Pehlivan and Fisher[111] obtained the following relationship between lacunary strong convergence and that of lacunary strong convergence using modulus function.

Proposition 1.3.1. *[111] Let f be a modulus function and $0 < \delta < 1$. Then for each $x \geq \delta$ we have $f(x) \leq 2f(1).\delta^{-1}x$.*

Theorem 1.3.2. *[111] Let f be a modulus function. Then $\lim_{t\to\infty} \frac{f(t)}{t} = \beta > 0$ then $N_\theta(f) = N_\theta$.*

Later in 2003, Colak[29] used modulus function to generalize the concept of Δ^m-lacunary strongly convergence as follows:

$$N_\theta(\Delta^m, f) = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f(|\Delta^m x_k - L|) = 0 \text{ for some } L \right\};$$

$$N_\theta^0(\Delta^m, f) = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f(|\Delta^m x_k|) = 0 \right\};$$

$$N_\theta^\infty(\Delta^m, f) = \left\{ x = (x_k) : \sup_r \frac{1}{h_r} \sum_{k\in I_r} f(|\Delta^m x_k|) < \infty \right\}.$$

Colak[29] proved the following inclusion relations between Δ^m-lacunary strongly convergence($N_\theta(\Delta^m)$) and Δ^m-lacunary strongly convergence ($N_\theta(\Delta^m, f)$) by using modulus function.

Theorem 1.3.3. *[29] Let f be a modulus function. Then $N_\theta(\Delta^m) \subset N_\theta(\Delta^m, f)$.*

Theorem 1.3.4. *[29] Let f be a modulus function. If $\lim_{t\to\infty} \frac{f(t)}{t} = \beta > 0$ then $N_\theta(\Delta^m, f) = N_\theta(\Delta^m)$.*

Theorem 1.3.5. *[29] Let $\theta = (k_r)$ be a lacunary sequence. If $1 < \liminf_r q_r \limsup_r q_r < \infty$, then $|\sigma_1|(\Delta^m, f) = N_\theta(\Delta^m, f)$.*

Et et al.[48] introduced new sequence spaces by using generalized de la vallèe-poussin mean and obtained some of their topological properties.

Definition 1.3.2. [48] For a modulus function f, the sequence spaces are defined as:

$$[V,\lambda,p,f]_L(\Delta^m,X)=\left\{x=(x_k):\lim_n \frac{1}{\lambda_n}\sum_{k\in I_n} f[(\|\Delta^m x_k-L\|)]^{p_k}=0 \text{ for some } L\right\};$$

$$[V,\lambda,p,f]_0(\Delta^m,X)=\left\{x=(x_k):\lim_n \frac{1}{\lambda_n}\sum_{k\in I_n} f[(\|\Delta^m x_k\|)]^{p_k}=0\right\};$$

$$[V,\lambda,p,f]_\infty(\Delta^m,X)=\left\{x=(x_k):\sup_n \frac{1}{\lambda_n}\sum_{k\in I_n} f[(\|\Delta^m x_k\|)]^{p_k}<\infty \text{ for some } L\right\}.$$

Theorem 1.3.6. *[48] Let $p=(p_k)$ be a bounded sequence of positive integers. Then, the sequence space $[V,\lambda,p,f]_{0,L,\infty}(\Delta^m,X)$ are linear spaces.*

Theorem 1.3.7. *[48] The inclusion*

$$[V,\lambda,f]_{0,L}(\Delta^{m-1},X)\subset[V,\lambda,f]_{0,L}(\Delta^m,X)$$

is strict for $m\geq 1$.

Theorem 1.3.8. *[48] For $m\geq 1$*

$$[V,\lambda,p,f]_{0,L}(\Delta^{m-1},X)\subset[V,\lambda,p,f]_{0,L}(\Delta^m,X).$$

Although, we have mentioned the introductory works on modulus function f, but this idea has attracted many mathematicians to work towards this direction. For more work towards this, we refer the papers of Altin and Et [8], Colak [29], Connor [31].

1.4 Sequence Spaces of Double Sequence

In this section, we shall give a brief background of double sequences, which has been used frequently in many branches of mathematics. In 1900, Pringsheim[114] defined some basic concepts related to double sequences. In the present thesis, the convergence of double sequences, we shall mean the convergence in Pringsheim's sense.

Definition 1.4.1. [114] A double sequence $x = (x_{ij})$ is said to be convergent in Pringsheim's sense if for every $\epsilon > 0$, there exists $n = n(\epsilon) \in \mathbb{N}$ such that

$$|x_{ij} - L| < \epsilon \quad \text{whenever} \quad i, j \geq n.$$

The number L is called Pringsheim limit of $x = (x_{ij})$ and we write it as $P - \lim_{ij\to\infty} x_{ij} = L$.

Definition 1.4.2. [114] A double sequence $x = (x_{ij})$ is said to be Cauchy sequence if for every $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$|x_{ij} - x_{mn}| < \epsilon \quad \text{whenever} \quad i \geq m \geq n_0 \quad \text{and} \quad j \geq n \geq n_0.$$

Definition 1.4.3. [114] A double sequence $x = (x_{ij})$ is bounded if there exists a positive integer M such that $|x_{ij}| < M$ for all i, j i.e. if $\|x\|_{(\infty,2)} = \sup_{i,j} |x_{ij}| < \infty$.

Many customary theorems like uniqueness of limit, linearity of limits, Sandwich principle, Cauchy convergence criterion etc., are direct analogue of corresponding theorems for single sequences and are therefore, easily extended to double sequences. But following example violates the important property of single sequences under which every convergent sequence is bounded.

Example 1.4.1. The sequence $x = (x_{ij})$ defined by

$$x_{ij} = \begin{cases} j, & \textit{if } i = 1, \textit{ for all } j \\ i, & \textit{if } j = 1, \textit{ for all } i \\ 0, & \textit{otherwise.} \end{cases}$$

is convergent with $\lim_{i,j\to\infty} x_{ij} = 0$, but is not bounded.

Now, we present some summability methods, which have been extended from single to double sequences. The following definitions of Cesàro summable double sequences has been taken from Moricz [94].

Definition 1.4.4. [94] A double sequence $x = (x_{ij})$ of real numbers is said to be Cesàro summable to L, if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1,j=1}^{m,n} x_{ij} = L.$$

The set of all Cesàro summable double sequences is denoted by $(C,1,1)$.

Similarly, we can define the strongly Cesàro summability and strongly p-Cesàro summability for double sequences as mentioned below:

Definition 1.4.5. [136] A double sequence $x = (x_{ij})$ is said to be strongly Cesàro summable to a number L if

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1,j=1}^{m,n} |x_{ij} - L| = 0.$$

The space of all strongly Cesàro summable double sequences is denoted by $|\sigma_{11}|$ i.e.

$$|\sigma_{11}| = \left\{ x = (x_{ij}) : \exists \text{ some } L\,,\, P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1,j=1}^{m,n} |x_{ij} - L| = 0 \right\}$$

Definition 1.4.6. [94] Let p be a positive real number. A double sequence $x = (x_{ij})$ is said to be strongly p-Cesàro summable to L if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1,j=1}^{m,n} |x_{ij} - L|^p = 0.$$

The space of all strongly p-Cesàro summable double sequences is denoted by w_p^2.

Mursaleen and Edley[100] in 2003 extended the concept of statistical convergence from single sequences to double sequences and obtained some interesting relationships between statistical convergence and strongly Cesàro double sequences. In order to prove the results they have used two dimensional analogue of natural density defined as below:

For $K \subset \mathbb{N} \times \mathbb{N}$, let $K(m,n)$ denotes the number of ordered pair (i,j) in K for which $i \leq n$ and $j \leq m$. Then, the two dimensional analogue of natural density is defined as follows:

The lower asymptotic density of $K \subset \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = \liminf_{n,m} \frac{K(n,m)}{nm}$$

In case the sequence $\frac{K(n,m)}{nm}$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm}$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0,$$

which implies, the set K has its double natural density zero, while the set $\{(i, 2j) : i, j \in \mathbb{N}\}$ has its double natural density as $\frac{1}{2}$.

Definition 1.4.7. [100] A double sequence $x = (x_{ij})$ of numbers is said to be statistically convergent to a number L if for every $\epsilon > 0$, the set

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : i \leq m, j \leq n : |x_{ij} - L| \geq \epsilon\}$$

has double natural density zero. In this case, we write $St_2 - \lim_{i,j\to\infty} x_{ij} = L$. The set of all statistically convergent double sequences is denoted by St_2.

Definition 1.4.8. [100] A double sequence $x = (x_{ij})$ of numbers is said to be statistically Cauchy sequence if for every $\epsilon > 0$, there exists $N = N(\epsilon)$ and $M = M(\epsilon)$ such that for all $i, m \geq N$, $j, n \geq M$,

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : i \leq m, j \leq n : |x_{ij} - x_{mn}| \geq \epsilon\},$$

the set has its double natural density zero.

Mursaleen and Edley[100] proved the following results for statistical convergence of double sequences.

Theorem 1.4.1. *[100] A real double sequence $x = (x_{ij})$ is statistically convergent to a number L if and only if there exists a subset $K = \{(i, j) \subseteq \mathbb{N} \times \mathbb{N}, i, j = 1, 2, \ldots, \}$ such that $\delta_2(K) = 1$ and*

$$\lim_{i,j \longrightarrow \infty} x_{ij} = L \text{ for } (i, j) \in K.$$

Theorem 1.4.2. *[100] A real sequence $x = (x_{ij})$ is statistically convergent if and only if x is statistically Cauchy.*

Theorem 1.4.3. *[100] Let $x = (x_{ij})$ be a double sequence and p be a positive real number. Then*

(i) x is statistically convergent to L if it is strongly $p-$Cesàro summable to L.

(ii) $w_p^2 \cap \ell_\infty^2 = St_2 \cap \ell_\infty^2$.

Theorem 1.4.4. *[100] The set $St_2 \cap \ell_\infty^2$ is nowhere dense in ℓ_∞^2.*

Patterson and Savas[[109], [135]] extended the ideas of lacunary statistical convergence and lacunary statistical Cauchy from single sequences to double sequences and proved their equivalency with the help of double lacunary sequence.

A double sequence $\theta_{rs} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing sequences of positive integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \longrightarrow \infty \text{ as } r \longrightarrow \infty$$

and

$$l_0 = 0,\ \overline{h}_s = l_s - l_{s-1} \longrightarrow \infty \text{ as } s \longrightarrow \infty$$

Let $h_{rs} = h_r\overline{h}_s$, $I_r = (k_{r-1}, k_r]$, $\overline{I}_s = (l_{s-1}, l_s]$ and $I_{rs} = \{(i,j) : k_{r-1} < i \leq k_r, l_{s-1} < j \leq l_s\}$.

Definition 1.4.9. [135] Let θ_{rs} be a double lacunary sequence. A double sequence $x = (x_{ij})$ of numbers is said to be lacunary statistically convergent to a number L in the Pringsheim sense (denoted by $S_{\theta_{rs}} - P - \lim_{i,j\to\infty} x_{ij} = L$) if for each $\epsilon > 0$,

$$P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| = 0.$$

The set of all lacunary statistically convergent double sequences is denoted by $S_{\theta_{rs}}$.

Definition 1.4.10. [135] Let θ_{rs} be a double lacunary sequence. A double sequence $x = (x_{ij})$ of numbers is said to be $N_{\theta_{rs}} - P-$convergent to a number L if

$$P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| = 0.$$

The space of all $N_{\theta_{rs}} - P-$convergent sequences is denoted by $N_{\theta_{rs}}$ i.e.

$$N_{\theta_{rs}} = \left\{ x = (x_{ij}) : \exists \text{ some } L,\ P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| = 0 \right\}.$$

Definition 1.4.11. [109] Let θ_{rs} be a double lacunary sequence. A double sequence $x = (x_{ij})$ of numbers is said to be $S_{\theta_{rs}}-$Cauchy double sequence if there exists a double subsequence $\{x_{k_r l_s}\}$ of x such that $\{k_r, l_s\} \in I_{rs}$ for each (r,s) $P-\lim_{(r,s)} x_{k_r l_s} = L$ and for every $\epsilon > 0$

$$P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} |\{(k,l) \in I_{rs} : |x_{ij} - x_{k_r l_s}| \geq \epsilon\}| = 0.$$

Savas and Patterson[135] obtained the following results concerning lacunary statistical convergence for double sequences.

Theorem 1.4.5. *[135] For any double lacunary sequence θ_{rs}, $St_2 - \lim x = L$ implies that $S_{\theta_{rs}} - \lim x = L$ if and only if* $\liminf q_r > 1$ *and* $\liminf \overline{q}_s > 1$.

Theorem 1.4.6. *[135] For any double lacunary sequence θ_{rs}, $S_{\theta_{rs}} - \lim x = L$ implies θ_{rs}, $St_2 - \lim x = L$ if and only if* $\limsup q_r < \infty$ *and* $\limsup \overline{q}_s < \infty$.

Theorem 1.4.7. *[135] Let θ_{rs} be a lacunary double sequence , then $St_2 = S_{\theta_{rs}}$ if and only if*

$$1 < \liminf q_r \leq \limsup q_r < \infty$$

$$\textit{and} \quad 1 < \liminf \overline{q}_s \leq \limsup \overline{q}_s < \infty,$$

then $St_2 - \lim x = L$ implies $S_{\theta_{rs}} - \lim x = L$.

Theorem 1.4.8. *[135] If $x \in St_2 \cap S_{\theta_{rs}}$, then $S_{\theta_{rs}} - \lim x = St_2 - \lim x = L$.*

Theorem 1.4.9. *[109] A double sequence $x = x_{ij}$ is $S_{\theta_{rs}}-$convergent if and only if x is an $S_{\theta_{rs}}-$Cauchy double sequence.*

Tripathy and Tripathy[147] extended the concepts of $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy from single sequences to double sequences. They also studied their different properties like solidity, symmetricity, completness and denseness etc. Further, Kumar[83] developed the idea of $\mathcal{I}^*$-convergence for double sequences and obtained some inclusion relations between these two notions. Belen and Yilldirim[13] introduced the notion of $\mathcal{I}_2$-statistical convergence or $S_2(\mathcal{I}_2)$-convergence for the double sequences and gave some inclusion relations. We recall some of the definitions and results of Tripathy and Tripathy[147], Kumar[83] and Belen and Yilldirim[13] as mentioned below:

Definition 1.4.12. [147] Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ be a non-trivial ideal. A double sequence $x = (x_{ij})$ is said to be $\mathcal{I}$-convergent in the Pringsheim sense to a number L, if for every $\epsilon > 0$

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in \mathcal{I}_2.$$

Definition 1.4.13. [83] A real number $x = (x_{ij})$ is said to be $\mathcal{I}^*-$convergent to a number L if and only if there exists a set $K = \{i,j), i, i = 1, 2, \cdots, \} \in F(\mathcal{I}_2)$ such that $\lim_{(i,j)\in K, i,j \longrightarrow \infty} x_{ij} = L$.

The set of all $\mathcal{I}^*-$convergent double sequences if denoted by $\mathcal{I}_2^*$.

Theorem 1.4.10. *[83] Let $\mathcal{I}$ be an admissible ideal such that $\mathcal{I}$ contains all sets of the form $H \times N$, $N \times H$ when H is a finite subset of $\mathbb{N}$. If $\mathcal{I}_2^* - \lim_{i,j\longrightarrow\infty} x_{ij} = L$, then $\mathcal{I}_2 - \lim_{i,j\longrightarrow\infty} x_{ij} = L$.*

Theorem 1.4.11. *[83] If the ideal $\mathcal{I}$ has the property(AP), then $\mathcal{I}-$convergene implies $\mathcal{I}^*-$convergene for real double sequences.*

Theorem 1.4.12. *[83] For an admissible ideal $\mathcal{I}$ in $\mathbb{N} \times \mathbb{N}$, $(\overline{\mathcal{I}_2^* \cap \ell_\infty^2}) = (\mathcal{I}_2 \cap \ell_\infty^2)$.*

Definition 1.4.14. [13] A double sequence $x = (x_{ij})$ of numbers is said to be $\mathcal{I}_2-$statistical convergent or $S_2(\mathcal{I}_2)-$convergent to L, if for each $\epsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - L| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $x_{ij} \to L(S_2(\mathcal{I}_2))$. The set of all $\mathcal{I}$-statistically convergent double sequences of numbers is denoted by $S_2(\mathcal{I})$

Theorem 1.4.13. *[13] $S_2(\mathcal{I}) \cap \ell_\infty^2$ is a closed linear subspace of the normed linear space ℓ_∞^2.*

Theorem 1.4.14. *[13] $S_2(\mathcal{I}) \subset S_{\lambda,\mu}(\mathcal{I})$ if and only if*

$$\liminf_m \frac{\lambda_m}{m} > 0 \text{ and } \liminf_n \frac{\mu_n}{n} > 0.$$

Various other authors including; Cakan and Altay[19], Hamilton[74], Hardy[75], Moricz([94], [95]), Mursaleen et al.[99], Orhan and Dirik[108], Patterson and Savas[110], Robison[118], Savas[125], Savas and Patterson[135] and Tripathy and Sarma([150], [151] and [152]) have also studied double sequences in different directions.

Subsequently, generalized statistical and ideal convergence of double sequences attracted many mathematicians to develop double sequence spaces using lacunary sequence, ideals, difference operator Δ, modulus function, invariant means and generalized de la vallee-poussin means. For detailed view in this direction, we refer([6], [13] and [47]).

1.5 Statistical Convergence and its Generalizations in 2-Normed Spaces

In this section, we first present 2-norm spaces and see how different generalizations of convergence have been extended in these spaces. We begin with the definition of 2-normed space, which was introduced by $\ddot{G}$ahler[63] in 1963 as a generalization of normed linear spaces.

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A real valued function $\|.,.\| : X \times X \to \mathbb{R}$ satisfying the following conditions:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(ii) $\|x, y\|$ is invariant under permutation.

(iii) $\|\alpha.x, y\| = |\alpha|.\|x, y\|$, for all $\alpha \in \mathbb{R}$

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

is called a 2-norm on X and the pair $(X, \|.,.\|)$ is called a 2-normed space.

Raymond et al.[116] developed the concept of usual convergence and Cauchy sequence in 2-normed space in the following way:

Definition 1.5.1. [116] Let $(X, \|.,.\|)$ be a 2-normed space. Then the sequence $x = (x_k)$ in X is said to be convergent to x in X if for every $z \in X$,

$$\lim_{k \longrightarrow \infty} \|x_k - x, z\| = 0.$$

In this case, we write $\lim_{k \longrightarrow \infty} \|x_k, z\| = \|x, z\|$.

Definition 1.5.2. [116] Let $(X, \|.,.\|)$ be a 2-normed space. The sequence $x = (x_k)$ in X is a Cauchy sequence if for every $\epsilon > 0$ and $z \in X$, there exists $k_0 \in \mathbb{N}$ such that

$$\|x_k - x_m, z\| < \epsilon \text{ for all } k, m \geq k_0.$$

Gürdal and Pehlivan[67] in 1995 extended statistical convergence in 2-normed spaces and studied its various properties. They also established equivalent condition for a sequence to be statistically Cauchy. Sahiner et al.[123] continued with the study of 2-normed spaces and extended $\mathcal{I}$-convergence in these spaces. They also defined some new sequence spaces using 2-norm. Gürdal[65] extended $\mathcal{I}$-limit points, $\mathcal{I}$-cluster points in 2-normed spaces and investigated relation between these two notions.

Later, Yamanc and Gurdal[155] united these ideas of statistical and ideal convergence to define and study S($\mathcal{I}$)-convergence in 2-normed spaces. We, now present some definitions and results of Sahiner et al.[123], Gurdal and Pehlivan[67] and Yamanc and Gurdal[155], which form the basis of chapter-III in the present thesis.

Definition 1.5.3. [67] Let $x = (x_k)$ in a 2-normed space $(X, \|.,.\|)$ is said to be statistically convergent to L, if for every $\epsilon > 0$

$$\lim_n \frac{1}{n}|\{k \leq n : \|x_k - L, z\| \geq \epsilon\}| = 0,$$

for each non-zero $z \in X$. In this case, we write $St - \lim_k \|x_k, z\| = \|L, z\|$.

Theorem 1.5.1. *[67] Let $(X, \|.,.\|)$ be a 2-normed space. A sequence $x = (x_k)$ in X is statistical convergent to x if and only if there exists $A \subseteq \mathbb{N}$ with $\delta(A) = 1$ and $\lim_{k\in A} x_k = x$.*

Theorem 1.5.2. *[67] Let $(X, \|.,.\|)$ be a 2-normed space. The sequence $x = (x_k) \in X$ is statistically convergent if and only if it is statistically Cauchy.*

Definition 1.5.4. [123] Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$ and $(X, \|.,.\|)$ is a 2-normed space. A sequence $x = (x_k)$ in X is said to be $\mathcal{I}-$convergent to $L \in X$ if for each $\epsilon > 0$ and nonzero $z \in X$, the set

$$A(\epsilon) = \{k \in \mathbb{N} : \|x_k - L, z\| \geq \epsilon\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - \lim_{k\to\infty} \|x_k, z\| = \|L, z\|$.

Definition 1.5.5. [155] Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. The sequence $x = (x_k)$ in X is said to be $\mathcal{I}-$statistically convergent to L, if for each $\epsilon > 0$ and $\delta > 0$ and non-zero $z \in X$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|x_k - L, z\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I}-st-\lim_{k\longrightarrow\infty}\|x_k-L,z\|=0$ or $\mathcal{I}-st-\lim_{k\longrightarrow\infty}\|x_k,z\|=\|L,z\|$.

Definition 1.5.6. [155] A sequence $x=(x_k)$ in 2-normed space $(X,\|.,.\|)$ is said to be $\mathcal{I}-$statistically Cauchy in X, if for each $\epsilon>0$, $\delta>0$ and every non-zero $z\in X$, there exists a number m such that

$$\left\{n\in\mathbb{N}:\frac{1}{n}|\{k\leq n:\|x_k-x_m,z\|\geq\epsilon\}|\geq\delta\right\}\in\mathcal{I}.$$

Theorem 1.5.3. *[155] Let $\mathcal{I}\subset\mathcal{P}(\mathbb{N})$ be an admissible ideal. A sequence $x=(x_k)$ in X is $\mathcal{I}-$statistically convergent to x in X if and only if $\mathcal{I}-st-\|x_k-x,u_i\|=0$ for every $i=1,2,\cdots d$.*

Theorem 1.5.4. *[155] Let $\mathcal{I}\subset\mathcal{P}(\mathbb{N})$ be an admissible ideal. A sequence $x=(x_k)$ in X is $\mathcal{I}-$statistically convergent to x if and only if $\delta_{\mathcal{I}}(A_n(\epsilon))=0$, where $(A_n(\epsilon))=\{k\leq n:x_k\notin B_u(x,t)\}$.*

Theorem 1.5.5. *[155] In any 2-normed space $(X,\|.,.\|)$, any $\mathcal{I}-$statistically Cauchy sequence is $\mathcal{I}-$statistically convergent if and only if any $\mathcal{I}-$statistically Cauchy sequence with respect to $\|.,.\|_\infty$ is $\mathcal{I}-$statistically convergent.*

For more works on 2-normed spaces, we refer([66], [68], [72], [78], [115], [132] and [133]).

1.6 Statistical Convergence in Probabilistic Normed Spaces

This section deals with brief study of Probabilistic normed space(briefly known as PN-spaces), which appeared in literature as a natural generalization of normed linear space.

In 1942, Manger[92] introduced the notion of probabilistic metric space(PM-space) to resolve the interpretative issue of quantum mechanics. He proposed transferring the probabilistic notions of quantum mechanics from physics to the underlying geometry. He replaced the distance between points p and q by a distribution function F_{pq} whose value $F_{pq}(x)$ at the real number x is interpreted as the probability that the distance between p and q is less than x. These spaces have been further developed by numerous authors. For an extensive view, we refer([35], [139] and [141]).

Later in 1962, $\hat{S}$erstnev[142] generalized PM-spaces and introduced the notion of probabilistic normed spaces. In these spaces, the norm of the vectors are represented by probability distribution functions rather than the numerical values. To proceed further, we recall some of the basic definitions. Let $\mathbb{R}$ denotes the set of reals and $\mathbb{R}_0^+ = [0, \infty)$.

Definition 1.6.1. [55] A function $f : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

Also a distance distribution function is a non-decreasing function F defined on $\mathbb{R}_0^+ = [0, \infty)$ which satisfies $F(0) = 0$ and $F(\infty) = 1$, and is left-continous on $(0, \infty)$.

Let D and D^+ denotes the set of all distribution functions and distance distribution functions respectively. The element of D^+ are partially ordered via $F \leq G$ if and only if $F(x) \leq G(x)$ for all $x \in \mathbb{R}$.

Definition 1.6.2. [140] A triangular norm (briefly, a t-norm) we mean a binary operation on $*$ on $[0, 1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, i.e., it is the continuous mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $a, b, c \in [0, 1]$:

(i) $a * 1 = a$,

(ii) $a * b = b * a$,

(iii) $a * (b * c) = (a * b) * c$,

(iv) $c * d \geq a * b$ if $c \geq a$ and $d \geq b$.

The $*$ operations defined by $a*b = \max\{a+b-1, 0\}$, $a*b = ab$, and $a*b = \min\{a, b\}$ on $[0, 1]$ are examples of t-norm.

Using Definition 1.6.1 and Definition 1.6.2, the notion of probabilistic normed space is defined as follows:

Definition 1.6.3. [55]A triplet $(X, F, *)$ is called a probabilistic normed space (briefly, a PN-Space) if X is a real vector space, F is a mapping from X into D and for $x \in X$, the the distribution function $F(x)$ is denoted by F_x, and $F_x(t)$ is the value of F_x at $t \in \mathbb{R}$ and $*$ is a t-norm satisfying the following conditions:

$(PN-1)$ $F_x(0) = 0$;

$(PN-2)$ $F_x(t) = 1$ for all $t > 0$ if and only if $x = 0$;

$(PN-3)$ $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R} - \{0\}$;

$(PN-4)$ $F_{x+y}(s+t) \geq F_x(s) * F_y(t)$ for all $x, y \in X$ and $s, t \in \mathbb{R}^+$.

We now give the convergence structure in PN-spaces. Let $(X, F, *)$ be a PN-Space.

Definition 1.6.4. [2] A sequence $x = (x_k)$ in X is said to be convergent to $x \in X$ with respect to the probabilistic norm F if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer k_0 such that $F_{x_k - x}(\epsilon) > 1 - \lambda$ whenever $k \geq k_0$. It is denoted by $F - \lim_{k \longrightarrow \infty} x_k = x$ or $x_k \longrightarrow x$ with respect to the probabilistic norm F.

Definition 1.6.5. [2] A sequence $x = (x_k)$ in X is called Cauchy sequence with respect to the probabilistic norm F if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer k_0 such that $F_{x_k - x_m}(\epsilon) > 1 - \lambda$, where $k, m \geq k_0$.

Karakus[77] generalized usual convergence in terms of statistical convergence in PN-Spaces and also developed the notions of statistical limit and statistical cluster point in these spaces.

Definition 1.6.6. [77] A sequence $x = (x_k)$ in X is said to be statistical convergent to some point $x_0 \in X$ with respect to the probabilistic norm F provided that for every $\epsilon > 0$ and $\lambda \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| = 0.$$

In this case, we write $S_F - \lim_{k\to\infty} x_k = x_0$.

Definition 1.6.7. [77] A sequence $x = (x_k)$ in X is said to be statistical Cauchy with respect to the probabilistic norm F provided that for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer m

$$\lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : F_{x_k - x_m}(\epsilon) \leq 1 - \lambda\}| = 0.$$

Definition 1.6.8. [77] A sequence $x = (x_k)$ in X is said to be statistically bounded with respect to the probabilistic norm F if there exists some $t \in \mathbb{R}$ and $\lambda \in (0,1)$ such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : F_{x_k}(t) \leq 1 - \lambda\}| = 0.$$

Karakus[77] proved the following results concerning statistical convergence in PN-Spaces:

Theorem 1.6.1. *[77] Let $(X, F, *)$ be a $PN-$space. If $F - \lim x = L$, then $S_F - \lim X = L$.*

Theorem 1.6.2. *[77] Let $(X, F, *)$ be a $PN-$space. Then, $S_F - \lim X = L$ if and only if there exists an increasing index sequence $K = \{k_n\}$ of natural numbers such that $\delta(K) = 1$ and $F - \lim_{n\in K} x_{k_n} = L$.*

Theorem 1.6.3. *[77] Let $(X, F, *)$ be a $PN-$space and $x = (x_k)$ be a sequence whose terms are in the vector space X. Then, the following conditions are equivalent.*
(i) x is a statistically Cauchy sequence with respect to probabilistic norm F.
(ii) There exists an increasing index sequence $k = (k_n)$ of natural numbers such that $\delta(k) = 1$ and the subsequence $\{x_{k_n}\}$ is a Cauchy sequence with respect to the probabilistic norm F.

Recently, Many ideas of generalized summability like $S_\theta-$convergence, ideal convergence, $\mathcal{I}-$limit and $\mathcal{I}-$cluster points have been extended on PN-spaces. For a detail view towards this direction, we refer([5], [83], [103], [104], [117] and [145])

1.7 Objectives of the study

The research objectives of the present study are defined as:

1. To extend certain generalized convergence notions and methods from single to double sequences.

2. To define and study statistical and ideal convergence with the help of difference sequences.

3. To study generalized convergence in some general abstract spaces such as probabilistic normed spaces, 2-normed spaces etc.

1.8 Thesis Organization

The present thesis consists of five chapters. Each chapter is further divided into various sections.

Chapter – II is devoted to the development of new sequence spaces $V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$, $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$ and $V_\lambda^f[\Delta_p^m, \mathcal{I}]_\infty$. These sequence spaces have been defined by the use of de la vallee poussin mean, modulus function f and the difference operator Δ. After making some early remarks on these spaces, some of their important properties and inclusion relations have been established in this chapter. Some concrete examples in support of our results have also been provided. Subsequently, the concept of $S_\lambda^{\Delta^m}(\mathcal{I})$-convergence has also been introduced and a condition is obtained under which $S_\lambda^{\Delta^m}(\mathcal{I})$-convergence coincides with above-mentioned sequence spaces.

In **Chapter – III**, we consider one of the generalized form of abstract spaces i.e. 2-norm space. The aim in this chapter is to extend the sequence spaces $V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$, $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$ and $V_\lambda^f[\Delta_p^m, \mathcal{I}]_\infty$ introduced in Chapter-II to the sequence spaces $V_\lambda^f[\|, \|, \Delta^m, \lambda, p]_0$, $V_\lambda^f[\|, \|, \Delta^m, \lambda, p]_L$ and $V_\lambda^f[\|, \|, \Delta^m, \lambda, p]_\infty$ in 2-norm spaces and have obtained some of their properties. A new class of generalized statistical convergent sequences has also been defined in this chapter with the help of an ideal and difference sequences. Further, we also establish a strong connection between this type of convergence and the aforesaid sequence space.

The aim of **Chapter – IV** is to introduce some new sequence spaces using ideals, modulus function and invariant means and examine some of their topological properties. We start with the introduction of the new sequence spaces $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$, $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_L$ and $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_\infty$ and prove that these are linear spaces. We also find some inclusion relations among these spaces by imposing some conditions on the modulus function f. Finally, a new kind of summability method called as $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})$-convergence for sequences of numbers has been presented in this chapter. Towards the end of this chapter, lacunary generalized convergence in probabilistic normed space has been developed.

The goal of **Chapter − V** is to introduce the concepts of $\mathcal{I}-$lacunary strongly Cesàro summability ($N_{\theta_{rs}}(\mathcal{I}_2)$) and $\mathcal{I}$-lacunary statistical convergence ($S_{\theta_{rs}}(\mathcal{I}_2)$) for double sequence of numbers with help of double lacunary sequence. Some important properties of these notions have been obtained in this chapter and it is proved that the set of all lacunary ideal statistically convergent bounded sequences is closed. Subsequently, a stronger form of $S_{\theta_{rs}}(\mathcal{I}_2)$-convergence has been defined with the help of ideals. Further, in this chapter some relevant connections between these notions have also been established.

Chapter 2

On Ideal Convergent Sequence Spaces and their topological properties

2.1 Introduction

The concept of statistical convergence was introduced by Fast[54] in 1951. This notion of statistical convergence is an interesting generalization of usual convergence and plays an important role, where the concept of classical convergence fails to exist. It has wide range of applications in various fields, such as number theory, summability theory, measure theory, general topology and some other areas(see for instance, [15], [16], [31], [122] and [146]). The idea of statistical convergence have been generalized in various forms by various authors.

In the year 2000, Mursaleen[97] used generalized de la vallèe-poussin means to define a new kind of statistical convergence, called as $\lambda-$statistical convergence and

Contents of this chapter have been published in **Atti della Fondazione Ronchi 2016, Anno LXXI:2** studied the corresponding sequence space S_λ. Later, Malkowsky and Savas[91] studied some new sequence spaces by using the concepts of generalized de la vallèe-poussin means and modulus function. More work in this direction can be found in ([4], [84], [127], [131] and [153]).

Recently, another generalization of statistical convergence namely, ideal convergence has appeared in literature. The credit goes to Kostyrko et al.[82], who first defined the notion of ideal convergence with the help of an admissible ideal $\mathcal{I}$ of subsets of $\mathbb{N}$, the set of positive integers and called it as $\mathcal{I}-$convergence. Continuing the study of Kostyrko et al.[82], Dems[40] introduced ideal analogue of Cauchy sequence and proved that in a Complete metric space (X, ρ), a Cauchy type condition is necessary and sufficient for $\mathcal{I}-$convergence. Following the work of Mursaleen[97] and Kostyrko et al.[82], Savas and Das [127] introduced the concept of $\mathcal{I}-$statistical convergence, $\mathcal{I} - S_\lambda-$convergence and obtained some interesting relations between these two notions. These ideas have further been explored by different authors to introduce new kind of sequence spaces. For an extensive view in this regard, see ([37], [46], [65] and [103]).

In 1981, Kizmaz[79] introduced the notion of difference sequence space as follows:

$$\Delta(X) = \{x = (x_k) : \Delta x_k \in X\},$$

for $X = \ell_\infty;\ c$ and c_0, where $\Delta x_k = x_k - x_{k+1}$.

Continuing on this way, the notion was further generalized by Et and Colak[51] by introducing the generalized difference sequence space as follows:

$$\Delta^m(X) = \{x = (x_k) : \Delta^m x_k \in X\},$$

for $X = \ell_\infty;\ c$ and c_0, where $m \in \mathbb{N}$ and where $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$.

Later, Et et al[48] in 2003 introduced λ_X^m-statistical convergence as another generalization of statistical convergence with the help of difference operator Δ. For extensive view in this area, we refer the papers ([49], [50], [90] and [149]).

The aim of the present chapter is to develop some new sequence spaces by using generalized de la vallèe poussin mean, modulus function and difference sequences. The contents of this chapter have been divided into four sections. Section 2.2 deals with some basic definitions and some results related to λ_X^m-statistical convergence and $\mathcal{I}-$lacunary statistical convergence have been obtained in this section. In section 2.3 the generalized sequence spaces $V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$, $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$ and $V_\lambda^f[\Delta_p^m, \mathcal{I}]_\infty$ have been introduced for an admissible ideal $\mathcal{I}$ and some of their important topological properties have been developed in this section. Further, some inclusion relations amongst the so defined sequence spaces have been established in this section. Finally, in the last section, a new class $(S_\lambda^{\Delta^m}(\mathcal{I}))$ of generalized statistical convergent sequences with the help of an ideal and difference sequences has been presented and a condition under which $(S_\lambda^{\Delta^m}(\mathcal{I}))$ coincides with the sequence spaces $V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$, $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$ and $V_\lambda^f[\Delta_p^m, \mathcal{I}]_\infty$ has been established.

2.2 Preliminaries

Throughout this chapter, $\mathbb{N}$ and and (X, $\|.\|$) will denote the set of all positive integers and a real normed linear space, respectively.

A family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an ideal in $\mathbb{N}$ if and only if

(i) $\emptyset \in \mathcal{I}$;

(ii) For each $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

(iii) For each $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$.

A non-empty family of sets $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ is called a filter on $\mathbb{N}$ if and only if

(i) $\emptyset \notin \mathcal{F}$;

(ii) For each $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;

(iii) For $A \in \mathcal{F}$ and $B \supset A \Rightarrow B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. It immediately implies that $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is a non-trivial ideal if and only if the class $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \in \mathcal{I}\}$ is a filter on $\mathbb{N}$. The filter $\mathcal{F} = \mathcal{F}(I)$ is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an admissible ideal in $\mathbb{N}$ if and only if it contains all singletons i.e. if and only if it contains $\{\{n\} : n \in \mathbb{N}\}$.
Now, we quote some definitions of Kostyrko et al.[82], Et et al.[48] and Savas and Das[127], which will form basis of newly introduced classes of generalized difference sequence spaces.

Definition 2.2.1. [82] Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$. A sequence $x = (x_k)$ in X is said to be $\mathcal{I}-$convergent to $\xi \in X$ if for each $\epsilon > 0$, the set

$$A(\epsilon) = \{k \in \mathbb{N} : \|x_k - \xi\| \geq \epsilon\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - \lim_{k\to\infty} x_k = \xi$.

Definition 2.2.2. [48] A sequence $x = (x_k)$ is said to be λ_X^m-statistically convergent to the number L if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| = 0.$$

In this case, we write $S_\lambda(\Delta^m, X) - \lim_{k\to\infty} x_k = L$.

Definition 2.2.3. [127] A sequence $x = (x_k) \in X$ is said to be $\mathcal{I}-$statistically convergent or $S(\mathcal{I})-$convergent to L, if for every $\epsilon > 0$, and every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|x_k - L\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(S(\mathcal{I}))$ or $S(\mathcal{I}) - \lim_{k\to\infty} x_k = L$.

Definition 2.2.4. [127] A sequence $x = (x_k) \in X$ is said to be $\mathcal{I} - [V, \lambda]$ summable to L, if for any $\delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \|x_k - L\| \geq \delta\right\} \in \mathcal{I},$$

where $I_n = [n - \lambda_n + 1, n]$.

Definition 2.2.5. [127] A sequence $x = (x_k)$ is said to be $\mathcal{I} - \lambda-$statistically convergent or $\mathcal{I} - S_\lambda$ convergent to L, if for every $\epsilon > 0$ and $\delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(\mathcal{I} - S_\lambda)$ or $\mathcal{I} - S_\lambda - \lim_{k\to\infty} x_k = L$.

2.3 Generalized Difference Sequence Spaces

In this section, we introduce certain new generalized difference sequence spaces defined by an admissible ideal $\mathcal{I}$, modulus function f and study some of their topological properties.

Definition 2.3.1. Let $\mathcal{I}$ be an admissible ideal, f be a modulus function and $p = (p_k)$ be any sequence of positive real numbers. For each $\delta > 0$, we define the following sequence spaces:

$$V_\lambda^f[\Delta_p^m, \mathcal{I}]_0 = \left\{x = (x_k) \in X : \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k\|)]^{p_k} \geq \delta\} \in \mathcal{I}\right\},$$

$$V_\lambda^f[\Delta_p^m, \mathcal{I}]_L = \left\{ x = (x_k) \in X : \exists\ L > 0, \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k - L\|)]^{p_k} \geq \delta\} \in \mathcal{I} \right\},$$

and

$$V_\lambda^f[\Delta_p^m, \mathcal{I}]_\infty = \left\{ x = (x_k) \in X : \exists\ K > 0, \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k\|)]^{p_k} \geq K\} \in \mathcal{I} \right\}.$$

It can be written as $x = (x_k) \in V_\lambda^f[\Delta_p^m, \mathcal{I}]_{0,L,\infty}$ or $x_k \to L(V_\lambda^f[\Delta_p^m, \mathcal{I}]_{0,L,\infty}$.

Remark 2.3.1. If we take $\mathcal{I} = \mathcal{I}_f = \{A \subset \mathbb{N} : \text{if } A \text{ is a finite subset}\}$, then we get the spaces $[V, \lambda, f, p]_L[\Delta^m, X]$ and $[V, \lambda, f, p]_0[\Delta^m, X]$ of Et et al.[48], instead of $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$ and $V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$.

Now, we prove the main results of this section.

Theorem 2.3.1. *If $p = (p_k)$ be a bounded sequence, then $V_\lambda^f[\Delta_p^m, \mathcal{I}]_{0,L,\infty}$ are linear spaces.*

Proof. We shall prove the assertion only for $V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$ and the proofs for the others follow the similar lines.

Let $x = (x_k)$ and $y = (y_k)$ be two sequences in $V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$, then for every $\delta > 0$, the sets

$$A_\delta(\lambda) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k\|)]^{p_k} \geq \frac{\delta}{2} \right\}, \tag{2.3.1}$$

$$B_\delta(\lambda) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m y_k\|)]^{p_k} \geq \frac{\delta}{2} \right\} \tag{2.3.2}$$

belong to $\mathcal{I}$.

Let α, β be two scalars. Since Δ^m is a linear operator, therefore, we have

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m(\alpha.x_k+\beta.y_k)\|)]^{p_k} \\
&\qquad\qquad \leq \frac{1}{\lambda_n}\sum_{k\in I_n}[f(|\alpha|.\|\Delta^m(x_k\|)+f(|\beta|.\|\Delta^m y_k\|)]^{p_k} \\
&\qquad\qquad \leq C.(K_\alpha)^H.\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k\|)]^{p_k}+C.(K_\beta)^H.\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m y_k\|)]^{p_k},
\end{aligned}$$

where $|\alpha|\leq K_\alpha$, $|\beta|\leq K_\beta$, and $H=\sup_k p_k$.

Thus for $\delta>0$, we have the following inclusion

$$\begin{aligned}
&\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m(\alpha.x_k+\beta.y_k)\|)]^{p_k}\geq\delta\right\} \\
&\qquad\qquad \subseteq\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k\|)]^{p_k}\geq\frac{\delta}{2C.(K_\alpha)^H}\right\} \\
&\qquad\qquad\qquad \cup\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m y_k\|)]^{p_k}\geq\frac{\delta}{2C.(K_\beta)^H}\right\}.
\end{aligned}$$

By the virtue of (2.3.1) and (2.3.2), the set

$$\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m(\alpha.x_k+\beta.y_k)\|)]^{p_k}\geq\delta\right\}$$

belongs to $\mathcal{I}$. This completes the proof of the theorem. □

Theorem 2.3.2. *The inclusion*

$$V_\lambda^f[\Delta_p^{m-1},\mathcal{I}]_{0,L,\infty}\subset V_\lambda^f[\Delta_p^m,\mathcal{I}]_{0,L,\infty}$$

is strict for $m\geq 1$.

Proof. We prove the result for $V_\lambda^f[\Delta_p^{m-1},\mathcal{I}]_0$ only. The others can be proved on the similar lines.

Suppose $x = (x_k) \in V_\lambda^f[\Delta_p^{m-1}, \mathcal{I}]_0$, then for each $\delta > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k\|)]^{p_k} \geq \delta \right\} \in \mathcal{I}. \tag{2.3.3}$$

By using the property of modulus function, we have the following inequality

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k\|)]^{p_k} \\
&\qquad \leq \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k\|) + f(\|\Delta^{m-1} x_{k+1}\|)]^{p_k} \\
&\qquad \leq C.\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k\|)]^{p_k} + C.\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_{k+1}\|)]^{p_k}.
\end{aligned}$$

Now for given $\delta > 0$, following containment holds

$$\begin{aligned}
&\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k\|)]^{p_k} \geq \delta \right\} \\
&\qquad \subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k\|)]^{p_k} \geq \frac{\delta}{2C} \right\} \\
&\qquad \cup \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_{k+1}\|)]^{p_k} \geq \frac{\delta}{2C} \right\}.
\end{aligned}$$

Also from (2.3.3), both the sets on the right hand side in the above containment belong to $\mathcal{I}$.Therefore, it follows that $x \in V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$. □

Remark 2.3.2. The inclusion is strict as the sequence $x = (x_k) = (k^{m-1})$ belongs to $x \in V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$, but does not belong to $V_\lambda^f[\Delta_p^{m-1}, \mathcal{I}]_0$ for $f(x) = x$, $\lambda_n = n$ and $p_k = 1$ for all $k \in \mathbb{N}$.

Theorem 2.3.3. *Let f' and f'' are modulus functions satisfying the property $\lim_{t\to\infty} \frac{f'(t)}{f''(t)} = M > 0$, then $V_\lambda^{f'}[\Delta_p^m, \mathcal{I}]_{0,L} \subset V_\lambda^{f''}[\Delta_p^m, \mathcal{I}]_{0,L}$.*

Proof. Let $x = (x_k) \in V_\lambda^{f'}[\Delta_p^m, \mathcal{I}]_L$. Given that $\lim_{t\to\infty} \frac{f'(t)}{f''(t)} = M$, therefore there exists a positive integer K such that

$$f'(t) \geq Kf''(t) \text{ for all } t \geq 0.$$

Therefore

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L\|)]^{p_k} \geq (K)^H.\frac{1}{\lambda_n}\sum_{k\in I_n}[f''(\|\Delta^m x_k - L\|)]^{p_k}.$$

For each $\delta > 0$, we have following relationship

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f''(\|\Delta^m x_k - L\|)]^{p_k} \geq \delta\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L\|)]^{p_k} \geq \delta.(K)^H\right\}.$$

Since $x = (x_k) \in V_\lambda^{f'}[\Delta_p^m, \mathcal{I}]_L$, therefore it follows that the set

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f''(\|\Delta^m x_k - L\|)]^{p_k} \geq \delta\right\}$$

belongs to $\mathcal{I}$, which proves the result. □

Theorem 2.3.4. *If f, f' and f'' are modulus functions, then*
(i) $V_\lambda^{f'}[\Delta_p^m, \mathcal{I}]_L \subset V_\lambda^{f\circ f'}[\Delta_p^m, \mathcal{I}]_L$,
(ii) $V_\lambda^{f'}[\Delta_p^m, \mathcal{I}]_L \cap V_\lambda^{f''}[\Delta_p^m, \mathcal{I}]_L \subset V_\lambda^{f'+f''}[\Delta_p^m, \mathcal{I}]_L$.

Proof. (i) Let $x = (x_k) \in V_\lambda^{f'}[\Delta_p^m, \mathcal{I}]_L$. For given $\epsilon > 0$, choose a $\delta \in (0, 1)$ such that

$$f(t) < \epsilon \text{ for all } 0 < t < \delta.$$

Since $x \in V_\lambda^{f'}[\Delta_p^m, \mathcal{I}]_L$, therefore

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L\|)]^{p_k} \geq \delta\right\} \in \mathcal{I}. \tag{2.3.4}$$

On the other hand,

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f\circ f'(\|\Delta^m x_k - L\|)]^{p_k}$$

$$= \frac{1}{\lambda_n}\sum_{k\in I_n \& f'(\|\Delta^m x_k - L\|)\geq\delta}[f\circ f'(\|\Delta^m x_k - L\|)]^{p_k}$$

$$+\frac{1}{\lambda_n}\sum_{k\in I_n \& f'(\|\Delta^m x_k - L\|)<\delta}[f\circ f'(\|\Delta^m x_k - L\|)]^{p_k}$$

$$\leq (\epsilon)^H + \max(1,(2.\frac{f(1)}{\delta})^H).\frac{1}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L\|)]^{p_k}$$

by the use of proposition(1.3.1).

Thus for any $\xi > 0$, we have the inclusion

$$\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f\circ f'(\|\Delta^m x_k - L\|)]^{p_k}\geq\xi\right\}$$

$$\subseteq\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L\|)]^{p_k}\geq\frac{(\xi-\epsilon)}{M}\right\},$$

where $M = \max(1,(2.\frac{f(1)}{\delta})^H)$.

Since $x = (x_k) \in V_\lambda^{f'}[\Delta_p^m, \mathcal{I}]_L$, so by (2.3.4) the latter set in the above inclusion belongs to $\mathcal{I}$. This proves that $x \in V_\lambda^{f\circ f'}[\Delta_p^m, \mathcal{I}]_L$

(ii) For the proof of part(ii), consider a sequence $x = (x_k) \in V_\lambda^{f'}[\Delta_p^m, \mathcal{I}]_L \cap V_\lambda^{f''}[\Delta_p^m, \mathcal{I}]_L$.

For every $\delta > 0$ the sets

$$K_1 = \left\{n\in\mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L\|)]^{p_k}\geq\frac{\delta}{2}\right\}$$

and

$$K_2 = \left\{n\in\mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f''(\|\Delta^m x_k - L\|)]^{p_k}\geq\frac{\delta}{2}\right\}$$

belong to $\mathcal{I}$.

On the other hand, the following inequality holds

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[(f'+f'')(\|\Delta^m x_k - L\|)]^{p_k}$$
$$\leq \frac{C}{\lambda_n}\sum_{k\in I_n}[f'(\|\Delta^m x_k - L\|)]^{p_k} + \frac{C}{\lambda_n}\sum_{k\in I_n}[f''(\|\Delta^m x_k - L\|)]^{p_k},$$

where $\sup_k p_k = H$ and $C = \max(1, 2^{H-1})$.

Therefore for any $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[(f'+f'')(\|\Delta^m x_k - L\|)]^{p_k} \geq \delta\right\} \subseteq K_1 \cup K_2.$$

This shows that $x = (x_k) \in V_\lambda^{f'+f''}[\Delta_p^m, \mathcal{I}]_L$. □

Theorem 2.3.5. *Let f be a modulus function and $p = (p_k)$ be a sequence of positive real numbers, then $V_\lambda[\Delta_p^m, \mathcal{I}]_L \subseteq V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$.*

The proof of this theorem follows on similar lines as in the proof of the part(i) of Theorem 2.3.4, so we omit it.

Theorem 2.3.6. *Let f be a modulus function. If $\lim_{t\to\infty}\frac{f(t)}{t} = M > 0$, then $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L \subseteq V_\lambda[\Delta_p^m, \mathcal{I}]_L$.*

Proof. Given that $\lim_{t\to\infty}\frac{f(t)}{t} = M > 0$, therefore there exists a positive integer K such that

$$f(t) \geq Kt \text{ for all } t \geq 0.$$

This implies that

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k - L\|)]^{p_k} \geq (K)^H \frac{1}{\lambda_n}\sum_{k\in I_n}\|\Delta^m x_k - L\|^{p_k},$$

where $\sup_k p_k = H$. Rest part of the result can be obtained by using the same technique as in Theorem 2.3.3. □

2.4 $S_\lambda^{\Delta^m}(\mathcal{I})-$Convergence

In this section, a new class of generalized statistical convergence ($S_\lambda^{\Delta^m}(\mathcal{I})-$convergence) has been introduced with the help of an ideal $\mathcal{I}$ and the difference operator Δ^m. Further, a condition under which $S_\lambda^{\Delta^m}(\mathcal{I})-$convergence coincides with the sequence spaces $V_\lambda^f[\Delta_p^m, \mathcal{I}]_0$, $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$ and $V_\lambda^f[\Delta_p^m, \mathcal{I}]_\infty$ has also been established in this section.

Definition 2.4.1. Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. A sequence $x = (x_k) \in X$ is said to be $S_\lambda^{\Delta^m}(\mathcal{I})-$convergent to a number L provided that for every $\epsilon > 0$ and $\delta > 0$

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(S_\lambda^{\Delta^m}(\mathcal{I}))$ or $S_\lambda^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty} x_k = L$. We shall denote the set of all $S_\lambda^{\Delta^m}(\mathcal{I})-$convergent sequences by $S_\lambda^{\Delta^m}(\mathcal{I})$.

Theorem 2.4.1. *Let $\mathcal{I}$ be an admissible ideal and $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers. If $S_\lambda^{\Delta^m} - \lim_{k\to\infty} x_k = L$, then $S_\lambda^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty} x_k = L$.*

Proof. Suppose $S_\lambda^{\Delta^m} - \lim_{k\to\infty} x_k = L$. Therefore, that for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| = 0.$$

This implies that for every $\delta > 0$, there exists a positive integer n_0 such that

$$\frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| < \delta \quad \text{for all} \quad n \geq n_0.$$

This shows that

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| \geq \delta\right\} \subset \{1, 2, \cdots, n_0 - 1\}.$$

Since $\mathcal{I}$ be an admissible ideal, therefore

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

It follows that $S_\lambda^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty} x_k = L$. □

Converse of this theorem does not hold true in general as can be seen in the following example:

Example 2.4.1. *Assume that $\mathcal{I}$ be an admissible ideal and choose a set $A \in \mathcal{I}$. Define a sequence $x = (x_k)$ as:*

$$x_k = \begin{cases} k^{m+1}, & \text{for } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n,\ n \notin A, \\ k^{m+1}, & \text{for } n - \lambda_n + 1 \leq k \leq n,\ n \in A, \\ 0, & \text{otherwise,} \end{cases}$$

for a fixed $m \in \mathbb{N}$.

For given $\epsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - 0\| \geq \epsilon\}|$$

$$= \lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : \|(-1)^{m+2}(m+1)![k + \frac{m}{2}] - 0\| \geq \epsilon\}| \leq \frac{[\sqrt{\lambda_n}]}{\lambda_n} \to 0$$

for $n \notin A$.

Now for every $\delta > 0$, there exists a positive integer n_0 such that

$$\frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - 0\| \geq \epsilon\}| < \delta \text{ for } n \notin A \text{ and } n \geq n_0.$$

Therefore, we have

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - 0\| \geq \epsilon\}| \geq \delta\right\} \subset \{A \cup (1, 2, \cdots n_0 - 1)\}.$$

Since $\mathcal{I}$ is an admissible ideal, it follows that $S_\lambda^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty} x_k = 0$.

But

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\|\Delta^m x_k - 0\| \to \infty \text{ as } n\to\infty.$$

This implies that $x = (x_k) \notin [V,\lambda]_0[\Delta^m, X]$. From [p-29, [48]], it follows that $S_\lambda^{\Delta^m} - \lim_{k\to\infty} x_k \neq 0$.

Theorem 2.4.2. *Let f be a modulus function and $p = (p_k)$ be a sequence of positive real numbers. If $0 < inf_k p_k = h \leq p_k \leq sup_k p_k = H < \infty$, then $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L \subset S_\lambda^{\Delta^m}(\mathcal{I})$.*

Proof. Assume that $x \in V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$ and let $\epsilon > 0$ be given. Then

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k - L\|)]^{p_k} \\
&\quad = \frac{1}{\lambda_n}\sum_{k\in I_n \& \|\Delta^m x_k - L\|\geq\epsilon}[f(\|\Delta^m x_k - L\|)]^{p_k} + \frac{1}{\lambda_n}\sum_{k\in I_n \& \|\Delta^m x_k - L\|<\epsilon}[f(\|\Delta^m x_k - L\|)]^{p_k} \\
&\quad \geq \frac{1}{\lambda_n}\sum_{k\in I_n \& \|\Delta^m x_k - L\|\geq\epsilon}[f(\|\Delta^m x_k - L\|)]^{p_k} \\
&\quad \geq \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\epsilon)]^{p_k} \\
&\quad \geq \frac{1}{\lambda_n}\sum_{k\in I_n}\min([f(\epsilon)]^h, [f(\epsilon)]^H) \\
&\quad \geq M\frac{1}{\lambda_n}|\{k\in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}|, \text{where } M = \min([f(\epsilon)]^h, [f(\epsilon)]^H).
\end{aligned}$$

Therefore for any $\delta > 0$, we have

$$\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}|\{k\in I_n : \|\Delta^m x_k - L\|\geq\epsilon\}| \geq \delta\right\}$$
$$\subseteq \left\{n\in\mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k - L\|)]^{p_k} \geq M.\delta\right\}.$$

Since $x_k \to L(V_\lambda^f[\Delta_p^m, \mathcal{I}]_L)$, then clearly

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I},$$

which proves the result. $\square$

Theorem 2.4.3. *Let f be a bounded modulus function and $0 < inf_k p_k = h \leq p_k \leq sup_k p_k = H < \infty$, then $S_\lambda^{\Delta^m}(\mathcal{I}) \subseteq V_\lambda^f[\Delta_p^m, \mathcal{I}]_L$.*

Proof. Suppose $x = (x_k) \in S_\lambda^{\Delta^m}(\mathcal{I})$. For $\epsilon > 0$, we can write

$$\frac{1}{\lambda_n}\sum_{k \in I_n}[f(\|\Delta^m x_k - L\|)]^{p_k}$$
$$= \frac{1}{\lambda_n}\sum_{k \in I_n \& \|\Delta^m x_k - L\| \geq \epsilon}[f(\|\Delta^m x_k - L\|)]^{p_k} + \frac{1}{\lambda_n}\sum_{k \in I_n \& \|\Delta^m x_k - L\| < \epsilon}[f(\|\Delta^m x_k - L\|)]^{p_k}$$

Since f is bounded, then the above equality becomes

$$\frac{1}{\lambda_n}\sum_{k \in I_n}[f(\|\Delta^m x_k - L\|)]^{p_k} \leq \frac{1}{\lambda_n}\sum_{k \in I_n \& \|\Delta^m x_k - L\| \geq \epsilon}\max([k]^h, [k]^H) + \frac{1}{\lambda_n}\sum_{k \in I_n}[f(\epsilon)]^{p_k}$$

$$\leq K.\frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\} + M,$$

where $K = \max([k]^h, [k]^H)$ and $M = \max([f(\epsilon)]^h, [f(\epsilon)]^H)$.

Consequently, we get

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k \in I_n}[f(\|\Delta^m x_k - L\|)]^{p_k} \geq M.\delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| \geq \frac{(\delta - M)}{K}\right\}.$$

Since $x = (x_k) \in S_\lambda^{\Delta^m}(\mathcal{I})$, it follows that

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| \geq \frac{(\delta - M)}{K}\right\} \in \mathcal{I}.$$

This completes the result. $\square$

Theorem 2.4.4. *If* $0 < inf_k p_k = h \leq p_k \leq sup_k p_k = H < \infty$*, then*

$$V_\lambda^f[\Delta_p^m, \mathcal{I}]_L = S_\lambda^{\Delta^m}(\mathcal{I})$$

if and only if f *is bounded.*

Proof. Let f be bounded. Then, by Theorem 2.4.2 and 2.4.3 we have $V_\lambda^f[\Delta_p^m, \mathcal{I}]_L = S_\lambda^{\Delta^m}(\mathcal{I})$.

Conversely*:* Suppose f is unbounded defined by $f(k) = k$ for all $k \in \mathbb{N}$. Take a fixed set $A \in \mathcal{I}$ and define $x = (x_k)$ as:

$$x_k = \begin{cases} k^{m+1}, & \text{for } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n,\ n \notin A, \\ k^{m+1}, & \text{for } n - \lambda_n + 1 \leq k \leq n,\ n \in A, \\ 0 & \text{otherwise,} \end{cases}$$

Obviously, $S_\lambda^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty} x_k = 0$.

If we take $p_k = 1$ for all $k = 1, 2, \cdots$, then $x_k \notin V_\lambda^f[\Delta_p^m, \mathcal{I}]$ by [148]. This contradicts the fact that $S_\lambda^{\Delta^m}(\mathcal{I}) = V_\lambda^f[\Delta_p^m, \mathcal{I}]$. $\square$

Chapter 3

Generalized Sequence Spaces in 2-Normed Space

3.1 Introduction

In 1963, Gähler[63] introduced the concept of 2-normed spaces as a generalization of normed linear space and studied their topological structures. Since then, Elumalai et al.[44], Freese et al[57], Ha et al[73] and many others have extensively developed the geometric and topological structure of 2-normed spaces. Recently, many generalizations of statistical convergence and ideal convergence have been extended in 2-normed space. Gurdal and Pehlivan[67] examined statistical convergence in 2-normed spaces. Gurdal[65] extended $\mathcal{I}-$limit and $\mathcal{I}-$cluster points in 2-normed spaces and obtained some relations between these two notions. Further, Esi and Hazarika[46] used ideals and difference sequences to introduce some new sequence spaces and investigated

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some of their algebric characterizations.

Nakano[106] in 1953, introduced the notion of modulus function and used it to resolve some structural problems of FK-spaces. Subsequently, many authors used modulus function to develop various sequence spaces. Maddox[88] introduced a new class of sequences, which are strongly Cesàro summable with respect to a modulus function by generalizing the concept of strong Cesàro summability. Savas[129] defined strong almost $A-$summability by using modulus function f and discussed their properties. For more details on sequence spaces defined by modulus function, we refer ([48], [64], [112] and [130]).

The aim of this chapter is to introduce some new sequence spaces by using modulus function and difference sequences in 2-normed space. Section 3.2 begins with a brief introduction of $V_\lambda-$summability, λ_X^m-statistical convergence and $\mathcal{I}-\lambda-$statistical convergence in 2-normed space. In section 3.3, new sequence spaces $V_f^{\mathcal{I}}[\|,\|,\,\Delta^m,\,\lambda,p]_0$, $V_f^{\mathcal{I}}[\|,\|,\,\Delta^m,\,\lambda,p]_L$ and $V_f^{\mathcal{I}}[\|,\|,\Delta^m,\lambda,p]_\infty$ have been defined by making use of modulus function and difference sequences in 2-normed spaces. Further, in this section some of the topological properties of these spaces have been established. Finally, in section 3.4, a new class $S_\lambda^{\Delta^m}(\|.,.\|,\mathcal{I})$ of generalized statistical convergent sequences has been defined and certain inclusion properties with the help of an ideal and difference sequences have been studied.

3.2 Preliminaries

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A real valued function $\|\bullet,\bullet\| : X \times X \to \mathbb{R}$ satisfying the following conditions:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(ii) $\|x, y\|$ is invariant under permutation,

(iii) $\|\alpha.x, y\| = |\alpha|.\|x, y\|$ for all $\alpha \in \mathbb{R}$,

(iv) $\|x, y+z\| \leq \|x, y\| + \|x, z\|$

(X, $\|.,.\|$) is called a 2-normed space.

Throughout this chapter, S_X^2 stands for the space of all sequences defined over 2-normed space (X, $\|.,.\|$).

Let $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_n = 1$ and the interval $I_n = [n - \lambda_n + 1, n]$. The set of all such sequences will be denoted by Ω.

Definition 3.2.1. [48] A sequence $x = (x_k)$ is said to be λ_X^m-statistically convergent to the number L if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\}| = 0.$$

In this case, we write $S_\lambda(\Delta^m, X) - \lim_{k\to\infty} x_k = L$.

Definition 3.2.2. [127] A sequence $x = (x_k) \in X$ is said to be $\mathcal{I} - [V, \lambda]$ summable to L, if for any $\delta > 0$, the set $\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \|x_k - L\| \geq \delta\right\}$ belongs to $\mathcal{I}$, where $I_n = [n - \lambda_n + 1, n]$.

Definition 3.2.3. [127]A sequence $x = (x_k)$ is said to be $\mathcal{I} - \lambda-$statistically convergent or $\mathcal{I} - S_\lambda$ convergent to L, if for every $\epsilon > 0$ and $\delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : \|x_k - L\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(\mathcal{I} - S_\lambda)$ or $\mathcal{I} - S_\lambda - \lim_{k\to\infty} x_k = L$.

By using above concepts, Gurdal[65] and Yamanc and Gurdal[155] defined the notions of $\mathcal{I}-$convergence and $\mathcal{I}-$statistical convergence of a sequence in 2-normed space as:

Definition 3.2.4. [65] Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$ and (X, $\|.,.\|$) is a 2-normed space. A sequence $x = (x_k)$ in X is said to be $\mathcal{I}-$convergent to $L \in X$ if for each $\epsilon > 0$ and non-zero $z \in X$, the set

$$A(\epsilon) = \{k \in \mathbb{N} : \|x_k - L, z\| \geq \epsilon\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - \lim_{k\to\infty} \|x_k, z\| = \|\text{Ł}, z\|$.

Definition 3.2.5. [155] Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. The sequence $x = (x_k)$ in X is said to be $\mathcal{I}-$statistically convergent to L, if for each $\epsilon > 0$, $\delta > 0$ and non-zero $z \in X$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|x_k - L, z\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - st - \lim_{k\to\infty} \|x_k - L, z\| = 0$ or $\mathcal{I} - st - \lim_{k\to\infty} \|x_k, z\| = \|L, z\|$.

3.3 $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$ - Summability

In this section, we introduce new sequence spaces $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_{0,L,\infty}$ and study some of their topological properties.

Definition 3.3.1. Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal, f be a modulus function and $p = (p_k)$ be a bounded sequence of positive (strictly) real numbers, then for each $\epsilon > 0$ and $z \in X$, we define the following sequence spaces:

$V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$

$$= \left\{x = (x_k) \in S_X^2 : \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq \delta\} \in \mathcal{I}\right\},$$

$V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_L$

$$= \left\{ x = (x_k) \in S_X^2 : \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \delta\} \in \mathcal{I} \right\}, \text{ for some } L > 0.$$

$V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_\infty$

$$= \left\{ x = (x_k) \in S_X^2 : \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq K\} \in \mathcal{I} \right\}, \text{ for } K > 0.$$

Symbolically, it is denoted as $x = (x_k) \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_{0,L,\infty}$.

Remark 3.3.1. If we take $f(x) = x$, then we obtain the sequence space $V^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_{0,L}$ instead of $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_{0,L}$.

Now, we present the main results of this section.

Theorem 3.3.1. *The sequence spaces* $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$, $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$ *and* $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_\infty$ *are linear spaces over* $\mathbb{C}$.

Proof. We assume $x = (x_k),\ y = (y_k) \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$ and let α, β be two scalars. Then, for any $\delta > 0$ and for each $z \in X$, the sets

$$A_\delta(\lambda) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq \frac{\delta}{2} \right\}, \tag{3.3.1}$$

$$B_\delta(\lambda) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m y_k, z\|)]^{p_k} \geq \frac{\delta}{2} \right\} \tag{3.3.2}$$

belong to $\mathcal{I}$. Since f be a modulus function and $\|.,.\|$ is a 2-norm function, therefore the following inequality holds

$\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\alpha.\Delta^m x_k + \beta.\Delta^m y_k, z\|]^{p_k}$

$$\leq \frac{1}{\lambda_n} \sum_{k \in I_n} [f(|\alpha|.\|\Delta^m x_k, z\|) + f(|\beta|.\|\Delta^m y_k, z\|]^{p_k}$$

$$\leq C.(M_\alpha)^H.\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k, z\|)]^{p_k} + C.(M_\beta)^H.\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m y_k, z\|)]^{p_k},$$

where M_α, M_β are positive integers such that $|\alpha| \leq M_\alpha$, $|\beta| \leq M_\beta$ and $H = \sup_k p_k$. For given $\delta > 0$ and for all $z \in X$, following containment holds:

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\alpha.\Delta^m x_k + \beta.\Delta^m y_k, z\|)]^{p_k} \geq \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k, z\|)]^{p_k} \geq \frac{\delta}{2C.(M_\alpha)^H}\right\}$$
$$\cup \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m y_k, z\|)]^{p_k} \geq \frac{\delta}{2C.(M_\beta)^H}\right\}.$$

By using (3.3.1)and(3.3.2), the set $\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\alpha.\Delta^m x_k + \beta.\Delta^m y_k, z\|)]^{p_k} \geq \delta\}$ belongs to $\mathcal{I}$. This completes the proof. $\square$

Theorem 3.3.2. *Let $p = (p_k)$ be a sequence of strictly positive real numbers, then the inclusion $V_f^{\mathcal{I}}[\|.,.\|, \Delta^{m-1}, \lambda, p]_{0,\infty} \subset V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_{0,\infty}$ is strict, for $m \geq 1$.*

Proof. We will prove the result for $V_f^{\mathcal{I}}[\|.,.\|, \Delta^{m-1}, \lambda, p]_0$ only. The others can be proved with similar arguments. Let $x \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^{m-1}, \lambda, p]_0$, therefore for $\delta > 0$ and for every $z \in X$, the set $\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^{m-1}x_k, z\|)]^{p_k} \geq \delta\}$ belongs to $\mathcal{I}$. By using the property(ii) in the definition of modulus function, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k, z\|)]^{p_k} \leq \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^{m-1}x_k, z\|) + f(\|\Delta^{m-1}x_{k+1}, z\|)]^{p_k}$$

$$\leq C.\frac{1}{\lambda_n}\left(\sum_{k\in I_n}[f(\|\Delta^{m-1}x_k, z\|)]^{p_k} + \sum_{k\in I_n}[f(\|\Delta^{m-1}x_{k+1}, z\|)]^{p_k}\right).$$

For any $\delta > 0$, the following inclusion holds

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k, z\|)]^{p_k} \geq \frac{\delta}{2C}\right\}$$
$$\cup \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_{k+1}, z\|)]^{p_k} \geq \frac{\delta}{2C}\right\},$$

for each $z \in X$. Since $x \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^{m-1}, \lambda, p]_0$, therefore, it follows that the sets on the right hand side in the above containment belong to $\mathcal{I}$. Hence $x \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$.

Following example shows that this inclusion is strict.

Take $f(x) = x$, $\lambda_n = n$ and consider a sequence $x = (x_k) = k^{m-1}$, then $x \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_0$ but $x \notin V_f^{\mathcal{I}}[\|.,.\|, \Delta^{m-1}, \lambda, p]_0$ for $p_k = 1$, $k \in \mathbb{N}$. This shows that the inclusion is strict. □

Theorem 3.3.3. *Let f', f'' are modulus functions. If $\limsup_{t\to\infty} \frac{f'(t)}{f''(t)} = P > 0$, then $V_{f'}^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p] \subset V_{f''}^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$.*

Proof. Suppose $\limsup_{t\to\infty} \frac{f'(t)}{f''(t)} = P$, therfore there exists $M > 0$ such that $f'(t) \geq M.f''(t)$ for all $t \geq 0$.

Therefore for each $z \in X$, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [f'(\|\Delta^m x_k - L, z\|)]^{p_k} \geq (M)^H . \frac{1}{\lambda_n} \sum_{k \in I_n} [f''(\|\Delta^m x_k - L, Z\|)]^{p_k}.$$

For every $\delta > 0$ and $z \in X$, following relationship holds

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f''(\|\Delta^m x_k - L, Z\|)]^{p_k} \geq \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f'(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \delta.(K)^H\right\}.$$

Since $x \in V^{\mathcal{I}}_{f'}[\|.,.\|, \Delta^m, \lambda, p]$, it follows that the set on left side in the above relationship belongs to $\mathcal{I}$, which implies that $x \in V^{\mathcal{I}}_{f''}[\|.,.\|, \Delta^m, \lambda, p]$. $\square$

Theorem 3.3.4. *If f, f' and f'' are modulus functions, then*
(i) $V^{\mathcal{I}}_{f'}[\|.,.\|, \Delta^m, \lambda, p] \subset V^{\mathcal{I}}_{f\circ f'}[\|.,.\|, \Delta^m, \lambda, p]$,
(ii) $V^{\mathcal{I}}_{f'}[\|.,.\|, \Delta^m, \lambda, p] \cap V^{\mathcal{I}}_{f''}[\|.,.\|, \Delta^m, \lambda, p] \subset V^{\mathcal{I}}_{f'+f''}[\|.,.\|, \Delta^m, \lambda, p]$.

Proof. (i) Let $x = (x_k) \in V^{\mathcal{I}}_{f'}[\|.,.\|, \Delta^m, \lambda, p]_L$, then for every $\epsilon > 0$ and for some $L > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_n} [f'(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \epsilon \right\} \in \mathcal{I} \tag{3.3.3}$$

for each $z \in X$.

For given $\epsilon > 0$, choose $\delta \in (0, 1)$ such that $f(t) < \epsilon$ for all $0 < t < \delta$.

On the other hand, we have

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k\in I_n} [f \circ f'(\|\Delta^m x_k - L, z\|)]^{p_k} \\
&\qquad = \frac{1}{\lambda_n} \sum_{k\in I_n \& f'(\|\Delta^m x_k - L, z\|)<\delta} [f \circ f'(\|\Delta^m x_k - L, z\|)]^{p_k} \\
&\qquad\qquad + \frac{1}{\lambda_n} \sum_{k\in I_n \& f'(\|\Delta^m x_k - L, z\|)\geq\delta} [f \circ f'(\|\Delta^m x_k - L, z\|)]^{p_k} \\
&\qquad \leq (\epsilon)^H + \max(1, (2.\frac{f(1)}{\delta})^H).\frac{1}{\lambda_n} \sum_{k\in I_n} [f'(\|\Delta^m x_k - L, z\|)]^{p_k},
\end{aligned}$$

where $H = \sup_k p_k$.

Thus for any $\xi > 0$, we have

$$\begin{aligned}
&\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_n} [f \circ f'(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \xi \right\} \\
&\qquad \subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_n} [f'(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \frac{(\xi - \epsilon)}{M} \right\},
\end{aligned}$$

where $M = \max(1, (2.\frac{f(1)}{\delta})^H)$. Since $x = (x_k) \in V^{\mathcal{I}}_{f'}[\|.,.\|, \Delta^m, \lambda, p]_L$, therefore by virtue of (3.3.3) the latter set belongs to $\mathcal{I}$. This shows that $x \in V^{\mathcal{I}}_{f\circ f'}[\|.,.\|, \Delta^m, \lambda, p]_L$.

(ii) The proof of this part is easy to obtain and so we omit it. □

Theorem 3.3.5. *Let $p = (p_k)$ be a sequence of positive real numbers, then*

$$V^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_L \subseteq V^{\mathcal{I}}_f[\|.,.\|, \Delta^m, \lambda, p]_L.$$

Proof. Proof of this theorem follows the same lines as in part(i) of the theorem 3.3.4.

□

Theorem 3.3.6. *Let f be a modulus function. If $\limsup_{t\to\infty} \frac{f(t)}{t} = K > 0$, then $V^{\mathcal{I}}_f[\|.,.\|, \Delta^m, \lambda, p] \subseteq V^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$.*

Proof. Let a sequence $x = (x_k) \in V^{\mathcal{I}}_f[\|.,.\|, \Delta^m, \lambda, p]_L$. Then, for every $\delta > 0$ the set

$$A(\delta) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \delta \right\} \in \mathcal{I}$$

for all $z \in X$.

Since $\limsup_{t\to\infty} \frac{f(t)}{t} = K > 0$, therefore there exists a constant $M > 0$ such that $f(t) \geq M.t$ for all $t \geq 0$.

Now, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq (M)^H . \frac{1}{\lambda_n} \sum_{k \in I_n} \|\Delta^m x_k - L, z\|^{p_k} \text{ for each } z \in X.$$

For given $\delta > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [\|\Delta^m x_k - L, z\|]^{p_k} \geq \delta \right\}$$
$$\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \delta.((M)^H \right\}.$$

Since $x = (x_k) \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_L$, therefore by definition of an ideal the set $\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [\|\Delta^m x_k - L, z\|]^{p_k} \geq \delta\right\} \in \mathcal{I}$. Hence $x = (x_k) \in V^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_L$.

□

3.4 $S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I})-$Convergence

The aim of present section is to introduce and develop $S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I})-$convergence in 2-normed space X. Further in this section, it has been shown that under a certain condition on f, the space $S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I})$ coincides with sequence space $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_L$ introduced in section 3.3 .

Definition 3.4.1. Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a non-trivial admissible ideal and $\lambda = (\lambda_n)$ be a non-decreasing sequence. A sequence $x = (x_k) \in X$ is said to be $S_\lambda^{\Delta^m}(\| .,. \| , \mathcal{I})-$convergent to L if for every $\epsilon > 0$, $\delta > 0$ and $z \in X$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^m x_k - L, z\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $S_\lambda^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty} \|x_k, z\| = \|L, z\|$.
Let $S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I})$ denotes the set of all $S_\lambda^{\Delta^m}(\mathcal{I})-$convergent sequences in 2-normed space X.
Main results of this section are the following theorems :

Theorem 3.4.1. *Let f be a modulus function and $0 < inf_k p_k = h \leq p_k \leq sup_k p_k = H < \infty$, then $V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_L \subset S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I})$.*

Proof. Suppose $x \in V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_L$ and $\epsilon > 0$ be given. Then, for every $z \in X$, we obtain

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k - L, z\|)]^{p_k} = \frac{1}{\lambda_n}\sum_{k\in I_n \& \|\Delta^m x_k - L, z\|\geq\epsilon}[f(\|\Delta^m x_k - L, z\|)]^{p_k}$$
$$+\frac{1}{\lambda_n}\sum_{k\in I_n \& \|\Delta^m x_k - L, z\|<\epsilon}[f(\|\Delta^m x_k - L, z\|)]^{p_k}$$

$$\geq \frac{1}{\lambda_n}\sum_{k\in I_n \& \|\Delta^m x_k - L, z\|\geq\epsilon}[f(\|\Delta^m x_k - L, z\|)]^{p_k}$$
$$\geq \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\epsilon)]^{p_k} \geq \frac{1}{\lambda_n}\sum_{k\in I_n}\min([f(\epsilon)]^h, [f(\epsilon)]^H)$$
$$\geq K.\frac{1}{\lambda_n}|\{k\in I_n : \|\Delta^m x_k - L, z\| \geq \epsilon\}|, \text{ where } K = \min([f(\epsilon)]^h, [f(\epsilon)]^H).$$

Therefore, for every $\delta > 0$ and $z \in X$, we have

$$\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}|\{k\in I_n : \|\Delta^m x_k - L, z\| \geq \epsilon\}| \geq \delta\right\}$$
$$\subseteq \left\{n\in\mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq K.\delta\right\}.$$

Since $x_k \to L(V_f^{\mathcal{I}}[\|.,.\|,\Delta^m,\lambda,p]_L)$, hence $S_\lambda^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty}\|x_k, z\| = \|L, z\|$. □

Converse of this theorem does not hold in general as can be seen in the following example:

Example 3.4.1. *Suppose f is unbounded defined by $f(k) = k$ for all $k \in \mathbb{N}$. We take a fixed set $A \in \mathcal{I}$, where $\mathcal{I}$ be an admissible ideal and define a sequence $x = (x_k)$ as:*

$$x_k = \begin{cases} k^{m+1}, & \text{for } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n,\ n \notin A, \\ k^{m+1}, & \text{for } n - \lambda_n + 1 \leq k \leq n,\ n \in A, \\ 0, & \text{otherwise.} \end{cases}$$

For a given $\epsilon > 0$ *and for each* $z \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^m x_k - 0, z\| \geq \epsilon\}| \leq \frac{[\sqrt{\lambda_n}]}{\lambda_n} \to 0$$

for $n \notin A$.

Hence, for $\delta > 0$, *there exists a positive integer* n_0 *such that*

$$\frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^m x_k - 0, z\| \geq \epsilon\}| < \delta \text{ for } n \notin A \text{ and } n \geq n_0.$$

Therefore, we have

$$\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^m x_k - 0, z\| \geq \epsilon\}| \geq \delta\} \subset \{A \cup (1, 2, \cdots n_0 - 1)\}.$$

Since $\mathcal{I}$ *is an admissible ideal, it follows that* $S_\lambda^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty} \|x_k, z\| \to 0$ *for each* $z \in X$.

On the other hand, if we take $p_k = 1$ *for all* $k = 1, 2, \cdots$, *then* $x_k \notin V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_L$. *This contradicts the fact* $S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I}) = V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_L$, *so our supposition is wrong.*

Theorem 3.4.2. *Let* $p = (p_k)$ *be a sequence of strictly positive real numbers and* f *be a bounded modulus function. If* $0 < inf_k p_k = h \leq p_k \leq sup_k p_k = H < \infty$, then $S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I}) \subset V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]_L$.

Proof. The proof of this theorem can be obtained in the similar way as in Theorem 2.4.3. □

Theorem 3.4.3. *If* f *be bounded and* $0 < inf_k p_k = h \leq p_k \leq sup_k p_k = H < \infty$, *then* $S_\lambda^{\Delta^m}(\|.,.\|, \mathcal{I}) = V_f^{\mathcal{I}}[\|.,.\|, \Delta^m, \lambda, p]$ *if and only if* f *is bounded.*

Proof. The proof of this result can be obtained by combining the proofs of Theorem 3.4.1 and Theorem 3.4.2. □

Chapter 4

On Generalized Difference Sequence Spaces defined by Invariant Means

4.1 Introduction

Fridy and Orhan [60] in 1993 generalized the concept of statistical convergence by using lacunary sequence and called it as lacunary statistical convergence or more briefly S_θ-convergence. The authors did so by replacing, the set $\{k : k \leq n\}$ in the definition of statistical convergence by the set $\{k : k_{r-1} < k \leq k_r\}$ for some lacunary sequence (k_r), which is an increasing sequence of positive integers such that $k_0 = 0$, $h_r = k_r - k_{r-1} \longrightarrow \infty$ as $r \longrightarrow \infty$. In yet another paper, Fridy and Orhan[61] in 1993 defined S_θ-analogue of Cauchy Criterion for convergence and proved that it is equivalent to S_θ-convergence. They also observed that the S_θ-summability

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method cannot be included by any matrix summability method. Later, Jinlu Li [87] studied the inclusion properties of lacunary statistical convergence by taking different lacunary sequences and solved the open problem of Fridy and Orhan [60] in which the authors left the general description of inclusion between two arbitrary lacunary methods as an open problem.

On the other hand, Schaefer[137] in 1972, developed the theory of σ-convergence based on the idea of invariant means or σ-means and studied some of their elementary properties. Mursaleen[96] extended it further to define the notion of strong σ-convergence and showed how σ-convergence leads to strong σ-convergence. Karakaya[76] combined the approaches of lacunary sequence with invariant means to introduce a new notion of lacunary strong σ-statistically convergence and gave some inclusion relations between lacunary strong convergence with respect to a sequence of Orlicz functions and lacunary strong σ-statistically convergence. For more work in this direction, we refer([70], [126] and [134]).

In the present chapter, we have generalized lacunary strong σ-statistical convergence with the help of difference sequences and an ideal $\mathcal{I}$. This new class of convergence is called as $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})-$convergence. This chapter is divided into five sections. In section 4.2 basic definitions and some related results have been presented. In section 4.3, certain new sequence spaces $w_\sigma^f\left[\Delta_p^m,\mathcal{I},\theta\right]_0$ $w_\sigma^f\left[\Delta_p^m,\mathcal{I},\theta\right]_L$ and $w_\sigma^f\left[\Delta_p^m,\mathcal{I},\theta\right]_\infty$ have been introduced and some properties of these spaces have been examined. Section 4.4 is devoted to a new kind of summability method, called as $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})-$convergence. In the last section, some new class of generalized convergent sequences have been developed by using lacunary sequence in Probabilistic Normed Spaces(PN-Saces).

4.2 Preliminaries

This section starts with a gentle introduction and background of the paper due to Karakaya[76], who introduced the concept of strong σ-lacunary statistically convergence for sequences of numbers.

Let σ be the mapping of the set of all positive integers into itself. A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or $\sigma-$mean if and only if

(i) $\varphi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$

(iii) $\varphi(x_{\sigma(n)}) = \varphi(x)$ for all $x \in \ell_\infty$.

For any sequence $x = (x_k)$, we set $Tx = (Tx_k) = (x_{\sigma(k)})$. It has been shown by Schaefer[137] that the set V_σ of all bounded $\sigma-$convergent sequences is given by

$$V_\sigma = \{x \in \ell_\infty : \lim_{k\to\infty} t_{k_m}(x) = L, \text{ uniformly in } m, \ \sigma - \lim x = L\},$$

where

$$t_{k_m}(x) = \frac{x_m + x_{\sigma(m)} + \cdots + x_{\sigma^k(m)}}{k+1}.$$

Definition 4.2.1. [76] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be strong $\sigma-$lacunary statistically convergent to L, if for every $\epsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |t_{k_m}(x - L)| \geq \epsilon\}| = 0, \text{ uniformly in } m,$$

The set of all strong $\sigma-$lacunary statistically convergent sequences is denoted by St_{θ_σ}.

Definition 4.2.2. [37] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}-$lacunay statistical convergent or $S_\theta(\mathcal{I})-$convergent to L provided that for every $\epsilon > 0$ and $\delta > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(S_\theta(\mathcal{I}))$ or $S_\theta(\mathcal{I}) - \lim_{k\to\infty} x_k = L$. The set of all $\mathcal{I}-$lacunary statistically convergent sequences is denoted by $S_\theta(\mathcal{I})$.

Definition 4.2.3. [37] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ of numbers is said to be $N_\theta(\mathcal{I})-$convergent to L, if for every $\epsilon > 0$ we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| \geq \epsilon\right\} \in \mathcal{I}.$$

Symbolically, It is written as $x_k \to L(N_\theta(\mathcal{I}))$.

4.3 $w_\sigma^f\left[\Delta_p^m, \mathcal{I}, \theta\right]$- Summability

In this section, we introduce some new kind of sequence spaces by using invariant means and modulus function and study some of their inclusion properties.

Definition 4.3.1. Let f be a modulus function and $p = (p_k)$ be any sequence of strictly positive real numbers. Then for each $\epsilon > 0$

$$w_\sigma^f\left[\Delta_p^m, \mathcal{I}, \theta\right]_0 = \left\{x \in w(X) : \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} \left[f\left(\left|t_{k_n}\left(\Delta^m x\right)\right|\right)\right]^{p_k} \geq \epsilon\right\} \in \mathcal{I}\right\},$$

$$w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta] = \left\{x \in w(X) : \exists\ \ell > 0, \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} [f(|t_{k_n}(\Delta^m x - \ell)|)]^{p_k} \geq \epsilon\right\} \in \mathcal{I}\right\},$$

$$w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_\infty = \left\{x \in w(X) : \exists K > 0, \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} [f(|t_{k_n}(\Delta^m x)|)]^{p_k} \geq K\right\} \in \mathcal{I}\right\}$$

uniformly in n.

Remark 4.3.1. In particular, if we take $\mathcal{I} = \mathcal{I}_f = \{E \subset \mathbb{N} : \text{if } E \text{ is a finite subset}\}$, $m = 0$, $f(x) = x$, $\theta = (2^r)$, and $p_k = 1$ for all $k \in \mathbb{N}$, then the space $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]$ reduces to the space $[w]_\sigma$, of all strongly almost $\sigma-$convergent sequences defined by Mursaleen et al.[102] and the space $\widehat{w}_\sigma^f[\mathcal{I}, p, \theta]$ reduces to the space $[\widehat{w}]$, of all strongly almost convergent sequences defined by Das and Sahoo[36].

Now, we present the main results of this section.

Theorem 4.3.1. *Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal and $p = (p_k)$ is a bounded sequence of strictly positive real numbers, then $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$, $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]$ and $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_\infty$ are linear spaces over $\mathbb{C}$.*

Proof. We prove the result for $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$ only and others can be proved in the similar manner. Suppose $x = (x_k)$, $y = (y_k) \in w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$. Then, for every $\epsilon > 0$ and uniformly in n

$$A_\theta\left(\frac{\epsilon}{2}\right) = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}[f(|t_{k_n}(\Delta^m x)|)]^{p_k} \geq \frac{\epsilon}{2}\right\},$$

and

$$B_\theta\left(\frac{\epsilon}{2}\right) = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r}[f(|t_{k_n}(\Delta^m y)|)]^{p_k} \geq \frac{\epsilon}{2}\right\}$$

belong to $\mathcal{I}$.

Since Δ^m is linear, so for any α, $\beta \in \mathbb{C}$, we have

$\frac{1}{h_r}\sum_{k \in I_r}[f(|t_{k_n}(\Delta^m(\alpha.x + \beta.y)|)]^{p_k}$

$$\begin{aligned} &= \frac{1}{h_r}\sum_{k \in I_r}[f(|t_{k_n}(\alpha.\Delta^m x + \beta.\Delta^m y)|)]^{p_k} \\ &\leq C.(K_\alpha)^H.\frac{1}{h_r}\sum_{k \in I_r}[f(|t_{k_n}(\Delta^m x)|)]^{p_k} + C.(K_\beta)^H.\frac{1}{h_r}\sum_{k \in I_r}[f(|t_{k_n}(\Delta^m y)|)]^{p_k}, \end{aligned}$$

where K_α, K_β are two positive numbers such that $|\alpha| \le K_\alpha$ and $|\beta| \le K_\beta$.

For given $\epsilon > 0$, we have the following relationship

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} [f(|t_{k_n}(\Delta^m(\alpha.x + \beta.y)|)]^{p_k} \ge \epsilon \right\}$$
$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} [f(|t_{k_n}(\Delta^m x)|)]^{p_k} \ge \frac{\epsilon}{2C.(K_\alpha)^H} \right\}$$
$$\cup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} [f(|t_{k_n}(\Delta^m y)|)]^{p_k} \ge \frac{\epsilon}{2C.(K_\beta)^H} \right\}$$

uniformly in n. Since $x, y \in w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$, it follows that the later sets belong to $\mathcal{I}$. By using the definition of an ideal, the set on the left hand side in the above expression also belongs to $\mathcal{I}$. This completes the proof. □

Theorem 4.3.2. *Let f be a modulus function and $p = (p_k)$ be a sequence of strictly positive real numbers, then, the inclusion $w_\sigma^f[\Delta_p^{m-1}, \mathcal{I}, \theta]_{0,\infty} \subset w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_{0,\infty}$ is strict for $m \ge 1$.*

Proof. Suppose $x = (x_k) \in w_\sigma^f[\Delta_p^{m-1}, \mathcal{I}, \theta]_0$, therefore for each $\epsilon > 0$ and uniformly in n the set $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} [f(|t_{k_n}(\Delta^{m-1} x_k)|)]^{p_k} \ge \epsilon \right\}$ belongs to $\mathcal{I}$. Since f is a modulus function, therefore we have the following inequality

$$\frac{1}{h_r} \sum_{k \in I_r} [f(|t_{k_n}(\Delta^m x_k)|)]^{p_k}$$
$$\le \frac{1}{h_r} \sum_{k \in I_r} [f(|t_{k_n}(\Delta^{m-1} x_k)|) + f(|t_{k_n}(\Delta^{m-1} x_{k+1})|)]^{p_k}$$
$$\le C.\frac{1}{h_r} \sum_{k \in I_r} [f(|t_{k_n}(\Delta^{m-1} x_k)|)]^{p_k} + C.\frac{1}{h_r} \sum_{k \in I_r} [f(|t_{k_n}(\Delta^{m-1} x_{k+1})|)]^{p_k}$$

uniformly in n.

Now for given $\epsilon > 0$, the following inclusion holds:

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}[f(|t_{k_n}(\Delta^m x_k)|)]^{p_k} \geq \epsilon\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}[f(|t_{k_n}(\Delta^{m-1} x_k)|)]^{p_k} \geq \frac{\epsilon}{2C}\right\}$$
$$\cup \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}[f(|t_{k_n}(\Delta^{m-1} x_{k+1})|)]^{p_k} \geq \frac{\epsilon}{2C}\right\},$$

uniformly in n. Both the sets on the right hand side in the above expression belong to ideal $\mathcal{I}$. It follows that $x \in w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$.

Since $\mathcal{I}$ be an admissible ideal and the sequence $x = (x_k) = (k^{m-1})$ belongs to $x \in w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$, but does not belong to $w_\sigma^f[\Delta_p^{m-1}, \mathcal{I}, \theta]_0$ for $f(x) = x$, $t_{0_n}(x) = (x_n)$, $\theta = (2^r)$ and $p_k = 1$ for all $k \in \mathbb{N}$. Hence, the inclusion is strict. □

Theorem 4.3.3. *Let $p = (p_k)$ is a bounded sequence of strictly positive real numbers and f', f'' are modulus functions. If $\limsup_{t\to\infty} \frac{f'(t)}{f''(t)} = M > 0$, then $w_\sigma^{f'}[\Delta_p^m, \mathcal{I}, \theta]_0 \subset w_\sigma^{f''}[\Delta_p^m, \mathcal{I}, \theta]_0$.*

Proof. Let $\limsup_{t\to\infty} \frac{f'(t)}{f''(t)} = M$. Therefore, there exists a positive number $K > 0$ such that $f'(t) \geq K.f''(t)$ for all $t \geq 0$.

It follows that

$$\frac{1}{h_r}\sum_{k\in I_r}[f'(|t_{k_n}(\Delta^m x)|)]^{p_k} \geq (K)^H.\frac{1}{h_r}\sum_{k\in I_r}[f''(|t_{k_n}(\Delta^m x)|)]^{p_k}$$

uniformly in n. Thus for any $\epsilon > 0$, we obtain the following containment

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}[f''(|t_{k_n}(\Delta^m x)|)]^{p_k} \geq \epsilon\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}[f'(|t_{k_n}(\Delta^m x)|)]^{p_k} \geq \epsilon.(K)^H\right\}$$

uniformly in n. This completes the proof. □

Theorem 4.3.4. *If f, f' and f'' are modulus functions, then*

(i) $w_\sigma^{f'}[\Delta_p^m, \mathcal{I}, \theta]_0 \subset w_\sigma^{f\circ f'}[\Delta_p^m, \mathcal{I}, \theta]_0$,

(ii) $w_\sigma^{f'}[\Delta_p^m, \mathcal{I}, \theta]_0 \cap w_\sigma^{f''}[\Delta_p^m, \mathcal{I}, \theta]_0 \subset w_\sigma^{f'+f''}[\Delta_p^m, \mathcal{I}, \theta]_0$.

Theorem 4.3.5. *If f be a modulus function and $p = (p_k)$ be a sequence of positive real numbers, then* $w_\sigma[\Delta_p^m, \mathcal{I}, \theta]_{0,\infty} \subseteq w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_{0,\infty}$.

Theorem 4.3.6. *Let f be a modulus function. If* $\limsup_{t\to\infty} \frac{f(t)}{t} = M > 0$, *then* $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta] \subseteq w_\sigma[\Delta_p^m, \mathcal{I}, \theta]$.

The proof of above theorems follow on similar lines as for the Theorems 4.3.2 and 4.3.3, so are omitted here.

4.4 $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})-$Convergence

In this section, we define more generalized notion of strong $\sigma-$lacunary statistical convergence introduced by [76] with the help of generalized difference operator Δ^m and study some analogous results.

Definition 4.4.1. Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. A sequence $x = (x_k)$ is said to be $\Delta^m(\mathcal{I})-$ strong $\sigma-$lacunary statistically convergent or $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})-$convergent to a number ℓ, provided that for every $\epsilon > 0$ and $\delta > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |t_{k_n}(\Delta^m x - \ell)| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}, \quad \text{uniformly in } n.$$

In this case, we write $x_k \to \ell(S_{\theta_\sigma}^{\Delta^m}(\mathcal{I}))$ or $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty} x_k = \ell$. We denote the set of all $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})-$convergent sequences by $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})$.

Theorem 4.4.1. *Let $p = (p_k)$ be a sequence of strictly positive real numbers. If* $0 < inf_k p_k = h \leq p_k \leq sup_k p_k = H < \infty$, *then* $w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0 \subset S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})$.

Proof. Assume $x \in w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$ and $\epsilon > 0$ be given. Then we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r}[f(|t_{k_n}(\Delta^m x)|)]^{p_k} &= \frac{1}{h_r}\sum_{k\in I_r \& |t_{k_n}(\Delta^m x)|\geq\epsilon}[f(|t_{k_n}(\Delta^m x)|)]^{p_k} \\
&\qquad + \frac{1}{h_r}\sum_{k\in I_r \& |t_{k_n}(\Delta^m x)|<\epsilon}[f(|t_{k_n}(\Delta^m x)|)]^{p_k} \\
&\geq \frac{1}{h_r}\sum_{k\in I_r \& |t_{k_n}(\Delta^m x)|\geq\epsilon}[f(|t_{k_n}(\Delta^m x)|)]^{p_k} \\
&\geq \frac{1}{h_r}\sum_{k\in I_r}[f(\epsilon)]^{p_k} \\
&\geq \frac{1}{h_r}|\{k \in I_r : |t_{k_n}(\Delta^m x)| \geq \epsilon\}|.\min([f(\epsilon)]^h, [f(\epsilon)]^H)
\end{aligned}$$

uniformly in n. Then, for every $\delta > 0$, we have the following relationship

$$\begin{aligned}
&\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |t_{k_n}(\Delta^m x)| \geq \epsilon\}| \geq \delta\right\} \\
&\qquad\qquad \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}[f(|t_{k_n}(\Delta^m x)|)]^{p_k} \geq K.\delta\right\}
\end{aligned}$$

uniformly in n, where $K = \min([f(\epsilon)]^h, [f(\epsilon)]^H)$. Since $x \in w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$, which implies that $x \in S_{\theta_\sigma}^{\Delta^m}(\mathcal{I})$. $\square$

Following example shows that the converse of the above theorem need not to be true.

Example 4.4.1. *Suppose f is unbounded function defined by $f(k) = k$ for all $k \in \mathbb{N}$ and $\theta = (2^r)$ be a lacunary sequence. Let $A \in \mathcal{I}$ be a fixed set and define a sequence $x = (x_k)$ as:*

$$x_k = \begin{cases} k^{m+1}, & \text{if } r \notin A \text{ and } 2^{r-1} + 1 \leq k \leq 2^{r-1} + [\sqrt{h_r}], \\ k^{m+1}, & \text{if } r \in A \text{ and } 2^{r-1} < k \leq 2^{r-1} + h_r, \\ 0, & \text{otherwise}, \end{cases}$$

where $I_r = (2^{r-1}, 2^r]$ *and* $h_r = 2^r - 2^{r-1}$.

For given $\epsilon > 0$, *we have*

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : |t_{k_n}(\Delta^m x - 0)| \geq \epsilon\}|$$

$$= \lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : |t_{k_n}((-1)^{m+2}(m+1)!(k+\frac{m}{2}) - 0)| \geq \epsilon\}| \leq \frac{[\sqrt{h_r}]}{h_r} \to 0$$

for all $r \notin A$ *and uniformly in* n.

For $\delta > 0$, *there exists a positive integer* r_0 *such that*

$$\frac{1}{h_r}|\{k \in I_r : |t_{k_n}(\Delta^m x - 0)| \geq \epsilon\}| < \delta \text{ for } r \notin A \text{ and } r \geq r_0.$$

Therefore,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |t_{k_n}(\Delta^m x - 0)| \geq \epsilon\}| \geq \delta\right\} \subset (A \cup (1, 2, \cdots r_0 - 1)).$$

Since $\mathcal{I}$ *be an admissible ideal, it follows that* $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I}) - \lim_{k\to\infty} x_k = 0$.

By taking $p_k = 1$ *for all* $k = 1, 2, \cdots$, *we have* $x_k \notin w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$. *This contradicts that* $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I}) \subseteq w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$.

In the next theorem, we impose a condition on f under which the converse of Theorem 4.4.1 holds.

Theorem 4.4.2. *Let* f *be a bounded modulus function and* $0 < inf_k p_k = h \leq p_k \leq sup_k p_k = H < \infty$, *then* $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I}) \subset w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$.

Proof. Proof of this theorem follows parallel lines as in Theorem 2.4.3. □

Theorem 4.4.3. *If* $0 < inf_k p_k = h \leq p_k \leq sup_k p_k = H < \infty$, *then* $S_{\theta_\sigma}^{\Delta^m}(\mathcal{I}) = w_\sigma^f[\Delta_p^m, \mathcal{I}, \theta]_0$ *if and only if* f *is bounded.*

Proof. Proof of this result is the direct consequence of Theorem 4.4.1 and Theorem 4.4.2. □

4.5 Generalized Statistical Convergence in PN-Space

In this section, we define a more generalized notion of convergence in probabilistic normed space(briefly PN-Space), introduced by $\hat{S}$erstnev[142] as a generalization of usual normed space. Various authors have shown their keen interest to study PN-Spaces in different aspects. For extensive view, we refer([2], [35], [141] and [139]). Quite recently, many summability methods have been extended in PN-paces. Analogous to Fast[54], Karakus[77] was the first one who used natural density to define statistical convergence in these spaces and studied its structural properties. After his pioneer work, many papers have appeared in literature on statistical convergence and related concepts, in PN-Spaces. For reference see([5], [84], [103], [104] and [117]). We begin with some basic facts and definitions concerning PN-Spaces in this section.

Definition 4.5.1. [2] Let $(X, F, *)$ be a PN-space. A sequence $x = (x_k)$ in X is said to be convergent to $x \in X$ with respect to the probabilistic norm F if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer k_0 such that $F_{x_k - x}(\epsilon) > 1 - \lambda$ whenever $k \geq k_0$. It is denoted by $F - \lim_{k\to\infty} x_k = x$ or $(x_k) \longrightarrow x$ with respect to the probabilistic norm F.

Definition 4.5.2. [77] A sequence $x = (x_k)$ in X is said to be statistical convergent to some point $x_0 \in X$ with respect to the probabilistic norm F if for every $\epsilon > 0$ and $\lambda \in (0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| = 0.$$

In this case, we write $S^F - \lim_{k\to\infty} x_k = x_0$.

Definition 4.5.3. [77] A sequence $x = (x_k)$ in X is said to be statistically Cauchy with respect to the probabilistic norm F provided that for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer m

$$\lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : F_{x_k - x_m}(\epsilon) \leq 1 - \lambda\}| = 0.$$

4.5.1 $S^F(\mathcal{I})$ and $S_\theta^F(\mathcal{I})-$Convergence

In this section, we introduce the concept of $\mathcal{I}$-statistical convergence ($S^F(\mathcal{I})$) and $\mathcal{I}$-lacunary statistical convergence($S_\theta^F(\mathcal{I})$) for sequences in PN-Space $(X, F, *)$ and study their topological properties.

Definition 4.5.4. Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a non-trivial ideal and $(X, F, *)$ be a PN-Space. A sequence $x = (x_k)$ in X is said to be $\mathcal{I}-$statistically convergent or $S^F(\mathcal{I})-$convergent to $x_0 \in X$ with respect to the probabilistic norm F, provided that for each $\epsilon > 0$, $\delta > 0$ and $\lambda \in (0, 1)$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to x_0(S^F(\mathcal{I}))$ or $S^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$.
We denote the set of all $\mathcal{I}-$statistically convergent sequences in PN-Space $(X, F, *)$ by $S^F(\mathcal{I})$.

Definition 4.5.5. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal, $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ in X is said to be $\mathcal{I}-$lacunary statistical convergent or $S_\theta^F(\mathcal{I})-$convergent to $x_0 \in X$ with respect to the probabilistic norm F if for each $\epsilon > 0$, $\delta > 0$ and $\lambda \in (0, 1)$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to x_0(S_\theta^F(\mathcal{I}))$ or $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$.

We denote the set of all $\mathcal{I}-$lacunary statistically convergent sequences in PN-Space $(X, F, *)$ by $S_\theta^F(\mathcal{I})$.

Remark 4.5.1. If we take $\mathcal{I} = \mathcal{I}_f = \{E \subset \mathbb{N} :$ if E is a finite subset$\}$, then $S^F(\mathcal{I})-$convergence and $S_\theta^F(\mathcal{I})-$convergence respectively coincide with statistical convergence of [77] and lacunary statistical convergence of [117] on PN-Spaces.

We, now present the main results of this section.

Theorem 4.5.1. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal and $\theta = (k_r)$ be a lacunary sequence. If $S_\theta^F - \lim_{k\to\infty} x_k = x_0$, then $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$. However converse may not be true.*

Proof. Suppose $S_\theta^F - \lim_{k\to\infty} x_k = x_0$, then for each $\epsilon > 0$ and $\lambda \in (0, 1)$ we have

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| = 0.$$

Thus, for each $\delta > 0$, we can find a positive integer r_0 (say) such that

$$\frac{1}{h_r} |\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| < \delta \ \text{ for all } \ r \geq r_0;$$

therefore, we have the containment

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| \geq \delta\right\} \subset \{1, 2, \cdots, r_0 - 1\}.$$

Since $\mathcal{I}$ be an admissible ideal, therefore the set on right hand side in the above containment belongs to $\mathcal{I}$, which implies that $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$. □

The converse of Theorem 4.5.1 does not hold in general as can be seen in the following example:

Example 4.5.1. *Let* $(\mathbb{R}, |.|)$ *denotes the space of real numbers with respect to the usual norm. For* $a, b \in [0,1]$*, if we define* $a * b = ab$ *and* $F_x(t) = \frac{t}{t+|x|}$ *for all* $x \in \mathbb{R}$ *and* $t \geq 0$*, then it is clear that* $(\mathbb{R}, F, *)$ *is a PN-Space. Let* $\theta = (2^r)$ *be a lacunary sequence and* $\mathcal{I} = \{K \subset \mathbb{N} : \delta(K) = 0\}$*, where* $\delta(K)$ *is an asymptotic density of subset* K*. Clearly,* $\mathcal{I}$ *is an admissible ideal. Now we choose a set* $K \in \mathcal{I}$ *such that* $K = \{n^2, n \in \mathbb{N}\}$*. Define a sequence* $x = (x_k)$ *as:*

$$x_k = \begin{cases} \sqrt{k}, & \text{if } r \notin K \text{ and } 2^{r-1} + 1 \leq k \leq 2^{r-1} + [\sqrt{h_r}], \\ \sqrt{k}, & \text{if } r \in K \text{ and } 2^{r-1} < k \leq 2^{r-1} + h_r, \\ 0, & \text{otherwise}, \end{cases}$$

where $I_r = (2^{r-1}, 2^r]$ and $h_r = 2^r - 2^{r-1}$.

For given $\epsilon > 0$ and $\lambda \in (0,1)$, we have

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : F_{x_k - 0}(\epsilon) \leq 1 - \lambda\}| \leq \lim_{r\to\infty} \frac{[\sqrt{h_r}]}{h_r} = 0 \quad \text{for } r \notin K.$$

Therefore, for $\delta > 0$, there exists a positive integer r_0 such that

$$\frac{1}{h_r} |\{k \in I_r : F_{x_k - 0}(\epsilon) \leq 1 - \lambda\}| < \delta \ for\ r \notin K\ and\ r \geq r_0.$$

This implies

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : F_{x_k - 0}(\epsilon) \leq 1 - \lambda\}| \geq \delta \right\} \subset \{K \cup (1, 2, \cdots r_0 - 1)\}.$$

Since $\mathcal{I}$ be an admissible ideal, it follows that $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k = 0$. Similarly, we can see that $\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : F_{x_k - 0}(\epsilon) \leq 1 - \lambda\}| \neq 0$, therefore $S_\theta^F - \lim_{k\to\infty} x_k \neq 0$. We, now establish some algebric characterization of this generalized summability method for the sequences in PN-Space.

Theorem 4.5.2. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal, $\theta = (k_r)$ be a lacunary sequence and $(X, F, *)$ be a PN-Space. Let $x = (x_k)$, $y = (y_k)$ be two sequences in X, then*

(i) If $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$ and $0 \neq c \in \mathbb{R}$, then $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} cx_k = cx_0$.

(ii) If $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$ and $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} y_k = y_0$, then $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty}(x_k + y_k) = (x_0 + y_0)$.

Proof. The proof of the theorem is very easy, so we omit it. □

Theorem 4.5.3. *Let $\theta = (k_r)$ be a lacunary sequence and $(X, F, *)$ be a PN-Space. For any sequence $x = (x_k)$ in X, $S^F(\mathcal{I}) \subset S_\theta^F(\mathcal{I})$ if and only if* $\liminf_r q_r > 1$.

Proof. We take $\liminf_r q_r > 1$, therefore there exists $\alpha > 0$ such that $q_r \geq 1 + \alpha$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\alpha}{1+\alpha}.$$

Let $S^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$. Therefore, for each $\epsilon > 0$, $\lambda \in (0,1)$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r}|\{k \leq k_r : F_{x_k - x_0}(\epsilon) \leq 1-\lambda\}| &\geq \frac{1}{k_r}|\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1-\lambda\}| \\ &= \left(\frac{h_r}{k_r}\right)\frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1-\lambda\}| \\ &\geq \left(\frac{\alpha}{1+\alpha}\right)\frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1-\lambda\}|;\end{aligned}$$

For any $\delta > 0$, we have the following inclusion

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1-\lambda\}| \geq \delta\right\}$$
$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{k_r}|\{k \leq k_r : F_{x_k - x_0}(\epsilon) \leq 1-\lambda\}| \geq \left(\frac{\alpha}{1+\alpha}\right)\delta\right\}.$$

Since $S^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$, therefore, it follows that $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$, by using the definition of ideal.

Conversely:- Let $\liminf_r q_r = 1$. Therefore, by the definition of limit inferior, we can pick a subsequence $(k_{r(j)})$ of (k_r) such that

$$\frac{k_{r_j}}{k_{r_j-1}} < 1 + \frac{1}{j} \quad \text{and} \quad \frac{k_{r_j-1}}{k_{r_{j-1}}} > j, \quad \text{where} \quad r_j \geq r_{j-1} + 2.$$

For a non-zero $x_0 \in X$, we define a subsequence $x = (x_k)$ as:

$$x_k = \begin{cases} x_0, & \text{if } k \in I_{r(j)} \text{for some } j = 1, 2, \cdots, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathcal{I}$ be an admissible ideal, therefore by using similar steps as in [[60], page 306], we obtain that $S^F(\mathcal{I}) - \lim_{k\to\infty} x_k = 0$ holds, but $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k$ does not exist. $\square$

Theorem 4.5.4. *Let $(X, F, *)$ be a probabilistic normed space. If ℓ_∞^F stands for bounded sequences in PN-Space, then $S_\theta^F(\mathcal{I}) \cap \ell_\infty^F$ is a closed linear subspace of ℓ_∞^F.*

Proof. By virtue of Theorem 4.5.2, $(S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)$ is a linear subspace of ℓ_∞^F. Since $(S_\theta^F(\mathcal{I}) \cap \ell_\infty^F) \subset \overline{(S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)}$ always holds, so it is sufficient to prove that $\overline{(S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)} \subset (S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)$.

Let $x \in \overline{(S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)}$ such that $B(x, r', t) \cap (S_\theta^F(\mathcal{I}) \cap \ell_\infty^F) \neq \phi$, we choose $y \in B(x, r', t) \cap (S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)$. Take $t > 0$ and $\epsilon \in (0, 1)$, choose $r' \in (0, 1)$ such that

$$(1 - r') * (1 - r') > (1 - \epsilon). \tag{4.5.1}$$

Since $y \in B(x, r', t) \cap (S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)$, then

$$F_{x_k - y_k}(\frac{t}{2}) > 1 - r' \tag{4.5.2}$$

for all $k \in \mathbb{N}$. We assume $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} y_k = x_0$, then for $0 < \delta < \frac{1}{5}$ the set

$$K = \left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : F_{y_k - x_0}(\frac{t}{2}) \leq 1 - r'\}| < \delta \right\} \in \mathcal{F}(\mathcal{I}).$$

We choose $r \in K$ such that $\frac{1}{h_r}|\{k \in I_r : F_{y_k-x_0}(\frac{t}{2}) \leq 1 - r'\}| < \delta < \frac{1}{5}$. Hence, there must be a $k \in I_r$ for which

$$F_{y_k - x_0}(\frac{t}{2}) > 1 - r' \tag{4.5.3}$$

holds.

By virtue of (4.5.1), (4.5.2) and (4.5.3), We have the inclusion

$$\left\{ k \in I_r : F_{y_k - x_0}(\frac{t}{2}) > 1 - r' \right\} \subseteq \{k \in I_r : F_{x_k - x_0}(t) > 1 - \epsilon\}$$

$$\text{and} \quad \frac{1}{h_r} |\{k \in I_r : F_{x_k - x_0}(t) \leq 1 - \epsilon\}| \leq \frac{1}{h_r} |\{k \in I_r : F_{y_k - x_0}(\frac{t}{2}) \leq 1 - r'\}|.$$

For $\delta > 0$, the following containment holds

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : F_{y_k - x_0}(\frac{t}{2}) \leq 1 - r'\}| < \delta \right\}$$
$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \, |\{k \in I_r : F_{x_k - x_0}(t) \leq 1 - \epsilon\} \, | < \delta \right\},$$

which implies $K \subseteq \{r \in \mathbb{N} : \frac{1}{h_r} \, |\{k \in I_r : F_{x_k - x_0}(t) \leq 1 - \epsilon\} \, | < \delta\}$.

It follows that $x = (x_k) \in (S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)$ and hence $\overline{(S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)} \subset (S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)$, which implies $\overline{(S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)} = (S_\theta^F(\mathcal{I}) \cap \ell_\infty^F)$. $\square$

After the introduction of $S^F(\mathcal{I})$ and $S_\theta^F(\mathcal{I})$-convergence, we now present the notion of $\mathcal{I}$-lacunary statistical Cauchy sequence in PN-Space.

Definition 4.5.6. Let $(X, F, *)$ be a PN-Space and $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ in X is said to be $\mathcal{I}-$lacunary statistically Cauchy or $S_\theta^F(\mathcal{I})-$Cauchy provided that for each $\epsilon > 0$, $\delta > 0$ and $\lambda \in (0,1)$, there exists a positive integer m such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_m}(\epsilon) \leq 1 - \lambda\}| \geq \delta\right\} \in \mathcal{I}.$$

Theorem 4.5.5. *Let $(X, F, *)$ be a probabilistic normed space and $\theta = (k_r)$ be a lacunary sequence. If a sequence $x = (x_k)$ is $S_\theta^F(\mathcal{I})-$convergent, then it is $S_\theta^F(\mathcal{I})-$Cauchy.*

Proof. Suppose $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$. Choose $\epsilon > 0$, $0 < \delta < \frac{1}{5}$ and $\lambda \in (0,1)$. Now, let $\Upsilon \in (0,1)$ such that

$$(1 - \Upsilon) * (1 - \Upsilon) > (1 - \lambda). \tag{4.5.4}$$

Since $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$, therefore, for each $\epsilon > 0$, $\delta > 0$ and $\lambda \in (0,1)$ we have

$$A = \left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\frac{\epsilon}{2}) \leq 1 - \Upsilon\}| \geq \delta\right\} \in \mathcal{I}$$

i.e.,

$$A^c = \left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\frac{\epsilon}{2}) \leq 1 - \Upsilon\}| < \delta\right\} \neq \phi.$$

belongs to $\mathcal{F}(\mathcal{I})$.

Choose $r \in A^c$ such that

$$\frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\frac{\epsilon}{2}) \leq 1 - \Upsilon\}| < \delta.$$

Since $h_r \to \infty$ as $r \to \infty$ and $A^c \in \mathcal{F}(\mathcal{I})$ being infinite, choose $r \in A^c$ so that $h_r > 5$. Hence, there must be an element $m \in I_r$ for which

$$F_{x_m - x_0}(\frac{\epsilon}{2}) > 1 - \Upsilon \tag{4.5.5}$$

holds.

Now, let $B = \left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_m}(\epsilon) \leq 1 - \lambda\}| \geq \delta\right\}$. In order to prove the result, it is sufficient to prove that $B \subseteq A$. For this purpose, we take $r \in B$. For particularly $r \in B$, we have the following inequality

$$\delta \leq \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_m}(\epsilon) \leq 1 - \lambda\}| \leq \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\frac{\epsilon}{2}) \leq 1 - \Upsilon\}|$$

and it follows that $r \in A$. Since $A \in \mathcal{I}$ and $B \subseteq A$, therefore $B \in \mathcal{I}$. This completes the proof. □

Theorem 4.5.6. *Let $(X, F, *)$ be a probabilistic normed space. If a subsequence of a $S_\theta^F(\mathcal{I})-$ Cauchy sequence $x = (x_k)$ in X is $S_\theta^F(\mathcal{I})-$convergent, then $x = (x_k)$ is also $S_\theta^F(\mathcal{I})-$convergent.*

Proof. For $\epsilon > 0$, $\delta > 0$ and $\lambda \in (0, 1)$ arbitrarily. Choose $\Upsilon \in (0, 1)$ such that

$$(1 - \Upsilon) * (1 - \Upsilon) > (1 - \lambda). \tag{4.5.6}$$

Since $x = (x_k)$ be a $S_\theta^F(\mathcal{I})-$ Cauchy sequence, so there exists a positive integer m such that

$$A = \left\{ r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_m}(\frac{\epsilon}{2}) \leq 1 - \Upsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

Suppose (x_{k_n}) is a subsequence of (x_k) such that $S_\theta^F(\mathcal{I}) - \lim_{n\to\infty} x_{k_n} = x_0$. It follows that the set

$$B = \left\{ r \in \mathbb{N} : \frac{1}{h_r}|\{k_n \in I_r : F_{x_{k_n} - x_0}(\frac{\epsilon}{2}) \leq 1 - \Upsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

Now, consider the set $C = \left\{ r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| \geq 2\delta \right\}$. In order to prove the result, it is sufficient to prove that $C \subseteq A \cup B$. If $r \in C$ such that

$\frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| \geq 2\delta$.

By using (4.5.6), we have the following inequality

$$2\delta \leq \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| \leq \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_{k_n}}(\frac{\epsilon}{2}) \leq 1 - \Upsilon\}| + \frac{1}{h_r}|\{k_n \in I_r : F_{x_{k_n} - x_0}(\frac{\epsilon}{2}) \leq 1 - \Upsilon\}|.$$

By virtue of above inequality, following containment holds:

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_0}(\epsilon) \leq 1 - \lambda\}| \geq 2\delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : F_{x_k - x_{k_n}}(\frac{\epsilon}{2}) \leq 1 - \Upsilon\}| \geq \delta\right\} \cup \left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k_n \in I_r : F_{x_{k_n} - x_0}(\frac{\epsilon}{2}) \leq 1 - \Upsilon\}| \geq \delta\right\},$$

i.e. $C \subseteq A \cup B$. Since $A, B \in \mathcal{I}$, therefore $C \in \mathcal{I}$. This proves that $S_\theta^F(\mathcal{I}) - \lim_{k\to\infty} x_k = x_0$. $\square$

Chapter 5

Generalized Convergence of Double Sequences

5.1 Introduction

Priengsheim[114] in 1900, initially studied the classical convergence for double sequences and observed that a convergent double sequence need not be bounded. This different behaviour of double sequences attracted many mathematicians to work towards this direction. Consequently, many concepts of single sequences have been developed for double sequences by various authors, out of which one was presented by Mursaleen and Edley[100] in 2003 by making use of two dimensional analogue of natural density and they reffered it as statistical convergence for double sequences. Patterson and Savas[136] defined statistical convergence for double sequences with

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the help of lacunary sequences and obtained interesting relationships between lacunary statistical convergence and statistical convergence. Apart from this, Tripathy and Tripathy[147] extended the ideas of $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy from single sequences to double sequences and studied some of their properties like solidity, symmetricity, completeness and denseness etc. Kumar[83] continued with this idea and introduced $\mathcal{I}^*$-convergence for double sequences and obtained some relations between $\mathcal{I}$ and $\mathcal{I}^*$-convergence for double sequences. In 2010, Mursaleen et al.[99] introduced λ-statistical convergence for double sequences. Later in 2012, Belen and Yilldirim[14] defined $\mathcal{I}_2$-statistical convergence or $S_2(\mathcal{I}_2)$-convergence for the double sequences and obtained some inclusion relations.

The purpose of this chapter is to extend some generalized methods of convergence for double sequences. Section 5.2 starts with a background on convergence of double sequences. In section 5.3, $\mathcal{I}$-lacunary Cesàro strong summability ($N_{\theta_{rs}}(\mathcal{I}_2)$) and $\mathcal{I}$-lacunary statistical convergence ($S_{\theta_{rs}}(\mathcal{I}_2)$-convergence) of double sequences have been defined and some of their inclusion relations have been established. In the last section, equality of the sets $S_2(\mathcal{I}_2)$ with $S_2(\mathcal{I}_2^*)$ and $S_{\theta_{rs}}(\mathcal{I}_2)$with $S_{\theta_{rs}}(\mathcal{I}_2^*)$ have been shown by using the condition (AP).

5.2 Preliminaries

In this section, We quote some definitions of Kostyrko et al.[82] and Mursaleen and Edley[100], which will be used later in this chapter.

Definition 5.2.1. [82] Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$ and (X, ρ) be a metric space. Then a sequence $x = (x_k)$ in X is said to be $\mathcal{I}-$convergent to ξ if for each $\epsilon > 0$, the set $A(\epsilon) = \{k \in \mathbb{N} : \rho(x_k, \xi) \geq \epsilon\}$ belongs to $\mathcal{I}$. In this case, we write $\mathcal{I} - \lim_{k\to\infty} x_k = \xi$.

Definition 5.2.2. [82] An admissible ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is said to be satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, A_3, \cdots\}$ belonging to $\mathcal{I}$ there exists a countable family $\{B_1, B_2, B_3, \cdots\}$ in $\mathcal{I}$ such that $A_i \triangle B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i \in \mathcal{I}$.

Definition 5.2.3. [100] A double sequence $x = (x_{ij})$ of numbers is said to be statistically convergent to a number L in the Pringshiem sense, if for every $\epsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn} |\{i \leq m, j \leq n : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case, we write $S_2 - P - \lim_{i,j\to\infty} x_{ij} = L$.
The set of all statistically convergent double sequences is denoted by S_2.

Before proceeding further, we now recall the definition of Savas and Patterson[136] on lacunary statistical convergence of double sequences.

A double sequence $\theta_{rs} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing sequences of positive integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \longrightarrow \infty \text{ as } r \longrightarrow \infty$$

and

$$l_0 = 0,\ \overline{h}_s = l_s - l_{s-1} \longrightarrow \infty \text{ as } s \longrightarrow \infty$$

Let $h_{rs} = h_r\overline{h}_s$, $I_r = (k_{r-1}, k_r]$, $\overline{I}_s = (l_{s-1}, l_s]$ and $I_{rs} = \{(i,j) : k_{r-1} < i \leq k_r, l_{s-1} < j \leq l_s\}$.

Definition 5.2.4. [136] Let θ_{rs} be a double lacunary sequence. A double sequence $x = (x_{ij})$ of numbers is said to be lacunary statistically convergent to a number L in the Pringsheim sense (denoted by $S_{\theta_{rs}} - P - \lim_{i,j\to\infty} x_{ij} = L$) if for each $\epsilon > 0$,

$$P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| = 0.$$

The set of all lacunary statistically convergent double sequences is denoted by $S_{\theta_{rs}}$.

Definition 5.2.5. [136] Let θ_{rs} be a double lacunary sequence. A double sequence $x = (x_{ij})$ of numbers is said to be $N_{\theta_{rs}} - P-$convergent to a number L if

$$P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| = 0.$$

Definition 5.2.6. [147] Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ be a non-trivial ideal. A double sequence $x = (x_{ij})$ of numbers is said to be $\mathcal{I}$-convergent in the Pringsheim sense to a number L, if for every $\epsilon > 0$

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in \mathcal{I}_2.$$

Definition 5.2.7. [13] Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ be a non-trivial ideal. A double sequence $x = (x_{ij})$ of numbers is said to be $\mathcal{I}_2-$statistical convergent or $S_2(\mathcal{I}_2)-$convergent to L, if for each $\epsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}|\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - L| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $x_{ij} \to L(S_2(\mathcal{I}_2))$ or $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$.

5.3 $S_{\theta_{rs}}(\mathcal{I}_2)$-Convergence

In this section, we introduce a more generalized class of statistical convergence called as $\mathcal{I}$-lacunary statistical convergence of double sequence.

Definition 5.3.1. Let θ_{rs} be a double lacunary sequence and $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. A double sequence $x = (x_{ij})$ of numbers is said to be $\mathcal{I}_2-$lacunary statistical convergent or $S_{\theta_{rs}}(\mathcal{I}_2)-$convergent to L, if for each $\epsilon > 0$ and $\delta > 0$

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2))$ or $S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$. The set of all $\mathcal{I}_2-$lacunary statistically convergent double sequences of numbers is denoted by $S_{\theta_{rs}}(\mathcal{I}_2)$.

Definition 5.3.2. Let θ_{rs} be a double lacunary sequence and $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. A double sequence $x = (x_{ij})$ of numbers is said to be $N_{\theta_{rs}}(\mathcal{I}_2)-$convergent to L, if for every $\epsilon > 0$, the set

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| \geq \epsilon\right\} \in \mathcal{I}_2.$$

We write it as $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$ or $N_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$. Symbolically, $N_{\theta_{rs}}(\mathcal{I}_2)$ denotes the set of all $N_{\theta_{rs}}(\mathcal{I}_2)-$convergent double sequences of numbers.

Following example shows that $S_{\theta_{rs}}(\mathcal{I}_2)$-convergence is more generalized than $S_{\theta_{rs}}$-convergence for the double sequences.

Example 5.3.1. *Let* $\mathcal{I}_2 = \{E \subset \mathbb{N} \times \mathbb{N} : E = (\mathbb{N} \times H) \cup (H \times \mathbb{N})$ *for some finite subset* H *of* $\mathbb{N}\}$ *and* $\theta_r = (2^r)$, $\theta_s = (3^s)$ *be two lacunary sequences. Consider a set* $A \in \mathcal{I}_2$ *and define a sequence* $x = (x_{ij})$ *by*

$$x_{ij} = \begin{cases} \sqrt{ij}, & \textit{for } (r,s) \notin A,\ 2^{r-1}+1 \leq i \leq 2^{r-1} + [\sqrt{h_r}] \textit{ and } 3^{s-1}+1 \leq j \leq 3^{s-1} + [\sqrt{\overline{h}_s}], \\ \sqrt{ij}, & \textit{for } (r,s) \in A,\ 2^{r-1} < i \leq 2^{r-1} + h_r \textit{ and } 3^{s-1} < j \leq 3^{s-1} + \overline{h}_s, \\ 0, & \textit{otherwise,} \end{cases}$$

where $I_r = (2^{r-1}, 2^r]$ *and* $I_s = (3^{s-1}, 3^s]$.

For each $\epsilon > 0$, *we have*

$$P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \leq \frac{[\sqrt{h_r}].[\sqrt{\overline{h}_s}]}{h_{rs}} \to 0$$

for $(r,s) \notin A$. *For* $\delta > 0$, *there exists a positive integer* r_0 *such that*

$$\frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| < \delta$$

for every $(r,s) \notin A$ *and* $r, s \geq r_0$.

Let $B = \{1, 2, \cdots r_0 - 1\}$ *and* $K = \{(r,s) \notin A : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \geq \delta\}$.

Then, by structure of the ideal $\mathcal{I}_2$ $K \subseteq (\mathbb{N} \times B) \cup (B \times \mathbb{N})$ *and* $K \in \mathcal{I}_2$.

Hence

$$\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \geq \delta\} \subset A \cup K.$$

It follows that $S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = 0$.

In the similar way, it can be proved that $P - \lim_{r,s\to\infty} \frac{1}{h_r \overline{h}_s} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \nrightarrow 0$, *so we omit it.*

We now present main results of this section, which are multidimensional analogue of Fridy and Orhan [60].

Theorem 5.3.1. *For an admissible ideal* $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$, $S_{\theta_{rs}}(\mathcal{I}_2)-$*limit of any double sequence if exists is unique.*

Theorem 5.3.2. *Let* $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ *be an admissible ideal and* θ_{rs} *be a double lacunary sequence. Then, the set* $S_{\theta_{rs}}(\mathcal{I}_2)$ *is closed under the operations of addition and scalar multiplication.*

Proofs of these two theorems are trivial, so we omit it.

Theorem 5.3.3. *Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be an admissible ideal and θ_{rs} be a double lacunary sequence. Then, (i) $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$ implies $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2))$;*

(ii) $N_{\theta_{rs}}(\mathcal{I}_2)$ is a proper subset of $S_{\theta_{rs}}(\mathcal{I}_2)$;

(iii) If $x = (x_{ij}) \in \ell^2_\infty$ and $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2))$, then $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$;

(iv) $S_{\theta_{rs}}(\mathcal{I}_2) \cap \ell^2_\infty = N_{\theta_{rs}}(\mathcal{I}_2) \cap \ell^2_\infty$; where ℓ^2_∞ denotes the space of all bounded double sequences.

Proof. (i) Suppose $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$. Then for $\epsilon > 0$,

$$\begin{aligned}\sum_{(i,j)\in I_{rs}} |x_{ij} - L| &\geq \sum_{(i,j)\in I_{rs} \& |x_{ij}-L|\geq\epsilon} |x_{ij} - L| \\ &\geq \epsilon |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}|,\end{aligned}$$

which implies

$$\frac{1}{\epsilon.h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| \geq \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}|.$$

Thus for any $\delta > 0$, we have the following containment

$$\begin{aligned}&\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| \geq \delta\right\} \\ &\qquad \subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| \geq \epsilon\,\delta\right\}.\end{aligned}$$

Since $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$, it follows that the later set belongs to $\mathcal{I}_2$ and hence the set $\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| \geq \delta\}$ belongs to $\mathcal{I}_2$. This shows that $x_{ij} \to L(S_{\theta_{r,s}}(\mathcal{I}_2))$.

(ii) Let the sequence $x = (x_{ij})$ be defined as follows:

$$x_{ij} = \begin{pmatrix} 1 & 2 & 3 & \dots & [\sqrt[3]{h_{rs}}] & 0 & \dots \\ 2 & 2 & 3 & \dots & [\sqrt[3]{h_{rs}}] & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & [\sqrt[3]{h_{rs}}] & \dots & \dots & [\sqrt[3]{h_{rs}}] & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is clear that (x_{ij}) is an unbounded double sequence. Moreover, for each $\epsilon > 0$

$$\frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \leq \frac{[\sqrt[3]{h_{rs}}]}{h_{rs}},$$

which immediately implies that for any $\delta > 0$,

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \geq \delta\right\}$$
$$\subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{[\sqrt[3]{h_{rs}}]}{h_{rs}} \geq \delta\right\}.$$

Since $P - \lim \frac{[\sqrt[3]{h_{r,s}}]}{h_{r,s}} = 0$, it follows that the set on the right side is finite and therefore belongs to $\mathcal{I}_2$. This shows that

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}_2$$

and therefore we have $x_{ij} \to 0(S_{\theta_{rs}}(\mathcal{I}_2))$.

On the other hand,

$$\frac{1}{h_{rs}} \sum_{(i,j) \in I_{rs}} |x_{ij} - 0| = \frac{1}{h_{rs}} \frac{[\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}] + 1))}{2} \to \frac{1}{2} \; as \; r, s \to \infty$$

which implies that the sequence $(\frac{1}{h_{rs}}[\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}] + 1))) \to 1 \; as \; r, s \to \infty.$

For $\epsilon = \frac{1}{4}$, we have

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(i,j) \in I_{rs}} |x_{ij} - 0| \geq \frac{1}{4}\right\}$$
$$= \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}[\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}]+1)) \geq \frac{1}{2}\right\} \in \mathcal{F}(\mathcal{I}_2).$$

This shows that $x_{ij} \to 0(N_{\theta_{rs}}(\mathcal{I}_2))$ does not hold.

(iii) Suppose that $x = (x_{ij}) \in \ell^2_\infty$ such that $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2))$. Then, there exists a number $M > 0$ such that $|x_{ij} - L| \leq M$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$.

Also for each $\epsilon > 0$, we can write

$$\frac{1}{h_{rs}} \sum_{(i,j) \in I_{rs}} |x_{ij} - L| = \frac{1}{h_{rs}} \sum_{(i,j) \in I_{rs}, |x_{ij} - L| \geq \frac{\epsilon}{2}} |x_{ij} - L| + \frac{1}{h_{rs}} \sum_{(i,j) \in I_{rs}, |x_{ij} - L| < \frac{\epsilon}{2}} |x_{ij} - L|$$
$$\leq \frac{M}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \frac{\epsilon}{2}\}| + \frac{\epsilon}{2}.$$

Consequently, we get

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(i,j) \in I_{rs}} |x_{ij} - L| \geq \epsilon\right\}$$
$$\subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \frac{\epsilon}{2}\}| \geq \frac{\epsilon}{2M}\right\}.$$

Since $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2))$, it follows that the later set belongs to $\mathcal{I}_2$, which immediately implies that $\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(i,j) \in I_{rs}} |x_{ij} - L| \geq \epsilon\right\} \in \mathcal{I}_2$. This means $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$.

(iv) Proof of this part is an immediate consequence of part(i) and (iii). $\square$

Theorem 5.3.4. *Let $\mathcal{I}_2$ be an admissible ideal satisfying the condition (AP) and θ_{rs} be a double lacunary sequence such that $\theta_{rs} \in \mathcal{F}(\mathcal{I}_2)$. If $x = (x_{ij}) \in S_2(\mathcal{I}_2) \cap S_{\theta_{rs}}(\mathcal{I}_2)$, then $S_2(\mathcal{I}_2) - P - \lim_{i,j \to \infty} x_{ij} = S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j \to \infty} x_{ij}$.*

Proof. Suppose $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$ and $S_{\theta_{r,s}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L'$, where $L \neq L'$. Select $0 < \epsilon < \frac{|L-L'|}{2}$.

Since $\mathcal{I}_2$ satisfies the condition (AP), so there is a set $M = \{(m_p, n_q) : p, q = 1, 2, \ldots\} \subseteq \mathbb{N} \times \mathbb{N}$ such that $M \in \mathcal{F}(\mathcal{I}_2)$ and

$$P - \lim_{p,q\to\infty} \frac{1}{m_p n_q} |\{i \leq m_p, j \leq n_q : |x_{ij} - L| \geq \epsilon\}| = 0.$$

Let $A = |\{i \leq m_p, j \leq n_q : |x_{ij} - L| \geq \epsilon\}|$ and $B = |\{i \leq m_p, j \leq n_q : |x_{ij} - L'| \geq \epsilon\}|$.

Then $m_p\, n_q = |A \cup B| \leq |A| + |B|$, which implies that $1 \leq \frac{|A|}{m_p\, n_q} + \frac{|B|}{m_p\, n_q}$.

Now, since $\lim_{p,q\to\infty} \frac{|B|}{m_p\, n_q} \leq 1$ and $\lim_{p,q\to\infty} \frac{|A|}{m_p\, n_q} = 0$, therefore $\lim_{p,q\to\infty} \frac{|B|}{m_p . n_q} = 1$.

Suppose $M^\star = M \cap \theta_{rs}$, then $M^\star \in F(\mathcal{I}_2)$ and is therefore an infinite set. Let $M^\star = \{(k_{\alpha_t}, l_{\beta_{t'}}) : t, t' = 1, 2, \ldots\}$ and consider the $(k_{\alpha_t}\, l_{\beta_{t'}})^{th}$ term of statistical limit expression $\frac{1}{m_p n_q} |\{i \leq m_p, j \leq n_q : |x_{ij} - L'| \geq \epsilon\}|$;

$$\frac{1}{k_{\alpha_t}\, l_{\beta_{t'}}} |\{i \leq k_{\alpha_t}, j \leq l_{\beta_{t'}} : |x_{ij} - L'| \geq \epsilon\}|$$

$$= \frac{1}{k_{\alpha_t}\, l_{\beta_{t'}}} |\{(i,j) \in \bigcup_{u=1,v=1}^{\alpha_t, \beta_{t'}} I_{uv} : |x_{ij} - L'| \geq \epsilon\}|$$

$$= \frac{1}{k_{\alpha_t}\, l_{\beta_{t'}}} \sum_{u=1,v=1}^{\alpha_t, \beta_{t'}} |\{(i,j) \in I_{uv} : |x_{ij} - L'| \geq \epsilon\}|$$

$$\leq \left(\frac{1}{\sum_{u=1,v=1}^{\alpha_t, \beta_{t'}} h_{uv}}\right) \sum_{u=1,v=1}^{\alpha_t, \beta_{t'}} h_{uv}\, z_{uv}, \tag{5.3.1}$$

where $z_{uv} = \frac{1}{h_{uv}} |\{(i,j) \in I_{uv} : |x_{ij} - L'| \geq \epsilon\}| \to 0(\mathcal{I}_2)$ as $S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L'$. Since θ_{rs} be a double lacunary sequence and (5.3.1) satisfies all the conditions for a four dimensional matrix transformation to map pringsheim null sequence into pringsheim null sequence [74] and therefore it is also $\mathcal{I}_2-$ convergent to zero as $t, t' \to \infty$ and so it has a subsequence which is convergent to zero since, $\mathcal{I}_2$ satisfies the (AP) condition.

Further, since this is also a subsequence of $(\frac{1}{mn}|\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - L'| \geq \epsilon\}|)_{(m,n)\in\mathbb{M}}$, so we infer that $\{\frac{1}{mn}|\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - L'| \geq \epsilon\}|\}$ does not converge to 1. Which is a contradiction to our supposition. Hence $L = L'$. □

We, now present two results on the closed-ness of the sets $S_{\theta_{rs}}(\mathcal{I}_2) \cap \ell_\infty^2$ and $S_2(\mathcal{I}_2) \cap \ell_\infty^2$. We shall give the proof of first and later can be established on the similar lines.

Theorem 5.3.5. *The set $S_{\theta_{rs}}(\mathcal{I}_2) \cap \ell_\infty^2$ is closed subset of ℓ_∞^2, the space of all bounded double sequences endowed with the superior norm.*

Proof. Let $x^{mn} = (x_{ij}^{mn})$ be a convergent sequence in $S_{\theta_{rs}}(\mathcal{I}_2) \cap \ell_\infty^2$ and suppose $x^{(mn)}$ converges to x. It is clear $x \in \ell_\infty^2$. Since $x^{(mn)} \in S_{\theta_{rs}}(\mathcal{I}_2)$, therefore there exists L_{mn} such that $S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim x_{ij}^{(mn)} = L_{mn}$ $(m, n = 1, 2, 3, \ldots)$. $x^{(mn)} \to x$, therefore it implies that $x^{(mn)}$ is a Cauchy sequence. So for each $\epsilon > 0$, there exists a positive integer n_0 such that for every $p \geq m \geq n_0, q \geq n \geq n_0$, we have

$$|x^{(pq)} - x^{(mn)}| < \frac{\epsilon}{3}. \tag{5.3.2}$$

Since $x_{ij}^{(mn)} \to L_{mn}(S_{\theta_{rs}}(\mathcal{I}_2))$, therefore for every $\epsilon > 0$ and $\delta > 0$, if we denote the sets

$$K_1 = \left\{(r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i, j) \in I_{rs} : |x_{ij}^{mn} - L_{mn}| \geq \frac{\epsilon}{3}\}| < \frac{\delta}{3}\right\}$$

and

$$K_2 = \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{pq} - L_{pq}| \geq \frac{\epsilon}{3}\}| < \frac{\delta}{3}\right\},$$

then $K_1 \cap K_2 \neq \phi$ belongs to $F(\mathcal{I}_2)$.

Let $(r,s) \in K_1 \cap K_2$, then we have

$$\frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{mn} - L_{mn}| \geq \frac{\epsilon}{3}\}| < \frac{\delta}{3} \quad \text{and}$$

$$\frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{pq} - L_{pq}| \geq \frac{\epsilon}{3}\}| < \frac{\delta}{3}.$$

which implies that

$$\frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{mn} - L_{mn}| \geq \frac{\epsilon}{3} \vee |x_{ij}^{pq} - L_{pq}| \geq \frac{\epsilon}{3}\}| < \delta < 1.$$

This shows that there exists a pair $(i_0, j_0) \in I_{r,s}$ for which $|x_{i_0j_0}^{mn} - L_{mn}| < \frac{\epsilon}{3}$ and $|x_{i_0j_0}^{pq} - L_{pq}| < \frac{\epsilon}{3}$.

Moreover, for $p \geq m \geq n_0$ and $q \geq n \geq n_0$, we have

$$|L_{pq} - L_{mn}| = |L_{pq} - x_{i_0j_0}^{pq}| + |x_{i_0j_0}^{pq} - x_{i_0j_0}^{mn}| + |x_{i_0j_0}^{mn} - L_{mn}| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.$$

Thus (L_{mn}) is a Cauchy double sequence in $\mathbb{R}$ (*or* $\mathbb{C}$) and consequently there is a number L such that $L_{mn} \longrightarrow L$ as $m,n \longrightarrow \infty$. In order to complete the proof of the theorem, it is sufficient to show that the sequence $x = (x_{ij}) \longrightarrow L(S_{\theta_{rs}}(\mathcal{I}_2))$. Since x^{mn} is convergent to $x \in \ell_\infty^2$ so by the structure of ℓ_∞^2, it is coordinate-wise convergent. Therefore for each $\epsilon > 0$, there exists a positive integer $n_1(\epsilon)$ such that

$$|x_{ij}^{mn} - x_{ij}| < \frac{\epsilon}{3}, \forall\ m,n \geq n_1(\epsilon). \tag{5.3.3}$$

Also $L_{mn} \to L$, so for each $\epsilon > 0$, we can find another positive integer $n_2(\epsilon)$ such that

$$|L_{mn} - L| < \frac{\epsilon}{3}, \forall\ m,n \geq n_2(\epsilon). \tag{5.3.4}$$

Choose $n_3(\epsilon) = \max\ \{n_1(\epsilon), n_2(\epsilon)\}$ and $m_0, n_0 \geq n_3(\epsilon)$. Then for any $(i,j) \in \mathbb{N} \times \mathbb{N}$

$$\begin{aligned} |x_{ij} - L| &\leq |x_{ij} - x_{ij}^{(m_0n_0)}| + |x_{ij}^{(m_0n_0)} - L_{m_0n_0}| + |L_{m_0n_0} - L| \\ &< \frac{\epsilon}{3} + |x_{ij}^{(m_0n_0)} - L_{m_0n_0}| + \frac{\epsilon}{3} \qquad (by\ (5.3.3) \text{ and } (5.3.4)). \end{aligned}$$

Therefore, the containment

$$\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\} \subseteq \{(i,j) \in I_{rs} : |x_{ij}^{m_0n_0} - L_{m_0n_0}| \geq \frac{\epsilon}{3}\} \text{ implies}$$
$$\frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| \leq \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{m_0n_0} - L_{m_0n_0}| \geq \frac{\epsilon}{3}\}|.$$

Further, for any $\delta > 0$ we have

$$\begin{aligned} &\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{m_0n_0} - L_{m_0n_0}| \geq \frac{\epsilon}{3}\}| < \delta\right\} \\ &\qquad \subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| < \delta\right\}. \end{aligned}$$

Since $\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{m_0n_0} - L_{m_0n_0}| \geq \frac{\epsilon}{3}\}| < \delta\right\} \in F(\mathcal{I}_2)$, so $\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{r,s} : |x_{ij} - L| \geq \epsilon\}| < \delta\right\} \in F(\mathcal{I}_2)$.
Which implies that $\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}_2$.
This shows that $x = (x_{ij}) \to L(S_{\theta_{rs}}(\mathcal{I}_2)$, which completes the proof. □

Theorem 5.3.6. *The set $S_2(\mathcal{I}_2) \cap \ell_\infty^2$ is closed subset of ℓ_∞^2, the space of all bounded double sequences endowed with the superior norm.*

5.4 $S_2(\mathcal{I}_2^*)$ and $S_{\theta_{rs}}(\mathcal{I}_2^*)-$Convergence

The aim of this section is to present stronger generalizations and extensions of several definitions and results obtained in the last section. We start by introducing the definitions of $\mathcal{I}_2^*$-statistical convergence ($S_2(\mathcal{I}_2^*)$) and $\mathcal{I}_2^*$-lacunary statistical convergence ($S_{\theta_{rs}}(\mathcal{I}_2^*)$) for the double sequences and prove some of their inclusion results.

Definition 5.4.1. A double sequence $x = (x_{ij})$ is said to be $\mathcal{I}_2^*-$ statistically convergent or $S_2(\mathcal{I}_2^*)-$ convergent to a number L, iff there exists a set $K = \{(m,n)\} \subset \mathbb{N} \times \mathbb{N}, m, n = 1,2,3,\cdots$ such that $K \in \mathcal{F}(\mathcal{I}_2)$ and

$$P - \lim_{(m,n)\in K, m,n\to\infty} \frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - L| \ge \epsilon\}| = 0.$$

We write it as $S_2(\mathcal{I}_2^*) - P - \lim_{i,j\to\infty} x_{ij} = L$ and $S_2(\mathcal{I}_2^*)$ denotes the set of all $\mathcal{I}_2^*-$ statistically convergent double sequences.

Definition 5.4.2. A double sequence $x = (x_{ij})$ is said to be $\mathcal{I}_2^*-$ lacunary statistically convergent or $S_{\theta_{rs}}(\mathcal{I}_2^*)-$ convergent to a number L if and only if there exists a set $K = \{(r,s) \in \mathbb{N} \times \mathbb{N}\}$ in $\mathcal{F}(\mathcal{I}_2)$ such that

$$P - \lim_{(r,s)\in K, r,s\to\infty} \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \ge \epsilon\}| = 0.$$

In this case, we write $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2^*))$ or $S_{\theta_{rs}}(\mathcal{I}_2^*) - P - \lim_{i,j\to\infty} x_{ij} = L$. The set of all $\mathcal{I}_2^*-$ lacunary statistically convergent double sequences is denoted by $S_{\theta_{rs}}(\mathcal{I}_2^*)$.

Theorem 5.4.1. *Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be an admissible ideal such that $\mathcal{I}_2$ contains all sets of the form $H \times \mathbb{N}$, $\mathbb{N} \times H$, where H is a finite subset of $\mathbb{N}$. For any double sequence $x = (x_{ij})$, if $S_2(\mathcal{I}_2^*) - P - \lim_{i,j\to\infty} x_{ij} = L$, then $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$.*

Proof. Suppose $S_2(\mathcal{I}_2^*) - P - \lim_{i,j\to\infty} x_{ij} = L$, which implies that there exists a set $K = \{(m,n) \in \mathbb{N} \times \mathbb{N}\}$ in $\mathcal{F}(\mathcal{I}_2)$ such that

$$P - \lim_{(m,n)\in K, m,n\to\infty} \frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - L| \ge \epsilon\}| = 0. \tag{5.4.1}$$

For given $\delta > 0$, there exists a positive integer m_0 such that for every $(m,n) \in K$ with $m, n \ge m_0$, we have

$$\frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - L| \ge \epsilon\}| < \delta.$$

Consider the set

$$M = \{(m,n) \in K : \frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - L| \ge \epsilon\}| \ge \delta\};$$

such that $M \subseteq (K_0 \times \mathbb{N}) \cup (\mathbb{N} \times K_0)$, where $K_0 = \{1, 2, \cdots m_0 - 1\}$.
Now, by structure of the ideal $\mathcal{I}_2$, $M \in \mathcal{I}_2$. Since the set $K \in \mathcal{F}(\mathcal{I}_2)$, so there exists a set $H \in \mathcal{I}_2$ such that $K = (\mathbb{N} \times \mathbb{N}) - H$. Thus, if we denote the set

$$A(\epsilon, \delta) = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - L| \ge \epsilon\}| \ge \delta\right\},$$

then $A(\epsilon, \delta) \subseteq M \cup H$. Hence, we conclude that $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$. $\square$

The converse of the above result does not hold true in general.

Example 5.4.1. *Let* $\mathbb{N} = \bigcup_{i=1}^{\infty} N_i$ *be a disjoint decomposition of* $\mathbb{N}$ *such that each* N_i *is an infinite set. Then* $\mathbb{N} \times \mathbb{N} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (N_i \times N_j)$ *is a disjoint decomposition of* $\mathbb{N} \times \mathbb{N}$.

Take $\mathcal{I}_2 = \{A \subset \mathbb{N} \times \mathbb{N} : A \subset (\mathbb{N} \times (\bigcup_{i=1}^{p} N_i)) \bigcup (\mathbb{N} \times (\bigcup_{i=1}^{q} N_i))\}$ *for some positive integers* p *and* q. *Obviously* $\mathcal{I}_2$ *is an admissible ideal in* $\mathbb{N} \times \mathbb{N}$ *and it contains the sets of the form* $H \times \mathbb{N}$, $\mathbb{N} \times H$ *where* H *is a finite subset of* $\mathbb{N}$.
For any set $A \in \mathcal{I}_2$ *and* $(m,n) \in N_i \times N_j$, *we define a sequences* $x = (x_{ij})$ *by*

$$x_{ij} = \begin{cases} \frac{1}{i} + \frac{1}{j}, & \text{if } (m,n) \notin A,\ m - [\sqrt{m}] + 1 \leq i \leq m, n - [\sqrt{n}] + 1 \leq j \leq n \\ \frac{1}{i} + \frac{1}{j}, & \text{if } (m,n) \in A,\ m - [\frac{m}{2}] + 1 \leq i \leq m, n - [\frac{n}{2}] + 1 \leq j \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Then for each $\epsilon > 0$, *we have*

$$P - \lim_{m,n\to\infty} \frac{1}{mn} |\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - 0| \geq \epsilon\}| \leq \frac{[\sqrt{m}].[\sqrt{n}]}{mn} \to 0$$

for $(m,n) \notin A$.
For each $\delta > 0$, *there exists a positive integer* n_0 *such that*

$$\frac{1}{mn} |\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - o| \geq \epsilon\}| < \delta$$

for $(m,n) \notin A$ *and* $m,n \geq n_0$.
Let $B = \{1, 2, \cdots n_0 - 1\}$ *and* $K = \{(m,n) \notin A : \frac{1}{mn} |\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - 0| \geq \epsilon\}| \geq \delta\}$.
Then clearly $K \subseteq (\mathbb{N} \times B) \cup (B \times \mathbb{N})$ *and* $K \in \mathcal{I}_2$ *by the structure of ideal* $\mathcal{I}_2$.
Hence

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - 0| \geq \epsilon\}| \geq \delta\} \subset A \cup K.$$

It follows that $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = 0$.

Now, we show that $S_2(\mathcal{I}_2^*) - P - \lim_{i,j\to\infty} x_{ij} \neq 0$. *Suppose* $S_2(\mathcal{I}_2^*) - P - \lim_{i,j\to\infty} x_{ij} = 0$, *by definition there exists a subset* $K = \{(m,n)\}, m,n = 1,2,3,\cdots \in \mathcal{F}(\mathcal{I}_2)$ *such that*

$$P - \lim_{(m,n)\in K, m,n\to\infty} \frac{1}{mn} |\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - 0| \geq \epsilon\}| = 0.$$

Since $K \in \mathcal{F}(\mathcal{I}_2)$, therefore there exists a set $H \in \mathcal{I}_2$ such that $K = \mathbb{N} \times \mathbb{N} - H$. By the structure of ideal $\mathcal{I}_2$, there exists positive integers p and q such that $H \subset (\mathbb{N} \times (\cup_{i=1}^{p} N_i)) \cup ((\cup_{j=1}^{q} N_j) \times \mathbb{N})$. This implies that all the sets $\{N_{p+1} \times N_{q+1}\} \subset K$. Since $\{N_{p+1} \times N_{q+1}\}$ is an infinite set so K contains $x_{ij} = \frac{1}{p+1} + \frac{1}{q+1}$ as infinite number of terms. This shows that

$$P - \lim_{(m,n)\in K, m,n\to\infty} \frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - 0| \ge \epsilon\}| \ne 0$$

and thus, we arrive at a contradiction, which proves the result.

Theorem 5.4.2. *Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be an admissible ideal satisfying the condition (AP). For any double sequence $x = (x_{ij})$, $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$ implies $S_2(\mathcal{I}_2^*) - P - \lim_{i,j\to\infty} x_{ij} = L$.*

Proof. Suppose that $\mathcal{I}_2$ satisfies the condition (AP). Let $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$, then for every $\epsilon > 0$ and $\delta > 0$, the set

$$A(\epsilon, \delta) = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - L| \ge \epsilon\}| \ge \delta\right\} \in \mathcal{I}_2.$$

Consider

$$K_1 = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - L| \ge \epsilon\}| \ge 1\right\}$$

and define

$$K_p = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{p} \le \frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - L| \ge \epsilon\}| < \frac{1}{p-1}\right\},$$

for $p \in \mathbb{N}$ and $p \ge 2$.

Since $\{K_p\}_{p\in\mathbb{N}}$ is a countable family of mutually disjoint sets belonging to $\mathcal{I}_2$, so by condition (AP) there exists a countable family of sets $\{B_1, B_2, \cdots\}$ such that $K_i \triangle B_i$ is a finite set for all $i \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{I}_2$.

Let $K = \{\mathbb{N} \times \mathbb{N}\} - B$, then $K \in \mathcal{F}(\mathcal{I}_2)$ and therefore to prove the result, it is sufficient to prove that

$$P - \lim_{(m,n)\in K, m,n\to\infty} \frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - L| \ge \epsilon\}| = 0.$$

Let $\eta > 0$ be given. Choose a positive integer q such that $\eta > \frac{1}{q+1}$. Then we have

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{1 \le i \le m, 1 \le j \le n : |x_{ij} - L| \ge \epsilon\}| \ge \eta\right\} \subset \cup_{i=1}^{q+1} K_i. \tag{5.4.2}$$

Since $K_i \triangle B_i$ is a finite set for each $i = 1, 2, \cdots q+1$, therefore there exists a positive integer p_0 such that

$$\left(\bigcup_{i=1}^{q+1} B_i\right) \bigcap \{(m,n) \in \mathbb{N}\times\mathbb{N} : m,n \geq p_0\} = \left(\bigcup_{i=1}^{q+1} K_i\right) \bigcap \{(m,n) \in \mathbb{N}\times\mathbb{N} : m,n \geq p_0\}.$$

If $m, n \geq p_0$ and $(m,n) \in K$, then $(m,n) \notin B$. This shows that $(m,n) \notin \cup_{i=1}^{q+1} B_i$ and therefore $(m,n) \notin \cup_{i=1}^{q+1} K_i$.
Hence for every $m, n \geq p_0$ with $(m,n) \in K$, we have

$$\frac{1}{mn}|\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - L| \geq \epsilon\}| < \eta.$$

by the use of (5.4.2).
This shows that

$$P - \lim_{(m,n)\in K, m,n\to\infty} \frac{1}{mn}|\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - L| \geq \epsilon\}| = 0.$$

and thus, the proof of the theorem is complete. $\square$

Next theorem gives a relationship between $S_2(\mathcal{I}_2)$ and $S_{\theta_{rs}}(\mathcal{I}_2)-$ convergence for the double sequences..

Theorem 5.4.3. *Let $\mathcal{I}_2$ be an admissible ideal and θ_{rs} be a double lacunary sequence. For any double sequence $x = (x_{ij})$, $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$ implies $S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$, if and only if,* $\liminf_r q_r > 1$ *and* $\liminf_s \overline{q}_s > 1$.

Proof. Suppose $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$; then there exists $\alpha > 0$ such that both $q_r \geq 1+\alpha$ and $\overline{q}_s \geq 1+\alpha$ for sufficient large r and s, which implies

$$\frac{h_r}{k_r} \geq \frac{\alpha}{1+\alpha} \quad \textit{and} \quad \frac{\overline{h_s}}{l_s} \geq \frac{\alpha}{1+\alpha}$$

If $x_{ij} \to L(S_2(\mathcal{I}_2))$, then for $\epsilon > 0$ as well as sufficiently large r and s, we have

$$\begin{aligned}
\frac{1}{k_{rs}}|\{i \leq k_r, j \leq l_s : |x_{ij} - L| \geq \epsilon\}| &\geq \frac{1}{k_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| \\
&= \left(\frac{h_{rs}}{k_{rs}}\right)\frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| \\
&\geq \left(\frac{\alpha}{1+\alpha}\right)^2 \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}|;
\end{aligned}$$

which implies, for any $\delta > 0$ the containment

$$\begin{aligned} A(\epsilon,\delta) &= \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}}|\{(i,j)\in I_{rs} : |x_{ij}-L|\geq\epsilon\}|\geq\delta\right\} \\ &\subseteq \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{k_{rs}}|\{i\leq k_r, j\leq l_s : |x_{ij}-L|\geq\epsilon\}|\geq\left(\frac{\alpha}{1+\alpha}\right)^2\delta\right\}. \end{aligned}$$

Since $x_{ij}\to L(S_2(\mathcal{I}_2))$, by definition it follows that $A(\epsilon,\delta)\in\mathcal{I}_2$ and therefore the sufficiency part is proved.
Conversely : suppose that $\liminf_r q_r = 1$ or $\liminf_s \overline{q}_s = 1$. Without loss of generality, we may assume $\liminf_r q_r = 1$; then proceeding as in [[56] ; p-510] we can select a subsequence (k_{r_m}) of the lacunary sequence θ_r such that

$$\frac{k_{r_m}}{k_{r_m-1}} < 1+\frac{1}{m} \quad\text{and}\quad \frac{k_{r_m-1}}{k_{r_{m-1}}} > m, \quad\text{where}\quad r_m \geq r_{m-1}+2.$$

Define a sequence $x = (x_{ij})$ as follows.

$$x_{ij} = \begin{cases} 1, & \text{if } i\in I_{r_m} \text{ and } j\in\mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then for any real L and $j\in\mathbb{N}$, we have

$$\frac{1}{h_{r_m s}}\sum_{(i,j)\in I_{r_m s}}|x_{ij}-L| = |1-L| \quad\text{for}\quad m=1,2,\ldots \quad\text{and}$$

$$\frac{1}{h_{rs}}\sum_{(i,j)\in I_{rs}}|x_{ij}-L| = |L| \quad\text{for}\quad r\neq r_m.$$

Therefore by [[37]; page 1513], each row of the sequence is in $S(\mathcal{I})$ but each row is such that $S(\mathcal{I}) \nsubseteq S_{\theta_r}(\mathcal{I})$. Since the double lacunary sequence θ_{rs} is factorable, therefore, we have $S_2(\mathcal{I}_2) \nsubseteq S_{\theta_r}(\mathcal{I}_2)$. □

Next two Theorems gives relations between $S_{\theta_{rs}}(\mathcal{I}_2)$ and $S_{\theta_{rs}}(\mathcal{I}_2^*)$-convergence and the proof of these relations can be obtained analogously to Theorems 5.4.1 and 5.4.2.

Theorem 5.4.4. *Let $\mathcal{I}_2\subseteq\mathcal{P}(\mathbb{N}\times\mathbb{N})$ be an admissible ideal such that $\mathcal{I}_2$ contain all sets of the form $H\times\mathbb{N}$, $\mathbb{N}\times H$ where H is a finite subset of $\mathbb{N}$. If $S_{\theta_{rs}}(\mathcal{I}_2^*) - P - \lim_{i,j\to\infty} x_{ij} = L$, then $S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$.*

Theorem 5.4.5. *Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be an admissible ideal satisfying the property (AP). Then $S_{\theta_{rs}}(\mathcal{I})-$ convergence implies $S_{\theta_{rs}}(\mathcal{I}_2^*)-$ convergence for real double sequences.*

We now give the following Theorem which is a natural combination of the Theorem 5.4.1, Theorem 5.4.3 and Theorem 5.4.5.

Theorem 5.4.6. *Let θ_{rs} be a double lacunary sequence and $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be an admissible ideal satisfying*

(i) $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$;

(ii) $\mathcal{I}_2$ contain all sets of the form $H \times \mathbb{N}$, $\mathbb{N} \times H$ where $H \subseteq \mathbb{N}$ is finite;

(iii)$\mathcal{I}_2$ satisfying the property (AP);

then $S_2(\mathcal{I}_2^*)-$ convergence implies $S_{\theta_{rs}}(\mathcal{I}_2^*)-$ convergence for double sequences of numbers.

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