A STUDY OF GAUSSIAN POLYNOMIAL

Dissertation submitted in partial fulfillment of the requirement for

the award of the degree of

Masters of Science

in

Mathematics and Computing

Submitted by

Richa Bansal

Roll No. 300903013

Under the esteemed guidance of

Dr. Meenakshi Rana

July 2011

School of Mathematics and Computer Applications

Thapar University

Patiala – 147004, Punjab

INDIA

DEDICATED

TO

MY PARENTS AND GOD

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"A Study of Gaussian Polynomial"** in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Meenakshi Rana**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

Richcha Bansal
Richa Bansal

Reg. No. 300903013

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

Meenakshi 14/7/11
Dr. Meenakshi Rana

Supervisor

Assistant Professor, SMCA

Thapar University, Patiala

Countersigned by

Dr. S.S. Bhatia 15/7/11

Professor and Head

SMCA

Thapar University, Patiala

Dr. S.K. Mohapatra

Dean of Academic Affairs

Thapar University

Patiala

ACKNOWLEDGEMENTS

First of all, I would like to thank the almighty for granting perseverance. I would like to express my gratitude to Dr. Meenakshi Rana, Assistant Professor, School of Mathematics and Computer Applications, Thapar University, Patiala, for their patient guidance and support throughout this work. I was truly very fortunate to have the opportunity to work under her as a student. I take this opportunity to express my sincere thanks to Prof. S.S. Bhatia, Head SMCA, Thapar University, Patiala, for their valuable support and help in completing this work.

My special thanks to Ms. Ankita jindal, Research Scholar, SMCA, for their valuable support in the completion of this work I do not find enough words with which I can express my feelings of thanks to the entire research scholars, faculty and staff of SMCA, Thapar University, Patiala, for their help, inspiration and moral support which went a long way in successful completion of my work. I thank all those who have contributed directly or indirectly to this work.

Lastly, and more importantly, I would like to thank my parents for their years of unyielding love and encouragement. They have always wanted the best for me and I admire my parent's determination and sacrifice to put me through college.

Richa Bansal

July 14, 2011

Richa Bansal

# ABSTRACT

In the application of Partition theory, such as in statistics mechanics, computer science, special functions, algebra, theoretical physics, combinatorics, we are often interested in restricted partitions, that is, partition in which several restrictions are imposed.

For example, partition in which the largest part is say, $\leq N$ and the number of parts is $\leq M$.So here in this thesis we study restricted partition as generating function of some q-series.

Chapter 1 deals with elementary definitions and results.

In Chapter 2, we study Gaussian Polynomial, the polynomial $G(N, M; q) = \dfrac{(q;q)_{N+M}}{(q;q)_N(q;q)_M}$ which were first studied by Gauss in 1863.

Lastly, in Chapter 3, we interpret some q-series as generating function of some restricted partition function.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Introduction

In this chapter we give some basic definitions and some results as preliminaries for Chapter 2 and Chapter 3.

Definition 1.1.1. [7] A **partition** of positive integer n is non-increasing of positive integers whose sum is n.

Notation. $p(n)$ denote the number of partitions of n.

Example 1.1.1. Take $n = 7$, the relevant partitions of 7 are :

$$7,$$

$$6 + 1,$$

$$5 + 2, 5 + 1 + 1,$$

$$4 + 3, 4 + 2 + 1, 4 + 1 + 1 + 1,$$

$$3 + 3 + 1, 3 + 2 + 2, 3 + 2 + 1 + 1, 3 + 1 + 1 + 1 + 1,$$

$$2 + 2 + 2 + 1, 2 + 2 + 1 + 1 + 1, 2 + 1 + 1 + 1 + 1 + 1,$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1.$$

So, $p(7) = 15$.

Note that $p(n) = 0$ for $n < 0$ because we cannot write a negative number into sum of positive integers.

Also, $p(0) = 1$.

Notation. $(a)_n$ is known as **pochammer symbol** defined as below:

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1)$$

Remark. $(\mathbf{a})_0 = 1$

$$(\mathbf{1})_n = (n)!$$

Definition 1.1.2. **Rising q-factorial** is define as $(a;q)_n$

where

$$(a;q)_n = \prod_{n=0}^{\infty} \frac{(1-aq^i)}{(1-q^{i+n})} = \prod_{i=0}^{(n-1)} (1-aq^i)$$

and when $a = q$,

$$(q;q)_n = \prod_{i=1}^{n} (1-q^i).$$

Definition 1.1.3. The **Generating function** $f(q)$ for the sequence $a_0, a_1, a_2, \cdots$ is

$$f(q) = \sum_{n=0}^{\infty} a_n q^n.$$

Definition 1.1.4. The **Generating function for p(n)** is given by:

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$$

$$= \frac{1}{(q;q)_\infty}$$

where $|q| < 1$ and $(q;q)_\infty$ is a rising q-factorial.

The reason for this is, if we expand $\prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$ then we have

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \prod_{n=1}^{\infty} (1+q^n+q^{2n}+\cdots)$$

$$= (1+q^{1.1}+q^{2.1}+q^{3.1}\cdots)(1+q^{1.2}+q^{2.2}+q^{3.2}\cdots)\cdots$$

$$= (1+q^{1.1})+(q^{1.2}+q^{2.1})+(q^{1.3}+q^{1.2+1.1}+q^{3.1})+(q^{1.4}+q^{1.3+1.1}+q^{2.2}+q^{1.2+2.1}+q^{4.1})$$

$$= p(0)+p(1)q+p(2)q^2+p(3)q^3+p(4)q^4+\cdots \tag{1.1.1}$$

note that the number of times q^n appear in (1.1.1) is same as the number of partitions on n.

Remark. Let $D(n)$ denote the number of partitions of n into distinct parts, then generating function of this is:

$$\sum_{n=0}^{\infty} D(n)q^n = \prod_{n=1}^{\infty}(1+q^n) = (-q;q)_\infty.$$

Let $O(n)$ denote the number of partitions of n into odd parts, then generating function of this is:

$$\sum_{n=0}^{\infty} O(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})} = \frac{1}{(q;q^2)_\infty}.$$

Let $E(n)$ denote the number of partitions of n into even parts, then generating function of this is:

$$\sum_{n=0}^{\infty} E(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})} = \frac{1}{(q^2;q^2)_\infty}.$$

Definition 1.1.5. The **Ferrers graph** of partition $t_1, t_2, t_3, \cdots, t_i$ of n is the set of i rows of equispaced dots alinged on left where j^{th} row has t_j dots.

Example 1.1.2. Let Π: 5+3+2 be any partition of 10, the ferrers graph of this partition is given below:

. . . . .

. . .

. .

read this graph horizontally, we see that first row has 5 dots, second row has 3 dots and third row has 2 dots.

Definition 1.1.6. In ferrers graph if we read the dots vertically instead of horizontally, we get the **Conjugate partition**.

We denote the conjugate partition of partition Π by Π^c.

Example 1.1.3. Let Π: 4+3+2 be any partition of 9, for finding its conjugate partition we have to draw its ferrers graph and then read it vertically, so graph is given below:

. . . .

. . .

. .

by reading it vertically we get Π^c: 3+3+2+1.

Definition 1.1.7. When the conjugate partition and the partition are same, we called the partition as **Self conjugate** .

Example 1.1.4. Let Π: 4+3+3+1 be any partition of 11, now we draw its ferrers graph

. . . .

. . .

. . .

.

now if we read this graph vertically we get the same partition Π: 4+3+3+1 so this partition is self conjugate partition.

In other words here $\Pi^c = \Pi$. So, it is self conjugate partition.

1.2 Basic theorems based on partitions

Theorem 1.2.1.

(a) $$(a)_n = \lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n}$$

(b) $$(1)_n = \lim_{q\to 1} \frac{(q;q)_n}{(1-q)^n} = n!$$

Theorem 1.2.2. (q-binomial Theorem) For $|q| < 1$, $|t| < 1$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} t^n = \frac{(at;q)_\infty}{(t;q)_\infty}$$

Theorem 1.2.3. The number of partitions of n into r parts is equal to the number of partitions of n in which largest part is r.

Theorem 1.2.4. The number of partitions of n into atmost r parts is equal to the number of partitions of n in which largest part does not exceed r.

Theorem 1.2.5. The number of partitions of n into distinct odd parts is equal to the number of self conjugate partitions of n.

Theorem 1.2.6. The number of partitions of n into distinct parts is equal to the number of partitions into odd parts.

In the next section we give detail proofs of the Theorems 1.2.1 - 1.2.6.

1.3 Proof of theorems

Proof of Theorem 1.2.1.(a)

$$\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = \lim_{q\to 1} \frac{(1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1})}{(1-q)^n}$$

$$= \lim_{q\to 1} \frac{(1-q^a)}{(1-q)} \frac{(1-q^{a+1})}{(1-q)} \cdots \frac{(1-q^{a+n-1})}{(1-q)}$$

$$= \lim_{q\to 1} (1+q+q^2+\cdots+q^{a-1})(1+q+q^2+\cdots+q^a)+ \cdots + (1+q+q^2+\cdots+q^{a+n-2})$$

$$= (1+1+\cdots+1)(1+1+\cdots+1)\cdots(1+1+\cdots+1)$$

$$= (a)(a+1)\cdots(a+n-1) \tag{1.3.1}$$

$$= (a)_n.$$

Proof of Theorem 1.2.1.(b)

Put $a = 1$ in Theorem 1.2.1.(a) we get,

$$(1)_n = \lim_{q\to 1} \frac{(q;q)_n}{(1-q)^n}$$

$$= \lim_{q\to 1} \frac{(1-q)}{(1-q)} \frac{(1-q^2)}{(1-q)} \cdots \frac{(1-q^n)}{(1-q)}$$

$$(1)_n = (1)(1+1)\cdots(1+1+\cdots+1) = n!$$

Proof of Theorem 1.2.2.

$$Let \quad F(t) = \prod_{n\geq 0} \frac{(1-atq^n)}{1-tq^n} = \sum_{n\geq 0} A_n t^n \tag{1.3.2}$$

now to prove

$$A_n = \frac{(at;q)_\infty}{(t;q)_\infty} = \frac{(a;q)_n}{(q;q)_n} \tag{1.3.3}$$

now,

$$F(t) = \prod_{n\geq 0} \frac{(1-atq^n)}{(1-tq^n)} = \frac{(1-at)}{(1-t)} \prod_{n\geq 1} \frac{(1-atq^n)}{(1-tq^n)}$$

put $n = n+1$

$$F(t) = \frac{(1-at)}{(1-t)} \prod_{n\geq 0} \frac{(1-atq^{n+1})}{(1-tq^{n+1})}$$

$$F(t) = \frac{(1-at)}{(1-t)} F(tq)$$

$$(1-t)F(t) = (1-at)F(tq) \tag{1.3.4}$$

by (1.3.2),

$$F(t) = \sum_{n\geq 0} A_n t^n$$

so, (1.3.4) becomes

$$\begin{aligned}(1-t)\sum_{n\geq 0} A_n t^n &= (1-at)\sum_{n\geq 0} A_n (qt)^n \\ &= (1-at)\sum_{n\geq 0} A_n q^n t^n\end{aligned}$$

equating the powers of t

$$\begin{aligned}A_n - A_{n-1} &= A_n q^n - a q^{n-1} A_{n-1} \\ A_n &= \frac{(1-aq^{n-1})}{(1-q^n)} A_{n-1} \qquad (1.3.5)\end{aligned}$$

iterating (1.3.5) n-times, we get,

$$A_n = \frac{(1-aq^{n-1})(1-aq^{n-2})\cdots(1-a)}{(1-q^n)(1-q^{n-1})\cdots(1-q)} A_0$$

we know that $A_0 = 1$

$$\begin{aligned}A_n &= \frac{(1-aq^{n-1})(1-aq^{n-2})\cdots(1-a)}{(1-q^n)(1-q^{n-1})\cdots(1-q)} \\ &= \frac{(a\ ;\ q)_n}{(q;q)_n}\end{aligned}$$

Corollary 1.3.1. For $|q| < 1$, $|t| < 1$

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} = \prod_{n=0}^{\infty}(1-tq^n)^{-1} = \frac{1}{(t;q)_\infty}.$$

Proof of Theorem 1.2.3. Consider a partition Π of n into r parts and draw its ferrers graph. Now read this ferrers graph vertically we get conjugate partition of Π which is a partition of n in which largest part is r.

. . . .

. . . . . .

. . . . .

and so on

. . . .

. . .

Consider a partition Π in which largest part is r. Draw its ferrers graph now read the graph vertically we get conjugate partition of Π which is a partition of n into r parts.

So, we get one-one correspondence between the partitions of n into r parts and the partition of n in which largest part is r.

Hence, we get the number of partitions of n into r parts is equal to the number of partitions of n in which largest part is r.

Proof of Theorem 1.2.4. Consider a partition Π of n into atmost r parts and draw its ferrers graph. Now read this ferrers graph vertically we get conjugate partition of Π which is a partition of n in which largest part does not exceed r.

. . . .

. . . . . .

. . . . .

and so on

. . . .

. . .

Consider a partition Π in which largest part does not exceed r parts. Draw its ferrers graph now read the graph vertically we get conjugate partition of Π which is a partition of n into atmost r parts.

So, we get one-one correspondence between the partitions of n into atmost r parts and the partition of n in which largest part does not exceed r.

Hence we get the number of partitions of n into atmost r is equal to the number of partitions of n in which largest part does not exceed r.

Proof of Theorem 1.2.5. Consider a self conjugate partition of n then make the ferrers graph corresponding to this partition and straight each bent along the south east direction and count the nodes in each row we will get the partition of n into distinct odd parts.

Conversly, consider a partition of n into distinct odd parts then each part can be written in the form of bent that is $2a + 1$ (at right angle) place them one after another in graph and we see that one durfee square and 2 equal tails are obtained.

Thus, by reading partition from this ferrers graph we get a self conjugate partition of n.

So, we get one-one correspondence between number of self conjugate partitions of n and the partition of n into distinct odd parts.

Hence we get the number of partitions of n into distinct odd parts is equal to the number of self conjugate partitions of n.

Proof of Theorem 1.2.6. Consider $D(n)$ denotes the number of partitions of n into distinct parts,

$$\begin{aligned}\sum_{n=1}^{\infty} D(n)q^n &= \prod_{n=1}^{\infty}(1+q^n)\\ &= \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^n}\\ &= \prod_{n=1}^{\infty}\frac{1-q^{2n}}{(1-q^{2n-1})(1-q^{2n})}\\ &= \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}\end{aligned}$$

which denotes the number of partitions of n into odd parts.

So, $\sum_{n=1}^{\infty} D(n)q^n = \sum_{n=1}^{\infty} O(n)q^n$

comparing the coefficients on both sides of above equation we get $D(n) = O(n)$ for all n

1.4 Restricted Partitions

Definition 1.4.1. Restricted Partitions are defined as below:

$p(N, M, n)$ is equal to the number of partitions of n into at most M parts in which each part $\leq N$.

Remark. (a) $p(N, M, n) = 0$ if $n > NM$

(b) $p(N, M, n) = 1$ if $n = NM$.

Example 1.4.1. $p(4, 4, 8)$ represents the number of partitions of 8 into at most 4 parts in which each part ≤ 4.
that is, $p(4, 4, 8) = 8$.
The relevant partitions are:

$$4 + 4,$$
$$4 + 3 + 1,$$
$$4 + 2 + 2,$$
$$4 + 2 + 1 + 1,$$
$$3 + 3 + 2,$$
$$3 + 3 + 1 + 1,$$
$$3 + 2 + 2 + 1,$$
$$2 + 2 + 2 + 2.$$

In Chapter 2 we study Gaussian Polynomial and their properties. In Chapter 3 we enumerate certain q-series using Gaussian Polynomial.

CHAPTER 2

Gaussian Polynomials

In the classical theory of partitions several restricted partition function have been studied $p(N, M, n)$ is one such function. Other restrictions are partitions into distinct parts, partitions into odd parts, partition satisfying certain modulo conditions etc. So, In general a restricted partition function is of the type $p(S, n)$ which counts the number of partitions of n that have all there parts in a set of positive integers S. Analogues to the $p(N, M, n)$, various other restricted partition are found in [4], [12] and [13].

2.1 The Generating function for restricted partition

Let $G(N, M; q) = \sum_{n \geq 0} p(N, M, n) q^n$
where $p(N, M, n)$ is the number of partitions of n into atmost M parts each $\leq N$.

Then $G(N, M; q)$ is the generating function for restricted partition $p(N, M, n)$, which is a polynomial in q of degree NM.

Theorem 2.1.1 Prove that:

$$G(N, M; q) = \begin{bmatrix} N + M \\ N \end{bmatrix} = \frac{(q\ ;\ q)_{N+M}}{(q\ ;\ q)_N (q\ ;\ q)_M}$$

Proof: Let $g(N, M; q) = \dfrac{(q; q)_{N+M}}{(q; q)_M (q; q)_N}$

$$g(0, M; q) = \frac{(q; q)_M}{(q; q)_M (q; q)_0} = \frac{1}{(q; q)_0} = 1$$

Similarly $g(N, 0; q) = 1$

so,

$$g(0, M; q) = g(N, 0; q) = 1 \tag{2.1.1}$$

Consider: $g(N, M; q) - g(N, M - 1; q)$

$$= \frac{(q;q)_{N+M}}{(q;q)_M (q;q)_N} - \frac{(q;q)_{N+M-1}}{(q;q)_{M-1}(q;q)_N}$$

$$= \frac{(q;q)_{N+M-1}}{(q;q)_M (q;q)_N}[(1 - q^{N+M}) - (1 - q^M)]$$

$$= \frac{(q;q)_{N+M-1}}{(q;q)_M (q;q)_N}[q^M - q^{N+M}]$$

$$= \frac{(q;q)_{N+M-1}}{(q;q)_M (q;q)_N}[q^M(1 - q^N)]$$

$$= q^M g(N - 1, M; q)$$

that is $g(N, M; q) - g(N, M - 1; q) = q^M g(N - 1, M; q)$

$$\Rightarrow g(N, M; q) = q^M g(N - 1, M; q) + g(N, M - 1; q) \tag{2.1.2}$$

By (2.1.1) we get g satisfies the initial conditions and by (2.1.2), it satisfies the recurrence relation. Now, we will show that (2.1.1) and (2.1.2) defines g uniquely. We will prove this uniqueness by double induction,
For $N = M = 0$
$g(N, M, q)$ is uniquely defined by (2.1.1)
Let for $N = 0, 1, 2, \cdots N - 1$ and $M = 0, 1, 2, \cdots M - 1$
$g(N, M; q)$ are uniquely defined by (2.1.1) *and* (2.1.2).
Now,

$$g(N, M - 1; q) + q^M g(N - 1, M; q) = \frac{(q;q)_{N+M-1}}{(q;q)_N (q;q)_{M-1}} + \frac{q^M (q;q)_{N+M-1}}{(q;q)_M (q;q)_{N-1}}$$

therefore, they are uniquely defined by double induction on $M$and N.

$$= \frac{(q;q)_{N+M-1}}{(q;q)_N (q;q)_M}[(1 - q^M) + q^M(1 - q^N)]$$

$$= \frac{(q;q)_{N+M-1}}{(q;q)_N(q;q)_M}(1-q^{M+N})$$

$$= \frac{(q;q)_{N+M}}{(q;q)_N(q\ \ q)_M} = g(N,M;q)$$

that is (2.1.1) and (2.1.2) defines g uniquely.

Now we will show that G satisfies (2.1.1) and (2.1.2).

We know that $p(0,M,n) = p(N,0,n) = 1$

so,

$$\sum_{n\geq 0} p(0,M,n)q^N = \sum_{n\geq 0} p(N,0,n)q^N = 1$$

that is

$$G(0,M;q) = G(N,0;q) = 1 \tag{2.1.3}$$

consider $p(N,M,n) - p(N,M-1,n)$

= no. of partitions of $n-M$ into atmost M parts where each part $\leq N-1$ (subtract 1 from each part) we get,

$$p(N,M,n) - p(N,M-1,n) = p(N-1,M,n)$$

multiply both sides by q^n and taking summation we get,

$$\sum_{n\geq 0} p(N,M,n)q^n - \sum_{n\geq 0} p(N,M-1,n)q^n = \sum_{n\geq 0} p(N-1,M,n)q^n$$

$$= \sum_{n\geq 0} p(N-1,M,n)q^{n+M}$$

from above we get,

$$G(N,M;q) - G(N,M-1;q) = q^M G(N-1,M;q). \tag{2.1.4}$$

so from (2.1.3) and (2.1.4), we see that G satisfies (2.1.1) and (2.1.2),which proves the theorem.

2.2 Gaussian polynomial

The polynomial $G(N, M; q)$ appearing in Section 2.1 were first studied by Gauss [11] in 1863 and have come to be known as Gaussian polynomial.
In this section we first give definition of Gaussian polynomial and then give some useful properties of these polynomial.

Definition 2.2.1 Gaussian Polynomials [cf[9], Def. 3.1] are defined as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \left\{ \begin{array}{cc} \dfrac{(q;q)_n}{(q;q)_m \, (q;q)_{n-m}} & \text{if } 0 \leq \text{m} \leq \text{n} \\ 0 & \text{otherwise} \end{array} \right\}$$

Degree of gaussian polynomial is $m(n-m)$.

Properties of Gaussian polynomials:

Property 2.2.1 $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1$

Property 2.2.2 $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix}$

Property 2.2.3 $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$

Property 2.2.4 $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m} \begin{bmatrix} n-1 \\ m \end{bmatrix}$

Property 2.2.5 $\lim_{q \to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \begin{pmatrix} n \\ m \end{pmatrix}$

In the next section we give the proofs of properties (2.2.1) - (2.2.5).

2.3 Proof of Properties of Gaussian Polynomials

Proof of Property 2.2.1

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \frac{(q;q)_n}{(q;q)_0\,(q;q)_{n-0}} = 1$$

$$\begin{bmatrix} n \\ n \end{bmatrix} = \frac{(q;q)_n}{(q;q)_n\,(q;q)_{n-n}} = 1.$$

Proof of Property 2.2.2

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q;q)_n}{(q;q)_m\,(q;q)_{n-m}}$$

$$\begin{bmatrix} n \\ n-m \end{bmatrix} = \frac{(q;q)_n}{(q;q)_{n-m}\,(q;q)_{n-(n-m)}}$$

$$= \frac{(q;q)_n}{(q;q)_{n-m}\,(q;q)_m}$$

which proves the property.

Proof of Property 2.2.3

$$\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n-1 \\ m \end{bmatrix} = \frac{(q;q)_n}{(q;q)_m\,(q;q)_{n-m}} - \frac{(q;q)_{n-1}}{(q;q)_m\,(q;q)_{n-m-1}}$$

$$= \frac{(q;q)_{n-1}}{(q;q)_m\,(q;q)_{n-m}}(1-q^n) - \frac{(q;q)_{n-1}}{(q;q)_m(q;q)_{n-m-1}}$$

$$= \frac{(q;q)_{n-1}}{(q;q)_{m-1}\,(q;q)_{n-m}}(\frac{(1-q^n)}{(1-q^m)} - \frac{(1-q^{n-m})}{(1-q^m)})$$

$$= \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} (\frac{(q^{n-m} - q^n)}{(1-q^m)})$$

$$= (q^{n-m}) \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$$

which proves the property.

Proof of Property 2.2.4

$$\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = \frac{(q;q)_n}{(q;q)_m\,(q;q)_{n-m}} - \frac{(q;q)_{n-1}}{(q;q)_{m-1}\,(q;q)_{n-m}}$$

$$= \frac{(q;q)_{n-1}}{(q;q)_m\,(q;q)_{n-m-1}} (\frac{(1-q^n) - (1-q^m)}{(1-q^{n-m})})$$

$$= \begin{bmatrix} n-1 \\ m \end{bmatrix} \frac{(q^m - q^n)}{(1-q^{n-m})}$$

$$= q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}$$

which proves the property.

Proof of Property 2.2.5

$$\lim_{q\to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \lim_{q\to 1} \frac{(q;q)_n}{(q;q)_m\,(q;q)_{n-m}}$$

$$= \lim_{q\to 1} \frac{\frac{(q\ ;\ q)_n}{(1-q)^n}}{\frac{(q\ ;\ q)_m}{(1-q)^m}\ \frac{(q\ ;\ q)_{n-m}}{(1-q)^{n-m}}}$$

$$= \lim_{q\to 1} \frac{(1)(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})}{(1)(1+q)\cdots(1+q+\cdots+q^{m-1})(1)(1+q)\cdots(1+q+\cdots+q^{n-m-1})}$$

$$= \frac{(1.\ 2.\ 3.\ .\ .\ .\ .\ .\ .\ n)}{(1.\ 2.\ 3.\ .\ .\ .\ .\ .\ .\ m)\ (1.\ 2.\ 3.\ .\ .\ .\ .\ .\ .\ (n-m))}$$

$$= \frac{(n)!}{(m)!\ \ (n-m)!}$$

$$= \left(\begin{array}{c} n \\ m \end{array} \right)$$

which proves the property.

2.4 Main Theorems

Theorem 2.4.1 Prove That

$$(z;q)_N \ = \ \sum_{n=0}^{N} \left[\begin{array}{c} N \\ n \end{array} \right] (-1)^n \ (z)^n \ (q)^{n(n-1)/2}$$

Theorem 2.4.2 Prove That

$$\frac{1}{(z;q)_N} \ = \ \sum_{n=0}^{\infty} \left[\begin{array}{r} N+n-1 \\ n \end{array} \right] (z)^n$$

Theorem 2.4.3 Prove That

$$\sum_{j=0}^{m} (-1)^j \left[\begin{array}{r} m \\ j \end{array} \right] \ = \ \left\{ \begin{array}{cc} (q\ ;\ q^2)_n & \text{if m = 2n} \\ 0 & \text{ifm is odd} \end{array} \right\}$$

Theorem 2.4.4 Prove that

$$\left[\begin{array}{r} n+m+1 \\ m+1 \end{array} \right] \ = \ \sum_{j=0}^{n} q^j \left[\begin{array}{r} n+j \\ m \end{array} \right]$$

2.5 Proof of Main Theorems

Proof of Theorem 2.4.1

Put $t = zq^N$, and $a = \dfrac{1}{q^N}$ in Theorem 1.2.2, we get,

$$(z;q)_N = \frac{(z;q)_\infty}{(zq^N;q)_\infty} = \sum_{n=0}^{N} \frac{(\frac{1}{q^N};q)_n (zq^N)^n}{(q;q)_n}$$

Now,

$$\sum_{n=0}^{N} \frac{(q^{-N};q)_n}{(q;q)_n} (zq^N)^n = \sum_{n=0}^{N} \frac{(1-q^{-N})(1-q^{-N+1})\cdots(1-q^{-N+n-1})}{(q;q)_n} z^n q^{Nn}$$

$$= \sum_{n=0}^{N} \frac{(-1)^n (q^{-N}-1)(q^{-N+1}-1)\cdots(q^{-N+n+1}-1)}{(q;q)_n} z^n q^{Nn}$$

$$= \sum_{n=0}^{N} \frac{(-1)^n q^{-N}\ q^{-N+1}\cdots q^{-N+n-1}(1-q^N)(1-q^{N-1})\cdots(1-q^{N-n+1})}{(q;q)_n} z^n q^{Nn}$$

$$= \sum_{n=0}^{N} \frac{(-1)^n q^{-Nn+(1+2+\cdots(n-1))}(1-q^N)(1-q^{N-1})\cdots(1-q^{N-n+1})}{(q;q)_n} z^n q^{Nn}$$

$$= \sum_{n=0}^{N} \frac{(-1)^n q^{-Nn+n(n-1)/2}(1-q^N)(1-q^{N-1})\cdots(1-q^{N-n+1})}{(q;q)_n} z^n q^{Nn}$$

$$= \sum_{n=0}^{N} \frac{(-1)^n q^{n(n-1)/2}(1-q^N)(1-q^{N-1})\cdots(1-q^{N-n+1})}{(q;q)_n} z^n$$

$$= \sum_{n=0}^{N} \frac{(-1)^n q^{n(n-1)/2}(q;q)_{N-n}(1-q^N)(1-q^{N-1})\cdots(1-q^{N-n+1})}{(q;q)_n (q;q)_{N-n}} z^n$$

$$= \sum_{n=0}^{N} \frac{(-1)^n q^{n(n-1)/2}(q;q)_N}{(q;q)_n (q;q)_{N-n}} z^n$$

$$= \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} (-1)^n q^{n(n-1)/2} z^n$$

which proves the theorem.

Proof of Theorem 2.4.2

$$\frac{1}{(a;q)_n} = \frac{(aq^n;q)_\infty}{(a;q)_\infty}$$

Put $t = z$, and $a = q^N$ in above and use theorem 1.2.2 we get,

$$\frac{1}{(z;q)_N} = \sum_{n=0}^{\infty} \frac{(q^N;q)_n}{(q;q)_n} z^n$$

$$= \sum_{n=0}^{\infty} \frac{(1-q^N)(1-q^{N+1})\cdots(1-q^{N+n-1})}{(q;q)_n} z^n$$

$$= \sum_{n=0}^{\infty} \frac{(q;q)_{N-1}(1-q^N)(1-q^{N+1})\cdots(1-q^{N+n-1})}{(q;q)_n(q;q)_{N-1}} z^n$$

$$= \sum_{n=0}^{\infty} \frac{(q;q)_{N+n-1}}{(q;q)_n(q;q)_{N-1}} z^n$$

$$= \sum_{n=0}^{\infty} \begin{bmatrix} N+n-1 \\ n \end{bmatrix} (z)^n$$

which proves the theorem.

Proof of Theorem 2.4.3

Let

$$f(m) = \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}$$

Multiply both sides by $\frac{z^m}{(q;q)_m}$ and summing we get,

$$\sum_{m=0}^{\infty} \frac{f(m) z^m}{(q\ ;\ q)_m} = \sum_{m=0}^{\infty} \frac{z^m}{(q\ ;q)_m} \sum_{j=0}^{m} \frac{(-1)^j (q\ ;\ q)_m}{(q\ ;\ q)_j (q\ ;\ q)_{m-j}}$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{(-1)^j z^m}{(q\ ;\ q)_j (q\ ;\ q)_{m-j}}$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j z^{m+j}}{(q\ ;\ q)_j (q\ ;\ q)_m}$$

$$= \sum_{m=0}^{\infty} \frac{z^m}{(q\ ;\ q)_m} \sum_{j=0}^{\infty} \frac{(-z)^j}{(q\ ;\ q)_j}$$

$$= \frac{1}{(z\ ;\ q)_\infty} \times \frac{1}{(-z\ ;\ q)_\infty}$$

$$= \frac{1}{(1-z)(1-zq)\cdots} \frac{1}{(1+z)(1+zq)\cdots}$$

$$= \frac{1}{(1-z^2)(1-z^2q^2)\cdots}$$

$$= \frac{1}{(z^2\ ;\ q^2)_\infty}$$

so

$$\sum_{m=0}^{\infty} \frac{f(m) z^m}{(q\ ;\ q)_m} = \frac{1}{(z^2\ ;\ q^2)_\infty} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(z^2\ ;\ q^2)_\infty} \qquad (By\ Corollary\ 1.3.1)$$

now when m is odd $f(m) = 0$, so on comparing the coefficients we get,

$$\frac{f(m)}{(q\ ;\ q)_m} = \frac{1}{(q^2\ ;\ q^2)_n}$$

therefore ,

$$f(m) = \frac{(q\ ;\ q)_m}{(q^2\ ;\ q^2)_n} \qquad (where\ \ m = 2n)$$

$$= \frac{(q\ ;\ q^2)_n (q^2\ ;\ q^2)_n}{(q^2\ ;\ q^2)_n}$$

$$= (q\ ;\ q^2)_n$$

Proof of Theorem 2.4.4

We will prove this theorem using induction on n,

if $n = 0$, left hand side $= \begin{bmatrix} m+1 \\ m+1 \end{bmatrix} = 1$

Right hand side $= \begin{bmatrix} n \\ n \end{bmatrix} = 1,$

so result holds for $n = 0$.

Assume that it is true for $n = 0, 1, 2, ..., n$.

We shall prove that it is true for $n + 1$

$$\begin{bmatrix} (n+1)+m+1 \\ m+1 \end{bmatrix} = \begin{bmatrix} n+m+2 \\ m+1 \end{bmatrix}$$

$$= \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix} + q^{n+1} \begin{bmatrix} n+m+1 \\ m \end{bmatrix} \qquad (using\ \ Property\ \ 2.2.3)$$

$$= \sum_{j=0}^{n} q^j \begin{bmatrix} n+j \\ m \end{bmatrix} + q^{n+1} \begin{bmatrix} (n+1)+m \\ m \end{bmatrix}$$

$$= \sum_{j=0}^{n+1} q^j \begin{bmatrix} n+j \\ m \end{bmatrix}.$$

Theorem 2.4.3 and Theorem 2.4.4 is due to Gauss [11] and was used by him in the treatment of Gaussian sums.

CHAPTER 3

Interpretations of q-series involving Gaussian Polynomials

3.1.Introduction

In this chapter, we give combinatorial interpretation of certain q-series involving Gaussian polynomial.

Some combinatorial interpretation of q-series using these polynomial are given in [3] and [4].

3.2 Main theorems

Theorem 3.2.1 Let $A_1(m, n, N)$ denote the number of partitions of N into exactly m parts where parts are $\leq n - m$ and differ by atleast 2 then

$$A_{m,n}(q) \ = \ \sum_{N=0}^{u} A_1(m, n, N) q^N$$

where $u \ = \ mn$ and $A_{m,n}(q) \ = \ q^{m^2} \begin{bmatrix} n \\ m \end{bmatrix}_q$.

Theorem 3.2.2 Let $A_2(m, n, N)$ denote the number of partitions of N into exactly m parts where parts are $\leq 2n - m + 1$ and differ by atleast 2 then

$$B_{m,n}(q) \ = \ \sum_{N=0}^{v} A_2(m, n, N) q^N$$

where $v \ = \ m(2n + 1)$ and $B_{m,n}(q) \ = \ q^{m^2} \begin{bmatrix} 2n + 1 \\ m \end{bmatrix}_q$.

Theorem 3.2.3 Let $A_3(m,n,N)$ denote the number of partitions of N into exactly m parts where parts are $\leq n-m$ and differ by atleast 2 then

$$C_{m,n}(q) \;=\; \sum_{N=0}^{w} A_3(m,n,N)q^N$$

where $w \;=\; n(2m+1)-m(m+1)$ and $C_{m,n}(q) \;=\; q^{m^2}\begin{bmatrix} n+m+1 \\ 2m+1 \end{bmatrix}_q$.

Theorem 3.2.4 Let $A_4(m,n,N)$ denote the number of partitions of N into exactly $m-1$ parts where parts are $\leq 2n-m$ and are distinct then

$$D_{m,n}(q) \;=\; \sum_{N=0}^{x} A_4(m,n,N)q^N$$

where $x \;=\; \binom{m}{2} + 2(m-1)(n-m)+(m-1)$

and $D_{m,n}(q) \;=\; q^{\binom{m}{2}}\begin{bmatrix} 2n-m \\ m-1 \end{bmatrix}_q$.

Theorem 3.2.5 Let$A_5(m,n,N)$ denote the number of partitions of N into $m-1$ distinct parts , where the value of each part is less than equal to $2n-m$,or the number of partitions of N into m distinct parts where each part has a value which is less than or equal to $2n-m+1$ then

$$E_{n,m}(q) \;=\; \sum_{N=0}^{y} A_5(m,n,N)q^N$$

where y $= 2nm-3\binom{m}{2}$

and

$$E_{n,m}(q) \;=\; \begin{bmatrix} 2n-m \\ m \end{bmatrix} q^{\binom{m}{2}} + (1+q^m)\begin{bmatrix} 2n-m \\ m-1 \end{bmatrix} q^{2n-2m+1+\binom{m}{2}}$$

3.3 Proofs of Main Theorems

Proof of Theorem 3.2.1

By definition of gaussian polynomial

$\begin{bmatrix} n \\ m \end{bmatrix}_q$ generates partitions into atmost m parts where parts are $\leq n-m$.

Multiplication of $\begin{bmatrix} n \\ m \end{bmatrix}_q$ by $q^{m^2} = q^{1+3+5+\cdots+(2m-1)}$ means that we are adding $2m-1$ to the largest part, $2m-3$ to next largest part, $\cdots$ and 1 to smallest part.

Since the largest part is less than or equal to $n-m+(2m-1) = n+m-1$.

So, $q^{m^2}\begin{bmatrix} n \\ m \end{bmatrix}_q$ generates the partition into exactly m parts where parts are less than equal to $n+m-1$ and differ by atleast 2.

and the degree of $A_{m,n}(q)$ is $m^2 + m(n-m) = nm$.

Proof of Theorem 3.2.2

By definition of gaussian polynomial

$\begin{bmatrix} 2n+1 \\ m \end{bmatrix}_q$ generates partitions into atmost m parts where parts are $\leq 2n-m+1$.

Multiplication of $\begin{bmatrix} 2n+1 \\ m \end{bmatrix}_q$ by $q^{m^2} = q^{1+3+5+\cdots+(2m-1)}$ means that we are adding $2m-1$ to the largest part, $2m-3$ to next largest part,$\cdots$ and 1 to smallest part.

Since the largest part is less than or equal to $2n-m+1+(2m-1) = 2n+m$.

So, $q^{m^2}\begin{bmatrix} 2n+1 \\ m \end{bmatrix}_q$ generates the partition into exactly m parts where parts are less than equal to $2n+m$ and parts differ by atleast 2.

and the degree of $B_{m,n}(q)$ is $m^2 + m(2n+1-m) = m(2n+1)$.

Proof of Theorem 3.2.3

By definition of gaussian polynomial

$\begin{bmatrix} n+m+1 \\ 2m+1 \end{bmatrix}_q$ generates partitions into atmost $2m+1$ parts where parts are $\leq n-m$.

Multiplication of $\begin{bmatrix} n+m+1 \\ 2m+1 \end{bmatrix}_q$ by $q^{m^2} = q^{1+3+5+\cdots+(2m-1)}$ means that we are adding $2m-1$ to the largest part, $2m-3$ to next largest part, $\cdots$ and 1 to smallest part.

Since the largest part is less than or equal to $n-m+(2m-1) = n+m-1$.

So, $q^{m^2}\begin{bmatrix} n+m+1 \\ 2m+1 \end{bmatrix}_q$ generates the partition into atmost$2m+1$ parts where parts are less than equal to $n+m-1$ and first m parts differ by atleast 2.

and the degree of $C_{m,n}(q)$ is $m^2+(2m+1)((n+m+1)-(2m+1)) = m^2+(2m+1)(n-m) = n(2m+1)-m(m+1)$.

Proof of Theorem 3.2.4

By definition of gaussian polynomial

$\begin{bmatrix} 2n-m \\ m-1 \end{bmatrix}_q$ generates partitions into atmost $m-1$ parts where parts are$\leq 2n-2m+1$.

Multiplication of $\begin{bmatrix} 2n-m \\ m-1 \end{bmatrix}_q$ by $q^{\binom{m}{2}} = q^{1+2+3+\cdots+(m-1)}$ means that we are adding $m-1$ to the largest part, $m-2$ to next largest part,$\cdots$ and 1 to smallest part.

Since the largest part is less than or equal to $2n-2m+1+(m-1) = 2n-m$.

So, $q^{\binom{m}{2}}\begin{bmatrix} n+m+1 \\ 2m+1 \end{bmatrix}_q$ generates the partition into exactly $m-1$ parts where parts are less than equal to $2n-m$ and all parts are distinct.

and the degree of $D_{m,n}(q)$ is $\begin{pmatrix} m \\ 2 \end{pmatrix} + (m-1)((2n-m)-(m-1)) = \begin{pmatrix} m \\ 2 \end{pmatrix} +$ $2(m-1)(n-m)+(m-1)$.

Proof of Theorem 3.2.5

$$E_{n,m}(q) = (\begin{bmatrix} 2n-m \\ m \end{bmatrix} + \begin{bmatrix} 2n-m \\ m-1 \end{bmatrix} q^{2n-2m+1}) q^{\begin{pmatrix} m \\ 2 \end{pmatrix}} + q^m \begin{bmatrix} 2n-m \\ m \end{bmatrix} q^{2n-2m+1+\begin{pmatrix} m \\ 2 \end{pmatrix}}$$

$$= (\begin{bmatrix} 2n-m \\ m \end{bmatrix} + \begin{bmatrix} 2n-m \\ m-1 \end{bmatrix} q^{2n-2m+1}) q^{\begin{pmatrix} m \\ 2 \end{pmatrix}} + \begin{bmatrix} 2n-m \\ m-1 \end{bmatrix} q^{2n-2m+1+m+\begin{pmatrix} m \\ 2 \end{pmatrix}}.$$

Now using property 2.2.3 by replacing n by $2n-m+1$, we get

$$= \begin{bmatrix} 2n-m+1 \\ m \end{bmatrix} q^{\begin{pmatrix} m \\ 2 \end{pmatrix}} + (\begin{bmatrix} 2n-m+1 \\ m \end{bmatrix} - \begin{bmatrix} 2n-m \\ m \end{bmatrix}) q^{\begin{pmatrix} m+1 \\ 2 \end{pmatrix}}$$

Since $E_{n,m}(q)$ is a polynomial of degree $\begin{bmatrix} 2n-m+1 \\ m \end{bmatrix} q^{\begin{pmatrix} m+1 \\ 2 \end{pmatrix}}$

which is $2nm - 3\begin{pmatrix} m \\ 2 \end{pmatrix}$.

3.4 Conclusion

The relevance of q-series identities in combinatorial problem has long been recognized such as in [8], [14], [3], [5], [6] and [10].

In particular, putting restriction on partition function makes the result more intersting. For example, the famous result of Euler, "the number of distinct partitions of n is equal to the number of odd partitions of n".

In this thesis we studied a particular restriction defined by $p(n, m, N)$ and discussed some results.

So, it is obvious and one's curiosty to study various other restricted partition function such as n-colour restricted partition function [4], restricted Frobenum partitions [13] and Weighed lattice paths [5].

References

[**1**] Agarwal A.K., "Properties of Recurring Sequence." The Fibonacci Quarterly, vol. 27(2), (1989), pp 169 - 175.

[**2**] Agarwal A.K., "The fibonacci Quarterly." vol. 29(2), (1991), pp 137 - 140.

[**3**] Agarwal A.K., "Identities and generating function for certain classes of F-partitions", ARS combinatoria, vol. 57, (2000), pp 65 - 75.

[**4**] Agarwal A.K., "n-colour Analogues of Gaussian Polynomial", ARS combinatoria, vol. 61, (2001), pp 97 - 117.

[**5**] Agarwal A.K. and Bressoud D.M., "Lattice paths and multiple basic hypergeometric series", Pacific J. Math., vol. 136(2), (1989), pp 209 - 228.

[**6**] Agarwal A.K. and Subbarao M.V., "Some Properties of Perfect partitions", Indian J. Pure Appl. Math., vol. 22(9), (1991), pp 737 - 743.

[**7**] Agarwal A.K., Padmavathamma, Subbarao M.V., "Partition Theory", Atma Ram and Sons, Chandigarh, India, (2005).

[**8**] Andrews G.E., "Enumerative proofs of certain q-identities," Glasgon Math. J., vol. 8, (1967), pp 33 - 40 .

[**9**] Andrews G.E., "The Theory of partition", Encyclopedia of mathematics and its applications, Addison-Wesley, Comb. Univ. Press, USA, (1986).

[**10**] Baxter R.J., "Exactly Solved Models in Statistical Mechanics," Academic Press, London, (1982).

[**11**] Gauss, C.F., Königliche Gesellschaft der Wissenschaften, Werke vol. 2, Göttingen. (1863).

[**12**] Padmavathamma, "The Hardey-Ramanujan-Rademacher Expansion for Restricted Generalized Frobenum Partitions", J. Ramanujan Math. Soc., vol. 3(2), (1988), pp 161 - 180.

[**13**] Padmavathamma, "Restricted Generalized Frobenum Partitions", Indian J. Pure and Appl. Math., vol. 19(1), (1988), pp 11 - 16.

[**14**] Subbarao, M.V., "Some Roger-Ramanujan type Partition theorems", Pacific J. Math., vol. 120, (1985), pp 431 - 435.